Two Stage Game Theoretic Modeling of Airline Frequency and Fare Competition

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Abstract

Airlines make decisions about pricing and service frequency in a competitive environment. These decisions significantly affect the performance and usability of aviation systems. Most existing game theoretic models of airline competition suffer from limited tractability in large-scale networks, limited empirical support, or limited consideration of the sequential nature of capacity and fare decisions. Meanwhile, statistical models of airline resource allocation used by aviation planners do not consider the competitive aspects of airline behavior. In order to capture the competitive and sequential nature of these decisions, we develop a two stage game theoretic model of airline competition, where airlines choose service frequencies during the first stage and set fares during the second stage. First, we show that a simplified two-player version of this game has properties that suggest credible and tractable solution algorithms for predicting player behavior. We then numerically approximate payoff functions to show that these properties and their implications extend to more realistic models, and demonstrate that we can solve large-scale systems in a computationally efficient way. Leveraging these properties, we develop and implement an approach to solve our game for an 11-airport, multi-airline network from the western United States, and empirically validate the frequency predictions of our model.

Subject Classifications: Transportation: Air. Games/group decisions: Noncooperative. Forecasting: Applications.

Area of Review: Transportation.

1. Background and Motivation

Airlines allocate capacity across a network of airports, and do so in the face of competition from other airlines. Capacity decisions, encompassing decisions about seats-per-flight and frequency of service, affect both the operating costs and revenues of the airlines making them. But because these decisions are made within a competitive environment, they are interdependent with the behavior of other airlines in overlapping markets. For this reason, these decisions have significant implications for the performance of the air transportation system as a whole. Over- and under-allocation of airline capacity has been shown to be responsible for billions of dollars in costs to airlines and passengers (Ball et al., 2010; Kahn, 1993), wastage of system resources (Morisset and Odoni, 2011), passenger inconvenience (Barnhart et al., 2014; Wittman, 2014) and environmental damages (Schumer and Maloney, 2008). Airline frequency competition in particular has been shown to be a major driver of increased airport congestion (Vaze and Barnhart, 2012a).

Decisions about capacity and fare are typically made sequentially, on different timelines. Capacity decisions are often made weeks or months in advance of the flights in question, with only an approximate understanding of future fare decisions. On the other hand, fare decisions can be made days or even minutes in advance, with complete knowledge of previously made capacity allocation decisions. The frequency component of capacity allocation is of particular interest in this paper for two reasons. First, frequency decisions significantly affect passenger demand, as more frequent daily flights offer greater scheduling flexibility to passengers. Second, they show far greater variability across time and location than do seating capacity decisions (see **Section 2**). Frequency and fare decisions both serve as tools in an airline's competitive arsenal, and hence neither can be modeled in isolation. A higher frequency, for example, often results in a more convenient schedule of flight offerings and may increase the attractiveness of an airline to passengers, allowing the airline to sell tickets at a higher price. A higher frequency or a lower fare than competing airlines can typically attract more passengers to an airline in a given market, while reducing the demand for competing offerings. Because of the importance of the frequency and fare decisions to the function of the air transportation system, it is important to develop tractable models that accurately describe their dynamics.

Methods for the analysis and forecasting of airline behavior are critical tools for decision-makers in the air transportation system. The FAA (Federal Aviation Administration) uses airport traffic forecasts in areas such as workforce staff planning (particularly for air traffic controllers), in the evaluation of current and future technological improvements at various airports, in the planning of airport capacity expansions, and in the evaluation of requests for federal funding for airport infrastructure improvements (Devoti, 2010; Federal Aviation Administration, 2016a, 2016b; Transportation Research Board, 2014). The decisions of individual airlines related to expansion and reduction of operations at various airports also must take into account predictions about the effects of these decisions. The effects of these forecast-based decisions are felt by taxpayers, airlines, and individual passengers: under-staffing and inadequate capacity expansion compromise safety and efficiency of air travel; over-staffing and unnecessary expansion add unnecessary costs. Leveraging an understanding of airline competition in forecasts and in decision-making represents a significant opportunity to improve their efficacy, and to increase our understanding of the incentives at play in the system. Vaze and Barnhart (2012a, 2012b), for example, studied the role of airline frequency competition in airport congestion, and used a frequency competition model to develop a strategy to reduce passenger and flight delays as well as to improve airline profits.

Game theory provides a mathematical framework to explore the strategic interactions of multiple autonomous agents. In this paper, we develop a two stage game theoretic model of competition between airlines, demonstrate its tractability across a range of assumptions and parameter values, and validate its predictions against observed airline behavior. Our game theoretic approach accounts for the interdependence of competing airlines' behaviors. The multistage nature of the model accounts the sequential nature of frequency and fare decisions as observed in practice. Frequency and fare decisions are made by very different departments within an airline, and these departments typically do not jointly optimize their decisions. Therefore, two stage rather than single stage game theoretic models are behaviorally much more adequate for describing these two kinds of airline decisions. Each airline is assumed to pick a frequency value in each nonstop segment during the first stage of the game, in order to maximize its own profit. In the second stage, each airline decides the fare to be charged in each market, again while maximizing its own profit.

We begin our analysis with a simplified model amenable to analytical examination: with two airlines competing in a single market with unlimited seating capacity and in the absence of a no-fly alternative for passengers. For this model, we prove that the second stage fare game has a unique equilibrium, and that the payoff function of each airline in the first stage frequency game is concave with respect to that player's frequency strategy. Additionally, we analytically prove that this game belongs to the class of *sub-modular* games, with the corollary that for two player games, changing the sign of the strategy space transforms the game into a *super-modular game*. Taken together, these properties suggest tractable games with credible solution concepts for predicting player behavior. We then extend our model by demonstrating that we can closely approximate second stage payoffs using polynomial functions of airline frequency strategies. This allows us to extend our model to more complicated and realistic scenarios. We demonstrate numerically that for a wide range of realistic values of model parameters, the properties proved for the simplified model still hold. These results enable us to formulate a suitable equilibrium solution concept and employ a tractable solution algorithm by leveraging the convergence of simple learning dynamics.

We use this algorithm to solve for a subgame perfect pure strategy Nash Equilibrium for multiple airlines making frequency decisions across a network of 11 airports in the western United States, using demand and aircraft size data available from the Bureau of Transportation Statistics (BTS) online database (BTS 2016a, 2016b, 2016c). The frequency estimates from our equilibrium solution are then compared to the observed frequencies of these airlines over the same period to calibrate airline payoff functions. In practice, the game converges to equilibrium quickly with simple learning algorithms, allowing for tractable payoff parameter estimation and experimentation. With our calibrated payoff functions, we then examine frequency predictions out-of-sample at various levels of aggregation, including individual carrier-segment pairs, individual segments, individual airports, individual carriers, individual carrier-segment types, and the overall network, for look-ahead horizons at various time scales. We find that our predictions match airline decisions with good accuracy. Decision-makers within the air transportation system may thus find refinements of our model useful for forecasting and scenario analysis at various scenario granularities of practical importance.

1.1 Literature Review

A number of previous studies have taken a game theoretic approach to modeling airline capacity and/or fare competition. Hansen (1990) solved an airline frequency competition game using successive airline profit optimizations for a network with 52 US airports and 28 airlines, though model predictions showed some significant divergences with empirical data. Adler (2001; 2005) modeled airline competition with respect to airport network construction, frequency, seats, and fares, with the last three chosen simultaneously in the second stage, and airport networks chosen in the first stage. Evans and Schäfer (2011) examined the impact of airport capacity constraints using a single stage frequency game in a 22-airport network in the US. Adler, Pels, and Nash (2010) solved a single stage simultaneous frequency, seats, and fare game for a large network of competitors in both rail and aviation. In a series of studies, Vaze and Barnhart (2012a; 2012b; 2015) studied single stage frequency-only competition, and found (in the 2012a study) a reasonable level of agreement between equilibrium predictions and observed frequencies.

Hong and Harker (1992) discussed a simultaneous frequency-fare game in the context of slot allocation, and solved the model for a three-airport, three-airline network. Pels, Nijkamp, and Rietveld

(2000) solved a three-airport, two-airline simultaneous game of frequency and fare analytically (for the symmetric case) and numerically (for the asymmetric case). Zito, Salvo, and La Franca (2011) considered a duopolistic frequency-fare game in the context of competition with ground transport. Hansen and Liu (2015) considered frequency and fare competition to compare two different market share models based respectively on the s-curve and schedule delay formulations of the frequency-market share relationship. Wei and Hansen (2007) considered a two-airline frequency and seats game. Brueckner (2010) considered a simple model of simultaneous frequency, seats, and fare competition, and solved for equilibrium analytically. Except for Hansen (1990), Evans and Schäfer (2011), and Vaze and Barnhart (2012a), none of these studies empirically validated their equilibrium predictions.

Treatment of two stage frequency-fare games is more limited, and focused almost exclusively on competition between two airlines in a single market. Dobson and Lederer (1993) used heuristic methods to solve a single market, two-airline game, with airline schedules decided before fares. Schipper, Rietveld, and Nijkamp (2003) analyzed the shift from monopoly to duopoly equilibria following airline deregulation by simulating a single market, two-airline two stage frequency-fare game. Brueckner and Flores-Fillol (2007) also focused on the properties of single market, two-airline games. They analytically compared simultaneous and sequential frequency-fare decisions. None of these three studies used real-world airline networks, nor did they empirically validate their models. Hansen and Liu (2015) noted the intractability of analytical approaches to single market, two-airline two stage games, and presented a small numerical example. Several studies have stressed the need to develop two stage frequency-fare game theoretic models in order to account for the sequential nature of these decisions (Dobson and Lederer, 1993; Norman and Strandenes, 1994; Schipper, Rietveld, and Nijkamp, 2003; Brueckner and Flores-Fillol, 2007; Hansen and Liu, 2015), but to our knowledge, no study has successfully bridged analytical, numerical, and empirical approaches to such models for a real-world transportation network.

Two stage game models, while behaviorally consistent, present several practical challenges. The major solution concept available for two stage games, that of subgame perfect Nash Equilibrium, can be difficult to analyze and intractable to compute in practice. Moreover, Nash equilibria may not exist, or multiple equilibria may exist, limiting the credibility of the solution concept as a predictive tool in these cases. Equilibria that do exist can be prohibitively expensive to compute for simple models, let alone for extended many-player networks. Furthermore, game theoretic models in this area can be difficult to calibrate as a result of their intractability. Because of this, accurate prediction of behavior is also a challenge. In the game theoretic models of airline competition that do attempt to validate equilibrium results and empirical behavior, predictions often significantly diverge from observed behavior (e.g. Hansen, 1990; Chao, 2016).

1.2 Contributions and Outline

The main contributions of this paper overcome the practical challenges of these two stage models from analytical, numerical, and empirical points of view, in a holistic way. First, we demonstrate that a simple single market two stage duopoly game has analytical properties that support a credible and tractable subgame perfect pure strategy Nash Equilibrium solution. Second, we show that polynomial approximations of payoff functions are able to accurately capture the general properties of these games in more complex scenarios. We use these approximations to show that the properties demonstrated analytically extend beyond the duopoly case and hold across a wide range of assumptions. Third, we show that we can solve this game for a large multi-airline network based on real world data in an efficient manner. Fourth, we provide a method for calibrating the model to historical airline behavior, and show that our calibrated model produces good approximations of airline frequency decisions both in and out-of-sample, at multiple levels of aggregation, across the entire network. Finally, we provide a general approach to bridging analytical evaluations within simplified game scenarios and efficient calibration and prediction in more realistic scenarios that we believe could be useful in other applications of multi-stage game theory that are otherwise intractable. In summary, we show that a model that accurately reflects the sequential nature of the airline decision-making process also possesses properties that make it tractable to solve, and we support the credibility of these solutions both theoretically and empirically. This theoretically justified methodological approach could be refined for practical use in scenario analysis and forecasting.

The subsequent sections of this paper develop these contributions in the order described above. Section 2 describes the details of our game theoretic model. Section 3 discusses our analytical results for the simplified duopoly case and their significance for model tractability. Section 4 covers our numerical approach to extending the model to more complex scenarios. Section 5 describes the application of our numerical models to a real world airline network. We conclude with a discussion of possible extensions of this work in applications within and beyond the airline industry in Section 6.

2. Game Theoretic Model

A game is described by a set of players, a set of possible strategies for each player, and a payoff function for each player that maps each set of strategies chosen by all players (a *strategy profile*) to a payoff for that player. In our two stage game theoretic framework, frequency decisions of all airlines in all nonstop segments are assumed to be made simultaneously in the first stage, while the average fare decisions for all airlines in all markets are assumed to be made simultaneously in the second stage. More formally, our game comprises a set A of airlines, with a cardinality $|A| = N$. The strategy of each airline a is denoted by $u_a = \{\boldsymbol{f}_a, \boldsymbol{p_a}\}$, and the payoff (i.e. profit) function of each airline a is denoted by $\pi_a(u_1, \dots u_N)$. $\boldsymbol{f}_a$ and $\boldsymbol{p}_a$ are vectors of service frequencies and ticket prices, respectively, of airline a across its network. Each strategy profile $\boldsymbol{u} = (u_1, \dots u_N)$ represents the frequencies and fares of all airlines competing in a given network of airports. We refer to the frequency component of a strategy profile $(\boldsymbol{f}_1, \dots \boldsymbol{f}_N)$ as a *frequency strategy profile.* While we model fares as part of the competitive decision-making process, we restrict our attention to predicting the frequency strategy profile alone.

Frequencies are of particular practical interest to us given their considerable effect on system performance metrics such as resource utilization and wastage, delays and disruptions, customer discomfort and inconvenience, as well as environmental damages. In other words, we seek a *solution concept* that predicts a stable outcome of the game, particularly with respect to service frequency, and an algorithm for reaching this solution. However, we want our model to capture the effects that a later, more dynamic competitive pricing decision process induces on airline frequency decisions.

Even though airline capacity allocation decisions in practice also include decisions about number of seats per flight (i.e., aircraft size), we focus only on frequency decisions as the critical component of capacity-based competition. While frequency and seats-per-flight decisions both affect an airline's passenger carrying capacity, frequency decisions also significantly affect the attractiveness of a particular airline to passengers, with higher frequencies providing passengers with more flexible and convenient travel schedules (Belobaba, 2009). In addition, major airlines typically serve short and medium haul US domestic segments using a fleet with low variability in the number of seats per flight, while frequency decisions in such markets often vary significantly both across years on the same segment and across segments in the same year. **Table 1** displays the average values of the coefficient of variation (the ratio of standard deviation to mean) of frequency and seats-per-flight across years (2005-2014) and across segments connecting the OEP-35 airports. OEP-35 (Operational Evolution Partnership) airports is a list of 35 commercial airports in the US with significant activity. Values are calculated using data from the Bureau of Transportation Statistics Air Carrier Statistics Database (BTS, 2016a).

Airline	**Across Years**		**Across Segments**	
	Seats-per-Flight	Frequency	Seats-per-Flight	Frequency
American Airlines (AA)	4.7%	12.7%	13.3%	60.2%
Delta Airlines (DL)	6.3%	21.7%	15.9%	67.9%
United Airlines (UA)	5.3%	27.3%	21.7%	67.1%
US Airways (US)	6.3%	16.9%	15.4%	58.6%
Southwest Airlines (WN)	3.5%	16.6%	1.8%	70.0%

Table 1: Variability in Seats-per-Flight and Frequency Measured as Average Values of Coefficients of Variation

In this paper, frequency decisions are estimated according to the equilibrium found by solving the interdependent two stage payoff maximization problems, one for each airline. A Subgame Perfect Nash Equilibrium (SPNE) is the most widely accepted and commonly used solution concept for solving such two stage games. The SPNE solution concept, in the context of our two stage frequency-fare game, states that for any given set of frequency decisions of all airlines, the fares are modeled in the second stage using the classical Nash equilibrium concept. At a Nash equilibrium, no airline can increase its profit by unilaterally changing its fares. Next, building on this idea, the SPNE concept dictates that the first stage frequency decisions of each airline are made to maximize that airline's own profit, while explicitly accounting for the corresponding fare decisions as dictated by the second stage fare equilibrium, as well as the frequency strategies of other players. Consistent with existing airline game theoretic literature, we focus on pure strategy Nash equilibria in this paper in order to maximize practical interpretability and to simplify analysis.

