AAE 590: Problem Set 03
Group Affine Systems, SE(2) Kinematics, and $\mathrm{SE}_2(2)$

Due: See Brightspace

Instructions

- Show all work for full credit. Derivations should be clear and complete.
- **Submit a single PDF to Gradescope** containing:
  - Derivations (handwritten and scanned, or typed in LaTeX)
  - Python code (as monospace text, screenshots, or `listings`/`minted` in LaTeX)
  - All plots and numerical verification results
- You may use AI tools, but you must understand your solutions (validated via in-class quizzes).
- **Prerequisite:** You should have working implementations of `se2_wedge`, `se2_vee`, `se2_exp`, `se2_log`, `se2_compose`, `se2_inverse`, and `se2_Ad` from PS02. You may reuse your PS02 Jacobian/Taylor helpers for the $\mathrm{SE}_2(2)$ Exp/Log and Ad/ad implementations.
- Typical time: ∼8–12 hours depending on how much of PS02 is already working.

Notation and Conventions (used throughout this assignment):

- $X \in \mathrm{SE}(2)$: $\begin{bmatrix} R(\theta) & \boldsymbol{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$, $\quad X \in \mathrm{SE}_2(2)$: $\begin{bmatrix} R(\theta) & \boldsymbol{v} & \boldsymbol{p} \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix}$. In both cases, $R(\theta)$ maps **body** $\to$ **world**.
- $\xi^\wedge$: vector $\to$ matrix (`wedge`). $\quad (\cdot)^\vee$: matrix $\to$ vector (`vee`).
- $\mathrm{Exp}(\xi^\wedge)$: Lie group exponential (closed-form). $\quad \mathrm{Log}(X)^\vee$: its inverse.
- Heading θ is CCW-positive. For SE(2): $\dot{\boldsymbol{p}} = R\boldsymbol{v}_{\text{body}}$. For $\mathrm{SE}_2(2)$: $\dot{\boldsymbol{p}} = \boldsymbol{v}$ (world-frame velocity).
- $\dot{X} = X\xi^\wedge$: left-invariant (body-frame twist). $\quad \dot{X} = \xi^\wedge X$: right-invariant (world-frame twist).

Problem 1: Fixed-Wing Velocity Kinematics on SE(2) (20 points)

A fixed-wing aircraft in coordinated (no-sideslip) flight at constant altitude can be modeled as a rigid body on SE(2), where the state $X \in \mathrm{SE}(2)$ encodes position (x, y) and heading θ. At this level, velocity and yaw rate are treated as direct control inputs.

(a) (8 pts) **Velocity-Level Kinematic Model:**

In coordinated flight, the body-frame twist is:

$$\xi = \begin{bmatrix} V \\ 0 \\ \omega \end{bmatrix}$$

where $V > 0$ is the airspeed and ω is the yaw rate. The kinematic equation is $\dot{X} = X\xi^\wedge$ (left-invariant on SE(2)).

- Expand $\dot{X} = X\xi^\wedge$ into scalar ODEs for $\dot{\theta}$, $\dot{x}$, and $\dot{y}$.
- Explain the non-holonomic constraint $v_y = 0$: what physical assumption does it encode?
- **Mixed-invariant decomposition:** Write the same dynamics as $\dot{X} = MX + XN$ with constant $M, N \in \mathfrak{se}(2)$ (both proper Lie algebra elements). For the standard unicycle, $M = 0$ and $N = \xi^\wedge$. Explain why any constant world-frame twist would contribute a non-zero M.

Remark: This "clean" form (both $M, N \in \mathfrak{g}$) will contrast with $\mathrm{SE}_2(2)$ in Problem 4, where the $\dot{\boldsymbol{p}} = \boldsymbol{v}$ coupling requires a coupling matrix $C \notin \mathfrak{g}$.

(b) (12 pts) **Bank-to-Turn, Trajectory, and Simulation:**

In a coordinated level turn, centripetal force balance gives $\omega = g \tan\phi / V$, where $g = 9.81$ m/s^2 and ϕ is the bank angle. The required bank angle for a desired turn rate is $\phi = \arctan(\omega V / g)$, and the turning radius is $r = V/|\omega| = V^2/(g \tan\phi)$.

