

CSE305 Concurrent Programming: N-Body simulation project

[GitHub Repository \(Public\)](#)

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1 Introduction

This document outlines the N-Body simulation project for CSE305, which includes how we approached the problem, our project structure, code breakdown and strategies, and encountered difficulties.

To get started quickly, try typing `make nbody` in the project directory, and generate a basic mp4 file. For information on how to execute the code, check our README file.

For this project, our aim is to be able to simulate systems of bodies with forces interacting with one another in 2D (such as orbiting planets around the solar system with gravitational forces, or with particles interacting with each other with Coulomb forces). This therefore includes two sections:

1. Simulation of the N bodies and their evolving states.
2. Visualization of recorded telemetries into an animated output.

Below is a general view of our project repository file structure.

Project directory summary

```
/
├── src .....Core files
│   ├── core.cpp/hpp .....Used throughout the entire proj
│   ├── simplesimulation.cpp/hpp
│   ├── barneshutt.cpp/hpp
│   ├── particlemesh.cpp/hpp
│   └── particlemeshthread.cpp/hpp
├── makefile .....Project builder
├── main.cpp .....Main execution file
├── mainparticlemesh.cpp .....Aux particle mesh run file
└── report (PDF, TeX)
```

The work has been split as follows. The arrangement is not defining, as we like to help each other out in different parts. Our files have still been mostly separated to properly separate who worked on what functionalities.

1. Martin handles general project structure, core classes, visualization, and the naive and its optimized algorithm implementation.
2. Ziyue handles the Barnes Hutt algorithm.
3. Oscar handles the Particle Mesh algorithm implementation (simple, thread based and cuda implementation)

2 Core components

2.1 Main elements

There are 3 primary classes defined used throughout our project: **Vector**, **Body**, **System**. These are defined in `core.cpp/hpp`.

1. **Vector** holds information on a pair of numbers, and also allows for operations like dot products with other vectors/scalars (as opposed to using `std::pair`).
2. **Body** holds information on a given particle or body, including its mass, coordinates, velocity, and acceleration. It also contains an update method which updates its position and velocity based on acceleration.
3. **System** stores a collection of bodies and its recorded telemetry from the simulations we are going to do. It also contains the visualization function, which takes its recorded telemetry and outputs an animated file.

Note: This organization is heavily inspired from assignment one from CSE306 Computer Graph-ics.

Elaborating more on the telemetry stored within the `System` class, this is stored as a vector of vector of `Vector`. What a mouthful! Our simulations creates different steps/frames. Each step/frame contains the positions of all bodies inside the system (this is a vector of `Vector`). To have the entire telemetry, we have a vector of frames, or the aforementioned data structure for the telemetry.

2.2 Visualization code

To visualize the code, we opt to create a gif animation after the telemetry is recorded, using the ImageMagick/Magick++ libraries. The bulk of the visualization code is located in `core.cpp` as a method for the `System` class. As of present, the visualization function works in two steps. First, it creates the frames for the animation, then merges the frames to an animation.

There are currently two version of the visualize function (`visualize`, `visualize2`). One implementation uses parallelization with `pragma omp`, while the other one doesn't. This distinction is made as some members working on Mac computers were unable to run the `omp` library. Moreover, we allow ourselves to use this parallelization solution as opposed to scheduling threads ourselves as to not spend too much time on the visualization, as this is not the primary focus of the project.

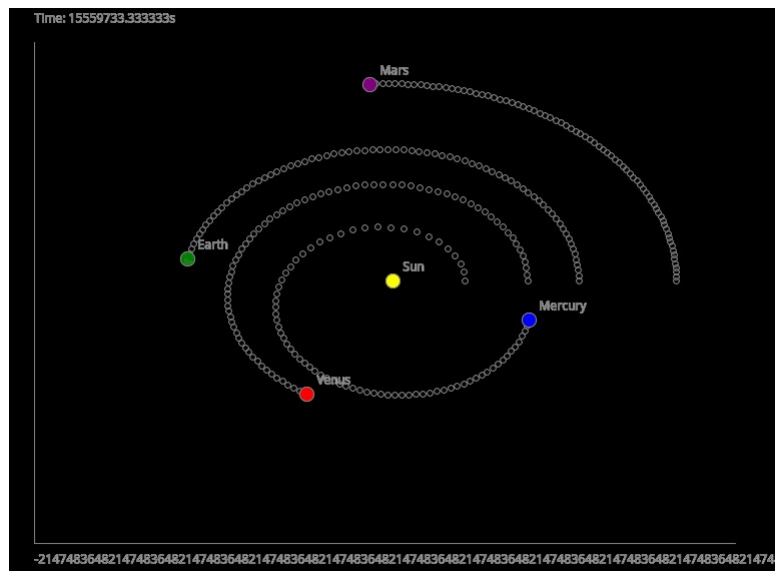


Figure 1: Early version of our visualization function.

