§1 January 20, 2022

§1.1 IMTC Winter 2022 Sample Problems

Problem Set.

I came out with an astounding 2/10 (I focused a little too much on the geo problems lol).

Problem 1.1 (IMTC Winter Sample 2022/1)

For how many positive integers n is the sum of the digits of n is equal to $\left\lfloor \frac{n}{10} \right\rfloor$?

Solution. The floor function just cuts off the last digit. We then check what we can put as the first digits. For 2 digit integers, the only thing we can do is set the one's digit to 0. Therefore, $10, 20, 30, \ldots, 90$ all work.

For 3 digit integers, we see that 109 works. Now we keep increasing the tens digit until we reach 199. Now we know $109, 119, \ldots, 199$ works.

Past this, we would need a number greater than 9 for the condition to be satisfied.

An easy way to count this is to realize that only the non-unit's digit matters to the value. We count from $10, 20, \ldots, 90, 109, 119, \ldots, 199$ to get $\boxed{019}$. □

Remark 1.2. I failed because I counted the final result incorrectly, lol.

Problem 1.3 (IMTC Winter Sample 2022/2)

Distinct real numbers r_1, r_2, r_3 are roots of the equation

$$12r_1x^3 + 72r_2x^2 + 432r_3x = 0 \tag{1.1}$$

Find $r_1^2 + r_2^2 + r_3^2$.

Solution. First divide 1.1 by 12 to get

$$r_1x^3 + 6r_2x^2 + 36r_3x = 0$$

Note that the polynomial has a root that is 0. So one of r_1, r_2, r_3 is 0. If r_1 or r_3 were 0, then we would have non-distinct roots. We conclude that $r_2 = 0$.

Now we plug in r_1 and r_3 into the equation, and set the results equal to each other.

$$r_1^3 + 36r_3 = r_1r_3^2 + 36r_3$$
$$r_1^2 = r_3^2$$

However, note that $r_1 \neq r_3$, so we conclude that $r_1 = -r_3$. Simply use Newton's sums past this:

$$r_1s_1 = 0 \implies s_1 = 0$$

$$r_1s_2 + 6r_2s_1 + 2 \cdot 36r_3 = 0 \implies r_1s_2 - 72r_1 = 0 \implies s_2 = \boxed{072}.$$

□

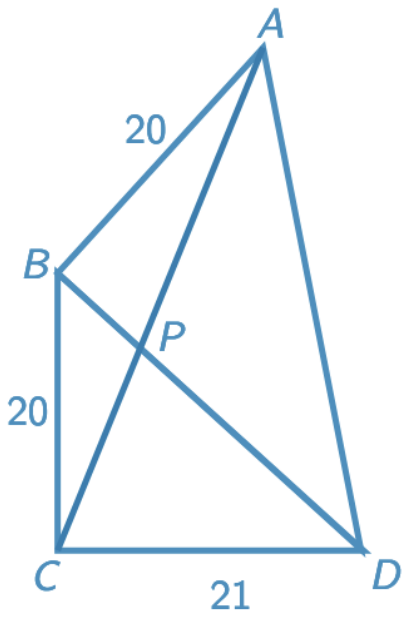

Figure 1: Problem 4

Problem 1.4 (IMTC Winter Sample 2022/4)

Quadrilateral $ABCD$ has side lengths $AB = 20, BC = 20$, and $CD = 21$, as shown. Furthermore, $\angle ABD = \angle BCD = 90°$. If P is the intersection of diagonals AC and BD, find AP^2.

Solution. Note that $BD = 29$. Since $\triangle ABC$ is isosceles, we set $\theta = \angle BAC = \angle BCA$.

$$\angle PCD = 90° - \theta,$$

but also,

$$\angle CPD = \angle APB = 90° - \theta.$$

We conclude that $\triangle PCD$ is isosceles. Then $PD = 21$, and $BP = BD - PD = 29 - 21 = 8$. Finding PA^2 is just Pythagorean theorem.

$$PA^2 = BP^2 + BA^2 = 8^2 + 20^2 = \boxed{464}.$$

□

Problem 1.5 (IMTC Winter Sample 2022/6)

Triangle ABC is inscribed in triangle ADE with right angles at B and D. Given that $CE = 1, CB = 3$, and $AB = 4$, the ratio $\frac{BD}{AD}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $10m + n$.

