CS 371

Condition of a problem

$$\kappa_A = \frac{||\Delta z||}{||\Delta x||}$$

$$\kappa_R = \frac{\left(\frac{||\Delta z||}{||z||}\right)}{\left(\frac{||\Delta x||}{||x||}\right)}$$

$$\Delta x = x - \hat{x} \quad \delta x = \frac{|\Delta x|}{x}$$

$$||x||_1 = \sum_{i=1}^{n} |x_i| \quad ||x||_p = \sqrt[p]{\sum_{i=1}^{n} x_i^p}$$

$$||x||_\infty = \max_{1\le i\le n} |x_i|$$

$|x \cdot y| \le ||x|| \times ||y||$ (Cauchy-Schwartz Inequality.)

$$||x||_\infty \le ||x||_2 \le ||x||_1 \le n||x||_\infty$$

$$||x||_\infty \le ||x||_p \le n||x||_\infty$$

$$e^x = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin(x) = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

Numerical Linear Algebra

$$\underbrace{A}_{n\times p}\overbrace{B}^{p\times m} \qquad \text{flops} = nm(p+p-1)$$

LU Factorization

$$P\mathbf{A} = \underbrace{LU}_{\text{Lower and upper triangular matrix}}$$

$$\mathbf{A} \underset{\text{row operations}}{\Rightarrow} U$$

$$\mathbf{I_n} \underset{\text{same row operations}}{\Rightarrow} L' \underset{\text{flip off-diagonal elements}}{\Rightarrow} L$$

$$\underset{\text{forward sub}}{L\vec{\mathbf{y}} = \vec{\mathbf{b}}} \Rightarrow \underset{\text{back sub}}{U\vec{\mathbf{x}} = \vec{\mathbf{y}}}$$

Computational cost

Decomposing: $\frac{2n^3}{3} + O(n^2)$
Forward and backward sub: $n^2 + O(n)$

Determinants

1. $\det(BC) = \det(B)\det(C)$
2. **U** upper triangular $\Rightarrow \det U = \prod_{i=1}^{n} u_{ii}$
3. **L** lower triangular $\Rightarrow \det U = \prod_{i=1}^{n} u_{ii}$
4. **P** Permutation matrix $\Rightarrow \det P_{\text{even}} = +1$, $\det P_{\text{odd}} = -1$
5. If $\det \mathbf{A} \ne 0$ then $\mathbf{A}x = b$ has **unique solution**
6. If $\det \mathbf{A} = 0$ then $\mathbf{A}x = b$ has 0 or infinite solutions.

Condition and Stability

$$\|\mathbf{A}\|_p = \max_{\|\vec{\mathbf{x}}\|_p \ne 0} \frac{\|\mathbf{A}\vec{\mathbf{x}}\|_p}{\|\vec{\mathbf{x}}\|_p}$$

$\|\mathbf{A}\|_1$ = maximum abs column sum

$\|\mathbf{A}\|_\infty$ = max maximum abs row sum

$$\|A\|_p = 0 \iff \|A\| = 0, \quad \|cA\|_p = |c|\|A\|_p,$$

$$\|A+B\|_p \le \|A\|_p + \|B\|_p$$

$\kappa_p(A) = \|A\|_p \left\|A^{-1}\right\|_p$ (Condition number)

$$\kappa_2(A) = \sqrt{\frac{\lambda_{\max}(A^TA)}{\lambda_{\min}(A^TA)}}$$

$\kappa(B) = \infty$ if B is singular

$\frac{1}{\kappa(B)}$ how close B is to a singular matrix

$\vec{\mathbf{r}} = \vec{\mathbf{b}} - A\vec{\mathbf{u}}$ (residual)

Root finding

Bisection Method make intervals $[a_k, b_k]$, $f(a_0)f(b_0) \le 0$ (Opposite signs)

$$a_k = \begin{cases} a_{k-1} & f(\frac{a_{k-1}+b_{k-1}}{2}) \cdot f(a_{k-1}) \le 0 \\ \frac{a_{k-1}+b_{k-1}}{2} & \text{otherwise} \end{cases}$$

$$b_k = \begin{cases} b_{k-1} & f(\frac{a_{k-1}+b_{k-1}}{2}) \cdot f(a_{k-1}) > 0 \\ \frac{a_{k-1}+b_{k-1}}{2} & \text{otherwise} \end{cases}$$

