# stat240 notes

Fall 2021

# 1 Counting

**1.1 Theorem.** Number of subsets of size $k$ The number of subsets of size $k$ selected from a set of size $n$ are:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**1.2 Theorem.** Counting with repeated symbols/objects If we have $n_i$ symbols of type $i$ with $i = 1, 2, \ldots, k$ with

$$n_1 + n_2 + \ldots + n_k = n$$

Then the number of arrangements using all the symbols is:

$$\begin{aligned}&\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k}\\ &= \frac{n!}{n_1!n_2!\cdots n_k!}\end{aligned}$$

**1.3 Theorem.** Hyper geometric identity

$$\sum_{x=0}^{\infty}\binom{a}{x}\binom{b}{n-x} = \binom{a+b}{n}$$

*Proof.*

$$\begin{aligned}(1+y)^{a+b} &= (1+y)^a(1+y)^b\\ &= \sum_{k=0}^{a+b}\binom{a+b}{k}y^k = \sum_{i=0}^{a}\binom{a}{i}y^i\sum_{j=0}^{b}\binom{b}{j}y^j\end{aligned}$$

The coefficient of $y^k$ on the left hand side is $\binom{a+b}{k}$ and the coefficient of $y^k$ on the right hand side is

$$\sum_{i=0}^{a}\binom{a}{i}\binom{b}{k-i}$$

both coefficients must be equal. $\square$

**1.4 Theorem. Exponential series**

$$e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}$$

# 2 Probability rules

**2.1 Theorem. Set properties**

(1) Distribution law:

$$A(B \cup C) = AB \cup AC$$

(2) Morgan's Laws:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

**2.2 Definition. Probability rules**

(1) $P(\mathcal{S}) = 1$ where $\mathcal{S}$ is the full sample space.

(2) For any event $A$, $0 \leq P(A) \leq 1$.

(3) If $A$ and $B$ are events with $A \subseteq B$ then $P(A) \leq P(B)$

(4) Addition Law:

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

(5) Complement events

$$P(A) = 1 - P(A^c)$$

**2.3 Definition. Independent events** 2 $A, B$ events are independent $\iff P(AB) = P(A)P(B)$

# 3 Conditional Probability

**3.1 Definition.** Probability $A$ occurs given $B$ has occurred

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

**3.2 Theorem. Multiplication Theorem** For $n$ events $A_1, A_2 \ldots, A_n$ we have

$$P(A_1A_2 \cdots A_n) = P(A_1)P(A_2|A_1)P(A_3|A_2A_1)P(A_4|A_3A_2A_1) \cdots P(A_n|A_1A_2 \cdots A_{n-1})$$

**3.3 Theorem. Total probability** For any event $A$ we have

$$P(A) = P(AB) + P(AB^c)$$

In general if we have a partition $A_1, A_2, \ldots,$ of $\mathcal{S}$, then

$$P(B) = P(B\mathcal{S} = \sum_{i=1}^{\infty} P(A_iB) = \sum_{i=1}^{\infty} P(A_i)P(B|A_i)$$

**3.4 Theorem.** Bayes formula

$$P(A_i|B) = \frac{P(A_iB)}{P(B)} = \frac{P(A_i)P(B|A_i)}{P(B)} = \frac{P(A_i)P(B|A_i)}{\sum\limits_{i=1}^{\infty} P(A_i)P(B|A_i)}$$

When $A_1, A_2, \ldots$ is a partition of $\mathcal{S}$ and $P(B) \neq 0$.

# 4 Discrete Random Variables

**4.1 Definition.** Random variable A random variable $X$ over $\mathcal{S}$ is a function

$$X : \mathcal{S} \to \mathbb{R}$$

$$X = X(\omega) \ , \ \omega \in \mathcal{S}$$

**4.2 Definition.** Probability mass function

$$f(x_i) = P[X = x_i] \ , \ i = 1, 2, \ldots$$

Some properties of *pmf*

$$\sum_{\text{all } x} f(x) = 1$$

**4.3 Definition.** cumulative distribution function

$$F(x) = P[X \leq x] \ \forall x \in \mathbb{R}$$

Where

$$(X \leq x) = \{\omega : X(\omega) \leq x \ , \ \omega \in \mathcal{S}\}$$

Some properties of *cdf*:

