Swarms on Sphere: A Programmable Swarm with Synchronous Behaviors like Oscillator Networks

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Abstract— In this paper, we introduce a programmable particle swarm evolving on a sphere called *swarm on sphere* system. This model is a polynomial dynamical system obtained from a consensus algorithm that exhibits a rich set of synchronous behaviors with close connections to the Kuramoto model of coupled oscillators, coalition formation in social networks, small-words, integer programming problems such as max-cut, and networks of self-synchronous oscillators with applications to synthetic biology. We prove the generalized Kuramoto model can be obtained as a special case of this model in dimension two. Moreover, we provide formal stability analysis of aligned, bipolar, and dispersed synchronous modes of the system. As a byproduct of this stability analysis, we obtain simple algorithms for programming the weights of the swarm enabling it to exhibit various desired patterns of synchrony. Simulation results are provided that demonstrate 3-D in-phase synchrony, coalition formation for interacting agents with mixed-sign couplings, and dispersal behavior with spatial order.

Index Terms— swarms, synchronization of oscillators, biologically-inspired systems, consensus algorithms, coalition formation, small-worlds, max-cut problem

I. INTRODUCTION

In this paper, we introduce a particle system that evolves on a sphere called *swarms on sphere* that exhibits broad range of self-organizing synchronous behaviors. The swarm on sphere (SOS) model is a network induced polynomial dynamical system with close connections to a variety of issues of interest in complex networked systems including continuous-time consensus problems [11], [15], [10], synchronization of networks of oscillators and the Kuramoto model [4], [18], flocking [20], [5], [9], coalition formation in social networks, combinatorial optimization problems such as the max-cut problem, positive polynomials and polynomial dynamical systems [13], programmable synchronous oscillators with connections to synthetic biology [2], and small-world networks [22].

The primary focus of most researchers is the study of in-phase synchronous behavior in networks of oscillators [17], [12], [6], [14]. Our main focus is on other modes of synchronous behavior that provide the capability of modeling coalition formation in social networks and biological synchrony that arise from interacting agents with *mixed-sign couplings*. The main contribution of this paper is to introduce a programmable self-organizing system in the form of a particle swarm that is capable of exhibiting a rich set of synchronous behaviors that are frequently encountered in complex engineering, biological, and social systems. In particular, three types of behaviors are analyzed: namely,

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alignment, coalition formation, and dispersal with or without spatial symmetry. As a byproduct of our stability analysis, we obtain simple algorithms for tuning the weights of the swarm that enables it to exhibit the aforementioned three modes of synchrony.

The closest counterparts of this *programmable swarm* with alternative modeling and conceptual frameworks are biologically-inspired *programmable self-assembly* [7] and *swarm intelligence* [1].

Here is an outline of the paper: The model of swarms on sphere and its synchronous modes of behavior are presented in Section II. The stability analysis of various modes and simple algorithms for programming the swarm are given in Section III. The max-cut problem and its connection to dispersal behavior of swarms on sphere is discussed in Section IV. Experimental results are given in Section V. Finally, the concluding remarks are stated in Section VI.

II. MODEL OF SWARMS ON SPHERE

Consider n agents (or particles) $x_i \in \mathbb{R}^m$ moving on a unit sphere in dimension $m \geq 2$ with interactions specified by a weighted graph $G = (V, E)$ of order n with adjacency matrix $A = [a_{ij}]$. The interaction couplings a_{ij} might be *non-negative*, *mixed-sign*, or *non-positive*. The set of *neighbors* of node i are denoted by $N_i = \{j \in V : a_{ij} \neq 0\}$. The following graph-induced polynomial dynamical system

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - \langle x_j, x_i \rangle x_i), \quad x_i \in \mathbb{R}^m \qquad (1)$$

will be called a *swarm on sphere* in dimension m (the notation $\langle \cdot, \cdot \rangle$ denotes the inner product of vectors in $\mathbb{R}^m$).

We shall see that the vector x_i represents opinion, belief, decision, phase, or velocity of agent i in various applications—thus, unlike flocking, there is nothing wrong with all particles coinciding or inter-agent collisions. Due to the identity

$$\frac{d}{dt}\langle x_i, x_i \rangle = 2\langle x_i, u_i \rangle = 0 \qquad (2)$$

the norm of $x_i(t)$ is preserved in time, and therefore, a swarm of n particles staring on an m-dimensional sphere with dynamics (1) evolves on the sphere for all time $t \geq 0$.

The model in (1) is the collective dynamics of a group of agents with *integrator on sphere* dynamics

$$\dot{x}_i = u_i - \langle u_i, x_i \rangle, \quad \|x_i(0)\| = 1 \qquad (3)$$

applying the following *consensus algorithm* [11]

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i), \quad x_i \in \mathbb{R}^m \qquad (4)$$

which can be expressed as $u = -Lx$ in terms of *graph Laplacian* $L = [l_{ij}] = D - A$ with elements

$$l_{ij} = \begin{cases} \sum_{j \neq i} a_{ij}, & i \neq j \\ -a_{ij}, & i = j. \end{cases} \quad (5)$$

One can readily show that the *collective dynamics of the integrator agents on sphere in* (3) *applying a consensus algorithm is system* (1). In other words, the swarm on sphere system consists of a group of integrator agents on sphere applying a consensus algorithm with a matrix of sign-indefinite couplings A. The importance of mixed-sign and non-positive couplings becomes clear in coalition formation and max-cut problems.