For simplicity, we begin by describing a single segment model in Section 2.1. Then we extend the model to include network effects in Section 2.2.

2.1 Single Segment Model

The payoff function for each airline is given as the difference between the total revenue across all markets and the total cost of operations across all segments. As above, we define a market as an origin-destination pair of airports for the passengers, and denote the set of markets in which airline a is present as K_a. Consider a single market where passengers can only fly nonstop. In this simple scenario, a market is the same as a nonstop segment. Revenue for an airline a in market m is then computed as

$$Rev_{a,m} = \min\left(M_m * MS_{a,m}, f_{a,m} * s_{a,m}\right) * p_{a,m} \quad \textbf{(1)}$$

Here M_m is the size of (i.e., total unconstrained demand in) market m, $s_{a,m}$ is the seating capacity per flight for airline a in market m, and $MS_{a,m}$ is the market share for airline a in market m. $f_{a,m}$ and $p_{a,m}$ are the frequency and fare of airline a in market m. We describe market share using a discrete choice multinomial logit model. Such formulations model the probability of a decision-maker (in this case a passenger) making a given selection out of a set of discrete alternatives (in this case a set of travel alternatives for an origin-destination pair) as a softmax function of explanatory variables related to choice outcomes. In other words, the probability that a passenger selects an airline product i is $P(i) = \exp(V_i) / \sum_{j \in C} \exp(V_j)$, where V_j is a function of explanatory variables related to choice j and C is the set of travel alternatives for the passengers in a given market. This formulation has the convenient interpretation that passengers select the utility maximizing travel option if utility is taken to be the sum of V_j and an unobserved (to the researcher), independently extreme value distributed error term (McFadden, 1974). If observed utility V_j is taken to be linear in parameters, such models also lend themselves to convenient statistical estimation. For these reasons, multinomial logit models are widely used in market share modeling in the airline industry and air travel demand literature (see e.g., Coldren et al., 2003; Adler, 2005; Barnhart et al., 2014; Lurkin et al., 2017).

In this paper, we explore two common multinomial logit formulations of airline market share as functions of airline frequency and fare, which we will call the *s-curve model* and the *schedule delay model.* In practice, market share is shaped by many other factors, including preference for particular airline policies (e.g., free checked luggage) and the heterogeneous sensitivities to schedule convenience, fare, and airline brand. Because we are particularly interested in capturing the effects of service frequency and fare, and because these two factors are particularly important in passenger decision-making, we will not explicitly model other decision factors or customer heterogeneity within a market.

The *s-curve model* is consistent with the often-cited "s-curve" relationship between frequency share and market share. Observed passenger utility is given by a linear combination of fare and a logarithmic transformation of frequency (Hansen and Liu, 2015; Vaze and Barnhart, 2015). The logarithmic transformation of frequency captures the diminishing returns to schedule convenience for higher frequency and the aggregated nature of the alternative in the context of a discrete choice setting (Hansen, 1990). Thus, for the set of airlines competing in m denoted $A_{m,}$ with the positive parameters α and β (coefficients of utility of log of frequency and fare, respectively), and exponential of the utility of the no-fly alternative denoted by N_m, the market share of carrier a can be expressed as

$$MS_{a,m} = \frac{e^{\alpha * ln(f_{a,m}) - \beta * p_{a,m}}}{N_m + \sum_{i \in A_m} e^{\alpha * ln(f_{i,m}) - \beta * p_{i,m}}} \quad \mathbf{(2)}$$

With equal fares (i.e., with $p_{i,m} = p_{j,m}\ \forall i, j \in A_m$) and in the absence of a no-fly alternative (i.e., with $N_m = 0$), market share in equation $(\mathbf{2})$ is a function of only the frequency share, and it follows an s-curve whose shape is modulated by α, with a linear relationship at $\alpha = 1$.

Market share can also be described using the so-called *schedule delay model*, which incorporates the concept of schedule delay, or the average difference between actual flight departure times and the departure times desired by passengers (Hansen and Liu, 2015). In this case, market share for airline a takes the following form:

$$MS_{a,m} = \frac{e^{-\varphi * f_{a,m}{}^{-r} - \beta * p_{a,m}}}{N_m + \sum_{i \in A_m} e^{-\varphi * f_{i,m}{}^{-r} - \beta * p_{i,m}}} \quad \mathbf{(3)}$$

Here, φ and r are positive parameters modulating the utility of frequency. Hansen and Liu (2015) argue that, unlike in the s-curve formulation, market share in this model depends on both frequency share and absolute competitor frequency, such that an airline cannot simply dominate the market share of an already high frequency market by arbitrarily increasing its own frequency. $r = 1$ describes a situation in which the flight departure times desired by the passengers are uniformly distributed and headways for scheduled flights are equal, though other values have been found empirically (Hansen and Liu, 2015).

Airline operating cost is typically modeled as a linear function of frequency in airline competition literature. An exception to this is the use of the Cobb-Douglas function (Adler and Berechman, 2001; Adler, 2001; 2005), expressed as $[\sum_{k \in Arc}(f_k)^{\alpha}]^{\beta}$, with Arc being the set of all legs in an airline's network, and f_k being the number of flights per week offered on leg k, for parameters α, β both > 0, estimated as $\alpha = 1.2$ and $\beta = 0.7$ by Adler (2005). With these values, in a single market situation, we found this function to be well approximated by a linear function with an R^2 of 0.9978 over a plausible range of frequencies. Here we have assumed the plausible range of frequencies to be between 1 flight per day and 20 flights per day.

Thus, with $c_{a,m}$ as cost per flight for airline a in market m, and the overall operating cost modeled as $c_{a,m} * f_{a,m}$ for airline *a* and market *m,* the payoff function (or profit) of airline a over a network of airports (assuming only nonstop passengers) is given by

$$\pi_a = \sum_{m \in K_a} \min\left(M_m * MS_{a,m},\ f_{a,m} * s_{a,m}\right) * p_{a,m} - c_{a,m} * f_{a,m} \qquad (\mathbf{4})$$

In practice, airlines have a limited number of aircraft that they can deploy throughout their network on a given day. Such a constraint will depend on an airline's daily frequencies on all segments, an indicator of aircraft types compatible with each segment, the number of available aircraft of each type, the average travel time for each carrier/segment/aircraft type combination, and the airport turnaround time for each airline/aircraft type/airport combination. We develop one possible form of this constraint in **Section 5.2**, when we examine a real world network. In our analytical work, we will simply assume a large but finite upper limit on the nonstop frequency of a segment under examination.

2.2 Network Model

We can relax the assumption of nonstop only passengers, with some additional model features. We begin by formulating a general optimization problem for an airline with a general market share function, which can be chosen to suit the network at hand. With connecting passengers, we must now differentiate between an airline's set of segments S_a, airports pairs between which an airline runs direct flights; an airline's set of markets K_a, airport pairs between which passengers wish to travel (by traversing one or more segments); and the set of routes R_a, the set of sequences of segments by which an airline connects its markets. δ_{ajm} = 1 if route j serves market m for airline a, and 0 otherwise. ω_{akj} = 1 if segment k is contained in route j for airline a, and 0 otherwise. In our subsequent models, we ignore routes with more than one stop, as these account for <2% of domestic passenger traffic in the US (BTS, 2016b). The capacity of a segment can be used for carrying passengers belonging to a number of different routes. Therefore, the airline can optimize its use of segment capacity to carry the most profitable passengers. The general (constrained) payoff maximization problem for airline a over a network of airports is given by

$$\max \pi_a = \sum_{j \in R_a} Pax_{a,j} * p_{a,j} - \sum_{k \in S_a} c_{a,k} * f_{a,k} \quad \textbf{(5)}$$

$$s.t.\ \ Pax_{a,j} \leq M_m * MS_{a,j}\ \forall\ m \in K_a, j \in R_a\ if\ \delta_{ajm} = 1$$

$$\sum_{j \in R_a} \omega_{akj} * Pax_{a,j} \leq f_{a,k} * s_{a,k}\ \forall\ k \in S_a$$

$$f_{a,k} \geq 0\ \forall k \in S_a, Pax_{a,j} \geq 0\ \forall\ j \in R_a$$

Revenues now sum over all routes of an airline, where each route may be priced differently. Costs sum over all individual segments. The variable $Pax_{a,j}$ is the number of passengers flying on a certain route j of airline a. The first constraint limits it to be at most the size of the market M_m multiplied by the proportional share of passengers $MS_{a,j}$ taken by route j, assuming route j serves market m. The second constraint sums over all passengers on routes j associated with segment k to ensure that the capacity allocated to that segment can support the passengers allocated.

The market share function $MS_{a,j}$ in equation $\mathbf{(5)}$ in its most general form might be a function of the frequencies of segments on relevant routes, fares, passenger preference for nonstop routes versus routes with one or more stops, and preference for shorter travel time (and thus, for connections with the shorter distances and layovers). As in equations $\mathbf{(2)}$ and $\mathbf{(3)}$, we use a softmax function of passenger utilities, this time on a per-route basis, such that

$$MS_{a,j} = \frac{e^{V_{a,j}}}{N_m + \sum_{i \in A_m} \sum_{q \in R_i | \delta_{iqm} = 1} e^{V_{i,q}}} \quad \textbf{(6)}$$

Here, $V_{a,j}$ is the observed passenger utility of airline a's route j and N_m is the exponential of the utility of the no-fly option in market m to which route j belongs. The remainder of the denominator sums over the exponential of the utilities of all other routes serving market m. Airlines may fly multiple routes serving a single market. To formulate utilities, we again focus on the frequency and price characteristics of a route. For nonstop routes, we thus retain the form of equations $(\mathbf{2})$ and $(\mathbf{3})$, with j replacing the index m, such that $V_{a,j} = \alpha * ln(f_{a,j}) - \beta * p_{a,j}$ for the s-curve model, and with the equivalent replacement for the schedule delay model. For one stop markets, we adapt a form from Hansen (1990), such that for the s-curve model, we have

$$V_{a,j} = \alpha_{min} * ln\left(\min\{f_{a,k} \mid \omega_{akj} = 1\}\right) + \alpha_{max} * ln\left(\max\{f_{a,k} \mid \omega_{akj} = 1\}\right) - \beta * p_{a,j} \quad (\mathbf{7})$$

Here, we keep the general form of the s-curve model, but replace a single market frequency utility parameter with separate parameters for the higher and lower frequencies of the two segments of the trip. By the same logic, we can formulate the observed utility of a route in the schedule delay market share model with one stop passengers as follows:

$$V_{a,j} = -\varphi_{min} * \min\{f_{a,k} \mid \omega_{kj} = 1\}^{-r_{min}} - \varphi_{max} * \max\{f_{a,k} \mid \omega_{kj} = 1\}^{-r_{max}} - \beta * p_{a,j} \quad (\mathbf{8})$$

3. Analytical Results for a Simplified Model

In this section, we motivate and describe an analytical approach to exploring the credibility and tractability of our model. As noted in **Section 1.1,** two-stage models of the type presented in **Section 2** above can be difficult to analyze. Subgame Perfect Nash Equilibrium (SPNE) is an attractive solution concept, in that no player has an incentive to unilaterally change its strategy at either stage of the game. However, it is not obvious that the interactions of less than perfectly rational and computationally capable airlines would reach such an outcome, even assuming that our payoff models capture most of the incentives at play. From the perspective of a researcher, a computationally intractable game would be difficult to analyze and calibrate to real world data. Furthermore, a game with difficult-to-interpret properties (for example, multiple fare equilibria) would be of questionable predictive utility. In other words, a solution concept should provide an outcome that can be reasonably uncovered by both the modelers and the agents being modeled.

One way of improving the credibility of a solution concept as the outcome of an interaction and demonstrating its tractability is to show that a learning dynamic (a process by which players adjust their strategies in response to the strategies played by others over time) converges in reasonable time to such an outcome. One such learning dynamic is the *myopic best response algorithm*. From an initial strategy profile, this algorithm iterates through players, each player in turn updating its strategy to maximize its payoff (i.e., playing their *best response* to opponent strategies) with the strategies of other players held constant. This process continues until no player changes their strategy from a previous round or all changes are smaller than an acceptable tolerance. At this point, all players are playing best responses to each other's strategies, and by definition have reached an Nash Equilibrium. Letting u_i denote a chosen strategy for player i and u_{-i} chosen strategies of all other players, this algorithm is described as follows:

<u>myopic best response algorithm</u>

initialize strategy profile $\boldsymbol{u} = (u_i , u_{-i})$ of strategies for all players

repeat

for player $i = 1{:}N$

$s_i :=$ a best response to s_{-i}

$\boldsymbol{s} := (s_i , s_{-1})$

end

until $\boldsymbol{s}$ has converged

This algorithm has been successfully employed in previous approaches to airline frequency competition, see for example Vaze and Barnhart (2012a), Evans and Schäfer (2011), Adler (2005), and Hansen (1990). Because airlines typically compete in markets with a stable set of competitors over significant time periods, they are able to adjust their strategies (fares in the short run, frequencies in the longer run) to maximize their profits, an environment intuitively suited for such learning dynamics. If we can demonstrate properties favorable to the convergence of a learning algorithm such as myopic best response, we can improve the credibility of our solution concept. At the same time, we also gain a convenient algorithm for solving the game ourselves.

The complexity of our model in its full form makes analytical approaches to evaluating such properties challenging. Instead, we begin by analyzing a simplified version of the model presented in **Section 2,** in order to examine the properties of our game, for either market share function $(\mathbf{2})$ or $(\mathbf{3})$. This analysis allows us to gain theoretical insights on the tractability of our model formulation and the credibility of our solution concept, with the aim of building intuition for numerical evaluation in more complex scenarios. Complete proofs of all propositions and the corollary in this section are provided in Appendix II.

We begin by assuming the payoff function $(\mathbf{4})$ in a single nonstop market served by two carriers, in the absence of a no-fly alternative (that is, $N_m = 0$) and with unlimited seating capacity on each flight, and positive fare and frequency. Frequencies are assumed to be continuous quantities, consistent with an interpretation of a frequency strategy as an average over a time period, e.g., over a week.

First, consistent with the standard procedure of backward induction which is used for finding an SPNE (Fudenberg and Tirole, 1991) we examine the second stage of our game, and demonstrate that it has a unique pure strategy Nash equilibrium.

Proposition 1: The second stage fare competition game always has unique pure strategy Nash equilibrium.

This existence and uniqueness result supports both the credibility and tractability of the fare competition component of our model, suggesting that pure strategy Nash Equilibrium is a reasonable solution concept for the second stage of this game. In particular, this means that for a given frequency strategy profile, we can specify a unique vector of equilibrium fares, and thus compute a single set of payoffs for a given set of frequency values in the first stage of the game. This in turn strengthens the credibility of the SPNE solution concept, as airlines can reasonably predict payoff outcomes from first stage frequency strategy profiles, allowing for a meaningful implementation of the myopic best response algorithm in the first stage. We will leverage this property in the subsequent sections of the paper when solving our game to make frequency predictions.

