Designing scalable networks with multiple personalities

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Abstract

In this paper we describe physical network topologies which readily support the establishment of overlays that greatly reduce and/or greatly increase the distances between nodes. Reducing pairwise distances (i.e. *compression*) implies that the overlay exhibits significantly lower mean latencies than the ambient physical network, implying that the overlay can be used to implement a "high-performance mode" that could be suddenly required in critical mission circumstances. Increasing pairwise distances on the other hand (i.e. *expansion*) implies that the overlay exhibits significantly higher mean latencies than the ambient physical network, a feature that is useful in settings where a malicious worm or virus has infected the network and where such an overlay would slow down the rate of infection propagation, allowing greater time for antidote generation.

1 Introduction

In the last decade, we have witnessed a surge in the application of peer-to-peer overlay networks. Each logical link in the overlay network consists of a multi-hop connection in the physical network. Data can travel along a logical link in the overlay without undergoing expensive routing or manipulation at the nodes. Overlay networks enable network designers to consider alternate network topologies whose properties deviate from those of the physical network. While modifying the physical network topology is time consuming and expensive, altering the overlay topology is, by comparison, fast and cheap.

Overlays have a profound impact on the objectives of physical network design. Historically network designers have sought physical networks that exhibit good performance metrics, yet struggled with the reality that these metrics are often in opposition with one another. We believe that the widespread acceptance of overlay technologies makes it advantageous to seek physical networks which support a *wide range of overlays*, each with its own merits vis-a-vis performance metrics.

Specifically, in this paper we focus on physical networks which support overlays that either greatly reduce or increase the distances between nodes. Reducing pairwise distances (i.e. *compression*) lowers mean latencies, meaning that such networks can be made to support a "high-performance mode" that may be required in mission-critical circumstances. On the other hand, increasing pairwise distances (i.e. *expansion*) may be useful when the network is discovered to have been infected by a malicious agent (e.g. a worm or virus), since this the use of such an overlay would slow down the rate at which the infection saturates the network's hosts, allowing greater time for antidote generation.

2 Mathematical Model

The **physical network** is represented by a graph G with vertices V and unweighted undirected edges $E \subset V \times V$.

A link in the overlay network is represented by a **walk** $p = (v_0, v_1, \cdots, v_l)$ in G, where (v_i, v_{i+1}) is in E, for each $i = 0, \cdots, l-1$. The **length** of p is denoted $|p| = l$, and the **boundary** of p is denoted $\partial p = \{v_0, v_l\}$. For each walk p, we define a corresponding **characteristic function** $\chi_p : E \to \mathbb{N}$ whose value at an edge e is defined to be the number of times e is traversed by p. For each $l \in \mathbb{Z}^+$, let $\mathcal{P}^l(G)$ to be the set of all walks in G whose length does not exceed l, and then put $\mathcal{P}^*(G) = \bigcup_{l \in \mathbb{Z}^+} \mathcal{P}^l(G)$.

An **overlay network** is a set of walks $Q \subset \mathcal{P}^*(G)$, which together implement a (multi) graph $Q^\circ = (V, \{\partial p \mid p \in Q\})$.

Definition 2.1. *Let G be a finite connected undirected graph, and $Q \subset \mathcal{P}^*(G)$ be a set of walks in G. The* **pointwise expansion** *and* **compression** *of Q on G are defined as*

$$\begin{aligned} \overline{\kappa}_0(G, Q) &= \max_{\substack{u,v \in V \\ u \neq v}} \frac{d_{Q^\circ}(u, v)}{d_G(u, v)}, \\ \underline{\kappa}_0(G, Q) &= \min_{\substack{u,v \in V \\ u \neq v}} \frac{d_{Q^\circ}(u, v)}{d_G(u, v)} = \max_{\substack{u,v \in V \\ u \neq v}} \frac{d_G(u, v)}{d_{Q^\circ}(u, v)}. \end{aligned}$$

For each $e \in E$, the **congestion of** Q **on** e *is the number of times e appears in Q*

$$\Phi_Q(e) = \sum_{p \in Q} \chi_p(e).$$

and the **congestion of** Q **on** G *is defined to be the largest congestion over all edges of* G

$$\tau_G(Q) \stackrel{def}{=} \max_{e \in E} \Phi_Q(e).$$

A set $Q \subset \mathcal{P}^*(G)$ *is called* **sparse** *if* $\tau_G(Q) = 1$. *Let* $S(G) \subset 2^{\mathcal{P}^*(G)}$ *be the collection of all sparse sets of paths in* G.

The next theorem shows that a set of sparse paths which exhibits nontrivial pointwise compression ($\underline{\kappa}_0(G, Q) < 1$) necessarily exhibits nontrivial pointwise expansion ($\overline{\kappa}_0(G, Q) > 1$).

Theorem 2.2 (No Free Lunch)**.** *Let* $G = (V, E)$ *be a non-trivial* ($|V| > 1$) *finite simple connected undirected graph, and let* Q *be a sparse set of walks for which* $\underline{\kappa}_0(G, Q) < 1$. *Then* $\overline{\kappa}_0(G, Q) > 1$.

