Two Tools of a Risk Quant: Time Series Analysis and Value at Risk

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Tools of the Risk Quant

- **Time Series**
  the lifeblood of finance

Source: **TradingView.com**

Tools of the Risk Quant

- **Value at Risk**
  the fundamental risk model

Business **Financial Markets** posted: 2 months ago

JPMorgan Tells SEC New VaR Model Didn't Require Prior Disclosure As Whale Impact Fades

JPMorgan Chase, facing criticism that it misled investors about a change to a risk model as trades backfired last year, told U.S. regulators that the bank wasn't obligated to disclose the move until May.

Bloomberg reports that while there was an 'interim change' to the lender's so-called value-at-risk model during the first three months of 2012, that adjustment had been reversed by the time the company filed its quarterly report in May, then-Chief Financial Officer Douglas Braunstein told the Securities and Exchange Commission in a December 3rd letter that was released Wednesday.

'As a result, the firm believes there was no model change within the meaning of' securities-disclosure laws, he wrote.

Motivation

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  - (Big words, big words, ...)

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    - How do we ensure that this translation is valid?

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  - Why the “Greeks” matter

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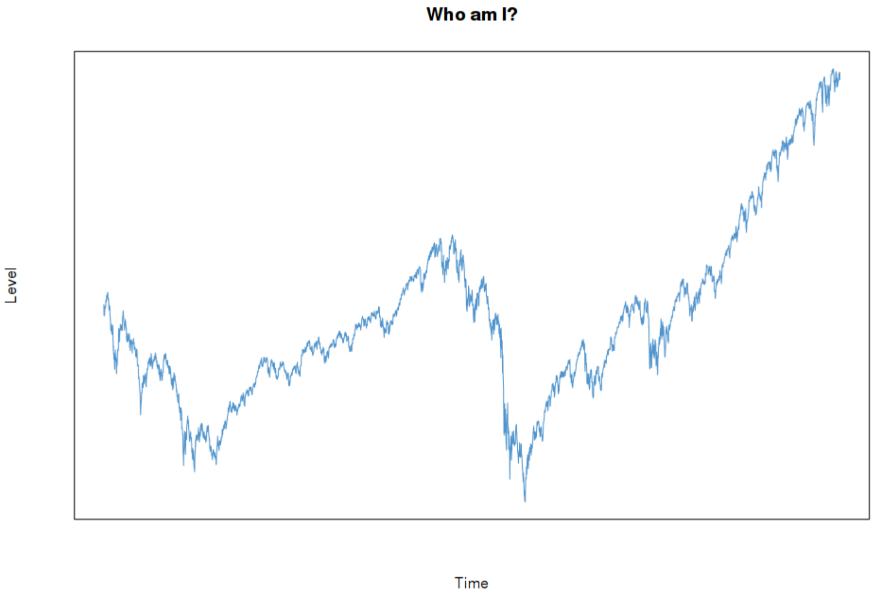
  - P&L is stationary
  - We can forecast P&L if it's stationary
  - Normal linear VaR ("Good old VaR")
  - Why the "Greeks" matter
  - At the end of the lecture we'll relate VaR to time-series analysis through a simple time-series model for the P&L forecast of a FX forward contract

Guess that Stock...



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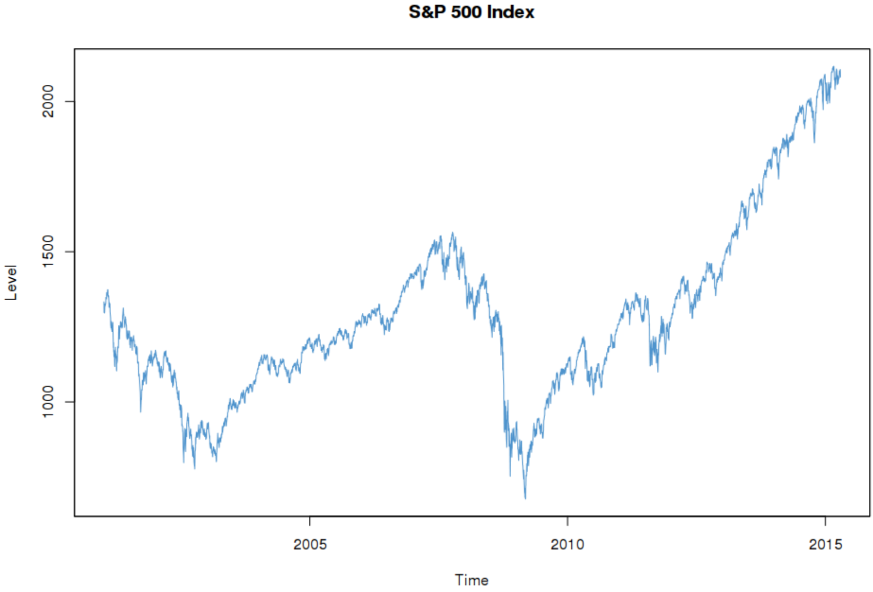

