AAE 590: Lie Group Methods for Control and Estimation

Mathematical Foundations

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Spring 2026

Lecture 2 Overview

Today's goals:

1. Review essential linear algebra
2. Define groups and verify examples
3. Understand matrix groups (GL, O, SO)
4. Introduce manifolds and tangent spaces
5. See how these connect to Lie groups

Philosophy:

- Build intuition, not just definitions
- Every abstract concept has a concrete example
- We're building toward practical algorithms

Linear Algebra Review: Vectors

Vectors in $\mathbb{R}^n$:

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Operations:

- Addition: $u + v$ (component-wise)
- Scalar multiplication: αv (scale each component)
- Dot product: $u \cdot v = u^T v = \sum_i u_i v_i$
- Norm: $\|v\| = \sqrt{v \cdot v}$

$\mathbb{R}^n$ **is a vector space:** Closed under $+$ and scalar $\times$, has zero vector.

Linear Algebra Review: Matrices

Matrix $A \in \mathbb{R}^{m \times n}$:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Key operations:

- Matrix-vector product: $Av \in \mathbb{R}^m$ (linear transformation)
- Matrix-matrix product: $(AB)_{ij} = \sum_k A_{ik} B_{kj}$
- Transpose: $(A^T)_{ij} = A_{ji}$

Square matrices $A \in \mathbb{R}^{n \times n}$:

- Trace: $\mathrm{tr}(A) = \sum_i a_{ii}$
- Determinant: $\det(A)$ (measures volume scaling)
- Inverse: A^{-1} exists iff $\det(A) \neq 0$

Matrix Properties We'll Use

Transpose properties:

- $(A^T)^T = A$
- $(AB)^T = B^T A^T$ (order reverses!)
- $(A + B)^T = A^T + B^T$

Inverse properties (when inverses exist):

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1} A^{-1}$ (order reverses!)
- $(A^T)^{-1} = (A^{-1})^T$

Determinant and trace:

- $\det(AB) = \det(A) \det(B)$
- $\det(A^{-1}) = 1/\det(A)$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (cyclic property)

Orthogonal Matrices

Definition

A matrix $Q \in \mathbb{R}^{n \times n}$ is **orthogonal** if $Q^T Q = I$.

Equivalent conditions:

- $Q^T Q = I$
- $QQ^T = I$
- $Q^{-1} = Q^T \quad$ (inverse is transpose!)
- Columns of Q are orthonormal
- Rows of Q are orthonormal

Key properties:

- $|\det(Q)| = 1$ (since $\det(Q^T Q) = \det(Q)^2 = 1$)
- Preserves lengths: $\|Qv\| = \|v\|$
- Preserves angles: $(Qu) \cdot (Qv) = u \cdot v$

Special Orthogonal Matrices

Orthogonal matrices have $\det = \pm 1$**:**

- $\det = +1$: **rotations** (orientation-preserving)
- $\det = -1$: **reflections** (orientation-reversing)

Definition

The **special orthogonal group** is:

$$\mathrm{SO}(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1\}$$

Physical meaning:

- $\mathrm{SO}(2)$: 2D rotations (rotate in the plane)
- $\mathrm{SO}(3)$: 3D rotations (rotate in space)

These are the rotation groups we study!

Skew-Symmetric Matrices

Definition

A matrix S is **skew-symmetric** (or antisymmetric) if $S^T = -S$.

Properties:

- Diagonal entries must be zero: $s_{ii} = -s_{ii} \Rightarrow s_{ii} = 0$
- $s_{ij} = -s_{ji}$ for all i, j
- In $\mathbb{R}^{3\times 3}$: only 3 free parameters

General form in 3D:

$$S = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$$

Preview

Skew-symmetric matrices form the Lie algebra $\mathfrak{so}(3)$!

The Wedge Map: Vectors to Skew-Symmetric Matrices

Given: $\omega = [\omega_1, \omega_2, \omega_3]^T \in \mathbb{R}^3$
Define:

$$\omega^\wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

Key property — cross product connection:

$$\omega^\wedge v = \omega \times v$$

The vee map is the inverse: $(\omega^\wedge)^\vee = \omega$

Isomorphism

$\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ is a vector space isomorphism.

