AAE 590: Lie Group Methods for Control and Estimation

SO(2): Lie Theory Fundamentals

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Lecture Overview

SO(2) — The Simplest Lie Group

- All Lie theory concepts in one dimension
- Lie algebra, exponential/logarithm, adjoint
- Foundation for everything that follows

$\mathrm{SO}(2)$ *is "too simple" — but perfect for learning the machinery!*

The Special Orthogonal Group SO(2) [1]

Definition

$\mathrm{SO}(2) = \{R \in \mathbb{R}^{2\times 2} : R^T R = I, \det(R) = 1\}$

Explicit form:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Verify it's a group (CAIN):

- **C**losure: $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ ✓
- **A**ssociativity: matrix multiplication ✓
- **I**nverse: $R(\theta)^{-1} = R(-\theta) = R(\theta)^T$ ✓
- **N**eutral: $R(0) = I$ ✓

SO(2) is Abelian

Special Property

SO(2) is **abelian** (commutative): $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$

Proof:

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta_2 + \theta_1) = R(\theta_2)R(\theta_1)$$

Warning

This is **NOT** true for SO(3)! 3D rotations do not commute.

SO(2) is "too simple" — many Lie theory concepts become trivial. But it's perfect for learning!

Why "$\mathrm{SO}(2)$" and not "S^1"?

Naming convention: The n in $\mathrm{SO}(n)$ is the dimension of the **space being rotated**, not the group.

Group	Acts on	Group dim.	Topology
$\mathrm{SO}(2)$	$\mathbb{R}^2$ (plane)	1	$\cong S^1$
$\mathrm{SO}(3)$	$\mathbb{R}^3$ (3D space)	3	$\cong \mathbb{RP}^3$

$\mathbb{RP}^3$ = **real projective 3-space**: *S^3 with opposite points identified ($x \sim -x$). This is why q and $-q$ represent the same rotation.*

S^1 **vs** $\mathrm{SO}(2)$**:**

- S^1: geometric shape (circle)
- $\mathrm{SO}(2)$: group structure (rotations)

Abstract vs Matrix:

- $\mathrm{SO}(2)$ is an abstract group
- 2×2 matrices = defining representation

SO(2) as a Manifold

Topology: $\mathrm{SO}(2) \cong S^1$ (the unit circle)

Notation: S^1 and $\cong$

S^1 denotes the **unit circle** $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. More generally, S^n is the n-dimensional sphere.

The symbol $\cong$ means **isomorphic to**. An **isomorphism** is a bijective (one-to-one and onto) map that preserves structure. When we write $A \cong B$, we mean A and B are "essentially the same" — they have identical structure, just different representations.

Key properties:

- 1-dimensional **manifold** (locally looks like $\mathbb{R}^1$)
- **Compact** (closed and bounded in $\mathbb{R}^n$)
- **Connected** (one piece)
- **Not simply connected** (has a hole)

Parameterization:

- $\theta \in [0, 2\pi)$ or $(-\pi, \pi]$
- Wrapping: $\theta + 2\pi \equiv \theta$
- No global "flat" coordinates!

Compact = contains all its limit points

Group Actions and Representations

Definition

A **group action** of G on a set X is a map $\cdot : G \times X \to X$ satisfying:

1. $e \cdot x = x$ (identity acts trivially)
2. $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ (composition rule)

Intuition: Each $g \in G$ "moves" points in X; composing group elements composes motions.

Definition

A **representation** is a group action on a **vector space** V via **linear** maps: a homomorphism $\rho : G \to \mathrm{GL}(V)$.

$\mathrm{GL}(V)$ *= invertible linear maps on V. For $V = \mathbb{R}^n$: invertible $n \times n$ matrices.*

Example: $\mathrm{SO}(2)$ acts on $\mathbb{R}^2$ by rotation: $R(\theta) \cdot \boldsymbol{v} = R(\theta)\boldsymbol{v}$

- Identity: $R(0) \cdot \boldsymbol{v} = I\boldsymbol{v} = \boldsymbol{v}$ ✓
- Composition: $R(\theta_1) \cdot (R(\theta_2) \cdot \boldsymbol{v}) = R(\theta_1 + \theta_2)\boldsymbol{v}$ ✓

Representations of $\mathrm{SO}(2)$

Different ways $\mathrm{SO}(2)$ can act on vector spaces:

1. **Standard (defining):** $\rho(\theta) = R(\theta)$ on $\mathbb{R}^2$ — 2D rotation matrices
2. **Trivial:** $\rho(\theta) = 1$ on $\mathbb{R}$ — everything maps to identity
3. **Complex exponential:** $\rho_n(\theta) = e^{in\theta}$ on $\mathbb{C}$ for any $n \in \mathbb{Z}$

Are They Isomorphic?

