2.a)

Question:
The set of 2×2 invertible matrices with real entries under matrix multiplication
Solution:

Closure All invertible matrices have nonzero determinants. $\det(AB) = \det A \det(B)$. Therefore, the only way for the determinant of the product of two matrices to be zero is if one of the matrices has a determinant of zero. However, both matrices are invertible so they have nonzero determinants so their product is a 2×2 invertible matrix with real entries. Thus, closure holds.

Associativity Matrix multiplication is always associative so this property is inherited.

Identity The identity matrix multiplied with any matrix is the matrix itself so the identity matrix is the identity for this set under matrix multiplication.

Inverse The set by definition contains only invertible matrices so every element can be inverted to find its multiplicative inverse.

Therefore, all group axioms are satisfied for this set-operation pair making it a group.

2.b)

Question:
The set of integers $\mathbb{Z}$ under addition
Solution:

Closure Adding two integers yields another integer so they are closed under addition.

Associativity Addition on integers is always associative so this property is satisfied.

Identity The identity element under addition is 0.

Inverse The additive inverse element can be found by negating the element which is still an integer so inverses always exist.

Therefore, all group axioms are satisfied for this set-operation pair making it a group.

2.c)

Question:
The set of $1, -1, i, -i$ under complex multiplication
Solution:

Closure

$$1^2 = 1,\ (-1)^2 = 1,\ i^2 = -1,\ (-i)^2 = -1$$
$$1 \times -1 = -1,\ 1 \times i = i,\ 1 \times -i = -i$$
$$-1 \times i = -i,\ -1 \times -i = i$$
$$i \times -i = 1$$

All of the products between elements of the set are still in the set, therefore, the set is closed under complex multiplication.

Associativity Complex multiplication is always associative so the property is inherited.

Identity Looking at the products above, it is clear that the identity element is 1.

Inverse Looking at the products above, the inverse of 1 is 1, the inverse of -1 is -1, the inverse of i is $-i$, and the inverse of $-i$ is i. Therefore, every element has an inverse.

This set and operation pair pass all of the axioms so it is a group.

2.d)

Question:
The set of 2×2 with determinant 1 under matrix addition
Solution:

Closure The identity matrix has a determinant of 1 so it is in the set. If the set is a group under matrix addition then twice the identity matrix should be in the set.

$$\det(I) = 1$$
$$\det(2I) = 2^2\det(I) = 4 \neq 1$$

Therefore, the set is not closed under matrix addition so it is not group with this choice of operation.

2.e)

Question:
The set of positive real numbers $\mathbb{R}^+$ under multiplication
Solution:

Closure The product of any two positive numbers is positive so closure is satisfied.

Associativity Real multiplication is always associative so the property is inherited.

Identity The product of any real number with 1 is itself so the property is inherited.

Inverse The product of any real number with its reciprical (which is always a real number) is 1 so there always exists an inverse element.

Therefore, the set and operation pair passes all of the axioms so it is a group.