Reading-14

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1 Comparing Cardinalities of Sets

Definition 1: Cardinality

2 sets have the same *cardinality* if there is a bijection $f : A \to B$. We write $|A| = |B|$ to indicate this.

This relation forms an equivalence relation. Shown in A04, so the equivalence relation $\mathcal{R}$ is given by:

$$A\mathcal{R}B \text{ if there is a bijection } f : A \to B$$

Definition 2: Comparing cardinality

If we have 2 sets A , B we say that the cardinality of A is less than or equal to $|B|$, then we write $|A| \leq |B|$, if there is an injective function $f : A \to B$

1.1 Properties of cardinality $\leq$

We want to prove that the $\leq$ relation behaves as an order relation. This requires some lemmas to be proven before the actual proof.

Lemma 1

Suppose we have A_1, B, A such that $A_1 \subseteq B \subseteq A$. If $|A_1| = |A|$ then $|B| = |A|$

Proof. The goal is to create a bijection $g : A \to B$.

As we are given $|A_1| = |A|$ so let $f : A \to A_1$ be a bijection. Now we use f to define a sequence of sets. We set $A_0 = A$ and $B_0 = B$ and for each $n \in \mathbb{N}$ we set:

$$A_{n+1} = f(A_n) \quad B_{n+1} = f(B_n)$$

Since f is bijection from A to A_1 we have $f(A_0) = A_1$. For each $n \in \mathbb{N}$ we can say $A_{n+1} \subseteq A_n$. This can be proven with induction as we already know that $A_1 \subseteq A_0$.

To define the bijection $g : A \to B$, it will map all elements of $A \setminus B$ to B but we need elements in B to map them to. To get this we define:

$$C_n = A_n \setminus B_n$$

and the set

$$C = \bigcup_{n=0}^{\infty} C_n$$

We claim that $f(C_n) = C_{n+1}$. First note that if $a \in f(C_n)$, then $a = f(c)$ for some $c \in C_n$. By definition $c \in A_n$ and $c \notin B_n$. Then $f(c) \in f(A_n) = A_{n+1}$. We also have $f(c) \notin f(B_n)$. If we have $f(c) = f(b)$ for some $b \in B_n$ as f is bijection we must have $c = b$ that means we have $c \in B_n$ which is a contradiction.

So now we have $f(c) \in A_{n+1} \setminus B_{n+1}$ that is $f(c) \in C_{n+1}$, this proves that $f(C_n) \subseteq C_{n+1}$. Now assume we have $a \in C_{n+1}$, then $c \in A_{n+1}$ and $a \notin B_{n+1}$. This leads to $a = f(a')$ for some $a' \in A_n$, since $a \notin B_{n+1}$ this means that $a' \in B_n$. This means that $a' \in C_n$ so that $a = f(a') \in f(C_n)$. This proves $C_{n+1} \subseteq f(C_n)$. Therefore we have $f(C_n) = C_{n+1}$.

The next claim is

$$f(C) = \bigcup_{n=1}^{\infty} C_n$$

Now assume we have $a \in f(C)$, then $a = f(c)$ for some $c \in C$ then $c \in C_n$ for some $n \in \mathbb{N}$. This means $a \in f(C) \in f(C_n) = C_{n+1}$ proving we have $a \in \bigcup_{n=1}^{\infty} C_n$. Now if we have $a \in \bigcup_{n=1}^{\infty} C_n$ then $a \in C_n$ for some n. Then if we have $n - 1 \in \mathbb{N}$ then we have $C_n = f(C_{n-1})$. So $a = f(c)$ for some $c \in C_{n-1}$ then we can write $a \in f(C)$. This proves the claim.

Finally, we define another set $D = A \setminus C$. We define our bijection g by defining it separately on the two sets C and D. All the elements of C are mapped to a smaller set and the elements of D are left where they are.

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ x & \text{if } x \in D \end{cases}$$

First we need to verify for all $x \in A$ we have $g(x) \in B$. If $x \in C$, then $f(x) \in C_n$ for some $n \geq 1$, which implies $f(x) \in A_n$. Now since $A_0 \supseteq A_1 \supseteq A_2 \cdots$. In particular that $f(x) \in A_1$. We have $f(x) \in A_1$ and $A_1 \subseteq B$, so $f(x) \in B$. Otherwise if $x \in D$, then $x \notin C$, in particular $x \notin C_0$ so we have $x \not in A \setminus B$ which means that it is ***not the case*** that $x \notin B$, then whenever $x \in D$ we have $x \in D$.

Now we need to show that g is a bijection. To see that it's one-to-one let $x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$. If x_1 and x_2 both belong to C then $f(x_1) = f(x_2)$ then $x_1 = x_2$. If both belong to D then we immediately have $x_1 = x_2$. If $x_1 \in C$ and $x_2 \in D$, then $f(x_1) = x_2$ but we must have $f(x_1) \in C$ so we have a contradiction. The same contradiction follows for $x_1 \in D$ and $x_2 \in C$. So we have shown g is injective.

To prove g is surjective, assume we are given $b \in B$. If $b \in f(C)$ then clearly there is $c \in C$ such that $f(c) = b$. Otherwise if $b \notin f(C)$. Then we can have $b \in C_0$ or $b \in D$. If $b \in C_0$ then $b \in A \setminus B$ but we already have $b \in B$ so this is a contradiction. Otherwise if we have $b \in D$ then $g(b) = b$ and we are done.

In conclusion the function $g : A \to B$ is a bijection so by definition we get $|A| = |B|$ □

Theorem 1: properties of $\leq$

(1) For all sets A, B, C, if $|A| \leq |B|$ and $|A| = |C|$ then $|C| \leq |B|$

(2) For all sets A, B, C, if $|A| \leq |B|$ and $|B| = |C|$ then $|A| \leq |C|$

(3) For all sets A, B, C, if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$

(4) (Cantor-Schroder-Bernstein Theorem) For all sets A B, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

Proof. The properties $(1), (2), (3)$ all follow from the same general fact if $f : A \to B$ and $gB \to C$ are both injective functions then so is $g \circ f : A \to C$

To prove (4) Cantor-Schroder-Bernstein theorem, suppose we have injective functions $f : A \to B$ and $g : B \to A$. Then the composition $g \circ f : A \to A$, which is also injective.
Now let $X = g(B)$ and $Y = g(f(A))$. Clearly $X \subset A$, and since $f(A) \subset B$, we get $f(A) \subset B$ so we get $Y \subset X$. Since $g \circ f$ is injective, it is also a bijective function function from A to Y, we get $|A| = |Y|$.

So we have $Y \subseteq X \subseteq A$ and $|Y| = |A|$, applying *Lemma 1* we can conclude $|A| = |X|$. But $X = g(B)$, and g is injective, so $g : B \to X$ is a bijection and $|X| = |B|$ so we can conclude $|B| = |A|$

$\square$

2 Finite Sets

Definition 3: Finite Sets

A set is A called *finite* if A has the same cardinality as n for some $n \in \mathbb{N}$. We write $|A| = n$ and we say A has n elements. A set is infinite if it is not finite.

For a finite set we want to have the cardinality to be well defined. We cannot have $|A| = m$ and $|A| = n$ for distinct m, n. The following lemma rules out that possibility.

Lemma 2

For any $n \in \mathbb{N}$, there is no injective mapping from n to a proper subset of $X \subset n$.

Proof. Assume this is not true. Then by the well ordering principle, we can find some least $n \in \mathbb{N}$ for which there is an injective mapping from n to one of its proper sets. Clearly $n \neq 0$ because there are no proper subsets of 0. Then there is no injective mapping from 0 to a proper subset of 0.

Now there are 2 cases: either $n-1 \in X$ or $n-1 \notin X$. If we have $n-1 \notin X$ then $X \subseteq (n-1)$, and $n-1 \in \mathbb{N}$ because $n \neq 0$. If $f : n \to X$ is an injective mapping, then we can now define a new mapping $g : n-1 \to X \setminus \{f(n-1)\}$ by taking $g(k) = f(k)$ when every $k \in n-1$. Now g is automatically an injection since f is, and g is mapping $n-1$ to a proper subset of $n-1$ since X is a proper subset of $n-1$ since we know that g atleast will mist $f(n-1) \in n-1$. This contradicts that n was minimal.
Now suppose $n-1 \in X$. Since f is injective, we know that $n-1 = f(k)$ for some unique $k \in n$. We can use this to define a new function $g : n-1 \to X \setminus \{n-1\}$ by taking:

$$g(i) = \begin{cases} f(i) & \text{if } i \neq k \\ f(n-1) & \text{if } i = k \end{cases}$$

This function is same as f but instead of $f(k)$ we map it to $f(n-1)$ to make sure that it maps into $X \setminus \{n-1\}$, g is injective and maps $n-1$ to a proper subset of $n-1$. Again this contradicts that n is minimal. □

This lemma has the following consequences:

Corollary 1

If A is finite set such that $|A| = n$ and $|A| = m$ then $n = m$ for all $n, m \in \mathbb{N}$

Proof. This implies that both n and m have the same cardinality. Assume that we have $n \neq m$. We have a bijection $f : n \to m$ if $n \neq m$ then either $n \subset m$ or $m \subset n$ in both cases we cannot have a bijection from n to m or the other way because of **Lemma 2** so this is a contradiction. □

Another consequence of **Lemma 2** is:

