AAE590 LGM HW2

1a: Explain why every subgroup of an abelian group is automatically normal

Abelian groups have the property that $a \cdot b = b \cdot a$

A subgrab $N \subseteq G$ is normal if $gNg^{-1} = N \ \forall g \in G$

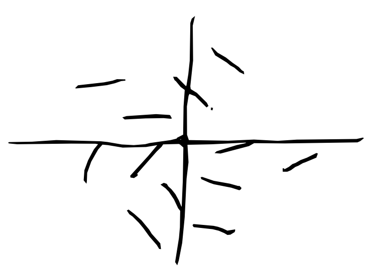
Subgroup retains abelian property

$$gNg^{-1} = Ngg^{-1} = N = N \ \forall g \in G \ \blacksquare$$

If H is a set of translations and g is a rotation what does gHg^{-1} represent?

gHg^{-1} represents rotating, translating, then rotating back to the original orientation

So this would look something like a twist in a vector field / spiral with some contraction or expansion



1b: Consider $\det: GL(2,\mathbb{R}) \to \mathbb{R}^*$ where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$
under multiplication

know from det properties that $\det(AB) = \det(A)\det(B)$

homomorphisms satisfy $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)$
φ

$\therefore$ det is homomorphism ■

$\ker(\det)$

Forget info on size, eigenvalues

- invariant to multiplication by matrices with determinant 1 ↑ in $SL(2)$
- Left multiplication by invertible matrix and right multiplication by its inverse

Know if full rank or not, if it switches directions, how much space is expanded

1c: First isomorphism theorem: $G/\ker(\phi) \cong \mathrm{im}(\phi)$

For $SE(2)$: if we define $\pi: SE(2) \to SO(2) \Rightarrow \mathrm{im}(\pi) = SO(2)$
by $\pi(t,R) = R$, what is $\ker(\pi)$?

$\ker(\pi) = T(2)$

What does First Isomorphism Theorem tell us about $SE(2)/\ker(\pi)$?

$$SE(2)/\ker(\pi) = \mathrm{im}(\pi)$$
$$SE(2)/T(2) = \mathrm{im}(\pi) = SO(2)$$

$$X = \begin{pmatrix} R & t \\ 0^T & 1 \end{pmatrix}$$

2a. • Derive the composition $X_1 X_2$ and inverse X^{-1} formulas using block matrix multiplication

$$X_1 X_2 = \begin{pmatrix} R_1 & t_1 \\ & 1 \end{pmatrix}\begin{pmatrix} R_2 & t_2 \\ & 1 \end{pmatrix} = \begin{pmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ & 1 \end{pmatrix}$$

$$X X^{-1} = I = \begin{pmatrix} R_1 R_2 & R_1 t_2 + t_1 \\ & 1 \end{pmatrix} = \begin{pmatrix} I & 1 \\ & 1 \end{pmatrix}$$

$$R_1 R_2 = I \Rightarrow R_2 = R_1^{-1} = R_1^T$$

$$R_1 t_2 + t_1 = 1 \Rightarrow t_2 = -R_1^T t_1$$

$$\Rightarrow X^{-1} = \begin{pmatrix} R^T & -R^T t \\ & 1 \end{pmatrix}$$

• Express in tuple form

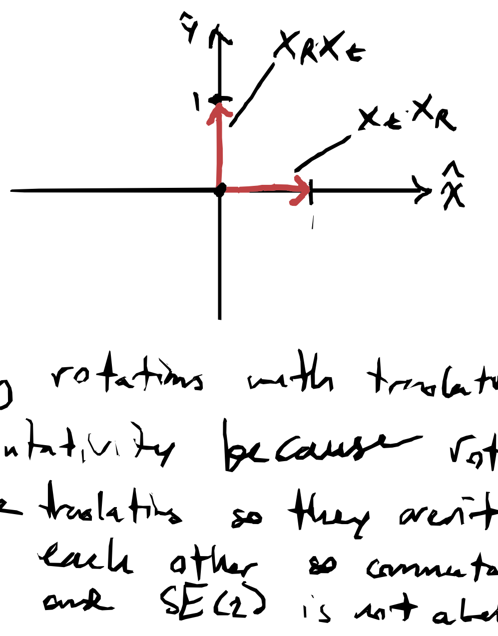
$$(t_1, R_1)\cdot(t_2, R_2) = (R_1 t_2 + t_1, R_1 R_2)$$

$$(t, R)^{-1} = (-R^T t, R^T)$$

2b. • Compute $X_R X_t$ and $X_t X_R$ where $X_R = (0, R_{90°})$, $X_t = \left(\begin{pmatrix}1\\0\end{pmatrix}, I\right)$

$$X_R X_t = \left(\begin{pmatrix}0\\1\end{pmatrix}, R_{90°}\right), \quad X_t X_R = \left(\begin{pmatrix}1\\0\end{pmatrix}, R_{90°}\right)$$

Not the same



• Coupling rotations with translation break commutativity because rotations affect the translations so they aren't independent of each other so commutativity is broken and SE(2) is not abelian.

