Polynomials

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December 1, 2020

1 Complex Numbers

Using localization similar to the construction of $\mathbb{Q}$ we can construct elements of $\mathbb{C}$ in terms of ordered pairs $(a, b) \in \mathbb{R} \times \mathbb{R}$. The addition is defined component wise,

$$(a, b) + (c, d) = (a + c, b + d)$$

and multiplication similar to the Gaussian integers is defined as

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc)$$

The addition is the same as the addition in $\mathbb{R}$ so we can already conclude that $\mathbb{C}, +$ is an abelian group. $(1, 0)$ is clearly the multiplicative identity and it is easy to check that multiplication is associative.

To check that every non-zero element in $\mathbb{C}$ has an multiplicative inverse:

$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

So we have $\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)$ this ordered pair is the inverse of (a, b). We have $a^2 + b^2 \neq 0$ when $(a, b) \neq (0, 0)$. Now consider

$$\begin{aligned}
&(a, b) \cdot ((c, d) + (e, f)) = (a, b) \cdot (c + e, d + f) \\
&= (a(c + e) - b(d + f), a(d + f) + b(c + e)) \\
&= (ac - bd + ae - bf, ad + bc + af + be) \\
&= (a, c) \cdot (e, f) + (a, b) \cdot (e, f)
\end{aligned}$$

So we can now say $\mathbb{C}$ is a field.

1.1 Complex Number Constructions and Properties

Definition 1

Let $z \in \mathbb{C}$ and we write $z = a + bi$ for some $a, b \in \mathbb{R}$

- The form $a + bi$ is called the standard form of z
- The number a is the *Real* part of z denoted by $\text{Re}(z)$
- The number b is the *imaginary* part of z denoted by $\text{Im}(z)$
- $\bar{z} = a - bi$ is called the *complex conjugate*
- $|z| = \sqrt{a^2 + b^2}$ is called the *absolute value* of z

Proposition 1

$\phi : \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \bar{z}$ is a ring homomorphism.

Proof. Let $z = a + bi$ and $w = c + di$

Consider $\phi(z + w) = \overline{z + w} = a + c - (b + d)i = a - bi + c - di = \phi(z) + \phi(w)$.

$\phi(zw) = \bar{z}\bar{w} = \overline{ac - bd + (ad + bc)i} = ac - bd - (ad + bc)i = ac - bd - adi - bci = (a - bi)(c - di) = \phi(z)\phi(w)$.

For $\phi(1 + 0i) = 1 - 0i = 1$. So ϕ is a ring homomorphism. $\square$

Proposition 2

For all $z \in \mathbb{C}$ we have $|zw| = |z||w|$

Proof. Let $z = a + bi$ and $w = c + di$ we have $zw = ac - bd + (ad + bc)i$ so we have

$$\begin{aligned}
|zw| &= \sqrt{(ac - bd)^2 (ad + bc)^2} \\
&= \sqrt{a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2} \\
&= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\
&= \sqrt{a^2(c^2 + d^2) + b^2(c^2 + d^2)} \\
&= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\
&= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\
&= |z||w|
\end{aligned}$$

$\square$

The *triangle inequality* also holds in $\mathbb{C}$

Theorem 1: Triangle inequality

For all $z, w \in \mathbb{C}$ we have $|z + w| \leq |z| + |w|$

Proof.

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} \\
&= (z + w)(\bar{z} + \bar{w}) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + |w|^2 + (z\bar{w} + \overline{z\bar{w}}) \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w})
\end{aligned}$$

Note that $\mathrm{Re}(z\bar{w}) \leq |z\bar{w}| = |z||\bar{w}| = |z||w|$. So we have

$$|z+w|^2 = |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2$$

Taking the positive square roots gives the triangle in equality. $\square$

1.2 Polar Form of a Complex Number