Course Name: Math for ML
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Assignment 1

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Question 1. *To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?*

My Answer 1.

A: event that someone is known to have Hansen's disease.

B: event that someone is known to have no Hansen's desease.

X: event that someone has positive result with Hansen's desease.

P(A) = 0.05

P(B) = 0.095

P(H/A) = 0.098

P(H/B) = 0.03

So, he probability that someone testing positive for Hansen's disease under this new test actually is:

$$P(A/H) = \frac{P(A) * P(H/A)}{P(H)}$$

$$P(A/H) = \frac{P(A) * P(H/A)}{P(H/A) * P(A) + P(H/B) * P(B)}$$

$$P(A/H) = \frac{0.098 * 0.05}{0.098 * 0.05 + 0.03 * 0.95}$$

$$P(A/H) \approx 0.632$$

Question 2. *Proof the following distributions are normalized then calculate the mean and standard deviation of these distribution:*

1. *Univariate normal distribution.*
2. *(Optional) Multivariate normal distribution.*

My Answer 2.

To prove that the univariate Gaussian distribution is normalized, we should show that it is normalized for a zero-mean Gaussian and extend that result to show that $\mathcal{N}(x|\mu, \sigma^2)$ is normalized.

The formula Gaussian distribution is given by:

$$\varphi(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}x^2\right) \quad -\infty < x < \infty. \tag{1}$$

To prove that the above expression is normalized, we have to show that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx = \sqrt{2\pi\sigma^2} \tag{2}$$

Proof. Let

$$I = \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2\right) dx \tag{3}$$

Squaring the above expression,

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}y^2\right) dx\, dy$$

To integrate this expression we make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by:

$$x = r\, cos\, \theta \tag{4}$$
$$y = r\, sin\, \theta \tag{5}$$

and using the trigonometric identity $cos^2\, \theta + sin^2\, \theta = 1$, we have $x^2 + y^2 = r^2$. Also the Jacobian of the change of variables is given by,

$$\begin{aligned}
\frac{\partial\,(x, y)}{\partial\,(r, \theta)} &= \begin{vmatrix} \frac{\partial\,(x)}{\partial\,(r)} & \frac{\partial\,(x)}{\partial\,(\theta)} \\ \frac{\partial\,(y)}{\partial\,(r)} & \frac{\partial\,(y)}{\partial\,(\theta)} \end{vmatrix} \\
&= \begin{vmatrix} cos\, \theta & -r\, sin\, \theta \\ sin\, \theta & r\, cos\, \theta \end{vmatrix} \\
&= r\, cos^2\, \theta + r\, sin^2\, \theta \\
&= r
\end{aligned}$$

using the same trigonometric identity $cos^2\, \theta + sin^2\, \theta = 1$. Thus equation (4) can be rewritten as

$$I^2 = \int_0^{2\pi}\int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\, dr\, d\theta \tag{6}$$
$$= 2\pi \int_0^{\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r\, dr \tag{7}$$
$$= 2\pi \int_0^{\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2}\, du \tag{8}$$
$$= \pi \left[\exp\left(-\frac{u}{2\sigma^2}\right) \left(-2\sigma^2\right)\right]_0^{\infty} \tag{9}$$
$$= 2\pi\sigma^2 \tag{10}$$

where we have used the change of variables $r^2 = u$. Thus

$$I = \left(2\pi\sigma^2\right)^{1/2}.$$

Finally to prove that $\mathcal{N}(\mathrm{x}|\mu,\, \sigma^2)$ is normalized, we make the tranformation $y = x - \mu$ so that,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathcal{N}(x \mid \mu, \sigma^2)\, dx &= \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} \int_{-\infty}^{\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy \\
&= \frac{I}{\left(2\pi\sigma^2\right)^{1/2}} \\
&= 1
\end{aligned}$$

as required.

My Answer 3.

CALCULATING VARIANCE

The normalization condition of the univariate Gaussian distribution is given by:

$$\int_{-\infty}^{\infty} \mathcal{N}\left(x \mid \mu, \sigma^2\right) dx = 1$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx = 1$$

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx = \left(2\pi\sigma^2\right)^{1/2}$$

Differentiating both sides with respect to σ^2,

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left\{\frac{\left(x-\mu\right)^2}{2}\right\}\left\{\frac{1}{\left(\sigma^2\right)^2}\right\} dx = \left(\frac{1}{2}\right)\left(2\pi\sigma^2\right)^{-1/2}\left(2\pi\right)$$

$$\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(x-\mu\right)^2 dx = \sigma^2\sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}\left(x-\mu\right)^2 dx = \sigma^2$$

It follows directly that

$$\mathbb{E}\left[\left(x-\mu\right)^2\right] = Varr\left[x\right] = \sigma^2$$

My Answer 4.

CALCULATING MEAN

The mean of the univariate Gaussian distribution is given by:

$$\mathbb{E}\left[x\right] = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} x\, dx$$

Changing the variables using $y = x - \mu$, the above expression becomes

$$\begin{aligned}\mathbb{E}\left[x\right] &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right)\left(y+\mu\right) dy \\ &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) y\, dy \\ &\quad + \int_{-\infty}^{\infty}\left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) \mu\, dy\end{aligned}$$

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) y\,dy \\
= & \int_{-\infty}^{0} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) y\,dy \\
& + \int_{0}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) y\,dy \\
= & \int_{\infty}^{0} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \left(\frac{1}{2}\right) \exp\left(-\frac{1}{2\sigma^2}u\right) du \\
& + \int_{0}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \left(\frac{1}{2}\right) \exp\left(-\frac{1}{2\sigma^2}u\right) du \\
= & \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \left(\frac{1}{2}\right) (-2\sigma^2) \\
& \left\{\left[\exp\left(-\frac{1}{2\sigma^2}u\right)\right]_{\infty}^{0} + \left[\exp\left(-\frac{1}{2\sigma^2}u\right)\right]_{0}^{\infty}\right\} \\
= & \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \left(\frac{1}{2}\right) (-2\sigma^2) \left\{\left[\frac{1}{e^0} - \frac{1}{e^\infty}\right] + \left[\frac{1}{e^\infty} - \frac{1}{e^0}\right]\right\} \\
= & \ 0
\end{aligned}$$

where we have used the change of variables $y^2 = u$ and hence $2y\,dy = du$.
So, we have:

$$\begin{aligned}
\mathbb{E}[x] = & \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) (y+\mu)\,dy \\
= & \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) y\,dy \\
& + \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) \mu\,dy \\
= & \ 0 + \mu * \int_{-\infty}^{\infty} \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left(-\frac{1}{2\sigma^2}y^2\right) dy \\
= & \ \mu
\end{aligned}$$