Trading the Untradable

Pricing Derivatives when Prices are not Observable

Thomas P. Harte (joint with Axel Buchner[†])

[†]University of Passau, Germany

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- More properly: *Untraded* assets

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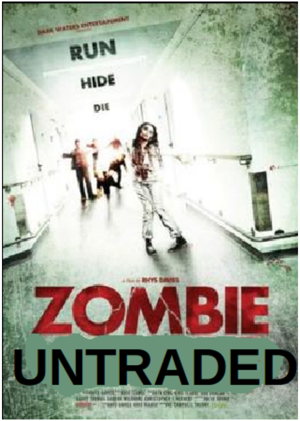
- 

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Options on a PE index

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Options on a PE index

- PE indexes already exist: the snag is that we can't trade them
- What if we built an index that could be replicated with publicly traded instruments?

Returns model

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  Returns can be decomposed into a component that can be replicated by traded factors (β) and an additional return premium (α) that is not spanned by publicly traded factors

- Assume that $r_{i,t}$, the return of an individual private equity fund i at any time t, is given by the following factor model:

$$r_{i,t} = r_{f,t} + \alpha_{i,t} + \beta_i^{\prime} F_t + \varepsilon_{i,t}, \tag{1}$$

  where $F_t = [F_{1,t}, F_{2,t}, ..., F_{J,t}]$ is a set of J common *tradable* factors in the public markets. $r_{f,t}$ is the return on the riskless asset. β_i contains the loadings on the common factors, F_t. $\alpha_{i,t}$ reflects the level of private-equity returns in excess of its systematic (and liquid) component of the return of fund i. $\varepsilon_{i,t}$ is a manager-specific latent factor with mean zero that is orthogonal to the traded factors, F_t. This idiosyncratic (and therefore "unsystematic") risk of fund i potentially makes fund i non-redundant in (and not representable by) the space of tradable assets.

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- The entire segment of private equity does not involve any remaining non-spannable risk
- This is almost identical to the usual assumptions in complete markets, *e.g.* for trading an option on a publicly traded stock
- This *differs* from the usual assumption in complete markets in the following ways: Under full spanning, the risk of the private-equity segment is traded in the market, but the private-equity segment can still earn a positive alpha, as shown in Equation (2). In contrast, under complete markets, this alpha would be arbitraged away. We assume that this arbitrage does not happen because the fund managers generate the alpha, and the investors can only earn it by investing in the private-equity segment along with the associated costs.

Index construction

- To illustrate, assume that private equity returns are generated by the standard market model:

$$r_t = r_{f,t} + \alpha_t + \beta_m(r_{m,t} - r_{f,t}), \tag{3}$$

where $r_{m,t}$ is the return of a traded stock-market index and β_m is the beta coefficient of the entire private-equity segment

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- Volatility of the stock market σ_m equals 20%
- With probability $p = 50\%$, the market either appreciates by

$$u - 1 = e^{+\sigma_m \sqrt{1}} - 1 = 22\% \tag{4}$$

or depreciates by

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- We further assume a constant riskless rate of $r_f = 2\%$ per period
- The risk-neutral probability of an up-movement of the index is given by

$$q = \frac{e^r - d}{u - d} = 0.5 \tag{6}$$

The stock-market index (which is publicly traded) starts at $S_0 = 1,000$ at $t = 0$.

Stock process

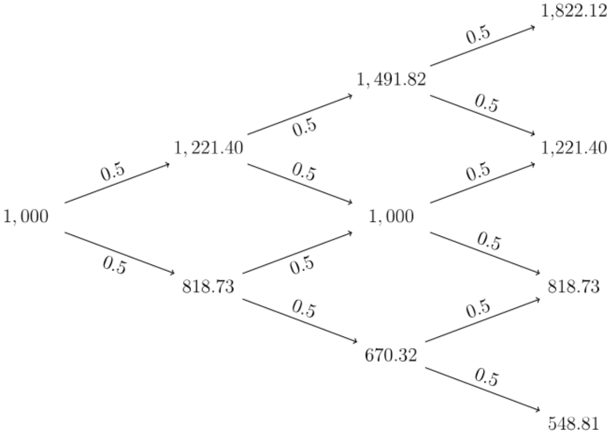

Figure: Stock-market process in a three-step binomial tree (we can trade this)

Cash-flow process

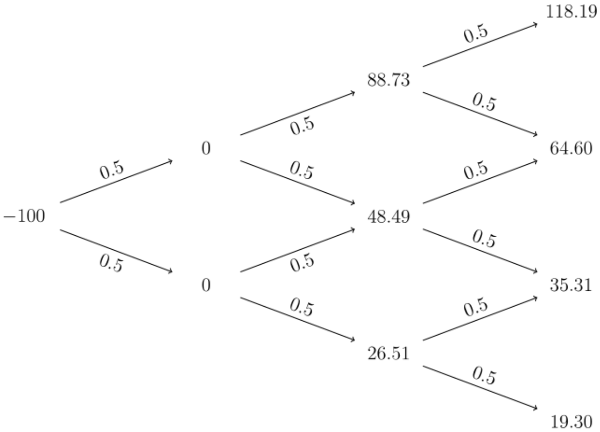

Figure: Cash-flow process for the private-equity segment with a market beta of 1.5 and an alpha equal to 3%, *i.e.* $\beta_m = 1.5$ and $\alpha = 0.03$.

Private-equity process

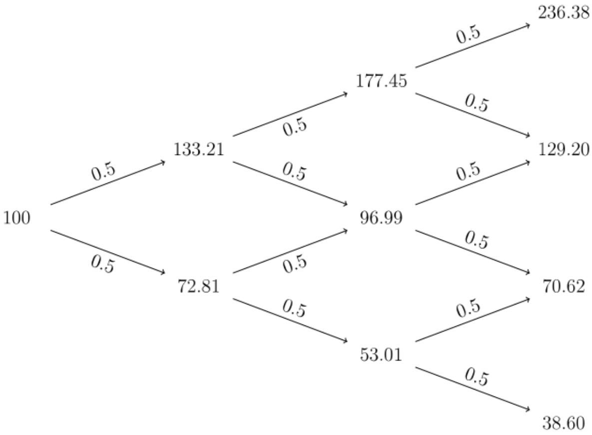

Figure: Private-equity process (the constructed index: we can't trade this!)

