Math 5825: Advanced Linear Algebra
Homework 1

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Problem A. Recall that for each prime p and $n \in \mathbb{N}$ there is a unique (up to isomorphism) field of order p^n. Construct a field with 4 elements. Give the tables for addition and multiplication. That is, pick four elements and make it into a field.
Solution:
A field with 4 elements, say $F = \{G = \{0, 1, \alpha, \beta\}, +, \cdot\}$ is a group with 4 elements under addition i.e $(G, +)$ and a group with 3 elements under multiplication i.e $(G - \{0\}, \cdot)$. A group with four elements is isomorphic to either $\mathbb{Z}_4$ or K_4. $(\mathbb{Z}_4, +)$ forms a group but $(\mathbb{Z}_4 - 0, \cdot)$ does not. So, in our example, $(G, +) \cong K_4$.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

Now, let define multiplication to form the group $(G - \{0\}, \cdot)$. Since the order of the group (non-trivial) is 3, $|\alpha| = |\beta| = 3$. So, $\alpha \cdot \alpha = \beta$, $\beta \cdot \beta = \alpha$ and $\alpha \cdot \beta = \beta \cdot \alpha = 1$.

$\cdot$	1	α	β
1	1	α	β
α	α	β	1
β	β	1	α

Problem B. Let F be a field. Let $(V, +, \cdot)$ be a vector space over F. Prove that $(-1) \cdot v = -v$ for all $v \in V$.
Proof:
Let $v \in V$. Consider the following:

$$v + (-1) \cdot v$$

$$= (1 + (-1)) \cdot v \text{ [Distributive axiom]}$$

$$= 0 \cdot v = \vec{0} \text{ [Theorem 1.5i]}$$

. Since $v + (-1) \cdot v = \vec{0}, (-1) \cdot v$ is the multiplicative inverse of v (denoted by $-v$). Hence, $(-1) \cdot v = -v$.

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Problem C. Let F be a field. Let V be a vector space over F. Prove that $0v = \vec{0}$ for all $v \in V$. Here 0 is the additive identity in F and $\vec{0}$ is the zero vector in V.
Proof:
Since 0 is an additive identity in F, $0 + 0 = 0$. This gives

$$(0+0)v = 0v$$

$$\Rightarrow 0v + 0v = 0v$$

Adding $-(0v)$ on both sides gives

$$0v + 0v - (0v) = 0v - (0v)$$

$$\Rightarrow 0v + (0v - 0v) = (0v - 0v)$$

$$\Rightarrow 0v + \vec{0} = \vec{0}.$$

$$\text{Hence, } 0v = \vec{0}.$$

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Problem D. Let V be a vector space. Prove that $-(-v) = v$ for all $v \in V$.
Proof:
Let $v \in V$. By the existence of the additive inverse, $-v \in V$. Now, using the conclusion from Problem B i.e. $(-1) \cdot v = -v$ for all $v \in V$, we get

$$-(-v) = (-1) \cdot (-v) = (-1)((-1)(v))$$

$$= (-1)(-1) \cdot v \text{ [Associative axiom]}$$

$$= 1 \cdot v = v.$$

Hence, $-(-v) = v$ for all $v \in V$.

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Problem E. In details describe how the set of $n \times m$ matrices with entries from $\mathbb{R}$ is a vector space over $\mathbb{Q}$.
Solution:
Let $A = [a_{ij}]_{m\times n},\ B = [b_{ij}]_{m\times n},\ C = [c_{ij}]_{m\times n}$ be $m \times n$ matrices with entries from $\mathbb{R}$. The following properties are inherited by matrices (with real entries) from the field $\mathbb{R}$.
(A1) $A + B = B + A$
(A2) $A + (B + C) = (A + B) + C$
(A3) $A + 0 = A$ where 0 is the $m \times n$ null matrix. (A4) For a matrix A, there exists its additive inverse $-A$ such that $A + (-A) = 0$.
(M1) For a scalar $a \in \mathbb{Q}$, $a(B + C) = aB + aC$.
(M2) For every pair of scalars $a, b \in \mathbb{Q}$, $(a + b)C = aC + bC$.
(M3) For every pair of scalars $a, b \in \mathbb{Q}$, $(ab)C = a(bC)$.
(M4) $1A = A$
Hence, the set of $n \times m$ matrices with entries from $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

Problem F. Let V be the set of all degree 2 polynomials with coefficients from $\mathbb{R}$. Why isn't V a vector space over $\mathbb{R}$? (with the obvious operations). How would you fix this so that V becomes a vector space?
Solution:
Let V be the set of all degree 2 polynomials with coefficients from $\mathbb{R}$. Consider two polynomials $p_1(x) = ax^2 + bx + c$ and $p_2(x) = -ax^2 + dx + e$.
Now, $(p_1 + p_2)(x) = (ax^2 + bx + c) + (-ax^2 + dx + e) = (b + d)x + (c + e)$ which is of degree 1. So, $(p_1 + p_2)(x) \notin V$. Hence, V is not a vector space over $\mathbb{R}$.
It could be fixed by redefining V as the set of all polynomials of degree 2 or less with coefficients from $\mathbb{R}$.