A reference trajectory can be defined as a **sequence of constant Lie algebra elements** (body-frame twists), each applied for a specified duration. Each segment produces a geodesic on SE(2).

Use the following race-track trajectory, defined as a table of segments (ξ_i, T_i), where each twist is $\xi_i = (V_i,\ v_{y,i},\ \omega_i)^T$:

Segment	V_i (m/s)	$v_{y,i}$	ω_i (rad/s)	T_i (s)
0	20	0	0	10.0
1	15	0	0.30	5.24
2	20	0	0	5.0
3	15	0	0.30	5.24
4	20	0	0	10.0
5	15	0	0.30	5.24
6	20	0	0	5.0
7	15	0	0.30	5.24

Notes: v_y is included for completeness; assume $v_y = 0$ (coordinated flight) throughout. The duration $T = 5.24 \approx (\pi/2)/0.3$ gives a 90° turn at $\omega = 0.3$ rad/s.

- **Plot** turning radius $r(V) = V^2/(g \tan\phi_0)$ vs. V for $V \in [10, 40]$ m/s with $\phi_0 = 30°$.
- Propagate the waypoints: $X_{i+1} = X_i \cdot \mathrm{Exp}((T_i \xi_i)^\wedge)$, starting from $X_0 = I$.
- To plot a smooth curve, sample within each segment: $X(t) = X_i \cdot \mathrm{Exp}((t_{\mathrm{seg}} \xi_i)^\wedge)$ for $t_{\mathrm{seg}} \in [0, T_i]$.

- **Plot** the reference trajectory in the (x, y) plane with heading arrows at each waypoint and segment numbers.
- **Feasibility check:** For each turning segment, compute the required bank angle $\phi_i = \arctan(\omega_i V_i / g)$. Verify $|\phi_i| \leq 45°$.
- **Simulate:** Implement a loop that propagates $X_{k+1} = X_k \cdot \text{Exp}((\Delta t\, \xi)^\wedge)$ with a **constant** twist (use $\Delta t = 0.01$ s). Run for 10 s with each of: $\xi = (20, 0, 0)^T$ (straight) and $\xi = (15, 0, 0.3)^T$ (turn). Verify you get a straight line and a circle, respectively.

Problem 2: Group Affine Systems and Mixed Invariant Vector Fields (25 points)

Group affine systems are a class of nonlinear systems on Lie groups whose error dynamics are **autonomous**—they depend only on the error, not on the absolute state. This property is the theoretical foundation for both geometric control and invariant filtering.

(a) (10 pts) **Deriving the Group Affine Property:**

Consider two trajectories $X(t)$ and $\hat{X}(t)$ evolving under the same input-dependent dynamics $\dot{X} = f_u(X)$ (same $u(t)$, different initial conditions). Define the left-invariant error $\eta = \hat{X}^{-1} X$.

- **Step 1:** Using the product rule and $\frac{d}{dt}(\hat{X}^{-1}) = -\hat{X}^{-1}\dot{\hat{X}}\hat{X}^{-1}$, show:

$$\dot{\eta} = -\hat{X}^{-1} f_u(\hat{X})\eta + \hat{X}^{-1} f_u(\hat{X}\eta)$$

- **Step 2:** Require that $\dot{\eta} = g(\eta)$ for some function g (autonomous in η). Because the autonomy requirement must hold for *all* $\hat{X}$, we can evaluate at $\hat{X} = I$ to identify $g(\cdot)$. Derive:

$$g(\eta) = f_u(\eta) - f_u(I)\eta$$

- **Step 3:** Substitute $g(\eta)$ from Step 2 into the expression from Step 1. Require equality for **all** $\hat{X}$. Renaming $\hat{X} \to X$ and $\eta \to Y$, derive the **group affine property**:

$$\boxed{f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y}$$

This is not an arbitrary definition—it is *derived* from requiring autonomous error dynamics.

(b) (8 pts) **Mixed Invariant Vector Fields:**

- A **mixed invariant** vector field has the form:

$$f_u(X) = \xi_R^\wedge X + X\xi_L^\wedge$$

where the vectors $\xi_L(u), \xi_R(u)$ depend only on inputs u, **not** on state X, and $\xi_L^\wedge, \xi_R^\wedge$ denote their matrix forms.

