Collins

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AS & A Level Further Mathematics

Further Mechanics

STUDENT'S BOOK: Worked solutions

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Worked solutions

1 Motion of a projectile

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

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Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 a Horizontal: $10\cos 23° = 9.21$ Vertical: $10\sin 23° = 3.91$

b Horizontal: $12.6\cos 45° = 8.91$ Vertical: $12.6\sin 45° = 8.91$

c Horizontal: $29\cos 65° = 12.3$ Vertical: $29\sin 65° = 26.3$

d Horizontal: $0.2\cos 29° = 0.175$ Vertical: $0.2\sin 29° = 0.0970$

2 a $v = u + at = 2 + 3 \times 4 = 14\,\text{m s}^{-1}$

b $v^2 = u^2 + 2as = 1^2 + 2 \times 2 \times 4 = 17$, $v = 4.12\,\text{m s}^{-1}$

c $u^2 = v^2 - 2as = 25^2 - 2 \times 10 \times 16 = 305$, $u = 17.5\,\text{m s}^{-1}$

d $s = ut + \frac{1}{2}at^2 = 5 \times 5 + 0.5 \times 4 \times 5^2 = 75\,\text{m}$

e $v = u + at = 10 + -2 \times 5 = 0\,\text{m s}^{-1}$

3 a $v^2 = u^2 + 2as = 0^2 + 2 \times 10 \times 10 = 200$, $v = 14.1\,\text{m s}^{-1}$

b $s = ut + \frac{1}{2}at^2$, $10 = 0 \times t + 0.5 \times 10 \times t^2$,

$5t^2 = 10$, $t = 1.41\,\text{s}$

4 a Maximum height, $v = 0$

$v = u + at$, $0 = 2 + -10 \times t$, $t = 0.2\,\text{s}$

b $v^2 = u^2 + 2as$, $0 = 2^2 + 2 \times -10 \times s$, $s = 0.2\,\text{m}$

Exercise 1.1A

1 a $u = 25$, $\theta = 45°$

$y = Ut\sin\theta - \frac{1}{2}gt^2$

$0 = 25t\sin 45° - \frac{1}{2} \times 10 \times t^2$

$t(25\sin 45° - 5t) = 0$

$t = 0$ or $t = 3.536$

Time in air $= 3.54\,\text{s}$

b Range, $x = ut\cos 45°$

$x = 25 \times 3.536 \times \cos 45°$

$x = 62.5\,\text{m}$

2 a $u = 40$, $\theta = 45°$

Time to maximum height

$0 = 40\sin 45° - 10t$

$t = 2.83\,\text{s}$

When $t = 2.828$

$y = 40 \times 2.828\sin 45° - \frac{1}{2} \times 10 \times 2.828^2$

Maximum height $= 40.0\,\text{m}$

b Time of flight $= 2 \times 2.828 = 5.656\,\text{s}$

Range $= 40 \times 5.656 \times \cos 45°$

Range $= 160\,\text{m}$

3 a Time of flight is found by finding t when $y = 0$

$0 = 8.8\sin 42°\, t - \frac{1}{2} \times 10\, t^2$

$t(8.8\sin 42° - 5t) = 0$

$t = 0$ or $t = 1.178$

Tiger will be in air for 1.18 s

b Range $= Ut\cos\theta$

$= 8.8 \times 1.178 \times \cos 42°$

$= 7.70\,\text{m}$

It will cross the river with 2.70 m to spare.

c Motion only occurs in two dimensions, air resistance is negligible, tiger is a point mass

4 a $x = Ut\cos\theta$

$x = 30 \times 3 \times \cos 53°$

$x = 54.2\,\text{m}$

b $y = Ut\sin\theta - \frac{1}{2}gt^2$

$y = 30 \times 3 \times \sin 53° - \frac{1}{2} \times 10 \times 3^2$

$y = 26.9\,\text{m}$

c $v_x = U\cos\theta$

$v_x = 30\cos 53°$

$= 18.05\,\text{m s}^{-1}$

$v_y = U\sin\theta - gt$

$v_y = 30 \times \sin 53° - 10 \times 3$

$= -6.041\,\text{m s}^{-1}$ i.e. downwards

So speed $= \sqrt{18.05^2 + (-6.041)^2} = 19.0\,\text{m s}^{-1}$

5 First find the time of flight:

$0 = 22t \sin\theta - \frac{1}{2} \times 10 \times t^2$

$0 = t(22 \sin\theta - 5t)$

$t = 0$ or $t = \frac{22\sin\theta}{5}$

Then find the range:

$x = Ut\cos\theta$

$12 = 22 \times \frac{22\sin\theta}{5} \times \cos\theta$

$\frac{30}{121} = \sin 2\theta$

$2\theta = 14.36°$

$\theta = 7.18°$

6 a When $t = 4$, $y = 0$

$0 = 25 \times 4 \times \sin\theta - \frac{1}{2} \times 10 \times 4^2$

$100 \sin\theta = 80$

$\sin\theta = \frac{80}{100} = \frac{4}{5}$

b If $\sin\theta = \frac{4}{5}$ then $\cos\theta = \frac{3}{5}$

Range $= Ut\cos\theta$

$x = 25 \times 4 \times \frac{3}{5}$

$= 60$

7 a Maximum height is when $v_y = 0$

$0 = 10 \sin 30° - 10t$

$t = 0.5$ s

Maximum height $= 10 \times 0.5 \times \sin 30° - \frac{1}{2} \times 10 \times 0.5^2$

$= 1.25$ m

b Range $= Ut\cos\theta$

$= 10 \times 0.5 \times \cos 30°$

$= 4.33$ m

c When $y = 1$

$1 = 10 \sin 30° t - 5t^2$

$5t^2 - 5t + 1 = 0$

Using the quadratic formula:

$t = 0.2764$ s and $t = 0.7236$ s

So it is above 1 m for 0.447 seconds.

Exercise 1.2A

1 a $u = 35$, $\theta = 20°$, $h = 45$

Time of flight is found by finding t when $y = 0$

$0 = 35t \sin 20° - \frac{1}{2} \times 10 \times t^2 + 45$

$5t^2 - (35 \sin 20°)t - 45 = 0$

Using the quadratic formula:

$t = -2.033$ or $t = 4.427$

So time of flight $= 4.43$ s

b Range $= 35 \times 4.427 \times \cos 20°$

$= 146$ m

c Motion only occurs in two dimensions, air resistance is negligible, stone is a point mass, gravity is constant

2 a $u = 20$, $\theta = 0°$, $h = 20$

Time of flight is found by finding t when $y = 0$

$0 = 20t \sin 0° - \frac{1}{2} \times 10 \times t^2 + 20$

$t = -2$ or $t = 2$

So time of flight $= 2$ s

b Range $= 20 \times 2 \times \cos 0°$

$= 40$ m

3 a $u = 29.4$, $\theta = 30°$, $h = 49$

Time of flight is found by finding t when $y = 0$

$0 = 29.4t \sin 30° - \frac{1}{2} \times 10 \times t^2 + 49$

$5t^2 - 29.4t \sin 30° - 49 = 0$

Using the quadratic formula:

$t = -1.988$ or $t = 4.928$

So particle will hit ground after 4.93 seconds.

b Range $= 29.4 \times 4.928 \times \cos 30°$

$= 125$ m

c $v_x = U\cos\theta$

$= 29.4 \cos 30°$

$= 25.46\ \text{m s}^{-1}$

$v_y = U\sin\theta - gt$

$= 29.4 \sin 30° - 10 \times 4.93$

$= -34.58\ \text{m s}^{-1}$ (i.e. downwards)

Speed $= \sqrt{25.46^2 + (-34.58)^2} = 42.9\ \text{m s}^{-1}$

d If θ is an angle between the ground and the direction of the particle as it hits the ground then

$\theta = \tan^{-1}\left(\frac{34.58}{25.46}\right) = 53.6°$

4 $u = 20$, $\theta = -10°$, $h = 50$

Time of flight is found by finding t when $y = 0$

$0 = 20t \sin(-10°) - \frac{1}{2} \times 10 \times t^2 + 50$

$5t^2 - 20t \sin(-10°) - 50 = 0$

Using the quadratic formula:

$t = -3.529$ or $t = 2.834$

So particle will hit the ground after 2.83 seconds.

Range $= 20 \times 2.834 \times \cos(-10°)$

$= 55.8$ m

5 a When the stone lands in the sea, $y = 0$

$0 = U \times 3 \times \sin 0° - \frac{1}{2} \times 10 \times 3^2 + h$

$0 = h - 45$

$h = 45\,\text{m}$

b Distance from base of cliff $= \sqrt{73.5^2 - 45^2}$

$= 58.11\,\text{m}$

$58.11 = U \times 3 \times \cos 0°$

$U = 19.4\,\text{m s}^{-1}$

6 Find the time taken for the ball to travel 4 m horizontally:

$4 = 25t\cos\theta$

$t = \dfrac{4}{25\cos\theta}$

The height when at this point must be 2.8 m.

$2.8 = 25 \times \dfrac{4}{25\cos\theta} \times \sin\theta - \dfrac{1}{2} \times 10 \times \left(\dfrac{4}{25\cos\theta}\right)^2$

$2.8 = 4\tan\theta - \dfrac{16}{125}\sec^2\theta$

$350 = 500\tan\theta - 16(1 + \tan^2\theta)$

$16\tan^2\theta - 500\tan\theta + 366 = 0$

Using the quadratic formula:

$\tan\theta = 0.75$ or $\tan\theta = 30.5$

So $\theta = 36.9°$ or $\theta = 88.1°$

Smallest possible angle is 36.9°.

Exercise 1.3A

1 a Time of flight $= \dfrac{2U\sin\theta}{g} = \dfrac{2U\sin 30°}{g} = \dfrac{2U \times \frac{1}{2}}{g} = \dfrac{U}{g}$

b Range $= \dfrac{U^2\sin 2\theta}{g} = \dfrac{U^2\sin 60°}{g} = \dfrac{U^2\frac{\sqrt{3}}{2}}{g} = \dfrac{U^2\sqrt{3}}{2g}$

c Maximum height $= \dfrac{U^2\sin^2\theta}{2g} = \dfrac{U^2\sin^2 30°}{2g}$

$= \dfrac{U^2 \times \left(\frac{1}{2}\right)^2}{2g} = \dfrac{U^2}{8g}$

2 For Mei, $\theta = 135°$ would be launching the particle backwards. Her range is correct but negative. She should have found the smallest value of 2θ for which $\cos 2\theta = 0$ which would have given an acute value of θ.

If $2\theta = 90°$, then $\theta = 45°$ and the range is $\dfrac{U^2}{g}$ m.

For Xing, although the graph of $\sin\theta$ has a maximum value for 90°, the graph of $\sin 2\theta$ has a maximum value for 45°, from which the maximum range is $\dfrac{U^2}{g}$ m.

3 a $x = ut\cos\theta$

$60 = 40t\cos\theta$

$t = \dfrac{3}{2}\sec\theta$

b $y = Ut\sin\theta - \frac{1}{2}gt^2$

$0 = 40t\sin\theta - \frac{1}{2}gt^2$

$80t\sin\theta - gt^2 = 0$

$t(80\sin\theta - gt) = 0$

$t = 0$ or $80\sin\theta - gt = 0$

$t = 0$ or $80\sin\theta = gt$

c $80\sin\theta = gt$ so $t = \dfrac{80\sin\theta}{g}$

Equating this with the expression from part **a**:

$\dfrac{80\sin\theta}{g} = \dfrac{3}{2}\sec\theta$

$160\sin\theta\cos\theta = 3g$

$2\sin\theta\cos\theta = \dfrac{3g}{80}$

$\sin 2\theta = \dfrac{3g}{80}$

4 $y = x\tan\theta - \dfrac{gx^2(1 + \tan^2\theta)}{2U^2}$

$y = 2x - \dfrac{gx^2(1 + 4)}{2U^2}$

$y = \dfrac{4xU^2}{2U^2} - \dfrac{5gx^2}{2U^2}$

$y = \dfrac{x(4U^2 - 5gx)}{2U^2}$

5 a $R = \dfrac{V^2\sin 2\theta}{g}$

$= \dfrac{2V^2\sin\theta\cos\theta}{g}$

$= \dfrac{2V^2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)}{g}$

$= \dfrac{24V^2}{25g}$ m

b Projected at α

$R = \dfrac{V^2\sin 2\alpha}{g}$

Projected at $(90° - \alpha)$

$R = \dfrac{V^2\sin 2(90° - \alpha)}{g}$

$\sin 2(90° - \alpha) = \sin(180° - 2\alpha)$

$\sin(180° - 2\alpha) = \sin(180°)\cos(2\alpha) - \sin(2\alpha)\cos(180°)$

$= \sin 2\alpha$

So $R = \dfrac{V^2\sin 2\alpha}{g}$

These ranges are the same.

6 Starting from the formula $y = x\tan\theta - \dfrac{gx^2(1+\tan^2\theta)}{2U^2}$:

For A, $\theta = 0$, $U = 14\,\text{m s}^{-1}$

$$y = x\tan 0° - \frac{gx^2(1+\tan^2 0°)}{2\times 14^2} = \frac{-10x^2}{392}$$

For B, $\theta = 45°$, $U = 28\,\text{m s}^{-1}$

$$y = x\tan 45° - \frac{gx^2(1+\tan^2 45°)}{2\times 28^2} = x - \frac{20x^2}{1568}$$

Since A is 60 m vertically above B, the particles collide if

$$x - \frac{20x^2}{1568} = \frac{-10x^2}{392} + 60$$

$1568x - 20x^2 = -40x^2 + 94\,080$

$20x^2 + 1568x - 94\,080 = 0$

Using the quadratic formula:

$x = -118$ or $x = 39.8$

So the particles collide after travelling a horizontal distance of 39.8 m.

Exam-style questions

1 a If it reaches its maximum height after 1.5 s, it will return to 1.5 m after 3 s.

$$t = \frac{2v\sin\theta}{g}$$

$$3 = \frac{2v\sin 45°}{10}$$

$v = 21.2$

b Maximum height $= \dfrac{v^2\sin^2\theta}{2g} + 1.5$

$$h = \frac{21.2^2\sin^2 45°}{2\times 10} + 1.5$$

$h = 12.7$

2 a $R = \dfrac{U^2\sin 2\theta}{g}$

$$25 = \frac{U^2\sin 70°}{10}$$

$$U^2 = \frac{250}{\sin 70°}$$

$U = 16.3\,\text{m s}^{-1}$

b When $t = 1$

$v_x = U\cos\theta = 16.31\cos 35° = 13.36$

$v_y = U\sin\theta - gt = 16.31\sin 35° - 10\times 1$
$= -0.6450\,\text{m s}^{-1}$

So speed $= \sqrt{(13.36)^2 + (-0.6450)^2} = 13.4\,\text{m s}^{-1}$

Direction $= \tan^{-1}\dfrac{0.6450}{13.36} = 2.76°$ below the horizontal

3 a $x = Ut\cos\theta$

$10 = 23t\sin\theta$

$$t = \frac{10}{23\sin\theta}$$

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 23t\sin\theta - 5t^2 + 20$

$$0 = 23\left(\frac{10}{23\sin\theta}\right)\sin\theta - 5\left(\frac{10}{23\sin\theta}\right)^2 + 20$$

$$0 = 30 - \frac{500}{529\sin^2\theta}$$

$15\,870\sin^2\theta = 500$

$$\sin\theta = \sqrt{\frac{500}{15\,970}}$$

$\theta = 10.2°$

b $y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 23t\sin 10.19° - 5t^2 + 20$

$5t^2 - (23\sin 10.19°)t - 20 = 0$

Using the quadratic formula:

$t = -1.63$ or $t = 2.45$

So the time of flight is 2.45 s.

4 a $v_x = U\cos\theta = 18\cos 20°$

$v_y = U\sin\theta - gt = 18\sin 20° - 10\times 3 = 18\sin 20° - 30$

Hits ground at angle of $\theta = \tan^{-1}\left(\dfrac{18\sin 20° - 30}{18\cos 20°}\right)$

$= 54.6°$

b speed $= \sqrt{(18\cos 20°)^2 + (18\sin 20° - 30)^2}$

$= 29.2\,\text{m s}^{-1}$

c Displacement after 3 s is the same as the height of the starting point.

$y = Ut\sin\theta - \frac{1}{2}gt^2$

$-h = 18\times 3\times\sin 20° - 0.5\times 10\times 3^2$

$h = 26.5\,\text{m}$

5 a $y = Ut\sin\theta - \frac{1}{2}gt^2$

$2 = (35\sin 38°)t - 5t^2$

$5t^2 - (35\sin 38°)t + 2 = 0$

Using the quadratic formula:

$t = 0.0949$ or $t = 4.215$

The ball hits the platform on the downward phase of its motion so this happens after 4.22 seconds.

b When $t = 4.215$

$v_x = U\cos\theta = 35\cos 38° = 27.58$

$v_y = U\sin\theta - gt = 35\sin 38° - 10\times 4.215$
$= -20.60$ (i.e. downwards)

speed $= \sqrt{27.58^2 + (-20.60)^2} = 34.42\,\text{m s}^{-1}$

The ball strikes the platform at a speed of $34.4\,\text{m s}^{-1}$.

6 **a** Find t when $y = 0$:

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 10 \times t \times \sin 30° - 5t^2 + 19$

$5t^2 - (10\sin 30°)t - 19 = 0$

Using the quadratic formula:

$t = -1.512$ or $t = 2.512$

$v_x = U\cos\theta = 10\cos 30° = 8.660$

$v_y = U\sin\theta - gt = 10\sin 30° - 10 \times 2.512 = -20.12$

speed $= \sqrt{(8.660)^2 + (-20.12)^2} = 21.9\,\text{m s}^{-1}$

Strikes the ground at an angle of $\theta = \tan^{-1}\left(\frac{20.12}{8.660}\right)$

$= 66.7°$

b $x = Ut\cos\theta = 10 \times 2.512 \times \cos 30° = 21.8\,\text{m}$

so $d = 21.8$

7 **a** Range $= \frac{u^2\sin 2\theta}{g}$

$15 = \frac{u^2\sin 2\theta}{10}$

$u^2\sin 2\theta = 150$

$u^2 = \frac{150}{\sin 2\theta}$

Maximum height $= \frac{u^2\sin^2\theta}{2g}$

$15 = \frac{u^2\sin^2\theta}{2 \times 10}$

$u^2\sin^2\theta = 300$

$u^2 = \frac{300}{\sin^2\theta}$

Equating:

$\frac{150}{\sin 2\theta} = \frac{300}{\sin^2\theta}$

$150\sin^2\theta = 300\sin 2\theta$

$150\sin^2\theta - 600\sin\theta\cos\theta = 0$

$150\sin\theta(\sin\theta - 4\cos\theta) = 0$

$150\sin\theta = 0$ or $\sin\theta - 4\cos\theta = 0$

$\theta = 0$

or $\tan\theta = 4$

$\theta = 76.0°$

b $u^2 = \frac{150}{\sin 2\theta}$

$= \frac{150}{\sin 152°}$

$u = 17.9$

8 **a** $y = Ut\sin\theta - \frac{1}{2}gt^2$

$1 = (12\sin 39°)t - 5t^2$

$5t^2 - (12\sin 39°)t + 1 = 0$

Using the quadratic formula:

$t = 0.1467$ or $t = 1.364$

So the ball is above 1 m for 1.22 s.

b When $t = 0.1467$

$v_x = U\cos\theta = 12\cos 39° = 9.326$

$v_y = U\sin\theta - gt = 12\sin 39° - 10 \times 0.1467 = 6.085$

speed $= \sqrt{(9.326)^2 + (6.085)^2} = 11.1\,\text{m s}^{-1}$

$\theta = \tan^{-1}\left(\frac{6.085}{9.326}\right) = 33.1°$

When $y = 1$, the speed is $11.1\,\text{m s}^{-1}$ at an angle of 33.1° above horizontal.

9 **a** Particle A

Range $= \frac{u^2\sin 2\theta}{g}$, $60 = \frac{u^2\sin 2\theta}{g}$, $600 = u^2\sin 2\theta$

Particle B

$x = Ut\cos\theta$, $75 = ut\cos\theta$, $t = \frac{75}{u\cos\theta}$

Time of flight $= \frac{75}{u\cos\theta}$ when $y = 0$.

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = u \times \frac{75}{u\cos\theta} \times \sin\theta - 5\left(\frac{75}{u\cos\theta}\right)^2 + 25$

$0 = 75\tan\theta - \frac{28125}{u^2\cos^2\theta} + 25$

From above, $u^2 = \frac{600}{\sin 2\theta} = \frac{300}{\sin\theta\cos\theta}$

$0 = 75\tan\theta - \frac{28\,125}{\left(\frac{300}{\sin\theta\cos\theta}\right)\cos^2\theta} + 25$

$0 = 75\tan\theta - \frac{28\,125}{\left(\frac{300\cos\theta}{\sin\theta}\right)} + 25$

$0 = 75\tan\theta - \frac{28\,125\tan\theta}{300} + 25$

$0 = 22\,500\tan\theta - 28\,125\tan\theta + 7500$

$5625\tan\theta = 7500$

$\theta = \tan^{-1}\left(\frac{7500}{5625}\right) = 53.13°$

The angle of projection of the particles is 53.1°.

b $600 = u^2\sin 2\theta$,

$u^2 = \frac{600}{\sin 106.26°}$

or, leave it as $\sin(2 \times 53.13)$ so as to not lose the precision

$u = 25\,\text{m s}^{-1}$

The initial speed of the particles is $25\,\text{m s}^{-1}$.

10 a $y = x\tan\theta - \dfrac{gx^2(1+\tan^2\theta)}{2U^2}$

$y = x\tan 30° - \dfrac{gx^2(1+\tan^2 30°)}{2\times 5^2}$

$y = \dfrac{\sqrt{3}}{3}x - \dfrac{10x^2\left(1+\frac{1}{3}\right)}{50}$

$y = \dfrac{\sqrt{3}}{3}x - \dfrac{4}{15}x^2$

b Range is when $y = 0$

$0 = \dfrac{\sqrt{3}}{3}x - \dfrac{4}{15}x^2$

$0 = x\left(\dfrac{\sqrt{3}}{3} - \dfrac{4}{15}x\right)$

So $x = 0$ or $\dfrac{\sqrt{3}}{3} - \dfrac{4}{15}x = 0$

$x = 0$ or $x = \dfrac{5\sqrt{3}}{4}$

So range = 2.17 m

c $y = \dfrac{\sqrt{3}}{3}x - \dfrac{4}{15}x^2 + 5$

11 a $y = 4x - 0.4x^2 + 1$

$\dfrac{dy}{dx} = 4 - 0.8x$

For a turning point, $\dfrac{dy}{dx} = 0$

So $x = 5$

So maximum height = $4(5) - 0.4(5)^2 + 1 = 11$ m.

b When $x = 0$, $y = 1$

So height of launch is 1 m.

c For range, $y = 0$

$0 = 4x - 0.4x^2 + 1$

$0 = 20x - 2x^2 + 5$

$2x^2 - 20x - 5 = 0$

Using the quadratic formula:

$x = -0.244$ or $x = 10.2$

So the range is 10.2 m.

12 a For time of flight, $y = 0$

Substitute into $y = Ut\sin\theta - \frac{1}{2}gt^2$:

$0 = Ut\sin\theta - \frac{1}{2}gt^2$

From which $0 = t(U\sin\theta - \frac{1}{2}gt)$

So $t = 0$ or $U\sin\theta - \frac{1}{2}gt = 0$

$t = 0$ is when the particle is launched so $t = \dfrac{2U\sin\theta}{g}$.

$x = Ut\cos\theta$

$x = U\times\dfrac{2U\sin\theta}{g}\times\cos\theta$

$x = \dfrac{2U^2\sin\theta\cos\theta}{g}$

Range $= \dfrac{U^2\sin 2\theta}{g}$

b Maximum range when $\sin 2\theta = 1$

$2\theta = 90°$

$\theta = 45°$

Maximum range occurs when angle of projection is 45°, so max range $= \dfrac{U^2}{g}$

13 a $R = \dfrac{U^2\sin 2\theta}{g}$

$2 = \dfrac{6.5^2\sin 2\theta}{10}$

$6.5^2\sin 2\theta = 20$

$\sin 2\theta = \dfrac{20}{42.25}$

$2\theta = 28.25$ or $2\theta = 151.8$

$\theta = 14.13$ or $\theta = 75.9$

The two possible angles are 14.1° or 75.9° above the horizontal.

b $t = \dfrac{2U\sin\theta}{g}$

$t = \dfrac{2\times 6.5\times\sin 14.13}{10} = 0.317$

The ball will be in the air for 0.317 s.

14 $y = x\tan\theta - \dfrac{gx^2\sec^2\theta}{40}$

$x = 1$, $y = 0.4$

$0.4 = \tan\theta - \dfrac{10\sec^2\theta}{40}$

$0.4 = \tan\theta - \dfrac{\sec^2\theta}{4}$

$1.6 = 4\tan\theta - (1 + \tan^2\theta)$

$1.6 = 4\tan\theta - 1 - \tan^2\theta$

$\tan^2\theta - 4\tan\theta + 0.6 = 0$

Using the quadratic formula:

$\tan\theta = 0.1561$ or $\tan\theta = 3.844$

$\theta = 8.87°$ or $\theta = 75.4°$

The two possible angles are 8.87° or 75.4° above the horizontal.

15 a Particle P:

$x = 20t\cos 60°$

$= 10t$

When $x = 15$, $t = 1.5$ s

b We need to know the height of P at $t = 1.5$ s

$y = 20t\sin 60° - \frac{1}{2}(10)t^2$

$y = 10\sqrt{3}t - 5t^2$

When $t = 1.5$

$y = 10\sqrt{3}(1.5) - 5(1.5)^2$

$y = 14.73$ m

The particles collide at $t = 1.5$ s at a height of 14.73 m

For Q

$y = Vt - 5t^2$

$14.73 = V(1.5) - 5(1.5)^2$

$V = 17.3$

c Max height of P

$$H = \frac{u^2\sin^2\theta}{2g}$$

$$= \frac{20^2\sin^2 60°}{20}$$

$= 15$ m

16 a Max height of P

$$H = \frac{u^2\sin^2\theta}{2g}$$

$$= \frac{32^2\sin^2 55°}{20}$$

$= 34.3557\ldots = 34.4$ m (to 3 s.f.)

b $\frac{1}{3}$ of max height $= 11.45$ m (to 4 s.f.)

$\frac{2}{3}$ of max height $= 22.90$ m (to 4 s.f.)

For P,

$y = 32t\sin 55° - 5t^2$

When $y = 11.45$,

$11.45 = 32t\sin 55° - 5t^2$

Solving using the quadratic formula gives:

$t = 0.4809$ or $t = 4.762$

When $y = 22.90$,

$22.90 = 32t\sin 55° - 5t^2$

Solving using the quadratic formula gives:

$t = 1.108$ or $t = 4.135$

So total time between this range $= (1.108 - 0.4809) + (4.762 - 4.135) = 1.25$ s (to 3 s.f.)

17 Range of P

$$R = \frac{12^2\sin 60°}{10} = \frac{36\sqrt{3}}{5}$$

Range of Q

$$R = \frac{V^2\sin 90°}{10} = \frac{V^2}{10}$$

Particles land 10 m apart

So $\frac{V^2}{10} = \frac{36\sqrt{3}}{5} + 10$

$V^2 = 224.7$

$V = 15.0\text{ m s}^{-1}$.

18 In order for the ball to pass through the window we need $2 < y < 3.5$ when the ball has travelled 12 m.

Time taken to travel 12 m?

$x = 14t\cos 35°$

$12 = 14t\cos 35°$

$t = 1.046$ s

At this time,

$y = 14 \times 1.046 \times \sin 35° - 5 \times 1.046^2 = 2.93$ m

This means the ball will pass through the window.

19 a $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$ which is the gradient of the curve

Since $y = 5x - x^2$, $\frac{dy}{dx} = 5 - 2x = \tan\theta$

b When $\theta = 10°$,

$5 - 2x = \tan 10°$

$x = \frac{1}{2}(5 - \tan 10°) = 2.412$

When $x = 2.412$, $y = 5(2.412) - (2.412)^2$

$= 6.24$ m

Mathematics in life and work

1 $u = 5$, $\theta = 30°$, with block, $h = 0.6$

Without block:

$$R = \frac{U^2\sin 2\theta}{g} = \frac{5^2\sin 60°}{10} = 2.165\text{ m}$$

With block:

Time of flight, $y = 0$

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = (5\sin 30°)t - 5t^2 + 0.6$

$5t^2 - 2.5t - 0.6 = 0$

Using the quadratic formula:

$t = -0.1772$ or $t = 0.6772$

So time of flight $= 0.677$ s.

$x = ut\cos\theta$

$= 5 \times 0.6772 \times \cos 30°$

$= 2.932$ m

So he can dive 0.767 m further using the starting block.

2 Without the block:

Time of flight = $\dfrac{2U\sin\theta}{g} = \dfrac{2\times 5\times \sin 30°}{10} = 0.5\,\text{s}$

So speed at landing:

$v_x = U\cos\theta$

$= 5\cos 30°$

$= 4.330\,\text{m s}^{-1}$

$v_y = U\sin\theta - gt$

$= 5\sin 30° - 10\times 0.5$

$= -2.5\,\text{m s}^{-1}$

Enters the water at an angle of $\tan^{-1}\dfrac{2.5}{4.33} = 30.0°$

With the block:

Time of flight = 0.6772 s

$v_x = U\cos\theta$

$= 5\cos 30°$

$= 4.330\,\text{m s}^{-1}$

$v_y = U\sin\theta - gt$

$= 5\sin 30° - 10\times 0.6772$

$= -4.272\,\text{m s}^{-1}$

Enters the water at an angle of $\tan^{-1}\dfrac{4.272}{4.330} = 44.6°$

He will enter the water at the smallest angle when diving from the poolside.

3 $u = 5.2$, $\theta = 30°$, $h = 0.6$

Time of flight, $y = 0$

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = (5.2\sin 30°)t - 5t^2 + 0.6$

$5t^2 - 2.6t - 0.6 = 0$

Using the quadratic formula:

$t = -0.1731$ or $t = 0.6931$

So time of flight = 0.693 s

$x = ut\cos\theta$

$= 5.2\times 0.6931\times\cos 30°$

$= 3.121\,\text{m}$

So he will be able to travel an extra 18.9 cm.

2 Equilibrium of a rigid body

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.
All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.
Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $W = mg$

$W = 0.125 \times 10 = 1.25\,\text{N}$

2 Horizontal component $= 9\cos 52° = 5.54\,\text{N}$

Vertical component $= 9\sin 52° = 7.09\,\text{N}$

3 $F = \mu R$

$\mu = \frac{F}{R} = \frac{7.2g}{18g} = 0.4$

4 **a** Resolving vertically, $R + 20\sin 30° = 5g$, $R = 40\,\text{N}$

b $F = \mu R = 0.25 \times 40 = 10\,\text{N}$

c Resultant force $= ma$

$20\cos 30° - 10 = 5a$

$a = 1.46\,\text{m}\,\text{s}^{-2}$

Exercise 2.1A

1 **a** **i** $6 \times 5 = 30\,\text{N}\,\text{m}$ **ii** $3.5 \times 12 = 42\,\text{N}\,\text{m}$

b **i** $28 \div 4 = 7\,\text{N}$ **ii** $15 \div 4 = 3.75\,\text{N}$

2 Clockwise: $12 \times 4 = 48\,\text{N}\,\text{m}$

Anticlockwise: $17 \times 3 = 51\,\text{N}\,\text{m}$

Overall turning moment $= 51 - 48 = 3\,\text{N}\,\text{m}$ anticlockwise

3 **a** $3 \times 9 = 27\,\text{N}\,\text{m}$ clockwise

b $5 \times 4 + 3 \times 5 = 35\,\text{N}\,\text{m}$ clockwise

c $5 \times 9 = 45\,\text{N}\,\text{m}$ clockwise

d $5 \times 15 - 3 \times 6 = 57\,\text{N}\,\text{m}$ clockwise

4 **a** Clockwise: $7 \times 3 = 21\,\text{N}\,\text{m}$

Anticlockwise: $7 \times 4 = 28\,\text{N}\,\text{m}$

Overall turning moment $= 28 - 21 = 7\,\text{N}\,\text{m}$ anticlockwise

b $7 \times 3 + 7 \times 4 = 49\,\text{N}\,\text{m}$ clockwise

c Clockwise: $7 \times 3 = 21\,\text{N}\,\text{m}$

Anticlockwise: $7 \times 4 = 28\,\text{N}\,\text{m}$

Overall turning moment $= 28 - 21 = 7\,\text{N}\,\text{m}$ anticlockwise

d $7 \times 3 + 7 \times 2\sin 45° = 30.9\,\text{N}\,\text{m}$ anticlockwise

5 Both methods are valid. Both methods lead to an answer of the form $Fd\sin\theta$. Chen's method is more efficient in this example.

6 **a** Clockwise: $8 \times 3\sin 30° = 12\,\text{N}\,\text{m}$

Anticlockwise: $10 \times 3\sin 60° = 25.98\,\text{N}\,\text{m}$

Overall turning moment $= 25.98 - 12 = 14.0\,\text{N}\,\text{m}$ anticlockwise

b The turning moment will still be anticlockwise but smaller ($11.0\,\text{N}\,\text{m}$).

7 **a** $13 \times 4 = 6 \times 5 + 11d$

$d = 2$

b $(15 - 11) \times 2 = 8\,\text{N}\,\text{m}$ anticlockwise

8 Let x m be distance of the man from the centre of the seesaw.

$60g \times x = 36g \times 2$

$x = 1.2\,\text{m}$

9 **a** Taking moments anticlockwise

$7 \times 4 + 9 \times 6 = 82\,\text{N}\,\text{m}$

Turning moment $= 82\,\text{N}\,\text{m}$ anticlockwise

b **i** Let force $= F\,\text{N}$

$2 \times F = 82$

$F = 41\,\text{N}$

F acts vertically downwards.

ii $R + 9 = 7 + 41$

$R = 39\,\text{N}$

10 The value of U is correct but the working is incorrect. She has taking moments incorrectly with cos and sin.

She has then also evaluated the compound angle formulae incorrectly.

a Resolving vertically:

$T\cos\alpha = U\cos\alpha$

$T = U$

$q = U$

b Taking moments about O:

$T \times 8\sin\alpha = U \times 3\cos\alpha$

$q \times 8\sin\alpha = q \times 3\cos\alpha$

$\tan\alpha = \frac{3}{8}$

$\alpha = \tan^{-1}\left(\frac{3}{8}\right) = 20.6°$

c Resolving horizontally:

$V = T\sin\alpha + U\sin\alpha$

$V = q\sin\alpha + q\sin\alpha$

$V = 2q\sin\alpha$

$V = 2q \times \frac{3}{\sqrt{73}} = \frac{6q\sqrt{73}}{73}$

Exercise 2.2A

1 **a** $3.7\bar{x} = 0.2 \times 2 + 0.7 \times 1 + 0.8 \times 0.5 + 1.2 \times 0.2$

$3.7\bar{x} = 1.74$

$\bar{x} = 0.470\,\text{m from } A$

b $9\bar{x} = 0.9 \times 2 + 2.1 \times 3 + 2.2 \times 4$

$9\bar{x} = 16.9$

$\bar{x} = 1.88\,\text{m from } A$

c $4m\bar{x} = 0.3 \times m + 1 \times m + 1.2 \times m + 1.3 \times m$

$4m\bar{x} = 3.8m$

$\bar{x} = 0.95\,\text{m from } A$

d $10.5m\bar{x} = 0.2 \times 2m + 0.7 \times 6m + 0.8 \times 2m + 1.2 \times 0.5m$

$10.5m\bar{x} = 6.8m$

$\bar{x} = 0.648\,\text{cm from } A$

2 $12\bar{x} = 0 \times 2 + 0.4 \times 2 + 0.8 \times 2 + 1.2 \times 2 + 1.6 \times 2 + 2 \times 2$

$12\bar{x} = 12$

$\bar{x} = 1\,\text{m from } A$

3 $(3m + 3.2) \times 1.2 = 0.2 \times m + 1.3 \times 2m + 2.5 \times 3.2$

$3.6m + 3.84 = 2.8m + 8$

$0.8m = 4.16$

$m = 5.2\,\text{kg}$

4 $450 \times 40 = 200 \times 10 + 100 \times 30 + 150 \times d$

$18\,000 = 150d + 5000$

$150d = 13\,000$

$d = 86.7\,\text{cm from } A$

5 $(3m + 3) \times 6 = m \times 1 + 2m \times 7 + 3 \times 9$

$18m + 18 = 15m + 27$

$3m = 9$

$m = 3\,\text{kg}$

Exercise 2.2B

1 **a** $8\bar{x} = 3 \times 4 + 5 \times 5$ $\quad 8\bar{y} = 3 \times 2 + 5 \times 5$

$\bar{x} = 4.625$ $\quad \bar{y} = 3.875$

So $(\bar{x}, \bar{y}) = (4.63, 3.88)$

b $10\bar{x} = 2 \times 5 + 5 \times 6 + 3 \times 5$

$10\bar{y} = 2 \times 3 + 5 \times 2 + 3 \times 0$

$\bar{x} = 5.5$ $\quad \bar{y} = 1.6$

So $(\bar{x}, \bar{y}) = (5.5, 1.6)$

c $7.6\bar{x} = 1.4 \times 5 + 4.3 \times 2 + 1.9 \times 1$

$7.6\bar{y} = 1.4 \times 1 + 4.3 \times 6 + 1.9 \times 1$

$\bar{x} = 2.30$ $\quad \bar{y} = 3.83$

So $(\bar{x}, \bar{y}) = (2.30, 3.83)$

d $680\bar{x} = 230 \times 2 + 342 \times 4 + 108 \times 9$

$\bar{x} = 4.12$

$680\bar{y} = 230 \times 3 + 342 \times 1 + 108 \times 2$

$\bar{y} = 1.84$

So $(\bar{x}, \bar{y}) = (4.12, 1.84)$

2 $12.5\bar{x} = 0 \times 5 + 4 \times 2.5 + 4 \times 2 + 0 \times 3$

$\bar{x} = 1.44$

$12.5\bar{y} = 0 \times 5 + 0 \times 2.5 + 2.5 \times 2 + 2.5 \times 3$

$\bar{y} = 1$

So $(\bar{x}, \bar{y}) = (1.44, 1)$

3 $4m\bar{x} = 2 \times m + 6 \times m + 2 \times m + 6 \times m$

$\bar{x} = 4$

$4m\bar{y} = 5 \times m + 5 \times m + 8 \times m + 8 \times m$

$\bar{y} = 6.5$

So $(\bar{x}, \bar{y}) = (4, 6.5)$

4 $(10 + m) \times 4 = 2 \times 1 + 3 \times 8 + 5 \times 4 + 2 \times m$

$40 + 4m = 46 + 2m$

$m = 3\,\text{kg}$

$13\bar{y} = 2 \times 1 + 3 \times 1 + 5 \times 5 + 3 \times 3$

$\bar{y} = 3$

5 $(5 + m + n) \times 1.6 = 3 \times 0 + 4 \times n + 4 \times 2 + 0 \times m$

$8 + 1.6m + 1.6n = 4n + 8$

$2.4n - 1.6m = 0$ $\quad$ 1

$(5 + m + n) \times 2.5 = 3 \times 0 + n \times 0 + 5 \times 2 + 5 \times m$

$12.5 + 2.5m + 2.5n = 10 + 5m$

$2.5n - 2.5m = -2.5$ $\quad$ 2

Solving simultaneously:

Substituting $m = 1.5n$ into 2

gives $n = 2$ and $m = 3$

Exercise 2.3A

1 **a**

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Large rectangle	54	6.5	5
Small rectangle	12	9	9.5
Lamina	66	$\bar{x}$	$\bar{y}$

$66\bar{x} = 54 \times 6.5 + 12 \times 9$ $\quad$ $66\bar{y} = 54 \times 5 + 12 \times 9.5$

$\bar{x} = 6.95$ (to 3 s.f.) $\quad$ $\bar{y} = 5.81$ (to 3 s.f.)

b

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Large rectangle	60	9	4.5
Medium rectangle	21	13.5	10.5
Small rectangle	15	4.5	9.5
Lamina	96	$\bar{x}$	$\bar{y}$

$96\bar{x} = 60 \times 9 + 21 \times 13.5 + 15 \times 4.5$

$96\bar{y} = 60 \times 4.5 + 21 \times 10.5 + 15 \times 9.5$

$\bar{x} = 9.28$ (to 3 s.f.) $\quad$ $\bar{y} = 6.59$ (to 3 s.f.)

c

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Large rectangle	90	9	8.5
Medium rectangle	10	11	8.5
Small rectangle	6	6	10.5
Lamina	74	$\bar{x}$	$\bar{y}$

$74\bar{x} = 90 \times 9 - 10 \times 11 - 6 \times 6$

$74\bar{y} = 90 \times 8.5 - 10 \times 8.5 - 6 \times 10.5$

$\bar{x} = 8.97$ (to 3 s.f.) $\quad$ $\bar{y} = 8.34$ (to 3 s.f.)

d

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Rectangle	60	8	7
Triangle	15	8	11
Lamina	75	$\bar{x}$	$\bar{y}$

$75\bar{x} = 60 \times 8 + 15 \times 8$ $\quad$ $75\bar{y} = 60 \times 7 + 15 \times 11$

$\bar{x} = 8$ $\quad$ $\bar{y} = 7.8$

e

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Large rectangle	72	9	7
Small rectangle	18	4.5	13
Circle	π	10	8
Lamina	$90 - \pi$	$\bar{x}$	$\bar{y}$

$(90 - \pi)\bar{x} = 72 \times 9 + 18 \times 4.5 - 10 \times \pi$

$(90 - \pi)\bar{y} = 72 \times 7 + 18 \times 13 - 8 \times \pi$

$\bar{x} = 8.03$ (to 3 s.f.) $\quad$ $\bar{y} = 8.21$ (to 3 s.f.)

f

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Rhombus	32	8	10
Rectangle	4	8	10.5
Lamina	28	$\bar{x}$	$\bar{y}$

$28\bar{x} = 32 \times 8 - 4 \times 8$ $\quad$ $28\bar{y} = 32 \times 10 - 4 \times 10.5$

$\bar{x} = 8$ $\quad$ $\bar{y} = 9.93$ (to 3 s.f.)