Problem H. Recall the definition of linear independence from class. Prove that this definition is equivalent to the one in the book.
Proof:
Let V be a vector space over F.
Definition from class: If set of vectors $\{v_1, v_2, ..., v_n\} \subseteq V$ is linearly independent, then $c_1v_1 + c_2v_2 + ... + c_nv_n = 0 \Rightarrow c_1 = c_2 = ... = c_n = 0$.
Definition from book: A finite sequence of vectors, $(v_1, v_2, ..., v_k)$ from a vector space V is linearly dependent if there are scalars $c_1, c_2, ..., c_k$,not all zero, such that $c_1v_1 + c_2v_2 + ... + c_kv_k = 0$. The sequence of vector is linearly independent if it is not linearly dependent.
Let's take the definition of linear dependence from book and negate it which yields if the vectors $\{v_1, v_2, ..., v_n\}$ are linearly independent, then $c_1v_1 + c_2v_2 + ... + c_kv_k = 0 \Rightarrow c_1 = c_2 = ... = c_n = 0$.

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Problem I. (This problem reviews concepts that you should already be familiar with.) Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{pmatrix}.$$

(I1) Find the RREF of A.
Solution:

$$\text{A} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow{R_2 - R_1 \to \text{R}_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -3 \\ 0 & 2 & -2 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3 \to \text{R}_3]{\frac{1}{3}R_2 \to \text{R}_2} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\xrightarrow[R_3 - R_2 \to \text{R}_3]{R_1 + R_2 \to \text{R}_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \text{RREF}(A).$$

(I2) Find a basis for the null space of A.
Solution: The basis of null space of A is equal to the basis of null space of RREF(A).
Consider the following:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives:

$$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3.$$
$$x_2 - x_3 = 0 \Rightarrow x_2 = x_3.$$

Let $x_3 = s$ (free variable) where $s \in \mathbb{R}$. Then, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s \\ s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

So, Basis for null space: $\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(I3) Find a basis for the column space of A.
Solution: The basis of column space of A is equal to the basis of column space of RREF(A).
Basis for column space: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \right\}$.

(I4) Find a basis for the row space of A.
Answer: The non-zero rows of RREF(A) are linearly independent.
So, Basis for the row space of A: $\{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}\}$.

(I5) What is the rank of A?
Answer: There are two non-zero (i.e. linearly independent) rows in RREF(A).
So, rank of A is 2.

(I6) What is the nullity of A?
Answer: Nullity, or the dimension of the null space, of A is 1.

Problem J. Let W be a subspace of V (finite dimensional). Prove that $\dim(V/W) = \dim(V) - \dim(W)$.
Proof:
Let the basis of W, with dimension k, be $\{v_1, v_2, ..., v_k\}$. Let the dimension of V be n. The basis of W can be extended to form the basis of V. Let the basis of V be $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$.
Since $V/W = \{v + W | v \in V\}$, we have a following claims.

Claim 1: $v_{k+1} + W, v_{k+2} + W, ..., v_n + W\}$ spans V/W.

Let $v + W \in V/W$ where $v \in V \Rightarrow v \in Span\{v_1, v_2, ..., v_n\} \Rightarrow v = a_1v_1 + a_2v_2 + ... + a_nv_n$ where a_i's are scalars for all i's.
Now, $v + W = a_1v_1 + a_2v_2 + ... + a_nv_n + W =$

$$\sum_{i=1}^{n} a_i v_i + W = \sum_{i=1}^{n} a_i(v_i + W)$$

. Here, $v_i \in W$ for $1 \leq i \leq k \Rightarrow v_i - 0 \in W \Rightarrow v_i + W = 0 + W$ for $1 \leq i \leq k$. So,

$$v + W = \sum_{i=1}^{k} a_i(v_i + W) + \sum_{i=k+1}^{n} a_i(v_i + W)$$

$$= \sum_{i=1}^{k} a_i(0 + W) + \sum_{i=k+1}^{n} a_i(v_i + W)$$

$$= \sum_{i=1}^{k}(0+W) + \sum_{i=k+1}^{n} a_i(v_i + W)$$

$$= \sum_{i=k+1}^{n} a_i(v_i + W)$$

. Hence, Claim 1 is true.

