Equivalence Relations, Equivalence Classes, and Partitions

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1 Equivalence Relations

Definition 1: Equivalence relation

Let R be be a binary relation on set A. We say that

- R is *reflexive* if , for all $a \in R$, we have aRa
- R is *symmetric* if , for all $a, b \in A$ if aRb then bRa
- R is *transitive* if , for all $a, b, c \in A$ if aRb and bRc then aRc.
- R is an equivalence relation if R is reflexive, symmetric and transitive.

1.1 Equivalence Classes

Definition 2: Equivalence Class

Suppose that E is an equivalence relation on some set A. Given an element $a \in A$ the equivalence class of a modulo E is the set

$$[a]_E = \{x \in A : aEx\}$$

So a equivalence relation splits up a set A into smaller equivalence classes. For any 2 elements in A their equivalence classes are identical or they are disjoint.

Theorem 1

Let E be an equivalence relation on set A.

(1) We have aEb if and only if $[a] = [b]$

(2) We have $(a, b) \notin E$ if and only if $[a] \cap [b] = \varnothing$

Proof. Assume that aEb. We have to prove that $[a] = [b]$.
Let $x \in [a]$ by definition we have aEx, since E is symmetric and we know aEb then bEa as well. As E is transitive and we have bEa and aEx this implies bEx. So we get $x \in [b]$ showing $[b] \subset [a]$ a symmetric argument for $[a] \subset [b]$ follows. Proving that $[a] = [b]$ is aEb.
Next, assume $[a] = [b]$. Since E is reflexive we know that bEb so we have $b \in [b]$ but we have $[b] = [a]$ then we get $b \in [a]$ by definition we get aEb.

We can prove (2) if we have $[a] \cap [b] = \varnothing$ then $[a] \neq [b]$ because both $[a]$ and $[b]$ are non empty sets. On the other hand assume $[a] \cap [b] \neq \varnothing$. Then there is some $x \in [a] \cap [b]$ so we have $x \in [a]$ and $x \in [b]$. That is by definition aEx and bEx using the property of equivalence relations we can get to aEb. By part (1) we have $[a] = [b]$. □

2 Partitions

Definition 3: Partitions

Given any set A, a *partition* $\mathcal{P}$ of A is a collection of non empty sets with the properties:

(1) For any two distinct sets $P_1, P_2 \in \mathcal{P}$, we have $P_1 \cap P_2 = \varnothing$

(2) $\bigcup \mathcal{P} = A$

Lemma 1.1 essentially shows that a every equivalence relation on A gives us a set A gives us a partition of that set. We have the notation A/E for the set $\{[a]_E : a \in A\}$.

Theorem 2

Let E be a equivalence relation on A. Then A/E is a partition of A.

Proof. Every equivalence class $[a]$ is non empty as we must have $a \in [a]$. According to **lemma 1.1** if 2 equivalence classes are not equal they are disjoint. Which verifies condition (1) for partitions. Condition (2) follows from the fact that every $a \in A$ belongs to its equivalence class $[a]$, so the union of all equivalence classes $[a]$ is the set whole set A $\square$

Therefore *every equivalence relation gives rise to a partition.* We can also show that the converse is true: That every partition can be used to define an equivalence relation on that set.

Theorem 3

Let A be a set , and let $\mathcal{P}$ be a partition on that set. We define a relation E on A by stating $a_1 E a_2$ if there is some $P \in \mathcal{P}$ such that $a_1 \in P$ and $a_2 \in P$. Then E is an equivalence relation on A.

Proof.

- **Reflexivity** Let $a \in A$ be arbitrary. Since $\mathcal{P}$ is a partition, there is some $P \in \mathcal{P}$ containing a. So clearly $a \in P$ and $a \in P$ then we have aEa for every $a \in A$.

- **Symmetry** Suppose we have $a_1, a_2 \in A$ such that $a_1 E a_2$. There there is some $P \in \mathcal{P}$ such that $a_1 \in P$ and $a_2 \in P$. Symmetrically we also have $a_2 \in P$ and $a_1 \in P$ leading to $a_2 E a_1$

- **Transitivity** Suppose we have $a_1, a_2, a_3 \in A$ such that $a_1 E a_2$ and $a_2 E a_3$. Then for some $P_1 \in \mathcal{P}$ we have $a_1 \in P_1$ and $a_2 \in P_1$, and also for some $P_2 \in \mathcal{P}$ we have $a_2 \in P_2$ and $a_3 \in P_2$. Since $\mathcal{P}$ is a partition if P_1 and P_2 were distinct we would have $P_1 \cap P_2 = \varnothing$. But this is not the case as a_2 is in both sets. Therefore we have $P_1 = P_2$. So a_1 and a_3 are in the same $P_1 \in \mathcal{P}$ leading to $a_1 E a_3$

$\square$

Therefore, we see that this correspondence runs both ways: every equivalence relation gives a partition, and every partition gives an equivalence relation.

Definition 4

Let A be a set and E be an equivalence relation on A. A set X is called the *set of representatives* for E if X contains exactly one element from each equivalence class.

In other words, for each $[a] \in A/E$ we have $X \cup [a] = \{\alpha\}$ for some $\alpha \in [a]$.