Remark 1. Unlike the linear consensus dynamics $\dot{x} = -Lx$ that becomes unstable for the case of non-positive couplings, the state of swarms on sphere always evolves on the sphere and remains bounded regardless of the sign of the weights. This why the particle system is chosen to evolve on a sphere.

A. Networked Oscillators and Swarms on Sphere

The generalized Kuramoto model (GKM) [10] of n networked oscillators is in the form

$$\dot{\theta}_i = \omega_i + \sum_{j \in N_i} a_{ij} \sin(\theta_j - \theta_i) \quad (6)$$

where θ_i and ω_i, respectively, denote the phase and frequency of the ith oscillator in a network with topology G and coupling matrix A. The special case of this model with a coupling matrix $A = (\kappa/n)J_n$ ($\kappa > 0$ and J_n is an $n \times n$ matrix of ones) is the celebrated *Kuramoto model* given by

$$\dot{\theta}_i = \omega_i + \frac{\kappa}{n} \sum_{j=1}^{n} \sin(\theta_j - \theta_i) \quad (7)$$

for n weakly connected phase oscillators [21].

The following theorem establishes the most direct connection between the consensus algorithm in (4) and the nonlinear model of n networked oscillators in (6) and (7).

Proposition 1. *The dynamics of particle swarms on a sphere in dimension* 2 *can be reduced to the generalized Kuramoto model of* n *oscillators with identical frequencies.*

Proof: A unit vector x_i on a circle can be represented by its phase θ_i as $x_i = (\cos(\theta_i), \sin(\theta_i))^T$. Let $x_i^{\perp} = \Omega x_i$ with

$$\Omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (8)$$

denote the orthogonal unit vector to x_i. We have

$$\dot{x}_i = \dot{\theta}_i x_i^{\perp} \implies \dot{\theta}_i = \langle x_i^{\perp}, \dot{x}_i \rangle = \sum_{j \in N_i} a_{ij} \langle x_j, x_i^{\perp} \rangle.$$

Expanding the right hand side of the last equation gives

$$\langle x_j, x_i^{\perp} \rangle = \sin(\theta_j)\cos(\theta_i) - \cos(\theta_j)\sin(\theta_i) = \sin(\theta_j - \theta_i),$$

and we obtain

$$\dot{\theta}_i = \sum_{j \in N_i} a_{ij} \sin(\theta_j - \theta_i) \quad (9)$$

that is equivalent to (6) after replacing θ_i by $\theta_i - \omega_i$ (note that $\omega_1 = \cdots = \omega_n$). $\qquad \square$

B. Synchronous Modes of Behavior and Measure of Order

System (1) exhibits a number of *synchronous modes* of behavior that are important classes of the equilibria of the system. But, we need to define a measure of order that quantifies the degree of behavioral synchrony in swarms.

Inspired by statistical physics, we define the (global) *order* ρ as the norm of the average position of n particles on a unit sphere, i.e. $\rho = \|\mu\|$ with $\mu = \frac{1}{n}\sum_i x_i$. Due to convexity of a sphere, μ stays inside a closed unit ball and therefore $0 \leq \rho \leq 1$. Given a partition $\{V_1, V_2\}$ of two *coalition* of nodes in V, one can define two centers $\mu_k = \frac{1}{|V_k|}\sum_{i \in V_k} x_i$ with $k = 1, 2$ and a *collective order* $\rho_c = \frac{1}{n}\sum_k n_k\|\mu_k\|$ with $n_k = |V_k|$ that quantifies average degree of synchrony across the coalitions. Here are some important equilibria of the swarm on sphere model:

i) *Aligned state:* This equilibrium corresponds to reaching a consensus and is in the form

$$x_1 = x_2 = \cdots = x_n \quad (10)$$

This is also known as the *in-phase cohesive state* [21] of the GKM associated with $\rho = 1$, or the maximum order.

ii) *Symmetric dispersed state:* This equilibrium is a *dispersal mode* of the system with the property $\sum_i x_i = 0$ that corresponds to $\rho = 0$ (zero order) with a spatial symmetry. This *spatial symmetry* in dimension m can be defined as follows. Let $S_i = \text{argmin}_{j \neq i}\|x_j - x_i\|$ denote the set of nearest-neighbors of agent i. Then, there exists $d > 0$ so that $\|x_j - x_i\| = d$ for all nodes and their nearest-neighbors j. This spatial symmetry is the same as the symmetry that exists in alpha-lattices [9] as the geometric model of flocks [9].

iii) *Asymmetric dispersed state:* This equilibria is a dispersal mode with $\mu = 0$ that does not satisfy the spatial symmetry condition.

