Sparse Periodic Goldbach Sets

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Abstract

In this paper, we consider sets of natural numbers $P \subseteq \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ which satisfy the property that every x in $\mathbb{N}$ is expressible as the arithmetic average of two (not necessarily distinct) elements from P. We call such sets "Goldbach sets", and demonstrate the existence of periodic Goldbach sets with arbitrarily small positive density in the natural numbers.

1 Introduction

Goldbach's original conjecture, as expressed by him in a June 7, 1742 letter to Euler, states: "it seems that every number that is greater than 2 is the sum of three[1] primes" [16]. This conjecture was later re-expressed by Euler in an equivalent form; the latter "binary" version of the Goldbach conjecture asserts that all positive even integers $\geqslant 4$ can be expressed as the sum of two primes.

From the measure-theoretic perspective, in 1938, Estermann [14] proved that almost all even numbers are the sums of two primes. Then, in 1939, Schnirelman [24] proved that every even number can be written as the sum of not more than 300,000 primes. Pogorzelski [22] claimed in 1977 to have proven the Goldbach conjecture, but his proof is not generally accepted by the mathematical community [25, pp. 30-31]. The following table [32] summarizes bounds n such that the Goldbach conjecture has been shown to be true for numbers $< n$.

[1]We remark that Goldbach considered the number 1 to be a prime, a convention that has since fallen out of favor.

1×10^4	Desboves [7]	1885
1×10^5	Pipping [21]	1938
1×10^8	Stein and Stein [27, 28]	1965
2×10^{10}	Granville et al. [17]	1989
4×10^{11}	Sinisalo [26]	1993
1×10^{14}	Deshouillers et al. [8]	1998
4×10^{14}	Richstein [23]	1999
2×10^{16}	Oliveira e Silva [12]	2003
6×10^{16}	Oliveira e Silva [13]	2003
2×10^{17}	Oliveira e Silva [11]	2005
3×10^{17}	Oliveira e Silva [10]	2005

The related conjecture that all odd numbers $\geqslant 9$ are the sum of three odd primes is called the "weak" Goldbach conjecture. Towards the weak conjecture, Vinogradov [29, 30, 31] proved, in 1937, that every sufficiently large odd number is the sum of three primes. Vinogradov's original "sufficiently large" $n \geqslant 3^{3^{15}} \approx 3.25 \times 10^{6846168}$ was reduced in 1989 to $e^{e^{11.503}} \approx 3.33 \times 10^{43000}$ by Chen and Wang [5]. Chen [3, 4] also showed that all sufficiently large even numbers are the sum of a prime and the product of at most two primes [18, 6].

As G. H. Hardy succinctly states in the beginning of his book [19, p. 19], "It is comparatively easy to make clever guesses; indeed there are theorems, like 'Goldbach's Theorem,' which have never been proved and which any fool could have guessed." Indeed, Faber and Faber [15] offered a $1,000,000 prize to anyone who proved Goldbach's conjecture between March 20, 2000 and March 20, 2002; the prize went unclaimed and the conjecture remains open.

Needless to say, *this paper does not seek to address the conjecture per se*. Rather, in this paper, we invert the classical problem: instead of considering the set of primes and attempting to determine the extent to which they satisfy the Goldbach conjecture, we study subsets of the natural numbers for which the Goldbach conjecture holds.

We follow the conventions of mathematical logic, taking $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and, additionally take 0 to be prime. While this decision might impact other statements in number theory, it is certainly a purely cosmetic change with regards to the binary Goldbach conjecture. Specifically, if an even number $2n \in \mathbb{N}$ is expressible as the sum of two primes p and q, where p or q is 0, then it follows that $n = 0$ or $n = 1$, since 0 and 2 are the only even primes. In short, taking $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and 0 as prime does not alter the Goldbach conjecture in any significant way. Our somewhat nonstandard conventions do, however, permit us to restate the Goldbach conjecture in the following succinct manner:

Every natural number n is expressible as the average of two primes.