We now examine the properties of the first stage of the game and prove the following result:

Proposition 2: In the first stage of a two stage frequency-fare game with no connecting passengers, assuming the absence of a no-fly alternative, infinite seating capacity on each flight and two carriers in a single market, profit functions are strictly submodular functions in the overall strategy space. That is,

$$\frac{\partial^2 \pi_i}{\partial f_1 \partial f_2} < 0 \text{ for } i \in \{1, 2\}$$

Corollary 1: By switching sign of the frequency strategy space of one of the two carriers, we can trivially convert the profit function into a *supermodular* function in the overall strategy space. That is,

$$\frac{\partial^2 \pi_i}{\partial f_1 \partial f_2} > 0 \text{ for } i \in \{1, 2\}$$

Intuitively, the submodularity of profit functions implies that an increase in one airline's frequency reduces the incentive of a competing airline to raise its own frequency: an increase in f_1 reduces $\partial \pi_2 / \partial f_2$. These properties demonstrate the utility of simple learning dynamics in solving for first stage frequency equilibrium. A game with compact scalar strategy spaces and continuous twice differentiable payoff functions with $\partial \pi_i / \partial f_i \partial f_j \geq 0 \; \forall \; i \neq j$ belongs to the class of *supermodular games* (Topkis, 1979). Taking untransformed frequency strategies (i.e., without switching their signs) to be in the range $[\epsilon, F]$, where ϵ is some small positive number, and F is some large positive number,[1] the supermodularity of the payoff functions after transformation puts the first stage game in this class. For these games, the existence of a pure strategy Nash equilibrium is guaranteed, even in the absence of concave payoffs, and a broad class of adaptive learning dynamics (including best response dynamics and fictitious play) converge to within the interval bounded by the largest and smallest pure strategy Nash Equilibria, as ordered by strategy profile (Milgrom and Roberts, 1990; Chen and Gazzale, 2004). Furthermore, initializing the myopic best response algorithm at a strategy profile larger than (or smaller than) the best responses of all players to each other's strategies will ensure that it converges monotonically downward (or upward, respectively) to a pure strategy Nash equilibrium (Vives, 1990). For example, in the context of our game, initialization at the infimum of the strategy profile space (ϵ, ϵ) would converge to the smallest pure strategy Nash Equilibrium, and initialization from the supremum (F, F) would converge to the largest. Thus, for our simplified model, we can guarantee the convergence of the myopic best response algorithm to a first stage pure strategy Nash equilibrium. We will use this algorithm in solving our game for the remainder of this paper.

Finally, we can demonstrate a further desirable property of the first stage frequency game in the next proposition.

Proposition 3: In the first stage of a two stage frequency-fare game with no connecting passengers, assuming the absence of a no-fly alternative, infinite seating capacity on each flight and two carriers in a single market, each airline's payoff π_i for $i \in \{1,2\}$ is strictly concave in airline i's frequency strategy, across plausible utility parameter ranges.[2] That is,

$$\frac{\partial^2 \pi_i}{\partial {f_i}^2} < 0 \text{ for } i \in \{1,2\}$$

Strictly concave first stage payoffs, not guaranteed in some one-stage models of airline frequency competition (e.g., Hansen 1990), ensure the existence of a first stage pure strategy Nash equilibrium even in the absence of supermodularity (Rosen 1965), and additionally ensure that first stage payoff maximization problems (as part of a myopic best response algorithm, for instance) are efficiently solvable and have a unique optimum, even for large-scale networks.

In summary, we have shown that a simplified version of our two-stage model has a unique second stage fare equilibrium, and submodular, concave (for plausible parameter ranges) payoffs with respect to first stage strategies. These properties imply that a two stage game approach to modelling frequency and fare competition induces properties in the payoff functions that improve the credibility and tractability of subgame perfect pure strategy Nash equilibrium. In other words, a more realistic approach to the sequential nature of airline decision-making also makes a game theoretic approach to analyzing airline decisions more attractive, both computationally and behaviorally. The existence of these properties in this simple case leads us to conjecture that more complex models *may* also show similar favorable properties. However, analytical approaches become substantially more difficult as the strong assumptions of this simple two player model are relaxed. We turn to numerical and computational approaches to extend our results to more realistic models. **Section 4** describes these approaches and their results.

4. Numerical Results for Extended Models

In this section, we present results of a series of numerical experiments for solving the more realistic full version of our model (as against the simplified version solved analytically). We now relax the strong assumptions made in **Section 3** (such as two players, absence of a no-fly alternative, absence of connecting passengers, and unlimited seating capacity per flight), and numerically test the existence, uniqueness, concavity, and submodularity results, similar to those we proved in **Section 3** for the simplified case, for a range of parameter values. In order to do this, we use polynomial approximations of second stage payoffs as functions of frequency strategies. These functions provide a good fit to the exact payoff functions, and allow for convenient evaluation of game properties. Additionally, since the real-world frequency decisions are anyway made with only an approximate knowledge of what fares will be in the future, a close approximation of payoffs that captures the gross properties of these functions seems justified.

4.1 Second Stage Game Solution

We first compute equilibrium fare vectors for the second stage game for a range of plausible (first stage) frequency strategy profiles. These profiles are generated on a grid of combinations of integer daily frequency strategies ranging from 1 flight per day to 20 flights per day. Second stage equilibria were computed by initializing fares for all players at \$100 (arbitrarily), and numerically iteratively optimizing each player's payoff (as defined in equations (4) and (5)) in turn with respect to its own fare, using the myopic best response algorithm. This algorithm was run until fares for each player converged to within a threshold of \$0.1, that is, until the change in fare from the previous iteration is less than \$0.1. This was done for both market share functions (s-curve and schedule delay); for 1, 2 and 3 player games; for varying numbers of seats per flight, both symmetric and asymmetric across players; with and without connecting passengers; for varying values of the exponential of the utility of the no-fly alternative (i.e., for varying values of N_m); and for varying values of the utility parameters for frequency and fare (α for the s-curve model, φ and r for the schedule delay model, and β for both models). Daily market size M was set at 1000 and cost per flight was set at \$10000.[3] For computational feasibility, parameters were varied one at a time for each number of players, with defaults set at $\alpha = 1.29$ (Hansen, 1990), $\beta = 0.0045$ (Hansen, 1990), $N = 0.5$ and number of seats $= 125$ for the s-curve model, and $r = 0.456$ (Douglas and Miller, 1974), $\varphi = 5.1$ (Hansen and Liu, 2016), $\beta = 0.012$ (Hansen and Liu, 2016), $N = 0.005$, and number of seats $=$ 125 for the schedule delay model. Values of N were chosen to be roughly on the same order of magnitude as the other (travel) alternatives, and accordingly differ between market share models to account for differing utility scales. Median seats-per-flight across all flights flown by major carriers (those who flew > 5% of all domestic passengers any given year between 2007 and 2015) ranged from 137 to 149 (BTS, 2016a), which motivated the default number of seats.

In all cases, this myopic best response heuristic converged to a pure strategy Nash equilibrium, suggesting that second stage fare equilibria for our model exist in practice across a broad range of scenarios. Moreover, upon varying the initial fare values (from the arbitrary \$100 value) across a wide range (\$0, \$50, \$100, \$200, \$1000), our experiments always led to the same equilibrium results in all tested scenarios. These results link the analytically demonstrated existence and uniqueness results of Proposition 1 with a broader landscape of more realistic but analytically intractable scenarios. Ranges of varied parameters were chosen to encompass values found in literature and in practice. Tables **2** and **3** list the ranges tested for each parameter and the increments by which these parameters were varied for the s-curve and schedule delay models respectively.

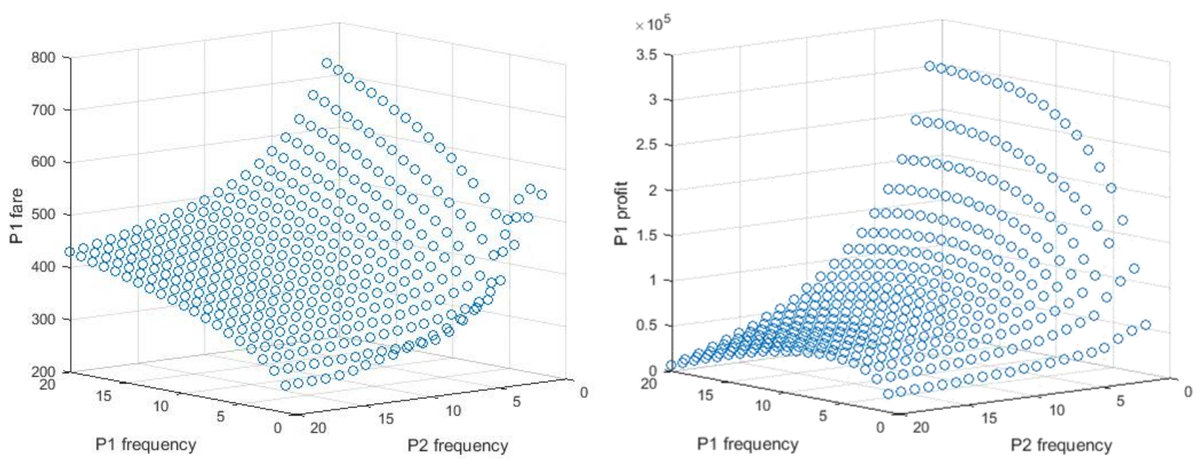
Parameter	Tested Range	Default Value	Testing Increments
$\boldsymbol{N}$	0.1 to 1	0.5	0.1
$\boldsymbol{\alpha}$	1 to 1.8	1.29	0.1
$\boldsymbol{\beta}$	0.001 to 0.01	0.0045	0.001
Seats-per-flight ($\boldsymbol{S}$)	25-250	125	25

Table 2: Parameter ranges tested against parameter defaults using s-curve model of market share

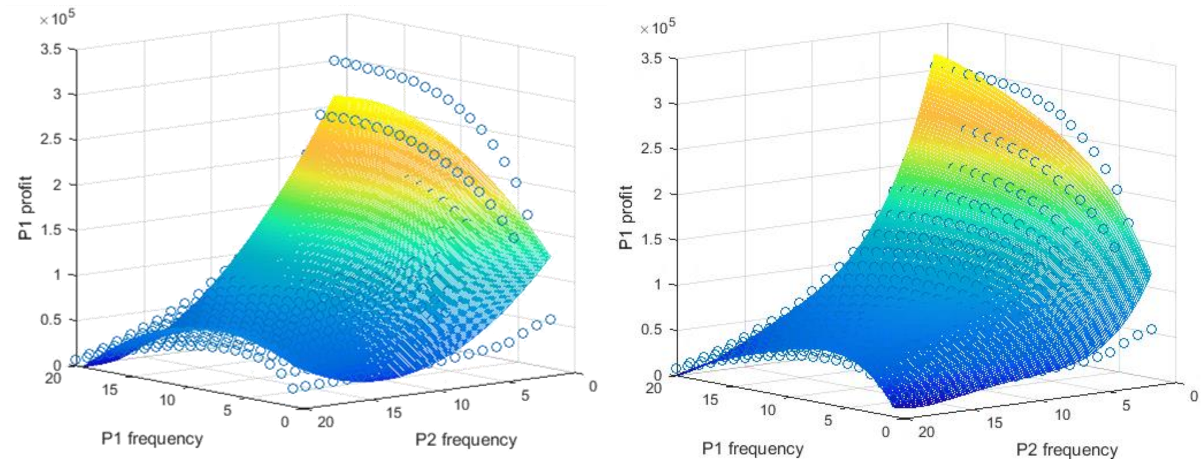
Parameter	Tested Range	Default Value	Testing Increments
$\boldsymbol{N}$	0.001 to 0.01	0.005	0.001
$\boldsymbol{r}$	0.1 to 1	0.456	0.1
$\boldsymbol{\varphi}$	2 to 10	5.1	1
$\boldsymbol{\beta}$	0.008 to 0.017	0.012	0.001
Seats-per-flight ($\boldsymbol{S}$)	25-250, and unlimited seating	125	25

Table 3: Parameter ranges tested against parameter defaults using schedule delay model of market share

Figure 1(a) shows the variation (against both players' frequencies) in player 1's second stage equilibrium fare values in a two-player game with the s-curve formulation and default parameter values.



(a) (b)



(c) (d)

Figure 1: Clockwise from top left: **(a)** Player 1's equilibrium fares in a two-player nonstop game for various frequency strategy profiles, with the s-curve formulation and default parameter values. **(b)** Player 1's equilibrium payoff for various frequency strategy profiles, in the two-player nonstop game from (a). **(c)** A quadratic polynomial approximation (with R^2 = 0.95) of second stage equilibrium payoffs, in the two-player nonstop game from (a). **(d)** A quartic polynomial approximation (with R^2 = 0.99) of second stage equilibrium payoffs, in the two-player nonstop game from (a).

4.2 First Stage Payoff Approximations

We used second stage equilibrium fares to calculate second stage equilibrium payoffs π_i for $i \in A$ (see **Figure 1(b)**). For each parameter combination and for each player, this generated 20 payoff data points for a single-player market, 400 data points for a two-player market, and 8000 data points for a three-player market. We then fit quadratic functions of frequency strategy profiles to their respective payoffs using linear (in parameter) regression by employing an ordinary least squares estimation process, for each type of market and for each tested parameter combination. The payoff function coefficients were estimated for the following functional forms.

- For a single-player nonstop market, the profit of carrier 1 (π_1) flying a daily frequency of f_1 was modeled as follows:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 {f_1}^2$$

- Likewise, for two-player nonstop markets a similar quadratic function was used:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 {f_1}^2 + \gamma_4 {f_2}^2 + \gamma_5 f_1 f_2$$

- Similarly, for three-player nonstop markets the following function was used:

$$\pi_1 \sim \gamma_0 + \gamma_1 f_1 + \gamma_2 f_2 + \gamma_3 f_3 + \gamma_4 {f_1}^2 + \gamma_5 {f_2}^2 + \gamma_6 {f_3}^2 + \gamma_7 f_1 f_2 + \gamma_8 f_1 f_3 + \gamma_9 f_2 f_3$$

- We approximated a two-player game with connecting passengers in a three-airport, two-segment, two-carrier network (**Figure 2**), using constrained payoff maximization problems described in equation $(\mathbf{5})$ and market share functions described in $(\mathbf{7})$ and $(\mathbf{8})$, with the following polynomial:

$$\pi_1 \sim \gamma_0 + \sum_{i,j \in \{1,2\}} \gamma_{i1,j1} f_{i,j} + \sum_{i,j \in \{1,2\}} \gamma_{i2,j2} {f_{ij}}^2 + \gamma_3 f_{11} f_{21} + \gamma_4 f_{21} f_{22}$$

Here, f_{ij} $\forall i \in \{1,2\}, \forall j \in \{1,2\}$ represents the frequency of player i on the j^{th} segment of the network. Three markets are described: nonstop markets between two airport pairs, and a one stop market connecting the third pair. **Figure 2** shows a visual representation of this network. Following Hansen (1990), we set the default values in equation (5) to $\alpha_{min} = 0.78$ and $\alpha_{min} = 0.33$. These are varied in increments of 0.1 within 0.3-1 and 0-0.7, respectively. Default market sizes were set at 700 for each nonstop market, and 300 for the one stop market, and varied in increments of 100 within 300-900 and 100-700, respectively. Seats per flight were set at 150 for both carriers on each leg, and varied in increments of 10 from 110 to 200. Other parameters were set and varied as in the experiments with the nonstop s-curve model.

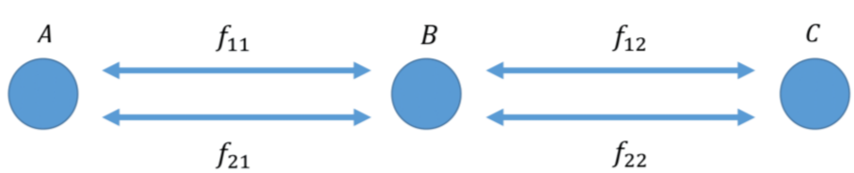


Figure 2: A three-airport, two-segment, two-carrier network. Airlines 1 and 2 run flights between airports A and B, and between airports B and C, transporting nonstop passengers between these airport pairs, as well as one stop passengers between A and C. The one stop market share of each airline is described by equation $(\mathbf{7})$.

