ECE-GY 9243 / ME-GY 7973
Optimal and Learning Control for Robotics

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Exercise Series 2
Linear Quadratic Problems

Exercise 1

Part a)

Definition of Stability: *The origin of a linear dynamical system is (globally asymptotically) stable if for any initial condition, the system dynamics converge to the origin. If there exist trajectories that diverge away from the origin, the fixed point is unstable.*

If the dynamics of the uncontrolled system is:

$$x_{n+1} = Ax_n$$

then the system is said to be Asymptotically stability if the norm of the eigenvalues of A is less than 1. Otherwise it is not stable.

This is due to the fact that at an n^{th} stage the state of the system will be:

$$x_n = A^n x_0$$

If x_0 is not the origin itself, then this system converges to the origin, only if the n^{th} power of the eigenvalues λ_k of A become zero; i.e. for $n \to \infty$ and $\lambda_k^n \to 0$, the value of $|\lambda_k|$ should be less than 1.

Dynamical system 1)

$$A = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

Solving on MATLAB for the eigenvalues of A you get $\lambda_1 = 1$, $\lambda_2 = 0.25 + 1.3919i$, and $\lambda_3 = 0.25 - 1.3919i$. Since λ_1 is not less than 1 the system is unstable. However since $\lambda_1 = 1$, if the system were to initiate at the origin, it will remain at the origin without diverging, given that there is no noise/disturbance of any kind.

Dynamical system 2)

$$A = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & -0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$$

Solving on MATLAB for the eigenvalues of A you get $\lambda_1 = 0.8774$, $\lambda_2 = 0.0613 + 0.3724i$, and $\lambda_3 = 0.0613 - 0.3724i$. The norm of all these eigenvalues is less than 1, the this system is asymptotically stable. Even if the system initiates at any point, it converges to the origin without any control input.

Dynamical system 3)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix}$$

Solving on MATLAB for the eigenvalues of A you get $\lambda_1 = 2$, $\lambda_2 = 0 + \sqrt{2}i$, and $\lambda_3 = 0 - \sqrt{2}i$. Since $\lambda_1 = 2$ the system is unstable. Even if the system initiates at the origin, the system will diverge to an undesired state.

Part b)

Definition of Controllability: *A linear system is controllable if it is possible to drive the system from any state to any other state with an appropriate choice of control inputs in a finite number of steps*

If the dynamics of the controlled discrete linear system is:

$$x_{n+1} = Ax_n + Bu_n$$

then the system is said to be controllable if the matrix $\left[\begin{array}{ccccc} \mathrm{A}^{s-1}B & \mathrm{A}^{s-2}B & \cdots & \mathrm{AB} & \mathrm{B} \end{array}\right]$, where s is the dimension of x, has a full rank.

Dynamical system 1)

$$x_{n+1} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & -2 \\ 4 & 2 & 1 \end{bmatrix} x_n + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_n$$

For this system,

$$\begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 1 & 0.75 & 0 & 0.5 & 0 & 0 \\ -4 & -2 & 0 & -2 & 1 & 0 \\ 2 & -1 & 2 & 1 & 0 & 1 \end{bmatrix}$$

This matrix has a rank 3, full rank; therefore the system is controllable.

Dynamical system 2)

$$x_{n+1} = \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 0 & -0.5 \\ 0.5 & 0.5 & 0.5 \end{bmatrix} x_n + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_n$$

For this system,

$$\begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 & 0 & 0.5 & 0 & 0 \\ -0.25 & -0.25 & 0 & -0.5 & 1 & 0 \\ 0.25 & 0.25 & 0.5 & 0.5 & 0 & 1 \end{bmatrix}$$

This matrix has a rank 3, full rank; therefore the system is controllable.

Dynamical system 3)

$$x_{n+1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 1 & 0 \end{bmatrix} x_n + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u_n$$

For this system,

$$\begin{bmatrix} A^2B & AB & B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{bmatrix}$$

This matrix has a rank 2, not full rank; therefore the system is not controllable.

Part c)

The LQR design for any system that is controllable, whether or not stable, can drive the system to origin and maintain its stability from any initial condition. However, the length of the horizon for the stabilization cannot be determined.

Dynamic System #	**Stability**	**Controllability**
1	Unstable	Controllable
2	Stable	Controllable
3	Unstable	Uncontrollable

Table 1: Summary