- **General parameterization:** Any mixed invariant system can be written as

$$\dot{X} = (M - C)X + X(N + C)$$

where M, N encode the natural physics and C is an arbitrary matrix that redistributes between left and right parts. The split is: $\xi_R^\wedge = M - C$ and $\xi_L^\wedge = N + C$. Note that $M + N = \xi_R^\wedge + \xi_L^\wedge$ is unchanged regardless of the choice of C.

- **Special cases:**
  - Left-invariant: $\xi_R^\wedge = 0$, so $\dot{X} = X\xi_L^\wedge$ (body-frame dynamics)
  - Right-invariant: $\xi_L^\wedge = 0$, so $\dot{X} = \xi_R^\wedge X$ (world-frame dynamics)
- **Prove:** Mixed invariant $\Rightarrow$ group affine. Verify $f_u(XY) = f_u(X)Y + Xf_u(Y) - Xf_u(I)Y$ for $f_u(X) = \xi_R^\wedge X + X\xi_L^\wedge$. Show the algebraic cancellation.

(c) (7 pts) **Autonomous Error Dynamics:**

- For the mixed invariant case $f_u(X) = \xi_R^\wedge X + X\xi_L^\wedge$, note that $f_u(I) = \xi_R^\wedge + \xi_L^\wedge$. Compute $g(\eta) = f_u(\eta) - f_u(I)\eta$. Show that:

$$g(\eta) = \eta\xi_L^\wedge - \xi_L^\wedge\eta = [\eta, \xi_L^\wedge]$$

  Only the **left-invariant part** $\xi_L^\wedge$ drives the left-invariant error dynamics! The right-invariant part $\xi_R^\wedge$ cancels completely.
- **Apply to Problem 1:** The SE(2) unicycle has $f_u(X) = X\xi^\wedge$ (left-invariant, $\xi_R^\wedge = 0$). Verify algebraically that this is group affine by showing $f_u(I) = \xi^\wedge$ and the key cancellation $f_u(X)Y - Xf_u(I)Y = 0$.
- **Verify numerically:** Choose random $X, Y \in \mathrm{SE}(2)$ and a twist $\xi = (15, 0, 0.3)^T$. Compute both sides of the group affine equation and confirm agreement to machine precision.
- **Error convention:** Explain why body-frame sensors (e.g., IMU) suggest left-invariant error $\eta = \hat{X}^{-1}X$, while world-frame sensors (e.g., GPS) suggest right-invariant error $\eta = X\hat{X}^{-1}$.

Problem 3: The Extended Pose Group $\mathrm{SE}_2(2)$ (30 points)

In lecture, $\mathrm{SE}_2(3) = \mathrm{SO}(3) \ltimes (\mathbb{R}^3 \times \mathbb{R}^3)$ was introduced for 3D inertial navigation, combining orientation, velocity, and position into a single Lie group. You will now derive the **2D analog**, $\mathrm{SE}_2(2) = \mathrm{SO}(2) \ltimes (\mathbb{R}^2 \times \mathbb{R}^2)$, which adds velocity to the SE(2) pose.

Matrix representation (4×4):

$$X = \begin{bmatrix} R & \boldsymbol{v} & \boldsymbol{p} \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

where $R \in \mathrm{SO}(2)$ (orientation), $\boldsymbol{v} \in \mathbb{R}^2$ (velocity in world frame), and $\boldsymbol{p} \in \mathbb{R}^2$ (position in world frame). **Dimension:** 5 (1 rotation + 2 velocity + 2 position).

(a) (5 pts) **Group Operations:**

- Derive the composition X_1X_2 and inverse X^{-1} using block matrix multiplication. Express in tuple form.
- You should get:

$$(R_1, \boldsymbol{v}_1, \boldsymbol{p}_1) \cdot (R_2, \boldsymbol{v}_2, \boldsymbol{p}_2) = (R_1R_2,\ R_1\boldsymbol{v}_2 + \boldsymbol{v}_1,\ R_1\boldsymbol{p}_2 + \boldsymbol{p}_1)$$
$$(R, \boldsymbol{v}, \boldsymbol{p})^{-1} = (R^T,\ -R^T\boldsymbol{v},\ -R^T\boldsymbol{p})$$

- Note: R acts on **both** $\boldsymbol{v}$ and $\boldsymbol{p}$ independently—this is the $\mathrm{SO}(2) \ltimes (\mathbb{R}^2 \times \mathbb{R}^2)$ semi-direct product structure.
- Implement `se22_compose(X1, X2)` and `se22_inverse(X)`.