(Isomorphism = structure-preserving bijection. Here: $\mathbb{R}^3$ and $\mathfrak{so}(3)$ are "the same" as vector spaces.)

Functions and Maps

Definition

A **function** (or **map**) $f : A \to B$ assigns to each element $a \in A$ exactly one element $f(a) \in B$.

Terminology:

- A is the **domain**
- B is the **codomain**
- $f(A) = \{f(a) : a \in A\}$ is the **image** (or range)

Examples:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ (squares a number)
- $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ (maps Lie algebra to group)
- $R \cdot v : \mathbb{R}^3 \to \mathbb{R}^3$ (rotation acting on vectors)

Injections, Surjections, and Bijections

Definition

A function $f : A \to B$ is:

- **Injective** (one-to-one) if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$
- **Surjective** (onto) if $\forall b \in B$, $\exists a \in A$ with $f(a) = b$
- **Bijective** if both injective and surjective

Intuition:

- Injective: different inputs give different outputs
- Surjective: every element in B is "hit" by some input
- Bijective: perfect one-to-one correspondence

Examples: Injective, Surjective, Bijective

$f(x) = x^2$ on $\mathbb{R} \to \mathbb{R}$:

- Not injective: $f(2) = f(-2) = 4$
- Not surjective: -1 has no preimage

$f(x) = x^3$ on $\mathbb{R} \to \mathbb{R}$:

- Injective: $x_1^3 = x_2^3 \Rightarrow x_1 = x_2$ ✓
- Surjective: every real has a cube root ✓
- **Bijective!**

$\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$:

- Surjective: every rotation has a log ✓
- Not injective globally: $\exp(\omega^\wedge) = \exp((\omega + 2\pi\hat{a})^\wedge)$
- **Locally bijective** (near identity)

Invertibility of Maps

Theorem

A function $f : A \to B$ has an inverse $f^{-1} : B \to A$ if and only if f is bijective.

When f is bijective:

- $f^{-1}(f(a)) = a$ for all $a \in A$
- $f(f^{-1}(b)) = b$ for all $b \in B$

For Lie groups:

- $\exp : \mathfrak{g} \to G$ is **locally** bijective
- $\log : G \to \mathfrak{g}$ is defined locally as its inverse
- Global behavior depends on the group's topology

Why This Matters

Bijection $\Leftrightarrow$ invertible $\Leftrightarrow$ can "undo" the map.

What is a Group?

Definition

A **group** $(G, \cdot)$ is a set G with a binary operation $\cdot$ satisfying:

(Binary operation = takes two elements and produces one, like + or ×)

1. **Closure:** $\forall a, b \in G\colon\ a \cdot b \in G$ *(result stays in the set)*
2. **Associativity:** $\forall a, b, c \in G\colon\ (a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. **Inverse:** $\forall a \in G,\ \exists a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$
4. **Neutral element:** $\exists e \in G$ such that $\forall a \in G\colon\ e \cdot a = a \cdot e = a$

Mnemonic: CAIN

Closure, **A**ssociativity, **I**nverse, **N**eutral element

Note

Commutativity is **NOT** required! Groups where $a \cdot b = b \cdot a$ for all a, b are called **abelian**.

Group Examples: Numbers

Abelian groups:

$(\mathbb{Z}, +)$: **Integers under addition**

- Closure: sum of integers is an integer ✓
- Associativity: $(a + b) + c = a + (b + c)$ ✓
- Inverse of n: $-n$ ✓
- Neutral element: 0 ✓

$(\mathbb{R}^*, \times)$: **Nonzero reals under multiplication**

- Closure: product of nonzero reals is nonzero ✓
- Associativity: $(ab)c = a(bc)$ ✓
- Inverse of x: $1/x$ ✓
- Neutral element: 1 ✓

Non-Examples: What Fails?

$(\mathbb{Z}, \times)$: **Integers under multiplication**

- Closure: ✓
- Associativity: ✓
- Inverse: 2 has no integer inverse! ✗
- Neutral element: 1 ✓

$(\mathbb{N}, +)$: **Natural numbers under addition**

- No inverse (no negative numbers) ✗
- No neutral element ($0 \notin \mathbb{N}$ in some definitions) ✗

$(\mathbb{Z}, -)$: **Integers under subtraction**

- Not associative: $(3 - 2) - 1 = 0 \neq 3 - (2 - 1) = 2$ ✗

Non-Abelian Groups

Definition

A group is **non-abelian** if there exist a, b with $a \cdot b \neq b \cdot a$.

