Page No.

<u>ReCap</u>:

Redundant Variable

$\hookrightarrow \beta_3 = 0 \qquad E(\beta_3) = 0.$

→ Non-redundant but omitted.

Model: $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + u_i$ MLRM CLRM.

" 2: $y_i = \beta_0 + \beta_1 x_{i1} + \varepsilon_i$

Estimated: $\tilde{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 x_{i1}$ - ③

$y_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2}$

1st Stage Reg: of x_{i2} on x_{i1}

$x_{i2} = \alpha + \delta x_{i1} + r_i$ [If x_{i2} not observable? PROXY

Substitute ⑤ in ①.

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2(\alpha + \delta x_{i1} + r_{i1}) + u_i$$

$$y_i = (\beta_0 + \beta_2\alpha) + (\beta_1 + \beta_2\delta)x_{i1} + (\beta_2 r_{i1} + u_i) \quad - ⑥$$

From ⑥.

$$\hat{y}_i = (\hat{\beta}_0 + \hat{\beta}_2\alpha) + (\hat{\beta}_1 + \hat{\beta}_2\tilde{\delta})x_{i1} \quad - ⑦$$

$$\tilde{\beta}_1 = \hat{\beta}_1 + \hat{\beta}_2\tilde{\delta}$$

$$E(\tilde{\beta}_1) = E(\hat{\beta}_1) + E(\hat{\beta}_2\tilde{\delta}) = \beta_1 + \beta_2\delta$$

$\boxed{E(\tilde{\beta}_1) - \beta_1 = \beta_2\delta}$ Amount of Bias.

$E(\tilde{\beta}_1) = \beta_1$ iff $\beta_2 = 0$ or $\tilde{\delta} = 0$.

If $E(\tilde{\beta}_1) \neq \beta_1$ then

	$\delta > 0$	$\delta < 0$
$\beta_2 > 0$	(+)	(−)
$\beta_2 < 0$	(−)	(+)

What is $\hat{\tilde{\delta}}$?

From 5, $\bar{x}_2 = \alpha + \delta \bar{x}_1 + \bar{r}_{i1}$ ⑧

subbract ⑤ − (8):

$(x_{i2} - \bar{x}_{i2}) = \delta x_{i1} - \delta \bar{x}_{i1} + r_{i1} - \bar{r}_{i1}$

$(x_{i2} - \bar{x}_{i2})(x_{i1} - \bar{x}_{i1}) = \delta(x_{i1} - \bar{x}_{i1}) + (r_{i1} - \bar{r}_{i1})(x_{i1} - \bar{x}_{i1})$

$\sum (x_{i2} - \bar{x}_2)(x_{i1} - \bar{x}_1) = \delta \sum (x_{i1} - \bar{x}_1) + (r_{i1} + \bar{r}_1)(x_{i1} - x_1)$

$= \dfrac{\sum (x_{i2} - \bar{x}_2)(x_{i1} - \bar{x}_1)}{\sum (x_{i1} - \bar{x}_1)^2} = \delta + \dfrac{(r_{i1} - \bar{r}_1)(x_{i1} - x_1)}{\sum (x_{i1} - x_1)}$

$\delta = \dfrac{\sum (x_{i2} - \bar{x}_2)(x_{i1} - \bar{x}_1)}{\sum (x_{i1} - \bar{x}_1)^2} = \dfrac{\text{Cov}(x_1, x_2)}{V(x_1)}$ (+, −)

	$\rho_{x_1 x_2} > 0$	$\rho_{x_1 x_2} < 0$ (+)
$\beta_2 > 0$	(+)	(−)
$\beta_2 < 0$	(−)	(+)

Standardization

Learning Outcome $= \beta_0 + \beta_1$ Year of Edu $+ \beta_2 X_2 + \dots \beta_n x_n$

$$y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + u_i$$

$$\bar{y} = \beta_0 + \beta_1 \bar{x}_1 + \beta_2 \bar{X}_2 + \bar{u}$$

$$(y_i - \bar{y}) = \cancel{\beta_0} + \beta_1 (X_{i1} - \bar{x}_1) + \beta_2 (x_{i2} - \bar{x}_2) + (u_i - \bar{u})$$