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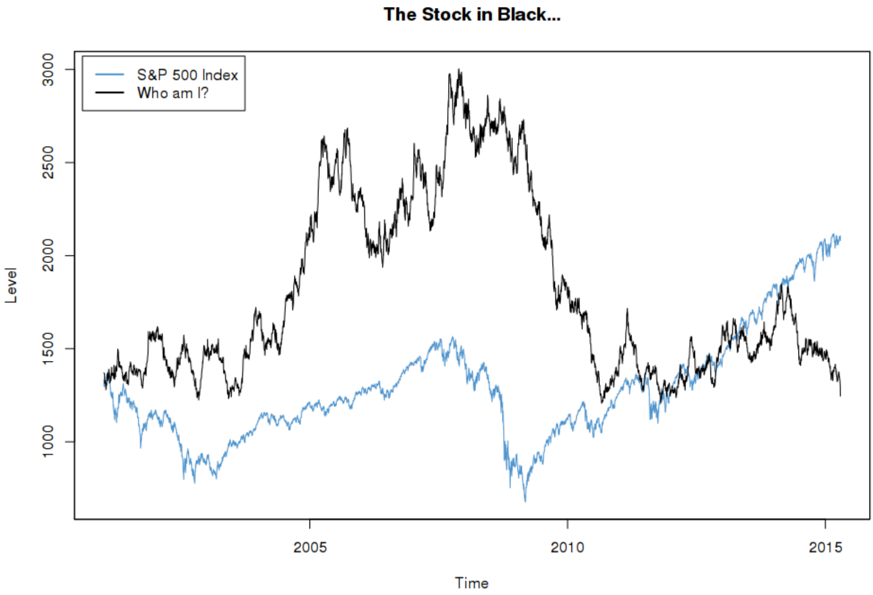
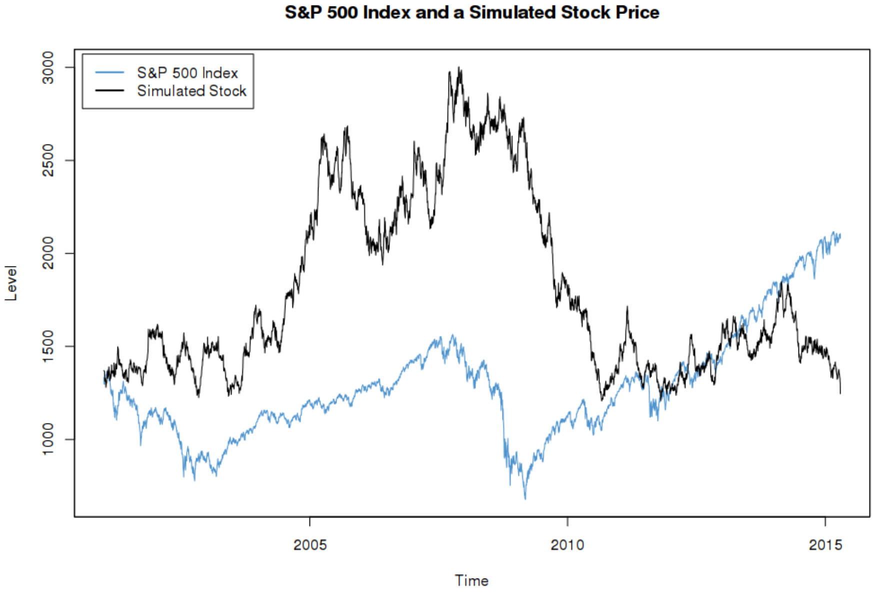
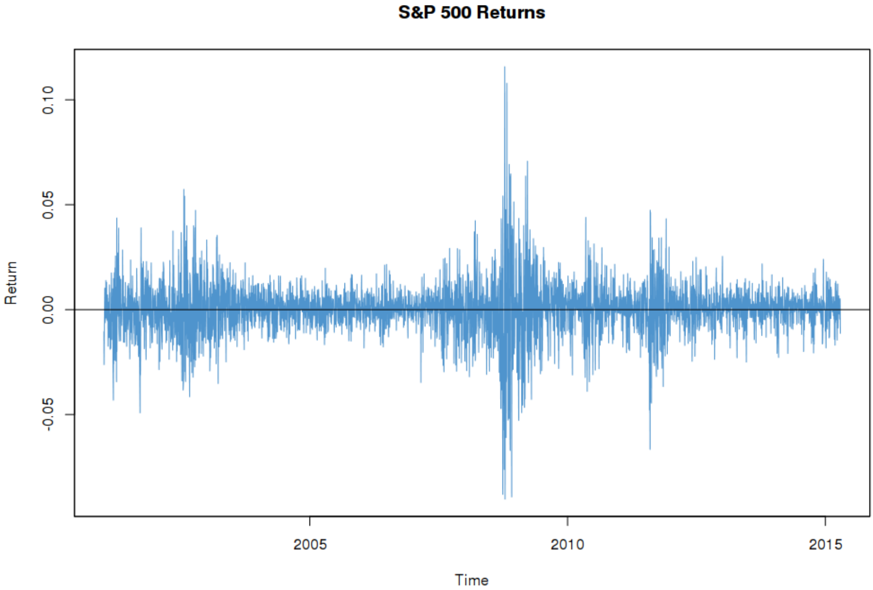