Example: $(\mathrm{GL}(n), \cdot)$ — **invertible matrices**

- In general, $AB \neq BA$ for matrices
- Inverse: A^{-1} (matrix inverse)
- Neutral element: I (identity matrix)

Example: $(\mathrm{SO}(3), \cdot)$ — **3D rotations**

- $R_x R_y \neq R_y R_x$ in general
- Order of rotations matters!

This is why rotation is hard!

The non-commutativity of $\mathrm{SO}(3)$ is the source of much complexity.

Non-Commutativity: Hands-On Demo

Try this with a book:

Experiment A:

1. Hold book facing you
2. Rotate 90° about x (pitch forward)
3. Rotate 90° about z (yaw left)
4. Note final orientation

Experiment B:

1. Hold book facing you
2. Rotate 90° about z (yaw left)
3. Rotate 90° about x (pitch forward)
4. Note final orientation

Observation

Final orientations are different!
$R_z(90°)R_x(90°) \neq R_x(90°)R_z(90°)$

What is a Lie Group?

Definition

A **Lie group** is a group G that is also a **smooth manifold**, where the group operations are smooth maps:

- Multiplication: $\mu : G \times G \to G$, $(g, h) \mapsto gh$
- Inversion: $\iota : G \to G$, $g \mapsto g^{-1}$

In plain terms:

- It's a group (CAIN axioms)
- Elements can be *continuously varied* (it's a manifold)
- Group operations depend *smoothly* on the elements

Key Insight

Lie groups blend **algebra** (group structure) with **geometry** (manifold structure). This lets us do calculus on groups!

Lie Group Examples

Lie groups (continuous symmetries):

- $(\mathbb{R}, +)$: translations on a line (1D manifold)
- $(\mathbb{R}^n, +)$: translations in n-space (n-dimensional manifold)
- $\mathrm{SO}(2)$: rotations in 2D (circle S^1)
- $\mathrm{SO}(3)$: rotations in 3D (3-dimensional manifold)
- $\mathrm{SE}(3)$: rigid body poses (6-dimensional manifold)

Not Lie groups:

- $(\mathbb{Z}, +)$: integers are discrete, not a manifold
- Finite groups (permutations, symmetries of polygons): no continuous structure

Why "Lie"? (pronounced "Lee")

Named after Sophus Lie (1842–1899), Norwegian mathematician who showed that continuous symmetry groups can be studied through their infinitesimal structure—the **Lie algebra**.

Matrix Lie Groups

Definition

A **matrix Lie group** is a Lie group that can be realized as a subgroup of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ for some n.

Examples we'll use:

- $\mathrm{SO}(2), \mathrm{SO}(3)$: rotation groups
- $\mathrm{SE}(2), \mathrm{SE}(3)$: rigid body motion groups
- $\mathrm{GL}(n)$: invertible matrices

Why Matrix Lie Groups?

- Concrete: elements are matrices we can compute with
- All operations (composition, inverse, exp, log) are matrix operations
- Sufficient for virtually all robotics and aerospace applications

Representation Theory and Ado's Theorem

Question: Are we missing anything by focusing on matrix Lie groups?

Definition

A **representation** of a Lie group G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ for some vector space V.

Theorem (Ado's Theorem)

Every finite-dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ has a faithful (injective) finite-dimensional representation.

Practical Implication

For our applications, we can **always** work with matrix Lie groups.
We're not losing generality by focusing on matrices!

Matrix Groups: GL(n)

Definition

The **general linear group** is:

$$\mathrm{GL}(n) = \{A \in \mathbb{R}^{n\times n} : \det(A) \neq 0\}$$

All $n \times n$ invertible real matrices.