2.3 Benchmarking

We aim to design and implement fast algorithms to simulate the gravitational forces between growing numbers of N within our system. Rather than generate N random bodies and perform random simulations of particles with very unexpected behavior, we decided to implement the [asteroid belt](#) within our simulation, increasing the numbers of asteroids for heavier workloads.

This allowed us to make a simple script to generate random asteroids in our `main.cpp` code, while also allowing us to generate cool orbit visualizations of the rocky planets of the solar system.

The asteroids are instantiated randomly, generated with random distances from the sun between 2.2 and 3.2 Astronomical units, random masses between 10^{13} kg and 10^{17} kg, and at random angles between 0 and 2π . Their initial velocities are all the same.

Adding asteroids also gave us the benefit of checking whether our simulations were correct for large numbers of bodies (if they started flying away, we'd know straight away that our simulation was incorrect!)

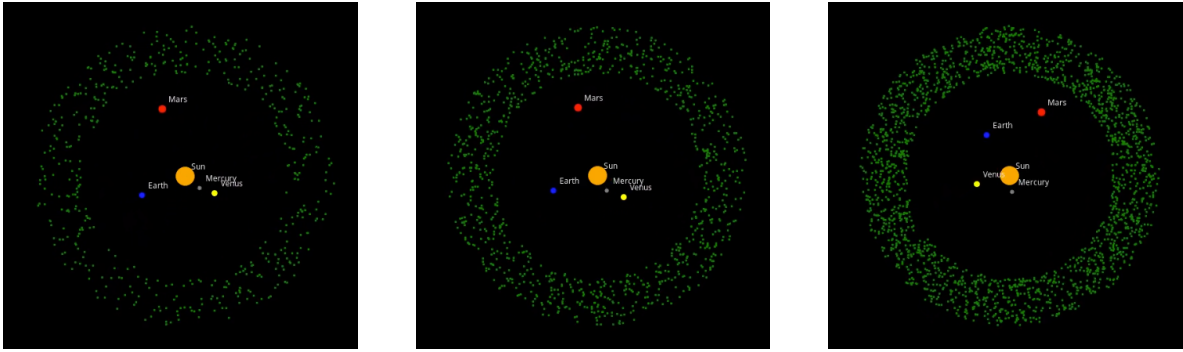


Figure 2: Visualizations with $N = 500, 1000, 2000$ asteroids, respectively.

3 Simple algorithm

Three approaches are proposed inside `simplesimulation.cpp`:

1. Naive implementation: Direct, sequential implementation
2. Parallelized implementation: directly imitating the sequential implementation, using auxiliary functions scheduled through `std::thread`. No atomic variables or mutexes used, but forces are computed twice.
3. Better parallelized implementation: Avoids use of mutexes and atomic variables, while also avoid computing forces twice at the cost of higher memory complexity.

3.1 Algorithms

The pseudocode for each algorithm's implementations is shown inside the appendix.

Simple, sequential algorithm: The first algorithm shows our simple implementation using Newtonian physics to update the positions of all bodies inside of the system. Note that this already manages to avoid computing the forces twice, by avoiding iterating over two same pairs (i, j) inside the loop. This algorithm is effective for small numbers of bodies.

Parallelized algorithm: The second algorithm parallelizes the first one, to some extent. By nature of the simulation, we cannot parallelize the simulation steps, as future states of the system depend on past ones. Therefore, we can only parallelize the inside of a simulation step. Thankfully, there is a lot to parallelize, including the forces calculation and the update step.

The second algorithm does a "dumb" parallelization, that is, does exactly what the sequential version does, just directly parallelized. To parallelize, the main race condition consideration was the computation of the acceleration vectors for each body, as several threads might update the same vector, causing issues.

