PMAT 32632 Tutorial 6 PS/2017/084

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1) $S \equiv 6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$

$a = 6,\ h = \frac{-5}{2},\ b = -6,\ g = 7,\ f = \frac{5}{2},\ c = 4$

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & \frac{-5}{2} & 7 \\ \frac{-5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} = 6\left(-24 - \frac{25}{4}\right) + \frac{5}{2}\left(-10 - \frac{35}{2}\right) + 7\left(\frac{-25}{4} + 42\right)$$

$$= 6\left(\frac{-121}{4}\right) + \frac{5}{2}\left(-\frac{55}{2}\right) + 7\left(\frac{143}{4}\right)$$

$$= \frac{-726 - 275 + 1001}{4} = \frac{-1001 + 1001}{4} = 0$$

Since $\Delta = 0$, $S = 0$ represents a pair of straight lines

$S' \equiv ax^2 + 2abxy + by^2 + 2x + 4y + 2 = 0$

Consider $S + \lambda S' = 0$ represents pairs of straight lines

$(6x^2 - 5xy - 6y^2 + 14x + 5y + 4) + \lambda(ax^2 + 2abxy + by^2 + 2x + 4y + 2) = 0$

$(6 + a\lambda)x^2 + (2\lambda ab - 5)xy + (b\lambda - 6)y^2 + (14 + 2\lambda)x + (5 + 4\lambda)y + 4 + 2\lambda = 0$

$$\Delta = \begin{vmatrix} 6 + \lambda a & \lambda ab - \frac{5}{2} & 7 + \lambda \\ \lambda ab - \frac{5}{2} & \lambda b - 6 & \frac{5}{2} + 2\lambda \\ 7 + \lambda & \frac{5}{2} + 2\lambda & 4 + 2\lambda \end{vmatrix} = 0$$

$$= (6 + \lambda a)\left[(\lambda b - 6)(4 + 2\lambda) - \left(\frac{5}{2} + 2\lambda\right)^2\right] - \left(\lambda ab - \frac{5}{2}\right)\left[\left(\lambda ab - \frac{5}{2}\right)(4 + 2\lambda) - (7 + \lambda)\left(\frac{5}{2} + 2\lambda\right)\right] + (7 + \lambda)\left[\left(\lambda ab - \frac{5}{2}\right)\left(\frac{5}{2} + 2\lambda\right) - (7 + \lambda)(\lambda b - 6)\right] = 0$$

$$= (\lambda a + 6)\left[(2b - 4)\lambda^2 + (4b - 22)\lambda - \frac{121}{4}\right] - \left(\lambda ab - \frac{5}{2}\right)\left[(2ab - 2)\lambda^2 + \left(4ab - \frac{43}{2}\right)\lambda - \frac{55}{2}\right] + (\lambda + 7)\left[(2ab - b)\lambda^2 + \left(\frac{5}{2}ab - 7b + 1\right)\lambda + \frac{143}{4}\right] = 0$$

$$= \left[a(2b - 4) - ab(2ab - 2) + 2ab - b\right]\lambda^3 + \left[6(2b - 4) + a(4b - 22) + \frac{5}{2}(2ab - 2) - ab\left(4ab - \frac{43}{2}\right) + 7(2ab - b) + \left(\frac{5}{2}ab - 7b + 1\right)\right]\lambda^2 + \left[6(4b - 22) - \frac{121}{4}a + \frac{5}{2}\left(4ab - \frac{43}{2}\right) + \frac{55}{2}ab + \frac{143}{4} + 7\left(\frac{5}{2}ab - 7b + 1\right)\right]\lambda + \left[\frac{-121}{4}(6) + \frac{5}{2}\left(-\frac{55}{2}\right) + 7\left(\frac{143}{4}\right)\right] = 0$$

$$= \left[-2a^2b^2 + 6ab - 4a - b\right]\lambda^3 + \left[-4a^2b^2 + 47ab - 22a - 2b - 28\right]\lambda^2 + \left[55ab - \frac{121}{4}a - 25b - 143\right]\lambda + 0 = 0$$

$$= \lambda\left[(-2a^2b^2 + 6ab - 4a - b)\lambda^2 + (-4a^2b^2 + 47ab - 22a - 2b - 28)\lambda + \left(55ab - \frac{121}{4}a - 25b - 143\right)\right] = 0$$

is a polynomial equation in λ

2. The given conics are,

$$S \equiv x^2+y^2-8x+4y+6=0 \quad \& \quad S' \equiv x^2+y^2+4x-6y+8=0$$

Equation of circle through the intersection of conics $S=0$ & $S'=0$ is,

$$S'' \equiv S+\lambda S'$$

$$S'' \equiv (x^2+y^2-8x+4y+6)+\lambda(x^2+y^2+4x-6y+8)=0$$

$$S'' \equiv (1+\lambda)x^2+(1+\lambda)y^2+(-8+4\lambda)x+(4-6\lambda)y+6+8\lambda=0 \longrightarrow (*)$$