(b) (3 pts) **Lie Algebra** $\mathfrak{se}_2(2)$**:**

The Lie algebra elements are 4×4 matrices tangent at the identity:

$$\xi^\wedge = \begin{bmatrix} \omega^\wedge & \boldsymbol{a} & \boldsymbol{b} \\ \mathbf{0}^T & 0 & 0 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

where $\omega^\wedge = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$, $\boldsymbol{a} \in \mathbb{R}^2$ (maps to velocity column via Exp), and $\boldsymbol{b} \in \mathbb{R}^2$ (maps to position column via Exp).

- **Convention (used for all** $\mathrm{SE}_2(2)$ **formulas in this assignment):** the Lie algebra vector is $\xi = (a_1, a_2, b_1, b_2, \omega)^T \in \mathbb{R}^5$ (translation-first ordering, matching the block structure of X).
- Implement `se22_wedge(xi)` and `se22_vee(Xi)`.

(c) (10 pts) **Exponential and Logarithm Maps:**

- **Derive** the exponential map from the power series $\mathrm{Exp}(\xi^\wedge) = I + \xi^\wedge + \frac{(\xi^\wedge)^2}{2!} + \cdots$ Show that $(\xi^\wedge)^k$ has the block structure:

$$(\xi^\wedge)^k = \begin{bmatrix} (\omega^\wedge)^k & (\omega^\wedge)^{k-1}\boldsymbol{a} & (\omega^\wedge)^{k-1}\boldsymbol{b} \\ \mathbf{0}^T & 0 & 0 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix} \quad \text{for } k \geq 1$$

  Summing the series gives:

$$\boxed{\mathrm{Exp}(\xi^\wedge) = \begin{bmatrix} R(\omega) & V(\omega)\boldsymbol{a} & V(\omega)\boldsymbol{b} \\ \mathbf{0}^T & 1 & 0 \\ \mathbf{0}^T & 0 & 1 \end{bmatrix}}$$

  where $R(\omega)$ is the SO(2) exponential and $V(\omega) = \frac{\sin\omega}{\omega} I + \frac{1-\cos\omega}{\omega} J$ is the **same** SO(2) left Jacobian from SE(2). Both translation columns use $V(\omega)$ because $\boldsymbol{a}$ and $\boldsymbol{b}$ decouple (they only interact through ω).

- **Derive the logarithm:** Given $X = (R, \boldsymbol{v}, \boldsymbol{p})$, extract $\omega = \mathrm{atan2}(R_{21}, R_{11})$, then $\boldsymbol{a} = V(\omega)^{-1}\boldsymbol{v}$ and $\boldsymbol{b} = V(\omega)^{-1}\boldsymbol{p}$. Use Taylor series for $\omega \approx 0$ (same as PS02).
- Implement `se22_exp(xi)` and `se22_log(X)`. Verify round-trip: $\mathrm{Exp}\big((\mathrm{Log}(X))^\wedge\big) = X$ for at least 3 random $X \in \mathrm{SE}_2(2)$ (i.e., `se22_exp(se22_log(X))` recovers X to machine precision).

(d) (10 pts) **Adjoint** Ad_X **and Small Adjoint** ad_ξ**:**

- **Derive** Ad_X from $(\mathrm{Ad}_X \xi)^\wedge = X \xi^\wedge X^{-1}$. Compute by block matrix multiplication. With translation-first ordering $\xi = (\boldsymbol{a}^T, \boldsymbol{b}^T, \omega)^T$, you should get:

$$\mathrm{Ad}_X = \begin{bmatrix} R & \mathbf{0} & \boldsymbol{v}^\odot \\ \mathbf{0} & R & \boldsymbol{p}^\odot \\ \mathbf{0}^T & \mathbf{0}^T & 1 \end{bmatrix} \in \mathbb{R}^{5\times 5}$$

where $\boldsymbol{v}^\odot = \begin{bmatrix} v_y \\ -v_x \end{bmatrix} = -J\boldsymbol{v} \in \mathbb{R}^2$ with $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (same $\odot$ operator as in SE(2) from PS02; the 2×1 column $\boldsymbol{v}^\odot$ fills the off-diagonal block so dimensions match).