Verify it's a group (CAIN):

1. **Closure:** $\det(AB) = \det(A)\det(B) \neq 0$ ✓
2. **Associativity:** Matrix multiplication is associative ✓
3. **Inverse:** $\det(A^{-1}) = 1/\det(A) \neq 0$ ✓
4. **Neutral element:** $I \in \mathrm{GL}(n)$ ✓

Dimension: n^2 parameters (all entries are free)

Matrix Groups: O(n) and SO(n)

Definition

The **orthogonal group** is:

$$\mathrm{O}(n) = \{Q \in \mathbb{R}^{n\times n} : Q^T Q = I\}$$

Verify O(n) is a subgroup of GL(n):

1. **Closure:** $(Q_1 Q_2)^T (Q_1 Q_2) = Q_2^T Q_1^T Q_1 Q_2 = Q_2^T Q_2 = I$ ✓
2. **Inverse:** $Q^{-1} = Q^T$ is also orthogonal ✓

Definition

The **special orthogonal group** is:

$$\mathrm{SO}(n) = \{R \in \mathrm{O}(n) : \det(R) = 1\}$$

Dimension of SO(n): $\frac{n(n-1)}{2}$ (e.g., SO(3) has dimension 3)

Matrix Groups: SE(n)

Definition

The **special Euclidean group** combines rotation and translation:

$$\mathrm{SE}(n) = \left\{ \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} : R \in \mathrm{SO}(n), t \in \mathbb{R}^n \right\}$$

Action on points:

$$\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} Rp + t \\ 1 \end{bmatrix}$$

Rotate by R, then translate by t.

Dimensions:

- $\mathrm{SE}(2)$: 3 DOF (1 rotation + 2 translation)
- $\mathrm{SE}(3)$: 6 DOF (3 rotation + 3 translation)

The Groups We'll Study

Group	Dim	Matrix Size	Physical Meaning
$\mathrm{SO}(2)$	1	2×2	Planar rotations
$\mathrm{SE}(2)$	3	3×3	Planar rigid motions
$\mathrm{SO}(3)$	3	3×3	3D rotations
$\mathrm{SE}(3)$	6	4×4	3D rigid motions

Hierarchy:

$$\mathrm{SO}(n) \subset \mathrm{O}(n) \subset \mathrm{GL}(n)$$

Dimension $=$ degrees of freedom $=$ independent parameters

Subgroups

Definition

A **subgroup** H of G is a subset that is itself a group under the same operation.

Subgroup test: $H \subseteq G$ is a subgroup if and only if:

1. $e \in H$ (contains identity)
2. $a, b \in H \Rightarrow ab \in H$ (closed under product)
3. $a \in H \Rightarrow a^{-1} \in H$ (closed under inverse)

Examples:

- $\mathrm{SO}(n)$ is a subgroup of $\mathrm{O}(n)$
- $\mathrm{O}(n)$ is a subgroup of $\mathrm{GL}(n)$
- $\{I\}$ is a subgroup of any group (trivial subgroup)
- G is a subgroup of itself

What is a Manifold?

Definition

A **manifold** M of dimension n is a topological space that is locally homeomorphic to $\mathbb{R}^n$.

Intuition: No matter where you stand on M, your immediate neighborhood looks like $\mathbb{R}^n$.

The Earth analogy:

- Earth's surface is a 2-manifold (locally looks like $\mathbb{R}^2$)
- Flat maps work locally (your city map)
- But no single flat map works globally (world map distortions)

Key Insight

Manifolds allow us to do calculus on curved spaces by working in local flat patches.

Manifold Examples

Circle S^1:

- Points satisfying $x^2 + y^2 = 1$ in $\mathbb{R}^2$
- 1-dimensional manifold
- Locally looks like $\mathbb{R}$ (a line segment)

Sphere S^2:

- Points satisfying $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$
- 2-dimensional manifold
- Locally looks like $\mathbb{R}^2$ (a flat disk)

Torus T^2:

- Donut shape
- 2-dimensional manifold
- Also locally looks like $\mathbb{R}^2$

SO(3) as a Manifold

$\mathrm{SO}(3)$ **is a 3-dimensional manifold:**

- It's a subset of $\mathbb{R}^{3\times 3} \cong \mathbb{R}^9$
- Constrained by $R^T R = I$ (6 equations)
- And $\det R = 1$ (1 equation, but redundant with orthogonality)
- Leaves $9 - 6 = 3$ degrees of freedom

Locally looks like $\mathbb{R}^3$:

- Near any rotation, small perturbations are 3-dimensional
- Can parameterize locally by 3 numbers
- But no global 3-parameter chart without singularities!

