Week 8

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1 The Rank of a Matrix and Matrix Inverses

Definition 1.1: Rank of a Matrix

Let $A \in M_{m\times n}(\mathbb{F})$ the rank of A denoted by $\operatorname{rank}(A)$ is rank of the linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$.

$$\operatorname{rank}(A) = \dim R(L_A) = \dim L_A(\mathbb{F}^n)$$

Remark 1.1. Let $A \in M_{m\times n}$ and let $a_1, \dots, a_n$ be column vectors of$A$ then

$$\operatorname{rank}(A) = \dim \operatorname{Col}(A)$$

Where $\operatorname{Col}(A)$ is the column space of A defined by

$$\operatorname{Col}(A) = \operatorname{span}(\{a_1, a_2, \dots, a_n\}$$

Proof. The range of L_A $R(L_A)$ is generated by $\{L_A(e_1), L_A(e_2), \dots, L_A(e_n)\}$. Therefore

$$\begin{aligned} R(L_A) &= \operatorname{span}(L_A(e_1), L_A(e_2), \dots, L_A(e_n)) \\ &= \operatorname{span}(Ae_1, Ae_2, \dots, Ae_n) \\ &= \operatorname{span}(a_1, a_2, \dots, a_n) \\ &= \operatorname{rank}(A) \end{aligned}$$

Since $Ae_i = a_i$. □

Remark 1.2. Let $A \in M_{m\times n}$ then $\operatorname{rank}(A) \le \min(m, n)$. Since $\{a_1, \dots, a_n\}$ generates A. Therefore $\dim R(L_A) \le n$. Since $R(L_A) \subset \mathbb{F}^m$ therefore $\dim R(L_A) \le m$.

Lemma 1.1

Let $T : V \to W$ be linear if V_0 is a subspace of V then

(1) $T(V_0) = \{T(x) \mid x \in V_0\}$ is a subspace of W

(2) If $\dim V_0 < \infty$ then $\dim V_0 = \dim T(V_0)$.

1.1 Rank of matrix products

Theorem 1.2

Let $A \in M_{m\times n}$ let $P \in M_{m\times m}$ and $Q \in M_{n\times n}$ be invertible. Then

(1) $\mathrm{rank}(AQ) = \mathrm{rank}(A)$.

(2) $\mathrm{rank}(PA) = \mathrm{rank}(A)$.

(3) $\mathrm{rank}(PAQ) = \mathrm{rank}(A)$.

Proof Sketch.

(1) Since Q is invertible $L_Q(\mathbb{F}^n) = \mathbb{F}^n$. Therefore

$$\dim R(L_{AQ}(\mathbb{F}^n)) = \dim R(L_A(\mathbb{F}^n)) = \mathrm{rank}(A)$$

(2) We can apply Lemma 1.1 with $T = L_P$ and $V = W = \mathbb{F}^n$ and $V_0 = L_A(\mathbb{F}^n) \subseteq V$. Then we have

$$\begin{aligned} \dim R(L_A(\mathbb{F}^n)) &= \dim L_P(L_A(\mathbb{F}^n)) \\ \mathrm{rank}(A) &= \dim L_P(L_A(\mathbb{F}^n)) \\ \mathrm{rank}(A) &= \mathrm{rank}(PA) \end{aligned}$$

□

Corollary 1.1.1: Invertible Matrix Theorem 2

Let $A \in M_{n\times n}$ is invertible if and only if $\mathrm{rank}(A) = n$.

Proof. ($\Rightarrow$) Using the above theorem we have $\mathrm{rank}(A) = \mathrm{rank}(AA^{-1}) = \mathrm{rank}(I_n) = n$.

($\Leftarrow$) If $\mathrm{rank}(A) = n$ then $n = \dim L_A(\mathbb{F}^n) = n$. Therefore L_A is onto $\to$ L_A is an isomorphism $\to$ A is invertible. □

Corollary 1.1.2

Elementary Row/Column operations are rank preserving.

Proof. Since $B = EA$ therefore $\mathrm{rank}(B) = \mathrm{rank}(EA) = \mathrm{rank}(A)$ because E is a square matrix. Same applies to column version. □

Theorem 1.3

Let $A \in M_{m\times n}$ then using a finite number of row and column operations we can transform A into the form

$$A' = \begin{bmatrix} I_r & O_1 \\ O_2 & O_3 \end{bmatrix}$$

Where O_1, O_2, O_3 are zero matrices and $r = \operatorname{rank}(A)$.