Table 4 gives an illustrative example of regression results for a two-player nonstop s-curve model. In this case, utility parameters are held at defaults ($\alpha = 1.29$, $\beta = 0.0045$, $N = 0.5$) and the number of seats per flight for both carriers are varied to encompass a wide range of aircraft sizes.

Seats, Player 1,2	γ_0	γ_1	γ_2	γ_3	γ_4	γ_5	R^2
Unlimited	122,200	18,135	-17,856	-494	686	-533	0.96
250	122,250	18,130	-17,861	-494	686	-533	0.96
225	122,400	18,115	-17,876	-494	687	-532	0.96
200	122,640	18,095	-17,901	-493	687	-532	0.96
175	123,470	18,030	-17,989	-492	690	-529	0.96
150	125,340	17,925	-18,214	-491	696	-523	0.96
125	129,430	17,885	-18,838	-496	716	-514	0.95
100	136,710	18,277	-20,301	-513	773	-518	0.94
75	142,620	20,224	-22,355	-567	865	-579	0.93
50	104,880	27,814	-18,929	-744	773	-847	0.93
25	-56,286	26,953	6,911.	-442	-210	-685	0.97

Table 4: Regression coefficients and model R^2 approximating Player 1 payoffs in a two-player nonstop game with number of seats per flight varied.

The coefficient of determination (R^2) for fitted models was at least 90% for nearly all tested parameter combinations, numbers of players, and nonstop versus one stop networks. Out of 339 scenarios tested, only 23 had an R^2 below 90% (see **Appendix III**). 22 of these 23 scenarios were one player markets (fit with only 20 data points), and of these, $R^2 > 80\%$ for 18 and > 75% for the remaining. The generally high R^2 values suggest that in nearly all cases, a quadratic function of player frequencies is able to capture a significant portion of the variation in equilibrium payoffs, and can provide a good numerical approximation of the payoff functions described in equation $(\mathbf{4})$ irrespective of whether we assume an s-curve model or a schedule delay model. This gives us a tool to probe the robustness of the concavity and submodularity properties described in Proposition 2 and 3 for these more realistic game situations.

Examining approximated payoff functions, we find that in all models with high R^2 (> 0.9), the signs of estimated coefficients are consistent with both submodularity and payoff concavity. For example, for the two-player nonstop market case, γ_3, the coefficient of the square of player 1's daily frequency, and γ_5, the coefficient of the interaction term $f_1 f_2$, are both negative, consistent with concavity and submodularity, respectively. Note that this is the case across the entire tested range of values of number of seats per flight in **Table 4. Figure 1(c)** shows a quadratic approximation surface fitted to second stage equilibrium payoffs (represented by points identical to those in **Figure 1(b)**) in a two-player game, with an R^2 of 0.95: note that the approximation captures the concavity of player 1's profit with respect to its own frequency strategy. For a more extensive enumeration of coefficient estimates and R^2 values for various parameter combinations, see Appendix III. While longer computational times precluded extensive parameter sensitivity tests for more than 3 players, limited testing of 4 player games revealed similar results: accurate quadratic function approximations and coefficient estimates consistent with concavity and submodularity (see Appendix IIID). We also examined higher order polynomial approximations for even closer fits to payoff functions: quartic (4^{th} order) polynomial approximations tested on several models retained submodularity and concavity properties (see **Figure 1(d)**). For the remainder of this paper, however, we will focus on quadratic approximations, as these allow for generally good approximations while remaining convenient for simple evaluation of function properties and keeping the number of parameters in check when calibrating models with real-word data.

4.3 Analytical Properties of the First Stage Game

The robustness of sub-modularity and concavity properties in approximated payoff functions across a wide range of scenarios and parameter values extends the analytical results of Propositions 2 and 3 to a much richer and more realistic class of models. These results suggest that in general, sub-game perfect pure strategy Nash equilibrium remains a highly tractable and credible solution concept for our game. That property of submodularity extends to more complex scenarios is consistent with the observation made by other researchers that games with this property tend to arise in strategic situations where there is competition for a resource (Roy and Sabarwal, 2012). In this case, that resource is market share. While we cannot extend supermodularity trivially beyond the two-player case (as we can no longer simply change the sign of one player's strategy space to flip the sign of all the cross derivatives consistently), analogy with the two-player case, as well the literature on aggregative games of strategic substitutes (e.g. Jensen, 2010) provide us with a starting point for further exploration of the convergence of learning dynamics. Concave payoffs maintain the guarantee of the existence of first stage frequency equilibrium, and our quadratic approximations provide a simple mechanism for checking the uniqueness of first stage equilibrium using Rosen's diagonal strict concavity condition (Rosen, 1965; Moulin, 1984). The condition states that for strictly concave, twice differentiable utility functions π_i, if the inequality

$$\left|\frac{\partial^2 \pi_i(\boldsymbol{f})}{\partial {f_i}^2}\right| > \sum_{j \neq i} \left|\frac{\partial^2 \pi_i(\boldsymbol{f})}{\partial f_i \partial f_j}\right| \qquad (\mathbf{9})$$

holds for all players i and j and for all strategy profiles $\boldsymbol{f}$, then there exists a unique pure strategy Nash Equilibrium, and the myopic best response algorithm will converge to it. In addition, *simultaneous best response dynamics*, where each player updates its strategy simultaneously in response to the previous strategy profile, also converges to a pure strategy equilibrium. We find that in all one player and two player scenarios, three player schedule delay scenarios with the exception of an extreme value of $\varphi = 10$, and some of the three player s-curve scenarios, estimated quadratic coefficients across tested parameter ranges ensure a guaranteed unique first stage equilibrium according to this sufficiency condition. Furthermore, concavity means that individual players' best responses have a unique and easily solved-for optimum, and the myopic best response algorithm can be deployed efficiently to find equilibria, even in large problem instances, assuming that concavity continues to hold for these larger instances. In the two-player scenarios, we can use supermodularity to guarantee the rapid convergence of a broad class of learning dynamics to Nash equilibrium in the first stage game (following Milgrom and Roberts, 1990) with our approximated payoff functions. For scenarios with larger number of players, we can ensure convergence using Moulin's inequality above where it applies. In scenarios where it does not, we can leverage concavity, submodularity, the quadratic nature of the approximated payoffs, and results from

Jensen (2010) to demonstrate the convergence of the myopic best response algorithm, as shown by Proposition 4:

Proposition 4: An N-player game (for all integer $N > 1$) with quadratic, strictly concave, and strictly submodular payoff functions, positive compact strategy spaces, and equal interaction coefficients $\delta_{i,j}$ between a player and their opponents, belongs to the class of quasi-aggregative games (as defined by Jensen, 2010), and the *myopic best response algorithm* converges to the set of pure strategy Nash Equilibria of such a game.

Proof Let the payoff function of player $i \in K$, for strategy profile $\boldsymbol{u}$**,** be

$$\pi_i(s) = \alpha_i + \sum_{k \in K} \beta_{i,k} u_k + \sum_{k \in K} \gamma_{i,k} u_k^2 + \sum_{\substack{j \in K \\ i \neq j}} \delta_i u_i u_j + \sum_{\substack{j,k \in K \\ i \neq j,\ i \neq k,\ j<k}} \theta_{i,j,k} u_j u_k \quad \textbf{(10)}$$

Let π_i be strictly concave and strictly submodular, such that $\beta_{i,i} < 0$ and $\delta_i < 0 \; \forall \; i \in K$.

A quasi-aggregative game, as defined in Jensen (2010), generalizes the notion of an aggregative game, where payoffs of a player depend only on some aggregate of opponent strategies. It is defined for a game with players $i \in \{1, \dots N\}$, pure finite dimensional strategies $u_i \in U_i$ (where we denote $U = U_1 \times \dots \times U_N$), strategy profiles as $\boldsymbol{u} = (u_1, \dots, u_N) \in U$, and $u_{-i} \in U_{-i}$ as a strategy profile excluding the strategy of player i, and upper semi-continuous payoff functions $\pi_i(s)$. In a quasi-aggregative game, there exist continuous functions $g: U \to \mathbb{R}$ (the *aggregator*), $F_i : \mathbb{R} \times U_i \to \mathbb{R}$, and $\sigma_i: U_{-i} \to \mathbb{R}$ such that we can write payoff functions as:

$$\pi_i(\boldsymbol{s}) = \pi_i(\sigma_i(u_{-i}), u_i) \quad \textbf{(11)}$$

And, for all $\boldsymbol{s} \in S$ and all i,

$$g(\boldsymbol{s}) = F_i(\sigma_i(u_{-i}), u_i) \quad \textbf{(12)}$$

Taking $\sigma_i(u_{-i}) = \sum_{j \neq i} u_j$, we can then write:

$$\pi_i(\boldsymbol{s}) = \pi_i\left(\sum_{j \neq i} u_j, u_i\right) = \alpha_i + \beta_{i,i} u_i + \gamma_{i,i} u_i^2 + \delta_i u_i \sum_{j \neq i} u_j \quad \textbf{(13)}$$

This is equivalent to our payoff functions, except for the linear and quadratic terms for other players, and the interaction terms between other players. These play no role in strategic decisions of player i, because they disappear when differentiating $\pi_i(\boldsymbol{u})$ with respect to u_i. Thus they do not appear in best response expressions, and can be ignored. We can then take $g(\boldsymbol{u}) = F_i(\sigma_i(u_{-i}), u_i) = \sigma_i(u_{-i}) + u_i = \sum_{k \in K} u_k$. The game is thus a quasi-aggregative game. Moreover, $\partial^2 \pi_i(\boldsymbol{s})/\partial(\sum_{j \neq i} u_j)\partial u_i = \delta_i < 0$, because the aggregator is a linear sum, and best responses are single valued (as payoffs are strictly concave), this game additionally satisfies the conditions given by Jensen (2010) such that the myopic best response algorithm converges to the set of pure strategy Nash Equilibria. ■

To summarize, we have the following results for games played with our approximated quadratic, strictly concave and strictly submodular payoff functions in single markets:

- For single player games, strict concavity of payoff functions ensures that optimization of the payoff function will bring us to a unique "equilibrium."
- For two player games, the ability to transform the game into a *supermodular* game means that we can ensure convergence of the myopic best response algorithm.
- For two or three player games in which inequality $(\mathbf{9})$ holds for all players i and j and for all strategy profiles $\boldsymbol{f}$, we can ensure that both the myopic best response algorithm and simultaneous best response dynamics will converge to a unique pure strategy Nash Equilibrium from any initial strategy profile.
- For N-player games where players have the same interaction term with all of their opponents, the myopic best response algorithm converges to a set of pure strategy Nash Equilibria. If the Nash Equilibrium is unique, myopic best response algorithm will always converge to it from any starting point.

These learning dynamics properties, supported by numerical testing of convergence with approximated payoff functions, are reassuring both from an intuitive and a computational perspective.

The introduction of a second stage fare game has given our payoff functions properties consistent with the convergence of various learning dynamics, and allows us to think of our airlines making decisions in less-than perfectly rational terms. In this sense, our model incorporates both a forward-looking aspect (approximate consideration of future fare decisions while making frequency decisions) and a backward-looking aspect (myopic adjustment to the past frequency decisions of other players, as approximated by learning dynamics) in the competitive capacity allocation decisions of airlines. From a computational point of view, fast convergence of easily implementable learning algorithms (due to a combination of submodularity, concavity and quasi-aggregativity of payoff functions) and easily solvable iterative payoff maximizations (due to concavity) enables efficient solutions, experimentation, and calibration of our model when comparing its decision predictions to observed behavior. The analytical framework provided by these properties provides potential future avenues for using polynomial payoff approximation functions of this type for otherwise analytically intractable multi-stage games. In the next section, we leverage these properties to apply our model to a real-world airline network.

5. Empirical Case Study

To test the tractability and predictive validity of our game theoretic model in practice, we apply it to a network of airports in the western United States. The test network consists of 11 airports, namely, Seattle-Tacoma International Airport (SEA), Portland International Airport (PDX), San Francisco International Airport (SFO), San Diego International Airport (SAN), Los Angeles International Airport (LAX), Las Vegas McCarran International Airport (LAS), Phoenix Sky Harbor International Airport (PHX), Oakland International Airport (OAK), Ontario California International Airport (ONT), Sacramento International Airport (SMF), and Mineta San Jose International Airport (SJC). Our dataset spans across the eight year period of 2007-2014. We estimate the daily flight frequencies of the major (non-regional) carriers in this network (in Q1 of 2007, these are Alaska Airlines - AS, United Airlines - UA, US Airways - US, and Southwest Airlines - WN) in the ordered airport pairs (i.e. segments) in which these carriers operate. Frequencies are estimated by computing pure strategy Nash equilibrium in their frequency strategies using concave, submodular, quadratic functions to approximate the payoffs described by equation $(\mathbf{4})$.

5.1 Data and Preprocessing

For this network consisting of short and medium haul segments, most passengers in most markets were nonstop passengers. Modelling of networks with larger proportions of connecting passengers will likely require more complicated game theoretic considerations, and is left for future work. For the rest of this section, we will use the term *market* to indicate nonstop markets, as in **Section 2.1**. Quadratic payoff functions are constructed for each valid carrier-market combination depending on the number of carriers in the market, based on actual operating cost and market size data taken from the Bureau of Transportation Statistics (BTS) records. In particular, payoff function approximations computed for default parameters and market sizes listed in **Section 4.1** were transformed by the operating costs and market sizes. It is a simple linear transformation given the functional form of equation $(\mathbf{4})$. Operating costs and air operating hours for different aircraft for different carriers were taken from the Schedule P-5.2 tables (BTS, 2016c). Data on market sizes, observed frequencies and flight distances was obtained from the T100 Segments tables (BTS, 2016a). Data from unidirectional markets containing the same airports were averaged, such that, for example PDX-SAN and SAN-PDX were treated identically for payoff function generation and frequency estimation purposes. This approach is reasonable because passenger flows, observed frequencies, and other data are generally quite close to each other for differently ordered airport pairs. For simplicity, carrier-market combinations with a carrier market share of less than 10% or average daily frequency of less than 0.5 were removed from consideration.

Finally, markets in which both a major airline and its regional partner were present at the same time were removed from consideration. For example, in Q1 of 2007, three markets (PDX-SJC, PDX-SFO, OAK-PDX) were removed from consideration, as they contain significant daily frequencies from both Alaska Airlines (AS) and its regional partner carrier Horizon Air (QX). Taking into account such alliances, which represent a cooperative relationship, in game theoretic frequency estimation will require additional modelling considerations and is left for future work. Frequencies of other regional carriers were held fixed at observed values but were accounted for in the payoff function expressions of major competing airlines.

5.2 Individual Airline Optimization Model

We now use the myopic best response algorithm, justified analytically and numerically in **Section 3** and **Section 4**, to solve the game model. Each airline iteratively modifies a vector of frequencies in all its markets, each time re-optimizing the sum of the payoff functions from each of these markets, until estimated frequencies converged within a tolerance threshold. Frequency choices were constrained by the estimated availability of various aircraft types to the airline within the network. A single aircraft type was assumed for each individual carrier-market combination. Where appropriate, multiple aircraft types were merged into synthetic aircraft type for the purposes of computing these constraints, such that each carrier-market appeared in only one synthetic aircraft type, and such that operating costs in each market reflected the proportional composition of aircraft used in that market. To estimate aircraft availability, carriers were assumed to generally utilize aircraft close to the limits of availability. Thus, the number of aircraft of a type k available to a certain airline a in the network was calculated as

$$F_{k,a} = \frac{\sum_{m \in M_{k,a}} 2 f_{k,a,m} \left(b_{a,m} + t \right)}{T} \quad \mathbf{(14)}$$

Here, $M_{k,a}$ is the set of markets where airline a uses aircraft type k, $f_{k,a,m}$ is the observed frequency of airline a in market m using aircraft type k, t is the minimum turnaround time of the aircraft (assumed to be 30 minutes in all cases), $b_{a,m}$ is the average number of block hours per flight of airline a on market m, the factor of 2 accounts for the two directed markets corresponding to a particular airport pair, and T is the number of hours of available flying time in the day, which we assume to be 18 on average. During each individual airline profit maximization problem, these fleet size restrictions were applied such that:

$$\sum_{m \in M_{k,a}} 2\hat{f}_{k,a,m}\left(b_{a,m} + t\right) \leq T * F_{k,a} \quad \mathbf{(15)}$$

Here, $\hat{f}_{k,a,m}$ is the model estimated frequency on market m for airline a using aircraft type $k.$ In all, for the Q1 of 2007, our network had 68 combinations of airlines and markets for which the frequency estimation was conducted. When the successive optimizations algorithm is run, the players are assumed to allocate flight frequencies across their respective networks by solving a constrained quadratic optimization problem during each iteration, continuing until convergence. The frequency decision vector for each player was initialized at 0. In practice, the model typically converged in 6 to 7 iterations of the myopic best response algorithm, and was solved in less than one second using quadratic programming functions in MATLAB. A convergence threshold of 0.01, the upper limit on the sum of all absolute differences between estimated frequencies in current and previous iterations, was used as the algorithm's stopping criterion.