σ_y - S.D of y

σ_{x_1} = S.D of x_1

σ_{x_2} = S.D of x_2

σ_{x_j} = S.D of x_i $\forall$ $j = 1(1)n$

$$\frac{(y_i - \bar{y})}{\sigma_y} = \beta_1 \left(\frac{x_{i1} - \bar{x}_1}{\sigma_{x_1}} \right) + \beta_2 \left(\frac{x_{i2} - \bar{x}_2}{\sigma_{x_2}} \right) + \left(\frac{u_i - \bar{u}}{\sigma_y} \right)$$

$$\frac{(y_i - \bar{y})}{\sigma_y} = \frac{\sigma_{x_1}}{\sigma_y} \beta_1 \left(\frac{x_{i1} - \bar{x}_1}{\sigma_{x_1}} \right) + \frac{\sigma_{x_2}}{\sigma_y} \beta_2 \left(\frac{x_{i2} - \bar{x}_2}{\sigma_{x_2}} \right) + \left(\frac{u_i - \bar{u}}{\sigma_y} \right)$$

$$Z_y = b_1 Z_{x_1} + b_2 Z_{x_2} + Z_u$$

Estimated $Z_y = b_1 Z_{x_1} + b_2 Z_{x_2}$

where $b_1 = \frac{\sigma_{x_1} \beta_1}{\sigma_y}$ $b_2 = \frac{\sigma_{x_2} \beta_2}{\sigma_y}$.

$\boxed{\text{Chapter 6/7 Wooldrige}}$

FABER-CASTELL
Date
Page No.

Scaling $y_i = \beta_0 + \beta_1 x_1 + u_i$

~~Birth rate~~

b-w Child Birth weight : y_i Gm → kg.
C-pd Cigeratte : x_i

$$\frac{\widehat{b\_wt}}{1000} = \frac{\hat\beta_0}{1000} + \frac{\hat\beta_1}{1000} c\_pd + \frac{\hat u_i}{1000}$$

Rescaling y - dependent variable

$$b\_wt = \hat\beta_0 + (12\hat\beta_1)\frac{c\_pd}{12}$$

Gauss Markov Theorem : $\boxed{\text{Wooldrige Appendix 3}}$
↳ B L U E

Unbiased: $E(\hat\beta_1) = \beta_1$. $E(\hat\beta_j) = \beta_j$.

Linear : $\hat\beta_1 = \frac{\sum (x_i - \bar x) y_i}{\sum (x_i - \bar x)^2}$

For simplicity.
Let $i = 1(1)2$

$$\hat\beta_1 = \frac{\sum (x_i - \bar x)}{\sum (x_i - \bar x)^2} y_1 + \frac{\sum (x_i - \bar x)}{\sum (x_i - \bar x)^2} y_2$$

$$\hat\beta_1 = w_1 y_1 + w_2 y_2 \qquad w_i = \frac{(x_i - x)}{\sum (x_i - x_1)^2}$$

$$= \sum_{i=1}^{n} w_i y_i$$

$$\hat\beta_j = \sum_{i=1} w_{ij} y_i \qquad j = 1(1)k.$$

Best : Minimum Variance

$$V(\hat{\beta}_{OLS}) \leq V(\tilde{\beta}_{OLS})$$

Scalar:

$$\tilde{\beta}_1 = \sum_i c_i y_i$$
$$= \sum_i c_i (\beta_0 + \beta_1 x_i + u_i)$$
$$= \beta_0 \sum_i c_i + \beta_1 \sum_i c_i x_i + \sum_i c_i u_i \quad -(1)$$

Now $E(\tilde{\beta}_1) = \beta_1$ requires, $\sum_i c_i = 0$ (2) & $\sum_i c_i x_i = 1$ -(3)

$$\text{Var}(\tilde{\beta}_1) = V(\sum c_i y_i) = V(\sum_i c_i u_i)$$
$$= \sigma^2 \sum_i c_i^2$$

$$\text{var}(c_1 u_1 + c_2 u_2) = c_1^2 \sigma^2 + c_2^2 \sigma^2 + 2\,\text{cov}(c_1 u_1, c_2 u_2)$$
$$= (c_1^2 + c_2^2) \cdot \sigma^2$$

