Week 7

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1 Invertibility and Isomorphisms

Definition 1.1: Invertible Matrix

A square matrix $A \in M_{n\times n}(\mathbb{F})$ is invertible if and only if there exists $B \in M_{n\times n}(\mathbb{F})$ such that $AB = I_n$.

Definition 1.2: Invertible Linear Transformation

Let $T : V \to W$, if there exists a mapping $U : W \to V$ such that $UT = I_V$ and $TU = I_W$ then T is invertible and U is the inverse of T.

Lemma 1.1: Inverse is Unique

Let T be invertible then T^{-1} the inverse of T is unique.

Theorem 1.2

$T : V \to U$ is invertible $\iff$ T is an isomorphism.

Lemma 1.3

If T is linear and invertible then T^{-1} is also linear.

Theorem 1.4: Linear transformations and invertible matrices

Let $T : V \to W$ be a linear transformation and let α be an ordered basis for V and let β be an ordered basis for W. Then

(1) T is an isomorphism $\iff$ $[T]_\alpha^\beta$ is invertible.

(2) $A \in M_{n\times n}(\mathbb{F})$ is invertible if and only if L_A is an isomorphism.

Proof Sketch. If T is an isomorphism then $T^{-1} : W \to V$ exists and

$$\begin{aligned}
[T]_\alpha^\beta [T^{-1}]_\beta^\alpha &= [TT^{-1}]_\beta^\beta \\
&= [I_W]_\beta^\beta \\
&= \begin{bmatrix} [I_W(w_1)]_\beta & [I_W(w_2)]_\beta \cdots \end{bmatrix} \\
&= \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} & \cdots \end{bmatrix} \\
&= I_n
\end{aligned}$$

The $(\Leftarrow)$ direction define $A = [T]_\alpha^\beta$ and then use A^{-1} to prove T is an isomorphism.

For (2) Let the standard basis for $\mathbb{F}^n$ be σ_n, Then $[L_A]_{\sigma_n}^{\sigma_n} = A$. The result follows from (1). ☐

Lemma 1.5: Properties of Inverse Matrices

Let A be invertible then

I. $(A^{-1})^{-1} = A$

II. $(cA)^{-1} = \frac{1}{c}A^{-1}$ for $c \neq 0$.

III. $(A^T)^{-1} = (A^{-1})^T$

IV. If A, B are invertible then AB is invertible with

$$(AB)^{-1} = B^{-1}A^{-1}$$

V. If AB is invertible then $A, B \in M_{n\times n}$ are invertible.

Theorem 1.6

If A is a square invertible matrix then the following are equivalent

(1) A is invertible

(2) $\exists C \in M_{n\times n}$ such that $AC = I_n$

(3) $\exists B \in M_{n\times n}$ such that $BA = I_n$

2 The Change of Coordinate Matrix

Theorem 2.1: Change of Coordinate

Let α and β be ordered basis for V then we define

$$Q = [I_V]_\alpha^\beta$$

Then

#1 Q is **invertible**.

#2 For $x \in V$ we have $[x]_\beta = Q[x]_\alpha$.

Proof Sketch. Q is invertible since I_V is an isomorphism and $[x]_\beta = [I(x)]_\beta = [I]_\alpha^\beta[x]_\alpha$. □

Remark 2.1. We can compute the change of coordinate matrix for $\alpha = \{v_1, \ldots, v_n\}$

$$[I_V]_\alpha^\beta = \begin{bmatrix} [v_1]_\beta & [v_2]_\beta & \cdots & [v_n]_\beta \end{bmatrix}$$

Theorem 2.2

Let $T : V \to V$ be linear and let α and β be ordered bases for T. Let $Q = [I_V]_\alpha^\beta$ then

$$[T]_\alpha^\alpha = Q^{-1}[T]_\beta^\beta Q$$

3 Elementary Matrix Operations and Elementary Matrices

Definition 3.1: Elementary Matrix Operations

(1) Interchanging rows or columns $R_i \leftrightarrow R_j$.

(2) Multiplying any row (column) by a *non-zero* scalar $R_i \leftarrow cR_i$.

(3) Adding any scalar multiple of rows (columns) $R_i \leftarrow R_i + cR_j$.

Definition 3.2: Elementary Matrix

An $n \times n$ elementary matrix is a matrix produced by performing elementary operations on I_n.

Theorem 3.1

Let $A \in M_{m\times n}(\mathbb{F})$ suppose B is obtained by performing elementary **row operations** on A then there exists $E \in M_{m\times m}(\mathbb{F})$ such that

$$B = EA$$

Where E is the matrix obtained by performing the same operations on I_m.

Theorem 3.2

Let $A \in M_{m\times n}(\mathbb{F})$ suppose B is obtained by performing elementary **column operations** on A then there exists $E \in M_{n\times n}(\mathbb{F})$ such that

$$B = AE$$

Where E is the matrix obtained by performing the same operations on I_n

Theorem 3.3

Elementary matrices are invertible and the inverse of an elementary matrix is an elementary matrix of the same type.

Proof Sketch. We have E such that $I_m = EA$ sine I_m is invertible A is invertible by Lemma 1.6. $\square$