AAE 590: Lie Group Methods for Control and Estimation

SE(2): Planar Rigid Body Motion

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Lecture Overview

SE(2) — Planar Rigid Body Motion

- Combining rotation and translation
- Semi-direct product structure
- Lie algebra $\mathfrak{se}(2)$ and twists
- Vehicle kinematics application

Our first "real" non-abelian Lie group!

From Rotation to Rigid Motion

$\mathrm{SO}(2)$ **describes:** Rotation only (orientation)

But rigid bodies also translate!

Rigid Body Motion in 2D

A planar rigid body has:

- Position: $(x, y) \in \mathbb{R}^2$
- Orientation: $\theta \in S^1$ (or $R \in \mathrm{SO}(2)$)

Total: 3 degrees of freedom

Question: How do we combine rotation and translation into a single group?

The Special Euclidean Group $\mathrm{SE}(2)$ [1]

Definition

$\mathrm{SE}(2)$ is the group of rigid motions in the plane:

$$\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$$

Matrix representation:

$$X = \begin{bmatrix} R & \boldsymbol{t} \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

where $R \in \mathrm{SO}(2)$ and $\boldsymbol{t} = (t_x, t_y)^T \in \mathbb{R}^2$.

Dimension: 3 (one for rotation, two for translation)

SE(2) Group Operations

Composition:

$$X_1 X_2 = \begin{bmatrix} R_1 & \boldsymbol{t}_1 \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_2 & \boldsymbol{t}_2 \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_1 R_2 & R_1 \boldsymbol{t}_2 + \boldsymbol{t}_1 \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Inverse:

$$X^{-1} = \begin{bmatrix} R^T & -R^T \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Neutral element:

$$I = \begin{bmatrix} I_{2\times 2} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Why $\mathrm{SE}(2)$ is NOT Abelian

Critical Difference from $\mathrm{SO}(2)$

$\mathrm{SE}(2)$ is **non-abelian**: $X_1X_2 \neq X_2X_1$ in general!

Example: Rotate then translate vs. translate then rotate

Let X_R = rotate 90°, X_t = translate by $(1, 0)$:

- X_RX_t: translate $(1, 0)$ then rotate $\Rightarrow$ end up at $(0, 1)$
- X_tX_R: rotate then translate $(1, 0)$ $\Rightarrow$ end up at $(1, 0)$

This is why Lie theory becomes non-trivial!

Semi-Direct Product: Formal Definition [2, 3]

Definition

Let H and N be groups, and let $\phi : H \to \mathrm{Aut}(N)$ be a homomorphism (i.e., H acts on N by automorphisms). The **semi-direct product** $N \rtimes_\phi H$ is the set $N \times H$ with multiplication:

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2),\ h_1 \cdot h_2)$$

Key ingredients:

- N is a **normal** subgroup of $N \rtimes H$ (invariant under conjugation—defined later)
- H "acts on" N: the action $\phi(h)$ twists N before combining
- If ϕ is trivial ($\phi(h) = \mathrm{id}$ for all h), we get the **direct product** $N \times H$

Notation

$N \rtimes H$ means H acts on N from the right. Some authors write $H \ltimes N$ for the same thing.

Semi-Direct Product: $\mathrm{SE}(2)$ Example

For $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$:

- $H = \mathrm{SO}(2)$ (rotations)
- $N = \mathbb{R}^2$ (translations, written additively)
- $\phi : \mathrm{SO}(2) \to \mathrm{Aut}(\mathbb{R}^2)$ is $\phi(R)(\boldsymbol{t}) = R\boldsymbol{t}$ (rotate the vector)

The multiplication formula:

$$(R_1, \boldsymbol{t}_1) \cdot (R_2, \boldsymbol{t}_2) = (R_1 R_2, \; \underbrace{R_1 \boldsymbol{t}_2}_{\phi(R_1)(\boldsymbol{t}_2)} + \boldsymbol{t}_1)$$

Engineer's intuition: When composing rigid motions, the second translation $\boldsymbol{t}_2$ gets rotated by the first rotation R_1 before being added. This is the "twist" that makes it a *semi*-direct product.

Key Point

The rotation "acts on" translations. This asymmetry is why $\mathrm{SE}(2)$ is non-abelian.