Claim 2: $v_{k+1}+W, v_{k+2}+W, ..., v_n+W\}$ is linearly independent.
Let, for scalars b_i's for all i's,

$$b_{k+1}(v_{k+1}+W) + b_{k+2}(v_{k+2}+W) + ... + b_n(v_n+W) = 0 + W$$

$$b_{k+1}v_{k+1} + W + b_{k+2}v_{k+2} + W + ... + b_nv_n + W = 0 + W$$

$$\Rightarrow (b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ... + b_nv_n) + W = 0 + W$$

$$\Rightarrow (b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ... + b_nv_n) - 0 \in W$$

$$\Rightarrow (b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ... + b_nv_n) \in W = Span\{v_1, v_2, ..., v_k\}$$

$$\Rightarrow b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ... + b_nv_n = b_1v_1 + b_2v_2 + ... + b_nv_n$$

$$\Rightarrow b_1v_1 + b_2v_2 + ... + b_nv_n - b_{k+1}v_{k+1} - b_{k+2}v_{k+2} - ... - b_nv_n = 0$$

Since $\{v_1, v_2, ..., v_k, v_{k+1}, ..., v_n\}$ is basis of V,

$$\_1v_1 + b_2v_2 + ... + b_nv_n - b_{k+1}v_{k+1} - b_{k+2}v_{k+2} - ... - b_nv_n = 0 \Rightarrow b_i = 0 \text{ for } 1 \le i \le n.$$

This gives $b_{k+1} = b_{k+2} = ... = b_n = 0$. So, Claim 2 is also true.
Since claim 1 and claim 2 are both true, $v_{k+1}+W, v_{k+2}+W, ..., v_n+W\}$ is indeed a basis of V/W which has dimension $n-k$. Hence, $\dim(V/W) = \dim(V) - \dim(W)$.

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Problem K. Consider the subspace $W = span\{x^2+x^3, 1+x^4\}$ of $V = \mathbb{R}_4[x]$. Find a basis for V/W.
Solution:
$1+x^4$ cannot be expressed as a scalar multiple of x^2+x^3 which implies they are linearly independent. So, $\{x^2+x^3, 1+x^4\}$ is the basis of W. The basis of W can be extended to form the basis of $V = \mathbb{R}_4[x]$.
Claim: $\{1, x, x^2, x^2+x^3, 1+x^4\}$ is the basis of $V = \mathbb{R}_4[x]$. This is because $a \cdot 1 + b \cdot x + c \cdot x^2 + d \cdot (x^2+x^3) + e \cdot (1+x^4) = 0 \Rightarrow (a+e) + bx + (c+d)x^2 + dx^3 + ex^4 = 0 \Rightarrow a = b = c = d = e = 0$.
So, the set is linearly independent and the claim is true.
Now, alluding to Problem J, the basis for V/W is $\{1+W, x+W, x^2+W\}$.

Section 1.2
11. Find all vector $v \in \mathbb{C}^2$ such that $(1+i)v + \begin{pmatrix} 2-i \\ 1+2i \end{pmatrix} = \begin{pmatrix} 6+i \\ 3+6i \end{pmatrix}$.

Solution:

$$(1+i)v + \begin{pmatrix} 2-i \\ 1+2i \end{pmatrix} = \begin{pmatrix} 6+i \\ 3+6i \end{pmatrix}$$

$$(1+i)v = \begin{pmatrix} 6+i \\ 3+6i \end{pmatrix} - \begin{pmatrix} 2-i \\ 1+2i \end{pmatrix}$$

$$(1+i)v = \begin{pmatrix} 4+2i \\ 2-4i \end{pmatrix}$$

$$(1-i)(1+i)v = (1-i)\begin{pmatrix} 4+2i \\ 2-4i \end{pmatrix}$$

$$2v = \begin{pmatrix} 4+2i \\ 2-4i \end{pmatrix}$$

$$\text{So, } v = \frac{1}{2}\begin{pmatrix} 6-2i \\ -2-6i \end{pmatrix} = \begin{pmatrix} 3-i \\ -1-3i \end{pmatrix}.$$

12. Find all vectors $\mathbb{F}_5^2$ such that $2v + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.
Solution:
Let $v = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a, b \in \mathbb{F}_5$.
Now, $2\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2a \\ 2b \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$.
This gives

$$2a = -2 = 8 \mod 5 \Rightarrow a = 4 \mod 5 \; [\because gcd(5,2) = 1].$$

$$2b = -1 = 4 \mod 5 \Rightarrow b = 2 \mod 5 \; [\because gcd(5,2) = 1].$$

Since $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ is a solution, $v = \begin{pmatrix} 4+5s \\ 2+5t \end{pmatrix}$ where $s, t \in \mathbb{Z}$.

