4.a)

Question:
Write the general form of an element $\Omega \in \mathfrak{so}(2)$
Solution:
2×2 skew symmetric matrices can be paramterized using $\theta \in \mathbb{R}$ as the following

$$\Omega = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \in \mathfrak{so}(2)$$

4.b)

Question:
Define the wedge operator $(\cdot)^\wedge : \mathbb{R} \to \mathfrak{so}(2)$ and its inverse, the vee operator $(\cdot)^\wedge : \mathfrak{so}(2) \to \mathbb{R}$
Solution:
For $SO2$ the operations are defined as the following:
The wedge operator is defined as $\theta^\wedge = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \Omega$
The vee operator is defined as $\Omega^\vee = \Omega_{21} = \theta$

4.c)

Question:
Compute the matrix exponential $\exp(\Omega)$ for $\Omega = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ using the power series definition. Show that this equals $R(\theta)$
Solution:
Define the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} \exp(\Omega) &= I + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \frac{1}{24}\Omega^4 + \dots \\ \Omega^2 &= \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = -\theta^2 I \\ &\implies \Omega^3 = -\theta^2 \Omega \\ \implies \exp(\Omega) &= (1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots)I + (\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots)J \\ \exp(\Omega) &= \cos(\theta)I + \sin(\theta)J = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = R(\theta) \\ &\therefore \exp(\Omega) = R(\theta) \end{aligned}$$

4.d)

Question:
What is the dimension of $\mathfrak{so}(2)$? How does this relate to the dimension of the of SO(2)?
Solution:
The dimension of $\mathfrak{so}(2)$ is 1 because the elements, 2×2 skew-symmetric matrices, are parametrized by a single number θ. The dimension of SO(2) is 1 because the 2 dimensional rotation matrices are also parametrized by a single number θ. The dimensions of the Lie group and Lie algebra are the same for SO(2) and $\mathfrak{so}(2)$. In fact, this holds for any Lie group.