(1) Non-decreasing $x_1 \leq x_2 \Rightarrow F(x_1) \leq F(x_2)$

(2) Bounded $0 \leq F(x) \leq 1$

(3) Limits using the monotone convergence theorem we have

$$\lim_{x\to\infty} F(x) = 1$$
$$\lim_{x\to-\infty} F(x) = 0$$

We can calculate $f(x)$ using $F(x)$ if $X$ takes integer values then

$$f(x) = P[X = x] = P[X \leq x] - P[X \leq x - 1] = F(x) - F(x - 1)$$
$$F(x) = \sum_{u \leq x} f(u)$$

## 4.1 Bernoulli trials and related random variables

**4.4 Definition.** Bernoulli trials Experiments with only 2 outcomes Success/Failure. Let $B$ = Sucess with $P(B) = p$. Then $B^c$ = Failure with $P(B^c) = 1 - p$.

**4.5 Definition.** Bernoulli Random Variable Let $X = X(\omega) = 1$ if $\omega = B$ otherwise $X(\omega) = 0$ if $\omega = B^c$. So the *pmf* of $X$ is

$$f(1) = p$$
$$f(0) = 1 - p$$

The distribution is

$$f(x) = p^x(1-p)^{1-x}$$

Where $x \in \{0, 1\}$

## 4.2 Binomial distribution

If we repeat independent Bernoulli trials with the same probability of success $p$, each $n$ times then let $X$ be the number of successes among $n$ trials.

**4.6 Theorem.** Binomial distribution If $X \sim \text{Binomial}(n, p)$ then the *pmf* is given by

$$f(x) = P[X = x] = \binom{n}{x} p^x(1-p)^{n-x}$$

## 4.3 Hypergeometric Distribution

If we have a collection of $N$ objects which can be classified into 2 types success ($S$) and failure. If there are $r$ successes and $N - r$ failures. Pick $n$ objects at random **without replacement**. Let $X$ be the number of successes obtained, $X$ has a hyper-geometric distribution.

$$X \sim \text{Hypergeometric}(N, r, n)$$

**4.7 Theorem.** hyper-geometric distribution If $X \sim \text{Hypergeometric}(N, r, n)$ then the *pmf* is given by

$$f(x) = P[X = x] = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$$

*Proof.* $\#\mathcal{S} = \binom{N}{n}$ total ways of choosing $n$ objects. First we choose $x$ successes from the total $r$ and then choose the rest $n - x$ from the $N - r$ failures. Using the product rule we get the *pmf*. $\square$

## 4.4 Negative Binomial Distribution

The experiment has two distinct outcomes $S$ and $F$ which is repeated independently with $P(S) = \mathbf{p}$. Continue doing the experiment until we get **k** successes. Then let $X$ be the number of failures before the $k-$th success is obtained then $X$ has a Negative Binomial Distribution.

$$X \sim \text{NBinomial}(k, p)$$

There will be $x + k$ trials where we get $x$ failures and $k$ successes then we stop. The last trial must be a success. So in the first $x + k - 1$ trials we need $x$ failures and $k - 1$ successes.
The number of ways to get that is $\binom{x+k-1}{x}$ we choose the $x$ failures and the rest must be successes.

**4.8 Theorem.** Negative Binomial Distribution If $X \sim \text{NBinomial}(k, p)$ then the *pmf* is given by

$$f(x) = P[X = x] = \binom{x + k - 1}{x} p^k (1 - p)^x$$

## 4.5 Geometric distribution

Consider the Negative Binomial Distribution with $k = 1$. That is we only need 1 success. So we repeat the trials until we get the first success.

$$X \sim \text{Geo}(p)$$

**4.9 Theorem.** Geometric Distribution If $X \sim \text{Geo}(p)$ then the *pmf* is given by

$$f(x) = P[X = x] = p(1 - p)^x$$

*Proof.* $k = 1$ in Negative Binomial Distribution. □

## 4.6 Poisson distribution

### 4.6.1 Limit of Binomial

The Poisson distribution happens as a limiting case of the binomial distribution. That is $n \to \infty$ and $p \to 0$. We can keep $\mu = np$ fixed while $n \to \infty$ this forces $p \to 0$. Then the limit for the *pmf* $f(x)$ is given by

$$f(x) = \frac{\mu^x e^{-\mu}}{x!}$$

### 4.6.2 Poisson Distribution from Poisson Process

Consider an event that occurs in random points in time with the following conditions:

- **Independence** The number of occurrences of the event in non-overlapping intervals is independent
- **Individuality** For a small interval $[T, T + \Delta t)$ we have

$$P[\text{ Two or more events in } [T, T + \Delta t)\ ] = o(\Delta t)$$

- **Homogeneity** events occur at a uniform rate $\lambda$ over time.

