

Lerdahl's tonal pitch space model and associated metric spaces

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(Received 3 May 2007; final version received 7 July 2010)

This paper explores the boundary separating a theoretically derived model of chord relations from an empirically derived model. Fred Lerdahl's tonal pitch space (TPS) model approximates cognitive perceptual relations by providing a combinatorial procedure for computing the distance value between any two chords in a key. If TPS posits a hypothesized model of perception, then we would like to know if, and the extent to which, it differs from experimental data it claims to approximate. To achieve such a comparison, we develop three conceptual tools. First, we develop normalized canonical representations of each model, thereby avoiding comparisons affected by design choices. Second, we develop a distance measure that allows us to accurately compare the TPS model with another model derived from perceived chord relations described by Bharucha and Krumhansl. Finally, we use the distance measure to inform the design of a third model. These three models are shown to create a metric space of metric tonal models. The proposed distance measure and the method of normalization are applicable to any model with formal properties described herein and have the potential to focus experimental design and strengthen the relationship between experimental data and analytic systems.

Keywords: Lerdahl; tonal pitch space; music cognition; perception; metric space

1. Introduction

The search for compelling representations of tonal hierarchy and its constituent harmonic relations has a long-standing history. Lerdahl [1] describes geometric approaches to this problem that involve the collection and modelling of data from experiments in music cognition. The multidimensional-scaling models of Bharucha and Krumhansl [2], and Deutsch [3], for example, seek to encode cognitive relationships between chords *within a single key area* (region) as Euclidean distances. The work of Heinichen [4], Kellner [5], and Weber [6], is more speculative and when formalized, develops geometric representations of relationships between different key areas (regions) through their placement within a multidimensional space.

Fred Lerdahl's tonal pitch space (TPS) model [1] approximates the cognitive perceptual relation between chords by providing a combinatorial procedure for computing a distance value between two triads. The procedure employed by the TPS model is informed by experimental data and plausible hypotheses about how we perceive tonal relations. The model is extraordinarily powerful

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and is able to describe relations both between chords within a region (e.g. Bharucha *et al.*) and between regions themselves (e.g. Heinichen *et al.*).¹

Because of the influence of experimental data on the TPS model, we would expect a high correlation between such data and analyses of intra-regional chord progressions generated by the TPS model. There are, in fact, important correlations between these two different kinds of data. However, if the TPS model posits a hypothesized model of perception, then we would like to know if and by how much it differs from the experimental data it claims to approximate. This article presents an inter-model comparative methodology that allows us to quantify these differences.² After discussing the features of the basic TPS model, we define a second, formally equivalent model based on experimental results.

The greatest obstacle inherent in quantifying a procedure for comparing these tonal models is negotiating the manner in which each constituent model codes its data. We resolve this challenge by developing canonical representations. Each canonical model serves as a representative of an entire equivalence class of tonal models and captures the most significant features of its class, while remaining agnostic to semantically insignificant design choices. We then go on to develop a similarity measure for tonal models. In practice, the similarity of two models is a metric that describes the distance between canonical representatives. By taking this approach, our similarity measure sidesteps the influence of idiosyncratic design choices. A third tonal model is introduced for comparative purposes and the similarity measure is used to connect all three models. This network of models is recursive insofar as it exemplifies the same geometric properties held by each tonal model.

2. The basic TPS model

Using a large body of empirical evidence, Lerdahl created an algebraic model for quantifying the distance between any two chords. He dwells mainly on triads, but considers other sonorities as well. Lerdahl’s basic TPS model assigns a number to each chord pair in a single key or region. That number is the intra-regional chord-pair ‘distance’ as defined by the chord distance rule.

Chord distance rule: $\delta(x \rightarrow y) = j + k$, where $\delta(x \rightarrow y)$ = the distance between chord x and chord y ; j = the number of applications of the chordal circle-of-fifths rule needed to shift x into y ; and k = the number of distinctive pitch classes (pcs) in the basic space of y compared to those in the basic space of x [1].

