

# The Structure of Automorphic Conjugacy in the Free Group of Rank Two

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**ABSTRACT.** In the study of the automorphism group of a free group  $F = F(X)$  on a set  $X$ , J. H. C. Whitehead introduced a graph whose vertices are elements of  $F$ , where two vertices are connected if and only if the corresponding elements of  $F$  are related by one of a specially chosen set of generators of  $\text{Aut}(F)$ . Here we give a precise structural description of Whitehead's graph for the case where  $F = F_2$  is the free group of rank two. This description allows us to quantify relationships between the natural length function  $||$  of  $F_2$ , and the action of  $\text{Aut}(F_2)$  on  $F_2$ . As an application, we show that Whitehead's algorithm for testing automorphic conjugacy in  $F_2$  runs in time that is at most quadratic in the length of the elements.

## 1. The Automorphism Graph of $F_2$

To start, let  $X$  denote a **base set** of elements  $\{a, b, \dots\}$ , and let  $X^{-1} = \{A, B, \dots\}$  be the set consisting of the corresponding formal inverses of elements from  $X$ . We call the elements of  $X \cup X^{-1}$  **letters**, and denote the **free group** on the set  $X$  as  $F(X)$ .

The elements of the free group can be taken as the set of freely reduced words of finite length over the alphabet  $X \cup X^{-1}$ , where by **freely reduced** we mean words which contain no subword of the form  $xx^{-1}$  or  $xx^{-1}$  for any  $x \in X$ . Multiplication of elements of  $F(X)$  is simply concatenation of words, followed by **free reduction**, which is to say repeated cancellation of all subwords of the form  $xx^{-1}$  or  $xx^{-1}$  for  $x \in X$ . The unique empty word of length 0 plays the role of the identity element. It is well-known that given two sets  $X$  and  $Y$  the free group  $F(X) \cong F(Y)$  if and only if  $|X| = |Y|$ . This justifies denoting such a free group as  $F_{|X|}$ , since upto isomorphism the group depends only on the cardinality of the base set. The group  $F_{|X|}$  is called *the* free group of **rank**  $|X|$ . This work considers  $F_2$ , the free group of rank two on the set  $X = \{a, b\}$ .

Recall that for any group  $G$ , the set of automorphisms of  $G$  again forms a group, denoted  $\text{Aut}(G)$ , in which composition of automorphisms plays the role of multiplication. Given a group  $G$ , two elements  $g, h \in G$  are said to be **automorphic conjugates** if there exists an automorphism  $\phi \in \text{Aut}(G)$  for which  $\phi(g) = h$ .

This work concerns the properties of  $\text{Aut}(F_2)$ , the group of automorphisms of  $F_2$ . In general, a structural description of the orbits of  $F_n$  under the action of

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