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On positive theories of groups with regular free length functions

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In this paper we discuss a general approach to positive theories of groups. As an application we get a robust description of positive theories of groups with regular free Lyndon length function. Our approach combines techniques of infinite words (see [27], [5]), cancellation diagrams introduced in [21], and Merzlyakov's method [22].

Contents

1. Introduction

Let L be a first-order language and $\mathcal{K}$ a class of L-structures. Recall that a set $\mathcal{E}$ of formulae of L is called an *elimination set* for $\mathcal{K}$ if for every formula ϕ of L there is a formula ϕ^* which is a boolean combination of formulae in $\mathcal{E}$, and ϕ is equivalent to ϕ^* in every structure in $\mathcal{K}$. The class $\mathcal{K}$ is said to admit *elimination of quantifiers* if there exists an elimination set for $\mathcal{K}$ consisting of quantifier-free formulae.

Quantifier elimination appeared in the context of the model theory of fields with two seminal results of Tarski [30] in 1951: (i) the class of algebraically closed fields admits elimination of quantifiers in the language of rings, whose symbols are $+, -, \cdot, 0, 1$, and (ii) the class of real-closed fields admits elimination of quantifiers in the language of ordered rings, whose symbols are $+, -, \cdot, 0, 1, \leq$. Since the formula $a \geq 0 \leftrightarrow \exists y(a = y^2)$ holds in every real closed field, it follows that the class of real-closed fields has an elimination set consisting of formulae of the type $\exists y(y^2 = p)$, where p is an arbitrary polynomial that does not contain y.

In this paper we consider quantifier elimination in the context of certain groups, but doing so requires reintroducing some standard terminology from logic. Recall that every first-order formula in a language L is logically equivalent to a formula

$$\phi = Q_1 y_1 Q_2 y_2 \ldots Q_m y_m \phi_0(x_1, \ldots, x_n, y_1, \ldots, y_m), \tag{1.1}$$

in which the $Q_i \in \{\forall, \exists\}$ are quantifiers, and ϕ_0 is a quantifier-free formula in L.

The standard classification of formulae is based either on the types of the quantifier prefix $Q_1 \ldots Q_m$ or on the type of ϕ_0. For example, a formula ϕ is called *positive* if the formula ϕ_0 in (1.1) does not contain symbols for negation, implication, or equivalence; ϕ is *existential* (*universal*) if all quantifiers Q_i in (1.1) are existential (universal); ϕ is $\forall\exists$-formula if the quantifier prefix of ϕ is in the form $\forall \ldots \forall \exists \ldots \exists$, etc. One can combine both types of restrictions: ϕ is *positive-primitive* if it is positive and ϕ_0 is a conjunction of atomic formulae (i.e., formulae which are equalities of terms); ϕ is an *identity* if it is universal and ϕ_0 is a conjunction of atomic formulae. Finally, ϕ is called a *sentence* if it contains no free variables (like x_i's above). The set of all sentences in L which hold in a given structure G is called the *elementary theory* of G, and we denote it by $Th(G)$. The set $Th^+(G)$ of all positive sentences from $Th(G)$ is called the *positive theory* of G.

In this terminology, Tarski's first result [30] implies that the class of real-closed fields (in the language of rings) admits an elimination set consisting of positive-primitive formulae. In the model theory of groups, the classical theorem due to Szmielew [29] (see also Eklof and Fisher [7]) states that the class of all abelian groups (in the language of groups with symbols $\cdot, ^{-1}, 1$) admits an elimination set consisting of $\forall\exists$-sentences and positive-primitive formulae. It follows that a fixed abelian group admits an elimination set of positive-primitive formulae.

Elimination of quantifiers for a class $\mathcal{K}$ has the potential to provide a powerful tool for the study of model-theoretic properties of structures in $\mathcal{K}$. Unfortunately, in practice, the elimination of quantifiers presents many obstacles. Broadly speaking, there are two approaches to achieving elimination of quantifiers in model-theoretic algebra: (i) one may try to find a "good" elimination set for a particular subset of formulae Φ in the language L, or (ii) one may try to construct for each structure $K \in \mathcal{K}$ a new structure K^*, such that the class of structures $\mathcal{K}^*$ admits a "good" elimination set.

Exemplary of approach (ii) is the classical construction where K^* is taken to be the so-called *skolemisation* K^s of the structure K. Given a formula in language L:

$$\phi(x_1, \ldots, x_n) = \exists y \psi(x_1, \ldots, x_n, y)$$

a function $f_\phi : K^n \to K$ is called a *Skolem function* for ϕ in K if the formula

$$\forall x_1 \ldots \forall x_n (\exists y \; \psi(x_1, \ldots, x_n, y) \to \psi(x_1, \ldots, x_n, f_\phi(x_1, \ldots, x_n))$$

is true in K. One can expand the language L to a larger language L^s by adding a new functional symbol f_ϕ for every such formula ϕ (see [10], for example, for details). Naturally, in adding the Skolem functions f_ϕ one can expand the structure K to a structure K^s in the language L^s, such that the formulae ϕ and $\psi(x_1, \ldots, x_n, f_\phi(x_1, \ldots, x_n))$ are equivalent in K^s. Thus one can, in K^s, eliminate the existential quantifiers from the original ϕ in L.

If skolemization appears at first to be a formal trick, the skepticism is bourn out in practice, since in most settings the structures K^s become too complex to be analyzed. The situation is dramatically different, however, when the Skolem functions f_ϕ can be defined using certain "simple" formulae in the initial language L. To this end we say that a Skolem function f_ϕ is *term-definable* in L when there exists a term $t(x_1, \ldots, x_n)$ in L such that $f_\phi(a_1, \ldots, a_n) = t(a_1, \ldots, a_n)$ for every $a_1, \ldots, a_n \in K$. Notice that if every formula ϕ has a term-definable Skolem function in K, then in fact K admits elimination of quantifiers in the initial language L.

In practice it is convenient to combine both approaches (i) and (ii) above. Although much of the exposition that follows could have been readily presented in a more general setting, we have chosen to restrict ourselves to the language of groups as this is subject we will ultimately be addressing.

Let G be a group, and denote by L_G the language of groups, augmented by constant symbols for the elements of G. Take Φ to be a set of first-order formulae in L_G. A *general elimination procedure* for Φ in G entails the following two steps:

1) Construct a new group G^* and an embedding $^* : G \to G^*$;
2) Define a set of formulae Φ^* in L_{G^*} and a map $^* : \Phi \to \Phi^*$ which maps $\phi(\bar{x}) \in \Phi$ into $\phi^*(\bar{x}) \in \Phi^*$ such that for any $g_1, \ldots, g_n \in G$

$$G \models \phi(g_1, \ldots, g_n) \Leftrightarrow G^* \models \phi^*(g_1^*, \ldots, g_n^*).$$

We say that the elimination procedure is *effective* if the maps $G \to G^*$, $\Phi \to \Phi^*$, as well as the relation $G^* \models \phi(g_1^*, \ldots, g_n^*)$, are effective.

Along these lines, Merzlyakov [22] made the remarkable discovery that every positive sentence in the language of group theory with constants from the free non-abelian group F, has term-definable Skolem functions in F:

Theorem (Merzlyakov). *Let $F = F(A)$ be a free non-abelian group with basis A and*

$$\phi = \forall x_1 \exists y_1 \forall x_2 \exists y_2 \cdots \forall x_k \exists y_k \ \phi_0(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$$

be a positive sentence in the language L_F. Then there exist words

$$q_1(x_1), q_2(x_1, x_2), \ldots, q_k(x_1, x_2, \ldots, x_k) \in F(A \cup X),$$

where $X = \{x_1, \ldots, x_k\}$, such that $F \models \phi$ if and only if

$$F(A \cup X) \models \phi_0(x_1, q_1(x_1), x_2, q_2(x_1, x_2), \ldots, x_k, q_k(x_1, x_2, \ldots, x_k)),$$

where on the right $x_1, \ldots, x_k$ are viewed as constants from $F(A \cup X)$.

We remark that proof of Merzlyakov's theorem hinges on standard Nielsen cancellation arguments, which are in turn based on properties of the canonical natural length function on F. The main contribution of this paper is to generalize Merzlyakov's result to *arbitrary groups possessing free regular Lyndon's length functions* (see Section 2 for definitions and examples). To this end we introduce the following:

Definition 1.1. *Let G be a group and $G[X] = G * F(X)$ be the free product of G and a free group on X. Take a sentence in L_G which holds in G:*

$$\phi = \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \ \phi_0(x_1, y_1, \ldots, x_k, y_k)$$

Then G is said to freely lift ϕ *(or ϕ* admits a free lift *over G) if*

$$G[X] \models \exists y_1 \ldots \exists y_k \ (\phi_0(x_1, y_1, \ldots, x_k, y_k) \bigwedge_{i=1}^{k} y_i \in G[X_i]),$$

where $X_i = \{x_1, \ldots, x_i\}$ and $x_1, \ldots, x_k$ are viewed as constants in $G[X]$.

For a set of sentences Φ in L_G we will say that G freely lifts Φ if G freely lifts every ϕ in Φ.

The main result of this paper is the following theorem.

Theorem A. *Every finitely generated non-abelian group G with a regular free Lyndon length function freely lifts every positive sentence of the language L_G which holds in G, i.e., G freely lifts its positive theory $Th^+(G)$.*

Observe that positive sentences are precisely those sentences which are preserved by epimorphisms (see [10], for example), i.e. if a structure H is a homomorphic image of a structure K then $Th^+(K) \subseteq Th^+(H)$. This, together with Theorem A and the universal property of free products, implies the following generalization of the Merzlyakov's result.

Theorem B. *Let G be a finitely generated non-abelian group with a regular free Lyndon length function. Then every positive sentence in the language L_G which* *holds in G has term-definable Skolem functions in G. Moreover, if G has a decidable word problem then such Skolem functions can be found effectively.*

We describe a general method to eliminate quantifiers from positive sentences.

A General Elimination Procedure for Positive Sentences.

Let G be a group with a finite (or countable) generating set A. We view G in the language L_A which contains elements from A as new constants. At the beginning of each step we specify any additional conditions on G that are required.