- **Derive** the Lie bracket $[\xi_1, \xi_2] = (\xi_1^\wedge \xi_2^\wedge - \xi_2^\wedge \xi_1^\wedge)^\vee$ and extract the 5×5 small adjoint:

$$\mathrm{ad}_\xi = \begin{bmatrix} \omega^\wedge & \mathbf{0} & \boldsymbol{a}^\odot \\ \mathbf{0} & \omega^\wedge & \boldsymbol{b}^\odot \\ \mathbf{0}^T & \mathbf{0}^T & 0 \end{bmatrix}$$

- Note: $\boldsymbol{a}$ and $\boldsymbol{b}$ do **not** interact directly—only through ω. This is the same pattern as $\mathrm{SE}_2(3)$ from lecture.
- Implement `se22_Ad(X)` and `se22_ad(xi)`. Verify numerically (for at least 3 random inputs each):
  - $\mathrm{Ad}_{X_1 X_2} = \mathrm{Ad}_{X_1} \mathrm{Ad}_{X_2}$
  - $[\xi_1, \xi_2] = \mathrm{ad}_{\xi_1} \xi_2$

(e) (2 pts) **Comparison Table:**

Fill in the "?" entries. The SO(2) column is provided as a worked example showing what each row means.

	SO(2)	SE(2)	$\mathrm{SE}_2(2)$	$\mathrm{SE}_2(3)$
Matrix size	2×2	?	?	?
DOF (group dim.)	1	?	?	?
ξ components	(ω)	?	?	?
$\mathrm{Exp}(\xi^\wedge)$ block form	$(R(\omega))$	?	?	?
Rotation left Jacobian	$V(\omega) \in \mathbb{R}^{2\times 2}$	?	?	?
Ad_X size	1×1	?	?	?

Hints: The "block form" row describes how the matrix columns of $\mathrm{Exp}(\xi^\wedge)$ are built from the Lie algebra components (e.g., $(R, V\boldsymbol{v})$ means the rotation block is R and the translation column is $V(\omega)\boldsymbol{v}$). Each translation-like column always uses the same rotation left Jacobian.

Problem 4: 2D Inertial Navigation on $\mathrm{SE}_2(2)$ (25 points)

A 2D fixed-wing aircraft carries a "2D IMU" that measures body-frame angular velocity ω and body-frame acceleration $\boldsymbol{a} = (a_x, a_y)^T$. Your task is to propagate the aircraft's state (orientation, velocity, position) using the $\mathrm{SE}_2(2)$ group structure—a 2D analog of the 3D IMU navigation problem from the $\mathrm{SE}_2(3)$ lecture.

(a) (8 pts) **Dynamics as Mixed Invariant on** $\mathrm{SE}_2(2)$**:**

The continuous-time dynamics for 2D inertial navigation are:

$$\begin{aligned} \dot{R} &= R\omega^\wedge && \text{(body-frame angular velocity)} \\ \dot{\boldsymbol{v}} &= R\boldsymbol{a} && \text{(body-frame acceleration rotated to world)} \\ \dot{\boldsymbol{p}} &= \boldsymbol{v} && \text{(kinematic coupling)} \end{aligned}$$

