MAT322β : Complex Variables.

Tutorial No: 02

(01)

a) i) an open set:

A set S is called open if every point of S has a neighborhood which consisting entirely in S.

ii) a connected set:

A set S is called connected if any two of its points can be joined by a chain of finitely many straight-line segments all of whose points belong to S.

iii) a domain:

An open and connected set is called a domain.

iv) A complex function:

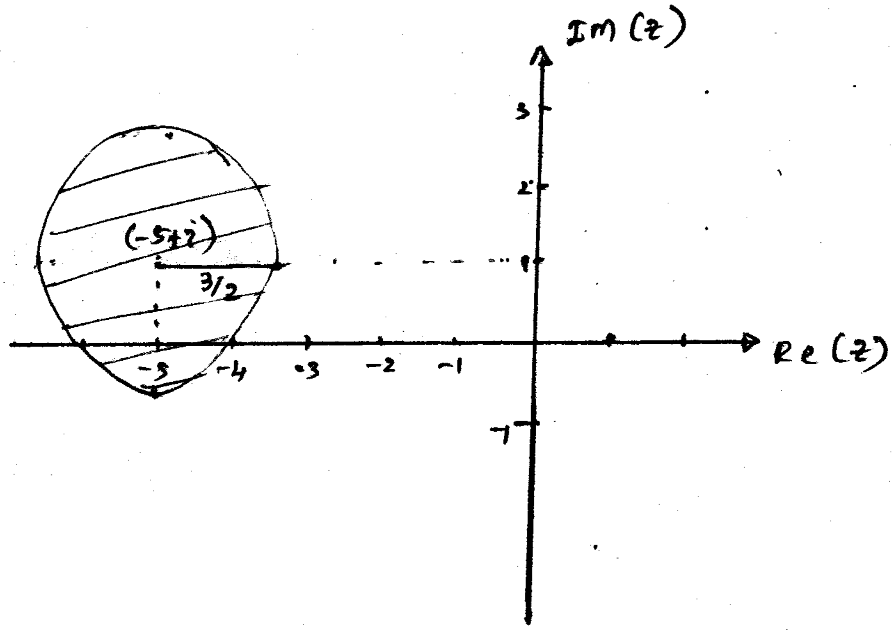
A function map from complex number to complex number is called a complex function. Further more the complex variable is denoted by $z = x + iy$, $f(z)$ is a function of a complex variable and is denoted by ω, $\omega = f(z)$, $\omega = u + iv$, where u & v are the real and the imaginary parts of $f(z)$.