Step 1. Existence of Term-Definable Skolem Functions
Suppose the group G freely lifts its positive theory. Now let

$$\phi = \forall x_1 \exists y_1 \cdots \forall x_k \exists y_k \ \phi_0(x_1, y_1, \ldots, x_k, y_k) \tag{1.2}$$

be an arbitrary positive sentence in the language L_A. If G freely lifts ϕ then ϕ holds in G if and only if the formula

$$\phi^* = \exists y_1 \ldots \exists y_k \ (\phi_0(x_1, y_1, \ldots, x_k, y_k) \bigwedge_{i=1}^{k} y_i \in G[X_i]), \tag{1.3}$$

holds in $G[X]$. If ϕ^* holds in $G[X]$ then must be solutions $y_i = q_i(x_1, \ldots, x_i) \in G[X_i]$ which satisfy

$$\phi_0(x_1, y_1, \ldots, x_k, y_k)$$

in $G[X]$. Given arbitrary $a_1, \ldots, a_k$ from G, any map sending $x_1 \mapsto a_1, \ldots, x_k \mapsto a_k$ can be extended to a unique homomorphism $f_a : G[X] \to G$ which is identical on G. Since positive formulae are stable under epimorphisms it follows that $y_i = q_i(a_1, \ldots, a_i)$ give a solution to $\phi_0(a_1, y_1, \ldots, a_k, y_k)$ in G, so $y_i = q_i(x_1, \ldots, x_i)$ are term-definable Skolem functions for ϕ in G. In other words,

$$G \models \phi \iff G \models \forall x_1 \ldots \forall x_k \ \phi_0(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k)).$$

Step 2. Finding Term-definable Skolem Functions Effectively
Suppose, in addition, that the Word Problem in G (w.r.t A) is decidable. It follows that the Word Problem in $G[X]$ is also decidable (with respect to the generating set $A \cup X$). Now, for an arbitrary formula ϕ of the type (1.2) we seek some solutions $y_i = q_i(x_1, \ldots, x_i) \in G[X_i]$ which make the formula $\phi_0(x_1, y_1, \ldots, x_k, y_k)$ true in $G[X]$. But if ϕ is in $Th^+(G)$, Step 1 showed us that the solutions y_i exist. We proceed by exhaustively substituting successively longer tuples of words

$$(q_1, \ldots, q_k) \in F[x_1] \times \ldots \times F[X]$$

into ϕ_0, and for each such tuple effectively deciding (using the Word Problem in $G[X]$) whether $\phi_0(x_1, q_1, \ldots, x_k, q_k)$ holds in $G[X]$ or not. It follows that if ϕ is in $Th^+(G)$ this algorithm will discover the term-definable skolem functions y_i in finitely many steps.

Step 3. (Decision algorithm)

Suppose that the positive existential theory of $G[X]$ in the language $L_{A\cup X}$ with constraints of the type $y_i \in G[X_i]$ is decidable. It follows that the Word Problem for G, as well as for $G[X]$, is decidable. Given this, one can effectively verify whether or not the formula ϕ^* from (1.3) is true in $G[X]$. Together with the Steps 1 and 2, this gives an effective elimination procedure for positive theory of G.

The discussion above proves the following:

Theorem C. *Let G be a group which freely lifts its positive theory. If the positive existential theory of $G[X]$ in the language $L_{G[X]}$ with constraints of type $y_i \in G[X_i]$ is decidable, then there exists an effective elimination procedure for positive theory of G in the language L_G.*

Makanin's proof (see [21]) of the decidability of the positive theory of free non-abelian groups uses an elimination procedure specializing the one above. He shows that Merzlyakov's theorem implies that free non-abelian groups freely lift their positive theories, and that their positive existential theories (with $y_i \in F(A \cup X_i)$ constraints) is decidable. We remark that Diekert [6] recently proved the decidability of positive existential theory of free groups with arbitrary *rational constraints.*

The previous discussion gives an indication of the far reaching consequences that occur when a group freely lifts its entire positive theory. On the other hand, we shall see in the last section of this paper that any given set of positive sentences is freely lifted by a wide class of groups. More precisely, the following is true.

Theorem D. *Let H be a group which freely lifts a subset $\Phi \subset Th^+(H)$. Suppose G is a group containing H as a subgroup, and possessing a retraction $\phi : G \longrightarrow H$ onto H, $\phi|_H = id_H$. Then $\Phi \subset Th^+(G)$ and G freely lifts Φ.*

2. Preliminaries

Here we introduce basic definitions and notation which will be used in this paper.

2.1. *Lyndon length functions*

Let G be a group and A be an ordered abelian group. Then a function $l : G \to A$ is called a *(Lyndon) length function* on G if the following conditions hold:

(L1) $\forall\, g \in G :\ l(g) \geqslant 0$ and $l(1) = 0$;
(L2) $\forall\, g \in G :\ l(g) = l(g^{-1})$;
(L3) $\forall\, g, f, h \in G :\ c(g, f) > c(g, h) \to c(g, h) = c(f, h)$,
where $c(g, f) = \frac{1}{2}(l(g) + l(f) - l(g^{-1}f))$.

In general $c(g, f) \notin A$, but $c(g, f) \in A_{\mathbb{Q}}$, so in the axiom (L3) we view A as a subgroup of $A_{\mathbb{Q}}$. The next two properties of length functions follow from (L1)-(L3):

- $\forall\, g, f \in G :\ l(gf) \leqslant l(g) + l(f)$;
- $\forall\, g, f \in G :\ 0 \leqslant c(g, f) \leqslant \min\{l(g), l(f)\}$.

Three additional axioms describe some special classes of Lyndon length functions.

(L4) $\forall\, g \in G: \; c(g, f) \in A$.

A length function $l : G \to A$ is called *free* if it satisfies

(L5) $\forall\, g \in G: \; g \neq 1 \to l(g^2) > l(g)$.

Notice that a group has a Lyndon length function which takes values in A and satisfies (L4)-(L5) if and only if it acts freely on an A-tree (see [4] for details).

The statement of the last axiom requires some notation. Given elements $g_1, \ldots, g_n \in G$ we write

$$g = g_1 \circ \cdots \circ g_n$$

if $g = g_1 \cdots g_n$ and $l(g) = l(g_1) + \cdots + l(g_n)$. For $\alpha \in A$ we write $g = g_1 \circ_\alpha g_2$ if $g = g_1 g_2$ and $c(g_1^{-1}, g_2) < \alpha$. If the length function l satisfies the *regularity* axiom:

(L6) $\forall g, f \in G, \exists u, g_1, f_1 \in G:$

$$g = u \circ g_1 \;\&\; f = u \circ f_1 \;\&\; l(u) = c(g, f)$$

then l is said to be *regular*.

2.2. *Infinite words*

In this subsection at first we recall some notions from the theory of ordered abelian groups (for details see the books [9] and [18]) and the construction of infinite words (see [27], for example).

Let A be an ordered abelian group and $a, b \in A$. The *closed segment* $[a, b]$ is defined by

$$[a, b] = \{c \in A \mid a \leqslant c \leqslant b\}.$$

Now a subset $C \subset A$ is called *convex* if for every $a, b \in C$ the set C contains $[a, b]$. In particular, a subgroup B of A is convex if $[0, b] \subset B$ for every positive $b \in B$. It is not hard to see that the set $\{A_i \mid i \in I_A\}$ of convex subgroups of an ordered abelian group A is linearly ordered by inclusion, that is, for any $i, j \in I_A,\ i \neq j$ we have $A_i < A_j$ whenever $i < j$ and

$$A = \bigcup_{i \in I_A} A_i.$$

An ordered abelian group A is called *discretely ordered* if A has a minimal positive element (we denote it by 1_A). In this event, for any $a \in A$:

1) $a + 1_A = \min\{b \mid b > a\}$,
2) $a - 1_A = \max\{b \mid b < a\}$.

Observe that if A is any ordered abelian group then $\mathbb{Z} \oplus A$ is discretely ordered with respect to the right lexicographic order.

Let A be a discretely ordered abelian group and let $X = \{x_i \mid i \in I\}$ be a set. Put $X^{-1} = \{x_i^{-1} \mid i \in I\}$ and $X^{\pm} = X \cup X^{-1}$. An *A-word* is a function of the type

$$w : [1_A, \alpha_w] \to X^{\pm},$$

where $\alpha_w \in A,\ \alpha_w \geqslant 0$. The element α_w is called the *length* $|w|$ of w.

By $W(A, X)$ we denote the set of all A-words. Observe, that $W(A, X)$ contains an empty A-word which we denote by ε. Concatenation uv of two A-words $u, v \in W(A, X)$ is an A-word of length $|u| + |v|$ and such that:

$$(uv)(a) = \begin{cases} u(a) & \text{if } 1_A \leqslant a \leqslant |u| \\ v(a - |u|) & \text{if } |u| < a \leqslant |u| + |v| \end{cases}$$

An A-word w is *reduced* if $w(\beta + 1_A) \neq w(\beta)^{-1}$ for each $1_A \leqslant \beta < |w|$. We denote by $R(A, X)$ the set of all reduced A-words. Clearly, $\varepsilon \in R(A, X)$.

For $u \in W(A, X)$ and $\beta \in [1_A, \alpha_u]$ by u_β we denote the restriction of u on $[1_A, \beta]$. If $u \in R(A, X)$ and $\beta \in [1_A, \alpha_u]$ then

$$u = u_\beta \circ \tilde{u}_\beta,$$

for some uniquely defined $\tilde{u}_\beta$.

An element $\mathrm{com}(u, v) \in R(A, X)$ is called the *(longest) common initial segment* of A-words u and v if

$$u = \mathrm{com}(u, v) \circ \tilde{u}, \quad v = \mathrm{com}(u, v) \circ \tilde{v}$$

for some (uniquely defined) A-words $\tilde{u}, \tilde{v}$ such that $\tilde{u}(1_A) \neq \tilde{v}(1_A)$.

Now, we can define the product of two A-words. Let $u, v \in R(A, X)$. If $\mathrm{com}(u^{-1}, v)$ is defined then

$$u^{-1} = \mathrm{com}(u^{-1}, v) \circ \tilde{u}, \quad v = \mathrm{com}(u^{-1}, v) \circ \tilde{v},$$

for some uniquely defined $\tilde{u}$ and $\tilde{v}$. In this event put

$$u * v = \tilde{u}^{-1} \circ \tilde{v}.$$

The product $*$ is a partial binary operation on $R(A, X)$.