This succinct statement now extends cleanly, making the inverted classical problem readily accessible: we seek subsets $P \subseteq \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ which satisfy the condition:

$$\forall x \in \mathbb{N}, \quad \exists p_x, q_x \in P, \quad x = \frac{p_x + q_x}{2}.$$

We begin by considering the problem in a finite setting and then extend our definitions in a natural way to their infinitary analogues.

2 Finite Initial Segments

Definition 2.1. *For $n \geqslant 1$ we define $I_n = \{0, \ldots, n-1\}$ as the finite initial segment of the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Consider a set $P \subseteq \mathbb{N}$. The density of P (in I_n) is defined as*

$$\delta_n(P) = \frac{|P \cap I_n|}{|I_n|}.$$

P is said to be **Goldbach in** I_n *if*

$$\forall\, x \in I_n, \exists p_x, q_x \in P \text{ where } 0 \leqslant p_x \leqslant x \leqslant q_x \text{ and } x = \frac{p_x + q_x}{2}.$$

P is said to be **strongly Goldbach in** I_n *if $P \cap I_n$ is also Goldbach in I_n.*

Remark 2.2. *If P is Goldbach in I_n then 0 is necessarily an element of P; additionally, if P is strongly Goldbach in I_n then $n-1$ is necessarily an element of P.*

In the classical (infinitary) setting, where P is the set of prime numbers, the two primes p_x and q_x are sometimes called a Goldbach partition of $2x$ [9]. To avoid confusion, here we shall refer to (p_x, q_x) as a **Goldbach pair** for x.

Definition 2.3. *For each natural number $n \geqslant 1$, define*

$$\begin{aligned} \Delta(n) &= \min\{\delta_n(P) \mid P \text{ is Goldbach in } I_n, \text{ and } P \subseteq \mathbb{N}\} \\ \Delta^*(n) &= \min\{\delta_n(P) \mid P \text{ is strongly Goldbach in } I_n, \text{ and } P \subseteq \mathbb{N}\}, \end{aligned}$$

to be the **Goldbach density** *and* **strong Goldbach density** *at n, respectively.*

To bound the separation between $\Delta^*(n)$ and $\Delta(n)$, it suffices to give a constructive procedure which converts any Goldbach set into a strongly Goldbach set. The next definition is required to describe one such constructive procedure.

Definition 2.4. *For any subset $X \subseteq \mathbb{N}$, define the* **reflection of** X **in** I_n *as*

$$ {}^n\bar{X} = \{(n-1) - x \mid x \in X\}. $$

We define the **symmetrization of** X **in** I_n *as*

$$ {}^nX^* = (X \cup {}^n\bar{X}) \cap I_n. $$

Note that ${}^nX^*$ is always a subset of I_n, regardless of whether X is or not. We shall denote ${}^n\bar{X}$ (resp. ${}^nX^*$) simply as $\bar{X}$ (resp. X^*) whenever n is clear from the context.

Lemma 2.5. $\Delta^*(n) \leqslant 2\Delta(n)$ *for all integers* $n \geqslant 1$.

The next theorem shows that if the Goldbach conjecture of number theory is true, then Goldbach density $\Delta(n)$ tends to zero as n tends to infinity, and hence motivates the property named in Definition 2.1.

Theorem 2.6. *If the Goldbach Conjecture is true, then there exists some constant c such that $\Delta(n) \leqslant c/\ln(n)$ for all n sufficiently large.*

Note that by applying Lemma 2.5 to Theorem 2.6,

$$ \Delta^*(n) \leqslant \delta_n(P^*) \leqslant 2\delta_n(P) \leqslant 2c/\ln(n), $$

where $P = \{0\} \cup \{p \in \mathbb{N} \mid p \text{ is prime}\}$. Thus if the Goldbach conjecture is true then strong Goldbach density also tends to zero as n tends to infinity. Since we will be concerned primarily with asymptotic behavior, in what follows we present all our results in terms of properties of either $\Delta(n)$ or $\Delta^*(n)$—whichever is more natural in the particular context. We remark that Lemma 2.5 can be leveraged to provide analogous assertions regarding the other quantity.

