The Epistemic Modal Logic

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Abstract

This write-up will explain in brief some important concepts of Epistemic Modal Logic. We will not be much concerned with doxatic modality here.

1 Introduction

Epistemic modal logic is a subfield of modal logic that is concerned with reasoning about knowledge. While epistemology has a long philosophical tradition dating back to Ancient Greece, epistemic logic is a much more recent development with applications in many fields, including philosophy, theoretical computer science, artificial intelligence, economics and linguistics. While philosophers since Aristotle have discussed modal logic, and Medieval philosophers such as Avicenna, Ockham, and Duns Scotus developed many of their observations, it was C. I. Lewis who created the first symbolic and systematic approach to the topic, in 1912. It continued to mature as a field, reaching its modern form in 1963 with the work of Kripke.

2 The Language / Syntax

2.1 Primitive Language

Agent : The subject whose knowledge and belief is being talked about. A_g is the set of such agents.
K_a : modal operator for knowledge. So, $K_a\psi$ is read as "agent a knows that ψ"
B_a : modal operator for Belief. So, $B_a\psi$ is read as "agent a believes that ψ"
O_p : set of such modal operators
A_t : set of primitive propositions (from PL)

$$\phi := p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi$$

where $p \in A_t$ and $\Box \in O_p$.

2.2 Some other modalities in EML

M_a : dual of K. also denoted as $\hat{K}_a$. defined as $\neg K_a \neg\phi$.
$E_A\phi$: everyone in A knows. defined as $\bigwedge_{a\in A} K_a\phi$
$\Box^k\phi$: $\Box^0\phi = \phi$ and $\Box^{k+1}\phi = \Box\,\Box^k\phi$
$\Box_A$: $A \subseteq A_g$. similar to $\Box_a$ but this time a group of agents know

2.3 Length and modal depth

the length $|\,\phi\,|$ and the modal depth $d(\phi)$ of a formula ϕ are both defined inductively as

$$\begin{aligned}
|p| = 1 \quad &and \quad d(p) = 0\\
|\neg\phi| = |\phi| + 1 \quad &and \quad d(\neg\phi) = d(\phi)\\
|(\phi \wedge \psi)| = |\phi| + |\psi| + 1 \quad &and \quad d(\phi \wedge \psi) = maxd(\phi), d(\psi)\\
|\Box_a\phi| = |\phi| + 1 \quad &and \quad d(\Box\phi) = 1 + d(\phi)
\end{aligned}$$

3 Semantics

3.1 Kripke model (K)

Given a set of A_t propositions and a set of A_g agents, a Kripke model is a structure M $= \langle S,\, R^{A_g},\, V^{A_t}\rangle$, where

1. $S \neq \emptyset$ is a set of states, denoted by domain of M, $D(M)$
2. R^{A_g} is a function, yielding an accessibility relation $R_a \subseteq S \times S$ for agent $a \in A_g$.
3. $V^{A_t} : S \rightarrow (A_t \rightarrow true, false)$ is function that, $\forall p \in A_t$ and $s \in S$, determines what truth value $V^{A_t}(s)(p)$ of p is in state s.

This model is assumed for knowledge modality only and can be easily extended to belief modality by replacing R^{A_g} by R^{O_p}.
A Kripke frame will then become $F = \langle S,\, R\rangle$.
(M, s) where s is a state is called *pointed model.*

Now, given the model, lets define truth value for this model. So, $M, s \models \phi$ inductively as follows :

$$\begin{array}{lll}
M, s \models p & iff & V(s)(p) = true\ for\ p \in A_t\\
M, s \models \phi \wedge \psi & iff & M, s \models \phi\ and\ M, s \models \psi\\
M, s \models \neg\phi & iff & M, s \not\models \phi\\
M, s \models K_a\phi & iff & M, t \models \phi\ \forall\ t\ st\ R_a st\\
M, s \models M_a\phi & iff & M, t \models \phi\ for\ some\ t\ st\ R_a st\\
M \models \phi & iff & M, s \models \phi\ \forall\ s \in\ S
\end{array}$$

Let χ be class of modals, i.e. , $\chi \subseteq K$,

- If $M \models \phi\ \forall\ M\ \in \chi$. then ϕ is valid in χ, i.e., $\chi \models \phi$.
- If $\exists\ M \in \chi$ and a state $s \in D(M)$ st $M, s \models \phi$, then ϕ is satisfiable in χ.