An element $v \in R(A, X)$ is termed *cyclically reduced* if $v(1_A)^{-1} \neq v(|v|)$. We say that an element $v \in R(A, X)$ admits a *cyclic decomposition* if $v = c^{-1} \circ u \circ c$, where $c, u \in R(A, X)$ and u is cyclically reduced. Observe that a cyclic decomposition is unique (whenever it exists). We denote by $CR(A, X)$ the set of all cyclically reduced words in $R(A, X)$ and by $CDR(A, X)$ the set of all words from $R(A, X)$ which admit a cyclic decomposition.

In what follows we refer to A-words as *infinite words* usually omitting A whenever it does not produce any ambiguity. The following result establishes the connection between infinite words and length functions.

Theorem 2.1. *[27] Let A be a discretely ordered abelian group and X be a set. Then any subgroup G of $CDR(A,X)$ has a free Lyndon length function with values in A – the restriction $L|_G$ on G of the standard length function L on $CDR(A,X)$.*

The converse of the theorem above was obtained by I.Chiswell [5].

Theorem 2.2. *[5] Let G have a free Lyndon length function $L : G \to A$, where A is a discretely ordered abelian group. Then there exists a length preserving embedding $\phi : G \to CDR(A,X)$, that is, $|\phi(g)| = L(g)$ for any $g \in G$.*

Corollary 2.1. *[5] Let G have a free Lyndon length function $L : G \to A$, where A is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \to CDR(A', X)$, where $A' = \mathbb{Z} \oplus A$ is discretely ordered with respect to the right lexicographic order and X is some set, such that, $|\phi(g)| = (0, L(g))$ for any $g \in G$.*

2.3. *Examples*

Theorems 2.1 and 2.2, and Corollary 2.2 show that a group has a free Lyndon length function if and only if it embeds into the set of infinite words and this embedding preserves the length. Moreover, it is not hard to show that this embedding also preserves regularity of the length function.

Theorem 2.3. *Let G have a free regular Lyndon length function $L : G \to A$, where A is an arbitrary ordered abelian group. Then there exists an embedding $\phi : G \to R(A', X)$, where A' is a discretely ordered abelian group and X is some set, such that, the Lyndon length function on $\phi(G)$ induced from $R(A', X)$ is regular.*

Proof. Since L is regular then for any $g, f \in G$ there exist $u, g_1, f_1 \in G$ such that

$$g = u \circ g_1 \ \&\ f = u \circ f_1 \ \&\ L(u) = c(g, f)$$

By Corollary 2.1 it follows that

$$|\phi(g)| = |\phi(u)| + |\phi(g_1)|,\ \ |\phi(f)| = |\phi(u)| + |\phi(f_1)|.$$

Indeed, if for example $|\phi(g)| < |\phi(u)| + |\phi(g_1)|$ then $L(g) < L(u) + L(g_1)$ - contradiction. So, we have

$$2c(\phi(g), \phi(f)) = |\phi(g)| + |\phi(f)| - |\phi(g^{-1}f)| = |\phi(g)| + |\phi(f)| - |\phi(g_1^{-1} \circ f_1)| = 2|\phi(u)|.$$

Hence, $c(\phi(g), \phi(f)) = |\phi(u)|$ and the length function on $\phi(G)$ induced from $R(A, X)$ is regular. $\square$

Notice that the converse of the theorem above is obviously true. Using this observation we present some examples of groups with regular free Lyndon length functions. Below we view the ring $\mathbb{Z}[t]$ of polynomials in a variable t with integer coefficients as an ordered abelian group.

Example 2.1. If $F = F(X)$ is a free group with finite basis X then

$$G(u, s) = \langle F, s \mid u^s = u \rangle$$

has the regular free Lyndon length function inherited from $CDR(\mathbb{Z}[t], X)$ (see [27]).

In fact, the methods presented in [27] provide more general results. For a free non-abelian group $F = F(X)$ with basis X one can consider a $\mathbb{Z}[t]$-*completion* $F^{\mathbb{Z}[t]}$ of F, which is called now the Lyndon's free $\mathbb{Z}[t]$-group.

Theorem 2.4. *[27] $F^{\mathbb{Z}[t]}$ is canonically embeddable into $CDR(\mathbb{Z}[t], X)$. Moreover, the length function on $F^{\mathbb{Z}[t]}$ induced from $CDR(\mathbb{Z}[t], X)$ is regular.*

The proof of this result takes advantage of the construction given in [24] which describes $F^{\mathbb{Z}[t]}$ as a union of a sequence of extensions of centralizers

$$F = G_0 < G_1 < \cdots < G_n < \cdots, \tag{2.1}$$

where G_{i+1} is obtained from G_i by extension of all cyclic centralizers in G_i by a free abelian group of countable rank.

As a corollary of the theorem above and results of [2,11,12] it follows that every finitely generated fully residually free group is embeddable into $CDR(\mathbb{Z}[t], X)$ (as a subgroup of $F^{\mathbb{Z}[t]}$). But observe that although $F^{\mathbb{Z}[t]}$ has a regular free Lyndon length function inherited from $CDR(\mathbb{Z}[t], X)$ it does not imply that all its subgroups do. Hence, the length function on a finitely generated fully residually free group induced from $CDR(\mathbb{Z}[t], X)$ may be not regular.

Example 2.2. Let $F = F(X)$ be a free group with a finite basis X. Consider an HNN-extension

$$G = \langle F, s \mid u^s = v \rangle,$$

where $u, v \in F$ and $|u| = |v|$. It was shown in [17] that G can be embedded into $CDR(\mathbb{Z}^2, X) \subset CDR(\mathbb{Z}[t], X)$, moreover the induced free Lyndon length function on G is regular.

Using Example 2.2 one can embed some finitely generated fully residually free groups into $CDR(\mathbb{Z}[t], X)$ directly - not as subgroups of $F^{\mathbb{Z}[t]}$.

Example 2.3. For any $n \geq 1$ a surface group of the type

$$G = \langle a_1, b_1, \ldots, a_n, b_n \mid \prod_{i=1}^{n} [a_i, b_i] = 1 \rangle$$

can be represented as an HNN-extension from Example 2.2, hence one can construct the embedding of G into $CDR(\mathbb{Z}[t], X)$ and the induced length function is regular. Indeed, consider the word

$$R(X) = x_1 \cdots x_{2n} x_1^{-1} \cdots x_{2n}^{-1}.$$

It is easy to see that $R(X)$ is quadratic and by Proposition 7.6 [20] there exists an automorphism ϕ of $F = F(x_1, \ldots, x_{2n})$ such that

$$R(X)^{\phi} = [y_1, y_2] \cdots [y_{2n-1}, y_{2n}],$$

where all $y_i, i \in [1, 2n]$ are different and $y_i \in \{x_1^{\pm 1}, \ldots, x_{2n}^{\pm 1}\}, i \in [1, 2n]$. It follows that G is isomorphic to

$$G' = \langle x_1, \ldots, x_{2n} \mid x_1 \cdots x_{2n} x_1^{-1} \cdots x_{2n}^{-1} \rangle,$$

which can be presented as an HNN-extension of the required form

$$G' = \langle F(x_2, \ldots, x_{2n}), x_1 \mid x_1 (x_2 \cdots x_{2n}) x_1^{-1} = x_{2n} x_{2n-1} \cdots x_2 \rangle,$$

since $|x_2 \cdots x_{2n}| = |x_{2n} x_{2n-1} \cdots x_2|$.

3. Properties

In this section we prove some technical results for groups with regular free length functions.

Remark 3.1. Observe that if G is a group with a free (regular) Lyndon length function then Corollary 2.1 (Theorem 2.3) makes it possible to view G as a subgroup of $CDR(A, X)$ for an appropriate discretely ordered abelian group A and some set X. Hence, we assume A and X to be fixed for the rest of this paper and without loss of generality we think of all groups with free Lyndon length functions appearing from now on in our considerations as subgroups of $CDR(A, X)$.

By Remark 3.1, A is a fixed discretely ordered abelian group and recall that the set $\{A_i \mid i \in I_A\}$ of convex subgroups of A is linearly ordered by inclusion, that is, for any $i, j \in I_A,\ i \neq j$ we have $A_i < A_j$ whenever $i < j$ and

$$A = \bigcup_{i \in I_A} A_i.$$

If a group G has a free Lyndon length function then we say that $g \in G$ *has the height* $i \in I_A$ and denote $ht(g) = i$ if $|g| \in A_i$ and $|g| \notin A_j$ for any $j < i$. Observe that this definition depends only on G since the complete chain of convex subgroups of A is unique.

It is easy to see that

$$ht(g_1 g_2) \leq \max\{ht(g_1), ht(g_2)\},$$

hence, if $G = \langle g_1, \ldots, g_k \rangle$ then we define

$$ht(G) = \max\{ht(g_1), \ldots, ht(g_k)\}.$$

Lemma 3.1. *If $g_1, \ldots, g_k$ and $f_1, \ldots, f_m$ are two generating sets of G then*

$$\max\{ht(g_1), \ldots, ht(g_k)\} = \max\{ht(f_1), \ldots, ht(f_m)\},$$

that is, $ht(G)$ does not depend on the choice of generators for G.

Proof. Since every g_i can be expressed as a word in the alphabet $f_1, \ldots, f_m$, that is, $g_i = w_i(f_1, \ldots, f_m)$, $i \in [1, k]$ then we have

$$ht(g_i) \leq \max\{ht(f_1), \ldots, ht(f_m)\}.$$

Thus, $\max\{ht(g_1), \ldots, ht(g_k)\} \leq \max\{ht(f_1), \ldots, ht(f_m)\}$. Symmetrically, one gets $\max\{ht(f_1), \ldots, ht(f_m)\} \leq \max\{ht(g_1), \ldots, ht(g_k)\}$ which proves the lemma. $\square$

It is easy to see that in case when G is finitely generated, $ht(G)$ is the minimal $\kappa \in I_A$ such that $ht(g) \leq \kappa$ for any $g \in G$.