5.2 Parameter Calibration

In order to calibrate our model, we adjusted payoff coefficients to minimize the Mean Absolute Percentage Error, or *MAPE*, between estimated and observed frequencies across the whole networks of all airlines. *MAPE* (Vaze and Barnhart, 2012; Cadarso et al., 2017) is calculated as

$$MAPE = \frac{\sum_{cm \in CM}\left|\hat{f}_{cm} - f_{cm}\right|}{\sum_{cm \in CM} f_{cm}} \quad \mathbf{(16)}$$

Here *CM* is the set of all carrier-market combinations. The estimated and actual frequencies for the carrier-market combination cm are denoted by $\hat{f}_{cm}$ and f_{cm} respectively. For the purposes of calibration, carrier-market combinations were classified into four groups: all the carrier-market combinations in three-player markets, all the carrier-market combinations in two-player markets where both airports are hubs for that carrier, all other carrier-market combinations in two-player markets, and all carrier-market combinations in one player markets. This grouping accounts for some of the variation in carrier-market type, while limiting the overall number of parameters in order to limit overfitting. Hubs for a carrier for a given quarter were defined to be those airports for which at least 70% of passengers arriving on that carrier were connecting passengers (40% in the case of low cost carriers Southwest, JetBlue and Virgin). This quantitative definition generally aligned with officially designated hubs or focus cities. This grouping resulted in 11 total coefficients of the payoff functions of these four groups: two for the one-player carrier-markets and three each for the other three groups. These 11 coefficients were adjusted simultaneously (before transformation by operating cost and market size data for each carrier-market combination) using a gradient approximation algorithm called SPSA (Simultaneous Perturbation Stochastic Approximation, from Spall, 1998) to minimize overall *MAPE*. Specifically, during each iteration of SPSA, a single game was solved, with payoff coefficients perturbed according to an approximated gradient of the *MAPE* loss function. SPSA was chosen for its ability to approximate gradient using only two measurements of the loss function, independent of the number of variables being optimized. The 11 coefficients were initialized with values obtained by fitting quadratic payoff functions of frequency, using the s-curve formulation with the following default parameter values: $\alpha = 1.29$, $\beta = 0.0045$, $N = 0.5$ and with unlimited seating capacity per flight. After initialization, the game was run repeatedly over the course of 10,000 iterations. The best performing coefficients across these 10,000 iterations were then used to estimate frequencies across the network, from which we evaluate in-sample and out-of-sample model performance.

Figure 3 compares actual frequencies (x-axis) and predicted frequencies (y-axis) as described in the previous paragraph. The 45^{O} line represents prefect predictions. Most market predictions are found to lie near this line. An overall in-sample *MAPE* of 18.4% is achieved: more concretely, this corresponds to 49% of absolute prediction errors (absolute differences between actual and predicted daily frequencies) being less than 1, and 78% being less than 2. Notable outliers were the three highest frequency markets, all ‘hub-to-hub’ airport markets flown by Southwest Airlines.

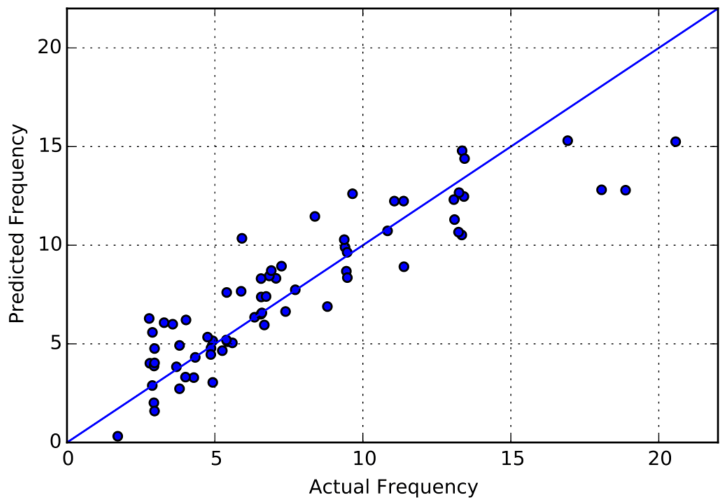


Figure 3: Model predicted vs. actual frequencies, for Q1 of 2007

5.3 Out-of-Sample Validation

We use these trained coefficients to make frequency predictions out-of-sample for future quarters. For example, coefficients trained using SPSA on data from Q1 of 2007 can be used to predict frequencies for Q4 of 2007. For this case, we find an out-of-sample testing *MAPE* of 20.6%, with 47% of absolute frequency errors being less than 1, and 73% being less than 2. However, we can leverage our knowledge of training errors when making out-of-sample predictions to further improve this performance. By adjusting our testing predictions for a given carrier-market by the error for that carrier-market in the calibration dataset (and simply performing no adjustment to carrier-markets that did not exist in the calibration set), we can substantially reduce our out-of-sample *MAPE. MAPE* in our Q4 2007 predictions falls to just 11.2%, with 72% of absolute frequency errors being less than 1, and 92% being less than 2. These *MAPE* results compare favorably with single airport *MAPEs* in the 14%-20% range from previous research that can be used as a benchmark (Vaze and Barnhart, 2012a).

A more concrete illustration of this model's out-of-sample prediction accuracy can be found by looking at a new nonstop market that arises in our dataset between Q1 and Q4 of 2007. PDX-SFO is not seen in the calibration dataset. Yet in Q4 it is a two-player market shared by AS and UA. The frequencies predicted by our model for this market are a good approximation for observed behavior, with an overall *MAPE* of 16.5%, as seen in **Table 5**. Note that for this market the adjusted and unadjusted predictions are identical because no adjustment can be made.

Carrier	Observed Frequency	Predicted Frequency	Absolute Error
UA	6.11	7.34	1.22
AS	3.02	2.74	0.28

Table 5: Out-of-sample performance on a new market – PDX-SFO predictions for Q4 of 2007 using coefficients calibrated on data from Q1 of 2007

We can take another view of out-of-sample prediction accuracy by looking at more aggregate measures of prediction performance, at the carrier, coefficient category (one-player, two-player hub-hub, other two-player, and three-player), market and airport levels. *MAPE* formulas for these values are provided in the second column of Table **6.** Frequency predictions are aggregated over various sets of carrier-markets: over carriers $a \in A$, coefficient categories $coef \in COEF$, markets $m \in K$ (K_a denotes the markets associated with a certain airline, as before), airports $ap \in AP$, and all carrier-markets $cm \in CM$. These can be used in conjunction to jointly specify a subclass of carrier-markets to be summed over: for example, $f_{ap,cm}$ specifies the observed number of flights in carrier market cm and airport ap.

Aggregated Measure	MAPE Calculation	Adjusted MAPE (Average Absolute Errors)
Carrier	$\frac{\sum_{a\in A}\left\lvert\sum_{m\in K_a}\hat{f}_{a,m}-\sum_{m\in K_a}f_{a,m}\right\rvert}{\sum_{a\in A}\sum_{m\in K_a}f_{a,m}}$	1.5% (2.11)
Coefficient Category	$\frac{\sum_{coef\in COEF}\left\lvert\sum_{cm\in CM}\hat{f}_{coef,cm}-\sum_{cm\in CM}f_{coef,cm}\right\rvert}{\sum_{coef\in COEF}\sum_{cm\in CM}f_{coef,cm}}$	2.5% (3.42)
Market	$\frac{\sum_{m\in K}\left\lvert\sum_{a\in A_m}\hat{f}_{a,m}-\sum_{a\in A_m}f_{a,m}\right\rvert}{\sum_{m\in K}\sum_{a\in A_m}f_{a,m}}$	6.3% (0.86)
Airport	$\frac{\sum_{ap\in AP}\left\lvert\sum_{cm\in CM}\hat{f}_{ap,cm}-\sum_{cm\in CM}f_{ap,cm}\right\rvert}{\sum_{ap\in aP}\sum_{cm\in CM}f_{ap,cm}}$	2.6% (2.59)
Total Frequency	$\frac{\left\lvert\sum_{cm\in CM}\hat{f}_{cm}-\sum_{cm\in CM}f_{a,m}\right\rvert}{\sum_{cm\in CM}f_{a,m}}$	2.5% (2.71)

Table 6: Formulas for MAPE calculation at various levels of aggregation

In our predictions for Q4 of 2007 based on calibration data from Q1 of 2007, we find excellent predictions at all of these levels, both unadjusted and adjusted according to training error. With respect to total frequencies allocated by each carrier, we find an *MAPE* of 2.0%, or 1.5% adjusted (corresponding to average absolute errors of 2.71 and 2.11 flights respectively). With respect to total frequencies allocated within each coefficient category, we find an *MAPE* of 3.0%, or 2.5% adjusted (corresponding to average absolute errors of 4.20 and 3.42 flights respectively. With respect to total frequency by market (across the 41 markets for which estimates were made in the network), we find an $MAPE$ of 14.4%, or 6.3% adjusted (corresponding to average absolute errors of 1.95 and 0.86 flights, respectively). With respect to total frequency by airport (across the 11 airports in the network), we find an $MAPE$ of 7.8%, or 2.6% adjusted (corresponding to average absolute errors of 7.75 and 2.59 flights, respectively). These adjusted values of *MAPEs* and average absolute errors at various levels of aggregation are listed in the third column of **Table 6**.

Figure 4 shows a visual representation of the error-adjusted predictions and actual total daily number of flights for Q4 of 2007 at an airport level. See Appendix I **Table A1** for a tabular representation of this data. Note that at airports where a significant increase in the number of flight operations was observed between Q1 and Q4 of 2007 (e.g. at LAS), our model was able to predict this increase, whereas in cases where the numbers of flight operations was relatively consistent across quarters (e.g. at SMF), our model predictions reflect this too. Predictions at each of these levels of aggregation may be of interest for airlines, airports and other policy makers. As discussed in **Section 1,** for instance, such airport-level traffic forecasts are used by the FAA in workforce staff planning, and in evaluation of airport capacity expansion and technological development (Devoti 2010; Federal Aviation Administration 2016a, 2016b; Transportation Research Board, 2014).

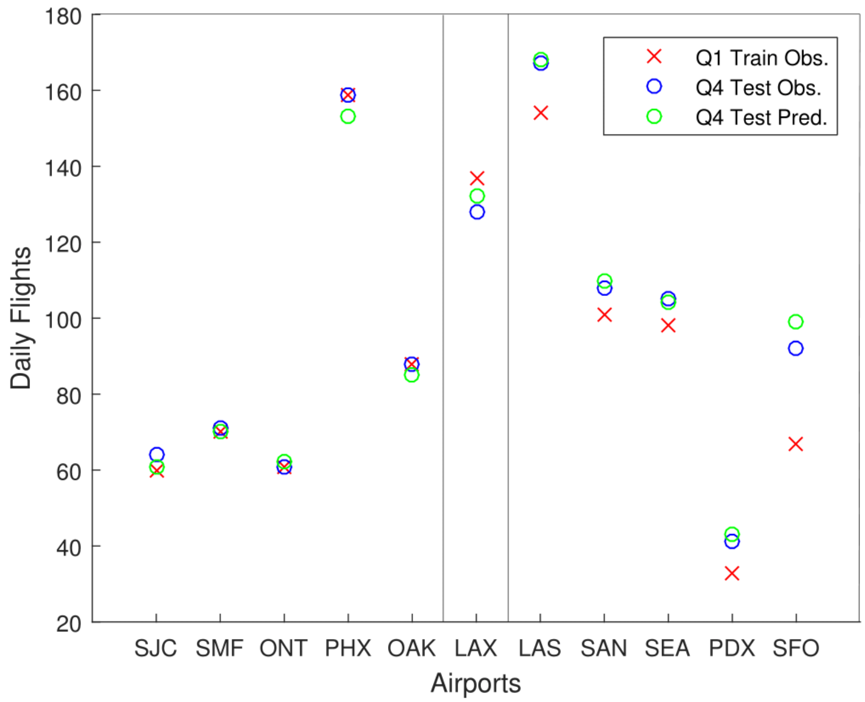
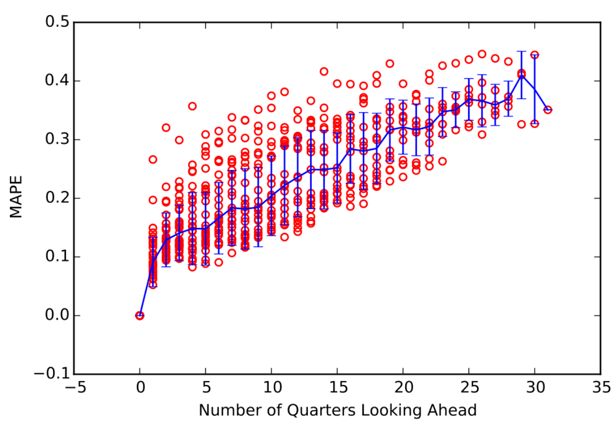


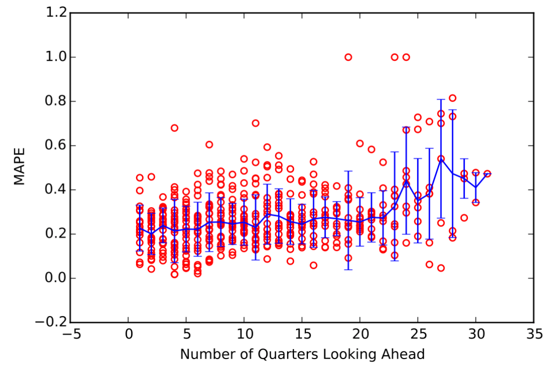
Figure 4: Left panel shows airports where observed airport level flight operations remained relatively consistent across quarters (crosses and dark circles for Q1 and Q4 respectively). The middle panel shows the airport where the number of flight operations decreased from Q1 to Q4. The right panel shows airports where the number of flight operations increased from Q1 to Q4. Predictions for Q4 are shown as light circles.

In order to examine the predictive accuracy of our model more broadly, we examined in-sample and out-of-sample prediction across years and for varying degrees of look-ahead in prediction. We calibrated our 11 coefficients on every quarter from 2007 to 2014, giving us 32 sets of coefficients. Using these coefficients, we predicted frequencies at every quarter after each of these calibration quarters. In this expanded dataset, we include new major carriers and new hubs in the network as appropriate for the quarter under consideration.