3.2 Frame Properties

Let R be the accessibility relation on domain of states S.

1. R is serial if $\forall$ s there is t st Rst. This class of models is denoted KD.
2. R is reflexive if $\forall$ s, Rss. This class of models is denoted KT.
3. R is transitive if $\forall$ s,t,u, if Rst and Rtu, then Rsu. This class of models is denoted $K4$.
4. R is euclidean if $\forall$ s,t,u, if Rst and Rsu, then Rtu. This class of models is denoted $K5$.
5. R is symmetric if $\forall$ s,t,u, if Rst, then Rts. This class of models is denoted KB.
6. We can combine the properties of relations:
   (a) The class of reflexive transitive models is denoted by $S4$.
   (b) The class of Euclidean transitive models is denoted by $K45$.
   (c) The class of serial transitive Euclidean models is denoted by $KD45$.
   (d) The class of models where relations are equivalent(reflexive, symmetric and transitive or reflexive or euclidean) is denoted by $S5$. (Major interest)

3.3 Valid Formulas

- If $\chi \models \phi \to \psi$ and $\chi \models \phi$ then $\chi \models \psi$.
  Modus Ponens
- If $\chi \models \phi$ then $\chi \models \; K\phi$.
  If a formula holds in a class of models, then all the agents should know this too.
- $\chi \models K\ (\phi \to \psi) \to (\ K\phi \to \psi)$.
  An agent can apply modus ponens to its knowledge.
- $KD \models K\phi \to M\phi$
  Instantiating the knowledge in serial models and an agent can't know a formula and its negation at same time.
- $KT \models K\phi \to \phi$
  In reflexive models, what an agent knows must be true.
- $K4 \models K\phi \to K\,K\phi$
  Positive introspection, an agent knows what it knows.
- $K5 \models \neg K\phi \to K\neg K\phi$
  Negative introspection, an agent knows what it doesn't know.
- $KB \models \phi \to K\,M\phi$
  If a formula is true, then agent knows that it is possible that formula is true
- If $\chi \subseteq A$ then $A \models \phi \implies \chi \models \phi$
  submodel property, common to most logics.

4 Group Knowledge

4.1 Notions

1. **Distributed Knowledge** : By now, we know E_A as the operator where all agents in A know. We interpret E_A as the necessity operator of relation $\bigcup_{a \in A} R_a$. Now, we introduce even stronger operator D_A named as distributed operator in which we basically allow agents to interact and share information. It can be interpreted as necessity operator of relation $\bigcap_{a \in A} R_a$.

2. **Common Knowledge** : Denoted by $C_A\phi$ that says that $\forall$ n, $E^n\phi$ holds. Or ϕ is false when someone knows that someone knows that ..(n-3 times).. someone knows that ϕ is false.

4.2 Semantics

Let $A \subseteq A_g$ be group of agents. Let $R_{E_A} = \bigcup_{a \in A} R_a$, $R_{D_A} = \bigcap_{a \in A} R_a$ and R_{C_A} be defined as transitive closure of $R_{E_A} = (\bigcup_{a \in A} R_a)^+$,i.e., there is some path s = $s_0, s_1, .., s_n = t$ from s to t st $n \geq 1$ and $\forall\ i\ <\ n$, there is some agent $a \in A$ for which $R_a\ s_i\ s_{i+1}$ holds.

- $(\,M, s\,) \models E_A\phi$ iff $\forall$ t st $R_{E_A}\, s\, t$, we have $(\,M, t\,) \models \phi$.
- $(\,M, s\,) \models D_A\phi$ iff $\forall$ t st $R_{D_A}\, s\, t$, we have $(\,M, t\,) \models \phi$.
- $(\,M, s\,) \models C_A\phi$ iff $\forall$ t st $R_{C_A}\, s\, t$, we have $(\,M, t\,) \models \phi$.

