AAE 590: Lie Group Methods for Control and Estimation

Group Affine Systems and Geometric Control

Dr. James Goppert

Purdue University
School of Aeronautics and Astronautics

Lecture Overview

Group Affine Systems and Geometric Control

- Group affine systems and autonomous error dynamics
- Geometric control and PID on $\mathrm{SE}(2)$
- Fixed-wing aircraft trajectory tracking (2D)
- Log-linear dynamic inversion

From theory to application!

Motivation: Linear Systems Have Autonomous Error Dynamics

Linear system: $\dot{x} = Ax + Bu, \quad \dot{\bar{x}} = A\bar{x} + B\bar{u}$
Error: $e = x - \bar{x} \quad \Rightarrow \quad \dot{e} = Ae + Bu_e$ ✓ *Autonomous in* e*!*

Nonlinear counterexample: **Unicycle** with $\dot{x} = u\cos\theta$, $\dot{y} = u\sin\theta$, $\dot{\theta} = \omega$
Error $e = (x - \bar{x}, y - \bar{y}, \theta - \bar{\theta})$:

$$\dot{e}_x = u\cos\theta - \bar{u}\cos\bar{\theta} = u\cos(\bar{\theta} + e_\theta) - \bar{u}\cos\bar{\theta}$$

This depends on $\bar{\theta}$, not just e! ✗ *Not autonomous!*

The Punchline

On $\mathrm{SE}(2)$: $\dot{X} = X\xi^\wedge$ with $\xi = (u, 0, \omega)^T$ is **left-invariant** $\Rightarrow$ **group affine**!
Same physical system, but error $\eta = \bar{X}^{-1}X$ has **autonomous** dynamics!

Group Affine Systems: Definition [1]

Definition

A vector field $f_u : G \to TG$ is **group affine** if for all $X, Y \in G$:

$$\boxed{f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y}$$

Running example — Unicycle on $\mathrm{SE}(2)$:

$$\dot{X} = X\xi^\wedge, \quad \xi = (u, 0, \omega)^T \quad \text{(left-invariant} \Rightarrow \text{group affine!)}$$

Where Does This Come From?

This definition isn't arbitrary—it's **derived** from requiring autonomous error dynamics. The next slides show Barrau & Bonnabel's elegant derivation.

Deriving the Group Affine Property (1/3) [1]

Goal: Find what property f_u must satisfy for error $\eta = \hat{X}^{-1}X$ to have **autonomous** dynamics $\dot{\eta} = g(\eta)$.

Setup: True and estimated states evolve by $\dot{X} = f_u(X)$ and $\dot{\hat{X}} = f_u(\hat{X})$.

Step 1: Differentiate the error using product rule

$$\begin{aligned}\dot{\eta} &= \frac{d}{dt}(\hat{X}^{-1}X) = \frac{d}{dt}(\hat{X}^{-1}) \cdot X + \hat{X}^{-1} \cdot \dot{X} \\ &= -\hat{X}^{-1}\dot{\hat{X}}\hat{X}^{-1}X + \hat{X}^{-1}\dot{X}\end{aligned}$$

Step 2: Substitute the dynamics

$$\dot{\eta} = -\hat{X}^{-1}f_u(\hat{X})\hat{X}^{-1}X + \hat{X}^{-1}f_u(X)$$

Since $\eta = \hat{X}^{-1}X$, we have $X = \hat{X}\eta$, so:

$$\boxed{\dot{\eta} = -\hat{X}^{-1}f_u(\hat{X})\eta + \hat{X}^{-1}f_u(\hat{X}\eta)}$$

Deriving the Group Affine Property (2/3) [1]

Key question: When does $\dot{\eta} = -\hat{X}^{-1} f_u(\hat{X})\eta + \hat{X}^{-1} f_u(\hat{X}\eta)$ depend **only on** η?

Step 3: Assume autonomous form exists

Suppose $\dot{\eta} = g(\eta)$ for some function g. We can find g by setting $\hat{X} = I$:

$$\begin{aligned} g(\eta) &= -I^{-1} f_u(I)\eta + I^{-1} f_u(I \cdot \eta) \\ &= -f_u(I)\eta + f_u(\eta) \end{aligned}$$

The Autonomous Error Dynamics

$$\boxed{g(\eta) = f_u(\eta) - f_u(I)\eta}$$

This is the form error dynamics **must** take if they are autonomous!