Consider a translation transformation

$$x = X + x_0$$
$$y = Y + y_0$$

Now the equation of $S''=0$ w.r.t. $O'X$ and $O'Y$

$$(1+\lambda)(X+x_0)^2+(1+\lambda)(Y+y_0)^2+(4\lambda-8)(X+x_0)+(4-6\lambda)(Y+y_0)+6+8\lambda=0$$

we choose (x_0,y_0) such that

$$\left.\frac{\partial S''}{\partial x}\right|_{x_0} = 2(1+\lambda)x_0+(-8+4\lambda)=0 \qquad\qquad x_0 = \frac{4-2\lambda}{1+\lambda}$$

$$\left.\frac{\partial S''}{\partial y}\right|_{y_0} = 2(1+\lambda)y_0+(4-6\lambda)=0 \qquad\qquad y_0 = \frac{3\lambda-2}{1+\lambda}$$

Then, center of the conic is $\left(\frac{4-2\lambda}{1+\lambda}, \frac{3\lambda-2}{1+\lambda}\right)$

Center lies on the line $y=x$

$$\frac{3\lambda-2}{1+\lambda} = \frac{4-2\lambda}{1+\lambda} \Rightarrow 5\lambda=6 \Rightarrow \lambda=\frac{6}{5}$$

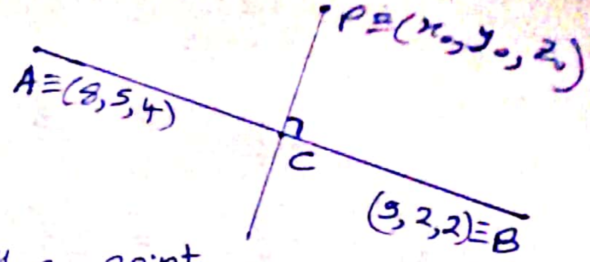
Then required equation is,

$$(*) \Rightarrow \left(1+\frac{6}{5}\right)x^2+\left(1+\frac{6}{5}\right)y^2+\left[-8+4\left(\frac{6}{5}\right)\right]x+\left[4-6\left(\frac{6}{5}\right)\right]y+6+8\left(\frac{6}{5}\right)=0$$

$$\frac{11}{5}x^2+\frac{11}{5}y^2-\frac{16}{5}x-\frac{16}{5}y+\frac{78}{5}=0$$

$$11x^2+11y^2-16x-16y+78=0$$

3 Let C be the midpoint of AB



Then, $C \equiv \left(\frac{8+5}{2}, \frac{5+2}{2}, \frac{4+2}{2}\right)$

$C \equiv \left(\frac{13}{2}, \frac{7}{2}, 3\right)$

Let P be the point equidistant to the point A & B. Then the equation of the locus of points equidistant to these point A & B is given by,

$$|AP| = |PB|$$

$$\sqrt{(x_0-8)^2+(y_0-5)^2+(z_0-4)^2} = \sqrt{(x_0-5)^2+(y_0-2)^2+(z_0-2)^2}$$

$$(x_0-8)^2+(y_0-5)^2+(z_0-4)^2 = (x_0-5)^2+(y_0-2)^2+(z_0-2)^2$$

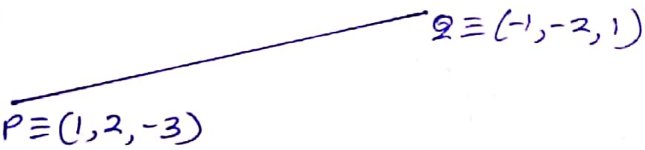
$$\left.\begin{array}{l} x_0^2-16x_0+64+y_0^2-10y_0+25 \\ +z_0^2-8z_0+16 \end{array}\right\} = x_0^2-10x_0+25+y_0^2-4y_0+4+z_0^2-4z_0+4$$

$$6x_0+6y_0+4z_0-72=0$$

Then the required equation is given by,

$$6x+6y+4z-72=0$$

4 (a)



