**Sparse Periodic Goldbach Sets**

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**Abstract**

In this paper, we consider sets of natural numbers $P \subseteq \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ which satisfy the property that every $x$ in $\mathbb{N}$ is expressible as the arithmetic average of two (not necessarily distinct) elements from $P$. We call such sets "Goldbach sets", and demonstrate the existence of periodic Goldbach sets with arbitrarily small positive density in the natural numbers.

# 1 Introduction

Goldbach's original conjecture, as expressed by him in a June 7, 1742 letter to Euler, states: "it seems that every number that is greater than 2 is the sum of three[1] primes" [16]. This conjecture was later re-expressed by Euler in an equivalent form; the latter "binary" version of the Goldbach conjecture asserts that all positive even integers $\geqslant 4$ can be expressed as the sum of two primes.

From the measure-theoretic perspective, in 1938, Estermann [14] proved that almost all even numbers are the sums of two primes. Then, in 1939, Schnirelman [24] proved that every even number can be written as the sum of not more than 300,000 primes. Pogorzelski [22] claimed in 1977 to have proven the Goldbach conjecture, but his proof is not generally accepted by the mathematical community [25, pp. 30-31]. The following table [32] summarizes bounds $n$ such that the Goldbach conjecture has been shown to be true for numbers $< n$.

[1]We remark that Goldbach considered the number 1 to be a prime, a convention that has since fallen out of favor.

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