2 From diagram, $\bar{x} = 6$

Using formula with $r = 2$, $\alpha = \frac{\pi}{2}$

$$\bar{y} = 2 + \frac{2 \times 2\sin\frac{\pi}{2}}{\frac{3\pi}{2}} = 2.85$$

3 Distance of centre of mass from (4, 2)

$$= \frac{2 \times 4 \times \sin\frac{\pi}{4}}{3 \times \frac{\pi}{4}} = 2.401$$

So $\bar{x} = 4 + 2.401\cos\frac{\pi}{4} = 5.70$

and $\bar{y} = 2 + 2.401\sin\frac{\pi}{4} = 3.70$

4 Using formula with $r = 10$, $\alpha = \frac{\pi}{12}$

Distance from O to centre of mass $= \dfrac{2 \times 10\sin\frac{\pi}{12}}{\frac{3\pi}{12}}$

$= 6.59$ cm

5 Length of median $= \sqrt{3^2 - 1.5^2} = 2.60$

So centre of mass lies $\frac{2}{3} \times 2.60 = 1.73$ cm along each median from the vertex.

Exercise 2.4A

1 $\frac{3}{8}r = \frac{3}{8} \times 4 = 1.5$

The centre of mass lies 1.5 cm above the centre of the base of the hemisphere.

2 a $\frac{3}{4}h = \frac{3}{4} \times 16 = 12$ so centre of mass lies 12 cm below the vertex. This means it is 4 cm above the base.

b $\frac{3}{4}h = \frac{3}{4} \times 16 = 12$ so centre of mass lies 12 cm below the vertex. This means it is 4 cm above the base.

c The centre of mass of a solid cone does not depend on its radius, only its height.

3 $\bar{x}$ and $\bar{y}$ will lie directly above the centre of the base of the cuboid.

Shape	Volume	z-coordinate of centre of mass
Cuboid	128	4
Pyramid	32	$8 + \frac{1}{4} \times 6 = 9.5$
Solid	160	$\bar{z}$

$160\bar{z} = 128 \times 4 + 32 \times 9.5$

$\bar{z} = 5.1$

The centre of mass is 5.1 cm above the base.

4 $\bar{x} = 0$ and $\bar{y} = 0$ due to the symmetry of the solid.

Shape	Volume	z-coordinate of centre of mass
Cone	261.8	$\frac{3}{4} \times 10 = 7.5$
Hemisphere	261.8	$10 + \frac{3}{8} \times 5 = 11.88$
Solid	523.6	$\bar{z}$

$523.6\bar{z} = 261.8 \times 7.5 + 261.8 \times 11.88$

$\bar{z} = 9.69$

5 $\frac{3}{8}r = 6$ so $r = 16$ cm

6 Assuming the origin is at the bottom left of the cross-section.

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Large rectangle	1500	15	25
Small rectangle	750	45	12.5
Cross-section	2250	$\bar{x}$	$\bar{y}$

$2250\bar{x} = 1500 \times 15 + 750 \times 45$

$2250\bar{y} = 1500 \times 25 + 750 \times 12.5$

$\bar{x} = 25$ $\bar{y} = 20.8$ (to 3 s.f.)

$\bar{z} = 25$ (due to the symmetry of the step)

7 Assuming the centre of the flat base of the cylinder is $(y, z) = (0, 0)$

Shape	Volume	x-coordinate of centre of mass
Cylinder	$\pi r^2\, r = \pi r^3$	$\frac{r}{2}$
Hemisphere	$\frac{2}{3}\pi r^3$	$2r + \frac{3}{8}r = \frac{19}{8}r$
Solid	$\frac{5}{3}\pi r^3$	$\bar{x}$

$$\frac{5}{3}\pi r^3\bar{x} = \pi r^3 \times \frac{r}{2} + \frac{2}{3}\pi r^3 \times \frac{19}{8}r$$

$$\frac{5}{3}\pi r^3\bar{x} = \frac{25}{12}\pi r^4$$

$$\bar{x} = \frac{13}{8}r$$

The centre of mass is $\frac{13}{8}r$ from the end of the sculpture.

8 $\bar{x} = 12.5$ and $\bar{y} = 12.5$ because of symmetry.

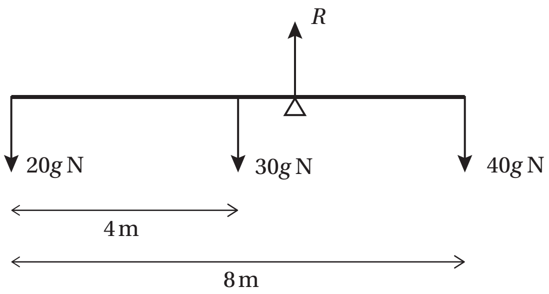
Shape	Area	z-coordinate of centre of mass
Large square	625	12.5
Small square	25	2.5
Cross-section	600	$\bar{z}$

$600\bar{z} = 625 \times 12.5 - 25 \times 22.5$

$\bar{z} = 12.1$ (to 3 s.f.)

Exercise 2.5A

1 a



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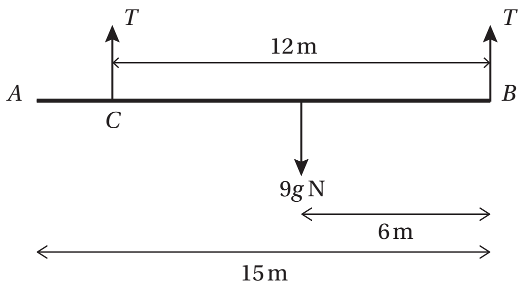

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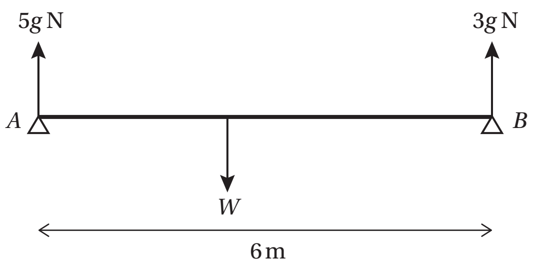

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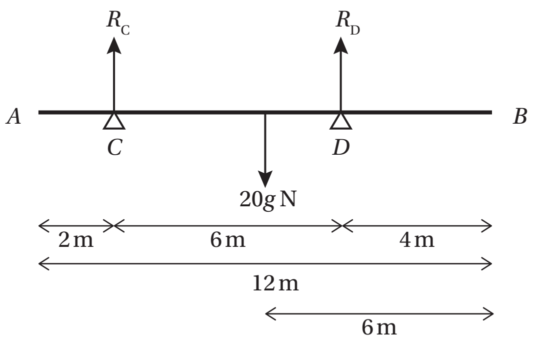

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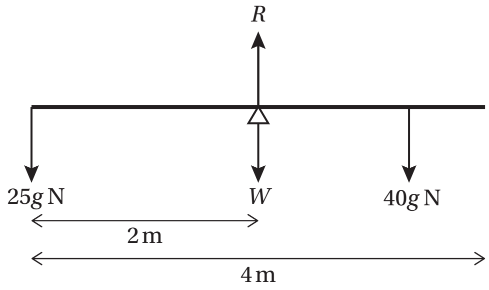

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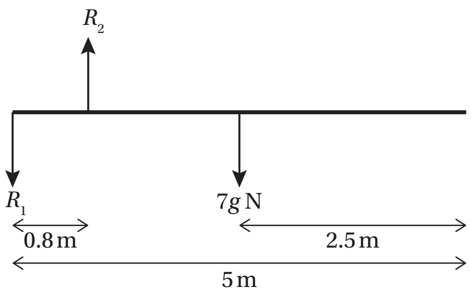


2 a It is rigid, straight and has uniform mass.

b Moments about C:

$50g \times 3 + 60g \times 0.5 = 30g \times 2 + R_D \times 2$

$R_D = 60g = 600\,\text{N}$

Resolving vertically:

$R_C + R_D = 30g + 60g + 50g$

$R_C + 60g = 140g$

$R_C = 80g = 800\,\text{N}$

3 a Let x m be the distance of the centre of mass from A.

Moments about C:

$2g \times 4 + 20g(x - 1) = 30g \times 1$

$20(x - 1) = 22$

$x - 1 = 1.1$

$x = 2.1\,\text{m}$

b Resolving vertically:

$R_C = 30g + 20g + 2g$

$R_C = 52g = 520\,\text{N}$

4 Let A be the end of the pole that he is holding and let B represent the position of his other hand.
Moments about A:

$5g \times 2 = R_B \times 0.8$

$R_B = 12.5g = 125\,\text{N}$

Resolving vertically:

R_A acts in the same direction as W (also described in the solution to 1f).

$R_A + 5g = 12.5g$

$R_A = 75\,\text{N}$

5 **a** Let $JL = x$ m

Moments about L:

$40g(12 - x) = 24gx + 16g(x - 6)$

$480 - 40x = 24x + 16x - 96$

$576 = 80x$

$x = 7.2$ m

b Resolving vertically:

$R_L = 24g + 16g + 40g$

$R_L = 80g = 800$ N

c The masses are concentrated at a single point and the centre of mass of the beam is at its midpoint.

6 **a** Let string be attached at A.

Moments about A:

$7g(0.375 - d) = 5gd + 3g(0.375 + d)$

$2.625 - 7d = 5d + 1.125 + 3d$

$1.5 = 15d$

$d = \frac{1}{10}$ m or 10 cm

b $T = 3g + 5g + 7g$

$T = 15g = 150$ N

7 **a** Resolving vertically:

$R_A + R_B = 64g + 16g$

$R_A + 3R_A = 80g$

$4R_A = 80g$

$R_A = 20g$

$R_B = 20g \times 3 = 60g = 600$ N

b Let $AB = x$

Moments about A:

$64g \times 3 + 16g \times 6 = 60gx$

$288 = 60x$

$x = 4.8$ m

c Resolving vertically:

$R_A + R_B = 64g + Mg$

$R_A + 5R_A = 64g + Mg$

$6R_A = 64g + Mg$

$R_A = \frac{1}{6}(64g + Mg)$

Moments about B:

$Mg \times 1.2 + R_A \times 4.8 = 64g \times 1.8$

$Mg \times 1.2 + \frac{1}{6}(64g + Mg) \times 4.8 = 64g \times 1.8$

$7.2M + 307.2 + 4.8M = 691.2$

$12M = 384$

$M = 32$

d The boulders are particles with their masses concentrated at a single point so that the perpendicular distances are exact.

8 **a** Let the weight of the rod be W and the distance of the centre of mass of the rod from C be x m.

For the 10 kg object:

Resolving vertically: $W + 10g = R_{A1} + R_{B1} = \frac{5}{4}R_{B1}$

$R_{B1} = \frac{4}{5}(W + 10g)$

Moments about A:

$W(x - 3) + 10g \times 7 = R_{B1} \times 5$

$W(x - 3) + 70g = \frac{4}{5}(W + 10g) \times 5$

$W(x - 3) + 70g = 4W + 40g$

$W(x - 3) = 4W - 30g$

For the 5kg object:

Resolving vertically: $W + 5g = R_{A2} + R_{B2} = \frac{7}{3}R_{B2}$

$R_{B2} = \frac{3}{7}(W + 5g)$

Moments about A:

$W(x - 3) = R_{B2} \times 5 + 5g \times 3$

$W(x - 3) = \frac{3}{7}(W + 5g) \times 5 + 15g$

$W(x - 3) = \frac{15}{7}(W + 5g) + 15g$

Hence $4W - 30g = \frac{15}{7}(W + 5g) + 15g$

$28W - 210g = 15W + 75g + 105g$

$13W = 390g$

$W = 30g = 300$ N

The weight of the rod is 300 N.

b $W(x - 3) = 4W - 30g$

$30g(x - 3) = 120g - 30g$

$30g(x - 3) = 90g$

$x - 3 = 3$

$x = 6$

The distance of the centre of mass from C is 6 m.

Exercise 2.5B

1 Since $T_K = 0$, the position of K is not required. Let the position of the particle from A be x m.

Moments about J:

$6g \times d = 9g(d - x)$

$6gd = 9gd - 9gx$

$9x = 3d$

$x = \frac{1}{3}d$

The particle is $\frac{1}{3}d$ m from A.

2 Let the distance of the centre of mass from A be x m.

$R_C = 0$

Moments about D:

$4g \times 0.7 = 10g(1.4 - x)$

$2.8 = 14 - 10x$

$10x = 11.2$

$x = 1.12$

The distance of the centre of mass from A is 1.12 m.

3 **a** Since the rod is on the point of tilting about D, the reaction force at E is 0 N.

b Moments about D:

$mg \times 6 + 70g \times 1.5 = 50g \times 3$

$6m + 105 = 150$

$6m = 45$

$m = 7.5$

4 **a** Moments about P:

$20g \times 2 = R_Q \times 5$

$R_Q = 8g = 80$ N

Resolving vertically:

$R_P + R_Q = 20g$

$R_P + 8g = 20g$

$R_P = 12g = 120$ N

b $R_P = 0$

Moments about Q:

$Mg \times 1 = 20g \times 3$

$M = 60$

5 **a** $R_V = 0$

Moments about U:

$5g \times 8 + W \times 3 = 70g \times 1$

$W = 10g = 100$

b Let the mass of the added load be M kg.

$R_U = 0$

Moments about V:

$(M + 5)g \times 3 = 10g \times 2 + 70g \times 6$

$3M + 15 = 20 + 420$

$3M = 425$

$M = \frac{425}{3}$

The mass of the load that has been added is $\frac{425}{3}$ kg.

6 Let x m be the distance of the rod's centre of mass from A.

Before block placed:

Moments about E:

$Mg(x - 2) = 18g \times 3$

$M(x - 2) = 54$

$Mx - 2M = 54$

$Mx = 2M + 54$

After block placed, $R_E = 0$.

Moments about F:

$72g \times 1.5 = Mg(5 - x)$

$108 = M(5 - x)$

$108 = 5M - Mx$

$Mx = 5M - 108$

Hence $2M + 54 = 5M - 108$

$162 = 3M$

$M = 54$

Substituting $M = 54$ into $Mx = 2M + 54$:

$54x = 108 + 54$

$54x = 162$

$x = 3$

The distance of the rod's centre of mass from A is 3 m.

Exercise 2.6A

1 Let the reaction force acting between the ladder and the floor be R_F, the friction force acting between the ladder and the floor be F_F and the reaction force acting between the ladder and the wall be R_W.

a Resolving vertically:

$R_F = Mg$ N

b $F = \mu R$

$F_F = 0.4R_F = 0.4Mg$ N

c Resolving horizontally:

$R_W = F_F = 0.4Mg$ N

d Let the length of the ladder be $2a$ m.

Moments about point of contact of ladder with floor:

$Mga \sin\alpha = R_W \times 2a \cos\alpha$

$Mg \sin\alpha = 0.4Mg \times 2 \cos\alpha$

$\tan\alpha = 0.8$

$\alpha = \tan^{-1}(0.8) = 38.7°$

2 Let distance of the centre of mass of the ladder from B be x.

Let the reaction force acting between the ladder and the ground be R_B, the friction force acting between the ladder and the ground be F_B and the reaction force acting between the ladder and the wall be R_A.

Resolving horizontally:

$R_A = F_B$

Resolving vertically:

$50g = R_B$

Moments about B:

$50g \times x\cos 70° = R_A \times 8\sin 70°$

$50g \times x\cos 70° = F_B \times 8\sin 70°$

$50g \times x\cos 70° = \frac{1}{4}R_B \times 8\sin 70°$

$50g \times x\cos 70° = \frac{1}{4} \times 50g \times 8\sin 70°$

$x = 2\tan 70°$

$x = 5.49\,\text{m}$

3 Let the reaction force acting between the ladder and the ground be R_G, the friction force acting between the ladder and the ground be F_G and the reaction force acting between the ladder and the wall be R_W.

Resolving horizontally:

$R_W = F_G$

Resolving vertically:

$36g + 45g = R_G$

$R_G = 81g$

Moments about G:

$45g \times 2\sin 30° + 36g \times 3\sin 30° = R_W \times 6\cos 30°$

$198g\sin 30° = 6R_W\cos 30°$

$33g\tan 30° = R_W$

$R_W = 11g\sqrt{3}$

$F_G = \mu R_G$

$\mu = \frac{F_G}{R_G} = \frac{R_W}{81g} = \frac{11g\sqrt{3}}{81g} = \frac{11\sqrt{3}}{81}$

4 Let the reaction force acting between the ladder and the ground be R_C, the friction force acting between the ladder and the ground be F_C, the reaction force acting between the ladder and the wall be R_D and the friction force acting between the ladder and the wall be F_D.

$F_D = \frac{1}{6}R_D$

Resolving horizontally:

$F_C = R_D = 6F_D$

Resolving vertically:

$F_D + R_C = 54g$

$F_C = 6(54g - R_C) = 324g - 6R_C$

Moments about D:

$54g \times a\cos\alpha + F_C \times 2a\sin\alpha = R_C \times 2a\cos\alpha$

$54g + 2F_C\tan\alpha = 2R_C$

$27g + F_C\tan\alpha = R_C$

Given that $\sin\alpha = \frac{4}{5}$, $\tan\alpha = \frac{4}{3}$.

$27g + \frac{4}{3}F_C = R_C$

$27g + \frac{4}{3}(324g - 6R_C) = R_C$

$27g + 432g - 8R_C = R_C$

$459g = 9R_C$

$R_C = 51g\,\text{N}$

$F_C = 324g - 6 \times 51g = 324g - 306g = 18g\,\text{N}$

$\mu_C = \frac{F_C}{R_C} = \frac{18g}{51g} = \frac{6}{17}$

5 Let the reaction force acting between the ladder and the ground be R_B, the friction force acting between the ladder and the ground be F_B and the reaction force acting between the ladder and the wall be R_A. Let the maximum distance the man can climb safely be x m.

Resolving horizontally:

$F_B = R_A$

Resolving vertically:

$R_B = 60g + 70g = 130g$

Moments about B:

$60g \times 1.75\cos\alpha + 70g \times x\cos\alpha = \mathrm{R}_A \times 3.5\sin\alpha$

$105g + 70gx = 3.5R_A\tan\alpha$

$105g + 70gx = \frac{70}{13}R_A$

$R_A = \frac{13}{70}(105g + 70gx)$

$R_A = 19.5g + 13gx$

$F_B \leqslant \mu R_B$

$F_B \leqslant \frac{2}{5}R_B$

$\frac{2}{5} \geqslant \frac{F_B}{R_B}$

$\frac{2}{5} \geqslant \frac{R_A}{130g}$

$\frac{2}{5} \geqslant \frac{19.5g + 13gx}{130g}$

$19.5g + 13gx \leqslant \frac{2}{5} \times 130g$

$19.5g + 13gx \leqslant 52g$

$13gx \leqslant 32.5g$

$x \leqslant 2.5$

The man can safely climb 2.5 m.

6 Let the reaction force acting between the ladder and the ground be R_C, the friction force acting between the ladder and the ground be F_C, the reaction force acting between the ladder and the wall be R_D and the friction force acting between the ladder and the ground be F_D.

$F_D = \mu R_D$ and $F_C = \mu R_C$

Resolving horizontally:

$F_C = R_D$

Resolving vertically:

$R_C + F_D = 40g$

$R_C + \mu R_D = 40g$

$R_C = 40g - \mu R_D$

$R_C = 40g - \mu F_C$

$R_C = 40g - \mu(\mu R_C)$

$R_C = 40g - \mu^2 R_C$

$(1 + \mu^2)R_C = 40g$

$R_C = \dfrac{40g}{1+\mu^2}$

Moments about D:

$40g \times a\cos\alpha + F_C \times 2a\sin\alpha = R_C \times 2a\cos\alpha$

$40g + 2F_C\tan\alpha = 2R_C$

$20g + F_C\tan\alpha = R_C$

Given that $\sin\alpha = \frac{4}{5}$, $\tan\alpha = \frac{4}{3}$.

$20g + \frac{4}{3}F_C = R_C$

$20g + \frac{4}{3}\mu R_C = R_C$

$20g = R_C\left(1 - \frac{4}{3}\mu\right)$

$R_C = \dfrac{20g}{1-\frac{4}{3}\mu}$

Hence $\dfrac{40g}{1+\mu^2} = \dfrac{20g}{1-\frac{4}{3}\mu}$

$40g\left(1-\frac{4}{3}\mu\right) = 20g(1+\mu^2)$

$2 - \frac{8}{3}\mu = 1 + \mu^2$

$6 - 8\mu = 3 + 3\mu^2$

$3\mu^2 + 8\mu - 3 = 0$

$(3\mu - 1)(\mu + 3) = 0$

$\mu = \frac{1}{3}$ or -3

Since $0 \leqslant \mu \leqslant 1$, $\mu = \frac{1}{3}$

Exercise 2.6B

1 Let the reaction at the ground be R_A and the friction at the ground be F_A.

a Moments about A:

$12g \times 0.9\cos 25° = R_C \times 1.2$

$R_C = 81.6\,\text{N}$

b Resolving horizontally:

$F_A = R_C\sin 25° = 81.57\sin 25° = 34.5\,\text{N}$

c Resolving vertically:

$R_A + R_C\cos 25° = 12g$

$R_A = 12g - 81.57\cos 25°$

$R_A = 46.07\,\text{N}$

d Magnitude $= \sqrt{34.47^2 + 46.07^2} = 57.5\,\text{N}$

Direction $= \tan^{-1}\left(\dfrac{46.07}{34.47}\right) = 53.2°$ to the horizontal

2 Let the magnitude of the components of the reaction at A be X and Y and of the tension in the cable be T.

a Moments about A:

$8g \times 0.8 = T\sin 60° \times 1.6$

$T\sin 60° = 4g$

$T = 46.2\,\text{N}$

b Resolving horizontally:

$X = T\cos 60° = 23.09\,\text{N}$

Resolving vertically:

$T\sin 60° + Y = 8g$

$4g + Y = 8g$

$Y = 4g$

Magnitude $= \sqrt{23.09^2 + (4g)^2} = 46.2\,\text{N}$

Direction $= \tan^{-1}\left(\dfrac{4g}{23.09}\right) = 60.0°$ to the horizontal

3 Let the friction at the wall be F N, the reaction at the wall be R N and the tension be T N.

a From the right-angled triangle, $\cos\alpha = \frac{12}{13}$, from which $\sin\alpha = \frac{5}{13}$.

Moments about A:

$Mg \times 4a = T\sin\alpha \times 12a$

$T\sin\alpha = \frac{1}{3}Mg$

$T \times \frac{5}{13} = \frac{1}{3}Mg$

$T = \frac{13}{15}Mg$

b Resolving vertically:

$F + T\sin\alpha = Mg$

$F + \frac{1}{3}Mg = Mg$

$F = \frac{2}{3}Mg$

Resolving horizontally:

$R = T\cos\alpha$

$R = \frac{13}{15}Mg \times \frac{12}{13} = \frac{4}{5}Mg$

$F \leqslant \mu R$

$$\mu \geqslant \frac{F}{R}$$

$$\mu \geqslant \frac{\frac{2}{3}Mg}{\frac{4}{5}Mg}$$

$$\mu \geqslant \frac{5}{6}$$

4 Let the magnitude of the components of the reaction at A be X and Y.

Moments about A:

$6g \times 5a = 2F \times 2a + F \times 10a$

$14F = 30g$

$F = \frac{300}{14}$ N

Resolving vertically:

$Y + F = 6g$

$Y = 6g - \frac{300}{14} = \frac{540}{14}$ N

Resolving horizontally:

$X = 2F = \frac{600}{14}$ N

Direction $= \arctan\left(\frac{540}{600}\right) = \arctan 0.9$ to the horizontal

5 Let the friction at the ground be F N and the reaction at the ground be R N.

a Moments about A:

$7.5g \times 2\cos\alpha = 49 \times 2.5$

$\cos\alpha = 0.8167$

$\alpha = 35.2°$

b Resolving horizontally:

$F = 49\sin\alpha = 28.28$ N

Resolving vertically:

$R + 49\cos\alpha = 7.5g$

$R = 34.98$ N

$\mu = \frac{F}{R} = \frac{28.27}{34.98} = 0.808$

6 Let the magnitude of the components of the reaction at A be X and Y and the thrust be T. Let AB be x m.

a Moments about A:

$0.6g \times \frac{1}{2}x = T\cos 75° \times \frac{3}{4}x$

$T\cos 75 = 0.4g$

$T = \frac{0.4g}{\cos 75°} = 15.5$ N

b Resolving vertically:

$Y + 0.4g = 0.6g$

$Y = 0.2g = 2.00$ N

Resolving horizontally:

$X = T\sin 75° = 14.93$ N

Magnitude $= \sqrt{2^2 + 14.92^2} = 15.1$ N

7 **a** Resolving horizontally:

$T_A\cos 20° = T_B\cos 50°$

$T_A = \frac{T_B\cos 50°}{\cos 20°}$

Resolving vertically:

$T_A\sin 20° + T_B\sin 50° = 2g$

$\frac{T_B\cos 50°}{\cos 20°}\sin 20° + T_B\sin 50° = 2g$

$T_B(\cos 50° \tan 20° + \sin 50°) = 2g$

$T_B = \frac{2g}{\cos 50° \tan 20° + \sin 50°} = 2g = 20.0$ N

$T_A = \frac{20\cos 50°}{\cos 20°} = 13.7$ N

b Moments about A:

$2gx = T_B\sin 50° \times 3$

$x = 3\sin 50° = 2.30$

Exercise 2.7A

1 **a** At the point of sliding, $F = \mu R$

$R = 12g$ so $F = 0.32 \times 12g = 38.40$ N

$P = F = 38.4$ N

b $R = 12g - P\sin 15°$

So $F = 0.32(12g - P\sin 15°)$

So $P\cos 15° = 0.32(12g - P\sin 15°)$

$P = \frac{0.32 \times 12g}{(\cos 15° + 0.32\sin 15°)} = 36.6$ N

2 **a** Let A be the bottom right corner, which is the pivot point.

Moments $(A) = 2g \times 5 - 20P$

At the point of toppling, moments about $A = 0$

So $10g - 20P = 0$

$P = 5$ N

b At the point of sliding, $F = \mu R$

$R = 2g$ so $F = 0.3 \times 2g = 6$ N

$P = F = 6$ N

c The cuboid would topple first.

3 **a** Sliding: At the point of sliding, $F = \mu R$

$F = 0.2 \times 2.5g = 5$ N

$P = F = 5$ N

Toppling: Let A be the bottom right corner, which is the pivot point.

Moments $(A) = 2.5g \times 20 - 50P$

At the point of toppling, moments about $A = 0$

$2.5g \times 20 - 50P = 0$

$P = 10$ N

So cuboid will slide first.

b Sliding: At the point of sliding, $F = \mu R$

$F = 0.2(2.5g + P\sin 10°)$

$P\cos 10° = 0.2(2.5g + P\sin 10°)$

$P = \dfrac{0.2 \times 2.5g}{\cos 10° - 0.2\sin 10°} = 5.26\,\text{N}$

Toppling: Let A be the pivot point.

Moments $(A) = 2.5g \times 0.2 - 0.5P\cos 10°$

At the point of toppling moments about $A = 0$

$2.5g \times 0.2 - 0.5P\cos 10° = 0$

$P = 10.2\,\text{N}$

So cuboid will slide first.

4 a Sliding:

Resolving horizontally: $F = 4g\sin\alpha$

Resolving vertically: $R = 4g\cos\alpha$

At the point of sliding, $F = \mu R$

$4g\sin\alpha = 0.3 \times 4g\cos\alpha$

$\tan\alpha = 0.3$

$\alpha = 16.7°$

b Toppling occurs when the centre of mass is directly above the lower corner of the cuboid.

This occurs when $\alpha = \arctan\left(\dfrac{2.5}{5}\right) = 26.6°$

c The cuboid will slide first.

5 a Sliding:

Resolving horizontally: $F = 12g\sin 20°$

Resolving vertically: $R = 12g\cos 20°$

At the point of sliding, $F = \mu R$

$12g\sin 20° = \mu \times 12g\cos 20°$

$\mu = \tan 20° = 0.364$

b Toppling occurs when the centre of mass is directly above the pivot point.

This occurs when $\alpha = \arctan\left(\dfrac{5}{12}\right) = 22.6°$

Exam-style questions

1 a Let the reaction force at C be R_C.

Resolving vertically:

$60g + 90g + 40g = R_C + 95g$

$R_C = 95g = 950\,\text{N}$

b Moments about C:

$90g(x - 1) + 40g \times 9 = 95g \times 6 + 60g \times 1$

$90(x - 1) + 360 = 570 + 60$

$90x - 90 + 360 = 630$

$90x = 360$

$x = 4$

2 Let the distance of the centre of mass from A be x m.

Moments about V:

$6mgd = 8mg(4d - x)$

$6d = 32d - 8x$

$8x = 26d$

$x = 3.25d$

3 Resolving vertically:

$R_X + R_Y = 7g + 5g + 9g$

$R_X + 2R_X = 7g + 5g + 9g$

$3R_X = 21g$

$R_X = 7g,\ R_Y = 14g$

Let the distance WY be x m.

Moments about W:

$5g \times 7 + 9g \times 14 = 7g \times 2 + 14gx$

$161 = 14 + 14x$

$14x = 147$

$x = 10.5$

$WY = 10.5\,\text{m}$

4 a The frustum is a cone of base 10 cm and height h with a cone of base 5 cm and height $h - 8$ removed.

By similarity, $h = 16$ cm.

Shape	Volume	z-coordinate of centre of mass
Large cone	1676 (4 s.f.)	4
Small cone	209.4	10
Frustum	1467	$\bar{z}$

$1467\bar{z} = 1676 \times 4 - 209.4 \times 10$

$\bar{z} = 3.14$ cm above the centre of the base.

b

Shape	Volume	z-coordinate of centre of mass
Frustum	1467	3.142
Hemisphere	261.8	9.875
New solid	1729	$\bar{z}$

$1729\bar{z} = 1467 \times 3.142 + 261.8 \times 9.875$

$\bar{z} = 4.161$ cm above the base.

So the centre of mass of the new solid is $4.161 - 3.142 = 1.02$ cm higher than the centre of mass of the old solid.

5 **a** The rope is light and inextensible.

Moments about D:

$W \times 5 + 28 \times 11 = T_C \times 7$

$T_C = \frac{5}{7}W + 44$

b Resolving vertically:

$T_C + T_D = W + 28$

$\frac{5}{7}W + 44 + T_D = W + 28$

$T_D = \frac{2}{7}W - 16$

$\frac{5}{7}W + 44 = 13\left(\frac{2}{7}W - 16\right)$

$\frac{5}{7}W + 44 = \frac{26}{7}W - 208$

$252 = 3W$

$W = 84$

6 Toppling will occur when the centre of mass is directly above bottom left corner.

This occurs when $\theta = \arctan\left(\frac{2.5}{4}\right) = 32.0°$

7 **a** Moments about C:

$1800 \times 8 = W(8 - x)$

$14\,400 = 8W - Wx$

$Wx = 8W - 14\,400$

Moments about B:

$1575 \times 8 = W(x - 2)$

$12\,600 = Wx - 2W$

$Wx = 2W + 12\,600$

$8W - 14\,400 = 2W + 12\,600$

$6W = 27\,000$

$W = 4500$

b $Wx = 2W + 12\,600$

$4500x = 9000 + 12\,600$

$x = 4.8$

8 **a** Let the bottom left of the diagram be the point (0,0).

Centre of mass of $AB = \left(1, \sqrt{3} + 2\right)$

Centre of mass of $BC = \left(2, \sqrt{3} + 1\right)$

Centre of mass of $CD = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$

Centre of mass of $CE = \left(1, \sqrt{3}\right)$

Centre of mass of $DE = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

Centre of mass of $AE = \left(0, \sqrt{3} + 1\right)$

Assume the mass of each rod is 1

Using $M\bar{x} = \sum_{i=1}^{n} m_i x_i$ you get

$6\bar{x} = 1 \times 1 + 1 \times 2 + 1 \times \frac{3}{2} + 1 \times 1 + 1 \times \frac{1}{2} + 1 \times 0$

$6\bar{x} = 6$

$\bar{x} = 1$ (this can also be found from the symmetry of the shape)

Using $M\bar{y} = \sum_{i=1}^{n} m_i y_i$ you get

$6\bar{y} = 1 \times \left(\sqrt{3} + 2\right) + 1 \times \left(\sqrt{3} + 1\right) + 1 \times \frac{\sqrt{3}}{2} + 1$

$\times \left(\sqrt{3} + 1\right) \times \frac{\sqrt{3}}{2} + 1 \times \left(\sqrt{3} + 1\right)$

$6\bar{y} = 5\sqrt{3} + 4$

$\bar{y} = \frac{5}{6}\sqrt{3} + \frac{2}{3}$

So centre of mass = (1, 2.11) (to 3 s.f.)

b The distance from centre of mass to centre of $AB = \sqrt{3} + 2 - 2.11 = 1.62$

So $\theta = \tan^{-1}\left(\frac{1.62}{1}\right) = 58.3°$

So angle between AB and vertical is 58.3°

9 Let the reaction force between the ladder and the wall be R_W, the reaction force between the ladder and the floor be R_F and the friction force between the ladder and the floor be F_F.

Let the angle between the ladder and the floor be α, where $\tan\alpha = 2$.

Moments about B:

$50g \times 1.75 \cos\alpha + 70gx \cos\alpha = R_W \times 3.5 \sin\alpha$

$87.5g + 70gx = 3.5R_W \tan\alpha$

$87.5g + 70gx = 3.5R_W \times 2 = 7R_W$

Resolving horizontally:

$R_W = F_F$

Resolving vertically:

$R_F = 120g$

$F_F \leqslant \mu R_F$

$R_W \leqslant \frac{1}{3} \times 120g$

$\frac{87.5g + 70gx}{7} \leqslant 40g$

$87.5g + 70gx \leqslant 280g$

$70gx \leqslant 192.5g$

$x \leqslant 2.75$

The person can climb 2.75 m before the ladder slides.

10 Let A be the point (0, 0)

Shape	Area	y-coordinate of centre of mass
Large circle	$4\pi r^2$	0
Small circle	πr^2	r
Lamina	$3\pi r^2$	$\bar{y}$

$$3\pi r^2\bar{y} = 0 \times 4\pi r^2 + \pi r^3$$

$$\bar{y} = \frac{r}{3}$$

The distance from A to the centre of mass is $\frac{r}{3}$

11 a Let the reaction at A be R_A and the friction at A be F_A

Moments about B:

$2.5g \times 3.6 \cos\alpha + F_A \times 7.2 \sin\alpha = R_A \times 7.2 \cos\alpha$

$2.5g + 2F_A \tan\alpha = 2R_A$

$F_A = \frac{1}{2}R_A$

$2.5g + R_A \tan\alpha = 2R_A$

Given that $\sin\alpha = \frac{3}{5}$, $\tan\alpha = \frac{3}{4}$

$2.5g + R_A \times \frac{3}{4} = 2R_A$

$2.5g = \frac{5}{4}R_A$

$R_A = 2g = 20\,\text{N}$

b Resolving vertically:

$R_A + T\sin\beta = 2.5g$

$T\sin\beta = 2.5g - 2g = 0.5g$

$0.28T = 0.5g$

$T = 17.9\,\text{N}$

c The rod is one-dimensional, straight and rigid, and the cable is a light inextensible string.

12 Sliding:

Resolving horizontally: $F = 5g\sin\alpha$

Resolving vertically: $R = 5g\cos\alpha$

At the point of sliding: $F = \mu R$

$5g\sin\alpha = 0.6 \times 5g\cos\alpha$

$\alpha = \arctan 0.6 = 31.0°$

Toppling: will occur when the centre of mass is directly above bottom right corner.

This occurs when $\alpha = \arctan\left(\frac{1}{4}\right) = 14.0$

So the solid will topple when the angle α reaches 14.0°.

13 Let $A = (0, 0)$

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Semi-circle	$\frac{\pi r^2}{2}$	$r + \frac{4r}{3\pi}$	0
Triangle	$\frac{r^2}{2}$	$\frac{2}{3}r$	$\frac{1}{3}r$
Lamina	$\frac{r^2}{2}(1+\pi)$	$\bar{x}$	$\bar{y}$

a $$\frac{r^2}{2}(1+\pi)\bar{x} = \frac{\pi r^2}{2}\left(r + \frac{4r}{3\pi}\right) + \frac{r^2}{2} \times \frac{2}{3}r$$

$$\frac{r^2}{2}(1+\pi)\bar{x} = \frac{\pi r^3}{2} + \frac{2r^3}{3} + \frac{r^3}{3}$$

$$\frac{r^2}{2}(1+\pi)\bar{x} = r^3\left(\frac{\pi}{2} + 1\right)$$

$$(1+\pi)\bar{x} = r(\pi + 2)$$

$$\bar{x} = \frac{r(2+\pi)}{(1+\pi)}$$

b $$\frac{r^2}{2}(1+\pi)\bar{y} = \frac{r^2}{2} \times \frac{1}{3}r$$

$$\frac{r^2}{2}(1+\pi)\bar{y} = \frac{r^3}{6}$$

$$(1+\pi)\bar{y} = \frac{r}{3}$$

$$\bar{y} = \frac{r}{3(1+\pi)}$$

c Let θ be the angle between the vertical and AO

$$\bar{x} = \frac{(1+\pi)(2+\pi)}{(1+\pi)} = 2 + \pi$$

$$\bar{y} = \frac{(1+\pi)}{3(1+\pi)} = \frac{1}{3}$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{3}}{2+\pi}\right) = 3.71°$$

14 a Let the weight of the rod be W N and the distance of the centre of mass from A be x m.

