Math 145

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November 6, 2020

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1 Axioms of Existence and Exensionality

Axiom 1: Axiom of Existence

There is a set containing no elements. For this set set A the statement $X \in A$ is false for all sets X.

Axiom 2: Axiom of Extensionality

If two sets have the same elements then they are equal. In other words for sets X, Y if $\forall x \in X : x \in Y$ and $\forall y \in Y : y \in X$ then $X = Y$.

This axiom useful in proving uniqueness of sets. A theorem that can be proved using these 2 axioms is:

Theorem 1: Unique Empty set

There is a unique set containing no elements. (Denoted by $(\varnothing)$)

Proof. The axiom of existence tells that a set with no elements exists. Assume we have 2 sets A_1 and A_2, each with no elements. The statement $\forall a \in A_1 \Rightarrow a \in A_2$ is true (*vacuously*). So each element of A_1 is an element of A_2, As A_2 is also empty. Similar argument follows. Therefore by *axiom 2 (Exensionality)* $A_1 = A_2$. Therefore the empty set is unique. $\square$

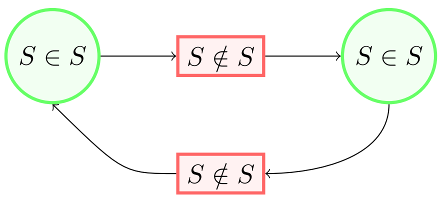
2 Comprehension

This axioms captures the intuition that we should be able to define a set with certain properties without getting into *Russell's paradox.*

Russell's paradox Let us define a set S with the following property:

$$S = \{X \mid X \notin X\}$$

This set is a set of sets that are not elements of themselves. The paradox arises when we ask $S \in S$?. If $S \in S$ by definition of$S$ we have $S \notin S$ which is a contradiction. But now that we have $S \notin S$ then by definition of S we must have $S \in S$. This leads to a paradox. The figure below shows the paradox.



The way around such a paradox is to define a set only when the sets come from a set already known to exist.

Axiom 3: Axiom Schema of Comprehension

Let $P(X)$ be a property of sets. For any set A, there is a set B defined by the property that $X \in B$ if and only if $X \in A$ and the statement $P(X)$ is true for all X.

Once we show that the set B defined by the property $P(X)$ and set A is unique. We can use the set builder notation to denote this set. The set B in the *axiom schema* will be written as

$$\{X \in A : P(X)\}$$

Theorem 2: Uniqueness of set in Axiom Schema of Comprehension

For every set A and property $P(X)$ of sets, the set B defined by A and $P(X)$ in (Axiom 3) is *unique*.

Proof. By Axiom 3 we know that such a set exists. We need to prove uniqueness. Let B_1 and B_2 be 2 sets defined by the property that set X is in these sets if and only if $X \in A$ and $P(X)$ is true.
So according to the definition any $X \in B_1$ also implies $X \in B_2$ by axiom of Exensionality we have $B_1 = B_2$. So the set B given the axiom schema of comprehension is *unique*. □

3 Pair, Union and Power Set

Axiom 4: Axiom of Pair

Given sets A and B, there is a set C whose elements are exactly A and B. In other words we have $X \in C$ if and only if $X = A$ or $X = B$.

We can also show this set C is unique by axiom of Exensionality.

Proof. Let sets C_1 and C_2 be defined as $X \in C_1, C_2$ if and only if $X = A$ or $X = B$ for sets A, B. By definition we have, for each $X \in C_1$ we have $X \in C_2$. Then by axiom of Exensionality we have $C_1 = C_2$ □

We will use the notation $\{A, B\}$ for the set containing A, B and in case $A = B$ we write $\{A\}$
The next axiom allows us to create larger sets from smaller ones.

Axiom 5: Axiom of Union

For any set S there exists a set U such that for any set X, $X \in U$ if and only if $X \in A$ for some $A \in S$.

Using this axiom we being with a set S and use it to construct U, which will represent the union of all elements in S (each element of S is a set). A set belongs to the union U exactly when it belongs to some member of S.

We can use Axiom of Extensionality to show that U is unique for a given set S. The notation used for this is:

$$\bigcup S$$

Represents the union of all elements of S. For example $\bigcup\{\varnothing, \{\varnothing\}\} = \{\varnothing\}$. As our definition states $x \in U$ if and only if $x \in A$ for some $A \in S$. We use the notation $A \cup B$ to represent $\bigcup\{A, B\}$

Axiom 6: Axiom of power set

For any set A, there exists a set $\mathcal{P}$, such that for any set X, $X \in \mathcal{P}$ if and only if $X \subseteq A$.

