AAE 590: Problem Set 02
SE(2): Planar Rigid Body Motions

Due: See Brightspace

Instructions

- Show all work for full credit. Derivations should be clear and complete.
- **Submit a single PDF to Gradescope** containing:
  - Derivations (handwritten and scanned, or typed in LaTeX)
  - Python code (included as text or screenshots)
  - All plots and numerical verification results
- You may use AI tools, but you must understand your solutions (validated via in-class quizzes).
- **Prerequisite:** You should have working implementations of `so2_wedge`, `so2_vee`, `so2_exp`, and `so2_log` from PS01.
- Estimated time: 6 hours

Problem 1: Normal Subgroups and Quotient Groups (15 points)

Before diving into SE(2), let's understand why normal subgroups matter.

(a) (5 pts) **Definition and Intuition:**

A subgroup $N \leq G$ is **normal** (written $N \trianglelefteq G$) if $gNg^{-1} = N$ for all $g \in G$.

- Explain why every subgroup of an abelian group is automatically normal.
- For non-abelian groups, conjugation can "twist" a subgroup. Give geometric intuition: if H is a set of translations and g is a rotation, what does gHg^{-1} represent?

(b) (5 pts) **A Matrix Group Example:**

Consider the determinant map $\det : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ under multiplication).

- Verify det is a homomorphism: $\det(AB) = \det(A)\det(B)$.
- What is ker(det)? (This group has a name—what is it?)
- What information about a matrix is "forgotten" when we apply det? What's preserved?

(c) (5 pts) **First Isomorphism Theorem (Preview):**

For a homomorphism $\phi : G \to H$:

- **Kernel:** $\ker(\phi) = \{g \in G : \phi(g) = e_H\}$ (elements mapped to identity)
- **Image (Range):** $\text{im}(\phi) = \{\phi(g) : g \in G\}$ (elements that ϕ "hits")

First Isomorphism Theorem: $G/\ker(\phi) \cong \text{im}(\phi)$.

In words: quotienting by what ϕ "kills" gives you what ϕ "sees."

- For SE(2): if we define $\pi : \text{SE}(2) \to \text{SO}(2)$ by $\pi(\boldsymbol{t}, R) = R$, what is $\ker(\pi)$?
- What does the First Isomorphism Theorem tell us about $\text{SE}(2)/\ker(\pi)$?

Problem 2: SE(2) and the Semi-Direct Product (35 points)

The group SE(2) of planar rigid motions is our first example of a **semi-direct product**. This structure—where one subgroup "twists" another—is fundamental to robotics.

The Translation Group $\text{T}(2)$**:** The set of 2D translations forms a group T(2) under composition. As a matrix Lie group:

$$\text{T}(2) = \left\{ \begin{bmatrix} I & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} : \boldsymbol{t} \in \mathbb{R}^2 \right\} \cong (\mathbb{R}^2, +)$$

This group is abelian (translations commute) and isomorphic to $\mathbb{R}^2$ with vector addition. Its Lie algebra is $\mathfrak{t}(2) \cong \mathbb{R}^2$.

(a) (5 pts) **Matrix Representation:** Write $X \in \text{SE}(2)$ as a 3×3 matrix:

$$X = \begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix}, \quad R \in \text{SO}(2),\ \boldsymbol{t} \in \mathbb{R}^2$$

Tuple notation: We can also write $X = (\boldsymbol{t}, R)$ as a compact tuple. The correspondence is:

$$(\boldsymbol{t}, R) \longleftrightarrow \begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

We order as $(\boldsymbol{t}, R)$ (translation first) to match the semi-direct product $\text{T}(2) \rtimes \text{SO}(2)$ convention. Both notations represent the same rigid motion: rotate by R, then translate by $\boldsymbol{t}$ (in the world frame).

- Derive the composition X_1X_2 and inverse X^{-1} formulas using block matrix multiplication. You don't need to expand 2×2 blocks—leave products like R_1R_2 as is.
- Express your results in tuple form: $(\boldsymbol{t}_1, R_1) \cdot (\boldsymbol{t}_2, R_2) = (?, ?)$ and $(\boldsymbol{t}, R)^{-1} = (?, ?)$.
- Implement `se2_compose(X1, X2)` and `se2_inverse(X)`.