First, a region is determined by affixing a major scale to the universal chromatic space. Second, triadic structures are overlaid on the diatonic space in a weighted fashion, reflecting the perceptual hierarchy of root, fifth, and third. This is the ‘basic space’ of a triad. Finally, triad structures are shifted to different positions on the diatonic space. The distance (δ) between two triads X and Y is the number of diatonic fifths the root has moved plus the number of pcs (p) in the basic space of a chord X that are unique to X plus those that intersect with Y . If $p \in X \cap Y$ (as in the case of 7), then only the ‘highest’ occurrence is counted; if $p \in X$ and $p \notin Y$, then every occurrence of p is counted.

For a single region, the pairwise chord distances are given in Table 1. Figure 1 shows an excerpt from the ‘Madamina’ aria from Mozart’s *Don Giovanni*. The progression, comprising the initial eight-bar parallel period, is annotated with the intra-regional chord-pair distance as defined by Lerdahl’s chord distance rule. By adding the distances together, we create an index, which we call the L distance, representing the total regional distance covered.³ The distance measure represents a cognitive relation between chords, where the shorter the distance the closer the two chords are in a perceptual sense.

Table 1. Theoretical harmonic relations from Ler Dahl [1].

	I	II	III	IV	V	VI	VII
I	0						
II	8	0					
III	7	8	0				
IV	5	7	8	0			
V	5	5	7	8	0		
VI	7	5	5	7	8	0	
VII	8	7	5	5	7	8	0

Is a mai - den fair and slen - der,
He will praise her for mo - dest sweet - ness.

Str.
p *fp*
Cor.

I IV V I

$$L \text{ Distance } I \xleftrightarrow{5} IV \xleftrightarrow{8} V \xleftrightarrow{5} I = 18$$

Figure 1. Madamina progression under L .

3. An experimental intra-regional model

Table 2 shows the results of the 1983 experiment by Bharucha and Krumhansl in which all possible pairs of diatonic triads from a single major mode key were judged in terms of ‘how well they sounded’ in succession [2,10].⁴ The higher the number between two chord pairs, the more strongly they are associated. Bharucha and Krumhansl’s experiments considered ordered sequences of chords.

The symmetrical regularity of the L model contrasts with the irregular findings of Bharucha and Krumhansl. Looking at Table 2 we can see some key differences between L and the BK data set. For example, whereas L produces symmetric relations between chord pairs, Bharucha and Krumhansl found an ordering effect. Another difference is the uniformity with which L treats fifth-related chords. In Ler Dahl’s basic model, interval-cycle seven plays a central organizing role, but a problem is created when perceptually important relations are generalized and used to model perceptually less-important relations. For example, is the relationship between vii and IV the same as ii to V as the L model claims, or do we hear them as two very different progressions? We must keep in mind that both models are products of different lines of questioning. The relations they

Table 2. Perceived harmonic relations from Bharucha and Krumhansl [2].

	I	II	III	IV	V	VI	VII
I	0	5.1	4.78	5.91	5.94	5.26	4.57
II	5.69	0	4.0	4.76	6.1	4.97	5.14
III	5.38	4.47	0	4.63	5.03	4.6	4.47
IV	5.94	5.0	4.22	0	6.0	4.35	4.79
V	6.19	4.79	4.47	5.51	0	5.19	4.85
VI	5.04	5.44	4.72	5.07	5.56	0	4.5
VII	5.85	4.16	4.16	4.53	5.16	4.19	0

Table 3. Symmetrized harmonic relations derived from Bharucha–Krumhansl.

	I	II	III	IV	V	VI	VII
I	0						
II	0.185	0					
III	0.197	0.236	0				
IV	0.165	0.205	0.226	0			
V	0.165	0.184	0.211	0.174	0		
VI	0.194	0.192	0.215	0.212	0.186	0	
VII	0.192	0.215	0.232	0.215	0.200	0.230	0

describe are the result of different methodologies designed for different reasons and therefore show different kinds of information. Nevertheless, they describe relations between the same set of objects. Furthermore, Lerdahl claims that there is a correlation between the relations his model describes and the relations that have been (and could be) described by experimental results. While some differences are readily apparent, quantifying those differences is a challenging task.