iv) *Bipolar state:* This mode has rarely been formally studied in the literature and corresponds to two sets of synchronous oscillators that oscillate in opposite phase of each other. This equilibrium can be described as follows: there exists a unit vector w and a partition $\{V_1, V_2\}$ of the set of nodes V into n_1 and n_2 nodes such that $n_1 \geq n_2$ with $n_1 + n_2 = n$ and

$$\begin{aligned} x_1 = \cdots = x_{n_1} &= w \\ x_{n_1+1} = \cdots = x_{n_1+n_2} &= -w \end{aligned} \quad (11)$$

One can explicitly calculate the order parameter as

$$\rho = \frac{n_1 - n_2}{n_1 + n_2} \geq 0. \quad (12)$$

This mode is called an (n_1, n_2) *bipolar state* at it has an import role in the study of coalition formation in social networks with mixed-sign couplings. Clearly, the global order depends on n_1, n_2 and is less than 1, but the collective order ρ_c is 1.

III. Stability Analysis of Swarms on Sphere

A formal analysis of global stability of the aligned state of the Kuramoto model was introduced in [21] and is known as Watanabe-Strogatz theorem [17]. The proof is rather lengthy and requires application of a singular change of coordinates. Here, we present a novel proof of this theorem. In addition, we provide stability analysis of the bipolar and dispersed equilibria of the swarms on sphere system. As a byproduct, we obtain algorithms that allow programming the swarm by tuning its weights that enable the swarm to exhibit a desired pattern of self-synchrony and to solve a combinatorial optimization problem.

A. Stability of Aligned State

Defining n centers as $\mu_i = \sum_j a_{ij} x_j$, the dynamics of the swarm can be rewritten as

$$\dot{x}_i = \mu_i - \langle \mu_i, x_i \rangle x_i. \tag{13}$$

The last equation in words can be stated as follows: *each agent moves towards the projected average position of its neighbors on a sphere*. This rule is almost identical to the flock centering rule of Reynolds [16] and its formal statement in [9]. The algorithm in (13) with non-negative couplings can be viewed as a rendezvous algorithm for particles on a sphere.

One can verify that only for complete graphs with identical weights, all centers are the same. In this case, the dynamics of the swarm is greatly simplified and can be expressed as

$$\dot{x}_i = \mu - \langle \mu, x_i \rangle x_i \tag{14}$$

Here is our version of global stability of the aligned state (i.e. the Watanabe-Strogatz theorem):

Proposition 2. *Consider system* (1) *with all-to-all positive couplings* $A = 1/n J_n$. *Let* U_0 *denote the set of initial states on a circle that are non-bipolar and have a non-zero order. Then, the aligned state is globally asymptotically stable for all initial conditions in* U_0.

Proof: See the Appendix. □

B. Stability of Bipolar State

Consider an (n_1, n_2) bipolar state and (without loss of generality) assume $V_1 = \{1, 2 \ldots, n_1\}$ and $V_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$. The *polarity* of node i is defined as

$$s_i = \begin{cases} +1, & i \in V_1 \\ -1, & i \in V_2 \end{cases} \tag{15}$$

and the polarity vector for the entire network is denoted by $s = (s_1, s_2, \ldots, s_n)^T \in \{-1, 1\}^n$. Let us focus on a graph with a *polar adjacency* matrix $P = [p_{ij}]$ with elements

$$p_{ij} = \begin{cases} +1, & s_i s_j = +1 \\ -1, & s_i s_j = -1 \end{cases} \tag{16}$$

This matrix is closely related to the *linearization* of the generalized Kuramoto model (9) with 0-1 weights that takes the following form

$$\dot{\theta} = -L_p \theta \tag{17}$$

where $L_p = \text{diag}(P \cdot \mathbf{1}) - P$ is the *Laplacian of the polar adjacency* matrix P. Apparently, L_p is the Laplacian of a graph with positive and negative weights and the system in (17) is, in general, unstable. We establish this fact in the following.

For a complete graph, a polar adjacency matrix is in the form $P = ss^T$ that can be expanded as

$$P = \begin{bmatrix} J_{n_1} & -J_{n_1, n_2} \\ -J_{n_2, n_1} & J_{n_2} \end{bmatrix}$$

where $J_{k,l}$ is a $k \times l$ matrix of ones. The Laplacian L_p of P takes the following form

$$L_p = (n_1 - n_2)\text{diag}(s) - ss^T.$$

Based on this property, the spectral characterization of Laplacian of polar matrices is provided in the following proposition.

Proposition 3. *(polar Laplacian spectrum) Let* L_p *be the Laplacian of a graph with a polar adjacency matrix* $P = ss^T$ *where* $s \in \{-1, 1\}^n$ *is a polarity vector with* n_1 *positives ones and* n_2 *negative ones. Then,* L_p *has eigenvalues* $\lambda_1 = 0$, $\lambda_2 = n$, $\lambda_3 = n_1 - n_2$ *with multiplicity* $n_1 - 1$, *and* $\lambda_4 = -(n_1 - n_2)$ *with multiplicity* $n_2 - 1$.