Price Levels are Non-stationary. . .



Returns are (mostly) Stationary. . .



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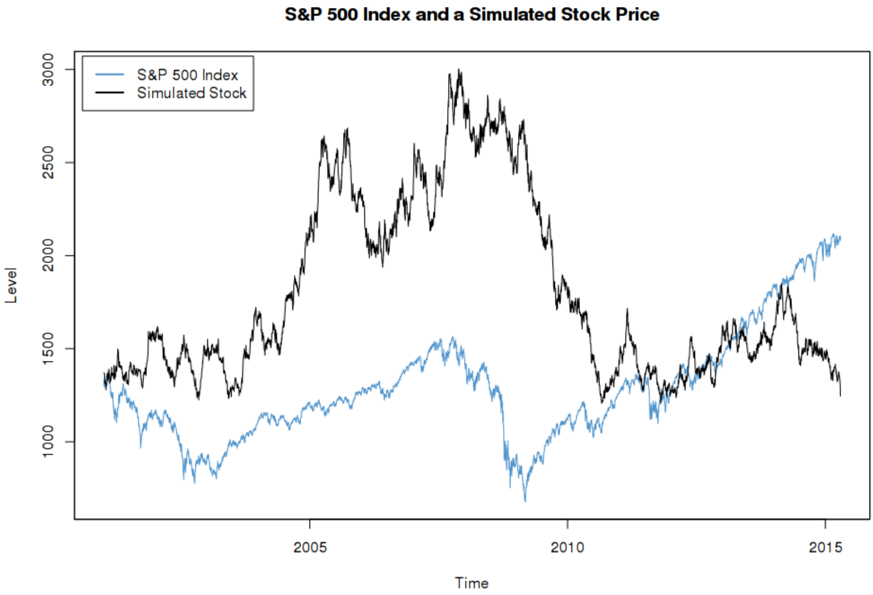

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- No obvious drift upon inspection, but we still say that Equation (2) has a *stochastic trend*.