On the point of tilting about C, $R_D = 0$.

$20g \times 4.8 + W(x - 1.2) = 100g \times 1.2$

$W(x - 1.2) = 24g$

$Wx - 1.2W = 24g$

$Wx = 1.2W + 24g$

On the point of tilting about D, $R_C = 0$.

$250g \times 0.8 = 20g \times 5.2 + W(5.2 - x)$

$96g = 5.2W - Wx$

$Wx = 5.2W - 96g$

Hence $1.2W + 24g = 5.2W - 96g$

$120g = 4W$

$W = 30g$

The mass of the rod is 30 kg.

b $Wx = 1.2W + 24g$

$30gx = 1.2 \times 30g + 24g$

$30x = 60$

$x = 2$

The centre of mass is 2 m from A.

15 a

Shape	Length	x-coordinate of centre of mass	y-coordinate of centre of mass
Semicircle	πr	0	$\frac{2r}{\pi}$ (below)
Straight rod	$2r$	0	0
Mass at A	$2r$	$-r$	0
Mass at B	πr	r	0
Lamina	$2r(\pi + 2)$	$\bar{x}$	$\bar{y}$

$2r(\pi + 2)\bar{x} = \pi r^2 - 2r^2$

$2r(\pi + 2)\bar{x} = r^2(\pi - 2)$

$\bar{x} = \frac{r^2(\pi - 2)}{2r(\pi + 2)}$

$\bar{x} = \frac{r(\pi - 2)}{2(\pi + 2)}$

$\bar{x} = \frac{r(\pi - 2)}{2\pi + 4}$

b $2r(\pi + 2)\bar{y} = \pi r \times \frac{2r}{\pi}$

$2r(\pi + 2)\bar{y} = 2r^2$

$\bar{y} = \frac{2r^2}{2r(\pi + 2)}$

$\bar{y} = \frac{r}{(\pi + 2)}$

c When $r = 5$,

$\bar{x} = \frac{5(\pi - 2)}{2(\pi + 2)}$ and $\bar{y} = \frac{5}{(\pi + 2)}$

So $\theta = \tan^{-1}\left(\frac{\bar{y}}{\bar{x}}\right) = \frac{2}{(\pi - 2)} = 60.3°$

16 a Let the friction force between the ladder and the wall be F_W, the reaction force between the ladder and the wall be R_W and the friction force between the ladder and the ground be R_G.

Resolving horizontally:

$R_W = F_G$

Resolving vertically:

$F_W + R_G = Mg$

$R_G = Mg - F_W$

$R_G = Mg - \mu_W R_W$

$R_G = Mg - \mu_W F_G$

$R_G = Mg - \mu_W \mu_G R_G$

$R_G(1 + \mu_W \mu_G) = Mg$

$R_G = \frac{Mg}{1 + \mu_W \mu_G}$

b Moments about the point of contact of the ladder with the wall:

$R_G \times 8a\cos\theta = F_G \times 8a\sin\theta + Mg \times 2a\cos\theta$

$4R_G\cos\theta = 4\mu_G R_G\sin\theta + Mg\cos\theta$

$4R_G = 4\mu_G R_G\tan\theta + Mg$

$R_G(4 - 4\mu_G\tan\theta) = Mg$

$R_G = \frac{Mg}{4 - 4\mu_G\tan\theta}$

c Hence $\frac{Mg}{1 + \mu_W\mu_G} = \frac{Mg}{4 - 4\mu_G\tan\theta}$

$1 + \mu_W\mu_G = 4 - 4\mu_G\tan\theta$

$\mu_W\mu_G = 3 - 4\mu_G\tan\theta$

$\mu_W = \frac{3 - 4\mu_G\tan\theta}{\mu_G}$

17 a Assuming origin is at centre of base of cross-section

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Rectangle	24	0	3
Triangle	6	0	7
Lamina	30	$\bar{x}$	$\bar{y}$

$30\bar{y} = 24 \times 3 + 6 \times 7$

$\bar{x} = 0 \qquad \bar{y} = 3.8$

Height of shape $= 6 + 3 = 9$

So centre of mass is $9 - 3.8 = 5.2$ below apex of triangle.

b Sliding:

Resolving horizontally: $F = 8g\sin\theta$

Resolving vertically: $R = 8g\cos\theta$

At the point of sliding: $F = \mu R$

$8g\sin\theta = 0.25 \times 8g\cos\theta$

$\theta = \arctan 0.25 = 14.0°$

Toppling will occur when the centre of mass is directly above the pivot point.

This occurs when $\theta = \arctan\left(\frac{2}{3.8}\right) = 27.8°$

So the solid will slide down the slope first, when the angle θ reaches 14.0°.

18 a Let $DC = x$, angle $OCD = 30°$

In triangle DOC, $\tan 30° = \frac{r}{x}$ so $x = \frac{r}{\frac{\sqrt{3}}{3}} = \frac{3r}{\sqrt{3}}$

Using Pythagoras' theorem on triangle BCD

$$BD^2 = \left(\frac{6r}{\sqrt{3}}\right)^2 - \left(\frac{3r}{\sqrt{3}}\right)^2$$

$$= \frac{36r^2}{3} + \frac{9r^2}{3}$$

$$= \frac{27r^2}{3}$$

$$= 9r^2$$

$$BD = \sqrt{9r^2} = 3r$$

b Let θ be the angle between ED and the vertical

$$\tan\theta = \frac{\bar{y}}{\bar{x}} = \frac{r}{\frac{3r}{2\sqrt{3}}} = \frac{2\sqrt{3}}{3}$$

So $\theta = 49.1°$

19 a

Shape	Volume	z-coordinate of centre of mass
Cylinder	502.7	5
Hemisphere	134.0	11.5
Solid	636.7	$\bar{z}$

$636.7\bar{z} = 502.7 \times 5 + 134.0 \times 11.5$

$\bar{z} = 6.37$

The centre of mass of the solid is 6.37 cm above the flat base.

b Sliding:

Resolving horizontally: $F = 5g\sin\theta$

Resolving vertically: $R = 5g\cos\theta$

At the point of sliding: $F = \mu R$

$5g\sin\theta = 0.32 \times 5g\cos\theta$

$\theta = 17.7°$

Toppling will occur when the centre of mass is directly above the pivot point.

This occurs when $\theta = \arctan\left(\frac{2}{6.37}\right) = 17.4°$

So the solid will topple first, when the angle θ reaches 17.4°.

20 Let $A = (0, 0)$

Shape	Area	x-coordinate of centre of mass	y-coordinate of centre of mass
Rectangle	60	5	3
Semicircle	39.27	5	8.122
Triangle	15.59	11.73	3
Lamina	114.86	$\bar{x}$	$\bar{y}$

a $114.86\bar{x} = 60 \times 5 + 39.27 \times 5 + 15.59 \times 11.73$

$114.86\bar{x} = 679.22$

$\bar{x} = 5.913$

Horizontal distance between centre of mass and $AE = 5.91$ cm

b $114.86\bar{y} = 60 \times 3 + 39.27 \times 8.122 + 15.59 \times 3$

$114.86\bar{y} = 545.72$

$\bar{y} = 4.751$

Vertical distance between centre of mass and $AB = 4.75$ cm

c Let θ be the angle between the vertical and AB

$\theta = \tan^{-1}\left(\frac{4.751}{5.913}\right) = 38.8°$

Mathematics in life and work

1 Let h be the height of the full pyramid.

By similarity, $2(h - 1.15) = h$ so $h = 2.3$ m.

2

Shape	Volume	z-coordinate of centre of mass
Full pyramid	0.7667	0.575
Small pyramid	0.09583	1.4375
Model	0.6709	$\bar{z}$

$0.6709\bar{z} = 0.7667 \times 0.575 - 0.09583 \times 1.4375$

$\bar{z} = 0.452$

The centre of mass of the model is 45.2 cm above the base.

3 Sliding: At the point of sliding, $F = \mu R$.

$R = 125g$ so $F = 0.4 \times 125g = 500\,\text{N}$

$P = F = 125\,\text{N}$

Toppling: Let A be the bottom right corner, which is the pivot point.

Moments $(A) = 125g \times 0.50 - 1.15P$

At point of toppling moments about $A = 0$.

So $125g \times 0.50 - 1.15P = 0$

$P = 543.5\,\text{N}$

So the solid will slide first.

3 Circular motion

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $720° = 4\pi$ radians

2 $5 \times 2\pi = 10\pi$ radians

3 $60° = \frac{\pi}{3}$ radians. Arc length $= r\theta = 12 \times \frac{\pi}{3} = 4\pi$ cm

4 $\text{PE} = mgh = 5 \times 10 \times 6 = 300\,\text{J}$

5 $\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 3000 \times 8^2 = 96\,000\,\text{J}$

6 Total energy = PE + KE

$= 500 \times 10 \times 10 + \frac{1}{2} \times 500 \times 15^2 = 106\,250\,\text{J}$

Exercise 3.1A

1 60 r.p.m. $= 60 \times 2\pi = 120\pi$ radians per minute

$= 2\pi$ radians per second

2 The second hand moves at a speed of 2π radians per minute. This is the same as $\frac{\pi}{30}$ radians per second.

3 $v = \omega r = \frac{\pi}{30} \times 0.1 = \frac{\pi}{300}\,\text{m s}^{-1} = 0.0105\,\text{m s}^{-1}$

4 **a** $\omega = \frac{15 \times 2\pi}{60} = \frac{\pi}{2}\,\text{rad s}^{-1}$

b $v = \omega r = \frac{\pi}{2} \times 1 = \frac{\pi}{2}\,\text{m s}^{-1} = 1.57\,\text{m s}^{-1}$

5 $\omega = \frac{5 \times 2\pi}{10} = \pi\,\text{rad s}^{-1}$

$v = \omega r = \pi \times 0.6 = 0.6\pi\,\text{m s}^{-1} = 1.88\,\text{m s}^{-1}$

6 **a** $t = \frac{60\pi}{5} = 12\pi$ seconds $= 37.7$ seconds

b $\omega = \frac{v}{r} = \frac{5}{30} = 0.167\,\text{rad s}^{-1}$

7 **a** Priya: $t = \frac{84\pi}{4.5} = 58.643$ seconds

Sanjay: $t = \frac{88\pi}{4.5} = 61.436$ seconds

Difference in time = 2.79 seconds

b Priya: $\omega = \frac{v}{r} = \frac{4.5}{42} = 0.10714\,\text{rad s}^{-1}$

Sanjay: $\omega = \frac{v}{r} = \frac{4.5}{44} = 0.10227\,\text{rad s}^{-1}$

Difference $= 0.00487\,\text{rad s}^{-1}$

8 Distance travelled $= v \times t = 15 \times 35$

$= 525$ m = circumference of the circle

$r = \frac{C}{2\pi} = \frac{525}{2\pi} = 83.6\,\text{m}$

9 $\omega = \frac{2\pi}{24 \times 60 \times 60} = \frac{\pi}{43\,200}\,\text{rad s}^{-1}$,

$v = \omega r = \frac{\pi}{43\,200} \times 6\,371\,000 = 463\,\text{m s}^{-1}$

Exercise 3.2A

1 $a = \frac{v^2}{r} = \frac{3^2}{0.25} = 36\,\text{m s}^{-2}$

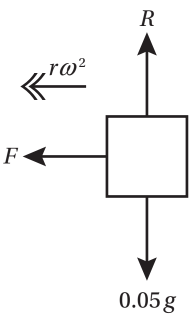
2 $a = \frac{v^2}{r} = \frac{4.5^2}{3} = 6.75\,\text{m s}^{-2}$

3 $a = \frac{v^2}{r} = \frac{\left(\frac{60 \times 1000}{60 \times 60}\right)^2}{45} = 6.17\,\text{m s}^{-2}$ (to 3 s.f.)

directed towards the centre of the circle

4 $F = \frac{mv^2}{r} = \frac{0.03 \times 4.8^2}{0.2} = 3.46\,\text{N}$ directed towards the centre of the circle

5 **a**



b Friction force $= mr\omega^2 = 0.05 \times 0.25 \times 3^2 = 0.113\,\text{N}$

6 Resolving vertically: $R = 2000g = 20\,000\,\text{N}$

Resultant force, $F = \frac{mv^2}{r} = \frac{2000 \times \left(\frac{50 \times 1000}{60 \times 60}\right)^2}{50}$

$= 7716\,\text{N}$

The car is in limiting equilibrium so $F = \mu R$:

$\mu R = 7716$ so $\mu = 0.386$

7 Resolving vertically: $R = 600g = 6000\,\text{N}$

Resultant force, $F = \frac{mv^2}{r} = \frac{600 \times v^2}{42}$

The motorbike is in limiting equilibrium so $F = \mu R$:

$\frac{600 \times v^2}{42} = 0.6 \times 6000$

So $v = 15.9\,\text{m s}^{-1}$

8 Resolving vertically: $R = 0.1g = 1\text{N}$

Resultant force, $F = \dfrac{mv^2}{r} = \dfrac{0.1 \times v^2}{0.25}$

The toy is in limiting equilibrium so $F = \mu R$:

$$\frac{0.1 \times v^2}{0.25} = 0.6 \times 1$$

$$v = 1.22\,\text{m s}^{-1}$$

The turntable must be rotating at a speed of $1.22\,\text{m s}^{-1}$ for the toy to slide off.

9 Resolving vertically: $R = 1500g = 15\,000\,\text{N}$

Resultant force, $F = \dfrac{mv^2}{r} = \dfrac{1500 \times \left(\dfrac{45 \times 1000}{60 \times 60}\right)^2}{r}$

The car is in limiting equilibrium so $F = \mu R$:

$$\frac{1500 \times \left(\dfrac{45 \times 1000}{60 \times 60}\right)^2}{r} = 0.2 \times 15000$$

$$r = 78.1\,\text{m}$$

10 Resolving vertically: $R = mg = 10m$

Resultant force, $F = mr\omega^2 = mx\omega^2$

For the object not to slide, $F \leqslant \mu R$

$$mx\omega^2 \leqslant 10m\mu$$

$$x\omega^2 \leqslant 10\mu$$

$$\omega^2 \leqslant \frac{10\mu}{x}$$

Exercise 3.3A

1 **a** $r = l\sin\theta = 0.5 \times \sin 30° = 0.25\,\text{m}$

b $t = 2\pi\sqrt{\dfrac{l\cos\theta}{g}} = 2\pi\sqrt{\dfrac{0.5\cos 30°}{10}} = 1.31\,\text{s}$

c Resolving vertically: $T\cos\theta = mg$

$$T\cos 30° = 2.5 \times 10$$

$$T = 28.9\,\text{N}$$

2 **a** $\sin\theta° = \dfrac{35}{55}$ so $\theta° = 39.5°$

b Resolving vertically: $T\cos\theta = mg$

$$T\cos 39.5212° = 0.4 \times 10$$

$$T = 5.19\,\text{N}$$

c $\omega = \sqrt{\dfrac{g}{l\cos\theta}} = \sqrt{\dfrac{10}{0.55\cos 39.5212°}} = 4.85\,\text{rad s}^{-1}$

$a = r\omega^2 = 0.35 \times 4.8549^2 = 8.25\,\text{m s}^{-2}$

d $v = r\omega = 0.35 \times 4.8549 = 1.70\,\text{m s}^{-1}$

3 **a** Resolving vertically: $T\cos\theta = mg$

$$T \times \frac{0.6}{0.9} = 0.6g$$

$$T = 9\,\text{N}$$

b $\omega = \sqrt{\dfrac{g}{l\cos\theta}} = \sqrt{\dfrac{10}{0.9 \times \dfrac{0.6}{0.9}}} = 4.08\,\text{rad s}^{-1}$

4 Resolving vertically: $R\sin 20° = 0.35g$

$$R = 10.233\,\text{N}$$

By Newton's second law (horizontally):

$$R\cos 20° = \frac{mv^2}{r}$$

$$10.233\cos 20° = \frac{0.35 \times 5^2}{r}$$

$$r = 0.910\,\text{m}$$

5 Let θ be the angle between the normal reaction force and the horizontal.

Resolving vertically: $R\sin\theta = 0.08g$

Resolving horizontally: $R\cos\theta = mrw^2$ where r is the radius of the circle

Radius of circle is $0.3\cos\theta$.

So $R\cos\theta = 0.08 \times 0.3\cos\theta \times 10^2$

$$R = 2.4\,\text{N}$$

This gives $\sin\theta = \dfrac{0.08g}{2.4}$ so $\theta = 19.5°$.

Perpendicular height of the ball below top of hemisphere, $h = 0.3\sin 19.5° = 0.100\,\text{m}$

So distance from base $= 0.3 - 0.1 = 0.2\,\text{m}$

6 Let θ be the angle between the normal reaction force and the horizontal.

So $\cos\theta° = \dfrac{0.4}{0.6}$, $\theta = 48.1897°$

Resolving vertically: $R\sin\theta = 0.3g$

$$R = 4.0249\,\text{N}$$

Resolving horizontally: $R\cos\theta = mr\omega^2$

$$\omega^2 = \frac{R\cos\theta}{mr} = \frac{4.0249\cos 48.1897°}{0.3 \times 0.4} = 22.4$$

$$\omega = 4.73\,\text{rad s}^{-1}$$

7 Let θ be the angle between the pendulum string and the vertical.

Resolving vertically: $T\cos\theta = mg$ (where T is the tension)

Resolving horizontally: $T\sin\theta = mr\omega^2$

Equating to eliminate T:

$$\frac{mg}{\cos\theta} = \frac{mr\omega^2}{\sin\theta}$$

$$\frac{g}{\cos\theta} = \frac{r\omega^2}{\sin\theta}$$

$$\omega^2 = \frac{g\sin\theta}{r\cos\theta}$$

Since $r = \sin\theta$ and $h = \cos\theta$:

$$\omega^2 = \frac{gr}{rh}$$

$$\omega = \sqrt{\frac{g}{h}}$$

8 Resolving vertically: $R\sin\theta = mg$

Resolving horizontally (and using Newton's second law):

$R\cos\theta = mrw^2$

Equating to eliminate R:

$$\frac{mg}{\sin\theta} = \frac{mr\omega^2}{\cos\theta}$$

$$\frac{g}{\sin\theta} = \frac{r\omega^2}{\cos\theta}$$

$$\omega^2 = \frac{g\cos\theta}{r\sin\theta}$$

$$\omega^2 = \frac{g}{r\tan\theta}$$

$$\omega = \sqrt{\frac{g}{r\tan\theta}}$$

Exercise 3.4A

1 **a** Total energy $= mgh + \frac{1}{2}mv^2$
$= 2 \times 10 \times 1.5 + 0.5 \times 2 \times 3^2 = 39\,\text{J}$

b At the lowest point the object will have no PE so:

$$\frac{1}{2}mv^2 = 39$$

$$\frac{1}{2} \times 2 \times v^2 = 39$$

$$v = 6.24\,\text{m s}^{-1}$$

2 At highest point of circle:

Total energy $= mgh + \frac{1}{2}mv^2$
$= 0.3 \times 10 \times 1.4 + 0.5 \times 0.3 \times 1.5^2 = 4.5375\,\text{J}$

At the lowest point, the object will have no PE so:

$$\frac{1}{2}mv^2 = 4.5375$$

$$\frac{1}{2} \times 0.3 \times v^2 = 4.5375$$

$$v = 5.50\,\text{m s}^{-1}$$

3 **a** Assuming the base of the swing has a height of 0 m:

Height of release point $= 2.4 - x$ where x is the perpendicular distance of the swing to the point where the rope is attached.

$x = 2.4\cos 30°$ so height of swing
$= 2.4 - 2.4\cos 30° = 0.32154\,\text{m}$

b Total energy $= mgh = 30 \times 10 \times 0.32154 = 96.5\,\text{J}$

c At lowest point: $\frac{1}{2}mv^2 = 96.46$

$$v = 2.54$$

The speed at the lowest point is $2.54\,\text{m s}^{-1}$.

4 **a** The maximum speed occurs at the bottom of the circle.

Assume $h = 0$ at base of circle.

Energy at start $= mgh = 3 \times 10 \times 2 = 60\,\text{J}$

At lowest point: $\frac{1}{2}mv^2 = 60$

$$v = 6.32\,\text{m s}^{-1}$$

b The maximum force occurs at the bottom of the circle.

$$F = \frac{mv^2}{r} = \frac{3 \times 40}{2} = 60\,\text{N}$$

5 **a** Height at point of release $= 2.5 - 2.5\cos 40°$
$= 0.58489\,\text{m}$

Total energy at point of release $= mgh$
$= 45 \times 10 \times 0.58489 = 263.2\,\text{N}$

Maximum speed occurs at the lowest point of the circle.

At lowest point: $\frac{1}{2}mv^2 = 263.2$

$$v = 3.42\,\text{m s}^{-1}$$

b When the angle made by the rope and the downward vertical is 25°, $h = 2.5 - 2.5\cos 25°$
$= 0.234\,\text{m}$

At this point $mgh + \frac{1}{2}mv^2 = 263.2$

$$45 \times 10 \times 0.23423 + \frac{1}{2} \times 45 \times v^2 = 263.2$$

$$v = 2.65\,\text{m s}^{-1}$$

6 **a** When the angle with downward vertical is 110°, height $= 0.75 + 0.75\sin 20° = 1.01\,\text{m}$

b Total energy at the end of the swing $= mgh$
$= 0.5 \times 10 \times 1.0652 = 5.0326\,\text{J}$

Energy at start: $mgh + \frac{1}{2}mv^2 = 5.0326$

$$0.5 \times 10 \times 0.75 + \frac{1}{2} \times 0.5 \times u^2 = 5.0326$$

$$u = 2.27\,\text{m s}^{-1}$$

7 **a** At the bottom of the circle, total energy $= \frac{1}{2}mv^2$
$= \frac{1}{2} \times m \times (4v)^2 = 8mv^2$

Total energy at top: $mgh + \frac{1}{2}mv^2$
$= mg(2l) + \frac{1}{2}mv^2$

Energy is conserved so $2mgl + \frac{1}{2}mv^2 = 8mv^2$

$$4gl + v^2 = 16v^2$$

$$4gl = 15v^2$$

$$l = \frac{15v^2}{4g}$$

b Maximum force is at the bottom of the circle.

$$F_{\text{maximum}} = \frac{mv^2}{r} = \frac{m(4v)^2}{\frac{15v^2}{4g}} = \frac{64gmv^2}{15v^2} = \frac{64gm}{15}\ \text{N}$$

Minimum force is at the top of the circle.

$$F_{\text{minimum}} = \frac{mv^2}{r} = \frac{mv^2}{\frac{15v^2}{4}} = \frac{4mv^2}{15v^2} = \frac{4m}{15}\ \text{N}$$

8 At the bottom of the circle, total energy
$= \frac{1}{2}mv^2 = \frac{1}{2}m\left(\sqrt{3gr}\right)^2 = \frac{3}{2}gmr\,\text{J}$

Height at any point $= r + r\sin(90° - \theta) = r(1 + \cos\theta)$

Total energy at any point:

$$mgh + \frac{1}{2}mv^2 = \frac{3}{2}gmr$$

$$mgr(1 + \cos\theta) + \frac{1}{2}mv^2 = \frac{3}{2}gmr$$

$$2gr(1 + \cos\theta) + v^2 = 3gr$$

$v^2 = 3gr - 2gr(1 + \cos\theta)$

$v^2 = 3gr - 2gr - 2gr\cos\theta$

$v^2 = gr - 2gr\cos\theta$

$v^2 = gr(1 - 2\cos\theta)$

$v = \sqrt{gr(1 - 2\cos\theta)}$

Exercise 3.5A

1 PE at top $= 2 \times 10 \times 1.5 = 30\,\text{J}$

For complete motion, total energy must be greater than 30 J when at the bottom to allow for the object to have some velocity at the top.

$\frac{1}{2}mv^2 > 30$

$\frac{1}{2} \times 2 \times v^2 > 30$

$v > 5.48\,\text{m s}^{-1}$

2 a Total energy $= mgh + \frac{1}{2}mv^2$

$= 0.1 \times 10 \times 0.6 + 0.5 \times 0.1 \times 5^2 = 1.85\,\text{J}$

At bottom: $\frac{1}{2}mv^2 = 1.85$

$v = 6.08\,\text{m s}^{-1}$

b The speed at the top of the circle must be greater than $0\,\text{m s}^{-1}$ and the reaction force must not decrease to zero before it reaches the top of the circle.

c At top: total energy

$= 0.1 \times 10 \times 1.2 + 0.5 \times 0.1v^2 = 1.85$

$v = 3.61\,\text{m s}^{-1}$

When $v = 3.61\,\text{ms}^{-1}$,

reaction force $= \frac{0.1 \times 3.61^2}{0.6} - 0.1 \times 10 > 0.$

This is greater than 0 so the marble will complete a full circle inside the pipe.

3 a Total energy $= \frac{1}{2}mv^2 = 0.5 \times 0.25 \times 4^2 = 2\,\text{J}$

When it comes to rest, it will have only PE.

$0.25 \times 10 \times h = 2$

$h = 0.8\,\text{m}$

b $\theta = \cos^{-1}\left(\frac{0.2}{1}\right) = 78.5°$

It will make an angle of 78.5° with the downward vertical when it comes to rest.

4 a It will oscillate around its starting point if it comes to rest before it reaches a height of 0.5 m.

Energy at start $= \frac{1}{2}mv^2 = 0.05u^2$

When $h = 0.5$, PE $= mgh = 0.1 \times 10 \times 0.5 = 0.5$

So $0.05u^2 = 0.5$, $u = 3.16\,\text{m s}^{-1}$

To oscillate around its starting point, $u \leqslant 3.16\,\text{m s}^{-1}$.

b It will complete a full circle if the speed at the top is greater than 0 and the reaction force does not decrease to zero before it reaches the top of the circle.

At top: PE $= mgh = 0.1 \times 10 \times 1 = 1\,\text{J}$

In order for it to have KE at top, total energy must be greater than 1 J.

So at bottom: $\frac{1}{2}mv^2 > 1$

$\frac{1}{2} \times 0.1 \times u^2 > 1$

$u > 4.47\,\text{m s}^{-1}$

Reaction force $= \frac{0.1 \times 4.47^2}{0.5} - 0.1 \times 10 > 0$

To complete a full circle, $u > 4.47\,\text{m s}^{-1}$.

5 Total energy at start $= mgh + \frac{1}{2}mv^2$

$= 0.04 \times 10 \times 0.6 + 0.5 \times 0.04 \times 2.3^2 = 0.3458\,\text{J}$

When $v = 0$, $mgh = 0.3458$ so $h = 0.865\,\text{m}$.

The particle will come to rest when $h = 0.865\,\text{m}$. This is above the horizontal so the particle will leave the circle and follow projectile motion.

6 a Total energy at bottom $= \frac{1}{2}mv^2$

$= 0.5 \times 0.4 \times 6^2 = 7.2\,\text{J}$

At top: $mgh + \frac{1}{2}mv^2 = 7.2$

$0.4 \times 10 \times 1 + \frac{1}{2} \times 0.4 \times v^2 = 7.2$

$v = 4\,\text{m s}^{-1}$

b Total force $= mg + \frac{mv^2}{r} = 0.4 \times 10 + \frac{0.4 \times 4^2}{0.5}$

$= 16.8\,\text{N}$

7 a Energy at top of bowl $= mgh + \frac{1}{2}mv^2$

$= 0.15 \times 10 \times 0.5 + 0.5 \times 0.15 \times 2.3^2 = 1.14675\,\text{J}$

The height at any point is $h = 0.75\cos\theta$.

So at any point $mgh + \frac{1}{2}mv^2 = 1.14675$

Total energy $= 0.15 \times 10 \times 0.75\cos\theta + 0.5 \times 0.15 \times v^2$

$= 1.125\cos\theta + 0.075v^2$ ($= 1.14675$)

b Applying Newton's second law towards the centre of the base:

$mg\cos\theta - R = \frac{mv^2}{r}$

When $R = 0$, $10\cos\theta = \frac{v^2}{0.75}$

From total energy equation, $v^2 = \frac{1.14675 - 0.75\cos\theta}{0.075}$

So $7.5\cos\theta = \frac{1.14675 - 1.125\cos\theta}{0.075}$

$0.5625\cos\theta = 1.14675 - 1.125\cos\theta$

$1.6875\cos\theta = 1.14675$

$\theta = \cos^{-1}\left(\frac{1.14675}{1.6875}\right) = 47.2°$

8 At the top of the hemisphere, total energy = mgh = $3 \times 10 \times 2 = 60$ J

Let θ be the angle between the line joining the ball and the centre of base of the sphere and the upwards vertical.

At any point on the hemisphere total energy

$= mgh + \frac{1}{2}mv^2 = 60$

$3 \times 10 \times 2\cos\theta + 0.5 \times 3 \times v^2 = 60$

$60\cos\theta + 1.5v^2 = 60$

$40\cos\theta + v^2 = 40$

Applying Newton's second law towards the centre of the base,

$mg\cos\theta - R = \frac{mv^2}{r}$

When $R = 0$, $10\cos\theta = \frac{v^2}{2}$

$v^2 = 40 - 40\cos\theta$

So $10\cos\theta = \frac{40 - 40\cos\theta}{2}$

$10\cos\theta = 20 - 20\cos\theta$

$30\cos\theta = 20$

$\theta = 48.2°$

The ball leaves the hemisphere when $\theta = 48.2°$.

At this point $v^2 = 40 - 40\cos 48.2°$

$v = 3.65\,\text{m s}^{-1}$

The ball leaves the hemisphere at a speed of $3.65\,\text{m s}^{-1}$.

9 **a** Total energy at bottom of circle $= \frac{1}{2}mv^2 = \frac{1}{2}mu^2$

At any point, total energy $= mgh + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$

At any point, height $= r - r\cos\theta = r(1 - \cos\theta)$

So $mgr(1 - \cos\theta) + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$

$2gr(1 - \cos\theta) + v^2 = u^2$

$v^2 = u^2 - 2gr(1 - \cos\theta)$

b Resolving towards the centre of the circle:

$T - mg\cos\theta = \frac{mv^2}{r}$

$T = mg\cos\theta + \frac{m}{r}(u^2 - 2gr(1 - \cos\theta))$

Exam style questions

1 **a** $\omega = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{10}{1.5\cos 19.5°}} = 2.66\,\text{rad s}^{-1}$

since $\theta = \sin^{-1}\left(\frac{0.5}{1.5}\right) = 19.5°$

$v = \frac{\omega}{r} = \frac{2.66}{0.5} = 5.32\,\text{m s}^{-1}$

b $t = 2\pi\sqrt{\frac{l\cos\theta}{g}} = 2\pi\sqrt{\frac{1.5\cos 19.5°}{10}} = 2.36$ s

2 **a** $\sin\theta = \frac{0.6}{1.3}$ so $\theta = 27.5°$

b Resolving vertically: $T\cos\theta = mg$

$T = \frac{1.5 \times 10}{\cos 27.5°} = 16.9$ N

3 **a** Total energy at top

$= 20 \times 10 \times 2 + 0.5 \times 20 \times 2^2 = 440$ J

Maximum speed is at the bottom.

$0.5 \times 20 \times v^2 = 440$

$v = 6.63\,\text{m s}^{-1}$

b $T - mg = \frac{mv^2}{r}$ so $T = 1080$ N

4 **a** $\omega = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{10}{5\cos 31.33°}} = 1.5302\,\text{rad s}^{-1}$

since $\theta = \sin^{-1}\left(\frac{2.6}{5}\right) = 31.33°$

$v = \frac{\omega}{r} = \frac{1.5302}{2.6} = 0.589\,\text{m s}^{-1}$

b Resolving vertically: $T\cos\theta = mg$

$T = \frac{50 \times 10}{\cos 31.3°} = 585$ N

5 **a** Resolving vertically: $R = 2500g = 25\,000$ N

Resultant force, $F = \frac{mv^2}{r} = \frac{2500 \times v^2}{80}$

The car is in limiting equilibrium so $F = \mu R$.

$\frac{2500 \times v^2}{80} = 0.7 \times 25\,000$

Maximum speed, $v = 23.7\,\text{m s}^{-1} = 85.2\,\text{km h}^{-1}$

b Resolving vertically: $R = 2500g = 25\,000$ N

Resultant force, $F = \frac{mv^2}{r} = \frac{2500 \times \left(\frac{45 \times 1000}{60 \times 60}\right)^2}{80}$

$= 4883$ N

The car is in limiting equilibrium so $F = \mu R$.

$\mu = \frac{F}{R} = \frac{4883}{25\,000} = 0.195$

6 **a** $\omega = \frac{90 \times 2\pi}{60} = 3\pi\,\text{rad s}^{-1}$

b Resolving vertically: $R = 0.2g = 2$ N

Resultant force, $F = mr\omega^2 = 0.2 \times d \times (3\pi)^2$

$= 17.765d$ N

When the object is in limiting equilibrium, $F = \mu R$.

$17.765d = 0.4 \times 2$

$d = 0.0450$ m

The maximum distance from centre that the object can be without sliding off the turntable is 4.50 cm.

7 **a** $\sin\theta = \frac{r}{1.5} = 0.28$

So $r = 1.5 \times 0.28 = 0.42$ m

b $\sin\theta = \frac{28}{100}$ or $\cos\theta = \frac{96}{100}$

$$\omega = \sqrt{\frac{g}{l\cos\theta}} = \sqrt{\frac{10}{1.5\times\frac{96}{100}}} = 2.64 \text{ rad s}^{-1}$$

c Resolving vertically,

$T\cos\theta = 3g$

$T = 31.25\,\text{N}$

8 $\omega = \frac{60\times 2\pi}{60} = 2\pi\,\text{rad s}^{-1}$

Resolving vertically: $R = 0.3g = 3\,\text{N}$

Resultant force, $F = mr\omega^2 = 0.3\times 0.1\times(2\pi)^2 = 1.184\,\text{N}$

When the object is in limiting equilibrium, $F = \mu R$.

$\mu = \frac{F}{R} = \frac{1.184}{3} = 0.395.$

The least possible value of μ is 0.395.

9 a Resolving vertically: $T = 2mg\sin\theta$ (where θ is angle between the horizontal and the string)

$$T = \frac{2mg}{\sin\theta} = \frac{2mg}{\left(\frac{4}{5}\right)} = \frac{100m}{4} = 25m\,\text{N}$$

b Resolving horizontally: $\frac{mv^2}{r} = 2mg\cos\theta$

$\frac{v^2}{3x} = 20\times\frac{3}{5}$

$v = \sqrt{36x}\ \text{m s}^{-1}$

10 a Height $= 4 - 4\cos 50° = 1.4289\,\text{m}$

Total energy $= mgh + \frac{1}{2}mv^2$

$= 0.05\times 10\times 1.4289 + 0.5\times 0.05\times 3^2 = 0.939\,\text{J}$

At the bottom: $\frac{1}{2}mv^2 = 0.939$

$v = 6.13\,\text{m s}^{-1}$

b When $v = 0$, $mgh = 0.939$ so $h = 1.88\,\text{m}$.

It will reach a height of 1.88 m before changing direction and returning towards its starting point.

11 a Total energy $= mgh + \frac{1}{2}mv^2$

$= 0.03\times 10\times 0.6 + 0.5\times 0.03\times 3^2 = 0.315\,\text{J}$

At θ below the horizontal

Total energy $= \frac{1}{2}\times 0.03\times v^2 + 0.03\times 10\times(0.6 - 0.6\sin\theta) = 0.315$

$0.015v^2 + 0.18 - 0.18\sin\theta = 0.315$

$0.015v^2 = 0.135 + 0.18\sin\theta$

$v^2 = 9 + 12\sin\theta$

b When the string becomes slack, $v = 0$ and the reaction force < 0, so it only has PE.

When $v = 0$, $mgh = 0.315$ so $h = 1.05\,\text{m}$.

So $\sin\theta = \frac{(1.05-0.6)}{0.6}$

So $\theta = 48.6°$

So string becomes slack at an angle of 48.6° above the horizontal.

12 a Total energy $= 0.5\times m\times 30x = 15mx$

At any point, total energy $= mgh + \frac{1}{2}mv^2 = \frac{1}{2}mu^2$

At any point, height $= x - x\cos\theta = x(1-\cos\theta)$

So $mgx(1-\cos\theta) + \frac{1}{2}mv^2 = 15mx$

$2gx(1-\cos\theta) + v^2 = 30x$

$v^2 = 30x - 20x(1-\cos\theta)$

$v^2 = 10x + 20x\cos\theta$

$v^2 = 10x(1+\cos\theta)$

Resolving towards the centre of the circle:

$T - mg\cos\theta = \frac{mv^2}{r}$

$T = mg\cos\theta + \frac{m}{x}(20x(1+\cos\theta))$

$T = 10m\cos\theta + 20m(1+\cos\theta))$

$T = 30m\cos\theta + 20m$

$T = 10m(3\cos\theta + 2)$

b When $T = 0$:

$3\cos\theta + 2 = 0$

$\cos\theta = -\frac{2}{3}$

$v^2 = 10x\left(1-\frac{2}{3}\right)$

$v^2 = \frac{10}{3}x$

13 a Let φ be the angle between the horizontal and OB.

Then $\sin\varphi = \frac{0.5}{0.7} = \frac{5}{7}$ but $\theta = 90 - \varphi$, so

$\sin(90° - \theta) = \frac{5}{7} = \cos\theta$

b Applying Newton's second law towards the centre of the base:

$mg\cos\theta - R = \frac{mv^2}{r}$

When $R = 0$:

$10\cos\theta = \frac{v^2}{2}$

$v^2 = 20\cos\theta$

$v = 3.78\,\text{m s}^{-1}$

c At the top of the hemisphere:

total energy $= 0.4\times 10\times 0.7 + 0.5\times 0.4\times u^2$

$= 0.2u^2 + 2.8$

At any point on hemisphere:

total energy $= mgh + \frac{1}{2}mv^2 = 0.2u^2 + 2.8$

$$0.4 \times 10 \times 0.7 \cos\theta + 0.5 \times 0.4 \times v^2 = 0.2u^2 + 2.8$$
$$2.8 \times \frac{5}{7} + 0.2v^2 = 0.2u^2 + 2.8$$
$$2 + 0.2v^2 = 0.2u^2 + 2.8$$
$$0.2v^2 = 0.2u^2 + 0.8$$
$$v^2 = u^2 + 4$$
$$u = 3.21\,\text{m s}^{-1}$$

14 Let the radius of the cone be r and the height be $\frac{4}{3}r$.

The half-angle at the vertex of the cone θ satisfies $\tan\theta = \frac{3}{4}$.

Resolving vertically: $R\sin\theta = mg$

$$R = \frac{10m}{\sin\theta} = \frac{10m}{\left(\frac{3}{5}\right)} = \frac{50m}{3}$$

Let x be the radius of the circle followed by the object.

Resolving horizontally and using Newton's second law:

$R\cos\theta = mx\omega^2$

$$R = \frac{mx\left(\frac{80}{3r}\right)}{\cos\theta} = \frac{\left(\frac{80xm}{3r}\right)}{\left(\frac{4}{5}\right)} = \frac{100xm}{3r}$$

Equating:

$$\frac{50m}{3} = \frac{100xm}{3r}$$

$150mr = 300xm$

$x = \frac{r}{2}$ (i.e. the radius of the circle traced out by the object is half the radius of the cone)

Using trigonometry, $d = \frac{r\tan\theta}{2} = \frac{3r}{4}$ so the perpendicular distance between the circle and the vertex of the cone is $\frac{3r}{4}$ where r is the radius of the cone.