b) i) $|z - i + 5| \leq 3/2$

$|z - (i - 5)| \leq 3/2$

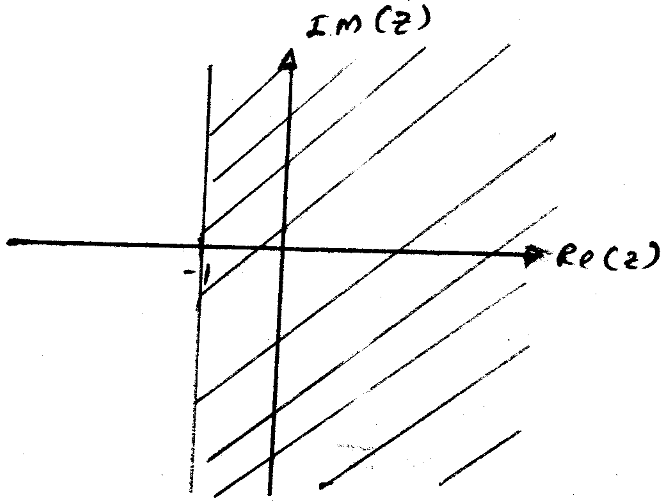
center : $-5 + i$

radius : $3/2$



(ii) $Re(z) \geq -1$

$x \geq -1$



(iii) $|z+i| \geq |z-i|$

Let $z = x+iy$

$|x+i(1+y)| \geq |x+i(y-1)|$

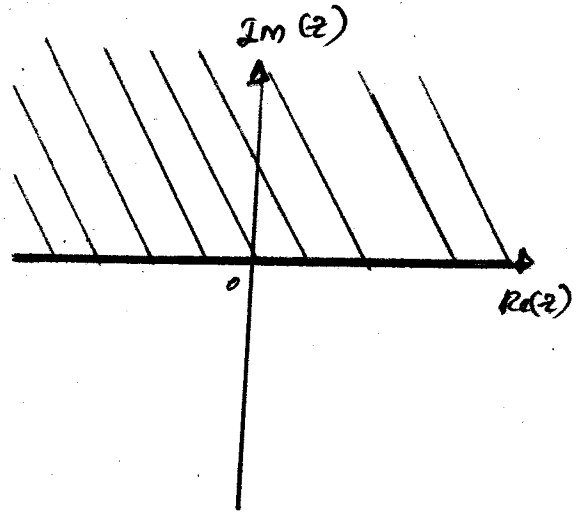
$x^2+(1+y)^2 \geq x^2+(y-1)^2$

$x^2+y^2+2y+1 \geq x^2+y^2-2y+1$

$2y+1 \geq -2y+1$

$4y \geq 0$

$y \geq 0$

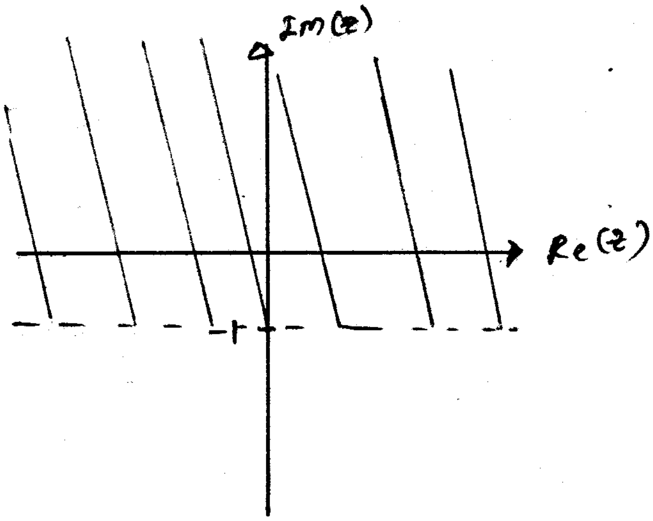


(iv) $Im(z) > -1$

let $z = x+iy$

$Im(z) = y$

$y > -1$



c) $w = f(z) = u + iv$.

$f(z) = 2iz + 8\bar{z}$

let $z = x + iy$, then $\bar{z} = x - iy$

$f(z) = 2i(x+iy) + 8(x - iy)$

$= 2xi - 2y + 8x - 8yi$

$= (8x - 2y) + i(2x - 8y)$

Here $u(x,y) = 8x - 2y$ and $v(x,y) = 2x - 8y$

- At $\frac{1}{2} + 3i$, $f(z)$

$f(z) = (8x - 2y) + i(2x - 8y)$

$f(\frac{1}{2} + 3i) = \left(8 \times \frac{1}{2} - 2(3)\right) + i\left(2 \cdot \frac{1}{2} - 8(3)\right)$

$= -2 - 23i$

d) $f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$

$f(z) = \bar{z} = x - iy$

$z = x + iy$

$\Delta z = \Delta x + i \Delta y$

$$f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + iy + \Delta x + i\Delta y) - f(x+iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{x + \Delta x - i(y + \Delta y) - (x - iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

considering the limit $\lim_{\Delta x \to 0}$, we have that

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \frac{0 - i\Delta y}{0 + i\Delta y}$$

$$f'(z) = -1$$

Again considering the limit, $\lim_{\Delta y \to 0}$, we have that

$$f'(z) = \lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \frac{\Delta x + i(0)}{\Delta x + i(0)}$$

$$f'(z) = 1$$

we have two different limits, which means that the limit of $f'(z)$ is not unique. That is, $f(z) = \bar{z}$ is not differentiable.

(e) The Cauchy-Riemann equations are defined by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Here $f(z) = u(x,y) + i\,v(x,y)$ where $z = x + iy$ and $u(x,y)$ and $v(x,y)$ are real valued functions.

$$f(z) = z^2$$

let $z = x + iy$

$$f(z) = (x+iy)^2$$

$$= x^2 + 2xyi + (iy)^2$$

$$= x^2 - y^2 + i(2xy)$$

Here $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

$$\frac{\partial u}{\partial x} = 2x \quad (1) \qquad -\frac{\partial v}{\partial x} = -2y \quad (3)$$

$$\frac{\partial u}{\partial y} = -2y \quad (2) \qquad \frac{\partial v}{\partial y} = 2x \quad (4)$$

Then

① = ④ $\quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ② = ③ $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f(z)$ satisfies the cauchy-Riemann equations for all x, y.
Therefore, $f(z)$ is an entire function. //

(f)

(i) $f(z) = x^2 + iy^2$

Here $u(x,y) = x^2$ and $v(x,y) = y^2$.

