Math 5825: Advanced Linear Algebra
Homework 2

Instructor: Dr. Thomas Madsen

Rabin Thapa

Problem A. Consider the map

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x-y \end{pmatrix}$$

(A1) Prove that T is a linear transformation.

Proof: Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ and $c \in \mathbb{R}$.

$$\begin{aligned}(i)\ T\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = T\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}\right) &= \begin{pmatrix} (x_1+x_2)+(y_1+y_2) \\ (x_1+x_2)-(y_1+y_2) \end{pmatrix} \\ &= \begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} + \begin{pmatrix} x_2+y_2 \\ x_2-y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.\end{aligned}$$

$$(ii)\ T\left(c\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) = T\begin{pmatrix} cx_1 \\ cy_1 \end{pmatrix} = \begin{pmatrix} cx_1+cy_1 \\ cx_1-cy_1 \end{pmatrix} = \begin{pmatrix} c(x_1+y_1) \\ c(x_1-y_1) \end{pmatrix} = c\begin{pmatrix} x_1+y_1 \\ x_1-y_1 \end{pmatrix} = cT\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$$

Since (i) and (ii) are true, T is a linear transformation.

■

(A2) What is the kernel of T?

Solution: Let $\begin{pmatrix} x \\ y \end{pmatrix} \in Kern(T)$. Then, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This gives

$x+y=0$ and $x-y=0 \Rightarrow x=0, y=0$. So, $Kern(T) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(A3) What is the range of T?

Solution: Let $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Now, consider

$\begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow x+y=a$ and $x-y=b$. Upon solving, we get $x=(a+b)/2$ and $y=(a-b)/2$.

$\forall\, a, b \in \mathbb{R}$, we can set $x = (a+b)/2$ and $y = (a-b)/2$ such that $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$.
Hence, $Range(T) = \mathbb{R}^2$.

(A4) Can you find a linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T \circ S$ and $S \circ T$ are identity maps? If you can, do it.
Solution: Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as

$$S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix}.$$

Appealing to parallelism with T, (A1) implies S is a linear transformation. Now,
$T \circ S\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} (x+y)/2 \\ (x-y)/2 \end{pmatrix} = \begin{pmatrix} (x+y)/2 + (x-y)/2 \\ (x+y)/2 - (x-y)/2 \end{pmatrix} = \begin{pmatrix} 2x/2 \\ 2y/2 \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}.$

$S \circ T\begin{pmatrix} x \\ y \end{pmatrix} = S\begin{pmatrix} x+y \\ x-y \end{pmatrix} = \begin{pmatrix} (x+y+x-y)/2 \\ (x+y-x+y)/2 \end{pmatrix} = \begin{pmatrix} 2x/2 \\ 2y/2 \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix}.$

Hence, $T \circ S$ and $S \circ T$ are identity maps.

Problem B. Let F be a field. Let A be an $n \times m$ matrix with entries from F. Define a map $T : F^m \to F^n$ by

$$T(v) = Av.$$

Prove that T is a linear transformation.
Proof: Let $v_1, v_2 \in F^m$ and $c \in F$. Now,
(i) $T(v_1 + v_2) = A(v_1 + v_2) = Av_1 + Av_2$ [Distributive property]
$\qquad = T(v_1) + T(v_2).$

(ii) $T(cv_1) = A(cv_1) = c(Av_1)$ [Matrix-scalar multiplication]
$\qquad = cT(v_1).$
Since (i) and (ii) are true, T is a linear transformation.

$\blacksquare$

Problem C. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{pmatrix}.$$

Consider the linear transformation $T = T_A : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(v) = Av$.

(C1) Find a basis for $\ker(T)$.

Solution: Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in Ker(T)$. So, $T(v) = Av = 0_{\mathbb{R}^3}$. In matrix form,

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \xrightarrow{R_2 - R_1 \to \mathrm{R}_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \xrightarrow[(1/2)R_3 \to \mathrm{R}_3]{(1/3)R_2 \to \mathrm{R}_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[R_1+R_2\to R_1]{R_3-R_2\to R_3}\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives:

$$\begin{aligned} x+z=0 &\Rightarrow x=-z=-t, \\ y-z=0 &\Rightarrow y=z=t, \\ z &= t \text{ (where } t\in\mathbb{R}). \end{aligned}$$

So, $v=\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -t \\ t \\ t \end{pmatrix}=t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. So, the basis for $Ker(T)$ is $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

(C2) Find a basis for range(T).