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- By using a first-order difference operator, or “differencing” for short

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  where $\varepsilon_t \overset{\text{i.i.d.}}{\sim} (0, \sigma^2)$
- In the random-walk model, let $y_t = \log P_t - \log P_{t-1} \approx \frac{P_t - P_{t-1}}{P_t}$, *i.e.* the return

$$\begin{aligned} y_t &= y_{t-1}\alpha + \varepsilon_t, \quad \varepsilon_t \sim \mathcal{N}\left(0,\ \sigma^2\right), \quad \text{for } t = 1, \ldots, N \\ y_t &\sim \mathcal{N}\left(y_{t-1}\alpha,\ \sigma^2\right) \end{aligned}$$

- AR(1) model is stable if $|\ \alpha\ | < 1$.
- If $|\ \alpha\ | < 1$, the AR(1) model is stationary.
- **Exercise** — prove that:

$$\mathbb{E}[y_t] = 0 \tag{6}$$

$$\mathbb{E}\left[y_t^2\right] = \frac{\sigma^2}{(1 - \alpha^2)} \tag{7}$$

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The fifth assumption concerns identifiability and can be simply thought of as ensuring that the least-squares estimate $\hat{\beta} = (X'X)^{-1}X'y$ can be computed, *viz.* that $(X'X)^{-1}$ exists.

$\hat{\beta}$ — the Regression Estimate

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The properties of the *theoretical*[2] estimator $\hat{\beta}$ will be now be derived to illustrate where the variables on which it depends come from.

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The matrix $X'X$ is non-singular, as guaranteed by Assumption (5), above.

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- From Assumption (2), above, we have $\mathbb{E}\left[\varepsilon^2\right] = \sigma^2$.

Theory versus Practice

- So far the properties of $\hat{\beta}$ are *theoretical* estimates, where it is assumed that what in practice are unknown quantities are, in fact, knowable.
- What, therefore, is a a practically realizable estimator $\hat{V}(\hat{\beta})$ for the theoretical covariance of $\hat{\beta}$, $V(\hat{\beta})$?
- We can find an estimator $\hat{V}(\hat{\beta})$ by first focusing on the vector of least-squares residuals:

$$\hat{\varepsilon} = y - X\hat{\beta}. \tag{19}$$

- The residuals $\hat{\varepsilon}$ (as opposed to the error term ε) are realizable, as y and X are our data, and are thus known, and we have already seen that $\hat{\beta} = (X'X)^{-1}X'y$, which is computed from our data.
- From Assumption (2), above, we have $\mathbb{E}\left[\varepsilon^2\right] = \sigma^2$.
- Recall that $V(\hat{\beta}) = \sigma^2(X'X)^{-1}$, where σ^2 is the *theoretical* variance of the error term ε.

Theory versus Practice (continued...)

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- All of the variables in Equation (23) we have in practice.

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- The value that is forecast is of theoretical (or unrealized) P&L

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- The Value at Risk of a portfolio — denoted by $\text{VaR}^{\$}_{\alpha,h}$ — expressed in dollar terms:

$$\Pr(D_{t,h}P_{t+h} - P_t < q^{\$}_{\alpha,h}) = \alpha \tag{25}$$

$$\text{VaR}^{\$}_{\alpha,h} = -q^{\$}_{\alpha,h}, \tag{26}$$

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where $q^{\$}_{\alpha,h}$ is the quantile of the discounted (theoretical, or unrealized) P&L distribution at time $t + h$.

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and

$$\mathrm{VaR}_{\alpha,h} = \begin{cases} -q_{\alpha,h} & \text{as a \% of portfolio value } P_t, \\ -q_{\alpha,h}P_t = \mathrm{VaR}^{\$}_{\alpha,h} & \text{in dollar terms,} \end{cases} \tag{29}$$

where $q_{\alpha,h}$ is the quantile of the discounted (theoretical, or realized) returns distribution at time $t + h$.

