QF620 Stochastic Modelling in Finance

G2 Group 7 Project Report

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Date of Submission: 20th November 2024 (12:00PM)

Part I - Analytical Option Formulae

This part lays the groundwork for subsequent parts by establishing the pricing functions for Black-Scholes, Bachelier, Black76 and Displaced-Diffusion, each of which implements a distinct stock-price process model.

1.1 Black-Scholes (BS) Model

BS Option Model	Pricing Valuation	
Vanilla Call	$S_0\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-Ke^{-rT}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	
Vanilla Put	$Ke^{-rT}\Phi\left(\frac{\log\frac{K}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-S_0\Phi\left(\frac{\log\frac{K}{S_0}-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	
Cash-Or-Nothing Call/Put	$e^{-rT}\Phi\left(\frac{\log\frac{S_0}{K}+\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$e^{-rT}\Phi\left(\frac{\log\frac{K}{S_0}-\left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$
Asset-Or-Nothing Call/Put	$S_0\Phi\left(\frac{\log\frac{S_0}{K}+\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$S_0\Phi\left(\frac{\log\frac{K}{S_0}-\left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

1.2 Bachelier Model

Bachelier Option Model	Pricing Valuation	
Vanilla Call	$e^{-rT}[(S_0-K)\Phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right)+\sigma S_0\sqrt{T}\,\phi\,(\frac{S_0-K}{\sigma S_0\sqrt{T}})]$	
Vanilla Put	$e^{-rT}[(K-S_0)\Phi\left(\frac{K-S_0}{\sigma S_0\sqrt{T}}\right)+\sigma S_0\sqrt{T}\,\phi\,(\frac{K-S_0}{\sigma S_0\sqrt{T}})]$	
Cash-Or-Nothing Call/Put	$e^{-rT}[\Phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right)]$	$e^{-rT}[\Phi\left(\frac{K-S_0}{\sigma S_0\sqrt{T}}\right)]$
Asset-Or-Nothing Call/Put	$S_0\Phi\left(\frac{S_0-K}{\sigma S_0\sqrt{T}}\right)+\sigma S_0\sqrt{T}\,\phi\,(\frac{S_0-K}{\sigma S_0\sqrt{T}})$	$S_0\Phi\left(\frac{K-S_0}{\sigma S_0\sqrt{T}}\right)-\sigma S_0\sqrt{T}\,\phi\,(\frac{K-S_0}{\sigma S_0\sqrt{T}})$

1.3 Black Model

Black Option Model	Pricing Valuation	
Vanilla Call	$e^{-rT}F_0\Phi\left(\frac{\log\frac{F_0}{K}+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-Ke^{-rT}\Phi\left(\frac{\log\frac{F_0}{K}-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	
Vanilla Put	$Ke^{-rT}\Phi\left(\frac{\log\frac{K}{F_0}+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)-e^{-rT}F_0\Phi\left(\frac{\log\frac{K}{F_0}-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	
Cash-Or-Nothing Call/Put	$e^{-rT}\Phi\left(\frac{\log\frac{F_0}{K}-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$e^{-rT}\Phi\left(\frac{\log\frac{K}{F_0}+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$
Asset-Or-Nothing Call/Put	$e^{-rT}F_0\Phi\left(\frac{\log\frac{F_0}{K}+\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$	$e^{-rT}F_0\Phi\left(\frac{\log\frac{K}{F_0}-\left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$

1.4 Displaced-Diffusion (D-D) Model

Displaced-Diffusion Model	Pricing Valuation
Vanilla Call/Put Cash-Or-Nothing Call/Put	$Black(\frac{F_0}{\beta}, K + \frac{1-\beta}{\beta} F_0, \sigma\beta, T)$
Asset-Or-Nothing Call	$e^{-rT}\frac{F_0}{\beta}\Phi\left(\frac{\log\frac{F_0}{F_0+\beta(K-F_0)} + \left(\frac{\beta^2\sigma^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right) - e^{-rT}\frac{1-\beta}{\beta}F_0\Phi\left(\frac{\log\frac{F_0}{F_0+\beta(K-F_0)} - \left(\frac{\beta^2\sigma^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right)$
Asset-Or-Nothing Put	$e^{-rT}\frac{F_0}{\beta}\Phi\left(\frac{\log\frac{F_0+\beta(K-F_0)}{F_0} - \left(\frac{\beta^2\sigma^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right) - e^{-rT}\frac{1-\beta}{\beta}F_0\Phi\left(\frac{\log\frac{F_0+\beta(K-F_0)}{F_0} + \left(\frac{\beta^2\sigma^2}{2}\right)T}{\beta\sigma\sqrt{T}}\right)$

