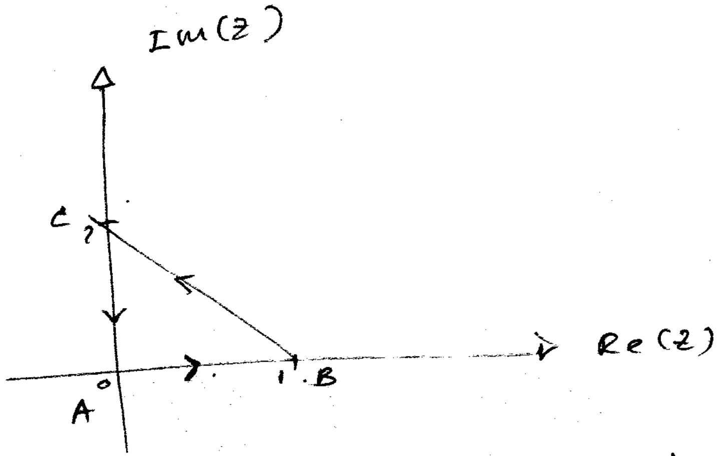
MAT 322β : Complex Variables

Tutorial No: 03

(Q1)

i)



$$\int_\gamma \bar z\, dz = \int_{AB} \bar z\, dz + \int_{BC} \bar z\, dz + \int_{CA} \bar z\, dz$$

let $z = x + iy$. Then $\bar z = x - iy$

$$\int_\gamma (x-iy)\,[dx + i dy] = \int_{AB} (x-iy)\,[dx+idy] + \int_{BC} (x-iy)\,[dx+idy]$$

$$+ \int_{CA} (x-iy)\,[dx+idy] \quad \text{——— } (1)$$

AB: $y = 0$, $dy = 0$, $x \in [0,1]$

BC: eq^n of the line $\dfrac{y-0}{x-1} = \dfrac{0-1}{1-0}$, $y = -x+1$, $x = 1-y$, $dx = -dy$, $y \in [0,1]$

CA: $x = 0$, $dx = 0$, $y \in [1,0]$

$(1-y-iy)(i-1)$
$= (1-y)i + iy + y - (1-y)$
$= i + 2y - 1$

Then the eq^n (1) becomes,

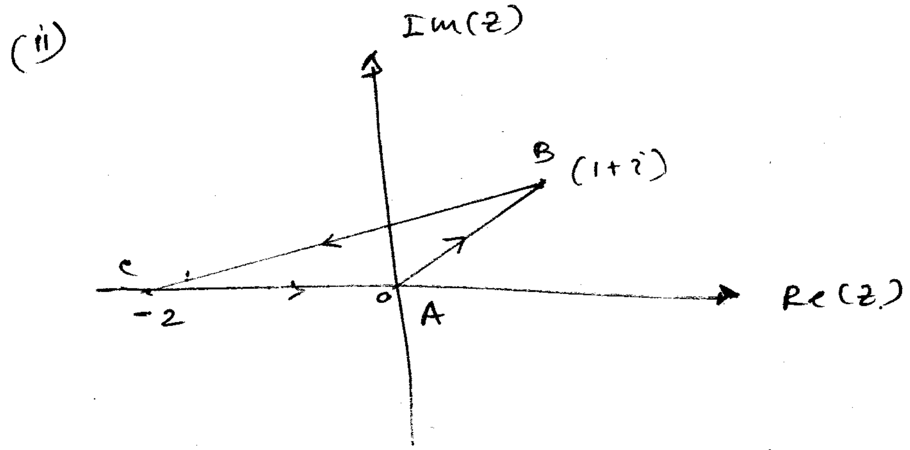
$$\int_\gamma \bar z\, dz = \int_0^1 x\,dx + \int_0^1 (1-y-iy)\,[-dy + idy] + \int_1^0 -iy\,(idy)$$

$$= \int_0^1 x\,dx + \int_0^1 (1-y-iy)\,(i-1)\,dy + \int_1^0 y\,dy$$

$$= \frac{x^2}{2}\Big|_0^1 + \int_0^1 [(2y-1)+i]\,dy + \frac{y^2}{2}\Big|_1^0$$

$$= \tfrac12 - 0 + [y^2 - y + iy]_0^1 + 0 - \tfrac12 = \underline{\underline{i}}$$

(ii)



$$\int_\gamma \bar{z}\,dz = \int_{AB} \bar{z}\,dz + \int_{BC} \bar{z}\,dz + \int_{CA} \bar{z}\,dz$$

$$= \int_{AB} (x-iy)(dx+idy) + \int_{BC} (x-iy)(dx+idy) + \int_{CA} (x-iy)(dx+idy) \quad \text{——— (2).}$$

AB

eq^n of the line

$\frac{y-0}{x-0} = \frac{0-1}{0-1}$

$y = x$

$dx = dy$

$x \in [0,1]$

BC

eq^n of the line.