Minimise $\sum_i c_i^2$ s.t. $\sum_i c_i = 0$ & $\sum c_i x_i = 1$

Using Lagrangian Multiplier :

$$\mathcal{L} = \sum_i c_i^2 + \lambda (\sum_i c_i) + \mu (\sum c_i x_i - 1)$$

$$\frac{\partial \mathcal{L}}{\partial c_i} = 2c_i + \lambda + \mu x_i = 0$$

$$c_i = \frac{-\lambda}{2} - \left(\frac{\mu}{2}\right) x_i$$

Let $c_i = A + B x_i$ where $A = \frac{-\lambda}{2}$ & $B = \frac{-\mu}{2}$

Now $\sum_i c_i = nA = B\sum_i x_i = 0.$

$$A = -\frac{B\sum_i x_i}{n} = -B\bar{x}$$

$$\sum_i c_i x_i = \sum_i (A + Bx_i)x_i = \sum_i Ax_i + \sum_i Bx_i^2$$
$$= 1$$

$$-B\bar{x}\sum_i x_i + B\sum_i x_i^2 = 1$$

$$B\sum_i x_i^2 - B\bar{x}\sum_i x_i = 1$$

$$B\sum_i x_i^2 - B\bar{x}\, n\bar{x} = 1$$

$$B\left(\sum_i x_i^2 - n\bar{x}^2\right) = 1$$

$$B = \frac{1}{\sum_i (x_i - \bar{x})^2}$$

$$c_i = -B\bar{x} + \underbrace{\left(\frac{1}{\sum_i (x_i-\bar{x})^2}\right)}_{B} x_i = B(x_i - \bar{x})$$
$$= \frac{(x_i - \bar{x})}{\sum_i (x_i - \bar{x})^2}$$

$$\sum_i c_i^2 = \frac{(x_i - x)^2}{\left(\sum_i (x_i - x)^2\right)^2}$$

$$\sum_i c_i^2 = \frac{\cancel{(x_i - \bar{x})}}{\sum_i (x_i - \bar{x})^2} = \frac{1}{\sum_i (x_i - x)^2}$$

$$\text{or } \sigma^2 \sum_i c_i^2 = \frac{\sigma^2}{\sum_i (x_i - \bar{x})^2}$$

$$V(\hat{\beta}_{1,OLS}) \leq V(\hat{\beta}_{OLS}).$$

$$\sigma^2 \sum_i c_i^2 = \frac{\sigma^2}{SST_x} = V(\hat{\beta}_{OLS}).$$

Scalar: Cauchy Schwartz inequality

$$\left(\sum_i u_i v_i\right)^2 \leq \left(\sum_i u_i^2\right)\left(\sum_i v_i^2\right)$$

Matrix

$$\tilde{\beta} = \underset{k\times n}{C}\ \underset{n(k+1)}{Y} \qquad \underset{n\times 1}{Y} = \underset{n\times(k+1)}{X}\ \underset{(k+1)\times 1}{\beta} + \underset{n\times 1}{u}.$$

$= C(X\beta + u).$

$= CX\beta + Cu.$

$E(\hat{\beta}) = \beta$, requires $CX = I$

Here $C = (X'X)^{-1}X'$

To satisfy $CX = (X'X)^{-1}X'X = I$

Let $C_0 = (X'X)^{-1}X'$

$C_0X = (X'X)^{-1}X'X = I$

$C_0X - CX = I - I = 0$

Define

$D = C - (X'X)^{-1}X'.$

$DX = CX - (X'X)^{-1}X'X = 0.$

$V(\tilde{\beta}) = V(CY) = V(C(X\beta + u)).$

$V[((X'X)^{-1}X' + D)(X\beta + u)]$

$V[(X'X^{-1})X'X\beta + DX\beta + (X'X)^{-1}X'u + Du].$

$V[(\beta + 0 + (X'X)^{-1}X'u + Du]$

$V((X'X)^{-1}X'u + Du]$

$= \sigma^2\, V(((X'X)^{-1}X' + D)) \quad [\because V(u) = \sigma^2]$

$(X(X'X)^{-1} + DD').$

$= \sigma^2 (X'X)^{-1}X'X(X'X)^{-1} + DD'$

$= \sigma^2 (X'X)^{-1}\cancel{X'X} + \sigma^2 DD' = V(\hat{\beta}) + \sigma^2 DD'$

positive semi definite matrix

$V(\tilde{\beta}) = V(\hat{\beta}) + \sigma^2 DD'$

$\rightarrow V(\tilde{\beta}) \geq V(\hat{\beta})$