Relevant properties of affine functions

- affine functions are concave and convex
- minimum of a sum of affine functions is concave

Relevant properties of concave functions

- sum of concave functions is concave
- minimum of a set of concave functions is concave

We are dealing with discrete $\mathcal{X}$ so the cost functions can be written as

$$\tilde{C}_t(\pi_t, u_t) = \int_{\mathcal{X}_T} C_t^{\text{exit}}(x_t)\pi_t(x_t)\mathrm{d}x_t = \begin{pmatrix} C_t(x_1, u_t) \\ C_t(x_2, u_t) \\ \vdots \\ C_t(x_{n_x}, u_t) \end{pmatrix}^T \begin{pmatrix} \pi_t(x_1) \\ \pi_t(x_2) \\ \vdots \\ \pi_t(x_{n_x}) \end{pmatrix}$$

$$\tilde{C}_T^{\text{exit}}(\pi_T) = \int_{\mathcal{X}_T} C_T^{\text{exit}}(x_T)\pi_T(x_T)\mathrm{d}x_T = \begin{pmatrix} C_T^{\text{exit}}(x_1) \\ C_T^{\text{exit}}(x_2) \\ \vdots \\ C_T^{\text{exit}}(x_{n_x}) \end{pmatrix}^T \begin{pmatrix} \pi_T(x_1) \\ \pi_T(x_2) \\ \vdots \\ \pi_T(x_{n_x}) \end{pmatrix}$$

Now it is obvious that $\tilde{C}_t(\pi_t, u_t)$ and $\tilde{C}_T^{\text{exit}}(\pi_T)$ are both affine in π_t.

$J_T(\pi_T) = \tilde{C}_T^{\text{exit}}(\pi_T)$ so $J_T(\pi_T)$ is also affine in π_t.

$$J_t(\pi_t) = \min_{u_t \in \mathcal{U}} \tilde{C}_t(\pi_t, u_t) + \int_{\pi_{t+1} \in \Delta(\mathcal{X})} J_{t+1}(\pi_{t+1})\tau(\pi_{t+1}|\pi_t, u_t)\mathrm{d}\pi_{t+1}$$

Let $\tau(u_t) = \begin{pmatrix} \tau_{u_t}(1|1) & \tau_{u_t}(1|2) & \dots & \tau_{u_t}(1|n_x) \\ \tau_{u_t}(2|1) & \tau_{u_t}(2|2) & \dots & \tau_{u_t}(2|n_x) \\ \vdots & \vdots & \ddots & \\ \tau_{u_t}(n_x|1) & \tau_{u_t}(n_x|2) & & \tau_{u_t}(n_x|n_x) \end{pmatrix} = \tau(\pi_{t+1}|\pi_t, u_t)$ where n_x is the number of states.

Now, it can be seen that

$$\pi_{t+1} = \tau(u_t)\pi_t$$

Focusing on $t = T - 1$

$$\int_{\pi_T \in \Delta(\mathcal{X})} J_T(\pi_T)\tau(\pi_T|\pi_{T-1}, u_{T-1})\mathrm{d}\pi_T = \begin{pmatrix} C_T^{\text{exit}}(x_1) \\ C_T^{\text{exit}}(x_2) \\ \vdots \\ C_T^{\text{exit}}(x_{n_x}) \end{pmatrix}^T \tau(u_{T-1}) \begin{pmatrix} \pi_{T-1}(x_1) \\ \pi_{T-1}(x_2) \\ \vdots \\ \pi_{T-1}(x_{n_x}) \end{pmatrix}$$

$$J_{T-1}(\pi_{T-1}) = \min_{u_{T-1} \in \mathcal{U}} \begin{pmatrix} C_{T-1}(x_1, u_{T-1}) \\ C_{T-1}(x_2, u_{T-1}) \\ \vdots \\ C_{T-1}(x_{n_x}, u_{T-1}) \end{pmatrix}^T \begin{pmatrix} \pi_{T-1}(x_1) \\ \pi_{T-1}(x_2) \\ \vdots \\ \pi_{T-1}(x_{n_x}) \end{pmatrix} + \begin{pmatrix} C_T^{\text{exit}}(x_1) \\ C_T^{\text{exit}}(x_2) \\ \vdots \\ C_T^{\text{exit}}(x_{n_x}) \end{pmatrix}^T \tau(u_{T-1}) \begin{pmatrix} \pi_{T-1}(x_1) \\ \pi_{T-1}(x_2) \\ \vdots \\ \pi_{T-1}(x_{n_x}) \end{pmatrix}$$

$$J_{T-1}(\pi_{T-1}) = \min_{u_{T-1} \in \mathcal{U}} (\begin{pmatrix} C_{T-1}(x_1, u_{T-1}) \\ C_{T-1}(x_2, u_{T-1}) \\ \vdots \\ C_{T-1}(x_{n_x}, u_{T-1}) \end{pmatrix}^T + \begin{pmatrix} C_T^{\text{exit}}(x_1) \\ C_T^{\text{exit}}(x_2) \\ \vdots \\ C_T^{\text{exit}}(x_{n_x}) \end{pmatrix}^T \tau(u_{T-1})) \begin{pmatrix} \pi_{T-1}(x_1) \\ \pi_{T-1}(x_2) \\ \vdots \\ \pi_{T-1}(x_{n_x}) \end{pmatrix}$$

It is obvious that $J_{T-1}(\pi_{T-1})$ is the minimum of a set of affine functions of π_{T-1} (one for each control input). This results in an a convex piecewise affine function of π_{T-1} with up to n_u affine regions.

For $t = T-2, \ldots, 1, 0$ the integration must be done by summing over the integrals of each affine region in $J_t(\pi_t)$ resulting in a sum of concave piecewise affine functions which when minimized is a concave piecewise affine function. Therefore, by induction J_t is piecewise affine and concave in $\pi_t \forall t = 0, 1, \ldots, T$.