Definition

Given a subgroup $H \subset G$ and element $g \in G$, the **left coset** is:

$$gH = \{g \cdot h : h \in H\}$$

where $g \cdot h$ denotes the group operation (often written as juxtaposition gh).

Intuition: A coset is H "shifted" by g. Cosets partition G into disjoint pieces.

Example: In $\mathrm{SE}(2)$, let $\mathcal{T}$ be the translations. The coset $(R, \boldsymbol{p})\mathcal{T}$ is:

$$(R, \boldsymbol{p})\mathcal{T} = \{(R, \boldsymbol{p})(I, \boldsymbol{t}) : \boldsymbol{t} \in \mathbb{R}^2\} = \{(R, \boldsymbol{p} + R\boldsymbol{t}) : \boldsymbol{t} \in \mathbb{R}^2\}$$

This is "all rigid motions with rotation R" — the translation part varies freely.

The Quotient Group G/H

Definition

The **quotient group** G/H is the set of all cosets with multiplication:

$$(g_1H)(g_2H) = (g_1g_2)H$$

Problem: This multiplication is only well-defined if H is **normal**!

Why? We need $(g_1h_1)(g_2h_2) \in (g_1g_2)H$ for any $h_1, h_2 \in H$.

$$\begin{aligned}(g_1h_1)(g_2h_2) &= g_1(h_1g_2)h_2 = g_1(g_2g_2^{-1}h_1g_2)h_2 \\ &= (g_1g_2)\underbrace{(g_2^{-1}h_1g_2)}_{\in H?}h_2\end{aligned}$$

This works iff $g_2^{-1}h_1g_2 \in H$ for all $g_2 \in G$, $h_1 \in H$ — the definition of normal!

Quotient Group Example: $\mathrm{SE}(2)/\mathcal{T}$

Since translations $\mathcal{T}$ are normal in $\mathrm{SE}(2)$ (we'll verify this), we can form:

$$\mathrm{SE}(2)/\mathcal{T} \cong \mathrm{SO}(2)$$

Elements of $\mathrm{SE}(2)/\mathcal{T}$: Cosets $(R, \boldsymbol{p})\mathcal{T}$
Each coset contains all rigid motions with the same rotation R, regardless of translation.

Multiplication:

$$[(R_1, \boldsymbol{p}_1)\mathcal{T}] \cdot [(R_2, \boldsymbol{p}_2)\mathcal{T}] = (R_1 R_2, \cdot)\mathcal{T}$$

The translation doesn't matter — it gets absorbed into $\mathcal{T}$!

Physical Interpretation

$\mathrm{SE}(2)/\mathcal{T}$ is "$\mathrm{SE}(2)$ with translations forgotten" — only orientation remains.

Normal Subgroups of $\mathrm{SE}(2)$

Definition

A subgroup $H \subset G$ is **normal** if $gHg^{-1} = H$ for all $g \in G$.

Translations are normal in $\mathrm{SE}(2)$:
Let $\mathcal{T} = \{(I, \boldsymbol{t}) : \boldsymbol{t} \in \mathbb{R}^2\} \cong \mathbb{R}^2$ be the translation subgroup.

Conjugation by $(R, \boldsymbol{p})$:

$$(R, \boldsymbol{p})(I, \boldsymbol{t})(R, \boldsymbol{p})^{-1} = (R, \boldsymbol{p})(I, \boldsymbol{t})(R^T, -R^T\boldsymbol{p}) = (I, R\boldsymbol{t})$$

The result $(I, R\boldsymbol{t})$ is still a pure translation! So $\mathcal{T} \triangleleft \mathrm{SE}(2)$.

Physical Interpretation

Rotating a translation changes its *direction* but not its *nature*—it remains a translation.

Rotations are NOT Normal

Is $\mathrm{SO}(2) \subset \mathrm{SE}(2)$ a normal subgroup?

Consider conjugating a rotation $(R, \mathbf{0})$ by a translation $(I, \boldsymbol{t})$:

$$(I, \boldsymbol{t})(R, \mathbf{0})(I, \boldsymbol{t})^{-1} = (I, \boldsymbol{t})(R, \mathbf{0})(I, -\boldsymbol{t}) = (R, \boldsymbol{t} - R\boldsymbol{t})$$

If $R \neq I$ and $\boldsymbol{t} \neq \mathbf{0}$, then $\boldsymbol{t} - R\boldsymbol{t} \neq \mathbf{0}$!