When designing this algorithm, there were two options, parallelize everything with one auxiliary function using mutexes and atomic variables, or use two functions without using any mutexes. We decide to opt for the second option. In our algorithm, we have an auxiliary function to compute forces, and one to update positions.

Bodies are split up in batches: each thread handles one batch and iterates over each body inside the batch. Inside each body iteration, it computes the force between this body and all other bodies inside the system, and only updates the acceleration vector for that body (updating acceleration vectors for bodies outside the batch would cause race conditions). Keeping this arrangement allows us to avoid race conditions without using mutexes or atomic variables, at the cost of computing everything twice.

Better parallelized algorithm: This third algorithm attempts to improve upon the second one, by having to remove the need to compute everything twice while still parallelizing. For this, we split the algorithm into three auxiliary functions: computing forces (and storing them to a matrix), computing accelerations, and updating the positions. For this, we introduce the force matrix:

$$\begin{pmatrix} f_{00} & f_{01} & f_{02} & \cdots & f_{0N} \\ f_{10} & f_{11} & f_{12} & \cdots & f_{1N} \\ f_{20} & f_{21} & f_{22} & \cdots & f_{2N} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ f_{N0} & f_{N1} & f_{N2} & \cdots & f_{NN} \end{pmatrix}$$

Each cell in the matrix f_{ij} represents the force exerted by Body i on Body j . Now, thanks to the force computation formula, we know that $f_{ii} = 0$ for all i , and $f_{ij} = -f_{ji}$. Therefore, it suffices to only compute either the upper or lower triangle of this matrix to obtain all the accelerations needed. To split the workload, split the columns from 1 to N , and allocate each column to each thread Round Robin style. For example, with 5 threads, we allocate all columns $i \bmod (5)$ to thread i . In each thread, we compute the forces above or below the diagonal, and store only one half of the matrix (thanks to the antisymmetric property).

Once the forces computed, we join all threads and compute the accelerations by summing each column of the matrix. Finally, after synchronising again, we compute the updated positions.

3.2 Results

With these algorithms in hand, we can begin testing. We start with hyper parameter testing. Using the university SSH lada computer, using `nproc` shows that we have 28 useable cores. We look for the number of threads that give the best performance when simulating 500 asteroids.

Number of Threads	Parallel simulation 1	Parallel simulation 2
5	22,398	10,780
7	18,105	10,471
10	15,147	10,643
13	13,502	10,501
16	13,278	12,043
20	13,747	13,249
23	15,300	15,046
27	15,646	16,650

Table 1: Execution time (in milliseconds) for varying numbers thread counts (N Bodies = 500). Tested on Lada SSH computer - Intel Core i7 14700K - lower times is better.

We find most favorable results around 13-14 threads, especially for the second algorithm's performance, where the time saved from parallelization does not make up for the overhead for setting up and synchronising multiple threads. So, using 13 threads, we now test the sequential, parallel and better parallel algorithms for increasing numbers of asteroid bodies simulated.

Number of Bodies	Sequential	Parallel	Better Parallel
0	4	6,405	9,598
10	41	5,250	7,824
50	468	6,567	9,752
100	1,689	5,739	8,151
200	6,573	4,747	5,566
500	38,384	13,565	10,554
1,000	148,840	27,047	22,326
2,000	593,866	125,044	75,797
3,000	1,340,074	270,069	191,026

Table 2: Execution time (in milliseconds) for varying numbers of bodies (N Threads = 13) Tested on Lada SSH computer - Intel Core i7 14700K - lower times is better.

I would also like to include the two tables below having tested my code on my home PC, to further compare with the obtained results from the SSH computer simulations.

Number of Bodies	Sequential	Parallel	Better Parallel
10	63	1,732	2,651
50	785	2,087	2,779
100	2,873	3,047	3,327
150	6,041	4,213	3,944
200	10,624	6,194	4,403
500	65,270	24,611	12,772
1,000	251,897	90,238	57,094
2,000	1,017,985	342,844	212,045
3,000	2,229,012	778,039	480,847

Table 3: Execution time (in milliseconds) for varying numbers of bodies (N Threads = 13) Tested on Lada SSH computer - Intel Core i7 14700K - lower times is better.

From the results, we can make several observations. Overall, we can say that these observations follow our expectations very well. On both computers, we can see that for very small number of bodies, it's better to run sequential as creating and synchronising threads is computationally expensive and gives little return. However, for huge amounts of bodies simulated, the sequential version was extremely slow while very strong improvements was shown by both parallel versions.

