Reading-18 Groups

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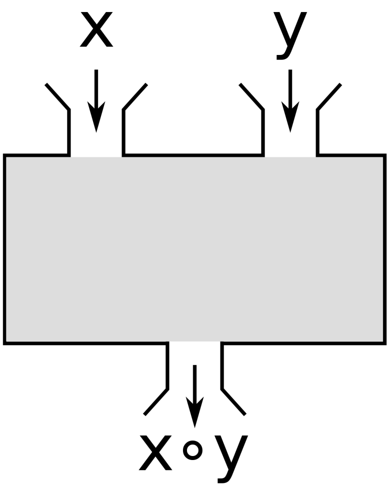
1 Binary operators

Definition 1: Binary operation

Let S be a set. A binary operator on S is a function from $S \times S$ to S. If $* : S \times S \to S$. For any 2 $a, b \in S$ we write $a * b$ to denote $*(a, b)$



Some examples of binary relations are:

Example 1 On the set $\mathbb{Z}$, the operations $+, -, \times$ all are binary operations. Division is not a binary operator on $\mathbb{Z}$ since it can output a number outside $\mathbb{Z}$.

Example 2 For any set A, the operations $\cup, \cap$ define binary operations on $\mathcal{P}(A)$, giving ways of taking pairs of subsets of A and defining new sets.

Definition 2: Types of Binary operators

Let S be a set and let $*$ be a binary operation on S. The binary operation is **associative** if for $a, b, c \in S$ we have $(a * b) * c = a * (b * c)$

Binary operation $*$ is **commutative** if for all $a, b \in S$ we have $a * b = b * a$.

The element $e \in S$ is said to be *unit* or *identity* element if $a * e = e * a = a$ for all $a \in S$

Associative property is very powerful because it the way we bracket the operations does not matter $(a * b) * (c * d) = (a * b * c) * d$ and many other ways. This can be formally proved.

Proposition 1

Let S be a set with a associative binary relation $*$. If $a_1, a_2, a_3, \ldots a_n$ with $n \geq 1$ are elements of S then the product

$$a_1 * a_2 * a_3 * \cdots * a_n$$

is well defined, regardless the choice of bracketing.

Proof. Notation:

We define recursively $\langle a_1 \rangle = a_1$, $\langle a_1, a_2 \rangle = a_1 * a_2$, and for $n \geq 3$ we have

$$\langle a_1, a_2, a_3, \ldots, a_n \rangle = \langle a_1, a_2, a_3, \ldots, a_{n-1} \rangle * a_n$$

We prove by strong induction on n that every product on n elements is equal to the standard product. Since there is no choice of bracketing on $n = 1$ and $n = 2$ those cases hold true. Now assume that $P(n)$ is true upto n, where $n \geq 2$. Now consider $P(n+1)$. The product $a_1 * a_2 * a_3 * \cdots * a_n * a_{n+1}$ can be expressed as $b * c$ which is the last application of $*$.

Where b is the product of some elements $a_1, a_2, a_3, \ldots a_k$ and c is the rest $a_{k+1}, a_{k+2}, \ldots a_{n+1}$.

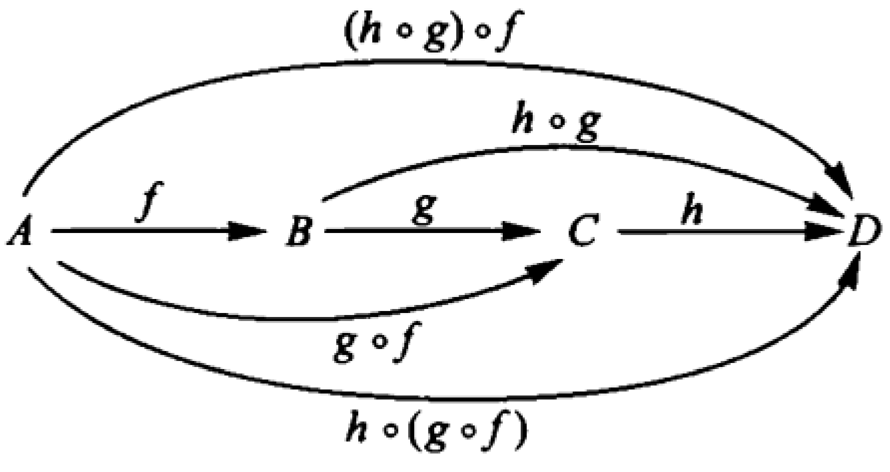
If we have $k = n$, then $c = a_{n+1}$, by inductive hypothesis $b = \langle a_1, a_2, \ldots, a_n \rangle$ is well defined. So we have $b * c = \langle a_1, a_2, \ldots, a_n \rangle * a_{n+1} = \langle a_1, a_2, \ldots, a_n, a_{n+1} \rangle$ by definition of the standard product. Otherwise if $k < n$, then $c = \langle a_{k+1}, a_{k+2}, \ldots, a_{n+1} \rangle = \langle a_{k+1}, a_{k+2}, \ldots, a_n \rangle * a_{n+1}$ and we have $b = \langle a_1, a_2, a_3, \ldots, a_k \rangle$.

$$b * c = \langle a_1, a_2, a_3, \ldots, a_k \rangle * (\langle a_{k+1}, a_{k+2}, \ldots, a_n \rangle * a_{n+1})$$

$$= (\langle a_1, a_2, a_3, \ldots, a_k \rangle * \langle a_{k+1}, a_{k+2}, \ldots, a_n \rangle) * a_{n+1}$$

$$= \langle\langle a_1, a_2, a_3, \ldots, a_n \rangle * a_{n+1} = \langle a_1, a_2, a_3, \ldots, a_{n+1} \rangle$$

$\square$



2 Monoids, Inverse Elements, and Groups

Definition 3: Monoid

Let S be a set with operation $*$. We call S a *monoid* if the operation $*$ is associative and has identity element $e \in S$ with respect to $*$.

Theorem 1: Uniqueness of identity for monoids

The identity element of a monoid is unique.

Proof. Let S be a monoid. Suppose S has 2 identity elements, $e_1, e_2 \in S$ for any $a \in S$ we know that $e_1 a = a e_1 = a$ and $e_2 a = a e_2 = a$. Using $a = e_1$ for the first equality $e_1 e_2 = e_2 e_1 = e_2$ and applying the second with $a = e_1$ we get $e_2 e_1 = e_1 e_2 = e_1$. Therefore we have $e_1 = e_2$. $\square$

Definition 4: Unit

Let S be a monoid. An element $a \in S$ is called *unit* if there exits some $b \in S$ such that $ab = ba = e$ where e is the identity element. If this is the case we call b the *inverse* of a.

Lemma 1: Uniqueness of inverse of any unit

In any monoid, the inverse of any unit is unique.

Proof. Let S be a monoid, let a be a unit. Suppose b_1 and b_2 are both inverse. Then $b_1 a = a b_1 = e$ and $b_2 a = a b_2 = e$. By definition of the identity element

$$b_1 = b_1 e = b_1(b_2 a) = (b_1 a) b_2 = e b_2 = b_2$$

$\square$

3 Groups

Now we can define groups.

Definition 5: Groups

A set G with the binary operation $*$ is a group if G is a *monoid* and every element in G is a *unit*. If $*$ is also commutative then G is called an *abelian group*.