Rotations about origin become screw motions!

Conjugating a rotation by a translation shifts the center of rotation, producing a rotation **plus** a translation.

Consequence: $\mathrm{SO}(2)$ (rotations about origin) is **not** a normal subgroup of $\mathrm{SE}(2)$.

The Normalizer

Definition

The **normalizer** of H in G is:

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

It's the largest subgroup of G in which H is normal.

For $\mathrm{SE}(2)$:

- $N_{\mathrm{SE}(2)}(\mathcal{T}) = \mathrm{SE}(2)$ (translations are normal in the whole group)
- $N_{\mathrm{SE}(2)}(\mathrm{SO}(2)) = \mathrm{SO}(2)$ (only rotations normalize rotations!)

Physical Meaning

To preserve "rotation about the origin" under conjugation, you can only use other rotations about the origin. Any translation shifts the center.

Summary: Subgroup Structure of $\mathrm{SE}(2)$

What we've learned:

Subgroup	Normal?	Normalizer
$\mathcal{T} \cong \mathbb{R}^2$ (translations)	Yes	$\mathrm{SE}(2)$
$\mathrm{SO}(2)$ (rotations about origin)	No	$\mathrm{SO}(2)$

Consequences:

- $\mathrm{SE}(2)/\mathcal{T} \cong \mathrm{SO}(2)$ is a valid quotient group
- $\mathrm{SE}(2)/\mathrm{SO}(2)$ is **not** well-defined (can't quotient by non-normal subgroup)
- The semi-direct product $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ encodes this asymmetry

Key Insight

The non-normality of $\mathrm{SO}(2)$ in $\mathrm{SE}(2)$ is precisely why "rotate then translate $\neq$ translate then rotate."

Worked Example: Conjugation in $\mathrm{SE}(2)$

Compute $(R, \boldsymbol{t}) \cdot (I, \boldsymbol{p}) \cdot (R, \boldsymbol{t})^{-1}$ *for* $R = R(\pi/2)$, $\boldsymbol{t} = (1, 0)^T$, $\boldsymbol{p} = (1, 1)^T$.

Solution: Conjugation in $\mathrm{SE}(2)$

Using $(R, \boldsymbol{t})^{-1} = (R^T, -R^T\boldsymbol{t})$ and tuple multiplication $(\boldsymbol{t}_1, R_1) \cdot (\boldsymbol{t}_2, R_2) = (R_1\boldsymbol{t}_2 + \boldsymbol{t}_1, R_1R_2)$:

$(R, \boldsymbol{t}) \cdot (I, \boldsymbol{p}) \cdot (R, \boldsymbol{t})^{-1} = (R\boldsymbol{p} + \boldsymbol{t}, R) \cdot (R^T, -R^T\boldsymbol{t}) = (R(-R^T\boldsymbol{t}) + R\boldsymbol{p} + \boldsymbol{t}, I) = (R\boldsymbol{p}, I)$

With $R = R(\pi/2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: $\quad R\boldsymbol{p} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Result: $((-1, 1)^T, I)$ — a pure translation! ($\mathcal{T}(2)$ is normal)

Action on Points

$\mathrm{SE}(2)$ **acts on points in** $\mathbb{R}^2$**:**

Using **homogeneous coordinates** $\tilde{\boldsymbol{p}} = (p_x, p_y, 1)^T$:
(Homogeneous coordinates append a 1 to represent a point, allowing translation to be expressed as matrix multiplication.)

$$X \cdot \tilde{\boldsymbol{p}} = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} R\boldsymbol{p} + \boldsymbol{t} \\ 1 \end{bmatrix}$$

Interpretation:

- X transforms point $\boldsymbol{p}$ by rotating then translating
- If X is "body in world" frame, then $X \cdot \boldsymbol{p}_{body} = \boldsymbol{p}_{world}$

The Lie Algebra $\mathfrak{se}(2)$ [1, 4]

Definition

The Lie algebra $\mathfrak{se}(2) = T_I\mathrm{SE}(2)$:

$$\mathfrak{se}(2) = \left\{ \begin{bmatrix} \omega^\wedge & \boldsymbol{v} \\ \mathbf{0}^T & 0 \end{bmatrix} : \omega \in \mathbb{R}, \boldsymbol{v} \in \mathbb{R}^2 \right\}$$

General element:

$$\xi^\wedge = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

where $\xi = (v_x, v_y, \omega)^T \in \mathbb{R}^3$ is the **twist**.