\# of steps for tolerance $t \ge \frac{|b_0 - a_0|}{2^N}$

Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

Secant Method

$$x_{k+1} = x_k - f(x_k)\left[\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}\right]$$

Rate of Convergence

Error: $e_k = x_k - x^*$
Order of convergence: x_k converges with order q if and only if $x_i \to x^*$

$$\lim c_i = N \in [0, \infty)$$

$$|e_{i+1}| = c_i|e_i|^q \qquad \begin{cases} \text{Bisection Method} & q = 1 \\ \text{Secant Method} & q = \frac{1+\sqrt{5}}{2} \\ \text{Newton's Method} & q = 2 \end{cases}$$

Fourier Series

$$\omega = 2\pi f$$

$$g_n(x) = \frac{a_0}{2} + \sum_{k=0}^{n}\left[a_k \cos\left(k\frac{2\pi}{b-a}\right) + b_k \sin\left(k\frac{2\pi}{b-a}\right)\right]$$

$$a_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(k\frac{2\pi x}{b-a}\right)$$

$$b_k = \frac{2}{b-a}\int_a^b f(x)\sin\left(k\frac{2\pi x}{b-a}\right)$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\exp(ikt)\,\mathrm{d}t \quad [a,b] = [-\pi, \pi]$$

$$c_k = \frac{1}{2}(a_k + ib_k)$$

$f(x) = f(-x)$ Even

$f(-x) = -f(x)$ Odd

$f(t)$ even $\Rightarrow b_k = 0 \ \forall k$ $(\cos(x) = \cos(-x))$

$f(t)$ odd $\Rightarrow a_k = 0 \ \forall k$ $(\sin(-x) = -sin(x))$

$$V = \left\{ f(x) : \sqrt{\int_a^b |f(x)|^2 \,\mathrm{d}x} < \infty \right\}$$

If $f(x) \in V$, Fourier series of $g_n(x) \to f(x)$ on $[a, b]$

$\bar{c_k} = c_{-k}$

$a_{-k} = a_k \ \ b_{-k} = -b_{-k}$

$$a_k = 2\,\mathrm{Re}(c_k)\ ,\ b_k = -2\,\mathrm{Im}(c_k)\ ,\ b_0 = 0\ ,\ c_0 = \frac{1}{2}a_0$$

Roots of unity

$W_N = \exp\left(\frac{2\pi i}{N}\right)$ $n-$th root of unity

$$W_N^k = \exp\left(\frac{2k\pi}{N}\right)$$

$$(W_N^k)^N = 1$$

$$(W_N)^{-k} = W_N^{N-k}$$

DFT

$f[i] = f(i)$ for $0 \leq i \leq N-1$ (Sampled for N points)

$$F[k] = DFT\{f[n]\} = \frac{1}{N}\sum_{n=0}^{N-1} f[n] W_N^{-kn}$$

$$f[n] = IDFT\{F[k]\} = \sum_{k=0}^{N-1} F[k] W_N^{kn}$$

$$\vec{F} = W\vec{f}$$

$W_{ij} = \frac{1}{N}W_N^{-ij}$ for $0 \leq i,j \leq N-1$

$W_{ij}^{-1} = W_N^{ij}$ for $0 \leq i,j \leq N-1$ $\quad W_{ij}^{-1} = N(\overline{W_{ij}})$

Properties of F

$F[k] = F[k+sN]$ for $s \in \mathbb{Z}$

$\overline{F[k]} = F[-k]$

$\text{Re}\{F[k]\}$ is even in k

$\text{Im}\{F[k]\}$ is odd in k

$f[n] = f[N-n] = f[-n]$ f is even in n

$f[n] = -f[N-n] = -f[-n]$ f is odd in n

$f[n]$ is even in $n \Rightarrow \text{Im}(F[k]) = 0$ (DFT is real)

$f[n]$ is odd in $n \Rightarrow \text{Re}(F[k]) = 0$ (DFT is purely imaginary)

Aliasing and the Sample Theorem

$f_s = \frac{N}{T}$ Sampling Rate

Sampling Theorem: If a function $f(t)$ is bandwidth limited to frequencies smaller than f_c (max frequency $\leq f_c$) and $f(t)$ is sampled at a rate $\boxed{f_s \geq 2f_c}$ then the function is completely determined by its samples $f[n]$.

