Homework5

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Homework 5

Choose independently two numbers B and C at random from the interval [0, 1] with uniform density. Prove that B and C are proper probability distributions. Note that the point (B,C) is then chosen at random in the unit square.

B and C are valid probability distributions

- The area under the unit square is 1
- P(B) and P(C) $\geq$ 0 for all B and C (B and C are bounded from [0,1])

Find the probability that

a) B + C < 1/2

B and C are continous random variables so we'll use integrals for these questions
B + C < 1/2 is computing $P(0 \leq B + C \leq 1/2)$ That is the lower half portion of the rectangle and the integral to solve is $\int_0^{1/2} \int_0^1 B + C \, dB \, dC$
Solving the first integral gives

$$\int_0^{1/2} \left[B^2/2 + BC \right]_0^1 dC = \int_0^{1/2} 1/2 + C \, dC =$$
$$\left[C/2 + C^2/2 \right]_0^{1/2} = 1/4 + 1/8 = 3/8 = 0.375$$

- This means that when B and C are picked at random, about 37.5% of time, the sum

of B and C will be within the lower half of the rectangle.

b) Find the probability that BC < 1/2

- Again using integrals and the definition of the probability density function

goal is to compute $P(0 \leq BC \leq 1/2)$ that is compute
$\int_0^{1/2} \int_0^1 BC \, dB \, dC$
that is

$$\int_0^{1/2} \int_0^1 BC\, dB\, dC =$$

$$\int_0^{1/2} \Big[B^2C/2\Big]_0^1 dC = \int_0^{1/2} C/2\, dC =$$

$$\Big[C^2/4\Big]_0^{1/2} = 1/16 = 0.0625$$

- This means that when B and C are picked at random, about 6.25% of time, the product

of B and C will be within the lower half of the rectangle.

c) Find the probability that |B - C| < 1/2

- With this one, note $\int_0^1 |B - C|\, dB = \Big[((B-C)|B-C|)/2\Big]_0^1$

which is $(2C^2 - 2C + 1)/2$ when you evaluate the limits. Another thing to note is that
|1-C| for C [0,1] always give a positive value which is the same result of 1-C and
|-C| is just C for all C.
Solving the integral $\int_0^{1/2} \int_0^1 |B - C|\, dB\, dC =$

$$1/2 \int_0^{1/2} (2C^2 - 2C + 1)\, dC = 1/2\Big[2C^3/3 - C^2 + C\Big]_0^1$$

$$= 1/2\Big[1/12 - 1/4 + 1/2\Big] = 1/6 = 0.16666$$

- About 16.6% of the time, taking the difference of B and C and computing their

absolute value the result is within the lower part of the box.

d) Find the Probability of max(B,C) < 1/2

- Let's note the max function; max(B,C) = {B if B $\geq$ C and C otherwise}

Computing the integral for max(B,C) results in two possibilities:

- if max is B, $\int max(B,C)dB = \int BdB = B^2/2 + constant$
- if max is C, $\int max(B,C)dB = \int CdB = BC + constant$
- now let us solve:

Case 1: max(B,C) = B

$$\int_0^{1/2} \int_0^1 B\, dB\, dC =$$

$$\int_0^{1/2} \Big[B^2/2\Big]_0^1 dC = \int_0^{1/2} 1/2\, dC =$$

$$\Big[C/2\Big]_0^{1/2} = 1/4 = 0.25$$

- Case 2: max(B,C) = C

$$\int_0^{1/2} \int_0^1 C\,dB\,dC =$$
$$\int_0^{1/2} \Big[BC\Big]_0^1 dC = \int_0^{1/2} C\,dC =$$
$$\Big[C^2/2\Big]_0^{1/2} = 1/8 = 0.125$$

- We see that there is a higher chance of B and C landing on the square if we compute

the max of B and C and B is larger.

e) Find the probability of min(B,C) < 1/2

- This is the same as comptuting the integral of a min(B,C) function doing both

cases results in the same computation and answer as d)