Dimension: 3 (matches the group)

Wedge and Vee for SE(2)

Wedge: $\mathbb{R}^3 \to \mathfrak{se}(2)$

$$\xi^\wedge = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}$$

Vee: $\mathfrak{se}(2) \to \mathbb{R}^3$

$$\begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}^\vee = \begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix}$$

Convention: $\xi = (v_x, v_y, \omega)^T$ (translation first)

Why Translation First?

With this ordering, Ad_X, J_l, J_r, and ad_ξ are all **block upper triangular**—matching the group matrix $X = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$. This structural consistency:

- Makes matrix algebra more intuitive (same block pattern everywhere)

Physical Interpretation of Twist

The twist $\xi = (v_x, v_y, \omega)^T$ represents:

- v_x: linear velocity in x direction (m/s)
- v_y: linear velocity in y direction (m/s)
- ω: angular velocity (rad/s)

Body frame interpretation:

- Velocities expressed in the body's local frame
- v_x is forward velocity
- v_y is lateral velocity
- ω is yaw rate

Perfect for vehicle kinematics!

Exponential Map for SE(2)

Closed form:

$$\mathrm{Exp}(\xi^{\wedge}) = \begin{bmatrix} R(\omega) & V(\omega)\boldsymbol{v} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where $R(\omega) = \mathrm{Exp}_{SO(2)}(\omega)$ and:

$$V(\omega) = \begin{cases} I & \text{if } \omega = 0 \\ \frac{\sin\omega}{\omega} I + \frac{1-\cos\omega}{\omega} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{otherwise} \end{cases}$$

Key Point

$V(\omega)$ "couples" the rotation and translation. It's not just R and $\boldsymbol{t}$ separately!

Special Cases of Exp

Pure rotation ($\boldsymbol{v} = 0$):

$$\mathrm{Exp}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}^{\wedge} = \begin{bmatrix} R(\omega) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Pure translation ($\omega = 0$):

$$\mathrm{Exp}\begin{pmatrix} v_x \\ v_y \\ 0 \end{pmatrix}^{\wedge} = \begin{bmatrix} I & \boldsymbol{v} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

General case ($\omega \neq 0$): The body moves along an **arc** (circular path), not a straight line!

Logarithmic Map for $\mathrm{SE}(2)$

Given $T = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$:

Step 1: Extract rotation angle

$$\omega = \operatorname{atan2}(R_{21}, R_{11})$$

Step 2: Compute velocity

$$\boldsymbol{v} = V(\omega)^{-1}\boldsymbol{t}$$

where:

$$V(\omega)^{-1} = \begin{cases} I & \text{if } \omega = 0 \\ \frac{\omega}{2}\cot\frac{\omega}{2} I + \frac{\omega}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{otherwise} \end{cases}$$

Result: $\mathrm{Log}(X) = (v_x, v_y, \omega)^{\wedge}$

Worked Example: $\mathrm{SE}(2)$ Exp and Log

Compute $\mathrm{Exp}((3,1,0)^\wedge)$ *(pure translation) and* $\mathrm{Exp}((0,0,\pi/2)^\wedge)$ *(pure rotation).*

Solution: $\mathrm{SE}(2)$ Exp and Log

Pure translation ($\omega = 0$): $V(0) = I$, so $\boldsymbol{t} = \boldsymbol{v}$

$$\mathrm{Exp}((3, 1, 0)^\wedge) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(move to } (3, 1)\text{, no rotation)}$$

Pure rotation ($\boldsymbol{v} = 0$): translation part is $V(\omega) \cdot \mathbf{0} = \mathbf{0}$

$$\mathrm{Exp}((0, 0, \pi/2)^\wedge) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(rotate 90 in place)}$$

The Adjoint for $\mathrm{SE}(2)$

Definition

$\mathrm{Ad}_X : \mathfrak{se}(2) \to \mathfrak{se}(2)$ acts on twists:

$$\mathrm{Ad}_X(\xi^\wedge) = X\xi^\wedge X^{-1}$$

Matrix form: $\mathrm{Ad}_X \in \mathbb{R}^{3\times 3}$ such that $\mathrm{Ad}_X \xi$ gives the transformed twist:

$$\mathrm{Ad}_X = \begin{bmatrix} \cos\theta & -\sin\theta & t_y \\ \sin\theta & \cos\theta & -t_x \\ 0 & 0 & 1 \end{bmatrix}$$

Block Upper Triangular!

