Week 5 Linear Transformations and Matrices

Thaqib Mo.

March 19, 2021

1 Linear Transformations

Definition 1.1: Linear Transformation

A function $T : V \to W$ is called linear if

#1 $T(x+y) = T(x) + T(y)$

#2 $T(cx) = cT(x)$

Proposition 1.1

A map $T : V \to W$ is linear **if and only if**

$$T(cx+y) = cT(x) + T(y)$$

Remark 1.1. The transformation $T : M_{m\times n}(\mathbb{F}) \to M_{n\times m}(\mathbb{F})$ defined by $T(A) = A^t$ (Transpose) is a linear transformation. The transpose is defined as

$$A^t_{ij} = A_{ji}$$

Remark 1.2. if T is linear then

- $T(0) = 0$

Theorem 1.2: Unique Linear Transformation

Let $\{v_1, \dots, v_n\}$ be a basis for a vector space V then let $\{w_1, \dots, w_n\} \in W$ then there exists a **unique** linear transformation such that

$$T(v_1) = w_1 \ \cdots \ T(v_n) = w_n$$

Definition 1.2: Null Space and Range

The null space or kernel of a linear transformation is defined as

$$N(T) = \{v \in V \mid T(v) = 0_W\}$$

The range is defined as

$$R(T) = \{T(v) \mid v \in V\}$$

Remark 1.3. $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Theorem 1.3: Rank-Nullity Theorem

$$\text{Rank}(T) + \text{Null}(T) = \dim V$$

proof sketch. Extending a basis $\{v_i\}_{i=1}^{k}$ of $N(T)$ to form a basis of V and removing forming a basis for $R(T)$ with $n-k$ vectors defined by $\{T(v_j)\}_{j=k+1}^{n}$. □

Definition 1.3: Isomorphism

A linear transformation is also called an isomorphism if it is also a bijection.

Lemma 1.4

Let T be linear then T is injective if and only if $N(T) = \{0_V\}$

Theorem 1.5: Basis for Isomorphic vector Spaces

Let $T : V \to W$ be linear. Let $\{v_1, \ldots, v_n\}$ be a basis for V then T is an isomorphism if and only if $\{T(v_1), \ldots, T(v_n)\}$ is a basis for W

Theorem 1.6: Isomorphism Condition

$V \cong W$ is and only if $\dim V = \dim W$ for $\dim V, W < \infty$

Theorem 1.7: Equivalent Conditions

Let $\dim W, \dim V < \infty$ and $\dim V = \dim W$. If $T : V \to W$ is linear then the following are **equivalent**

(1) T is one-to-one.

(2) T is onto.

(3) $\text{rank}(T) = \dim V$.

2 Matrices and Linear Transformations

2.0.1 Set of all linear transformations

Definition 2.1: $\mathcal{L}$

Let V, W be vector spaces then $\mathcal{L}(V, W)$ is the set of all linear transformation $T : V \to W$.

Remark 2.1. For any sets A, B the set A^B is the set of all functions $B \to A$. We can define the operations on this set by

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x)$$

If A, B are vector spaces then A^B is also a vector space with these operations.

Theorem 2.1

$\mathcal{L}(V, W)$ is a subspace of W^V

Definition 2.2: Matrix-Vector Multiplication

For a column vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

The product $A\mathbf{x}$ is defined as

$$\begin{bmatrix} \sum_{i=1}^{n} a_{1i}x_i \\ \sum_{i=1}^{n} a_{2i}x_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi}x_i \end{bmatrix}$$

Remark 2.2. We can also define the product $A\mathbf{x}$ in terms of linear combination of the column vectors of A

$$x_1 \begin{bmatrix} a_1 \end{bmatrix} + x_2 \begin{bmatrix} a_2 \end{bmatrix} + \dots x_n \begin{bmatrix} a_n \end{bmatrix}$$

Where a_i is the $i-$th column vector. The result is a $m \times 1$ vector. For $e_j \in \mathbb{F}^n$ the product Ae_j is the $j-$th column vector of A.