Recall that any element $g \in CDR(A, X)$ admits cyclic decomposition

$$g = u_g^{-1} \circ \overline{g} \circ u_g,$$

where u_g, $\overline{g} \in R(A, X)$ and $\overline{g}$ is cyclically reduced. Obviously, if $g \in G$ and G has a regular free Lyndon length function then u_g, $\overline{g} \in G$.

Lemma 3.2. *If G is a finitely generated group with a regular free Lyndon length function then G contains a cyclically reduced element u such that $ht(u) = ht(G)$.*

Proof. Take any $a \in G$ such that $ht(a) = ht(G)$. If a is cyclically reduced than we are done. Suppose on the contrary that $a = a_1^{-1} \circ c_1 \circ a_1$ and $a_1 \neq \varepsilon$. Since the length function on G is regular it follows that $a_1, c_1 \in G$.

If $ht(c_1) = ht(a)$ then we are done. Hence we assume $ht(c_1) < ht(a)$, that is, $ht(a_1) = ht(a)$. If a_1 is cyclically reduced then we are done also. If not then we have $a_1 = a_2^{-1} \circ c_2 \circ a_2$ and $a_2 \neq \varepsilon$. Hence,

$$a = (a_2^{-1} \circ c_2^{-1} \circ a_2) \circ c_1 \circ (a_2^{-1} \circ c_2 \circ a_2)$$

and it follows that

$$(c_1 \circ a_1) * (c_1 \circ a_1) = (c_1 \circ a_1) \circ (c_1 \circ a_1),$$

that is, $c_1 \circ a_1$ is cyclically reduced. Finally, observe that $ht(c_1 \circ a_1) = ht(a) = ht(G)$. $\square$

The following lemma was proved in [27] (see, Lemma 6.4) for subgroups of $CDR(\mathbb{Z}[t], X)$. It turns out that some restricted version of the statement holds for any group with a regular free Lyndon length function.

Lemma 3.3. *Let G be a group with a regular free Lyndon length function and $f, h \in G$ be cyclically reduced. If $ht(f) = ht(h)$ and $c(f^m, h^n) \geqslant |f| + |h|$ for some $m, n > 0$ then $[f, h] = \varepsilon$.*

Proof. Suppose $|h| \geqslant |f|$ and $c(f^m, h^n) \geqslant |f| + |h|$ for some $m, n > 0$. Observe that since $ht(f) = ht(h)$ then there exists $k \in \mathbb{N}$ such that $|h| \geq k|f|$, $|h| \leq (k+1)|f|$.

We have $h = f^k \circ h_1, |f| > |h_1|, k \geqslant 1$ and $f = h_1 \circ f_1$. Since $c(f^m, h^n) \geqslant |f| + |h|$ one has $(f^k \circ h_1) \circ f = f^{k+1} \circ h_1$. So, $h_1 \circ h_1 \circ f_1 = h_1 \circ f_1 \circ h_1$ and $f = h_1 \circ f_1 = f_1 \circ h_1$. It follows that $[f_1, h_1] = \varepsilon$, hence, $[h_1, f] = \varepsilon$ and $[f, h] = \varepsilon$. $\square$

The following result is analogous to Lemma 6.9 [27].

Lemma 3.4. *Let G be a finitely generated group with a regular free Lyndon length function. If $u, v \in G$ are such that $[u, v] \neq \varepsilon$, $ht(u) = ht(G)$ and u is cyclically reduced then there exists $r \in \mathbb{N}$ such that for all $m, n \geqslant r$ the following holds:*

$$u^m * v * u^n = u^{m-r} \circ (u^r * v * u^r) \circ u^{n-r}.$$

Proof. Suppose that the statement of the lemma does not hold for a pair (u, v). Then the lemma fails for any pair $(u, v * u^k)$, where $k \in \mathbb{Z}$. Since $ht(u) = ht(G)$ there exists $k \in \mathbb{Z}$ such that $(v * u^k) * u = (v * u^k) \circ u$. Hence, replacing v by $v * u^k$, we may assume from the beginning that $v * u = v \circ u$.

Similarly, there exists $k \in \mathbb{N}$ such that $u * (u^k * v) = u \circ (u^k * v)$. If v does not cancel completely in $u^k * v$ then

$$u^m * v * u^n = u^{m-k} \circ (u^k * v) \circ u^n$$

for any $m > k$, a contradiction. Hence, v cancels completely and $v = u^p \circ v_1, |v_1| \leqslant |u|$. So, we can assume $v = v_1$ and $|v| \leqslant |u|$. Thus $u = u_1 \circ v^{-1}$. It follows that $c((u_1^{-1} \circ v)^n, u^m) \geqslant 2|u| = |u_1^{-1} \circ v| + |u|$ for big enough $m, n > 0$. Since both $u_1^{-1} \circ v$ and u are cyclically reduced then by Lemma 3.3, $[u_1^{-1} \circ v, v] = \varepsilon$. Hence, $u_1^{-1} \circ v = u$ and we have $u_1^{-1} \circ v = u_1 \circ v^{-1}$. It follows that $u_1^{-1} = u_1$ and $v = v^{-1}$ - contradiction.

This shows that our assumption that the pair (u, v) does not satisfy the statement of the lemma is false. □

Now, we can prove one of the most important technical results of this paper.

Lemma 3.5. *If G is a finitely generated non-abelian group with a regular free Lyndon length function then there exist $a, b \in G$ such that*

(1) $[a, b] \neq \varepsilon$,
(2) $a * b = a \circ b,\ b * a = b \circ a$,
(3) $ht(a^k) = ht(G)$ *for any* $0 \neq k \in \mathbb{Z}$.

Proof. Obviously there exists $u \in G$ such that $ht(u) = ht(G)$. In view of Lemma 3.2 we can assume u to be cyclically reduced. Since G is non-abelian then there exists $v \in G$ such that $[u, v] \neq \varepsilon$. Now, the pair (u, v) satisfies all the conditions listed in Lemma 3.4, hence there exists $r \in \mathbb{N}$ such that for all $m, n \geqslant r$ the following holds:

$$u^m * v * u^n = u^{m-r} \circ (u^r * v * u^r) \circ u^{n-r}.$$

Observe that

(1) $[u, u^r * v * u^r] \neq \varepsilon$, and
(2) $ht((u^r * v * u^r)^k) = ht(G)$ for any $0 \neq k \in \mathbb{Z}$.

Finally, we complete the proof by setting $a = u^r * v * u^r, \ b = u$. $\square$

Corollary 3.1. *If G is a finitely generated non-abelian group with a regular free Lyndon length function then there exist $a, b \in G$ such that*

1) $[a, b] \neq \varepsilon$,
2) $b * a^k * b = b \circ a^k \circ b$ *for any* $k \in \mathbb{N}$,
3) *For any $g \in G$ there exists $n_g \in \mathbb{N}$ such that g does not contain $b \circ a^k \circ b$ as a subword for any $k \geq n_g$.*

Proof. By Lemma 3.5 it follows that there exist $a, b \in G$ such that

(i) $[a, b] \neq \varepsilon$,
(ii) $a * b = a \circ b, \ b * a = b \circ a$,
(iii) $ht(a^k) = ht(G)$ for any $0 \neq k \in \mathbb{Z}$.

Hence, 1) and 2) follow immediately from (i) and (ii). Finally, 3) follows from (iii) and from the fact that for any $g \in G$ there exists $n_g \in \mathbb{N}$ such that $|g| \leq |a^k| = k|a|$ for any $k \geq n_g$. $\square$

4. Cancellation diagrams

In this section, we extend the standard notion of a cancellation diagram from the theory of free groups, to groups with regular free Lyndon length functions. The construction follows a procedure which is analogous to the familiar cancellation process in free groups (see, for example, [14]).

We fix a subgroup G which has a regular free Lyndon length function and by Remark 3.1 we assume it to be a subgroup of $CDR(A, X)$ for some fixed A and X.

The concatenation $uv, \ u, v \in G$ is said to be *pinchable* if $|c(u^{-1}, v)| > 0$, that is, in other words, $u * v \neq u \circ v$.

The previous definition extends to tuples of elements $\bar{w} = (w_i)_{i=0,\ldots,k-1}$ from G. That is we call the concatenation $w_0 \cdots w_{k-1}$ *pinchable* if there exists a pair of consecutive elements $w_j, w_{j+1}, \ j \in [0, k-2]$ such that, $w_j w_{j+1}$ is pinchable.

Suppose we are given a tuple of elements $w_0, \ldots, w_{k-1}$ from G. If a concatenation of any pair of successive elements in this sequence is found to be pinchable, we make a maximal pinch at the boundary between the two A-words. The process continues in this way and is seen to terminate after at most finitely many pinches. Below, we formally describe the procedure, and in the course of the exposition, we introduce nomenclature that will be used in later sections.

Suppose $u, v \in G$ and consider the concatenation uv. We define a *pinch decomposition* of uv to be

$$l_1(u, v) \ [com(u^{-1}, v)]^{-1} \ com(u^{-1}, v) \ l_2(u, v)$$

where l_1 and l_2 are A-words defined implicitly by the following equalities:

$$l_1(u,v) \circ [com(u^{-1},v)]^{-1} = u$$
$$com(u^{-1},v) \circ l_2(u,v) = v.$$

Observe, that in the definition above we have $l_1(u,v), com(u^{-1},v), l_2(u,v) \in G$ because the length function on G is regular.

Now, given a length k sequence of elements $\bar{w} \in G^k$,

$$\bar{w} = (w_i)_{i=0,\ldots,k-1}.$$

The associated directed *cancellation graph* $\Gamma_{\bar{w}}(V,E)$ and a path $\rho_{\bar{w}}$ (in $\Gamma_{\bar{w}}$) are defined as follows.

The construction of $\Gamma_{\bar{w}}$ and $\rho_{\bar{w}}$ proceeds in stages. At each stage i, we obtain an intermediate graph $\Gamma^i_{\bar{w}} = (V_i, E_i)$ and path $\rho^i_{\bar{w}}$ in $\Gamma^i_{\bar{w}}$. In addition, at each stage the edges of $\Gamma^i_{\bar{w}}$ will be partitioned into two classes $E_i = C_i \sqcup L_i$.