So a set G with operation is called a *group* if:

1 For all $a, b, c \in G$ we have $a(bc) = (ab)c$. (Associativity).

2 There is some $e \in G$ such that $ae = ea = a$ for all $a \in G$. (Identity)

3 For all $a \in G$, there is $b \in G$ such that $ab = ba = e$. (Inverse).

- If G is an abelian group the optional 4th condition applies.

4 For all $a, b \in G$ we have $ab = ba$. (Commutative).

Some examples of groups.

Example 1 The set $\mathbb{Z}$ with the operation $+$ is an abelian group. 0 is the identity element and $-a$ is the inverse of any element a. Similarly the set $\{1, -1\}$ of units in $\mathbb{Z}$ with respect to the multiplication operation $\times$ is an example of finite abelian group.

Theorem 2

Let M be a monoid and let M^* be the units of M, then M^* is a group called the group of units of M

Proof. (Closed) Assume we have $a, b \in M^*$, we know that $a^{-1}, b^{-1} \in M^*$, by definition, $aa^{-1} = a^{-1}a = e = bb^{-1} = b^{-1}b$. Then get have

$$(ab)(b^{-1}a^{-1}) = aea^{-1} = aa^{-1} = e$$

$$(b^{-1}a^{-1})(ab) = beb^{-1} = bb^{-1} = e$$

This shows that ab has an inverse $(ab)^{-1} = a^{-1}b^{-1}$. So M^* is closed.

Associativity, and Inverse is trivial by construction of M^* and the identity element $e \in M$ is its own inverse so $ee = e$ therefore we have $e \in M^*$.

$\square$

3.1 Exponent notation for Groups

For a group G and $g \in G$ we set $g^0 = e$ the identity element, $g^1 = g$ and g^m to be the product of m copies of g.

$$g^m = \underbrace{ggg \cdots ggg}_{m \text{ coppies}}$$

Theorem 3

Let G be a group and $g, h \in G$

(1) For all $n, m \in \mathbb{Z}$ we have $g^{n+m} = g^n g^m$

(2) For all $n, m \in \mathbb{Z}$ $(g^n)^m = g^{mn}$

(3) If g and h commute ($gh = hg$) then $(gh)^n = g^n h^n$

Proof. By induction on n

(1) $P(0)$ $g^{0+m} = g^0 g^m = eg^m = g^m$ base case holds. Assume $P(n)$ is true. Now consider $P(n+1)$

$$g^{n+1} g^m = (g \cdot g^n) \cdot g^m = g \cdot (g^n \cdot g^m) = g \cdot g^{n+m}$$

If $n + m \geq 0$ then the result is multiplying $m + n$ copies of g with one more copy which is g^{m+n+1} by definition. If $m + n = -1$ then $gg^{-1} = e$ which is $g^{1-1} = 0$. Finally if $n + m \leq 2$ then it is multiplying $|m + n|$ copies of g^{-1} and one copy of g, which is $|n + m| - 1$ copies of g^{-1}. Thus (1) holds for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ applying the same argument for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ proves (1). ■

(2) For $n = 0$ we have $(g^m)^0 = e$, and $g^{0 \times m} = g^0 = e$. Now assume it holds for some n. Consider $n + 1$, we can apply (1)

$$(g^m)^{n+1} = (g^m)^n \cdot g^m = g^{mn} \cdot g^m = g^{m(n+1)}$$

If n is a negative integer let $n = -l$ then $(g^m)^n = (g^m)^{-l}$. For any $h \in G$, h^{-l} is product of h^{-1} l times. So we have

$$(g^m)^{-l} = ((g^m)^l)^{-1} = (g^{ml})^{-1} = g^{-ml} = g^{m(-l)} = g^{mn} \qquad ■$$

(3) For $n = 0$ we have $(gh)^0 = e = ee = g^0 h^0$, so base case holds true. Assume (3) holds for some n and Now consider $P(n+1)$

$$(gh)^{n+1} = (gh)^n(gh) = g^n(h^n g)h = (g^n g)(h^n h) = g^{n+1} \cdot h^{n+1}$$

If n is in the form $n = -l$ for some $l \in \mathbb{N}$ we get

$$(gh)^n = (gh)^{-l} = ((gh)^l)^{-1} = (g^l h^l)^-1 = h^{-l} g^{-l} = h^n g^n = g^n h^n$$

□

4 Orders of Elements and Cyclic Groups

Theorem 4

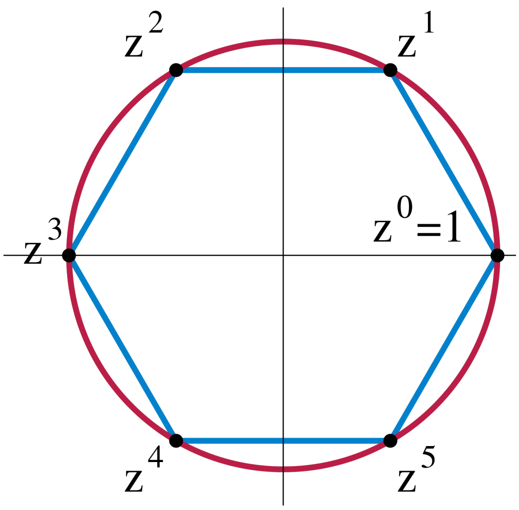
Let G be a group and let $g \in G$ be an element. Then

$$\langle g \rangle = \{g^k : k \in \mathbb{Z}\} \text{ is a group}$$

Proof. Closure, for any $g^m, g^n \in \langle g \rangle$ consider $g^m \cdot g^n$ by the above theorem we have $g^m \cdot g^n = g^{m+n} \in \langle g \rangle$. Associative is automatic since G is a group. We have $g^0 = e$ by definition so we have $g^0 \in \langle g \rangle$ therefore the identity property is also satisfied. Finally for any $g^m \in \langle g \rangle$ we also have $g^{-m} \in \langle g \rangle$ so the inverse property is also satisfied. Therefore $\langle g \rangle$ is a group. □

Definition 6: Cyclic Group

If $g \in G$ is an element , the set $\langle g \rangle$ is a subgroup of G generated by g. If $G = g$ for some element in g we say that G is a *cyclic group* and that g is a *generator* of G.



Example The set of integers modulo n. This is denoted by $\mathbb{Z}/n\mathbb{Z}$ where n is a positive integer. To define the group we take the set of equivalence classes of $\mathbb{Z}$ under the modulo relation. Which is defined as follows

$$a \equiv b \ (\text{mod} n) \text{ if } n|a-b$$

Here $n|a-b$ means n divides $a-b$, if there is some $k \in \mathbb{Z}$ such that $a-b=kn$. The equivalence classes are $[0],[1],[2],\dots[n-1]$. The addition can be defined via:

$$[a]+[b]=[a+b]$$

For any $a,b \in \mathbb{Z}$. We can prove this is well defined and the identity element is $[0]$ and inverse of $[a]$ is $[-a]$. We can show that this forms a *finite abelian group*.

