MATH 6995: Homework 7

Instructor: Dr. Alexis Byers

Rabin Thapa

1. Show that if $\binom{n}{t}2^{1-\binom{t}{2}} < 1$, then $R(t,t) > n$.
Proof:

Let S denote the probability space of red-blue colorings of K_n. Let T be a t-element subset of $V(K_n)$ and A_T denote the event that the subgraph of K_n induced by T is a red K_t or a blue K_t. Since the probability of a randomly chosen edge in K_n to be blue is $\frac{1}{2}$, the probability that the subgraph of K_n induced by T is a blue K_t is $\left(\frac{1}{2}\right)^{\binom{t}{2}}$. Similarly, the probability that the subgraph of K_n induced by T is a red K_t is $\left(\frac{1}{2}\right)^{\binom{t}{2}}$.

So, the probability that the subgraph of K_n induced by T is a monochromatic K_t is

$$P[A_T] = \left(\frac{1}{2}\right)^{\binom{t}{2}} + \left(\frac{1}{2}\right)^{\binom{t}{2}} = 2^{1-\binom{t}{2}}.$$

Let $\mathcal{T}$ denote the set of all $\binom{n}{t}$ t -element subsets of $V(K_n)$ and consider the event $\cup_{T\in\mathcal{T}} A_{\mathcal{T}}$.

$$P\Big[\bigcup_{T\in\mathcal{T}} A_T\Big] \leq \sum_{T\in\mathcal{T}} P[A_T] = \binom{n}{t}2^{1-\binom{t}{2}}.$$

But, it is given that $\binom{n}{t}2^{1-\binom{t}{2}} < 1$.

So, $P\Big[\bigcup_{T\in\mathcal{T}} A_T\Big] < 1 \Rightarrow P\Big[\overline{\bigcup_{T\in\mathcal{T}} A_T}\Big] > 0$ which gives $P\Big[\bigcap_{T\in\mathcal{T}} \overline{A_T}\Big] > 0$.

Thus, $\cap_{T\in\mathcal{T}}\overline{A_{\mathcal{T}}} \neq \phi$ which means there is at least an element in the probability space S that belongs to $\cap_{T\in\mathcal{T}}\overline{A_{\mathcal{T}}}$. So, there is a red-blue coloring of K_n that prevents a monochromatic K_t. Hence, $R(t,t) > n$.

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2. Prove that every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable.
Proof:

Let $e = \{v_1, v_2, \ldots v_k\}$ be an edge of a k-uniform hypergraph, H with order $|H|$. In a random 2-coloring of H, the probability that e is monochromatic, $P[A_e] = \left(\frac{1}{2}\right)^k + \left(\frac{1}{2}\right)^k = 2^{1-k}$. Let $\bigcup_{e\in E(H)} A_e$ denote the cases in which at least one of the edges of H is monochromatic. Then,

$$P\Big[\bigcup_{e\in E(H)} A_e\Big] \leq \sum_{e\in E(H)} P[A_e] = |H| \cdot 2^{1-k}.$$

But, $|H| < 2^{k-1}$. So,

$$P\Big[\bigcup_{e\in E(H)} A_e\Big] < 2^{k-1}\cdot 2^{1-k} = 1.$$

Since $P\Big[\bigcup_{e\in E(H)} A_e\Big] < 1$, $P\overline{\Big[\bigcup_{e\in E(H)} A_e\Big]} > 0 \Rightarrow P[\bigcap_{e\in E(H)} \overline{A_e}] > 0$. So, $\bigcap_{e\in E(H)} \overline{A_e}] \neq \phi$ which implies there is a 2-coloring of H such that no edges in H is monochromatic. Hence, every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable.

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3. For all integers n, k with $n \geq k \geq 2$, show that the probability of $G \in G(n,p)$ has a set of k independent vertices is at most $\binom{n}{k}(1-p)^{\binom{k}{2}}$.

Proof:

Let G be a graph with vertex set $\{v_1, v_2, \ldots v_n\}$ and $P[v_iv_j \in E(G)] = p$. So, $P[v_iv_j \notin E(G)] = (1-p)$. Let T be a fixed k-element subset of $V(G)$ and I_T denote the event that the subgraph of G induced by T is $\overline{K_t}$ i.e. $v_mv_n \notin E(G)\ \forall\ v_m, v_n \in T$. So, $P[I_T] = (1-p)^{\binom{k}{2}}$.

Let $\mathcal{T}$ denote the set of all $\binom{n}{k}$ k -element subsets of $V(G)$ and consider the event $\cup_{T\in\mathcal{T}} I_{\mathcal{T}}$.

$$P\Big[\bigcup_{T\in\mathcal{T}} I_T\Big] \leq \sum_{T\in\mathcal{T}} P[I_T] = \binom{n}{k}(1-p)^{\binom{k}{2}}.$$

So, the probability that $G \in G(n,p)$ has a set of k independent vertices is at most $\binom{n}{k}(1-p)^{\binom{k}{2}}$.

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4. Determine the expected number of monochromatic triangles in a random 2-coloring of $E(K_6)$. Can you generalize this?

Solution:

Let T be a 3-element subset of $V(K_6)$. Since the probability of a randomly chosen edge in K_6 to be red is $\frac{1}{2}$, the probability that the subgraph of K_6 induced by T is a red K_3 is $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$. Similarly, the probability that the subgraph of K_6 induced by T is a blue K_3 is $\frac{1}{8}$. So, the probability that the subgraph of K_6 induced by T is a monochromatic K_3 is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Since any three vertices in a K_6 forms a unique triangle, the number of triangles in K_6 is $\binom{6}{3} = 20$. So, the expected number of monochromatic triangles in a random 2-coloring of $E(K_6)$ is $\left(\frac{1}{4}\right) \cdot 20 = 5$.

Generalization: The probability that the subgraph of K_n induced by a fixed t-element subset is a monochromatic K_t is $2^{1-\binom{t}{2}}$. And, there are $\binom{n}{t}$ choices for such t-element subsets. So, the expected number of monochromatic K_t's in a random coloring of K_n $(n \geq t)$ is $\binom{n}{t}2^{1-\binom{t}{2}}$. Thus, the expected number of monochromatic triangles in a random 2-coloring of K_n $(n \geq 3)$ is $\binom{n}{3}2^{1-\binom{3}{2}} = \binom{n}{3}2^{-2} = \frac{1}{4}\binom{n}{3}$.