As we shall see in lemma 4.1, each stage i of the construction process maintains the following properties:

P1. Each directed edge of $e \in E_i$ is labelled by some element $\tilde{e} \in G$.

P2. $C_i = \bar{C}_i$, i.e., if $e = (u,v) \in C_i$, then $\bar{e} = (v,u) \in C_i$. The edges in C_i are referred to as *cancelled* edges.

P3. $L_i \cap \bar{L}_i = \emptyset$, i.e., if $e = (u,v) \in L_i$, then $\bar{e} = (v,u) \notin L_i$. The edges in L_i are referred to as *live* edges.

P4. The path $\rho^i_{\bar{w}} = (e^i_0, e^i_1, e^i_2, \ldots e^i_{m_i})$ is an Euler path in $\Gamma^i_{\bar{w}} = (V_i, E_i)$.

P5. The set of edges in L_i forms a single directed path

$$\ell_i = (f^i_0, f^i_1, \ldots f^i_{r_i})$$

Edge f^i_j appears before f^i_{j+1} in $\rho^i_{\bar{w}}$, and all intermediate edges of $\rho^i_{\bar{w}}$ belong to C_i.

P6. The Euler path $\rho^i_{\bar{w}} = (e^i_0, e^i_1, e^i_2, \ldots e^i_{m_i})$ at stage i satisfies the following equality:

$$\tilde{e}^i_0 * \tilde{e}^i_1 * \cdots * \tilde{e}^i_{m_i} = w_0 * w_1 * \cdots * w_{k-1}.$$

More precisely, there exists a strictly increasing function $\alpha_i : \{0,\ldots,k\} \to \{0,\ldots,m_i+1\}$ satisfying $\alpha_i(0) = 0$ and $\alpha_i(k) = m_i + 1$, for which

$$\tilde{e}^i_{\alpha_i(j)} \circ \tilde{e}^i_{\alpha_i(j)+1} \circ \cdots \circ \tilde{e}^i_{\alpha_i(j+1)-1} = w_j,$$

for $j = 0,\ldots,k-1$.

P7. Let $T^i_{\bar{w}}$ be the *undirected* graph on vertices V_i, and edges D_i, where D_i consists of all live edges L_i and for each edge $(u,v) \in C_i$ either $(u,v) \in D_i$ or $(v,u) \in D_i$ (but not both). Then $T^i_{\bar{w}}$ is a tree.

We begin by defining $\Gamma^0_{\bar{w}}$. Take $V_0 = \{v_0,\ldots,v_k\}$ as the set of vertices; then for each pair of consecutively indexed vertices (v_i, v_{i+1}) add an edge e_i, labelled by $\tilde{e}_i = w_i$. Put $E_0 = \{e_i \mid i = 0, \ldots, k-1\}$, and specify the partition $C_0 = \emptyset$, $L_0 = E_0$. The path $\rho^0_{\bar{w}}$ is defined to be the sequence $(e_0, e_1, \ldots, e_{k-1})$.

Suppose now that we have constructed $\Gamma^i_{\bar{w}}$.

If $L_i = \emptyset$, we define $\Gamma^{i+1}_{\bar{w}} = \Gamma^i_{\bar{w}}$, $\rho^{i+1}_{\bar{w}} = \rho^i_{\bar{w}}$, and say that at stage i the construction process converged to $\Gamma_{\bar{w}} = \Gamma^i_{\bar{w}}$ and $\rho_{\bar{w}} = \rho^i_{\bar{w}}$.

Otherwise $L_i \neq \emptyset$; then since $\Gamma^i_{\bar{w}}$ satisfies property $P4$, we consider the concatenation of A-words which appear as labels of live edges along the unique maximal non-trivial directed path ℓ_i:

$$\tilde{\ell}_i = \tilde{f}^i_0 \tilde{f}^i_1 \cdots \tilde{f}^i_{r_i}$$

- If $\tilde{\ell}_i$ is not pinchable, we define $\Gamma^{i+1}_{\bar{w}} = \Gamma^i_{\bar{w}}$, $\rho^{i+1}_{\bar{w}} = \rho^i_{\bar{w}}$, and say that at stage i the construction process converged to $\Gamma_{\bar{w}} = \Gamma^i_{\bar{w}}$ and $\rho_{\bar{w}} = \rho^i_{\bar{w}}$.
- Otherwise $\tilde{\ell}_i$ is pinchable. We proceed to construct $\Gamma^{i+1}_{\bar{w}}$ and $\rho^i_{\bar{w}}$ as follows.

  Since $\Gamma^i_{\bar{w}}$ satisfies property $P1$, the pinch in $\tilde{\ell}_i$ must occur in $\tilde{f}^i_j \tilde{f}^i_{j+1}$, for some $j \in \{0, \ldots, r_i - 1\}$. Note that if there are several candidates for j, we choose the smallest among them. Suppose $f^i_j = (r, s)$ and $f^i_{j+1} = (s, t)$.

  Split vertex s into two vertices s' and v'. Vertex s' inherits all the live edges that were incident to s, while vertex v' inherits all the cancelled edges that were incident to s. Put $V_{i+1} = (V_i - \{s\}) \cup \{s', v'\}$. Then, using the pinch decomposition of $\tilde{f}^i_j \tilde{f}^i_{j+1}$:

  - We add two cancelled edges: (s', v') labelled by $[com([\tilde{f}^i_j]^{-1}, \tilde{f}^i_{j+1})]^{-1}$ and (v', s') labelled by $com([\tilde{f}^i_j]^{-1}, \tilde{f}^i_{j+1})$. We relabel live edges (r, s') by $l_1(\tilde{f}^i_j, \tilde{f}^i_{j+1})$ and (s', t) by $l_2(\tilde{f}^i_j, \tilde{f}^i_{j+1})$.
  - We obtain $\rho^{i+1}_{\bar{w}}$ from $\rho^i_{\bar{w}}$ by replacing the edge f^i_j with the sequence of edges $(r, s')(s', v')$, and replacing f^i_{j+1} by $(v', s')(s', t)$.
  - Finally, we check if either (r, s') or (s', t) have labels of length 0. If so, we contract these offending edges, and remove them from the path ρ^{i+1}.

  Figure 1 shows one step in the inductive construction process described below. Cancelled edges are depicted darkened, while live edges appear light. The Euler path is apparent in the structure of the directed edges in the graph.

Since the inductive step of the previous constructive procedure assumed that properties $P1$-$P7$ were maintained at the prior stage, we need the following lemma to show that $\Gamma_{\bar{w}}(V, E)$ and $\rho_{\bar{w}}$ are well-defined.

Lemma 4.1. *Let G be a group with a regular free Lyndon length function, and take $\bar{w} = (w_i)_{i=0,\ldots,k-1} \in G^k$. Then for all $i \in \mathbb{N}$, the pair $(\Gamma^i_{\bar{w}}, \rho^i_{\bar{w}})$ satisfies properties $P1$-$P7$ listed above.*

Proof. It is easy to verify that $(\Gamma^0_{\bar{w}}, \rho^0_{\bar{w}})$ satisfies properties $P1$-$P7$ listed above.

Suppose that $(\Gamma^i_{\bar{w}}, \rho^i_{\bar{w}})$ satisfies properties $P1$-$P7$. If the process converges at stage i then $(\Gamma_{\bar{w}}, \rho_{\bar{w}}) = (\Gamma^i_{\bar{w}}, \rho^i_{\bar{w}})$, so the claim is trivial. If the process does not converge at stage i, then we verify the properties $P1$-$P7$ individually for $(\Gamma^{i+1}_{\bar{w}}, \rho^{i+1}_{\bar{w}})$.

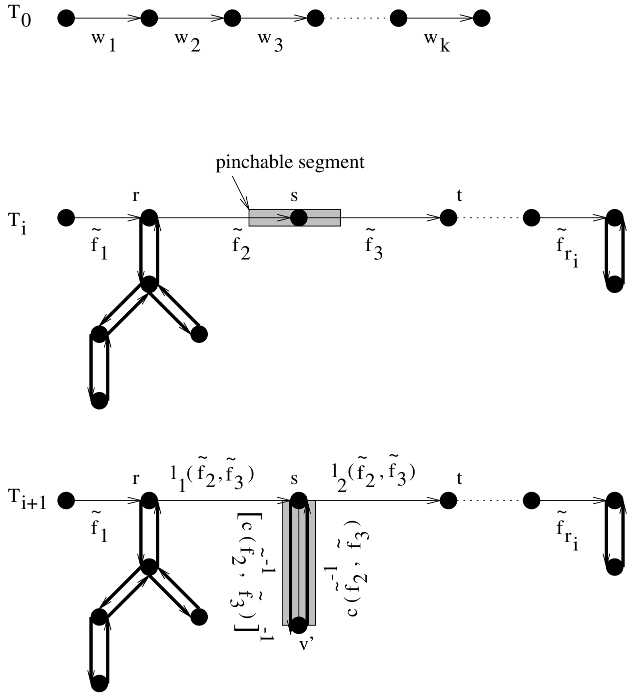

Fig. 1. One stage in the construction of $\Gamma_{\bar{w}}$ and $\rho_{\bar{w}}$.

Note that $\Gamma_{\bar{w}}^{i+1}$ is derived from $\Gamma_{\bar{w}}^{i}$ manipulating edges $f_j^i = (r,s)$ and $f_{j+1}^i = (s,t)$ to obtain (at most) four edges: (r,s'), (s,v'), (v',s'), (s',t).

P1. By inductive hypothesis, the edges $f_j^i = (r,s)$ and $f_{j+1}^i = (s,t)$ are each labelled by non-trivial elements $\tilde{e} \in G$. (r,s'), (s',v') [resp. (v',s'), (s',t)] are labelled by initial or terminal subsegments of $\tilde{f}_j^i$ [resp. $\tilde{f}_{j+1}^i$]. The fact that G has a regular free length function and the way we relabel edges on each stage guarantee that the labels of edges (r,s'), (s',v') [resp. (v',s'), (s',t)] are also elements of G. Since $c([\tilde{f}_j^i]^{-1}, \tilde{f}_{j+1}^i) > 0$, the labels of (s',v') and (v',s') are non-trivial A-words. Finally, if (r,s') or (s',t) were found to be labelled by trivial words during the construction of $\Gamma_{\bar{w}}^{i+1}$, these offending edge(s) were deleted. It follows that $\Gamma_{\bar{w}}^{i+1}$ satisfies property $P1$.