Definition 7: Order

Given any group G and element $g \in G$, the order of g denoted by $o(g)$ is the smallest integer $n \geq 1$ such that $g^n = e$, if such integer exists. Otherwise if $g^n \neq e$ for any $n \geq 1$ we write $o(g) = \infty$

Proposition 2

For a group G and $g \in G$ we have $o(g) = o(g^{-1})$

Proof. If we have $o(g) = \infty$ then we have no integer $n \geq 1$ such that $g^n = e$ taking the inverse on both sides $(g^{-1})^n \neq e$ so $o(g^{-1}) = \infty$. Otherwise if we have $o(g) = n$ for some $n \geq 1$. Then we have

$$g^n = e$$

Taking the inverse of both sides $g^{-n} = e^{-1} = (g^{-1})^n = e$. Now assume there is some $m < n$ such that $g^{-m} = e$ again taking the inverse of both sides $g^m = e^{-1} = e$ which contradicts the definition of $o(g)$. □

Theorem 5

Let G be a group and let $g \in G$ be an element of finite order n. Then:

(1) $g^k = g^m$ if and only if $k \equiv m \bmod n$

(2) We have $\langle g \rangle = \{e, g, g^2, \dots g^{n-1}\}$, In particular the size of $\langle g \rangle$ is $o(g)$.

Proof. Suppose $k \equiv m \bmod n$ for some $k, m \in \mathbb{Z}$. By definition we have $k - m = \ell n$ for some $\ell \in \mathbb{Z}$. Since $g^n = e$ we also have $g^{\ell n} = e^\ell = e$. Therefore $g^{k-m} = e$. Multiplying by g^m gives $g^k = g^m$. Conversely suppose $g^k = g^m$. This implies $g^{k-m} = e$. To show that $n|k-m$, consider the division with remainder. We have

$$k - m = nq + r$$

But then

$$e = g^{k-m} = g^{nq+r} = g^{nq} \cdot g^r = e^q g^r = g^r$$

. Since $r < n$ the definition of $o(g)$ forces $r = 0$ therefore we have $k - m = qn$ and $k = m \bmod n$. Thus $g^k = g^m$.

To prove (2) we clearly have $\{e, g, g^2, \ldots g^{n-1}\} \subseteq \langle g \rangle$. In the other direction suppose $k \in \mathbb{Z}$, and using division by remainder to write $k = nq + r$ we have shown $g^k = g^r$ and we must have $0 \leq n < r$ which shows $g^k \subseteq \{e, g, g^2, \ldots, g^{n-1}\}$. So we have $\langle g \rangle = \{e, g, g^2, \ldots g^{n-1}\}$ □

5 Subgroups

Definition 8: Subgroups

If $(G, *)$ is a group then $H \subset G$ is a subgroup if $(H, *)$ itself is also a group.

The first example is the trivial subgroup for any group, which is $\{e\}$ just the identity element. Other examples are $\mathbb{Z}$ is a subgroup of $\mathbb{Q}$ under addition and $\mathbb{Q}$ is a subgroup of $\mathbb{R}$ under addition.

Theorem 6: Subgroup Test

Let G be a group and let H be a non empty subset of G. Then H is a subgroup of G if and only if for all $a, b \in H$ we have $ab^{-1} \in H$

Proof. Suppose H is a subgroup of G. For each $a, b \in H$ we know that $b^{-1} \in H$ since H is a group. Then since H is closed with respect to the operation we get $a \cdot b \in H$.

Conversely, suppose that for all $a, b \in H$ we have $ab^{-1} \in H$. Since H is non empty taking $a = b$ we get $aa^{-1} \in H$ so the identity element is in H. Let $b \in H$ and let $a = e$ then $eb^{-1} = b^{-1} \in H$ so for any $b \in H$ its inverse is also in H. The operation is associative automatically. To see the operation is closed we know that $b^{-1} \in H$ and so $a(b^{-1})^{-1} = ab \in H$.

If we let e_H to be the identity of H and e_G to be the identity of G, then we have shown that $e_G \in H$ and by uniqueness of identity we have $e_G = e_H$. Similarly let $h \in H$ we have shown that h^{-1} its inverse in G also belongs to H. So h has inverse in H. By uniqueness of inverse the inverse of h in G and H is same. $\square$

5.1 Examples of Subgroup test

Example 1 (Center of a group)

Let G be a group. We define the ***center*** of the group to be:

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$$

We can verify that $Z(G)$ is an abelian group. First $e \in Z(G)$ since $eg = ge = g$. Now suppose $a, b \in Z(G)$. We must verify $ab^{-1} \in Z(G)$. Now assume we have some $b \in Z(G)$ then $gb = bg$ applying b^{-1} on both sides

$$b^{-1}gbb^{-1} = b^{-1}bgb^{-1}$$

$$b^{-1}g = gb^{-1}$$

Therefore $b^{-1} \in Z(G)$, now

$$(ab^{-1})g = a(b^{-1}g) = (ag)b^{-1} = (ga)b^{-1} = g(ab^{-1})$$

Hence we have $ab^{-1} \in Z(G)$ therefore $Z(G)$ is an subgroup. To see why it is abelian, consider $a, b \in Z(G)$ then assume we have $a, b \in Z(G)$ then we also have $ab \in Z(G)$ since we know that $ag = ga$ then set $g = b$ to get $ab = ba$ hence $Z(G)$ is commutative.

Example 2 Conjugate

Suppose H is a subgroup of G, and for $g \in G$ we define the conjugate of H in G by g to be

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

We know that gHg^{-1} is empty because H is non empty. So let $a, b \in gHg^{-1}$ then $a = gh_1g^{-1}$ and $b = gh_2g^{-1}$ for $h_1, h_2 \in H$.

$$\begin{aligned} ab^{-1} &= gh_1g^{-1}(gh_2g^{-1})^{-1} \\ &= gh_1g^{-1}(gh_2^{-1}g^{-1}) \\ &= gh_1h_2^{-1}g^{-1} \end{aligned}$$

Since $h_1h_2^{-1} \in H$ by definition of the group we have $ab^{-1} \in gHg^{-1}$ so it is a group by the subgroup test.

Example 3

Let C_4 be the cyclic group with 4 elements. $C_4 = \{e, a, a^2, a^3\}$. The trivial subgroup $\{e\}$ is obviously a subgroup. If we construct a subgroup H, assume we have $a \in H$ then we must have $a^{-1} = a^3 \in H$ and similarly we should also have $a \cdot a = a^2 \in H$, and we must also have the identity so $H = C_4$. So H is not a proper subgroup of C_4, it is the similar case if we start with a^3, but if we start with a^2, assume $a^2 \in H$ then we must also have $e \in H$ and $(a^2)^{-1} = a^{-2} = a^4 = e$ so $H = \{e, a^2\}$ is a subgroup of C_4. Only the trivial subgroup and $\{e, a^2\}$ are subgroups of C_4.

6 Symmetric Groups

The set of bijections from a set A to it self is a group, with compositions ($\circ$) as the group operation. If we restrict A to a finite set that means we have $|A| = n$ Here we can treat the bijections from $A \to A$ as the set of bijections from $\{1, 2, 3, \ldots n\}$ to itself. This is denoted by S_n and it is called the *Symmetric group of degree n*. We can represent the elements of this group using a matrix.