$\frac{y-1}{x-1} = \frac{1-0}{1-(-2)}$

$3y-3 = x-1$

$3y-2 = x.$

$3dy = dx$

$y \in [1,0]$

CA

$y=0$

$dy = 0$

$x \in [-2,0]$

$(3y-2-iy)(3+i)$

$= 3(3y-2) + y + i(3y-2-3y)$

$= (10y-6) - 2i$

Then the eq^n (2) becomes,

$$\int_\gamma \bar{z}\,dz = \int_0^1 (x-ix)(dx+idx) + \int_1^0 (3y-2-iy)(3dy+idy) + \int_{-2}^0 x\,dx$$

$$= \int_0^1 x\underbrace{(1-i)(1+i)}_{=(1-i^2)}\,dx + \int_1^0 (3y-2-iy)(3+i)\,dy + \int_{-2}^0 x\,dx.$$

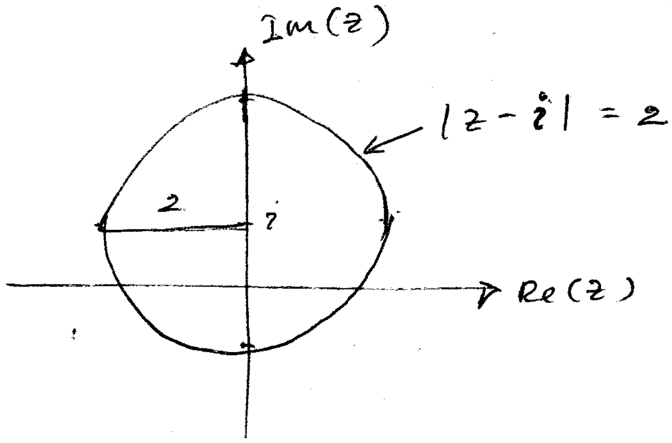
$$= \int_0^1 x(2)\,dx + \int_1^0 (10y-6)-2i\,dy + \int_{-2}^0 x\,dx$$

$$= \frac{2x^2}{2}\Big|_0^1 + \left(\frac{10y^2}{2} - 6y - 2iy\right)\Big|_1^0 + \frac{x^2}{2}\Big|_{-2}^0$$

$$= 1 + (-(5-6-2i)) + 0 - 2$$

$$= \underline{\underline{2i}}.$$

(iii)



Let

$z = i + 2e^{i\theta}$.

$dz = 2ie^{i\theta}d\theta$.

$z = i + 2(\cos\theta + i\sin\theta)$

$= 2\cos\theta + i(1 + 2\sin\theta)$

$\bar{z} = 2\cos\theta - i(1 + 2\sin\theta)$

$= -i + 2[\cos\theta - i\sin\theta]$

$= -i + 2e^{-i\theta}$.

$$\int_\gamma \bar{z}\,dz = \int_0^{2\pi} (-i + 2e^{-i\theta})\, 2ie^{i\theta} d\theta$$

$$= \int_0^{2\pi} 2i(-ie^{i\theta} + 2)\, d\theta.$$

$$= 2i\left[\frac{-ie^{i\theta}}{i} + 2\theta.\right]_0^{2\pi}$$

$$= 2i\left[-1 + 4\pi - (-1)\right]$$

$$= \underline{\underline{8\pi i}}$$

Q2 $f(z) = \dfrac{z^2-1}{z}$

a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$

$$\int_C f(z)\,dz = \int_0^{\pi} \frac{[(2e^{i\theta})^2 - 1]}{2e^{i\theta}} (2e^{i\theta} i)\,d\theta.$$

$$= i\int_0^{\pi} 4e^{2\theta i} - 1 \quad d\theta.$$

$$= i\left[\frac{4e^{2\theta i}}{2i} - \theta\right]_0^{\pi}$$

$$= i\left[\frac{4}{2i} - \pi - \left(\frac{4}{2i} - 0\right)\right]$$

$$= \underline{\underline{-\pi i}}$$

b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$

$$\int_C f(z)\,dz = \int_{\pi}^{2\pi} \frac{[(2e^{i\theta})^2 - 1]}{2e^{i\theta}} (2e^{i\theta} i)\,d\theta.$$