Solution. The presence of a $3 - 4 - 5$ triangle inspires us to find similar triangles. Firstly, we set $\angle BAD = \varphi$, which implies

$$\angle ABD = 90° - \varphi \implies \angle CBE = 180° - 90° - (90° - \varphi) = \varphi$$

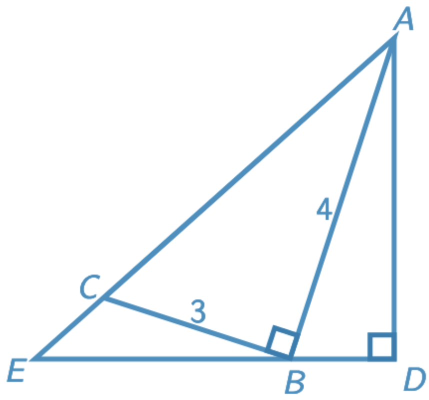

Figure 2: Problem 6

Now we can create a similar triangle! If we drop a perpendicular foot from C to ED, call it P, we come to two important realizations.

$$\triangle CPB \sim \triangle BDA \qquad \text{and} \qquad \triangle CEP \sim \triangle AED$$

Therefore, it suffices to find the ratio between $\frac{CP}{PB}$. Using the fact that $\triangle CEP \sim \triangle AE$, and $CA = 5, CE = 1$, we find that

$$CP = \frac{1}{6}AD.$$

Now using the fact that $\triangle CPB \sim \triangle BDA$, and that their ratio of side lengths is $4:3$,

$$PB = \frac{3}{4}AD.$$

Then our desired answer is

$$\frac{m}{n} = \frac{CP}{PB} = \frac{\frac{1}{6}AD}{\frac{3}{4}AD} = \frac{2}{9} \implies 10m + n = 20 + 9 = \boxed{029}.$$

□

Problem 1.6 (IMTC Winter Sample 2022/8)

Let ABCD be an isosceles trapezoid such that $AB < CD$ and let E be the orthocenter of $\triangle BCD$. Given that $\triangle EAD$ is an equilateral triangle with side length 12, the area of trapezoid $ABCD$ can be expressed as $a + b\sqrt{c}$ for positive integers a, b, c with c having no square prime divisors, find $a + b + c$.

Solution. My general idea was to split the trapezoid into easier triangles and sum their areas. We first draw the altitudes from B and D, and call them P and F respectively. Now we do a lot of angle chasing :D! Let $\angle EBF = \alpha$, then $\angle BEF = 90^\circ - \alpha$, and then since D, E, F are collinear,

$$\angle BEA = 180^\circ - \angle DEA - \angle BEF = 30^\circ + \alpha,$$

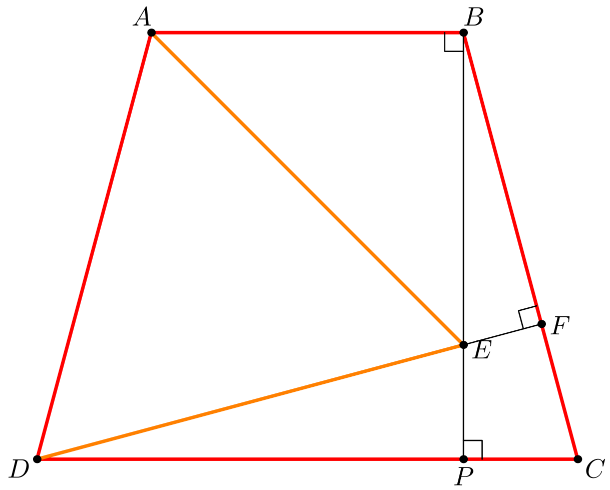

Figure 3: Problem 8

then because $\triangle ABE$ is right, $\angle BAE = 60^\circ - \alpha$. We know that the top two angles are equal because we have an isosceles trapezoid, so

$$\begin{aligned} \angle DAE &= \angle CBA \\ 60^\circ + 60^\circ - \alpha &= 90^\circ + \alpha \\ \implies \alpha &= 15^\circ \end{aligned}$$

But then $\triangle BAE$ is a $45-45-90$ triangle! This gives us pretty much every angle in the diagram, The finishing step before we calculate is to note that $\triangle DPE \cong BPC$. Angle chasing gives that they are both $15-75-90$ triangles, and since $ABCD$ is isosceles, $DE = BC = 12$, so this is true. Then from trig we know that $EP = 12\sin 15^\circ$ and $DP = 12\cos 15^\circ$. To finish, sum up all of our triangles (noting double angle identity $\sin 2\theta = 2\cos\theta\sin\theta$):

$$\begin{aligned} [ABCD] &= [\triangle AED] + [\triangle ABE] + [\triangle BPC] + [\triangle DEP] \\ &= \frac{12^2\sqrt{3}}{4} + \frac{(6\sqrt{2})^2}{2} + \frac{1}{2}(12 \cdot \sin 15^\circ \cdot 12 \cdot \cos 15^\circ) + \frac{1}{2}(12 \cdot \sin 15^\circ \cdot 12 \cdot \cos 15^\circ) \\ &= \frac{12^2\sqrt{3}}{4} + \frac{(6\sqrt{2})^2}{2} + 144 \cdot \sin 15^\circ \cdot \cos 15^\circ \\ &= \frac{12^2\sqrt{3}}{4} + \frac{(6\sqrt{2})^2}{2} + 72 \cdot \sin 30^\circ \\ &= 36\sqrt{3} + 36 + 36 \\ &= 72 + 36\sqrt{3}. \end{aligned}$$

Then

$$a + b + c = 72 + 36 + 3 = \boxed{111}.$$

$\square$