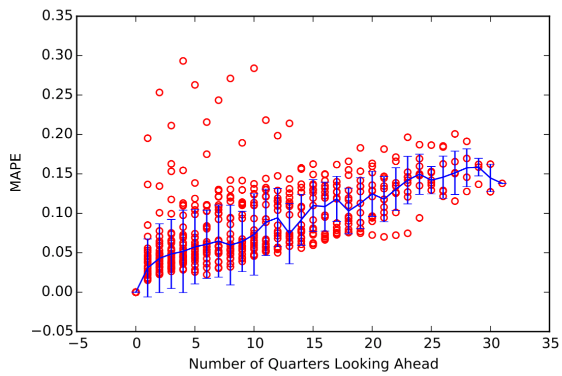
Examining the unadjusted *MAPE* at varying look-ahead values (i.e. number of quarters ahead for which the prediction is made with respect to the quarter on which the data is calibrated), we find an almost monotonic increase in median error. Median *MAPE* remains below 25% for 15 quarters or smaller look-ahead durations, suggesting reasonable predictive accuracy in the short and medium term. However, by adjusting predictions by calibration errors in the manner described earlier in this section, we can achieve significant improvements in *MAPE*. Median *MAPE* in this case remains below 15% for 5 quarters into the future, though a similar overall increase in median *MAPE* with increasing look-ahead is seen. **Figure 5a** shows error-adjusted *MAPE* for each possible calibration-validation combination (circles), and the median *MAPE* for that look-ahead value as the blue line (with error bars at one standard deviation). As might be expected, the utility of the error adjustment diminishes with increasing look-ahead, with differences between adjusted and unadjusted *MAPE* of 13% with one quarter of look-ahead, 10% with 2 quarters of look ahead, and essentially no improvement beyond 10 quarters of look-ahead. We can also look at median *MAPE* for new markets (**Figure 5b**), i.e. nonstop markets not seen in the calibration data. Finally, **Figure 5c** shows *MAPE* aggregated at the airport level (as shown in **Figure 4** for a single calibration-validation quarter pair). Outliers in low look-ahead values in **Figure 5c** represent certain quarters toward the end of our dataset's timespan for which calibration did not lead to very effective predictions for remaining quarters. In general, median out-of-sample $MAPE$ remains below 15% across all markets, remains around 20% for the new markets (suggesting that the new market displayed in **Table 7** is not an abnormal short term prediction), and remains below 10% for the airport level aggregation, for look-ahead durations of several quarters, suggesting reasonable predictive accuracy in the short and medium terms.



5(a)



5(b)



5(c)

Figure 5: MAPE for different metrics at varying look-ahead durations. **(a)** Calibration error-adjusted MAPE at carrier-market level **(b)** MAPE at carrier-market level for new markets. **(c)** Calibration error-adjusted MAPE at airport level of aggregation.

6. Conclusions and Future Research Directions

This study investigates a two stage frequency-fare game theoretic model of airline competition which is behaviorally consistent with the sequential nature of airline capacity and fare decisions. For simple cases, the analytical qualities of this model indicate well behaved and tractable games, with unique equilibria and convergence properties. Using polynomial payoff function approximations, these properties can be shown numerically to extend to more realistic formulations of the game. In practice, when applied to a real airline network, the model converges quickly and generates daily frequency predictions that closely approximate actual airline decisions, both in sample and out-of-sample within a short-to-medium term time horizon.

We believe that our model presents multiple avenues for application and future research. To the best of our knowledge, this is the first study to investigate the favorable properties discussed within the context of a two stage model of airline competition, providing analytical, numerical, and empirical results for a game theoretic approach that has received limited attention in the airline competition literature. We hope that our results presented here can serve as a foundation for further research into multi-stage models of airline decision making under competition. Furthermore, the predictive performance of our model on real world data suggests that similar game theoretic approaches may be worth exploring in the context of planning, forecasting and policy-making decision support. The tractability of this model and the flexibility with which different scenarios can be tested suggest its potential for rapid and interpretable experimentation even in large-scale airline networks. Here we have considered a relatively simple model of airline competition, without taking into account factors such as market segmentation between business and leisure passengers, passenger loyalty, behavioral differences between airlines and between their other characteristics beyond frequency and fare, and characteristics of markets beyond operating cost, observed passenger flows, and hub presence. The fact that our simple model provides a good approximation of airline frequency allocation suggests that more flexible parameterizations in calibration and prediction could be promising avenues for practitioners.

More broadly, the general approach presented in this paper could be usefully employed in other domains of applied game theory. Here we have presented a theory-motivated framework for informing a game theoretic predictive model with data. Game theoretic models often become intractable for analytical exploration when they are extended to increasingly realistic scenarios. Polynomial approximations of profit functions may provide a convenient method for extending the analytical results of simple game theoretic models to more realistic scenarios, and provide a bridge from theoretical results to models that can be easily calibrated using increasingly available data on strategic behavior. The polynomial approximation approach may provide simple ways to evaluate the properties of otherwise inscrutable models by providing simple windows into properties such as concavity, sub/supermodularity and aggregation. We hope that such an approach, within the two stage framework in particular, could be extended to other applied domains where capacity and pricing decisions are made on different time scales. Earlier work has explored such two stage capacity and price decisions in models motivated by the telecommunications industry (Acemoglu et al., 2006): we hope that parallels with such domains and the work here could be fruitfully explored in the future studies.

Notes

1. These assumptions are made in order to guarantee compactness of strategy spaces and continuity of profit functions. They can be justified by the notion that an airline must have some presence in a market to be considered a player in that market, and that airlines have a finite (though possibly large) supply of aircraft they can deploy.
2. For the s-curve model, this holds for all cases where $\alpha < 2.4456$, a very conservative bound with respect to empirically estimated values (typically 1.3-1.7), see Belobaba (2009). For the schedule delay model, concavity guarantees depend on both φ and r. See proof of theorem 1 in Appendix II for a discussion of these values.
3. Across the years 2007-2015, median costs per hour of flying time in the US ranged from \$3895 to \$5304 (BTS, 2016c), suggesting that this default flight cost is reasonable for a flight in the range of 1-3 hours of air time. Across the same time period, 30%-41% of passengers flew in markets with more than 1000 bi-directional passengers a day (BTS, 2016b), so this market size baseline is also reasonable.

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Appendix I: Airport-level Model Predictions

Airport	Observed Flights Q1	Observed Flights Q4	Predicted Flights Q4
LAX	137	128	132
SJC	60	64	61
LAS	154	167	168
SAN	101	108	110
SMF	70	71	70
SEA	98	105	104
PDX	33	41	43
SFO	67	92	99
ONT	61	61	62
PHX	159	159	153
OAK	88	88	85

Table A1: Out-of-sample predictions of daily number of flight operations at all airports in network, for Q4 of 2007, calibrated on data from Q1 of 2007.

Appendix II: Full Proofs for s-curve and schedule delay models

Here we give proofs for Propositions 1, 2, and 3 in **Section** 3. We present proofs for the s-curve and schedule delay formulations in parallel, bifurcating where appropriate because they generally follow a similar structure. Where similar steps are repeated for both formulations, numerical labels are appended with a decimal, $.\mathbf{1}$ for the s-curve formulation and $.\mathbf{2}$ for the schedule delay formulation. For example, $(\mathbf{1.1})$ and $(\mathbf{1.2})$ are used for a corresponding equation $(\mathbf{1})$ of the respective formulations. Proofs are presented in a different order than in the main paper, for analytic convenience.

Here we begin with a proof that in a market with two competing airlines, no connecting passengers, infinite seating capacity, and in the absence of a no-fly alternative, profit π_i for each airline $i \in \{1, 2\}$ is a concave function of the frequency of airline i in that market.

IIA. Proof of Concavity (Proposition 3)

With f_i as the positive frequency and p_i as the fare of each airline and with α and β as constants, the market share under the s-curve formulation for airline 1 is given by:

$$MS_1 = \frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}$$

For the schedule delay formulation, with r and φ as additional positive parameters, it is given as

$$MS_1 = \frac{e^{-\varphi f_1{}^{-r} - \beta p_1}}{e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}}$$

With M as market size, and c as the operating cost of a flight, the profit of airline 1 can then be written as

$$\pi_1 = M * p_1 * MS_1 - cf_1 \quad (\mathbf{1})$$

Now, we will invoke the first order conditions (i.e., $\frac{\partial \pi_1}{\partial p_1} = 0$ and $\frac{\partial \pi_2}{\partial p_2} = 0$) for the second stage fare equilibrium. Differentiating, and substituting MS_1 in terms of its complementary market share as $1 - MS_2$, we get

$$\frac{\partial \pi_1}{\partial p_1} = M(MS_1) + Mp_1 \frac{\partial}{\partial p_1}(1 - MS_2)$$

Expanding for the s-curve formulation, we then have:

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) - Mp_1\beta\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1} . e^{\alpha\, ln(f_2) - \beta p_2}}{(e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2})^2}\right)$$

At $\frac{\partial \pi_1}{\partial p_1} = 0$,

$$M\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = Mp_1\beta\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1} . e^{\alpha\, ln(f_2) - \beta p_2}}{(e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2})^2}\right)$$

$$\left(\frac{e^{\alpha\, ln(f_2) - \beta p_2}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = \frac{1}{\beta p_1} = MS_2 \quad (\mathbf{2a.1})$$

Repeating the process for π_2, we also have

$$\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_1) - \beta p_1} + e^{\alpha\, ln(f_2) - \beta p_2}}\right) = \frac{1}{\beta p_2} = MS_1 \quad (\mathbf{2b.1})$$

Following the same procedure for the schedule delay formulation, we have:

$$\frac{\partial \pi_1}{\partial p_1} = M\left(\frac{e^{-\varphi f_1{}^{-r} - \beta p_1}}{e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}}\right) - Mp_1\beta\left(\frac{e^{-\varphi f_1{}^{-r} - \beta p_1} . e^{-\varphi f_2{}^{-r} - \beta p_2}}{\left(e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}\right)^2}\right)$$

At $\frac{\partial \pi_1}{\partial p_1} = 0$,

$$M\left(\frac{e^{-\varphi f_1{}^{-r} - \beta p_1}}{e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}}\right) = Mp_1\beta\left(\frac{e^{-\varphi f_1{}^{-r} - \beta p_1} . e^{-\varphi f_2{}^{-r} - \beta p_2}}{\left(e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}\right)^2}\right)$$

$$\left(\frac{e^{-\varphi f_2{}^{-r} - \beta p_2}}{e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}}\right) = \frac{1}{\beta p_1} = MS_2 \quad (\mathbf{2a.2})$$

Repeating this process for π_2, we also have

$$\left(\frac{e^{-\varphi f_1{}^{-r} - \beta p_1}}{e^{-\varphi f_1{}^{-r} - \beta p_1} + e^{-\varphi f_2{}^{-r} - \beta p_2}}\right) = \frac{1}{\beta p_2} = MS_1 \quad (\mathbf{2b.2})$$

Additionally, $MS_1 + MS_2 = 1$ in the absence of a no-fly alternative. Therefore,

$$\frac{1}{\beta p_1} + \frac{1}{\beta p_2} = 1 \Rightarrow \frac{1}{p_1} + \frac{1}{p_2} = \beta \Rightarrow p_1 + p_2 = \beta p_1 p_2$$

$$\frac{p_1}{p_2} = \beta p_1 - 1 \quad (\mathbf{3})$$

Substituting $(\mathbf{2}\boldsymbol{b})$ into $(\mathbf{1})$**,** we have

$$\pi_1 = \frac{Mp_1}{\beta p_2} - cf_1$$

Substituting $(\mathbf{3})$ into the above,

$$\pi_1 = \frac{M}{\beta}(\beta p_1 - 1) - cf_1 = Mp_1 - \frac{M}{\beta} - cf_1 \quad (\mathbf{4})$$

Thus, $sgn\left(\frac{\partial^2 \pi_1}{\partial {f_1}^2}\right) = sgn\left(\frac{\partial^2 \mathrm{p}_1}{\partial {f_1}^2}\right) \quad (\mathbf{5})$

For the s-curve model, dividing $(\mathbf{2}\boldsymbol{b}.\mathbf{1})$ by $(\mathbf{2}\boldsymbol{a}.\mathbf{1})$, we also get

$$\left(\frac{e^{\alpha\, ln(f_1) - \beta p_1}}{e^{\alpha\, ln(f_2) - \beta p_2}}\right) = e^{\alpha(ln(f_1) - ln(f_2)) - \beta(p_1 - p_2)} = \frac{p_1}{p_2}$$

Taking the log of both sides and substituting $\frac{p_1}{p_2} = \beta p_1 - 1$ from $(\mathbf{3})$,

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta(p_1 - p_2) + ln(\beta p_1 - 1)$$

Substituting $\frac{p_1}{\beta p_1 - 1} = p_2$ from $(\mathbf{3})$,

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta\left(p_1 - \frac{p_1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1) \quad (\mathbf{6}.\mathbf{1})$$

Differentiating both sides with respect to f_1 ,

$$\alpha\left(\frac{f_2}{f_1}\right)\frac{1}{f_2} = \frac{\partial p_1}{\partial f_1}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$\frac{\alpha}{f_1} = \left(\frac{\beta^2 p_1}{(\beta p_1 - 1)^2} + \beta\right)\frac{\partial p_1}{\partial f_1}$$

$$\frac{\alpha}{\beta}\frac{1}{f_1} = \left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1} \quad (\mathbf{7}\boldsymbol{a}.\mathbf{1})$$

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)} \quad (\mathbf{7}\boldsymbol{b}.\mathbf{1})$$

To find the corresponding $(\mathbf{7}\boldsymbol{a}.\mathbf{2})$ and $(\mathbf{7}\boldsymbol{a}.\mathbf{2})$ for the schedule delay formulation, we follow similar steps.

Dividing $(\mathbf{2}\boldsymbol{b}.\mathbf{2})$ by $(\mathbf{2}\boldsymbol{a}.\mathbf{2})$, we also get

$$\left(\frac{e^{-\varphi {f_1}^{-r} - \beta p_1}}{e^{-\varphi {f_2}^{-r} - \beta p_2}}\right) = e^{-\varphi({f_1}^{-r} - {f_2}^{-r}) - \beta(p_1 - p_2)} = \frac{p_1}{p_2}$$

Taking the log of both sides,

$$ln\left(\frac{p_1}{p_2}\right) = -\varphi({f_1}^{-r} - {f_2}^{-r}) - \beta(p_1 - p_2)$$

Substituting $\frac{p_1}{p_2} = \beta p_1 - 1$ from $(\mathbf{3})$,

$$-\varphi({f_1}^{-r} - {f_2}^{-r}) = \beta(p_1 - p_2) + ln(\beta p_1 - 1)$$

Substituting $\frac{p_1}{\beta p_1 - 1} = p_2$ from $(\mathbf{3})$,

$$-\varphi({f_1}^{-r} - {f_2}^{-r}) = \beta\left(p_1 - \frac{p_1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1)$$

$$-\varphi({f_1}^{-r} - {f_2}^{-r}) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1) \quad (\mathbf{6.2})$$

Differentiating both sides with respect to f_1 ,

$$\varphi r {f_1}^{-(r+1)} = \frac{\partial p_1}{\partial f_1}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$\varphi r {f_1}^{-(r+1)} = \left(\frac{\beta^2 p_1}{(\beta p_1 - 1)^2} + \beta\right)\frac{\partial p_1}{\partial f_1}$$

$$\frac{\varphi r {f_1}^{-(r+1)}}{\beta} = \left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1} \quad (\mathbf{7a.2})$$

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\varphi r}{\beta}{f_1}^{-(r+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)} \quad (\mathbf{7b.2})$$

From $(\mathbf{3})$, we note that $\beta p_1 > 1$, because the price ratio must be positive. In equation $(\mathbf{7a.1})$ and $(\mathbf{7a.2})$, the left-hand side and the multiplicands on the right-hand side are positive. Additionally, differentiating the left multiplicand of the right hand side of $(\mathbf{7a.1})$ and $(\mathbf{7a.2})$with respect to p_1, and given that $\beta p_1 > 1$, we get

$$\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) = \beta\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4} < 0$$