Proof.

$$A \rightsquigarrow A' = \left[\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline 0 & & & \\ 0 & & B & \\ \vdots & & & \end{array}\right] \rightsquigarrow \left[\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline 0 & I_k & & O_1 \\ 0 & & & \\ \vdots & O_2 & & O_3 \end{array}\right]$$

Using some elementary row / column operations we can transform A to A' then inductively we can transform B' in the form. Then we have

$$\operatorname{rank}(A) = \dim \operatorname{span}(e_1, e_2, \ldots, e_r, 0, 0 \ldots) = r$$

□

Theorem 1.4

Let $\operatorname{rank}(A) = r$, then using finite row/column operations we can transform A into the D_{upper} form.

$$D_{upper} = \begin{bmatrix} 1 & d_{12} & d_{13} & \cdots & d_{1,r} & d_{1,r+1} & \cdots & d_{1n} \\ 0 & 1 & d_{23} & \cdots & d_{2,r} & d_{2,r+1} & \cdots & d_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & d_{r,r+1} & \cdots & d_{rn} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proof. It is clear that $\operatorname{rank} D_{\text{upper}} = r$. □

1.2 Rank of Matrix Properties

Corollary 1.2.1

Let $A \in M_{m\times n}$ then there exists invertible matrices with size $m \times m$ and $n \times n$ such that

$$D = BAC$$

where

$$D = \begin{bmatrix} I_r & O_1 \\ O_2 & O_3 \end{bmatrix}$$

Proof. Since we are applying a finite number of row/column operations we have

$$D = \underbrace{R_k R_{k-1} \dots R_1}_{B} A \underbrace{C_1 C_2 \dots C_p}_{C}$$

B and C are both invertible since they are product of elementary matrices. □

Corollary 1.2.2

1. $\text{rank}(A) = \text{rank}(A^t)$

Proof. Let $D = BAC$ where B, C are invertible matrices. Then

$$D^t = (BAC)^t = C^t A^t B^t$$

Since B^t and C^t are invertible we have $\text{rank}(A^t) = \text{rank}(D^t)$.

$$D^t = \begin{bmatrix} I_r^t & O_1^t \\ O_2^t & O_3^t \end{bmatrix}$$

Since D^t is in the same form as D we have $\text{rank}(D^t) = r = \text{rank}(D)$. □

Corollary 1.2.3

$\text{rank}(A)$ is the dimension of subspace generated by the rows of A.

Proof. The columns of A are rows of A^t and using the previous corollary we have the result. □

1.3 Rank of Matrix Product

Theorem 1.5

Let A, B be matrices such that the product AB is defined then

$$\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$$

Proof.

$$R(L_{AB}) = \{AB\mathbf{x} \mid x \in \mathbb{F}^p\} \subseteq \{A\mathbf{y} \mid y \in \mathbb{F}^n\}$$

Therefore $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$. For B we have

$$\operatorname{rank}(AB) = \operatorname{rank}((AB)^t) = \operatorname{rank}(B^t A^t) \leq \operatorname{rank}(B^t) = \operatorname{rank}(B).$$

□

2 Four Fundamental Subspaces of a Matrix

Definition 2.1: Four Fundamental Subspaces

Let $A \in M_{m\times n}(\mathbb{F})$

#1 $\mathrm{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{F}^n\} \subseteq \mathbb{F}^m$. Span of the column vectors of A. Column Space of A

#2 $\mathrm{Row}(A) = \mathrm{Col}(A^t)$. Span of the row vectors of A. Row Space of A

#3 $\mathrm{Null}(A) = \{\mathbf{x} \in \mathbb{F}^n \mid A\mathbf{x} = \mathbf{0}\}$ Null Space of A

#4 $\mathrm{Null}(A^T) = \{\mathbf{x} \in \mathbb{F}^m \mid A^T\mathbf{x} = \mathbf{0}\}$ Left Null Space of A

Theorem 2.1

Let $A \in M_{m\times n}(\mathbb{F})$

(1) $\mathrm{Col}(A)$ and $\mathrm{Null}(A)$ are subspaces of $\mathbb{F}^m$; $\mathrm{Row}(A)$ and $\mathrm{Null}(A^t)$ are subspaces of $\mathbb{F}^n$.

(2) $\mathrm{rank}(A) = \dim \mathrm{Col}(A) = \dim \mathrm{Row}(A)$.