Option process

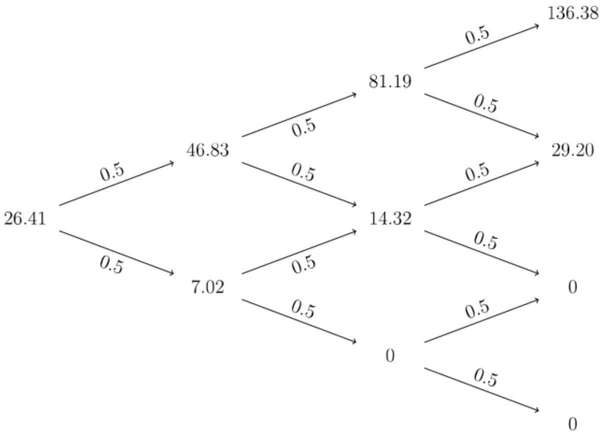

Figure: Cash-flow process for a European call struck at $K = \$100$ with maturity $T = 3$ written on the private-equity process (*i.e.* on the constructed index: we *can* trade this!)

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- The value of this replicating portfolio in case of two up-movements in the tree equals \$81.19. Working backwards in the tree, the initial value of the call turns out to be \$26.41
- Note that we can replicate the option here with an investment into the traded stock-market index (and *not* the untraded private equity index) and an investment into the riskless asset even though the private equity index has an alpha of 3% and a beta equal to 1.5

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  - no arbitrage
  - cash-flow replication
  - perfect hedging
  - complete markets
  - unique prices

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- Can we create new opportunities to trade?

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The valuation of uncertain income streams and the pricing of options

Mark Rubinstein

Assistant Professor
Graduate School of Business Administration
University of California, Berkeley

A simple formula is developed for the valuation of uncertain income streams consistent with rational risk averse investor behavior and equilibrium in financial markets. Applying this formula to the pricing of an option as a function of its associated stock, the Black-Scholes formula is derived even though investors can only trade at discrete points in time.

Rubinstein's model (the Stochastic Discount Factor)

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- Markets are frictionless
- An investor is infinitely lived and chooses lifetime consumption and investment plans, subject to budget constraints, to maximize lifetime expected utility U, as represented by the time-separable power utility function:

$$U = \sum_{t=0}^{\infty} \delta^t E\left[\frac{C_t^{1-\gamma}}{1-\gamma}\right], \tag{9}$$

  where C_t denotes consumption at time t, $\delta > 0$ is a measure of the investor's time preference, and $\gamma \geq 0$ is the investor's constant coefficient of relative risk aversion.

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where C_t denotes consumption at time t, $\delta > 0$ is a measure of the investor's time preference, and $\gamma \geq 0$ is the investor's constant coefficient of relative risk aversion.

- Rubinstein shows that the present value PV_0 of any security that generates an uncertain future cash flow stream X_t can be found by

$$PV_0 = \sum_{t=1}^{\infty} \frac{E[X_t] - \lambda_t \operatorname{corr}[X_t, -(1+r_{m,t})^{-\gamma}]\operatorname{std}[X_t]}{1+r_{f,t}}, \tag{10}$$

where $r_{f,t}$ and $r_{m,t}$ are, respectively, the returns of a riskless asset and of the market portfolio over the time interval $(0, t]$; $\operatorname{corr}[x, y]$ is the correlation of x and y; $\operatorname{std}[\cdot]$ is standard deviation; and $\lambda_t = \operatorname{std}[(1+r_{m,t})^{-\gamma}]/E[(1+r_{m,t})^{-\gamma}]$

Rubinstein's model (the Stochastic Discount Factor)

- Inserting λ_t into (10) yields

$$PV_0 = \sum_{t=1}^{\infty} \frac{E[X_t]E[(1+r_{m,t})^{-b}] - \text{corr}[X_t, -(1+r_{m,t})^{-\gamma}]\text{std}[X_t]\text{std}[(1+r_{m,t})^{-\gamma}]}{(1+r_{f,t})E[(1+r_{m,t})^{-\gamma}]}. \tag{11}$$

By noting that $\text{corr}[X_t, -(1+r_{m,t})^{-\gamma}] = -\text{corr}[X_t, (1+r_{m,t})^{-\gamma}]$, the second term in the numerator can be expressed in terms of the covariance $\text{cov}[X_t, (1+r_{m,t})^{-\gamma}]$. This gives

$$PV_0 = \sum_{t=1}^{\infty} \frac{E[X_t]E[(1+r_{m,t})^{-\gamma}] + \text{cov}[X_t, (1+r_{m,t})^{-\gamma}]}{(1+r_{f,t})E[(1+r_{m,t})^{-\gamma}]}. \tag{12}$$

Using the identity,
$E[X_t]E[(1+r_{m,t})^{-\gamma}] + \text{cov}[X_t, (1+r_{m,t})^{-\gamma}] \equiv E[X_t/(1+r_{m,t})^{\gamma}]$, Equation (12) simplifies to

$$PV_0 = \sum_{t=1}^{\infty} \frac{E[X_t/(1+r_{m,t})^{\gamma}]}{(1+r_{f,t})E[(1+r_{m,t})^{-\gamma}]}. \tag{13}$$

Rubinstein's model (the Stochastic Discount Factor)

- Since Equation (13) also holds for the market portfolio, Rubinstein shows that $1 + r_{f,t} = E[(1 + r_{m,t})^{1-\gamma}]/E[(1 + r_{m,t})^{-\gamma}]$. Inserting this into Equation (13) gives

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- Equation (14) is a simple and practical formula for the valuation of uncertain income streams, consistent with rational risk averse investor behavior and equilibrium in financial markets. The formula does not require any assumption on the covariance $\mathrm{cov}[X_t, (1 + r_{m,t})^{-\gamma}]$, since this covariance is implicitly contained in the expectation $E[X_t/(1 + r_{m,t})^{\gamma}]$. The valuation thus reduces to computing expectations, *e.g.* for a call option struck at K:

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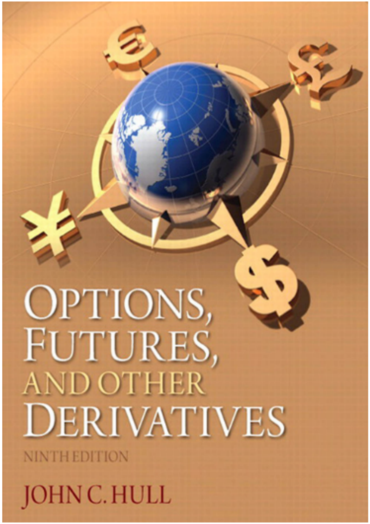
$$\hat{C}_n = \frac{\frac{1}{n}\sum_{i=1}^{n} \max(S_T^{(i)} - K, 0)/(1 + r_{m,T}^{(i)})^{\gamma}}{\frac{1}{n}\sum_{i=1}^{n}(1 + r_{m,T}^{(i)})^{1-\gamma}} \tag{15}$$

Canonical Black Scholes



Canonical Black Scholes

15.8 BLACK–SCHOLES–MERTON PRICING FORMULAS

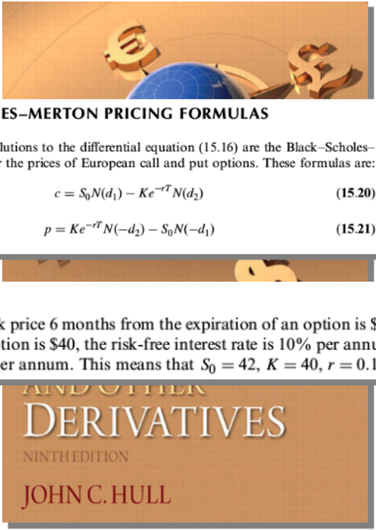
The most famous solutions to the differential equation (15.16) are the Black–Scholes–Merton formulas for the prices of European call and put options. These formulas are:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{15.20}$$

and

$$p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \tag{15.21}$$

Canonical Black Scholes



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and

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1) \quad \textbf{(15.21)}$$

Example 15.6

The stock price 6 months from the expiration of an option is \$42, the exercise price of the option is \$40, the risk-free interest rate is 10% per annum, and the volatility is 20% per annum. This means that $S_0 = 42$, $K = 40$, $r = 0.1$, $\sigma = 0.2$, $T = 0.5$,

Canonical Black Scholes in R

```r
`bs.test`<- function() {
    # Hull (2009), "Options, Futures and Other Derivative Securities",
    # 9th Edition, p. 338, et sqq

    d1<- bs.d1(S0=42, K=40, T=0.5, sigma=0.2, r=0.1)
    d1.out<- sprintf("%.4f", d1)
    cat(sprintf("d1 = %s\n", d1.out))
    assert(d1.out=="0.7693")

    d2<- bs.d2(S0=42, K=40, T=0.5, sigma=0.2, r=0.1)
    d2.out<- sprintf("%.4f", d2)
    cat(sprintf("d2 = %s\n", d2.out))
    assert(d2.out=="0.6278")

    c<- bs.call(S0=42, K=40, T=0.5, sigma=0.2, r=0.1)
    c.out<- sprintf("%.2f", c)
    cat(sprintf("c = %s\n", c.out))
    assert(c.out=="4.76")

    p<- bs.put(S0=42, K=40, T=0.5, sigma=0.2, r=0.1)
    p.out<- sprintf("%.2f", p)
    cat(sprintf("p = %s\n", p.out))
    assert(p.out=="0.81")
}
bs.test()
```

```
d1 = 0.7693
d2 = 0.6278
c = 4.76
p = 0.81
```

Call price = \$4.76. Put price = \$0.81.

Price the call with Rubinstein's model

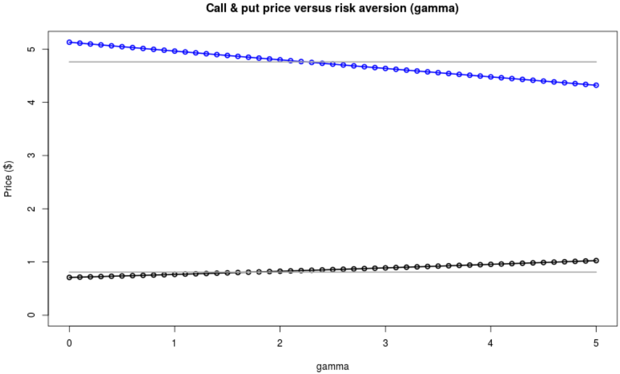


Figure: Using the Rubinstein model to estimate the value of a put (bottom, in black) and a call (top, in blue) in Hull's [4, Example 15.6, *p.* 338, *et sqq.*], where $S_0 = \$40, K = \$42, r = 0.1, \sigma = 0.2, T = 0.5$. The gray lines represent the closed-form solutions given by the Black Scholes model [4, *p.* 339], *viz.*: Call = \$4.76; put = \$0.81.

Stochastic Volatility

Stochastic volatility violates Black Scholes:

$$dS_t = \mu_S S_t dt + \sqrt{V_t} S_t dB_{S,t} \tag{16}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_{V,t} \tag{17}$$

where $B_{S,t}$ and $B_{V,t}$ are correlated standard Brownian motions, *i.e.*, $dB_{S,t} dB_{V,t} = \rho_{SV} dt$

Stochastic Volatility

Stochastic volatility violates Black Scholes:

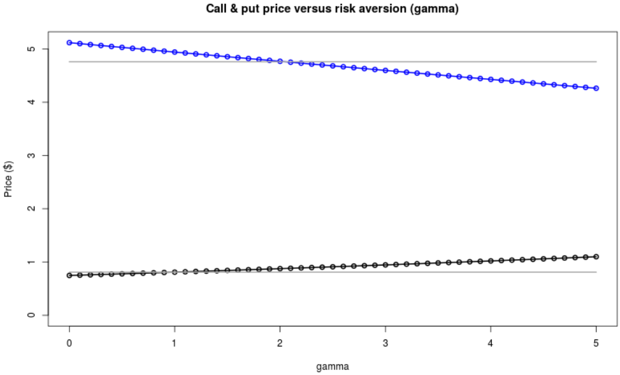


Figure: Using the Rubinstein model to estimate the value of a put (bottom, in black) and a call (top, in blue) in Hull's [4, Example 15.6, *p.* 338, *et sqq*.], where $S_0 = \$40, K = \$42, r = 0.1, \sigma = 0.2, T = 0.5$; here the stochastic process S has stochastic volatility but with parameters $\theta = \sigma_S^2, \kappa = 1$, and $\sigma_V = 10^{-5}$, *i.e.*, parameters that render the volatility almost constant (this lets us compare to the constant-volatility case in Figure 5). The gray lines represent the closed-form solutions given by the Black Scholes model [4, *p.* 339], *viz.*: Call = \$4.76; put = \$0.81.