Definition 2.3: L_A function

Let $A \in M_{m\times n}(\mathbb{F})$ then L_A denotes the function $\mathbb{F}^n \to \mathbb{F}^m$ defined by

$$L_A(\mathbf{x}) = A\mathbf{x}$$

Corollary 2.0.1: Matrix Equality Theorem

Let $A, B \in M_{m\times n}(\mathbb{F})$ then $A = B$ if and only if for all $\mathbf{x} \in \mathbb{F}^n$ we have

$$A\mathbf{x} = B\mathbf{x}$$

Theorem 2.2: Linearity of Matrix Vector Multiplication

Let $A \in M_{m\times n}(\mathbb{F})$ then the linear transformation $L_A : \mathbb{F}^n \to \mathbb{F}^m$ is liner

proof sketch. Prove using $L_A(c\mathbf{x} + \mathbf{y}) = cL_A(\mathbf{x}) + L_A(\mathbf{y})$ and the linear combination of column vector version of matrix vector multiplication. □

Proposition 2.3

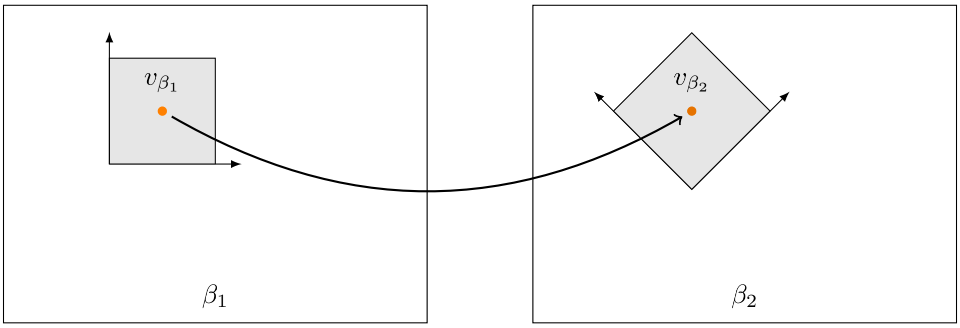
Let $\mathsf{L} : M_{m\times n}(\mathbb{F}) \to \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ be defined as

$$\mathsf{L}(A) = L_A$$

Then L is linear and one-to-one. (Injective).

Remark 2.3. L is actually an isomorphism from $M_{m\times n}(\mathbb{F}) \to \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$

3 Coordinates



Definition 3.1: Ordered Basis

An ordered basis for V is a basis set $\{v_1, \ldots, v_n\}$ with a specific order.

Definition 3.2: Coordinate Vector

Let β be an **ordered basis** for V then the coordinate vector for $x \in V$ is $[x]_\beta$ is

$$[x]_\beta = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

Where $x = \sum_{k=1}^{n} a_k v_k$ where $v_k \in \beta$.

Theorem 3.1

The map $[\ ]_\beta : V \to \mathbb{F}^n$ where $\dim V = n$ is an isomorphism.

4 Matrix Representation of a Linear Transformation

Definition 4.1

Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be linear and $\beta = \{v_1, \ldots, v_n\}$ be an ordered basis for V and γ be an ordered basis for W. Then the matrix representation of T is given by

$$[T]_\beta^\gamma = \begin{bmatrix} [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix}$$

If $T : V \to V$ then $[T]_\beta$ denotes $[T]_\beta^\beta$.

Remark 4.1. If $A = [T]_\beta^\gamma$ then $A \in M_{m\times n}(\mathbb{F})$ where $\dim W = m$ and $\dim V = n$.

Theorem 4.1

Let $T : V \to W$ be linear and β, γ be ordered basis for V, W then

$$[T(x)]_\gamma = [T]_\beta^\gamma [x]_\beta$$

Proposition 4.2

(1) For $T, U \in \mathcal{L}(V, W)$ we have

$$[cT + U]_\beta^\gamma = c[T]_\beta^\gamma + [U]_\beta^\gamma$$

(2) For every $C \in M_{m\times n}(\mathbb{F})$ there exists $T \in \mathcal{L}(V, W)$ such that $[T]_\beta^\gamma = C$.