We can show the uniqueness using another axioms. The notation $\mathcal{P}(A)$ is used to denote the power set of A. Examples $\mathcal{P}(\{x_1, x_2\}) = \{\varnothing, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$. So in general for any set A and the empty set $\varnothing$ are both subsets of A so both are in $\mathcal{P}(A)$.

Set Constructions Some of the common set constructions using axioms are:

\# $A \cap B$, This can be constructed using Axiom 3. We define it as following:

$$A \cap B = \{x \in A : x \in B\}$$

\# $A \setminus B$, This can also be constructed using axiom of comprehension.

$$A \setminus B = \{x \in A : x \notin B\}$$

4 Natural Numbers and Successors

We define natural numbers by the following property, every natural number n is a set with exactly n elements. As $\varnothing$ has no elements, it forces us to choose $0 = \varnothing$.

For sets with one element, all these sets have a single element: $\{\varnothing\}$, $\{\{\varnothing, \{\varnothing\}\}\}$, and $\{\{\varnothing\}\}$. More generally, for any set x, the set $\{x\}$ has one element.

One natural way to proceed is to define natural numbers based on the previously defined natural numbers. So $1 = \{0\}$, and $2 = \{1, 0\}$ and so on. This leads to the recursive definition of a natural number n:

$$n = \{0, 1, 2, \dots n - 1\}$$

.

Another way to define a number $n + 1$ (given numbers until n are defined) is

$$n + 1 = n \cup \{n\}$$

Since the elements of $n + 1$ are exactly the elements of $\{n\}$ and n itself.

Definition 1: Successor

For any set x, the *successor* of x is the set $S(x) = x \cup \{x\}$

So the natural numbers are what we get by applying successor to the empty set a finite number of times.

4.1 Inductive Sets and the Axiom of Infinity

until now we do not have a way of explicitly defining the natural numbers based on the axioms we have so far. For this we need to define *inductive set*

Definition 2: Inductive set

A set I is called *inductive* if it has the following properties:

1. $0 \in I$.
2. if $n \in I$, then $S(n) \in I$

Axiom 7: Axiom of infinity

An inductive set exists.

Using this definition we can build $\mathbb{N}$. We let $\mathcal{I}$ to be the set said to exist by axiom of infinity, and use axiom of comprehension we have:

$$\mathbb{N} = \{x \in \mathcal{I} : x \in I \text{ for all inductive sets } I\}$$

Theorem 3

$\mathbb{N}$ is *inductive*

Proof. We to justify the 2 properties of inductive sets. First, $0 \in \mathbb{N}$, for every inductive set I $0 \in I$ by definition. Now assume that an element $n \in \mathbb{N}$, then by definition we have $n \in I$, for each such set I we know that $n+1 \in I$. As $n + 1$ belongs to every inductive set, hence it belongs to $\mathbb{N}$ as well. $\square$

To complete the construction of natural numbers we define a way to order the natural numbers.

Definition 3: Ordering on $\mathbb{N}$

Let $m, n \in \mathbb{N}$. We say that $m < n$ if $m \in n$

5 Ordered Pairs, Relations, and Cartesian Products

5.1 Ordered Pairs

An ordered paid (x, y) as the name suggest is ordered unlike sets. So an ordered paid $(x, y) \neq (y, x)$ the ordering of the elements matter. We can only have 2 ordered pairs equal $(x, y) = (x', y') \iff x = x' \quad y = y'$.

Definition 4: Ordered pair

Given any 2 sets x and y, the *ordered paid* (x, y) is defined to be the set

$$\{\{x\}, \{x, y\}\}$$

Theorem 4: Uniqueness of ordered pairs

For any sets x, y, x_1, y_1 we have $(x, y) = (x_1, y_1)$ if and only if $x = x_1$ and $y = y_1$.

Proof. ($\Leftarrow$)

Assume $x = x_1$ and $y = y_1$ Then we have:

$$
\begin{aligned}
(x, y) &= \{\{x\}, \{x, y\}\} \\
&= \{\{x_1\}, \{x_1, y_1\}\} \\
&= (x_1, y_1)
\end{aligned}
$$

($\Rightarrow$) Now assume that $(x, y) = (x_1, y_1)$ due to the assumption we have

$$\{\{x\}, \{x, y\}\} = \{\{x_1\}, \{x_1, y_1\}\}$$

By axiom of Extensionality, 2 sets are equal if and only if they have the same elements.