Some more validities :

- $\kappa \models (C_A\phi \rightarrow E_A\phi) \wedge (E_A\phi \rightarrow K_a\phi) \wedge (K_a\phi \rightarrow D_A\phi)$ where $a \in A$. Common knowledge leads to whole crowd knowing ϕ, everyone knowing leads to an agent knowing ϕ and an agent knowing ϕ leads to distributed knowledge of ϕ.

- $T \models D_a\phi \rightarrow \phi$, i.e., distributed knowledge in single agent case is similar to reflexive models.

5 Bisimulation

Given models M = (S,R,V) and M' = (S',R',V'), a non-empty relation $Z \subseteq S \times S'$ is a bisimulation between M and M' iff for all $s \in S$ and $s' \in S'$ with $(s, s') \in Z$:

- V(s)(p) = V'(s')(p) for all $p \in A_t$.

- for all $a \in A_g$ and all $t \in S$, if $R_a\, s\, t$, then there is a $t' \in S'$ st $R'_a\, s'\, t'$ and $(t, t') \in Z$. (forth)

- for all $a \in A_g$ and all $t' \in S'$, if $R'_a\, s'\, t'$, then there is a $t \in S$ st $R_a\, s\, t$ and $(t, t') \in Z$. (back)

We write it as $(M, s) \underline{\leftrightarrow} (M', s')$ and say (M,s) and (M',s') are bisimilar.

5.1 Preservation under bisimulation

Theorem: Suppose $(M, s)\underline{\leftrightarrow}(M', s')$. Then, for all formulas $\phi \in L_{CK}$, we have

$$M, s \models \phi \Leftrightarrow M', s' \models \phi$$

Proof: We will apply induction on the length of formula ϕ here.

- $\phi = p \in A_t$ or $\phi = \neg\psi$ or $\phi = \psi \wedge \phi$
  By definition of bisimulation and using contradiction for second case.

- $\phi = B_a\psi$
  Given $(M, s) \models B_a\psi$, To prove $(M', s') \models B_a\psi$. Other way will be similar.

  $$\Longrightarrow (M, s) \models B_a\psi$$
  $$\Longrightarrow \exists\ t\ st\ R_a\, s\, t\ and\ (M, t) \models \psi$$

  Using given condition of bisimulation, we get

  $$\Longrightarrow \exists\ t'\ st\ R'_a\, s'\, t'\ and\ (t', t) \in\ Z$$
  $$\Longrightarrow (M', t') \models \psi$$
  $$\Longrightarrow (M', s') \models B_a\psi$$

- $\phi = C_A\psi$
  For simplicity, we will prove only one side and call the agent for which R_{C_A} holds 'a'.

  $$\Longrightarrow (M, s) \models C_A\psi$$
  $$\Longrightarrow \exists\ t\ st\ there\ is\ some\ path\ s\ = s_0,\ ...\ s_n =\ t\ st\ n \geq 1$$
  $$and\ \forall\ i\ <\ n\ , R_a\, s_i\, s_{i+1}\ and\ (M, t) \models \psi$$

  Now, we need to show that if $(M, s_i)\underline{\leftrightarrow}(M', s'_i)$ then $(M, s_{i+1})\underline{\leftrightarrow}(M', s'_{i+1})$ where $R_a\, s_i\, s_{i+1}$ and $R'_a\, s'_i\, s'_{i+1}$. But this is true by forth condition of bisimulation. So, by applying this condition, we get $(M', t') \models \psi$. Now, by applying definition of C_A operator, we get $(M', s') \models C_A\psi$

HP. This preservation can be extended to group of agents also.

6 Decidability

Finite model property :
For all classes of models χ and language L, we have, for all $\phi \in L$

$$\chi \models \phi\ iff\ Fin(\chi) \models \phi$$

,i.e., a formula is satisfiable in a model in χ iff it is satisfiable in finite model in χ.

