

## Week2

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### 1

$$p_{i|j} = \frac{\exp(-\|x_i - x_j\|^2 / 2\sigma^2)}{\sum_{k \neq i} \exp(-\|x_i - x_k\|^2 / 2\sigma^2)}$$

$$q_{j|i} = \frac{\exp(-\|y_i - y_j\|^2)}{\sum_{k \neq i} \exp(-\|y_i - y_k\|^2)}$$

with  $p_{i|i} = q_{i|i} = 0$

**The Loss:**

We have:

$$p_{i|j} = q_{j|i} = \frac{(1 + \|y_i - y_j\|)^{-1}}{\sum_{k, l \neq k} (1 + \|y_i - y_k\|)^{-1}} = \frac{E_{ij}^{-1}}{\sum_{k, l \neq k} E_{kl}^{-1}} = \frac{E_{ij}^{-1}}{Z}$$

Notice that  $E_{ij} = E_{ji}$ . The loss function is defined as

$$C = KL(P||Q) = \sum_i \sum_j p_{ij} \log \frac{p_{ji}}{q_{ji}}$$

$$\sum_{k, l \neq k} p_{lk} \log \frac{p_{ji}}{q_{ji}} = \sum_{k, l \neq k} p_{lk} \log p_{lk} - p_{lk} \log q_{lk} = \sum_{k, l \neq k} p_{lk} \log p_{lk} - p_{lk} \log E_{kl}^{-1} + p_{lk} \log Z$$

Drive with respect to  $y_i$

$$\frac{\delta C}{\delta y_i} = \sum_{k, l \neq k} -p_{lk} \log E_{kl}^{-1} + \sum_{k, l \neq k} p_{lk} \log Z$$

$$\sum_{k, l \neq k} -p_{lk} \frac{\delta C \log E_{kl}^{-1}}{\delta C y_i} = -2 \sum_{j \neq i} -p_{ji} \frac{\delta C \log E_{ij}^{-1}}{\delta C y_i}$$

Since  $\frac{\delta C \log E_{ij}^{-1}}{\delta C y_i} = E_{ij}^{-2}(-2(y_i - y_j))$ , then

$$-2 \sum_{j \neq i} -p_{ji} \frac{E_{ij}^{-2}}{E_{ij}^{-1}} (-2(y_i - y_j)) = 4 \sum_{j \neq i} -p_{ji} E_{ij}^{-1} (y_i - y_j)$$

$$\sum_{k, l \neq k} p_{kl} \frac{\delta \log Z}{\delta y_i} = \frac{1}{Z} \sum_{k', l' \neq k'} \frac{\delta E_{kl}^{-1}}{\delta y_i} = 2 \sum_{j \neq i} \frac{\delta E_{ji}^{-2}}{Z} (-2(y_j - y_i)) = -4 \sum_{j \neq i} q_{ij} E_{ji}^{-1} (y_i - y_j)$$

So:

$$\frac{\delta C}{\delta y_i} = 4 \sum_j (p_{ji} - q_{ji}) E_{ji}^{-1} (y_i - y_j)$$

$$\frac{\delta C}{\delta y_i} = 4 \sum_j (p_{ji} - q_{ji}) \left(1 + \|y_i - y_j\|^2\right)^{-1} (y_i - y_j)$$