In order to make the following comparison of L and BK meaningful, we begin by asking ‘are L and BK considering the same kinds of things in the same way?’ The comparative treatment of analytic systems raises questions about the *equivalence* of such systems. Pre-developed analytic systems are not designed *per se* to be compared with each other. In order to achieve a conceptually relevant comparison of BK to L , we must first establish an equivalence between the two models. Taking BK, we approximate L ’s unordered harmonic relationships by considering the symmetrized magnitudes of the relationships they reported. To wit, the (i, j) entry in what we call BK (Table 3) is obtained by averaging the (i, j) and (j, i) entries of BK (Table 2). The values are proportionally inverted and symmetrized (averaged) so, like L , the cognitively closest pair is represented by the smallest distance. All ensuing discussions of the model BK refer to the tonal model we derived from the BK data set.

4. Measuring distance between models

Formally, L and BK are metric spaces over a tonal region R with a global distance function (i.e. a metric) d , that, for every two chords (c_1, c_2) in R , gives the distance between them as a *non-negative real number* [11]. The reader can verify that both L and BK have the following properties:

- (1) $d(c_1, c_2) \geq 0$
- (2) Identity of indiscernibles: $d(c_1, c_2) = 0 \iff c_1 = c_2$
- (3) Symmetry: $d(c_1, c_2) = d(c_2, c_1)$
- (4) Triangle inequality: $d(c_1, c_2) + d(c_2, c_3) \geq d(c_1, c_3)$.

By extension, let \mathcal{M}_R be the set consisting of all models of R . Since we are interested in how BK and L differ from each other, we define a distance measure between any two models in \mathcal{M}_R that focuses how differently each model represents specific chord pairs. In the following discussion, we use the term ‘metric models’ specifically to mean tonal models that have the properties of a metric space.

Given a model M_R from \mathcal{M}_R , we denote the *minimal* (respective *maximal*) separation as $\sigma_R^{\min}(M_R)$ (respective $\sigma_R^{\max}(M_R)$) and define these quantities by

$$\sigma_R^{\min}(M_R) \stackrel{\text{def}}{=} \min_{c_1 \neq c_2} d(c_1, c_2),$$

$$\sigma_R^{\max}(M_R) \stackrel{\text{def}}{=} \max_{c_1 \neq c_2} d(c_1, c_2).$$

Fix a region R consisting of a set of chords C , where $|C| \geq 2$. Recall that \mathcal{M}_R is the set of *all metric models*. Let $M_R^1 = (C, d_1)$ and $M_R^2 = (C, d_2)$ be any two metric models in \mathcal{M}_R . Define:

$$\mu(M_R^1, M_R^2) \stackrel{\text{def}}{=} \max_{\substack{c_1, c_2 \in C \\ c_1 \neq c_2}} \left\{ \frac{d_2(c_1, c_2)}{d_1(c_1, c_2)} \right\},$$

$$\epsilon(M_R^1, M_R^2) \stackrel{\text{def}}{=} |\log[\mu(M_R^1, M_R^2)]|.$$

We extend ϵ and define the distance between models M_R^1 and M_R^2 to be

$$\delta(M_R^1, M_R^2) \stackrel{\text{def}}{=} \epsilon(M_R^1, M_R^2) + \epsilon(M_R^2, M_R^1).$$

Intuitively, $\delta(M_R^1, M_R^2)$ interprets the distance between two models as the sum of two values: (i) the log of the maximum ratio of pairwise distances found in switching from M_R^1 to M_R^2 and (ii) the log of the maximum ratio of pairwise distances found in switching from M_R^2 to M_R^1 . Clearly, $\delta : \mathcal{M}_R \times \mathcal{M}_R \rightarrow \mathbb{R}_{\geq 0}$.