Another point of interest is the performance difference between both parallel implementations. Here, we can see that the better parallel performs better asymptotically, as expected. As N grows very large, we can see that indeed, the ratio in computation time between the two is close to 2, given how the better parallel implementation doesn't compute forces twice over.

4 Particle Mesh based algorithm

I have implemented a sequential Particle mesh algorithm, as well as a thread based. In the annexe below is the pseudo-code for the sequential implementation as well as for the thread-based implementation. For both implementations the assignment of the mass for each grid was done using a method called the Nearest Grid Point method. This means that the mass was assigned to the grid in which the particle is situated. Other methods for grid-assignment are known such as the "Cloud-In-Cell" or the "Triangle Shaped Cloud" but showed suboptimal results in parallel implementations. Assigning mass in to a space is later used for the computation of the gravitational potential, which is essential to compute forces between the bodies(particles).

4.1 Time performance of the particle-mesh simulation (sequential and parallel) and benchmarking

The simulation was done on MacBook air with M2 chip containing 8 cores. All simulations were done with grid size = 10, a spatial extent R of 10000 and time increment $dt = 0.2$. Additionally, each simulation includes asteroids with high radius for their orbit (between 5 and 3000 as radius) and the rest being small asteroids with smaller radius for their orbit (between 0.5 and 5 as radius). The reason why I included big differences in radius is to allow for uniformity in the space domain. The code to simulate my particlemesh implementation is included in the `mainparticlemesh.cpp` file and is structured similarly to the `main.cpp`. However, instead of the main mass to be the Sun, the central mass is 900 kg.

Parallel computation was done through parallelizing via the bodies for the mass assignment, the computation of the acceleration as well as for the update step. Each body's mass was assigned in the grid via multiple threads and the computation of their acceleration was also done through parallelization.

The Fast Fourier transforms as well as the computation of the potential was not done in parallel since, assigning a thread would have been very inefficient if multiple bodies were assigned on the same grid. Indeed, locking the cells of each grid to avoid race conditions would have given a performance similar to a sequential one. In order to parallelize the computation of the gravitational potential, one needs smaller grids, and therefore divide the space into a higher value of grids. This would however mean higher number of "low-density" grids (grids with no or few bodies inside them) and therefore additional inefficient computation.

However, one can still implement Fast-Fourier transform in parallel within the library `fftw3`. Unfortunately, the performance observed was suboptimal.

Time performance of the parallel particle-mesh simulation

Number of Bodies	Sequential	Threads = 5	Threads = 7	Threads = 10
4	52	1,661	—	—
5	54	2,209	2,848	4,058
50	109	2,295	2,906	3,263
100	157	2,298	3,034	3,165
1,000	941	3,436	3,800	5,006
2,000	3,013	5,075	5,865	6,595
5,000	7,331	7,576	6,244	6,757
10,000	14,053	18,230	12,306	13,881

Table 4: Execution time (in milliseconds) for varying numbers of bodies and thread counts (grid size = 10).

Looking at performance we see that until 2000 bodies, sequential time-performance is better than for the parallel implementation. However, for higher number of bodies, the parallel implementation with 7 threads displays better performance. 7 threads is particularly optimal as it is very close to the number of cores in the machine on which we simulate (8 cores), allowing optimal use of the CPU.

5 Barnes–Hut Algorithm

The Barnes–Hut algorithm reduces the complexity of the classical $O(N^2)$ N-body force computation to approximately $O(N \log N)$ by hierarchically clustering distant bodies. Our implementation follows these main stages:

Algorithm 1 Barnes–Hut Simulation Outline

Require: System of bodies with masses m_i , positions \vec{r}_i , velocities \vec{v}_i

Require: Time step Δt , number of steps N , opening angle θ

```
1: telemetry  $\leftarrow \emptyset$ 
2: telemetry.append(initial positions)
3: for step  $\leftarrow 0$  to  $N - 1$  do
4:   [minB, maxB]  $\leftarrow$  computeBounds(universe)
5:   root  $\leftarrow$  createRootNode([minB, maxB])
6:   for each body  $b_i$  do
7:     insertBody(root,  $b_i$ )
8:   end for
9:   computeMassDistribution(root)
10:  for each body  $b_i$  do
11:     $\vec{a}_i \leftarrow$  forceOnBody( $b_i$ , root,  $\theta$ )/ $m_i$ 
12:  end for
13:  updateBodies(universe,  $\Delta t$ )
14:  freeQuadTree(root)
15:  telemetry.append(current positions)
16: end for
Ensure: Position history for all bodies stored in telemetry
```

1. **Compute Simulation Bounds.** We first call

- `computeBounds(const System&)` to find an axis-aligned square enclosing all bodies (with a small padding).