$$PQ = \sqrt{[1-(-1)]^2+[2-(-2)]^2+(-3-1)^2}$$

$$PQ = \sqrt{4+16+16} = \sqrt{36} = 6$$

Direction cosines are,

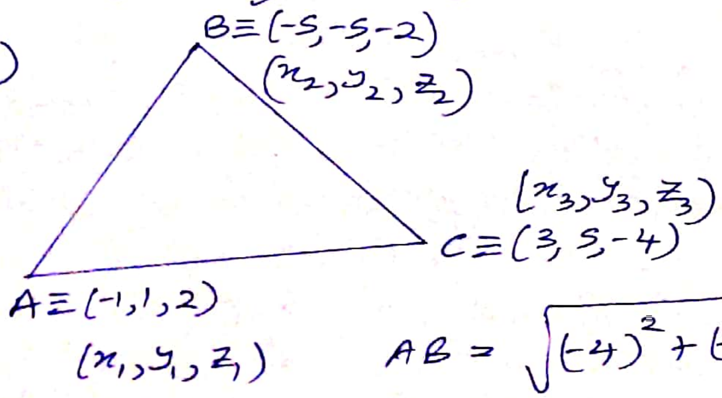
$$\cos\alpha = \frac{x_2-x_1}{PQ} = \frac{(-1)-1}{6} = \frac{-1}{3} \quad \text{or} \quad \cos\alpha = \frac{1}{3}$$

$$\cos\beta = \frac{y_2-y_1}{PQ} = \frac{-2-2}{6} = \frac{-2}{3} \quad \text{or} \quad \cos\beta = \frac{2}{3}$$

$$\cos\gamma = \frac{z_2-z_1}{PQ} = \frac{1-(-3)}{6} = \frac{2}{3} \quad \text{or} \quad \cos\gamma = \frac{-2}{3}$$

$$\left(\frac{-1}{3}, \frac{-2}{3}, \frac{2}{3}\right) \quad \text{or} \quad \left(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}\right)$$

(b)



Direction ratios of AB

$$x_2-x_1 : y_2-y_1 : z_2-z_1$$

$$-5-(-1) : -5-1 : -2-2$$

$$-4 : -6 : -4$$

$$AB = \sqrt{(-4)^2+(-6)^2+(-4)^2} = \sqrt{16+36+16} = \sqrt{68}$$

$$= 2\sqrt{17}$$

4 (b) $l = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$ or $l = \frac{2}{\sqrt{17}}$

$m = \frac{-6}{2\sqrt{17}} = \frac{-3}{\sqrt{17}}$ or $m = \frac{3}{\sqrt{17}}$

$n = \frac{-4}{2\sqrt{17}} = \frac{-2}{\sqrt{17}}$ or $n = \frac{2}{\sqrt{17}}$

Then the direction cosines of AB are

$\left(\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$ or $\left(\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}\right)$

Direction ratios of BC $\Rightarrow (x_3-x_2):(y_3-y_2):(z_3-z_2)$

$8:10:-2$

$BC = \sqrt{8^2+10^2+(-2)^2} = \sqrt{64+100+4} = \sqrt{168} = 2\sqrt{42}$

The direction cosines of BC are,

$\left(\frac{8}{2\sqrt{42}}, \frac{10}{2\sqrt{42}}, \frac{-2}{2\sqrt{42}}\right) \Rightarrow \left(\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$

Direction ratios of AC, $\Rightarrow (x_3-x_1):(y_3-y_1):(z_3-z_1)$

$4:4:-6$

$AC = \sqrt{4^2+4^2+(-6)^2} = \sqrt{16+16+36} = \sqrt{68} = 2\sqrt{17}$

The direction cosines of AC are,

$\left(\frac{4}{2\sqrt{17}}, \frac{4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}\right) = \left(\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}\right)$

(c) Let the direction cosines of a line which makes angles of α with each of coordinate axes

