Reading-12

Thaqib Mo.

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1 Order Relations

Definition 1: Antisymmetric Relation

Let R be a binary relation on set A. We say that R if, whenever we have $a, b \in A$ such that aRb and bRa, it follows that $a = b$. A relation $\preceq$ is an order relation on A if if it is reflexive , *antisymmetric*, and transitive.

1.1 Chains and Extremal Elements

Definition 2: Total ordering

If $\preceq$ is a partial ordering on set A we say that 2 elements $a, b \in A$ are comparable if either $a \preceq b$ or $b \preceq a$. A partial order in which every pair of elements is comparable is called a *total ordering* or *linear ordering*.

Example The relation $\leq$ on $\mathbb{Z}$ is a total ordering as every pair $a, b \in \mathbb{Z}$ are comparable.

Another important notion which comes is the notion of a *chain*:

Definition 3: Chain

If $\preceq$ is a partial order on set A, a subset C of A is called a *chain* if every pair of C are comparable. In particular, if $\preceq$ is a total order on A then A itself is a chain in A.

Because not every element of a set must be comparable we have to be careful in distinguishing the greatest and the least elements of of a given set. The distinction is given by the following definitions.

Definition 4: Maximal, minimal, greatest, least elements

Let A be a set with partial order relation $\preceq$. Given a subset B of A we say that

- An element $b \in B$ is the **least element** of B if we have $b \preceq b'$ for all $b' \in B$. Similarly, an element is the **greatest element** if $b' \preceq b$ for all $b' \in B$.
- An element $b \in B$ is the **minimal element** of B if there are no smaller elements, that is if $b' \preceq b$ **for some** $b' \in B$ then $b = b'$. Similarly an element is the **maximal element** if there are no larger elements, that is if $b \preceq b'$ **for some** $b' \in B$ then $b' = b$.

2 Bounds on Sets, Suprema, and Infima

Definition 5: Bounds, Suprema, and Infima

Suppose A is a set with order relation $\preceq$ and with a subset B.

- An element $a \in A$ is a lower bound for B if $a \preceq b$ **for all** $b \in B$. Similarly $a \in A$ is a upper bound for B if $b \preceq a$ **for all** $b \in B$.
- An element $a \in A$ is the infimum (Greatest lower bound) of B if a is the greatest element of the set of al lower bounds for B.
- Am element $a \in A$ is the supremum (Least upper bound) of B if a is the least element of the set of all upper bounds for B.

  If they exist, we use $sup\ B$ and $inf\ B$ to denote the supremum and infimum of a set B, respectively.