Normal Linear VaR

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- Suppose that
$$X_{t+h} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu_h, \sigma_h^2), \tag{30}$$
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where μ_h is the expected return and σ_h^2 is the returns variance for holding period h.
- We can re-write Equation (28):

$$\begin{aligned}
\Pr(X_{t+h} < q_{\alpha,h}) &= \Pr\left(\frac{X_{t+h} - \mu_h}{\sigma_h} < \frac{q_{\alpha,h} - \mu_h}{\sigma_h}\right) && (31)\\
&= \Pr\left(Z < \frac{q_{\alpha,h} - \mu_h}{\sigma_h}\right) && (32)\\
&= \Pr\left(Z < \Phi^{-1}(\alpha)\right) && (33)\\
&= \alpha, && (34)
\end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$, and Φ^{-1} is the inverse normal distribution function (*e.g.* $\Phi^{-1}(0.01) \approx 2.33$).

Normal Linear VaR, continued. . .

Normal Linear VaR, continued...

- From Equation (32) and (33) we have

$$\frac{q_{\alpha,h} - \mu_h}{\sigma_h} = \Phi^{-1}(\alpha), \tag{35}$$

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- Equation (36) is therefore an analytic expression for the VaR under the assumption of Equation (30), and it is this analytic expression that makes the normal linear model so tractable.

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$$
\begin{aligned}
V^{\$}_{t,T-t} \Leftrightarrow V^{\$}(K, S_t, r_d, r_f, T-t) &= \pm N^{\mathrm{c}} \times e^{-r_d(T-t)} \times (\mathbb{E}_t[S_T] - K) \\
&= \pm N^{\mathrm{c}} \times e^{-r_d(T-t)} \times (F_{t,T-t} - K) \\
&= \pm N^{\mathrm{c}} \times e^{-r_d(T-t)} \times (S_t + f_{t,T-t} - K) \\
&= \pm N^{\mathrm{c}} \times e^{-r_d(T-t)} \times (S_t e^{(r_d - r_f)(T-t)} - K) \\
&= \pm N^{\mathrm{c}} \times \left(S_t e^{-r_f(T-t)} - K e^{-r_d(T-t)}\right)
\end{aligned}
$$

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\end{aligned}
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which, in vector form, becomes

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- Risk measures:

$$\begin{aligned}
\Delta V^{\$}_{t,\textsc{Spot}} &= \frac{\partial V^{\$}_t}{\partial S_t} \times \Delta S = \pm N^{\mathrm{C}} \times e^{-r_f(T-t)} \times \Delta S \\
\Delta V^{\$}_{t,r_d} &= \frac{\partial V^{\$}_t}{\partial r_d} \times \Delta r_d = \pm N^{\mathrm{C}} \times K(T-t) \times \Delta r_d \\
\Delta V^{\$}_{t,r_f} &= \frac{\partial V^{\$}_t}{\partial r_f} \times \Delta r_f = \pm N^{\mathrm{C}} \times -S_t(T-t) \times \Delta r_f \\
\Delta V^{\$}_{t,\tau} &= \frac{\partial V^{\$}_t}{\partial \tau} \times \Delta \tau = \pm N^{\mathrm{C}} \times \Big(-r_f S_t + r_d K\Big) \times \Delta \tau
\end{aligned}$$

which, in vector form, becomes

$$\Delta V^{\$}_t = \underbrace{\begin{bmatrix} \frac{\partial V^{\$}_t}{\partial S_t} \\ \frac{\partial V^{\$}_t}{\partial r_d} \\ \frac{\partial V^{\$}_t}{\partial r_f} \\ \frac{\partial V^{\$}_t}{\partial \tau} \end{bmatrix}}_{\nabla V^{\$}_t \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon}$$

The Humble FX Forward Contract, continued. . .

The Humble FX Forward Contract, continued...

$$\Delta V_t^{\$} = \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon}$$

The Humble FX Forward Contract, continued...

$$\Delta V_t^{\$} = \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \pm N^{C} \times \underbrace{\begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon}$$

The Humble FX Forward Contract, continued...

$$\Delta V_t^{\$} = \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \underbrace{\pm N^C \times \begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon}$$

$$= \nabla V_t^{\$} \odot \varepsilon$$

The Humble FX Forward Contract, continued...

$$\Delta V_t^{\$} = \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \underbrace{\pm N^C \times \begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon}$$

$$= \nabla V_t^{\$} \odot \varepsilon$$

What is the aggregate effect of a perturbation to the value of the contract?