Deriving the Group Affine Property (3/3) [1]

Step 4: Derive the constraint on f_u

For $\dot{\eta} = g(\eta) = f_u(\eta) - f_u(I)\eta$ to hold for **all** $\hat{X}$, substitute back:

$$f_u(\eta) - f_u(I)\eta = -\hat{X}^{-1} f_u(\hat{X})\eta + \hat{X}^{-1} f_u(\hat{X}\eta)$$

Multiply both sides by $\hat{X}$ and rearrange:

$$f_u(\hat{X}\eta) = f_u(\hat{X})\eta + \hat{X} f_u(\eta) - \hat{X} f_u(I)\eta$$

Renaming $\hat{X} \to X$ and $\eta \to Y$:

$$\boxed{f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y}$$

The Punchline

The group affine property is **exactly** what's needed for autonomous error dynamics—not an arbitrary definition, but a **derived necessity**!

Worked Example: Unicycle is Group Affine

Show the unicycle $\dot{X} = X\xi^{\wedge}$ *with* $\xi = (u, 0, \omega)^T$ *satisfies:* $f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y$.

Solution: Unicycle is Group Affine

For $f_u(X) = X\xi^\wedge$ (left-invariant):

LHS: $f_u(XY) = (XY)\xi^\wedge$

RHS: $f_u(X)Y + Xf_u(Y) - Xf_u(I)Y = X\xi^\wedge Y + XY\xi^\wedge - X\xi^\wedge Y = XY\xi^\wedge$ ✓

Key: $f_u(I) = \xi^\wedge$, so the correction term $-Xf_u(I)Y$ cancels $f_u(X)Y$.

The Unicycle Takeaway

The "nonlinear" unicycle is actually **linear in the Lie algebra** and **group affine**!

Mixed Invariant Vector Fields [1]

Definition

A **mixed invariant** vector field: $f_u(X) = X\xi_L(u) + \xi_R(u)X$ where $\xi_L(u), \xi_R(u) \in \mathfrak{g}$ depend only on inputs u, **not on state** X.

Special Cases and Error Frames

- **Left-invariant**: $f_u(X) = X\xi_L \Rightarrow$ error $\eta = \hat{X}^{-1}X$ in **body frame**
- **Right-invariant**: $f_u(X) = \xi_R X \Rightarrow$ error $\eta = X\hat{X}^{-1}$ in **world frame**

Choose Intelligently!

Pick the error definition that matches your sensor frame:

- IMU (body-frame measurements) $\rightarrow$ left-invariant error
- GPS/landmarks (world-frame) $\rightarrow$ right-invariant error

Mixed Invariant $\Rightarrow$ Group Affine [1]

Theorem: If $f_u(X) = X\xi_L + \xi_R X$, then f_u is group affine.
Proof: Verify $f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y$:

$$\begin{aligned}
\text{LHS: } f_u(XY) &= XY\xi_L + \xi_R XY \\
\text{RHS: } &= (X\xi_L + \xi_R X)Y + X(Y\xi_L + \xi_R Y) - X(\xi_L + \xi_R)Y \\
&= X\xi_L Y + \xi_R XY + XY\xi_L + X\xi_R Y - X\xi_L Y - X\xi_R Y \\
&= XY\xi_L + \xi_R XY \quad \checkmark
\end{aligned}$$

Open Question

Converse (group affine $\Rightarrow$ mixed invariant) is conjectured but not proven.

Autonomous Error Dynamics [1]

Theorem: If f_u is group affine, error $\eta = \hat{X}^{-1}X$ has **autonomous** dynamics.
Setup: $\dot{X} = f_u(X)$, $\dot{\hat{X}} = f_u(\hat{X})$, $X = \hat{X}\eta$. **Error derivative:**

$$\dot{\eta} = -\hat{X}^{-1} f_u(\hat{X})\eta + \hat{X}^{-1} f_u(X)$$

Apply group affine to $f_u(X) = f_u(\hat{X}\eta)$:

$$f_u(\hat{X}\eta) = f_u(\hat{X})\eta + \hat{X} f_u(\eta) - \hat{X} f_u(I)\eta$$

Substitute and simplify:

$$\dot{\eta} = -\hat{X}^{-1} f_u(\hat{X})\eta + \hat{X}^{-1}\left[f_u(\hat{X})\eta + \hat{X} f_u(\eta) - \hat{X} f_u(I)\eta\right] = \boxed{f_u(\eta) - f_u(I)\eta}$$