It is an matrix with 2 rows with the top row containing the numbers $1, 2, \ldots n$ and for $\sigma \in S_n$ the bottom row contains $\sigma(1), \sigma(2) \ldots$. For example the identity element ε with $\varepsilon(x) = x$

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}$$

Similarly a bijection τ that sends 1 to 2, 2 to 3 $\ldots n$ to 0 can be written as:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 3 & 4 & \cdots & 1 \end{pmatrix}$$

Proposition 3: Number of elements in symmetric groups

$|S_n| = n!$

For the first element we have n choices and the next one we have $n-1$ then $n-2$ up until 1 to the total number of choices is $n \times (n-1) \times \cdots \times 1 = n!$. Another proposition that this leads to is:

Proposition 4

For $n \geq 3$, the group S_n is not non-abelian

Proof. Assume for $n \geq 3$, S_n is abelian. We define 2 bijections:

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 3 & 2 & 1 & \cdots & n \end{pmatrix} \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 1 & 3 & \cdots & n \end{pmatrix}$$

So σ just swaps 1 and 2 and τ cycles through $1, 2, 3$. Both the bijections leave $4, 5, 6 \ldots n$ fixed. So we have $(\sigma \circ \tau)(1) = \sigma(3) = 3$ but $(\tau \circ \sigma)(3) = \tau(3) = 1$ therefore we have $(\tau \circ \sigma) \neq (\sigma \circ \tau)$ therefore the group is non abelian. $\square$

7 Homomorphisms and Cosets

7.1 Group Homomorphisms

Definition 9: Group Homomorphisms

Let G_1 and G_2 be groups a function $\phi : G_1 \to G_2$ is called homomorphism if for all elements $a, b \in G_1$ we have

$$\phi(ab) = \phi(a) \cdot \phi(b)$$

Here $\cdot$ is the binary operation in G_2

Examples

Example 1

For any 2 groups there is always a homomorphism called the trivial homomorphism $\phi : G_1 \to G_2$ given by $\phi(a) = e_{G_2}$ for all $a \in G_1$ here e_{G_2} is the identity element of G_2.

Example 2

For any positive integer n, the "Reduction modulo n" map $\phi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ defined by $\phi(m) = [m]$ is a homomorphism with respect to $+$ operation.

$$\phi(m_1 + m_2) = [m_1 + m_2]$$

From the example on Page 6 we know that (by definition)

$$[m_1 + m_2] = [m_1] + [m_2]$$

Here the $+$ denotes different operations one is in $\mathbb{Z}$ and other one is in $\mathbb{Z}/n\mathbb{Z}$

Theorem 7: Properties of Homomorphisms

Let $\phi : G_1 \to G_2$ be a homomorphism. Then

(1) ϕ preserves the identity: $\phi(e_{G_1}) = e_{G_2}$

(2) ϕ preserves the inverse: $\phi(g^{-1}) = \phi(g)^{-1}$

(3) ϕ preserves the powers: $\phi(g^m) = \phi(g)^m$

(4) The composition of 2 homomorphisms, $\psi : G_2 \to G_3$ and $\phi : G_1 \to G_2$ then $\psi \circ \phi : G_1 \to G_3$ is also an homomorphism.

Proof. (1)

$$\phi(e_{G_1}) = \phi(e_{G_1} \cdot e_{G_1}) = \phi(e_{G_1}) \cdot \phi(e_{G_1})$$

Then

$$\phi(e_{G_1})^{-1}\phi(e_{G_1}) = \phi(e_{G_1})^{-1}\phi(e_{G_1}) \cdot \phi(e_{G_1})$$

$$e_{G_2} = \phi(e_{G_1})$$

(2) For each $g \in G$ we have

$$\phi(g)\phi(g^{-1}) = \phi(gg^{-1}) = \phi(e_{G_1}) = e_{G_2}$$

Similarly we have $\phi(g^{-1})\phi(g) = e_{G_2}$. By the uniqueness of inverse this shows that $\phi(g^{-1}) = \phi(g)^{-1}$

(3) For $k = 0$ we have $\phi(g^0) = \phi(e_{G_1}) = e_{G_2} = \phi(g)^0$. Now assuming it holds for some $k \in \mathbb{N}$, consider $k + 1$

$$\phi(g^{k+1}) = \phi(g^k \cdot g) = \phi(g^k)\phi(g) = \phi(g)^k\phi(g) = \phi(g)^{k+1}$$

If $k < 0$ then $k = -m$

$$\phi(g^{-m}) = \phi(g^{-1})^m = (\phi(g)^{-1})^m = \phi(g)^{-m}$$

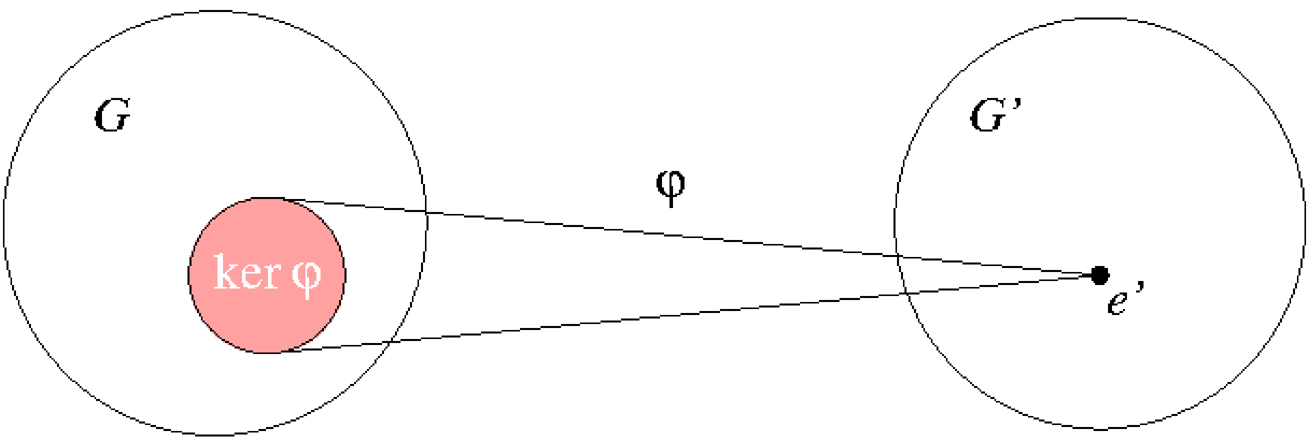
(4) Consider $\psi \circ \phi(g \cdot h)$ We have $\psi(\phi(g) \cdot \phi(h)) = \psi(\phi(g)) \cdot \psi(\phi(h))$ since ϕ and ψ are both homomorphisms we by definition $\psi \circ \phi$ is also an homomorphism.

□

Definition 10: kernel

If $\phi : G_1 \to G_2$ is a homomorphism then the *kernel* $\ker\phi$ is the set

$$\ker\phi = \{g \in G_1 : \phi(g) = e_{G_2}\}$$



Theorem 8

(1) The set $\ker \psi$ is a subgroup of G_1

(2) The function ψ is injective if and only if $\ker \psi = \{e_{G_1}\}$

Proof. (1) We can apply the subgroup test from Theorem 6, First $\ker \phi$ is non empty because e_{G_1} always belongs to the kernel. Now if $a, b \in \ker \phi$ then we need to show that $ab^{-1} \in \ker \phi$. Using the properties of homomorphisms

$$\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)\phi(b)^{-1} = e_{G_2} \cdot e_{G_2} = e_{G_2}$$

(2) Suppose $\ker \phi$ is injective. Now suppose $g \in \ker \phi$ and let $\phi(g) = e_{G_2} = \phi(e_{G_1})$ since ϕ is injective we have $\phi = e_{G_2}$, so if ϕ is injective then $\ker \phi = \{e_{G_1}\}$.

