MPS21XH - Counting Principle and Combinations Class Examples
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1.) Given the letters a,b,c,d,e and f. How many ways are there to form a three letter sequence under the following conditions:

a.) With no repetition allowed?

b.) With repetition allowed?

c.) Without repetition **and** containing e?

d.) With repetition **and** containing e?

2.) In how many ways can a 4-person subcommittee be chosen from a 7-person committee? What if a particular person must be on the subcommittee?

3.) If $\dfrac{\dbinom{n}{4}}{\dbinom{n}{3}} = \frac{3}{4}$, then compute the value of n.

4.) How many ways can you rearrange the letters in the word LINGUINI?

5.) A set of n points is chosen on the circumference of a circle so that the number of different triangles with all three vertices among the points is equal to the number of pentagons with all 5 vertices in the set. Compute the value of n.

Solutions:

1.) We will explore each one of the parts individually:

a.) Without repetition, the answer is just $6 * 5 * 4 = 120$ (6 choices for 1st letter, 5 for the 2nd and 4 for the 3rd).

b.) With repetition, each space can be filled with 6 choices, so the answer is $6 * 6 * 6 = 216$.

For the next two parts, it helps to start by making a diagram displaying the positions in the sequence (which can also be applied to constructing a number of a certain number of digits) as follows: ___

The reason for that is it helps focus on choices involving the positions. So let's look at the next two parts with this idea in mind.

c.) Since e can go in any one of the 3 positions, let's look at it in terms of our diagram above:
e__ , or _ e _, or __ e, where each one of these diagrams represents a different case. For the first open position, there are 5 choices and for the second open position, there are 4 choices.

$\therefore$ there are $3 * 5 * 4 = 60$ total sequences.

d.) Here we have to be careful because all of the letters, including e can be used multiple times. The trap students fall into is to say there are 3 places to put the e, and then another 6 * 6 = 36 choices for the other two positions, giving a final answer of 108, which is actually an overcount. To deal with this issue, the problem has to be broken into cases that ensure distinct outcomes. Since repetition of all the letters can be allowed, our cases should be based on where the 1st e occurs.

Case 1: the first e occurs in position 1: e __ $\Rightarrow$ there are 36 ways to fill in the other two positions.

Case 2: the first e occurs in position 2: _ e _ $\Rightarrow$ the 1st open spot has 5 choices (no e), but the last position has 6 choices, so there are 30 total sequences.

Case 3: the first e occurs in position 3: __ e $\Rightarrow$ each of the open positions has 5 choices (no e), so there are 25 total sequences here.

$\therefore$ the total number of sequences is $36 + 30 + 25 = 91$.

2.) The first part is quite straightforward $\therefore \binom{7}{4} = 35$ ways.

The second part requires one little extra step. First, lock in that one person onto the committee, then pick 3 people from the remaining 6 $\Rightarrow \binom{6}{3} = 20$ ways.

3.) First let's setup a proportion by applying the formula for $\binom{n}{k}$:

$$\frac{\frac{n!}{(n-4)!4!}}{\frac{n!}{(n-3)!3!}} = \frac{3}{4}$$

Rather than cross multiply at this point, let's first simplify the left hand side so as to make our last couple steps much more manageable:

$$\Rightarrow \frac{n!}{(n-4)!4!} * \frac{3!(n-3)!}{n!} = \frac{3}{4}$$

$$\Rightarrow \frac{n-3}{4} = \frac{3}{4}$$

$$\Rightarrow 4n - 12 = 12$$

$$\therefore n = 6$$

4.) Rather than designate the 3 I's and 2 N's as distinct letters and work with over-counting, choose final positions for the repeated letters, then proceed with the distinct letters.

$\Rightarrow$ There are $\binom{8}{3} = 56$ ways to place the 3 I's. Now, with 5 remaining spots and 2 N's, there are $\binom{5}{2} = 10$ ways to place the N's. That leaves the 3 distinct letters and 3 spots, or 3! ways to arrange them. Thus, the final answer is $56 * 10 * 6 = 3360$ ways to rearrange the letters.

5.) To form triangles, choose 3 points from the set of n points, which is $\binom{n}{3}$. Similarly, to form pentagons, the number of ways is $\binom{n}{5}$

$$\Rightarrow \binom{n}{3} = \binom{n}{5}$$

$$\Rightarrow \frac{n!}{(n-3)!3!} = \frac{n!}{(n-5)!5!}$$

$$\Rightarrow \frac{(n-3)!}{(n-5)!} = \frac{5!}{3!}$$

$$\Rightarrow (n-3)(n-4) = 20$$

$$\Rightarrow n^2 - 7n - 8 = 0$$
$$\therefore n = 8$$

This example actually serves to prove our first combinatorial identity: $\binom{n}{k} = \binom{n}{n-k}$

Think about why without just trying to verify it algebraically - To choose k things from a set of n, one could just as well choose the $(n-k)$ things they don't want.