Solution: The basis for column space of A is $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right\}$. So,the basis for range(T) is $\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}\right\}$.

(C3) What is the rank of T?
Solution: The rank of T is the dimension of $Range(T)$ which is 2.

(C4) What is the nullity of T?
Solution: Here, rank(T) + nullity(T) = dim(T) = dim($\mathbb{R}^3$) = 3. So, nullity(T) = 3 - 2 = 1.

Problem D. Recall that the vector spaces $M_n(\mathbb{R})$ of $n\times n$ matrices with entries in $\mathbb{R}$ has dimension n^2. Consider the function $T: M_2(\mathbb{R})\to M_2(\mathbb{R})$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a+b & d \\ c-d & c \end{pmatrix}$$

(D1) Prove that T is a linear transformation.
Proof: Let $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\in M_2(\mathbb{R})$ and $s\in\mathbb{R}$.

$$\begin{aligned} (i)\ T\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}+\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) &= T\begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}=\begin{pmatrix} a_1+a_2+b_1+b_2 & d_1+d_2 \\ c_1+c_2-d_1-d_2 & c_1+c_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1+b_1 & d_1 \\ c_1-d_1 & c_1 \end{pmatrix}+\begin{pmatrix} a_2+b_2 & d_2 \\ c_2-d_2 & c_2 \end{pmatrix}=T\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}+T\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \end{aligned}$$

$$(ii)\ T\left(s\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\right)=T\begin{pmatrix} sa_1 & sb_1 \\ sc_1 & sd_1 \end{pmatrix}=\begin{pmatrix} sa_1+sb_1 & sd_1 \\ sc_1-sd_1 & sc_1 \end{pmatrix}=s\begin{pmatrix} a_1+b_1 & d_1 \\ c_1-d_1 & c_1 \end{pmatrix}=sT\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

So, T is a linear transformation.

∎

(D2) Find a basis for the kernel of T.

Solution: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Kern(T)$. So, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & d \\ c-d & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

This gives $c = d = 0$ and $a + b = 0 \Rightarrow b = -a$. So, $A = \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} = a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$.

Thus, $Kern(T) = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right\}$.

(D3) Find a basis for the range of T.

Solution: For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & d \\ c-d & c \end{pmatrix} = (a+b)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Since $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ is a linearly independent set, it is the basis for the range of T.

(D4) Find a basis for $M_2(\mathbb{R})/\ker(T)$.

Solution: $M_2(\mathbb{R})/\ker(T) = \{M + \ker(T) | M \in M_2(\mathbb{R})\}$.

Here, $\ker(T) = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \right\}$ and the standard basis for $M_2(\mathbb{R}$ is $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$.

So by HW1, the basis for $M_2(\mathbb{R})/\ker(T)$ is $\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \ker(T), \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \ker(T), \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \ker(T) \right\}$

i.e. $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}$.

This is in accordance with the First Isomorphism Theorem since $M_2(\mathbb{R})/\ker(T) \cong Range(T)$ implies $dim(M_2(\mathbb{R})/\ker(T) = dim(Range(T))$.

Problems from the book

Section 2.1

1. Define $T : \mathbb{F}^3 \to \mathbb{F}_2[x]$ by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b-2c) + (a-b)x + (a-c)x^2$. Prove that T is a linear transformation.

Proof: Let $\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \in \mathbb{F}^3$ and $s \in \mathbb{F}$. Then,

(i) $T\left(\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} \right) = T\begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}$

$= (a_1 + a_2 + b_1 + b_2 - 2c_1 - 2c_2) + (a_1 + a_2 - b_1 - b_2)x + (a_1 + a_2 - c_1 - c_2)x^2$

$= (a_1 + b_1 - 2c_1) + (a_1 - b_1)x + (a_1 - c_1)x^2 + (a_2 + b_2 - 2c_2) + (a_2 - b_2)x + (a_2 - c_2)x^2$

$= T\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} + T\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$.

