Graphic Arithmetic II:
$+$ Irreducibility, Canonical Decompositions and Cancellation Laws

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Abstract

In this paper, we continue with our investigation of natural arithmetic properties on the set of all flow graphs, that is, the set of all finite directed connected multigraphs having a pair of distinguished vertices(see [1]). We introduce a graph operation called *-Deletion and study some of its properties. This leads to a notion of splitting vertex and w-Splitting of a flow graph. We study some properties of splitting vertices in a flow graph and show that a flow graph is $+$ reducibile if and only if it has a splitting vertex. We define a concept of splitting vertex ranking and use it to develop a theory of canonical + decompositions.

1 Introduction

In [1], we generalized classical arithmetic defined over the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, to the set F consisting of all *flow graphs*: finite directed connected multigraphs [1] in which a pair of distinguished vertices is designated as the *source* and *target* vertex. The proposed model exhibits the property that the natural numbers appear as a submodel, with the directed path of length n playing the role of the standard integer n. We discussed basic features including associativity, distributivity, and various identities relating the order relation to addition and multiplication.

In this section, we review some definitions and results from [1].

Definition 1.1 (Flow graph)**.** *We define a* flow graph A *to be a triple* (G_A, s_A, t_A)*, where* G_A *is a finite*[2] *directed connected multigraph and* $s_A, t_A \in V[G_A]$ *are called the* source *and the*

[1] By multigraph we mean graphs in which parallel and loop edges are permitted.
[2] In this paper, we focus on finite flow graphs, although many of our results continue to hold in formulation which considers infinite flow graphs as well.

Now if $|\chi(A)| = 0$ the theorem is trivially true. Suppose $|\chi(A)| = m > 0$. Then $\phi(u_{m-1}) = v_{m-1}$ and so $\phi(A^s_{u_{m-1}}) = B^s_{v_{m-1}}$. Since $|\chi(A^s_{u_{m-1}})| = m-1$ the inductive hypothesis applies and ϕ maps $A_0+A_1+\cdots+A_{2(m-1)}$ componentwise onto $B_0+B_1+\cdots+B_{2(m-1)}$. Since A_{m-1}, B_{m-1} are infinitesimal or trivial, and A_m, B_m are non-infinitesimal or trivial (but not both can be trivial, it follows that ϕ maps A_{m-1} onto B_{m-1} and A_m onto B_m. $\square$

3 Conclusions and Future Work

As we have seen, some theorems that are true in $\mathcal{N}$ continue to hold in $\mathcal{F}$, while others fail. Our future research program will proceed to explore further properties of flow graphs.

Some questions we are presently considering are listed below.

1. Characterize -commuting pairs, i.e. under what conditions on flow graphs A and B does $AB = BA$?

2. *Graph Prime Factorization Conjecture.* Every flow graph is uniquely expressible (up to some well-defined reordering) as the product of prime flow graphs.

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