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Additive Operations on Flow Graphs

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Abstract

This paper extends the theory of *graphic arithmetic*, an extension of classical arithmetic on $\mathbb{N}$ to a larger model $\mathcal{F}$ defined on the set of all flow graphs—finite directed connected multigraphs having a pair of distinguished vertices. We give a graph-theoretic characterization for a flow graph to be reducible into a proper sum, and use this to develop a canonical decomposition of flow graphs into irreducible summands. Such decompositions enable us to characterize commutativity conditions for addition, which in turn, reveal the structure of additive centralizers in $\mathcal{F}$.

Keywords: multi-graphs, flow graphs, canonical decomposition, graph arithmetic.

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1. Introduction

There have been previous attempts to define algebraic structures on the set of all graphs, notably via ordinals and partially ordered sets [3, 4, 11, 13, 8], the classical operations on graphs [15], including graph products [6]. In considering binary operations on graphs, considerable prior work has addressed the Cartesian product [12], Kronecker products [14], tensor products [1], zig-zag and sandwich products [10], and the rooted product [5], and others. For each of these, there is, of course, a corresponding nascient algebraic theory. With respect to Cartesian products, for example, Li considers the problem of enumerating prime graphs [9], while Kaveh and Laknejadi recently addressed issues surrounding factorization [7], In our own prior work [2], we generalized arithmetic (classically defined over

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