Any process with these 3 conditions is called a **Poisson process**.

**4.10 Theorem.** Poisson Process In a poission process with a rate of occurrence $\lambda$ the number of occurrences $X$ in a time interval $t$ has a Poisson distribution. The *pmf* is given by

$$f(x) = P[X = x] = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$$

# 5 Expected Value and Variance

**5.1 Definition.** Expected Value Let $X$ be a discrete random variable with range $A$ and *pmf* $f(x)$ then the expected value of $X$ is

$$E(X) = \sum_{x \in A}^{x} f(x)$$

**5.2 Theorem.** *Law of unconscious statistician*

$$E(g(X)) = \sum_{x \in A} g(x) f(x)$$

**5.3 Theorem.** Linearity of Expectation

$$E(a \cdot g(X) + b) = aE(g(X)) + b$$

**5.4 Definition.** Variance Let $\mu = E(X)$ then the variance is given by

$$\mathrm{Var}(X) = E[(X - \mu)^2]$$

**5.5 Theorem.** *Variance properties* Let $\sigma^2 = \mathrm{Var}(X)$ then

$$\sigma^2 = E(X^2) - E(X)^2 \tag{1}$$

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X) \tag{2}$$

**5.6 Theorem.** Expected and Variance value for common distributions

(1) $X \sim \text{Bernoulli}(p)$ then $E(X) = p$ and $\mathrm{Var}(X) = p(1-p)$

(2) $X \sim \text{Binomial}(n, p)$ then $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$

(3) $X \sim \text{Geometric}(p)$ then $E(X) = \frac{1-p}{p}$ and $\mathrm{Var}(X) = \frac{(1-p)^2}{p^2} + \frac{1-p}{p}$

(4) $X \sim \text{NB}(k, p)$ then $E(X) = \frac{k(1-p)}{p}$ and $\mathrm{Var}(X) = \frac{k(1-p)}{p^2}$

(5) $X \sim \text{Hypergeometric}(N, M, n)$ then $E(x) = \frac{nM}{N}$

(6) $X \sim \text{Poisson}(\mu)$ then $E(x) = \mu$ and $\mathrm{Var}(X) = \mu$.

# 6 Multivariate Distributions

## 6.1 Joint and marginal pmf

Joint probability functions for more then 1 variables are defined as:

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= P[X_1 = x_1 \wedge X_2 = x_2 \wedge \cdots \wedge X_n = x_n] \\ &= P[X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n] \end{aligned}$$

$f(x_1, \ldots, x_n)$ is called the joint probability function of $(X_1, \ldots, X_n)$

**6.1 Definition.** Properties of Joint Probability function

$$(1) \qquad \sum_{\text{all}(x_1,\ldots,x_n)} f(x_1, \ldots, x_n) = 1$$

$$(2) \qquad f(x_1, \ldots, x_n) \geq 0 \ \ \forall (x_1, \ldots, x_n)$$

**6.2 Definition.** Marginal probability function Marginal probability function for a single variable $X$ or $Y$ denoted by $f_X(x)$ and $f_Y(y)$ are

$$f_X(x) = \sum_{\text{all}(y)} f(x, y)$$

$$f_Y(y) = \sum_{\text{all}(yx} f(x, y)$$

where $f(x, y)$ is the joint probability function for $X, Y$.

### 6.1.1 Conditional pmf

**6.3 Definition.** Conditional pmf The conditional *pmf* of $X$ given $Y = y$ is

$$f_X(x|y) = \frac{f(x, y)}{f_Y(y)}, \text{ Given } f_Y(y) > 0$$

### 6.1.2 Independent random variables

**6.4 Definition.** Independent random variables Two random variables $X, Y$ are independent if

$$P[X = x, Y = y] = P[X = x] \cdot P[Y = y]$$

Or $f(x, y) = f_X(x) f_Y(y)$ for all $(x, y)$.