No! $\rho_1(\theta) = e^{i\theta}$ and $\rho_2(\theta) = e^{2i\theta}$ are fundamentally different—ρ_2 "winds twice" around the circle as θ goes from 0 to 2π.

The Tangent Space at Identity

Question: What does "infinitesimal rotation" look like?

Taylor expansion:

$$R(\epsilon) = \begin{bmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{bmatrix} \approx \begin{bmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{bmatrix} = I + \epsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Derivative at identity:

$$\frac{d}{d\theta} R(\theta)\bigg|_{\theta=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Notation: $\frac{d}{d\theta}$ *is the* **total derivative** *(one variable). Use* $\frac{\partial}{\partial\theta}$ *for* **partial derivatives** *(multiple variables).*

This matrix generates all rotations!

Vector Spaces

Definition

A **vector space** over $\mathbb{R}$ is a set V with two operations:

- **Addition:** $+ : V \times V \to V$
- **Scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$

satisfying: commutativity, associativity, identity ($\mathbf{0}$), inverses, and distributivity.

Key examples:

- $\mathbb{R}^n$ with standard addition and scalar multiplication
- $\mathbb{R}^{m \times n}$ (matrices) with matrix addition and scalar multiplication
- Polynomials, continuous functions, etc.

Why This Matters

In a vector space we can **add**, **scale**, and do **calculus**. Lie groups are NOT vector spaces (can't add rotations!), but their Lie algebras ARE.

Morphisms (Structure-Preserving Maps) [2]

Definition

A **homomorphism** $\phi : G \to H$ between groups preserves the group operation:

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2)$$

Definition

An **isomorphism** is a **bijective** homomorphism. If $\phi : G \to H$ is an isomorphism, we write $G \cong H$ ("G is isomorphic to H").

Definition

An **automorphism** is an isomorphism from a structure to **itself**: $\phi : G \to G$. The set of all automorphisms of G forms a group $\mathrm{Aut}(G)$.

Example: For $G = \mathbb{R}^2$, the map $\phi_R(\boldsymbol{v}) = R\boldsymbol{v}$ (rotation) is an automorphism.

Homeomorphisms (Topology)

Definition

A **homeomorphism** is a **continuous bijection** with a **continuous inverse**.

Homeomorphism $\approx$ "same shape" (topology)
Isomorphism $\approx$ "same algebraic structure"

Examples:

- $\mathrm{SO}(2) \cong S^1$ (circle) — homeomorphic *and* isomorphic as Lie groups
- Coffee mug $\approx$ donut — homeomorphic (both have one hole)
- $[0, 1)$ and S^1 are **not** homeomorphic (circle is compact)

For Lie Groups

We care about **Lie group isomorphisms**: smooth maps that preserve both the group structure *and* the manifold structure.

Worked Example: Group Properties of $\mathrm{SO}(2)$

Verify CAIN for $\mathrm{SO}(2)$*: show* $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$ *and* $R(\theta)^{-1} = R(-\theta)$.

Solution: Group Properties of $\mathrm{SO}(2)$

$$R(\theta_1)R(\theta_2) = \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} = \begin{bmatrix} c_1 c_2 - s_1 s_2 & -c_1 s_2 - s_1 c_2 \\ s_1 c_2 + c_1 s_2 & -s_1 s_2 + c_1 c_2 \end{bmatrix} = \begin{bmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{bmatrix} = R(\theta_1 + \theta_2)$$

$$R(\theta)^{-1} = R(\theta)^T = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} = \begin{bmatrix} c_{-\theta} & -s_{-\theta} \\ s_{-\theta} & c_{-\theta} \end{bmatrix} = R(-\theta) \quad \text{(using } \cos(-\theta) = \cos\theta,\ \sin(-\theta) = -\sin\theta\text{)}$$

The Lie Algebra $\mathfrak{so}(2)$ [1, 3]

Definition

The **Lie algebra** $\mathfrak{so}(2)$ is the tangent space at identity:

$$\mathfrak{so}(2) = T_I \mathrm{SO}(2) = \left\{ \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} : \omega \in \mathbb{R} \right\}$$

Notation: $T_I G$ means the **tangent space** *to G at the identity I—the set of all "velocity vectors" at that point.*

Properties:

- $\mathfrak{so}(2)$ is a **vector space** (1-dimensional)
- Elements are **skew-symmetric** matrices: $A^T = -A$
- Isomorphic to $\mathbb{R}$: each element determined by single scalar ω

Key Insight

The Lie algebra is a **flat** vector space where we can do calculus!