(Lots of) stochastic volatility

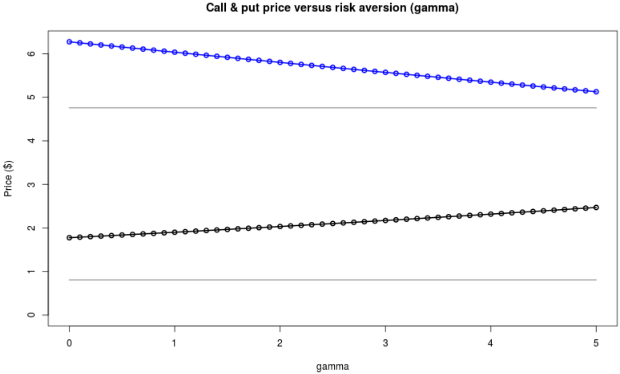


Figure: Using the Rubinstein model to estimate the value of a put (bottom, in black) and a call (top, in blue) in Hull's [4, Example 15.6, *p.* 338, *et sqq*.], where $S_0 = \$40, K = \$42, r = 0.1, \sigma = 0.2, T = 0.5$. Here the stochastic process S has stochastic volatility with parameters $\theta = 0.1, \kappa = 0.5$, and $\sigma_V = 0.05$. The gray lines represent the closed-form solutions given by the Black Scholes model [4, *p.* 339], *viz.*: Call = \$4.76; put = \$0.81.

Contents

One simulation

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
1	7.80	0.00	7.70	88.50	0.50	-8.30	-1	0.00	0.44	0.00
2	17.40	0.03	13.80	88.50	1.00	-10.11	-1	0.00	0.54	0.00
3	22.40	0.42	16.60	88.50	1.50	-5.04	-1	0.00	0.28	0.03
4	23.60	0.42	19.90	88.50	2.00	-1.70	-1	0.00	0.07	0.00
5	31.70	0.42	25.10	88.50	2.50	-8.69	-1	0.00	0.46	0.00
6	35.10	0.42	30.90	88.50	3.00	-3.84	-1	0.00	0.19	0.00
7	41.90	3.37	34.70	88.50	3.50	-4.40	-1	0.00	0.38	0.24
8	45.90	3.37	50.80	88.50	4.00	-4.48	-1	0.00	0.22	0.00
9	51.90	10.07	66.50	88.50	4.50	0.19	-1	0.00	0.34	0.54
10	54.20	10.65	93.40	88.50	5.00	-2.14	-1	0.00	0.12	0.05
11	54.20	25.58	90.10	88.50	5.50	14.43	-1	0.00	0.00	1.19
12	54.20	36.07	81.10	88.50	6.00	9.99	-1	0.00	0.00	0.84
13	54.20	42.39	54.30	88.50	6.50	5.82	-1	0.00	0.00	0.51
14	55.60	47.60	61.20	88.50	7.00	3.28	-1	0.00	0.08	0.42
15	55.60	52.71	69.50	88.50	7.50	4.61	-1	0.00	0.00	0.41
16	56.30	64.79	54.40	88.50	8.00	10.83	0.00	0.00	0.04	0.97
17	60.00	65.98	41.30	88.50	8.50	-2.96	-1	0.00	0.20	0.09
18	60.90	74.73	48.20	88.50	9.00	7.31	0.04	0.00	0.05	0.70
19	62.70	90.32	35.80	88.50	9.50	13.33	0.12	0.00	0.10	1.25
20	62.80	93.12	31.40	88.50	10.00	2.25	0.14	0.00	0.02	0.22
21	62.80	93.12	39.60	88.50	10.40	-0.44	0.13	0.00	0.00	0.00
22	69.50	99.13	56.40	75.10	10.90	-1.20	0.13	0.00	0.39	0.48
23	69.50	103.55	40.40	69.20	11.30	4.05	0.15	0.71	0.00	0.35
24	71.10	107.88	31.50	61.80	11.60	2.45	0.16	0.71	0.09	0.34
25	72.10	120.34	22.60	37.30	11.90	11.07	0.20	4.07	0.06	0.99
26	77.90	123.59	24.20	32.00	12.10	-2.72	0.19	4.07	0.32	0.26
27	78.40	125.96	27.30	28.80	12.30	1.72	0.19	5.19	0.03	0.18
28	79.90	130.01	30.00	24.60	12.40	2.46	0.20	5.19	0.08	0.32
29	81.60	131.25	27.00	23.50	12.50	-0.61	0.20	5.19	0.10	0.09
30	82.40	138.79	31.10	17.00	12.60	6.62	0.21	7.76	0.05	0.60
[...]										
35	84.70	158.20	17.20	8.20	13.00	1.52	0.24	11.33	0.00	0.12
36	84.70	161.50	20.20	6.60	13.00	3.26	0.25	12.30	0.00	0.26
37	85.60	166.45	25.00	5.00	13.00	4.04	0.25	12.30	0.05	0.39
38	86.00	169.14	17.20	4.50	13.10	2.17	0.26	13.83	0.03	0.21
39	86.30	172.57	17.90	3.60	13.10	3.17	0.26	13.83	0.01	0.27
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

A single sample drawn from 100,000 Monte Carlo simulations of a leveraged-buyout fund structure with $C_0 = \$100$. Columns are as follows:

- `Qtr`: the calendar quarter (1 thru 40)
- `D`: the cumulative drawdowns (in $)
- `R`: the cumulative distributions (in $)
- `NIC`: the net invested capital (in $)
- `MF`: the GP's (fixed) management fees (in $)
- `dNCF`: the instantaneous change in net cash flow (in $)
- `IRR`: the internal rate of return for this quarter; a value of -1 indicates that the IRR is not available for that quarter
- `CI`: the cumulative GP's carried interest (in $); the carried-interest is called when IRR > 8%
- `dTF`: the instantaneous change in the GP's carried interest (in $)
- `dMOF`: the instantaneous change in the GP's monitoring fees (in $)