This is Why We Need Lie Theory

To work on $\mathrm{SO}(3)$ without coordinate singularities.

The Hairy Ball Theorem

Theorem (Hairy Ball Theorem)

There is no continuous non-vanishing vector field on the sphere S^2.

Intuition: If you try to "comb" a hairy ball flat, there must be at least one cowlick (singularity).

Consequence for rotations:

- Any 3-parameter representation of $\mathrm{SO}(3)$ defines a map from $\mathbb{R}^3$ (or a subset)
- This map cannot be continuous and surjective simultaneously
- There **must** be singularities—this is topology, not bad engineering!

Key Takeaway

Gimbal lock isn't a flaw in Euler angles—it's an unavoidable consequence of topology. **Any** 3-parameter representation has singularities.

Charts and Atlases

Definition

A **chart** (U, ϕ) is an open set $U \subset M$ with a homeomorphism $\phi : U \to \mathbb{R}^n$.

Think: A chart is a local coordinate system.

Definition

An **atlas** is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ that cover M.

Think: An atlas is a complete set of maps.

Example — Circle S^1:

- Chart 1: $\theta \in (-\pi, \pi)$ covers most of circle
- Chart 2: $\theta \in (0, 2\pi)$ covers most of circle
- Together they cover everything (atlas)

The Circle: Coordinate Singularity

$S^1 = \{(\cos\theta, \sin\theta) : \theta \in \mathbb{R}\}$

Single chart attempt: $\theta \in [0, 2\pi)$

- Works everywhere on the circle
- But $\theta = 0$ and $\theta = 2\pi$ are the same point!
- Discontinuity when crossing $\theta = 0$

This is a coordinate singularity:

- Nothing special happens at the point $(\cos 0, \sin 0) = (1, 0)$
- The singularity is in our **coordinates**, not the manifold

Same Phenomenon in SO(3)

Euler angles have singularities; SO(3) does not.

Tangent Vectors: Intuition

Question: What does "velocity" mean on a curved space?

On $\mathbb{R}^n$: Velocity is just another vector in $\mathbb{R}^n$.
On a manifold: Velocities live in the **tangent space**.

Intuition:

- Consider a curve $\gamma(t)$ passing through point p at $t = 0$
- The velocity $\dot{\gamma}(0)$ is a tangent vector at p
- All such velocities form the tangent space T_pM

Visual: Think of a plane tangent to a sphere at a point.

Tangent Space: Definition

Definition

The **tangent space** T_pM at $p \in M$ is the vector space of all tangent vectors at p.

Key properties:

- T_pM is a **vector space** of dimension $\dim(M)$
- Even though M is curved, T_pM is flat!
- We can do linear algebra in T_pM

Examples:

- T_pS^1: a line tangent to the circle at p (1D)
- T_pS^2: a plane tangent to the sphere at p (2D)
- $T_R\mathrm{SO}(3)$: tangent to rotation group at R (3D)

Tangent Space of the Circle

Circle: $S^1 = \{(\cos\theta, \sin\theta)\} \subset \mathbb{R}^2$

Curve on circle: $\gamma(t) = (\cos(\theta_0 + t), \sin(\theta_0 + t))$
Velocity: $\dot{\gamma}(0) = (-\sin\theta_0, \cos\theta_0)$

Observation:

- Position: $p = (\cos\theta_0, \sin\theta_0)$
- Velocity: $\dot{\gamma} = (-\sin\theta_0, \cos\theta_0)$
- These are perpendicular: $p \cdot \dot{\gamma} = 0$

Tangent space:

$$T_pS^1 = \{v \in \mathbb{R}^2 : v \perp p\}$$

A 1-dimensional subspace of $\mathbb{R}^2$ (a line).

Tangent Space of SO(3)

Curve on SO(3): $R(t)$ with $R(0) = I$

Constraint: $R(t)^T R(t) = I$ for all t
Differentiate: $\dot{R}^T R + R^T \dot{R} = 0$
At $t = 0$**:** $\dot{R}(0)^T + \dot{R}(0) = 0$

Conclusion

$\dot{R}(0)$ must be **skew-symmetric**!