Wedge and Vee Operators

Definition

Wedge ($\wedge$): $\mathbb{R} \to \mathfrak{so}(2)$ — scalar to matrix

$$\omega^\wedge = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

Definition

Vee ($\vee$): $\mathfrak{so}(2) \to \mathbb{R}$ — matrix to scalar

$$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}^\vee = \omega$$

These are inverses: $(\omega^\wedge)^\vee = \omega$ and $(A^\vee)^\wedge = A$

Notation Convention

We'll use ω for both the scalar and $\omega^\wedge$ for the matrix form.

The Exponential Map

Definition

The **exponential map** $\mathrm{Exp} : \mathfrak{so}(2) \to \mathrm{SO}(2)$:

$$\mathrm{Exp}(\omega^\wedge) = e^{\omega^\wedge} = \sum_{k=0}^{\infty} \frac{(\omega^\wedge)^k}{k!}$$

Key property: $(\omega^\wedge)^2 = -\omega^2 I$

Closed form:

$$\begin{aligned}
\mathrm{Exp}(\omega^\wedge) &= I + \omega^\wedge + \frac{(\omega^\wedge)^2}{2!} + \frac{(\omega^\wedge)^3}{3!} + \cdots \\
&= I\left(1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \cdots\right) + \omega^\wedge\left(1 - \frac{\omega^2}{3!} + \frac{\omega^4}{5!} - \cdots\right) \\
&= I\cos\omega + \omega^\wedge \frac{\sin\omega}{\omega} = R(\omega)
\end{aligned}$$

Exp **maps from the tangent space (flat) to the manifold (curved)**

Lie algebra $\mathfrak{so}(2)$:

- Linear space $\cong \mathbb{R}$
- ω is angular velocity
- Can add/subtract freely

Lie group $\mathrm{SO}(2)$:

- Curved manifold $\cong S^1$
- $R(\omega)$ is rotation
- Composition, not addition

$$\boxed{\omega \in \mathbb{R} \xrightarrow{\text{wedge}} \omega^\wedge \in \mathfrak{so}(2) \xrightarrow{\mathrm{Exp}} R(\omega) \in \mathrm{SO}(2)}$$

The Logarithmic Map

Definition

The **logarithmic map** $\mathrm{Log} : \mathrm{SO}(2) \to \mathfrak{so}(2)$ is the inverse of Exp:

$$\mathrm{Log}(R) = \omega^\wedge \quad \text{where} \quad R = R(\omega)$$

For $\mathrm{SO}(2)$:

$$\mathrm{Log}(R) = \mathrm{atan2}(R_{21}, R_{11})^\wedge$$

Multi-valuedness

Log returns $\omega \in (-\pi, \pi]$. But $R(\omega) = R(\omega + 2\pi k)$ for any $k \in \mathbb{Z}$ (integers).

For $\mathrm{SO}(2)$: Exp is **surjective** (onto: every rotation is reached), Log is only locally defined.

Why Exp/Log Matter

Problem: How to interpolate between rotations R_0 and R_1?

Wrong (Euclidean):

$$R(t) = (1 - t)R_0 + tR_1 \quad \text{NOT a rotation matrix!}$$

Correct (Lie group):

$$R(t) = R_0 \cdot \mathrm{Exp}\left(t \cdot \mathrm{Log}(R_0^{-1} R_1)\right)$$

- Compute relative rotation: $\Delta R = R_0^{-1} R_1$
- Map to algebra: $\Delta\omega = \mathrm{Log}(\Delta R)^\vee$
- Scale in algebra: $t \cdot \Delta\omega$
- Map back: $\mathrm{Exp}((t \cdot \Delta\omega)^\wedge)$
- Apply from R_0

Worked Example: $\mathrm{SO}(2)$ Exp and Log

Compute $\mathrm{Exp}((\pi/4)^\wedge)$*, verify* $R^T R = I$*, then* $\mathrm{Log}(R)$ *to recover* $\pi/4$*.*

Solution: $\mathrm{SO}(2)$ Exp and Log

$$\mathrm{Exp}((\pi/4)^\wedge) = R(\pi/4) = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Verify: $R^T R = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} + \frac{1}{2} \end{bmatrix} = I \checkmark$

$$\mathrm{Log}(R)^\vee = \mathrm{atan2}(R_{21}, R_{11}) = \mathrm{atan2}(\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}) = \pi/4 \checkmark$$

The Adjoint Representation Ad

Definition

The **adjoint map** $\mathrm{Ad}_R : \mathfrak{so}(2) \to \mathfrak{so}(2)$ is conjugation:

$$\mathrm{Ad}_R(\omega^\wedge) = R\,\omega^\wedge\,R^{-1}$$

For $\mathrm{SO}(2)$: Since $\mathrm{SO}(2)$ is abelian:

$$\mathrm{Ad}_R(\omega^\wedge) = R\,\omega^\wedge\,R^T = \omega^\wedge$$

$\mathrm{SO}(2)$ is trivial!