$l = \cos\alpha$, $m = \cos\alpha$, $n = \cos\alpha$

$l^2 + m^2 + n^2 = 1$

$(\cos\alpha)^2 + (\cos\alpha)^2 + (\cos\alpha)^2 = 1$

$3(\cos\alpha)^2 = 1$

$\cos\alpha = \frac{\pm 1}{\sqrt{3}}$

Hence the direction cosines are,

$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$

5 Consider the two planes

$x+2y-3z+4=0 \longrightarrow ①$ and $2x+5y+4z+1=0 \longrightarrow ②$

Direction ratios of the normal of ① is $1:2:-3$

Direction ratios of the normal of ② is $2:5:4$

$$l_1l_2+m_1m_2+n_1n_2 = 1(2)+2(5)+(-3)4$$
$$= 2+10-12 = 0$$

Planes are perpendicular

Now consider

$x+2y-3z+4=0 \longrightarrow ①$

$4x+7y+6z+2=0 \longrightarrow ③$

Direction ratios of normal of ③ is $4:7:6$

$$l_1l_2+m_1m_2+n_1n_2 = 1(4)+2(7)+(-3)6$$
$$= 4+14-18 = 0$$

Planes are perpendicular

equation ① plane is perpendicular to each of the planes ② & ③

6 $\dfrac{x-1}{-3} = \dfrac{y-2}{2k} = \dfrac{z-3}{2} \longrightarrow ①$ and

$\dfrac{x-1}{3k} = \dfrac{y-5}{1} = \dfrac{z-6}{-5} \longrightarrow ②$

Direction ratios of the equation ① is $-3:2k:2$

Direction ratios of the equation ② is $3k:1:-5$

If the two lines are perpendicular

$$(-3)(3k)+2k(1)+2(-5) = 0$$
$$-9k+2k-10 = 0$$
$$k = \frac{-10}{7}$$

7 Let $u \equiv 2x-3y-2z-9=0$

$V \equiv 3x-6y-2z-15=0$

$u+\lambda V=0$; $\lambda \in \mathbb{R}$ represents the equation of any plane passing through the line of intersecting of the 2 planes,

Consider,

$$u-\frac{1}{2}V = 2x-2z-9-\frac{3}{2}x+z+\frac{15}{2}=0$$
$$x-2z-3=0 \longrightarrow ①$$

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$\boxed{7}$ $u - \frac{2}{3}v = -3y - 2z - 9 + 4y + \frac{4}{3}z + 10 = 0$

$$y - \frac{2}{3}z + 1 = 0$$

$$3y - 2z + 3 = 0 \longleftrightarrow (2)$$

$(1) \Rightarrow z = \frac{x-3}{2}$, $(2) \Rightarrow \frac{3y+3}{2}$

$$\Rightarrow \frac{x-3}{2} = \frac{3y+3}{2} = z$$

$$\frac{x-3}{2} = \frac{y+1}{2/3} = \frac{z-0}{1}$$

$\boxed{8}$ Let $u \equiv x + y + z - 6 = 0$

$$v \equiv 2x + 3y + 4z + 5 = 0$$

Equation of any plane which is passing through the intersection of $u = 0$ and $v = 0$ given by,

$$u + \lambda v = 0$$

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$$

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - 6 + 5\lambda = 0$$

The plane passing through the point $(1, 1, 1)$

$$(1 + 2\lambda)(1) + (1 + 3\lambda)(1) + (1 + 4\lambda)(1) - 6 + 5\lambda = 0$$

$$14\lambda - 3 = 0$$

$$\lambda = \frac{3}{14}$$

The required equation of the plane

$$(x + y + z - 6) + \frac{3}{14}(2x + 3y + 4z + 5) = 0$$

$$20x + 23y + 26z - 69 = 0$$

$\boxed{9}$ (a) $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} \longrightarrow (1)$

$$\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7} \longrightarrow (2)$$

coordinates of any point on the line (1)

$$= (-1 + 3t, -3 + 5t, -5 + 7t)$$

If two lines intersect, the coordinates should satisfy the equation (2)

$$(2) \Rightarrow \frac{-1 + 3t - 2}{1} = \frac{-3 + 5t - 4}{4} = \frac{-5 + 7t - 6}{7}$$

$$\frac{3t - 3}{1} = \frac{5t - 7}{4} = \frac{7t - 11}{7}$$

9) (a) From $\frac{3t-3}{1} = \frac{5t-7}{4}$

$$12t - 12 = 5t - 7 \Rightarrow t = \frac{5}{7}$$

From $\frac{5t-7}{4} = \frac{7t-11}{7}$

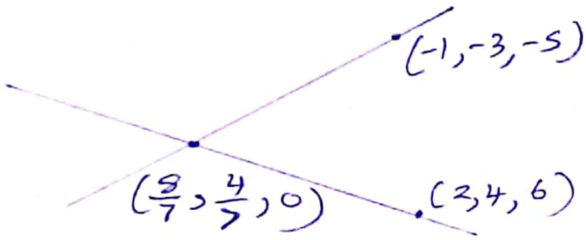
$$35t - 49 = 28t - 44 \Rightarrow t = \frac{5}{7}$$

Both equations gives $t = \frac{5}{7}$

∴ coordinates of the point of intersection

$$t = \frac{5}{7} \Rightarrow \equiv \left(-1+3\left(\frac{5}{7}\right), -3+5\left(\frac{5}{7}\right), -5+7\left(\frac{5}{7}\right)\right)$$

$$= \left(\frac{8}{7}, \frac{4}{7}, 0\right)$$



The equation of any plane passing through the line given by (1) is,

$$\left(\frac{x+1}{3} - \frac{y+3}{5}\right) + \lambda\left(\frac{y+3}{5} - \frac{z+5}{7}\right) = 0 \longrightarrow (*)$$

