Linear Programming Problem (LPP)

Linear programming (LP) is a mathematical modeling technique designed to optimize the usage of available limited resources.

LP problems have a linear objective function and linear constraints, which may include both equalities and inequalities. The feasible set is a polytope, a convex, connected set with flat polygonal faces. The contours of the objective fns are planar.

General LPP

Let z be a linear fn on $\mathbb{R}^n$ defined by

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \quad \text{--- (1)}$$

where c_j's are scalars. Let $A_{m\times n} = (a_{ij})$ be a real matrix and let $\{b_1, b_2, \ldots, b_m\}$ be a set of scalars such that

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}\right\} \text{--- (2)}$$

and let $x_j \geq 0, \; j = 1, 2, \ldots, n$ --- (3)

The problem of determining an n-tuple $(x_1, x_2, \ldots, x_n)$ which optimizes (max or min) z and which satisfies (2) and (3) is called the general LPP.

Constraints: The inequations (2) are called the constraints of the GLPP

Non-negative restrictions: The set of inequations (3) is known as the non-negative restrictions of the GLPP

Solution: An n-tuple $(x_1, x_2, \dots, x_n)$ of real numbers which satisfies the constraints of a GLPP.

Feasible solution: Any solution to a GLPP which satisfies the non-negative restrictions.

Optimum solution: Any feasible solution which optimizes (maximizes or minimizes) the objective function of a GLPP

Slack variables: Let the constraints of GLPP be

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i , \quad i = 1, 2, \dots, k$$

Then the non-negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i , \quad i = 1, 2, \dots, k$

are called slack variables.

Surplus variables: Let the constraints of the GLPP be $\sum_{j=1}^{n} a_{ij} x_j \geq b_i , \quad i = k+1, k+2, \dots, l.$

Then, the non-negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i , \quad i = k+1, k+2, \dots, l$

are called surplus variables.

Standard form of LPP

Maximize $z = c^T x$

subject to the constraints

$$Ax = b, \quad x \geq 0$$

- $x, c \in \mathbb{R}^n$
- z is a linear fn on $\mathbb{R}^n$
- A is an $m \times n$ real matrix of rank m (full rank)

Canonical form of LPP

Maximize $z = c^T x$

subject to the constraints

$$Ax \leq b, \quad x \geq 0.$$

Basic solution

Given a system of m simultaneous linear equations in n unknowns

$$Ax = b, \quad x \in \mathbb{R}^n$$

where A is an $m \times n$ matrix of rank m $(m < n)$. Let B be any $m \times m$ submatrix formed by m linearly independent columns of A. Then a solution obtained by setting $n-m$ variables not associated with the columns of B equal to zero, and solving the resulting system is called a basic solution to the given system of equations.

Note 1: The m variables associated with the columns of B are called basic variables

Note 2: The $m \times m$ non-singular submatrix B is called ~~the~~ a basis matrix, the columns of B are called as basis vectors.

Degenerate solution:

A basic solution to the system $Ax=b$ is called degenerate if one or more of the basic variables vanish.

Find all the basic solutions and check if it has a degenerate solution.

a) $x_1 + 2x_2 + x_3 = 4$
$2x_1 + x_2 + 5x_3 = 5$

(b) $2x_1 + x_2 - x_3 = 2$
$3x_1 + 2x_2 + x_3 = 3$

(a) $B_a = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

$B_1 x = b \Rightarrow x_1 = 2,\ x_2 = 1$ basic soln
$x_3 = 0$ non-basic variable

Non-degenerate soln.

$B_2 = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$

$B_2 x = b \Rightarrow x_1 = 5,\ x_3 = -1$ basic soln
$x_2 = 0$ non-basic variable

non-degenerate soln

$B_3 = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$

$B_3 x = b \Rightarrow x_2 = 5/3,\ x_3 = 2/3$ basic soln
$x_1 = 0$ non-basic variable

(b) $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$; $x_1 = 1,\ x_2 = 0$, nonbasic $x_3 = 0$, Degenerate
$(1,0,0)$

$\begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$; $x_2 = 5/3,\ x_3 = -1/3$ non-basic $x_1 = 0$ non-degenerate
$(0, 5/3, -1/3)$

$\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$; $x_1 = 1,\ x_3 = 0$, nonbasic $x_2 = 0$ degenerate
$(1,0,0)$

Basic feasible solution (BFS)

A feasible solution to an LPP which is also a basic solution to the problem is called a BFS to the LPP.

Previous eg (b)

(x_1, x_2, x_3)	Nature of soln.
$(0, 5/3, -1/3)$	Non-degenerate, infeasible, basic
$(1, 0, 0)$	Degenerate, basic feasible
$(1, 0, 0)$	

Associated cost vector

Let x_B be a BFS to the LPP:

$$\text{Max } z = c^T x$$

subject to

$$Ax = b, \quad x \geq 0.$$

Then the vector

$$c_B = (c_{B_1}, c_{B_2}, \dots, c_{B_m})$$

where c_{B_i} are components of c associated with the basic variables is called the cost vector associated with the BFS x_B.

Note: The value of the objective function for the basic feasible soln x_B is

$$z_0 = c_B^T x_B$$

Improved BFS (IBFS)

Let X_B and X_B' be two BFS to the LPP. Then X_B' is said to be an IBFS as compared to X_B if

$$c_B' X_B' \geq c_B X_B$$

where c_B' is the associated cost vector corresponding to X_B'.

Optimum BFS (OBFS)

A BFS to the LPP

$$\max z = c^T x$$

subject to

$$Ax = b, \quad x \geq 0$$

is called an OBFS if

$$z_0 = c_B^T X_B \geq z^*$$

where z^* is the value of the objective function for any feasible solution.

Find all the BFS without using the simplex algorithm and choose the one which maximizes z.

$$\max z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

subject to $2x_1 + 3x_2 - x_3 + 4x_4 = 8$, $\quad x_i \geq 0,\ i = 1,2,3,4.$

$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

x_1	x_2	x_3	x_4	
1	2	$\boxed{0}$	$\boxed{0}$	← nonbasic
22/9	$\boxed{0}$	$\boxed{0}$	7/9	
$\boxed{0}$	45/16	7/16	$\boxed{0}$	← Maximizes $z = 28.8$
$\boxed{0}$	$\boxed{0}$	44/17	45/17	