Now assume that $\ker \phi = \{e_{G_1}\}$ suppose we have $a, b \in G_1$ such that $\phi(a) = \phi(b)$.

$$\phi(b)^{-1}\phi(a) = \phi(b)\phi(b)^{-1} = e_{G_2}$$

The right hand side simplifies to $\phi(ab^{-1})$ since we have $ab^{-1} \in \ker \phi$ then $ab^{-1} = e_{G_1}$ this means we have $a = b$. Thus if $\ker \phi = \{e_{G_1}\}$ then ϕ is injective.

□

7.2 Cosets

We can describe equivalence classes on group elements using homomorphisms. If we have $\phi : G_1 \to G_2$, then let

$$H = \ker \phi$$

Then for $a, b \in G$ we have $a \sim b$ if and only if $\phi(a) = \phi(b)$ but from the subgroup test we also know that $\phi(ab^{-1}) = e_{G_2}$ which is because we have $ab^{-1} \in H$.

This condition can be generalized to subgroups other than $\ker \phi$

Theorem 9

Let G be a group and let H be a subgroup of G. We define the relation $\sim$ on G. We have $a \sim b$ if and only if $ab^{-1} \in H$. Then $\sim$ is an equivalence relation and the equivalence class of an element $a \in G$ is the set $Ha = \{ha : h \in H\}$ which is called the right coset of H generated by a.

Proof.

- **Reflexive** For any $a \in G$ we have $a \sim a$ since $aa^{-1} = e \in H$ since H is subgroup.

- **Symmetric** Assume we have $a \sim b$, so we have $ab^{-1} \in H$. Since H is closed $(ab^{-1})^{-1} = ba^{-1} \in H$ which implies $b \sim a$.

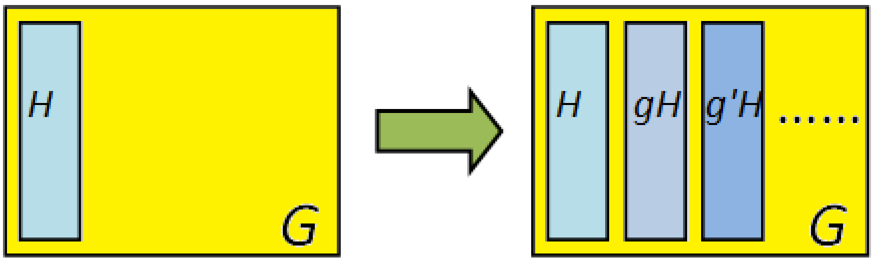
- **Transitive** Assume we have $a \sim b$ and $b \sim c$ that means we have $ab^{-1} \in H$ and $bc^{-1} \in H$ since H is closed
$$ab^{-1}bc^{-1} = aec^{-1} = ac^{-1} \in H$$
Therefore we have $a \sim c$

This proves that $\sim$ is an equivalence relation. Now choose $a \in G$ by definition we have:

$$\begin{aligned}[a] &= \{g \in G : g \sim a\}\\ &= \{g \in G : ga^{-1} \in H\}\\ &= \{g \in G : ga^{-1} \in H\}\\ &= \{g \in G : ga^{-1} = h \in H\}\\ &= \{g \in G : g = ha \in H\}\\ &= Ha\end{aligned}$$

This shows that the equivalence class of $[a]$ is the right coset generated by a. $\square$

We can define another relation $\sim_L$ with $a \sim_L b$ if $b^{-1}a \in H$, resulting in the equivalence class of the form $aH = \{ah : h \in H\}$ which is called the *left coset* of a. If G is abelian,then $aH = Ha$.



An important fact used for proving theorems involving costs is

Proposition 5

$Ha = Hb \iff ab^{-1} \in H$

Proof. ($\Rightarrow$) Let $Ha = Hb$ then $a = hb$ for some $h \in H$ then applying b^{-1} leads to $ab^{-1} = h \in H$. For the ($\Leftarrow$) part assume we have $ab^{-1} \in H$. Then $ab^{-1} = h \rightsquigarrow a = hb \rightsquigarrow a \in Hb \Rightarrow Ha = Hb$ (Since a is arbitrary and the argument is symmetric in a, b). $\square$

Example

Let $G = \mathbb{Z}$. Let n be a positive integer, the set

$$H = n\mathbb{Z} = \{m \in \mathbb{Z} : m = nk \text{ for some } k \in \mathbb{Z}\}$$

This is a subgroup of $\mathbb{Z}$. Since the group operation is $+$ we can define the cosets as $n\mathbb{Z} + a$. Moreover, if we have $b \in n\mathbb{Z} + a$ if and only if $b \sim a$ this holds true if $b - a \in n\mathbb{Z}$ this is same as $n|(b - a)$ so b belongs to the equivalence class of $[a] = n\mathbb{Z} + a$ if and only if $b \equiv a \bmod n$, thus this equivalence class here is the equivalence class of congruence modulo n

8 Normal Subgroups and Quotient Groups

Definition 11: Normal Subgroup

Let G be a group, a subgroup H is called a *normal* subgroup if $gHg^{-1} = H$ defined by

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

In this case the notation $H \lhd G$ is used to denote that H is a normal subgroup of G.

The conjugate subgroup shown to be a sub group in in Example 2. If G is abelian then we have the following:

Proposition 6

Every subgroup of an abelian group is normal.

Proof. Consider a subgroup H of G an abelian group. Now for any $h \in H$ consider

$$ghg^{-1} = g(hg^{-1}) = (gg^{-1})h = eh = h$$

So we have

$$gHg^{-1} = H$$

therefore H is normal and we have, $H \lhd G$. $\square$

Another important theorem that follows is:

Theorem 10

Let G be a group. For every subgroup H of G the product $(Ha)(Hb) = Hab$ is well defined multiplication of cosets if and only if $H \lhd G$.

Proof. ($\Rightarrow$) Let $(Ha)(Hb) = Hab$. Now consider some $h \in H$ and the cosets

$$Hh = He \text{ and } Hg = Hg$$
$$\rightsquigarrow (Hg)(Hh) = (Hg)(He)$$
$$Hgh = Hge$$
$$\text{Since } H \text{ is a subgroup } (gh)(ge)^{-1} \in H \Rightarrow ghg^{-1} \in H$$

This proves that $gHg^{-1} \subseteq H$, taking g^{-1} in place of g above we get $g^{-1}Hg \subseteq H$. This directly implies $H \subseteq gHg^{-1}$

($\Leftarrow$) Conversely, assume that $H \lhd G$. Suppose we have $a, b, a_1, b_1 \in G$ such that $Ha = Ha_1$ and $Hb = Hb_1$ then we have $aa_1^{-1} \in H$ and $bb_1^{-1} \in H$. We need to show $(Ha)(Hb) = (Ha_1)(Hb_1)$. This is equivalent to $(ab)(a_1b_1)^{-1} \in H$. Then we have

$$abb_1^{-1}a_1^{-1} = a(bb_1^{-1})a_1^{-1} = (a(bb_1^{-1})a^{-1})(aa_1^{-1})$$

Now, $aHa^{-1} = H$ and $bb_1^{-1} \in H$, so we have $(a(bb_1^{-1})a^{-1}) \in aHa^{-1} = H$. Also $aa_1^{-1} \in H$ then we get that $(a(bb_1^{-1})a^{-1})(aa_1^{-1}) = a(bb_1^{-1})a_1^{-1} \in H$. Thus multiplication of right cosets is closed. $\square$

Some important properties of subgroups are:

Theorem 11: Properties of Quotient groups

Let G be a group and $H \lhd G$ then

(1) The set G/H if right costs of H is a group under the operation $(Ha)(Hb) = Hab$ called the *quotient* group of G by H.

(2) The function $\phi : G \to G/H$ given by $\phi(g) = Hg$ is a surjective homomorphism, called the *quotient mapping.*

(3) If G is abelian, then G/H is abelian.

(4) If G is cyclic then G/H is cyclic.

Proof. (1) Theorem 10 tells us that the the operation is well defined when H is normal. For associativity consider

$$((Ha)(Hb))(Hc) = (Hab)(Hc) = H(ab)c = Ha(bc) = (Ha)(Hbc) = (Ha)((Hb)(Hc))$$

The identity element is the coset $H = He$ and inverse of Ha is Ha^{-1}. Thus G/H is a group.

(2) Consider

$$\phi(ab) = Hab = (Ha)(Hb) = \phi(a)\phi(b)$$

Hence, ϕ is a homomorphism. For surjective,for any coset $Ha \in G/H$ we have $\phi(a) = Ha$.

(3) If G is abelian,then consider

$$(Ha)(Hb) = Hab = Hba = (Hb)(Ha)$$

Hence the operation on G/H is commutative.

(4) If we have $G = \langle g \rangle$ for some $g \in G$ then every element of G is in the form g^k for some $k \in \mathbb{Z}$, thus given $Ha \in G/H$ we know that $a = g^k$ for some $k \in \mathbb{Z}$. Then $Ha = Hg^k = \phi(g^k) = \phi(g)^k = (Hg)^k$ where ϕ is the quotient homomorphism. Since ϕ is surjective we have $G/H = \langle Hg \rangle$, so G/H is also cyclic.

$\square$

An example of quotient group is the group $\mathbb{Z}/n\mathbb{Z}$ it has elements in the form $n\mathbb{Z}+a$ which are the equivalence classes under the relation of congruence modulo n.

Example

Consider the group $(\mathbb{Q}, +)$, then $\mathbb{Z}$ is a group. Since $\mathbb{Q}$ is abelian, $\mathbb{Z}$ is automatically a subgroup. The elements of the quotient group $\mathbb{Q}/\mathbb{Z}$ are of the form $\mathbb{Z} + q$ for $q \in \mathbb{Q}$.
Every element of $\mathbb{Q}/\mathbb{Z}$ has a unique representative of the form $\mathbb{Z} + \delta$ where $0 \leq \delta \leq 1$. For any $q \in \mathbb{Q}$ we can round down $||$ using the floor function $\lfloor q \rfloor$ then we have

$$0 \leq q - \lfloor q \rfloor \leq 1$$

So we can set $\delta = q - \lfloor q \rfloor$. If we had δ' such that $\delta + \mathbb{Z} = \delta'\mathbb{Z}$ then $\delta - \delta' \in \mathbb{Z}$. but due to the given constrains on δ we have $-1 < \delta - \delta' < 1$ the only integer is 0 so we have $\delta = \delta'$.
The group is countably infinite but every element has a finite order. For any given coset $\mathbb{Z} + q$, we write $q = \frac{a}{b}$ where a, b are integers and $b \neq 0$. Note that then $b(\mathbb{Z} + q) = \mathbb{Z} + bq = \mathbb{Z} + a = \mathbb{Z} + 0$ So the order of the coset $\mathbb{Z} + \frac{a}{b}$ is at most b.

9 Lagrange's Theorem

An important definition is needed before the main theorem

Definition 12: Index

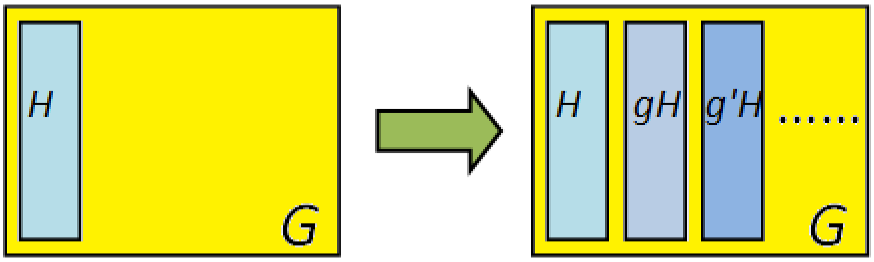
Let G be a group and let H be a subgroup. The *index* of H in G denoted by $|G : H|$ is the number of distinct right cosets of H in G.

In particular if $H \lhd G$ then $|G : H|$ is the number of elements in G/H. Now consider Lagrange's Theorem.

Theorem 12: Lagrange's Theorem

Let G be a finite group, and let H be a subgroup of G. Then $|H|$ divides $|G|$ and $|G : H| = \frac{|G|}{|H|}$.

Proof. Since the right cosets of H in G form a partition of G in the form Ha_1, $Ha_2, \ldots Ha_n$ The union of



$Ha_1, Ha_2, \ldots$ is G (they are equivalence classes) and $Ha_i \cap Ha_j = \varnothing$ if $i \neq j$. For any a_i we have $|H| = |Ha_i|$ since mapping h to ha_i is a bijection.

Thus G is a disjoint union of $n = |G : H|$ cosets, each with size $|H|$. So we have $|G : H||H| = |G|$ and hence $\frac{|G|}{|H|} = |G : H|$ is an integer. In particular $|G|$ is a multiple of $|H|$. $\square$

The theorem immediately leads to the following

Corollary 1

If G is finite group and $g \in G$ then $o(g)$ divides G

Proof. Consider the subgroup $H = \langle g \rangle$ and we know that $|H| = o(g)$ the corollary follows immediately from Lagrange's Theorem. $\square$

Corollary 2

If G is a finite group with $|G| = n$ then for all $g \in G$ we have $g^n = e$

Proof. Let $g \in G$ and $k = o(g)$ divides n by the above corollary. Thus $n = k\ell$ for integer ℓ then $g^n = g^{k\ell} = (g^k)^\ell = e^\ell = e$ $\square$

Corollary 3

If $|G| = p$ where p is prime then every G is cyclic. For any non-identity element we have $G = \langle g \rangle$

Proof. Let G be a group with $|G| = p$. Since $p \geq 2$ there is a non-identity element $g \in G$. We set $H = \langle g \rangle$. Then $|H| > 1$ since the generator of H has order larger than 1. But $|H|$ must divide $|G|$ by Lagrange's Theorem. The only multiples of $|G| = p$ are 1 and p itself since p is prime. Thus we must have $|H| = p = |G|$, so $H = G = \langle g \rangle$ showing G is cyclic. $\square$

10 Introduction to Rings

Definition 13: Ring

A ring is a structure equipped with two binary operations denoted by addition $(+)$ and multiplication. With the following conditions

#1 The set $(R, +)$ is an abelian group. (The identity is denoted by 0).

#2 The set $(R, \cdot)$ is a monoid. (The identity of the monoid is denoted by 1).

#3 Left and right distributive laws hold. That is for all $a, b, c \in R$

$$a(b + c) = ab + ac$$

$$(a + b)c = ac + bc$$

If $\cdot$ is commutative then R is called a *commutative ring.* Some examples are:

Example 1 The sets $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ with $\times, +$ are all commutative rings.