The equation of plane passing through the point (2,4,6) is

$$\left(\frac{2+1}{3} - \frac{4+3}{5}\right) + \lambda\left(\frac{4+3}{5} - \frac{6+5}{7}\right) = 0$$

$$\left(1 - \frac{7}{5}\right) + \lambda\left(\frac{7}{5} - \frac{11}{7}\right) = 0$$

$$\lambda\left(\frac{49-55}{35}\right) = \frac{2}{5} \Rightarrow \lambda = \frac{-7}{3}$$

By substituting $\lambda = \frac{-7}{3}$ in (*)

$$\left(\frac{x+1}{3} - \frac{y+3}{5}\right) - \frac{7}{3}\left(\frac{y+3}{5} - \frac{z+5}{7}\right) = 0 \longrightarrow (**)$$

If both lines are coplanar $\left(\frac{8}{7}, \frac{4}{7}, 0\right)$ should satisfy (**)

$$L.H.S. \equiv \left(\frac{\frac{8}{7}+1}{3} - \frac{\frac{4}{7}+3}{5}\right) - \frac{7}{3}\left(\frac{\frac{4}{7}+3}{5} - \frac{0+5}{7}\right)$$

$$\equiv \left(\frac{15}{21} - \frac{25}{35}\right) - \frac{7}{3}\left(\frac{25}{35} - \frac{5}{7}\right) \equiv \left(\frac{5}{7} - \frac{5}{7}\right) - \frac{7}{3}\left(\frac{5}{7} - \frac{5}{7}\right)$$

$$\equiv 0 - \frac{7}{3}(0) \equiv 0 \equiv R.H.S.$$

∴ Two lines are co-planar

Equation of the plane $\equiv x - 2y + z = 0$

9 (b) $\frac{x+1}{4} = \frac{y-5}{5} = \frac{z+2}{-1} \longrightarrow (1)$

$3x + 4y + 5z - 5 = 0 \longrightarrow (2)$

coordinates of any point on the line (1)

$\equiv (-1+4t,\ 5+5t,\ -2-t)$

The line intersect the plane

$\therefore$ coordinates lie on the plane

$3(-1+4t) + 4(5+5t) + 5(-2-t) - 5 = 0$

$27t + 2 = 0 \Rightarrow t = \frac{-2}{27}$

$\therefore$ coordinates of intersection $\equiv \left(-1+4\left(\frac{-2}{27}\right),\ 5+5\left(\frac{-2}{27}\right),\ -2-\left(\frac{-2}{27}\right)\right)$

$\equiv \left(\frac{-35}{27},\ \frac{125}{27},\ \frac{-52}{27}\right)$

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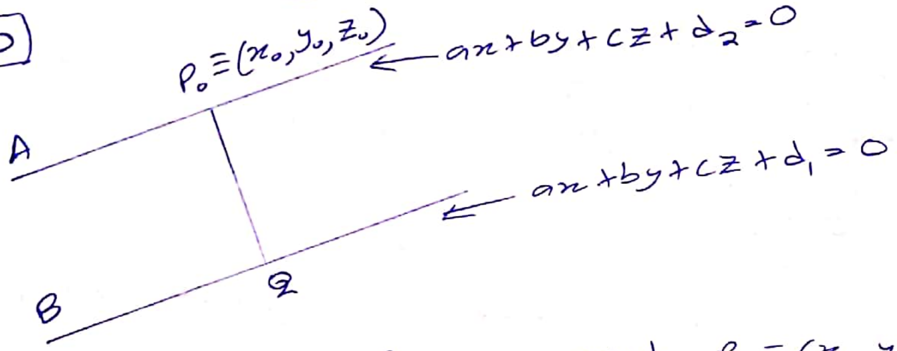

The distance from a point $P_0 \equiv (x_0, y_0, z_0)$ to the plane $ax+by+cz+d_1=0$ is given by

$$P_0Q = \frac{|ax_0+by_0+cz_0+d_1|}{\sqrt{a^2+b^2+c^2}} \longrightarrow (*)$$

The point P_0 lies on the plane A

so, $ax_0+by_0+cz_0+d_2=0$

$d_2 = -(ax_0+by_0+cz_0)$

$$(*) \Rightarrow P_0Q = \frac{|-d_2+d_1|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|-(d_2-d_1)|}{\sqrt{a^2+b^2+c^2}}$$

$$P_0Q = \frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$$