THEOREM 4.1 (\mathcal{M}_R, δ) is a metric space.

The theorem follows immediately from Propositions 4.2 to 4.4, proven below, which verify that each of the defining properties for a metric space hold in (\mathcal{M}_R, δ) .

PROPOSITION 4.2 (Reflexivity) Let $M_R^1 = (C, d_1)$ and $M_R^2 = (C, d_2)$ be any two metric models in \mathcal{M}_R . Then

$$\delta(M_R^1, M_R^2) = 0 \Leftrightarrow M_R^1 = M_R^2.$$

Proof (\Leftarrow) If $M_R^1 = M_R^2$ then $d_1 \equiv d_2$ as functions. Hence for all $c_1, c_2 \in C$, $d_1(c_1, c_2) = d_2(c_1, c_2)$. Hence $\epsilon(M_R^1, M_R^2) = |\log 1| = 0$. By a symmetric argument, $\epsilon(M_R^2, M_R^1) = 0$. Hence $\delta(M_R^1, M_R^2) = 0$.

(\Rightarrow) Suppose $\delta(M_R^1, M_R^2) = 0$. Since $\delta(M_R^1, M_R^2)$ is the sum of two non-negative quantities, it follows that $|\log[\mu(M_R^2, M_R^1)]| = |\log[\mu(M_R^1, M_R^2)]| = 0$. Hence $\mu(M_R^2, M_R^1) = \mu(M_R^1, M_R^2) = 1$. It follows that for each $c_1, c_2 \in C$, $d_1(c_1, c_2) = d_2(c_1, c_2)$, and so $M_R^1 = M_R^2$. ■

PROPOSITION 4.3 (Symmetry) Let $M_R^1 = (C, d_1)$ and $M_R^2 = (C, d_2)$ be any two metric models in \mathcal{M}_R . Then

$$\delta(M_R^1, M_R^2) = \delta(M_R^2, M_R^1).$$

Proof Immediate, since

$$\begin{aligned}\delta(M_R^1, M_R^2) &= \epsilon(M_R^1, M_R^2) + \epsilon(M_R^2, M_R^1) \\ &= \epsilon(M_R^2, M_R^1) + \epsilon(M_R^1, M_R^2) \\ &= \delta(M_R^2, M_R^1)\end{aligned}$$

i.e. δ is symmetric in its arguments. ■

PROPOSITION 4.4 (Triangle inequality) *Let $M_R^1 = (C, d_1)$, $M_R^2 = (C, d_2)$, and $M_R^3 = (C, d_3)$ be any three metric models in \mathcal{M}_R . Then*

$$\delta(M_R^1, M_R^3) \leq \delta(M_R^1, M_R^2) + \delta(M_R^2, M_R^3).$$

Proof Let φ be vertices c_1 and c_2 . Then

$$\begin{aligned}\frac{d_1(\varphi)}{d_3(\varphi)} &= \left[\frac{d_1(\varphi)}{d_2(\varphi)} \right] \cdot \left[\frac{d_2(\varphi)}{d_3(\varphi)} \right] \\ &\leq \mu(M_R^1, M_R^2) \cdot \mu(M_R^2, M_R^3).\end{aligned}$$

Since $\mu(M_R^1, M_R^3)$ is the maximum value of $d_1(\varphi)/d_3(\varphi)$ (maximized over all distinct φ in C), we see that

$$\mu(M_R^1, M_R^3) \leq \mu(M_R^1, M_R^2) \cdot \mu(M_R^2, M_R^3).$$

Taking logarithms and appealing to convexity of absolute values, it follows that

$$\begin{aligned}|\log[\mu(M_R^1, M_R^3)]| &\leq |\log[\mu(M_R^1, M_R^2)]| + |\log[\mu(M_R^2, M_R^3)]| \\ &\leq |\log[\mu(M_R^1, M_R^2)]| + |\log[\mu(M_R^2, M_R^3)]|.\end{aligned}$$