Example 2 For any positive integer n, the set of integers modulo n $\mathbb{Z}/n\mathbb{Z}$. For the addition operation the previously defined addition will be used

$$[a] + [b] = [a + b]$$

For the multiplication operation it is defined as follows

$$[a] \cdot [b] = [ab]$$

Suppose we have a, a', b, b' such that $[a] = [a']$ and $[b] = [b']$. So we have $a - a' = kn$ and $b - b' = \ell n$ we want to show $[ab] = [a'b']$, so $ab - a'b'$ is a multiple of n.

$$ab - a'b' = ab - ab' + ab' + a'b' = a(b - b') + b'(a - a') = a(\ell n) + b'(kn) = n(a\ell + b'k)$$

This proves that $ab \equiv a'b' \bmod n$ so $[ab] = [a'b']$. The identity is $[1]$ and it is easy to check that this is commutative. We need to check for the distributive laws.
Let $[a], [b], [c] \in \mathbb{Z}/n\mathbb{Z}$. Then notice that

$$[a] \cdot ([b] + [c]) = [a] \cdot ([b + c]) = [a(b + c)] = [ab + ac] = [ab] + [ac] = [a][b] + [a][c]$$

Since $\cdot$ is commutative the other case would be symmetric. So $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring.

Example 3 Let $(G, +)$ be a group. We let $End(G)$ denote the set of homomorphism $G \to G$. This is called the set of ***endomorphisms*** of G. We can define addition on this set as follows:

$$(\phi + \psi)(g) = \phi(g) + \psi(g)$$

We can show this is closed

$$\begin{aligned}
(\phi + \psi)(g + h) &= \phi(g + h) + \psi(g + h) \\
&= \phi(g) + \phi(h) + \psi(g) + \psi(h) \\
&= (\phi(g) + \psi(g)) + (\phi(h) + \psi(h)) \\
&= (\phi + \psi)(g) + (\phi + psi)(h)
\end{aligned}$$

This uses the fact that G is abelian. The identity element is the $\mathbf{0} : G \to G$ given by $\mathbf{0}(g) = 0$ since for any $\phi(g)$ we have $(\phi + \mathbf{0})(g) = \phi(g) + \mathbf{0}(g) = \phi(g) + 0 = \phi(g)$. For inverse the inverse is $-\phi(g)$.

$$(\phi + (-\phi))(g) = \phi(g) - \phi(g) = 0$$

For associativity consider $(\phi + \psi + \pi)(g)$

$$((\phi + \psi)(g) + \pi(g)) = (\phi(g) + \psi(g) + \pi(g)) = (\phi(g) + (\psi + \pi)(g))$$

Multiplication is defined as

$$\phi\psi = \phi \circ \phi$$

Since composition of homomorphism is a homomorphism we get $\phi\psi \in Eng(G)$. The identity is given by $\iota(g) = g$. To check for distributive law consider:

$$\begin{aligned}
\phi(\psi + \pi)(g) &= \phi(\psi(g) + \pi(g)) \\
&= \phi(\psi(g)) + \phi(\pi(g)) \\
&= \phi\psi(g) + \phi\pi(g) \\
&= (\phi\psi + \phi\pi)(g)
\end{aligned}$$

10.1 Properties of Rings and Definitions

Theorem 13

Let 0 be the additive identity for any $a \in R$ we have

$$a \cdot 0 = 0 = 0 \cdot a$$

Proof.

$$a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$$
$$a \cdot 0 - a \cdot 0 = a \cdot 0 + a \cdot 0 - a \cdot 0$$
$$0 = a \cdot 0$$

Similar argument for $0 \cdot a$ will apply. $\square$

Theorem 14

Let R be a ring and let $a, b \in R$ then

$$(-a)b = a(-b) = -(ab)$$

$$(-a)(-b) = (ab)$$

Proof. Since additive inverse of an element is unique we need show that both $a(-b)$ and $(-a)(b)$ are inverses of (ab).

$$(-a)(b) + (ab) = (-a + a)b = 0 \cdot b = 0$$

The same argument will apply for $(a)(-b)$ thus by uniqueness of inverse $(-a)b = a(-b) = -(ab)$.

Now we can apply the first part to get

$$(-a)(-b) = -(a(-b)) = -(-(ab)) = ab$$

$\square$

Definition 14: Characteristic of ring

The characteristic of a ring denoted by $CharR$ is the order of multiplicative identity 1 in the group under addition. If the order of 1 is not finite then $CharR = 0$

For the rings $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ all of them have characteristic 0. While any since there is no positive integer n such that $n \cdot 1 = 0$. For the ring $\mathbb{Z}/n\mathbb{Z}$ it has characteristic n since $n[1] = [n] = [0]$.

Theorem 15

If $Char(R) > 0$ then for $r \in R$ we have $k \cdot r = 0$ if and only if $n|k$. If $Char(R) = 0$ then $k \cdot r = 0$ if and only if $k = 0$.

Proof. Suppose $n = Char(R)$ and let k such that $n|k$. Let $r \in R$. We have $k = mn$ for some m. Then $(mn) \cdot r = ((mn) \cdot 1) \cdot r$ by distributivity. Since we have $n \cdot 1 = 0$ we get

$$r((mn) \cdot 1) = r(m(n \cdot 1)) = r(m \cdot 0) = r0 = 0$$

Now suppose k is an integer such that for all $r \in R$ we have $r \cdot k = 0$. Then $k \cdot 1 = 0 = n \cdot 0$. So R must be a multiple of $o(1)$ in the additive group. Using the Theorem 5 for groups in additive notation we have $k = 0 \mod n$ therefore $n|k$

Finally, suppose $Char R = 0$ if $k = 0$ then $0 \cdot r = 0$. Conversely if $k \cdot r = 0$ then $k \cdot 1 = 0$ since $o(1) = \infty$ this only happens when $k = 0$.

□

11 Subrings and Homomorphisms

Definition 15: Sub ring

Let R be a ring. S is a subring of R, if addition and multiplication on R restrict to binary operations on S, and S is a ring with respect to those operations. Moreover

$$\mathbf{1}_R = \mathbf{1}_S$$

Their multiplicative identities are the same.

The following example shows why we needed the $1_R = 1_S$ condition.

Example 1 Consider the Ring $\mathbb{Z}/6\mathbb{Z}$. The subset $S = \{[0], [2], [4]\}$ of even equivalence classes. $(S, +)$ is an abelian group moreover it is a cyclic group generated by $\langle [2] \rangle$. It is also closed under multiplication. Here $[4]$ acts as the identity for any $a \in S$ we have $4 \times [a] = [a]$. But this is not a subring since $\mathbf{1}_{\mathbb{Z}} \neq \mathbf{1}_{\mathbb{Z}/n\mathbb{Z}}$.

The set $\mathbb{Z}$ is a subring of $\mathbb{Q}, \mathbb{R}$. It satisfies all the conditions and the multiplicative identity is same in all 2 sets.

Theorem 16: Sub-Ring Test

Let S be a sub set of $(R, +, \times)$. S is a sub ring if the following conditions hold:

- $1_R \in S$
- if $a, b \in S$ then $a - b \in S$
- if $a, b \in S$ then $ab \in S$.

Proof. Assume $S \subseteq R$ satisfies all the conditions. Then by the second condition $(S, +)$ is a group (Sub group test). Using the third condition $S, \times$ is closed under addition. Associatively holds because it held in R. Since we have $1_R \in S$ then $1_R \cdot s = s \cdot 1_R = s$ so by uniqueness of identity element $1_R = 1_S$. The distributive law holds since it holds in R.