Section 1.3
4. Let V be a vector space. Prove the following cancellation property: for vectors v, x, y, if v + x = v + y, then x = y.
Proof:
Let $v \in V$ and V be a vector space. By the existence of additive inverse, there exists $-v \in V$ such that $v + (-v) = \vec{0}$. Adding $(-v)$ to each side of $v + x = v + y$ gives

$$(v + x) + (-v) = (v + y) + (-v)$$

$$x + (v + (-v)) = y + (v + (-v))$$

$$x + \vec{0} = y + \vec{0}$$

$$x = y$$

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11. Define $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} | a, b \in \mathbb{R}^+ \right\}$. Define "addition" on V by $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix}$. Further define "scalar multiplication" for $c \in \mathbb{R}$ by $c \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^c \\ b^c \end{pmatrix}$. Prove that V is a vector space over $\mathbb{R}$ where $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the zero vector and $-\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{1}{a} \\ \frac{1}{b} \end{pmatrix}$.

Proof:

Let $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \in V$ where $a_i, b_i, c_i \in \mathbb{R}^+$ for $i = 1, 2, 3$.

(A1) $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} = \begin{pmatrix} a_2 a_1 \\ b_2 b_1 \end{pmatrix} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

(A2) $\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right) + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 a_2 a_3 \\ b_1 b_2 b_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 a_3 \\ b_2 b_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \left(\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \right)$.

(A3) There exists $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in V$ such that $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \cdot a_1 \\ 1 \cdot b_1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

(A4) For every $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in V$, there exists $-\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{a_1} \\ \frac{1}{b_1} \end{pmatrix}$ such that $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \left(-\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right) = \begin{pmatrix} a_1 \cdot (1/a_1) \\ b_1 \cdot (1/b_1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(M1) For a scalar c, $c\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \right) = \begin{pmatrix} (a_1 a_2)^c \\ (b_1 b_2)^c \end{pmatrix} = \begin{pmatrix} (a_1)^c \\ (b_1)^c \end{pmatrix} + \begin{pmatrix} (a_2)^c \\ (b_2)^c \end{pmatrix} = c \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + c \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$.

(M2) For every pair of scalars c, d, $(c + d) \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} {a_1}^{(c+d)} \\ {b_1}^{(c+d)} \end{pmatrix} = \begin{pmatrix} {a_1}^c {a_1}^d \\ {b_1}^c {b_1}^d \end{pmatrix} = \begin{pmatrix} {a_1}^c \\ {b_1}^c \end{pmatrix} + \begin{pmatrix} {a_1}^d \\ {b_1}^d \end{pmatrix} = c \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + d \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

(M3) For every pair of scalars c, d, $(cd) \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} {a_1}^{cd} \\ {b_1}^{cd} \end{pmatrix} = \begin{pmatrix} {a_1}^{d^c} \\ {b_1}^{d^c} \end{pmatrix} = c \begin{pmatrix} {a_1}^d \\ {b_1}^d \end{pmatrix} = c\left(d \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \right)$.

(M4) $1 \cdot \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} {a_1}^1 \\ {b_1}^1 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

Hence, V is a vector space over $\mathbb{R}$.

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Section 1.4

3. Set $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid 3x - 2y + 4z = 0 \right\}$. Prove that W is a subspace in $\mathbb{R}^3$.

Proof:

Let $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \in W$. So, $3x_1 - 2y_1 + 4z_1 = 0$ and $3x_2 - 2y_2 + 4z_2 = 0$.

Now, $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \in W$ since $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{pmatrix} \in \mathbb{R}^3$ and

$3(x_1+x_2)s - 2(y_1+y_2) + 4(z_1+z_2) = (3x_1 - 2y_1 + 4z_1) + (3x_2 - 2y_2 + 4z_2) = 0 + 0 = 0.$

And for a scalar c, $c\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix} \in W$ because $\begin{pmatrix} cx_1 \\ cy_1 \\ cz_1 \end{pmatrix} \in \mathbb{R}^3$ and $3(cx_1) - 2(cy_1) + 4(cz_1) = c(3x_1 - 2y_1 + 4z_1) = c \cdot 0 = 0.$

Hence, by subspace test, W is a subspace in $\mathbb{R}^3$.

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13. Let $X, Y.Z$ be subspaces of vector space V and assume that $Y \subset X$. Prove that $X \cap (Y + Z) = Y + (X \cap Z)$. This is known as the *modular law* of subspaces.

