3.a)

Question:
Prove that $R(\theta)^T R(\theta) = I \forall \theta$
Solution:

$$R(\theta)^T R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$
$$\therefore R(\theta)^T R(\theta) = I \forall \theta$$

3.b)

Question:
Prove that $\det(R(\theta)) = 1 \forall \theta$
Solution:

$$\det(R(\theta)) = \det(\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}) = \cos^2(\theta) + \sin^2(\theta) = 1$$
$$\therefore \det(R(\theta)) = 1 \forall \theta$$

3.c)

Question:
Derive the inverse $R(\theta)^{-1}$ and show it equals $R(-\theta)$
Solution:
The inverse of $R(\theta)$ is the rotation matrix that when multiplied with $R(\theta)$ results in the identtity matrix. This inverse is the transpose of $R(\theta)$ as seen by part 3.a.

$$R(\theta)^{-1} = R(\theta)^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$R(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = R(\theta)^{-1}$$
$$\therefore R(\theta)^{-1} = R(-\theta)$$

3.d)

Question:
Prove that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$
Solution:

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix} = R(\theta_1 + \theta_2)$$
$$\therefore R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$$

3.e)

Question:
Is SO(2) an abelian group?
Solution:
An abelian group is defined by a group where the group operation commutes. If SO(2) commutes then $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$

$$\text{from 3.d we know } R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2) = R(\theta_2 + \theta_1) = R(\theta_2)R(\theta_1)$$

Therefore, SO(2) commutes so it is an abelian group.