Proof: See the Appendix. □

For the special case with $n = n_1$ and $n_2 = 0$, we obtain the well-known result that the Laplacian of a complete graph has a zero eigenvalue and $\lambda_k = n$ for $k = 2, \ldots, n$.

Corollary 1. *Every* (n_1, n_2) *bipolar state is a saddle point (or unstable equilibrium) of the linearization of the Kuramoto model with identical positive weights.*

Proof: For any (n_1, n_2) bipolar state with $n_1 > n_2$, the matrix $-L_p$ has at least one positive eigenvalue at $\lambda = n_1 - n_2$ and at least one negative eigenvalue at $\lambda = -(n_1 - n_2)$. Thus, any bipolar state with $n_2 > 0$ corresponds to a saddle point of the linearization of (9). □

Here is the main question regarding programmability of oscillator networks: *How do we design the weights of a network so that a bipolar state becomes an asymptotically stable equilibrium of the GKM?* We have already shown that the standard Kuramoto model, or the generalized model with non-negative weights do not solve this problem. It turns out that the desired weights have to be necessarily a mix of positive and negative weights.

Let us consider the linearization of the GKM model in (9). Define the adjacency matrix $Q = [q_{ij}]$ with elements

$$q_{ij} = a_{ij} \cos(\theta_j - \theta_i),$$

and let L_q denote the Laplacian of Q. Then, the linearization of (9) around an (n_1, n_2) bipolar state θ^* is in the form of a consensus dynamics

$$\dot{\tilde{\theta}} = -L_q \tilde{\theta} \tag{18}$$

where $\tilde{\theta} = \theta - \theta^*$ and the elements of θ^* are given by

$$\theta_i^* = \begin{cases} 0, & i \in V_1; \\ \pi, & i \in V_2. \end{cases} \tag{19}$$

Furthermore, we have $\cos(\theta_j^* - \theta_i^*) = s_i s_j$ where s is the polarity vector of the nodes. From consensus theory, the stability of system (18) can be guaranteed if the weight matrix Q is a non-negative adjacency matrix. This can be achieved by setting the weights to

$$a_{ij} = \cos(\theta_j^* - \theta_i^*) = s_i s_j, \qquad (20)$$

or $A = ss^T$. In other words, a network of oscillators with a polar adjacency matrix ss^T exhibits a (locally) *stable bipolar synchronous behavior* associated with the polarity vector s, i.e. the nodes with similar polarity are in-phase and the nodes with opposite polarities are out-of-phase. Here is our main stability result for the bipolar states:

Proposition 4. *Let $U_0 = \{x \in S_1^n : 0 < \rho < 1\}$ be the set of initial conditions of a swarm on a circle with an interaction coupling matrix $A = \frac{1}{n}ss^T$ associated with a polarity vector $s \in \{-1, 1\}^n$. Then, any bipolar mode of type s is locally asymptotically stable for (1).*

Proof: For any arbitrary (n_1, n_2) bipolar mode with polarity vector s, $a_{ij} = \frac{1}{n}s_i s_j$ and

$$q_{ij} = a_{ij}s_i s_j = \frac{1}{n}s_i^2 s_j^2 = \frac{1}{n},$$

or $Q = \frac{1}{n}J_n$. Hence, L_q is a positive semidefinite matrix with one isolated zero eigenvalue, i.e. $\tilde{\theta} = 0$ is a stable equilibrium of system (18) for any arbitrary polarity vector s. $\qquad \square$

Similar to the case of the aligned state in Proposition 2, the author believes that the above result holds globally based on what experimental results indicate.

Conjecture 1. *Given the assumptions of Proposition 4, any bipolar state of type s is globally asymptotically stable for a swarm on circle.*

C. Stability of Dispersed State

The symmetric dispersed state in dimension $m = 2$ can be simply defined using n phases $\theta_k = 2\pi k/n$ for $k = 1, \ldots, n$ corresponding to n evenly dispersed particles on a ring. Convergence to a dispersed mode usually occurs when all couplings are negative. The following result justifies why a network with all-negative couplings is useful for reaching disorder with $\rho = 0$.

Proposition 5. *Consider the swarm in (1) with the coupling matrix $A = -\kappa J_n$ and $\kappa > 0$. Let U_0 be the set of initial states of n particles on a sphere with an order less than 1 that are not bipolar states. Then, the order ρ is monotonically non-increasing. Moreover, the system can never asymptotically converge to a bipolar or aligned state in dimension two.*

Proof: The first part follows in a similar way along the the lines of the proof of Proposition 2. Defining $r = \rho^2$, we get (see the appendix)

$$\dot{r} = -2\kappa r \sum_i \sin^2(\psi_i) \le 0.$$

Thus, r is monotonically non-increasing. To show the second part, note that the linearization of the system around a bipolar state of type s has the following Laplacian matrix

$$L = -\kappa(n_1 - n_2)\text{diag}(s) + \kappa ss^T.$$

Based on Proposition 3, this matrix always has at least one negative eigenvalue. Therefore, no bipolar state can be asymptotically stable for this choice of the coupling matrix. Finally, in an aligned state $\rho = 1$. Since $\rho(t) < \rho(0) < 1$, for all $t > 0$, the system can never asymptotically converge to an aligned state either. $\qquad \square$

From the above proposition, the intuitive conclusion is that for initial states that are neither bipolar, nor aligned, the system globally converges to a dispersed mode with $\rho = 0$. The proof (or disproof) of this claim remains open.