$\frac{\partial u}{\partial x} = 2x$ $\quad -\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\quad \frac{\partial v}{\partial y} = 2y$

The cauchy Riemann equations are satisfied when $x = y$.
Therefore the function is differentiable when the point set is $\{x+iy \in \mathbb{Z}, x = y\}$ //

(ii) $f(z) = 2xy + i(y+x)^2$.

Here $u(x,y) = 2xy$ and $v(x,y) = (y+x)^2$

$\frac{\partial u}{\partial x} = 2y$ $\quad -\frac{\partial v}{\partial x} = -2(y+x)$

$\frac{\partial u}{\partial y} = 2x$ $\quad \frac{\partial v}{\partial y} = 2(y+x)$

~~The function is to be differentiable~~
The cauchy Riemann equations should be satisfied to the function is to be differentiable.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.
$2y = 2(y+x)$
$\Rightarrow x = 0$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
$2x = 2(y+x)$
$\Rightarrow y = 0$

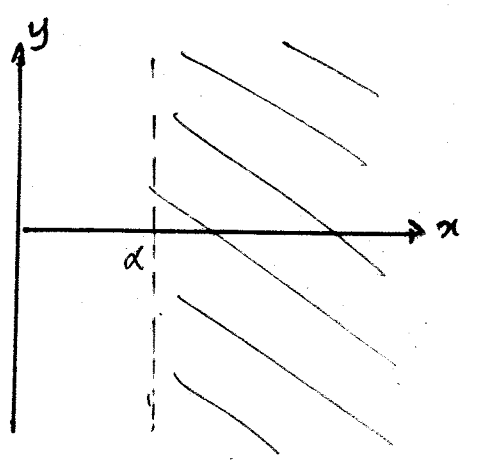
Therefore the function is ~~to be~~ differentiable when the point set is $\{x+iy \in \mathbb{Z}, x = 0 \text{ and } y = 0\}$.
i.e. The fun^n is differentiable only at $z = 0$.

02.

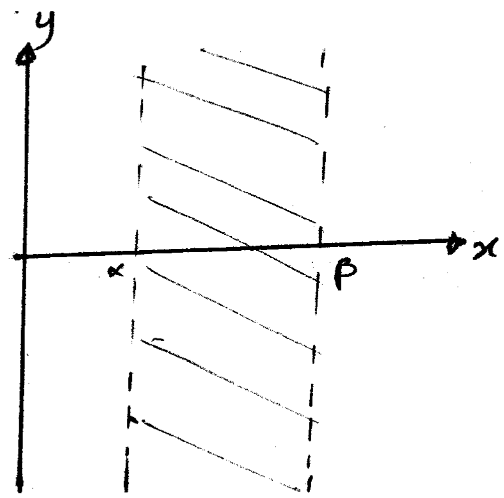
(a) $A = \{ z = x + iy \in \mathbb{C},\ x, y \in \mathbb{R} \text{ and } x > \alpha \}$

$B = \{ z = x + iy \in \mathbb{C},\ x, y \in \mathbb{R} \text{ and } \alpha < x < \beta \}$

(i)



set A



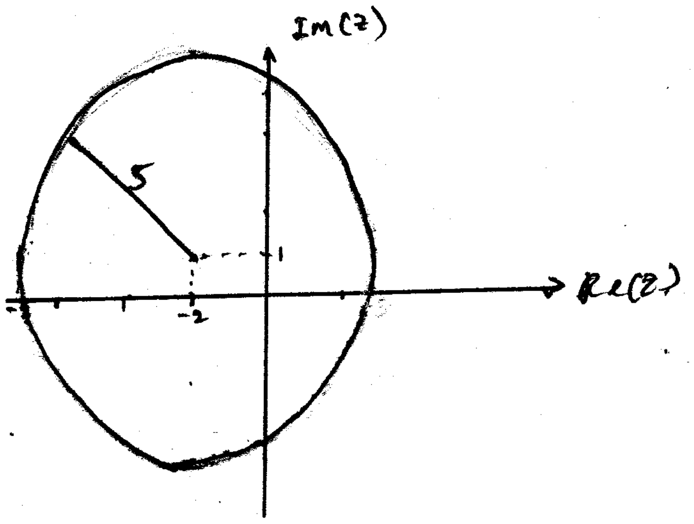
set B

(ii) a) Set A and B are open.
(every point in A is an interior point of A).

b) Set A and B are connected.
(every two points $z_1, z_2 \in A$ can be joined by the union of a finite number of line segments lying in A.)