The next theorem provides a lower bound on strong Goldbach density.

Theorem 2.7. $\Delta^*(n) \geqslant 1/\sqrt{n}$ *for all integers* $n \geqslant 1$.

Applying Lemma 2.5 to Theorem 2.7, we conclude

Corollary 2.8. $\Delta(n) \geqslant \frac{1}{2\sqrt{n}}$ *for all integers* $n \geqslant 1$.

Given a set that is strongly Goldbach in I_n, it is natural to attempt to construct sets which are strongly Goldbach in I_{kn} for $k > 1$. The next definition is required to describe one such construction.

Definition 2.9. *Given a set $X \subseteq \mathbb{Z}$ and an integer $d \in \mathbb{Z}$, define the* **translation of** X **by** d *as:*

$$ (d + X) = \{d + x \mid x \in X\}. $$

The next proposition shows the relationship of $\Delta^*(kn)$ and $\Delta^*(n)$ for integers $n, k \geqslant 1$.

Proposition 2.10. *$\Delta^*(kn) \leqslant \Delta^*(n)$ for all integers $n, k \geqslant 1$.*

The next lemma is a natural complement to Lemma 2.5, providing another relationship between Goldbach density and strong Goldbach density.

Lemma 2.11. *$\Delta(n) \leqslant \frac{1}{2}\Delta^*(\lfloor \frac{n}{2} \rfloor)$ for all integers $n \geqslant 1$.*

The next proposition shows the relationship of $\Delta(2n)$ and $\Delta(n)$ for integers $n \geqslant 1$.

Proposition 2.12. *$\Delta(2n) \leqslant \Delta(n)$, for every integer $n \geqslant 1$.*

Note that if P be strongly Goldbach in I_{kn} then P is also Goldbach in I_{kn}. If we consider a set P satisfying $\delta_{kn}(P) = \Delta^*(kn)$, then

$$\Delta(kn) \leqslant \delta_{kn}(P) = \Delta^*(kn) \leqslant \Delta^*(n) \leqslant 2\Delta(n)$$

where the last two inequalities are implied by Proposition 2.10 and Lemma 2.5, respectively. We have shown:

Corollary 2.13. *$\Delta(kn) \leqslant 2\Delta(n)$ for all integers $n, k \geqslant 1$.*

We remark that while Propositions 2.10 and 2.12 provide a sense of the behavior of $\Delta^*(n)$ and $\Delta(n)$ respectively, they do not necessarily imply that the functions are non-increasing. We conjecture, however, that there exists an integer N such that for all $n > N$ sufficiently large, $\Delta^*(n+1) \leqslant \Delta^*(n)$ and $\Delta(n+1) \leqslant \Delta(n)$.

2.1 Infinitary Analogues

We generalize the previously stated Definition 2.1 to sets which are not segments of natural numbers as follows:

Definition 2.14. *Let $P, U \subseteq \mathbb{Z}$ be sets of integers. The upper density of P in U as*

$$\delta_U(P) = \limsup_{t \to \infty} \frac{|U \cap P \cap [-t, t]|}{|U \cap [-t, t]|}.$$

We say P is **Goldbach in** *U if*

$$\forall x \in U, \exists p_x, q_x \in P \text{ where } p_x \leqslant x \leqslant q_x \text{ and } x = \frac{p_x + q_x}{2}.$$

P is said to be **strongly Goldbach in** *U if $P \cap U$ is also Goldbach in U.*

The above definition connects with those in the previous section in a natural way: it is easy to verify that Definition 2.14 coincides with Definition 2.1 when $U = I_n$ and $P \subseteq \mathbb{N}$.

Note that if $P \subseteq \mathbb{N}$ is Goldbach in I_n for every $n \in \mathbb{N}\backslash\{0\}$, then P is strongly Goldbach in $\mathbb{N}$. Considering the converse, if P is Goldbach in $\mathbb{N}$, then P is Goldbach in I_n for every $n \in \mathbb{N}\backslash\{0\}$. Note, however, that in this latter scenario, P need not be *strongly* Goldbach in I_n for all $n \in \mathbb{N}\backslash\{0\}$: for example, the set $P = \mathbb{N}\backslash\{2\}$ is Goldbach in $\mathbb{N}$ and Goldbach in I_n for all $n \in \mathbb{N}\backslash\{0\}$, but P is not *strongly* Goldbach in I_3.