Proof. Suppose, towards contradiction, that $\overline{\kappa}_0(G, Q) \leqslant 1$. Then for u, v in V,

$$\frac{d_{Q^\circ}(u, v)}{d_G(u, v)} \leqslant \max_{\substack{u,v \in V \\ u \neq v}} \frac{d_{Q^\circ}(u, v)}{d_G(u, v)} = \overline{\kappa}_0(G, Q) \leqslant 1.$$

It follows that $d_{Q^\circ}(u, v) \leqslant d_{G^\circ}(u, v)$ for each u, v in V. Since G is non-trivial and connected, consider any u and v distinct adjacent vertices in G. By (1) $d_{Q^\circ}(u, v) \leqslant 1$. Now the simplicity of G mandates that every path in Q must have distinct endpoints, and hence $d_{Q^\circ}(u, v) = 1$. It follows every edge in G appears as a length one path in Q, and hence $Q \subseteq E$. Now since Q is sparse it must be that $|Q| = |E|$, and so we conclude that $Q = E$. This, however, implies that $\underline{\kappa}_0(G) = 1$, a contradiction. $\square$

In contrast with pointwise compression which is defined by considering distances between corresponding vertices in G and Q°, the next definition considers the aggregate property of diameter.

Definition 2.3. *Let* G *be a finite simple undirected connected graph, and consider* $Q \subset \mathcal{P}^*(G)$ *a set of walks in* G. *Then the* **diametric compression** *of* Q *on* G *is defined to be*

$$\underline{\kappa}_1(G, Q) \quad = \quad \frac{Diam(Q^\circ)}{Diam(G)}.$$

Building on this, we define

$$\begin{aligned} \overline{\kappa}_1(G) &= \max_{Q \in S(G)} \underline{\kappa}_1(G, Q), \\ \underline{\kappa}_1(G) &= \min_{Q \in S(G)} \underline{\kappa}_1(G, Q). \end{aligned}$$

We now extend our definitions of pointwise/diametric compression and expansion to sequences of graphs.

Definition 2.4. *Let $\mathcal{G} = (G_j \mid j \in \mathbb{N})$ be a sequence of graphs. Define*

$$\begin{aligned} \overline{\kappa}_i(\mathcal{G}) &= \limsup_{j\in\mathbb{N}} \underline{\kappa}_i(G_j), \\ \underline{\kappa}_i(\mathcal{G}) &= \liminf_{j\in\mathbb{N}} \underline{\kappa}_i(G_j) \end{aligned}$$

for $i = 0, 1$. The sequence of graphs $\mathcal{G}$ is designated:

pointwise ultraexpandible	*if* $\overline{\kappa}_0(\mathcal{G}) = \infty$,
pointwise ultracompressible	*if* $\underline{\kappa}_0(\mathcal{G}) = 0$,
diametrically ultraexpandible	*if* $\overline{\kappa}_1(\mathcal{G}) = \infty$,
diametrically ultracompressible	*if* $\underline{\kappa}_1(\mathcal{G}) = 0$.

The next four theorems show that it is possible to construct sequences of graphs that are ultracompressible and ultracompressible in both the pointwise and diametric sense.

Theorem 2.5. *There exists a pointwise ultracompressible sequence of graphs.*

Proof. For any $m \in N$, we construct bridge graph B_m as shown in Figure 1. Our overlay network consists of the sparse set of paths Q_m shown in Figure 2. Consider the sequence of graphs $\mathcal{G} = (B_{2^j} \mid j \in \mathbb{N})$.

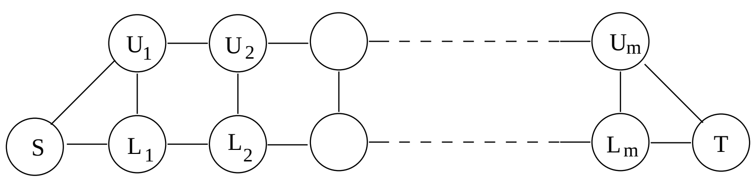

Figure 1: The physical network B_m

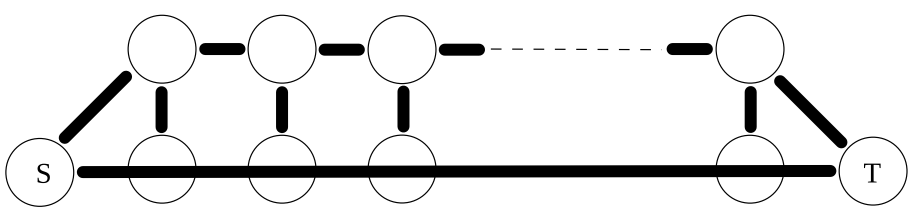

Figure 2: The overlay network Q_m on B_m

To see that $\mathcal{G}$ is pointwise ultracompressible, note that $d_{Q^\circ_{2^j}}(S,T) = 1$ while $d_{B_{2^j}}(S,T) = 2^j + 1$, and hence

$$\underline{\kappa}_0(\mathcal{G}) = \liminf_{j\in\mathbb{N}} \underline{\kappa}_0(B_{2^j}) \leqslant \liminf_{j\in\mathbb{N}} \underline{\kappa}_0(B_{2^j}, Q_{2^j}) \leqslant \liminf_{j\in\mathbb{N}} \frac{1}{2^j+1} = 0.$$

□

Theorem 2.6. *There is a diametrically ultracompressible sequence of graphs.*

Proof. Consider the chain of bridges $C_{n,m}$ as seen in Figure 3.

