Homework8

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11. A company buys 100 lightbulbs, each of which has an exponential lifetime of 1000 hours.

What is the expected time for the first of these bulbs to burn out?

We are given the expected lifetime of a bulb is 1000 hours $= \mu$

let X_i be the independent random variable for light bulb i.

then the X_i's all follow a exponential distribution with rate parameter λ_i

Expected value for light bulb i: $E[X_i] = \mu = 1000 = 1/\lambda_i$ so $\lambda_i = 1/1000$

summing up the lambda values (they're all equal) gives $100 * \lambda = 100 * (1/1000) = 1/10$

we can use the updated λ value that is the sum of all lambda values and show that

the expected time is $E[min(X_1,x_2,... X_100)] = 1/(1/10) = 10 hours.

Exercise 14 (from Section 7.2) Assume that X_1 and X_2 are independent random variables, each having an

exponential density with parameter λ. Show that $Z = X_1 - X_2$ has density

$f_Z(z) = (1/2)\lambda e^{-\lambda|z|}$

Using convolution we are going to compute the difference of two expoential distributions

note that $f_{X_1}(x_1) = f_{X_2}(x_1) = \lambda e^{-\lambda z}$ for $x_1 \geq 0$ and 0 otherwise.

First, we can rewrite Z as $Z = X_1 + (-X_2)$ and set $-X_2 = Z - X_1$

The convolution of X_1 and $-X_2$ is

1. $f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{-X_2}(z - x_1) dx_1$

note that $f_{-X_2}(Z - X_1) = f_{X_2}(X_1 - Z)$ (switch the signs)

That is

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X_1}(x_1) f_{X_2}(x_1 - z) dx_1 =$$

we set the limits in the integral from 0 to positive infinity)

$$\int_0^{\infty} \lambda e^{-\lambda x_1} \lambda e^{-\lambda(x_1 - z)} dx_1 =$$

$$\lambda^2 \int_0^{\infty} e^{-\lambda(2x_1 - z)} dx_1 =$$

$$\lambda^2 \int_0^{\infty} e^{\lambda z} e^{-2\lambda x_1} dx_1 =$$

$$\lambda^2 e^{\lambda z} \int_0^{\infty} e^{-2\lambda x_1} dx_1 =$$

$$\lambda^2 e^{\lambda z} \big[-(1/2) e^{-2\lambda x_1} \big]_0^{\infty} =$$

$$(1/2)\lambda e^{\lambda z}$$

Through this long exercise we notice that when $z < 0$, we get $f_Z(z) = (1/2)\lambda e^{-\lambda z}$ and for $z \geq 0$, we get $f_Z(z) = (1/2)\lambda e^{\lambda z}$ we can use the absolute value of z as $f_Z(z) = f_Z(-z)$ that is $f_Z(z) = (1/2)\lambda e^{-\lambda|z|}$ which is a distrubution for all z.

Exercise 1 (from Section 8.2): Let X be a continuous random variable with mean $\mu = 10$ and variance

$\sigma^2 = 100/3$**. Using Chebyshev's Inequality, find an upper bound for the following**

probabilities:

(a) $P(|X - 10| \geq 2)$

(b) $P(|X - 10| \geq 5)$

(c) $P(|X - 10| \geq 9)$

(d) $P(|X - 10| \geq 20)$

(From the textbook which will be used to solve these exercises)
(Chebyshev Inequality) Let X be a continuous random variable with density function $f(x)$. Suppose X has a finite expected value $\mu = E(X)$ and finite variance $\sigma^2 = V(X)$. Then for any positive number $\epsilon > 0$ we have
$P(|X - \mu| \geq \epsilon) \leq \sigma^2/\epsilon^2$

a) $\epsilon = 2$ and the upper bound is $\sigma^2/\epsilon^2 = (100/3)/4 = 25/3$

b) $\epsilon = 5$ and the upper bound is $\sigma^2/\epsilon^2 = (100/3)/25 = 4/3$

However for a) and b), the upper bound is greater than 1 which cannot be true for a probability value. Hence the upper bound is 1.

c) $\epsilon = 9$ and the upper bound is $\sigma^2/\epsilon^2 = (100/3)/81 = 100/243$

d) $\epsilon = 20$ and the upper bound is $\sigma^2/\epsilon^2 = (100/3)/400 = 1/12$