A PE fund's life cycle

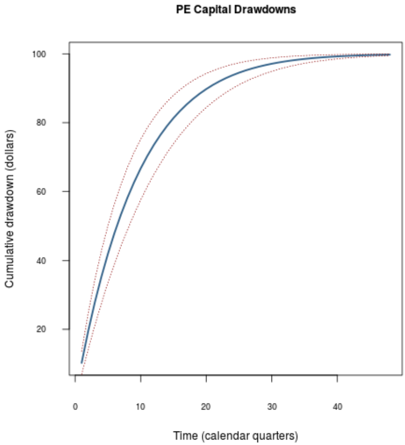
1. GP forms a new fund
2. GP raises capital from LPs
3. LP commits C_0 in capital for T_L
4. GP draws on each LP's C_0 for T_I, where $I \leq L$
5. GP invests in portfolio companies throughout T_I
6. GP harvests investments at any time $0 < t \leq T_L$
7. GP exacts fees from LPs' committed capital (some fixed, some variable)
8. GP distributes proceeds according to the fund's waterfall
9. GP fully liquidates the fund at some time $0 \leq t \leq T_L$

A PE fund's life cycle

- Capital drawdowns, or calls
- Capital distributions, or returns
- Fund value
- Net cash distribution

A PE fund's life cycle

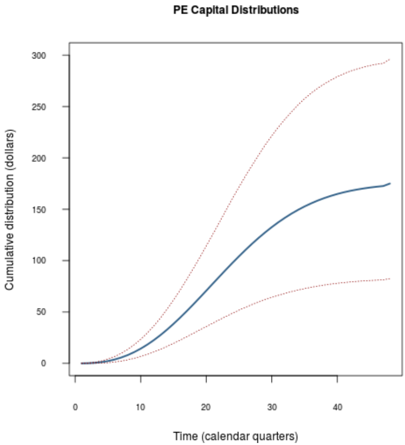
- Capital drawdowns, or calls



- Capital distributions, or returns
- Fund value

A PE fund's life cycle

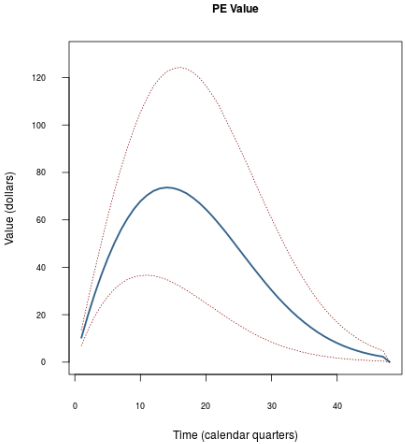
- Capital drawdowns, or calls
- Capital distributions, or returns



- Fund value

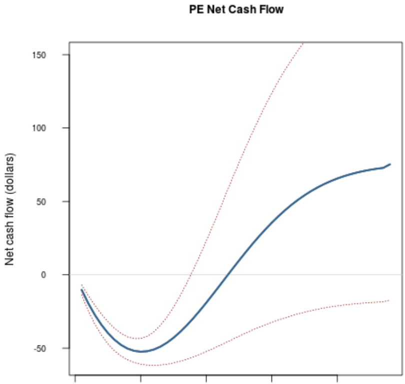
A PE fund's life cycle

- Capital drawdowns, or calls
- Capital distributions, or returns
- Fund value



A PE fund's life cycle

- Capital drawdowns, or calls
- Capital distributions, or returns
- Fund value
- Net cash distribution



Fund value

- Let V_t denote the value of the fund at time t
- Let D_t denote the cumulative capital drawdowns from the LPs up to time t
- Let R_t denote the cumulative capital distributions to the LPs up to time t
- $B_{M,t}$ is a standard Brownian motion driving aggregate stock market returns, such that $r_{M,t} = \mu_M + \sigma_M \, \mathrm{d}B_{M,t}$, where μ_M is the mean rate of return of the aggregate stock market ("the market"), and σ_M is the returns volatility of the market
- $B_{\varepsilon,t}$ is a second Brownian motion, representing idiosyncratic shocks to the fund, where $\mathrm{d}B_{M,t} \, \mathrm{d}B_{\varepsilon,t} = 0$, the mean rate of return of the idiosyncratic shocks is zero, and σ_ε is the volatility of the idiosyncratic shocks

Assumption

The dynamics of the fund value, V_t, under the real-world probability measure $\mathbb{P}$, can be described by the stochastic process $\{V_t,\ 0 \leq t \leq T_L\}$:

$$\mathrm{d}V_t = V_t\Big(\mu_V \mathrm{d}t + \beta_V \sigma_M \mathrm{d}B_{M,t} + \sigma_\varepsilon \mathrm{d}B_{\varepsilon,t}\Big) + \mathrm{d}D_t - \mathrm{d}R_t, \tag{16}$$

where $\mu_V > 0$ is the mean rate of return of the fund, and β_V is the market beta of the fund

Capital drawdowns

- Let I_0 be the capital available for investment, *i.e.* C_0 less fees. For simplicity we can at first assume that $I_0 = C_0$

Assumption

The dynamics of the cumulative capital drawdowns, D_t, can be described by the ordinary differential equation:

$$\mathrm{d}D_t = \delta_t (I_0 - D_t)\mathbf{1}_{\{0 \le t \le T_I\}}\mathrm{d}t, \tag{17}$$

where $\mathbf{1}_{\{\cdot\}}$ is an indicator function. The fund's drawdown rate δ_t is assumed to follow a stochastic process $\{\delta_t, 0 \le t \le T_I\}$ given by:

$$\delta_t = \delta + \sigma_\delta B_{\delta,t}, \tag{18}$$

where $\delta > 0$ is the mean of the drawdown rate, $\sigma_\delta > 0$ is the volatility of the drawdown rate; $B_{\delta,t}$ is a third standard Brownian motion for which it is assumed that $\mathrm{d}B_{\delta,t}\mathrm{d}B_{M,t} = \rho_\delta \mathrm{d}t$, where ρ_δ is the correlation between drawdown rate and stock market returns, and $\mathrm{d}B_{\delta,t}\mathrm{d}B_{\varepsilon,t} = 0$. In order to avoid negative drawdown rates, we use $\delta_t^+ = \max(\delta_t, 0)$ in the model implementation.