Armed with Definition 2.14, we are ready to begin considering cases where both P and U are infinite. We will begin this by considering periodic infinite sets.

Definition 2.15. *A set $S \subseteq \mathbb{Z}$ is* **periodic** *if for some integer $n \geqslant 1$ we have:*

$$x \in S \Leftrightarrow x + n \in S.$$

The minimal such $n \geqslant 1$ is called the **period** *of S.*

Periodic sets that are Goldbach in $U \subseteq \mathbb{Z}$ have the property that it is always possible to find Goldbach pairs that are “close together”. The next lemma makes this assertion precise.

Lemma 2.16. *Given sets $P, U \subseteq \mathbb{Z}$ where P is Goldbach in U and* periodic *with period n. Then*

$$\forall x \in U, \exists p_x, q_x \in P \text{ where } p_x \leqslant x \leqslant q_x \text{ and } x = \frac{p_x + q_x}{2} \text{ and } 0 \leqslant x - p_x = q_x - x < n.$$

The next definition gives us a compact notation for representing periodic sets of $\mathbb{Z}$ (and their induced subsets in $\mathbb{N}$).

Definition 2.17. *Let $A \subseteq I_n$ and define*

$$\begin{aligned} \mathbb{Z}_A^n &= \bigcup_{a \in A} \{a + kn \mid k \in \mathbb{Z}\} \\ \mathbb{N}_A^n &= \bigcup_{a \in A} \{a + kn \mid k \in \mathbb{N}\} = \mathbb{Z}_A^n \cap \mathbb{N}. \end{aligned}$$

We seek to construct sets which are strongly Goldbach in $\mathbb{N}$, but frequently will find it easier to construct sets that are Goldbach in $\mathbb{Z}$. The next lemma indicates precisely when a set that is Goldbach in $\mathbb{Z}$ gives rise to an induced subset of $\mathbb{N}$ that is strongly Goldbach in $\mathbb{N}$.

Lemma 2.18. *Let $P \subseteq \mathbb{Z}$ be Goldbach in $\mathbb{Z}$ and* periodic *with period n. Then the following statements are equivalent:*

I. *P is strongly Goldbach in $\mathbb{N}$.*

II. For all $x \in I_n$, there exists a Goldbach pair p_x, q_x for which $p_x \geqslant 0$.

Note that there are many subsets of $\mathbb{Z}$ which are Goldbach in $\mathbb{Z}$ but not strongly Goldbach in $\mathbb{N}$. The simplest such example is the set $\mathbb{Z}\backslash\{0\}$. However, the previous lemma provides some insight into how a periodic set that is Goldbach in $\mathbb{Z}$ can be augmented, so that the resulting set is strongly Goldbach in $\mathbb{N}$. The next corollary demonstrates the augmentation:

Corollary 2.19. *For any set $P \subseteq \mathbb{Z}$ which is Goldbach in $\mathbb{Z}$ and* periodic *with period n, the set $\hat{P} = P \cup I_n$ is strongly Goldbach in $\mathbb{N}$, and moreover $\delta_{\mathbb{N}}(\hat{P}) = \delta_{\mathbb{Z}}(P)$.*

Lemma 2.18 and its corollary above indicate that in searching for sets that are strongly Goldbach in $\mathbb{N}$, it suffices to find periodic Goldbach subsets of $\mathbb{Z}$.

Proposition 2.20. *Take*

$$\begin{aligned} n_0 = 2, &\quad A_0 = \{0\}. \\ n_1 = 6, &\quad A_1 = \{0, 2\}. \\ n_2 = 18, &\quad A_2 = \{0, 2, 6, 8, 14\}. \end{aligned}$$

Then $S_i = \mathbb{N}_{A_i}^{n_i}$ is strongly Goldbach in $\mathbb{N}$, for $i = 0, 1, 2$.