Sampling Theorem(ii): For a fixed sample $f[n]$ with a fixed sampling rate f_s. Then if the maximum frequency of a signal $f_c \leq \frac{f_s}{2}$ that can be deconstructed from $f[n]$ such that is free of aliasing errors (DFT if free of aliasing errors).

Fast Fourier Transform

`flops` $F[k] = 2N \Rightarrow$ `flops` $\vec{F} = 2N^2$ (Using sums)

$E[k]$ (DFT of even indexed part)

$O[k]$ (DFT of odd indexed part)

$$F[k] = \frac{1}{2}\left(E[k] + W_N^{-k}O[k]\right) \text{ for } k = 0,\dots,\frac{N}{2}-1$$

$$F[k+\frac{N}{2}] = \frac{1}{2}\left(E[k] - W_N^{-k}O[k]\right) \text{ for } k = 0,\dots,\frac{N}{2}-1$$

`flops` $\vec{F} = \frac{5}{2}N\log_2(N)$

Power Spectrum and Parseval's Theorem

Power spectrum of $f(t)$ or $f[n]$ is $|F[k]|^2$

Parseval's Theorem (Continous)

$$\frac{1}{b-a}\int_a^b f(t)^2\,\mathrm{d}t = \sum_{k=-\infty}^{\infty}|c_k|^2$$

Parseval's Theorem (Discrete)

$$\frac{1}{N}\sum_{k=0}^{N-1}|f[n]|^2 = \sum_{k=0}^{N-1}|F[k]|^2$$

Interpolation

For a basis $B = \{\phi_j(x)\}_{j=0}^m$ interpolating points $\{(x_i, f_i)\}_{i=0}^n$

$$y(x) = \sum_{i=0}^m a_i\phi_i(x)$$

$$\Phi = \begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \cdots & \phi_m(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \cdots & \phi_m(x_1) \\ \vdots & \vdots & \dots & \vdots \\ \phi_0(x_n) & \phi_1(x_n) & \cdots & \phi_m(x_n) \end{pmatrix}$$

$\Phi\vec{a} = \vec{f}$ (Solves for a_i)

Vandermonde Polynomial:

$y(x) = a_0 + a_1x + a_2x^2 + \dots a_nx^n$ for $(n+1)$ points (x_i, f_i).

$$\Phi = V = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$\det(V) = \prod_{0 \leq i < j \leq n}(x_j - x_i)$$

Computing V takes $\Theta(N^3)$ `flops`

Computing $y(x)$ takes $3N$ `flops`

Lagrange Interpolation The Lagrange basis functions are defined as

$$\ell_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$\ell_i(x) = \prod_{j=0, j\neq i}^n \frac{x - x_j}{x_i - x_j} \qquad \text{(basis)}$$

$\Phi = L = I_{n\times n}$

Computing I takes 0 `flops`

Computing pre-computed terms takes $2N^2 + 2N$ `flops`

Computing $y(x)$ takes $5N$ `flops`

Newton Interpolation

$$\pi_j(x) = \prod_{i=0}^{j-1}(x - x_i) \quad \pi_0(x) = 1 \qquad \text{(basis)}$$

$\pi_j(x_i) = 0$ if $i < j$

$$\mathbf{\Pi} = \begin{pmatrix} \pi_0(x_0) & \cdots & \mathbf{0} \\ \vdots & \ddots & \mathbf{0} \\ \pi_n(x_n) & \cdots & \pi_n(x_n) \end{pmatrix} \qquad (\Pi \text{ is lower triangular})$$