Thus, differentiating both sides of $(\mathbf{7a.1})$ with respect to f_1 a second time, we get

$$-\frac{\alpha}{\beta}\frac{1}{{f_1}^2} = \frac{\partial}{\partial f_1}\left[\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1}\right]$$

$$-\frac{\alpha}{\beta}\frac{1}{{f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\left(\frac{\partial p_1}{\partial f_1}\right)^2$$

We can rearrange $(\mathbf{7a.1})$ as $\frac{\partial p_1}{\partial f_1} = \frac{\alpha}{\beta}\frac{1}{f_1}\left(\frac{(\beta p_1 - 1)^2}{\beta^2 {p_1}^2 - \beta p_1 + 1}\right)$, and plugging this expression into the equation above, we have

$$-\frac{\alpha}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\alpha^2}{\beta^2 {f_1}^2}\left(\frac{(\beta p_1 - 1)^4}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)$$

$$-\frac{\alpha}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^{\,2}}\right)\frac{\alpha^2}{\beta {f_1}^2}$$

$$-\frac{\alpha}{\beta {f_1}^2} - \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^{\,2}}\right)\frac{\alpha^2}{\beta {f_1}^2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)$$

$$\underbrace{\frac{-\frac{\alpha}{\beta {f_1}^2}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}}_{A}\left[1 + \alpha\underbrace{\left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^{\,2}}\right)}_{B}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2} \quad (\mathbf{8.1})$$

To get a corresponding expression $(\mathbf{8.2})$ for the schedule delay formulation, we differentiate both sides of $(\mathbf{7}\boldsymbol{a}.\mathbf{2})$ with respect to f_1 a second time, and get

$$\frac{-\varphi r(r+1){f_1}^{-(r+2)}}{\beta} = \frac{\partial}{\partial f_1}\left[\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)\frac{\partial p_1}{\partial f_1}\right]$$

$$\frac{-\varphi r(r+1){f_1}^{-(r+2)}}{\beta} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\left(\frac{\partial p_1}{\partial f_1}\right)^2$$

We can rearrange $(\mathbf{7}\boldsymbol{a}.\mathbf{2})$ as $\frac{\partial p_1}{\partial f_1} = \frac{\varphi r}{\beta}{f_1}^{-(r+1)}\left(\frac{(\beta p_1 - 1)^2}{\beta^2 {p_1}^2 - \beta p_1 + 1}\right)$, and plugging this expression into the equation above, we have

$$\frac{-\varphi r(r+1){f_1}^{-(r+2)}}{\beta}$$

$$= \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \beta\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\varphi^2 r^2}{\beta^2}{f_1}^{-2(r+1)}\left(\frac{(\beta p_1 - 1)^4}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)$$

$$\frac{-\varphi r(r+1){f_1}^{-(r+2)}}{\beta} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) + \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\varphi^2 r^2}{\beta}{f_1}^{-2(r+1)}$$

$$\frac{-\varphi r(r+1){f_1}^{-r-2}}{\beta} - \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\varphi^2 r^2}{\beta}{f_1}^{-2r-2} = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)$$

$$\frac{-\varphi r(r+1){f_1}^{-r-2}}{\beta}\left[1 + \left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^2}\right)\frac{\varphi \mathrm{r}}{r+1}{f_1}^{-2r-2+r+2}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)$$

$$\underbrace{\frac{-\varphi r(r+1){f_1}^{-r-2}}{\beta\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}}_{A}\left[1 + \frac{\varphi \mathrm{r}}{r+1}{f_1}^{-r}\underbrace{\left(\frac{1 - \beta^2 {p_1}^2}{(\beta^2 {p_1}^2 - \beta p_1 + 1)^{\,2}}\right)}_{B}\right] = \frac{\partial^2 p_1}{\partial {f_1}^2} \quad (\mathbf{8.2})$$

The expression to the left of the square brackets in $(\mathbf{8.1})$ and $(\mathbf{8.2})$ (labeled A in both cases) is always negative. Expression B in $(\mathbf{8.1})$ and $(\mathbf{8.2})$ has the form $\left(\frac{1-x^2}{(x^2-x+1)^2}\right)$, which has a global minimum of -0.408894 at $x = \beta p_1 = 1.53209$.

Thus, in the case of the s-curve formulation, for any $\alpha < \frac{1}{0.408894} = 2.4456$ (which is an extreme value for this parameter),

$$\frac{\partial^2 p_1}{\partial {f_1}^2} < 0$$

From (**5**)**,** this implies

$$\frac{\partial^2 \pi_1}{\partial {f_1}^2} < 0$$

The profit function of player 1 is thus concave with respect to the frequency of player 1 in the s-curve market formulation for $\alpha < 2.4456$.

In the case of the schedule delay formulation, we have a slightly more involved evaluation. Let $K = \frac{\varphi r}{r+1} {f_1}^{-r}$. If K is < 2.4456, $\frac{\partial^2 p_1}{\partial {f_1}^2} < 0$ and from (**5**)**,** this implies $\frac{\partial^2 \pi_1}{\partial {f_1}^2} < 0$, and thus that the profit function of player 1 is guaranteed to be concave with respect to the frequency strategy of player 1.

Using $\varphi = 5.1$ estimated by Hansen and Liu (2015), and any value of r less than 0.92, the expression K is less than 2.4456 for all values of $f_1 \geq 1$. r has been cited in literature as 0.456 by Douglas and Miller (1974), a value also used by Hansen and Liu (2015). Abrahams (1983) takes $r = 1$, in which case concavity holds for all values of $f_1 > 1.043$. Taking the significantly larger $\varphi = 8.71$ derived by Hansen and Liu (2015) for business passengers (who are particularly sensitive to schedule convenience), concavity holds for all values of $f_1 > 1.115$ for $r = 0.456$, and $f_1 > 1.781$ for $r = 1$. Thus, even for relatively extreme parameter combinations, concavity holds for a broad frequency interval. **Figure A.1** displays K for values of player 1 frequency, for different values of r and $\varphi = 5.1$. The horizontal line represents the value of K below which concavity of player 1's profit function holds. ■

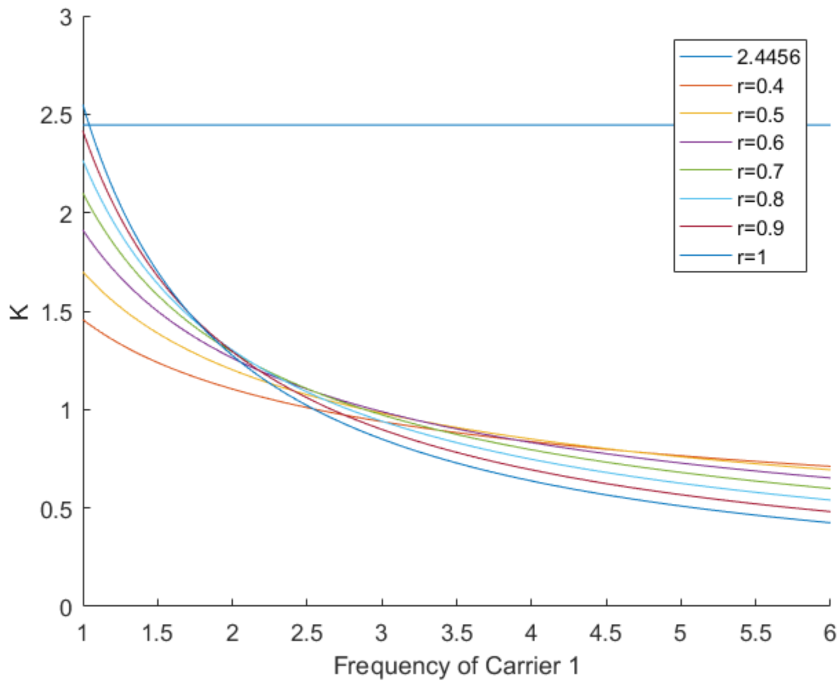

Figure A1: Values of K for different frequency and m values, with $\phi = 5.1$. Concavity is guaranteed to hold when K is less than 2.4456 (horizontal line).

IIB. Proof of Unique Pure Strategy Fare Equilibrium in the Second Stage (Proposition 1)

From $(\mathbf{6.1})$, we have

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \beta p_1\left(1 - \frac{1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1) \quad (\mathbf{9.1})$$

Likewise, from $(\mathbf{6.2})$,

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \beta p_1\left(1 - \frac{1}{\beta p_1 - 1}\right) + ln(\beta p_1 - 1) \quad (\mathbf{9.2})$$

Denoting the right hand side of this equation by $F(\beta p_1)$ and differentiating with respect to (βp_1), we get

$$\frac{\partial F(\beta p_1)}{\partial(\beta p_1)} = \frac{(\beta p_1)^2 - 2\beta p_1 + 2}{(\beta p_1 - 1)^2} + \frac{1}{\beta p_1 - 1} = 1 + \frac{1}{(\beta p_1 - 1)^2} + \frac{1}{\beta p_1 - 1}$$

Since $\beta p_1 > 1$ (due to $(\mathbf{3})$),

$$\frac{\partial F(\beta p_1)}{\partial(\beta p_1)} > 0 \;\forall\; \beta p_1 \in (1, \infty) \quad (\mathbf{10})$$

Since β is a positive constant, $F(\beta p_1)$ is monotonically increasing in p_1 over $p_1 \in \left(\frac{1}{\beta}, \infty\right)$.

In addition,

$$\lim_{\beta p_1 \to \infty} F(\beta p_1) = \infty$$

This is true because $\lim_{\beta p_1 \to \infty} ln(\beta p_1 - 1) = \infty$, and using L'Hopital's rule,

$$\lim_{\beta p_1 \to \infty} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to \infty} \frac{2\beta p_1 - 2}{1} = \infty$$

Also,

$$\lim_{\beta p_1 \to 1^+} F(\beta p_1) = -\infty$$

This is true because $\lim_{\beta p_1 \to 1^+} ln(\beta p_1 - 1) = -\infty$, and

$$\lim_{\beta p_1 \to 1^+} \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) = \lim_{\beta p_1 \to 1^+}[(\beta p_1)^2 - 2\beta p_1] \lim_{\beta p_1 \to 1^+} \frac{1}{\beta p_1 - 1} = -\infty$$

Thus, denoting the left hand side of $(\mathbf{6.1})$ or $(\mathbf{6.2})$ by $G(f_1, f_2)$, for any given $\{f_1, f_2\}$, there exists an F^{-1} such that $p_1 = \frac{1}{\beta} F^{-1}(G(f_1, f_2))$ with $\beta p_1 > 1$. By symmetry, the same argument applies for p_2. Thus, for the second stage game, there exists a unique fare vector $({p_1}^*, \, {p_2}^*)$. ■

IIC. Proof of Submodularity (Proposition 2 and Corollary 1)

For the s-curve formulation, from $(\mathbf{7}\boldsymbol{b}.\mathbf{1})$, we have

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}$$

Differentiating both sides with respect to f_2, we get

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\alpha}{\beta}\frac{1}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2} \frac{\partial}{\partial f_2}\left[\frac{\beta p_1}{(\beta p_1 - 1)^2}\right] = \frac{-\frac{\alpha}{f_1}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2}\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\partial p_1}{\partial f_2}$$

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\alpha}{f_1}}{(\beta p_1 + (\beta p_1 - 1)^2\,)^2}\left[(1 - \beta^2 {p_1}^2)\frac{\partial p_1}{\partial f_2}\right] \quad (\mathbf{15.1})$$

Likewise, for the schedule delay formulation, from $(\mathbf{7}\boldsymbol{b}.\mathbf{2})$, we have

$$\frac{\partial p_1}{\partial f_1} = \frac{\frac{\varphi m}{\beta}{f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)}$$

Differentiating both sides with respect to f_2, we get

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\frac{\varphi m}{\beta}{f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2} \frac{\partial}{\partial f_2}\left[\frac{\beta p_1}{(\beta p_1 - 1)^2}\right] = \frac{-\varphi m {f_1}^{-(m+1)}}{\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right)^2}\left(\frac{1 - \beta^2 {p_1}^2}{(\beta p_1 - 1)^4}\right)\frac{\partial p_1}{\partial f_2}$$

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} = \frac{-\varphi m {f_1}^{-(m+1)}}{(\beta p_1 + (\beta p_1 - 1)^2\,)^2}\left[(1 - \beta^2 {p_1}^2)\frac{\partial p_1}{\partial f_2}\right] \quad (\mathbf{15.2})$$

On the right-hand side of $(\mathbf{15.1})$**,** the left multiplicand is < 0 because α and f_1 are both positive. Likewise, on the right-hand side of $(\mathbf{15.2})$**,** the left multiplicand is < 0 because φ, m and f_1 are all positive.

Now we must check the sign of $(1-\beta^2{p_1}^2)\frac{\partial p_1}{\partial f_2}$. From $(\mathbf{6.1})$, we have

$$\alpha\, ln\left(\frac{f_1}{f_2}\right) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

Differentiating both sides with respect to f_2 ,

$$-\alpha\left(\frac{f_2}{f_1}\right)\frac{f_1}{{f_2}^2} = \frac{\partial p_1}{\partial f_2}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$-\frac{\alpha}{f_2} = \beta\frac{\partial p_1}{\partial f_2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) \quad (\mathbf{16.1})$$

Likewise, from $(\mathbf{6.2})$, we have

$$-\varphi({f_1}^{-m} - {f_2}^{-m}) = \frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)$$

Differentiating both sides with respect to f_2,

$$-\varphi m{f_2}^{-(m+1)} = \frac{\partial p_1}{\partial f_2}\frac{\partial}{\partial p_1}\left(\frac{\beta p_1}{\beta p_1 - 1}(\beta p_1 - 2) + ln(\beta p_1 - 1)\right)$$

$$-\varphi m{f_2}^{-(m+1)} = \beta\frac{\partial p_1}{\partial f_2}\left(\frac{\beta p_1}{(\beta p_1 - 1)^2} + 1\right) \quad (\mathbf{16.2})$$

Thus in both $(\mathbf{16.1})$ and $(\mathbf{16.2})$, $\frac{\partial p_1}{\partial f_2} < 0$, and hence $(1-\beta^2{p_1}^2)\frac{\partial p_1}{\partial f_2} > 0$ if $\beta p_1 > 1$ (which is the case at the second stage fare equilibrium). So from both $(\mathbf{15.1})$ and $(\mathbf{15.2})$ we have

$$\frac{\partial^2 p_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{17})$$

From $(\mathbf{4})$, we have

$$\pi_1 = Mp_1 - \frac{M}{\beta} - cf_1$$

So $(\mathbf{17})$ also implies

$$\frac{\partial^2 \pi_1}{\partial f_1 \partial f_2} < 0 \quad (\mathbf{18})$$

Thus we have a 2-player submodular game. **(Corollary 1)** This can be converted to a supermodular game by converting one airline's strategy $f_1 \rightarrow -f_1$. ■

Appendix III: Numerical demonstrations of concavity, sub-modularity, and the goodness of fit of quadratic approximations: s-curve and schedule delay formulations

Following the discussion in **Section 4,** the following tables provide values of the parameters being varied, estimated coefficients γ relevant to concavity and submodularity, and R^2 values as measures of approximation goodness of fit. For parameters in which related results are available for both s-curve models and schedule delay models, columns are split for parameter values, coefficients and R^2 values, with s-curve values on the left and schedule delay values on the right, indicated by a heading of SC and SD respectively. Parameters not being varied are held at defaults in **Table 2/3**, except for parameters of mixed nonstop-one stop market network payoffs, which are held at default values given in **Section 4.2.**