The Magic: Trajectory Independence [1]

Result: For group affine systems: $\boxed{\dot{\eta} = f_u(\eta) - f_u(I)\eta}$

Unicycle example: $f_u(X) = X\xi^\wedge$, so $f_u(I) = \xi^\wedge$ and:

$$\dot{\eta} = \eta\xi^\wedge - \xi^\wedge\eta = [\eta, \xi^\wedge] \quad \text{(commutator!)}$$

Why This Matters

Error dynamics depend **only on** the error η and input ξ.
Not on the true pose X or estimate $\hat{X}$ — unlike Cartesian coordinates!

Consequences for estimation:

- Linearized error dynamics are **state-independent**
- EKF Jacobians don't depend on state estimate
- Filter covariance is **consistent** (doesn't grow incorrectly)

This is the IEKF Foundation

The Invariant EKF exploits this property for superior convergence!

Geodesics and Group Affine Systems

Definition

A **geodesic** on Lie group G: $\gamma(t) = \gamma(0) \cdot \mathrm{Exp}((t\xi)^\wedge)$ for fixed $\xi \in \mathfrak{g}$. The "straightest" curves on G.

Key: Geodesics $=$ constant Lie algebra velocities.
For left-invariant $\dot{X} = X\xi^\wedge$: if ξ constant, trajectory is geodesic.

Caveat

Autonomous error requires both trajectories follow **same** f_u. For time-varying $\xi(t)$, not geodesics, but group affine still ensures state-independent linearization.

Explicit Error Dynamics for Mixed Invariant [1]

For mixed invariant $f_u(X) = X\xi_L + \xi_R X$, substitute into $\dot{\eta} = f_u(\eta) - f_u(I)\eta$:

$$f_u(\eta) = \eta\xi_L + \xi_R\eta, \quad f_u(I) = \xi_L + \xi_R, \quad f_u(I)\eta = \xi_L\eta + \xi_R\eta$$

Therefore:

$$\dot{\eta} = \eta\xi_L + \xi_R\eta - \xi_L\eta - \xi_R\eta = \boxed{\eta\xi_L - \xi_L\eta = [\eta, \xi_L]}$$

Key Result

For **left-invariant** systems ($\xi_R = 0$): error dynamics are the **commutator** $[\eta, \xi_L]$—linear in the Lie algebra near identity!

Why "Linear in the Lie Algebra"?

Group error dynamics $\dot{\eta} = \eta\xi^\wedge - \bar{\xi}^\wedge\eta$ are nonlinear on G, but...
Parameterize near identity: $\eta = \mathrm{Exp}(\epsilon^\wedge)$, $\epsilon \in \mathfrak{g}$
Lie algebra error dynamics:

$$\dot{\epsilon} = A(\bar{\xi})\epsilon + (\xi - \bar{\xi}) + O(\|\epsilon\|^2)$$

The Key Property

$A(\bar{\xi})$ depends **only on inputs**, not on state! This gives:

- **Linear** error propagation (for small errors)
- **State-independent** Jacobians
- **Consistent** filter covariance

SE(2) Kinematics is Group Affine

Fixed-wing aircraft (unicycle) kinematics:

$$\dot{X} = X \underbrace{\begin{bmatrix} 0 & -\omega & v \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\xi^\wedge}$$

This is **purely left-invariant**: $\dot{X} = X\xi^\wedge$ with $\xi_R = 0$

Why left-invariant?

- $\xi = (v, 0, \omega)^T$ is the **body-frame** velocity
- Aircraft moves forward relative to its own heading
- Control inputs (v, ω) are body-referenced

Group Affine Structure

Since ξ depends only on inputs (v, ω), not on state X, this system is group affine!