Based on experimental observations, for large values of n, the state of system (1) with identical couplings $a_{ij} = -1/n$ generically converges to an asymmetric dispersed state. The following proposition provides a distributed algorithm so that the swarm can be programmed to converge to the symmetric dispersed state on a circle.
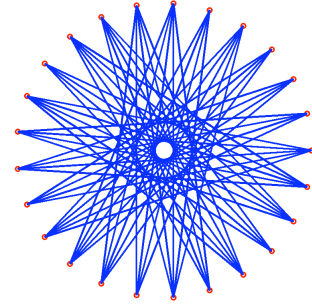

Fig. 1. Network for symmetric dispersal.

Proposition 6. *The symmetric dispersed state is rendered locally asymptotically stable for system (1) in dimension two using a regular network (Fig. 1) with at most $nk + \lfloor n/2 \rfloor$ links and non-positive couplings a_{ij} defined by*

$$a_{ij} = \begin{cases} \cos(\theta_j - \theta_i), & |\pi - |\theta_j - \theta_i|| < \delta; \\ 0, & \text{otherwise}; \end{cases} \qquad (21)$$

with $\delta = (2k + 1)\pi/n \le \pi/2$, $\theta_i = 2\pi i/n$ for $k \ge 1$ and $n \ge 6$ provided that the initial state of the system is neither aligned, nor bipolar (here, $\lfloor \cdot \rfloor$ denotes the floor function).

Proof: See the Appendix. $\qquad \square$

IV. SOLVING MAX-CUT

The max-cut problem has a long history in combinatorial optimization and semidefinite programming [3], [19]. Let $G = (V, E)$ be a graph with a matrix of sign-indefinite weights $W = [w_{ij}]$. A *cut* is a partition of the nodes of the graph into the sets S and $\bar{S} = V \setminus S$. The *cut value* is the sum of the weights of the edges that go between S and $\bar{S}$, i.e.

$$\Psi(S) = \sum_{i \in S, j \in \bar{S}} w_{ij}.$$

The *max-cut problem* is to find the cut $\{S, \bar{S}\}$ with the maximum cut value. The connections between max-cut and *Ising spin-glass models* in statistical physics are well-known.

Let $s \in \{-1, 1\}^n$ be the polarity vector of the nodes so that $s_i = 1, \forall i \in S$ and $s_i = -1, \forall i \in \bar{S}$. Then, the cut value is closely related to the *disagreement function* $\varphi(s) = \frac{1}{2} s^T L s$ [11] of a group of agents with communication topology G and decision/opinions s_i. This relationship is explained in the following

$$\varphi(s) = \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij}(s_j - s_i)^2, \qquad (22)$$

$$= \sum_{\{i,j\} \in E} w_{ij}(1 - s_i s_j), \qquad (23)$$

$$= \sum_{\{i,j\} \in E} w_{ij} + \sum_{i \in S, j \in \bar{S}} a_{ij}, \qquad (24)$$

$$= C_0 + \Psi(S). \qquad (25)$$

The first term is fixed and does not depend on the choice of S, but the second term is the cut value. Therefore, solving the max-cut is equivalent to finding a set of decisions s_i by the agents that maximizes the disagreement function $\varphi(s)$. Maximizing such a quadratic function is known as integer programming and is an important area of combinatorial optimization [19].

A heuristic approach to maximizing disagreement is to use the following *dispersal protocol*

$$u_i = \sum_{j \in N_i} -w_{ij}(x_j - x_i) \qquad (26)$$

and the set of agents with integrator on sphere dynamics

$$\dot{x}_i = u_i - \langle x_i, u_i \rangle x_i$$

The collective dynamics of this system is a swarm on sphere with the coupling matrix $A = -W$ with sign-indefinite elements.

After convergence to a dispersed state, a consensus is reached among the agents regarding a unit vector $\mathbf{c} \in \mathbb{R}^m$ that is then used to *decode* the decision of each agent using the following non-deterministic rule

$$s_i = \operatorname{sgn}_r(\mathbf{c}^T x_i) \qquad (27)$$

where $\operatorname{sgn}_r(z)$ is the *random sign function*. This function takes 1 for $z > 0$, -1 for $z < 0$, and either of these values with probability of $1/2$ for $z = 0$.

For $m = 2$, $\mathbf{c}$ can be chosen as one of the l unit vectors $(\cos(\frac{2\pi k}{l}), \sin(\frac{2\pi k}{l}))^T$ for $k = 0, \ldots, l$ that maximizes $\varphi(s)$. In higher dimensions, one can use the k-means clustering algorithm to create two coalitions and set $\mathbf{c}$ to the normalized center of one of clusters with highest order. A distributed version of this algorithm requires solving an average-consensus plus a max-consensus [11].

