Week 3 and 4

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February 12, 2021

1 Bases and Dimention

Definition 1.1

Let V be a vector space. A sibset S of V is called a basis for V if it satifies the following conditions:

- S is linearly independent.
- $\text{Span}(S) = V$

If S is a basis for V we also say S forms a basis for V.

Example 1.1.
- The empty set $\varnothing$ is a bais for the zero vector space since $\text{Span}(\varnothing) = \{\mathbf{0}\}$.
- In $\mathbb{F}^n$ the subset $\{e_1, e_2, \ldots, e_n\}$ where for each $e_j \in \mathbb{F}^n$ is the vector whose $j-$th coordinate is 1. If

$$c_1e_1 + c_2e_2 + \ldots + c_ne_n = \mathbf{0}$$

  Then $(c_1, c_2, \ldots, c_n) = 0$. Therefore this set is linearly independent and spans the entire $\mathbb{F}^n$. S is called the standard basis for $\mathbb{F}^n$
- $\{E_{ij} \in M_{m\times n}(\mathbb{F}) : 1 \leq i \leq m,\ 1 \leq j \leq n\} \subset M_{m\times n}(\mathbb{F})$. E_{ij} is an $m \times n$ matrix whose $i-$th row and $j-$th column is 1 and rest are 0.

Theorem 1.1: Basis vectors

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for a vector space V then every $x \in V$ can be **uniquely expressed** as a linear combinator of $v_1, \ldots, v_n$.

Proof.
- Existence: Since $S = \{v_1, \ldots, v_n\}$ is a basis for V, so any $x \in V$ can be expressed as a linear combination of vectors in S by definition.
- Uniqueness:

$$x = a_1v_1 + a_2v_2 + \ldots + a_nv_n = b_1v_1 + b_2v_2 + \ldots b_nv_n$$

$$(a_1 - b_1)v_1 + (a_1 - b_1)v_2 + \ldots + (a_n - b_n)v_n = 0$$

  Since S is linearly independent we must have $a_i - b_i = 0 \rightsquigarrow a_i = b_i$

$\square$

Theorem 1.2

If a vector space is generated by a countable set S then some subset of S is a basis for V.

Proof. Proved using the construction

$$v_{i_k} \text{ such that } v_{i_k} \notin \mathrm{Span}(\{v_{i_1}, v_{i_2}, v_{i_3}, \ldots, v_{i_{k-1}}\})$$

□

Theorem 1.3

Every vector space has a basis.

Proof. maximal linearly independent subsets and Zorn's lemma □

Theorem 1.4: Replacement Theorem

Let V be a vector space with a finite spanning set S. Let T be linearly independent in V then

- $|T| \leq |S|$
- There exists $H \subset S$ such that exactly $|H| = |S| - |T|$ and $H \cup T$ is a basis for V.

Proof. By induction on the size of $|T| = m$ □

Definition 1.2: Dimension

A vector field is finite dimensional if it has a basis containing a finite number of vectors. This is denoted by $\dim V$ and $\dim\{\mathbf{0}\} = 0$.

Corollary: 1.9.1

Suppose $\dim V < \infty$ then all bases for V are finite and have the same number of elements.

Corollary: 1.9.2

Let V be such that $\dim V = n$ then

#1 Any spanning set of V contains **atleast** n vectors.

#2 Any linearly independent set contains **at most** n vectors.

#3 Any linearly independent set with n vectors is a basis for V

#4 Any linearly independent set can be extended to form a basis for V

#5 If W is a subspace of V with $\dim W \leq \dim V$

2 Quotient Spaces

Definition 2.1: Quotient Spaces

Let W be a subspace of V then for $x \in V$

$$x + W = \{x + w : w \in W\}$$

This is a **coset** of W in V and x is called a representative of $x + W$.

- If $x - y \in W$ then $x \cong y \bmod W$
- The set V/W is the collection of all cosets.

$$V/W = \{x + W : x \in V\}$$

Proposition 2.1

- $x \in x + W$
- $x + W = y + W \iff x - y \in W$

Remark 2.1. $x \cong y \bmod W$ is an equivalence relation.

The operations on V/W are defined by

$$(x + W) + (y + W) = (x + y) + W$$
$$a(x + W) = ax + W$$
$$\text{for } a \in \mathbb{F}\ x \in V$$

Theorem 2.2: Quotient Space

The set V/W with the operations defined above is a vector space over $\mathbb{F}$.

Theorem 2.3: Basis for Quotient Space

Let $\{v_1, \ldots, v_n\}$ be a basis for V such that $\{v_1, \ldots, v_k\}$ with $k \leq n$ is a basis for W. Then

$$\{v_{k+1} + W, \ldots, v_n + W\} \text{ Is a basis for } V/W$$
$$\dim(V/W) = \dim V - \dim W$$

3 Sums and Internal Direct Sums of Subspaces

The sum of 2 subspaces is defined as

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1 \; w_2 \in W_2\}$$

- If $W_1 \cap W_2 = \{\mathbf{0}\}$ then we write $W_1 \oplus W_2$.
- If $W_1 \oplus W_2 = V$ then W_2 is the complementary subspace of W_1.

Lemma 3.1

#1 $W_1 \cap W_2$ is a subspace of W_1, W_2, V

#2 $W_1 + W_2$ is the smallest subspace containing W_1, W_2

#3 If $V = W_1 \oplus W_2$ if and only if there exists unique w_1, w_2 such that $v = w_1 + w_2$ for all $v \in V$.

3.1 Dimension

Theorem 3.2: Dimension Theorem

If W_1 and W_2 are finite dimensional then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

If $V = W_1 \oplus W_2$ then

$$\dim(W_1) + \dim(W_2) = \dim(V)$$

Proposition 3.3: Existence of Complementary Subspaces

et $\dim V < \infty$, and let W be a subspace then $\exists U$ such that $U \oplus W = V$

Proof. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for W. We can extended this to $\{v_1, v_2, \ldots, v_n\}$ for a basis of V. Now we have $U = \mathrm{Span}(\{v_{k+1}, \ldots, v_n\})$ is a subspace such that $V = U + W$ and $U \cap W = \{\mathbf{0}\}$. $\square$