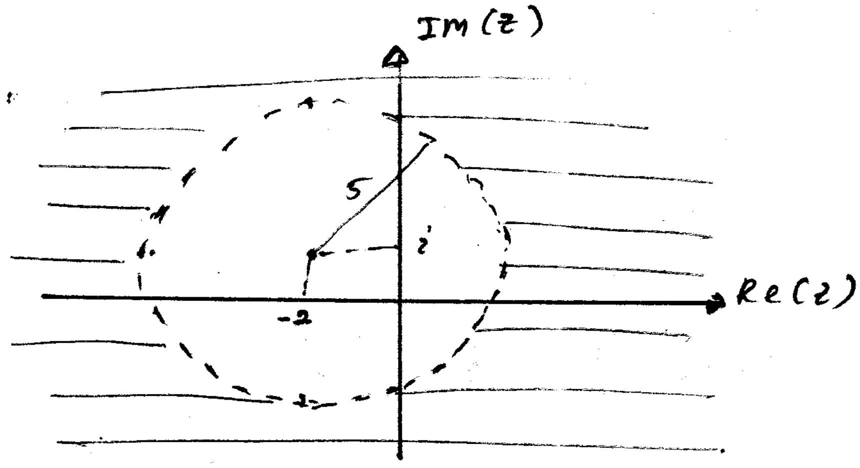
c) Set A and B are domains.
(An open connected set is called a domain).

(b) (i) $|z - i + 2| = 5$ is a circle of radius 5 centered at $i - 2$.

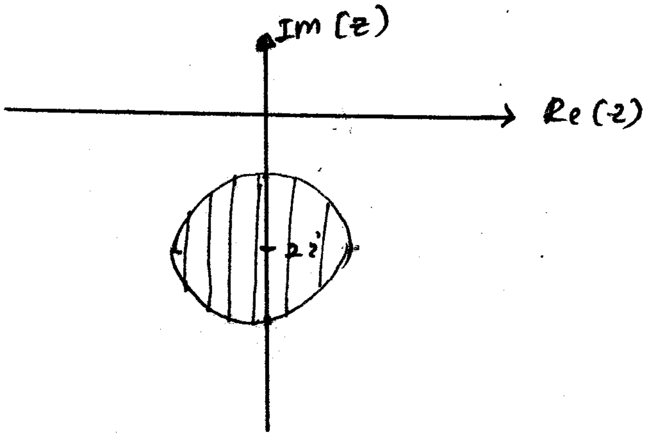


(ii) $|z - i + 2| > 5$

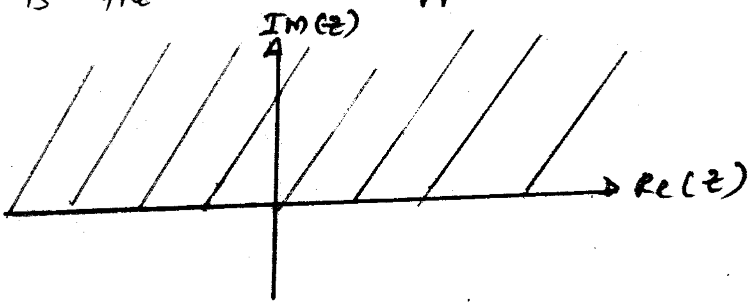
complement of the closed circle of radius 5 and centered at $i-2$.



(iii) $|z + 2i| \leq 1$ is a closed circle with radius and centered at $-2i$.



(iv) $Im(z) \geqslant 0$. is the closed upper half plane.



(c) Any complex number a can be written as

$a = re^{i\theta}$ with $0 \le r = |a| < 1$

$$|a^n| = |r^n e^{in\theta}| = r^n$$

Since $\lim_{n\to\infty} r^n = 0 \quad (\because r < 1)$

It implies that $\lim_{n\to\infty} |a^n| = 0.$

Therefore $\lim_{n\to\infty} a^n = 0.$ //

If $|a| > 1$

we write $a = re^{i\theta}$ with $|a| = r$

$$|a^n| = r^n$$

$|a^n| \longrightarrow \infty$ as $n \longrightarrow \infty$. so the limit

$\lim_{n\to\infty} a^n$ does not exist. //

d) Let $z = re^{i\theta}$.