- Case 1: $x = y$ Then we have $\{\{x\}, \{x, y\}\} = \{\{x\}\}$ by the above equality we must have $\{x\} = \{x_1\} \iff x = x_1$, and $\{x_1, y_1\} = \{x\}$ this is true if and only if $x = x_1$ and $x = y_1 = y$. There we have established $x = x_1$ and $y = y_1$
- Case 2: $x \neq y$ Given the equality above $\{x, y\} = \{x_1\}$ is impossible then we must have $\{x, y\} = \{x_1, y_1\}$. Then it must be the case that $\{x\} = \{x_1\} \iff x = x_1$.

  $$\{x, y\} = \{x_1, y_1\} = \{x, y_1\}$$

  Then we must have $y_1 = y$ this completes the proof.

$\square$

5.2 Cartesian Products

Informally the product of 2 sets $X \times Y$ is the collection of all ordered pairs. With first coordinate from X and second from Y. In order to construct this product we need to supply a set with all elements of $X \times Y$. This can be defined as

$$X \times Y = \{w \in Z : w = (x, y) \text{ for some } x \in X, y \in Y\}$$

The problem is what is this set Z?

First we need a set that contains the elements of X and Y this can be done with axiom of union. By constructing $X \cup Y$. Certainly $\{x\}$ and $\{x, y\}$ are both subsets of $X \cup Y$. So the elements of the set $\{x, \{x, y\}\}$ can be taken to belong to the power set $\mathcal{P}(X \cup Y)$

Now we have $\{x\} \in \mathcal{P}(X \cup Y)$ and $\{x, y\} \in \mathcal{P}(X \cup Y)$. but the set $\{x, \{x, y\}\}$ is not a subset of $X \times Y$ so we cannot take $Z = \mathcal{P}(X \cup Y)$.

To get a set of *subsets* of $\mathcal{P}(X \cup Y)$ we need to consider the power set of this set. We need an element of $\mathcal{P}(\mathcal{P}(X \cup Y))$. As $\{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$. So we can take $Z = \mathcal{P}(\mathcal{P}(X \cup Y))$ The formal definition is :

Definition 5: Product of sets

Given any two sets X and Y , the Cartesian product of X and Y is the set

$$X \times Y = \{w \in \mathcal{P}(\mathcal{P}(X \cup Y)) \ : \ w = (x, y) \text{ for some } x \in X, y \in Y\}$$

5.3 Relations and Functions

Definition 6: Binary Relation

Given 2 sets a A, B *binary* relation from A to B is a subset of $A \times B$. More generally a set R is called a *relation* if all the elements of R are ordered pairs. Rather than writing $(x, y) \in R$ we write xRy.

Definition 7: Function

A *function* is a relation such if (a, b_1) and (a, b_2) in f then we must have $b_1 = b_2$. If $(a, b) \in f$ we write $f(a) = b$

A function from the set A to B is denoted by

$$f : A \to B$$

5.4 Terminology Around Relations and Functions

Definition 8: Domain and Range

The **domain** of the relation R is the set of all such x such that $(x, y) \in R$ for some y.

The **range** of R is the set of all such y such that $(x, y) \in R$ for some x. We use $\mathrm{ran}(R)$ and $\mathrm{dom}(R)$ to denote the domain and range of R. The set $\mathrm{ran}(R) \cup \mathrm{dom}(R)$ is called the **field** of R.

All these definitions also apply to functions.

Definition 9: Image, Inverse Image

Let R be a binary relation. The *image* of set A under R is the set

$$R(A) = \{b \in \mathrm{ran}(R) : (a, b) \in R \text{ for some } a \in A\}$$

Similarly given set B the *inverse image* of B under R is defined as

$$R^{-1}(B) = \{a \in \mathrm{dom}R : (a, b) \in R \text{ for some } a \in A\}$$

Given 2 relations their composition $R_2 \circ R_1$ is defined as:

