CSCI 5870: Data Structures and Algorithms
Homework 1

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Part I. 10 points each.

1. [2.3 - 3]
Use mathematical induction to show that when n is an exact power of 2, the solution of the recurrence

$$\boldsymbol{T(n) = \begin{cases} 2 & \text{if } n = 2 \\ 2T(n/2) + n, & \text{if } n = 2^k \text{ for } k > 1 \end{cases}}$$

is $\boldsymbol{T(n) = n \lg n}$.

Proof (by induction): We need to prove $T(n) = n \lg n$ where $n = 2^k$ for $k \geq 1$.

Base cases: For $n = 2^1$, $T(2) = 2 = 2 \lg 2$.

For $n = 2^2$, $T(2^2) = 2T(2^2/2) + 2^2 = 2T(2) + 2^2 = 2(2) + 2^2 = 2^2(2) = 2^2 \lg 2^2$.

Now, assume for $n = 2^k$, $T(n) = n \lg n = 2^k \lg 2^k$. Then, we need to show $T(n) = n \lg n$ is true for $n = 2^{k+1}$.

$$\begin{aligned} T(2^{k+1}) = 2T(2^{k+1}/2) + 2^{k+1} = 2T(2^k) + 2^{k+1} &= 2(2^k \lg 2^k) + 2^{k+1} \\ &= 2^{k+1}(\lg 2^k + 1) = 2^{k+1}(\lg 2^k + \lg 2) = 2^{k+1}(\lg 2^{k+1}). \end{aligned}$$

Hence, by mathematical induction $T(n) = n \lg n$ where $n = 2^k$ for $k \geq 1$.

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4. [3.1 - 4]
Is $\boldsymbol{2^{n+1} \in O(2^n)}$? Is $\boldsymbol{2^{2n} \in O(2^n)}$?
Since $0 \leq 2^{n+1} = 2 \cdot 2^n \leq 3 \cdot 2^n \ \forall n \in \mathbb{N}$, $2^{n+1} \in O(2^n)$.
Now, for the sake of contradiction, let's assume $\exists\ c > 0$ such that $0 \leq 2^{2n} \leq c \cdot 2^n$ for all $n \geq n_0$. But, dividing by 2^n on all sides yields $0 \leq 2^n \leq c$ which cannot possibly hold for arbitrary large n, since c is a constant. So, $2^{2n} \notin O(2^n)$.

2. [2.3 - 7]
Describe a $\boldsymbol{\Theta(n \lg n)}$ algorithm that, given a set $\boldsymbol{S}$ of $\boldsymbol{n}$ integers and another integer $\boldsymbol{x}$, determines whether or not there exist two elements in $\boldsymbol{S}$ whose sum is exactly $\boldsymbol{x}$.

The set S can be sorted in $\Theta(n \lg n)$ time using Merge sort. Then, the sorted list can be scanned in $\Theta(n)$ time using the following algorithm.

Pre-conditions: S is a sorted list of n integers.
Post-conditions: *true* if there exists two integers in S such that their sum is x, otherwise *false*

Set $i = 1, j = n$
while $i \leq j$ **do**
  if $S[i] + S[j] = x$
    return true
  if $S[i] + S[j] < x$
    $i \leftarrow i + 1$
  else
    $j \leftarrow j - 1$
return false

Hence, time complexity of the entire algorithm is $\theta(n \lg n) + \Theta(n)$ which is $\Theta(n \lg n)$.

3. [BvG 1.5]
Show that $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$ for integers $n \geq 1$. Hint: Group values of n into ranges of the form $2^k \leq n < 2^{k+1}$.
Proof: We need to show $\lceil \lg(n+1)e \rceil = \lfloor \lg n \rfloor + 1$ for integers $n \geq 1$. Let's group values of n into ranges of the form $2^k \leq n < 2^{k+1}$ for integer $k \geq 0$. Here, the set $\{n \mid 2^k \leq n < 2^{k+1}, k \geq 0\} = \mathbb{N}$. Now for any n, we get

$$2^k \leq n < 2^{k+1} \Rightarrow 2^k < n+1 \leq 2^{k+1} \Rightarrow k < \lg(n+1) \leq k+1$$
$$\Rightarrow \lceil \lg(n+1) \rceil = k+1 \ [\because \lceil x \rceil = p \Leftrightarrow p-1 < x \leq p].$$

