stat333 Notes

Thaqib M

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1 Week 1

1.1 Definition (Stochastic Process). Let $(X_t)_{t\in T}$ be a collection of random variables this is called a Stochastic Process. T is the *index set*.

1.2 Example (Simple Random Walk on $\mathbb{Z}$)**.** Let $X_i \sim$ iid where $X_i \in \{-1, 1\}$ with

$$P(X_i = 1) = \frac{1}{2}$$
$$P(X_i = -1) = -\frac{1}{2}$$

now let

$$S_n = \sum_{i=0}^{n} X_i$$

Then $(S_i)_{k=0}^{\infty}$ is a stochastic process.

1.3 Definition (Transition Probability). Given $(X_s)_{s\leq t}$ we need the probability for X_{t+1}.

$$P(X_{(t+1)} = x_{t+1} | X_1 = x_1, X_2 = x_2, \ldots X_t = x_t)$$

1.4 Note. Conditional Probability Properties

$$P(A|B) = \frac{P(AB)}{P(B)}\ P(B) > 0$$
$$P(ABC) = P(A|BC) \cdot P(B|C) \cdot P(C)$$

1.5 Example. Transition Probabilities for SRW on $\mathbb{Z}^d$

$$P\left(\|X_{t+1} - X_t\| \mid (X_s)_{s\leq t}\right) = \frac{1}{2d}$$

1.1 Markov Chains

1.6 Definition (Markov Property)**.** A process has the Markov property if:

$$P(X_{t+1} = x_{t+1} \mid (X_s)_{s \leq t}) = P(X_{t+1} = x_{t+1} \mid X_t = x_t)$$

(Next outcome only depends on the previous outcome)

1.7 Note (Markov Chain)**.** A stochastic process that satisfies the Markov property is called a Markov chain.

1.8 Definition (Time Homogeneous Markov Chain)**.** A Markov Chain is called time homogeneous if the following is true

$$P(X_{t+1} = x_t \mid X_t = x_t) = P(X_1 = x_1 \mid X_0 = x_0)$$

1.9 Definition (Stochastic Matrix)**.** A matrix $\mathbf{P}$ is called stochastic if

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} & \cdots \\ p_{10} & p_{11} & \cdots \\ \vdots & \ddots & \end{pmatrix}$$

$$0 \leq p_{ij} \leq 1$$

$$\sum_{all(j)} p_{i_0 j} = 1 \text{ for fixed } i_0$$

1.10 Definition (Transition Matrix)**.** Let $\mathbf{P}$ be a Stochastic matrix and let p_{ij} = value in $i-$th row and $j-$th column. We define p_{ij} as

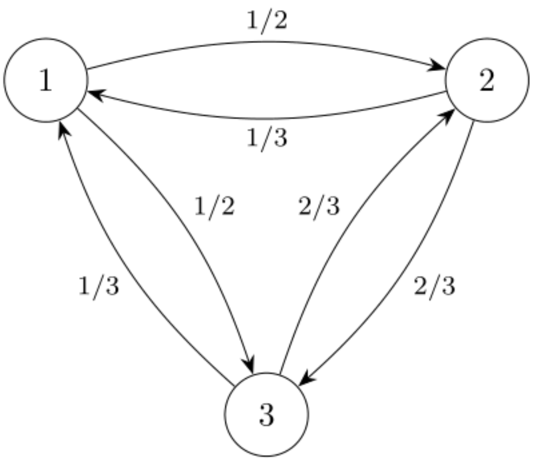
$$p_{ij} = P(X_t = j \mid X_{t-1} = i)$$

This is called the transition matrix for $(X_t)_{t \in T}$.

1.11 Example. Transition Matrix Consider this transition matrix The transition matrix for this Markov Chain is

$$\begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix} \end{array}$$

this can be visualized as:



1.1.1 Multistep Transition Probabilities

1.12 Definition.

$$[P(n, n+m)]_{xy} = P(X_{n+m} = y \mid X_n = x)$$

1.13 Theorem. *Multistep Transition Probability Matrix Let $(X_t)_{t\in T}$ be a stochastic process satisfying the Markov property and let $\mathbf{P}$ be the transition matrix.*

$$[P(n, n+m)]_{xy} = \mathbf{P}^m_{xy}$$

1.14 Lemma.

$$[P(n, m+1+n)]_{xy} = \sum_{\text{all}(z)} [P(n, m+n)]_{xz} P_{zy}$$

Proof. To go from state $x \to y$ we must add up all probabilities of going to an intermediate state $\mathbf{z}$, $x \to \mathbf{z} \to y$ we add possibilities of $\mathbf{z}$.

$[P(n, m+1+n)]_{xy} = P(X_{m+1+n} = y \mid X_n = x)$

$= \sum_{\text{all}(z)} P(X_{m+1+n} = y, X_{n+m} = z \mid X_n = x)$ Marginal probability function (stat240)

$= \sum_{\text{all}(z)} P(X_{m+1+n} = y \mid X_{n+m} = z, X_n = x) P(X_{n+m} = z \mid X_n = x)$ conditional probability

Since X_t satisfies the Markov property we get

$$= \sum_{\text{all}(z)} P(X_{m+1+n} = y \mid X_{n+m} = z) P(X_{n+m} = z \mid X_n = x)$$

By definition we have $P(X_{m+1+n} = y \mid X_{n+m} = z) = P_z y$ and $P(X_{n+m} = z \mid X_n = x) = [P(n, n+m)]_{xz}$. □

Using Lemma 1.14 we can prove the Theorem 1.13.

Since 1.14's result is the definition of matrix multiplication we get

$$[P(n, m+1+n)]_{xy} = [P(n, m+n)P]_{xy}$$

by induction on m with base case $P(n, n+1) = P$ we get

$$[P(n, m+1+n)]_{xy} = \mathbf{P}^m$$