The Humble FX Forward Contract, continued...

$$
\begin{aligned}
\Delta V_t^{\$} &= \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \underbrace{\pm N^C \times \begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} \\
&= \nabla V_t^{\$} \odot \varepsilon
\end{aligned}
$$

What is the aggregate effect of a perturbation to the value of the contract?

$$
V_t^{\$}(K, S_t + \Delta S, r_d + \Delta r_d, r_f + \Delta r_f, T - t + \Delta\tau) \approx V_t^{\$}(K, S_t, r_d, r_f, T - t) + (\nabla V_t^{\$})'\varepsilon
$$

The Humble FX Forward Contract, continued...

$$
\begin{aligned}
\Delta V_t^{\$} &= \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \underbrace{\pm N^C \times \begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} \\
&= \nabla V_t^{\$} \odot \varepsilon
\end{aligned}
$$

What is the aggregate effect of a perturbation to the value of the contract?

$$
\underbrace{V_t^{\$}(K, S_t + \Delta S, r_d + \Delta r_d, r_f + \Delta r_f, T - t + \Delta \tau)}_{\text{Level (hypothesized from unrealized)}} \approx \underbrace{V_t^{\$}(K, S_t, r_d, r_f, T - t)}_{\text{Level (unrealized)}} + (\nabla V_t^{\$})' \varepsilon
$$

The Humble FX Forward Contract, continued...

$$
\begin{aligned}
\Delta V_t^{\$} &= \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \pm N^C \times \underbrace{\begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} \\
&= \nabla V_t^{\$} \odot \varepsilon
\end{aligned}
$$

What is the aggregate effect of a perturbation to the value of the contract?

$$
\underbrace{V_t^{\$}(K, S_t + \Delta S, r_d + \Delta r_d, r_f + \Delta r_f, T - t + \Delta \tau)}_{\text{Level (hypothesized from unrealized)}} \approx \underbrace{V_t^{\$}(K, S_t, r_d, r_f, T - t)}_{\text{Level (unrealized)}} + (\nabla V_t^{\$})' \varepsilon
$$

or

The Humble FX Forward Contract, continued...

$$
\begin{aligned}
\Delta V_t^{\$} &= \underbrace{\begin{bmatrix} \frac{\partial V_t^{\$}}{\partial S_t} \\ \frac{\partial V_t^{\$}}{\partial r_d} \\ \frac{\partial V_t^{\$}}{\partial r_f} \\ \frac{\partial V_t^{\$}}{\partial \tau} \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} = \underbrace{\pm N^{C} \times \begin{bmatrix} e^{-r_f(T-t)} \\ K(T-t) \\ -S_t(T-t) \\ -r_f S_t + r_d K \end{bmatrix}}_{\nabla V_t^{\$} \Leftrightarrow \text{“Greeks”}} \odot \underbrace{\begin{bmatrix} \Delta S \\ \Delta r_d \\ \Delta r_f \\ \Delta \tau \end{bmatrix}}_{\varepsilon} \\
&= \nabla V_t^{\$} \odot \varepsilon
\end{aligned}
$$

What is the aggregate effect of a perturbation to the value of the contract?

$$
\underbrace{V_t^{\$}(K, S_t + \Delta S, r_d + \Delta r_d, r_f + \Delta r_f, T - t + \Delta \tau)}_{\text{Level (hypothesized from unrealized)}} \approx \underbrace{V_t^{\$}(K, S_t, r_d, r_f, T - t)}_{\text{Level (unrealized)}} + (\nabla V_t^{\$})' \varepsilon
$$

or

$$
\underbrace{V_t^{\$}(K, S_t + \Delta S, r_d + \Delta r_d, r_f + \Delta r_f, T - t + \Delta \tau) - V_t^{\$}(K, S_t, r_d, r_f, T - t)}_{\text{Hypothesized P\&L — the first-order difference — is stationary}} \approx (\nabla V_t^{\$})' \varepsilon
$$