1.5 Use Cases and Comparison of Pricing Models

After deriving the option prices using various models, we have drawn several key takeaways regarding their applicability and performance across various financial assets. Each model exhibits distinct strengths and limitations, influenced by its underlying assumptions. These insights allow us to better understand how to align model selection with the characteristics of the instruments being priced and the market environment.

	Black Scholes Model	Bachelier Model	Black Model	Displaced-Diffusion Model
Key Mathematical Assumptions	▪ Log-normal price distribution and implied volatility	▪ Normal price distribution and implied volatility	▪ Log-normal price distribution and implied volatility	▪ Log-normal price distribution and implied volatility
Strengths	▪ Works well when volatility curve is flat and no significant price jumps are expected	▪ Suitable for pricing assets which can have negative prices	▪ Suitable for pricing forwards/futures	▪ Handles volatility skew and negative asset prices asset price distributions more flexibly
Limitations	▪ Cannot handle negative prices or significant skew in returns.	▪ Allows for negative prices which may be unsuitable for most assets	▪ Assumes log-normal prices, not always realistic for interest rates	▪ Requires complex calibration such as SABR model
Pricing Use Cases	▪ Suitable for pricing most assets (Equity and FX options)	▪ Derivatives with underlying that may assume negative values ▪ Fixed income	▪ Forward/futures contracts	▪ Skewed distributions ▪ Commodities ▪ Exotics

The option pricings ensure that no arbitrage opportunities exist and risk preferences should already be reflected in the underlying asset price. Exotics payoffs can be replicated using portfolios of simpler options. The payoffs of an asset-or-nothing call can be replicated by a portfolio of combining a vanilla call and a cash-or-nothing call option where the cash payoff equals to the strike price (based on the payoff diagram). Consequently, the price of an asset-or-nothing call equals to the sum of the prices of a vanilla call and a cash-or-nothing call which has cash payout equal to strike price, assuming they all have the same underlying, strike price, and maturity.

The cash-or-nothing and asset-or-nothing option prices can also be derived using the *expectation approach* from the PV of expected call/put price at expiration in the risk-neutral measure using r_f. The European call is represented by a long position in an asset-or-nothing call and shorting cash-or-nothing call at the same strike price and payoff. Alternatively, the put is represented by buying cash-or-nothing put and selling asset-or-nothing put. The cash-or-nothing options can be structured to pay different amount than the strike, K. The parameter K will then be replaced with the expected amount to be paid but d_2 will still be calculated using the original strike price.

Additional analyses through the iteration of some valuation inputs yielded the following:

- In general, the Bachelier model produced lower call prices and higher put prices than the other models using the same parameter inputs, especially for options closer to at-the-money.
- In general, the Bachelier model is the most sensitive to changes in maturity duration, keeping other variables constant. However, when the option is far enough out of the money, the relationship flips and the other models become more sensitive to changes in maturity duration – presumably due to the larger Brownian motion steps scaled by stock price increasing the probability of becoming in the money.
- Option prices generally increase with risk-free rate (r_f) for all models except the Bachelier model.
- The above insights can be explained by the fact the Black76 and Displaced-Diffusion models were formulated off the BS model, hence sharing similarities in price process such as assuming log-normal price distribution, while the Bachelier's model assumes normal asset price distribution.
- Prices calculated using the Displaced-Diffusion model are more sensitive to changes in other variables when β values are extreme (closer to 0 or 1).