$$= i\int_{\pi}^{2\pi} (4e^{i2\theta} - 1)\,d\theta.$$

$$= i\left[\frac{4e^{i2\theta}}{2i} - \theta\right]_{\pi}^{2\pi}$$

$$= i\left[\frac{4}{2i} - 2\pi - \left(\frac{4}{2i} - \pi\right)\right]$$

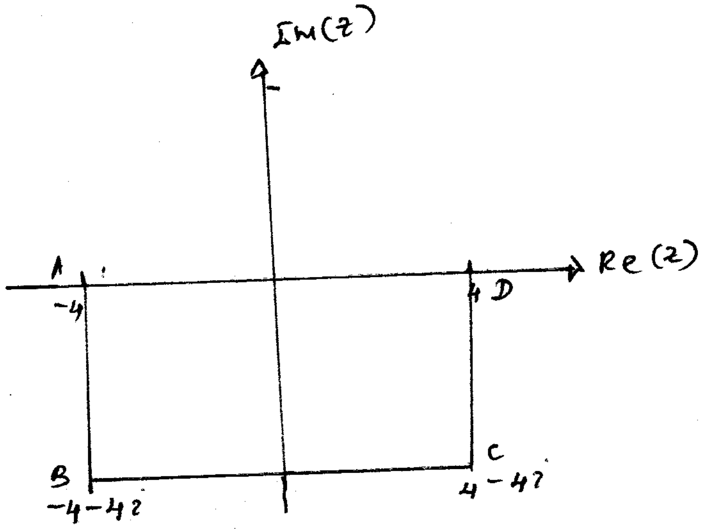
$$= \underline{\underline{-\pi i}}$$

c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$

$$\int_C f(z)\,dz = \int_0^{2\pi} f(z)\,dz = \underbrace{\int_0^{\pi} f(z)\,dz}_{(a)} + \underbrace{\int_{\pi}^{2\pi} f(z)\,dz}_{(b)} = \underline{\underline{-2\pi i}}$$

(3)

a)



Let $z = x + iy$

$f(z) = Re(z) = x$

AB: $x = -4$, $dx = 0$, $y \in [0, -4]$

BC: $y = -4$, $dy = 0$, $x \in [-4, 4]$

CD: $x = 4$, $dx = 0$, $y \in [-4, 0]$

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz$$

$$= \int_{AB} x\,[dx + dy\,i] + \int_{BC} x\,[dx + dy\,i] + \int_{CD} x\,[dx + dy\,i]$$

$$= \int_0^{-4} -4\,[0 + dy\,i] + \int_{-4}^{4} x\,[dx + 0] + \int_{-4}^{0} 4\,[0 + dy\,i]$$

$$= \int_0^{-4} -4i\,dy + \int_{-4}^{4} x\,dx + \int_{-4}^{0} 4i\,dy$$

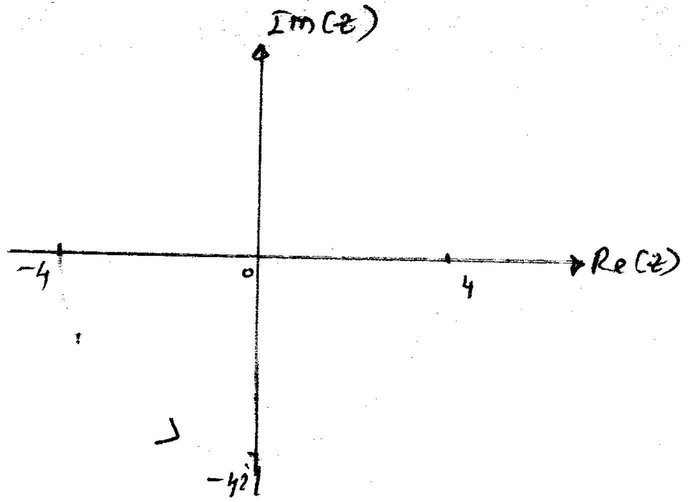
$$= -4iy\Big|_0^{-4} + \frac{x^2}{2}\Big|_{-4}^{4} + 4iy\Big|_{-4}^{0}$$

$$= 16i - 0 + \frac{16}{2} - \left(\frac{16}{2}\right) + 0 - (-16i)$$

$$= \underline{\underline{32i}}$$

d) We have seen that the integral along each contour has a different value. The reason is that the function $f(z) = Re(z)$ is not analytic on any domain containing any of the contours discussed in parts (a), (b) and (c). In fact, this function is nowhere analytic.