Capital distributions

Assumption

The dynamics of the cumulative capital distributions, R_t, can be described by:

$$\mathrm{d}R_t = \nu_t V_t dt, \text{ for } t < T_L, \quad \text{and} \quad R_t = V_t \mathbf{1}_{\{t=T_L\}} + \int_0^t \nu_u V_u \mathrm{d}u, \text{ for } t \leq T_L \quad (19)$$

The fund's distribution rate ν_t is assumed to follow a stochastic process $\{\nu_t,\ 0 \leq t \leq T_L\}$ given by:

$$\nu_t = \nu t + \sigma_\nu B_{\nu,t}, \quad (20)$$

where ν is the mean distribution rate, and $\sigma_\nu > 0$ is the volatility of the distribution rate; $B_{\nu,t}$ is a fourth standard Brownian motion for which it is assumed that $\mathrm{d}B_{\nu,t}\mathrm{d}B_{M,t} = \rho_\nu \mathrm{d}t$, where ρ_ν is the correlation between the drawdown rate and stock market returns, and $\mathrm{d}B_{\nu,t}\mathrm{d}B_{\varepsilon,t} = 0$. In order to avoid negative distributions rates, we use $\nu_t^+ = \max(\nu_t, 0)$ in the model implementation.

Manager compensation

GPs typically receive three types of compensation for managing the investments:

1. A performance-related component called "carried interest" or simply "carry". Carry ranges from 0% to 50%, but sharply peaked around 20% (ample data to support)
2. A (typically fixed) fee called the "management fee". The fixed fee is usually charged quarterly; annualized, the fee ranges from 1% to 3%, but it is sharply peaked around 2% (Ample data to support vanilla flat fees, not so the more exotic combinations)
3. A fixed fee for setting up the fund (anecdotal evidence:[1] usually a flat fee—up to 1% of the committed capital?)
4. Fees charged to the portfolio companies (Leveraged Buyout Funds):
   - transaction fees (anecdotal evidence: 1.37%)[2]
   - monitoring fees (anecdotal evidence: 2%)[3]

[1] Metrick, A. and Yasuda, A. (2010) "The Economics of Private Equity Funds", *Review of Financial Studies*, **23** (6), p. 2315. The fund may cap this fee (also known as the "establishment cost") at a flat $1 MM.

[2] *ibid.* p. 2319, *et seq.*

[3] *ibid.* p. 2319, *et seq.*

Management fees

- The management fee is levied against a basis: this is usually either the committed capital, C_0, or the net invested capital ("NIC"),[4] and it is one of four different types that is specified in the limited partnership agreement ("LPA"):
  1. flat fee
  2. tapered fee: tapers after the investment period, $T_I < t \leq T_L$
  3. change basis to NIC after investment period with flat fee[5]
  4. change basis to NIC after investment period with tapered fee
- Let MF_t denote the cumulative management fees up to some time $t \in [0, T_L]$.
- **Fixed Management Fees:** If management fees are defined as a percentage c_{MF} of the committed capital C_0 and are paid continuously, the dynamics are given by:

$$\mathrm{dMF}_t = c_{\mathrm{MF}} C_0 \mathrm{d}t \tag{21}$$

- **Management Fees with Change in Basis**: Latterly, tapered management fees appear to be gaining in popularity. The tapering typically begins after the investment period, *i.e.* for $T_I < t \leq T_L$, and reflects the fact that less time is required by the GP in managing the activities of the portfolio companies. Many funds change the fee basis from committed capital (during the commitment period) to NIC capital (after the commitment period).

[4] Invested capital minus the cost basis of exited investments, *ibid.* p. 2315, *et seq.*

[5] *ibid.* p. 2315, *et seq.*

Management fees: basis change to NIC requires *ex ante* computation

- If *ab initio* the basis for management-fee calculation is agreed to change from committed capital, C_0, for $0 \leq t \leq T_I$, to NIC for $T_I < t \leq T_L$, then how do GPs determine I_C, the capital available for investment, for $t \leq T_I$? Is it specified in the LPA?
- We use an iterative algorithm to arrive at the NIC (convergence is rapid):
  1. Set the initial guess for NIC to C_0
  2. Subtract the fixed management fees applicable for $t \leq T_I$, which we know at $t = 0$ to follow
     $$\mathrm{dMF}_t = c_{\mathrm{MF}} C_0 \mathrm{d}t \, \mathbf{1}_{0 \leq t \leq T_I} \tag{22}$$
     the value of NIC_t for $t = T_I$ is then initialized to $C_0 - \mathrm{MF}_{T_I}$
  3. The dynamics of management fees for $T_I < t \leq T_L$ are assumed to follow:
     $$\mathrm{dMF}_t = c_{\mathrm{MF}} \mathrm{NIC}_t \mathrm{d}t \, \mathbf{1}_{T_I < t \leq T_L} \tag{23}$$
  4. The fund's distribution rate, ν_t, is assumed to follow a stochastic process $\{\nu_t, \ 0 \leq t \leq T_L\}$ given by $\nu_t = \nu t + \sigma_\nu B_{\nu,t}$, as per Equation 20, and this rate is applied to the NIC to give its dynamics as:
     $$\mathrm{dNIC}_t = \nu_t \mathrm{NIC}_t \mathrm{d}t \tag{24}$$
  5. Finally, we can solve for the invested capital I_C, by noting[6] that at $t = 0$ it must be the case that $I_C = C_0 - \mathrm{NPV}(\mathrm{MF}_{T_I}) - \mathrm{NPV}(\mathrm{MF}_{T_L})$, where the last term can be expressed as $x \times I_C$ for some fraction x

[6] As Metrick & Yasuda suggest, *ibid.* p. 2309, *et seq.*

Carried interest (I)

