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Lerdahl's tonal pitch space model and associated metric spaces

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This paper explores the boundary separating a theoretically derived model of chord relations from an empirically derived model. Fred Lerdahl's tonal pitch space (TPS) model approximates cognitive perceptual relations by providing a combinatorial procedure for computing the distance value between any two chords in a key. If TPS posits a hypothesized model of perception, then we would like to know if, and the extent to which, it differs from experimental data it claims to approximate. To achieve such a comparison, we develop three conceptual tools. First, we develop normalized canonical representations of each model, thereby avoiding comparisons affected by design choices. Second, we develop a distance measure that allows us to accurately compare the TPS model with another model derived from perceived chord relations described by Bharucha and Krumhansl. Finally, we use the distance measure to inform the design of a third model. These three models are shown to create a metric space of metric tonal models. The proposed distance measure and the method of normalization are applicable to any model with formal properties described herein and have the potential to focus experimental design and strengthen the relationship between experimental data and analytic systems.

Keywords: Lerdahl; tonal pitch space; music cognition; perception; metric space

1. Introduction

The search for compelling representations of tonal hierarchy and its constituent harmonic relations has a long-standing history. Lerdahl [1] describes geometric approaches to this problem that involve the collection and modelling of data from experiments in music cognition. The multidimensional-scaling models of Bharucha and Krumhansl [2], and Deutsch [3], for example, seek to encode cognitive relationships between chords *within a single key area* (region) as Euclidean distances. The work of Heinichen [4], Kellner [5], and Weber [6], is more speculative and when formalized, develops geometric representations of relationships between different key areas (regions) through their placement within a multidimensional space.

Fred Lerdahl's tonal pitch space (TPS) model [1] approximates the cognitive perceptual relation between chords by providing a combinatorial procedure for computing a distance value between two triads. The procedure employed by the TPS model is informed by experimental data and plausible hypotheses about how we perceive tonal relations. The model is extraordinarily powerful

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models. This served as a similarity measure allowing us to articulate specific ways the two models differed. Third, we showed how to derive canonical representatives. Comparing canonical representatives allowed us to critique analytic claims made by L against *BK* without the interference of arbitrary design choices. Fourth, we showed how the distance between canonical representatives provided the point of reference used to inform our design of a third model, F. Finally, we showed how these three tonal models are members of three different equivalence classes whose representatives each come from a metric space of tonal models. Since every subset of a metric space is a metric space, the three models form a metric space. In closing, it is important to mention that the choice of canonical representatives clearly influences the resulting distance judgement. In principle, it could be the case that where one choice would show $[\mathcal{M}_R^1]$ as closer to $[\mathcal{M}_R^2]$, than to $[\mathcal{M}_R^3]$, another choice might show $[\mathcal{M}_R^1]$ as closer to $[\mathcal{M}_R^3]$. The strategy we present above was designed to facilitate intuitive comparisons by choosing canonical representatives with the same minimal and maximal separations. Nevertheless, the choice of canonical representatives is an important issue that we hope to explore more fully in a future paper.

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Notes

1. Noll and Garbers [7] provide an excellent critique of Lerdahl's TPS model in the context of theoretical problems associated with his attempt to combine a principle of hierarchy with a principle of shortest path. Their discussion is comprehensive and addresses issues outside the scope of what we present here.
2. An earlier form of this methodology was introduced in [8] in a broader context and in [9].
3. We identify the basic model as L to distinguish it from the larger TPS framework.
4. The data in Table 2 is shown in [10, Table 8.2]. However, [2] is given as the source.

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