Contemporary Mathematics

The Structure of Automorphic Conjugacy in the Free Group of Rank Two

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ABSTRACT. In the study of the automorphism group of a free group $F = F(X)$ on a set X, J. H. C. Whitehead introduced a graph whose vertices are elements of F, where two vertices are connected if and only if the corresponding elements of F are related by one of a specially chosen set of generators of $Aut(F)$. Here we give a precise structural description of Whitehead's graph for the case where $F = F_2$ is the free group of rank two. This description allows us to quantify relationships between the natural length function $|\ |$ of F_2, and the action of $Aut(F_2)$ on F_2. As an application, we show that Whitehead's algorithm for testing automorphic conjugacy in F_2 runs in time that is at most quadratic in the length of the elements.

1. The Automorphism Graph of F_2

To start, let X denote a **base set** of elements $\{a, b, \ldots\}$, and let $X^{-1} = \{A, B, \ldots\}$ be the set consisting of the corresponding formal inverses of elements from X. We call the elements of $X \cup X^{-1}$ **letters**, and denote the **free group** on the set X as $F(X)$.

The elements of the free group can be taken as the set of freely reduced words of finite length over the alphabet $X \cup X^{-1}$, where by **freely reduced** we mean words which contain no subword of the form xx^{-1} or xx^{-1} for any $x \in X$. Multiplication of elements of $F(X)$ is simply concatenation of words, followed by **free reduction**, which is to say repeated cancellation of all subwords of the form xx^{-1} or xx^{-1} for $x \in X$. The unique empty word of length 0 plays the role of the identity element. It is well-known that given two sets X and Y the free group $F(X) \cong F(Y)$ if and only if $|X| = |Y|$. This justifies denoting such a free group as $F_{|X|}$, since upto isomorphism the group depends only on the cardinality of the base set. The group $F_{|X|}$ is called *the* free group of **rank** $|X|$. This work considers F_2, the free group of rank two on the set $X = \{a, b\}$.

Recall that for any group G, the set of automorphisms of G again forms a group, denoted $Aut(G)$, in which composition of automorphisms plays the role of multiplication. Given a group G, two elements $g, h \in G$ are said to be **automorphic conjugates** if there exists an automorphism $\phi \in Aut(G)$ for which $\phi(g) = h$.

This work concerns the properties of $Aut(F_2)$, the group of automorphisms of F_2. In general, a structural description of the orbits of F_n under the action of