It follows, by the definition of ϵ , that

$$\epsilon(M_R^1, M_R^3) \leq \epsilon(M_R^1, M_R^2) + \epsilon(M_R^2, M_R^3). \quad (1)$$

A symmetric argument yields that

$$\epsilon(M_R^3, M_R^1) \leq \epsilon(M_R^3, M_R^2) + \epsilon(M_R^2, M_R^1). \quad (2)$$

By combining corresponding sides in expressions (1) and (2), we conclude that $\delta(M_R^1, M_R^3) \leq \delta(M_R^1, M_R^2) + \delta(M_R^2, M_R^3)$, as claimed. ■

5. Normalizations and canonical representatives

5.1. Normalizations

Having defined a distance measure between models, we would like to apply it to study the relationship between specific models, such as Lerdahl’s basic-TPS model (L) and the symmetrized Bharucha–Krumhansl model (BK). As it stands, we can compare the two models as they are presented in Tables 1 and 3, by using the metric δ . The question is, how meaningful is this as a similarity measure? It is clear that our distance measure would be unduly influenced by arbitrary choices in each model’s representational design.

Why, for example, should our comparison be constrained by *BK*'s choice of a scale ranging between integers 1 and 7, when it may as well have been -10 to 70 , or real numbers between 0 and 1. When we compare these two spaces, we must make every effort to ensure that we are comparing their most basic essence.

First we give a formal interpretation to the notion that models incorporate ‘arbitrary choices’ in their design. Given a model $M_R = (C, d)$ and two real numbers α, β , where $\alpha \in (0, +\infty)$ and $\beta \in (-\infty, \sigma_R^{\min}(M_R))$, we denote the (α, β) -normalization of M_R as

$$\langle M_R \rangle_{\alpha, \beta} \stackrel{\text{def}}{=} (C, \langle d \rangle_{\alpha, \beta}),$$

where $\langle d \rangle_{\alpha, \beta} : C \times C \rightarrow \mathbb{R}^{\geq 0}$ is defined to be

$$\langle d \rangle_{\alpha, \beta}(c_1, c_2) \stackrel{\text{def}}{=} \begin{cases} \alpha[d(c_1, c_2) - \beta] & \text{if } c_1 \neq c_2 \\ 0 & \text{otherwise,} \end{cases}$$

for every c_1, c_2 in C .

Intuitively, the (α, β) -normalization of M_R represents a linear translation of all positive distances by $-\beta$ followed by a rescaling by a factor of α . Normalization parameters must be considered when assessing the similarity of two models, since the implicit $\alpha = 1, \beta = 0$ choices come from arbitrary choices in the formal underpinnings or experimental design. These arbitrary choices are unimportant when models are considered *in isolation*, but when we want to measure the similarity *between* models, the choices exert undue influence.

In particular, given two models $M_R^1 = (C, d_1)$ and $M_R^2 = (C, d_2)$, and specific

$$\begin{aligned} \alpha_1, \alpha_2 &\in (0, +\infty) \\ \beta_1 &\in (-\infty, \sigma_R^{\min}(M_R^1)) \\ \beta_2 &\in (-\infty, \sigma_R^{\min}(M_R^2)), \end{aligned}$$

it is difficult to make any general assertions (i.e. independent of our choices of $\alpha_1, \alpha_2, \beta_1, \beta_2$) regarding the relationship between the distance between the two original models and the distance between their normalizations.

The two models, M_R^1 and M_R^2 , are said to be *normalization-equivalent* if one model is merely a linear normalization of the other, i.e.

$$M_R^1 \simeq M_R^2 \Leftrightarrow \langle M_R^1 \rangle_{\alpha_1, \beta_1} = M_R^2,$$

for some $\alpha_1 \in (0, +\infty), \beta_1 \in (-\infty, \sigma_R^{\min}(M_R^1))$. It is easy to see that \simeq defines an equivalence relation on \mathcal{M}_R .