Autonomous Error Dynamics on $\mathrm{SE}(2)$

Consider: Actual trajectory $X(t)$ and reference $\bar{X}(t)$, both satisfying:

$$\dot{X} = X\xi^\wedge, \quad \dot{\bar{X}} = \bar{X}\bar{\xi}^\wedge$$

Define error: $\eta = \bar{X}^{-1}X$ (left-invariant error)

Error dynamics:

$$\begin{aligned}
\dot{\eta} &= \frac{d}{dt}(\bar{X}^{-1}X) = -\bar{X}^{-1}\dot{\bar{X}}\bar{X}^{-1}X + \bar{X}^{-1}\dot{X} \\
&= -\bar{X}^{-1}\bar{X}\bar{\xi}^\wedge\bar{X}^{-1}X + \bar{X}^{-1}X\xi^\wedge \\
&= -\bar{\xi}^\wedge\eta + \eta\xi^\wedge
\end{aligned}$$

Result:

$$\boxed{\dot{\eta} = \eta\xi^\wedge - \bar{\xi}^\wedge\eta}$$

Why Autonomous Error Dynamics Matter

$$\dot{\eta} = \eta\xi^{\wedge} - \bar{\xi}^{\wedge}\eta$$

Key observation: Error dynamics depend on:

- The error η itself
- The control input ξ
- The reference input $\bar{\xi}$

NOT on the absolute states X or $\bar{X}$!

Consequences

- **Control design:** Same controller works everywhere on the manifold
- **Stability analysis:** Error equilibrium at $\eta = I$ (identity)
- **Linearization:** Jacobians are state-independent
- **Filtering:** Consistent covariance propagation (Week 9)

Worked Example: Error Dynamics on $\mathrm{SE}(2)$

Robot at $(4, 3), 0$*; reference at* $(2, 1), 0$*. Compute* $\eta = \bar{X}^{-1}X$ *and* $\xi_e = \mathrm{Log}(\eta)^\vee$*.*

Solution: Error Dynamics on $\mathrm{SE}(2)$

$$\bar{X} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{X}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\eta = \bar{X}^{-1} X = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(pure translation error)}$$

$\omega = 0, \quad \boldsymbol{t}_{error} = (2, 2)^T$. Since $\omega = 0$: $V(0)^{-1} = I$

$\xi_e = \mathrm{Log}(\eta)^\vee = (2, 2, 0)^T$ (2 units right, 2 units up in reference frame)

Linearized Error Dynamics

Near the identity, $\eta \approx I + \epsilon^\wedge$ where $\epsilon \in \mathbb{R}^3$ is small.

Linearizing $\dot{\eta} = \eta\xi^\wedge - \bar{\xi}^\wedge\eta$**:**

$$\dot{\epsilon}^\wedge \approx (I + \epsilon^\wedge)\xi^\wedge - \bar{\xi}^\wedge(I + \epsilon^\wedge)$$

To first order:

$$\dot{\epsilon}^\wedge = \xi^\wedge - \bar{\xi}^\wedge + \epsilon^\wedge\xi^\wedge - \bar{\xi}^\wedge\epsilon^\wedge$$

In vector form: $\dot{\epsilon} = A\epsilon + (\xi - \bar{\xi})$
where A depends on $\bar{\xi}$ but **NOT on** X **or** $\bar{X}$!

State-Independent Jacobians

This is why PID control works uniformly across all of $\mathrm{SE}(2)$—the linearization looks the same everywhere.

Group Affine: The Big Picture

This week: Group affine structure on $\mathrm{SE}(2)$

- Definition and key property
- Autonomous error dynamics
- Foundation for control (next section)

Weeks 8–9: Group affine systems for estimation

- Extended pose group $\mathrm{SE}_2(3)$ with mixed invariance
- Invariant EKF exploits autonomous error dynamics
- State-independent Jacobians $\Rightarrow$ consistent filter

Key Takeaway

Group affine structure is the theoretical foundation for both geometric control **and** invariant filtering.

The Control Problem [2]

Goal

Design a controller to track a reference trajectory $\bar{X}(t) \in \mathrm{SE}(2)$

Given:

- Current pose: $X \in \mathrm{SE}(2)$
- Reference pose: $\bar{X}(t) \in \mathrm{SE}(2)$
- Control inputs: forward velocity v, yaw rate ω

Design: Feedback law $(v, \omega) = f(X, \bar{X})$

Challenge

How do we define "error" when states live on a curved manifold?

Why Euclidean Error Fails

Naive approach: $e = (x - \bar{x}, y - \bar{y}, \theta - \bar{\theta})$

Problems:

1. **Angle wrapping:** $\theta = 179^\circ$ and $\bar{\theta} = -179^\circ$ gives $e_\theta = 358^\circ$!
2. **Frame inconsistency:** Position error in world frame, but control in body frame
3. **Ignores geometry:** Doesn't respect the group structure

Key Insight

The error should be computed **using the group operation**, not subtraction.