P2. In deriving $\Gamma_{\bar{w}}^{i+1}$, the only cancelled edges that were added were (s',v') and (v',s'), both of which were of cancelled type; no cancelled edges were deleted. Thus $C_{i+1} = \bar{C}_{i+1}$.

P3. In deriving $\Gamma_{\bar{w}}^{i+1}$, the only change to live edges that may have occurred is that (r,s') or (s',t) may have been deleted; no live edges were added. Thus $L_{i+1} \cap \bar{L}_{i+1} = \emptyset$.

P4. Appealing to properties $P4$ and $P5$ at stage i, we see that replacing edges f_j^i and f_{j+1}^i in $\rho_{\bar{w}}^i$ by the sequences $(r,s')(s',v')$ and $(v',s')(s',t)$ respectively, one gets an Euler path of $\Gamma_{\bar{w}}^{i+1}$.

P5. In deriving $\Gamma_{\bar{w}}^{i+1}$, the deletion of live edges occurred by contraction; no live edges were added. Since the live edges of L_i form a directed path in $\Gamma_{\bar{w}}^{i}$ (by inductive hypothesis), it follows that this property also holds in $\Gamma_{\bar{w}}^{i+1}$.

Using inductive hypothesis $P4$ we conclude that (r, s') appears before (s', t) in $\rho_{\bar{w}}^{i+1}$. Moreover since (s', v') and (v', s') are cancelled edges, all edges between (r, s') and (s', t) in $\rho_{\bar{w}}^{i+1}$ belong to C_{i+1}.

P6. By induction hypothesis the Euler path $\rho_{\bar{w}}^{i} = (e_0^i, e_1^i, \ldots e_{m_i}^i)$ at stage i connects the endpoints of ℓ_i, and satisfies the following equality:

$$\tilde{e}_1^i * \tilde{e}_2^i * \cdots * \tilde{e}_{m_i}^i = w_1 * w_2 * \cdots * w_k.$$

Replacing edges f_j^i and f_{j+1}^i in $\rho_{\bar{w}}^{i}$ by the sequences $(r, s')(s', v')$ and $(v', s')(s', t)$ respectively. It follows that $\tilde{e}_0^{i+1} \tilde{e}_1^{i+1} \cdots \tilde{e}_{m_{i+1}}^{i+1} = \tilde{e}_0^{i} \tilde{e}_1^{i} \cdots \tilde{e}_{m_i}^{i}$, and that claimed α_{i+1} can be constructed from α_i.

P7. Appealing to the inductive property $P7$, we see that splitting vertex s into s', v' produces two disjoint trees. Subsequent addition of edges (s', v') and (v', s') merges these again into a single tree.

The lemma is proved. □

Having defined the cancellation graph $\Gamma_{\bar{w}}(V, E)$ and Euler path $\rho_{\bar{w}}$ associated with a tuple $\bar{w} \in G^k$, we now show that constructive process by which these objects are defined converges in a finite number of stages.

Lemma 4.2. *Let G be a group with a regular free Lyndon length function, and take $\bar{w} = (w_i)_{i=0,\ldots,k-1} \in G^k$. Then the sequence*

$$(\Gamma_{\bar{w}}^0, \rho_{\bar{w}}^0) \rightsquigarrow (\Gamma_{\bar{w}}^1, \rho_{\bar{w}}^1) \rightsquigarrow (\Gamma_{\bar{w}}^2, \rho_{\bar{w}}^2) \rightsquigarrow \cdots$$

converges in at most $k-1$ stages. Furthermore, the number of vertices in $\Gamma_{\bar{w}}$ is bounded above by $2k$, and the number of edges is at most $3k-2$.

Proof. Consider $\bar{w} \in G^k$. If the previously described construction converges for $\bar{w}$ let us define $n(\bar{w})$ to be the stage at which convergence is achieved; otherwise, if the process does not converge we agree to put $n(\bar{w}) = +\infty$. Take

$$N(k) = \max_{\bar{w} \in G^k} n(\bar{w})$$

Clearly, $N(1) = 0$. Suppose that at stage i, consideration of the concatenation $\tilde{f}_1^i \tilde{f}_2^i \cdots \tilde{f}_{r_i}^i$ along ℓ leads us to make a pinch in $\tilde{f}_j^i \tilde{f}_{j+1}^i$. Then after performing the pinch, the label $l_1(\tilde{f}_j^i, \tilde{f}_{j+1}^i) l_2(\tilde{f}_j^i, \tilde{f}_{j+1}^i)$ of the subpath $(r, s), (s, t)$ represents a reduced A-word $z = l_1(\tilde{f}_j^i, \tilde{f}_{j+1}^i) \circ l_2(\tilde{f}_j^i, \tilde{f}_{j+1}^i)$. It follows that the number of stages required beyond stage i before convergence is achieved cannot be more than $n(\tilde{f}_1^i \tilde{f}_2^i \cdots \tilde{f}_{j-1}^i z \tilde{f}_{j+1}^i \cdots \tilde{f}_{r_i}^i)$, which is at most $N(i-1)$. Thus $N(i) \leqslant N(i-1) + 1$. It follows that $N(k) \leqslant k - 1$.

Since initially $\Gamma^0_{\bar{w}}$ has $k+1$ vertices and k edges. Each stage of the process introduces at most one new vertex and two new edges. It follows that the final number of vertices is at most $2k$ and the final number of edges is at most $3k-2$. The lemma is proved. $\square$

Lemma 4.3. *Let G be a group with a regular free Lyndon length function, and take $\bar{w} = (w_i)_{i=0,\dots,k-1} \in G^k$. The product $w_1 * w_2 * \cdots * w_k$ is trivial if and only if the Euler path $\rho_{\bar{w}}$ is actually an Euler tour of $\Gamma_{\bar{w}}$.*

Proof. When the process converges the concatenation of labels of edges along ℓ is not pinchable, i.e., it is the reduced A-word representing the product $w_1 * w_2 * \cdots * w_k$. Thus, if $w_1 * w_2 * \cdots * w_k = \varepsilon$, this implies that the concatenation of labels of edges along ℓ is ε. Since $\rho_{\bar{w}}$ is an Euler path in $\Gamma_{\bar{w}}$ which connects the endpoints of ℓ, it follows that $\rho_{\bar{w}}$ is actually closed; in other words, $\rho_{\bar{w}}$ is an Euler tour of $\Gamma_{\bar{w}}$. The lemma is proved. $\square$

5. Equations

Recall that the category of G-groups is the class of all groups which contain a distinguished copy of G as a subgroup. Let $X = \{x_1, \dots, x_n\}$ and $G[X]$ be a free G-group with basis X. A *system of equations* $S(X) = 1$ over G is a subset of $G[X]$. As an element of the free product the left side of every equation in $S(X) = 1$ can be written as a product of elements from $X \cup X^{-1}$ called *variables* and elements from G called *constants*. A *solution* of the system $S(X) = 1$ over a group G is a tuple of elements $g_1, \dots, g_n \in G$ such that after replacement of each x_i by g_i the left hand side of every equation in $S(X) = 1$ is the trivial element of G.

Remark 5.1. Let G be a group with a regular free Lyndon length function. By Remark 3.1 we assume it to be a subgroup of $CDR(A, X)$ for some fixed A and X.Here we slightly abuse notation denoting by X the set of variables which appear in equations over G, while X was already reserved for the purpose of embedding of G in $CDR(A, X)$. On the other hand it is a common habit to denote the set of variables by X and it will be always clear from the context below what meaning is attributed to X.

Let $w(X) = 1, w(X) \in G[X]$ be a single equation. Clearly w decomposes as

$$w(X) : g_0\ x_{i_0}^{\epsilon_0}\ g_1 \cdots x_{i_{n-1}}^{\epsilon_{n-1}}\ g_n = 1,$$

where $g_j \in G$ for $(j = 0, \dots, n)$ and $x_{i_k} \in X^{\pm 1}$ for $(k = 0, \dots, n)$.

The *relaxed form* of $w(X)$ is the following equation in variables $X \cup Y$, obtained by treating the constants $g_0, \dots, g_n$ as new variables $y_0, \dots, y_n$:

$$\hat{w}(X, Y) : y_0\ x_{i_0}^{\epsilon_0}\ y_1 \cdots x_{i_{n-1}}^{\epsilon_{n-1}}\ y_n = 1.$$

Definition 5.1. A *cancellation scheme* for equation $w(X) = 1$ is a triple (Γ, ρ, α) where

- $\Gamma = (V, E)$ is a directed graph, enjoying the following properties:
  (i) $n \leqslant |V| \leqslant 4n + 2$,
  (ii) $E = \bar{E}$,
  (iii) considered as an undirected graph Γ is a tree.
- $\rho = (e_0, e_1, \ldots, e_m)$ is an Euler path in Γ.
- $\alpha : \{0, 1, \ldots, 2n+2\} \to \{0, 1, \ldots, m+1\}$ is a non-decreasing function satisfying $\alpha(0) = 0$, $\alpha(2n + 2) = m + 1$.

The following lemma is important for our considerations.