The equivalence class of a model M_R under the equivalence relation \simeq is denoted:

$$[M_R] = \{ \langle M_R \rangle_{\alpha, \beta} \mid \alpha \in (0, +\infty), \beta \in (\sigma_R(M_R), +\infty) \}.$$

Viewed as a subset of \mathcal{M}_R , $[M_R]$ is the set of all linear normalizations of the model M_R . We extend this idea and define the set of all equivalence classes in \mathcal{M}_R as

$$[\mathcal{M}_R] = \{ [M_R] \mid M_R \in \mathcal{M}_R \}.$$

Lerdahl's model L is in fact only one member of an infinite collection $[L]$ of related models, each of which corresponds to a different normalization of L . The position of L within that set $[L]$ reflects specific choices in the representational design of L . Indeed, both L and *BK* are laden with choices that influence the value of $\delta(L, BK)$. The following discussion presents a way to measure distances between models that is insensitive to design choices.

5.2. Canonical representation of L and BK spaces

We now choose a member of $[M_R]$ to represent the equivalence class. We call this member \bar{M}_R and define it as the member of $[M_R]$ whose α, β normalization is the model whose $\sigma_R^{\max} = 7$ and $\sigma_R^{\min} = 1$.

Our final step in the comparative methodology is our definition of λ as a distance function such that $\lambda([M_R^1], [M_R^2]) = \delta(\bar{M}_R^1, \bar{M}_R^2)$ where $([M_R^1], [M_R^2])$ are equivalence classes and members of $[\mathcal{M}_R]$, the set of all equivalence classes. Clearly, $\lambda : [\mathcal{M}_R] \times [\mathcal{M}_R] \rightarrow \mathbb{R}^{\geq 0}$, and the reader will confirm that the distance measure enables us to define $([\mathcal{M}_R], \lambda)$ as a metric space. The distance measure λ refines the notion of δ and is specific to the canonical representatives of each equivalence class. In the case of $[L]$ and $[BK]$, the distance $\lambda([L], [BK]) = \delta(\bar{L}, \bar{BK}) = 4.727$.

Looking at Table 4, we see that the largest points of divergence occur at IV–V and iii–vii. In terms of \bar{BK} , \bar{L} greatly overrates the distance between iii and vii and underrates the distance between IV and V. These discrepancies summarize the difference between the two models. Lerdahl’s algorithmic approach privileges all fifth-related harmonies by giving them the lowest value (i.e. representing the ‘closest’ cognitive relations). By doing so, he distorts some important relations, the most important being the IV–V progression. We see this reflected in the interpretation of the Madamina progression using the canonical forms of L and BK shown in Figure 2. The difference is clear. The overall \bar{L} distance covered is 9, while the \bar{BK} distance is 6.212.

Table 4. Canonical representatives for $[L]$ and $[BK]$.

	\bar{L}	\bar{BK}	\bar{L}/\bar{BK}	\bar{BK}/\bar{L}
I–ii	7	2.690	2.602	0.384
I–iii	5	3.704	1.350	0.741
ii–iii	7	7.000	1.000	1.000
iii–IV	7	6.155	1.137	0.879
ii–IV	5	4.380	1.141	0.876
iii–V	5	4.887	1.023	0.977
iii–vi	1	5.225	0.191	5.225
iii–vii	1	6.662	0.150	6.662
I–IV	1	3.451	0.290	3.451
ii–V	1	2.606	0.384	2.606
ii–vi	1	3.282	0.305	3.282
ii–vii	5	5.225	0.957	1.045
I–V	1	1.000	1.000	1.000
I–vi	5	3.451	1.449	0.690
I–vii	7	3.282	2.133	0.469
IV–V	7	1.761	3.976	0.252
IV–vi	5	4.972	1.006	0.994
IV–vii	1	5.225	0.191	5.225
vi–vii	7	6.493	1.078	0.928
V–vi	7	2.775	2.523	0.396
V–vii	5	3.958	1.263	0.792

Note: Bold values indicate the largest points of divergence between the two models.

$$\bar{L} \text{ Distance} \quad I \xleftrightarrow{1} IV \xleftrightarrow{7} V \xleftrightarrow{1} I = 9$$

$$\bar{BK} \text{ Distance} \quad I \xleftrightarrow{3.451} IV \xleftrightarrow{1.761} V \xleftrightarrow{1} I = 6.212$$

Figure 2. Madamina progression under canonical forms L and BK .