b)



In this case, the contour C is defined by

$$z = 4e^{i\theta} \Rightarrow dz = 4e^{i\theta} i$$
$$= 4(\cos\theta + i\sin\theta)$$
$$= 4\cos\theta + i(4\sin\theta) \quad \text{with} \quad \pi \leq \theta \leq 2\pi.$$

$$\int_C \mathrm{Re}(z)\,dz = \int_{\pi}^{2\pi} 4\cos\theta \,(4ie^{i\theta})\,d\theta$$

$$= 16i \int_{\pi}^{2\pi} \cos\theta\, e^{i\theta}\,d\theta$$

$$= 16i \int_{\pi}^{2\pi} \cos\theta\,[\cos\theta + i\sin\theta]\,d\theta.$$

$$= 16i \int_{\pi}^{2\pi} [\cos^2\theta + i\sin\theta\cos\theta]\,d\theta.$$

$$= 16i \left[\int_{\pi}^{2\pi} \cos^2\theta\,d\theta + \frac{i}{2}\int_{\pi}^{2\pi} \sin 2\theta\,d\theta.\right]$$

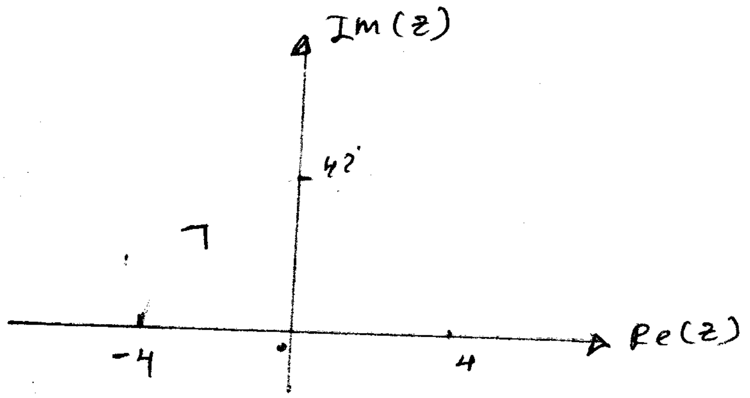
$$= 16i \left[\int_{\pi}^{2\pi} \left(\frac{1+\cos 2\theta}{2}\right)d\theta + \frac{i}{2}\left(-\frac{\cos 2\theta}{2}\right)\Big|_{\pi}^{2\pi}\right]$$

$$= 16i \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_{\pi}^{2\pi} - \frac{i}{2}\left(\frac{1}{2} - \frac{1}{2}\right)\right]$$

$$= 16i\left(\pi - \frac{\pi}{2} - \frac{i}{2}(0)\right)$$

$$= \underline{\underline{8\pi i}}$$

c)



In this case, the contour C is defined by

$$z = 4e^{i\theta} \Rightarrow dz = 4e^{i\theta}.i$$

$$= 4(\cos\theta + i\sin\theta)$$

$$= 4\cos\theta + i(4\sin\theta) \quad \text{with } \pi \leq \theta \leq 0$$

$$\int_C Re(z)\, dz = \int_{\pi}^{0} 4\cos\theta \cdot 4e^{i\theta} i\, d\theta$$

$$= 16i \int_{\pi}^{0} e^{i\theta}\cos\theta\, d\theta.$$

$$= 16i \int_{\pi}^{0} (\cos\theta + i\sin\theta)\cos\theta\, d\theta.$$

$$= 16i \int_{\pi}^{0} \cos^2\theta + i\sin\theta\cos\theta\, d\theta$$

$$= 16i \left[\int_{\pi}^{0} \cos^2\theta\, d\theta + \frac{i}{2}\int_{\pi}^{0} \sin 2\theta\, d\theta \right].$$

$$= 16i \left[\int_{\pi}^{0} \frac{1+\cos 2\theta}{2}\, d\theta + \frac{i}{2}\left(\frac{-\cos 2\theta}{2}\right)_{\pi}^{0} \right].$$

$$= 16i \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_{\pi}^{0} + \frac{i}{2}\left(-\frac{1}{2} - \left(-\frac{1}{2}\right)\right) \right]$$