Lemma 5.1. *For any equation $w(X) = 1, w(X) \in G[X]$, the number of distinct cancellation schemes is finite.*

Proof. The number of graphs on $\leqslant 4n + 2$ vertices is finite. For one of these graphs Γ, the number of Euler paths ρ in Γ is finite. Considering one of these paths ρ (say of length m), there are only a finite number of distinct functions α between $\{0, 1, \ldots, 2n + 2\}$ and $\{0, 1, \ldots, m + 1\}$. $\square$

Definition 5.2. Let (Γ, ρ, α) be a cancellation scheme for equation $w(X) = 1$. We introduce a set of variables $Z = \{z_e \mid e \in E[\Gamma]\}$; to each edge $e \in E[\Gamma]$ we associate a distinct variable z_e. The associated *inverse edge equations* are defined to be

$$S_{INV}(Z) = \{z_e^{-1} = z_{\bar{e}} \mid e \in E[\Gamma]\}.$$

The associated *constant segment equations* are taken as

$$S_{CONST}(Z) = \{g_i = z_{e_{\alpha(2i)}} z_{e_{\alpha(2i)+1}} \cdots z_{e_{\alpha(2i+1)-1}} \mid i = 0, \ldots, n\}.$$

With the jth variable occurrence in w ($j = 0, \ldots, n-1$) we associate the Z-word

$$P_j(Z) = z_{e_{\alpha(2j+1)}} z_{e_{\alpha(2j+1)+1}} \cdots z_{e_{\alpha(2j+2)-1}}.$$

Finally, the associated *variable segment equations* are defined to be

$$S_{VAR}(Z) = \{P_j^{\epsilon_j}(Z) = P_k^{\epsilon_k}(Z) \mid i_j = i_k, 0 \leqslant j \neq k \leqslant n-1\}.$$

Given a cancellation scheme (Γ, ρ, α) for the equation $w(X) = 1$ we define the *associated system of equations* as

$$S_{w,(\Gamma,\rho,\alpha)}(Z) = S_{INV}(Z) \cup S_{CONST}(Z) \cup S_{VAR}(Z).$$

Each cancellation scheme (Γ, ρ, α) for an equation $w(X) = 1$ gives rise to an associated system of equations according to the previous definition.

Note that solutions of $w(X) = 1$ are precisely those solutions of $\hat{w}(X, Y) = 1$ in G for which $y_j = g_j$ for $j = 0, \ldots, n+1$.

Suppose $\bar{h} \in G^n$ is a solution of $w(X)$. By Lemma 4.2, the cancellation diagram T for the word $w(\bar{h})$ is a directed graph of at most $3|w| - 2 = 6n + 1$ edges and at most $2|w| = 4n+2$ vertices. Since $\bar{h}$ is a solution of w, it follows that $w(\bar{h}) = 1$ in G and so by Lemma 4.3 the Euler path ρ associated with T is a closed Euler tour.

Lemma 5.2. *Let G be a group with a regular free Lyndon length function. Let $w(X,g) = 1$ be an equation over G, where g is a tuple of constants. Let $ASE(w) = \{\Omega_1(Z_1), \ldots, \Omega_r(Z_r)\}$ be the set of associated systems of equations for all cancellation schemes of $w(X,g) = 1$. Then*

(1) For any solution $h \in G^n$ of the equation $w(X,g) = 1$ there exists $\Omega_i(Z_i)$ and a solution U of $\Omega_i(Z_i)$ such that $h = P(U)$, where $P = (P_1, \ldots, P_n)$, and this equality is graphical;
(2) If a system $\Omega_i(Z_i) \in ASE(w)$ has a solution U in any G-group H then $P(U)$ is a solution of $w(X,g) = 1$ in H.

Proof. Follows from the definition of a system associated with a cancellation scheme, from Lemma 5.1 and discussion above. $\square$

Now, suppose we have a finite system of equations $S(X) = 1, S(X) \in G[X]$, that is, $S(X) = \{w_1(X), \ldots, w_m(X)\},\ m \in \mathbb{N}$. Suppose

$$ASE(w_i) = \{\Omega_1^i(Z_1^i), \ldots, \Omega_{r_i}^i(Z_{r_i}^i)\},\ i \in [1,m]$$

is the set of associated systems of equations for all cancellation schemes of $w_i(X) = 1$. For each $i \in [1,m]$ take $\Omega_{j_i}^i(Z_{j_i}^i) \in ASE(w_i)$ and consider the union

$$U(j_1, \ldots, j_m) = \{\Omega_{j_1}^1(Z_{j_1}^1), \ldots, \Omega_{j_m}^m(Z_{j_m}^m)\}.$$

Observe that in this union the sets of variables $Z_{j_i}^i,\ i \in [1,m]$ are disjoint but every original variable $x \in X^{\pm 1}$ which appears in $S(X)$ is expressed in terms of each set $Z_{j_i}^i$ in finitely many ways and all these expressions give rise to the system $S_{VAR}(Z_{j_i}^i) \subseteq \Omega_{j_i}^i(Z_{j_i}^i)$. Hence, to bind all $\Omega_{j_i}^i(Z_{j_i}^i)$ together in the union one has to add finitely many equations in which both sides are expressions of the same variable $x \in X^{\pm 1}$ in terms of different sets $Z_{j_i}^i,\ i \in [1,m]$, that is, form the system

$$S_{CONNECT}(j_1, \ldots, j_m) = \{P_{k,x}^{\epsilon_k}(Z_{j_k}^k) = P_{p,x}^{\epsilon_p}(Z_{j_p}^p) \mid x \in X,\ 1 \leqslant k \neq p \leqslant m\},$$

where $P_{k,x}^{\epsilon_k}(Z_{j_k}^k)$ is a left(right) hand side of an equation in $S_{VAR}(Z_{j_k}^k) \subseteq \Omega_{j_k}^k(Z_{j_k}^k)$ and $P_{p,x}^{\epsilon_p}(Z_{j_p}^p)$ is a left(right) hand side of an equation in $S_{VAR}(Z_{j_p}^p) \subseteq \Omega_{j_p}^p(Z_{j_p}^p)$. Hence,

$$\Omega_{j_1,\ldots,j_m}(Z_{j_1}^1, \ldots, Z_{j_m}^m) = \{\Omega_{j_1}^1(Z_{j_1}^1), \ldots, \Omega_{j_m}^m(Z_{j_m}^m)\} \cup S_{CONNECT}(j_1, \ldots, j_m)$$

is the system of equations in $Z_{j_1}^1, \ldots, Z_{j_m}^m$ which is associated with a combination of cancellation schemes corresponding to $\Omega_{j_i}^i(Z_{j_i}^i)$ for each $w_i(X) = 1,\ i \in [1,m]$. Finally, we define the set of associated systems of equations $ASE(S)$ for $S(X)$ as the set of all possible $\Omega_{j_1,\ldots,j_m}(Z_{j_1}^1, \ldots, Z_{j_m}^m)$, that is,

$$ASE(S) = \{\Omega_{j_1,\ldots,j_m}(Z_{j_1}^1, \ldots, Z_{j_m}^m) \mid j_k \in [1, r_k],\ k \in [1,m]\}.$$

The following result which is similar to Lemma 5.2 follows automatically.

Lemma 5.3. *Let G be a group with a regular free Lyndon length function and let $S(X) = 1$ be a finite system of equations over G, where $S(X) = \{w_1(X), \dots, w_m(X)\}$, $m \in \mathbb{N}$. Let $ASE(S)$ be the set of associated systems of equations for all combinations of cancellation schemes of all $w_i(X) = 1$, $i \in [1, m]$. Then*

(1) For any solution $h \in G^n$ of the system $S(X) = 1$ there exists $\Omega_{j_1,\dots,j_m}(Z^1_{j_1}, \dots, Z^m_{j_m}) \in ASE(S)$ and a solution U of $\Omega_{j_1,\dots,j_m}(Z^1_{j_1}, \dots, Z^m_{j_m})$ such that $h = P(U)$, where $P = (P_1, \dots, P_n)$, and this equality is graphical;
(2) If a system $\Omega_{j_1,\dots,j_m}(Z^1_{j_1}, \dots, Z^m_{j_m}) \in ASE(S)$ has a solution U in any G-group H then $P(U)$ is a solution of $S(X) = 1$ in H.

6. Main results

In this section we present proofs of Theorem A, Theorem B, and Theorem D, which were stated in the introduction. In particular, Theorem A extends Merzlykov's Theorem on elimination of quantifiers for positive sentences to the class of groups with free regular Lyndon length functions.

Let G be a group and let ψ be a positive sentence in the language $\mathcal{L}_G$. Then ψ is equivalent to the sentence

$$\forall x_1 \exists y_1 \dots \forall x_k \exists y_k \;\; \psi_0(x_1, y_1, \dots, x_k, y_k)$$

written in standard form in which ψ_0 is a disjunction of conjunctions of equations in $X \cup Y$ with constants from G, where $X = \{x_1, \dots, x_k\}, Y = \{y_1, \dots, y_k\}$. It follows that ψ is equivalent to the sentence

$$\forall x_1 \exists y_1 \dots \forall x_k \exists y_k \;\; (S_1(X, Y) = 1 \vee \dots \vee S_n(X, Y) = 1)$$

for some natural n, where each $S_i(X, Y) = 1$, $i \in [1, n]$ is a system of equations in $X \cup Y$ with constants from G.

Theorem *A. Every finitely generated non-abelian group with a regular free Lyndon length function freely lifts its positive theory.*

Proof. By Remark 3.1 we assume G to be a subgroup of $CDR(A, X)$ for some fixed A and X.

Let $\phi \in Th^+(G)$. Then ϕ is equivalent to a formula of the type

$$\psi : \forall x_1 \exists y_1 \dots \forall x_k \exists y_k \;\; \psi_0(x_1, y_1, \dots, x_k, y_k),$$

where ψ_0 is

$$S_1(x_1, y_1, \dots, x_k, y_k) = 1 \vee \dots \vee S_n(x_1, y_1, \dots, x_k, y_k) = 1$$

for some natural n and each $S_i(X,Y) = 1,\ i \in [1,n]$ is a system of equations in $X \cup Y$ with constants from G. So, let $G \models \psi$.

Let $ASE(S_i) = \{\Omega_1^i(Z_1^i), \ldots, \Omega_{r_i}^i(Z_{r_i}^i)\},\ i \in [1,n]$ be the set of associated systems of equations for all combinations of cancellation schemes of equations from $S_i(X,Y) = 1$ (since each S_i contains only finitely many equations then from Lemma 5.1 it follows that $ASE(S_i)$ is finite). Denote $\rho_j^i = |Z_j^i|$ the number of variables in $\Omega_j^i(Z_j^i)$ and let $\rho = \max_{i=1}^n\{\rho_1^i, \ldots, \rho_{r_i}^i\}$. Also, if $S_i(X,Y) = \{w_{i,1}(X,Y), \ldots, w_{i,k_i}(X,Y)\}$ then let $L = \max_{i=1}^n\{|w_{i,1}(X,Y)|, \ldots, |w_{i,k_i}(X,Y)|\}$ the maximal syllable length of equations $w_{i,j}(X,Y),\ i \in [1,n],\ j \in [1,k_i]$ as an elements of a free product.