**6.5 Theorem.** Vandermonde's Convolution Formula

$$\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$$

# 7 Multivariate Distributions

## 7.1 Trinomial Distribution

There are 3 possible outcomes $A, B, C$ with

$$P[A] = p_1, P[B] = p_2, P[C] = p_3$$
$$p_1 + p_2 + p_3 = 1$$

The joint *pmf* when $x_1 + x_2 + x_3 = n$ is given by

$$P[X_1 = x_1, X_2 = x_2, X_3 = x_3] = \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$

the marginal distribution is

$$X_1 \sim \text{Binomial}(n, p_1); X_2 \sim \text{Binomial}(n, p_2); X_3 \sim \text{Binomial}(n, p_3);$$

## 7.2 Multinomial Distribution

Each trial for $k \geq 2$ has possible outcomes $A_1, \ldots, A_k$ with $P[A_i] = p_i$. Such that $\sum p_i = 1$. If we have

$$\sum X_i = n$$

Then the joint *pmf* is

$$(X_1, \ldots, X_k) \sim \text{Multinomial}(n, p_1, \ldots, p_k)$$
$$P[X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \ldots p_k^{x_k}$$

# 8 Multivariate Expectation

**8.1 Theorem.** The law of unconscious statistician If $(X_1, X_2) \sim f(x_1, x_2)$ then

$$E(g(X_1, X_2)) = \sum_{x_1} \sum_{x_2} g(x_1, x_2) f(x_1, x_2)$$

**8.2 Theorem.** Properties of Expectation

(1) $E(X + Y) = E(X) + E(Y)$

(2) $E(aX + bY + c) = aE(X) + bE(Y) + c$

(3) if $X$ and $Y$ are independent then

$$E[h(X)g(Y)] = E[h(X)]E[g(Y)]$$

## 8.1 Covariance

**8.3 Definition.** Covariance

$$\begin{aligned} \text{Cov}(X, Y) &= E[(X - EX)(Y - EY)] \\ &= E(XY) - (EX)(EY) \end{aligned}$$

**8.4 Definition.** Varriance

$$\begin{aligned} &\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + \text{Cov}(X, Y) \\ &\text{Var}(aX + bY + c) = a^2\text{Var}(X) + 2ab\text{Cov}(X, Y) + b^2\text{Var}(Y) \end{aligned}$$

**8.5 Theorem.** Multinational co-variance If $(X_1, \ldots, X_k) \sim \text{Multinomial}(n, p_1, \ldots, p_k)$ then

$$\text{Cov}(X_i, X_j) = -np_ip_j$$

## 8.2 Correlation coefficient

**8.6 Definition.**

$$\rho = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}}$$

$|\rho| \leq 1$ and $|\rho| = 1$ if and only if $Y = aX + b$.

## 8.3 Expectation and Variance results

**8.7 Theorem.** General result

(1) $E(\sum c_i X_i) = \sum c_i E(X_i)$

(2) $\text{Var}(\sum c_i X_i) = \sum c_i^2 \text{Var}(X_i) + \sum_{i<j} c_i c_j \text{Cov}(X_i, X_j)$

(3) If $X_1, \ldots, X_n$ are independent then $\text{Var}(\sum c_i X_i) = \sum c_i^2 \text{Var}(X_i)$

# 9 Continuous probability distributions

**9.1 Definition.** Probability density function (PDF) $X$ has *pdf*, where

$$\begin{aligned} &f(x) \geq 0 && x \in (-\infty, \infty) \\ &P[a \leq X \leq b] = \int_a^b f(x)\mathrm{d}x && a \leq b \end{aligned}$$

**9.2 Definition.** Cumulative distribution function (CDF) $X$ has *cdf*, where

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(t)\mathrm{d}t$$

(1) $0 \leq F(x) \leq 1$

(2) $F(x_1) \leq F(x_2)$ if $x_1 < x_2$

(3) If $f(x)$ is continuous at $x$ then $\frac{\mathrm{d}F(x)}{\mathrm{d}x} = f(x)$

## 9.1 Change of variables

Suppose $X \sim f(x)$ and let $Y = h(X)$. If $h(X)$ has a *unique* inverse or $h(\cdot)$ is one-to-one then the *pdf* of $Y$ is given by

$$g(y) = f(h^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}h^{-1}(y)\right|$$

*Proof.*

$$g(y) = \frac{\mathrm{d}}{\mathrm{d}y}F(h^{-1}(y)) = F'(h^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}h^{-1}(y)\right| = f(h^{-1}(y))\left|\frac{\mathrm{d}}{\mathrm{d}y}h^{-1}(y)\right|$$