Check your work: $(\boldsymbol{t}_1, R_1) \cdot (\boldsymbol{t}_2, R_2) = (\boldsymbol{t}_1 + R_1\boldsymbol{t}_2,\ R_1R_2)$ and $(\boldsymbol{t}, R)^{-1} = (-R^T\boldsymbol{t},\ R^T)$.

(b) (5 pts) **Why SE(2) is Non-Abelian:**

- Compute X_RX_t and X_tX_R where $X_R = (\boldsymbol{0}, R_{90^\circ})$ and $X_t = ((1, 0)^T, I)$.
- Where does the origin end up under each? Sketch both results to see why order matters.

- Why does coupling rotations with translations break commutativity? (Hint: SO(2) alone is abelian.)
- **Action on points:** For $X = (\boldsymbol{t}, R)$, the action on a point is $X \cdot \boldsymbol{p} = R\boldsymbol{p} + \boldsymbol{t}$. Note that $(X_1 X_2) \cdot \boldsymbol{p} = X_1 \cdot (X_2 \cdot \boldsymbol{p})$ (the right factor acts first). Using your result for $X_R X_t$, find the world position of a sensor at body position $\boldsymbol{p}_b = (0.5, 0)^T$.

(c) (10 pts) **The Semi-Direct Product Structure:**

SE(2) is the semi-direct product $\mathrm{SE}(2) = \mathrm{T}(2) \rtimes \mathrm{SO}(2)$, where rotations *act on* translations. For a general semi-direct product $N \rtimes_\phi H$, the group operation is:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2),\ h_1 h_2)$$

where $\phi : H \to \mathrm{Aut}(N)$ describes how H acts on N.

- For $\mathrm{SE}(2) = \mathrm{T}(2) \rtimes \mathrm{SO}(2)$: identify N, H, and the action ϕ. What does $\phi(R)$ do to a translation $\boldsymbol{t}$?
- Apply the semi-direct product formula to show $(\boldsymbol{t}_1, R_1) \cdot (\boldsymbol{t}_2, R_2) = (\boldsymbol{t}_1 + R_1\boldsymbol{t}_2,\ R_1 R_2)$. Verify this matches your composition formula from part (a).
- **Key insight:** In a direct product $N \times H$, neither factor affects the other. In a semi-direct product, H "twists" N. Explain why "translate then rotate" $\neq$ "rotate then translate."
- Why is SE(2) written as $\mathrm{T}(2) \rtimes \mathrm{SO}(2)$ and not $\mathrm{SO}(2) \rtimes \mathrm{T}(2)$? (Hint: which subgroup is normal?)

(d) (15 pts) **Normal Subgroups and the Quotient** $\mathrm{SE}(2)/\mathrm{T}(2) \cong \mathrm{SO}(2)$**:**

The translation subgroup is $\mathrm{T}(2) = \{(\boldsymbol{t}, I) : \boldsymbol{t} \in \mathbb{R}^2\} \subset \mathrm{SE}(2)$, i.e., elements of the form:

$$(\boldsymbol{t}, I) \iff \begin{bmatrix} I & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- Compute the conjugation $(\boldsymbol{p}, R) \cdot (\boldsymbol{t}, I) \cdot (\boldsymbol{p}, R)^{-1}$ using the tuple formula. Also verify by matrix multiplication. Is the result a pure translation?
- Based on your result, explain why $\mathrm{T}(2) \trianglelefteq \mathrm{SE}(2)$ (i.e., translations form a normal subgroup).
- Now try: is $\mathrm{SO}(2) = \{(\mathbf{0}, R)\}$ a normal subgroup? Compute $(\boldsymbol{t}, I) \cdot (\mathbf{0}, R) \cdot (\boldsymbol{t}, I)^{-1}$.
- Describe the cosets of T(2) in SE(2). What do two elements in the same coset share?
- Define $\pi : \mathrm{SE}(2) \to \mathrm{SO}(2)$ by $\pi(\boldsymbol{t}, R) = R$. Verify π is a homomorphism with $\ker(\pi) = \mathrm{T}(2)$.
- Apply the First Isomorphism Theorem: $\mathrm{SE}(2)/\mathrm{T}(2) \cong \mathrm{SO}(2)$. Interpret: "forgetting position leaves orientation."
- **Predicting the Adjoint Structure (will verify in Problem 3):**

  Your conjugation results reveal the structure of Ad_X before you derive it!

