Homework9

Jonathan Hernandez

October 27, 2018

1. The price of one share of stock in the Pilsdorff Beer Company is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_365 is

a. ≥ 100

b. ≥ 110

c. ≥ 120

we can assume the X_n's are nearly normal and compute the sum of the independent random variables

$$\begin{aligned} S_n = X_1 + X_2 + ... + X_n \\ = (Y_2 - Y_1) + (Y_3 - Y_2) + ... + (Y_{n+1} - Y_n) \\ = Y_{n+1} - Y_1 \\ = Y_{n+1} - 100 \end{aligned}$$

The $Y_2, Y_3 upto Y_n$'s all cancel out so it's only $Y_{n+1} - Y_1$ that survives.

The mean of S_n is computed as $n\mu = 0$

The variance of S_n is computed as $n\sigma^2 = 364 * 0.25 = 91$

The standard deviation of S_n is computed as $\sqrt{n\sigma^2} = \sqrt{91}$

(n = 364 as $Y_{364+1} = Y_{364} + X_{364} = Y_{365}$)

$S_{364} = Y_{365} - 100$ and $Y_{365} = S_{364} + 100$

for a.

$P(Y_{365} \geq 100) = P(S_{364} + 100 \geq 100) = P(S_{364} \geq 0)$

As a nearly normal distribution with mean 0 and is symmetric around 0, $P(S_{364} \geq 0) \approx 0.5$

for b.

$P(Y_{365} \geq 110) = P(S_{364} + 100 \geq 110) = P(S_{364} \geq 10) = P(S^*_{364} \geq 10/\sqrt{91})$

for c.

$P(Y_{365} \geq 120) = P(S_{364} + 100 \geq 120) = P(S_{364} \geq 20) = P(S^*_{364} \geq 20/\sqrt{91})$

we can use R to assist us in estimating the probablities

b.

```r
sd <- sqrt(91)
prob <- 10/sd

pnorm(prob, lower.tail = FALSE)
```

```
## [1] 0.1472537
```

c.

```r
sd <- sqrt(91)
prob <- 20/sd

pnorm(prob, lower.tail = FALSE)
```

```
## [1] 0.01801584
```

2. Calcuate the expected value and variance of the binomial distribution using the moment generating function

Answer:

Binomial Distribution: $p(x) = \binom{n}{j} p^j (q)^{n-j}$

Moment Generating Function:

$$\begin{aligned} g(t) &= \sum_{j=0}^{n} e^{jt} p(x) \\ &= \sum_{j=0}^{n} e^{jt} \binom{n}{j} p^j (q)^{n-j} \\ &= \binom{n}{j} (pe^t)^j q^{n-j} \\ &= (pe^t + q)^n \end{aligned}$$

The last part of the equality is that if you let $a = pe^t$, you have the binomial theorem that is $\sum_{j=0}^{n} \binom{n}{j} (a)^j q^{n-j} = (a+q)^n$ and substitute back, you get the momenting generating function to be $(pe^t + q)^n$

$\sigma^2 = E(X^2) - E(X)^2$

$E(X) = g'(0) = n(pe^t + q)^{n-1} pe^t \big|_{t=0} = np$

(When you set t = 0, you get p+q in the base and q = 1-p so p+1-p = 1)

$E(X)^2 = (np)^2$

$E(X^2) = g''(0) = np\Big[e^t\big(p(n-1)(pe^t+q)^{n-2} + (pe^t+q)^{n-1}\big)\Big]\Big|_{t=0} = np^2(n-1) + np$

$\sigma^2 = np^2(n-1) + np - (np)^2 = np(1-p)$

The $E(X^2)$ is using the product rule for dervatives and evaluating at t =0.

σ^2 is derived using factoring.

3. Calcuate the expected value and variance of the exponential distribution using the moment generating function

Answer:

Exponential Distribution: $f(x) = \lambda e^{-\lambda x}$

Moment Generating Function:

$$\begin{aligned} M_X(t) &= \int_{-\infty}^{\infty} e^{xt} f(x) dx \\ &= \int_{0}^{\infty} e^{xt} \lambda e^{-\lambda x} dx \\ &= \lambda \int_{0}^{\infty} e^{x(t-\lambda)} dx \\ &= \lambda(e^x/(t-\lambda))\Big|_0^\infty (t < \lambda) \\ &= \lambda/(\lambda - t) \end{aligned}$$

Note that $t < \lambda$ otherwise the moment generating function would not converge.

Calculating the variance using a moment generating function:

$\sigma^2 = E(X^2) - E(X)^2$

$E(X) = M_X^{'}(0)$ that is the first moment is equal to the derivative of the moment generating function with respect to t evaluated at 0 (t=0)

$E(X^2) = M_X^{''}(0)$ that is the second moment if equal to the 2nd derivative of the moment generating function with respect to t evaluated at 0 (t=0)

$E(X) = M_X^{'}(0) = \lambda * -1 * -1(\lambda - 0)^{-2} = 1/\lambda$

$E(X)^2 = 1/\lambda^2$

$E(X^2) = M_X^{''}(0) = 2\lambda/(\lambda - 0)^{-3} = 2/\lambda^2$

$\sigma^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$