Remark 2. This swarm-based max-cut algorithm has been tested on several benchmark graphs for the max-cut problem and gives a satisfactory cut value that is close to the bounds found by semidefinite programming. These experimental results are extensive and need to be presented in a separate paper.

V. EXPERIMENTAL RESULTS

To integrate a particle system on a unit sphere with integrator dynamics $\dot{x}_i = u_i$, we can use the following approximation

$$x_i(k+1) = \frac{x_i(k) + \epsilon u_i(k)}{\|x_i(k) + \epsilon u_i(k)\|}, \ x_i(k) \in \mathbb{R}^m \qquad (28)$$

with an initial condition $\|x_i(0)\| = 1$ and a small step-size $\epsilon > 0$. Note that $\|x_i(k)\| = 1$ for all $k \geq 0$ and

$$\|x_i(k) + \epsilon u_i(k)\| = (\|x_i(k)\|^2 + \epsilon^2 \|u_i(k)\|^2)^{1/2} \geq 1$$

due to orthogonality of $x_i(k)$ and $u_i(k)$. Thus, the ratio in the discrete-time system (28) is always well-defined.

Fig. 2 demonstrates the convergence of a 3-D swarm of 100 agents to an *aligned state*. The network topology is a directed small-world network that consists of a n points on a ring where node i is linked to nodes $i+1$ and $i+2$ (mod n). Then, $l = 30$ shortcuts are randomly added to the initial directed lattice on a ring (See [22], [8]) for details). This small-world network has 230 links in comparison to ≈ 5000 links of a complete graph.
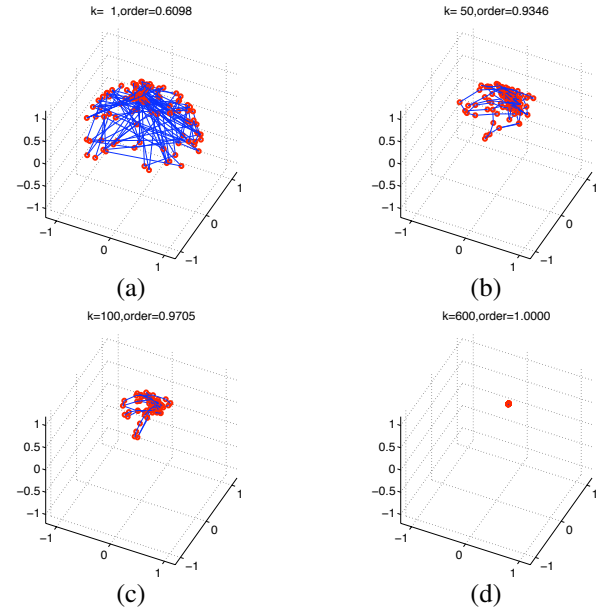

Fig. 2. Convergence to aligned state for a 3-D swarm of $n = 100$ agents.

Fig. 3 shows evolution of the state of a swarm with mixed couplings on a circle for $n = 100$ agents with the choice of $(n_1, n_2) = (55, 45)$. The nodes with positive polarity are red circles and the ones with negative polarity are green squares. Note that the limiting order satisfies our analytical prediction of $\rho = \frac{n_1 - n_2}{n - 1 + n_2} = 0.1$.

Finally, Fig. 4 shows the dispersal mode of 2-D swarms on sphere for two cases: 1) with non-positive couplings in equation (21) for $n = 100$ agents with a star-shaped network topology similar to Fig. 1 with $k = 8$ (i.e. $k = O(\log(n))$) and 2) with identical all-to-all weights of $-1/n$ (Fig. 4 (c) and (d)).
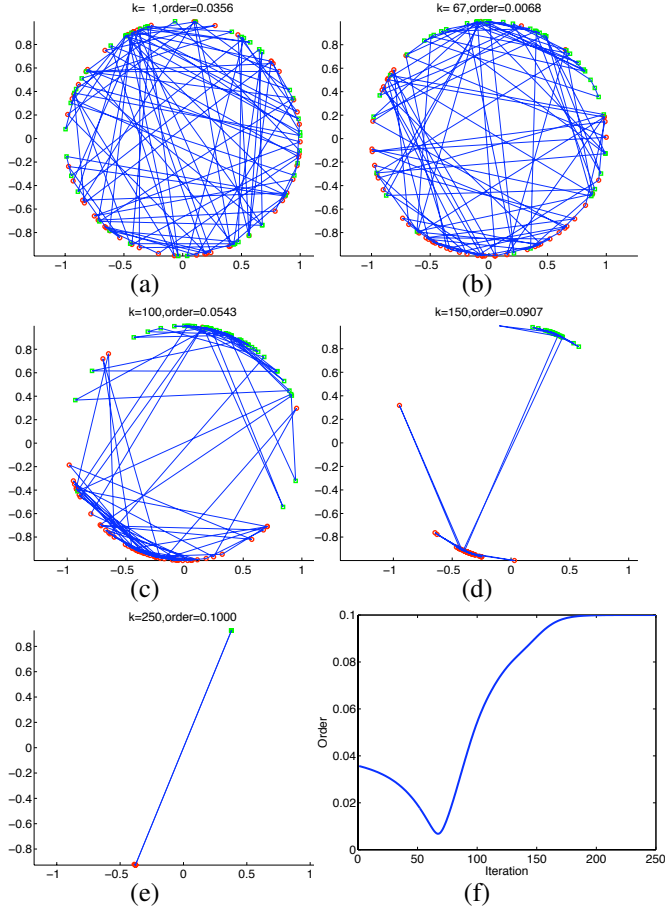
Fig. 3. Coalition formation for $n = 100$ agents: (a) through (e), evolution of the state of the swarm to a $(55, 45)$ bipolar state with order 0.1 and (f) evolution of order parameter ρ.