Part II - Model Calibration

Data for SPX (European) and SPY (American) options, along with discount rates, has been provided. Multiple steps of data pre-processing were undertaken to streamline the data for the application of various option pricing models. This included scaling the discount rates and strike prices to ensure consistency with the underlying assets' magnitudes. It was necessary to scale the discount rates and options strike prices provided to account for the fact that rates are in *percentages* and have the strike price which matches the *magnitude* of the underlying.

Before calibrating the model, we had to find the implied volatility of the options using the BS model. For each expiry date, we filter for out-of-the-money options (more liquid as compared to in-the-money options) and utilize Brent's method using the *SciPy* library to obtain the implied volatility.

The calibration process for the D-D model involved minimizing the sum of squared errors between the implied volatilities from market prices using the BS model and the implied volatilities from the D-D model. Using the BS ATM volatility as an input to the D-D formula, the option price was computed. Based on this computed option price, the implied volatility of the D-D model was determined.

Across all tenors, the β parameter for both SPX (European) and SPY (American) options was near 0. The σ parameter for SPY options was marginally higher compared to SPX options of the same tenor. This aligns with expectations, as American options would be more valuable than European due to their additional flexibility of early exercise, which the option holder might exercise on deep ITM options for dividend paying stocks, hence the higher implied volatility. Despite these results, the model fitted poorly to the market implied volatility, as the D-D model failed to capture skew or smile effects of the implied volatility. The fitted implied volatility curves for SPX and SPY remained below market curves for all three tenors, except for a small section near the spot price.

Tenor	SPX (European)		SPY (American)	
	Sigma, σ	Beta, β	Sigma, σ	Beta, β
17	0.17448	0	0.20090	0
45	0.18490	0	0.19721	0
80	0.19374	0	0.20024	0

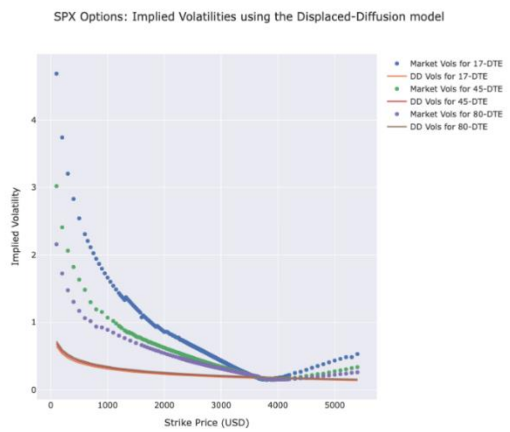


Figure 1 Implied Volatilities using the D-D Model (SPX Options)

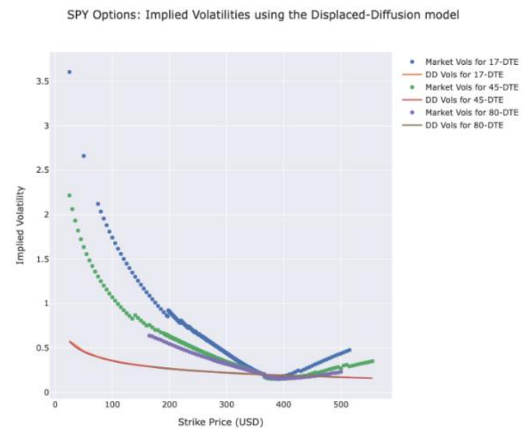


Figure 2 Implied Volatilities using the D-D Model (SPY Options)

The SABR model provides a better fit for market-implied volatilities due to its greater flexibility with higher degrees of freedom, characterized by three key parameters. Firstly, α which determines the overall level of implied volatility, increases with time to maturity for both SPX and SPY options, consistent with the expectation that longer-dated options exhibited higher implied volatilities. Secondly, ρ which governs the skewness of the implied volatility curve and becomes more negative with longer maturities, indicating more pronounced skewness for options with longer durations. Lastly, ν which controls kurtosis, affecting the shape of the implied volatility smile which decreases with increasing maturity, resulting in flatter volatility smiles over time.

The SABR model calibration followed a similar procedure, where the sum of squared errors between the market-implied volatilities and the SABR-implied volatilities was minimized. Unlike the D-D model, the SABR model does not require the BS ATM volatility as an input.