- Verify that these dynamics can be written as $\dot{X} = \xi_R^\wedge X + X\xi_L^\wedge$ where:
  - $\xi_L^\wedge = \begin{bmatrix} \omega^\wedge & \boldsymbol{a} & \mathbf{0} \\ \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix}$ encodes body-frame quantities $(\boldsymbol{a}, \omega)$. The $+1$ in the bookkeeping block (row 3, col 4) implements the coupling $\dot{\boldsymbol{p}} = \boldsymbol{v}$: when you compute $X\xi_L^\wedge$, the velocity column $\boldsymbol{v}$ of X is copied into the position column of $\dot{X}$.
  - $\xi_R^\wedge = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & -1 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix}$ has -1 in the same bookkeeping position. The product $\xi_R^\wedge X$ places -1 in row 3, col 4 of $\dot{X}$, exactly cancelling the $+1$ artifact that $X\xi_L^\wedge$ introduced there. The result: the bottom two rows of $\dot{X}$ remain zero, as required for $\dot{X}$ to be tangent to $\mathrm{SE}_2(2)$.
- **Key observation: Neither** $\xi_L^\wedge$ nor $\xi_R^\wedge$ is in $\mathfrak{se}_2(2)$! Both have non-zero entries in position $(3, 4)$, which is outside the Lie algebra. Using the $(M-C)X + X(N+C)$ framework from Problem 2:
  - The "physics" matrix $N = \begin{bmatrix} \omega^\wedge & \boldsymbol{a} & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix} \in \mathfrak{se}_2(2)$ and $M = 0$.
  - The coupling matrix $C = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 0 & 1 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix} \notin \mathfrak{se}_2(2)$ handles $\dot{\boldsymbol{p}} = \boldsymbol{v}$.
  - Then $\xi_R^\wedge = M - C = -C$ and $\xi_L^\wedge = N + C$, with $C \notin \mathfrak{se}_2(2)$ pulling both outside the algebra.

  This is the structural difference from SE(2) (Problem 1), where $C = 0$ and both $M, N \in \mathfrak{g}$.
- This mirrors $\mathrm{SE}_2(3)$ from lecture (week 06), where the gravity and velocity coupling terms similarly require $C \notin \mathfrak{se}_2(3)$.
- Verify by expanding $\xi_R^\wedge X + X\xi_L^\wedge$ that you recover $\dot{R} = R\omega^\wedge$, $\dot{\boldsymbol{v}} = R\boldsymbol{a}$, and $\dot{\boldsymbol{p}} = \boldsymbol{v}$.
- Despite $\xi_R^\wedge \notin \mathfrak{g}$, the system is **still group affine**. Verify numerically: choose random $X, Y \in \mathrm{SE}_2(2)$ and inputs $(\boldsymbol{a}, \omega)$, check $f(XY) = f(X)Y + Xf(Y) - Xf(I)Y$.

(b) (5 pts) **Adding Wind (World-Frame):**

A constant wind acceleration $\boldsymbol{w}_a \in \mathbb{R}^2$ (in the world frame) acts on the aircraft, analogous to gravity in $\mathrm{SE}_2(3)$. For example, a persistent headwind creates a drag-like deceleration in the world frame. The dynamics become:

$$\dot{\boldsymbol{v}} = R\boldsymbol{a} + \boldsymbol{w}_a, \qquad \dot{\boldsymbol{p}} = \boldsymbol{v}, \qquad \dot{R} = R\omega^\wedge$$

Separately, the airspeed (velocity relative to the air mass) in the body frame is $\boldsymbol{v}_{\text{air}} = R^T(\boldsymbol{v} - \boldsymbol{w}_v)$ where $\boldsymbol{w}_v$ is the wind velocity.

- Show that the wind acceleration $\boldsymbol{w}_a$ enters M, giving:

$$M = \begin{bmatrix} 0 & \boldsymbol{w}_a & \mathbf{0} \\ \mathbf{0}^T & 0 & 0 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix}, \quad \xi_R^\wedge = M - C = \begin{bmatrix} 0 & \boldsymbol{w}_a & \mathbf{0} \\ \mathbf{0}^T & 0 & -1 \\ \mathbf{0}^T & 0 & 0 \end{bmatrix}$$

  while $\xi_L^\wedge = N + C$ is unchanged. The left-invariant part captures body-frame / airspeed-relative quantities; the right-invariant part captures world-frame quantities (wind, coupling).
- Verify the group affine property numerically with $\boldsymbol{w}_a = (3, -2)^T$ m/s^2.
- For non-holonomic flight, $|\boldsymbol{v}_{\text{air}}|$ is the airspeed and we assume $\boldsymbol{v}_{\text{air}} \approx (V_{\text{air}}, 0)^T$ (no sideslip in body frame).