By Corollary 3.1 there exist $a, b \in G$ such that

1) $[a,b] \neq \varepsilon$,
2) $b * a = b \circ a,\ a * b = a \circ b$,
3) for any $g \in G$ there exists $n_g \in \mathbb{N}$ such that g does not contain $b \circ a^k \circ b$ as a subword for any $k \geq n_g$.

Take $g_1 \in G$ such that

$$g_1 = b \circ a^{m_{11}} \circ b \circ a^{m_{12}} \circ b \cdots a^{m_{1n_1}} \circ b,$$

where $m_{11} < m_{12} < \cdots < m_{1n_1}$ and $\rho L < n_1$. Then since $G \models \psi$ there exists $h_1 \in G$ such that

$$G \models \forall x_2 \exists y_2 \cdots \forall x_k \exists y_k\ (S_1(g_1, h_1, x_2, y_2, \ldots, x_k, y_k) = 1 \vee \cdots$$

$$\vee S_n(g_1, h_1, x_2, y_2, \ldots, x_k, y_k) = 1).$$

Suppose now that elements $g_1, h_1, \ldots, g_{i-1}, h_{i-1} \in G$ are given. We find $g_i \in G$ such that

$$g_i = b \circ a^{m_{i1}} \circ b \circ a^{m_{i2}} \circ b \cdots a^{m_{in_i}} \circ b,$$

where $m_{i1} < m_{i2} < \ldots < m_{in_i}$, $\rho L < n_i$ and no subword of the type $b \circ a^{m_{ij}} \circ b$ occurs in any of $g_l, h_l, l < i$. Observe that Corollary 3.1 ensures that such g_i exists.

Then there exists an element $h_i \in G$ such that

$$G \models \forall x_{i+1} \exists y_{i+1} \cdots \forall x_k \exists y_k\ (S_1(g_1, h_1, \ldots, g_i, h_i, x_{i+1}, y_{i+1}, \ldots, x_k, y_k) = 1 \vee \cdots$$

$$\vee S_n(g_1, h_1, \ldots, g_i, h_i, x_{i+1}, y_{i+1}, \ldots, x_k, y_k) = 1).$$

Hence, we have obtained inductively elements $g_1, h_1, \ldots, g_k, h_k \in G$ and there exists at least one index $i_0 \in [1,n]$ such that

$$S_{i_0}(g_1, h_1, \ldots, g_k, h_k) = 1$$

in G. For simplicity we denote $S(X,Y) = S_{i_0}(X,Y)$.

By Lemma 5.3 there exists $\Omega_m(Z_m) \in ASE(S)$, words $P_i(Z_m), Q_i(Z_m) \in F[Z_m],\ i \in [1,k]$ of length not more than $|Z_m| \leq \rho$, and a solution $U = (u_1, \ldots, u_p)$ of $\Omega_m(Z_m)$ in G such that the following equalities hold

$$g_i = P_i(U),\ h_i = Q_i(U),\ i \in [1,k].$$

Since $n_i > \rho L$ and $P_i(U) = y_1 \cdots y_q$ with $y_j \in U^{\pm 1},\ q \leqslant \rho$ the graphical equalities

$$g_i = b \circ a^{m_{i1}} \circ b \circ \cdots \circ a^{m_{in_i}} \circ b = P_i(U),\ i \in [1,k] \tag{6.1}$$

show that there exists a subword $v_i = b \circ a^{m_{ij}} \circ b$ of g_i such that every occurrence of this subword in (6.1) is an occurrence inside some $u_j^{\pm 1}$. For each i fix such a subword $v_i = b \circ a^{m_{ij}} \circ b$ in g_i. Observe that by the choice of $g_i,\ i \in [1,k]$ above, v_i does not occur in any of the words $g_j (j \neq i), h_s (s < i)$, moreover, in g_i it occurs precisely once. Denote by $j(i)$ the unique index such that v_i occurs inside $u_{j(i)}^{\pm 1}$ in $P_i(U)$ from (6.1) (and v_i occurs in it precisely once). From this argument it follows that the variable $z_{j(i)}$ in the system $\Omega_m(Z_m)$ does not occur in expressions of $x_t (t \neq i),\ y_s (s < i)$ in terms of Z_m.

We "mark" the unique occurrence of v_i (as $v_i^{\pm 1}$) in $u_{j(i)},\ i \in [1,k]$. Now we are going to mark some other occurrences of v_i in $u_1, \ldots, u_p$ as follows. Suppose some u_d has a marked occurrence of some v_i.

If Ω_m contains an equation of the type $z_d^{\epsilon} = z_r^{\delta}$ then $u_d^{\epsilon} = u_r^{\delta}$. Hence u_r has an occurrence of subword $v_i^{\pm 1}$, which correspond to the marked occurrence of $v_i^{\pm 1}$ in u_d. We mark this occurrence of $v_i^{\pm 1}$ in u_r.

Suppose Ω_m contains an equation of the type

$$[z_{\alpha_1} \cdots z_{\beta_1 - 1}]^{\epsilon_1} = [z_{\alpha_2} \ldots z_{\beta_2 - 1}]^{\epsilon_2},$$

such that z_d occurs in it, say in the left. Then

$$[u_{\alpha_1} \ldots u_{\beta_1 - 1}]^{\epsilon_1} = [u_{\alpha_2} \ldots u_{\beta_2 - 1}]^{\epsilon_2}.$$

Since $v_i^{\pm 1}$ is a subword of u_d it occurs also in the right of the equality above, say in some u_r. We mark this occurrence of $v_i^{\pm 1}$ in u_r. The marking process will terminate in finitely many steps. Observe, that one and the same u_r can have several marked occurrences of some $v_i^{\pm 1}$.

Now in all words $u_1, \ldots, u_p$ we replace every marked occurrence of $v_i = b \circ a^{m_{ij}} \circ b$ with a new word $b \circ a^{m_{ij}} \circ x_i \circ b$ from the group $G[X]$. Observe that for any $x_i \in G$ we have $b \circ a^{m_{ij}} \circ x_i \circ b \in G$. Denote the resulting words from $G[X]$ by $\bar{u}_1, \ldots, \bar{u}_p$. It follows from the description of the marking process that the tuple $\bar{U} = (\bar{u}_1, \ldots, \bar{u}_p)$ is a solution of the system Ω_m in the group $G[X]$. Indeed, all the equations in Ω_m are graphically satisfied by the substitution $z_i \to u_i$ hence the substitution $u_i \to \bar{u}_i$ makes them graphically equal. Now by Lemma 5.3, $X = P(\bar{U}), Y = Q(\bar{U})$ is a solution of the system $S(X,Y) = 1$ over $G[X]$. Hence, $Y = Q(\bar{U})$ gives rise to functions $q_i(x_1, \ldots, x_i),\ i \in [1,k]$ such that

$$G[X] \models (S(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k)) = 1)$$

and eventually

$$G[X] \models \psi_0(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k))$$

as desired. $\square$

In the proof of Theorem D we use the following well-known result.

Proposition 6.1. *[10] Let $\theta : G \longrightarrow G^\theta$ be a group homomorphism. Let $\phi = \phi(X, G)$ be any positive formula with free variables X and constants from G. Then $\forall \bar{g} \in G^{|X|}$,*

$$G \models \phi(\bar{g}, G) \Leftrightarrow G^\theta \models \phi(\bar{g}^\theta, G^\theta)$$

Theorem D. *Let H be a group freely lifts a subset $\Phi \subset Th^+(H)$. Suppose G is a group containing H as a subgroup, and possessing a retraction $\phi : G \longrightarrow H$ onto H, $\phi|_H = id_H$. Then*

$$\Phi \subset Th^+(G)$$

and G freely lifts Φ.

Proof. Let ψ be a sentence in Φ. $G \models \psi \Longrightarrow H \models \psi$ follows from Proposition 6.1.

We prove $H \models \psi \Longrightarrow G \models \psi$. Let

$$\psi = \forall x_1 \exists y_1 \ldots \forall x_k \exists y_k \ \ \psi_0(x_1, y_1, \ldots, x_k, y_k),$$

where ψ_0 is quantifier-free. Since H freely lifts ψ there exist $q_1(x_1)$, $q_2(x_1, x_2), \ldots,$ $q_k(x_1, \ldots, x_k) \in H[X]$ such that

$$H[X] \models \psi_0(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k)).$$

Since ψ_0 is equivalent to the disjunction

$$S_1(X, Y) = 1 \vee \cdots \vee S_n(X, Y) = 1$$

for some natural n and each $S_j(X, Y) = 1$, $j \in [1, n]$ is a system of equations in $X \cup Y$ with constants from G then there exists $j_0 \in [1, n]$ such that

$$H[X] \models (S_{j_0}(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k)) = 1).$$

Now, let $h_1, \ldots, h_k \in G$. Then $\lambda : x_i \longrightarrow h_i$, $i \in [1, k]$ is a homomorphism from $H[X]$ to G, so

$$S_{j_0}(x_1, q_1(x_1), \ldots, x_k, q_k(x_1, \ldots, x_k))^\lambda = S_{j_0}(x_1^\lambda, q_1(x_1^\lambda), \ldots, x_k^\lambda, q_k(x_1^\lambda, \ldots, x_k^\lambda)) =$$

$$= S_{j_0}(h_1, q_1(h_1), \ldots, h_k, q_k(h_1, \ldots, h_k)) = 1.$$

Now we have for all $h_1, \ldots, h_k \in G$,

$$S_{j_0}(h_1, q_1(h_1), \ldots, h_k, q_k(h_1, \ldots, h_k)) = 1.$$

Since q_i in $H[X]$, and G contains H as a subgroup, it follows that $q_i(h_1, \dots, h_i) \in G,\ i \in [1, k]$. Thus

$$G \models \forall x_1 \exists y_1 \dots \forall x_k \exists y_k \ \ S_{j_0}(x_1, y_1, \dots, x_k, y_k) = 1,$$

so,

$$G \models \forall x_1 \exists y_1 \dots \forall x_k \exists y_k \ \ \psi_0(x_1, y_1, \dots, x_k, y_k).$$

Hence $\Phi \subseteq Th^+(G)$.

Finally, since $H[X] \leqslant G[X]$, it follows that G freely lifts Φ. $\square$

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