15 a Initially, total energy $= m \times 10 \times 0.5 = 5m$

The maximum speed occurs at the bottom of the circle where $h = 0$

At this points, KE $= \frac{1}{2}mv^2 = 5m$

So $v^2 = 10$

$v = 3.16\,\text{m s}^{-1}$.

So max angular speed $\omega = \frac{v}{r} = \frac{3.16}{0.5} = 6.32\,\text{rad s}^{-1}$.

b At top: loss in KE = gain in GPE

$0.5m(u^2 - v^2) = 0.5mg$

$u^2 - v^2 = g$

so $v^2 = u^2 - g$ ①

Also $T + mg = \frac{mv^2}{r}$ but $r = 0.5$ so

$T + mg = 2mv^2$ ②

Substituting ① in ② gives

$T + mg = 2m(u^2 - g)$

$T + mg = 2mu^2 - 2mg$

$T = 2mu^2 - 3mg$

For complete circles, $T \geqslant 0$ so $2mu^2 - 3mg \geqslant 0$

Therefore $2u^2 \geqslant 3g$ and $u \geqslant \sqrt{\frac{3g}{2}}$.

c Let min speed (at top of circle) be v.

This means the max speed (at bottom of circle) is $2v$.

The speed of projection is u.

At top: loss in KE = gain in GPE

$0.5m(u^2 - v^2) = 0.5mg$

$u^2 - v^2 = g$ ①

At bottom: gain in KE = loss in GPE

$0.5m((2v)^2 - u^2) = 0.5mg$

$0.5m(4v^2 - u^2) = 0.5mg$

$4v^2 - u^2 = g$ ②

① + ② gives $3v^2 = 2g$

$v^2 = \frac{2g}{3}$

So from ① $u^2 = \frac{2g}{3} + g$

$u^2 = \frac{5g}{3}$

$u = \sqrt{\frac{5g}{3}}$

16 a Tension in the string $= 8g = 80\,\text{N}$

$T\cos\theta = 4g$

$\cos\theta = \frac{40}{80} = 0.5$

b Resolving vertically

$T\sin\theta = \frac{mv^2}{r}$

$8g\sin\theta = \frac{4v^2}{2} = 2v^2$

If $\cos\theta = \frac{1}{2}$, then $\sin\theta = \frac{\sqrt{3}}{2}$

$\frac{8g\sqrt{3}}{2} = 2v^2$ so $v^2 = 2\sqrt{3}g$

17 At lowest point, total energy $= 0.5mv^2$

At highest point, total energy $= 0.5mv + 1.5mg$

Energy is conserved so

$0.5mv^2 = 0.5mv + 1.5mg$

$0.5v^2 = 0.5v + 15$

$v^2 = v + 30$

$v^2 - v - 30 = 0$

$(v - 6)(v + 5) = 0$

$v = 6$ or $v = -5$

So $v = 6\,\text{m s}^{-1}$.

So maximum speed is $6\,\text{m s}^{-1}$.

18 Let x = radius of circle and r = radius of bowl.
Let θ be the angle between OP and the circular face.

a $\sin\theta = \frac{0.5r}{r} = 0.5$. Therefore $\theta = 30°$

b Resolving vertically, $10\sin\theta = mg$

Therefore $m = \frac{10 \times 0.5}{10} = 0.5\,\text{kg}$

c Resolving horizontally, $10\cos\theta = \frac{mv^2}{x}$

$\cos\theta = \frac{\sqrt{3}}{2}$ and $x = \sqrt{(r^2 - (0.5r)^2)}$

$x = \frac{\sqrt{3}r}{2}$

Therefore $\frac{10\sqrt{3}}{2} = \frac{0.5v^2}{\frac{\sqrt{3}r}{2}}$

This gives $v^2 = 15r$ so $v = \sqrt{15r}$

19 **a** The height of P at any point $= l - l\cos\theta$
$= l(1 - \cos\theta)$

Total energy $= 0.5 \times 2 \times 10^2 = 100\,\text{J}$

So at any point,

Energy $= 0.5 \times 2 \times v^2 + 2 \times 10 \times l(1 - \cos\theta)$
$= v^2 + 20l(1 - \cos\theta)$

So $v^2 + 20l(1 - \cos\theta) = 100$

So $v^2 = 100 - 20l(1 - \cos\theta)$

Resolving towards O

$T - 20\cos\theta = \frac{2v^2}{l}$

$T = 20\cos\theta + \frac{2v^2}{l}$

$T = 20\cos\theta + \frac{200 - 40l + 40l\cos\theta}{l}$

$T = 20\cos\theta + \frac{200}{l} - 40 + 40\cos\theta$

$T = 60\cos\theta + \frac{200}{l} - 40$

b The max l for complete circular motion would require $v = 0$ at top of circle and reaction force to be greater than 0.

So $mgh = 100$

$2 \times 10 \times 2l = 100$

$40\,l = 100$

$l = 2.5\,\text{m}$

20 **a** Let T be the tension in the string

Resolving vertically:

$T\cos\theta = 2mg$

$T = \frac{2mg}{\cos\theta}$

Resolving these horizontally, by Newton's second law:

$T\sin\theta = 2ma$

$T\sin\theta = 2mrw^2$

$T = \frac{2mr\omega^2}{\sin\theta}$

Equating the above two equations:

$\frac{2mg}{\cos\theta} = \frac{2mr\omega^2}{\sin\theta}$

Since $\sin\theta = \frac{r}{3}$.

$\frac{g}{\cos\theta} = \frac{r\omega^2}{\frac{r}{3}}$

$\cos\theta = \frac{g}{3\omega^2}$

b $\omega = \sqrt{\frac{10}{3\cos\theta}}$

Min value $= 1.83\,\text{rad s}^{-1}$

As θ increases towards 90°, ω increases.

So range of ω values is $\omega \geqslant 1.83\,\text{rad s}^{-1}$

Mathematics in life and work

1 Resolving vertically: $R = mg = 10m\,\text{N}$

Resultant force, $F = \frac{mv^2}{r} = \frac{m \times v^2}{25}$

The go-kart is in limiting equilibrium so $F = \mu R$.

$\frac{m \times v^2}{25} = 0.8 \times 10m$

$\frac{v^2}{25} = 8$

$v = 14.1$

The maximum speed at which the go-karts can travel without sliding is $14.1\,\text{m s}^{-1}$ or $50.8\,\text{km h}^{-1}$.

2 When wet $\frac{m \times v^2}{25} = 0.5 \times 10m$

$\frac{v^2}{25} = 5$

$v = 11.2$

The maximum speed at which the go-karts can travel without sliding is $11.2\,\text{m s}^{-1}$ or $40.2\,\text{km h}^{-1}$.

So the speed would decrease by $2.9\,\text{m s}^{-1}$ or $10.6\,\text{km h}^{-1}$.

4 Hooke's law

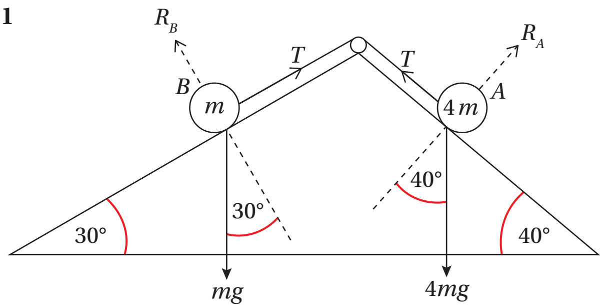
Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1



a $R_A = 4mg\cos 40°$

$R_B = mg\cos 30°$

$F = ma$ for

$4mg\sin 40° - T - 0.1mg\cos 40° = 4ma$ (1)

$T - mg\sin 30° - 0.15mg\cos 30° = ma$ (2)

Adding equations (1) and (2) together

$4mg\sin 40° - 0.1mg\cos 40 - mg\sin 30°$
$- 0.15mg\cos 30° = 5ma$

$40\sin 40 - \cos 40° - 10\sin 30° - 1.5\cos 30° = 5a$

$a = 3.73\,\text{ms}^{-2}$

b Using equation 2:

$T - mg\sin 30° - 0.15mg\cos 30° = 3.73m$

$T = 3.73m + 10m\sin 30° + 1.5m\cos 30°$

$T = 10.03m\,\text{N}$

c Using parallelogram of forces and the cosine rule:

$F^2 = (10.03m)^2 + (10.03m)^2$
$- 2 \times 10.03m \times 10.03m\cos 70$

$F = 11.5m\,\text{N}$

2 a Resolve vertically at Z:

$2T\sin\theta = 4mg$

$T\sin\theta = 2mg$

$T = \dfrac{10mg}{3}$

b Resolve vertically at X:

$R_x = mg + T\sin\theta$

$R_x = 3mg$

Resolve horizontally at X:

$3mg\mu = T\cos\theta$

$3mg\mu = \dfrac{8mg}{3}$

$\mu = \dfrac{8}{9}$

3 Carriage has lost both kinetic energy and potential energy since it has both slowed and dropped

Loss in potential energy = $mgh = 250 \times 10 \times 4 = 10\,000\,\text{J}$

Loss in kinetic energy = $\frac{1}{2}mu^2 - \frac{1}{2}mv^2$

(where u is speed at start and v final speed)

$= \frac{1}{2} \times 250 \times 2^2 - \frac{1}{2} \times 250 \times 1^2 = 375\,\text{J}$

Work done against resistances = 400×100
$= 40\,000\,\text{N}$

Work done by carriage is $40\,000 - 10\,000 - 375 = 29\,625\,\text{J}$ (as the energy lost by the motion provides some of the work required to overcome the resistances).

Exercise 4.1A

1 $T = \dfrac{40 \times 0.2}{3}$

$T = 2.67\,\text{N}$

2 $l = \dfrac{\lambda x}{T}$

$l = 60 \times \dfrac{1.4 - l}{37}$

$37l = 84 - 60l$

$97l = 84$

$l = 86.6\,\text{cm}$

3 $33 = \lambda \times \dfrac{0.3}{0.9}$

$\lambda = 33 \times 3 = 99\,\text{N}$

4 $T = \dfrac{32 \times 0.75l}{l}$

$T = 24\,\text{N}$

5 The percentage of the compression of the spring

$= \dfrac{x}{l} \times 100$

$1.4g = \dfrac{120 \times (l - x)}{l}$

$14l = 120l - 120x$

$120x = 106l$

$\frac{x}{l} = \frac{106}{120}$

Therefore percentage of the compression of the spring is 88.3%.

6 $T = 2g$

$T = 20\,\text{N}$

$l = 30 \times \frac{1.5 - l}{20}$

$20l = 45 - 30l$

$50l = 45$

$l = 90\,\text{cm}$

7 $5.5 \times 10 = \frac{350 \times x}{0.2}$

$x = 3.14\,\text{cm}$

8 The tension acts throughout the strings and is equal to $0.5g = 5\,\text{N}$.

For S_1: $5l_1 = 17 \times x_1$

$5 \times 1 = 17x_1$

$x_1 = 0.294\,\text{m}$

For S_2: $5l_2 = 20x_2$

$5 \times 0.6 = 20x_2$

$x_2 = 0.15\,\text{m}$

Total length is $1 + 0.294 + 0.6 + 0.15 = 2.04\,\text{m}$

9 Resolving vertically: $2g + T_2 = T_1$

Let x_1 be the extension in SP and x_2 be the extension in TP.

$1.5 + x_1 + x_2 = 2.7$

$x_1 + x_2 = 1.2$

$T_1 = \frac{45x_1}{0.75} = 60x_1$

$T_2 = \frac{45x_2}{0.75} = 60x_2$

$2g + 60x_2 = 60x_1$

$20 + 60x_2 = 60(1.2 - x_2)$

$120x_2 = 52$

$x_2 = 0.433\,\text{m}$

$x_1 = 0.767\,\text{m}$

10 Let x be the extension in string attached to X.

Let y be the extension in string attached to Y.

$x + y = 0.7$

$T_X = \frac{\lambda x}{0.5}$

$T_Y = \frac{3\lambda y}{0.3}$

As the particle is in equilibrium $T_X = T_Y$ so:

$\frac{\lambda x}{0.5} = \frac{3\lambda y}{0.3}$

$2x = 10y$

$2x = 10(0.7 - x)$

$12x = 7$

$x = 0.583\,\text{m}$

$y = 0.117\,\text{m}$

11 $T = \frac{130 \times 0.1}{0.5}$

$T = 26\,\text{N}$

Resolving parallel to the plane:

$26 = 4g \sin\theta$

$\sin\theta = \frac{26}{40}$

$\theta = 40.5°$

12 Let x be the extension in the string attached to point A.

Let y be the extension in the string attached to point B.

$2 + x + 3 + y = 7$

$x + y = 2$

$T_x = \frac{16x}{2} = 8x$

$T_y = \frac{14y}{3}$

$8x = \frac{14y}{3}$

$24x = 14y$

$24x = 14(2 - x)$

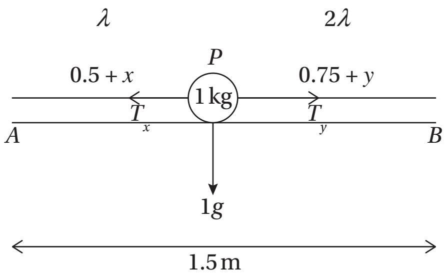
$38x = 28$

$x = 0.737\,\text{m}$

$y = 1.26\,\text{m}$

13 Let x be the extension in string AP.

Let y be the extension in string BP.



$y = 0.15\,\text{m}$ (given in the question)

$x = 1.5 - 0.5 - 0.75 - 0.15 = 0.1\,\text{m}$

First calculate the tensions in the strings so you know in which direction friction will be acting.

$T_x = \frac{\lambda \times 0.1}{0.5} = 0.2\lambda$

$T_y = \frac{2\lambda \times 0.15}{0.75} = 0.4\lambda$

Given that $T_y > T_x$ friction will act to oppose force T_y.

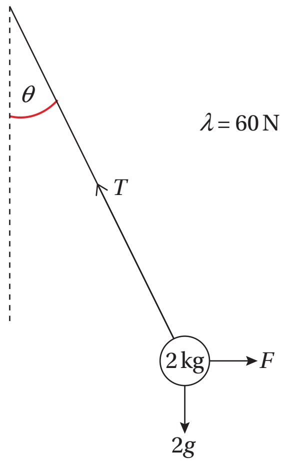
$$T_x + \mu R = T_y$$
$$0.2\lambda + 0.3 \times 1 \times 9.8 = 0.4\lambda$$
$$0.2\lambda = 2.94$$
$$\lambda = 14.7\,\text{N}$$

14 $T = \dfrac{0.1 \times 60}{0.3} = 20\,\text{N}$

Resolve vertically: $20 \cos\theta = 2g$

$\cos\theta = 0.98$

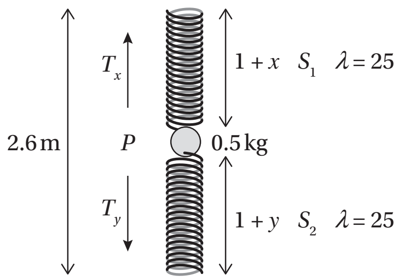
$\theta = 11.478\ldots = 11.5°$



Resolve horizontally: $20 \sin 11.48° = F$

$F = 3.98\,\text{N}$

15



Let x be the extension in S_1.

Let y be extension in S_2.

$x + y = 0.6$

$T_x = 25x$

$T_y = 25y$

Resolving vertically: $25x = 25y + 0.5g$

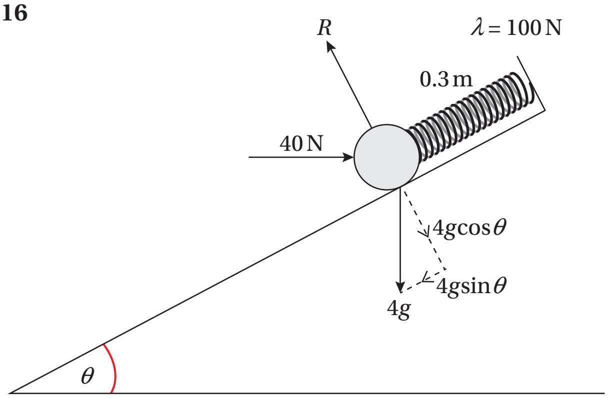
$25x = 25(0.6 - x) + 5$

$50x = 20$

$x = 0.4\,\text{m}$

$y = 0.2\,\text{m}$

16



If $\sin\theta = 0.8$ then $\cos\theta = 0.6$.

Thrust in compressed spring $= \dfrac{100 \times 0.2}{0.5} = 40$

Resolving perpendicular to the slope:

$R = 40 \sin\theta + 4g \cos\theta = 40 \times 0.8 + 39.2 \times 0.6$

$R = 55.52\,\text{N}$

Resolving parallel to the slope:

$4g \sin\theta + 40 = 40 \cos\theta + \mu \times R$

$40 \times 0.8 + 40 = 40 \times 0.6 + \mu \times 55.52$

$72 = 24 + 55.52\mu$

$\mu = 0.865$

17 Let T be the force in the stretched string.

$F = ma$

$T - \mu R = 0.5 \times 0.25$

$T - 0.5 \times 4.9 = 0.125$

$T = 2.575$

$2.575 = 40 \times \dfrac{x}{0.2}$

$x = 0.013$

Length of string $= 0.2 + 0.013 = 0.213\,\text{m}$

Exercise 4.2A

1 $W = \dfrac{8 \times 0.5^2}{2 \times 0.5} = 2\,\text{J}$

2 $\text{EPE} = \dfrac{7 \times 0.5^2}{2 \times 0.7} = 1.25\,\text{J}$

3 $T = \dfrac{\lambda x}{l} = 15x = 2g$

$20 = 15x$

$x = 1.33$

$\text{EPE} = \dfrac{15 \times 1.33^2}{2 \times 1} = 13.3\,\text{J}$

4 EPE $= \dfrac{50 \times 1.5^2}{2 \times 2} = 28.125\,\text{J}$

Using conservation of energy:

KE gain = EPE loss

$\frac{1}{2}mv^2 = 28.125$

$v^2 = 56.25$

$v = 7.5\,\text{m s}^{-1}$

5 $30 = \dfrac{2000 \times x^2}{2 \times 1}$

$x^2 = 0.03$

$x = 0.173$

The stretched length of the string is 1.17 m.

6 $60 = \dfrac{48 \times (3l)^2}{2 \times l}$

$60 = 216l$

$l = 0.278\,\text{m}$

7 Gain in EPE $= \dfrac{100 \times 1.9^2}{2 \times 2} - \dfrac{100 \times 1.4^2}{2 \times 2} = 41.25\,\text{J}$

8 EPE $= \dfrac{50 \times 4^2}{2 \times 1} = 400\,\text{J}$

By conservation of energy:

Loss of EPE = gain in KE + work done against friction

$400 = \frac{1}{2} \times 2.5 \times v^2 + 0.3 \times 2.5 \times 10 \times 4$

$v^2 = 296$

$v = 17.2\,\text{m s}^{-1}$

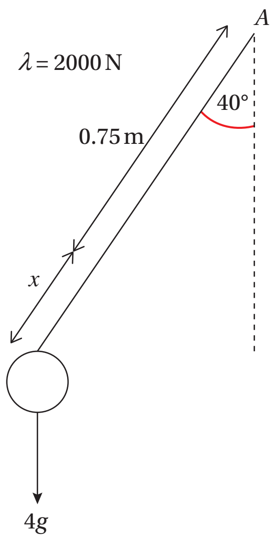
9 Resolving vertically:

$T\cos 40° = 4g$

$\dfrac{x \times 2000}{0.75}\cos 40° = 40$

$x = 0.0196$

The stretched length of the string is 0.770 m.



$F = ma$

$T\sin\theta = mr\omega^2$

where

$T = \dfrac{0.019581 \times 2000}{0.75} = 52.22$

$r = 0.7696\sin 40° = 0.4947$

$52.22\sin 40° = 4 \times 0.4947 \times \omega^2$

$\omega^2 = 16.96$

$\omega = 4.12\,\text{rad s}^{-1}$

10 loss of EPE = gain in KE

$\dfrac{5 \times 1.3^2}{2 \times 1.7} = \dfrac{1}{2} \times 2 \times v^2$

$v = 1.58\,\text{m s}^{-1}$

11 As at lowest point $v = 0$

loss of EPE = PE gain

$1.3g(0.5 + x) = \dfrac{20 \times x^2}{2 \times 1.5}$

$20x^2 - 39x - 19.5 = 0$

$x = 2.36$ or -0.413

So $x = 2.36$ and the lowest point is 3.86 m below X.

Let the extension at equilibrium point be y.

$1.3g = \dfrac{20 \times y}{1.5}$

$y = 0.975\,\text{m}$

KE gained + EPE gained = PE lost

$\frac{1}{2} \times 1.3v^2 + \dfrac{20 \times 0.975^2}{2 \times 1.5} = 1.3 \times 10 \times 1.48$

$0.65v^2 + 6.3375 = 19.24$

$v = 4.46\,\text{m s}^{-1}$

12 WD $= \dfrac{1000 \times 40^2}{2 \times 300} = 2670\,\text{J}$

13 Gain in EPE = loss in GPE

$\dfrac{\lambda \times 1.5^2}{2 \times 0.5} = 0.7 \times 10 \times 2$

$\lambda = 6.22\,\text{N}$

14 Model the doll as a particle and assume that motion is vertical (not side to side).

At start total energy is EPE and PE:

EPE $= \dfrac{80 \times 0.2^2}{2 \times 0.3} = 5.33\,\text{J}$

PE $= 0.6 \times 10 \times 0.1 = 0.6\,\text{J}$

At maximum distance the spring will have extended by x so will still contain EPE, $v = 0$ so KE = 0 and it will have gained height so will also have gained PE.

EPE $= \dfrac{80 \times x^2}{2 \times 0.3} = 133.3x^2$

PE $= 0.6 \times 10 \times (x + 0.2) = 6x + 1.2$

By conservation of energy:

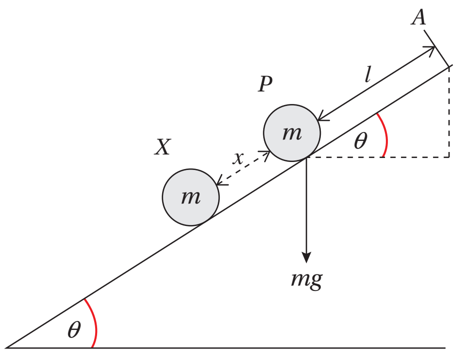
$133.3x^2 + 6x + 1.2 = 5.93$

$133.3x^2 + 6x - 4.73 = 0$

$x = 0.167$ or -0.212 (this is the start position)

So the doll rises a maximum distance of $0.3 + 0.167 = 0.467$ m.

15



Relative to A:

At point P: EPE = 0

$\text{GPE} = -mgl\sin\theta = -\frac{5}{13}mgl$

At point X: $\text{EPE} = \frac{\lambda \times x^2}{2 \times l}$

$\text{GPE} = -mg(l + x)\sin\theta = -\frac{5}{13}mg(l + x)$

At both points KE = 0 as at rest.

By conservation of energy:

$$-\frac{5}{13}mgl = -\frac{5}{13}mgl - \frac{5}{13}mgx + \frac{\lambda \times x^2}{2 \times l}$$

$$\frac{5}{13}mgx = \frac{\lambda \times x^2}{2 \times l}$$

$$x = \frac{10lmg}{13\lambda}$$

16 At start: $\text{EPE} = \frac{8 \times 2^2}{2 \times 2} = 8$ J

After the car has travelled 2 m and the string becomes slack:

EPE lost = gain in KE + work done against resistance

$8 = \frac{1}{2} \times 0.2v^2 + 2 \times 1$

$v^2 = 60$

$v = 7.75\ \text{m s}^{-1}$

So the velocity at the point that the string stops pulling car is $7.75\ \text{m s}^{-1}$.

Calculate the deceleration:

$F = ma$

$1 = 0.2a$

$a = 5\ \text{m s}^{-2}$ (deceleration)

$v^2 = u^2 + 2as$

$v^2 = 60 - 2 \times 5 \times 2$

$v = 6.32\ \text{m s}^{-1}$

The speed at which the car crashes into the wall is $6.32\ \text{m s}^{-1}$.

17 Assume that man falls exactly 70 m and starts 70 m above ground.

At start: GPE = 86 × 10 × 70 = 60 200

At bottom: gain in EPE = loss of GPE

$\frac{5000 \times x^2}{2 \times l} = 60\,200$

$120\,400l = 5000x^2$

Given that the stretched length of rope must not exceed 70 m: $x + l = 70$

So the equation becomes:

$24.08l = x^2$

$24.08l = (70 - l)^2$

$l^2 - 164.08l + 4900 = 0$

$l = 125$ or 39.3

Hence maximum value for the rope's natural length is 39.3 m.

18 Length of elastic $= 2 \times (0.075^2 + 0.30^2)^{0.5} = 0.61847$ m

Loss of EPE = gain in KE

$\frac{50 \times 0.41847^2}{2 \times 0.2} = \frac{1}{2} \times 0.1 \times v^2$

$v^2 = 437.8$

$v = 20.9\ \text{m s}^{-1}$

19 The model plane starts at equilibrium point.

Resolve vertically: $0.4g = \frac{15x}{0.4}$

$x = 0.107$ m

When plane has been pulled down:

$\text{EPE} = \frac{15 \times 0.6^2}{2 \times 0.4} = 6.75$ J

Find the speed when string becomes slack after release:

Loss of EPE = gain in KE + gain in GPE

$6.75 = \frac{1}{2} \times 0.4 \times v^2 + 0.4 \times 10 \times 0.6$

$6.75 = 0.2v^2 + 2.4$

$v = 4.66\ \text{m s}^{-1}$

Using this as u for motion under gravity, calculate how far the plane would travel if the ceiling was not present.

$v^2 = u^2 + 2as$

$0 = 21.75 + 2(-10)s$

$s = 1.09$ m

So the plane would hit the ceiling.

20 At top: GPE = $5 \times 10 \times 3 = 150$ J

After falling 2 m as the string is still slack:

Loss in GPE = gain in KE

$150 - 5 \times 10 \times 1 = \frac{1}{2} \times 5 \times v^2$

$v = 6.32\,\text{m s}^{-1}$

Loss in GPE = gain in EPE

$150 - 5 \times 10 \times (1 - x) = \frac{9400 \times x^2}{2}$

$100 + 50x = 4700x^2$

$4700x^2 - 50x - 100 = 0$

$x = 0.15$ or -0.14

So the length of the string when the particle first comes to instantaneous rest is 1.15 m.

21 a KE = $\frac{1}{2} \times 300 \times 10^2 = 15\,000$ J

b By conservation of energy: EPE transferred to elastic rope = 15 000 J

$\frac{2000 \times x^2}{2 \times 5} = 15\,000$

$x = 8.7$ m

Distance from O will be 13.7 m.

c The calculation above assumes no resistance forces.

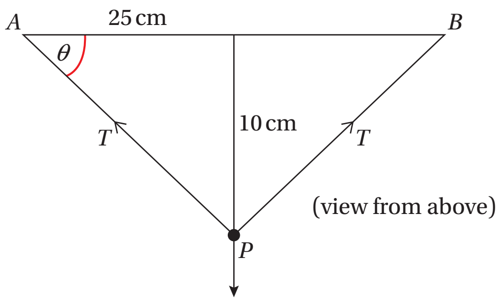
Exam-style questions

1 a $T = \frac{\lambda x}{l}$

$300 = \frac{\lambda \times 10}{40}$

$\lambda = 1200$ N

b



$T = \frac{1200 \times 6.926}{20} = 416$

Resolving perpendicular to the line AB

$2T\sin\theta = 2a$

$2 \times 416 \times \frac{10}{26.926} = 2a$

$a = 154\,\text{m s}^{-2}$

c By conservation of energy

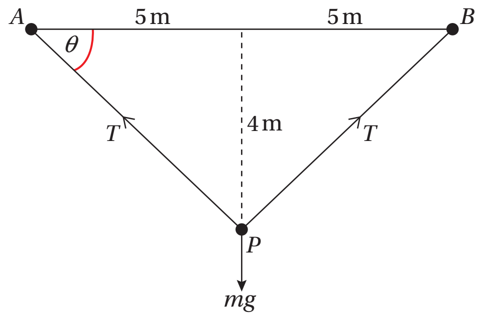
EPE at start = EPE at O + KE at O

$\frac{1200 \times 0.1385^2}{2 \times 0.4} = \frac{1200 \times 0.1^2}{2 \times 0.4} + \frac{1}{2} \times 2 \times v^2$

$v^2 = 13.8$

$v = 3.71\,\text{m s}^{-1}$

2



a Length $AP = \sqrt{5^2 + 4^2} = 6.403$ m

$T = \frac{60 \times 1.403}{3} = 28.1$ N

Resolve vertically: force up = force down as in equilibrium

$2T\sin\theta = mg$

$mg = 2 \times 28.1 \times \frac{4}{\sqrt{41}} = 35.1$ N

$m = 3.51$ kg

b EPE = $\frac{60 \times 2.806^2}{2 \times 6} = 39.4$ J

3 a Loss in GPE = gain in KE + gain in EPE

$mg(l + x) = \frac{1}{2} \times m \times v^2 + \frac{2mg \times x^2}{2 \times l}$

$2g(l + x) = v^2 + \frac{2g \times x^2}{l}$

$v^2 = 2g(l + x) - \frac{2gx^2}{l}$

b $2v\frac{dv}{dx} = 2g - \frac{4gx}{l}$

$\frac{dv}{dx} = 0, 2g - \frac{4gx}{l} = 0$

$x = \frac{1}{2}l$

$v^2 = 2g\left(l + \frac{1}{2}l\right) - \frac{2g\left(\frac{1}{2}l\right)^2}{l}$

$v^2 = \frac{5gl}{2}$

$v = \sqrt{\frac{5gl}{2}}$

c When $v = 0$:

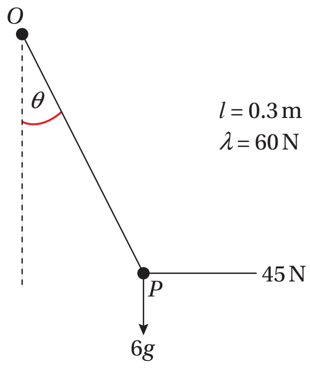
$2g(l + x) - \frac{2gx^2}{l} = 0$

$l + x - \frac{x^2}{l} = 0$

$x^2 - xl - l^2 = 0$

$x = \frac{(1 + \sqrt{5})l}{2}$

4



Resolve vertically: $T\cos\theta = 60$

Resolve horizontally: $T\sin\theta = 45$

$\tan\theta = 0.75$, so $\sin\theta = 0.6$

$T \times 0.6 = 45$

$T = 75\text{ N}$

Compare the equation for EPE and tension in an extended string:

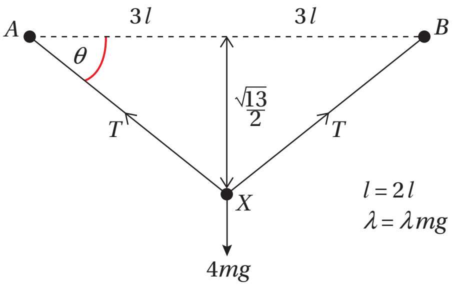
$\text{EPE} = \dfrac{xT}{2}$

$8 = 37.5x$

$x = 0.213\text{ m}$

So the extended length of the string is $0.3 + 0.213 = 0.513\text{ m}$.

5 a $AX = \sqrt{(3l)^2 + \left(\dfrac{\sqrt{13}l}{2}\right)^2} = \dfrac{7l}{2}$



$2T\sin\theta = 4mg$

$T\dfrac{\frac{\sqrt{13}}{2}}{\frac{7}{2}} = 2mg$

$T = \dfrac{14mg}{\sqrt{13}}$

$\dfrac{14mg}{\sqrt{13}} = \dfrac{\lambda mg\frac{3l}{2}}{2l}$

$\lambda = \dfrac{56}{3\sqrt{13}}$

b $AX = \sqrt{(3l)^2 + (12l)^2} = 12.369l$

Extension in each string is $10.369l$.

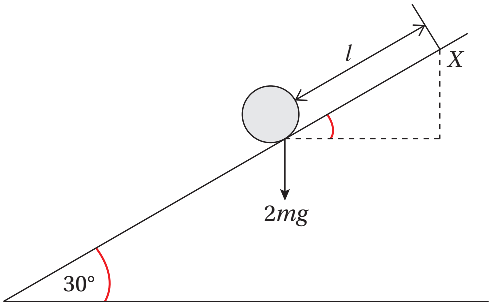
EPE lost = gain in GPE + gain in KE

$$2 \times \frac{\frac{56}{3\sqrt{13}}mg \times (10.369l)^2}{2 \times 2l} = 4mg \times 7l + \frac{1}{2}mv^2$$

$$\frac{56}{3\sqrt{13}}g \times 107.52l = 56gl + v^2$$

$$v^2 = 501gl$$

6 a GPE lost = EPE gained



$$2mg \times \sin 30° = \frac{5mgx^2}{2l}$$

$$2l = 5x^2$$

$$x = \sqrt{0.4l}$$

b Greatest speed is when $a = 0$.

This is when force up the slope = force down slope.

$T = mg\sin 30°$

$\dfrac{5mgx}{l} = mg\sin 30°$

$x = \dfrac{l}{10}$

Gain in KE + gain in EPE = loss in PE

$$\frac{1}{2}mv^2 + \frac{5mg\left(\frac{l}{10}\right)^2}{2l} = mg\frac{l}{10}\sin 30°$$

$$v^2 = \frac{lg}{10} - \frac{lg}{20}$$

$$v = \sqrt{\frac{lg}{10}}$$

7 a When at rest $v = 0$, hence kinetic energy = 0.

EPE gained = GPE lost

$\dfrac{19.6x^2}{4} = (2 + x) \times 0.9g$

$19.6x^2 - 36x - 72 = 0$

$x = 3.04$ (or -1.21)

$OA = 5.04\text{ m}$

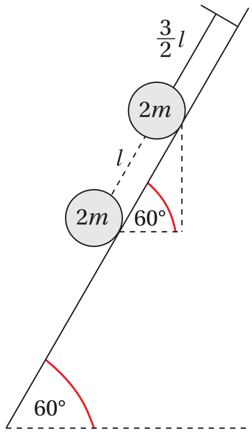
b Resolve vertically:

$F = ma$

$\dfrac{19.6 \times 3.04}{2} - 0.9 \times 10 = 0.9a$

$a = 23.1\text{ m s}^{-2}$

8 **a** KE at start + EPE at start = EPE at end – loss in GPE

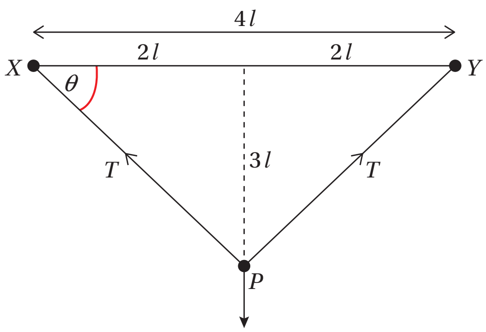


$$\frac{1}{2}\times 2m\times U^2+\frac{3mg\left(\frac{l}{2}\right)^2}{2l}=\frac{3mg\left(\frac{3l}{2}\right)^2}{2l}$$
$$-\,2mgl\sin 60°$$
$$U^2=\frac{27gl}{8}-\frac{8\sqrt{3}gl}{8}-\frac{3gl}{8}$$
$$U^2=gl\left(3-\sqrt{3}\right)$$
$$U=\sqrt{gl\left(3-\sqrt{3}\right)}$$

b Gain in KE + gain in PE = loss of EPE

$$\frac{1}{2}\times 2m\times v^2+2mg\times 1.5l\sin 60=\frac{3mg\left(\frac{3l}{2}\right)^2}{2l}$$
$$v^2=\frac{27gl}{8}-\frac{12\sqrt{3}gl}{8}$$
$$v=\sqrt{\frac{27gl}{8}-\frac{12\sqrt{3}gl}{8}}$$

9



$$XP=\sqrt{(2l)^2+(3l)^2}=\sqrt{13}l$$
$$2T\sin\theta=2mg$$
$$T\times\frac{3}{\sqrt{13}}=mg$$
$$T=\frac{\sqrt{13}}{3}mg$$

Using Hooke's law:

$$T=\frac{\lambda\left(2\sqrt{13}-5\right)}{5}=\frac{\sqrt{13}}{3}mg$$
$$\lambda=\frac{5\sqrt{13}}{3\left(2\sqrt{13}-5\right)}mg=\frac{65mg}{78-15\sqrt{13}}$$

10 **a** EPE released = KE gained

$$\frac{\lambda\left(\frac{l}{2}\right)^2}{2l}=\frac{m2gl}{2}$$
$$\lambda=8mg$$

b EPE released = gain in PE + gain in KE

$$\frac{8mg\left(\frac{3l}{4}\right)^2}{2l}=mg\frac{3l}{4}+\frac{1}{2}mv^2$$
$$\frac{9lg}{4}=\frac{3lg}{4}+\frac{2v^2}{4}$$
$$v^2=3lg$$
$$v=\sqrt{3lg}$$

11 **a** Resolve vertically:

$$0.9g=\frac{0.45\lambda}{1.3}$$
$$\lambda=26\,\text{N}$$

b Original EPE = EPE at instantaneous rest – loss in GPE

$$\frac{26\times 0.4^2}{2\times 1.3}=\frac{26\times x^2}{2\times 1.3}-0.9g(x+0.4)$$
$$4.16=26x^2-23.4x-9.36$$
$$26x^2-23.4x-13.52=0$$
$$x=1.3\ (\text{or }-0.4)$$
$$AB=2.6\,\text{m}$$

12 **a** Resolving vertically:

$$\frac{200x}{0.5}=20g$$
$$x=0.5\,\text{m}$$

Initial extension = 1 m

EPE = $\frac{200\times 1^2}{2\times 0.5}$ = 200 J at a distance of 0.5 m below equilibrium point.

b EPE at start = gain in GPE + gain in KE + EPE when particle first comes to rest

$$200=20\times 10\times(1.5-x)+\frac{1}{2}\times 20\times v^2$$
$$+\frac{200\times(1.5-x)^2}{2\times 0.5}$$
$$200=300-200x+10v^2+200x^2-600x+450$$
$$v^2=20x^2-80x+55$$

$v = 0$ as particle is at rest.

$$x=0.882\,\text{m}$$

c Assumptions: no air resistance, and that the string remained taut and therefore had some EPE.

13 a In equilibrium

$$\frac{\lambda \times 0.5}{2} = 1.5 \times g$$

$$\lambda = 60\text{ N}$$

b From conservation of energy (taking point A as $h = 0$)

PE when raised = EPE at 2 m + KE at 2 m + GPE at 2 m

At 2 m, extension of spring = 0, therefore EPE = 0

$$1.5 \times g \times (0.8 + x) = \frac{60 \times (0)^2}{2 \times 2} + \frac{1}{2} \times 1.5 \times v^2 + 1.5 \times g \times x$$

$12 + 15x = 0.75v^2 + 15x$

$0.75v^2 = 12$

$v^2 = 16$

From equations of motion

$v^2 = u^2 + 2as$

At A, $v = 0$, $a = -10$

$0 = 16 + (2 \times -10 \times x)$

$20x = 16$

$x = 0.8\text{ m}$

So $OA = 0.8 + 2 = 2.8$ m

14 a Let x be the extension in spring one and, let y be the extension in spring 2.