6. Using λ to inform the design of a new model

The disagreement between pairwise distances in \bar{L} and their partners in \overline{BK} invites us to question the algorithm used to generate \bar{L} . With a clearer view of the differences between these two models that the distance measure affords, the question is: how can we use this information to create a new a chord distance algorithm that minimizes the differences encountered switching between \bar{L} and \overline{BK} ? The following section provides one answer.

The progression of the subdominant (IV) to the dominant (V) needs no real introduction. For Hugo Riemann, the relation between these harmonies was paramount. The theory of tonal functions of chords outlined by Riemann [12,13] in *Simplified Harmony* and pre-empted by *Systematische Modulationslehre* focuses on three primary triads, described in the former as *T* (for tonic; I), *S* (subdominant; IV), and *D* (dominant; V). The later treatise describes a four-stage cadence model [14].

- (1) Tonic: *opening assertion*.
- (2) Subdominant: *conflict*.
- (3) Dominant: *resolution of the conflict*.
- (4) Tonic: *confirmation, conclusion*.

Riemann emphasizes the meaning of individual chords (e.g. the subdominant is ‘conflict’) and thus moves away from earlier dialectic conceptions of chord pairs while still highlighting the importance of the IV–V progression. A metric space comprises pairwise distances, bundling chords together and resisting individual identities. Nevertheless, there is a correlation between Riemann’s cadence model and *BK*’s model as *BK* assigns the lowest (e.g. perceptually closest) distances to IV–V and V–I.

The four-stage cadence model cited above suggests a shortest path that agrees more with *BK* than it does with *L*. Using the three harmonic functions, we categorize all intra-regional chords as either strongly or weakly belonging to either *T*, *S*, or *D*. We assign strong function chords the value 1 and weak function chords the value 2. Pairwise distance between any two chords is assigned the metric which is the sum the values of the two chords. Chords I, IV, and V strongly belong to *T*, *S*, and *D*, respectively. Chords iii and vi weakly belong to *T*, chord ii weakly belongs to *S*, and vii weakly belongs to *D*. Table 5 shows the pairwise chord distances of the new intra-regional model we call *F*.

The distance measure $\lambda([F], [BK]) = 2.880$. This confirms that there is less distance between *F* and *BK* than there is between *L* and *BK*. This concisely supports the claim that *F* better correlates with *BK* than does *L*.

The Madamina progression is reinterpreted in terms of all three models in Figure 3 and we see that *F* does exactly what we have asked of it: interpret the progression more like *BK* than *L* was able to do. By comparative extension, we find the $\lambda([F], [L]) = 5.615$.

Table 5. Functional harmonic distances.

	I	II	III	IV	V	VI	VII
I	0						
II	3	0					
III	3	4	0				
IV	2	3	3	0			
V	2	3	3	2	0		
VI	3	4	3	3	2	0	
VII	3	4	4	3	3	4	0

$$\begin{aligned}
\overline{L} \text{ Distance} \quad & \text{I} \xleftrightarrow{1} \text{IV} \xleftrightarrow{7} \text{V} \xleftrightarrow{1} \text{I} \\
\overline{BK} \text{ Distance} \quad & \text{I} \xleftrightarrow{3.451} \text{IV} \xleftrightarrow{1.761} \text{V} \xleftrightarrow{1} \text{I} \\
\overline{F} \text{ Distance} \quad & \text{I} \xleftrightarrow{2} \text{IV} \xleftrightarrow{2} \text{V} \xleftrightarrow{2} \text{I}
\end{aligned}$$

Figure 3. Madamina progression under canonical forms L , BK , and F .Table 6. Canonical representatives for $[F]$ and $[BK]$.

