XXIX Semana do IME/UFG e VI Seminário de Pesquisa e Pós-Graduação do IME/UFG

Formalização de Teoremas em Assistentes de Prova

Section 1: Lógica e Dedução Formal

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Oct 6 -8 , 2021

Talk's Plan

1. Section 1
   - Formalizing Mathematics
   - Gentzen's Calculus
   - The Prototype Verification System (PVS)
   - Exercises - propositional logic
   - Gentzen Deductive Rules vs PVS Proof Commands

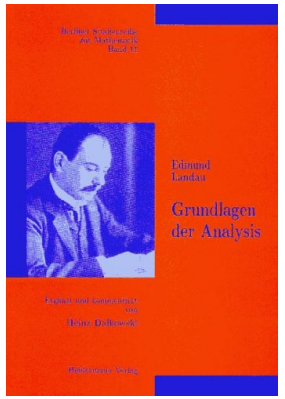
Formalizing Mathematics

Since the early development of computers, implementing mathematical deduction was a very important challenge:

Nicolaas Govert de Bruijn (1918-2012).
Dutch mathematician leader of the
Automath project.

Automath started in 1967:

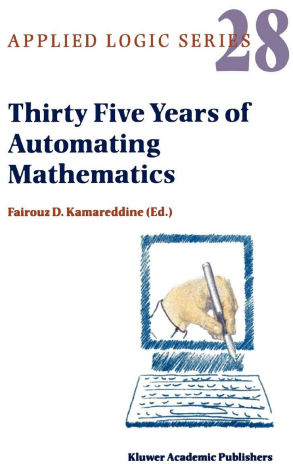
Mechanical verification of the famous Edmund Landau's (1877-1938) book *Grundlagen der Analysis*, Leipzig 1930.

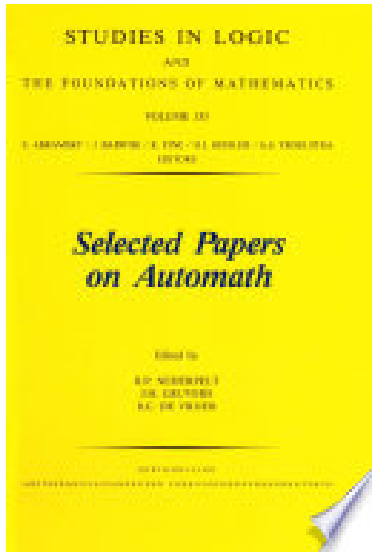


Formalizing Mathematics

https://www.win.tue.nl/automath/

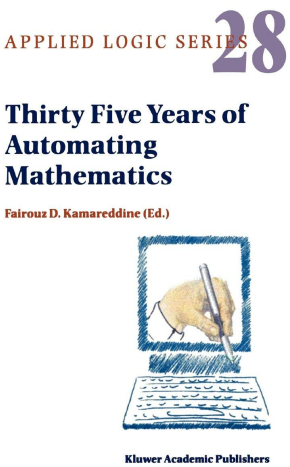
Automath is considered predecessor of modern proof assistants as: Coq, Nurpl, Isabelle, PVS ...





Formalizing Mathematics

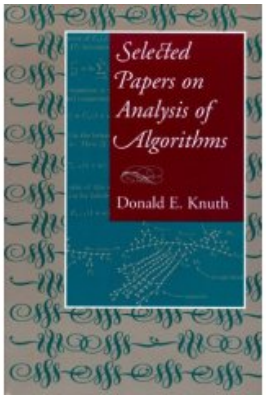
In Automath N.G. de Bruijn developed the first formalization of λ-calculus with intuitionistic types and explicit substitutions.



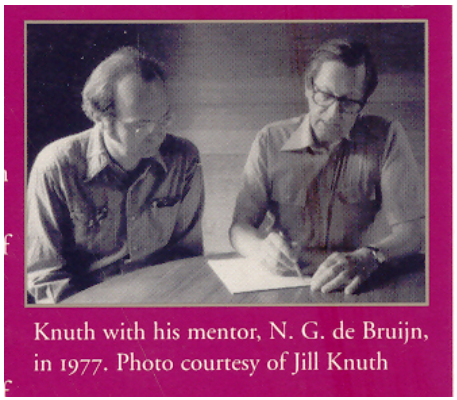
N.G. de Bruijn was a well established mathematician before deciding in 1967 at the age of 49 to work on a new direction related to Automating Mathematics. In the 1960s he became fascinated by the new computer technology and decided to start the new Automath project where he could check, with the help of the computer, the correctness of books of mathematics. Through his work on Automath, de Bruijn started a revolution in using the computer for verification, and since, we have seen more and more proof-checking and theorem-proving systems.

