# Reading-13

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November 6, 2020

# 1 Axiom of choice

Suppose $\mathcal{C}$ is a non empty collection of sets then the Cartesian product of all sets in $\mathcal{C}$ is written as:

$$\prod_{C \in \mathcal{C}} C$$

If $\mathcal{C}$ is an infinite collection of sets then we need infinite ordered tuple of elements. Then we need a function $\alpha : \mathcal{C} \to \bigcup \mathcal{C}$. Then for each $C \in \mathcal{C}$ we have $\alpha(C) \in C$. The function $\alpha(C)$ should give the $C$-th coordinate of the infinite tuple. We can formally define this as:

**Definition 1: Cartesian Product for infinite sets**

Let $\mathcal{C}$ denote a non empty collection of sets. The Cartesian product $\prod_{C \in \mathcal{C}} C$ is the set of all functions $\alpha : \mathcal{C} \to \bigcup \mathcal{C}$ with the property $\alpha(C) \in C$.

We can use this definition to redefine the Cartesian product, even for finitely many sets. We can define an ordered pair $(a, b) \in A \times B$ as $(\alpha(A), \alpha(B)) \in A \times B$.

The problem arises when for an infinite number of sets in $\mathcal{C}$ we cannot show (using the standard axioms of set theory) that:

$$\prod_{C \in \mathcal{C}} C \neq \varnothing$$

For this we need a new axiom.

**Axiom 1: Axiom of Choice**

The Cartesian product of any non-empty collection of non-empty sets is non-empty.

This axiom is also equivalent to the existence of a choice function for every non-empty collection of set $\mathcal{C}$.

## 2 Zorn's Lemma and the Well-Ordering Theorem

Another statement about partially ordered set is logically equivalent to Axiom of choice.

**Theorem 1: Zorn's Lemma**

Let $A$ be a partially ordered set with order relation $\preceq$. Suppose that every chain in $A$ has an upper bound in $A$, then $A$ has a maximal element.

A Corollary that follows from Zorn's Lemma is:

**Corollary 1**

Suppose that $A$ is a partially ordered set with order relation $\preceq$ then every chain $\mathcal{C}$ in $A$ is contained in a maximal chain $\mathcal{M}$.

*Proof.* Let $\mathcal{C}$ be an arbitrary chain in $A$. Consider the set $\Gamma$ of all chains $\mathcal{C}$ in $A$. The subset relation $\subseteq$ relation on $\mathcal{P}(A)$ giving the order relation between elements of $\Gamma$. Now suppose we have a chain $\mathcal{D}$ in the ordered set $\Gamma$. We can show that $\mathcal{D}$ is an upper bound in $\gamma$

If we define $\mathcal{C}_0 = \bigcup \mathcal{D}$, the union of all the, clearly for each $\mathcal{C}' \in \mathcal{D}$ we have $C' \subseteq \mathcal{C}_0$. Since $C_0$ contains every chain of $\mathcal{D}$, each of which contains $\mathcal{C}$ we know that $\mathcal{C}_0$ contains $\mathcal{C}$. Therefore $C_0$ is an upper bound on $\mathcal{D}$ by definition. If we can verify $C_0 \in \Gamma$ that means $C_0$ is a chain in $A$

Suppose we have $a_1, a_2 \in \mathcal{C}_0$. By construction each element of $a_1$ and $a_2$ belong to a chain in $\mathcal{D}$, so we can write $a_1 \in \mathcal{C}_1$ and $a_2 \in \mathcal{C}_2$. Since $\mathcal{D}$ is a chain with respect to the relation $\subseteq$, we can either have $\mathcal{C}_1 \subseteq \mathcal{C}_2$ or we can have $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Without loss of generality we say $\mathcal{C}_1 \subseteq \mathcal{C}_2$ . Thus both $a_1, a_2 \in C_2$. Since $\mathcal{C}_2$ is a chain, the elements in $\mathcal{C}_2$ are comparable with $\preceq$, this verifies that $\mathcal{C}_0$ is a chain in $A$, as $C_0 \in \Gamma$.

Now if we apply Zorns lemma to $\Gamma$ which is ordered by $\subseteq$, there is a chain $\mathcal{M} \in \Gamma$, maximal with respect to the containment relation $\subseteq$. This is a chain in $A$ with respect to the ordering $\preceq$, as it is maximal with with respect to $\subseteq$ all chains $\mathcal{C}$ of $A$ are contained in $\mathcal{M}$ and it is not properly contained in any chain.

□

The well ordering theorem is another statement known to be equivalent to axiom of choice. We can define the WOP of $\mathbb{N}$ more generally.

**Definition 2: Well ordering**

Suppose $A$ is a set with order relation $\preceq$. The ordered set $A$ is said to be well ordered if every non empty subset of $A$ has a least element with respect to the relation $\preceq$

This leads to the well ordering theorem:

**Theorem 2: Well-Ordering Theorem**

Every non empty set has a well ordering.

In other words, if $A$ is a non empty set then there is a an order relation $\preceq$ on $A$, such that $A$ is well ordered with respect to $\preceq$. This means that the set $\mathbb{R}$ has as well ordering, not with respect to the relation $\leq$ as this fails for any open interval in $\mathbb{R}$ but according to *well ordering theorem* there exits a relation $\preceq$ on $\mathbb{R}$ on which every subset of $\mathbb{R}$ will have a least element with respect to $\preceq$ which is not the relation $\leq$.

The 3 statements Axiom of choice, Well-Ordering Theorem, Zorn's Lemma are all logically equivalent.