Also,

$$2^k \leq n < 2^{k+1} \Rightarrow k \leq \lg n < k+1 \Rightarrow \lfloor \lg n \rfloor = k \ [\because \lfloor x \rfloor = p \Leftrightarrow p-1 \leq x < p]$$
$$\Rightarrow \lfloor \lg n \rfloor + 1 = k+1.$$

Hence, $\lceil \lg(n+1) \rceil = \lfloor \lg n \rfloor + 1$ for integers $n \geq 1$.

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5. [3.2 - 3]
Prove equation (3.19). Also prove that $n! \in \omega(2^n)$ and $n! \in o(n^n)$.
Proof(3.19): Equation (3.19) states $\lg(n!) = \Theta(n)$. To prove it, we use Sterling's approximation.

$$\lg(n!) = \lg\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n}\right) \text{ where } \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$$

Taylor expansion of e^{α_n} gives

$$e^{\alpha_n} = 1 + \alpha_n + \frac{1}{2!}\alpha_n^2 + \frac{1}{3!}\alpha_n^3 \ldots.$$

Since $\alpha = \Theta\Big(\frac{1}{n}\Big)$, $e^{\alpha_n} = 1 + \Theta\Big(\frac{1}{n}\Big)$.
Now,

$$\begin{aligned}\lg(n!) = \lg\left(\sqrt{2\pi n}\Big(\frac{n}{e}\Big)^n\Big(1+\Theta\Big(\frac{1}{n}\Big)\Big)\right) &= \frac{1}{2}\lg(2\pi n) + n\lg\Big(\frac{n}{e}\Big) + \lg\Big(1+\Theta\Big(\frac{1}{n}\Big)\Big)\\ &= \frac{1}{2}\lg(2\pi n) + n\lg n - n\lg e + \Theta\Big(\lg\Big(1+\frac{1}{n}\Big)\Big).\end{aligned}$$

For sufficiently large values of n, the dominant term in the expansion is $n\lg n$, others being $\Theta(\lg n)$, $\Theta(n)$ and $\Theta\Big(\frac{1}{n}\Big)$. Hence, $\lg(n!) = \Theta(n)$.

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Also, we need to prove:

$$(a)\ n! \in \omega(2^n),$$
$$(b)\ n! \in o(n^n).$$

Proof(a): Consider the following:

$$\frac{n!}{2^n} = \frac{n\cdot(n-1)\cdot(n-2)\ ...\ 2\cdot 1}{2\cdot 2\cdot 2\ ...\ 2\cdot 2} = \Big(\frac{n}{2}\Big)\Big(\frac{n-1}{2}\Big)\Big(\frac{n-2}{2}\Big)\ ...\ \Big(\frac{2}{2}\Big)\Big(\frac{1}{2}\Big)$$

$$\geq \Big(\frac{n}{2}\Big)(1)(1)\ ...\ (1)\Big(\frac{1}{2}\Big) = \frac{n}{4}.$$

Now, since $\frac{n!}{2^n} \geq \frac{n}{4}$,

$$\lim_{x\to\infty}\frac{n!}{2^n} \geq \lim_{x\to\infty}\frac{n}{4} \Rightarrow \lim_{x\to\infty}\frac{n!}{n^n} \geq \infty$$

$$\Rightarrow \lim_{x\to\infty}\frac{n!}{n^n} = \infty.$$

Hence $n! \in \omega(2^n)$.