IIIA. 1 Player Markets

N		γ_0		γ_1		γ_2		**R²**	
SC	SD	SC	SD	SC	SD	SC	SD	SC	SD
0.1	**0.001**	115910	45865	73791	21371	-2474	-1031	0.94	0.77
0.2	**0.002**	96026	38244	58832	15795	-1947	-831	0.95	0.8
0.3	**0.003**	83375	33385	50674	12760	-1662	-722	0.95	0.83
0.4	**0.004**	73912	29733	45192	10724	-1472	-650	0.96	0.86
0.5	**0.005**	66372	26810	41130	9218	-1333	-597	0.96	0.89
0.6	**0.006**	60078	24366	37944	8038	-1224	-555	0.96	0.91
0.7	**0.007**	54657	22258	35350	7078	-1137	-522	0.97	0.92
0.8	**0.008**	498823	20399	33180	6277	-1064	-494	0.97	0.94
0.9	**0.009**	456378	18734	31324	5593	-1002	-470	0.97	0.95
1	**0.01**	41852	17233	29709	5000	-949	-449	0.97	0.95

Table A2: N varied

α	**φ**	γ_0		γ_1		γ_2		**R²**	
SC	SD	SC	SD	SC	SD	SC	SD	SC	SD
1	**2**	76055	72499	27112	13610	-979	-834	0.92	0.80
1.1	**3**	73019	57985	31773	12024	-1096	-750	0.94	0.82
1.2	**4**	69656	43250	36620	10592	-1218	-673	0.95	0.85
1.3	**5**	659923	28308	41640	9334	-1346	-604	0.96	0.89
1.4	**6**	62053	13306	46822	8264	-1478	-543	0.97	0.92
1.5	**7**	57858	-325	52156	7164	-16145	-484	0.97	0.95
1.6	**8**	53419	-9554	57631	5581	-1756	-411	0.98	0.98
1.7	**9**	48708	-14370	63240	3573	-1901	-325	0.98	0.99
1.8	**10**	43741	-16057	68975	1383	-2051	-238	0.98	0.99

Table A.3: α varied for s-curve and ϕ varied for schedule delay market share formulations

β		γ_0		γ_1		γ_2		**R²**	
SC	SD	SC	SD	SC	SD	SC	SD	SC	SD
-0.001	**-0.008**	428238	40214	198072	18827	-896	-7165	0.99	0.79
-0.002	**-0.009**	214119	35746	94036	15624	-796	-3582	0.99	0.8
-0.003	**-0.01**	142746	32172	59357	13061	-716	-2388	0.98	0.83
-0.004	**-0.011**	107060	29247	42018	10965	-651	-1791	0.98	0.86
-0.005	**-0.012**	85648	26810	31615	9218	-597	-1433	0.97	0.89
-0.006	**-0.013**	71373	24747	246789	7739	-551	-1194	0.95	0.91
-0.007	**-0.014**	61177	22980	19725	6472	-512	-1024	0.94	0.93
-0.008	**-0.015**	53530	21448	16009	5374	-478	-896	0.91	0.94
-0.009	**-0.016**	47582	20107	13119	4413	-448	-796	0.88	0.96
-0.01	**-0.017**	42824	18924	10807	3565	-421	-716	0.87	0.96

Table A4: β varied

$Seats$	γ_0		γ_1		γ_2		R^2	
SC/SD	SC	SD	SC	SD	SC	SD	SC	SD
250	93653	35106	36515	7861	-1159	-547	0.97	.90
225	92282	35053	36767	7871	-1169	-547	0.97	.90
200	89852	34615	37202	7948	-1186	-550	0.97	.90
175	85564	33481	37957	8144	-1215	-558	0.96	.90
150	78456	31137	39169	8534	-1261	-572	0.96	.89
125	66372	26810	41130	9218	-1333	-597	0.97	.89
100	45878	19186	44145	10306	-1436	-634	0.97	.90
75	12697	6741	47810	11641	-1533	-668	0.98	.93
50	-24046	-7769	45542	11067	-1281	-586	1.00	.99
25	-6874	-3359	18537	1393	-138	-144	1.00	1.00

Table A5: Seats per flight varied

r	γ_0	γ_1	γ_2	R^2
0.1	25330	-5778	-119	1.00
0.2	25260	-1286	-247	0.99
0.3	25466	3167	-387	0.97
0.4	26194	7200	-524	0.92
0.5	27408	10674	-651	0.87
0.6	29103	13565	-763	0.83
0.7	31142	15917	-859	0.80
0.8	33437	17797	-940	0.78
0.9	35900	19276	-1006	0.77
1	38451	20423	-1060	0.75

Table A6: r varied with schedule delay market share formulation

IIIB. 2 Player Markets

N		γ_3		γ_5		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
0.1	**0.001**	-515	-274	-589	-143	0.87	0.91
0.2	**0.002**	-526	-279	-564	-132	0.91	0.94
0.3	**0.003**	-516	-278	-547	-123	0.93	0.96
0.4	**0.004**	-507	-272	-530	-122	0.94	0.97
0.5	**0.005**	-496	-268	-514	-120	0.95	0.97
0.6	**0.006**	-483	-263	-503	-117	0.96	0.98
0.7	**0.007**	-471	-258	-492	-113	0.96	0.98
0.8	**0.008**	-460	-254	-480	-110	0.96	0.98
0.9	**0.009**	-449	-249	-469	-106	0.96	0.98
1	**0.01**	-438	-244	-458	-103	0.97	0.98

Table A7: Payoff coefficients relevant for concavity (γ_3) and submodularity (γ_5) with N varied

α	φ	γ_3		γ_5		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
1	**2**	-425	-180	-299	-72	0.94	0.93
1.1	**3**	-452	-215	-367	-77	0.94	0.94
1.2	**4**	-477	-244	-442	-91	0.95	0.96
1.3	**5**	-497	-266	-523	-116	0.95	0.97
1.4	**6**	-514	-275	-612	-147	0.95	0.97
1.5	**7**	-527	-261	-706	-164	0.95	0.98
1.6	**8**	-537	-230	-804	-171	0.95	0.99
1.7	**9**	-545	-187	-906	-170	0.95	0.99

1.8	**10**	-550	-139	-1011	-162	0.95	0.99

Table A8: α varied

$\boldsymbol{\beta}$		γ_3		γ_5		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
-0.001	**-0.008**	-2230	-402	-2313	-2313	0.97	0.93
-0.002	**-0.009**	-1115	-357	-1157	-1157	0.97	0.95
-0.003	**-0.01**	-743	-322	-771	-771	0.96	0.96
-0.004	**-0.011**	-557	-292	-578	-578	0.95	0.96
-0.005	**-0.012**	-446	-268	-463	-463	0.95	0.97
-0.006	**-0.013**	-372	-247	-386	-386	0.95	0.98
-0.007	**-0.014**	-319	-230	-330	-330	0.95	0.98
-0.008	**-0.015**	-279	-214	-289	-289	0.96	0.98
-0.009	**-0.016**	-248	-201	-257	-257	0.97	0.99
-0.01	**-0.017**	-223	-189	-240	-85	0.97	0.99

Table A9: β varied

$\boldsymbol{Seats}$	γ_3		γ_5		R^2	
SC/SD	SC	SD	SC	SD	SC	SD
250	-494	-270	-533	-124	0.96	0.97
225	-494	-270	-533	-124	0.96	0.97
200	-494	-270	-532	-124	0.96	0.97
175	-493	-270	-532	-124	0.96	0.97
150	-492	-269	-529	-123	0.96	0.97
125	-491	-268	-523	-120	0.95	0.97
100	-496	-272	-514	-118	0.94	0.97
75	-513	-291	-518	-147	0.93	0.96
50	-567	-354	-579	-246	0.93	0.96
25	-744	-247	-847	-242	0.97	0.98

Table A10: Seats per flight varied

$\boldsymbol{Seats}$, *Player 1*	$\boldsymbol{Seats}$, *Player 2*	γ_3		γ_5		R^2	
SC/SD	**SC/SD**	SC	SD	SC	SD	SC	SD
225	**25**	-833	-378	-852	-226	0.98	0.97
200	**50**	-563	-294	-565	-146	0.96	0.98
175	**75**	-509	-275	-491	-116	0.95	0.97
150	**100**	-497	-268	-493	-110	0.94	0.97
125	**125**	-496	-268	-514	-120	0.95	0.97
100	**150**	-503	-271	-546	-129	0.95	0.97
75	**175**	-528	-279	-585	-146	0.94	0.97
50	**200**	-587	-302	-684	-180	0.94	0.97
25	**225**	-422	-234	-619	-172	0.98	0.99

Table A11: Seats per flight varied and asymmetric between players

$\boldsymbol{r}$	γ_3	γ_5	R^2
0.1	-76	-14	1.00
0.2	-142	-45	1.00
0.3	-199	-80	0.99
0.4	-246	-109	0.98
0.5	-283	-126	0.96
0.6	-311	-132	0.95

0.7	-332	-131	0.93
0.8	-348	-128	0.92
0.9	-358	-123	0.91
1	-365	-116	0.90

Table A12: r varied with schedule delay market share formulation

Nonstop M, Segment 1	Nonstop M, Segment 2	One stop M	γ_3/γ_6	γ_{10}/γ_{11}	R^2
900	**900**	**100**	-408	-422	0.97
800	**800**	**200**	-402	-384	0.97
700	**700**	**300**	-396	-346	0.97
600	**600**	**400**	-390	-308	0.96
500	**500**	**500**	-384	-270	0.96
400	**400**	**600**	-376	-233	0.95
300	**300**	**700**	-369	-197	0.94
900	**900**	**100**	-408	-422	0.97
800	**800**	**200**	-402	-384	0.97

Table A13: Nonstop-one stop network with market sizes varied, s-curve market share formulation

α_{min}	γ_3/γ_6	γ_{10}/γ_{11}	R^2
0.4	-342	-327	0.97
0.5	-356	-331	0.97
0.6	-371	-336	0.97
0.7	-385	-341	0.97
0.8	-399	-347	0.97
0.9	-412	-354	0.97
1	-425	-362	0.96
1.1	-437	-370	0.96
1.2	-448	-379	0.95

Table A14: Nonstop-one stop network with α_{min} varied, s-curve market share formulation

α_{max}	γ_3/γ_6	γ_{10}/γ_{11}	R^2
0	-399	-337	0.96
0.1	-400	-340	0.96
0.2	-399	-342	0.96
0.3	-397	-345	0.97
0.4	-393	-348	0.97
0.5	-389	-351	0.97
0.6	-385	-355	0.97
0.7	-379	-358	0.97

Table A15: Nonstop-one stop network with α_{max} varied, s-curve market share formulation

α	γ_3/γ_6	γ_{10}/γ_{11}	R^2
0.8	-313	-135	0.97
0.9	-334	-168	0.97
1	-353	-207	0.97
1.1	-370	-251	0.97
1.2	-384	-299	0.97
1.3	-397	-351	0.97
1.4	-407	-407	0.97

1.5	-415	-465	0.97
1.6	-421	-526	0.97
1.7	-424	-589	0.96

Table A16: Nonstop-one stop network with α varied, s-curve market share formulation

β	γ_3/γ_6	γ_{10}/γ_{11}	R^2
-0.001	-1781	-1557	0.98
-0.002	-886	-776	0.98
-0.003	-594	-519	0.97
-0.004	-443	-388	0.97
-0.005	-355	-311	0.97
-0.006	-295	-259	0.97
-0.007	-253	-222	0.98
-0.008	-222	-194	0.98
-0.009	-197	-173	0.98
-0.01	-177	-155	0.98

Table A17: Nonstop-one stop network with β varied, s-curve market share formulation

$Seats$	γ_3/γ_6	γ_{10}/γ_{11}	R^2
110	-400	-329	0.95
120	-394	-327	0.96
130	-391	-332	0.96
140	-391	-339	0.97
150	-396	-346	0.97
160	-397	-349	0.97
170	-400	-353	0.96
180	-402	-356	0.96
190	-402	-356	0.96
200	-402	-356	0.96

Table A18: Nonstop-one stop with seats per flight varied, s-curve market share formulation

N	γ_3/γ_6	γ_{10}/γ_{11}	R^2
0.3	-386	-343	0.96
0.4	-393	-347	0.97
0.5	-396	-346	0.97
0.6	-392	-340	0.97
0.7	-389	-333	0.97
0.8	-384	-327	0.97
0.9	-378	-320	0.97
1	-372	-312	0.97

Table A19: Nonstop-one stop network with N varied, s-curve market share formulation

IIIC. 3 Player Markets

N		γ_4		γ_7		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
0.1	**0.001**	-253	-167	-252	-58	0.94	0.98
0.2	**0.002**	-253	-167	-259	-60	0.95	0.99
0.3	**0.003**	-250	-165	-261	-61	0.96	0.99

0.4	**0.004**	-246	-163	-261	-61	0.96	0.99
0.5	**0.005**	-242	-161	-260	-60	0.96	0.99
0.6	**0.006**	-238	-159	-258	-59	0.96	0.99
0.7	**0.007**	-233	-157	-255	-59	0.97	0.99
0.8	**0.008**	-229	-155	-252	-58	0.97	0.99
0.9	**0.009**	-224	-153	-249	-57	0.97	0.99
1	**0.01**	-220	-150	-246	-56	0.97	0.99

Table A20: N varied

α	**φ**	γ_4		γ_7		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
1	**2**	-237	-98	-160	-6	0.97	0.99
1.1	**3**	-242	-128	-193	-20	0.97	0.99
1.2	**4**	-243	-151	-227	-39	0.97	0.99
1.3	**5**	-242	-161	-263	-58	0.96	0.99
1.4	**6**	-237	-160	-300	-75	0.96	0.99
1.5	**7**	-231	-150	-338	-89	0.96	0.99
1.6	**8**	-222	-132	-377	-98	0.95	0.99
1.7	**9**	-211	-108	-416	-104	0.95	1.00
1.8	**10**	-198	-79	-456	-104	0.94	1.00

Table A21: α varied

β		γ_4		γ_7		R^2	
SC	**SD**	SC	SD	SC	SD	SC	SD
-0.001	**-0.008**	-1089	-242	-1168	-90	0.98	0.98
-0.002	**-0.009**	-544	-215	-584	-80	0.97	0.98
-0.003	**-0.010**	-363	-193	-389	-72	0.96	0.99
-0.004	**-0.011**	-272	-176	-292	-66	0.96	0.99
-0.005	**-0.012**	-218	-161	-234	-60	0.97	0.99
-0.006	**-0.013**	-181	-149	-195	-56	0.97	0.99
-0.007	**-0.014**	-155	-138	-167	-52	0.98	0.99
-0.008	**-0.015**	-136	-129	-146	-48	0.98	0.99
-0.009	**-0.016**	-121	-121	-130	-45	0.99	1.00
-0.01	**-0.017**	-109	-114	-117	-43	0.99	1.00

Table A22: β varied

$Seats$	γ_4		γ_7		R^2	
SC/SD	SC	SD	SC	SD	SC	SD
250	-243	-161	-261	-61	0.96	0.99
225	-243	-161	-261	-61	0.96	0.99
200	-243	-161	-261	-61	0.96	0.99
175	-243	-161	-261	-61	0.96	0.99
150	-243	-161	-261	-61	0.96	0.99
125	-242	-161	-260	-60	0.96	0.99
100	-240	-160	-255	-59	0.96	0.99
75	-240	-160	-243	-55	0.95	0.99
50	-274	-172	-241	-61	0.91	0.98
25	-614	-278	-764	-236	0.90	0.97

Table A23: Seats per flight varied

r	γ_4	γ_7	R^2
0.1	-55	-8	1.00
0.2	-95	-27	1.00
0.3	-126	-45	1.00
0.4	-150	-57	0.99
0.5	-169	-62	0.99
0.6	-185	-63	0.99
0.7	-198	-60	0.98
0.8	-208	-57	0.98
0.9	-216	-52	0.97
1	-221	-47	0.97

Table A.24: r varied with schedule delay market share formulation

IIID. 4 Player Markets

α	Squared player 1 frequency coefficient	Player 1 frequency interaction coefficient	R^2
1	-335	-228	0.94
1.2	-301	-313	0.94
1.4	-246	-402	0.95
1.6	-174	-493	0.94
1.8	-91	-584	0.94

Table A.25: α varied with the s-curve market share formulation and market size of 2000