Math 146 Week 2

Thaqib Mo.

January 30, 2021

1 Linear Combinations and Systems of Linear Equations

Definition 1.1: linear combination

Let V be a vector space over $\mathbb{F}$ and let S be a non-empty subset of V. Then $x \in V$ is a linear combination of S if there are a **finite number** of $u_1, u_2, \ldots, u_n \in S$ and $a_i \in \mathbb{F}$ such that

$$x = a_1 u_1 + a_2 u_2 + \ldots + a_n u_n$$

Definition 1.2: span(S)

Let V be a vector space and $S \subseteq V$ with $S \neq \varnothing$. Then $\text{span}(S)$ is all the set of all Linear combinations of S. We define $\text{span}(\varnothing) = \{\mathbf{0}\}$.

Example 1.1. Let $S = \{(0,1,0),(1,0,0)\}$ for $V = \mathbf{R}^3$. Then $\text{span}(S) = \{a(0,1,0) + b(1,0,0) : a, b \in \mathbf{R}\} = \{(a,b,0) : a, b \in \mathbf{R}\}$. This is the $xy-$plane.

Example 1.2. Let $V = M_{2\times 2}(\mathbf{R})$. and we define

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Then let $x = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$. Does $x \in \text{span}(S)$? Then we must have

$$a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + c \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ -a+b & b \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

This system has no solutions. So $x \notin \text{span}(S)$.

Theorem 1.1

Let S be a subset of V. Then $\text{span}(S)$ is a subspace of V. Moreover, $\text{span}(S)$ is the smallest subspace of V containing S.

- $\text{span}(S)$ is a subspace of V containing S
- If W is any other subspace of V containing S then $\text{span}(S) \subseteq W$.

Proof. Let $S \neq \varnothing$ since the $\varnothing$ case is trivial and $\{0\}$ is a subspace of V. The first case of subspaces is satisfied. Let $u \in S$, then $0 \cdot u = \mathbf{0} \in \text{span}(S)$. So $\text{span}(S) \neq \varnothing$. Now consider $x \in \text{span}(S)$ and let $c \in \mathbb{F}$. Then consider $cx = c(\sum_{i=1}^{n} a_i u_i)$ for $u_i \in S$ and $a_i \in \mathbb{F}$. Then using the generalized distributive property we have $cx = ca_1 u_1 + ca_2 u_2 + \ldots + ca_n u_n \in \text{span}(S)$.

Now for closed under addition. Let $x, y \in \text{span}(S)$. By definition we have $u_1, u_2 \dots u_n \in S$ x, y can be written as linear combinations of u_i.

$$x = a_1u_1 + \ldots + a_nu_n \quad y = b_1u_1 + b_2u_2 + \ldots + b_nu_n$$

Then we have using associative, commutative and distributive property of V we have

$$x + y = (a_1 + b_1)u_1 + \ldots + (a_n + b_n)u_n \in \text{span}(S)$$

So for all $x \in S$ we have $x \in \text{span}(S)$ since $1 \cdot x = x$. Assume W is a W is a subspace of V and $S \subset W$. Let x be a linear combination of $u_i \in S$. So we have $x = a_1u_1 + \ldots + a_nu_n$. Since $S \subseteq W$. Then $u_1, u_2, \ldots \in W$. Since W is a subspace then $a_1u_1 + \ldots a_nu_n \in W$ (Closed under addition and scalar multiplication). Therefore $\text{span}(S) \subseteq W$. $\square$

Definition 1.3: Spans

Let $S \subseteq V$ we say S **generates/spans** V if $\text{span}(S) = V$.

Since for any vector space and set S we always $\text{span}(S) \subset V$ showing $V \subset \text{span}(S)$ is enough to show $\text{span}(S) = V$.

2 Linear Dependence and Linear Independence

Definition 2.1: Linear Dependence and Independence

A set S is called *linearly dependent* if there exists a **finite number** of **distinct** vectors $u_{i=0}^n \in S$ and $c_{i=0}^n \in \mathbb{F}$, such that all c_i are **not zero** such that

$$c_1u_1 + c_2u_2 + \ldots + c_nu_n = \mathbf{0}$$

A set S is called *Linearly independent* if it's not linearly dependent.

Note 2.1. If can show that $c_1u_1 + c_2u_2 + \ldots + c_nu_n = \mathbf{0} \iff c_{i=0}^n = 0$ for all vectors $\{u_1, u_2, \ldots, u_n\} = S$, then S is linearly independent.

Theorem 2.1: Linear Dependence Condition

Let $S \subseteq V$ of a vector space V of $\mathbb{F}$. Then S is linearly dependent if and only if $S = \{0\}$ or some $u \in S$ which is the linear combination of other vectors in S.

Proof. ($\Leftarrow$) $S = \{0\}$ is already linearly dependent. Now assume some $x \in S$ is a linear combination of other vectors in S. So let

$$x = c_1u_1 + c_2u_2 + \dots c_nu_n$$
$$\rightsquigarrow c_1u_1 + \dots + c_nu_n - x = \mathbf{0}$$
$$c_1u_1 + \dots + c_nu_n(-1)x = \mathbf{0}$$

By definition S is linearly dependent. ($\Rightarrow$) Assume S is linearly dependent. By definition we have

$$c_1u_1 + c_2u_2 + \dots + c_nu_n = \mathbf{0}$$

For distinct u_i and $(c_1, c_2, \dots, c_n) \neq (0, 0, \dots, 0)$. Now without loss of generality we can assume all c_i are non zero and we can remove any 0. So now we have 2 cases $n = 1$ and $n \geq 2$.

Case I If $n = 1$ then $c_1u_1 = \mathbf{0}$ then c_1^{-1} exits and multiplying it on both sides gives $u_1 = \mathbf{0}$ so $\mathbf{0} \in S$. Now if $S \neq \{\mathbf{0}\}$ then we pick $v \in S \setminus \{\mathbf{0}\}$. Then we have $\mathbf{0} = 0 \cdot v_1$ so $\mathbf{0}$ is a linear combination of another vector in S.

Case II $n \geq 2$ Since $a_n \neq 0$ the multiplicative inverse exists.

$$c_1u_1 + c_2u_2 + \dots = -c_nu_n$$
$$-c_1u_2 - c_2u_2 + \dots = c_nu_n$$

Since c_n is non-zero all of them have a multiplicative inverse of $a_n \in \mathbb{F}$ exits, therefore we have

$$u_n = (c_n^{-1})(-c_1)u_1 + (c_n^{-1})(-c_2)u_2 + \dots$$

So u_n is a linear combination of vectors in S.

$\square$