Proof:

We try to show $X \cap (Y+Z) \subseteq Y + (X \cap Z)$ and $X \cap (Y+Z) \supseteq Y + (X \cap Z)$.

Let $w \in X \cap (Y+Z) \Rightarrow w \in X \wedge (w \in Y \vee w \in Z)$ [Distributive law].

$$\Rightarrow (w \in X \wedge w \in Y) \vee (w \in X \wedge w \in Z)$$

$$\Rightarrow w \in Y \vee (w \in X \wedge w \in Z) \ [\because Y \subset X]$$

$$\Rightarrow w \in Y + (X \cap Z).$$

So, $X \cap (Y+Z) \subseteq Y + (X \cap Z)$.

Now, let $w \in Y + (X \cap Z) \Rightarrow w \in Y \vee (w \in X \wedge w \in Z)$ [Distributive law]

$$\Rightarrow (w \in Y \vee w \in X) \wedge (w \in X \vee w \in Z)$$

$$\Rightarrow w \in X \wedge (w \in X \vee w \in Z) \ [\because Y \subset X]$$

$$\Rightarrow w \in X \cap (Y+Z).$$

So, $X \cap (Y+Z) \supseteq Y + (X \cap Z)$. Hence $X \cap (Y+Z) = Y + (X \cap Z)$

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Section 1.5

1. Let X, Y be sequences or subsets of a vector space V. Assume $X \subset Span(Y)$ and $Y \subset Span(X)$. Prove that $Span(X) = Span(Y)$.

Proof:

Since $X \subset Span(Y), Span(X) \subset Span(Span(Y)) = Span(Y)$ [By Corollary 1.1i]

Again, $Y \subset Span(X) \Rightarrow Span(Y) \subset Span(Span(X)) = Span(X)$ [By Corollary 1.1i]

So, $Span(X) \subset Span(Y)$ and $Span(X) \supset Span(Y) \Rightarrow Span(X) = Span(Y)$.

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6. Let u, v be non-zero vectors. Prove that (u, v) is linearly dependent if and only if the vectors are scalar multiples of one another.

Proof:

($\Rightarrow$) Let the non-zero vectors u, v be linearly dependent. Then, by definition of linear dependence, there exists scalars c_1, c_2 (both non-zero) such that

$$c_1 u + c_2 v = 0 \Rightarrow u = \frac{-c_2}{c_1} v = cv$$

where c is a scalar.
($\Leftarrow$) Let a non-zero vector u be a scalar (non-zero) multiple of another non-zero vector v i.e.

$$u = cv \Rightarrow u - cv = 0 \Rightarrow 1u + (-c)v = 0.$$

So, the non-trivial linear combination of u and v results to 0 which means the vector are linearly dependent.

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Section 1.8
1. a) Verify that $F = (1 + x, 1 + x^2, 1 + 2x - 2x^2)$ is a basis of $F_{(2)}[x]$.
Proof:
Claim 1: $\{1 + x, 1 + x^2, 1 + 2x - 2x^2\}$ is linearly independent.
Let $a(1 + x) + b(1 + x^2) + c(1 + 2x - 2x^2) = 0 \Rightarrow (a + b + c) + (a + 2c)x + (b - 2c)x^2 = 0.$ for some scalars a, b, c.
This gives $a + b + c = 0,\ a + 2c = 0,\ b - 2c = 0 \Rightarrow a = b = c = 0$. So, claim 1 is true.
Claim 2: $\{1 + x, 1 + x^2, 1 + 2x - 2x^2\}$ spans $F_{(2)}[x]$.
We know the standard basis for $F_{(2)}[x]$ is $\{1, x, x^2\}$. Each element in the basis can be obtained by the linear combination of $\{1 + x, 1 + x^2, 1 + 2x - 2x^2\}$ in the following way:

$$-2(1 + x) + 2(1 + x^2) + 1(1 + 2x - 2x^2) = 1,$$

$$3(1 + x) - 2(1 + x^2) - 1(1 + 2x - 2x^2) = x,$$

$$2(1 + x) - 1(1 + x^2) - 1(1 + 2x - 2x^2) = x^2.$$

So, $\{1 + x, 1 + x^2, 1 + 2x - 2x^2\}$ spans $F_{(2)}[x]$. Hence is the basis of $F_{(2)}[x]$.
b) Compute the coordinate vectors $[1]_F, [x]_F, [x^2]_F$.
From a, we get

$$[1]_F = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix},$$

$$[x]_F = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix},$$

$$[1]_F = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.$$