With translation-first ordering, Ad_X has the same block structure as X.

The Lie Bracket for $\mathrm{SE}(2)$

The Lie bracket $[\xi_1^\wedge, \xi_2^\wedge] = \xi_1^\wedge \xi_2^\wedge - \xi_2^\wedge \xi_1^\wedge$

For $\xi_1 = (v_{1x}, v_{1y}, \omega_1)^T$ and $\xi_2 = (v_{2x}, v_{2y}, \omega_2)^T$:

$$[\xi_1, \xi_2] = \begin{bmatrix} \omega_2 v_{1y} - \omega_1 v_{2y} \\ \omega_1 v_{2x} - \omega_2 v_{1x} \\ 0 \end{bmatrix}$$

Key observations:

- Angular components don't interact (both contribute 0)
- Rotation-translation coupling produces new translation
- This is the infinitesimal version of "rotate then translate $\neq$ translate then rotate"

Worked Example: $\mathrm{SE}(2)$ Adjoint and Bracket

For $X = ((2,1)^T, \pi/2)$*: compute* Ad_X *and transform* $\xi = (1,0,0)^T$*.*

Solution: $\mathrm{SE}(2)$ Adjoint and Bracket

$$\mathrm{Ad}_X = \begin{bmatrix} R & \boldsymbol{t}^{\odot} \\ \mathbf{0}^T & 1 \end{bmatrix} \text{ where } \boldsymbol{t}^{\odot} = (t_y, -t_x)^T = (1, -2)^T$$

$$\mathrm{Ad}_X = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Ad}_X \xi = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Interpretation: Body “forward” $(1, 0, 0)^T$ becomes world “left” $(0, 1, 0)^T$ (robot is facing up!)

Kinematics on $\mathrm{SE}(2)$

Body-frame velocity:

$$\dot{X} = X\xi^{\wedge}$$

where $\xi = (v_x, v_y, \omega)^T$ is the body-frame twist.

Expanded form:

$$\begin{bmatrix} \dot{R} & \dot{\boldsymbol{t}} \\ \mathbf{0}^T & 0 \end{bmatrix} = \begin{bmatrix} R & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \omega^{\wedge} & \boldsymbol{v} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

This gives:

$$\dot{R} = R\omega^{\wedge}$$
$$\dot{\boldsymbol{t}} = R\boldsymbol{v}$$

Vehicle Kinematics Example

Unicycle model: Forward velocity v, yaw rate ω

Body-frame twist:

$$\xi = \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}$$

(No lateral velocity — can't move sideways)

State evolution:

$$\dot{X} = X \begin{bmatrix} 0 & -\omega & v \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is the kinematic model for a fixed-wing aircraft in 2D!

Fixed-Wing Aircraft on $\mathrm{SE}(2)$: Setup

State: $X \in \mathrm{SE}(2)$ (pose of aircraft in world frame)

Control inputs:

- v: airspeed (throttle)
- ω: yaw rate (bank angle / coordinated turn)

Kinematics:

$$\dot{X} = X \begin{bmatrix} v \\ 0 \\ \omega \end{bmatrix}^{\wedge}$$

Goal: Track a reference trajectory $X_{ref}(t) \in \mathrm{SE}(2)$

Next Week

We'll design a PID controller in the Lie algebra to track $X_{ref}(t)$!

Error on $\mathrm{SE}(2)$

How do we measure “error” between two poses?

Wrong (Euclidean):

$$e = X_{ref} - X \quad \text{Not meaningful for Lie group elements!}$$

Correct (Lie group):

$$\eta = X_{ref}^{-1} X \in \mathrm{SE}(2) \quad \text{(left error)}$$

or

$$\eta = X X_{ref}^{-1} \in \mathrm{SE}(2) \quad \text{(right error)}$$

At zero error: $\eta = I$ (identity)
Error in algebra: $\xi_e = \mathrm{Log}(\eta)^\vee \in \mathbb{R}^3$

Preview: PID in Lie Algebra

Error: $\eta = X_{ref}^{-1} X$
Error twist: $\xi_e = \mathrm{Log}(\eta)^\vee = (x_e, y_e, \theta_e)^T$

PID control law:

$$u = -K_p \xi_e - K_i \int \xi_e \, dt - K_d \dot{\xi}_e$$

where $K_p, K_i, K_d \in \mathbb{R}^{3\times3}$ are gain matrices.

