Reading-9-10

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1 Ordered Pairs, Relations, and Cartesian Products

1.1 Ordered Pairs

An ordered paid (x, y) as the name suggest is ordered unlike sets. So an ordered paid $(x, y) \neq (y, x)$ the ordering of the elements matter. We can only have 2 ordered pairs equal $(x, y) = (x', y') \iff x = x' \quad y = y'$.

Definition 1: Ordered pair

Given any 2 sets x and y, the *ordered paid* (x, y) is defined to be the set

$$\{\{x\}, \{x, y\}\}$$

Theorem 1: Uniqueness of ordered pairs

For any sets x, y, x_1, y_1 we have $(x, y) = (x_1, y_1)$ if and only if $x = x_1$ and $y = y_1$.

Proof. $(\Leftarrow)$

Assume $x = x_1$ and $y = y_1$ Then we have:

$$\begin{aligned} (x, y) &= \{\{x\}, \{x, y\}\} \\ &= \{\{x_1\}, \{x_1, y_1\}\} \\ &= (x_1, y_1) \end{aligned}$$

$(\Rightarrow)$ Now assume that $(x, y) = (x_1, y_1)$ due to the assumption we have

$$\{\{x\}, \{x, y\}\} = \{\{x_1\}, \{x_1, y_1\}\}$$

By axiom of Extensionality, 2 sets are equal if and only if they have the same elements.

- Case 1: $x = y$ Then we have $\{\{x\}, \{x, y\}\} = \{\{x\}\}$ by the above equality we must have $\{x\} = \{x_1\} \iff x = x_1$, and $\{x_1, y_1\} = \{x\}$ this is true if and only if $x = x_1$ and $x = y_1 = y$. There we have established $x = x_1$ and $y = y_1$

- Case 2: $x \neq y$ Given the equality above $\{x, y\} = \{x_1\}$ is impossible then we must have $\{x, y\} = \{x_1, y_1\}$. Then it must be the case that $\{x\} = \{x_1\} \iff x = x_1$.

  $$\{x, y\} = \{x_1, y_1\} = \{x, y_1\}$$

  Then we must have $y_1 = y$ this completes the proof.

$\square$

1.2 Cartesian Products

Informally the product of 2 sets $X \times Y$ is the collection of all ordered pairs. With first coordinate from X and second from Y. In order to construct this product we need to supply a set with all elements of $X \times Y$. This can be defined as

$$X \times Y = \{w \in Z : w = (x, y) \text{ for some } x \in X, y \in Y\}$$

The problem is what is this set Z?

First we need a set that contains the elements of X and Y this can be done with axiom of union. By constructing $X \cup Y$. Certainly $\{x\}$ and $\{x, y\}$ are both subsets of $X \cup Y$. So the elements of the set $\{x, \{x, y\}\}$ can be taken to belong to the power set $\mathcal{P}(X \cup Y)$

Now we have $\{x\} \in \mathcal{P}(X \cup Y)$ and $\{x, y\} \in \mathcal{P}(X \cup Y)$. but the set $\{x, \{x, y\}\}$ is not a subset of $X \times Y$ so we cannot take $Z = \mathcal{P}(X \cup Y)$.

To get a set of *subsets* of $\mathcal{P}(X \cup Y)$ we need to consider the power set of this set. We need an element of $\mathcal{P}(\mathcal{P}(X \cup Y))$. As $\{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$. So we can take $Z = \mathcal{P}(\mathcal{P}(X \cup Y))$ The formal definition is :

Definition 2: Product of sets

Given any two sets X and Y , the Cartesian product of X and Y is the set

$$X \times Y = \{w \in \mathcal{P}(\mathcal{P}(X \cup Y)) \ : \ w = (x, y) \text{ for some } x \in X, y \in Y\}$$

1.3 Relations and Functions

Definition 3: Binary Relation

Given 2 sets a A, B *binary* relation from A to B is a subset of $A \times B$. More generally a set R is called a *relation* if all the elements of R are ordered pairs. Rather than writing $(x, y) \in R$ we write xRy.

Definition 4: Function

A *function* is a relation such if (a, b_1) and (a, b_2) in f then we must have $b_1 = b_2$. If $(a, b) \in f$ we write $f(a) = b$

A function from the set A to B is denoted by

$$f : A \to B$$

1.4 Terminology Around Relations and Functions

Definition 5: Domain and Range

The **domain** of the relation R is the set of all such x such that $(x, y) \in R$ for some y.

The **range** of R is the set of all such y such that $(x, y) \in R$ for some x. We use $\text{ran}(R)$ and $\text{dom}(R)$ to denote the domain and range of R. The set $\text{ran}(R) \cup \text{dom}(R)$ is called the **field** of R.

All these definitions also apply to functions.

Definition 6: Image, Inverse Image

Let R be a binary relation. The *image* of set A under R is the set

$$R(A) = \{b \in \text{ran}(R) : (a, b) \in R \text{ for some } a \in A\}$$

Similarly given set B the *inverse image* of B under R is defined as

$$R^{-1}(B) = \{a \in \text{dom}R : (a, b) \in R \text{ for some } a \in A\}$$

Given 2 relations their composition $R_2 \circ R_1$ is defined as:

$$R_2 \circ R_1 = \{z \in \text{ dom}(R_2)\times \text{ ran}(R_1)\} : z = (a, c) \text{ such that } \exists b(a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

2 Invertibility, Injectivity, and Surjectivity

A function is called Injective if for all $x_1, x_2 \in$dom$f$ such that $x_1 \neq x_2$. We have $f(x_1) \neq f(x_2)$.

If a function from A to B is called Surjective (onto) if $\text{ran}(f) = B$. In other words, for each $b \in B$ there exits $a \in A$ such that $f(a) = b$.

A function is called a bijection if it is both injective and surjective.