2. **Quadtree Construction.** We represent space by a pointer-based quadtree of `QuadNode` objects:

- `createRootNode(const Bounds&)` allocates the root covering the full region.
- `insertBody(QuadNode *, Body&)` recursively subdivides nodes so that each leaf contains at most one body.

3. **Mass–Center Distribution.** A post-order traversal aggregates mass and center-of-mass at every internal node:

- `computeMassDistribution(QuadNode *)` computes `totalMass` and `centerOfMass`.

4. **Force Computation.** For each body b_i , we traverse the tree and apply the opening-angle criterion θ :

- `forceOnBody(const Body&, QuadNode, double theta)` approximates distant clusters as single masses, or recurses into children when closer.

5. **Parallelization (Shared-Memory).** To exploit multicore CPUs, we compute forces in parallel:

- `computeForcesParallel(System&, QuadNode, double theta)` spawns a pool of `std::threads`, partitions the bodies into chunks, and each thread calls `forceOnBody` on its subset.
- Accelerations are stored per-thread and then written back, avoiding fine-grained locks.

6. **Time Integration.** Once all accelerations are known, we update velocities and positions in one pass:

- `updateBodies(System&, double dt)` applies

$$\mathbf{v} \leftarrow \mathbf{v} + \mathbf{a} \Delta t, \quad \mathbf{x} \leftarrow \mathbf{x} + \mathbf{v} \Delta t.$$

7. **Cleanup.** The quadtree is deallocated to avoid memory leaks:

- `freeQuadTree(QuadNode *)` recursively deletes all nodes.

Concurrency Aspects

- **Multithreading in C++:** we illustrate basic shared-memory concurrency by partitioning the force computation across threads. This uses standard `std::thread` and per-thread buffers, avoiding complex locking.
- **Concurrent Data Structures:** insertion into the quadtree could be parallelized by feeding bodies into a thread-safe queue; our current version remains serial for clarity but is structured to allow a concurrent `insertBody` using a mutex per node or a lock-free pointer array.
- **GPU / PRAM Illustration (Optional):** under `USE_CUDA`, we provide `simulateBruteForceGPU(System&,double)` which launches an $O(N^2)$ CUDA kernel. Each GPU thread computes one body's net force, demonstrating the PRAM model in practice.
- **Correctness & Testing:** we compare parallel results to a single-threaded reference on small N , and use tools like ThreadSanitizer to detect data races. Telemetry outputs (positions over time) are also verified for physical invariants (e.g. center-of-mass motion).

A Appendix: Algorithms

A.1 Simple simulation

Algorithm 2 Naive simulation outline

Require: System of bodies with masses m_i , initial positions \vec{r}_i , velocities \vec{v}_i **Require:** Time step Δt , number of steps N

```
for step  $\leftarrow 0$  to  $N - 1$  do
  for each body  $i$  do
     $\vec{a}_i \leftarrow \vec{0}$  ▷ Reset accelerations
  end for
  for  $i \leftarrow 0$  to  $n - 1$  do
    for  $j \leftarrow i + 1$  to  $n - 1$  do
       $\vec{F}_{ij} \leftarrow \text{ComputeGravitationalForce}(\text{body}_i, \text{body}_j)$ 
       $\vec{a}_i \leftarrow \vec{a}_i + \vec{F}_{ij}/m_i$ 
       $\vec{a}_j \leftarrow \vec{a}_j - \vec{F}_{ij}/m_j$ 
    end for
  end for
  for each body  $i$  do
     $\vec{v}_i \leftarrow \vec{v}_i + \vec{a}_i \Delta t$  ▷ Update velocity
     $\vec{r}_i \leftarrow \vec{r}_i + \vec{v}_i \Delta t$  ▷ Update position
  end for
end for
```

