§16 January 11, 2021

§16.1 AIME PSET 3

Problem 16.1 (AIME II 2002/10)

While finding the sine of a certain angle, an absent-minded professor failed to notice that his calculator was not in the correct angular mode. He was lucky to get the right answer. The two least positive real values of x for which the sine of x degrees is the same as the sine of x radians are $\frac{m\pi}{n-\pi}$ and $\frac{p\pi}{q+\pi}$, where m, n, p, and q are positive integers. Find $m+n+p+q$.

Solution. We want the solutions to

$$\sin x = \sin \frac{x\pi}{180}.$$

By graphing both of these, we see that the first two intersection points happen on the way down and second time up on $\sin x$ (easy to verify yourself). We want the angles to be equal, so we have the equations

$$\pi - x = \frac{x\pi}{180} \implies x = \frac{180\pi}{\pi + 180}$$
$$x - 2\pi = \frac{x\pi}{180} \implies x = \frac{360\pi}{180 - \pi}$$

The second equation required a few attempts, but you can see that if we subtract 2π we basically go back a cycle, so we just check if the angles are equal. Sum of the numbers is $\boxed{900}$. □

Problem 16.2 (AIME I 2010/7)

Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1,2\},\{2,3\},\{1,3,4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1,2,3,4,5,6,7\}$. Find the remainder when N is divided by 1000.

Solution. Construct a Venn diagram. The first condition basically says that we need 1 element in each intersection in the diagram. The second condition says nothing can be in the intersection of A B and C. Finally, the remaining elements can be in any of the 3 areas in the diagram that are only members of one set, or outside of the circles (not counted). So we have 7, 6, and 5 ways to put the first three elements in the intersections. Then there are 4 choices for each remaining element: either exclusively in set A, B or C, or not in any set.

$$7 \cdot 6 \cdot 5 \cdot 4^4 = \boxed{760}.$$

□

Problem 16.3 (AIME I 2006/13)

For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

Solution. Note that $g(2k) = 2g(k)$, and we can just let $g(1) = 1$. Let's find the first few S_n's.

$$\begin{aligned}
S_1 &= 2(1) = 2 \\
S_2 &= 2(1+2) = 6 \\
S_3 &= 2(1+2+1+4) = 16 \\
S_4 &= 2(1+2+1+4+1+2+1+8) = 40
\end{aligned}$$

I think you can see the pattern in the sums. In fact, it's fairly simple to define a recursion. The second "half" of the sum $S_n = 2(1+2+1+\ldots 2^n)$ is just the first half, copied, and then we add 2^n to the end! So our recursion is

$$S_{n+1} = 2S_n + 2^n.$$

To get an explicit definition, we just continue the recursion; hopefully something clean comes out of it!

$$\begin{aligned}
S_{n+1} &= 2S_n + 2^n \\
&= 2(2S_{n-1}) + 2 \cdot 2^n \\
&= 2(2(2S_{n-2})) + 3 \cdot 2^n
\end{aligned}$$

Notice that as we recur it n times (to get our sum expressed through S_1), we find the explicit formula is

$$S_{n+1} = 2^n S_1 + n2^n = 2^n(n+2).$$

This is *far* easier to check its squareness. Notice that n and $n+2$ have the same parity. If n is even, then it must be square (since 2^n is a square). If n is odd, then $n+2$ must be 2 times a square. However, recall that $n+2$ is also odd! So no n can be odd. The largest even square less than 1000 is 900. Setting $n+2 = 900$, we find that $n+1 = \boxed{899}$ is our answer. □