THE METRIC SPACE OF CONNECTED SIMPLE GRAPHS

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Abstract

We introduce a sequence $e_n^\ell(\ell = 1, 2, \dots)$ of real-valued functions on $\mathcal{G}_n \times \mathcal{G}_n$, where $\mathcal{G}_n$ is the set of all simple, connected, undirected graphs of order n up to isomorphism. The functions e_n^ℓ arise naturally from the consideration of graph embeddings. The binary relations $\succ_\ell$ on $\mathcal{G}_n$ are then defined to be exactly the zeroes of e_n^ℓ. We observe that the relation $\succ_1$ coincides precisely with the classical subgraph relation, and that $\succ_1, \succ_2, \dots$ is a sequence of weaker binary relations on $\mathcal{G}_n$. Motivated by this, we define $d_n^\ell(H,K) = e_n^\ell(H,K)+e_n^\ell(K,H)$, thereby obtaining a sequence of symmetric, real-valued functions $d_n^\ell(\ell = 1, 2, \dots)$ on $\mathcal{G}_n \times \mathcal{G}_n$. When $\ell = 1$, (G_n, d_n^1) is a totally disconnected metric space of graphs that embodies the classical notion of graph isomorphism: $H \cong K \Leftrightarrow d_n^1(H,K) = 0$, $H \ncong K \Leftrightarrow d_n^1(H,K) = \infty$. As ℓ increases in the range $2, \dots, n-2$, the value of d_n^ℓ decreases for every distinct pair of graphs in $\mathcal{G}_n$ although distinct graphs in G_n do not collapse. Further, we show that the graphs H for which $d_n^\ell(H, K_n)$ is finite, are precisely those graphs $H \in \mathcal{G}_n$ which have diameter $\leqslant \ell$. Finally, when $\ell \geqslant n-1$, the functions d_n^ℓ form a stationary sequence whose elements equip $\mathcal{G}_n$ once again with full metric structure. We refer to this space as (G_n, d_n^*), "The metric space of connected simple graphs." Unlike (G_n, d_n^1), the space (G_n, d_n^*) is connected. A number of open questions concerning the geodesic structure of (G_n, d_n^*) are also presented.

1 Introduction

We begin with an informal, brief description of the concepts and motivation behind this work. The statements here will be formulated precisely in the subsequent sections.

Given two connected, simple graphs H, K of order n, begin by considering a one-to-one mapping $\phi : V[K] \to V[H]$. Upon fixing ϕ, to each edge $e = (u, v) \in E[K]$ we associate a walk p_e between $\phi(u)$ and $\phi(v)$ in H. We view ϕ together with the chosen set of walks $Q = \{p_e | e \in E[K]\}$ as a topological embedding ϕ' of K into H. If all the walks in Q are of length $\leqslant \ell$, then ϕ' is called an ℓ-topological

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