$\mathrm{Ad}_R = I$ for all $R \in \mathrm{SO}(2)$. The adjoint does nothing.

This will be **much more interesting** for $\mathrm{SO}(3)$ and $\mathrm{SE}(2)$!

The Small Adjoint ad

Definition

The **small adjoint** $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket:

$$\mathrm{ad}_{\omega_1}(\omega_2) = [\omega_1^\wedge, \omega_2^\wedge] = \omega_1^\wedge \omega_2^\wedge - \omega_2^\wedge \omega_1^\wedge$$

For $\mathrm{SO}(2)$:

$$[\omega_1^\wedge, \omega_2^\wedge] = 0 \quad \text{(always zero!)}$$

Abelian $\Rightarrow$ Trivial Bracket

The Lie bracket measures "non-commutativity." For abelian groups, it vanishes.

Relationship: ad is the derivative of Ad at identity.

Baker-Campbell-Hausdorff (BCH) Formula

Question: Given $\operatorname{Exp}(\xi_1^\wedge)$ and $\operatorname{Exp}(\xi_2^\wedge)$, what is $\operatorname{Exp}(\xi_1^\wedge) \cdot \operatorname{Exp}(\xi_2^\wedge)$?

Theorem (BCH Formula)

$$\operatorname{Exp}(\xi_1^\wedge) \cdot \operatorname{Exp}(\xi_2^\wedge) = \operatorname{Exp}\left(\left(\xi_1 + \xi_2 + \frac{1}{2}[\xi_1, \xi_2] + \frac{1}{12}[\xi_1, [\xi_1, \xi_2]] + \cdots\right)^\wedge\right)$$

For $\mathrm{SO}(2)$: All brackets vanish, so:

$$\operatorname{Exp}(\omega_1^\wedge) \cdot \operatorname{Exp}(\omega_2^\wedge) = \operatorname{Exp}((\omega_1 + \omega_2)^\wedge)$$

i.e., $R(\omega_1)R(\omega_2) = R(\omega_1 + \omega_2)$ — just addition!

For non-abelian groups: BCH is much more complex.

Kinematics on $\mathrm{SO}(2)$

Angular velocity: $\omega(t) \in \mathbb{R}$

Kinematic equation:

$$\dot{R} = R\,\omega^\wedge$$

Interpretation:

- $\omega^\wedge \in \mathfrak{so}(2)$ is the **body-frame** angular velocity
- Right multiplication: velocity expressed in body frame
- This is a **left-invariant** vector field on $\mathrm{SO}(2)$

$\mathrm{SO}(2)$ is Abelian: Left = Right Invariant

Since $\mathrm{SO}(2)$ is abelian, $R\omega^\wedge = \omega^\wedge R$, so left and right invariant forms are **equivalent**. This is **not** true for $\mathrm{SE}(2)$ or $\mathrm{SO}(3)$!

Solution (constant ω):

$$R(t) = R(0) \cdot \mathrm{Exp}(t\,\omega^\wedge) = R(0) \cdot R(t\omega)$$

SO(2) Summary

Concept	SO(2)
Group	2×2 rotation matrices
Lie algebra $\mathfrak{so}(2)$	2×2 skew-symmetric
Dimension	1
Wedge $\omega^\wedge$	$\begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$
$\mathrm{Exp}(\omega^\wedge)$	$R(\omega)$
$\mathrm{Log}(R)$	$\mathrm{atan2}(R_{21}, R_{11})^\wedge$
Ad_R	Identity (trivial)
$[\cdot, \cdot]$	Zero (abelian)

$\mathrm{SO}(2)$ **is "too nice"** — but now we have all the machinery for harder groups!

Looking Ahead: SE(2)

Next lecture: $\mathrm{SE}(2)$ — combining rotation AND translation

- Non-abelian (rotations and translations don't commute)
- Semi-direct product structure
- Non-trivial adjoint representation
- Foundation for rigid body control

References I

[1] Brian C Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. 2nd. Springer, 2015.

[2] David S Dummit and Richard M Foote. *Abstract Algebra*. 3rd. Wiley, 2004.

[3] John Stillwell. *Naive Lie Theory*. Springer, 2008.