$(0.6 + x) + (0.9 + y) = 2$

$x + y = 0.5$

$y = 0.5 - x$

In equilibrium so:

$$\frac{3x}{0.6} = \frac{2y}{0.9}$$

$$\frac{3x}{0.6} = \frac{1 - 2x}{0.9}$$

$x = 0.15$

$AP = 0.75\text{ cm}$

b EPE at start = EPE in AP + EPE in PB

$$\frac{3 \times 0.2^2}{2 \times 0.6} + \frac{2 \times 0.7^2}{2 \times 0.9} = \frac{29}{45}$$

When particle is next at instantaneous rest $v = 0$ and let distance of particle from $B = y$

From conservation of energy

$$\frac{2 \times (0.9 - y)^2}{2 \times 0.9} + \frac{3 \times (1.4 - y)^2}{2 \times 0.6} = \frac{29}{45}$$

$$\frac{3.24 - 7.2y + 4y^2}{3.6} + \frac{17.64 - 25.2y + 9y^2}{3.6} = \frac{29}{45}$$

$13y^2 - 32.4y + 18.56 = 0$

$y = 1.6$ or 0.89

$y = 1.6$ is start position so $AC = 2 - 0.89 = 1.11\text{ m}$

15 a Resolving vertically:

$$\frac{50x}{0.3} = 2.4g$$

$x = 0.144\text{ m}$

$$\text{EPE} = \frac{50 \times 0.144^2}{2 \times 0.3} = 1.73\text{ J}$$

b Mass comes to rest when $v = 0$.

EPE before = EPE when next at rest + gain in PE

$$1.728 = \frac{50 \times x^2}{2 \times 0.3} + 1.2 \times 10 \times (0.144 + x)$$

$50x^2 + 7.2x = 0$

$x = 0$, i.e. the spring has returned to its natural length.

16 a EPE at X = work done against friction + EPE at Y

$$\frac{36 \times 3.1^2}{2 \times 0.9} = 6F + \frac{36 \times 1.1^2}{2 \times 0.9}$$

$6F = 168$

$F = 28\text{ N}$

b At Y tension $= \dfrac{1.1 \times 36}{0.9} = 44\text{ N}$ which is greater than 28 N so the particle starts to move again.

When particle is next at rest the work done against friction has used up all of the EPE from Y.

EPE at Y = work done against friction

$$\frac{36 \times 1.1^2}{2 \times 0.9} = 28s$$

$s = 0.86\text{ m}$

Total distance = 6 + 0.86 = 6.86 m

17 a $l = 0.8$, $\lambda = 30\text{ N}$ $m = 0.3\text{ kg}$

Let x be extension at equilibrium

At equilibrium

$$0.3g = \frac{30x}{0.8}$$

$$x = \frac{0.3 \times 10 \times 0.8}{30} = 0.08\text{ m}$$

b Particle is extended by a further distance y

$$\text{EPE at extension} = \frac{30 \times (0.08 + y)^2}{2 \times 0.8}$$

When particle is released and has returned to original length, by conservation of energy

EPE loss = gain in KE + gain in GPE

$\frac{30 \times (0.08 + y)^2}{2 \times 0.8} = \frac{1}{2} \times 0.3 \times v^2 + 0.3 \times 10\,(0.08 + y)$

$18.75(0.0064 + 0.16y + y^2) = 0.15v^2 + 0.24 + 3y$

$0.15v^2 = 18.75y^2 - 0.12$

$v^2 = 125y^2 - 0.8$

Then motion under gravity $u^2 = 125y^2 - 0.8$, $v = 0.1$, $s = 0.8$, $a = -10$

$v^2 = u^2 + 2as$

$0.01 = 125y^2 - 0.8 + 2 \times (-10) \times 0.8$

$125y^2 = 16.81$

$y = 0.367\,\text{cm}$

So $AB = 0.8 + 0.08 + 0.367 = 1.25\,\text{m}$

18 $l = 1$, modulus of elasticity $= \lambda$

$\text{EPE} = \frac{\lambda \times 2^2}{2 \times 1} = 2\lambda$

When string is no longer stretched by conservation of energy

EPE lost = gain in KE + gain in GPE

$2\lambda = \frac{1}{2} \times 0.5 \times v^2 + 0.5 \times 10 \times 2$

$2\lambda = 0.25v^2 + 10$

$v^2 = 8\lambda - 40$

Motion under gravity

Assuming that it reaches $v = 0$ when $s > 1$, $u^2 = 8\lambda - 40$, $a = -10$

$v^2 = u^2 + 2as$

$0 < 8\lambda - 40 + 2 \times (-10) \times 1$

$8\lambda > 60$

$\lambda > 7.5$

19 a $T = \frac{\lambda mg \times 0.3L}{L} = 0.3\lambda mg$

b Resolving vertically:

$mg = T\cos 60°$

$mg = 0.3\lambda mg \times 0.5$

$\lambda = \frac{20}{3}$

c $T = 2mg$

$F = ma$

$2mg\sin 60° = mr\omega^2$

$\sqrt{3}g = (1.3L\sin 60°)\,\omega^2$

$\omega = \sqrt{\frac{2g}{1.3L}}$

20 a Loss in GPE = Gain in KE + gain in EPE

$10m(l + x) = \frac{1}{2}mv^2 + \frac{30mx^2}{2 \times l}$

$20(l + x) = v^2 + \frac{30x^2}{l}$

$v^2 = 20(l + x) - \frac{30x}{l}$

Maximum velocity occurs when particle passes though equilibrium point.

$mg = \frac{3mgx}{l}$

which is when

$l = 3x$

$v^2 = 20(l + x) - \frac{30x^2}{l}$

$v^2 = 20(3x + x) - \frac{30x^2}{3x}$

$= 80x - 10x$

$= 70x$

$v = \sqrt{70x}$

$= \sqrt{\frac{70l}{3}}$

b At A $v = 0$ so KE = 0

so by conservation of energy

$\frac{3mgx^2}{2 \times l} = mg(l + x)$

$\frac{3x^2}{2l} = (l + x)$ (Note this equation can also be found by setting $v^2 = 0$ in equation from part **a**)

$3x^2 = 2l^2 + 2lx$

$3x^2 - 2lx - 2l^2 = 0$

$x = \frac{2l \pm \sqrt{\left((-2l)^2 - 4 \times 3 \times (-2l)^2\right)}}{6}$

$x = \frac{2l \pm \sqrt{(52l^2)}}{6}$

$x = \frac{l \pm 2l\sqrt{13}}{3}$

So distance $AB = l + \frac{l + 2l\sqrt{13}}{3}$

$= \frac{l\left(4 + \sqrt{13}\right)}{3}$

Mathematics in life and work

1 When speed is at a maximum, $a = 0$ and jumper is in equilibrium.

Let x be the distance from platform.

Resolving vertically:

$\frac{3645(x - 36)}{36} = 88g$

$x = 44.69\,\text{m}$

Extension of cord at maximum speed is 8.69 m.

Gain in EPE + KE = loss in GPE

$\frac{3645\,(x - 36)^2}{2 \times 36} + \frac{88v^2}{2} = 88gx$

$v^2 = 2gx - \frac{405(x - 36)^2}{352}$

$v^2 = 806.9$

$v = 28.4\,\text{m}\,\text{s}^{-1}$

2 From equation above:

$$\frac{3645\,(x-36)^2}{2\times 36}+\frac{88v^2}{2}=88gx$$

At maximum extension, $v = 0$

$50.625(x-36)^2 = 880x$

$50.625(x^2 - 72x + 1296) = 880x$

$50.625x^2 - 3645x + 65\,610 = 880x$

$50.625x^2 - 4525x + 65\,610 = 0$

$x = 71.2$ m (or 18.2 m)

$x = 71$ m to nearest metre

So assuming the man is less than 3 m tall, the minimum value of y is 74 m.

3 No air resistance, always assumed extension is greater than 36 m, assume man is less than 3 m tall. Modelled man as particle until final stage.

5 Linear motion under a variable force

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 $\frac{x}{x+1} = 1 - \frac{1}{x+1}$

2 $\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int x\,dx$

$\ln y = \frac{1}{2}x^2 + c$

$2\ln y = x^2 + d$

3 $\frac{dy}{dx} = \frac{\sin x}{y}$

$\int y\ dy = \int \sin x\ dx$

$\frac{1}{2}y^2 = -\cos x + c$

$y^2 = -2\cos x + d$

4 $\frac{dy}{dx} = \frac{5}{e^x}$

$\int 1\ dy = \int 5e^{-x}\ dx$

$y = -5e^{-x} + c$

5 $\frac{dy}{dx} = 4x^2 - 25$

$\int 1\ dy = \int (4x^2 - 25)\,dx$

$y = \frac{4}{3}x^3 - 25x + c$

6 $x\frac{dy}{dx} = 4x^2 - 25$

$\int 1\,dy = \int \frac{4x^2 - 25}{x}\,dx$

$\int 1\,dy = \int \left(4x - \frac{25}{x}\right)dx$

$y = 2x^2 - 25\ln x + c$

7 $x\frac{dy}{dx} = y^2$

$\int y^{-2}\,dy = \int \frac{1}{x}\,dx$

$\frac{-1}{y} = \ln x + c$

When $y = 4$, $x = 1$

$-\frac{1}{4} = \ln 1 + c$

$c = -\frac{1}{4}$

So $\frac{-1}{y} = \ln x - \frac{1}{4}$

$\frac{1}{y} = \frac{1}{4} - \ln x$

8 $x\frac{dy}{dx} = e^y$

$\int e^{-y}\,dy = \int \frac{1}{x}\,dx$

$-e^{-y} = \ln x + c$

$y = 0$ when $x = 4$

$-e^{-0} = \ln 4 + c$

$c = -\ln 4$

$-e^{-y} = \ln x - \ln 4$

$-e^{-y} = \ln\frac{x}{4}$

$e^{-y} = \ln\frac{4}{x}$

Exercise 5.1A

1 a $v = \int t^2 - 2t\ dt$

$v = \frac{1}{3}t^3 - t^2 + c$

When $t = 0$, $v = 14$

So $c = 14$

$v = \frac{1}{3}t^3 - t^2 + 14$

b $s = \int \frac{1}{3}t^3 - t^2 + 14\ dt$

$s = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 14t + c$

When $t = 0$, $s = 0$

So $c = 0$

$s = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 14t$

2 a $a = \frac{dv}{dt}$

$v = 5(5t+2)^{-1}$

$\frac{dv}{dt} = -5 \times 5(5t+2)^{-2}$

$a = \frac{-25}{(5t+2)^2}$

b $s = \int \frac{5}{5t+2}\,dt$

$s = \ln(5t+2) + c$

When $t = 0$, $s = 0$

So $c = -\ln 2$

$s = \ln(5t+2) - \ln 2$

$s = \ln\left(\frac{5t+2}{2}\right)$

3 **a** $a = 5e^{-2t}$

$v = \int 5e^{-2t}\,dt$

$v = -2.5e^{-2t} + c$

When $t = 0$, $v = 50$

So $c = 50 + 2.5 = 52.5$

So $v = 52.5 - 2.5e^{-2t}$

b When $t = 0$, $v = 50$

As $t \to \infty$, $v \to 52.5 - 0 = 52.5$

So $50 \leqslant v < 52.5$

4 **a** $x = 5\sin 2\pi t$

$v = \frac{dx}{dt} = 10\pi\cos 2\pi t$

$a = \frac{d^2x}{dt^2} = -20\pi^2 \sin 2\pi t$

When $t = 5$, $a = 0\,\text{m s}^{-2}$

b Maximum magnitude of acceleration $= 20\pi^2\,\text{m s}^{-2}$

5 **a** $a = 3x - 2$

$v\frac{dv}{dx} = 3x - 2$

$\int v\,dv = \int (3x-2)\,dx$

$\frac{1}{2}v^2 = \frac{3}{2}x^2 - 2x + c$

When $x = 0$, $v = 0$, $c = 0$

$v^2 = 3x^2 - 4x$

$v = \sqrt{3x^2 - 4x}$

b When $x = 3$

$v = \sqrt{3(3)^2 - 4(3)}$

$v = \sqrt{15} = 3.87\,\text{m s}^{-1}$

6 $a = 10x$

$v\frac{dv}{dx} = -10x$

$\int v\,dv = \int -10x\,dx$

$\frac{1}{2}v^2 = -5x^2 + c$

When $x = 0$, $v = 10$, $c = 50$

$\frac{1}{2}v^2 = 50 - 5x^2$

$v^2 = 100 - 10x^2$

Changes direction when $v = 0$.

$0 = 100 - 10x^2$

$x = \sqrt{10} = 3.16\,\text{m}$

7 $a = 10e^{0.5x}$

$v\frac{dv}{dx} = 10e^{0.5x}$

$\int v\,dv = \int 10e^{0.5x}\,dx$

$\frac{1}{2}v^2 = 20e^{0.5x} + c$

When $x = 0$, $v = 2$.

$2 = 20 + c$

So $c = -18$

$\frac{1}{2}v^2 = 20e^{0.5x} - 18$

$v = \sqrt{40e^{0.5x} - 36}$

When $x = 5$, $v = 21.2\,\text{m s}^{-1}$.

8 **a** $a = x^3(3-x)$

$v\frac{dv}{dx} = x^3(3-x)$

$\int v\,dv = \int 3x^3 - x^4\,dx$

$\frac{1}{2}v^2 = \frac{3}{4}x^4 - \frac{1}{5}x^5 + c$

When $x = 0$, $v = 0$ so $c = 0$

$v^2 = \frac{3}{2}x^4 - \frac{2}{5}x^5$

$v = \sqrt{\frac{3}{2}x^4 - \frac{2}{5}x^5}$

b When $v = 0$

$0 = \frac{3}{2}x^4 - \frac{2}{5}x^5$

$x^4\left(\frac{3}{2} - \frac{2}{5}x\right) = 0$

$x = 0$ or $\frac{3}{2} - \frac{2}{5}x = 0$

So the object is at rest when $x = 0\,\text{m}$ and when $x = 3.75\,\text{m}$.

9 **a** $10a = 2x - 3x^2$

$a = 0.2x - 0.3x^2$

$v\frac{dv}{dx} = 0.2x - 0.3x^2$

$\int v\,dv = \int \left(0.2x - 0.3x^2\right)dx$

$\frac{1}{2}v^2 = 0.1x^2 - 0.1x^3 + c$

When $x = 0$, $v = 0$ so $c = 0$

$v^2 = 0.2x^2 - 0.2x^3$

$v = \sqrt{0.2x^2 - 0.2x^3}$

b When $v = 0$

$0.2x^2 - 0.2x^3 = 0$

$x^2 - x^3 = 0$

$x^2(1 - x) = 0$

So $x = 0$ or $x = 1$

The object will travel 1m before instantaneously coming to rest.

Exercise 5.2A

1 $3a = -21v$

$a = -7v$

$\frac{dv}{dt} = -7v$

$\int \frac{1}{v}\, dv = \int -7\, dt$

$\ln v = -7t + c$

When $t = 0$, $v = 30$ so $c = \ln 30$

$t = \frac{1}{7}\ln\left(\frac{30}{v}\right)$

When $v = 15$

$t = \frac{1}{7}\ln\left(\frac{30}{15}\right) = 0.0990$ s

2 $a = -15v$

$v\frac{dv}{dx} = -15v$

$\int 1\, dv = \int -15\, dx$

$v = -15x + c$

When $x = 0$, $v = 20$ so $c = 20$

$v = -15x + 20$

When $v = 5$

$5 = -15x + 20$

$x = 1$m

3 $0.5a = \frac{500}{v} - 4v$

$0.5\frac{dv}{dt} = \frac{500 - 4v^2}{v}$

$\frac{1}{2}\int \frac{v}{500 - 4v^2}\, dv = \int 1\, dt$

$-\frac{1}{16}\int \frac{-8v}{500 - 4v^2}\, dv = \int 1\, dt$

$-\frac{1}{16}\ln(500 - 4v^2) = t + c$

When $t = 0$, $v = 10$

$c = -\frac{1}{16}\ln(500 - 4 \times 10^2)$

$c = -\frac{1}{16}\ln(100)$

$t = \frac{1}{16}\ln\left(\frac{100}{500 - 4v^2}\right)$

When $t = 5$

$5 = \frac{1}{16}\ln\left(\frac{100}{500 - 4v^2}\right)$

$\frac{100}{500 - 4v^2} = e^{80}$

$500 - 4v^2 = 100e^{-80}$

$v = 11.2\,\text{m s}^{-1}$

4 **a** $70\,\text{km h}^{-1} = \frac{70 \times 1000}{60 \times 60} = 19.4\,\text{m s}^{-1}$

b $800a = \frac{1600}{v} - 4v$

$800\frac{dv}{dt} = \frac{1600 - 4v^2}{v}$

$800\int \frac{v}{(1600 - 4v^2)}\, dv = \int 1\, dt$

$-100\int \frac{-8v}{(1600 - 4v^2)}\, dv = \int 1\, dt$

$-100\ln(1600 - 4v^2) = t + c$

When $t = 0$, $v = 15$

$c = -100\ln(1600 - 4 \times 15^2)$

$c = -100\ln(700)$

So $t = 100\ln\left(\frac{700}{1600 - 4v^2}\right)$

When $v = 19.444$

$t = 100\ln\left(\frac{700}{1600 - 4 \times 19.444^2}\right)$

$t = 208$ s

5 $1000a = \frac{4900}{v} - 0.4v^2$

$1000\,v\frac{dv}{dx} = \frac{4900}{v} - 0.4v^2$

$1000\,v\frac{dv}{dx} = \frac{4900 - 0.4v^3}{v}$

$1000\,\frac{v^2}{4900 - 0.04v^3}\frac{dv}{dx} = 1$

$-\frac{25000}{3}\int \frac{-0.12v^2}{4900 - 0.04v^3}\, dv = \int 1\, dx$

$-\frac{25\,000}{3}\ln(4900 - 0.04v^3) = x + c$

When $x = 0$, $v = 5$

$c = -\frac{25\,000}{3}\ln(4900 - 5)$

$c = -\frac{25\,000}{3}\ln(4895)$

So $x = \dfrac{25\,000}{3}\ln\left(\dfrac{4895}{4900 - 0.04v^3}\right)$

When $v = 12$

$x = \dfrac{25000}{3}\ln\left(\dfrac{4895}{4900 - 0.04\times 12^3}\right)$

$= 110\,\text{m}$ (3sf)

6 a $F = \mu R$

$= \mu mg$

$= 0.6 \times 0.02 \times 10$

$= 0.12$

b $0.02a = -(5v^2 + 0.12)$

$a = -(250v^2 + 6)$

$v\dfrac{dv}{dx} = -(250v^2 + 6)$

$\displaystyle\int \frac{v}{250v^2 + 6}\,dv = \int -1\,dx$

$\displaystyle\frac{1}{500}\int \frac{500v}{250v^2 + 6}\,dv = \int -1\,dx$

$\dfrac{1}{500}\ln(250v^2 + 6) = -x + c$

When $x = 0$, $v = 10$

$c = \dfrac{1}{500}\ln(250 \times 10^2 + 6)$

$c = \dfrac{1}{500}\ln(25\,006)$

So $x = \dfrac{1}{500}\ln\left(\dfrac{25\,006}{250v^2 + 6}\right)$

When $v = 5$

$x = \dfrac{1}{500}\ln\left(\dfrac{25\,006}{250\times 5^2 + 6}\right)$

$x = 0.002\,77\,\text{m}$

7 a $450a = \dfrac{45}{v+3}$

$450v\dfrac{dv}{dx} = \dfrac{45}{v+3}$

$\displaystyle\int 450v(v+3)\,dv = \int 45\,dx + c$

$\displaystyle\int \left(450v^2 + 1350v\right)dv = \int 45\,dx + c$

$\dfrac{450v^3}{3} + \dfrac{1350v^2}{2} = 45x + c$

$150v^3 + 675v^2 = 45x + c$

When $x = 0$, $v = 0$ so $c = 0$

$x = \dfrac{1}{45}(150v^3 + 675v^2)$

When $v = 4$, $x = 453\,\text{m}$

b $450a = \dfrac{45}{v+3}$

$450\dfrac{dv}{dt} = \dfrac{45}{v+3}$

$\displaystyle\int 450(v+3)\,dv = \int 45\,dt + c$

$\dfrac{450v^2}{2} + 1350v = 45t + c$

When $t = 0$, $v = 0$ so $c = 0$

$t = \dfrac{1}{45}(225v^2 + 1350v)$

When $v = 4$, $t = 200\,\text{s}$

Exercise 5.3A

1 $0.3a = 0.3 \times 10 - (2 + 0.5v^2)$

$0.3a = 1 - 0.5v^2$

For terminal velocity: $1 - 0.5v^2 = 0$

$v = 1.41\,\text{m s}^{-1}$

2 $0.5a = -0.5 \times 10 - 0.6v$

$0.5a = -(5 + 0.6v)$

$0.5v\dfrac{dv}{dx} = -(5 + 0.6v)$

$\displaystyle\int \frac{0.5v}{5 + 0.6v}\,dv = \int -1\,dx$

$\displaystyle\int \left(\frac{5}{6} - \frac{25}{6(5 + 0.6v)}\right)dv = \int -1\,dx$

$\displaystyle\frac{5}{6}\int \left(1 - \frac{5}{5 + 0.6v}\right)dv = \int -1\,dx$

$\displaystyle\frac{5}{6}\int \left(1 - \frac{25}{3}\,\frac{0.6}{5 + 0.6v}\right)dv = \int -1\,dx$

$\dfrac{5}{6}v - \dfrac{125}{18}\ln(5 + 0.6v) = -x + c$

When $x = 0$, $v = 20$

So $c = \dfrac{50}{3} - \dfrac{125}{18}\ln(17)$

So $x = \dfrac{50}{3} - \dfrac{125}{18}\ln(17) - \dfrac{5}{6}v + \dfrac{125}{18}\ln(5 + 0.6v)$

Maximum height is when $v = 0$.

So maximum height $= 8.17\,\text{m}$.

3 $0.8a = -0.8 \times 10 - 0.5v$

$0.8\dfrac{dv}{dt} = -(8 + 0.5v)$

$\displaystyle\int \frac{0.8}{8 + 0.5v}\,dv = \int -1\,dt$

$\displaystyle\frac{8}{5}\int \frac{0.5}{8 + 0.5v}\,dv = \int -1\,dt$

$\dfrac{8}{5}\ln(8 + 0.5v) = -t + c$

When $t = 0$, $v = 15$

So $c = \dfrac{8}{5}\ln(15.5)$

So $t = \dfrac{8}{5}\ln\left(\dfrac{15.5}{8 + 0.5v}\right)$

When $v = 5$

$t = 0.623$ seconds

4 $0.3a = -(0.3 \times 10 + 0.5v^2)$

$0.3v\dfrac{dv}{dx} = -(3 + 0.5v^2)$

$\displaystyle\int \frac{0.3v}{3 + 0.5v^2}\,dv = \int -1\,dx$

$\dfrac{5}{3}\ln(3 + 0.5v^2) = -x + c$

When $x = 0$, $v = 10$

So $c = \dfrac{5}{3}\ln(53)$

So $x = \dfrac{5}{3}\ln\left(\dfrac{53}{3 + 0.5v^2}\right)$

Maximum height is when $v = 0$.

So maximum height = 4.79 m.

5 **a** $7a = 7 \times 10 - 20v$

$7\dfrac{dv}{dt} = 70 - 20v$

$\displaystyle\int \frac{7}{70 - 20v}\,dv = 1\,dt$

$-\dfrac{7}{20}\ln(70 - 20v) = t + c$

When $t = 0$, $v = 0$

So $c = -\dfrac{7}{20}\ln(70)$

So $t = \dfrac{7}{20}\ln\left(\dfrac{7}{70 - 20v}\right)$

When $t = 3$

$\ln\left(\dfrac{7}{70 - 20v}\right) = \dfrac{60}{7}$

$\dfrac{7}{70 - 20v} = e^{\frac{60}{7}}$

$70 - 20v = 7e^{-}$

$v = 3.50\,\text{m s}^{-1}$

b For terminal velocity, $a = 0$

So $70 - 20v = 0$

So $v = 3.5\,\text{m s}^{-1}$

6 $a = -(10 + 2v^2)$

$\dfrac{dv}{dt} = -(10 + 2v^2)$

$\displaystyle\int \frac{1}{10 + 2v^2}\,dv = \int -1\,dt$

$\dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{v}{\sqrt{5}}\right) = -t + c$

When $t = 0$, $v = 25$

So $c = \dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{25}{\sqrt{5}}\right)$

So $t = \dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{25}{\sqrt{5}}\right) - \dfrac{1}{2\sqrt{5}}\tan^{-1}\left(\dfrac{v}{\sqrt{5}}\right)$

Maximum height is when $v = 0$.

So time to reach maximum height is 0.331 seconds.

7 **a** $75a = 75 \times 10 - 25v$

$75a = 750 - 25v$

$3\dfrac{dv}{dt} = 30 - v$

$\displaystyle\int \frac{3}{30 - v}\,dv = \int 1\,dt$

$-3\ln(30 - v) = t + c$

When $t = 0$, $v = 0$

So $c = -3\ln 30$

So $t = 3\ln\left(\dfrac{30}{30 - v}\right)$

When $t = 4$

$4 = 3\ln\left(\dfrac{30}{30 - v}\right)$

$\dfrac{4}{3} = \ln\left(\dfrac{30}{30 - v}\right)$

$\dfrac{30}{30 - v} = e^{\frac{4}{3}}$

$30 - v = 30e^{-\frac{4}{3}}$

$v = 22.1\,\text{m s}^{-1}$

b Terminal velocity is when $a = 0$.

$75a = 75 \times 10 - 20v - 40v^2$

$0 = 750 - 20v - 40v^2$

$v = -4.59$ or $v = 4.09$

So her terminal velocity is $4.09\,\text{m s}^{-1}$.

Exam-style questions

1 **a** $a = \dfrac{16}{(x + 2)^2}$

$v\dfrac{dv}{dx} = 16(x + 2)^{-2}$

$\displaystyle\int v\,dv = \int 16(x + 2)^{-2}\,dx$

$\dfrac{1}{2}v^2 = -16(x + 2)^{-1} + c$

$\dfrac{1}{2}v^2 = \dfrac{-16}{(x + 2)} + c$

When $x = 2$, $v = 0.5\,\text{m s}^{-1}$

$\dfrac{1}{8} = -4 + c$

So $c = \dfrac{33}{8}$

So $\dfrac{1}{2}v^2 = \dfrac{-16}{(x + 2)} + \dfrac{33}{8}$

$v = \sqrt{\dfrac{33}{4} - \dfrac{32}{x + 2}}$

b As $x \to \infty$, $v \to \sqrt{\dfrac{33}{4}} = 2.87\,\text{m s}^{-1}$

2 **a** $0.4a = -0.4 \times 10 - 0.8v$

$0.4v\dfrac{dv}{dx} = -(4 + 0.8v)$

$$\int \frac{0.4v}{4+0.8v}\,dv = \int -1\,dx$$

$$\int \frac{v}{10+2v}\,dv = \int -1\,dx$$

$$\int \frac{1}{2} - \frac{5}{10+2v}\,dv = \int -1\,dx$$

$$\frac{1}{2}v - \frac{5}{2}\ln(10+2v) = -x + c$$

When $x = 0$, $v = 25$

So $c = \frac{25}{2} - \frac{5}{2}\ln(60)$

So $x = \frac{25}{2} - \frac{1}{2}v + \frac{5}{2}\ln\left(\frac{10+2v}{60}\right)$

Maximum height is when $v = 0$.

So maximum height = 8.02 m.

b $0.4\frac{dv}{dt} = -(4 + 0.8v)$

$$\int \frac{0.4}{4+0.8v}\,dv = \int -1\,dt$$

$$\frac{1}{2}\ln(4+0.8v) = -t + c$$

When $t = 0$, $v = 25$

So $c = \frac{1}{2}\ln(24)$

So $t = \frac{1}{2}\ln\left(\frac{24}{4+0.8v}\right)$

Maximum height is when $v = 0$.

So time to reach maximum height = 0.896 seconds.

3 a $2000a = \frac{60\,000}{v} - 5v^2$

$$2000v\frac{dv}{dx} = \frac{60\,000}{v} - 5v^2$$

$$400v\frac{dv}{dx} = \frac{12\,000}{v} - v^2$$

$$400v\frac{dv}{dx} = \frac{12\,000 - v^3}{v}$$

$$400\int \frac{v^2}{12\,000 - v^3}\,dv = \int 1\,dx$$

$$-\frac{400}{3}\ln(12\,000 - v^3) = x + c$$

When $x = 0$, $v = 4$

So $c = -\frac{400}{3}\ln(11\,936)$

So $x = \frac{400}{3}\ln\left(\frac{11\,936}{12\,000 - v^3}\right)$

$$\frac{3}{400}x = \ln\left(\frac{11\,936}{12\,000 - v^3}\right)$$

$$e^{\frac{3}{400}x} = \frac{11\,936}{12\,000 - v^3}$$

$$12\,000 - v^3 = 11\,936e^{-\frac{3}{400}x}$$

$$v^3 = 12\,000 - 11\,936e^{-\frac{3}{400}x}$$

$$v = \sqrt[3]{12\,000 - 11\,936e^{-\frac{3}{400}x}}$$

b When $v = 15$

$$x = \frac{400}{3}\ln\left(\frac{11\,936}{12\,000 - 15^3}\right)$$

$x = 43.3$ m

4 a $x = \int v\,dt$

$$x = \int e^{-5t} + 5t^2\,dt$$

$$x = -\frac{1}{5}e^{-5t} + \frac{5}{3}t^3 + c$$

When $t = 0$, $x = 3$

So $c = \frac{16}{5}$

So $x = -\frac{1}{5}e^{-5t} + \frac{5}{3}t^3 + \frac{16}{5}$

When $t = 5$

$$x = \frac{1}{5}e^{-25} + \frac{625}{3} + \frac{16}{5}$$

$= 211.533\ldots = 212$ (3 s.f.)

b $a = \frac{dv}{dt}$

$$v = e^{-5t} + 5t^2$$

$$\frac{dv}{dt} = -5e^{-5t} + 10t$$

When $t = 3$

$a = 30.0\ \text{m s}^{-2}$

5 a $a = \frac{1}{3}e^{-5t}$

$$v = \int \frac{1}{3}e^{-5t}\,dt$$

$$v = -\frac{1}{15}e^{-5t} + c$$

When $t = 0$, $v = 5$

$$5 = -\frac{1}{15}e^{-5\times 0} + c$$

So $c = 5 + \frac{1}{15} = \frac{76}{15}$

So $v = \frac{76}{15} - \frac{1}{15}e^{-5t}$

b When $t = 3$

$v = 5.07\ \text{m s}^{-1}$

c $5 \leqslant v \leqslant \frac{76}{15}$

6 a $a = -(1 \times 10 + 3v^2)$

$$v\frac{dv}{dx} = -(10 + 3v^2)$$

$$\int \frac{v}{10+3v^2}\,dv = \int -1\,dx$$

$\frac{1}{6}\ln(10+3v^2) = -x + c$

When $x = 0$, $v = 25$

So $c = \frac{1}{6}\ln(1885)$

So $x = \frac{1}{6}\ln\left(\frac{1885}{10+3v^2}\right)$

Maximum height is when $v = 0$.

So $x = 0.873\,\text{m}$

So maximum height of object $= 1.2 + 0.873$
$= 2.07\,\text{m}$.

b $\frac{dv}{dt} = -(10+3v^2)$

$\int \frac{1}{10+3v^2}\,dv = \int -1\,dt$

$\int \frac{1}{\left(\sqrt{10}\right)^2 + \left(\sqrt{3}v\right)^2}\,dv = \int -1\,dt$

$\frac{1}{\sqrt{30}}\tan^{-1}\left(\sqrt{\frac{3}{10}}v\right) = -t + c$

When $t = 0$, $v = 25$

So $c = \frac{1}{\sqrt{30}}\tan^{-1}\left(\sqrt{\frac{1875}{10}}\right)$

So $t = \frac{1}{\sqrt{30}}\tan^{-1}\left(\sqrt{\frac{1875}{10}}\right) - \frac{1}{\sqrt{30}}\tan^{-1}\left(\sqrt{\frac{3}{10}}v\right)$

Maximum height is when $v = 0$.

So time to reach maximum height = 0.273 seconds.

7 a $a = \frac{1}{3}e^{-\frac{1}{2}x}$

$v\frac{dv}{dx} = \frac{1}{3}e^{-\frac{1}{2}x}$

$\int v\,dv = \int \frac{1}{3}e^{-\frac{1}{2}x}\,dx$

$\frac{1}{2}v^2 = -\frac{2}{3}e^{-\frac{1}{2}x} + c$

When $x = 0$, $v = 2$

So $c = \frac{1}{2}\times 4 + \frac{2}{3} = \frac{8}{3}$

$\frac{1}{2}v^2 = \frac{8}{3} - \frac{2}{3}e^{-\frac{1}{2}x}$

$v = \sqrt{\frac{16}{3} - \frac{4}{3}e^{-\frac{1}{2}x}}$

b As $x \to \infty$, $v \to 2.31\,\text{m}\,\text{s}^{-1}$.

8 a $1500a = 9000 - 3000 - 350v$

$30a = 120 - 7v$

$30\frac{dv}{dt} = 120 - 7v$

$\int \frac{30}{120-7v}\,dv = \int 1\,dt$

$-\frac{30}{7}\ln(120-7v) = t + c$

When $t = 0$, $v = 0$

So $c = -\frac{30}{7}\ln(120)$

So $t = \frac{30}{7}\ln\left(\frac{120}{120-7v}\right)$

$\ln\left(\frac{120}{120-7v}\right) = \frac{7}{30}t$

$\frac{120}{120-7v} = e^{\frac{7}{30}t}$

$120 - 7v = 120e^{-\frac{7}{30}t}$

$v = \frac{120}{7}\left(1 - e^{-\frac{7}{30}t}\right)$

b As $t \to \infty$, $v \to \frac{120}{7} = 17.1\,\text{m}\,\text{s}^{-1}$.

9 $50a = 50 \times 10 - 0.08v^2$

$50v\frac{dv}{dx} = 500 - 0.08v^2$

$\int \frac{50v}{500-0.08v^2}\,dv = \int 1\,dx$

$-\frac{50}{0.16}\ln(500-0.08v^2) = x + c$

$-\frac{625}{2}\ln(500-0.08v^2) = x + c$

When $x = 0$, $v = 0$

So $c = -\frac{625}{2}\ln(500)$

So $x = \frac{625}{2}\ln\left(\frac{500}{500-0.08v^2}\right)$

When $x = 50$

$50 = \frac{625}{2}\ln\left(\frac{500}{500-0.08v^2}\right)$

$\frac{4}{25} = \ln\left(\frac{500}{500-0.08v^2}\right)$

$\frac{500}{500-0.08v^2} = e^{\frac{4}{25}}$

$500 - 0.08v^2 = 500e^{-\frac{4}{25}}$

$v = 30.4\,\text{m}\,\text{s}^{-1}$

10 a $0.2a = e^x - (1+2x)$

$0.2a = e^x - 1 - 2x$

$a = 5e^x - 10x - 5$

$v\frac{dv}{dx} = 5e^x - 10x - 5$

b $\int v\,dv = \int\left(5e^x - 10x - 5\right)dx$

$\frac{1}{2}v^2 = 5e^x - 5x^2 - 5x + c$

When $x = 0$, $v = 4$

$8 = 5 + c$

So $c = 3$

So $\frac{1}{2}v^2 = 5e^x - 5x^2 - 5x + 3$

When $x = 10$

$\frac{1}{2}v^2 = 5e^{10} - 500 - 50 + 3$

$v = 468\,\text{m s}^{-1}$

11 a $a = 0.2e^{0.2x}$

$v\frac{dv}{dx} = 0.2e^{0.2x}$

$\int v\,dv = \int 0.2e^{0.2x}\,dx$

$\frac{1}{2}v^2 = e^{0.2x} + c$

When $x = 0$, $v = 0$

So $c = -1$

So $\frac{1}{2}v^2 = e^{0.2x} - 1$

$v = \sqrt{2e^{0.2x} - 2}$

b When $x = 10$

$v = 3.57\,\text{m s}^{-1}$

c $a = 0$

12 a $F = 6a = \frac{120}{kv^2}$

$a = \frac{20}{kv^2}$

$\frac{dv}{dt} = \frac{20}{kv^2}$

$\int v^2\,dv = \int \frac{20}{k}\,dt$

$\frac{v^3}{3} = \frac{20t}{k} + c$

$v^3 = \frac{60t}{k} + d$

When $t = 0$, $v = 1$ and when $t = 1$, $v = 2$

$1 = 0 + d$ so $d = 1$

$v^3 = \frac{60t}{k} + 1$

$8 = \frac{60}{k} + 1$

$7 = \frac{60}{k}$

$k = \frac{60}{7}$

b $v^3 = \frac{60t}{k} + 1$

$v^3 = \frac{60t}{\frac{60}{7}} + 1$

$v^3 = 7t + 1$

c $a = \frac{7}{3v^2}$

$v\frac{dv}{dx} = \frac{7}{3v^2}$

$v^3\frac{dv}{dx} = \frac{7}{3}$

$\int v^3\,dv = \int \frac{7}{3}\,dx$

$\frac{v^4}{4} = \frac{7}{3}x + c$

$v^4 = \frac{28}{3}x + d$

When $x = 0$, $v = 1$

$1^4 = 0 + d$ so $d = 1$

$v^4 = \frac{28}{3}x + 1$

13 a $F = 60a = \frac{10\,\text{cosec}\,v}{\cos v}$

$a = \frac{\text{cosec}\,v}{6\cos v}$

$\frac{dv}{dt} = \frac{\text{cosec}\,v}{6\cos v}$

$\int \frac{6\cos v}{\text{cosec}\,v}\,dv = \int 1\,dt$

$\int 6\cos v \sin v\,dv = \int 1\,dt$

$\int 3\sin 2v\,dv = \int 1\,dt$

$-\frac{3}{2}\cos 2v = t + c$

When $t = 0$, $v = \frac{\pi}{6}$

So $c = -\frac{3}{2}\cos\frac{\pi}{3}$

$c = -\frac{3}{4}$

So $t = \frac{3}{4} - \frac{3}{2}\cos 2v$

$4t = 3 - 6\cos 2v$

$6\cos 2v = 3 - 4t$

$\cos 2v = \frac{3 - 4t}{6}$

b When $t = 1$

$\cos 2v = -\frac{1}{6}$

$2v = 1.738$ or $2v = 4.454$

Since $0 < v < \frac{\pi}{2}$

$2v = 1.738$

$v = 0.869$

So $v = 0.869\,\text{m s}^{-1}$ when $t = 1$

14 a $F = ma = 0.6a = 0.5v^2 + 1 - 0.25v^2 - 0.5$

$0.6a = 0.25v^2 + 0.5$

$0.6v\frac{dv}{dx} = 0.25v^2 + 0.5$

$\int \frac{v}{0.25v^2 + 0.5} dv = \int \frac{5}{3} dx$

$2\ln(0.25v^2 + 0.5) = \frac{5}{3}x + c$

When $x = 0$, $v = \sqrt{2}$.

$2\ln(1) = c = 0$

$2\ln(0.25v^2 + 0.5) = \frac{5}{3}x$

$\ln(0.25v^2 + 0.5) = \frac{5}{6}x$

$0.25v^2 + 0.5 = e^{\frac{5}{6}x}$

$v^2 = 4\left(e^{\frac{5}{6}x} - 0.5\right)$

$v = \sqrt{4(e^{\frac{5}{6}x} - 0.5)}$

b When $x = 1.5$

$v = \sqrt{4(e^{\frac{5}{6}x} - 0.5)}$

$v = 3.46\,\text{m s}^{-1}$.