Formalizing Mathematics

N.G. De Bruijn's influence in computing is not restricted to Automath.



Donald Knuth dedicates his book to his mentor, N. G. de Bruijn.



Knuth with his mentor, N. G. de Bruijn, in 1977. Photo courtesy of Jill Knuth

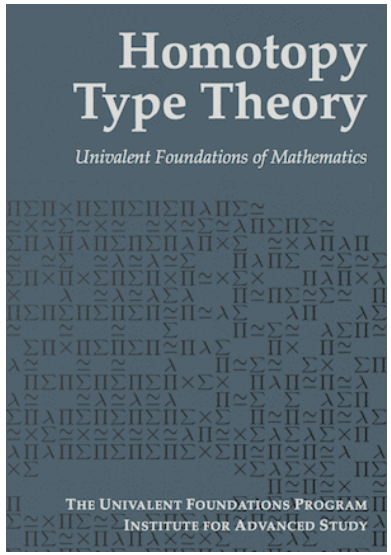
... I'm dedicating this book to N.G. "Dick" de Bruijn because his influence can be felt on every page. Ever since the 1960s he has been my chief mentor, the main person who would answer my questions when I was stuck on a problem that I had not been taught how to solve. I originally wrote Chapter 26 for his $(3 \cdot 4 \cdot 5)$th birthday; now he is 3^4 years young as I gratefully present him with this book.

Donald E. Knuth

Formalizing Mathematics

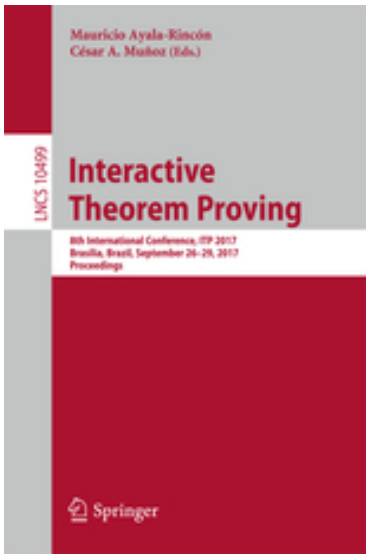
Vladimir Voevodsky (1966-2017) ( 2002) popularised the Univalent Foundations that use classical predicate logic as the underlying deductive sytem, categorical approaches, and intuitionistic types, indeed the so called

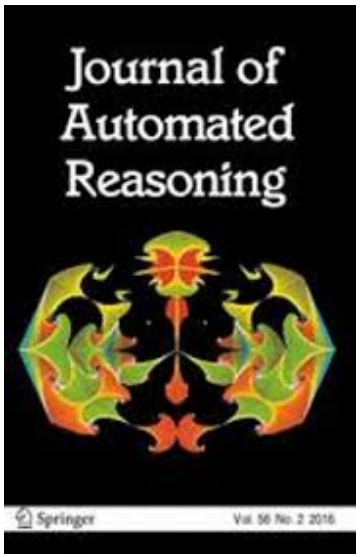
`https://homotopytypetheory.org`

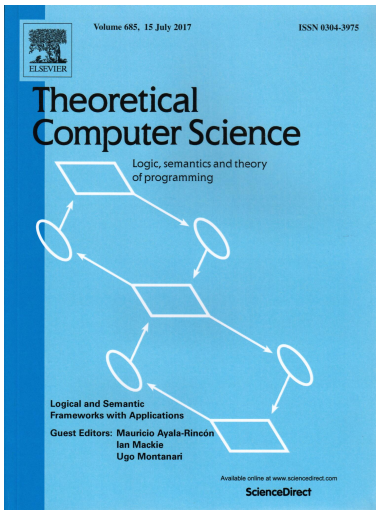


Formalizing Mathematics

Some related conferences/journals:







Formalized Mathematics by GTC members

- Rewriting Theory `https://github.com/nasa/pvslib/tree/master/TRS`
- Termination `https://github.com/nasa/pvslib/tree/master/PVS0`
- Nominal equational reasoning `nominal.cic.unb.br`
- Group and Ring's Theories

  `https://github.com/nasa/pvslib/tree/master/algebra`

Formalized Mathematics by GTC members:

Term Rewriting `trs.cic.unb.br`

- Newmann, Yokohuchi, Rosen Confluence Theorems — André Galdino (PhD Math UnB 2008), Ana Cristina Rocha Oliveira (PhD Inf UnB 2016)

  JFR (2008) *"A Formalization of Newman's and Yokouchi's Lemmas in a Higher-Order Language"*