$(ii)\ T\left(s\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}\right)=T\begin{pmatrix}sa_1\\sb_1\\sc_1\end{pmatrix}$
$=(sa_1+sb_1-2sc_1)+(sa_1-sb_1)x+(sa_1-sc_1)x^2$
$=s((a_1+b_1-2c_1)+(a_1-b_1)x+(a_1-c_1)x^2)=sT\begin{pmatrix}a_1\\b_1\\c_1\end{pmatrix}.$
Since (i) and (ii) are true, T is a linear transformation.

■

2. Define $T:\mathbb{F}_3[x]\to\mathbb{F}^2$ by $T(a_3x^3+a_2x^2+a_1x+a_0)=\begin{pmatrix}a_2a_3\\a_0+a_1\end{pmatrix}$. Show that T is not a linear transformation.
Proof: Let $s\in\mathbb{F}$ and $P(x)=ax^3+bx^2+cx+d\in\mathbb{F}_3[x]$. Then,

$$T(s(ax^3+bx^2+cx+d))=T(sax^3+sbx^2+scx+sd)=\begin{pmatrix}(sa_2)(sa_3)\\sa_0+sa_1)\end{pmatrix}=s\begin{pmatrix}sa_2a_3\\a_0+a_1)\end{pmatrix}$$
$$\neq s\begin{pmatrix}a_2a_3\\a_0+a_1)\end{pmatrix}=sT(ax^3+bx^2+cx+d).$$

Since $T(sP(x))\neq sT(P(x))$, T is not a linear transformation.

■

4. Define $V=\left\{\begin{pmatrix}a\\b\end{pmatrix}|a,b\in\mathbb{R}^+\right\}$. Define "addition" on V by $\begin{pmatrix}a_1\\b_1\end{pmatrix}+\begin{pmatrix}a_2\\b_2\end{pmatrix}=\begin{pmatrix}a_1a_2\\b_1b_2\end{pmatrix}$. Further define "scalar multiplication" for $c\in\mathbb{R}$ by $c\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a^c\\b^c\end{pmatrix}$. And define $T:\mathbb{R}^2\to V$ by $T\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}e^x\\e^y\end{pmatrix}$.
Prove that T is a linear transformation.
Proof: Let $\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}\in\mathbb{R}^2$ and $c\in\mathbb{R}$. Then,

$$(i)\ T\left(\begin{pmatrix}x_1\\y_1\end{pmatrix}+\begin{pmatrix}x_2\\y_2\end{pmatrix}\right)=T\begin{pmatrix}x_1+x_2\\y_1+y_2\end{pmatrix}=\begin{pmatrix}e^{x_1+x_2}\\e^{y_1+y_2}\end{pmatrix}_V=\begin{pmatrix}e_1^xe_2^x\\e_1^ye_2^y\end{pmatrix}_V$$
$$=\begin{pmatrix}e_1^x\\e_1^y\end{pmatrix}_V+\begin{pmatrix}e_2^x\\e_2^y\end{pmatrix}_V=T\begin{pmatrix}x_1\\y_1\end{pmatrix}+T\begin{pmatrix}x_2\\y_2\end{pmatrix}.$$

$$(ii)\quad T\left(c\begin{pmatrix}x_1\\y_1\end{pmatrix}\right)\quad=\quad T\begin{pmatrix}cx_1\\cy_1\end{pmatrix}\quad=\quad\begin{pmatrix}e^{cx_1}\\e^{cy_1}\end{pmatrix}_V\quad=\quad\begin{pmatrix}(e^{x_1})^c\\(e^{y_1})^c\end{pmatrix}_V\quad=\quad c\begin{pmatrix}e^{x_1}\\e^{y_1}\end{pmatrix}_V\quad=\quad cT\begin{pmatrix}x_1\\y_1\end{pmatrix}.$$

Since (i) and (ii) are satisfied, T is a linear transformation.