15 a $4a = v^3 e^{5x-4}$

$4v\frac{dv}{dx} = v^3 e^{5x-4}$

$\int \frac{4}{v^2} dv = \int e^{5x-4} dx$

$-\frac{4}{v} = \frac{1}{5}e^{5x-4} + c$

When $x = 0.8$, $v = 2$

$-2 = \frac{1}{5}e^0 + c$

$c = -\frac{11}{5}$

$-\frac{4}{v} = \frac{1}{5}e^{5x-4} - \frac{11}{5}$

$-\frac{20}{v} = e^{5x-4} - 11$

$v = -\frac{20}{e^{5x-4} - 11}$

$v = \frac{20}{11 - e^{5x-4}}$

b When $x = 0.5$

$v = \frac{20}{11 - e^{5(0.5)-4}}$

$v = 1.86\,\text{m s}^{-1}$

16 a $80a = 80 \times 10 - 30v$

$80a = 800 - 30v$

$8a = 80 - 3v$

$8\frac{dv}{dt} = 80 - 3v$

$\int \frac{8}{80 - 3v} dv = \int 1\, dt$

$-\frac{8}{3}\ln(80 - 3v) = t + c$

When $t = 0$, $v = 0$

So $c = -\frac{8}{3}\ln(80)$

So $t = \frac{8}{3}\ln\left(\frac{80}{80 - 3v}\right)$

When $t = 5$

$\ln\left(\frac{80}{80 - 3v}\right) = \frac{15}{8}$

$\frac{80}{80 - 3v} = e^{\frac{15}{8}}$

$80 - 3v = 80e^{-\frac{15}{8}}$

$v = \frac{80}{3}\left(1 - e^{-\frac{15}{8}}\right)$

$v = 22.6\,\text{m s}^{-1}$

b For terminal velocity, $a = 0$

$80a = 800 - 30v - 45v^2$

$0 = 800 - 30v - 45v^2$

$0 = 9v^2 + 6v - 160$

$v = -4.56$ or 3.90

So terminal velocity $= 3.90\,\text{m s}^{-1}$

17 a $ma = -\left(mg + \frac{mv^2}{4\lambda^2}\right)$

$a = -\left(10 + \frac{v^2}{4\lambda^2}\right)$

$a = -\left(\frac{40\lambda^2 + v^2}{4\lambda^2}\right)$

$v\frac{dv}{dx} = -\left(\frac{40\lambda^2 + v^2}{4\lambda^2}\right)$

$\int \frac{4\lambda^2 v}{40\lambda^2 + v^2} dv = \int -1\, dx$

$2\lambda^2 \ln(40\lambda^2 + v^2) = -x + c$

When $x = 0$, $v = 40\lambda$

So $c = 2\lambda^2 \ln(40\lambda^2 + (40\lambda^2))$

$x = 2\lambda^2 \ln\left(\frac{40\lambda^2 + (40\lambda)^2}{40\lambda^2 + v^2}\right)$

Maximum height is when $v = 0$.

$x = 2\lambda^2 \ln\left(\frac{40\lambda^2 + (40\lambda)^2}{40\lambda^2}\right)$

$x = 2\lambda^2 \ln(1 + 40)$

$x = 2\lambda^2 \ln(41)$

b $a = -\left(10 + \dfrac{v^2}{4\lambda^2}\right)$

$a = -\left(\dfrac{40\lambda^2 + v^2}{4\lambda^2}\right)$

$\dfrac{dv}{dt} = -\left(\dfrac{40\lambda^2 + v^2}{4\lambda^2}\right)$

$\int \dfrac{4\lambda^2}{40\lambda^2 + v^2}\, dv = \int -1\, dt$

$\dfrac{4\lambda^2}{\lambda\sqrt{40}}\tan^{-1}\left(\dfrac{v}{\lambda\sqrt{40}}\right) = -t + c$

$\dfrac{4\lambda}{\sqrt{40}}\tan^{-1}\left(\dfrac{v}{\lambda\sqrt{40}}\right) = -t + c$

When $t = 0$, $v = 40\lambda$

So $c = \dfrac{4\lambda}{\sqrt{40}}\tan^{-1}\left(\sqrt{40}\right)$

$t = \dfrac{4\lambda}{\sqrt{40}}\tan^{-1}\left(\sqrt{40}\right) - \dfrac{4\lambda}{\sqrt{40}}\tan^{-1}\left(\dfrac{v}{\lambda\sqrt{40}}\right)$

When $v = 0$

$t = \dfrac{4\lambda}{\sqrt{40}}\tan^{-1}\left(\sqrt{40}\right)$

18 a $ma = -(mg + mkv^2)$

$a = -(g + kv^2)$

$v\dfrac{dv}{dx} = -(g + kv^2)$

$\int \dfrac{v}{g + kv^2}\, dv = \int -1\, dx$

$\dfrac{1}{2k}\ln(g + kv^2) = -x + c$

When $x = 0$, $v = u$

So $c = \dfrac{1}{2k}\ln(g + ku^2)$

$x = \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g + kv^2}\right)$

Maximum height is when $v = 0$.

Maximum height $= \dfrac{1}{2k}\ln\left(\dfrac{g + ku^2}{g}\right)$

b $ma = -(mg + mkv^2)$

$a = -(g + kv^2)$

$\dfrac{dv}{dt} = -(g + kv^2)$

$\int \dfrac{1}{g + kv^2}\, dv = \int -1\, dt$

$\dfrac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\dfrac{k}{g}}\right) = -t + c$

When $t = 0$, $v = u$

So $c = \dfrac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\dfrac{k}{g}}\right)$

$t = \dfrac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\dfrac{k}{g}}\right) - \dfrac{1}{\sqrt{gk}}\tan^{-1}\left(v\sqrt{\dfrac{k}{g}}\right)$

Maximum height is when $v = 0$.

So time to maximum height

$= \dfrac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\dfrac{k}{g}}\right)$

19 a $0.5a = \dfrac{x}{\sec\left(x^2\right)}$

$0.5a = x\cos(x^2)$

$0.5v\dfrac{dv}{dx} = x\cos(x^2)$

$\int 0.5v\; dv = \int x\cos\left(x^2\right) dx$

$\dfrac{1}{4}v^2 = \int x\cos\left(x^2\right) dx$

$v^2 = \int 4x\cos\left(x^2\right) dx$

b Let $u = x^2$

$\dfrac{du}{dx} = 2x$

So $v^2 = \int 4x\cos(u)\dfrac{1}{2x}\, du$

$v^2 = \int 2\cos(u)\; du$

$v^2 = 2\sin(u) + c$

$v^2 = 2\sin(x^2) + c$

When $x = 0$, $v = 4$

So $c = 16$

$v = \sqrt{2\sin(x^2) + 16}$

c Max of $2\sin(x^2) + 16 = 18$

Min of $2\sin(x^2) + 16 = 14$

So max of v is $\sqrt{18} = 3\sqrt{2}$

Min of v is $\sqrt{14}$

20 a $10a = x^2 e^{-v}$

$10v\dfrac{dv}{dx} = x^2 e^{-v}$

$\int 10ve^{v}\, dv = \int x^2\; dx$

$\dfrac{x^3}{3} = \int 10ve^{v}\, dv$

$x^3 = 30\int ve^{v}\, dv$

b $\int ve^{v}\, dv = ve^{v} - \int e^{v}\, dv$

$= ve^{v} - e^{v} + c$

$= e^{v}(v - 1) + c$

So $x^3 = 30e^{v}(v - 1) + c$

When $x = 0$, $v = 0$ so $c = 30$

So $x^3 = 30e^v(v - 1) + 30$

c When $v = 2$

$x^3 = 30e^2(2 - 1) + 30$

$x^3 = 251.7$

$x = 6.31\,\text{m}$

Mathematics in life and work

1 $180a = 180 \times 10 - 25v$

$180\dfrac{dv}{dt} = 1800 - 25v$

$36\dfrac{dv}{dt} = 360 - 5v$

$\displaystyle\int \frac{36}{360 - 5v}\,dv = \int 1\,dt$

$-\dfrac{36}{5}\ln(360 - 5v) = t + c$

When $t = 0$, $v = 0$

So $c = -\dfrac{36}{5}\ln(360)$

So $t = \dfrac{36}{5}\ln\left(\dfrac{360}{360 - 5v}\right)$

When $t = 45$

$\ln\left(\dfrac{360}{360 - 5v}\right) = \dfrac{225}{36}$

$\dfrac{360}{360 - 5v} = e^{\frac{225}{36}}$

$360 - 5v = 360e^{-\frac{225}{36}}$

$v = 71.861\ldots = 71.9\,\text{m}\,\text{s}^{-1}$

2 $36v\dfrac{dv}{dx} = 360 - 5v$

$\displaystyle\int \frac{36v}{360 - 5v}\,dv = \int 1\,dx$

$\dfrac{36}{5}\displaystyle\int \frac{v}{72 - v}\,dv = \int 1\,dx$

$\dfrac{36}{5}\displaystyle\int \frac{v}{72 - v}\,dv = \int 1\,dx$

$\dfrac{36}{5}\displaystyle\int -1 + \frac{72}{72 - v}\,dv = \int 1\,dx$

$\dfrac{36}{5}(-v - 72\ln(72 - v)) = x + c$

When $x = 0$, $v = 0$

So $c = -\dfrac{2592}{5}\ln(72)$

$x = \dfrac{2592}{5}\ln\left(\dfrac{72}{72 - v}\right) - \dfrac{36}{5}v$

When $v = 71.861$, $x = 2723\,\text{m}$

When she opens her parachute she will be $3200 - 2723 = 477\,\text{m}$ above the ground.

3 Terminal velocity, $a = 0$

$180\dfrac{dv}{dt} = 1800 - (20v + 28v^2)$

$28v^2 + 20v - 1800 = 0$

$v = -8.38$ or 7.67

So her maximum possible speed is $7.67\,\text{m}\,\text{s}^{-1}$.

6 Momentum

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Prerequisite knowledge

1 Momentum $= mv = 2.6 \times 3 = 7.8\,\text{kg m s}^{-1}$

2 Momentum $= mv = 0.56 \times 2 = 1.12\,\text{kg m s}^{-1}$

3 $2 \times 5 + 3 \times 2 = 2 \times v_1 + 3 \times 4$

$v_1 = 2\,\text{m s}^{-1}$ in the same direction as before

4 $4 \times 2 + 4.5 \times -3 = 4 \times -1 + 4.5 \times v_2$

$v_2 = -\frac{1}{3}\,\text{m s}^{-1}$ i.e. $0.333\,\text{m s}^{-1}$ in opposite direction

5 **a)** Add both equations together

$2x = 4$

$x = 2$

Substituting $x = 2$ into the first equation gives:

$2 + y = 5$ so $y = 3$

b) Add both equations together

$5x = 2.5$

$x = 0.5$

Substituting $x = 0.5$ into the second equation gives:

$1.5 - y = 1.2$ so $y = 0.3$

Exercise 6.1A

1 $e = \frac{v}{u} = \frac{2.6}{4} = 0.65$

2 $e = \frac{v}{u} = \frac{10}{12} = 0.833$

3 $u = \frac{v}{e} = \frac{3.2}{0.55} = 5.81\,\text{m s}^{-1}$

4 **a** $v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 1.5$

$= 30$

$v = 5.48\,\text{m s}^{-1}$

b $v = eu = 0.4 \times 5.48 = 2.19\,\text{m s}^{-1}$

c $s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 2.19^2}{2 \times -10} = 0.24\,\text{m}$

5 $v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 1.42$

$= 28.4$

$v = 5.33\,\text{m s}^{-1}$

Speed immediately before impact is $5.33\,\text{m s}^{-1}$

$u^2 = v^2 - 2as$

$= 0^2 - 2 \times -10 \times 0.75$

$= 15$

$u = 3.87\,\text{m s}^{-1}$

Speed immediately after impact is $3.87\,\text{m s}^{-1}$

So $e = \frac{3.87}{5.33} = 0.726$

6 $u^2 = v^2 - 2as$

$= 0^2 - 2 \times -10 \times 2.1$

$= 42$

$u = 6.48\,\text{m s}^{-1}$

Speed immediately after impact is $6.48\,\text{m s}^{-1}$

Speed immediately before impact is

$u = \frac{v}{e} = \frac{6.48}{0.76} = 8.53\,\text{m s}^{-1}$

$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 8.53^2}{2 \times -10} = 3.64\,\text{m}$

7 Speed before first impact:

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 3$

$= 60$

$v = 7.75\,\text{m s}^{-1}$

Speed immediately after first impact

$= 7.75 \times 0.7 = 5.42\,\text{m s}^{-1}$

Speed immediately after second impact

$= 5.42 \times 0.7 = 3.80\,\text{m s}^{-1}$

Speed immediately after third impact

$= 3.80 \times 0.7 = 2.66\,\text{m s}^{-1}$

Height reached after third impact:

$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 2.66^2}{2 \times -10} = 0.353\,\text{m}$

So the maximum height reached after the third bounce is 35.3 cm.

8 **a** Speed after impact with wall $= 15 \times 0.6 = 9\,\text{m s}^{-1}$

b $U = 9\,\text{m s}^{-1}$, $\theta = 0°$, $a_y = 10\,\text{m s}^{-2}$, $y = 0\,\text{m}$, $h = 1.2\,\text{m}$

Substitute into $y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 9\sin 0° \times t - \frac{1}{2} \times 10 \times t^2 + 1.2$

Simplify:

$5t^2 - 1.2 = 0$

$t^2 = 0.24$

$t = 0.490$ or -0.490

The time of flight for the ball is 0.490 seconds.

c $U = 9\,\text{m}\,\text{s}^{-1}$, $\theta = 0°$, $t = 0.4899\,\text{s}$

Substitute into $x = Ut\cos\theta$

$x = 9 \times 0.490 \times \cos 0$

$= 4.41$

The ball will hit the ground 4.41 m from the base of the wall.

9 $U = ?\,\text{m}\,\text{s}^{-1}$, $\theta = 0°$, $a_y = 10\,\text{m}\,\text{s}^{-2}$, $y = 0\,\text{m}$, $h = 2\,\text{m}$, $t = ?$

Substitute into $y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = Ut\sin 0° - 0.5 \times 10 \times t^2 + 2$

$t^2 = 0.4$

$t = 0.63246$

$U = ?\,\text{m}\,\text{s}^{-1}$, $\theta = 0°$, $t = 0.63246$

Substitute into $x = Ut\cos\theta$

$5 = U \times 0.63246 \times \cos 0°$

$0.63246U = 5$

$U = 7.906$

The ball leaves the wall with a speed of $7.906\,\text{m}\,\text{s}^{-1}$.

So the speed before impact is $\frac{7.906}{0.8} = 9.88\,\text{m}\,\text{s}^{-1}$.

Exercise 6.2A

1 a $m_A = 3\,\text{kg}$, $u_A = 5\,\text{m}\,\text{s}^{-1}$, $m_B = 4\,\text{kg}$, $u_B = 2\,\text{m}\,\text{s}^{-1}$, $e = 0.4$

By conservation of momentum:

$3 \times 5 + 4 \times 2 = 3 \times v_A + 4 \times v_B$

$3v_A + 2v_B = 23$

By Newton's experimental law:

$v_B - v_A = 0.4(5 - 2)$

$v_B - v_A = 1.2$

Solving these simultaneous equations:

$7v_B = 26.6$ so $v_B = 3.8\,\text{m}\,\text{s}^{-1}$

$3.8 - v_A = 1.2$ so $v_A = 2.6\,\text{m}\,\text{s}^{-1}$

b $m_A = 1\,\text{kg}$, $u_A = 0.5\,\text{m}\,\text{s}^{-1}$, $m_B = 1\,\text{kg}$, $u_B = -0.9\,\text{m}\,\text{s}^{-1}$, $e = 0.8$

By conservation of momentum:

$1 \times 0.5 + 1 \times -0.9 = 1 \times v_A + 1 \times v_B$

$v_A + v_B = -0.4$

By Newton's experimental law:

$v_B - v_A = 0.8(0.5 - -0.9)$

$v_B - v_A = 1.12$

Solving these simultaneous equations:

$2v_B = 0.72$ so $v_B = 0.36\,\text{m}\,\text{s}^{-1}$

$0.36 - v_A = 1.12$ so $v_A = -0.76\,\text{m}\,\text{s}^{-1}$ i.e. $0.76\,\text{m}\,\text{s}^{-1}$ in the opposite direction.

c $m_A = 5\,\text{kg}$, $u_A = 6\,\text{m}\,\text{s}^{-1}$, $m_B = 3\,\text{kg}$, $u_B = -5\,\text{m}\,\text{s}^{-1}$, $e = 0.2$

By conservation of momentum:

$5 \times 6 + 3 \times -5 = 5 \times v_A + 3 \times v_B$

$5v_A + 3v_B = 15$

By Newton's experimental law:

$v_B - v_A = 0.2(6 - -5)$

$v_B - v_A = 2.2$

Solving these simultaneous equations:

$8v_B = 26$ so $v_B = 3.25\,\text{m}\,\text{s}^{-1}$

$3.25 - v_A = 2.2$ so $v_A = 1.05\,\text{m}\,\text{s}^{-1}$

d $m_A = 4\,\text{kg}$, $u_A = 2\,\text{m}\,\text{s}^{-1}$, $m_B = 3\,\text{kg}$, $u_B = -6\,\text{m}\,\text{s}^{-1}$, $e = 1$

By conservation of momentum:

$4 \times 2 + 3 \times -6 = 4 \times v_A + 3 \times v_B$

$4v_A + 3v_B = -10$

By Newton's experimental law:

$v_B - v_A = 1(2 - -6)$

$v_B - v_A = 8$

Solving these simultaneous equations:

$7v_B = 22$ so $v_B = 3.14\,\text{m}\,\text{s}^{-1}$

$3.14 - v_A = 8$ so $v_A = -4.86\,\text{m}\,\text{s}^{-1}$ i.e. $4.86\,\text{m}\,\text{s}^{-1}$ in the opposite direction

2 $m_A = 2\,\text{kg}$, $u_A = 3\,\text{m}\,\text{s}^{-1}$, $m_B = 3\,\text{kg}$, $u_B = 0\,\text{m}\,\text{s}^{-1}$, $e = 0.7$

By conservation of momentum:

$2 \times 3 + 3 \times 0 = 2 \times v_A + 3 \times v_B$

$2v_A + 3v_B = 6$

By Newton's experimental law:

$v_B - v_A = 0.7(3 - 0)$

$v_B - v_A = 2.1$

Solving these simultaneous equations:

$5v_B = 10.2$ so $v_B = 2.04\,\text{m}\,\text{s}^{-1}$ in the same direction that A was originally moving in.

3 $m_A = 4m\,\text{kg}$, $u_A = 5\,\text{m}\,\text{s}^{-1}$, $m_B = 5m\,\text{kg}$, $u_B = -5\,\text{m}\,\text{s}^{-1}$, $e = 0.55$

By conservation of momentum:

$4m \times 5 + 5m \times -5 = 4m \times v_A + 5m \times v_B$

$4v_A + 5v_B = -5$

By Newton's experimental law:

$v_B - v_A = 0.55(5 - -5)$

$v_B - v_A = 5.5$

Solving these simultaneous equations:

$9v_B = 17$ so $v_B = 1.89\,\text{m s}^{-1}$

$1.89 - v_A = 5.5$ so $v_A = -3.61\,\text{m s}^{-1}$

So both balls are moving in the opposite direction compared to before the collision with $v_A = -3.61\,\text{m s}^{-1}$ and $v_B = 1.89\,\text{m s}^{-1}$.

4 $m_A = 4\,\text{kg}$, $u_A = 5\,\text{m s}^{-1}$, $m_B = 3\,\text{kg}$, $u_B = 0\,\text{m s}^{-1}$, $e = 0.6$

By conservation of momentum:

$4 \times 5 + 3 \times 0 = 4 \times v_A + 3 \times v_B$

$4v_A + 3v_B = 20$

By Newton's experimental law:

$v_B - v_A = 0.6(5 - 0)$

$v_B - v_A = 3$

Solving these simultaneous equations:

$7v_B = 32$ so $v_B = 4.57\,\text{m s}^{-1}$

$4.57 - v_A = 3$ so $v_A = 1.57\,\text{m s}^{-1}$

So both balls are moving in the same direction with $v_A = 1.57\,\text{m s}^{-1}$ and $v_B = 4.57\,\text{m s}^{-1}$.

5 **a** $m_A = 2m\,\text{kg}$, $u_A = u\,\text{m s}^{-1}$, $m_B = 3m\,\text{kg}$, $u_B = -3u\,\text{m s}^{-1}$, $e = e$

By conservation of momentum:

$2m \times u + 3m \times -3u = 2m \times v_A + 3m \times v_B$

$2mv_A + 3mv_B = -7mu$

$2v_A + 3v_B = -7u$

By Newton's experimental law:

$v_B - v_A = e(u - -3u)$

$v_B - v_A = 4eu$

Solving these simultaneous equations:

$2v_A + 3v_B = -7u$

$2v_B - 2v_A = 8eu$

$5v_B = 8eu - 7u$

$v_B = \frac{u}{5}(8e - 7)$

b $v_B - v_A = 4eu$

$\frac{u}{5}(8e - 7) - v_A = 4eu$

$8eu - 7u - 5v_A = 20eu$

$5v_A = -12eu - 7u$

$v_A = \frac{-u}{5}(12e + 7)$

6 First impact:

$m_A = m\,\text{kg}$, $u_A = 4\,\text{m s}^{-1}$, $m_B = m\,\text{kg}$, $u_B = 0\,\text{m s}^{-1}$, $e = 0.6$

By conservation of momentum:

$m \times 4 + m \times 0 = m \times v_A + m \times v_B$

$v_A + v_B = 4$

By Newton's experimental law:

$v_B - v_A = 0.6(4 - 0)$

$v_B - v_A = 2.4$

Solving these simultaneous equations:

$2v_B = 6.4$

$v_B = 3.2\,\text{m s}^{-1}$

Second impact:

$m_B = m\,\text{kg}$, $u_B = 3.2\,\text{m s}^{-1}$, $m_C = m\,\text{kg}$, $u_C = 0\,\text{m s}^{-1}$, $e = 0.6$

By conservation of momentum:

$m \times 3.2 + m \times 0 = m \times v_B + m \times v_C$

$v_B + v_C = 3.2$

By Newton's experimental law:

$v_C - v_B = 0.6(3.2 - 0)$

$v_C - v_B = 1.92$

Solving these simultaneous equations:

$2v_C = 5.12$

$v_C = 2.56\,\text{m s}^{-1}$

Exercise 6.3A

1 Parallel to the surface:

Speed before $= 4.5\cos 30°$, speed after $= 4.5\cos 30°$

Perpendicular to the surface:

Speed before $= 4.5\sin 30°$,
speed after $= 0.45 \times 4.5\sin 30° = 2.025\sin 30°\,\text{m s}^{-1}$

Velocity after $= \sqrt{(4.5\cos 30°)^2 + 1.0125^2} = 4.03\,\text{m s}^{-1}$

Direction $\theta = \tan^{-1}\left(\frac{2.025\sin 30°}{4.5\cos 30°}\right) = 14.6°$ above the horizontal

2 Parallel to the surface:

Speed before $= 12\cos 50°$, speed after $= 12\cos 50°$

Perpendicular to the surface:

Speed before $= 12\sin 50°$,
speed after $= 0.4 \times 12\sin 50° = 4.8\sin 50°$

Velocity after $= \sqrt{(12\cos 50°)^2 + (4.8\sin 50°)^2}$
$= 8.55\,\text{m s}^{-1}$

Direction $\theta = \tan^{-1}\left(\frac{4.8\sin 50°}{12\cos 50°}\right) = 25.5°$ above the horizontal

3 Parallel to the surface:

Speed before $= 2.5\cos 80°$,

speed after $= 2.5\cos 80°$

Perpendicular to the surface:

Speed before = 2.5 sin 80°,
speed after = 0.7 × 2.5 sin 50° = 1.75 sin 50°

Velocity after = $\sqrt{(2.5\cos 80°)^2 + (1.75\sin 80°)^2}$
$= 1.78\text{ m s}^{-1}$

Direction $\theta = \tan^{-1}\left(\frac{1.75\sin 80°}{2.5\cos 80°}\right) = 75.9°$
to the wall.

4 Parallel to the cushion:

Speed before = 4 cos 72°, speed after = 4 cos 72°

Perpendicular to the cushion:

Speed before = 4 sin 72°,
speed after = 0.78 × 4 sin 72° = 3.12 sin 72°

Velocity after = $\sqrt{(4\cos 72°)^2 + (3.12\sin 72°)^2}$
$= 3.21\text{ m s}^{-1}$

Direction $\theta = \tan^{-1}\left(\frac{3.12\sin 72°}{4\cos 72°}\right) = 67.4°$ between
the direction of the ball and the cushion

5 a Let α be the angle at which the particle hit the surface.

tan 30° = 0.3 tan α so α = 62.5°

b Parallel to the surface:

Speed after = 3.2 cos 30°,
speed before = 3.2 cos 30°

Perpendicular to the surface:

Speed after = 3.2 sin 30°,
speed before = 3.2 sin 30° ÷ 0.3 = 10.67 sin 30°

Velocity before = $\sqrt{(3.2\cos 30°)^2 + (10.67\sin 30°)^2}$
$= 6.01\text{ m s}^{-1}$

6 a Parallel to the surface:

Speed before = 3.2 cos 36°, speed after = 3.2 cos 36°

Also speed after = 2.8 cos α so

$\alpha = \cos^{-1}\left(\frac{3.2\cos 36°}{2.8}\right) = 22.4°$

b tan 22.393° = e tan 36°

e = 0.567

7 Let α be the angle between the wall and the ball before the collision.

Then 0.88 tan α = tan 50°

So $\alpha = \tan^{-1}\left(\frac{\tan 50°}{0.88}\right)$

α = 53.6°

8 a Parallel to the surface:

Speed before = 3.2 cos 40°, speed after = 3.2 cos 40°

Perpendicular to the surface:

Speed before = 3.2 sin 40°,
speed after = 0.6 × 3.2 sin 40°
= 1.92 sin 40°

Velocity after = $\sqrt{(3.2\cos 40°)^2 + (1.92\sin 40°)^2}$
$= 2.74\text{ m s}^{-1}$

Direction $\theta = \tan^{-1}\left(\frac{1.92\sin 40°}{3.2\cos 40°}\right) = 26.7°$
above the horizontal

b Maximum height = $\frac{U^2\sin^2\theta}{2g}$

$= \frac{2.7445^2\sin^2 26.7234°}{20}$

= 0.0762 m or 7.62 cm

c Range = $\frac{U^2\sin 2\theta}{g}$

$= \frac{2.7445^2\sin 53.45°}{10}$

= 0.605 m or 60.5 cm

9 a Time of flight:

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 5.1t\sin 30° - 0.5 \times 10 \times t^2 + 2$

$5t^2 - 2.55t - 2 = 0$

$t = \frac{2.55 \pm \sqrt{(-2.55)^2 - 4\times 5\times -2}}{2\times 5}$

$t = -0.427$ or $t = 0.937$

So the ball is in air for 0.937 seconds before it hits the ground.

So the distance to point of impact is
$x = Ut\cos\theta = 5.1 \times 0.937 \times \cos 30° = 4.14$ m.

At point of impact:

Horizontal velocity = 5.1 cos 30° = 4.42 m s^{-1}

Vertical velocity = $U\sin\theta - gt$

= 5.1 sin 30° − 10 × 0.937

= −6.82 m s^{-1}

i.e. 6.82 m s^{-1} downwards

Horizontal velocity before bounce = 4.42 m s^{-1}
so horizontal velocity after bounce = 4.42 m s^{-1}

Vertical velocity before bounce = 6.82 m s^{-1}
so vertical velocity after bounce = 0.7 × 6.82
= 4.77 m s^{-1}

Resultant velocity = $\sqrt{4.42^2 + 4.77^2} = 6.50\text{ m s}^{-1}$

Angle after bounce = $\tan^{-1}\left(\frac{4.77}{4.42}\right) = 47.2°$

Distance to highest point

= 0.5 × range = $\frac{U^2\sin 2\theta}{2g}$

$= \frac{6.50^2\sin 94.4°}{20}$

= 2.11 m

So total horizontal distance = 4.14 + 2.11
= 6.25 m

b Time of flight to bounce = 0.937 s

Time from bounce to catch

$= 0.5 \times \frac{2U\sin\theta}{g} = \frac{13\sin 47.2°}{20}$

= 0.477 seconds

So total time of flight = 0.937 + 0.477 = 1.41 s

c Motion occurs only in two dimensions.

Air resistance is negligible.

Gravity remains constant.

There is no spin applied to the ball.

10 After each bounce, the horizontal velocity will remain $10\,\text{m s}^{-1}$.

Find time of first flight by substituting $y = 0$ into

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 10t\sin 0° - 0.5 \times 10 \times t^2 + 5$

$5t^2 = 5$

$t = 1\,\text{s}$

Vertical velocity $= U\sin\theta - gt$

$= 10\sin 0° - 10 \times 1 = 10\,\text{m s}^{-1}$ downwards

After first bounce: $v_x = 10\,\text{m s}^{-1}$,

$v_y = 0.65 \times 10 = 6.5\,\text{m s}^{-1}$

Resultant velocity $= \sqrt{10^2 + 6.5^2} = 11.927\,\text{m s}^{-1}$

Angle after bounce $= \tan^{-1}\left(\frac{6.5}{10}\right) = 33.024°$

Maximum height after bounce

$= \frac{U^2\sin^2\theta}{2g} = \frac{11.927^2\sin^2 33.024°}{20} = 2.11\,\text{m}$, which is greater than 0.5 m

Second stage of journey:

Immediately before second bounce:

$v_x = 10\,\text{m s}^{-1}$, $v_y = 6.5\,\text{m s}^{-1}$

After second bounce: $v_x = 10\,\text{m s}^{-1}$,

$v_y = 0.65 \times 6.5 = 4.225\,\text{m s}^{-1}$

Resultant velocity $= \sqrt{10^2 + 4.225^2} = 10.856\,\text{m s}^{-1}$

Angle after bounce $= \tan^{-1}\left(\frac{4.23}{10}\right) = 22.904°$

Maximum height after bounce

$= \frac{U^2\sin^2\theta}{2g} = \frac{10.856^2\sin^2 22.904°}{20} = 0.893\,\text{m}$, which is greater than 0.5 m

Third stage of journey:

Immediately before third bounce: $v_x = 10\,\text{m s}^{-1}$,

$v_y = 4.225\,\text{m s}^{-1}$

After third bounce: $v_x = 10\,\text{m s}^{-1}$,

$v_y = 0.65 \times 4.225 = 2.746\,\text{m s}^{-1}$

Resultant velocity $= \sqrt{10^2 + 2.746^2} = 10.37\,\text{m s}^{-1}$

Angle after bounce $= \tan^{-1}\left(\frac{2.746}{10}\right) = 15.355°$

Maximum height after bounce

$= \frac{U^2\sin^2\theta}{2g} = \frac{10.37^2\sin^2 15.355°}{20} = 0.377\,\text{m}$, which is less than 0.5 m

So the maximum height of the ball is less than 50 cm for the first time after the third bounce.

Exercise 6.4A

1 a Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision = 3 sin 52°

$= 2.364\,\text{m s}^{-1}$

Sphere B: speed after collision $= 0\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$3\cos 52° = v_A + v_B$

By Newton's experimental law:

$v_B - v_A = 0.6 \times 3\cos 52°$

$v_B - v_A = 1.8\cos 52°$

Solving simultaneously:

$2v_B = 4.8\cos 52°$

$v_B = 1.478\,\text{m s}^{-1}$

$v_A + 1.478 = 3\cos 52°$

$v_A = 0.369\,\text{m s}^{-1}$ to 3 s.f.

Sphere A:

Velocity after impact $= \sqrt{2.364^2 + 0.369^2}$

$= 2.39\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{2.364}{0.369}\right) = 81.1°$ above the line joining the centres of the two spheres.

Sphere B:

Sphere B is travelling at a velocity of $1.48\,\text{m s}^{-1}$ parallel to the line joining the centres of the two spheres.

b Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision = 10 sin 45°

$= 7.0711\,\text{m s}^{-1}$

Sphere B: speed after collision = 8 sin 30°

$= 4\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$10\cos 45° - 8\cos 30° = v_A + v_B$

$v_A + v_B = 0.1429$

By Newton's experimental law:

$v_B - v_A = 0.5 \times (10 \cos 45° - (-8 \cos 30°))$

$v_B - v_A = 6.9996$

Solving simultaneously:

$2v_B = 7.1425$

$v_B = 3.5713\,\text{m s}^{-1}$

$v_A + 3.5713 = 0.1429$

$v_A = -3.4284\,\text{m s}^{-1}$

Sphere A:

Velocity after impact $= \sqrt{7.0711^2 + (-3.4284)^2}$
$= 7.86\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{7.0711}{3.4284}\right) = 64.1°$ below the line joining the centres of the two spheres.

Sphere B:

Velocity after impact $= \sqrt{4^2 + 3.5713^2} = 5.36\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{3.5713}{4}\right) = 41.8°$ below the line joining the centres of the two spheres in the opposite direction to B.

c Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision $= -3 \sin 25°$ $= -1.2679\,\text{m s}^{-1}$ (to the left)

Sphere B: speed after collision $= 2 \sin 30°$ $= 1\,\text{m s}^{-1}$ (to the right)

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$3 \cos 25° - 2 \cos 30° = v_A + v_B$

$v_A + v_B = 0.9869$

By Newton's experimental law:

$v_B - v_A = 0.9 \times (3 \cos 25° - (-2 \cos 30°))$

$v_B - v_A = 4.0059$

Solving simultaneously:

$2v_B = 4.9928$

$v_B = 2.4964\,\text{m s}^{-1}$

$v_A + 2.4964 = 0.9869$

$v_A = -1.5095\,\text{m s}^{-1}$

Sphere A:

Velocity after impact $= \sqrt{(-1.2679)^2 + (-1.5095)^2}$
$= 1.97\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{1.2679}{1.5095}\right) = 40.0°$ to the line joining the centres of the two spheres (downwards)

Sphere B:

Velocity after impact $= \sqrt{1^2 + 2.4964^2} = 2.69\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{1}{2.5}\right) = 21.8°$ to the line joining the centres of the two spheres in the opposite direction to B (upwards).

d Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision $= 5 \sin 60°$ $= 4.3301\,\text{m s}^{-1}$ (to the left)

Sphere B: speed after collision $= 4 \sin 40°$ $= 2.5712\,\text{m s}^{-1}$ (to the right)

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$5 \cos 60° - 4 \cos 40° = v_A + v_B$

$v_A + v_B = -0.5642$

By Newton's experimental law:

$v_B - v_A = 0.3 \times (5 \cos 60° - (-4 \cos 40°))$

$v_B - v_A = 1.6693$

Solving simultaneously:

$2v_B = 1.1051$

$v_B = 0.5526\,\text{m s}^{-1}$

$v_A + 0.5526 = -0.5642$

$v_A = -1.1168\,\text{m s}^{-1}$

Sphere A:

Velocity after impact $= \sqrt{(4.3301)^2 + (-1.1168)^2}$ $= 4.47\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{4.3301}{1.1168}\right) = 75.5°$ to the line joining the centres of the two spheres

Sphere B:

Velocity after impact $= \sqrt{2.5712^2 + 0.5526^2}$
$= 2.63\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{2.57}{0.56}\right) = 77.9°$ to the line joining the centres of the two spheres in the opposite direction to B.

2 Perpendicular to the line joining the centres of the spheres:

Red ball: speed after collision $= 0.5 \sin 15° = 0.129\,\text{m s}^{-1}$

Blue ball: speed after collision $= 0\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$0.5 \cos 15° = v_R + v_B$

$v_R + v_B = 0.4830$

By Newton's experimental law:

$v_B - v_R = 0.3 \times (0.5 \cos 15° - 0)$

$v_B - v_R = 0.1449$

Solving simultaneously:

$2v_B = 0.6279$

$v_B = 0.314\,\text{m s}^{-1}$

$v_R + 0.314 = 0.483$

$v_R = 0.169\,\text{m s}^{-1}$

Red ball:

Velocity after impact $= \sqrt{(0.129)^2 + (0.169)^2}$
$= 0.213\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{0.129}{0.169}\right) = 37.4°$ to the line joining the centres of the two spheres

Blue ball moves with a velocity of $0.314\,\text{m s}^{-1}$ parallel to the line joining the centres.

3 Perpendicular to the line joining the centres of the spheres:

Marble A: speed after collision $= 2 \sin 25°$
$= 0.8452\,\text{m s}^{-1}$

Marble B: speed after collision $= 1.5 \sin 30°$
$= 0.75\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$u_A + u_B = -2 \cos 25° + 1.5 \cos 30°$

$u_A + u_B = -0.5136$

By Newton's experimental law:

$1.5 \cos 30° - (-2 \cos 25°) = 0.4 \times (u_A - u_B)$

$0.4u_A - 0.4u_B = 3.1117$

$u_A - u_B = 7.7791$

Solving simultaneously:

$2u_A = 7.2655$

$u_A = 3.633\,\text{m s}^{-1}$

$u_B + 3.633 = -0.5136$

$u_B = -4.147\,\text{m s}^{-1}$

Marble A:

Velocity before impact $= \sqrt{(0.845)^2 + (3.633)^2}$
$= 3.73\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{0.845}{3.633}\right) = 13.1°$ to the line joining the centres of the two marbles

Marble B:

Velocity before impact $= \sqrt{0.75^2 + (-4.147)^2}$
$= 4.21\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{0.75}{4.147}\right) = 10.3°$ to the line joining the centres of the two marbles.

Before the collision there was an angle of 157° between their directions of motion.

4 Perpendicular to the line joining the centres of the spheres:

Sphere A: speed before collision $= 0\,\text{m s}^{-1}$

Sphere B: speed after collision $= u\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$u = v_A + v_B$

By Newton's experimental law:

$v_B - v_A = -0.5u$

Solving simultaneously:

$2v_B = 0.5u$

$v_B = 0.25u\,\text{m s}^{-1}$

$v_A + 0.25u = u$

$v_A = 0.75u\,\text{m s}^{-1}$

Sphere A will travel at $0.75u\,\text{m s}^{-1}$ parallel to the line joining the centres of the two spheres.

Sphere B:

Velocity after impact $= \sqrt{u^2 + (0.25u)^2} = 1.03u\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{u}{0.25u}\right) = 76.0$ (3 s.f.) to the line joining the centres of the two spheres.

5 If $\tan\theta = \frac{5}{4}$ then $\cos\theta = \frac{4}{\sqrt{41}}$ and $\sin\theta = \frac{5}{\sqrt{41}}$.

Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision $= -16 \sin\theta$
$= -12.4939\,\text{m s}^{-1}$

Sphere B: speed after collision $= 10 \sin\theta = 7.8087\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$4 \times 16 \cos\theta - 10 \cos\theta = 4v_A + v_B$

$4v_A + v_B = 33.7335$

By Newton's experimental law:

$v_B - v_A = 0.22 \times (16 \cos\theta - (-10 \cos\theta))$

$v_B - v_A = 3.5733$

Solving simultaneously:

$5v_B = 48.0267$

$v_B = 9.605\,\text{m s}^{-1}$

$4v_A + 9.605 = 33.7335$

$v_A = 6.032\,\text{m s}^{-1}$

Sphere A:

Velocity after impact $= \sqrt{(-12.494)^2 + (6.032)^2}$
$= 13.9\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{12.5}{6.03}\right) = 64.2°$ to the line joining the centres of the two spheres.

Sphere B:

Velocity after impact $= \sqrt{7.809^2 + 9.605^2}$

$= 12.4\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\frac{7.81}{9.60}\right) = 39.1°$ to the line joining the centres of the two spheres in the opposite direction to B.

After the collision there will be an angle of 76.7° between their directions of motion.

Exam-style questions

1 $m_A = 5m$ kg, $u_A = 2u\,\text{m s}^{-1}$, $m_B = 3m$ kg, $u_B = -4u$, $e = e$

By conservation of linear momentum

$10mu - 12mu = 5mv_A + 3mv_B$

$-2u = 5v_A + 3v_B$

By Newton's experimental law:

$v_B - v_A = e(2u + 4u)$

$v_B - v_A = 6eu$

Solving these simultaneous equations:

$8v_B = 30eu - 2u$

$v_B = \frac{u}{4}(15e - 1)$

For B to travel in the opposite direction we need v_B to be greater than 0.

