Q: How do we expand $(a+b)^n$?

$(a+b)^1 = a+b$
$(a+b)^2 = a^2+2ab+b^2$
$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

} Special Cases

Patterns that we can notice?

- There are $(n+1)$ terms, the first being a^n and the last being b^n
- The exponents of a decrease by 1, while the exponents of b increase by 1 (from term to term)
- The sum of the exponents of a and b in each term is n.

General Expansion of $(a+b)^n$:

$(a+b)^n = (a+b)(a+b)(a+b) \dots (a+b)$ (n times).

From each term, we can pick either an a or a b.

Suppose I wanted the coefficient of a^2b^{n-2} (or $a^{n-2}b^2$)

From the n terms, I pick ~~the fo~~ ~~them~~ 2 of a from 2 of them and b from the remaining $(n-2)$ factors.

$\Rightarrow$ Coefficient of $a^2b^{n-2} = \binom{n}{2}$

What about $a^{n-2}b^2$?

$\Rightarrow$ Pick $(n-2)$ a's and 2 b's $\Rightarrow \binom{n}{n-2}$

$\Rightarrow$ Coefficient of $a^{n-2}b^2 = \binom{n}{n-2} = \binom{n}{2}$

~~(a+b)~~

In general, the expansion of $(a+b)^n$ can be thought of, term by term, as picking k a's and $(n-k)$ b's, $0 \le k \le n$.

$$\Rightarrow \boxed{(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n}$$

or w/ Sigma Notation:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \quad \text{or} \quad \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$* \binom{n}{k} = \binom{n}{n-k}$

Binomial Theorem

Expand $(x+2)^3$ and $(2x-1)^4$

$\Rightarrow (x+2)^3 = \binom{3}{0}x^3(2)^0 + \binom{3}{1}x^2(2)^1 + \binom{3}{2}x(2)^2 + \binom{3}{3}x^0(2)^3$

$= \boxed{x^3 + 6x^2 + 12x + 8}$

$(2x-1)^4 = \binom{4}{0}(2x)^4 + \binom{4}{1}(2x)^3(-1) + \binom{4}{2}(2x)^2(-1)^2 + \binom{4}{3}(2x)(-1)^3 + \binom{4}{4}(2x)^0(-1)^4$

$= \boxed{16x^4 - 32x^3 + 24x^2 - 8x + 1}$

The Binomial Theorem can also be applied to find specific terms in the expansion w/o doing the whole expansion.

~~Expand~~

Example: Find the coefficient of x^2y^3 in the expansion of $(2x-3y)^5$

Solution: General Term in $(a+b)^5$: $\binom{5}{k}(a)^k(b)^{n-k}$

Let $a = 2x$, $b = -3y$, $n = 5$, $k = 2$

$\Rightarrow \binom{5}{2}(2x)^2(-3y)^3$

$\therefore \binom{5}{2}(2)^2(-3)^3 = 10 \cdot 4 \cdot -27$

$= \boxed{-1080}$

Example: Find the coefficient of x^4 and the constant term ~~of~~ in the expansion of $\left(3x + \frac{2}{3x}\right)^6$

Solution: General Term: $\binom{6}{k}(3x)^k\left(\frac{2}{3x}\right)^{6-k}$

$= \binom{6}{k}(3x)^k\left(\frac{2}{3}x^{-1}\right)^{6-k}$

$= \binom{6}{k}(3)^k\left(\frac{2}{3}\right)^{6-k} x^{2k-6}$

x^4: $2k-6=4$

$\Rightarrow k=5$

$\Rightarrow \binom{6}{5}(3)^5\left(\frac{2}{3}\right) = 6 \cdot 243 \cdot \frac{2}{3}$

$= 4 \cdot 243$

$= \boxed{972}$

Constant: $2k-6=0$

$\Rightarrow k=3$

$\Rightarrow \binom{6}{3}(3)^3\left(\frac{2}{3}\right)^3$

$= \frac{6\cdot5\cdot4}{6} \cdot 27 \cdot \frac{8}{27}$

$= 20 \cdot 8$

$= \boxed{160}$

Find the coefficient of y^4 in the expansion of $(\sqrt{2}+y)^{12}$

$\Rightarrow \binom{12}{k}(\sqrt{2})^k(y)^{12-k}$

$\Rightarrow 12-k=4$

$\Rightarrow k=8$

$\Rightarrow \binom{12}{8}(\sqrt{2})^8 = \overset{5}{\frac{12\cdot11\cdot10\cdot9}{24}} \cdot (16)$

$= \cancel{1456/16}\ 495 \cdot 16 = \boxed{7920}$

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