  - You showed $(\boldsymbol{p}, R) \cdot (\boldsymbol{t}, I) \cdot (\boldsymbol{p}, R)^{-1} = (R\boldsymbol{t}, I)$. The adjoint Ad_X is the linearization (derivative at the identity) of the conjugation map $Y \mapsto XYX^{-1}$.
  - **Predict:** Based on this, what 2×2 block should appear in Ad_X acting on the velocity components (v_x, v_y)? Explain your reasoning, and also predict whether translation affects angular velocity (Hint: is SO(2) normal?).

You'll verify your predictions in Problem 3(d).

Problem 3: SE(2) Lie Algebra, Exp/Log, and Adjoint (35 points)

The Lie algebra $\mathfrak{se}(2)$ consists of twists. We use **translation-first** ordering $\xi = (v_x, v_y, \omega)^T \in \mathbb{R}^3$, which yields block upper-triangular matrices.

Notation: Exp denotes the matrix exponential. It takes a matrix $\xi^\wedge \in \mathfrak{se}(2)$ and returns a group element in SE(2).

(a) (5 pts) **Lie Algebra Structure:**

With $\xi = (v_x, v_y, \omega)^T = (\boldsymbol{v}^T, \omega)^T$, the wedge map is:

$$\xi^\wedge = \begin{bmatrix} \omega^\wedge & \boldsymbol{v} \\ \mathbf{0}^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

- Verify this is block upper-triangular (rotation block in upper-left, translation in upper-right).
- Implement `se2_wedge(xi)` and `se2_vee(Xi)`.

(b) (8 pts) **Exponential Map:** The closed-form expression is:

$$\mathrm{Exp}(\xi^\wedge) = \begin{bmatrix} R(\omega) & V(\omega)\boldsymbol{v} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where $V(\omega) = \frac{\sin\omega}{\omega} I + \frac{1-\cos\omega}{\omega} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

- Derive by hand using the matrix exponential power series (show key steps).
- Implement `se2_exp(xi)` with Taylor series for $\omega \approx 0$ (use 5 terms).

(c) (7 pts) **Logarithm Map:**

The inverse of $V(\omega)$ is:

$$V(\omega)^{-1} = \frac{\omega}{2} \cot \frac{\omega}{2} I + \frac{\omega}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Implement `se2_log(X)`. Use Taylor series for $\omega \approx 0$ (use 5 terms).
- Verify numerically: $\mathrm{Exp}(\mathrm{Log}(X)) = X$ for random $X \in \mathrm{SE}(2)$.

(d) (10 pts) **Adjoint Representation Ad_X:**

The adjoint $\mathrm{Ad}_X : \mathfrak{se}(2) \to \mathfrak{se}(2)$ is defined by $(\mathrm{Ad}_X \xi)^\wedge = X \xi^\wedge X^{-1}$.

Note: You don't need to multiply out 2×2 blocks explicitly. Showing block form (e.g., $R\omega^\wedge R^T$, $R\boldsymbol{v}$) is sufficient.

Translation-first ordering $\xi = (v_x, v_y, \omega)^T$:

- Compute $X\xi^\wedge X^{-1}$ explicitly for $X = (\boldsymbol{t}, R)$.
- Extract the 3×3 matrix Ad_X such that $\mathrm{Ad}_X \xi$ gives the transformed twist.
  You should get: $\mathrm{Ad}_X = \begin{bmatrix} R & \boldsymbol{t}^\odot \\ \mathbf{0}^T & 1 \end{bmatrix}$ where $\boldsymbol{t}^\odot = \begin{bmatrix} t_y \\ -t_x \end{bmatrix} = -J\boldsymbol{t}$ with $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Rotation-first ordering $\xi = (\omega, v_x, v_y)^T$:

- Repeat the derivation with this ordering. The wedge map produces the same 3×3 matrix, but extracting Ad_X requires matching to the new vector ordering.
- You should get: $\mathrm{Ad}_X = \begin{bmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{t}^{\odot} & R \end{bmatrix}$ (block *lower*-triangular).