Algorithm 3 Optimized Thread-Parallel N-Body Algorithm

Require: System of bodies with masses m_i , positions \vec{r}_i , velocities \vec{v}_i **Require:** Time step Δt , number of steps N , number of threads T

```
 $block\_size \leftarrow n/T$  ▷ Divide work among threads
for step  $\leftarrow 0$  to  $N - 1$  do
  // Phase 1: Parallel Force Computation
  for each block  $B_t$  of bodies (1 block per thread) do
    for each body  $i \in B_t$  do
      Compute forces between body  $i$  and all other bodies in system
      Update  $\vec{a}_i$  accordingly
    end for
  end for
  Synchronize threads
  // Phase 2: Parallel Position Update
  for each block  $B_t$  of bodies (1 block per thread) do
    for each body  $i \in B_t$  do
      Update velocity and position using  $\vec{a}_i$  and  $\Delta t$ 
      Store updated position in telemetry array
    end for
  end for
  Synchronize threads
  Record positions for current timestep
end for
```

Algorithm 4 Optimized N-Body Algorithm with Force Matrix

Require: System of bodies with masses m_i , positions \vec{r}_i , velocities \vec{v}_i

Require: Time step Δt , number of steps N , number of threads T

Initialize $N \times N$ force matrix F with zero vectors

for step $\leftarrow 0$ to $N - 1$ **do**

 // Phase 1: Parallel Force Matrix Computation

for each thread t **do**

for body $i \leftarrow t$ to $N - 1$ with stride T **do**

 ▷ Round-robin distribution

for body $j \leftarrow 0$ to $i - 1$ **do**

$F_{ij} \leftarrow \text{ComputeGravitationalForce}(\text{body}_i, \text{body}_j)$

end for

end for

end for

 Synchronize threads

 // Phase 2: Parallel Acceleration Computation

for each block B_t of bodies **do**

for each body $i \in B_t$ **do**

 Sum forces from upper triangle: $\vec{a}_i \leftarrow \sum_{j < i} F_{ij}$

 Sum forces from lower triangle: $\vec{a}_i \leftarrow \vec{a}_i - \sum_{j > i} F_{ji}$

$\vec{a}_i \leftarrow \vec{a}_i / m_i$

end for

end for

 Synchronize threads

 // Phase 3: Parallel Position Update

for each block B_t of bodies **do**

for each body $i \in B_t$ **do**

 Update velocity and position using \vec{a}_i and Δt

 Store updated position in telemetry array

end for

end for

 Synchronize threads

 Record positions for current timestep

end for

A.2 Barnes Hutt Algorithms

A.3 Particle Mesh Algorithm

Algorithm 5 Particle-Mesh Simulation, using Nearest-Grid-Point (NGP)

Require: System *universe*, time step Δt , grid size N , spatial extent R

```
1: telemetry  $\leftarrow \emptyset$ 
2: telemetry.append(initial positions)
3: boundaries  $\leftarrow [-R, R]$ 
4: Compute cell size  $h \leftarrow \frac{2R}{N}$ 
5: Initialize mass density grid  $M[G][G] \leftarrow 0$ 
6: Initialize potential grid  $\Phi[G][G] \leftarrow 0$ 
7: Initialize FFTW input/output arrays and plans
8: for each time step  $s = 1$  to  $S$  do
9:   Clear mass density grid  $M$ 
10:  for each body  $b$  in universe do
11:     $i \leftarrow \lfloor \frac{x_b - \min_x}{h} \rfloor$ 
12:     $j \leftarrow \lfloor \frac{y_b - \min_y}{h} \rfloor$ 
13:    if  $(i, j)$  in bounds then
14:       $M[i][j] \leftarrow M[i][j] + \text{body.mass}$ 
15:    end if
16:  end for
17:  Copy grid mass to FFTW input array
18:  Compute FFT of mass density using forward FFT
19:  for each  $(i, j)$  in frequency domain do
20:    Compute wave numbers  $(k_x, k_y)$ 
21:    Compute  $k^2 \leftarrow k_x^2 + k_y^2$ 
22:    if  $k^2 > 0$  then
23:      Multiply by  $-\frac{G}{k^2}$  in frequency domain
24:    else
25:      Set value to zero
26:    end if
27:  end for
28:  Compute inverse FFT to obtain gravitational potential
29:  Normalize inverse FFT result and store in potential grid  $\Phi$ 
30:  for each body  $b$  in universe do
31:     $i \leftarrow \lfloor \frac{x_b - \min_x}{h} \rfloor$ 
32:     $j \leftarrow \lfloor \frac{y_b - \min_y}{h} \rfloor$ 
33:    if  $(i, j)$  is valid and not at boundary then
34:      Compute force via central difference of potential:
35:       $\vec{a}_i \leftarrow - \left( \frac{\Phi[i+1][j] - \Phi[i-1][j]}{2h}, \frac{\Phi[i][j+1] - \Phi[i][j-1]}{2h} \right)$ 
36:    else
37:       $\vec{a}_i \leftarrow (0, 0)$ 
38:    end if
39:  end for
40:  for each body  $i$  do
41:     $\vec{v}_i \leftarrow \vec{v}_i + \vec{a}_i \Delta t$   $\triangleright$  Update velocity
42:     $\vec{r}_i \leftarrow \vec{r}_i + \vec{v}_i \Delta t$   $\triangleright$  Update position
43:  end for
44:  telemetry.append(current positions)
45: end for
46: Cleanup FFTW plans and memory
```