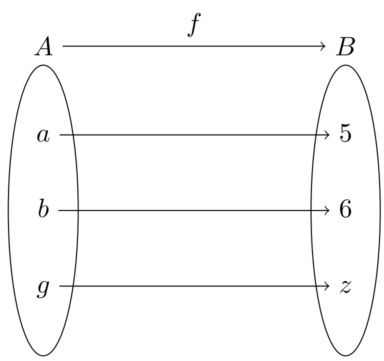
$$R_2 \circ R_1 = \{z \in \ \mathrm{dom}(R_2) \times \ \mathrm{ran}(R_1)\} : z = (a, c) \text{ such that } \exists b(a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

6 Invertibility, Injectivity, and Surjectivity

A function is called Injective if for all $x_1, x_2 \in$dom$f$ such that $x_1 \neq x_2$. We have $f(x_1) \neq f(x_2)$.

If a function from A to B is called Surjective (onto) if $\mathrm{ran}(f) = B$. In other words, for each $b \in B$ there exits $a \in A$ such that $f(a) = b$.

A function is called a bijection if it is both injective and surjective.



7 Equivalence Relations

Definition 10: Equivalence relation

Let R be be a binary relation on set A. We say that

- R is *reflexive* if , for all $a \in R$, we have aRa
- R is *symmetric* if , for all $a, b \in A$ if aRb then bRa
- R is *transitive* if , for all $a, b, c \in A$ if aRb and bRc then aRc.
- R is an equivalence relation if R is reflexive, symmetric and transitive.

7.1 Equivalence Classes

Definition 11: Equivalence Class

Suppose that E is an equivalence relation on some set A. Given an element $a \in A$ the equivalence class of a modulo E is the set

$$[a]_E = \{x \in A : aEx\}$$

So a equivalence relation splits up a set A into smaller equivalence classes. For any 2 elements in A their equivalence classes are identical or they are disjoint.

Theorem 5

Let E be an equivalence relation on set A.

(1) We have aEb if and only if $[a] = [b]$

(2) We have $(a, b) \notin E$ if and only if $[a] \cap [b] = \varnothing$

Proof. Assume that aEb. We have to prove that $[a] = [b]$.

Let $x \in [a]$ by definition we have aEx, since E is symmetric and we know aEb then bEa as well. As E is transitive and we have bEa and aEx this implies bEx. So we get $x \in [b]$ showing $[b] \subset [a]$ a symmetric argument for $[a] \subset [b]$ follows. Proving that $[a] = [b]$ is aEb.

Next, assume $[a] = [b]$. Since E is reflexive we know that bEb so we have $b \in [b]$ but we have $[b] = [a]$ then we get $b \in [a]$ by definition we get aEb.

We can prove (2) if we have $[a] \cap [b] = \varnothing$ then $[a] \neq [b]$ because both $[a]$ and $[b]$ are non empty sets. On the other hand assume $[a] \cap [b] \neq \varnothing$. Then there is some $x \in [a] \cap [b]$ so we have $x \in [a]$ and $x \in [b]$. That is by definition aEx and bEx using the property of equivalence relations we can get to aEb. By part (1) we have $[a] = [b]$. □

8 Partitions

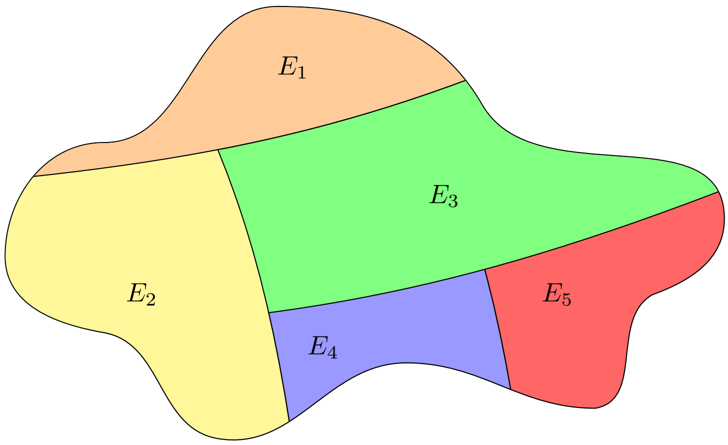
Definition 12: Partitions

Given any set A, a *partition* $\mathcal{P}$ of A is a collection of non empty sets with the properties:

(1) For any two distinct sets $P_1, P_2 \in \mathcal{P}$, we have $P_1 \cap P_2 = \varnothing$

(2) $\bigcup \mathcal{P} = A$

Theorem 5 essentially shows that a every equivalence relation on A gives us a set A gives us a partition of that set. We have the notation A/E for the set $\{[a]_E : a \in A\}$.



