AAE 590: Problem Set 01
Abstract Algebra and SO(2)

Due: See Brightspace

Instructions

- Show all work for full credit.
- Python code should be well-commented and submitted as a separate .py file or Jupyter notebook.
- You may use AI tools, but you must understand your solutions (validated via in-class quizzes).

Problem 1: Linear Algebra Review (20 points)

This problem reviews essential linear algebra concepts used throughout the course.

(a) For matrices $A, B \in \mathbb{R}^{n\times n}$, prove that $(AB)^T = B^T A^T$.

(b) A matrix Q is *orthogonal* if $Q^T Q = I$. Prove that if Q is orthogonal, then $|\det(Q)| = 1$.

(c) Prove that the product of two orthogonal matrices is orthogonal.

(d) The *trace* of a matrix is the sum of its diagonal elements: $\mathrm{tr}(A) = \sum_i A_{ii}$. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$.

(e) A matrix S is *skew-symmetric* if $S^T = -S$. Show that for any skew-symmetric matrix, the diagonal elements must be zero.

Problem 2: Group Axioms (20 points)

A *group* $(G, \cdot)$ is a set G with a binary operation $\cdot$ satisfying: (1) Closure, (2) Associativity, (3) Identity exists, (4) Inverses exist.

For each of the following, determine whether it forms a group under the given operation. If yes, identify the identity element and describe inverses. If no, state which axiom(s) fail.

(a) The set of 2×2 invertible matrices with real entries under matrix multiplication.

(b) The set of integers $\mathbb{Z}$ under addition.

(c) The set $\{1, -1, i, -i\}$ under complex multiplication.

(d) The set of 2×2 matrices with determinant 1 under matrix addition.

(e) The set of positive real numbers $\mathbb{R}^+$ under multiplication.

Problem 3: SO(2) Properties (20 points)

Consider the rotation matrix $R(\theta) \in \mathrm{SO}(2)$:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(a) Prove that $R(\theta)^T R(\theta) = I$ for all θ.

(b) Prove that $\det(R(\theta)) = 1$ for all θ.

(c) Derive the inverse $R(\theta)^{-1}$ and show it equals $R(-\theta)$.

(d) Prove that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$.

(e) Is SO(2) an abelian group? Justify your answer.

Problem 4: Lie Algebra of SO(2) (15 points)

The Lie algebra $\mathfrak{so}(2)$ consists of 2×2 skew-symmetric matrices.

(a) Write the general form of an element $\Omega \in \mathfrak{so}(2)$.

(b) Define the wedge operator $(\cdot)^\wedge : \mathbb{R} \to \mathfrak{so}(2)$ and its inverse, the vee operator $(\cdot)^\vee$.

(c) Compute the matrix exponential $\exp(\Omega)$ for $\Omega = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$ using the power series definition. Show that this equals $R(\theta)$.

(d) What is the dimension of $\mathfrak{so}(2)$? How does this relate to the dimension of SO(2)?

Problem 5: Python Implementation (25 points)

Implement the following functions in Python using NumPy:

(a) `so2_wedge(theta)`: Returns the 2×2 skew-symmetric matrix $\theta^\wedge$.

(b) `so2_vee(Omega)`: Returns the scalar θ from a skew-symmetric matrix.

(c) `so2_exp(theta)`: Returns the rotation matrix $R(\theta)$ using the closed-form formula.

(d) `so2_log(R)`: Returns the angle $\theta \in (-\pi, \pi]$ from a rotation matrix. *Hint: use `numpy.arctan2`.*

(e) Write unit tests that verify:

- $\exp(\log(R)) = R$ for random rotation matrices
- $\log(\exp(\theta)) = \theta$ for $\theta \in (-\pi, \pi]$
- $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$