Virial expansion for the partition function of hard disks

Settings

System:

N disks of radius σ in a square of side L and surface $V = L^2$.

Density:

$$\eta = \frac{\text{occupied surface}}{\text{total surface}} = \frac{N\pi\sigma^2}{L^2} \tag{1}$$

Probability distribution (not normalized) of the centers $(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1})$ of the disks:

$$\pi(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}) = \begin{cases} 1 & \text{if there is no overlap between disks} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Partition function at density η

$$Z(\eta) = \int d\mathbf{x}_0 \ldots d\mathbf{x}_{N-1}\ \pi(\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}) \tag{3}$$

Expansion of the partition function

Rewriting of $Z(\eta)$ with the non-overlapping constraint made explicit:

$$Z(\eta) = \int d\mathbf{x}_0 \ldots d\mathbf{x}_{N-1}\ \underbrace{\big[1 - \Upsilon(\mathbf{x}_0, \mathbf{x}_1)\big]\big[1 - \Upsilon(\mathbf{x}_0, \mathbf{x}_2)\big] \ldots \big[1 - \Upsilon(\mathbf{x}_{N-2}, \mathbf{x}_{N-1})\big]}_{\frac{1}{2}N(N-1)\text{ factors}} \tag{4}$$

The integral runs over the $2N$ individual coordinates, each of them running on $[0, L]$.
Indicator function:

$$\Upsilon(\mathbf{x}_k, \mathbf{x}_l) = \begin{cases} 1 & \text{if } \mathrm{dist}(\mathbf{x}_k, \mathbf{x}_l) < 2\sigma \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

0^{th} order term of the expansion of the product of [...]'s in (4):

$$\int d\mathbf{x}_0 \ldots \mathbf{x}_{N-1} = Z(0) = V^N \tag{6}$$

1^{st} order terms of this expansion all include an integral of the form:

$$\int d\mathbf{x}_k d\mathbf{x}_l \Upsilon(\mathbf{x}_k, \mathbf{x}_l) = V \underbrace{\int d\mathbf{x}_k \Upsilon(\mathbf{x}_k, \mathbf{x}_l)}_{\substack{\text{volume ot the} \\ \text{excluded region for } \mathbf{x_k}}} = V \cdot 4\pi\sigma^2 \tag{7}$$

Gathering the term of 0^{th} order and those of 1^{st} order [there are $\frac{1}{2}N(N-1)$ of those]:

$$\begin{aligned} Z(\eta) &= V^N - \frac{1}{2}N(N-1) \cdot \overbrace{V \cdot 4\pi\sigma^2}^{(7)} \cdot \overbrace{V^{N-2}}^{\text{other integrals than the } \int d\mathbf{x}_k d\mathbf{x}_l \text{ of } (7)} && (8) \\ &= V^N\Big(1 - 4\pi\sigma^2 \frac{N(N-1)}{2V} + \ldots\Big) && (9) \\ &\simeq \underbrace{V^N}_{Z(0)}\ \underbrace{\exp\big[-2(N-1)\eta\big]}_{p_{\text{accept}}(\eta)} && (10) \end{aligned}$$

Taking the logarithm:

$$\log Z(\eta) = N \log V - 2(N-1)\eta + \dots \tag{11}$$

Differentiating with respect to the volume V:

$$\frac{\partial \log Z(\eta)}{\partial V} = \frac{N}{V} + 2N(N-1)\pi\sigma^2 \frac{1}{V^2} + \dots \tag{12}$$

Multiplying by V/N:

$$\frac{V}{N}\frac{\partial \log Z(\eta)}{\partial V} = 1 + 2(N-1)\pi\sigma^2 \frac{1}{V} + \dots \tag{13}$$

The expansion to higher order, corresponding to products of two Υ's, three Υ's, etc., when expanding the product of [...]'s in (4), takes the form:

$$\frac{V}{N}\frac{\partial \log Z(\eta)}{\partial V} = 1 + 2(N-1)\pi\sigma^2 \frac{1}{V} + (...)\frac{1}{V^2} + (...)\frac{1}{V^3} + \dots \tag{14}$$