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Proof(b): Consider the following:

$$\frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot(n-2)\ ...\ 1}{n\cdot n\cdot n\ ...\ n} = \Big(\frac{n}{n}\Big)\Big(\frac{n-1}{n}\Big)\Big(\frac{n-2}{n}\Big)\ ...\ \Big(\frac{1}{n}\Big)$$

$$\leq (1)(1)(1)\ ...\ \Big(\frac{1}{n}\Big) = \frac{1}{n}.$$

Now, since $0 \leq \frac{n!}{n^n} \leq \frac{1}{n}$,

$$\lim_{x\to\infty} 0 \leq \lim_{x\to\infty}\frac{n!}{n^n} \leq \lim_{x\to\infty}\frac{1}{n} \Rightarrow 0 \leq \lim_{x\to\infty}\frac{n!}{n^n} \leq 0$$

$$\to \lim_{x\to\infty}\frac{n!}{n^n} = 0 \text{ [by Squeeze Theorem]}.$$

Hence $n! \in o(n^n)$.

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Part II. 25 points each.

6. [Problem 3-3 (modified)]

Rank the following functions by order of growth; that is, find an arrangement $g_1, g_2, ..., g_29$ of the functions satisfying $g_1 \in \Omega(g_2), g_2 \in \Omega(g_3), ..., g_{28} \in \Omega(g_{29})$. Partition your list into equivalence classes such that $f(n)$ and $g(n)$ are in the same class if and only if $f(n) \in \Theta(g(n))$.

$\lg(\lg^* n)$	$2^{\lg^* n}$	$(\sqrt{2})$	n^2	$n!$	$(\lg n)!$
$(\frac{3}{2})^n$	n^3	$\lg^2 n$	$\lg(n!)$	2^{2^n}	$n^{1/\lg n}$
$\ln\ln n$	$\lg^* n$	$n \cdot 2^n$	$n^{\lg\lg n}$	$\ln n$	1
$2^{\lg n}$	$(\lg n)^(\lg n)$	e^n	$4^{\lg n}$	$(n+1)!$	$\sqrt{\lg n}$
$\lg^*(\lg n)$		n	2^n	$n \lg n$	$2^{2^{n+1}}$

The following table ranks the functions by order of growth (decreasing from top to bottom). Two functions $f(n)$ and $g(n)$ are in the same class (row) if and only if $f(n) \in \Theta(g(n))$.

$2^{2^{n+1}}$	
2^{2^n}	
$(n+1)!$	
$n!$	
e^n	
$n2^n$	
2^n	
$\left(\frac{3}{2}\right)^n$	
$n^{\lg\lg n}$	$(\lg n)^{\lg n}$
$(\lg n)!$	
n^3	
n^2	$4^{\lg n}$
$2^{\lg n}$	n
$(\sqrt{2})^{\lg n}$	
$n \lg n$	$\lg(n!)$
$\lg^2 n$	
$\ln n$	
$\sqrt{\lg n}$	
$\ln\ln n$	
$2^{\lg^* n}$	
$\lg^* n$	$\lg^*(\lg n)$
$\lg(\lg^* n)$	
$n^{1/\lg n}$	1

7. [Problem 3-4]
Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.
a. $(n) \in O(g(n))$ implies $g(n) \in O(f(n))$.
Counterexample: Let $f(n) = n$ and $g(n) = n^2$. Then, $f(n) \in O(g(n))$ but $g(n) \notin O(f(n))$. So, the statement is false.

b. $f(n) + g(n) \in \Theta(\min(f(n), g(n)))$.
Counterexample: Let $f(n) = n$ and $g(n) = n^n$. Then, $\min(f(n), g(n)) = \min(n, n^n) = n$. But $f(n) + g(n) = n + n^n \notin \Theta(n)$. So, the statement is false.

c. $f(n) \in O(g(n))$ implies $\lg f(n) \in O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n.
Proof: Since $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n,

$$f(n) \in O(g(n)) \Rightarrow \exists\ c, n_0 > 0 \ni 1 \leq f(n) \leq cg(n)\ \forall n \geq n_0.$$

Taking lg on all sides (which preserve the inequality since all sides are greater or equal to 1), we get

$$\begin{aligned} &\exists\ c, n_0 > 0 \ni \lg 1 \leq \lg f(n) \leq \lg(cg(n))\ \forall n \geq n_0 \\ \Rightarrow &\exists\ c, n_0 > 0 \ni 0 \leq \lg f(n) \leq \lg c + \lg(g(n))\ \forall n \geq n_0 \end{aligned}$$