• Find $X_R X_t P_b$, $P_b = \begin{pmatrix}0.5\\0\end{pmatrix}$

$$X_t \cdot P_b = \begin{pmatrix}1.5\\0\end{pmatrix}, \quad X_R \cdot (X_t P_b) = \begin{pmatrix}0\\1.5\end{pmatrix}$$

2c. For $SE(2) = T(2) \rtimes SO(2)$ identify N, H and action ϕ. What does $\phi(R)$ do to a translation t?

N is $T(2)$, H is $SO(2)$, ϕ rotates a translation vector

$\phi(R)$ rotates a translation t by R

• Show using semidirect product formula composition of SE(2)

$$(n_1, h_1)\cdot(n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2), h_1 h_2)$$

$n_1 = t_1$, $h_1 = R_1$, $n_2 = t_2$, $h_2 = R_2$, $\phi(h_1) = R_1$

$\Rightarrow (t_1, R_1)(t_2, R_2) = (t_1 + R_1 t_2, R_1 R_2)$ ✓ matches

• SE(2) is a semi direct product so SO(2) affects the translation when elements are composed if it were a direct product SO(2) and T(2) would be totally separate.

SO(2) is not a normal subgroup of SE(2) because of the semi direct product

• Written as $T(2) \rtimes SO(2)$ because the first one has to be normal subgroup and T(2) is but SO(2) isn't

2d. • Compute $(p, R)\cdot(t, I)\cdot(p, R)^{-1}$

tuple: $(p, R)\cdot(t, I)\cdot\underbrace{(p, R)^{-1}}_{(-R^T p,\, R^T)} = (p, R)\cdot(t - R^T p, R^T) = (Rt, I)$

matrix:

$$\begin{pmatrix} R & p \\ & 1 \end{pmatrix}\begin{pmatrix} I & t \\ & 1 \end{pmatrix}\begin{pmatrix} R^t & -R^T p \\ & 1 \end{pmatrix} = \begin{pmatrix} R & p \\ & 1 \end{pmatrix}\begin{pmatrix} R^t & -R^t p + t \\ & 1 \end{pmatrix}$$

$$= \begin{pmatrix} I & Rt \\ & 1 \end{pmatrix}$$

~~Aft~~ a pure translation, t gets rotated by R but is in T(2) so is pure...

• T(2) forms a normal subgroup because conjugation results in element of T(2) still and it's in T(2)

• $(t, I)\cdot(0, R)\cdot(t, I)^{-1} = (t, I)\cdot(-Rt, R) = ((I - R)t, R)$

SO(2) Not a normal subgroup of SE(2)

• cosets of T(2) in SE(2) share same translation but could have different rotation

• Define $\pi: SE(2) \to SO(2)$ by $\pi(t, R) = R$

$$\pi(X_1 X_2) = \pi((R_1 t_2 + t_1, R_1 R_2)) = R_1 R_2 = \pi(X_1)\cdot\pi(X_2) = R_1 \cdot R_2 \checkmark$$

∴ homomorphism

$$\pi(\underbrace{(t, I)}_{\in T(2)}) = I \text{ so } T(2) = \ker(\pi) \checkmark$$

• $SE/T(2) = SO(2)$ as shown by ↑

so removing position knowledge leaves translation

• Predict Adjoint structure

I predict based upon $(p, R)\cdot(t, I)(p, R)^{-1} = (Rt, I)$

and that $\left.\frac{d}{d\theta} R\right|_{\theta=0} = \left.\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

that the two by two block acting on $\begin{pmatrix} v_x \\ v_y \end{pmatrix}$ is

Translation will not affect angular velocity because SO(2) is normal (it is abelian)

$$[\omega^\wedge]^2$$

3a: $\xi^\wedge = \begin{bmatrix} \omega^\wedge & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \omega^\wedge & v \\ \hline 0 & 0 \end{array}\right]$

$\omega^\wedge = \omega J$

$\omega^{\wedge 2} = -\omega^2 I,\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\omega^{\wedge 3} = -\omega^2 \omega^\wedge = -\omega^3 J$

block upper triangular ✓

3b: $\exp(\xi^\wedge) = I + \xi^\wedge + \frac{1}{2}\xi^{\wedge 2} + \frac{1}{6}\xi^{\wedge 3} + \frac{1}{24}\xi^{\wedge 4} + \dots$

$$\xi^{\wedge 2} = \begin{pmatrix} \omega^\wedge & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega^\wedge & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \omega^{\wedge 2} & \omega^\wedge v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 I & \omega J v \\ 0 & \end{pmatrix}$$