Clearly $\delta_{\mathbb{N}}(S_i) = \frac{|A_i|}{n_i}$, and thus we determine the densities of S_0, S_1 and S_2:

$$\begin{aligned} \delta_{\mathbb{N}}(S_0) &= 1/2 = 0.5 \\ \delta_{\mathbb{N}}(S_1) &= 2/6 = 0.\bar{3} \\ \delta_{\mathbb{N}}(S_2) &= 5/18 = 0.2\bar{7} \end{aligned}$$

Note that

$$\begin{aligned} A_0 &= \{0\}, \\ A_1 &= A_0 \cup (n_0 + A_0) \cup (2n_0 + A_0) \setminus \{4\}, \\ A_2 &= A_1 \cup (n_1 + A_1) \cup (2n_1 + A_1) \setminus \{12\}. \end{aligned}$$

Thus, the construction of S_0, S_1 and S_2 in the previous proposition generalizes naturally to an inductive construction of S_i for each $i \in \mathbb{N}$, as the following lemma shows:

Lemma 2.21. *Take*

$$\begin{aligned} n_0 &= 2, \quad A_0 = \{0\}, \\ n_i &= 3n_{i-1}, \\ A_i &= A_{i-1} \cup (n_{i-1} + A_{i-1}) \cup (2n_{i-1} + A_{i-1}) \setminus \{2n_{i-1}\}. \end{aligned}$$

Then for each $i \in \mathbb{N}$, the set $S_i = \mathbb{N}_{A_i}^{n_i}$ is strongly Goldbach in $\mathbb{N}$.

The previous sequence of sets $A_i \subseteq I_{n_i}$ have the property that $|A_i| = \frac{3^i-1}{2} + 1$, and $n_i = 2 \cdot 3^i$. Since $\delta_{\mathbb{N}}(S_i) = \frac{|A_i|}{n_i}$ and

$$\lim_{i:0\to\infty} \frac{\frac{3^i-1}{2}+1}{2\cdot 3^i} = 1/4,$$

the previous lemma yields that there are sets that are strongly Goldbach and have density arbitrarily close to $1/4$. We have proved:

Theorem 2.22. *For each $\epsilon > 0$ there exists a deterministically constructible subset $P_\epsilon \subseteq \mathbb{N}$ which is strongly Goldbach in $\mathbb{N}$ and has density $\delta_{\mathbb{N}}(P_\epsilon) < 0.25 + \epsilon$.*

2.2 Randomized Constructions

We will now describe a randomized procedure to construct a sequence of periodic sets each of which is Goldbach in $\mathbb{Z}$ and whose densities tend, asymptotically, to zero.

Theorem 2.23. *There exists a sequence of sets $X_1, X_2, \ldots X_n, \ldots$ satisfying:*

A. For each $n \in \mathbb{N}$ $n \geqslant 1$, the set X_n is a subset of I_n and $\mathbb{Z}^n_{X_n}$ is Goldbach in $\mathbb{Z}$.

B. $\lim_{n\to\infty} E[\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n})] = 0$.

The proof of the theorem requires us to introduce some definitions and basic results.

Definition 2.24. *Given a set $C \subseteq I_n$, an element $v \in I_n$ is called an n-**violator** of C if $\forall\delta \in \{0, 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$*

$$(v + \delta \notin C) \quad \vee \quad (v - \delta \notin C).$$

The set of all violators of C is denoted

$$V_n(C) = \{v \in I_n \mid v \textit{ is a violator of } C\}.$$

Lemma 2.25. *Let $C \subseteq I_n$ be a subset of cardinality k, chosen uniformly at random from the set of all k-subsets of I_n. Then*

$$E[|V_n(C)|] = n\left(1 - \frac{k}{n}\right)^{2-(n \bmod 2)} \cdot \left(1 - \left(\frac{k}{n}\right)^2\right)^{\lceil n/2 \rceil - 1}.$$

Definition 2.26. *Given a set $C \subseteq I_n$, we define the **Goldbach closure** of C in I_n as*

$$G_n(C) = C \cup V_n(C).$$

Since C and $V_n(C)$ are disjoint sets, the following statement follows immediately from Lemma 2.25.