Group Error: Left vs. Right

Two natural choices:

Left-invariant error:

$$\eta_L = \bar{X}^{-1} X$$

"Error from reference's perspective" (body frame of reference)

Right-invariant error:

$$\eta_R = X \bar{X}^{-1}$$

"Error from current pose's perspective" (body frame of vehicle)

Both are valid!

Choice depends on where you want to express the error.

Understanding Left-Invariant Error

Left-invariant error: $\eta_L = \bar{X}^{-1} X$

Interpretation:

- η_L describes how to get from $\bar{X}$ to X
- Expressed in the **reference frame**
- At zero error: $\eta_L = I$ (identity)

If $\eta_L = \begin{bmatrix} R_e & \boldsymbol{t}_e \\ 0 & 1 \end{bmatrix}$**:**

- R_e: orientation error (how much to rotate)
- $\boldsymbol{t}_e$: position error in reference frame

Understanding Right-Invariant Error

Right-invariant error: $\eta_R = X\bar{X}^{-1}$

Interpretation:

- η_R describes how to get from $\bar{X}$ to X
- Expressed in the **current body frame**
- At zero error: $\eta_R = I$ (identity)

If $\eta_R = \begin{bmatrix} R_e & \boldsymbol{t}_e \\ 0 & 1 \end{bmatrix}$**:**

- R_e: same orientation error
- $\boldsymbol{t}_e$: position error in current body frame

Which Error to Use?

For vehicle control: Right-invariant is often preferred

Why?

- Controls (v, ω) are in body frame
- Body-frame error directly maps to control action
- "I need to go forward and turn left" makes sense in body frame

Alternative view: Use left-invariant with reference-frame controls, then transform.

Important: This is a Choice!

Both conventions are valid and used in the literature:

- **Left-invariant**: $\eta_L = \bar{X}^{-1} X$ (error in reference frame)
- **Right-invariant**: $\eta_R = X \bar{X}^{-1}$ (error in body frame)

We use **left-invariant** in this course. Be consistent and explicit!

Error in the Lie Algebra

Group error: $\eta = \bar{X}^{-1} X \in \mathrm{SE}(2)$
Algebra error: $\xi_e = \mathrm{Log}(\eta)^\vee \in \mathbb{R}^3$

$$\xi_e = \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix}$$

Properties:

- $\xi_e = 0$ when $X = \bar{X}$ (perfect tracking)
- ξ_e is a vector — we can do linear algebra!
- Automatically handles angle wrapping
- Captures position error in reference frame

Computing the Error Twist

Given:

$$X = \begin{bmatrix} R & \boldsymbol{t} \\ 0 & 1 \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} \bar{R} & \bar{\boldsymbol{t}} \\ 0 & 1 \end{bmatrix}$$

Step 1: Compute group error

$$\eta = \bar{X}^{-1} X = \begin{bmatrix} \bar{R}^T R & \bar{R}^T(\boldsymbol{t} - \bar{\boldsymbol{t}}) \\ 0 & 1 \end{bmatrix}$$

Step 2: Apply logarithm

$$\xi_e = \mathrm{Log}(\eta)^\vee = \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix}$$

where $\theta_e = \text{atan2}(\eta_{21}, \eta_{11})$ and $(x_e, y_e)^T = V(\theta_e)^{-1}\eta_{1:2,3}$

Proportional Control

Simplest controller: $u = -K_p \xi_e$

$$\begin{bmatrix} v_x \\ v_y \\ \omega \end{bmatrix} = -K_p \begin{bmatrix} x_e \\ y_e \\ \theta_e \end{bmatrix}$$

For a unicycle (no lateral motion):

$$\omega = -k_\theta \theta_e$$
$$v = -k_x x_e$$

Issue: What about y_e?

Unicycle Constraint

Unicycles can't move sideways! $v_y = 0$ always. We need a smarter approach.

Dealing with Non-Holonomic Constraints

Unicycle: $v_y = 0$ (can't move laterally)

Solution: Use rotation to eliminate lateral error

Intuition:

- If target is to the left, turn left first
- Drive forward to reduce x_e
- Turning handles y_e indirectly

Modified control law:

$$\omega = -k_\theta \theta_e - k_y y_e$$
$$v = -k_x x_e$$

The $k_y y_e$ term turns toward the target!