□

## 9.2 Commonly used continuous distributions

## 9.3 Exponential distribution

$X \sim \text{EXP}(\lambda)$

(1) The *pdf* is $f(x) = \frac{\lambda}{e^{\lambda x}}$ $x \geq 0, \lambda > 0$

(2) The *cdf* is

$$F(x) = \int_{-\infty}^{x} \lambda e^{-\lambda t} \mathrm{d}t = 1 - e^{-\lambda x} \qquad x > 0$$

$$F(x) = 0 \qquad x \leq 0$$

(3) Memory less property

$$P[X > t + s | X > s] = P[X > t]$$

(4) Can be used to model waiting time. Suppose $\lambda$ is the intensity parameter of a Poisson process. Let $X$ be the waiting time for the next event.

$$F(x) = 1 - P(X > x) = 1 - P[\text{No events in } [0, x)] = 1 - \frac{e^{-\lambda x}(\lambda x)^0}{0!} = 1 - e^{-\lambda x}$$

## 9.4 Gamma distribution

$X \sim \text{GAM}(\alpha, \beta)$

(1) The *pdf* is

$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha - 1} \exp\left(\frac{-x}{\beta}\right)$$

## 9.5 Uniform distribution

$X \sim \text{UNIF}(a, b)$ then the *pdf* is

$$f(x) = \frac{1}{b - a}$$

the *cdf* is

$$F(x) = \frac{x - a}{b - a}$$

### 9.6 Beta distribution

Beta function:

$$B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

(1) pdf is given by

$$f(x) = \frac{1}{B(\alpha_1, \alpha_2)} x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}$$

## 10 Expectation and Variance of continuous random variables

**10.1 Definition.** Expectation and variance

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) \mathrm{d}x \\ \mathrm{Var}(X) &= \int_{-\infty}^{\infty} [x - E(X)]^2 f(x) \mathrm{d}x \end{aligned}$$

# 11 Moments and moment generating function

**11.1 Definition.** If $X$ is a random variable then the moment generating function is

$$M_X(t) = E(e^{tX}) \qquad \text{Given it exists for } t \in (-h, h)$$

## 11.1 Joint MGF

**11.2 Definition.** Joint MGF For two random variables $X, Y$ the joint MGF is

$$\begin{aligned} M_{X,Y}(t_1, t_2) &= E(e^{t_1 X + t_2 Y}) \\ &= \iint_{\mathbb{R}^2} e^{t_1 x} e^{t_2 y} f(x, y) \mathrm{d}x \mathrm{d}y \end{aligned}$$

if the joint expectation exists for $t_1 \in (-h_1, h_1)$ and $t_2 \in (-h_2, h_2)$ $M_X(t) = M_{X,Y}(t, 0)$ and $M_Y(t) = M_{X,Y}(0, t)$

**11.3 Theorem.** MGF of linear function Suppose $X$ has *mgf* $M_X(t)$ for $t \in (-h, h)$. Let $Y = aX + b$ then

$$M_Y(t) = e^{bt} M_X(at) \qquad t \in \left(-\frac{h}{|a|}, \frac{h}{|a|}\right)$$

# 12 Properties of MGF

**12.1 Theorem.** Moments Suppose $X$ has MGF $M_X(t)$. Then $M_X(0) = 1$ and

$$E(X^k) = M_X^{(k)}(0)$$

**12.2 Theorem.** Joint Moments Suppose $X, Y$ are random variables with joint MGF $M_{X,Y}(t_1, t_2)$. Then

$$E(X^jY^k) = \frac{\partial^{j+1}}{\partial t_1^j \partial t_2^k} M_{X,Y}(t_1, t_2)\Bigg|_{(t_1,t_2)=(0,0)}$$

**12.3 Theorem.** MGF Independence Two random variables $X, Y$ are independent **if and only if**

$$M_{X,Y}(t_1, t_2) = M_X(t_1)M_Y(t_2)$$

**12.4 Theorem.** Sum of independent random variables If $Y = X_1 + X_2$ then

$$M_Y(t) = M_1(t)M_2(t)$$

# 13 Convergence in probability

**13.1 Definition.** Convergence in probability $X_n \xrightarrow{p} b$ if for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} P[|X_n - b| \geq \varepsilon] = 0$$