Corollary 2.27. *Let $C \subseteq I_n$ be a subset of cardinality k, chosen uniformly at random from the set of all k-subsets of I_n. Then*

$$E[|G_n(C)|] = k + n\left(1 - \frac{k}{n}\right)^{2-(n \bmod 2)} \cdot \left(1 - \left(\frac{k}{n}\right)^2\right)^{\lceil n/2 \rceil - 1}$$

Lemma 2.28. *For any $S \subseteq I_n$, the set $\mathbb{Z}^n_{G_n(S)}$ is Goldbach in $\mathbb{Z}$.*

The proof of **Theorem 2.23** (see pp. 8) proceeds by fixing $\epsilon \in (0, \frac{1}{2})$ and taking C_n be a random subset of I_n having cardinality $n^{\frac{1}{2}+\epsilon}$, chosen uniformly at random from the set of all subsets having cardinality $n^{\frac{1}{2}+\epsilon}$. Take $X_n = G_n(C_n)$ to be the Goldbach closure of C_n. By Lemma 2.28, we know that $\mathbb{Z}^n_{X_n}$ is Goldbach in $\mathbb{Z}$. By Corollary 2.27, the expected density $\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n})$ may be calculated, and taking the limit of both sides, as n tends to infinity, we see that $\lim_{n\to\infty} E[\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n})] = 0$.

We can now leverage the previous randomized construction to show the existence of arbitrarily sparse sets that are strongly Goldbach in $\mathbb{N}$.

Since $\lim_{n\to\infty} E[\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n})] = 0$, there exists an n_ϵ such that for all integers $n > n_\epsilon$, $E[\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n})] < \epsilon$. Thus there exists at least one random set X_n, for which $\delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n}) < \epsilon$. Now, applying Corollary 2.19, we get an augmented set $\hat{\mathbb{Z}}^n_{X_n} = \mathbb{Z}^n_{X_n} \cup I_n$ that is strongly Goldbach in $\mathbb{N}$ and for which $\delta_{\mathbb{N}}(\hat{\mathbb{Z}}^n_{X_n}) = \delta_{\mathbb{Z}}(\mathbb{Z}^n_{X_n}) < \epsilon$. Taking $A_\epsilon = \hat{\mathbb{Z}}^n_{X_n} \cap \mathbb{N}$, we have shown the existence of a set having the required properties:

Corollary 2.29. *Given any $\epsilon > 0$, there exists set $A_\epsilon \subseteq \mathbb{N}$ which is strongly Goldbach in $\mathbb{N}$ with $\delta_{\mathbb{N}}(A_\epsilon) < \epsilon$.*

3 Open Questions and Future Directions

In the finite case, we would like to give better upper and lower bounds for $\Delta(n)$ and $\Delta^*(n)$, and to understand the behavior of the function $\Delta^*(n) - \Delta(n)$. We seek to confirm or refute the conjecture that $\Delta^*(n)$ and $\Delta(n)$ are eventually non-increasing functions (this remains open in spite of Propositions 2.10 and 2.12). In the infinitary case, we wish to devise a deterministic algorithm which produces density $\epsilon > 0$ sets which are strongly Goldbach in $\mathbb{N}$—perhaps derandomization techniques [20] can be adapted to the randomized construction we have presented. While Corollary 2.29 demonstrates the existence of sets which are strongly Goldbach in $\mathbb{N}$ and have arbitrarily low (constant) density $\epsilon > 0$, it remains open *whether or not one* *can show the existence of a density-zero set that is strongly Goldbach in* $\mathbb{N}$ *without presuming the Goldbach Conjecture*; and if so, whether an explicit construction can be given. We are also considering several related prime number conjectures using similar techniques, including the "weak conjecture", and Levy's conjecture which asserts every odd number $\geqslant 7$ can be expressed as the sum of a prime plus twice a prime.

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