MATH 6995: Homework 5 (Redo)

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5. Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$. Show that

$$\frac{d}{dx}\phi(G, x) = \sum_{i=1}^{n} \phi(G - v_i, x).$$

This means that the characteristic polynomial of a fraph is almost reconstructive form the characteristic polynomial of its vertex-deleted subgraphs (except for the constant).

Proof:

Let G be a graph of order n with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$, and adjacency matrix $A = [a_{ij}]_{n\times n}$. For a vertex deleted subgraph $G - v_i$, the adjacency matrix is (i, j) minor of A i.e. $Adj(G - v_i) = M_{i,i}$. So, $\phi(G - v_i, x) = \det(xI_{n-1} - M_{i,i})$.

$$\text{And, } \phi(G, x) = \det(xI_n - A) = \det \begin{bmatrix} x & -a_{12} & -a_{13} & \dots & a_{1n} \\ -a_{21} & x & -a_{23} & \dots & a_{2n} \\ . & & & & \\ . & & & & \\ . & & & & \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & x \end{bmatrix} = \begin{bmatrix} \kappa_1(x) & \kappa_2(x) & \dots & \kappa_n(x) \end{bmatrix},$$

where $\kappa_j(x)$ represents the j^{th} column of $(I_n - A)$.

Now,

$$\frac{d}{dx}\phi(G, x) = \dot{\phi}(G, x) = \det \begin{bmatrix} \kappa_1(x) & \kappa_2(x) & \dots & \kappa_n(x) \end{bmatrix}$$

Using the property of derivative of determinant of a matrix, we get

$$\dot{\phi}(G, x) = \det \begin{bmatrix} \dot{\kappa}_1(x) & \kappa_2(x) & \dots & \kappa_n(x) \end{bmatrix} + \begin{bmatrix} \kappa_1(x) & \dot{\kappa}_2(x) & \dots & \kappa_n(x) \end{bmatrix} + \dots + \begin{bmatrix} \kappa_1(x) & \kappa_2(x) & \dots & \dot{\kappa}_n(x) \end{bmatrix}$$

Let's consider $\begin{bmatrix} \kappa_1(x) & \dots & \dot{\kappa}_j(x) & \dots & \kappa_n(x) \end{bmatrix}$. The elements in j^{th} column are either one of the following:

- 0 if $i \neq j$ because in this case, $a_{ij} = 0$ or 1 which implies $\frac{d}{dx}(-a_{ij}) = 0$
- 1 if $i = j$ because in this case, $a_{ij} = x$ which implies $\frac{d}{dx}(x) = 1$.

So, $\dot{\phi}(G,x) = \det \begin{bmatrix} 1 & -a_{12} & -a_{13} & \dots & a_{1n} \\ 0 & x & -a_{23} & \dots & a_{2n} \\ . \\ . \\ . \\ 0 & -a_{n2} & -a_{n3} & \dots & x \end{bmatrix} + \det \begin{bmatrix} x & 0 & -a_{13} & \dots & a_{1n} \\ -a_{21} & 1 & -a_{23} & \dots & a_{2n} \\ . \\ . \\ . \\ -a_{n1} & 0 & -a_{n3} & \dots & x \end{bmatrix} +$

$$\dots + \begin{bmatrix} x & -a_{12} & -a_{13} & \dots & 0 \\ -a_{21} & x & -a_{23} & \dots & 0 \\ . \\ . \\ . \\ -a_{n1} & -a_{n2} & -a_{n3} & \dots & 1 \end{bmatrix} = \sum_{i=1}^{n} \det(D_i)$$

$$\text{where } \det(D_i) = \det \begin{bmatrix} x & \dots & -a_{1i} = 0 & \dots & -a_{1n} \\ -a_{21} & \dots & -a_{2i} = 0 & \dots & -a_{2n} \\ . \\ -a_{i1} & & 1 & & -a_{in} \\ . \\ -a_{n1} & \dots & -a_{ni} = 0 & \dots & x \end{bmatrix}_{n \times n} .$$

But, upon expanding along i^{th} column,

$$\det(D_i) = (-1)^{i+i} \cdot 1 \cdot \det \begin{bmatrix} x & \dots & -a_{1,i-1} & -a_{1,i+1} & \dots & -a_{1,n} \\ -a_{2,1} & \dots & & & . & -a_{2,n} \\ . \\ -a_{i-1,1} & & x \\ . \\ -a_{i+1,1} & & & x \\ . \\ -a_{n,1} & \dots & & & . & x \end{bmatrix}_{(n-1)\times(n-1)}$$

$$= (-1)^{2i} \cdot \det(xI_{n-1} - M_{i,i})$$

where

$$M_{i,i} = \begin{bmatrix} 0 & \dots & a_{1,i-1} & a_{1,i+1} & \dots & a_{1,n} \\ a_{2,1} & \dots & & & . & a_{2,n} \\ . \\ a_{i-1,1} & & 0 \\ . \\ a_{i+1,1} & & & 0 \\ . \\ a_{n,1} & \dots & & & . & 0 \end{bmatrix}_{(n-1)\times(n-1)} .$$

So, $\det(D_i) = \det(xI_{n-1} - M_{i,i}) = \phi(G - v_i, x)$.

Thus,

$$\dot{\phi}(G,x) = \sum_{i=1}^{n} \det(D_i) = \sum_{i=1}^{n} \phi(G - v_i, x).$$

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