§1 January 19, 2022

Update: It's been a week, I've been working on school work and more AIME prep, but it is mostly geo and constructions. They don't necessarily translate well into latex, especially because of my low skill in Asymptote. Just thought I would do something a little more fun for today. "Tips to be a better problem solver" was the last episode in 3Blue1Brown's online lecture series. The link to the video can be found here. **NOTE: there will be 1 "very purposeful mistake".**

§1.1 Tips to be a better problem solver

Problem 1.1

Prove the inscribed angle theorem.

Tip 1: Always use the defined features of the setup. a.k.a. seek out what we know!

Drawing radii seems useful.

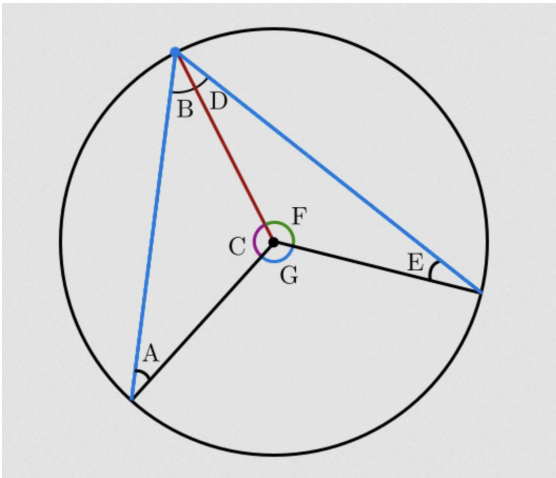

Figure 1: Problem 2

Exercise 1.2 (Problem 2; Sub-problem of 1.1)**.** In the diagram, the dot in the middle is the center of the circle and the 7 labels A through G represent angle values. Which of the following is true?
A. $A+B+D+E=C+F+G$ **B.** $A=B$ and $D=E$ **C.** $C=F$

Solution. Isosceles triangles $\implies$ $\boxed{\textbf{(B)}}$. **A** and **C** are only true in certain circumstances, it is easy to find counterexamples. □

Now we can start working on the proof for the inscribed angle theorem. First we use the sums of triangles:

$$2B+C=\pi$$

$$2D+F=\pi$$

$$C+F+G=2\pi$$

Now we can try to cancel out the C and F:

$$G-2B-2D=0$$

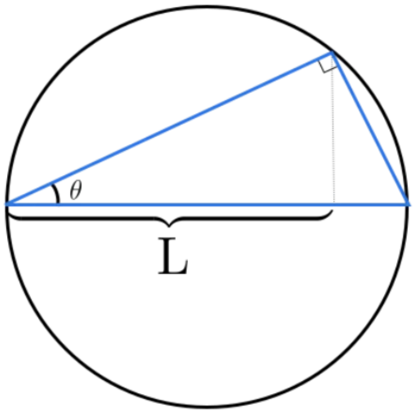

Figure 2: Problem 3

$$\implies B + D = \frac{G}{2}$$

Tip 2: Give things meaningful names!

Assigning the small (θ_S) and large angle (θ_L).

$$\theta_S = \frac{\theta_L}{2}$$

□

Tip 3: Leverage Symmetry.

Tip 4: Describe one object in multiple ways.

Exercise 1.3. Prove that

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}.$$

We make a diagram, taking some unit circle triangle formed by $\cos\theta$ and $\sin\theta$. We can draw a perpendicular to the radius, and then we separate the triangle into lengths $\cos^2\theta$ and $\sin^2\theta$. Now how about we inscribe the triangle, since we want something with 2θ?

Exercise 1.4 (Problem 3; Sub-problem of 1.3)**.** What is the length of L in 2?

Solution. We start by drawing the the radius, which passes through L because of inscribed angle theorem. Then L is split into two parts, one is the radius, the other has the length equal to the radius times $\cos(2\theta)$. The radius is $\frac{1}{2}$ because the hypotenuse of a right triangle is the diameter of the circle by a special case of the inscribed angle theorem. Therefore,

$$L = \boxed{\frac{1 + \cos(2\theta)}{2}}$$

□

Problem 1.5

[Problem 1] Suppose two numbers are chosen at random from the range $[0, 1]$, according to a uniform distribution. Suppose p is the probability that the ratio of the first number to the second rounds down to an even number. What's your best guess for where the value of p falls?

A. $p < 0.3$ **B.** $0.3 \leq p < 0.4$ **C.** $0.4 \leq p < 0.5$
D. $0.5 \leq p < 0.6$ **E.** $0.6 \leq p < 0.7$ **F.** $0.7 \leq p$

Tip 5: Make a drawing.

Plot x vs. y on $[0, 1] \times [0, 1]$.

Tip 6: Ask a simpler variant of the problem.

Exercise 1.6 (Problem 4; Sub-problem of 1.5)**.** What is the probability that $\left\lfloor \frac{x}{y} \right\rfloor = 0$?

Solution. $x < y$ suffices. Therefore any part of the graph over $x = y$. This has area $\frac{1}{2}$. □

Exercise 1.7 (Problem 5; Sub-problem of 1.5)**.** What is the probability that $\left\lfloor \frac{x}{y} \right\rfloor = 2$?

Solution.

$$2 \leq \frac{x}{y} < 3$$
$$2y \leq x < 3y$$

So we have inequalities:

$$2y \leq x \qquad 3y > x$$

Graphing gives us the area between $y = \frac{x}{2}$ and $y = \frac{x}{3}$. □

For the finishing blow, we need find the area of the triangles as we continue for even numbers. We use the bases for every triangle past the first one as the right side. The height therefore always stays as 1.

$$\text{Area} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right)$$

Lemma 1.8

$\ln x$ can be defined as:

$$\ln(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

additionally,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2.$$

Solution. Turn the RHS into an integral by taking the derivative.

$$\int_0^1 1 - x + x^2 + \dots$$

Then note the geometric series $1 - x + x^2 + \cdots = \frac{1}{1+x}$.

$$\int_0^1 \frac{1}{1+x} = \ln(1+x)\Big|_0^1 = \ln 2.$$

□

Now this brings up the hard truth:

Tip 7: Read a lot, and think about problems a lot.

This is probably the best way to get used to solving problems. Reflecting on this, I agree!

Grant concludes that

$$\text{Area} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right) = \frac{1}{2}\ln 2.$$

Tip 8: Always gut check your answer.

The sum evaluates to ≈ 0.35. Wait, but when we check our diagram of all solutions we find that it must be at least $\frac{1}{2}$ (from the large triangle)! Here is the issue: the part inside the parenthesis was not actually $\ln 2$. We are motivated to re-evaluate this as:

$$\text{Area} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \dots\right) = \frac{1}{2}(2 - \ln 2) \approx 0.653.$$

That seems more reasonable! We conclude that the answer is $\boxed{\textbf{(E)}}$.

Tip 9: Know at least a bit of programming.

It helps you view a problem from a new perspective when you write code. It also teaches you how to define stuff. Since computers work in finite memory, you cannot just let it run for an infinite time, there has to be some reasonable way of approximating results (esp. in combinatorics).