Tenor	SPX (European)			SPY (American)		
	Alpha, α	**Rho, ρ**	**Nu, ν**	**Alpha, α**	**Rho, ρ**	**Nu, ν**
17	1.2122	-0.30090	5.4597	0.66540	-0.41189	5.2499
45	1.8165	-0.40430	2.7901	0.90813	-0.48877	2.7285
80	2.1401	-0.57493	1.8417	1.1209	-0.63293	1.7422

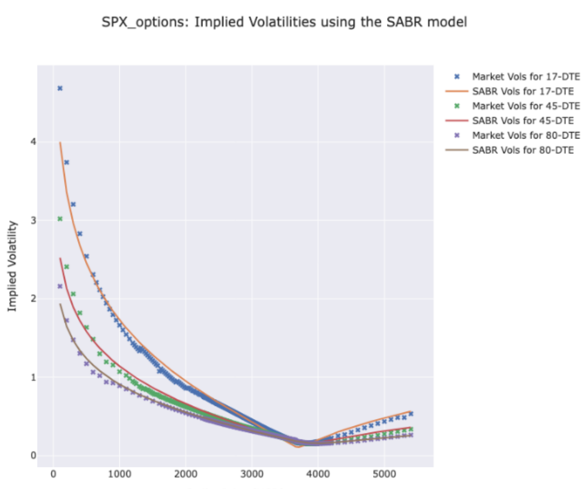


Figure 3 Implied Volatilities using the SABR Model (SPX Options)

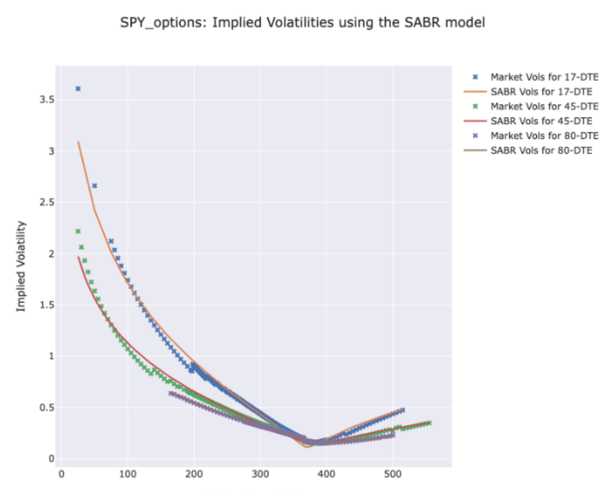


Figure 4 Implied Volatilities using the SABR Model (SPY Options)

Under D-D model, the parameter β controls the weighting between the lognormal model and normal model. A near-zero β emphasizes the normal model, resulting in pronounced skewness for out-of-the-money (OTM) put options. This causes higher implied volatilities for OTM puts compared to OTM calls.

For the SABR model, the parameter ρ influences skewness, more negative ρ values result in higher implied volatilities for OTM puts relative to OTM calls. The parameter ν controls kurtosis, determining the curvature of the smile. Higher ν values lead to steeper smiles, while lower ν values result in flatter smiles as maturity increases.

The volatility smile reflects market expectations of asset price movements and captures the limitations of the BS model in modelling true stochastic volatility. The SABR model offers better flexibility for capturing the smile by incorporating parameters that control skewness and kurtosis. Traders often express the smile in terms of implied volatility associated with the option's Delta and give insights into which options are most/least expensive after accounting for differences related to moneyness, expiration and option type.