Theorem 2

The set $\mathbb{N}$ is infinite

Consider the bijection $f : \mathbb{N} \to \mathbb{N}$ given by $d(n) = 2n$ this is a bijection so $\mathbb{N}$ is infinite, by definition as it maps $\mathbb{N}$ to a proper subset of $\mathbb{N}$. $\square$

2.1 Properties of Finite sets

Theorem 3: Subsets of finite sets are finite

If A is a *finite* set and $B \subseteq A$ then $|B| \leq |A|$ and B is *finite*

Proof. We can define a function $\iota : B \to A$ by $\iota(b) = b$. Clearly this is an injective mapping so we have $|B| \leq |A|$. Now since we know that A is finite we have $|A| = n$ for some $n \in \mathbb{N}$. For $n = 0$ we already have that B is finite. Consider $n \geq 1$ we can index the elements of A since there is a bijection from A to n, so we can write :

$$A = \{a_0, a_1, \ldots, a_{n-1}\}$$

Given that B is a subset of A if B is empty we are done. Otherwise, there is some least index i for which $a_i \in B$. We call that b_0. If we have $B = \{b_0\}$ it is finite and we are done. Otherwise there is another index $i_1 > i$ for which $a_{i_1} \in B$ and $B = \{b_0, b_1\}$. If we keep repeating this process we must end at some point as A is finite. So at the end we get a sequence of indices $i_0 < i_1 < i_2 < \ldots$ for which $a_{i_0}, a_{i_1}, a_{i,2}, \ldots \in B$. So for each element in B in the form a_{i_j} we can map it to j creating a bijection from B to a natural number. $\square$

Note: Every application of Axiom Schema of Comprehension leads to a subset of X. So if only finite sets are allowed to exist **Theorem 3** shows that Axiom of Comprehension can only derive more finite sets.

3 Countable Sets

Definition 4: Countable sets

A set A is called countable if $|A| = |\mathbb{N}|$, it is called *most countable* if $|A| \leq |\mathbb{N}|$. If $|A|$ is countable, we write $A = \aleph_0$

For any countable set we have a bijection from A to $\mathbb{N}$ so we can list the elements as an infinite sequence of $A = \{a_0, a_1, \ldots\}$ conversely if we can list the elements as an infinite sequence it is countable.

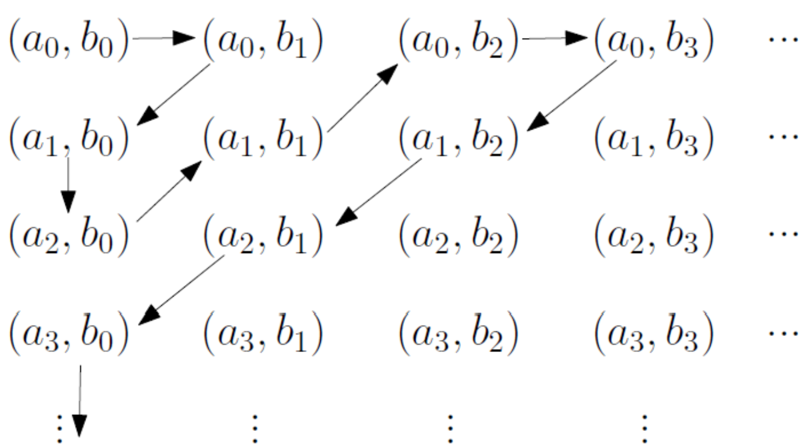
Lemma 3: Subsets of countable sets

Every subset of a countable set are countable or finite

Proof. Suppose we have A and $B \subseteq B$. If B is finite we are done. So assume B is infinite. We can write the elements of A as a sequence $A = \{a_0, a_1, a_2, \ldots\}$ we can define elements of B recursively. First choose the smallest index k_0 such that $a_{k_0} \in B$ then choose the next $k_1 > k_0$ the continuing for each natural number i, we let $k_i > k_{i-1}$ such that $a_{k_i} \in B$ and taking $b_i = a_{k_i}$. We can do this because $B \setminus \{b_0, b_1, b_2, \ldots\}$ will always be non empty given that it is infinite. So we can enumerate the elements of B as a subsequence of $\{a_n\}$ namely $\{a_{k_n}\}$ This proves that B is countable. $\square$

Lemma 4: Cartesian Product of Countable sets

If A and B are countable then $A \times B$ is also countable.



$\square$

We can use induction to prove for finitely many countable sets $A_0, A_1, \ldots A_n$ the product $\prod_{i=0}^{n} A_i$ is also finite.