Conversely assume S is a sub ring of R. First since $(S, +)$ is a subgroup by the subgroup test the condition holds. Since S is closed under multiplication the third condition holds. Finally we know that $1_S = 1_R$ so we must have $1_R \in S$ to the first condition holds as well.

□

Example 2 Center of a ring

11.1 Ring Homomorphisms

Definition 16: Ring Homomorphism

A function $\phi : R \to S$ for rings S, R is called a *ring homomorphism* if the following conditions hold:

$$\phi(a + b) = \phi(a) + \phi(b) \tag{1}$$

$$\phi(ab) = \phi(a)\phi(b) \tag{2}$$

$$\phi(\mathbf{1_R}) = \mathbf{1_S} \tag{3}$$

———EXAMPLES———

Theorem 17: Properties of Ring homomorphisms

Let $\phi : R_1 \to R_2$ be a ring homomorphism. Then the following hold:

$$\phi(0) = 0 \tag{1}$$

$$\phi(-r) = -\phi(r) \tag{2}$$

$$\phi(kr) = k\phi(r) \text{ For } k \in \mathbb{Z} \tag{3}$$

$$\phi(r^n) = \phi(r)^n \text{ For } n \in \mathbb{N} \tag{4}$$

$$\phi(r^k) = \phi(r)^k \text{ For all } k \in \mathbb{Z} \text{ if } r \text{ is a } \textit{unit} \tag{5}$$

12 Ideals and Quotient Rings

12.1 Quotient Rings

For defining quotient rings. We need to prove fundamental result about the relationship between kernels and normal subgroups.

Theorem 18

Let G be a group

(1) If G_1 is any group and $\phi : G \to G_1$ is a homomorphism then $\ker\phi$ is a normal subgroup of G.

(2) If H is a normal subgroup of G then there is a group homomorphism $\phi : G \to G_1$ such that $H = \ker\phi$.

Proof. (1) Suppose we have a homomorphism $\phi : G \to G_1$ and the set $K = \ker\phi$. We already know that K is a subgroup of G. Suppose we have $h \in gKg^{-1}$ then $h = gkg^{-1}$ for some $k \in K$. Then we get

$$\phi(h) = \phi(gkg^{-1}) = \phi(g)\phi(k)\phi(g^{-1}) = \phi(g)e\phi(g^{-1}) = \phi(g)\phi(g)^{-1} = e$$

Therefore we have $h \in K$. So we know that $gKg^{-1} \subseteq K$. Taking g^{-1} in place of g we get $g^{-1}Kg \subseteq K$ this implies $K \subseteq gKg^{-1}$ hence we have $K = gKg^{-1}$ by definition $K = \ker\phi \lhd G$

(2) Suppose $H \lhd G$, let $G_1 = G/H$ and consider the quotient homomorphism $q : G \to G/H$ given by $q(g) = Hg$ we have $g \in \ker q$ if and only if $Hg = He$ that implies $e^{-1}g \in H$ that means we have $g \in H$. So we have $H = \ker q$

□

This theorem shows that normal subgroups of G same as the kernels of homomorphisms of G.

Definition 17: Kernel of Rings

Let R, S be rings and let $\phi : R \to S$ be a group homomorphism the kernel is given by

$$\ker \phi = \{r \in R : \phi(r) = 0_S\}$$

Since the way ring homomorphisms are constructed the kernel of a ring homomorphism with domain R is automatically a additive subgroup of R.

12.2 Ideals

Definition 18: Ideals

Let R be a ring. A subset I of R is called an ideal if

#1 I is a subgroup of the additive group R.

#2 I *absorbs* multiplication. That is, if $r \in I$ and $a \in R$, then $ra, ar \in I$.

Example 1 If R is a ring then both R and $\{0\}$ are ideals of R. The set $\{0\}$ is the trivial subgroup of $(R, +)$ and for any $r \in R$ we have $r \cdot 0 = 0 \cdot r = 0 \in \{0\}$. This is called the *zero ideal* of R.

Example 2 For any $n \in \mathbb{N}$ the additive groups $n\mathbb{Z}$ are ideals of $\mathbb{Z}$. We know that $n\mathbb{Z}$ is a subgroup. To check for the absorption property, for any $m \in n\mathbb{Z}$ we have $m = nk$ it follows that

$$\ell m = m\ell = (nk)\ell = n(k\ell) \in n\mathbb{Z}$$

Theorem 19

Let R be a ring and $I \subseteq R$ be an ideal. Then the set of right cosets R/I can be given the structure of a ring with addition defined as $(I + a) + (I + b) = I + a + b$ and multiplication being $(I + a)(I + b) = I + ab$

Proof. We know that I is an additive group of R, since $(R, +)$ is an abelian group, every subgroup is normal and R/I is also an abelian group.

Now we need to check that multiplication is well defined, suppose $a, b, a', b' \in R$ such that $I + a = I + a'$ and $I + b = I + b'$, this means that we have $a - a' \in I$ and $b - b' \in I$ we have to show that $I + ab = I + a'b'$ which means we have to show $ab - a'b' \in I$

$$ab - a'b' = ab - a'b + a'b - a'b' = (a - a')b + a'(b - b')$$

Since $a - a' \in I$ and it absorbs multiplication we have $(a - a')b \in I$ and same applies for $a'(b - b') \in I$. Finally I is closed under addition so we get $ab - a'b' \in I$

The element $I+1$ is the multiplicative identity since $(I+a)(I+1) = I+a\cdot 1 = I+a$ we can easily check that this operation is associative. Now for the distributive property

$$\begin{aligned}((I+a)+(I+b))(I+c) &= (I+a+b)(I+c)\\ &= (I+c(a+b))\\ &= I+ca+cb\\ &= (I+ca)+(I+cb)\\ &= (I+a)(I+c)+(I+b)(I+c)\end{aligned}$$

□

The theorem connecting kernels with groups and normal subgroup also holds for rings and ideals.

Theorem 20

Let R be a ring

(1) Let S be any ring and $\phi : R \to S$ be a ring homomorphism then $\ker\phi$ is an ideal of R.

(2) Let I be an ideal of R then there is a ring R_1 such that $\phi : R \to R_1$ is a ring homomorphism and $I = \ker\phi$

Proof. (1) Let $K = \ker\phi$, K is closed under multiplication since we have let $a \in K$ and $b \in R$ so we get $\phi(ab) = \phi(a)\phi(b) = 0_S \cdot 0_S$ therefore $ab \in K$ absorbs multiplication. we already know that $K \lhd (R,+)$ so I is an ideal.

(2) Let I be an ideal of R consider the quotient mapping $q : R \to R/I$ given by $q(a) = I+a$, we can check by definition of the operations on R/I that q is a ring homomorphism. Now let $a \in \ker q$ which means we have $q(a) = I+a = I+0$, which holds if $a \in I$ therefore $I = \ker q$

□

Example The sets $n\mathbb{Z}$ are ideals for the ring $\mathbb{Z}$ then the coset $n\mathbb{Z}+m$ corresponds to the equivalence class $[m]$ under congruence modulo n. The multiplication

$$(n\mathbb{Z}+m_1)(n\mathbb{Z}+m_2) = n\mathbb{Z}+m_1m_2$$

This corresponds to the multiplication $[m_1][m_2] = [m_1m_2]$