- Let CI_t denote the cumulative carried interest up to some time $t \in [0, T_L]$
- **Carried Interest**: Let the carried interest level be given by c_{CI} and let h denote the hurdle rate. The **dynamics of carried interest** are given by:

$$\mathrm{dCI}_t = c_{\mathrm{CI}} \max\Bigg\{\underbrace{\mathrm{d}R_t - \mathrm{d}D_t - \mathrm{dMF}_t}_{\text{net cash flow } = \, \mathrm{dNCF}_t},\ 0\Bigg\} \mathbf{1}_{\{\mathrm{IRR}_t > h\}}$$

  where $\mathbf{1}_{\{\mathrm{IRR}_t > h\}}$ indicates that carried interest is only payable at time t if the internal rate of return of the fund at t, IRR_t, exceeds the hurdle rate h
- **Catch-up provision**: Most LPAs that contain a hurdle rate also include a provision that provides the GPs with a greater share of the profits once the hurdle rate has been paid and until the carry level has been reached

Carried Interest (II)

- **Carried interest with catch-up**: If the carried interest is paid with a 100% catch-up provision, then its dynamics are given by:

$$\mathrm{dCI}_t = \begin{cases} c_{\mathrm{CI}} \max\Big\{\mathrm{dNCF}_t,\, 0\Big\} \mathbf{1}_{\{\mathrm{IRR}_t > h\}}, & \text{if } \mathrm{CI}_t/(R_t - C_0) = c_{\mathrm{CI}} \\ \min\Big\{c_{\mathrm{CI}}(R_t - C_0) - \mathrm{CI}_t,\, \mathrm{dNCF}_t\Big\} \mathbf{1}_{\{\mathrm{IRR}_t > h\}}, & \text{if } \mathrm{CI}_t(R_t - C_0) < c_{\mathrm{CI}} \end{cases} \tag{25}$$

where $\mathrm{dNCF}_t = \mathrm{d}R_t - \mathrm{d}D_t - \mathrm{dMF}_t$

Portfolio company fees—monitoring fees

- Let MoF_t denote the cumulative monitoring fees paid up to time $t \in [0, T_L]$ and assume that monitoring fees are paid at exit as a fraction c_{MoF} of the total firm value
- If s_F denotes the (average) share the fund holds in its portfolio companies, the **dynamics of the monitoring fees** can be modeled by:

$$\mathrm{dMoF}_t = c_{\mathrm{MoF}} \mathrm{d}R_t \times \left(\frac{1 + l}{s_F} \right) \tag{26}$$

- We use the typical sharing rule and allocate 20% of the monitoring fees to the GP and 80% to the LPs, *i.e.* $\mathrm{dMoF}_t^{(\mathrm{LP})} = 0.8 \times \mathrm{dMoF}_t$ and $\mathrm{dMoF}_t^{(\mathrm{GP})} = 0.2 \times \mathrm{dMoF}_t$

Calibrated model parameters

Table: Parameters used in model

Parameter	Notation	Value
Life of the PE fund investment (years)	T_L	10
Simulation frequency (years)	$\mathrm{d}t$	1/4
Committed capital (US dollars)	C_0	100
Risk-free rate	r_f	0.05%
Expected return of stock market	μ_M	0.10
Volatility of stock market returns	σ_M	0.15
Alpha of PE funds	α	0.075
Market beta of PE funds	β_M	1.30
Idiosyncratic volatility of PE fund returns	σ_ε	0.35
Drawdown rate of PE funds	δ	0.34%
Volatility of the drawdown rate	σ_δ	0.27
Correlation between drawdown rate and stock market returns	ρ_δ	0.30
Average distribution rate	ν	0.07%
Volatility of the distribution rate	σ_ν	0.16
Correlation between distribution rate and stock market returns	ρ_ν	0.50
Management fee	c_{MF}	2%
Hurdle rate	h	8%
Carried interest	c_{CI}	20%
Transaction fees	c_{TF}	1.4%
Monitoring fees	c_{MoF}	2%
Leverage (debt-to-equity ratio)	l	3

Model parameters are stated as annualized units, except where indicated

Contents

Back to that simulation

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
1	7.80	0.00	7.70	88.50	0.50	-8.30	-1	0.00	0.44	0.00
2	17.40	0.03	13.80	88.50	1.00	-10.11	-1	0.00	0.54	0.00
3	22.40	0.42	16.60	88.50	1.50	-5.04	-1	0.00	0.28	0.03
4	23.60	0.42	19.90	88.50	2.00	-1.70	-1	0.00	0.07	0.00
5	31.70	0.42	25.10	88.50	2.50	-8.69	-1	0.00	0.46	0.00
6	35.10	0.42	30.90	88.50	3.00	-3.84	-1	0.00	0.19	0.00
7	41.90	3.37	34.70	88.50	3.50	-4.40	-1	0.00	0.38	0.24
8	45.90	3.37	50.80	88.50	4.00	-4.48	-1	0.00	0.22	0.00
9	51.90	10.07	66.50	88.50	4.50	0.19	-1	0.00	0.34	0.54
10	54.20	10.65	93.40	88.50	5.00	-2.14	-1	0.00	0.12	0.05
11	54.20	25.58	90.10	88.50	5.50	14.43	-1	0.00	0.00	1.19
12	54.20	36.07	81.10	88.50	6.00	9.99	-1	0.00	0.00	0.84
13	54.20	42.39	54.30	88.50	6.50	5.82	-1	0.00	0.00	0.51
14	55.60	47.60	61.20	88.50	7.00	3.28	-1	0.00	0.08	0.42
15	55.60	52.71	69.50	88.50	7.50	4.61	-1	0.00	0.00	0.41
16	56.30	64.79	54.40	88.50	8.00	10.83	0.00	0.00	0.04	0.97
17	60.00	65.98	41.30	88.50	8.50	-2.96	-1	0.00	0.20	0.09
18	60.90	74.73	48.20	88.50	9.00	7.31	0.04	0.00	0.05	0.70
19	62.70	90.32	35.80	88.50	9.50	13.33	0.12	0.00	0.10	1.25
20	62.80	93.12	31.40	88.50	10.00	2.25	0.14	0.00	0.02	0.22
21	62.80	93.12	39.60	88.50	10.40	-0.44	0.13	0.00	0.00	0.00
22	69.50	99.13	56.40	75.10	10.90	-1.20	0.13	0.00	0.39	0.48
23	69.50	103.55	40.40	69.20	11.30	4.05	0.15	0.71	0.00	0.35
24	71.10	107.88	31.50	61.80	11.60	2.45	0.16	0.71	0.09	0.34
25	72.10	120.34	22.60	37.30	11.90	11.07	0.20	4.07	0.06	0.99
26	77.90	123.59	24.20	32.00	12.10	-2.72	0.19	4.07	0.32	0.26
27	78.40	125.96	27.30	28.80	12.30	1.72	0.19	5.19	0.03	0.18
28	79.90	130.01	30.00	24.60	12.40	2.46	0.20	5.19	0.08	0.32
29	81.60	131.25	27.00	23.50	12.50	-0.61	0.20	5.19	0.10	0.09
30	82.40	138.79	31.10	17.00	12.60	6.62	0.21	7.76	0.05	0.60
[...]										
35	84.70	158.20	17.20	8.20	13.00	1.52	0.24	11.33	0.00	0.12
36	84.70	161.50	20.20	6.60	13.00	3.26	0.25	12.30	0.00	0.26
37	85.60	166.45	25.00	5.00	13.00	4.04	0.25	12.30	0.05	0.39
38	86.00	169.14	17.20	4.50	13.10	2.17	0.26	13.83	0.03	0.21
39	86.30	172.57	17.90	3.60	13.10	3.17	0.26	13.83	0.01	0.27
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