Theorem 6

Let E be a equivalence relation on A. Then A/E is a partition of A.

Proof. Every equivalence class $[a]$ is non empty as we must have $a \in [a]$. According to **lemma 1.1** if 2 equivalence classes are not equal they are disjoint. Which verifies condition (1) for partitions. Condition (2) follows from the fact that every $a \in A$ belongs to its equivalence class $[a]$, so the union of all equivalence classes $[a]$ is the set whole set A □

Therefore *every equivalence relation gives rise to a partition.* We can also show that the converse is true: That every partition can be used to define an equivalence relation on that set.

Theorem 7

Let A be a set , and let $\mathcal{P}$ be a partition on that set. We define a relation E on A by stating $a_1 E a_2$ if there is some $P \in \mathcal{P}$ such that $a_1 \in P$ and $a_2 \in P$. Then E is an equivalence relation on A.

Proof.

- **Reflexivity** Let $a \in A$ be arbitrary. Since $\mathcal{P}$ is a partition, there is some $P \in \mathcal{P}$ containing a. So clearly $a \in P$ and $a \in P$ then we have aEa for every $a \in A$.
- **Symmetry** Suppose we have $a_1, a_2 \in A$ such that $a_1 E a_2$. There there is some $P \in \mathcal{P}$ such that $a_1 \in P$ and $a_2 \in P$. Symmetrically we also have $a_2 \in P$ and $a_1 \in P$ leading to $a_2 E a_1$
- **Transitivity** Suppose we have $a_1, a_2, a_3 \in A$ such that $a_1 E a_2$ and $a_2 E a_3$. Then for some $P_1 \in \mathcal{P}$ we have $a_1 \in P_1$ and $a_2 \in P_1$, and also for some $P_2 \in \mathcal{P}$ we have $a_2 \in P_2$ and $a_3 \in P_2$. Since $\mathcal{P}$ is a partition if P_1 and P_2 were distinct we would have $P_1 \cap P_2 = \varnothing$. But this is not the case as a_2 is in both sets. Therefore we have $P_1 = P_2$. So a_1 and a_3 are in the same $P_1 \in \mathcal{P}$ leading to $a_1 E a_3$

□

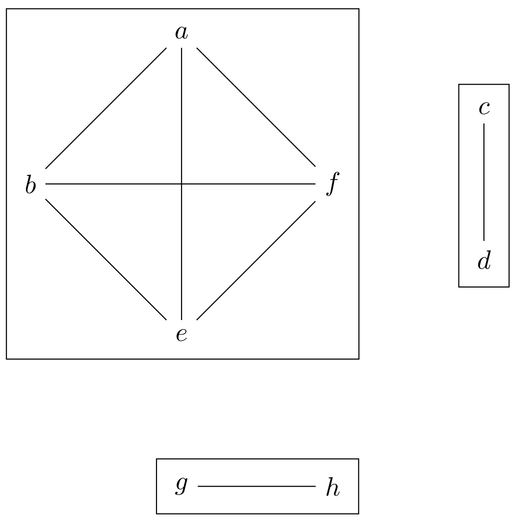
Therefore, we see that this correspondence runs both ways: every equivalence relation gives a partition, and every partition gives an equivalence relation.

Definition 13

Let A be a set and E be an equivalence relation on A. A set X is called the *set of representatives* for E if X contains exactly one element from each equivalence class.

In other words, for each $[a] \in A/E$ we have $X \cup [a] = \{\alpha\}$ for some $\alpha \in [a]$.

These equivalence classes can be visualized using this figure:



9 Order Relations

Definition 14: Antisymmetric Relation

Let R be a binary relation on set A. We say that R if, whenever we have $a, b \in A$ such that aRb and bRa, it follows that $a = b$. A relation $\preceq$ is an order relation on A if if it is reflexive , *antisymmetric*, and transitive.

9.1 Chains and Extremal Elements

Definition 15: Total ordering

If $\preceq$ is a partial ordering on set A we say that 2 elements $a, b \in A$ are comparable if either $a \preceq b$ or $b \preceq a$. A partial order in which every pair of elements is comparable is called a *total ordering* or *linear ordering*.