PID Control in Lie Algebra [2, 3]

Full PID law:

$$u = -K_p\xi_e - K_i \int_0^t \xi_e(\tau)d\tau - K_d\dot{\xi}_e$$

Gain matrices: (with $\xi_e = (v_{ex}, v_{ey}, \omega_e)^T$, $u = (v, \omega)^T$)

$$K_p = \begin{bmatrix} k_{p,x} & 0 & 0 \\ 0 & k_{p,\theta y} & k_{p,\theta} \end{bmatrix}, \quad K_i, K_d \text{ similar}$$

For unicycle: Output is $(v, \omega)^T \in \mathbb{R}^2$, input is $\xi_e \in \mathbb{R}^3$

Key insight: The $K_{p,\theta y}$ term couples yaw control to lateral error!

Computing $\dot{\xi}_e$

For the derivative term, we need $\dot{\xi}_e$

Option 1: Numerical differentiation

$$\dot{\xi}_e \approx \frac{\xi_e(t) - \xi_e(t - \Delta t)}{\Delta t}$$

Option 2: Analytical (more complex)

$$\dot{\xi}_e = J_l^{-1}(\xi_e)\left(\xi - \mathrm{Ad}_{\eta^{-1}}\bar{\xi}\right)$$

where J_l is the left Jacobian of the exponential map.

In practice: Numerical differentiation with filtering works well.

Stability Analysis

Linearization at $\eta = I$:

Near identity, $\mathrm{Log}(\eta) \approx \eta - I$ and dynamics become linear:

$$\dot{\xi}_e \approx A\xi_e + Bu$$

For proportional control:

$$\dot{\xi}_e \approx (A - BK_p)\xi_e$$

Stability: Eigenvalues of $(A - BK_p)$ in left half-plane

Local Stability

With proper gain selection, the system is locally asymptotically stable around $\eta = I$.

Worked Example: Linearized Stability

For P-control on $\mathrm{SE}(2)$ *with* $K_p = diag(1, 1, 2)$*, find eigenvalues of closed-loop system.*

Solution: Linearized Stability

For fully actuated system: $\dot{\xi}_e = -K_p \xi_e$ (ignoring ref feedforward)

Closed-loop: $A_{cl} = -K_p = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -2$

All eigenvalues $< 0 \Rightarrow$ **asymptotically stable**

Heading converges $2\times$ faster than position (by design with K_p).

Why Lie Algebra PID Works

Key properties:

1. **Well-defined error:** $\xi_e = 0 \Leftrightarrow X = \bar{X}$
2. **Smooth near identity:** Log is smooth for small errors
3. **Coordinate-free:** No gimbal lock or singularities
4. **Natural gains:** Physical interpretation of K_p entries

Limitations

- Large errors: Log has limited domain
- Global stability not guaranteed (only local)
- Non-holonomic constraints need special handling

Feedforward + Feedback

Better approach: Feedforward + feedback

Reference trajectory provides:

- $\bar{X}(t)$: desired pose
- $\bar{\xi}(t)$: desired twist (feedforward)

Control law:

$$u = u_{ff} + u_{fb} = \bar{\xi} - K_p \xi_e - K_d \dot{\xi}_e$$

Interpretation:

- $\bar{\xi}$: "what we should be doing"
- $-K_p \xi_e$: "correction for position error"
- $-K_d \dot{\xi}_e$: "damping"

Body Frame vs. Reference Frame

Our error ξ_e is in reference frame
Controls are in body frame

Transform: Use the adjoint!

$$\xi_{body} = \mathrm{Ad}_{\eta}^{-1} \xi_{ref\_frame}$$

Or equivalently: The error twist in body frame is:

$$\xi_e^{body} = \mathrm{Log}(X^{-1}\bar{X})^{\vee} = -\mathrm{Log}(\eta^{-1})^{\vee}$$

Practical Note

For small errors, the difference between frames is small. Many implementations ignore this.