■

16. Assume $T : V \to W$ is a linear transformation, $(v_1, ..., v_k)$ a sequence of vectors from V, and set $w_i = T(v_i), i = 1, ..., k$. Assume $(w_1, ..., w_k)$ is linearly independent. Prove that $(v_1, ..., v_k)$ is linearly independent.
Proof: Since $(w_1, ..., w_k)$ is linearly independent, the following is true for $c_i \in \mathbb{R}$.

$$c_1w_1 + c_2w_2 + ... + c_nv_n = 0_W \Rightarrow c_i = c_2 = ... = c_n = 0.$$

But $T(c_1v_1 + c_2v_2 + ... + c_nv_n) = c_1w_1 + c_2w_2 + ... + c_nv_n$. So,

$$T(c_1v_1 + c_2v_2 + ... + c_nv_n) = 0_W \Rightarrow c_i = c_2 = ... = c_n = 0.$$
$$\text{or, } c_1v_1 + c_2v_2 + ... + c_nv_n = 0_v \Rightarrow c_i = c_2 = ... = c_n = 0.$$

Hence, $(v_1, ..., v_k)$ is linearly independent.

■

Section 2.2

2. Let $a \neq b \in \mathbb{F}$. Define a linear transformation $T : \mathbb{F}_3[x] \to \mathbb{F}^2$ by $T(f) = \begin{pmatrix} f(a) \\ f(b) \end{pmatrix}$. Describe the kernel of T (find a basis) and find the rank and nullity of T.
Solution:

3. Let $T : \mathbb{R}_3[x] \to \mathbb{R}^4$ be a linear transformation given by

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a + 2b + 2d \\ a + 3b + c + d \\ a + b - c + d \\ a + 2b + 2d \end{pmatrix}$$

Determine bases for the range and kernel of T and use these to compute the rank and nullity of T.
Solution:

$$\begin{aligned} T(a + bx + cx^2 + dx^3) &= \begin{pmatrix} a + 2b + 2d \\ a + 3b + c + d \\ a + b - c + d \\ a + 2b + 2d \end{pmatrix} \\ &= (a + 2b + 2d) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + (a + 3b + c + d) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + (a + b = c + d) \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \end{aligned}$$

Since $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is linearly independent and it spans *Range*(T), it is the basis for the range of T.
Now, let $a + bx + cx^2 + dx^3 \in \ker(T)$. Then,

$$T(a + bx + cx^2 + dx^3) = \begin{pmatrix} a + 2b + 2d \\ a + 3b + c + d \\ a + b - c + d \\ a + 2b + 2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In augmented matrix form,

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 1 & 3 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 2 & 0 \end{array}\right) \xrightarrow[R_4-R_1]{R_2-R_1,\, R_3-R_1} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 2 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_1-2R_2]{R_3+R_2} \left(\begin{array}{cccc|c} 1 & 0 & -2 & 4 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

So,

$$\begin{aligned} -2d = 0 &\Rightarrow d = 0, \\ b + c - d = 0 &\Rightarrow b = -c, \\ a - 2c + 4d = 0 &\Rightarrow a = 2c. \end{aligned}$$

Hence, $\ker(T) = \{2c, -cx, cx^2, 0\}$ and basis of $\ker(T) = \{2, -x, x^2\}$
As a result, nullity of T is 3 and rank of T is $4 - 3 = 1$.

4. Show that the linear transformation $T : \mathbb{F}^4 \to \mathbb{F}_{(2)}[x]$ given by $T\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = (a-d) + (b-d)x + (c-d)x^2$ is surjective. Then explain why T is not an isomorphism.

Proof: The standard basis of $\mathbb{F}^4$ is $\left\{ f_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, f_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

So, $T(f_1) = 1$, $T(f_2) = x$, $T(f_3) = x^2$, $T(f_4) = -1 - x - x^2$.
Now, $Span\{T(f_1), T(f_2), T(f_3), T(f_4)\} = Span\{1, x, x^2, -1 - x - x^2\} = \mathbb{F}_{(2)}[x] \supseteq Range(T)$. Since $Range(T) \subseteq Span\{T(f_1), T(f_2), T(f_3), T(f_4)\}$, T is surjective.
However, $dim(\mathbb{F}^4) = 4 \neq \mathbb{F}_{(2)}[x] = 3$. So, T is not an isomorphism.

■