(c) (10 pts) **2D INS Simulator:**

Implement a simulator that propagates the $\text{SE}_2(2)$ state given "2D IMU" measurements.

- **Input:** A time series of body-frame measurements $\{(\omega_k, \boldsymbol{a}_k)\}$ at timestep $\Delta t = 0.01$ s.
- **Propagation** (using the $\text{SE}_2(2)$ group exponential, matching the $\text{SE}_2(3)$ lecture):

$$X_{k+1} = X_k \cdot \text{Exp}_{\text{SE}_2(2)}\big((\xi_k \, \Delta t)^\wedge\big) \cdot \Gamma(\Delta t)$$

  where the body-frame twist $\xi_k = (a_{x,k},\ a_{y,k},\ 0,\ 0,\ \omega_k)^T$ has zero position components, and the **coupling matrix**

$$\Gamma(\Delta t) = \begin{bmatrix} I_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & 1 & \Delta t \\ \mathbf{0}^T & 0 & 1 \end{bmatrix}$$

  handles the $\dot{\boldsymbol{p}} = \boldsymbol{v}$ kinematic coupling, analogous to the $\Gamma(\boldsymbol{g}, \Delta t)$ matrix from the $\text{SE}_2(3)$ lecture (but without gravity in 2D). The group exponential handles the body-frame dynamics (rotation and acceleration); Γ applies the non-algebraic coupling exactly over the step. After the matrix product, extract R from rows 1–2, columns 1–2; $\boldsymbol{v}$ from rows 1–2, column 3; and $\boldsymbol{p}$ from rows 1–2, column 4. Then reset rows 3–4 to [0 0 1 0; 0 0 0 1] to restore the $\text{SE}_2(2)$ structure.
- **Test scenario:** Use the following synthetic flight profile with 4 phases:
  i. **Acceleration:** $\boldsymbol{a} = (2, 0)^T$ m/s^2, $\omega = 0$ for 5 s (straight-line speedup)
  ii. **Cruise:** $\boldsymbol{a} = (0, 0)^T$, $\omega = 0$ for 5 s (constant velocity)
  iii. **Banked turn:** $\boldsymbol{a} = (0, 0)^T$, $\omega = 0.3$ rad/s for $\pi/0.3 \approx 10.5$ s (180° turn). Note: the vehicle yaws at constant rate while maintaining the inertial speed from the prior phase—no centripetal acceleration is modeled here.
  iv. **Deceleration:** $\boldsymbol{a} = (-1, 0)^T$, $\omega = 0$ for 5 s (slow down)

  Start from $X_0 = I$ (origin, heading east, zero velocity).
- Propagate the state and produce the following plots:
  1. (x, y) trajectory with heading arrows at key points
  2. Speed $|\boldsymbol{v}(t)|$ vs. time
  3. Heading $\theta(t)$ vs. time
- **Spiral test:** Run 10 s with constant $\omega = 0.5$ rad/s, $\boldsymbol{a} = (1, 0)^T$ m/s^2 from $X_0 = I$. Verify the (x, y) trajectory is a spiral (accelerating turn).

(d) (2 pts) **Discussion:** SE(2) **vs.** $SE_2(2)$**:**

- The SE(2) model (Problem 1) commands velocity/yaw-rate directly: instantaneous velocity response.
- The $SE_2(2)$ model commands acceleration: velocity has transients (more realistic).
- Discuss: when is the SE(2) velocity-level model sufficient? When do you need the $SE_2(2)$ acceleration-level model?

Submission Checklist

Submit a **single PDF to Gradescope** containing:

- ☐ Problem 1: Fixed-wing kinematics on SE(2), mixed-invariant form, bank-to-turn, trajectory plot, simulation
- ☐ Problem 2: Group affine derivation, mixed invariant proof, error dynamics, **numerical verification** of group affine identity on SE(2)
- ☐ Problem 3: $SE_2(2)$ group operations, Exp/Log derivations, Ad/ad derivations, comparison table, code, **numerical verification** of Exp/Log round-trip and Ad product property
- ☐ Problem 4: Mixed invariant dynamics on $SE_2(2)$, wind model, **numerical verification** of group affine on $SE_2(2)$, 2D INS simulator, plots, SE(2) vs $SE_2(2)$ discussion