Example The relation $\leq$ on $\mathbb{Z}$ is a total ordering as every pair $a, b \in \mathbb{Z}$ are comparable.

Another important notion which comes is the notion of a *chain*:

Definition 16: Chain

If $\preceq$ is a partial order on set A, a subset C of A is called a *chain* if every pair of C are comparable. In particular, if $\preceq$ is a total order on A then A itself is a chain in A.

Because not every element of a set must be comparable we have to be careful in distinguishing the greatest and the least elements of of a given set. The distinction is given by the following definitions.

Definition 17: Maximal, minimal, greatest, least elements

Let A be a set with partial order relation $\preceq$. Given a subset B of A we say that

- An element $b \in B$ is the **least element** of B if we have $b \preceq b'$ for all $b' \in B$. Similarly, an element is the **greatest element** if $b' \preceq b$ for all $b' \in B$.
- An element $b \in B$ is the **minimal element** of B if there are no smaller elements, that is if $b' \preceq b$ **for some** $b' \in B$ then $b = b'$. Similarly an element is the **maximal element** if there are no larger elements, that is if $b \preceq b'$ **for some** $b' \in B$ then $b' = b$.

10 Bounds on Sets, Suprema, and Infima

Definition 18: Bounds, Suprema, and Infima

Suppose A is a set with order relation $\preceq$ and with a subset B.

- An element $a \in A$ is a lower bound for B if $a \preceq b$ **for all** $b \in B$. Similarly $a \in A$ is a upper bound for B if $b \preceq a$ **for all** $b \in B$.
- An element $a \in A$ is the infimum (Greatest lower bound) of B if a is the greatest element of the set of al lower bounds for B.
- Am element $a \in A$ is the supremum (Least upper bound) of B if a is the least element of the set of all upper bounds for B.

  If they exist, we use $sup\ B$ and $inf\ B$ to denote the supremum and infimum of a set B, respectively.

11 Axiom of choice

Suppose $\mathcal{C}$ is a non empty collection of sets then the Cartesian product of all sets in $\mathcal{C}$ is written as:

$$\prod_{C \in \mathcal{C}} C$$

If $\mathcal{C}$ is an infinite collection of sets then we need infinite ordered tuple of elements. Then we need a function $\alpha : \mathcal{C} \to \bigcup \mathcal{C}$. Then for each $C \in \mathcal{C}$ we have $\alpha(C) \in C$. The function $\alpha(C)$ should give the C-th coordinate of the infinite tuple. We can formally define this as:

Definition 19: Cartesian Product for infinite sets

Let $\mathcal{C}$ denote a non empty collection of sets. The Cartesian product $\prod_{C \in \mathcal{C}} C$ is the set of all functions $\alpha : \mathcal{C} \to \bigcup \mathcal{C}$ with the property $\alpha(C) \in C$.

We can use this definition to redefine the Cartesian product, even for finitely many sets. We can define an ordered pair $(a, b) \in A \times B$ as $(\alpha(A), \alpha(B)) \in A \times B$.
The problem arises when for an infinite number of sets in $\mathcal{C}$ we cannot show (using the standard axioms of set theory) that:

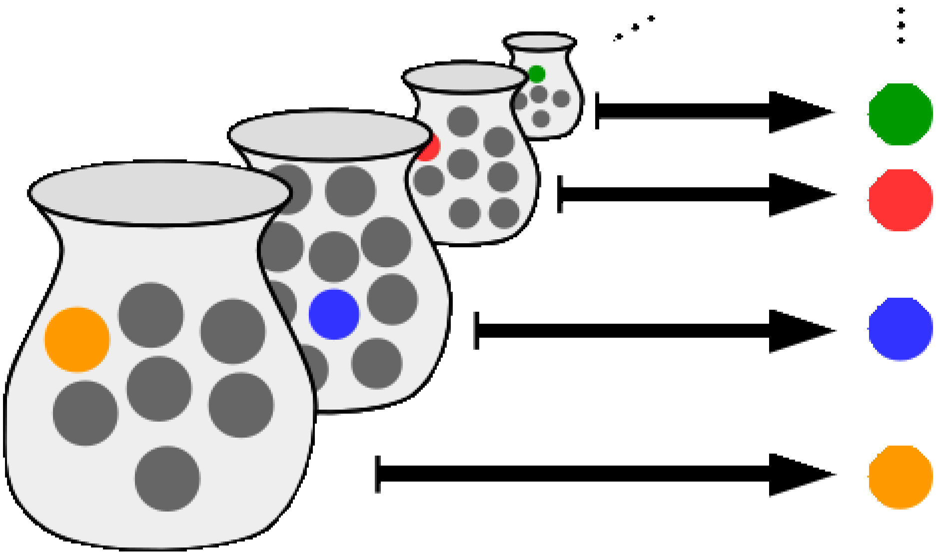
$$\prod_{C \in \mathcal{C}} C \neq \varnothing$$

For this we need a new axiom.