$$\xi^{\wedge 3} = \begin{pmatrix} \omega^\wedge & v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega^\wedge & v \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} \omega^{\wedge 3} & \omega^{\wedge 2} v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\omega^3 J & -\omega^2 v \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \exp(\xi^\wedge) = \begin{pmatrix} (I - \frac{1}{2}\omega^2 + \frac{1}{24}\omega^4 - \dots)I + (\omega - \frac{1}{6}\omega^3 + \frac{1}{120}\omega^5 - \dots)J & ((\omega I - \frac{1}{6}\omega^3 I + \dots) + (\frac{1}{2}\omega^2 J + \frac{1}{24}\omega^4 J - \dots))\frac{v}{\omega} \\ 0 & \end{pmatrix}$$

$$(\omega I + \tfrac{1}{2}\omega^2 J - \tfrac{1}{6}\omega^3 I + \tfrac{1}{24}\omega^4 J - \dots)\frac{v}{\omega}$$

$$((\omega I - \tfrac{1}{6}\omega^3 I + \dots) + (\tfrac{1}{2}\omega^2 J + \tfrac{1}{24}\omega^4 J - \dots))\frac{v}{\omega}$$

$$(\sin\omega\, I + (1-\cos\omega) J)\frac{v}{\omega}$$

$$V(\omega) = \sin\omega\, I + (1-\cos\omega)J \checkmark$$

$$\Rightarrow \exp(\xi^\wedge) = \begin{pmatrix} R(\omega) & V(\omega)v \\ & 1 \end{pmatrix}$$ ← SE(2) exp map

2c: $V(\omega)^{-1} = \frac{\omega}{2}\cot\frac{\omega}{2} I + \frac{\omega}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

cot taylor series is

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + \frac{x^7}{4725} - \dots$$

2d: $X\xi^\wedge X^{-1} = \begin{pmatrix} R & t \\ & 1 \end{pmatrix}\begin{bmatrix} \omega^\wedge & v \end{bmatrix}\begin{pmatrix} R^T & -R^T t \\ & 1 \end{pmatrix} = \begin{pmatrix} R & t \\ & 1 \end{pmatrix}\begin{pmatrix} \omega^\wedge R^T & -\omega^\wedge R^T t + v \\ & 1 \end{pmatrix}$

$$= \begin{pmatrix} R\omega^\wedge R^T & -R\omega^\wedge R^T t + Rv + t \\ & 1 \end{pmatrix}$$

That-ing? $\begin{pmatrix} R & -Jt \\ & 1 \end{pmatrix}\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} Rv - \omega^\wedge t \\ \omega \end{pmatrix}$

At angle $R\omega^\wedge R^T = \omega^\wedge$

and $-R\omega^\wedge R^T t + Rv + t = -\omega^\wedge t + Rv$

Now with $\begin{pmatrix} \omega \\ v \end{pmatrix}$ ordering

$$\omega' = \omega$$
$$v' = Rv - Jt\omega$$

$$\Rightarrow \mathrm{Ad}_X \xi = \begin{pmatrix} 1 & 0 \\ -Jt & R \end{pmatrix}\begin{pmatrix} \omega \\ v \end{pmatrix} \checkmark$$

Look at v', R acts on the velocity component as predicted

3e: Compute $[\xi_1^\wedge, \xi_2^\wedge] = \xi_1^\wedge \xi_2^\wedge - \xi_2^\wedge \xi_1^\wedge$

$$\xi_1^\wedge \xi_2^\wedge = \begin{pmatrix} \omega_1^\wedge & v_1 \end{pmatrix}\begin{pmatrix} \omega_2^\wedge & v_2 \end{pmatrix} = \begin{pmatrix} \omega_1^\wedge \omega_2^\wedge & \omega_1^\wedge v_2 \end{pmatrix}$$

$$\Rightarrow [\xi_1^\wedge, \xi_2^\wedge] = \begin{pmatrix} \cancel{\omega_1^\wedge\omega_2^\wedge - \omega_2^\wedge\omega_1^\wedge}\ 0 & \omega_1^\wedge v_2 - \omega_2^\wedge v_1 \end{pmatrix}$$

$\begin{pmatrix} v_y \\ -v_x \end{pmatrix} = -Jv$

$-\omega_2^\wedge v_1 = v_1^\odot \omega_2$

want $[\xi_1, \xi_2^\wedge] = \mathrm{ad}_{\xi_1}\xi_2$

$$\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} = \begin{pmatrix} -ab & 0 \\ 0 & -ab \end{pmatrix} = \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

$$\Rightarrow [\xi_1^\wedge, \xi_2^\wedge] = \begin{pmatrix} 0 & \omega_1^\wedge v_2 + v_1^\odot \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1^\wedge & v_1^\odot \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_2 \\ \omega_2 \end{pmatrix}$$

result is $\begin{pmatrix} \omega_1^\wedge v_2 + v_1^\odot \omega_2 \\ 0 \end{pmatrix}$, $\mathrm{ad}_\xi = \begin{pmatrix} \omega^\wedge & v^\odot \\ 0 & 0 \end{pmatrix}$ ✓