Size of Models :
For a finite model M $= \langle S, R^{A_g}, V^{A_t} \rangle$. the size of M, denoted as $\|M\|$ is $|S|\ +\ |R_a|$, where $|S|$ is number of states in S and $|R_a|$ is number of pairs

of $|R_a|$.

Proposition :
For all $\phi \in L$, ϕ is satisfiable in χ iff there is a model $M \in \chi$ st $|D(M| \leq 2^{|\phi|}$ and ϕ is satisfiable in M.

6.1 Model Checking

Denoted as $MODCHECK_\chi$. Given a formula $\phi \in L$ and a pointed model (M,s) with $M \in \chi$ and $s \in DM$, is it case that $M,\, s \models \phi$? It is a decidable problem.

Theorem : Model checking formulas in L(A_t,O_p,A_g), with $O_p = \{K_a |\; a \in A_g\}$, in finite models is in P.
Proof : We describe an algorithm that, given a model M $= \langle S,\, R^{A_g},\, V^{A_t} \rangle$ and a formula $\phi \in L$, determines in time polynomial in $|\phi|$ and $\|M\|$ whether $M,\, s \models \phi$. Given ϕ, we order the subformulas ϕ_1, ϕ_2, ..., ϕ_m of ϕ st if ϕ_i is subformula of ϕ_j then $i < j$. Also, $m \leq |\phi|$. We claim that:

(*) for every $k \leq m$, we can label each state s in M with either ϕ_j (if ϕ_j is true at s) or $\neg\phi_j$(otherwise), for every $j \leq k$, in $k\|M\|$ steps.

We continue the proof by induction on k.

- k = 1
  So, ϕ_1 is primitive proposition, we need $|M| \leq \|M\|$ steps to label all the states.

- Holds for k, i.e., collection of formulas ϕ_1, ϕ_2, ..., ϕ_k can be labelled in $k\|M\|$ steps. To show for k+1.
  If ϕ_{k+1} is primitive proposition ,then above reason holds.
  If ϕ_{k+1} is a negation formula, then it must be of the form $\neg\phi_j$ for some $j \leq k$. To label this formula, we just need to label it in states where ϕ_j doesn't hold, Hence it will take atmost $\|M\|$ steps more. So, total steps are atmost $(k+1)\|M\|$.
  Similar analysis for $\wedge$ case.
  If ϕ_{k+1} is of the form $K_a\phi_j$ for some $j \leq k$, we label a state s with $K_a\phi_j$ iff each state t with $R_a\, s\, t$ is labelled with ϕ_j. This will take extra $R_a(s) \leq \|M\|$ steps.

HP.

6.2 Satisfiability

Denoted as SAT_χ. Given a formula $\phi \in L$, does there exist a model $M \in \chi$ and a state $s \in D(M)$ st $M,\, s \models \phi$? It is a decidable problem.

Theorem : The complexity of satisfiability problem is

1. NP-complete if $\chi \in \{KD45,\, S5\}$ and L = L(A_t, O_p, $A_{g=1}$).

2. PSPACE-complete if

- $\chi \in \{K, T, S4\}$ and L = L(A_t, O_p, $A_{g\geq 1}$) or
- $\chi \in \{KD45, S5\}$ and L = L(A_t, O_p, $A_{g\geq 2}$)

3. EXPTIME-complete if

   - $\chi \in \{K, T\}$ and L = L(A_t, $O_p \cup \{C\}$, $A_{g\geq 1}$) or
   - $\chi \in \{S4, KD45, S5\}$ and L = L(A_t, $O_p \cup \{C\}$, $A_{g\geq 2}$)

6.3 Validity

Denoted as VAL_χ. Given a formula $\phi \in L$, is it the case that $\chi \models \phi$? Because of the equivalence that ϕ is valid in χ iff $\neg\phi$ is not satisfiable in χ. Therefore, it is also a decidable problem and complexity is similar to Satisfiability problem.