Verification:

- Verify numerically: $\mathrm{Ad}_{X_1 X_2} = \mathrm{Ad}_{X_1}\mathrm{Ad}_{X_2}$ for random $X_1, X_2 \in \mathrm{SE}(2)$.
- Verify numerically: $\mathrm{Ad}_X^{-1} = \mathrm{Ad}_{X^{-1}}$ (the adjoint respects inverses).
- **Connection to Problem 2:** Confirm that R acts on velocity components, as you predicted from the normal subgroup structure.

(e) (5 pts) **Lie Bracket and ad_ξ:**

The Lie bracket on vectors is defined by $[\xi_1, \xi_2]^\wedge := \xi_1^\wedge \xi_2^\wedge - \xi_2^\wedge \xi_1^\wedge$ (i.e., compute the matrix commutator, then apply vee). The small adjoint ad_ξ is the 3×3 matrix such that $[\xi_1, \xi_2] = \mathrm{ad}_{\xi_1} \xi_2$.

Translation-first ordering $\xi = (v_x, v_y, \omega)^T$:

- Compute $[\xi_1^\wedge, \xi_2^\wedge]$ for $\xi_i = (v_{ix}, v_{iy}, \omega_i)^T$. Extract the result as a vector.
- Derive by hand the 3×3 matrix ad_ξ.

  You should get: $\mathrm{ad}_\xi = \begin{bmatrix} \omega^\wedge & \boldsymbol{v}^{\odot} \\ \mathbf{0}^T & 0 \end{bmatrix}$ where $\boldsymbol{v}^{\odot} = \begin{bmatrix} v_y \\ -v_x \end{bmatrix} = -J\boldsymbol{v}$.

Problem 4: SE(2) Kinematics Simulation (15 points)

The kinematic equation $\dot{X} = X\xi^\wedge$ describes rigid body motion on SE(2).

(a) (5 pts) **Unicycle Model:** A unicycle has forward velocity v and yaw rate ω. In the body frame, there is no lateral motion ($v_y = 0$), so $v_x = v$.

- Write the body-frame twist $\xi = (v_x, v_y, \omega)^T$ for the unicycle.
- Expand $\dot{X} = X\xi^\wedge$ to derive the ODEs: $\dot{\theta} =?$, $\dot{x} =?$, $\dot{y} =?$
- For constant $v > 0$ and $\omega > 0$: What shape is the trajectory? What is the radius?

(b) (10 pts) **Simulation:** Implement two integrators:

- **Euler:** $X_{k+1} = X_k(I + \Delta t\, \xi_k^\wedge)$
- **Lie group:** $X_{k+1} = X_k \mathrm{Exp}((\Delta t\, \xi_k)^\wedge)$

Simulate with $v = 1$ m/s, $\omega = 0.5$ rad/s, $\Delta t = 0.1$ s for 20 seconds. Start at $X_0 = I$.

- **Plot 1:** (x, y) trajectories for both methods on the same figure.
- **Plot 2:** $\|R^T R - I\|_F$ vs time. Which method preserves $R^T R = I$? Why?
- **Exact solution:** For constant twist ξ, the exact solution is $X(t) = \mathrm{Exp}((t\xi)^\wedge)$. Verify that your Lie group integrator matches $\mathrm{Exp}((t\xi)^\wedge)$ to numerical precision at $t = 20$s. (This confirms the exponential map's meaning!)

Submission Checklist

Submit a **single PDF to Gradescope** containing:

- ☐ Problem 1: Normal subgroup concepts and examples
- ☐ Problem 2: Semi-direct product structure and normal subgroup proofs
- ☐ Problem 3: Exp/Log, Ad_X, and ad_ξ derivations with code
- ☐ Problem 4: Kinematics derivation and simulation plots