Computing Π takes $\Theta(N^2)$ `flops`

Computing pre-computed terms takes $\Theta(N^2)$ `flops`

Computing $y(x)$ takes $\Theta(N)$ `flops`

Extending newton polynomial with additional point (x_{n+1}, f_{n+1})

$$y_{n+1}(x) = y_n(x) + a_{n+1}\pi_{n+1}(x) \qquad a_{n+1} = \frac{f_{n+1} - y_n(x_{n+1})}{\pi_{n+1}(x)}$$

Hermite Interpolation Given $\{(x_i, f_i, f_i')\}_{i=0}^n$ interpolating polynomial has degree $\boxed{2n+1}$.

Chebyshev Points on interval $[a,b]$ the $n+1$ points are

$$x_j = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2j+1}{2n+2}\pi\right) \text{ for } j = 0,\dots,n$$

Cubic Splines *Smoothness conditions:* $y_j'(x_j) = y_{j+1}'(x_j)$ and $y_j''(x_j) = y_{j+1}''(x_j)$ for cubic splines *free boundary*: $y_1''(x_0) = 0, y_n''(x_n) = 0$, *clamped boundary*: $y_1''(x_0) = f_0', y_n''(x_n) = f_n'$, *periodic boundary* if $f_0 = f_n$: $y_1'(x_0) = y_n'(x_n)$ and $y_1''(x_0) = y_n''(x_n)$. Cubic splines can be solved in $\Theta(N)$ `flops`

Regression

Φ with fewer basis function $m+1 < n+1$. the system is over-determined we can find solution $\vec{a}$ such that the residue $\min\left\|\Phi\vec{a} - \vec{f}\right\|_2^2$ is minimized.

$$\underbrace{\Phi}_{(n+1)\times(m+1)}\ \underbrace{\vec{a}}_{(m+1)\times 1} + \vec{r} = \vec{f}$$

$$\boxed{\Phi^T\Phi\vec{a} = \Phi^T\vec{f}} \qquad \text{(normal equation)}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{(n+1)\sum x_i^2 - (\sum x_i)^2}\begin{pmatrix} (\sum x_i^2)(\sum f_i) - (\sum x_i)(\sum x_if_i) \\ (n+1)(\sum x_if_i) - (\sum x_i)(\sum f_i) \end{pmatrix}$$

Numerical Integration

$$I_M = \sum_{i=1}^n (x_i - x_{i-1}) f\left(\frac{x_i - x_{i-1}}{2}\right) \qquad \text{(Midpoint rule)}$$

$$I_T = \sum_{i=1}^n \frac{(x_i - x_{i-1})}{2}\left(f(x_{i-1}) + f(x_i)\right) \qquad \text{(Trapezoid Rule)}$$

$$I_S = \sum_{i=1}^n \frac{x_i - x_{i-1}}{6}\left[f(x_{i-1}) + 4f(x_{i-\frac{1}{2}}) + f(x_i)\right]$$

Where $x_{i-\frac{1}{2}} = \frac{x_{i-1} + x_i}{2}$ (Simpsons Rule)

Error Bounds

$$|E_M| \leq \frac{(b-a)^3}{24n^2}\max_{x\in[a,b]}\left|f''(x)\right| \qquad \text{(Midpoint rule error bound)}$$

$$|E_T| \leq \frac{(b-a)^3}{12n^2}\max_{x\in[a,b]}\left|f''(x)\right| \qquad \text{(Trapezoid rule error bound)}$$

$$|E_S| \leq \frac{(b-a)^5}{2880n^4}\max_{x\in[a,b]}\left|f^{(4)}(x)\right| \qquad \text{(Simpsons rule error bound)}$$

Degree of precision: $\hat{I}$ has degree of precision m if $E = I - \hat{I} = 0$ for polynomial $p(x)$ such that $\deg p \leq m$. Or $\hat{I}$ integrates any polynomial p with $\deg p \leq m$ *exactly*. Midpoint and Trapezium rule: $\deg_p = 1$, Simpsons rule: $\deg_p = 3$.