**13.2 Theorem.** Convergence of Sum and product If $X_n \xrightarrow{p} a$ and $Y_n \xrightarrow{p} b$ then

$$X_n + Y_n \xrightarrow{p} a + b$$
$$X_n Y_n \xrightarrow{p} ab$$

**13.3 Theorem.** Markov's inequality For random variable $X$ with finite $E(|X|^k)$,

$$P[|X| \geq c] \leq \frac{E[|X|^k]}{c^k}$$

## 13.1 Weak Law of large numbers

**13.4 Theorem.** Weak Law of large numbers Suppose $X_i'$s are **iid** with $EX_i = \mu$ and $\text{Var}X_i = \sigma^2 < \infty$ then

$$\bar{X} = \sum_{n=1}^{n} \frac{X_i}{n}$$
$$\bar{X} \xrightarrow{p} \mu$$

# 14 Convergence in distribution

**14.1 Definition.** Convergence in distribution $X_n \xrightarrow{d} X$ if

$$\lim_{n\to\infty} F_n(x) = F(x) \qquad \text{All all continuity points of } F$$

where $F_n(x) = P[X_n \le x]$ and $F$ is the $CDF$ of $X$

## 14.1 e limit

**14.2 Theorem.** *e* limit If $b$ is a real constant and $\lim_{n\to\infty} \psi(n) = 0$ then

$$\lim_{n\to\infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^n = e^b$$

## 14.2 MGF for limiting distributions

**14.3 Theorem.** MGF Convergence theorem Let $X_1, X_2, \ldots, X_n$ and let $M_1(t), M_2(t), \ldots, M_n(t), \ldots$ be the MGF. Let $X$ be a random variable with MGF $M(t)$ if there exists $h > 0$ such that

$$\lim_{n\to\infty} M_n(t) = M(t) \qquad t \in (-h, h)$$

then $X_n \xrightarrow{d} X$

# 15 Central limit theorem

**15.1 Theorem.** Central limit theorem Suppose $X_i$ are *iid* with $EX_i = \mu$ and $\text{Var}X_i = \sigma^2 < \infty$ then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$$

Equivalently

(1) $S_n = \sum X_i \xrightarrow{d} N(n\mu, n\sigma^2)$

(2) $\bar{X} = \frac{1}{n} \sum X_i \xrightarrow{d} N(\mu, \frac{\sigma^2}{n})$

# CDF Not in notes

| Name | *PDF* | $E(X)$ | Var$(X)$ | MGF $(M(t))$ |
|---|---|---|---|---|
| Exponential $(X \sim EXP(\lambda))$ | $\lambda e^{-\lambda x}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda^2}$ | $\frac{1}{1-\frac{t}{\lambda}}$ for $t < \lambda$ |
| Gamma $(X \sim Gamma(\alpha, \beta))$ | $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{\frac{-x}{\beta}}$ | $\alpha \cdot \beta$ | $\alpha \cdot \beta^2$ | $\frac{1}{(1-\beta t)^\alpha}$ for $t < \frac{1}{\beta}$ |
| Beta $(X \sim Beta(\alpha, \beta))$ | $f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ | $\frac{\alpha}{\alpha+\beta}$ | $\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$ | - |

# Useful

IID for distribution

# 1 If $Y \sim BN(n, p)$ then

$$Y = \sum_{i=1}^{n} X_i \qquad X_i \overset{\text{iid}}{\sim} Bernoulli(p)$$

# 2 If $Y \sim \text{POI}(n)$ then

$$Y = \sum_{i=1}^{n} X_i \qquad X_i \overset{\text{iid}}{\sim} \text{POI}(1)$$

# 3 If $Y \sim GAMMA(\alpha, \beta)$ then

$$Y = \sum_{i=1}^{n} X_i \qquad X_i \overset{\text{iid}}{\sim} GAMMA(1, \beta)$$

# 4 If $Y \sim NB(k, p)$ then

$$Y = \sum_{i=1}^{k} X_i \qquad X_i \overset{\text{iid}}{\sim} Geo(p)$$

Assignment Theorems:

**15.2 Theorem. A4Q7** If $\{X_1, X_2, \ldots, X_n, \ldots\}$ is a sequence of random variables and $E(X_n) = \mu$ and $\text{Var}(x_n) = \sigma_n^2$ with

$$\lim_{n\to\infty} \mu_n = \mu$$

$$\lim_{n\to\infty} \sigma_n^2 = 0$$

Then

$$X_n \overset{p}{\to} X = \mu$$