Algorithm 6 Parallel Particle-Mesh Simulation using Nearest-Grid-Point (NGP)

Require: System *universe*, time step Δt , grid size N , spatial extent R , number of threads T

```
1: telemetry  $\leftarrow \emptyset$ 
2: telemetry.append(initial positions)
3: boundaries  $\leftarrow [-R, R]$ 
4: Compute cell size  $h \leftarrow \frac{2R}{N}$ 
5: Initialize mass density grid  $M[N][N] \leftarrow 0$ 
6: Initialize potential grid  $\Phi[N][N] \leftarrow 0$ 
7: Initialize FFTW input/output arrays and plans
8: for each time step  $s = 1$  to  $S$  do
9:   Clear mass density grid  $M$ 
10:  Parallel for each thread  $t = 1$  to  $T$ 
11:    Assign a chunk of bodies to thread  $t$ 
12:    for each body  $b$  assigned to thread  $t$  do
13:       $i \leftarrow \lfloor \frac{x_b - \min_x}{h} \rfloor$ 
14:       $j \leftarrow \lfloor \frac{y_b - \min_y}{h} \rfloor$ 
15:      if  $(i, j)$  in bounds then
16:        Lock  $M[i][j]$ 
17:         $M[i][j] \leftarrow M[i][j] + \text{body.mass}$ 
18:        Unlock  $M[i][j]$ 
19:      end if
20:    end for
21:  End parallel for
22:  Copy mass grid to FFTW input array
23:  Compute FFT of mass density using forward FFT
24:  for each  $(i, j)$  in frequency domain do
25:    Compute wave numbers  $(k_x, k_y)$ 
26:     $k^2 \leftarrow k_x^2 + k_y^2$ 
27:    if  $k^2 > 0$  then
28:      Multiply by  $-\frac{G}{k^2}$  in frequency domain
29:    else
30:      Set value to zero
31:    end if
32:  end for
33:  Compute inverse FFT to obtain gravitational potential
34:  Normalize result and store in potential grid  $\Phi$ 
35:  Parallel for each thread  $t = 1$  to  $T$ 
36:    Assign a chunk of bodies to thread  $t$ 
37:    for each body  $b$  assigned to thread  $t$  do
38:       $i \leftarrow \lfloor \frac{x_b - \min_x}{h} \rfloor$ 
39:       $j \leftarrow \lfloor \frac{y_b - \min_y}{h} \rfloor$ 
40:      if  $(i, j)$  valid and not at boundary then
41:        Compute force via central difference:
42:         $\vec{a}_b \leftarrow - \left( \frac{\Phi[i+1][j] - \Phi[i-1][j]}{2h}, \frac{\Phi[i][j+1] - \Phi[i][j-1]}{2h} \right)$ 
43:      else
44:         $\vec{a}_b \leftarrow (0, 0)$ 
45:      end if
46:    end for
47:  End parallel for
48:  Parallel for each thread  $t = 1$  to  $T$ 
49:    Assign a chunk of bodies  $t = 1$  to  $T$ 
50:    for each body  $b$  assigned to thread  $t$  do
51:       $\vec{v}_b \leftarrow \vec{v}_b + \vec{a}_b \Delta t$ 
52:       $\vec{r}_b \leftarrow \vec{r}_b + \vec{v}_b \Delta t$ 
53:    end for
54:  End parallel for
55:  telemetry.append(current positions)
56: end for
57: Cleanup FFTW plans and memory
```