Tangent space at identity:

$$T_I \mathrm{SO}(3) = \mathfrak{so}(3) = \{\Omega \in \mathbb{R}^{3\times 3} : \Omega^T = -\Omega\}$$

This is the **Lie algebra**!

The Lie Algebra

Definition

For a Lie group G, the **Lie algebra** $\mathfrak{g}$ is the tangent space at identity:

$$\mathfrak{g} = T_e G$$

For our groups:

- $\mathfrak{so}(2) = T_I \mathrm{SO}(2)$: 1-dimensional
- $\mathfrak{so}(3) = T_I \mathrm{SO}(3)$: 3-dimensional (skew-symmetric matrices)
- $\mathfrak{se}(2) = T_I \mathrm{SE}(2)$: 3-dimensional
- $\mathfrak{se}(3) = T_I \mathrm{SE}(3)$: 6-dimensional

Key Point

The Lie algebra is a **vector space**! We can do linear algebra here.

What is an "Algebra"?

Why is it called a Lie algebra?

Definition

An **algebra** (over $\mathbb{R}$) is a vector space V equipped with a **bilinear product**:

$$\cdot : V \times V \to V$$

that distributes over addition and scalar multiplication.

Examples you know:

- $\mathbb{R}^3$ with cross product: $u \times v$ (gives another vector in $\mathbb{R}^3$)
- $\mathbb{R}^{n\times n}$ with matrix multiplication: AB
- Polynomials with polynomial multiplication

Key Idea

An algebra $=$ vector space $+$ multiplication-like operation.

The Lie Bracket: What Makes It a Lie Algebra

A Lie algebra has a special product called the Lie bracket:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

Properties of the Lie bracket:

1. **Bilinearity:** $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
2. **Antisymmetry:** $[X, Y] = -[Y, X]$
3. **Jacobi identity:** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

For matrix Lie algebras, the bracket is the commutator:

$$[A, B] = AB - BA$$

Physical Meaning

The Lie bracket measures how much two infinitesimal transformations fail to commute.

Why "Lie"? Historical Motivation

Named after Sophus Lie (1842–1899), Norwegian mathematician.

Lie's inspiration: Évariste Galois (1811–1832) showed that *discrete* groups explain why there's no general formula for quintic polynomials.

Lie's question: Can *continuous* groups do the same for differential equations?

His insight: Yes! Symmetries of a differential equation form a continuous group, and this group structure helps solve or simplify the equation.

The Big Idea

Just as Galois used discrete symmetries to understand polynomials,
Lie used **continuous symmetries** to understand differential equations.

The Lie–Klein Connection

Lie collaborated with Felix Klein (of Klein bottle fame).

Klein's Erlangen Program (1872): Classify geometries by their symmetry groups.

- Euclidean geometry: symmetries $=$ rigid motions $(\mathrm{SE}(3))$
- Projective geometry: symmetries $=$ projective transformations
- Different geometry $\Leftrightarrow$ different symmetry group

Lie's contribution: Study the groups themselves by looking at *infinitesimal* transformations—the Lie algebra.

For Us

We use Lie's machinery for a different purpose: representing rotations and poses in robotics and aerospace. But the mathematical tools are the same ones Lie developed to solve differential equations!

The Lie Group–Lie Algebra Connection

The connection:

- The Lie group G captures *finite* transformations
- The Lie algebra $\mathfrak{g}$ captures *infinitesimal* transformations
- The Lie bracket encodes how the group fails to be commutative

For $\mathfrak{so}(3)$

The Lie bracket $[\omega_1^\wedge, \omega_2^\wedge] = (\omega_1 \times \omega_2)^\wedge$ is related to the cross product!
This is why angular velocities combine via cross products.

Lie's legacy: His theory is now foundational in physics (quantum mechanics, particle physics) and engineering (robotics, navigation, control).

Tangent Space at Other Points

Question: What's the tangent space at $R \neq I$?

For curves through R**:** $\gamma(t)$ with $\gamma(0) = R$
Write $\gamma(t) = R \cdot \eta(t)$ where $\eta(0) = I$.
Then $\dot{\gamma}(0) = R \cdot \dot{\eta}(0)$.

