Trace property: For a gaussian random variable a with mean $\bar{a}$ and covariance A

$$\mathbb{E}(a^T M a) = \bar{a}^T M \bar{a} + Tr(MA)$$

Looking at $t = T-1$

$$J_t(\pi_t) = \min_{\mu_{t:T-1}} \sum_{k=t}^{T-1} \mathbb{E}(||x_k||^2_{Q_k} + ||u_k||^2_{R_k}) + \mathbb{E}||x_T||^2_{Q_T}$$

$$J_{T-1}(\pi_{T-1}) = \mathbb{E}(||x_k||^2_{Q_k} + ||u_k||^2_{R_k}) + \mathbb{E}||X_t||^2_{Q_t}$$

$$Q_T = S_T \implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(||x_k||^2_{Q_k} + ||u_k||^2_{R_k}) + \mathbb{E}(x_T^T S_T x_T)$$

$$x_{t+1} = A_t x_t + B_t u_t + w_t \implies \mathbb{E}(x_T^T S_T x_T) = \mathbb{E}((A_{T-1}x_{T-1} + B_{T-1}u_{T-1} + w_{T-1})^T S_T (A_{T-1}x_{T-1} + B_{T-1}u_{T-1} + w_{T-1}))$$

$$\mathbb{E}(w_t) = 0 \implies \mathbb{E}(x_T^T S_T x_T) = \mathbb{E}((A_{T-1}x_{T-1} + B_{T-1}u_{T-1})^T S_T (A_{T-1}x_{T-1} + B_{T-1}u_{T-1})) + Tr(S_T W_{T-1})$$

$$\implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(x_{T-1}^T Q_{T-1} x_{T-1} + u_{T-1}^T R_{T-1} u_{T-1} + (A_{T-1}x_{T-1})^T S_T (A_{T-1}x_{T-1}) + 2(B_{T-1}u_{T-1})^T S_T (A_{T-1}x_{T-1})$$
$$+ (B_{T-1}u_{T-1})^T S_T B_{T-1} u_{T-1}) + Tr(S_T W_{T-1})$$

$$M_t = B_t^T S_t B_t + R_t \implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(x_{T-1}^T Q_{T-1} x_{T-1} + u_{T-1}^T M_{T-1} u_{T-1} + (A_{T-1}x_{T-1})^T S_T (A_{T-1}x_{T-1})$$
$$+ 2(B_{T-1}u_{T-1})^T S_T (A_{T-1}x_{T-1})) + Tr(S_T W_{T-1})$$

Completing the square

$$u_{T-1}^T M_{T-1} u_{T-1} + 2(B_{T-1}u_{T-1})^T S_T (A_{T-1}x_{T-1})$$
$$= (u_{T-1} + M_{T-1}^{-1} B_{T-1}^T S_T A_{T-1} x_{T-1})^T M_{T-1} (u_{T-1} + M_{T-1}^{-1} B_{T-1}^T S_T A_{T-1} x_{T-1})$$
$$- (B_{T-1}^T S_T A_{T-1} x_{T-1})^T M_{T-1}^{-1} (B_{T-1}^T S_T A_{T-1} x_{T-1})$$

know $S_t = A_t^T S_{t+1} A_t - A_t^T S_{t+1} B_t M_t^{-1} B_t^T S_{t+1} A_t + Q_t$

$$\implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(x_{T-1}^T S_{T-1} x_{T-1} + (u_{T-1} + M_{T-1}^{-1} B_{T-1}^T S_{T-1} A_{T-1} x_{T-1})^T M_{T-1} (u_{T-1} + M_{T-1}^{-1} B_{T-1}^T S_{T-1} A_{T-1} x_{T-1}))$$
$$+ Tr(S_T W_{T-1})$$

know $u^* = -M_t^{-1} B_t^T S_t A_t \hat{x}_t$

$$\implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(x_{T-1}^T S_{T-1} x_{T-1}$$
$$+ (M_{T-1}^{-1} B_{T-1}^T S_{T-1} A_{T-1} (\hat{x}_{T-1} - x_{T-1}))^T M_{T-1} (M_{T-1}^{-1} B_{T-1}^T S_{T-1} A_{T-1} (\hat{x}_{T-1} - x_{T-1})))$$

know $\Theta_t = K_t^T M_t K_t$ and $K_t = -M_t^{-1} B_t^T S_t A_t$

$$\implies J_{T-1}(\pi_{T-1}) = \mathbb{E}(x_{T-1}^T S_{T-1} x_{T-1} + (\hat{x}_{T-1} - x_{T-1})^T \Theta_{T-1} (\hat{x}_{T-1} - x_{T-1})) + Tr(S_T W_{T-1})$$

From trace rules $\mathbb{E}(x_{T-1}^T S_{T-1} x_{T-1}) = \hat{x}_{T-1}^T S_{T-1} \hat{x}_{T-1} + Tr(S_{T-1} P_{T-1|T-1})$

$$\implies J_{T-1}(\pi_{T-1}) = \hat{x}_{T-1}^T S_{T-1} \hat{x}_{T-1} + Tr(S_{T-1} P_{T-1|T-1}) + Tr(\Theta_{T-1} P_{T-1|T-1}) + Tr(S_T W_{T-1})$$

Now we need to get the value function for $t = T-2, T-3, \ldots, 1, 0$. This can be done in a similer way to in exercise 2.4) Finally,

$$J_t(\pi_t) = ||\hat{x}_t||^2_{S_t} + Tr(S_t P_{t|t}) + \sum_{k=1}^{T-1} (Tr(W_k S_{k+1}) + Tr(\Theta_k P_{k|k}))$$