Math 146 Week 1

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1 Fields and Vector Spaces

Definition 1.1: Fields

A ring $\mathbb{F}$ is called a field if it is *commutative* and if every non zero element in F has a multiplicative inverse.

Example 1.0.1. $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}/\mathbf{Z}_p$ *is* p *is prime.*

Definition 1.2: Vector Space

A *vector space* over a field $\mathbb{F}$ is a set V with two operations

#1 Addition $+ : V \times V \to V$

#2 Scalar multiplication, $F \times V \to V$, $(a, x) \to ax$

$(V, +)$ is an Abelian group, and the following properties of scalar multiplication hold

#1 $1x = x$

#2 $(ab)x = a(bx)$

#3 $a(x + y) = ax + ay$

#4 $(a + b)x = ax + bx$

The elements of $\mathbb{F}$ are called scalars and elements of V are called vector.

Example 1.0.2. $\mathbf{R}^2, \mathbf{R}^3$ *are examples of vector spaces over* $\mathbf{R}$.

Proposition 1.1

For any field $\mathbb{F}$ the set $\mathbb{F}^n$ the set of $n-$tuples with entries from $\mathbb{F}$ form a vector space.

$$\mathbb{F}^n = \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{F}\}$$

Proof. The additive identity is the zero vector $\mathbf{0} = (0, 0, 0 \ldots, 0)$ so we have $\mathbf{0} + a = a$ for any $a \in F^n$. Addition of components is commutative since they are commutative in $\mathbb{F}$, same logic would apply for Associativity. For inverse we have $-a = (-a_1, -a_2, \ldots, -a_n)$. The properties for scalar multiplication is trivial. $\square$

Note 1.0.1. $\mathbf{Q}^n$ *is not a vector space over* $\mathbf{R}$ *because scalar multiplication may not be closed. Similarly* $\mathbf{R}^n$ *is not a vector space over* $\mathbf{C}$. *But* $\mathbf{R}^n$ *is a vector space over* $\mathbf{Q}$ *and* $\mathbf{C}^n$ *is a vector space over* $\mathbf{R}$.

Definition 1.3: Matrices

Let $\mathbb{F}$ be a field and let $m, n \geq 1$, an $m \times n$ matrices with entries from $\mathbb{F}$ is a array with n columns and m rows in the form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

Where $a_{ij} \in \mathbb{F}$. We can write (a_{ij}) where $1 \leq i \leq m$ and $1 \leq j \leq n$. So a_{ij} is the entry at $i-th$ row and $j-th$ column.

The space of $m \times n$ matrices denoted my $M_{m\times n}(\mathbb{F})$ forms a vector space, with component wise addition and scalar multiplication.

Definition 1.4: Polynomials over a Field

Let F be a field then we define $F[x]$ to be the field of polynomials with coefficients in F to be

$$F[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n : n \geq 0, a_i \in F \text{ for each } i\}$$

The degree of the polynomial $f \in F[x]$ is denoted by $\deg(f)$ is the largest index i such that $a_i \neq 0$. $P_n(\mathbb{F})$ is the set of all polynomials with $\deg(f) \leq n$. Addition and scalar multiplication is defined by:

$$f + g = \sum_{i=0}^{\max(m,n)} (a_i + b_i)x$$

$$af = \sum_{i=0}^{n} a a_i x^i$$

$F[x]$ is a vector space over F.

1.1 Basic Properties

Theorem 1.2: Basic Properties

- Cancellation Law for Vector Addition

$$x + y = z + y \to x = z$$

- Additive Inverse and Identity $\mathbf{0}$ in $(V, +)$ are unique.
- $0 \cdot x = \mathbf{0}$
- $a \cdot \mathbf{0} = \mathbf{0}$
- $(-a)x = -(ax) = a(-x)$ ($(-a)x$ is the additive inverse of ax).

Proof. The first 2 are a consequence of $(V, +)$ being a commutative group and the 3rd and 4th are trivial. For the last property consider

$$(-a)x + ax = (-a + a)x = \mathbf{0}$$

So $(-a)x$ is the additive inverse of ax which is unique so $(-a)x = -ax$ $\square$

2 Subspaces

Definition 2.1: Subspace

$W \subseteq V$ with V being a vector space over a field $\mathbb{F}$ is called a subspace of V if

(S 1) $W \neq \varnothing$

(S 2) W is closed under addition.

(S 3) W is closed under scalar multiplication.

Theorem 2.1: Subspace theorem

If W is a subspace of vector space V over $\mathbb{F}$ then W is a vector field over $\mathbb{F}$.

Proof. All conditions trivially hold for any $W \subset V$ expect the zero vector and the additive inverse condition. So let $x, y, z \in W$.

Since $W \neq \varnothing$, let $x \in W$ and by definition of subspaces W is closed under scalar multiplication so $0 \cdot x = \mathbf{0} \in W$. So the additive identity property holds. Now W again we let $u = (-1)x \in W$ and we know that $x + u = x + (-x) = 0$. So W is a vector field over $\mathbb{F}$. $\square$