Contemporary Mathematics

Positively generated subgroups of free groups and the Hanna Neumann conjecture

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Abstract. The *Hanna Neumann conjecture* states that if F is a free group, then for all subgroups $H, K \leqslant F$,

$$\text{rank}(H \cap K) - 1 \leqslant [\text{rank}(H) - 1]\,[\text{rank}(K) - 1]$$

Previous research on the conjecture has proceeded largely by "translating" the group-theoretic properties of subgroups of free groups into the graph-theoretic properties of their corresponding foldings or finite automata. This paper attempts to elaborate the reverse.

In particular, in this paper we give group-theoretic interpretation of the well-known graph-theoretic property of strong connectivity. Specifically, we show that strong connectivity of a subgroup's folding corresponds exactly to the property that the subgroup is *positively generated* (i.e. is generated by a set of words containing no negative exponents). To accomplish this, we present the notion of a strong directed trail decomposition of a directed graph; this decomposition provides a useful computational tool, and facilitates inductive arguments about the properties of positively generated subgroups of free groups.

As an example application of directed trail decomposition techniques, we prove that if a subgroup $H \leqslant F$ is positively generated, or if its associated folding Γ_H has no source or sink vertices, then for all subgroups $K \leqslant F$, the Hanna Neumann conjecture holds for the pair (H, K). We also show that if a subgroup of a free group is positively generated, then it has a positive basis. Finally, we describe an algorithm which decides whether an arbitrary finitely generated subgroup of a free group is positively generated, and if so, outputs a positive basis for the subgroup.

1. Introduction

Improving Howson's earlier bound [**5**] on the rank of intersections of finitely generated (f.g.) subgroups of free groups, H. Neumann proved in [**9**] that any $H, K \leqslant_{\text{f.g.}} F$ must satisfy

$$rank(H \cap K) - 1 \leqslant 2[rank(H) - 1][rank(K) - 1]$$

The stronger assertion obtained by omitting the factor of 2 has come to be known as the Hanna Neumann conjecture. In [**1**], Burns improved H. Neumann's bound

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