Part III - Static Replication

We price two exotic options on 1st Dec 2020 with 45 days to expiry using different pricing models. For the BS and Bachelier models, we used the implied volatility of ATM options (average of ATM call and put implied volatility) from the respective models

Key Assumptions

	SPY ETF	SPX Index
Underlying (USD)	366.02	3662.45
Maturity (Years)	0.1233 years (45 DTE)	0.1233 years (45 DTE)
Black Scholes Volatility (%)	19.684%	18.747%
Bachelier Volatility (%)	19.682%	18.747%
Risk-free Rate (%)	0.2051%	0.2051%

Option 1: Exotic option with payoff function

$$S_T^{\frac{1}{3}} + 1.5 \times \log S_T + 10$$

1. <u>Pricing formula using BS model</u>

Under BS, the stock price process is as follows:

$$S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma W_T}$$

Through taking expectation of the payoff function with the BS price process and expanding it into integral of the payoff function multiplied by the standard normal distribution function, the following pricing formula was obtained:

$$V_0 = e^{-rT}\left(S_0^{\frac{1}{3}} \cdot e^{\frac{1}{3}\left(r-\frac{1}{3}\sigma^2\right)T} + 1.5\log S_0 + 1.5\left(r - \frac{1}{2}\sigma^2\right)T + 10\right)$$

With the above assumptions, the price of the exotic option as predicted under the BS model would be as follows:

Underlying	SPY ETF	SPX Index
Exotic Option 1 price (USD)	26.00	37.70

2. <u>Pricing formula using Bachelier's model</u>

Under the Bachelier's model, the stock price process is as follows:

$$S_T = S_0 + \sigma W_T$$

With the same approach previously, but under the Bachelier's price process instead, the following pricing formula was obtained:

$$V_0 = e^{-rT}\left(S_0^{\frac{1}{3}} + 1.5\log S_0 + 10 - \frac{1}{9}S_0^{-\frac{5}{3}}\sigma^2 T - \frac{3}{4}\frac{\sigma^2 T}{S_0^2}\right)$$

Using the Bachelier's implied volatility value for the respective ETFs, the prices obtained were:

Underlying	SPY ETF	SPX Index
Exotic Option 1 price (USD)	26.00	37.70

3. <u>Pricing through static replication</u>

On plotting the payoff function of the exotic option, we see that the payoff function is continuous, and the equation of the payoff function is clearly differentiable twice. Hence, we are able to use the result of the Carr-Madan Equation to replicate its payoff.

Thus, we find that the twice differential of the payoff function is:

$$h''(S_T) = \frac{2}{9{S_T}^{\frac{5}{3}}} - \frac{3}{2{S_T}^2}$$

Call and put integrand functions were set up using the SABR model calibrated with the α, ρ, ν output values calibrated on 45 DTE options from question 2 and choice β of 0.7 to obtain a volatility value. The integrand functions then feed this volatility value into the BS model divided by square of the strike, multiplied by the twice differential to price the corresponding portion of the option according to the Carr-Madan formula.

The final static replication formula obtained was:

$$V_0 = e^{-rT}\left(F^{\frac{1}{3}} + 1.5\log F + 10 + \int_0^F h''(K)\frac{BS\ Call(K)}{K^2}dK + \int_F^\infty h''(K)\frac{BS\ Put(K)}{K^2}dK\right)$$

Underlying	SPY ETF	SPX Index
Exotic Option 1 price (USD)	26.00	37.71

Key Insight

The 3 different pricing methods have resulted in basically the same prices, with the American option being priced higher than the European option likely due to early exercise premium.

Option 2: "Model-free" integrated variance

$$\sigma_{MF}^2\, T = E\left[\int_0^T \sigma_t^2 dt\right]$$

The model free integrated variance function can be expressed as the following function.

$$E\left[\int_0^T \sigma_t^2 dt\right] = 2rT - 2E\left[\log\frac{S_T}{S_0}\right]$$

1. Pricing formula using BS model

Using the same method used to price option 1, the following pricing formula was obtained:

$$V_0 = \sigma^2 T$$

Using the ATM volatility for the respective ETFs, we obtain option prices of

Underlying	SPY ETF	SPX Index
Exotic Option 2 price (USD)	0.00478	0.00433

2. Pricing formula using Bachelier's model

The log term can be expressed using the Bachelier stock price process. Taking expectation of the log term, we arrive at the following pricing formula:

$$E\left[\int_0^T \sigma_t^2 dt\right] = 2rT - 2\int_{-\infty}^{\infty} \log\left(\frac{S_0 + \sigma\sqrt{T}x}{S_0}\right) e^{-\frac{x^2}{2}}\, dx$$

