1.a) Definition and Intuition:

Question:
A subgroup $N \leq G$ is **normal** (written $N \trianglelefteq G$) if $gNg^{-1} = N$ for all $g \in G$.

- Explain why every subgroup of an abelian group is automatically normal.
- For non-abelian groups, conjugation can "twist" a subgroup. Give geometric intuition: if H is a set of translations and g is a rotation, what does gHg^{-1} represent?

Solution:

1.b) A Matrix Group Example:

Question:
Consider the determinant map $\det : \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ under multiplication).

- Verify det is a homomorphism: $\det(AB) = \det(A)\det(B)$.
- What is $\ker(\det)$? (This group has a name—what is it?)
- What information about a matrix is "forgotten" when we apply det? What's preserved?

Solution:

1.c) First Isomorphism Theorem (Preview):

Question:
For a homomorphism $\phi : G \to H$:

- **Kernel:** $\ker(\phi) = \{g \in G : \phi(g) = e_H\}$ (elements mapped to identity)
- **Image (Range):** $\mathrm{im}(\phi) = \{\phi(g) : g \in G\}$ (elements that ϕ "hits")

First Isomorphism Theorem: $G/\ker(\phi) \cong \mathrm{im}(\phi)$.
In words: quotienting by what ϕ "kills" gives you what ϕ "sees."

- For SE(()2): if we define $\pi : \mathrm{SE}(()2) \to \mathrm{SO}(()2)$ by $\pi(\mathbf{t}, R) = R$, what is $\ker(\pi)$?
- What does the First Isomorphism Theorem tell us about $\mathrm{SE}(()2)/\ker(\pi)$?

Solution: