AAE 590LGM: Lie Group Methods

Problem Set 1

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Contents

Code Listing

P1) Linear Algebra Review

This problem reviews essential linear algebra concepts used throughout the course.

1.a)

Question:
For matrices $A, B \in \mathbb{R}^{n\times n}$, prove that $(AB)^T = B^T A^T$
Solution:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$
$$\text{Transpose swaps the indices}$$
$$\implies (AB)^T_{ij} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} B_{ki}A_{jk} = \sum_{k=1}^{n} B^T_{ik}A^T_{kj} = B^T A^T$$
$$\therefore (AB)^T = B^T A^T$$

1.b)

Question:
A matrix Q is orthogonal if $Q^T Q = I$. Prove that if Q is orthogonal, then $|\det(Q)| = 1$.
Solution:

$$\text{for any matrix, } \det\{Q^T\} = \det\{Q\}$$
$$1 = \det(I) = \det(Q^T Q) = \det(Q^T)\det(Q) = \det(Q)^2$$
$$\therefore \det(Q) = \pm 1$$
$$\Leftrightarrow |\det(Q)| = 1$$

1.c)

Question:
Prove that the product of two orthogonal matrices is orthogonal
Solution:

$$A^T A = I = B^T B$$
$$B^T A^T A = B^T$$
$$B^T A^T AB = B^T B = I$$
$$(AB)^T AB = I$$
$$\therefore AB \text{ is orthogonal}$$

1.d)

Question:
Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$
Solution:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$$

$$(AB)_{ll} = \sum_{k=1}^{n} A_{lk} B_{kl}$$
$$\mathrm{tr}(AB) = \sum l = 1^n \sum_{k=1}^{n} A_{lk} B_{kl} = \sum_{l=1}^{n} \sum_{k=1}^{n} A_{kl} B_{lk}$$
$$\mathrm{tr}(BA) = \sum l = 1^n \sum_{k=1}^{n} B_{lk} A_{kl} = \mathrm{tr}(AB)$$
$$\therefore \mathrm{tr}(AB) = \mathrm{tr}(BA)$$

1.e)

Question:
Show that for any skew-symmetric matrix, the diagonal elements must be zero
Solution:

$$S^T = -S$$
$$S_{ij}^T = -S_{ij}$$
$$\text{look at only diagonal elements} S_{ii}^T = -S_{ii} \Leftrightarrow S_{ii} = -S_{ii}$$
$$S_{ii} = -S_{ii} \text{ is only possible if } S_{ii} = 0$$
$$\therefore \text{ the diagonal elements are } 0$$

P2) Group Axioms

For each of the following, determine whether it forms a group under the given operation. If yes, identify the element and describe inverses. If no, state which axiom(s) fail.

2.a)

Question:
The set of 2×2 invertible matrices with real entries under matrix multiplication
Solution:

Closure All invertible matrices have nonzero determinants. $\det(AB) = \det A \det(B)$. Therefore, the only way for the determinant of the product of two matrices to be zero is if one of the matrices has a determinant of zero. However, both matrices are invertible so they have nonzero determinants so their product is a 2×2 invertible matrix with real entries. Thus, closure holds.

Associativity Matrix multiplication is always associative so this property is inherited.

Identity The identity matrix multiplied with any matrix is the matrix itself so the identity matrix is the identity for this set under matrix multiplication.

Inverse The set by definition contains only invertible matrices so every element can be inverted to find its multiplicative inverse.

Therefore, all group axioms are satisfied for this set-operation pair making it a group.

2.b)

Question:
The set of integers $\mathbb{Z}$ under addition
Solution:

Closure Adding two integers yields another integer so they are closed under addition.

Associativity Addition on integers is always associative so this property is satisfied.

Identity The identity element under addition is 0.

Inverse The additive inverse element can be found by negating the element which is still an integer so inverses always exist.

Therefore, all group axioms are satisfied for this set-operation pair making it a group.

2.c)

Question:
The set of $1, -1, i, -i$ under complex multiplication
Solution:

Closure

$$1^2 = 1,\ (-1)^2 = 1,\ i^2 = -1,\ (-i)^2 = -1$$
$$1 \times -1 = -1,\ 1 \times i = i,\ 1 \times -i = -i$$
$$-1 \times i = -i,\ -1 \times -i = i$$
$$i \times -i = 1$$

All of the products between elements of the set are still in the set, therefore, the set is closed under complex multiplication.

Associativity Complex multiplication is always associative so the property is inherited.

Identity Looking at the products above, it is clear that the identity element is 1.

Inverse Looking at the products above, the inverse of 1 is 1, the inverse of -1 is -1, the inverse of i is $-i$, and the inverse of $-i$ is i. Therefore, every element has an inverse.

This set and operation pair pass all of the axioms so it is a group.

2.d)

Question:
The set of 2×2 with determinant 1 under matrix addition
Solution:

Closure The identity matrix has a determinant of 1 so it is in the set. If the set is a group under matrix addition then twice the identity matrix should be in the set.

$$\det(I) = 1$$
$$\det(2I) = 2^2 \det(I) = 4 \neq 1$$

Therefore, the set is not closed under matrix addition so it is not group with this choice of operation.

2.e)

Question:
The set of positive real numbers $\mathbb{R}^+$ under multiplication
Solution:

Closure The product of any two positive numbers is positive so closure is satisfied.

Associativity Real multiplication is always associative so the property is inherited.

Identity The product of any real number with 1 is itself so the property is inherited.

Inverse The product of any real number with its reciprical (which is always a real number) is 1 so there always exists an inverse element.

Therefore, the set and operation pair passes all of the axioms so it is a group.

P3) SO(2) Properties

Consider the rotation matrix $R(\theta) \in \mathrm{SO}(2)$:

$$R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

3.a)

Question:
Prove that $R(\theta)^T R(\theta) = I \forall \theta$
Solution:

$$R(\theta)^T R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} \cos^2(\theta) + \sin^2(\theta) & -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) \\ -\sin(\theta)\cos(\theta) + \sin(\theta)\cos(\theta) & \cos^2(\theta) + \sin^2(\theta) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$
$$\therefore R(\theta)^T R(\theta) = I \forall \theta$$

3.b)

Question:
Prove that $\det(R(\theta)) = 1 \forall \theta$
Solution:

$$\det(R(\theta)) = \det(\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}) = \cos^2(\theta) + \sin^2(\theta) = 1$$
$$\therefore \det(R(\theta)) = 1 \forall \theta$$

3.c)

Question:
Derive the inverse $R(\theta)^{-1}$ and show it equals $R(-\theta)$
Solution:
The inverse of $R(\theta)$ is the rotation matrix that when multiplied with $R(\theta)$ results in the identtity matrix. This inverse is the transpose of $R(\theta)$ as seen by part 3.a.

$$R(\theta)^{-1} = R(\theta)^T = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$
$$R(-\theta) = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = R(\theta)^{-1}$$
$$\therefore R(\theta)^{-1} = R(-\theta)$$

3.d)

Question:
Prove that $R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2)$
Solution:

$$R(\theta_1)R(\theta_2) = \begin{pmatrix} \cos(\theta_1) & -\sin(\theta_1) \\ \sin(\theta_1) & \cos(\theta_1) \end{pmatrix} \begin{pmatrix} \cos(\theta_2) & -\sin(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) - \sin(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\sin(\theta_2) - \sin(\theta_1)\cos(\theta_2) \\ \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\sin(\theta_2) + \cos(\theta_1)\cos(\theta_2) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix} = R(\theta_1+\theta_2)$$

$$\therefore R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2)$$

3.e)

Question:
Is SO(2) an abelian group?
Solution:
An abelian group is defined by a group where the group operation commutes. If SO(2) commutes then $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$

$$\text{from 3.d we know } R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2) = R(\theta_2+\theta_1) = R(\theta_2)R(\theta_1)$$

Therefore, SO(2) commutes so it is an abelian group.

P4) Lie Algebra of SO(2)

The Lie algebra $\mathfrak{so}(2)$ consists of 2×2 skew-symmetric matrices

4.a)

Question:
Write the general form of an element $\Omega \in \mathfrak{so}(2)$
Solution:
2×2 skew symmetric matrices can be paramterized using $\theta \in \mathbb{R}$ as the following

$$\Omega = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \in \mathfrak{so}(2)$$

4.b)

Question:
Define the wedge operator $(\cdot)^\wedge : \mathbb{R} \to \mathfrak{so}(2)$ and its inverse, the vee operator $(\cdot)^\wedge : \mathfrak{so}(2) \to \mathbb{R}$
Solution:
For $SO2$ the operations are defined as the following:
The wedge operator is defined as $\theta^\wedge = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = \Omega$
The vee operator is defined as $\Omega^\vee = \Omega_{21} = \theta$

4.c)

Question:
Compute the matrix exponential $\exp(\Omega)$ for $\Omega = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ using the power series definition. Show that this equals $R(\theta)$
Solution:
Define the matrix $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\exp(\Omega) = I + \Omega + \frac{1}{2}\Omega^2 + \frac{1}{6}\Omega^3 + \frac{1}{24}\Omega^4 + \dots$$
$$\Omega^2 = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = -\theta^2 I$$
$$\implies \Omega^3 = -\theta^2\Omega$$
$$\implies \exp(\Omega) = (1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 + \dots)I + (\theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 + \dots)J$$
$$\exp(\Omega) = \cos(\theta)I + \sin(\theta)J = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = R(\theta)$$
$$\therefore \exp(\Omega) = R(\theta)$$

4.d)

Question:
What is the dimension of $\mathfrak{so}(2)$? How does this relate to the dimension of the of SO(2)?
Solution:
The dimension of $\mathfrak{so}(2)$ is 1 because the elements, 2×2 skew-symmetric matrices, are parametrized by a single number θ. The dimension of SO(2) is 1 because the 2 dimensional rotation matrices are also parametrized by a single number θ. The dimensions of the Lie group and Lie algebra are the same for SO(2) and $\mathfrak{so}(2)$. In fact, this holds for any Lie group.

P5) Python Implementation

Implement the following functions in Python using NumPy:

5.a-d) SO(2) Mapping Functions

Code Snippet 1: SO(2) Mapping Functions

```
import numpy as np

def so2_wedge(theta):
    """
    Maps from so2 real number parametrization to so2 Lie algebra

    :param theta: angle in R
    """
    Omega = np.block([[0, -theta],
                      [theta,  0]]).reshape((2, 2))

    return Omega

def so2_vee(Omega):
    """
    Maps from so2 Lie algebra to its real number parametrization

    :param Omega: 2x2 skew symmetric matrix (in so2 Lie algebra)
    """

    theta = Omega[1, 0]

    return theta

def so2_exp(theta):
    """
    Maps from parametrization of so2 Lie algebra to SO2 Lie group

    :param theta: angle in R
    """

    R = np.array([[np.cos(theta), -np.sin(theta)],
                  [np.sin(theta),  np.cos(theta)]]).reshape((2, 2))

    return R

def so2_log(R):
   """
   Maps from SO2 Lie group to the parametrization of its Lie algebra so2

   :param R: 2D rotation matrix
   """
   theta = np.arctan2(R[1, 0], R[0, 0])

   return theta
```

5.e) Unit Tests

Code Snippet 2: Unit Tests Using Pytest

```
import pytest as pt
import numpy as np
from Code.SO2.SO2_maps import so2_wedge, so2_vee, so2_exp, so2_log

def test_so2_exp_log():
    """
    Test that exp(log(R)) = R for random rotation matrices
    """
```

```python
    rng = np.random.default_rng()
    theta_random = rng.uniform(-10, 10, 100)

    checks = np.zeros([theta_random.size, 1])
    for i in range(theta_random.size):
        R_random = so2_exp(theta_random[i])
        checks[i] = np.all(so2_exp(so2_log(R_random)) == pt.approx(R_random))

    assert np.all(checks)

def test_so2_log_exp():
    """
    Test that log(exp(theta)) = theta for all theta in (-pi, pi]
    """
    theta_array = -np.linspace(-np.pi, np.pi, 100)

    checks = np.zeros([theta_array.size, 1])
    for i in range(theta_array.size):
        checks[i] = np.all(so2_log(so2_exp(theta_array[i])) == pt.approx(theta_array[i]))

    assert np.all(checks)

def test_so2_commutativity():
    """
    Test that R(theta_1)R(theta_2) = R(theta_1 + theta_2)
    """
    iter = 100

    rng = np.random.default_rng()
    theta_1 = rng.uniform(-10, 10, iter)
    theta_2 = rng.uniform(-10, 10, iter)

    checks = np.zeros([iter, 1])
    for i in range(iter):
        checks[i] = so2_exp(theta_1[i]) @ so2_exp(theta_2[i]) == pt.approx(so2_exp(theta_1[i] + theta_2[i]))

    assert np.all(checks)
```

Figure 1: Pytest unit test output showing all 3 tests passing

```
PS C:\Users\thatf\OneDrive\Documents\Purdue Classes\AAE 590LGM\Lie Group Methods> pytest
=================================================== test session starts ===================================================
platform win32 -- Python 3.10.11, pytest-9.0.2, pluggy-1.6.0
rootdir: C:\Users\thatf\OneDrive\Documents\Purdue Classes\AAE 590LGM\Lie Group Methods
plugins: anyio-3.6.2, typeguard-2.13.3
collected 3 items

Tests\SO2\test_SO2.py ...                                                                                            [100%]

==================================================== 3 passed in 0.37s ====================================================
```