Evaluating the integral, using the respective ETFs implied volatility, we obtain option prices of:

Underlying	SPY ETF	SPX Index
Exotic Option 2 price (USD)	0.00481	0.00436

3. Pricing through Static Replication

Adopting a Static Replication method, the "Model-free" integrated variance replication function can be reduced to:

$$E\left[\int_0^T \sigma_t^2 dt\right] = 2e^{rT}\int_0^F \frac{P(K)}{K^2} dK + 2e^{rT}\int_F^{\infty} \frac{C(K)}{K^2} dK$$

Implementing the static replication formula with the same α, ρ, ν output values and choice β value as in pricing option 1, we arrive at the final prices of:

Underlying	SPY ETF	SPX Index
Exotic Option 2 price (USD)	0.00602	0.00635

Part IV - Dynamic Hedging

4.1 Methodology for Dynamic Hedging

- The BS model is an imperfect model in many respects with the assumption that future stock evolution is lognormal with a known volatility and trading can be carried out continuously
- **For Part IV**, we investigate the effect of dropping only one of the key BS assumptions of the possibility of continuous hedging using the constraint that only a discrete number of rebalancing trades monthly (N=21 and N=84) at regular intervals are allowed
    - Re-hedging at discrete, evenly spaced time intervals as the underlying stock changes with final P&L expected to be exactly zero for portfolio to be self-financing

$$Final\ P\&L = V_T{}^{*}(Dynamic\ B - S) - V_T(S_T - K)$$

Using BS model to simulate the stock price (GBM) and shorting the ATM call option, the dynamic hedging will be executed N times during T = $\frac{1}{12}$ year time period. The dynamic hedging strategy for the call option is represented by C_t based on its processes **(a)** $\emptyset_t$ (delta hedge) and **(b)** ψ_t (bond component).

$$C_t = \emptyset_t S_t - \psi_t B_t$$

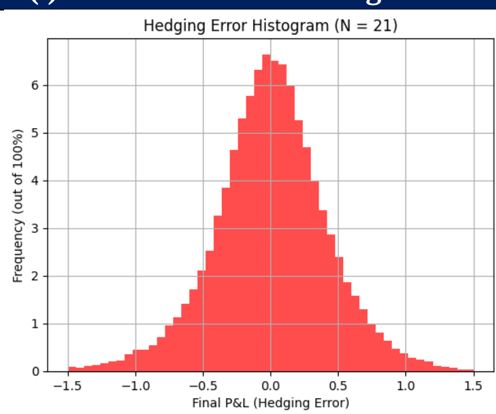
$$(\boldsymbol{a})\ \emptyset_t = \frac{\delta C}{\delta S} = \Phi\left(\frac{\log\left(\frac{S_t}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\ \textbf{(b)}\ \psi_t B_t = -Ke^{-r(T-t)}\Phi\left(\frac{\log\left(\frac{S_t}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

4.2 Results and Interpretation

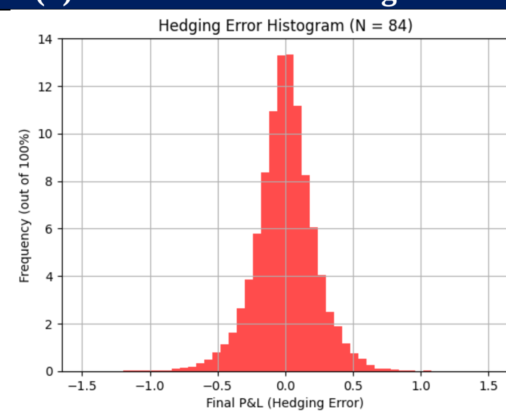
Hedging per month, N	Mean of Final P&L	Std Dev of Final P&L
N=21 (Daily)	-0.00019%	0.42650%
N=84 (QID)	-0.00030%	0.21818%

- When the re-balancing trades are done at discrete intervals, the hedge is imperfect and gives rise to replication error from either a positive or negative μ around 0%. Since the σ is lower for N=84, the hedging error has been observed to be minimized with increased hedging rebalances.