This will be when $e > \frac{1}{15}$

2 Parallel to the surface:

Speed before = 5 cos 45°, speed after = 5 cos 45°

Perpendicular to the surface:

Speed before = 5 sin 45°, speed after = 5e sin 45°

Velocity after = $\sqrt{(5\cos 45°)^2 + (5e\sin 45°)^2} = 4$

$(5\cos 45°)^2 + (5e\sin 45°)^2 = 16$

$12.5 + 12.5e^2 = 16$

$12.5(1 + e^2) = 16$

$e = 0.529$

3 Parallel to the wall:

Before = 16 cos θ, speed after = 16 cos θ

Perpendicular to the wall:

Speed before = 16 sin θ, speed after = 9.6sin θ

After the collision $v = 12\,\text{m s}^{-1}$

So $\sqrt{(16\cos\theta)^2 + (9.6\sin\theta)^2} = 12$

$256\cos^2\theta + 92.16\sin^2\theta = 144$

$256(1 - \sin^2\theta) + 92.16\sin^2\theta = 144$

$256 - 163.84\sin^2\theta = 144$

$163.84\sin^2\theta = 112$

$\sin\theta = 0.8268$

$\theta = 55.8°$

4 $m_A = 3m$ kg, $u_A = u\,\text{m s}^{-1}$, $m_B = 5m$ kg, $u_B = 0$, $e = e$

By conservation of momentum:

$3m \times u + 5m \times 0 = 3m \times v_A + 5m \times v_B$

$3v_A + 5v_B = 3u$

By Newton's experimental law:

$v_B - v_A = e(u - 0)$

$v_B - v_A = eu$

Solving these simultaneous equations:

$8v_B = 3u(1 + e)$ so $v_B = \frac{3}{8}u(1 + e)$

$v_B - v_A = eu$

$\frac{3}{8}u(1 + e) - v_A = eu$

$v_A = \frac{3}{8}u + \frac{3}{8}eu - eu$

$v_A = \frac{3}{8}u + \frac{3}{8}eu - eu$

$v_A = \frac{3}{8}u - \frac{5}{8}eu$

$v_A = \frac{1}{8}u(3 - 5e)$

5 **a** Parallel to the surface:

Speed before = 5.2 cos 42°, speed after = 5.2 cos 42°

Perpendicular to the surface:

Speed before = 5.2 sin 42°, speed after = 5.2e sin 42°

Velocity after = $\sqrt{(5.2\cos 42°)^2 + (5.2e\sin 42°)^2} = 3.9$

$14.933 + 12.107e^2 = 15.21$

$e = 0.151$

b $\tan\theta = 0.1513 \times \tan 42°$

$\theta = 7.76°$

6 **a** Speed before first impact:

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 2$

$= 40$

$v = 6.32\,\text{m s}^{-1}$

b Speed immediately after first impact :

$v^2 = u^2 + 2as$

$0^2 = u^2 + 2 \times -10 \times 1.2$

$u^2 = 24$

$u = 4.90\,\text{m s}^{-1}$

c $e = \frac{4.90}{6.32} = 0.775$

7 a Speed before first impact:

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 6$

$= 120$

$v = 10.9545\,\mathrm{m\,s^{-1}}$

Speed immediately after first impact

$= 10.9545 \times 0.8 = 8.764\,\mathrm{m\,s^{-1}}$

Height reached after first impact:

$$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 8.764^2}{2 \times -10} = 3.84\,\mathrm{m}$$

b Time between bounces $= 2 \times \dfrac{v-u}{a}$

$$= 2 \times \frac{0 - 8.764}{-10} = 1.75\,\mathrm{s}$$

c Speed immediately before second impact is $8.76\,\mathrm{m\,s^{-1}}$

8 a Speed after impact $= 0.7 \times 12 = 8.4\,\mathrm{m\,s^{-1}}$

b Lands when $y = 0$

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 8.4 \times t \times \sin 0 - \frac{1}{2} \times 10 \times t^2 + 2$

$5t^2 = 2$

$t = 0.632\,\mathrm{s}$

c At time of landing:

$v_x = U\cos\theta = 8.4\cos 0 = 8.4\,\mathrm{m\,s^{-1}}$

$v_y = U\sin\theta - gt = 8.4\sin 0° - 10 \times 0.63246$

$= 6.3246\,\mathrm{m\,s^{-1}}$ downwards

After impact:

$v_x = 8.4\,\mathrm{m\,s^{-1}}$

$v_y = 0.5 \times 6.3246 = 3.1623\,\mathrm{m\,s^{-1}}$

So speed $= \sqrt{8.4^2 + 3.1623^2} = 8.976\,\mathrm{m\,s^{-1}}$

$\theta = \tan^{-1}\left(\dfrac{3.1623}{8.4}\right) = 20.63°$

$T = \dfrac{2 \times 8.976 \times \sin 20.63}{10} = 0.633\,\mathrm{s}$

9 a $m_A = m$ kg, $u_A = 4\,\mathrm{m\,s^{-1}}$, $m_B = m$ kg, $u_B = 0\,\mathrm{m\,s^{-1}}$, $e = 0.7$

By conservation of momentum:

$m \times 4 + m \times 0 = m \times v_A + m \times v_B$

$v_A + v_B = 4$

By Newton's experimental law:

$v_B - v_A = 0.7(4 - 0)$

$v_B - v_A = 2.8$

Solving these simultaneous equations:

$2v_B = 6.8$

$v_B = 3.4\,\mathrm{m\,s^{-1}}$

$3.4 - v_A = 2.8$

$v_A = 0.6\,\mathrm{m\,s^{-1}}$

b Speed of B before impact with wall $= 3.4\,\mathrm{m\,s^{-1}}$

Speed of B after impact with wall

$= 0.5 \times 3.4 = 1.7\,\mathrm{m\,s^{-1}}$

10 a This is motion in a straight line.

$m_A = 0.05$ kg, $u_A = 4\,\mathrm{m\,s^{-1}}$, $m_B = 0.04$ kg, $u_B = 0\,\mathrm{m\,s^{-1}}$, $e = ?$, $v_A = 1\,\mathrm{m\,s^{-1}}$

By the conservation of linear momentum:

$4 \times 0.05 + 0 \times 0.04 = 0.05 \times 1 + 0.04v_B$

$v_B = \dfrac{0.15}{0.04} = 3.75\,\mathrm{m\,s^{-1}}$

b By Newton's experimental law:

$3.75 - 1 = e(4 - 0)$

$e = 0.688$

11 a $\tan 30° = e\tan 60°$

$e = \dfrac{1}{3}$

b Parallel to the surface:

Speed before $= 5u\cos 60°$, speed after $= 5u\cos 60°$

Perpendicular to the surface:

Speed before $= 5u\sin 60°$, speed after $= \frac{5}{3}u\sin 60°$

$$\text{Speed after} = \sqrt{(5u\cos 60°)^2 + \left(\frac{5}{3}u\sin 60°\right)^2} = \lambda u$$

$$\frac{25}{4}u^2 + \frac{25}{12}u^2 = \lambda^2 u^2$$

$$\frac{25}{4} + \frac{25}{12} = \lambda^2$$

$\lambda = 2.89$

12 a Time of flight:

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 4t\sin 40° - 0.5 \times 10 \times t^2 + 3$

$5t^2 - 2.57115t - 3 = 0$

$$t = \frac{2.57115 \pm \sqrt{(-2.57115)^2 - 4 \times 5 \times -3}}{2 \times 5}$$

$t = -0.559\,\mathrm{s}$ or $t = 1.07327\,\mathrm{s}$

So the ball is in air for 1.07327 seconds before it hits the ground.

At point of impact:

Horizontal velocity $= 4\cos 40° = 3.0642\,\mathrm{m\,s^{-1}}$

Vertical velocity $= U\sin\theta - gt$

$= 4\sin 40° - 10 \times 1.07327 = -8.1615\,\mathrm{m\,s^{-1}}$

i.e. $8.1615\,\mathrm{m\,s^{-1}}$ downwards

Horizontal velocity before bounce $= 3.0642\,\mathrm{m\,s^{-1}}$ so horizontal velocity after bounce $= 3.0642\,\mathrm{m\,s^{-1}}$

Vertical velocity before bounce $= 8.1615\,\mathrm{m\,s^{-1}}$ so vertical velocity after $= 0.6 \times 8.1615 = 4.8969\,\mathrm{m\,s^{-1}}$

Resultant velocity after bounce

$= \sqrt{3.0642^2 + 4.8969^2} = 5.78\,\mathrm{m\,s^{-1}}$

Angle after bounce $= \tan^{-1}\left(\dfrac{4.88}{3.06}\right) = 58.0°$

b Time of flight to first bounce = 1.07327 s

Distance travelled to first bounce = $Ut\cos\theta$

$= 4 \times 1.07327 \times \cos 40° = 3.289\,\text{m}$

Time of flight to second bounce $= \dfrac{2U\sin\theta}{g}$

$= \dfrac{2 \times 5.7766 \sin 57.964°}{10} = 0.9794\,\text{s}$

Distance travelled to first bounce = $Ut\cos\theta$

$= 5.7766 \times 0.9794 \times \cos 57.964° = 3.001\,\text{m}$

Total horizontal distance = 6.29 m

13 Time to first bounce:

$v^2 = u^2 + 2as$

$= 0^2 + 2 \times 10 \times 10$

$v = 10\sqrt{2}$

Time to reach ground after dropping:

$t = \dfrac{v-u}{a} = \dfrac{10\sqrt{2}}{10} = \sqrt{2}$

Speed after first bounce = $e\sqrt{2}$

Speed after second bounce = $e^2\sqrt{2}$, etc.

Time of flight $= \dfrac{2u}{g}$

So total time of flight

$= \sqrt{2} + \dfrac{20e\sqrt{2}}{10} + \dfrac{20e^2\sqrt{2}}{10} + \dfrac{20e^3\sqrt{2}}{10} + \ldots$

$= \sqrt{2} + \sqrt{2}\left(2e + 2e^2 + 2e^3 + \ldots\right)$

$= \sqrt{2} + \sqrt{2} \times \dfrac{2e}{1-e}$

$= \sqrt{2}\left(1 + \dfrac{2e}{1-e}\right)$

$= \sqrt{2}\left(\dfrac{(1-e)}{(1-e)} + \dfrac{2e}{(1-e)}\right)$

$= \sqrt{2}\left(\dfrac{1+e}{1-e}\right)$

14 a Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision = 3.5 sin 35°
$= 2.0075\,\text{m s}^{-1}$

Sphere B: speed after collision = 2.9 sin 40°
$= 1.8641\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$2m \times 3.5\cos 35° - 3m \times 2.9\cos 40° = 2mv_A + 3mv_B$

$2v_A + 3v_B = -0.93052$

By Newton's experimental law:

$v_B - v_A = 0.2 \times (3.5\cos 35° - (-2.9\cos 40°))$

$v_B - v_A = 1.01771$

Solving simultaneously:

$5v_B = 1.1049$

$v_B = 0.22098\,\text{m s}^{-1}$

$0.22098 - v_A = 1.01771$

$v_A = -0.79673\,\text{m s}^{-1}$

Sphere A:

Velocity after impact $= \sqrt{(2.0075)^2 + (-0.7967)^2}$

$= 2.16\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\dfrac{2.0075}{0.7967}\right) = 68.4°$ to the line joining the centres of the two spheres

Sphere B:

Velocity after impact $= \sqrt{(1.8641)^2 + (0.221)^2}$

$= 1.88\,\text{m s}^{-1}$

$\alpha = \tan^{-1}\left(\dfrac{1.8641}{0.221}\right) = 83.2°$ to the line joining the centres of the two spheres

b Sphere B will hit the wall at an angle of 83.24° between the wall and the direction of travel with a speed of $1.877\,\text{m s}^{-1}$.

Parallel to the surface:

Speed before = 1.877 cos 83.24° = $0.2209\,\text{m s}^{-1}$, speed after = $0.2209\,\text{m s}^{-1}$

Perpendicular to the surface:

Speed before = 1.877 sin 83.24° = $1.864\,\text{m s}^{-1}$, speed after = 0.6 × 1.864 = $1.118\,\text{m s}^{-1}$

Velocity after $= \sqrt{0.2209^2 + 1.118^2} = 1.14\,\text{m s}^{-1}$

Direction $= \tan^{-1}\left(\dfrac{1.118}{0.2209}\right) = 78.8°$ between the wall and the direction of travel

15 a $m_A = 3\,\text{kg}$, $u_A = 4\,\text{m s}^{-1}$, $m_B = 4\,\text{kg}$, $u_B = -2\,\text{m s}^{-1}$, $e = 0.2$

By conservation of momentum:

$3 \times 4 + 4 \times -2 = 3 \times v_A + 4 \times v_B$

$3v_A + 4v_B = 4$

By Newton's experimental law:

$v_B - v_A = 0.2(4 - -2)$

$v_B - v_A = 1.2$

Solving these simultaneous equations:

$7v_B = 7.6$

$v_B = 1.09\,\text{m s}^{-1}$

$1.086 - v_A = 1.2$

$v_A = -0.114\,\text{m s}^{-1}$

b Time ball B is in the air:

$s = ut + \frac{1}{2}at^2$

$0.8 = 0 \times t + \frac{1}{2} \times 10 \times t^2$

So $t^2 = 0.16$

So $t = 0.4\,\text{s}$

Horizontal range:

$x = 1.086 \times 0.4 \times \cos 0°$

$= 0.434\,\text{m}$

So the ball will land 0.434 m away from the base of the table.

16 a Perpendicular to the line joining the centres of the spheres:

Sphere A: speed after collision = $u \sin 50°$

Sphere B: speed after collision = $0\,\text{m s}^{-1}$

Parallel to the line joining the centres of the spheres:

By the conservation of linear momentum:

$u \cos 50° = v_A + v_B$

By Newton's experimental law:

$v_B - v_A = e \times (u \cos 50°)$

$v_B - v_A = eu \cos 50°$

Solving simultaneously:

$2v_B = u \cos 50°\ (1 + e)$

$v_B = \frac{u}{2}\cos 50°(1 + e)$

$v_A = u \cos 50° - \frac{u}{2}\cos 50°(1 + e)$

$v_A = \frac{u}{2}\cos 50°(1 - e)$

Velocity of A after impact

$= \sqrt{u^2\sin^2 50° + \frac{u^2}{4}\cos^2 50°(1-e)^2}$

b Velocity of B after impact $= \frac{u}{2}\cos 50°(1 + e)$

17 $m_A = 5\,\text{kg}$, $m_B = 3\,\text{kg}$, $m_C = 1\,\text{kg}$

Collision between A and B

By conservation of linear momentum

$5 \times 0.5 + 3 \times 0 = 5v_A + 3v_B$

$5v_A + 3v_B = 2.5$

By Newton's experimental law:

$v_B - v_A = 0.4\ (0.5)$

$v_B - v_A = 0.2$

$5v_B - 5v_A = 1$

Solving simultaneously,

$8v_B = 3.5$

$v_B = 0.4375\,\text{m s}^{-1}$.

Collision between B and C

By conservation of linear momentum

$3 \times 0.4375 + 1 \times 0 = 3v_B + v_C$

$3v_B + v_C = 1.3125$

By Newton's experimental law:

$v_C - v_B = 0.8\ (0.4375)$

$v_C - v_B = 0.35$

$3v_C - 3v_B = 1.05$

Solving simultaneously,

$4v_C = 2.3625$

$v_C = 0.591\,\text{m s}^{-1}$

The speed of C after the collision with B is $0.591\,\text{m s}^{-1}$.

18 a By conservation of linear momentum

$5 \times 4 + 3 \times 0 = 5v_A + 4v_B$

$5v_A + 4v_B = 20$

By Newton's experimental law:

$v_B - v_A = 0.8\ (4)$

$v_B - v_A = 3.2$

$5v_B - 5v_A = 16$

Solving simultaneously,

$9v_B = 36$

$v_B = 4\,\text{m s}^{-1}$.

$4 - v_A = 3.2$

$v_A = 0.8\,\text{m s}^{-1}$ (i.e. in the same direction as before the collision)

b Parallel to the surface:

Speed before = $4 \cos 45°$, speed after = $4 \cos 45°$

Perpendicular to the surface:

Speed before = $4 \sin 45°$, speed after = $4e \sin 45°$

Velocity after $= \sqrt{(4\cos 45°)^2 + (4e \sin 45°)^2}$

$= 2.9$

$\sqrt{8 + 8e^2} = 2.9$

$8e^2 = 0.41$

$e = \sqrt{0.05125}$

$= 0.226$

c θ = angle between direction of B and wall after collision

$0.226 \tan 45° = \tan\theta$

So $\theta = 12.7°$

19 Collision between A and B

By the conservation of linear momentum

$1 \times 2 + 1 \times 0 = (1 + 1)\, v_{AB}$

$2 = 2v_{AB}$

$v_{AB} = 1\,\text{m s}^{-1}$.

Collision between AB and C

KE of AB before $= \frac{1}{2} \times 2 \times 1^2 = 1\,\text{J}$

KE of AB after $= \frac{1}{3} \times 1 = \frac{1}{3}\,\text{J}$

So velocity of AB after collision $= \sqrt{\dfrac{\frac{1}{3}}{\frac{1}{2} \times 2}}$
$= 0.5774\,\text{m s}^{-1}$.

Collision between AB and C

By the conservation of linear momentum

$2 \times 1 + 1 \times 0 = 2v_{AB} + v_C$

$2v_{AB} + v_C = 2$

By Newton's experimental law

$v_C - v_{AB} = e\,(1)$

$2v_C - 2v_{AB} = 2e$

Solving simultaneously,

$3v_C = 2 + 2e$

$v_C = \frac{2}{3}(1 + e)$

$v_{AB} = v_C - e$

$= \frac{2}{3}(1 + e) - e$

$= \frac{2}{3} - \frac{1}{3}e$

From above, $v_{AB} = 0.5774$

$\frac{2}{3} - \frac{1}{3}e = 0.5774$

$e = 0.268$

Mathematics in life and work

1 Time of flight until first bounce:

$y = Ut\sin\theta - \frac{1}{2}gt^2 + h$

$0 = 10t\sin 0° - 0.5 \times 10 \times t^2 + 2$

$5t^2 = 2$

$t = 0.632456\,\text{s}$

So the ball is in air for 0.632 seconds before it hits the ground.

At point of impact:

Horizontal velocity $= 10\,\text{m s}^{-1}$

Vertical velocity $= U\sin\theta - gt$
$= 10\sin 0° - 10 \times 0.632456 = -6.32456\,\text{m s}^{-1}$
i.e. $6.32456\,\text{m s}^{-1}$ downwards

Horizontal velocity before bounce $= 10\,\text{m s}^{-1}$ so horizontal velocity after bounce $= 10\,\text{m s}^{-1}$

Vertical velocity before bounce $= 6.32456\,\text{m s}^{-1}$ so vertical velocity after $= 0.8 \times 6.32456 = 5.0596\,\text{m s}^{-1}$

Resultant velocity after bounce $= \sqrt{10^2 + 5.0596^2}$
$= 11.21\,\text{m s}^{-1}$

Angle after bounce $= \tan^{-1}\left(\frac{5.0596}{10}\right) = 26.84°$

To reach the assistant's mouth, $y = 1.1$

$1.1 = 11.21t\sin 26.84° - \frac{1}{2} \times 10 \times t^2$

$5t^2 - 5.0613t + 1.1 = 0$

$t = 0.316$ or $t = 0.696$

Total time taken to reach the assistant's mouth is either 0.948 s or 1.33 s.

2 At these times, the distance since the bounce is:

distance $= 11.21 \times 0.316 \times \cos 26.84° = 3.161\,\text{m}$

and distance $= 11.21 \times 0.696 \times \cos 26.84° = 6.962\,\text{m}$

Distance between release and first bounce
$= 10 \times 0.6325 = 6.325\,\text{m}$

So distance between the two entertainers is 9.49 m or 13.3 m.

Summary Review

Please note: Full worked solutions are provided as an aid to learning, and represent one approach to answering the question. In some cases, alternative methods are shown for contrast.

All sample answers have been written by the authors. Cambridge Assessment International Education bears no responsibility for the example answers to questions taken from its past question papers, which are contained in this publication.

Non-exact numerical answers should be given correct to 3 significant figures, or 1 decimal place for angles in degrees, unless a different level of accuracy is specified in the question.

Warm-up questions

1 $\tan\alpha = \frac{8}{15}$ so $\cos\alpha = \frac{15}{17}$ and $\sin\alpha = \frac{8}{17}$

Resolving vertically:

$T \times \frac{15}{17} = 3$

$T = 3.4\,\text{N}$

Resolving horizontally:

$F = 3.4 \times \frac{8}{17} = 1.6$

2 **i** Using $F = \mu R$ and resolving horizontally, $R = W$

So $F = 1.25W$

As $W < F$, the minimum vertical force to move the block is less than the minimum horizontal force to move the block.

ii Using Newton's second law:

$P - 1.25 \times 60 = 6 \times 4$

$P = 99$

3 **i** Work done $= Fd$

$= (30\cos\alpha + 40\cos\beta) \times 20$

$= (30 \times 0.6 + 40 \times 0.8) \times 20 = 1000\,\text{J}$

ii Constant speed so friction is equal to driving force and $F = \mu R$.

Resolving vertically: $R = W$

$F = \mu R$.

$50 = \frac{5}{8}W$

$W = 80\,\text{N}$

A Level questions

1 $\sin\alpha = \frac{2}{7}$ so $\cos\alpha = \frac{\sqrt{45}}{7}$

Sliding:

Resolving horizontally:

$P\cos\alpha = F$

$F = \frac{\sqrt{45}}{7}P$ (1)

Resolving vertically:

$R + P\sin\alpha = 5g$

Since $F = 0.42R$

$\frac{F}{0.42} + P \times \frac{2}{7} = 5g$ (2)

Substituting (1) in (2) and making P the subject:

$P = \frac{50}{\left(\frac{\frac{\sqrt{45}}{7}}{0.42} + \frac{2}{7}\right)} = 19.47\,\text{N}$

Toppling:

$\sqrt[3]{4096} = 16\,\text{cm} = 0.16\,\text{m}$

Moments about A:

$5g \times 0.08 = P\sin\alpha \times 0.16 + P\cos\alpha \times 0.16$

$4 = \frac{8P}{175} + \frac{0.16\sqrt{45}P}{7}$

$P = 20.10\,\text{N}$

P for sliding is less than P for toppling, so the cube will slide first.

2 **i** $F = ma$

$0.4v^{\frac{1}{2}} = 0.6v\frac{dv}{dx}$

Dividing by $0.2v^{\frac{1}{2}}$:

$2 = 3v^{\frac{1}{2}}\frac{dv}{dx}$

ii $2\int dx = 3\int v^{\frac{1}{2}}dv$

$2x = 3 \times \frac{2}{3}v^{\frac{3}{2}} + c$

$2x = 2v^{\frac{3}{2}} + c$

When $x = 1$, $v = 1$

$2 = 2 + c$

So $c = 0$

$x = v^{\frac{3}{2}}$

$v = x^{\frac{2}{3}}$

iii $\frac{dx}{dt} = x^{\frac{2}{3}}$

$\int x^{-\frac{2}{3}}\,dx = \int dt$

$3x^{\frac{1}{3}} = t + c$

When $t = 0$, $x = 1$

So $c = 3$

$3x^{\frac{1}{3}} = t + 3$

At B, $x = 8$

$3 \times 2 = t + 3$

$t = 3$

3 Let v_A and v_B be the speeds of A and B after the collision.

Conservation of momentum:

$2mu = 2mv_A + mv_B$

$2u = 2v_A + v_B \quad (1)$

Newton's experimental law:

$eu = v_B - v_A \quad (2)$

Solving simultaneously:

$2u - eu = 3v_A$

$v_A = \dfrac{u(2-e)}{3}$

$2u + 2eu = v_B + 2v_B$

$2u(1 + e) = 3v_B$

$v_B = \dfrac{2u(1+e)}{3}$

Newton's experimental law for B bouncing off wall:

$0.4\left(\dfrac{2u(1+e)}{3}\right) = \dfrac{u(2-e)}{3}$

$0.8(1 + e) = 2 - e$

$e = \dfrac{2}{3}$

Speed = distance ÷ time

For B (from first collision to wall):

$\dfrac{10u}{9} = \dfrac{d}{t}$

$t = \dfrac{9d}{10u}$

So the distance travelled by A during that time is given by d_A where:

$\dfrac{4u}{9} = \dfrac{d_A}{\frac{9d}{10u}}$

$\dfrac{4u}{9} = \dfrac{10ud_A}{9d}$

$d_A = \dfrac{2d}{5}$

When B reaches the wall, A has travelled $\dfrac{2d}{5}$ metres. From the wall to the second collision, the particles are travelling towards each other at the same constant speed, so (by symmetry), they will meet in the middle. The distance between A and B is $\dfrac{3d}{5}$, so B will be $\dfrac{3d}{10}$ metres away from the wall when the particles collide again.

4 **i** At point of collision P and Q must have travelled the same distance horizontally.

As acceleration = 0 horizontally, use $S = Ut$

$V\cos 45°\ t = V\cos 60°\ (t + 1)$

$\cos 45°\ t = \cos 60°\ (t + 1)$

$\dfrac{t}{\sqrt{2}} = \dfrac{t+1}{2}$

$\dfrac{\sqrt{2}t}{2} = \dfrac{t+1}{2}$

$\left(\sqrt{2} - 1\right)t = 1$

$t = \sqrt{2} + 1 = 2.414\,213\,5 = 2.414$ seconds to 3 decimal places, as required.

Or using your calculator from line 2 in the above working:

$0.707\,11t = 0.5(t + 1)$

Make sure you write sufficient decimal places to show the answer is correct to 3 d.p.

As before, $t = 2.414$ to 3 decimal places as required.

ii At point of collision P and Q must have travelled the same distance vertically.

Use $s = ut + \frac{1}{2}at^2$ vertically.

Vertical distance =

$V\sin 45°\,t - \dfrac{gt^2}{2} = V\sin 60°\,(t+1) - \dfrac{g(t+1)^2}{2}$

$V\sin 45°\,t - V\sin 60°\,(t+1) = \dfrac{gt^2}{2} - \dfrac{g(t+1)^2}{2}$

Substitute in for t:

$V\,1.707 - V\,2.957 = \dfrac{10(2.414)^2}{2} - \dfrac{10(3.414)^2}{2}$

$-1.250\ V = 29.13698 - 58.27698$

$V = 23.3$

iii Greatest height,

$H = \dfrac{U^2(\sin\theta)^2}{2g} = \dfrac{23.3^2\sin^2 60°}{20} = 20.35$

At point of collision, use $S = Ut + \frac{1}{2}at^2$

$h = 23.3 \times \sin 60° \times 3.414 - 5 \times (3.414)^2$

$= 68.89 - 58.28 = 10.61$

Vertical distance below greatest height = 9.74 m

NB: You may have slightly different figures depending on the number of decimal places you work with.

5 **i** $\tan\theta = \dfrac{5}{12}$ so $\cos\theta = \dfrac{12}{13}$ and $\sin\theta = \dfrac{5}{13}$

Conservation of energy (with bottom of circle as zero potential energy):

$mg \times 2 = mg(2 - 2\cos\theta) + \frac{1}{2}mv^2$

$20 = 20 - 20 \times \dfrac{12}{13} + \dfrac{1}{2}v^2$

$\dfrac{1}{2}v^2 = \dfrac{240}{13}$

$v = \sqrt{\dfrac{480}{13}}$

ii At B, the particle is h m above the ground, where $h = 3 - 2\cos\theta = \frac{15}{13}$.

Let v_G be the speed at impact with the ground.

Conservation of energy (with ground as zero potential energy):

Energy at B = energy at impact with the ground

$$10m \times \frac{15}{13} + \frac{1}{2}m \times \frac{480}{13} = \frac{1}{2}mv_G^2$$

$$\frac{150}{13} + \frac{240}{13} = \frac{1}{2}v_G^2$$

$$v_G^2 = \frac{780}{13}$$

$$v_G = \sqrt{\frac{780}{13}}\,\text{m s}^{-1}$$

iii Speed = distance ÷ time, so the horizontal displacement of the particle at time t is:

$$x = \left(\sqrt{\frac{480}{13}} \times \frac{12}{13}\right)t$$

$$t = \left(\frac{13}{12}\sqrt{\frac{13}{480}}\right)x$$

Using $s = ut + \frac{1}{2}at^2$, the vertical displacement of the particle at time t is:

$$y = \left(\sqrt{\frac{480}{13}} \times \frac{5}{13}\right)t + \frac{1}{2}(-10)t^2$$

$$y = \left(\frac{5}{13}\sqrt{\frac{480}{13}}\right)t - 5t^2$$

Substituting for t:

$$y = -5\left(\frac{13}{12}\right)^2\left(\frac{13}{480}\right)x^2 + \frac{5}{13}\sqrt{\frac{480}{13}} \times \frac{13}{12}\sqrt{\frac{13}{480}}x$$

$$y = -\frac{2197}{1152}x^2 + \frac{5}{12}x$$

$$y = \frac{480}{1152}x - \frac{2197}{1152}x^2$$

$$y = \frac{x}{1152}(480 - 2197x)$$

6 Let x be the extension when the particle is initially at rest.

By Hooke's law:

$$T = \frac{28x}{1.6} = 17.5x$$

Resolving vertically:

$$17.5x = 0.35g$$

$$x = 0.2\text{ m}$$

Conservation of energy:

$$\frac{28 \times 0.2^2}{2 \times 1.6} + \frac{1}{2} \times 0.35 \times 1.8^2 = 0.35 \times 10 \times 0.2 + \frac{1}{2} \times 0.35v^2$$

$$0.35 + 0.567 = 0.7 + 0.175v^2$$

$$v^2 = 1.24$$

$$v = 1.11\text{ m s}^{-1}$$

7 i Let m be the mass of the particle.

$T = 3mg$

Resolving vertically:

$T\cos\theta = mg$

$3mg\cos\theta = mg$

$\cos\theta = \frac{1}{3}$ so $\sin\theta = \frac{\sqrt{8}}{3}$

By Newton's second law:

$T\sin\theta = mr\omega^2$

$3mg \times \frac{\sqrt{8}}{3} = mr \times 5^2$

$r = 1.13\text{ m}$

Using trigonometry on the triangle:

$\sin\theta = \frac{r}{l}$

$\frac{\sqrt{8}}{3} = \frac{1.13\ldots}{l}$

$l = 1.2\text{ m}$

ii $v = r\omega$

$v = 1.13\ldots \times 5 = 5.66\text{ m s}^{-1}$

8 i

Shape	Arc	Rod	Total
Mass (kg)	1.8π	3.6	$1.8\pi + 3.6$
Distance of centre of mass from O (m)	$\frac{1.8\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}$	0	$\bar{y}$

Centre of mass:

$$1.8\pi \times \frac{1.8 \times 2}{\pi} = (1.8\pi + 3.6)\bar{y}$$

$$\frac{6.48}{1.8\pi + 3.6} = \bar{y}$$

$$\bar{y} = 0.700\text{ m}$$

ii

Shape	Frame	Lamina	Total
Mass (kg)	M	2.75	$M + 2.75$
Distance of centre of mass from O (m)	0.70…	$\frac{2 \times 1.8\sin\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}} = \frac{2.4}{\pi}$	$\bar{y}$

$\bar{y} = 1.8\tan 22° = 0.7272$

Centre of mass:

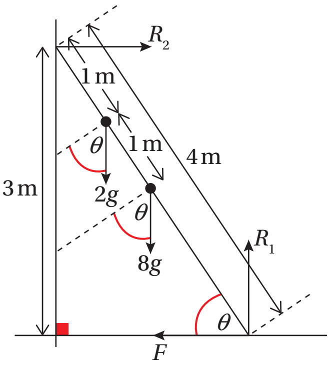
$$0.7002M + \frac{2.75 \times 2.4}{\pi} = (2.75 + M) \times 0.7272$$

$$0.7002M + \frac{6.6}{\pi} = 2.000 + 0.7272M$$

$$M = \frac{\frac{6.6}{\pi} - 2.000}{0.7272 - 0.7002} = 3.74\text{ kg}$$

So the weight of the frame is 37.4 N.

9



$\sin\theta = \frac{3}{4}$ and $\cos\theta = \frac{\sqrt{7}}{4}$

Resolving vertically:

$R_1 = 10g = 100\,\text{N}$

$F = \mu R_1 = 100\mu$

Resolving horizontally:

$F = R_2 = 100\mu$

Moments about the base of the ladder:

$2 \times 8g\cos\theta + 3 \times 2g\cos\theta = 4R_2\sin\theta$

$220 \times \frac{\sqrt{7}}{4} = 4 \times \frac{3}{4} \times 100\mu$

$220\sqrt{7} = 1200\mu$

$\mu = \frac{11\sqrt{7}}{60}$

10 i Using $v^2 = u^2 + 2as$

$u = 2\sin 60° = \sqrt{3}\,\text{m s}^{-1}$, $a = -10\,\text{m s}^{-2}$, $s = -0.5$, $v = ?$

$v^2 = 3 + 2 \times (-10) \times (-0.5) = 13$

$v = \pm\sqrt{13}$

As the ball is moving towards the ground, $v = -\sqrt{13}\ \text{m s}^{-1}$.

Horizontal motion is a constant speed of $2\cos 60° = 1\,\text{m s}^{-1}$.

Let v_G be the speed on impact with the ground.

By Pythagoras:

$v_G^2 = 1^2 + 13$

$v_G = \sqrt{14}\,\text{m s}^{-1}$

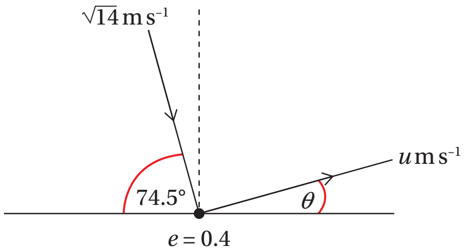
By trigonometry:

$\tan\alpha = \sqrt{13}$

$\alpha = 74.5°$

The ball hits the ground at a speed of $\sqrt{14}\,\text{m s}^{-1}$ at an angle of 74.5° to the horizontal ground.

ii



Newton's experimental law:

$0.4 \times \sqrt{14} \times \sin 74.5° = u\sin\theta$

$u\sin\theta = 1.4422$ (1)

Motion parallel to the ground:

$\sqrt{14}\cos 74.5° = u\cos\theta$

$u\cos\theta = 1$ (2)

Square and add (1) and (2):

$u^2 = 1.4422^2 + 1^2 = 3.080$

$u = 1.75\ \text{m s}^{-1}$ (3 sf)

Substituting in (1):

$\sin\theta = \frac{1.4422}{1.7550} = 0.82178$

$\theta = 55.3°$ (3 sf)

For vertical motion after the bounce: using $v^2 = u^2 + 2as$

$u = 1.755\sin 55.26°\ \text{m s}^{-1}$, $a = -10\,\text{m s}^{-2}$, $s = ?$, $v = 0\,\text{m s}^{-1}$

$0 = 1.4422 + 2 \times (-10) \times s$

$s = \frac{1.4422}{20} = 0.0721\,\text{m} = 7.21\,\text{cm}$ (3 sf)

11 For complete circular motion, tension $\geqslant 0$ at the top of the circle.

By Newton's second law:

$T + mg = \frac{mu^2}{a}$

$T = \frac{mu^2}{a} - mg$

But $T \geqslant 0$, so

$\frac{mu^2}{a} - mg \geqslant 0$

$\frac{u^2}{a} \geqslant g$

$u \geqslant \sqrt{ag}$

By conservation of energy:

$mg(2a) + \frac{1}{2}mu^2 = \frac{1}{2}mv^2$

$2ag + \frac{1}{2}ag = \frac{1}{2}v^2$

$\frac{5}{2}ag = \frac{1}{2}v^2$

$v^2 = 5ag$

$v = \sqrt{5ag}$ (where v is the speed of P before the collision)

Let v_C be the speed of the combined particle after the collision.

By conservation of momentum:

$m\sqrt{5ag} = \frac{5m}{4} \times v_C$

$v_C = \frac{4}{5}\sqrt{5ag}$

By conservation of energy:

$\frac{1}{2}\left(\frac{5m}{4}\right) \times \frac{16}{25} \times 5ag = \frac{5mg}{4}(a + a\cos\theta) + \frac{1}{2}\left(\frac{5m}{4}\right)v^2$

$2mga = \frac{5mga}{4} + \frac{5mga\ \cos\theta}{4} + \frac{5mv^2}{8}$

$6mga = 10mga\ \cos\theta + 5mv^2$ ①

By Newton's second law:

$T + \frac{5mg\cos\theta}{4} = \frac{5m}{4}\frac{v^2}{a}$

$4aT + 5mga\cos\theta = 5mv^2$ ②

② in ①:

$6mga = 10mga\ \cos\theta + 4aT + 5mga\ \cos\theta$

$4aT = 6mga - 15mga\ \cos\theta$

$T = \frac{3mg}{2} - \frac{15mg\cos\theta}{4}$

The string becomes slack when $T = 0$:

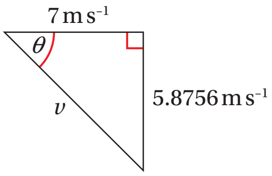
$\frac{3}{2} - \frac{15}{4}\cos\theta = 0$

$\cos\theta = \frac{6}{15} = \frac{2}{5}$

12 i Horizontally acceleration = 0 so velocity is constant and = 14cos 60° = 7

Vertically use $v = u + at$

$v = 14\sin 60° - 1.8g = -5.8756$



$v^2 = 7^2 + 5.8756^2 = 83.52$

$v = 9.139\ \text{m s}^{-1}$

$\tan\theta = 5.8756/7$

$\theta = 40.0°$

Speed = 9.14 m s^{-1} in direction 40.0° below the horizontal

ii Use $s = ut + \frac{1}{2}at^2$ in vertical direction

$-2 = (14\sin 60°)t - \frac{1}{2}gt^2$

$5t^2 - 12.124t - 2 = 0$

Use the quadratic formula, taking only the positive root (the negative root is before the motion started)

$t = 2.58$ s

13 i Take moments about A

$10 \times 1.2\sin(\theta - 30°) = 6 \times 0.8\sin\theta$

Use:

$\sin(\theta - 30°) = \sin\theta\cos 30° - \cos\theta\sin 30° = \frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta$

$12 \times \left(\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta\right) = 6 \times 0.8\sin\theta$

$\sqrt{3}\sin\theta - \cos\theta = 0.8\sin\theta$

$\left(\sqrt{3} - 0.8\right)\sin\theta = \cos\theta$

$\tan\theta = \frac{1}{\left(\sqrt{3} - 0.8\right)} = 1.0729$

$\theta = 47.0°$

ii Resolve horizontally

Reaction at A = 10 sin 30° (as the rod is in equilibrium)

Resolve vertically

Friction force at A = 10cos 30° – 6

Use $F \leqslant \mu R$, Limiting equilibrium, so $F = \mu R$

$10\cos 30° - 6 = \mu \times 10\sin 30°$

$\mu = \frac{(10\cos 30° - 6)}{10\sin 30°} = 0.532$

14 i At A: KE $= \frac{1}{2}mv^2 = \frac{1}{2}m\frac{3}{10}ga = \frac{3}{2}ma$,

PE $= mgh = 10ma$

At bottom: KE $= \frac{1}{2}mv^2$

Conservation of energy:

$\frac{3}{2}ma + 10ma = \frac{1}{2}mv^2$

$3a + 20a = v^2$

$v^2 = 23a$

Acceleration towards centre $= \frac{v^2}{r} = \frac{23a}{a} = 23$

$F = ma = 23m$

$R = 23m + mg = 33m$

ii At B: $PE = mgh = 10m\left(a - a\sin\theta\right) = \frac{30}{4}ma$,

KE $= \frac{1}{2}mv^2$

Conservation of energy:

$\frac{1}{2}m \times 23a = \frac{30}{4} + \frac{1}{2}mv^2$

$46ma = 30ma + 2mv^2$

$v^2 = 8a$

$v = \sqrt{8a} = 2.83\sqrt{a}$

iii Vertical component of speed at $B = \sqrt{8a}\cos\theta$

$\sin\theta = \frac{1}{4}$, $\therefore\ \cos\theta = \frac{\sqrt{15}}{4}$ (Note: as θ is acute $\sin^2\theta + \cos^2\theta = 1$)

Use $v^2 = u^2 + 2aS$ for vertical motion of projectile.