But $\lg c + \lg(g(n)) \leq (\lg c)\lg(g(n)) + \lg(g(n)) = (1 + \lg c)\lg(g(n))$ $[\because \lg(g(n)) \geq 1]$.
Let $c_1 = (1 + \lg c)$. So,

$$\exists\ c_1, n_0 > 0 \ni 0 \leq \lg f(n) \leq c_1 \lg(g(n))\ \forall n \geq n_0$$

which implies $\lg f(n) \in O(\lg(g(n)))$. Hence, the statement is true.

d. $f(n) \in O(g(n))$ implies $2^{f(n)} \in O(2^{g(n)})$.
Counterexample: Let $f(n) = 2n$ and $g(n) = n$. Then, $2n \in O(n)$ but $2^{2n} \notin O(2^n)$ (from Problem 4). Hence, the statement is false.

e. $f(n) \in O(f(n)^2)$.
Counterexample: Let $f(n) = \frac{1}{\sqrt{n}} > 0$ for all n. We want to show $\frac{1}{\sqrt{n}} \notin O((\frac{1}{\sqrt{n}})^2)$ *i.e.* $\frac{1}{\sqrt{n}} \notin O(\frac{1}{n})$.
Assume for the sake of contradiction,

$$\exists\ c, n_0 > 0 \ni 0 \leq \frac{1}{\sqrt{n}} \leq c\Big(\frac{1}{n}\Big)\ \forall\ n \geq n_0$$

$$\Rightarrow \exists\ c, n_0 > 0 \ni 0 \leq \sqrt{n} \leq c\ \forall\ n \geq n_0$$

which cannot possibly hold for arbitrary large n, since c is a constant. So, $\frac{1}{\sqrt{n}} \notin O(\frac{1}{n})$. Hence, the statement is false.

f. $f(n) \in O(g(n))$ implies $g(n) \in \Omega(f(n))$.
Proof:

$$\begin{aligned} f(n) \in O(g(n)) &\Rightarrow \exists\ c, n_0 > 0 \ni 0 \leq f(n) \leq cg(n)\ \forall n \geq n_0 \\ &\Rightarrow \exists\ c, n_0 > 0 \ni 0 \leq (\frac{1}{c})f(n) \leq g(n)\ \forall n \geq n_0 \\ &\Rightarrow \exists\ c_1, n_0 > 0 \ni 0 \leq c_1 f(n) \leq g(n)\ \forall n \geq n_0 \end{aligned}$$

where $c_1 = \frac{1}{c} > 0$. So, $g(n) \in \Omega(f(n))$.

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g. $f(n) \in \Theta(f(n/2))$.
Counterexample: Let $f(n) = 2^{2n}$, then $f(n/2) = 2^{2(n/2)} = 2^n$. But from Problem 4, $2^{2n} \notin O(2^n)$. Hence, the statement is false.

h. $f(n) + o(f(n)) \in \Theta(f(n))$.
Proof: Let $g \in o(f(n)$. Then by definition, for any positive constant c,

$$\exists\ n_0 > 0 \ni 0 \leq g(n) < cf(n)\ \forall\ n \geq n_0$$

$$\Rightarrow \exists\ n_0 > 0 \ni f(n) \leq f(n) + g(n) < cf(n) + f(n)\ \forall\ n \geq n_0$$

$$\Rightarrow \exists\ n_0 > 0 \ni 0 \leq f(n) \leq f(n) + g(n) < (c+1)f(n)\ \forall\ n \geq n_0.$$

Then, for some $n \geq n_1, \exists\ c_1,\ 0 \leq c_1 \leq c+1$, such that $f(n) + g(n) \leq c_1 f(n)$. So, we get

$$\Rightarrow \exists\ n_1 > 0 \ni 0 \leq f(n) \leq f(n) + g(n) < c_1 f(n)\ \forall\ n \geq n_1.$$

But g is an arbitrary function in $o(f(n)$. Hence, $f(n) + o(f(n)) \in \Theta(f(n))$.

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