Fig. 4. Phase transition from order to disorder: (a),(b) symmetric dispersal and (c),(d) asymmetric dispersal.

VI. CONCLUSIONS

We introduced a programmable swarm that evolves on a sphere and is capable of exhibiting various forms of self-synchronous behavior including in-phase synchrony (or rendezvous), bipolar synchrony and coalition formation in social networks with mixed-sign couplings, and symmetric dispersal. This model is the collective dynamics of integrator agents on a sphere with nonlinear dynamics that apply a consensus algorithm. We showed that the Kuramoto model is a special case of swarms on sphere and provided a novel proof of Watanabe-Strogatz theorem. It was demonstrated that this system is capable of approximately solving the max-cut problem. Several simulation results were provided that are consistent with our theoretical predictions.

APPENDIX

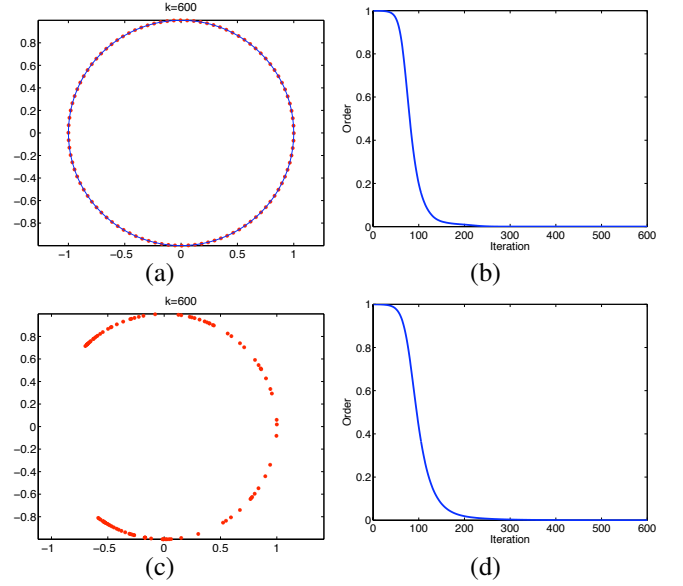*Proof:* (Prop. 2) Consider $r = 1 - \rho^2 \geq 0$ as a candidate Lyapunov function for system (14) that takes its minimum at $\rho = 1$ (i.e. aligned state). By direct differentiation, we get

$$
\begin{aligned}
\dot{r} &= -2\langle\dot{\mu}, \mu\rangle \\
&= -\frac{2}{n^2}\sum_{i,j}\langle\dot{x}_i, x_j\rangle \\
&= -\frac{2}{n^2}\sum_{i,j}\langle\mu - \langle\mu, x_i\rangle x_i, x_j\rangle \\
&= -\frac{2}{n}\sum_i\left\{\sum_j\langle\mu, \frac{1}{n}\sum_j x_j\rangle - \langle\mu, x_i\rangle\langle x_i, \frac{1}{n}\sum_j x_j\rangle\right\} \\
&= -\frac{2}{n}\sum_i(\|\mu\|^2 - \langle\mu, x_i\rangle^2) \\
&= -\frac{2\rho^2}{n}\sum_i\sin^2(\psi_i) \leq 0
\end{aligned}
$$

where ψ_i is the angle between x_i and μ, or $\cos(\psi_i) = \langle\mu, x_i\rangle$. Hence, r satisfies $\dot{r} = -2r/n\sum_i\sin^2(\psi_i)$. This implies that the order ρ is monotonically non-decreasing along the solutions of (14) if initially the order is non-zero (otherwise, a state with order zero is an equilibrium).

Based on LaSalle's invariance principle, we need to determine the largest invariant set Ω_0 satisfying $\dot{r} = 0$. For this set, $\sin(\psi_i) = 0$ for all i which implies either $\psi_i = 0$, or $\psi_i = \pi$ (up to $2k\pi$ for $k \in \mathbb{Z}$). In other words, this invariant set consists of the aligned state and all possible bipolar states. Hence, using r as a Lyapunov function is inconclusive, but it guarantees $\rho > 0$ (or $\mu \neq 0$) is monotonically non-decreasing for all initial states starting in U_0.