(3) $\mathrm{nullity}(A^T) = m - \mathrm{rank}(A)$ and $\mathrm{nullity}(A) = n - \mathrm{rank}(A)$

Proof Sketch. (3) The $\mathrm{nullity}(A)$ is the dimension of $\mathrm{Null}(A)$ so if we apply Rank-Nullity Theorem on $L_A : \mathbb{F}^m \to \mathbb{F}^n$ we have

$$\mathrm{rank}(L_A) + \mathrm{nullity}(L_A) = n$$
$$\rightsquigarrow \mathrm{rank}(A) + \mathrm{nullity}(A) = n$$

The same applies to A^t and using $\mathrm{rank}(A) = \mathrm{rank}(A^T)$. $\square$

3 The Inverse of a Matrix

Theorem 3.1: Invertible Matrix Theorem 3

Let $A \in M_{n\times n}(\mathbb{F})$ then the following are equivalent

#1 A is invertible

#2 $\mathrm{Col}(A) = \mathbb{F}^n$

#3 $\mathrm{Row}(A) = \mathbb{F}^n$

#4 A is a product of elementary matrices.

Proof. (1) $\Leftrightarrow$ (2). Since A is invertible $\mathrm{rank}(A) = \dim \mathrm{Col}(A) = n$ therefore $\mathrm{Col}(A) = \mathbb{F}^n$. On the other hand if $\mathrm{Col}(A) = \mathbb{F}^n$ then again we have A is invertible. Similarly we have prove (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (4) Let

$$A = E_1 E_2 \cdots E_p$$

Then since each E_i is elementary the inverse exists and we define

$$A^{-1} = E_p^{-1} E_{p-1}^{-1} \cdots E_1^{-1}$$

(4) $\Rightarrow$ (1) Assume A is invertible then $\mathrm{rank}(A) = n$ and by Corollary 1.2.1 we have

$$D = \begin{bmatrix} I_r & O_1 \\ O_2 & O_3 \end{bmatrix}$$

Since $r = \mathrm{rank}(A)$ we have $r = n$. Therefore $D = I_n$. So we have

$$\begin{aligned} I_n &= BAC \\ B^{-1} I_n C^{-1} &= A \end{aligned}$$

Since B, C were product of elementary matrices their inverse is a product of elementary matrix. Therefore $A = B^{-1}C^{-1}$ is a product of elementary matrix. $\square$

Remark 3.1. The Let $A \in M_{n\times n}(\mathbb{F})$ then notation $B = (A \mid I_n)$ represents the extended matrix of dimension $(n \times 2n)$. where the first n columns $(b_i)_{i=1}^n = a_i$ and $(b_i)_{i>n} = I_i$.

Theorem 3.2: Algorithm to find Inverse

If A is an *invertible* matrix then we can transform

$$(A \mid I_n) \overset{\mathcal{R}}{\rightsquigarrow} (I_n \mid A^{-1})$$

Where $\mathcal{R}$ is a **finite** sequence of **row** operations.

Conversely if $A \in M_{n\times n}(\mathbb{F})$ and there exists B such that

$$(A \mid I_n) \overset{\mathcal{R}}{\rightsquigarrow} (I_n \mid B)$$

Where $\mathcal{R}$ is a **finite** sequence of **row** operations. Then $B = A^{-1}$.

Proof. $(\Rightarrow)$ We have

$$A^{-1}(A \mid I_n) = (A^{-1}A \mid A^{-1}I_n) = (I_n \mid A^{-1})$$

This holds because A^{-1} is multiplied with ever column. By the invertible matrix theorem since A^{-1} is invertible it is a product of elementary matrices. We let

$$A^{-1} = E_p E_{p-1} \cdots E_1$$

Therefore

$$E_p E_{p-1} \cdots E_1 (A \mid I_n) = (I_n \mid A^{-1})$$

So if we perform the row operations $E_1, E_2, \ldots, E_p$ we can transform the matrices.

$(\Leftarrow)$ Let $G_1 \ldots G_p$ be the elementary matrices obtained by applying the finite row operations on $(A \mid I_n)$. Therefore we have

$$\begin{aligned} G_p \ldots G_1(A \mid I_n) &= (I_n \mid B) \\ G(A \mid I_n) &= (I_n \mid B) \\ (GA \mid GI_n) &= (I_n \mid B) \\ (GA \mid G) &= (I_n \mid G) \end{aligned}$$

So $GA = I_n$ and $G = B$, therefore $BA = I_n$ and by the invertible matrix theorem **2** we have $B = A^{-1}$. $\square$