A single sample drawn from 100,000 Monte Carlo simulations of a leveraged-buyout fund structure with $C_0 = \$100$. Columns are as follows:

- `Qtr`: the calendar quarter (1 thru 40)
- `D`: the cumulative drawdowns (in $)
- `R`: the cumulative distributions (in $)
- `NIC`: the net invested capital (in $)
- `MF`: the GP's (fixed) management fees (in $)
- `dNCF`: the instantaneous change in net cash flow (in $)
- `IRR`: the internal rate of return for this quarter; a value of -1 indicates that the IRR is not available for that quarter
- `CI`: the cumulative GP's carried interest (in $); the carried-interest is called when IRR > 8%
- `dTF`: the instantaneous change in the GP's carried interest (in $)
- `dMOF`: the instantaneous change in the GP's monitoring fees (in $)

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.09 ($\pm$\$0.0032)
  - Put: \$0.32 ($\pm$\$0.0119)

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.09 ($\pm$\$0.0032)
  - Put: \$0.32 ($\pm$\$0.0119)

- Price of a call and a put on the carried interest struck at $K = \$10$ with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.09 ($\pm$\$0.0032)
  - Put: \$0.32 ($\pm$\$0.0119)

- Price of a call and a put on the carried interest struck at $K = \$10$ with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.44 ($\pm$\$0.0183)
  - Put: \$10.27 ($\pm$\$0.1882)

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.09 ($\pm$\$0.0032)
  - Put: \$0.32 ($\pm$\$0.0119)
- Price of a call and a put on the carried interest struck at $K = \$10$ with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.44 ($\pm$\$0.0183)
  - Put: \$10.27 ($\pm$\$0.1882)
- But we can, in fact, price anything.

Options on a leveraged buyout fund

Qtr	D	R	V	NIC	MF	dNCF	IRR	CI	dTF	dMOF
40	86.60	198.06	0.00	0.00	13.10	25.14	0.28	19.61	0.02	2.03

- Price of a call and a put on the cash multiple (*i.e.* R/C_0) struck at 1.0 with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.09 ($\pm$\$0.0032)
  - Put: \$0.32 ($\pm$\$0.0119)

- Price of a call and a put on the carried interest struck at $K = \$10$ with maturity in $T = T_L = 10$ years (where the coefficient of risk aversion is $\gamma = 5$):
  - Call: \$0.44 ($\pm$\$0.0183)
  - Put: \$10.27 ($\pm$\$0.1882)

- But we can, in fact, price anything. At any maturity.

Price a call and a put on a PE fund's cash multiple

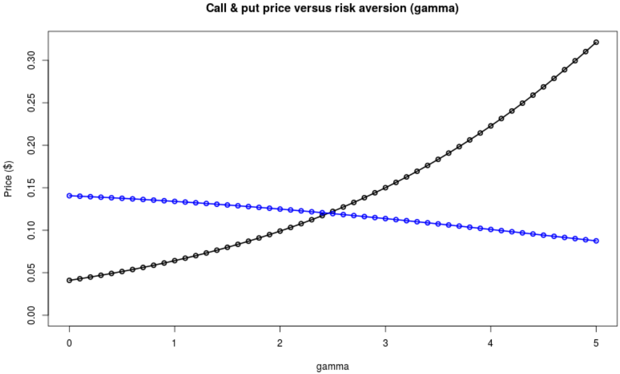


Figure: Using the Rubinstein model to estimate the value of a put (in black) and a call (in blue) on the cash multiple (*i.e.* the ratio of the cumulative returns R to the initial investment C_0: cash multiple $= R/C_0$) of the leveraged-buyout fund parameterized in Table 1.

Price a call and a put on a PE fund's carried interest (compound option)

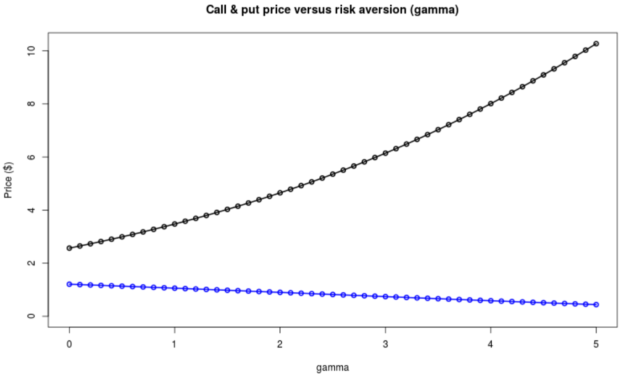


Figure: Using the Rubinstein model to estimate the value of a put (in black) and a call (in blue) on the carried interest of the leveraged-buyout fund parameterized in Table 1. This is a compound option, *i.e.* an option on an option, because the carried interest is itself an option (see Equation 25).

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Conclusion

- With the right tools we can price (just about) anything in the marketplace

Conclusion

- With the right tools we can price (just about) anything in the marketplace
- Can we put on this private-equity trade?

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References

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