Based on this fact that μ never vanishes for all $t > 0$, one can construct a rotating frame of coordinates with orthogonal bases $e_1 = \mu/\rho$ and $e_2 = \mu^\perp/\rho$. In the rotating frame, μ is a fixed vector equal to $(\rho, 0)^T$. Let z denote the position of any arbitrary particle on the unit circle with a phase ψ with

respect to μ. Then, the following equation

$$\dot{z} = \mu - \langle \mu, z \rangle z \qquad (29)$$

describes the motion of the particle on the circle with the condition that $\psi(0) \neq (2k+1)\pi$ with $k \in \mathbb{Z}$. Note that $z = (\cos\psi, \sin\psi)^T$. Now, we show how the dynamics of this particle can be reduced to a stable scalar system. We have $\dot{z} = z^\perp \dot{\psi}$ and $z^\perp = (-\sin\psi, \cos\psi)^T$, thus (29) yields

$$z^\perp \dot{\psi} = (\rho - \rho\cos^2\psi, -\rho\cos\psi\sin\psi)^T. \qquad (30)$$

Multiplying both sides of the last equation by the unit vector $(z^\perp)^T$ gives a one-dimensional system

$$\dot{\psi} = -\rho\sin\psi \qquad (31)$$

For all $\psi(0) \in (-\pi, \pi)$, $\psi = 0$ is globally asymptotically stable for this scalar system with a Lyapunov function $W(\psi) = \frac{1}{2}\psi^2$ satisfying

$$\dot{W} = -\rho\psi\sin(\psi), \ \forall \psi \in (-\pi, \pi) \setminus \{0\}. \qquad (32)$$

Therefore, every particle asymptotically has a zero phase ψ_i and all particles coincide. $\qquad \square$

Proof: (Prop. 3) To prove this statement, we explicitly calculate the characteristic polynomial $p(\lambda) = \det(\lambda I_n - L_p)$ of L_p. This can be done using a determinant identity $\det(M + uv^T) = \det(M) + v^T\mathrm{adj}(M)u$ where u, v are n-vectors and M is a square matrix. Let

$$D(\lambda) = \mathrm{diag}((\lambda - (n_1 - n_2)s_1, \ldots, \lambda - (n_1 - n_2)s_n)),$$

we have

$$
\begin{aligned}
p(\lambda) &= \det(D(\lambda) - ss^T) \\
&= \det(D(\lambda)) - s^T\mathrm{adj}(D(\lambda))s \\
&= (\lambda - n_1 + n_2)^{n_1}(\lambda + n_1 - n_2)^{n_2} \\
&\quad - \sum_{i=1}^{n}\prod_{j\neq i}(\lambda - (n_1 - n_2)s_j) \\
&= (\lambda - n_1 + n_2)^{n_1}(\lambda + n_1 - n_2)^{n_2} \\
&\quad - (\lambda - n_1 + n_2)^{n_1-1}(\lambda + n_1 - n_2)^{n_2-1} \\
&\quad \times [n_1(\lambda - n_1 + n_2) + n_2(\lambda + n_1 - n_2)] \\
&= (\lambda - n_1 + n_2)^{n_1-1}(\lambda + n_1 - n_2)^{n_2-1} \\
&\quad \times (\lambda^2 - (n_1 + n_2)\lambda) \\
&= \lambda(\lambda - n)(\lambda - n_1 + n_2)^{n_1-1}(\lambda + n_1 - n_2)^{n_2-1}
\end{aligned}
$$

which proves L_p has four eigenvalues with the multiplicity stated in the question. $\qquad \square$

Proof: (Prop. 6) According to (21), each node has $2k$ neighbors for odd values of n and $2k+1$ neighbors for even values of n. In both cases, the network is a d-regular graph with at most $\lfloor nd/2 \rfloor = \max\{nk, \lfloor n(2k+1)/2 \rfloor\} = nk + \lfloor n/2 \rfloor$ links (showing the minor details of this fact are left to the reader). Let $Q = [q_{ij}]$ be the adjacency matrix corresponding to Laplacian of the linearized system of (9). Then, for every link $\{i, j\}$ of the network

$$q_{ij} = a_{ij}\cos(\theta_j - \theta_i) = \cos^2(\theta_j - \theta_i) > 0, \ a_{ij} < 0.$$

This property is due to the fact that $\pi/2 < |\theta_j - \theta_i| < 3\pi/2$. Moreover, for n evenly distanced points on a circle, as shown in Fig. 1, node i and node $i + 1$ are connected via either node $i + n/2$ (n even), or node $i + (n-1)/2$ (n odd)—all node indices are modulo n. Therefore, the entire network is connected. Thus, Q is a non-negative adjacency matrix of a connected network and its Laplacian L is a positive semi-definite matrix with one isolated zero eigenvalue. In other words, the linearized system $\dot{\theta} = -L\theta$ is stable for the couplings in (21). $\qquad \square$

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