MPS21XH - Counting Principles and Review of Permutations and Combinations
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Fundamental Counting Principle of Addition: Let there be n_1 different objects in set 1, n_2 different objects in set 2, ..., and n_k different objects in set k **and** if the different sets are **disjoint** or mutually exclusive, then the number of ways to select an object from one of the k sets is $n_1 + n_2 + \ldots + n_k$.

Fundamental Counting Principle of Multiplication: Suppose a procedure can be broken into k successive (ordered) stages, with n_1 different outcomes in stage 1, n_2 different outcomes in stage 2, ..., and n_k different outcomes in stage k. If the number of outcomes at each stage is independent of the choices in previous stages and if the **composite outcomes are all distinct**, then the total procedure has $n_1 * n_2 * \ldots n_k$ different composite outcomes.

Example: Find the number of divisors of 540.

Solution: The standard method is to find the prime factorization of 540, which is $2^2 * 3^3 * 5$. Any number that divides 540 can only have 2,3 or 5 as a prime factor, so all factors of 540 are of the form $2^a * 3^b * 5^c$.

A divisor of 540 can't have more than 2 2's (for example, 8 does not divide 540), so a can be 0,1, or 2, meaning it has 3 different values. Using a similar argument for 3 and 5, there are 4 different values of b and 2 different values of $c \Rightarrow$ there are $3*4*2 = 24$ divisors of 540.

Example: How many ways can a row of 4 seats be filled from a set of 7 people? Can we generalize this?

Solution: For the specific problem, we have 7 options to fill the 1st seat, 6 for the 2nd, 5 for the 3rd and 4 for the 4th seat $\Rightarrow 7 * 6 * 5 * 4 = 840$

In general, suppose we wanted to fill a row of k seats from a set of n people. Line them up in a similar fashion $\Rightarrow n * (n-1) * (n-2) * \ldots * (n-k+1)$

Notice that all the integers from 1 to $(n-k)$ are not included.

We can simplify as follows: $n * (n-1) * (n-2) * \ldots * (n-k+1) = \frac{n!}{(n-k)!}$. This is called a **permutation** of k objects of a set of n objects, denoted as ${}_nP_k$. A special case is ${}_nP_n$, which is simply $n!$

Example: In how many ways can Antoinette give one piece of candy each to two children if she has 3 different red candies, 4 different green candies and 5 different yellow candies and the two children insist upon having candies of different colors?

Solution: For this problem, we can separate individual cases, find the number for each and then add them up. In this problem, there are 6 cases: red-green, green-red, red-yellow, yellow-red, green-yellow and yellow-green.

For red-green (and green-red), there are 3 * 4 = 12 ways.
For red-yellow, there are 3 * 5 = 15 ways (same for yellow-red)
For yellow-green (and green-yellow), there are 4 * 5 = 20 ways.

The total answer is $2(12 + 15 + 20) = 94$ ways to distribute the candies.

Example: There are 5 different flavors of ice cream. In how many ways can I choose 3 of the 5 flavors to eat?

Solution: It is important to note that in this case, the order in which the flavors are chosen does not matter. However, let's pretend that it does for a moment - then the answer is $5 * 4 * 3 = 120$. If order does not matter, we have considerably over-counted.

Let's say that the three flavors I've chosen are cookies and cream, chocolate and strawberry. No matter how you order, you still have the same three flavors. In fact, you have counted every arrangement of the three flavors. As you recall, there are $3 * 2 * 1 = 6$ arrangements of the 3 flavors.

Thus, the answer to this problem is $\frac{5*4*3}{3*2*1} = 10$ ways to choose the 3 flavors.

This is an example of finding the number of **combinations** of k objects from a set of n objects, denoted as $\binom{n}{k}$, or ${}_nC_k$.

To derive a formula, start with the formula for ${}_nP_k$, which is $\frac{n!}{(n-k)!}$. Recall that this assumes choosing the objects in order. If order does not matter, you need to divide out by the number of ways to rearrange the k objects, or $k!$.

$\therefore \binom{n}{k} = \frac{{}_nP_k}{k!} = \frac{n!}{(n-k)!k!}$

In general, permutations mean that order matters, which is not the case with combinations.