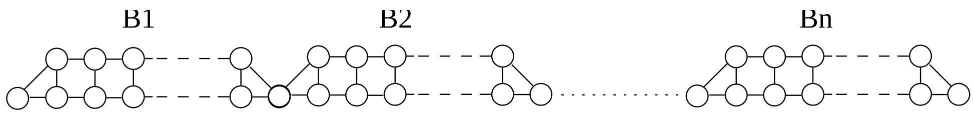

Figure 3: The Chain of Bridges Graph, $C_{n,m}$

Then $Diam(C_{n,m}) = n(m+1)$. We replicate the overlay network shown in Figure 2 n times. Then for $m > 4$, $n > 1$, $Diam(C_{n,m})^{\circ} = [2(1 + \lfloor \frac{m}{2} + 1 \rfloor) + (n-2)]$. Thus $\underline{\kappa}_1(C_{n,m}) = \frac{2(1+\lfloor \frac{m}{2}+1\rfloor)+(n-2)}{n(m+1)}$. It follows that the sequence of graphs $\mathcal{G} = (C_{j,j} \mid j \in \mathbb{N})$ is a diametrically ultracompressible family of graphs. □

Theorem 2.7. *There is a pointwise ultraexpandible sequence of graphs.*

Proof. Consider the sequence of graphs $\mathcal{G} = (K_j \mid j \in \mathbb{N})$ in which the jth element is the complete graph on j vertices. Take Q_j be a set of edges forming a Hamiltonian path in K_j, and let S_j and T_j be the start and end vertices of this path.

Note that $d_{Q_j^{\circ}}(S_j, T_j) = j - 1$ while $d_{K_j}(S_j, T_j) = 1$, and hence

$$\underline{\kappa}_0(\mathcal{G}) = \liminf_{j \in \mathbb{N}} \underline{\kappa}_0(K_j) \leqslant \liminf_{j \in \mathbb{N}} \underline{\kappa}_0(K_j, Q_j) \leqslant \liminf_{j \in \mathbb{N}} \frac{j-1}{1} = \infty.$$

□

Since in the proof of Theorem 2.8, the vertices S_j and T_j are the unique maximally separated pair in Q_j°, it follows by the same proof that:

Theorem 2.8. *There is a diametrically ultraexpandible sequence of graphs.*

Definition 2.9. *A sequence of graphs that is both ultracompressible and ultraexpandible is said to exhibit* multiple personalities *with respect to diameter.*

To provide a succinct description of the construction of such a graph, we require the following definition:

Definition 2.10. *Let $G = (V, E)$ be a graph and take $n \geqslant 0$ an integer. Take*

$$G * K_n = (V \cup (V \times \{1, 2, \ldots, n\}), E \cup E^*)$$

where $(u, (u, i))$ and $((u, i), (u, j))$ are edges in E^* for every $u \in V$ and $i, j \in \{1, 2, \ldots, n\}$.

Finally we present a sequence of graphs with multiple personalities.

Theorem 2.11. *The sequence of graphs* $\mathcal{G} = (C_{j,j} * K_j \mid j \in \mathbb{N})$ *is both ultraexpandible and ultracompressible.*

3 Conclusion

Ultraexpandible sequences of graphs provide a recipe for building networks that support extreme expansion of pairwise distances between their constituent nodes using overlays. Such networks possess overlays that exhibit significantly higher mean latencies than the ambient physical network, a feature that is useful in settings where a malicious worm or virus has infected the network, where such an overlay would slow down the rate of infection propagation and allow greater time for antidote generation. Ultracompressible sequences of graphs, on the other hand, provide a recipe for building networks that support the extreme reduction of pairwise distances between their constituent nodes using overlays. Such networks possess overlays that exhibit significantly lower mean latencies than the ambient physical network, a feature that is useful for implementing a "high performance mode" for the physical network that could be suddenly required in critical mission circumstances. In this paper we have described the mathematical formalisms surrounding these two classes of networks and provided some examples of each of these two important classes, as well as shown that the two classes have sequences in common.

In our future work we plan to further explore the relationships between the properties of graph compressibility and expandibility. We will also seek to extend the notion of multiple personalities using overlays to metrics other than those based on graph diameter and pairwise distances that we considered here.

References

[1] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2nd ed., 2001.