Why This Works

- Error lives in $\mathbb{R}^3$ (vector space) — PID makes sense
- At $\eta = I$: $\xi_e = 0$ — equilibrium is at identity
- Gains can weight position vs. orientation errors differently

Worked Example: Error Computation

Robot at $(3, 2), 0$*; target at* $(0, 0), 0$*. Compute* $\eta = X_{ref}^{-1} X$ *and* $\xi_e = \mathrm{Log}(\eta)^{\vee}$*.*

Solution: Error Computation

$X_{ref} = I$ (at origin, heading 0), so $X_{ref}^{-1} = I$

$$\eta = X_{ref}^{-1} X = X = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(pure translation, } \omega = 0\text{)}$$

Since $\omega = 0$: $V(0)^{-1} = I$, so $\boldsymbol{v} = \boldsymbol{t} = (3, 2)^T$

$\xi_e = \mathrm{Log}(\eta)^\vee = (3, 2, 0)^T$ (3 units right, 2 units up, no heading error)

SE(2) Summary

Concept	$\mathrm{SE}(2)$
Group	3×3 transformation matrices
Structure	$\mathrm{SO}(2) \ltimes \mathbb{R}^2$ (semi-direct product)
Lie algebra $\mathfrak{se}(2)$	Twists $\xi = (v_x, v_y, \omega)^T$
Dimension	3
Abelian?	**No!**
$\mathrm{Exp}(\xi^\wedge)$	Rotation + coupled translation
Ad_X	Block upper triangular 3×3 matrix
$[\xi_1, \xi_2]$	Rotation-translation coupling

$\mathrm{SE}(2)$ is our first "real" Lie group:

- Non-abelian structure
- Non-trivial adjoint and bracket
- Direct application to robotics and vehicles

Looking Ahead

Week 3: Differentials on Lie Groups

- Pushforward/pullback, left/right invariant vector fields
- Jacobians of the exponential map (J_l, J_r)
- Group affine systems and autonomous error dynamics
- PID controller in Lie algebra
- Fixed-wing aircraft trajectory tracking
- **PS03: Implement aircraft trajectory controller**

Week 4: SO(3)

- 3D rotations — non-abelian from the start!
- Rodrigues formula
- Quaternion connection

Appendix: Homomorphism Terminology

Definition

For a group homomorphism $\phi : G \to H$:

- **Kernel:** $\ker(\phi) = \{g \in G : \phi(g) = e_H\}$
  The elements that ϕ "kills" (maps to identity). Always a normal subgroup of G.
- **Range/Image:** $\mathrm{range}(\phi) = \mathrm{im}(\phi) = \{\phi(g) : g \in G\} \subseteq H$
  The elements that ϕ "hits." Always a subgroup of H. (Notation varies: range, im, Im)

Example

For $\pi : \mathrm{SE}(2) \to \mathrm{SO}(2)$ defined by $\pi(\boldsymbol{t}, R) = R$:

- $\ker(\pi) = \{(\boldsymbol{t}, I) : \boldsymbol{t} \in \mathbb{R}^2\} = \mathcal{T}(2)$ (translations)
- $\mathrm{range}(\pi) = \mathrm{SO}(2)$ (all rotations)

Appendix: First Isomorphism Theorem

Theorem (First Isomorphism Theorem)

If $\phi : G \to H$ is a homomorphism, then:

$$G/\ker(\phi) \cong \mathrm{range}(\phi)$$

In words: Quotienting G by what ϕ kills gives you what ϕ sees.

Example

For $\pi : \mathrm{SE}(2) \to \mathrm{SO}(2)$, $\pi(\boldsymbol{t}, R) = R$:

$$\mathrm{SE}(2)/\mathcal{T}(2) \cong \mathrm{SO}(2)$$

Interpretation: If we "forget" position (quotient by translations), only orientation remains.

Why it matters: This theorem explains how group structure "factors" through homomorphisms—essential for understanding semi-direct products like $\mathrm{SE}(2)$.

References I

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