It implies that $z \to 0 \to r \to 0$.

we have

$$\lim_{z\to 0} \frac{Re(z^2)}{|z|^2} = \lim_{r\to 0} \frac{r^2\cos 2\theta}{r^2} = \cos 2\theta.$$

$\cancel{Re}(z^2 = (re^{i\theta})^2$

$= r^2 e^{i2\theta}$

$= r^2(\cos 2\theta + i\sin 2\theta)$

$= r^2\cos 2\theta + ir^2\sin 2\theta.$

$|z|^2 = (\sqrt{x^2+y^2})^2$

$= x^2+y^2$

$= r^2$

which depends on θ.
since the limit does not exist at $z = 0$.
Then function is not continuous at $z = 0$.

03

(a) Let $f(z) = u(r,\theta) + iv(r,\theta)$, $z = re^{i\theta}$

we have

$x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$

using the chain rule of differentiation, we get

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= (-r\sin\theta)\frac{\partial u}{\partial \theta} + (r\cos\theta)\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial v}{\partial x} + \sin\theta\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial v}{\partial x} + r\cos\theta\frac{\partial v}{\partial y}$$

using the cauchy-Riemann equations in cartesian coordinates $u_x = v_y$ and $u_y = -v_x$ we can write.

$$\frac{\partial v}{\partial r} = -\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial u}{\partial x}$$

$$= -\frac{1}{r}\left[-r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}\right]$$

$$= -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$$\frac{\partial v}{\partial \theta} = -r\sin\theta\left(-\frac{\partial u}{\partial y}\right) + r\cos\theta\frac{\partial u}{\partial x}$$

$$= r\left[\cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}\right]$$

$$= r\frac{\partial u}{\partial r}$$

Therefore the cauchy-Riemann equations in polar fo[illegible] are

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right)$$

- $f(z) = z^n$

let $z = re^{i\theta}$, $r \neq 0$

$$z^n = r^n e^{in\theta}$$
$$= r^n(\cos n\theta + i\sin n\theta)$$
$$= u(r,\theta) + i\,v(r,\theta)$$

$u(r,\theta) = r^n\cos n\theta$, $\quad v(r,\theta) = r^n \sin n\theta$..

we have

$$u_r = nr^{n-1}\cos n\theta = \frac{1}{r}v_\theta$$

$$u_\theta = -nr^n \sin n\theta = -r\,v_r$$

b) $u(x,y) = x + y^3 - 3x^2y$

$\frac{\partial u}{\partial x} = 1 - 6xy$ $\qquad$ $\frac{\partial u}{\partial y} = 3y^2 - 3x^2$

$\frac{\partial^2 u}{\partial x^2} = -6y$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = 6y$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0.$$

since $\nabla^2 u = 0$, $u(x,y)$ is harmonic.

From C-R. equ^{ns}

$$v_y = u_x$$
$$v_y = 1 - 6xy$$
$$\frac{\partial v}{\partial y} = 1 - 6xy$$

By integrating partially with respect to y.

$$v = \int 1 - 6xy \, dy$$

$$= y - \frac{6xy^2}{2}$$

$$v = y - 3xy^2 + \phi(x) \quad \text{——} \ (1)$$

$\phi(x)$ is an arbitrary function of x.

Again by C-R equations.

$$v_x = -u_y$$

$$\frac{\partial v}{\partial x} = -(3y^2 - 3x^2) \quad \text{——} \ (2)$$

From (1), we have

$$\frac{\partial v}{\partial x} = -3y^2 + \phi'(x) \quad \text{——} \ (3)$$

(2) = (3)

$$\frac{\partial v}{\partial x} = -3y^2 + \phi'(x) = -3y^2 + 3x^2$$

$$\phi'(x) = 3x^2$$

Integrating w.r.t. x

$$\phi(x) = x^3 + c$$

where c is an arbitrary constant.

$$\Rightarrow \underline{\underline{v = y - 3xy^2 + x^3 + c.}}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$= x + y^3 - 3x^2y + i(y - 3xy^2 + x^3 + c)$$

$$= x + iy + y^3 - 3x^2y - 3xy^2 i + x^3 i + ci$$

$$= \underline{\underline{z + i(z^3) + iC}}$$