$$= 16i \left[0 - \left(\frac{\pi}{2} + 0\right) + \frac{i}{2}(0) \right]$$

$$= \underline{\underline{-8\pi i}}$$

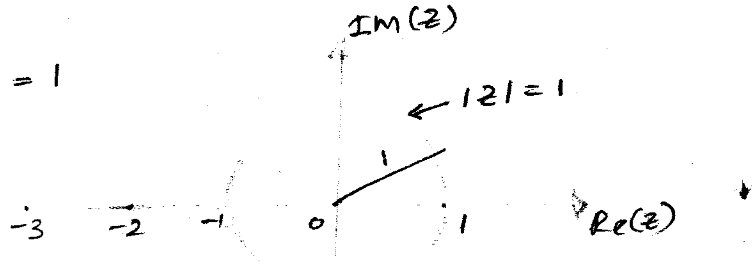
04) a) Cauchy - Goursat theorem

Let $f(z)$ be analytic in a simply connected domain D. Then $\oint_c f(z)\,dz = 0$ along every simply, smooth closed curve c contained in D.

b)

i) $f(z) = \dfrac{z^3}{z^2+5z+6}$; $|z| = 1$

$= \dfrac{z^3}{(z+2)(z+3)}$



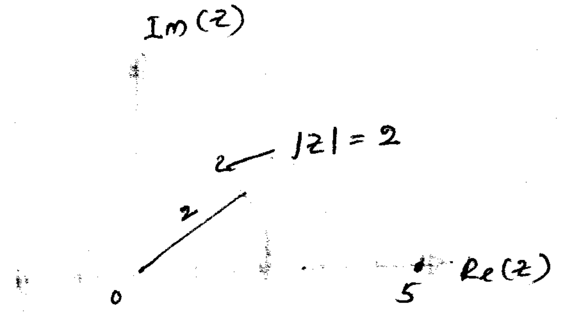
- $z = -2, -3$ are not in the circle.
- Therefore $f(z)$ is analytic in $|z| \leq 1$.

- Simply connected domain.
- Simple & closed.

By the cauchy integral theorem,

$$\oint_c f(z)\,dz = \oint_c \frac{z^3}{z^2+5z+6}\,dz = \underline{\underline{0}}$$

ii) $f(z) = \dfrac{z^2}{z-5}$; $|z| = 2$



- $z = 5$ is not in the circle.
- Therefore $f(z)$ is analytic in $|z| \leq 2$.

- simply connected domain.
- Simple & closed.

By the cauchy integral theorem,

$$\oint_c f(z)\,dz = \oint_c \frac{z^2}{z-5}\,dz = \underline{\underline{0}}$$

Extra Problems

(05) The simplest representation for the unit circle is the exponential form

$$z = e^{i\theta} \quad \text{and} \quad dz = ie^{i\theta} d\theta$$

where we have noted that $r=1$ for a unit circle with the center at the origin.

Then we have

$$\oint \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{i\theta} d\theta}{e^{i\theta}}$$

$$= \int_0^{2\pi} i \, d\theta$$

$$= \underline{\underline{2\pi i}}$$

(06) The unit circle is in the exponential form $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$.

we then have, if $n > 1$,

$$\oint \frac{dz}{z^n} = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{ni\theta}} d\theta$$

$$= i \int_0^{2\pi} e^{i\theta(1-n)} d\theta$$

$$= \frac{i e^{i\theta(1-n)}}{i(1-n)} \Big|_0^{2\pi}$$

$$= \frac{1}{1-n} (1-1)$$

$$= \underline{\underline{0}}$$

(07)

a) The arc is $C: z = 1 + e^{i\theta}$ $(\pi \leq \theta \leq 2\pi)$.

Then

$$\int_{C} (z-1)\, dz = \int_{\pi}^{2\pi} (1 + e^{i\theta} - 1)\, i e^{i\theta} d\theta$$

$$= i \int_{\pi}^{2\pi} e^{2\theta i} d\theta$$

$$= i \left[\frac{e^{i2\theta}}{2i} \right]_{\pi}^{2\pi}$$

$$= \frac{1}{2} (e^{4\pi i} - e^{2\pi i})$$

$$= \frac{1}{2} (1 - 1)$$

$$= \underline{\underline{0}}$$

b) Here $C: z = x$ $(0 \leq x \leq 2)$. Then

$$\int_{C} (z-1)\, dz = \int_{0}^{2} (x-1)\, dx$$

$$= \left[\frac{x^2}{2} - x \right]_{0}^{2}$$

$$= 2 - 2$$

$$= \underline{\underline{0}}$$