At greatest height above point of projection $v = 0$

$0 = 8a\cos^2\theta - 20S$

$20S = \frac{15a}{2}$

$S = \frac{15a}{40} = \frac{3a}{8}$

At moment of projection let h be the distance of the bead below the level of O.

$\sin\theta = \frac{h}{a}$

Greatest height of bead above the level of O

$= \frac{3a}{8} - \frac{1}{4}a = \frac{1}{8}a$ as required.

15 i At point of release PE = $mgh = 6h$, KE = 0,

$\text{EE} = \frac{\lambda x^2}{2l} = \frac{45\times 1}{2\times 1.5} = 15$

When P comes to rest PE = 0, KE = 0,

$\text{EE} = \frac{\lambda x^2}{2l} = \frac{45\times h^2}{2\times 1.5} = 15h^2$

Conservation of energy,

$15h^2 = 15 + 6h$

$5h^2 - 2h - 5 = 0$

Solve for h, taking the positive solution

$h = 1.22$ m

ii Greatest speed occurs at equilibrium position.

Let x be the distance between the point of release and the equilibrium position.

Resolving vertically using Hooke's Law

$\frac{\lambda x}{l} = \frac{\lambda(1-x)}{l} + mg$

$\frac{45x}{1.5} = \frac{45(1-x)}{1.5} + 6$

$45x = 45 - 45x + 9$

$90x = 54$

$x = 0.6$

Distance below $A = 0.6 + 1.5 = 2.1$m

Energy at equilibrium position

$= \frac{1}{2}mv^2 + \frac{\lambda x^2}{2l} + \frac{\lambda(1-x)^2}{2l}$

$= \frac{1}{2}0.6v^2 + \frac{45\times 0.6^2}{2\times 1.5} + \frac{45(1-0.6)^2}{2\times 1.5}$

$= 0.3v^2 + 5.4 + 2.4$

$= 7.8 + 0.3v^2$

Conservation of energy:

$7.8 + 0.3v^2 = 0.6g\times 0.6 + \frac{45\times 1}{2\times 1.5}$

$7.8 + 0.3v^2 = 3.6 + 15$

$0.3v^2 = 10.8$

$v^2 = 36$

$v = 6\ \text{m s}^{-1}$

iii At top position, resultant force

$= mg + \frac{\lambda x}{l} = 6 + \frac{45\times 1}{1.5} = 36$

$F = ma$, $0.6a = 36$, $a = 60\ \text{m s}^{-2}$

At bottom position, resultant force

$= \frac{\lambda x}{l} - mg = \frac{45\times 1.22}{1.5} - 6 = 30.6$

$F = ma$, $0.6a = 30.6$, $a = 51\ \text{m s}^{-2}$

Greatest magnitude of the acceleration of P is $60\ \text{m s}^{-2}$

16 i Vertically, resultant force $= 0.6g - 3v$

Use $F = ma$ with acceleration as $\frac{dv}{dt}$ to find t

$0.6g - 3v = 0.6\frac{dv}{dt}$

$0.6\frac{dv}{dt} = 0.6g - 3v$

Divide both sides by 0.6

$$\frac{dv}{dt} = g - 5v$$

$$\int\frac{1}{10-5v}dv = \int dt$$

$-\frac{1}{5}\int\frac{-5}{10-5v}dv = \int dt$

$-\frac{1}{5}\ln(10-5v) = t + c$

Substitute initial conditions, i.e. $V = 0$ when $t = 0$

$-\frac{1}{5}\ln(10) = c$

$$t = \frac{1}{5}\ln(10) - \frac{1}{5}\ln(10 - 5\times 1.95)$$

$$t = \frac{1}{5}\ln(10) - \frac{1}{5}\ln(0.25)$$

$$t = \frac{1}{5}\ln(40)$$

$t = 0.738$ seconds

ii Use $F = ma$ with acceleration as $v\frac{dv}{dx}$ to find x

$-3v = 0.6v\frac{dv}{dx}$

$\frac{dv}{dx} = -5$

$\int dv = -\int 5dx$

$v = -5x + c$

Substitute initial conditions, i.e. $v = 1.95$ and $x = 0$

$c = 1.95$

$v = -5x + 1.95$

P comes to rest when $v = 0$

$5x = 1.95$

$x = 0.39$ m

17 Let v_A and v_B be velocities of spheres A and B after impact, respectively.

Momentum before = momentum after

$$4mu = 4mv_A + 2mv_B$$

$$2u = 2v_A + v_B \qquad (1)$$

$e \times$ (speed of approach) = speed of separation

$$eu = v_B - v_A$$

$$v_B = eu + v_A \quad ②$$

Substitute ② into Eqn ①

$$2u = 2v_A + eu + v_A = eu + 3v_A$$

$$2u = eu + 3v_A$$

$$(2 - e)u = 3v_A \quad ③$$

Consider the change in KE of sphere A

$$\frac{1}{2}(4m)v_A^2 = \frac{1}{4} \times \frac{1}{2}(4m)u^2$$

$$v_A^2 = \frac{1}{4}u^2 \quad ④$$

Substitute ④ into the square of ③

$$(2 - e)^2 u^2 = 9(v_A)^2$$

$$(2 - e)^2 u^2 = 9\frac{1}{4}u^2$$

$$4(2 - e)^2 = 9$$

$$4e^2 - 16e + 7 = 0$$

$$e^2 - 4e + \frac{7}{4} = 0$$

$$e = \frac{1}{2} \text{ or } e = \frac{7}{2}$$

As $0 \leqslant e \leqslant 1$, $e = \frac{1}{2}$

From above $v_A{}^2 = \frac{1}{4}u^2$, so $v_A = \frac{1}{2}u$

$$eu = v_B - v_A$$

$$\frac{1}{2}u = v_B - \frac{1}{2}u, \text{ so } v_B = u$$

Consider the collision between spheres B and C

Let the speeds of spheres B and C after collision be v'_B and v_C, respectively

$e \times$ (speed of approach) = speed of separation

$$e \times \frac{1}{2}u = v_C - v'_B$$

$$v'_B - v_C = -\frac{1}{4}u \quad ⑤$$

Conservation of momentum

$$2mu + \frac{1}{2}mu = 2mv'_B + mv_C$$

$$\frac{5}{2}u = 2v'_B + v_C \quad ⑥$$

Solve ⑤ and ⑥ simultaneously

$$2v'_B + v_C = \frac{5}{2}u$$

$$v'_B - v_C = -\frac{1}{4}u$$

$$3v'_B = \frac{9}{4}u$$

$$v'_B = \frac{3}{4}u$$

As $v'_B = \frac{3}{4}u$ and $v_A = \frac{1}{2}u$ and $\frac{3}{4}u > \frac{1}{2}u$, spheres A and B will not collide again, so no further collisions between the spheres.

18 i Horizontally, $s = ut$, $x = (15\cos 41°)t$

Vertically, $s = ut + \frac{1}{2}at^2$, $y = (15\sin 41°)t - 5t^2$

Substitute $t = \dfrac{x}{15\cos 41°}$

$$y = (15\sin 41°) \times \frac{x}{15\cos 41°} - 5\left(\frac{x}{15\cos 41°}\right)^2$$

$$y = 0.869x - 0.0390x^2$$

ii When $x = 1.5$

$y = 0.869 \times 1.5 - 0.0390 \times 1.5^2 = 1.216$

Remember to add the height of O above the horizontal.

Height of fence, $h = 1.6 + 1.216 = 2.82$ m

Ball lands when $y = -1.6$

$$-1.6 = 0.869x - 0.0390x^2$$

$$0 = 1.6 + 0.869x - 0.0390x^2$$

Solve using the quadratic formula

$$x = 24.0$$

Distance from fence to $A = 24.0 - 1.5 = 22.5$ m

19 i For triangle, centre of mass lies on the line of symmetry.

Height of triangle $= 0.36 \times 2 \div 0.6 = 1.2$

Distance of centre of mass from base of triangle $= 1.2 \div 3 = 0.4$

For semicircle, centre of mass lies on line of symmetry at a distance of $\frac{4r}{3\pi}$ from the straight edge: $\frac{4r}{3\pi} = \frac{0.8}{\pi}$

Let mass per sq metre be m

Take moments about the axis OB

If the centre of mass of the lamina lies on OB then these moments will be equal.

For triangle: $mg \times 0.36 \times 0.4 = 0.144mg$

For semicircle: $mg \times \dfrac{\pi 0.6^2}{2} \times \dfrac{0.8}{\pi} = 0.144mg$

Centre of mass lies on OB.

ii Let x be the distance of the centre of mass of the lamina from the point O.

Take moments about the axis OC:

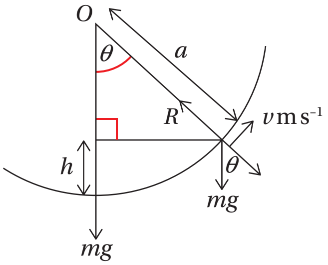
For the parts: $0.36mg \times 0.3$. (Note: centre of mass of semicircle is on this axis.)

For the whole: $(0.36mg + \pi \times \dfrac{0.6^2}{2}mg) \times x$

$$0.36mg \times 0.3 = (0.36mg + \pi \times \frac{0.6^2}{2}mg) \times x$$

$$0.36 \times 0.3 = (0.36 + \pi \times \frac{0.6^2}{2}) \times x$$

$$x = 0.117 \text{ m}$$

20 

Initially energy = KE = $\frac{1}{2}m \times \frac{7}{2}ga = \frac{7}{4}mga$

After motion has started, P also has PE due to the vertical height gained.

Conservation of energy:

$\frac{7}{4}mga = mgh + \frac{1}{2}mv^2$

$\frac{7}{4}mga = mg(a - a\cos\theta) + \frac{1}{2}mv^2$

$v^2 = \frac{7}{2}ga - 2ga + 2ga\cos\theta$

$v^2 = \frac{3}{2}ga + 2ga\cos\theta$

Acceleration towards centre = $\frac{v^2}{r}$

$F = ma = \frac{mv^2}{r} = \frac{mv^2}{a}$

Resolving forces towards centre: $R = \frac{mv^2}{a} + mg\cos\theta$

Substituting for v^2

$R = \frac{m}{a} \times \left(\frac{3}{2}ga + 2ga\cos\theta\right) + mg\cos\theta$

$R = \frac{3mg}{2} + 2mg\cos\theta + mg\cos\theta$

$R = \frac{3mg}{2} + 3mg\cos\theta$

$R = \frac{3mg}{2}(1 + 2\cos\theta)$ as required

i It loses contact with the sphere when $R = 0$

$1 + 2\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$v^2 = \frac{3}{2}ga + 2ga\cos\theta = \frac{3}{2}ga - ga = \frac{1}{2}ga$

$v = \sqrt{\frac{1}{2}ga}$

ii Consider energy when P passes through the horizontal plane level with O.

$\frac{1}{2}mv^2 + mga$

Conservation of energy:

$\frac{7}{4}mga = \frac{1}{2}mv^2 + mga$

$\frac{7}{4}ga = \frac{1}{2}v^2 + ga$

$v^2 = 2 \times \left(\frac{7}{4}ga - ga\right) = \frac{3}{2}ga$

$v = \sqrt{\frac{3}{2}ga}$

21 i Consider forces vertically.

Resultant force, $F = mg + T$

$T = \frac{\lambda x}{l} = \frac{19.2 \times 1.5}{1.2}$

$F = 0.4 \times 10 + \frac{19.2 \times 1.5}{1.2} = 28$

Use $F = ma$

$28 = 0.4a$

$a = \frac{28}{0.4} = 70\ \text{m s}^{-2}$

ii Initial energy = PE + EE

$= mgh + \frac{\lambda x^2}{2l}$

$= 0.4 \times 10 \times 2.7 + \frac{19.2 \times 1.5^2}{2 \times 1.2}$

$= 10.8 + 18 = 28.8$

At A, energy = $\frac{1}{2}mv^2$

$= 0.2v^2$

Conservation of energy:

$0.2v^2 = 28.8$

$v^2 = \frac{28.8}{0.2} = 144$

$v = 12\ \text{m s}^{-1}$

22 i Use $F = ma$

$4e^{-x} - 2.4x^2 = 0.8 \times v\frac{dv}{dx}$

$\frac{4e^{-x}}{0.8} - \frac{2.4x^2}{0.8} = v\frac{dv}{dx}$

$v\frac{dv}{dx} = 5e^{-x} - 3x^2$

ii $v\frac{dv}{dx} = 5e^{-x} - 3x^2$

Separate and integrate.

$\int v\,dv = \int (5e^{-x} - 3x^2)dx$

$\frac{v^2}{2} = -5e^{-x} - x^3 + c$

Substitute initial conditions, i.e. $v = 6$, $x = 0$

$18 = -5 + c$, $c = 23$

$\frac{v^2}{2} = -5e^{-x} - x^3 + 23$

Now substitute $x = 2$ to find the required value of v.

$\frac{v^2}{2} = -5e^{-2} - 2^3 + 23$

$\frac{v^2}{2} = 14.32$

$v^2 = 28.64$

$v = 5.35\ \text{m s}^{-1}$

23 First collision:

Let v_A and v_B be the speeds of A and B, respectively, after the collision.

Momentum before = momentum after

$2mu = 2mv_A + mv_B$

$2u = 2v_A + v_B$

$e \times$ (speed of approach) = speed of separation

$\frac{-2}{3}u = v_A - v_B$

$2u = 2v_A + v_B$

Adding gives:

$\frac{4}{3}u = 3v_A \qquad v_A = \frac{4}{9}u$

$2u = 2v_A + v_B$

$2u = 2 \times \frac{4}{9}u + v_B$

$\frac{10}{9}u = v_B \qquad v_B = \frac{10}{9}u$

Second collision (sphere B and the barrier):
Let the speed of B be v'_B after the collision.

$e \times$ (speed of approach) = speed of separation

$e\frac{10}{9}u = v'_B$

After A and B collide again, they move either in the same direction or in opposite directions.

If they move in the same direction:

Conservation of Momentum:

$$\frac{8mu}{9} - \frac{10meu}{9} = 2mw_A + 5mw_A$$

$$\frac{8u}{9} - \frac{10eu}{9} = 7w_A \qquad (1)$$

Newton's Experimental Law:

$$\frac{2}{3}\left(\frac{4u}{9} + \frac{10eu}{9}\right) = 5w_A - w_A = 4w_A$$

$$\frac{1}{6}\left(\frac{4u}{9} + \frac{10eu}{9}\right) = w_A \qquad (2)$$

(2) in (1)

$$\frac{8u}{9} - \frac{10eu}{9} = \frac{7}{6}\left(\frac{4u}{9} + \frac{10eu}{9}\right)$$

$48u - 60eu = 28u + 70eu$

$130eu = 20u$

$e = \frac{2}{13}$

If they move in opposite directions:

Conservation of Momentum:

$$\frac{8mu}{9} - \frac{10meu}{9} = -2mw_A + 5mw_A$$

$$\frac{8u}{9} - \frac{10eu}{9} = 3w_A \qquad (1)$$

Newton's Experimental Law:

$$\frac{2}{3}\left(\frac{4u}{9} + \frac{10eu}{9}\right) = 5w_A + w_A = 6w_A$$

$$\frac{1}{3}\left(\frac{4u}{9} + \frac{10eu}{9}\right) = 3w_A \qquad (2)$$

(2) in (1)

$$\frac{8u}{9} - \frac{10eu}{9} = \frac{1}{3}\left(\frac{4u}{9} + \frac{10eu}{9}\right)$$

$24u - 30eu = 4u + 10eu$

$40eu = 20u$

$e = \frac{1}{2}$

24 i Horizontally, use $s = ut \quad x = u\cos 45°t = \frac{20}{\sqrt{2}}t$

$x = 10\sqrt{2}\,t$

Vertically, use $s = ut + \frac{1}{2}at^2 \quad y = 20\sin 45°t - 5t^2$

$y = 10\sqrt{2}t - 5t^2$

ii $x = 10\sqrt{2}\,t \;\therefore\; t = \frac{x}{10\sqrt{2}}$

Substitute this into $y = 10\sqrt{2}t - 5t^2$

$$y = 10\sqrt{2} \times \frac{x}{10\sqrt{2}} - 5\left(\frac{x}{10\sqrt{2}}\right)^2 = x - \frac{5x^2}{200}$$

$$y = x - \frac{x^2}{40}$$

iii Ball strikes the ground when $y = 0$

$$0 = x - \frac{x^2}{40} = x\left(1 - \frac{x}{40}\right)$$

$x = 40$, so ball first strikes the ground 40 m from O.

25 Centre of mass: distance from $O = \frac{2r\sin\alpha}{2\alpha}$ on line/plane of symmetry

$$\frac{2r\sin\alpha}{3\alpha} = \frac{2r\sin\left(\frac{\pi}{3}\right)}{3\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{\pi}r$$

Take moments about O:

For 15N force $\quad 15r\cos\left(\frac{\pi}{3}\right) = 7.5r$

For centre of mass $\quad 20\cos\left(\frac{\pi}{3} - \theta\right) \times \frac{\sqrt{3}}{\pi}r$

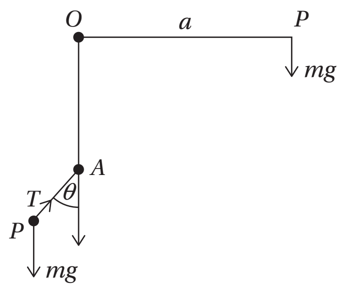
At limit of equilibrium: $\quad 7.5r = 20\cos\left(\frac{\pi}{3} - \theta\right) \times \frac{\sqrt{3}}{\pi}r$

$$\cos\left(\frac{\pi}{3} - \theta\right) = \frac{7.5 \times \pi}{20\sqrt{3}} = 0.6802$$

$$\left(\frac{\pi}{3} - \theta\right) = 0.8228$$

$\theta = 0.224$ radians

26



Initial energy = PE = mga

At base of motion energy = KE = $\frac{1}{2}mv^2$

Conservation of energy :

$\frac{1}{2}mv^2 = mga$

$v^2 = 2ga$

Circular motion about A, when angle is θ

Energy $= \frac{1}{2}mv^2 + mg(a - x)(1 - \cos\theta)$

Conservation of energy:

$\frac{1}{2}mv^2 + mg(a-x)(1-\cos\theta) = mga$

$v^2 = 2gx + 2ga\cos\theta - 2gx\cos\theta$

$= 2gx + 2g(a-x)\cos\theta$

Acceleration towards centre $= \frac{v^2}{r}$

$F = ma = m\frac{v^2}{r} = \frac{mv^2}{(a-x)}$

Resolving forces towards centre:

$T = mg\cos\theta + \frac{mv^2}{(a-x)}$

$T = mg\cos\theta + \frac{m(2gx + 2g(a-x)\cos\theta)}{(a-x)}$

$T = mg\cos\theta + \frac{2mgx}{a-x} + 2mg\cos\theta$

$T = 3mg\cos\theta + \frac{2mgx}{a-x}$

$T = mg(3\cos\theta + \frac{2x}{a-x})$

P completes a circle if $T \geqslant 0$ at the top of the motion; for minimum value, $T = 0$ at the top of the circle, which is when $\theta = \pi$ radians.

$0 = mg(3\cos\theta + \frac{2x}{a-x})$

$3 = \frac{2x}{a-x}$

$3a - 3x = 2x$

$3a = 5x$

$\frac{x}{a} = \frac{3}{5}$

27 i Tension $= 0.4g = 4$ N

Use Hooke's law $T = \frac{\lambda x}{l}$

$4 = \frac{16 \times x}{0.8}$

$x = 0.2$ m

ii When P comes to rest all energy is elastic,

$\text{EPE} = \frac{\lambda x^2}{2l} = \frac{16 \times (0.6)^2}{2 \times 0.8}$

When P is projected, PE $= mgh = 0.4 \times 10 \times 0.4$
$= 1.6$

$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.4u^2$

$\text{PE} = \frac{\lambda x^2}{2l} = \frac{16 \times (0.2)^2}{2 \times 0.8} = 0.4$

Conservation of energy

$3.6 = 1.6 + 0.2u^2 + 0.4$

$1.6 = 0.2u^2$

$u^2 = 8$

$u = \sqrt{8} = 2.828427\ldots = 2.83\ \text{m s}^{-1}$

iii When string becomes slack,

PE $= mgh = 0.4 \times 10 \times (1.4 - 0.8) = 2.4$

$\text{KE} = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.4 \times v^2 = 0.2v^2$

EPE $= 0$ (as there is no extension $x = 0$)

Conservation of energy

$3.6 = 2.4 + 0.2v^2$

$v^2 = \frac{1.2}{0.2} = 6$

$v = 2.45\ \text{m s}^{-1}$

28 i Resultant force $= mg - 0.8v = 2 - 0.8v$

Use $F = ma$

$2 - 0.8v = 0.2a$

$a = \frac{2}{0.2} - \frac{0.8v}{0.2} = (10 - 4v)\ \text{m s}^{-2}$

ii Use $a = \frac{\text{d}v}{\text{d}t}$

$\frac{\text{d}v}{\text{d}t} = (10 - 4v)$

$\frac{1}{-4}\int \frac{-4}{10-4v}\text{d}v = \int \text{d}t$

$\frac{-1}{4}\ln(10 - 4v) = t + c$

$v = 0$ when $t = 0$

$\frac{-1}{4}\ln(10) = c$

$\frac{-1}{4}\ln(10 - 4v) = t - \frac{1}{4}\ln(10)$

When $t = 0.6$

$\frac{-1}{4}\ln(10 - 4v) = 0.6 - \frac{1}{4}\ln(10)$

$10 - 4v = 0.9072$

$v = 2.27\ \text{m s}^{-1}$

29 Conservation of momentum

Momentum before = momentum after

$8mu - 3mu = 2mv_A + v_B$

$5mu = 2mv_A + v_B$ (1)

Coefficient of restitution

$e \times$ (speed of approach) = speed of separation

$-e(4u + 3u) = v_A - v_B$

$-7eu = v_A - v_B$ (2)

(1) + (2)

$5mu - 7eu = 3v_A$

$v_A = \frac{1}{3}(5u - 7ue)$

$v_A < 0$ if $5u - 7ue < 0$

$5u < 7ue$

$e > \frac{5}{7}$

Extension questions

1 Let h be the 'height' of the cone.

Volume of the hemisphere $= \frac{2}{3}\pi(0.1)^3 = \frac{2\pi}{3000}$

Volume of the cone $= \frac{1}{3}\pi(0.1)^2 h = \frac{\pi(h)}{300}$

Mass = density × volume

Moments about the common circle:

$D \times \left(\frac{2\pi}{3000}\right) \times \left(\frac{3}{8} \times 0.1\right) = D \times \left(\frac{\pi h}{300}\right) \times \left(\frac{h}{4}\right)$

$\frac{0.6}{24\,000} = \frac{h^2}{1200}$

$h^2 = 0.03$

$h = 0.1732\,\text{m}$

For the hemisphere end:

$r = 0.1$ so $v = 0.1 \times 4 = 0.4\,\text{m s}^{-1}$

For the cone end:

$r' = 0.1732$

$v = 0.1732 \times 4 = 0.693\,\text{m s}^{-1}$

2 **i** Before collision for A initially: PE = mgl

Vertically below O, before collision KE = $\frac{1}{2}mv^2$

Conservation of energy:

$\frac{1}{2}mv^2 = mgl$

$v^2 = 2gl$

$v = \sqrt{2gl}$

Similarly for B PE = $3mgl$

KE = $\frac{1}{2}(3m)v^2$

$\frac{1}{2}(3m)v^2 = 3mgl$ and similarly $v = \sqrt{2gl}$

ii e(speed of approach) = speed of separation

$0.4\left(2\sqrt{2gl}\right) = v_A + v_B$

$0.8\sqrt{2gl} = v_A + v_B$

Conservation of momentum

(left as positive)

Momentum before = momentum after

$-m\sqrt{2gl} + 3m\sqrt{2gl} = mv_A - 3mv_B$

$2\sqrt{2gl} = v_A - 3v_B$

Solving the two equations leads to

$v_B = -0.3\sqrt{2gl}$

$v_A = 1.1\sqrt{2gl}$

iii For particle B

After collision KE = $\frac{1}{2}3m(0.3\sqrt{2gl})^2$

At point at which it comes to rest PE = $3mgh$ where h = height above point of collision

$h = l(1 - \cos\theta)$

Conservation of energy:

$\frac{1}{2} \times 3m(0.3\sqrt{2gl})^2 = 3mgh$

$0.09l = h$

$0.09l = l(1 - \cos\theta)$

$\cos\theta = 0.91$

$\theta = 24.5°$

For particle A

After collision KE $= \frac{1}{2}m\,(1.1\sqrt{2gl})^2$

At point at which it comes to rest PE = mgh where h = height above point of collision

$h = l(1 - \cos\theta)$

Conservation of energy:

$\frac{1}{2} \times m(1.1\sqrt{2gl})^2 = mgh$

$1.21l = h$

$1.21l = l(1 - \cos\theta)$

$\cos\theta = -0.21$

$\theta = 102°$

3 For the initial fall: using $v^2 = u^2 + 2as$

$u = 0\,\text{m s}^{-1}$, $a = 10\,\text{m s}^{-2}$, $s = 20\,\text{m}$, $v = ?$

$v^2 = 2 \times 10 \times 20 = 400$

$v = 20\,\text{m s}^{-1}$

The speed at which the ball leaves the ground is equal to the speed at which the ball hits the ground again when it comes back down.

For a height of $\leqslant 5\,\text{m}$:

$u = ?$, $a = -10\,\text{m s}^{-2}$, $s \leqslant 5\,\text{m}$, $v = 0\,\text{m s}^{-1}$

$0^2 \leqslant u^2 + 2 \times (-10) \times 5$

$u^2 \geqslant 100$

$u \geqslant 10\,\text{m s}^{-1}$

By Newton's experimental law: $v = 0.85u$

So after n bounces, the speed of the ball immediately after the bounce is given by:

$v_n = 20 \times 0.85^n$

For the height of the bounce is to remain below 5 m, $v_n \geqslant 10$.

$20 \times 0.85^n \geqslant 10$

$0.85^n \geqslant 0.5$

$\ln 0.85^n \geqslant \ln 0.5$

$n \ln 0.85 \geqslant \ln 0.5$

$n \geqslant \frac{\ln 0.5}{\ln 0.85}$

$n \geqslant 4.265$

So the ball will remain below 5 m after 5 bounces.

4 Use $F = ma$

$\frac{x^2}{\sin\left(\frac{v}{2}\right)} = 2v\frac{\mathrm{d}v}{\mathrm{d}x}$

Separate and integrate

$\int x^2\,\mathrm{d}x = 2\int v\sin\left(\frac{v}{2}\right)\mathrm{d}v$

Integrate the RHS using integration by parts

$$\frac{x^3}{3} = 2\left[-2v\cos\left(\frac{v}{2}\right) - \int -2\cos\left(\frac{v}{2}\right)\mathrm{d}v\right]$$

$$\frac{x^3}{3} = 2\left[-2v\cos\left(\frac{v}{2}\right) + 4\sin\left(\frac{v}{2}\right)\right] + c$$

$$\frac{x^3}{3} = -4v\cos\left(\frac{v}{2}\right) + 8\sin\left(\frac{v}{2}\right) + c$$

Substitute in the given initial conditions, i.e. when $t = 0,\ v = \pi,\ x = 0$

$$0 = -4\pi\cos\left(\frac{\pi}{2}\right) + 8\sin\left(\frac{\pi}{2}\right) + c$$

$c = -8$

$$\frac{x^3}{3} = -4v\cos\left(\frac{v}{2}\right) + 8\sin\left(\frac{v}{2}\right) - 8$$

$$x^3 = 3\left[-4v\cos\left(\frac{v}{2}\right) + 8\sin\left(\frac{v}{2}\right) - 8\right]$$

$$x = \sqrt[3]{-24 + 24\sin\left(\frac{v}{2}\right) - 12v\cos\left(\frac{v}{2}\right)}$$

5 Initially, elastic energy =

$$\frac{\lambda x^2}{2l} = \frac{10 \times 0.25 \times 0.25}{1} = 0.625$$

At point of projection all energy is converted to KE

$$\frac{1}{2}mv^2 = 0.625$$

$v^2 = 12.5$

$v = \sqrt{12.5}$

For motion from point of projection

Horizontally $x = \sqrt{12.5} \times \cos 45° \times t = 2.5t$

$x = \frac{5}{2}t$, or $t = \frac{2}{5}x$

Vertically using $s = ut + \frac{1}{2}at^2$

$$y = \sqrt{12.5}\sin 45°\, t - \frac{1}{2}10\, t^2$$

$y = 2.5t - 5t^2$

Substituting for t, $y = x - \frac{4}{5}x^2$

This is the equation of motion for the particle from the point of launch.

For maximum speed, acceleration = 0

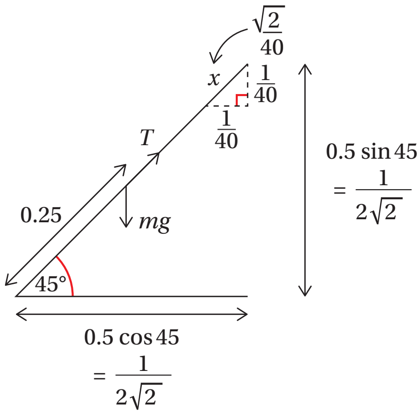
$F = ma$

$T - mg\cos\theta = 0$

$$\frac{\lambda x}{l} = mg\cos\theta$$

$$\frac{10x}{0.5} = 0.1 \times 10 \times \frac{1}{\sqrt{2}}$$

$$x = \frac{1}{20\sqrt{2}} = \frac{\sqrt{2}}{40}$$



Relative to O, point of launch has coordinates

$$\left(\frac{1}{2\sqrt{2}} - \frac{1}{40},\ \frac{1}{2\sqrt{2}} - \frac{1}{40}\right)$$

Therefore equation of trajectory relative to O is

$$y = \left(x - \frac{1}{2\sqrt{2}} + \frac{1}{40}\right) - \frac{4}{5}\left(x - \frac{1}{2\sqrt{2}} + \frac{1}{40}\right)^2 + \frac{1}{2\sqrt{2}} - \frac{1}{40}$$

6 **i** If P and Q collide, $l < 4.2$ m

ii For particle P:

Energy at $P = \frac{\lambda x^2}{2l} = \frac{1}{6}$

Energy at $A = \frac{1}{2}mv^2 = \frac{1}{6}$ due to conservation of energy

$$v = \frac{\sqrt{3}}{6}$$

For particle Q:

Energy at $Q = \frac{\lambda x^2}{2l} = \frac{5}{6}$

Energy at $B = \frac{1}{2}mv^2 = \frac{5}{6}$

$$v = \frac{\sqrt{5}}{3}$$

e(speed of approach) = (speed of separation)

$$0.7\left(\frac{\sqrt{3}}{6} + \frac{\sqrt{5}}{3}\right) = v_A + v_B$$

Momentum before = momentum after

$$\frac{\sqrt{3}}{6} \times 4 - \frac{\sqrt{5}}{3} \times 3 = 3v_B - 4v_A$$

$$4 \times 0.7\left(\frac{\sqrt{3}}{6} + \frac{\sqrt{5}}{3}\right) = 4v_A + 4v_B$$

solve simultaneously for v_B leads to

$v_B = 0.2591$

For Q: energy after collision = KE

$$= \frac{1}{2}mv^2 = \frac{1}{2} \times 3 \times (0.2591)^2 = 0.1007$$

When Q stops moving all energy is elastic,

$$\frac{\lambda x^2}{2l} = \frac{2x^2}{2.4} = 0.1007$$

$x = 0.3476$

Distance from $B = 1.2 + 0.34766\ldots$

$= 1.55$ m (3 s.f.)

7 Use $F = ma$

$$e^{2x}\sin 2x = 4v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Separate and integrate

$$\int 4v\mathrm{d}v = \int e^{2x}\sin 2x\mathrm{d}x$$

For the RHS use integration by parts twice.

$$2v^2 = \frac{e^{2x}\sin 2x}{4} - \frac{e^{2x}\cos 2x}{4} + c$$

Substitute initial conditions, $x = 0,\ t = 0,\ v = 2$

$$8 = 0 - \frac{1}{4} + c,\ c = 8.25$$

$$2v^2 = \frac{e^{2x}\sin 2x}{4} - \frac{e^{2x}\cos 2x}{4} + 8.25$$

$$v = \sqrt{\frac{e^{2x}\sin 2x}{8} - \frac{e^{2x}\cos 2x}{8} + \frac{33}{8}}$$

8 Initial energy

$$= \frac{1}{2}mv^2 + mgh = \frac{1}{2}\times 100\times u^2 + 100\times 10\times(5\sin 30° + 5)$$

$$= 50u^2 + 7500$$

At point of collision

$$\text{energy} = \frac{1}{2}mv^2 = 0.5\times 100\times v^2 = 50v^2$$

Conservation of energy

$$50v^2 = 50u^2 + 7500$$

$$v^2 = u^2 + 150$$

e(speed of approach) = speed of separation

$$e\sqrt{u^2+150} = v_Q + v_P$$

$$e\sqrt{u^2+150} = 5u + v_P$$

Conservation of momentum

Before = after

$$100\sqrt{u^2+150} = 5v_Q - 100v_P$$

$$100\sqrt{u^2+150} = 25u - 100v_P$$

Eliminate v_P using the equation above for e.

$$100\sqrt{u^2+150} = 25u - 100\left[e\sqrt{u^2+150} - 5u\right]$$

$$100\sqrt{u^2+150} = 25u + 500u - 100e\sqrt{u^2+150}$$

$$100\sqrt{u^2+150} + 100e\sqrt{u^2+150} = 525u$$

$$(1+e)100\sqrt{u^2+150} = 525u$$

$$(1+e) = \frac{525u}{100\sqrt{u^2+150}}$$

$$e = \frac{525u}{100\sqrt{u^2+150}} - 1$$

9 Use $s = ut + \frac{1}{2}at^2$

At collision height, for P:

$$s = uT - 5T^2$$

At collision height, for Q:

$$s = 2u(T-2) - 5(T-2)^2$$

At collision height

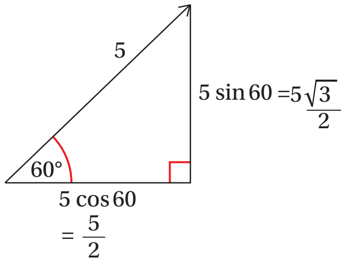
$$uT - 5T^2 = 2u(T-2) - 5(T-2)^2$$

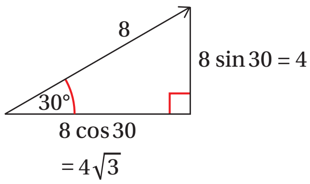
$$uT - 5T^2 = 2uT - 4u - 5T^2 + 20T - 20$$

$$4u + 20 = uT + 20T = T(u+20)$$

$$T = \frac{4u+20}{u+20}$$

10





Use $s = ut$ horizontally and $s = ut + \frac{1}{2}at^2$ vertically

For P horizontally $x = \frac{5}{2}t$

vertically $y = \frac{5\sqrt{3}}{2}t - 5t^2$

For Q horizontally $x = 4\sqrt{3}t$

vertically $y = 4t - 5t^2$

When P and Q collide the time of flight for Q is t and time of flight for P is $t + T$

$$\begin{pmatrix} 4\sqrt{3}t \\ 4t - 5t^2 \end{pmatrix} = \begin{pmatrix} \frac{5}{2}(t+T) \\ \frac{5\sqrt{3}}{2}(t+T) - 5(t+T)^2 \end{pmatrix}$$

Using the x component $(t+T) = \frac{2}{5}4\sqrt{3}t$

Substitute this into the y component

$$4t - 5t^2 = \frac{5\sqrt{3}}{2}\left(\frac{2}{5}4\sqrt{3}t\right) - 5\left(\frac{2}{5}4\sqrt{3}t\right)^2$$

$$4t - 5t^2 = 12t - \frac{4}{5}\times 48t^2$$

$$0 = 8t - 33.4t^2$$

$$0 = t(8 - 33.4t)$$

Solving, $t = 0.2395$

Substitute into $(t+T) = \frac{2}{5}4\sqrt{3}t$

$T = \left(\frac{2}{5} \times 4\sqrt{3} - 1\right)t$

$T = 0.424\,\text{s}$

11 i For a projectile the range of flight $\frac{u^2\sin 2\alpha}{g}$

For particle A, range of flight $= \frac{u^2}{10}$

For particle B, range of flight $= \frac{u^2}{10}$

Given that the particles collide, $0 < l < \frac{u^2}{5}$

ii Horizontal speed before collision $= 10\cos 45°$

$= \frac{10}{\sqrt{2}}$

Horizontally:

e(speed of approach) = speed of separation

$0.5\left(2 \times \frac{10}{\sqrt{2}}\right) = v_P + v_Q \qquad v_P = v_Q = \frac{5}{\sqrt{2}}$

Collision takes place 7 m from O

Consider horizontal motion

$s = ut \quad 7 = u\cos 45°t,\ 7 = \frac{10}{\sqrt{2}}t \quad \text{i.e. } t = \frac{7\sqrt{2}}{10}$

Collision takes place $\frac{7\sqrt{2}}{10}$ seconds after particles are projected.

Total time of flight – consider vertical motion

$s = ut + \frac{1}{2}at^2$

$0 = 10\sin 45°t - 5t^2 = t\left(\frac{10}{\sqrt{2}} - 5t\right)$

$t = 0$, or $5t = \frac{10}{\sqrt{2}}$ ie $t = \sqrt{2}$ seconds

Time of flight left after collision takes place

$= \sqrt{2} - \frac{7\sqrt{2}}{10}$

Horizontal distance covered in this time

$= \left(\sqrt{2} - \frac{7\sqrt{2}}{10}\right) \times \frac{5}{\sqrt{2}} = 5 - \frac{35}{10} = 1.5$

Both particles cover this distance, ∴ distance between A and B is 3 m

12 $a = \frac{4}{\sqrt{x^2 - 9}}$

$v\frac{dv}{dx} = \frac{4}{\sqrt{x^2 - 9}}$

Separate and integrate

$\frac{1}{4}\int v\,dv = \int \frac{1}{\sqrt{x^2 - 9}}dx$

$\frac{v^2}{8} = \ln\left(x + \sqrt{x^2 - 9}\right) + c$

Substitute $v = 4$ when $x = 3\sqrt{65}$

$2 = \ln\left(3\sqrt{65} + 24\right) + c$

$c = 2 - \ln(24 + 3\sqrt{65})$

$\frac{v^2}{8} = \ln\left(x + \sqrt{x^2 - 9}\right) + 2 - \ln(24 + 3\sqrt{65})$

When $x = 6$

$\frac{v^2}{8} = \ln\left(6 + \sqrt{6^2 - 9}\right) + 2 - \ln(24 + 3\sqrt{65})$

$v^2 = 4.324$

$v = \pm 2.08\,\text{m s}^{-1}$