These results can be used to show that $\mathbb{Q}$ is countable. We can write any number in the form $\frac{a}{b}$ so it can be represented as (a, b) of integers where $b \neq 0$. We know that $\mathbb{Z} \times \mathbb{Z}$ is countable. So we can treat $\mathbb{Q}$ as a subset of $\mathbb{Z} \times \mathbb{Z}$. Every subset of a countable set is either countable or finite. $\mathbb{Q}$ contains the infinite set $\mathbb{Z}$ so $\mathbb{Q}$ is not finite therefore is is countable.

3.1 Building Sets from Countable Sets

Lemma 5: Union of Countable sets

Let A, B countable sets then $A \cup B$ is countable.

Proof. Since A, B are both countable, let $A = \{a_n\}$ and $B = \{b_n\}$ we can define a sequence whose range is $A \cup B$, which we call $c_0, c_1, c_2, \ldots$, by taking $c_{2k} = a_k$ and $c_{2k+1} = b_k$ for each $k \in \mathbb{N}$. The sequence $\{c_n\}$ might have some duplicates, after removing the duplicates we end up with a sequence $c_0, c_1, c_2, \ldots$ which enumerates $A \cup B$, this sequence has a bijection to one of it's proper subsets to it is infinite and Lemma 3 shows that this is countable. $\square$

This can be extended to union of finite number of countable sets this can be proved using induction. So we have for the set:

$$\bigcup_{i=0}^{n} A_i \text{ Is countable}$$

To prove the case where we have a countable collection of countable sets. Consider:

$$\mathcal{C} = \{A_0, A_1, A_2, \ldots\}$$

Here $\mathcal{C}$ is a countable collection of countable sets. We want to show that $\bigcup \mathcal{C}$ is also countable. Using axiom of choice we can enumerate all the countable sets at once. Let $A_i = \{a_{i,0}, a_{i,1}, a_{i,2}, \ldots\}$ for each $i \in \mathbb{N}$. Then similar to Lemma 4 we can list $a_{0,0}$ first then $a_{1,0}$ then $a_{0,1}$ and continuing all the indices whose sum is 1 then 2 then 3 .... This might have some duplicates, removing them and re indexing the sequence will yield the union $\bigcup \mathcal{C}$ as enumerated by an infinite sequence. Thus union of countable collection of countable sets is countable. This yields the following theorem.

Theorem 4: Countable Union of Countable sets

For a of a countable collection of countable sets $\mathcal{C} = \{A_0, A_1, A_2, \ldots,\}$ the union

$$\bigcup_{i=0}^{\infty} A_i \text{ Is Countable}$$

4 Uncountable Sets

4.1 The Real Numbers $\mathbb{R}$ are Uncountable

Proposition: $|(0,1)| = |\mathbb{R}|$

Proof. Consider the function $f : (0,1) \to \mathbb{R}$ defined as:

$$f(x) = \frac{1-2x}{x(x-1)}$$

It can be shown that this function is both injective and surjective. Therefore this is a bijection, leading to $|(0,1)| = |\mathbb{R}|$ $\square$

Theorem 5: $\mathbb{R}$ is uncountable

The set of real numbers is uncountable

Proof. We can try and enumerate all the real number between $(0,1)$ as all the numbers have a decimal expansion we can list with as

n	$f(n)$												
1	0	.	3	1	4	1	5	9	2	6	5	3	...
2	0	.	3	7	3	7	3	7	3	7	3	7	...
3	0	.	1	4	2	8	5	7	1	4	2	8	...
4	0	.	7	0	7	1	0	6	7	8	1	1	...
5	0	.	3	7	5	0	0	0	0	0	0	0	...
$\vdots$	$\vdots$												

Now we can write the sequence as:

$$\begin{aligned}
r_1 &= 0.\boxed{b_{1,1}}b_{1,2}b_{1,3}b_{1,4}\dots \\
r_2 &= 0.b_{2,1}\boxed{b_{2,2}}b_{2,3}b_{2,4}\dots \\
r_3 &= 0.b_{3,1}b_{3,2}\boxed{b_{3,3}}b_{3,4}\dots \\
r_4 &= 0.b_{4,1}b_{4,2}b_{4,3}\boxed{b_{2,4}}\dots \\
&\vdots \qquad\qquad\qquad \ddots
\end{aligned}$$

Selecting the diagonals we can make a new number r as

$$r = 0.c_1c_2c_3\dots$$

For each c_i we have $c_i = \begin{cases} 4 & \text{If } b_{i,i} \neq 4 \\ 5 & \text{If } b_{i,i} = 4 \end{cases}$ we know that $r \in (0,1)$ but r is not in the sequence r_n since it differs at the i^{th} decimal position for each r_n. So no matter how we try to enumerate $(0,1)$ we cannot do it. So $(0,1)$ is uncountable and so is $\mathbb{R}$ $\square$

Theorem 6: Power set of $\mathbb{N}$ and $\mathbb{R}$

$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$, $\mathcal{P}(\mathbb{N})$ is uncountable.