Remark 4.2. $\Big[\quad\Big]_\beta^\gamma$ is an isomorphism $\mathcal{L}(V, W) \to M_{m\times n}(\mathbb{F})$. Where $m = \dim W$ and $n = \dim V$.

5 Matrix Multiplication

Definition 5.1: Matrix Multiplication

Let $A \in M_{m\times n}$ and $B \in M_{n\times p}$ then the product $AB = C \in M_{m\times p}$ is defined by

$$C_{ij} = \sum_{t=1}^{n} a_{it}b_{tj}$$

If numbers of columns of A does not equal number of rows in B then AB is not defined.

Remark 5.1. We can also compute AB by considering the products Ab_i where b_i it the $i-$th column vector of B, then

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

Lemma 5.1: Properties of matrix Multiplication

1. $A(B + C) = AB + AC$ where $A \in M_{m\times n}$ and $B, C \in M_{n\times p}$
2. $(D + E)A = AD + AE$ where $D, E \in M_{m\times n}$ and $A \in M_{n\times p}$
3. $\alpha(AB) = (\alpha A)B = A(\alpha B)$ where $\alpha \in \mathbb{F}$
4. $(AB)^t = B^tA^t$
5. $I_mA = AI_n = A$
6. $AO_{n\times p} = O_{m\times p}$ and $O_{q\times m}A = O_{q\times n}$

Theorem 5.2: Matrix of Composition of Linear Transformations

Let V, W, Z be finite dimensional vector spaces. With ordered bases $\alpha = \{v_1, \dots, v_p\}$, $\beta = \{w_1, \dots, w_n\}$ and $\gamma = \{z_1, \dots, z_m\}$. Now let

$$T : V \to W \quad U : W \to Z$$

Be linear. Then the corresponding matrices are $[T]_\alpha^\beta \in M_{m\times p}$ and $[U]_\beta^\gamma \in M_{n\times p}$ and let $C = [UT]_\alpha^\gamma \in M_{m\times p}$. Then $C = AB$, that is

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$$

Proof sketch. $[U]_\beta^\gamma [T(v_j)]_\beta$ is the $j-$th column of $[U]_\beta^\gamma [T]_\alpha^\beta$ and $[U]_\beta^\gamma [T(v_j)]_\beta = [U(x)]_\gamma$ where $x = T(v_j)$. Therefore $[U(T(v_j))]_\gamma = [UT(v_j)]_\gamma$. So both matrices have the same $j-$th column. □

Proposition 5.3

Let $A \in M_{m\times n}(\mathbb{F})$ and consider $L_A : \mathbb{F}^n \to \mathbb{F}^m$. Let β be the standard ordered basis for $\mathbb{F}^n$ and let γ be the standard ordered basis for $\mathbb{F}^m$ then

$$[L_A]_\beta^\gamma = A$$

Corollary 5.0.1

#1 $L_{AB} = L_A L_B$ where $A \in M_{m\times n}$ and $B \in M_{n\times p}$

#2 $A(BC) = (AB)C$

Proof sketch. Let α be standard ordered basis for $\mathbb{F}^p$, β be the standard ordered basis for $\mathbb{F}^n$ and let γ be the standard ordered basis for $\mathbb{F}^m$. Then we know that $[L_A B]_\alpha^\gamma = AB$ and

$$\begin{aligned} [L_A L_B]_\alpha^\gamma &= [L_A]_\alpha^\beta [L_B]_\alpha^\beta \\ &= AB \end{aligned}$$

Since $[\ ]_\alpha^\beta$ is one-to-one we get $L_A L_B = L_{AB}$.

(#2) Using the function L defined Proposition 2.3 since it's injective, $L_{ABC} = L_A L_B L_C$ and function composition is associative. $\square$