Result:

$$T_R \mathrm{SO}(3) = \{R\Omega : \Omega \in \mathfrak{so}(3)\}$$

Left translation:

- Tangent vectors at R are related to those at I by L_R
- $L_R : G \to G$ defined by $L_R(g) = Rg$
- This is why we can always work in the Lie algebra!

The Exponential Map: Definition

How do we go from Lie algebra to Lie group?

Definition

The **matrix exponential** is:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Convergence: This series converges for any matrix A.

For scalars: Reduces to familiar $e^a = 1 + a + a^2/2! + \cdots$

The Exponential Map: Properties

Key properties:

1. $\exp(0) = I$
2. $\frac{d}{dt} \exp(tA)\big|_{t=0} = A$
3. $\exp((s+t)A) = \exp(sA)\exp(tA)$ (when A commutes with itself)
4. $\exp(A)^{-1} = \exp(-A)$

Warning: In general, $\exp(A+B) \neq \exp(A)\exp(B)$!
(Unless A and B commute: $AB = BA$)

For Lie Groups

If $\Omega \in \mathfrak{so}(n)$, then $\exp(\Omega) \in \mathrm{SO}(n)$.
The exponential maps tangent vectors to group elements!

Exponential Map: SO(3)

For $\omega^\wedge \in \mathfrak{so}(3)$:

Series: $\exp(\omega^\wedge) = I + \omega^\wedge + \frac{(\omega^\wedge)^2}{2!} + \frac{(\omega^\wedge)^3}{3!} + \cdots$

Closed form (Rodrigues' formula):

$$\exp(\omega^\wedge) = I + \frac{\sin \|\omega\|}{\|\omega\|}\omega^\wedge + \frac{1 - \cos \|\omega\|}{\|\omega\|^2}(\omega^\wedge)^2$$

Physical interpretation:

- ω is axis-angle: direction is axis, magnitude is angle
- $\exp(\omega^\wedge)$ is the rotation matrix for that rotation
- Angular velocity $\times$ time $\mapsto$ rotation

The Logarithm Map

Definition

The **logarithm map** is the inverse of exp (where it exists):

$$\log : G \to \mathfrak{g}$$

For SO(3):

$$\theta = \arccos\left(\frac{\mathrm{tr}(R) - 1}{2}\right)$$

$$\omega^\wedge = \frac{\theta}{2\sin\theta}(R - R^T)$$

Caveat: Not globally one-to-one

- At $R = I$: $\omega = 0$
- At $\theta = \pi$: Multiple solutions (axis ambiguous)

Why This All Matters

The Lie group framework gives us:

1. **Singularity-free representation:** Work with $R \in \mathrm{SO}(3)$ directly
2. **Linear calculus:** Differentiate and integrate in $\mathfrak{so}(3) \cong \mathbb{R}^3$
3. **Exp/log bridge:** Move between algebra and group
4. **Consistent error:** Define errors that respect the geometry
5. **Better algorithms:** IEKF, Equivariant Filter, geometric control

Coming Up

Next week: SO(2) in detail—the simplest non-trivial example!

Course Roadmap

Where we're headed:

1. **Weeks 2–3:** SO(2), SE(2), Group Affine Systems, Control
2. **Weeks 4–6:** SO(3) — 3D rotations, Attitude Control
3. **Week 7:** SE(3) — 6-DOF rigid body poses
4. **Weeks 8–9:** Kalman Filter, $\mathrm{SE}_2(3)$ Extended Pose
5. **Weeks 10–12:** IEKF — Invariant Extended Kalman Filter
6. **Weeks 13–14:** Equivariant Filter
7. **Weeks 15–16:** Advanced topics and projects

PS01: Assigned this week (group axioms, SO(2), matrix exponential)

Summary: Week 1

Lecture 1 — Motivation:

- Rotation representations all have trade-offs
- Minimal $\Leftrightarrow$ singularities; no singularities $\Leftrightarrow$ redundancy
- This is topology, not engineering
- Lie groups let us work with the natural structure

Lecture 2 — Foundations:

- Groups: closure, associativity, identity, inverses
- Matrix groups: GL(n), O(n), SO(n), SE(n)
- Manifolds: locally Euclidean spaces
- Tangent spaces $\rightarrow$ Lie algebras
- Exponential map: $\mathfrak{g} \rightarrow G$

Next week: SO(2) — rotations in the plane