Summary: Geometric Control Recipe

Step 1: Define error

$$\eta = \bar{X}^{-1} X$$

Step 2: Map to algebra

$$\xi_e = \mathrm{Log}(\eta)^{\vee}$$

Step 3: Apply PID

$$u = \bar{\xi} - K_p \xi_e - K_i \int \xi_e - K_d \dot{\xi}_e$$

Step 4: Handle constraints

- Project to feasible controls
- Couple ω with lateral error for non-holonomic vehicles

Log-Linear Dynamic Inversion [4]

Lin, Goppert, Hwang (2024) — IEEE Trans. Automatic Control

Key insight: Group affine structure enables exact linearization via Jacobians!

Error dynamics on group: $\dot{\eta} = \eta\xi^{\wedge} - \bar{\xi}^{\wedge}\eta$

Error dynamics in algebra: Let $z = \mathrm{Log}(\eta)^{\vee}$, then:

$$\dot{z} = J_l^{-1}(z) \cdot \underbrace{(\xi - \mathrm{Ad}_{\eta}^{-1}\bar{\xi})}_{\text{“effective input”}}$$

The magic: Choose control to cancel the Jacobian:

$$\xi = \bar{\xi} + J_l(z) \cdot v \quad \Rightarrow \quad \dot{z} = v$$

LTI Closed-Loop!

Set $v = -Kz$ and get $\dot{z} = -Kz$ — **exactly** linear, not just linearized!
This works because group affine $\Rightarrow$ Jacobians depend only on error z, not on state X.

Worked Example: Log-Linear Dynamic Inversion

For group affine system $\dot{\eta} = \eta\xi^\wedge - \bar{\xi}^\wedge\eta$*, design control so error dynamics become LTI:* $\dot{z} = -Kz$*.*

Solution: Log-Linear Dynamic Inversion

Step 1: Error in Lie algebra: $z = \mathrm{Log}(\eta)^\vee \in \mathbb{R}^3$

Step 2: Differentiate using chain rule with Jacobian:

$$\dot{z} = J_l^{-1}(z) \cdot (\text{body velocity of } \eta) = J_l^{-1}(z) \cdot (\xi - \mathrm{Ad}_\eta^{-1}\bar{\xi})$$

Step 3: Near identity (z small): $\mathrm{Ad}_\eta^{-1} \approx I - \mathrm{ad}_z$, so:

$$\dot{z} \approx J_l^{-1}(z) \cdot (u + \mathrm{ad}_z\bar{\xi}) \quad \text{where } u = \xi - \bar{\xi}$$

Step 4: Log-linear dynamic inversion — choose u to cancel nonlinearity:

$$\boxed{u = J_l(z) \cdot (-Kz) - \mathrm{ad}_z\bar{\xi}}$$

Result: $\dot{z} = -Kz$ (LTI closed-loop!) With $K = 2I$: $z(t) = z_0 e^{-2t}$

Summary

This lecture we learned:

- Group affine systems and autonomous error dynamics
- Left/right invariant vector fields: body vs world frame
- Geometric control: PID in the Lie algebra
- Log-linear dynamic inversion for exact linearization

Key takeaway:

Lie Algebra = Natural Control Space

Working in the Lie algebra gives us a vector space where PID makes sense, while respecting the geometry of the configuration space.

Next Week: SO(3)

Week 4: The Special Orthogonal Group $\mathrm{SO}(3)$

- 3D rotations — non-abelian!
- Rodrigues formula for Exp
- Axis-angle and quaternion representations
- Connection to aerospace attitude

Preview: $\mathrm{SO}(3)$ is like $\mathrm{SO}(2)$ but with non-trivial Ad and Lie bracket. Everything we learned for $\mathrm{SE}(2)$ applies, but in 3D!

References I

[1] Axel Barrau and Silvere Bonnabel. “The invariant extended Kalman filter as a stable observer”. In: *IEEE Transactions on Automatic Control* 62.4 (2017), pp. 1797–1812.

[2] Francesco Bullo and Andrew D Lewis. *Geometric Control of Mechanical Systems*. Springer, 2005.

[3] Joan Solà, Jeremie Deray, and Dinesh Atchuthan. “A micro Lie theory for state estimation in robotics”. In: *arXiv preprint arXiv:1812.01537* (2018).

[4] Li-Yu Lin, James Goppert, and Inseok Hwang. “Log-Linear Dynamic Inversion Control With Provable Safety Guarantees in Lie Groups”. In: *IEEE Transactions on Automatic Control* 69.8 (2024), pp. 5591–5597. DOI: 10.1109/TAC.2024.3369549.