1.a)

Question:
For matrices $A, B \in \mathbb{R}^{n\times n}$, prove that $(AB)^T = B^T A^T$
Solution:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

Transpose swaps the indices

$$\implies (AB)^T_{ij} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = B^T A^T$$

$$\therefore (AB)^T = B^T A^T$$

1.b)

Question:
A matrix Q is orthogonal if $Q^T Q = I$. Prove that if Q is orthogonal, then $|\det(Q)| = 1$.
Solution:

$$\text{for any matrix, } \det\{Q^T\} = \det\{Q\}$$

$$1 = \det(I) = \det(Q^T Q) = \det(Q^T)\det(Q) = \det(Q)^2$$

$$\therefore \det(Q) = \pm 1$$

$$\Leftrightarrow |\det(Q)| = 1$$

1.c)

Question:
Prove that the product of two orthogonal matrices is orthogonal
Solution:

$$A^T A = I = B^T B$$

$$B^T A^T A = B^T$$

$$B^T A^T AB = B^T B = I$$

$$(AB)^T AB = I$$

$$\therefore AB \text{ is orthogonal}$$

1.d)

Question:
Prove that $\text{tr}(AB) = \text{tr}(BA)$ for any $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$
Solution:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

$$(AB)_{ll} = \sum_{k=1}^{n} A_{lk}B_{kl}$$

$$\mathrm{tr}(AB) = \sum l = 1^n \sum_{k=1}^{n} A_{lk} B_{kl} = \sum_{l=1}^{n} \sum_{k=1}^{n} A_{kl} B_{lk}$$

$$\mathrm{tr}(BA) = \sum l = 1^n \sum_{k=1}^{n} B_{lk} A_{kl} = \mathrm{tr}(AB)$$

$$\therefore \mathrm{tr}(AB) = \mathrm{tr}(BA)$$

1.e)

Question:
Show that for any skew-symmetric matrix, the diagonal elements must be zero
Solution:

$$S^T = -S$$

$$S^T_{ij} = -S_{ij}$$

$$\text{look at only diagonal elements} S^T_{ii} = -S_{ii} \Leftrightarrow S_{ii} = -S_{ii}$$

$$S_{ii} = -S_{ii} \text{ is only possible if } S_{ii} = 0$$

$$\therefore \text{ the diagonal elements are } 0$$