(i) Cumulative P&L using N = 21



(ii) Cumulative P&L using N = 84



- Histograms for the final P&L for a one-month at-the-money call hedged at discrete times respectively to expiration and the fair BS value of the call option is $C_0 = 2.512$
- Both of the frequencies (out of 100%) were obtained from normalizing the histogram by dividing the bin counts by the total number of paths (50,000 scenario) and multiplying by 100

Key Insights from Results

1. Mean of the Final P&L: The average final P&L is approximately zero which means that dynamic hedging does not bias the outcome in either direction when all the other parameters are correctly specified
2. Reducing Hedging Error: Increasing N reduces errors because the hedging portfolio better tracks the instantaneous changes in the underlying. Minimizing Gamma exposure reduces second-order risks, which arise due to the convexity of option payoffs.

3. Trade-offs of Frequent Rebalancing: While theoretically continuous hedging minimizes errors, real-world constraints like transaction fees, bid-ask spreads, and liquidity considerations mean that the optimal hedge frequency must balance error reduction with execution costs.

4.3 Additional Analysis and Insights

a. Approximate Interpretation for $\sigma_{Finall\ P\&L}$ by Goldman Sachs Equity Derivatives Research[1]

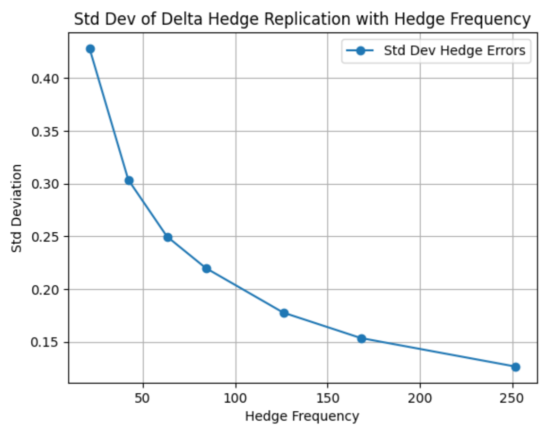
The following simple intuitive derived representation is a reasonable approximation for $\sigma_{P\&L}$ due to its similar qualitative behaviour with the full formula. The estimation is close for pricing parameters for $T = \frac{1}{12}\ Year$ (< 1 year), r = 5% (< 10%) and $K = S_0 = \$100$ (strike price close enough to spot price)

$$\sigma_{P\&L} = \sqrt{\frac{\pi}{4}}(\kappa)\frac{\sigma}{\sqrt{N}} \qquad where\ \kappa = S_0\sqrt{T}\frac{e^{-d_1^2}}{\sqrt{2\pi}}$$

Hedging per month, N	Approximation of $\sigma_{P\&L}$
N=21 (Daily)	0.443%
N=84 (QID)	0.222%

- Both estimates are close to the simulation results of $\sigma_{Final\ P\&L}$(N=21) = 0.42650% ≈ 0.44% and $\sigma_{Final\ P\&L}$ (N=84) = 0. 21818% ≈ 0.22%. The $\sigma_{P\&L}$ is an approximate measure of the true volatility of the underlying price when sampling the underlying price discretely. The discrete sampling will introduce the hedging error analogous to statistical fluctuations that are observed when flipping a fair coin N time with variance of $\frac{1}{\sqrt{N}}$ around the mean. These errors decrease with higher N leading to reduced variability in the final P&L.

b. Hedging improvements diminishes at higher levels of time stamps



- We observed that as hedge frequency increases, the standard deviation of hedging errors declines significantly. This reflects the theoretical underpinning of the BS model, where continuous hedging minimizes errors by dynamically adjusting the portfolio's Delta. However, once the intervals are sufficiently small, further reductions have diminishing benefits, as remaining hedging errors are dominated by second-order effects such as Gamma or nonlinear price movements. This insight is critical in practice, as pursuing very high hedge frequencies may offer minimal additional benefit while incurring disproportionately higher transaction costs and operational complexities

[1] Derman, E. (1999, January 1). When You Cannot Hedge Continuously: The Corrections of Black-Scholes.