	\bar{F}	\overline{BK}	\bar{F}/\overline{BK}	\overline{BK}/\bar{F}
I–ii	4	2.690	1.487	0.673
I–iii	4	3.704	1.080	0.926
ii–iii	7	7.000	1.000	1.000
iii–IV	4	6.155	0.650	1.539
ii–IV	4	4.380	0.913	1.095
iii–V	4	4.887	0.818	1.222
iii–vi	4	5.225	0.765	1.306
iii–vii	7	6.662	1.051	0.952
I–IV	1	3.451	0.290	3.451
ii–V	4	2.606	1.535	0.651
ii–vi	7	3.282	2.133	0.469
ii–vii	7	5.225	1.340	0.746
I–V	1	1.000	1.000	1.000
I–vi	4	3.451	1.159	0.863
I–vii	4	3.282	1.219	0.820
IV–V	1	1.761	0.568	1.761
IV–vi	4	4.972	0.805	1.243
IV–vii	4	5.225	0.765	1.306
vi–vii	7	6.493	1.078	0.928
V–vi	4	2.775	1.442	0.694
V–vii	4	3.958	1.011	0.989

Note: Bold values indicate the largest points of divergence between the two models.

Table 7. The metric space $([\mathcal{M}_R], \lambda)$.

	$[L]$	$[BK]$	$[F]$
$[L]$	0		
$[BK]$	4.727	0	
$[F]$	5.615	2.880	0

Because we define a single, symmetric distance between metric models (δ), and their equivalence classes (λ), we can extend the notional of a metric space to the set of all equivalence classes $([\mathcal{M}_R])$. Table 7 shows the recursive ‘metric space of metric spaces’ as defined by $([\mathcal{M}_R], \lambda)$. Here, $[L]$, $[BK]$, and $[F]$ are compared in a single view.

7. Conclusion

This paper has achieved five music-analytic goals. First, we showed how certain tonal models meet the requirements of a metric space. Second, we defined a distance measure between pairs of

models. This served as a similarity measure allowing us to articulate specific ways the two models differed. Third, we showed how to derive canonical representatives. Comparing canonical representatives allowed us to critique analytic claims made by L against BK without the interference of arbitrary design choices. Fourth, we showed how the distance between canonical representatives provided the point of reference used to inform our design of a third model, F . Finally, we showed how these three tonal models are members of three different equivalence classes whose representatives each come from a metric space of tonal models. Since every subset of a metric space is a metric space, the three models form a metric space. In closing, it is important to mention that the choice of canonical representatives clearly influences the resulting distance judgement. In principle, it could be the case that where one choice would show $[\mathcal{M}_R^1]$ as closer to $[\mathcal{M}_R^2]$, than to $[\mathcal{M}_R^3]$, another choice might show $[\mathcal{M}_R^1]$ as closer to $[\mathcal{M}_R^3]$. The strategy we present above was designed to facilitate intuitive comparisons by choosing canonical representatives with the same minimal and maximal separations. Nevertheless, the choice of canonical representatives is an important issue that we hope to explore more fully in a future paper.

Acknowledgement

The authors wish to thank the reviewers for their comments and suggestions, many of which were incorporated into the final submission.

Notes

1. Noll and Garbers [7] provide an excellent critique of Lerdahl's TPS model in the context of theoretical problems associated with his attempt to combine a principle of hierarchy with a principle of shortest path. Their discussion is comprehensive and addresses issues outside the scope of what we present here.
2. An earlier form of this methodology was introduced in [8] in a broader context and in [9].
3. We identify the basic model as L to distinguish it from the larger TPS framework.
4. The data in Table 2 is shown in [10, Table 8.2]. However, [2] is given as the source.

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