Axiom 8: Axiom of Choice

The Cartesian product of any non-empty collection of non-empty sets is non-empty.

This axiom is also equivalent to the existence of a choice function for every non-empty collection of set $\mathcal{C}$.



12 Zorn's Lemma and the Well-Ordering Theorem

Another statement about partially ordered set is logically equivalent to Axiom of choice.

Theorem 8: Zorn's Lemma

Let A be a partially ordered set with order relation $\preceq$. Suppose that every chain in A has an upper bound in A, then A has a maximal element.

A Corollary that follows from Zorn's Lemma is:

Corollary 1

Suppose that A is a partially ordered set with order relation $\preceq$ then every chain $\mathcal{C}$ in A is contained in a maximal chain $\mathcal{M}$.

Proof. Let $\mathcal{C}$ be an arbitrary chain in A. Consider the set Γ of all chains $\mathcal{C}$ in A. The subset relation $\subseteq$ relation on $\mathcal{P}(A)$ giving the order relation between elements of Γ. Now suppose we have a chain $\mathcal{D}$ in the ordered set Γ. We can show that $\mathcal{D}$ is an upper bound in γ

If we define $\mathcal{C}_0 = \bigcup \mathcal{D}$, the union of all the, clearly for each $\mathcal{C}' \in \mathcal{D}$ we have $C' \subseteq \mathcal{C}_0$. Since C_0 contains every chain of $\mathcal{D}$, each of which contains $\mathcal{C}$ we know that $\mathcal{C}_0$ contains $\mathcal{C}$. Therefore C_0 is an upper bound on $\mathcal{D}$ by definition. If we can verify $C_0 \in \Gamma$ that means C_0 is a chain in A

Suppose we have $a_1, a_2 \in \mathcal{C}_0$. By construction each element of a_1 and a_2 belong to a chain in $\mathcal{D}$, so we can write $a_1 \in \mathcal{C}_1$ and $a_2 \in \mathcal{C}_2$. Since $\mathcal{D}$ is a chain with respect to the relation $\subseteq$, we can either have $\mathcal{C}_1 \subseteq \mathcal{C}_2$ or we can have $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Without loss of generality we say $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Thus both $a_1, a_2 \in C_2$. Since $\mathcal{C}_2$ is a chain, the elements in $\mathcal{C}_2$ are comparable with $\preceq$, this verifies that $\mathcal{C}_0$ is a chain in A, as $C_0 \in \Gamma$.

Now if we apply Zorns lemma to Γ which is ordered by $\subseteq$, there is a chain $\mathcal{M} \in \Gamma$, maximal with respect to the containment relation $\subseteq$. This is a chain in A with respect to the ordering $\preceq$, as it is maximal with with respect to $\subseteq$ all chains $\mathcal{C}$ of A are contained in $\mathcal{M}$ and it is not properly contained in any chain.

□

The well ordering theorem is another statement known to be equivalent to axiom of choice. We can define the WOP of $\mathbb{N}$ more generally.

Definition 20: Well ordering

Suppose A is a set with order relation $\preceq$. The ordered set A is said to be well ordered if every non empty subset of A has a least element with respect to the relation $\preceq$

This leads to the well ordering theorem:

Theorem 9: Well-Ordering Theorem

Every non empty set has a well ordering.

In other words, if A is a non empty set then there is a an order relation $\preceq$ on A, such that A is well ordered with respect to $\preceq$. This means that the set $\mathbb{R}$ has as well ordering, not with respect to the relation $\leq$ as this fails for any open interval in $\mathbb{R}$ but according to *well ordering theorem* there exits a relation $\preceq$ on $\mathbb{R}$ on which every subset of $\mathbb{R}$ will have a least element with respect to $\preceq$ which is not the relation $\leq$.

The 3 statements Axiom of choice, Well-Ordering Theorem, Zorn's Lemma are all logically equivalent.