  
  (2017) *"Confluence of Orthogonal Term Rewriting Systems in the Prototype Verification System"*

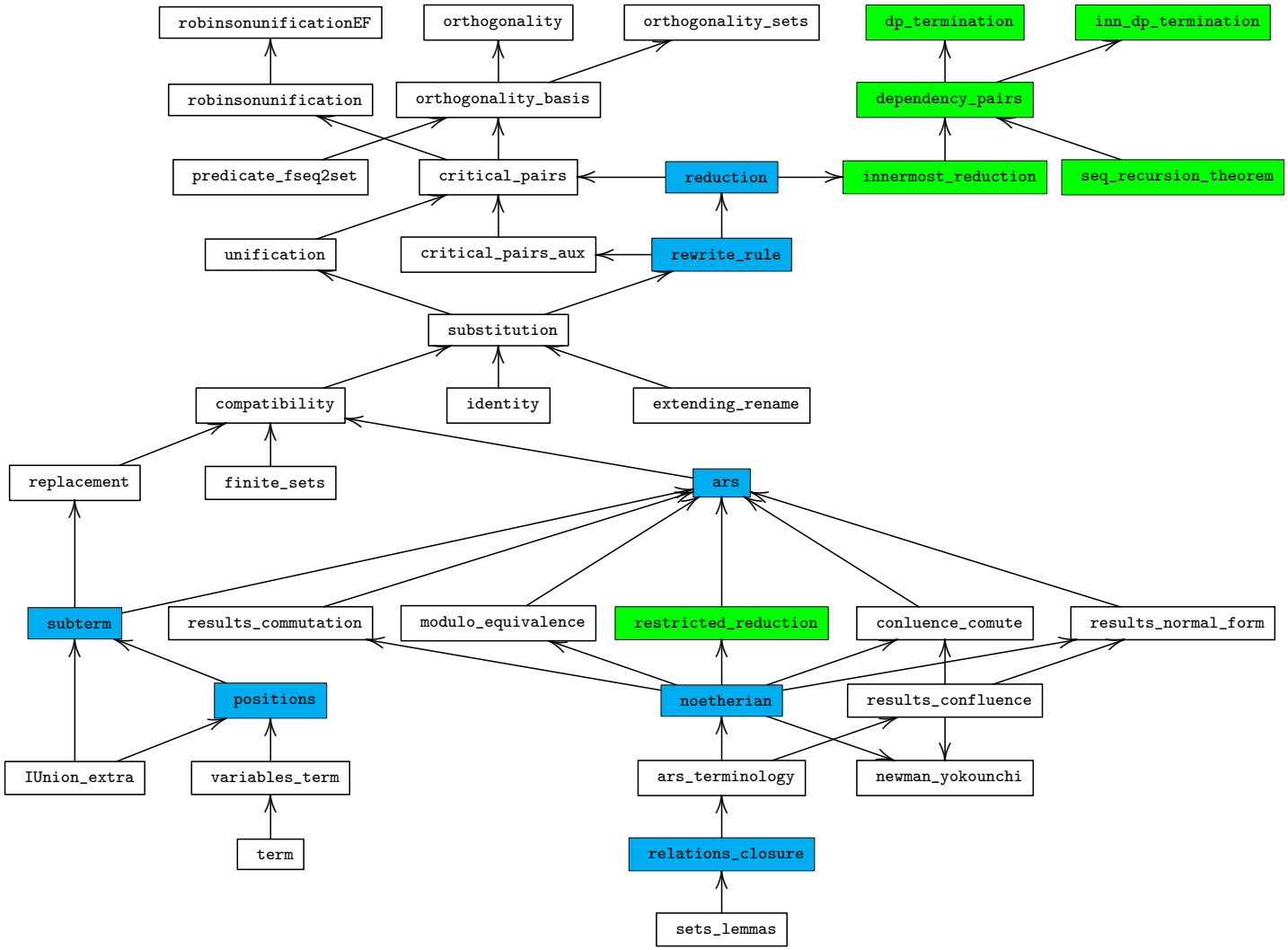
- Knuth-Bendix Critical Pairs Theorem — André Galdino (PhD Math UnB 2008)

  
  (2010) *"A Formalization of the Knuth-Bendix(-Huet) Critical Pair Theorem"*

- Existence of First-order Unification Theorem — Andréia Borges Avelar (PhD Math UnB 2014)

  
  (2014) *"First-order unification in the PVS proof assistant"*

PVS TRS Theory (Around 1051 theorems)



trs.cic.unb.br

Formalized Mathematics by GTC members: Termination

https://github.com/nasa/pvslib

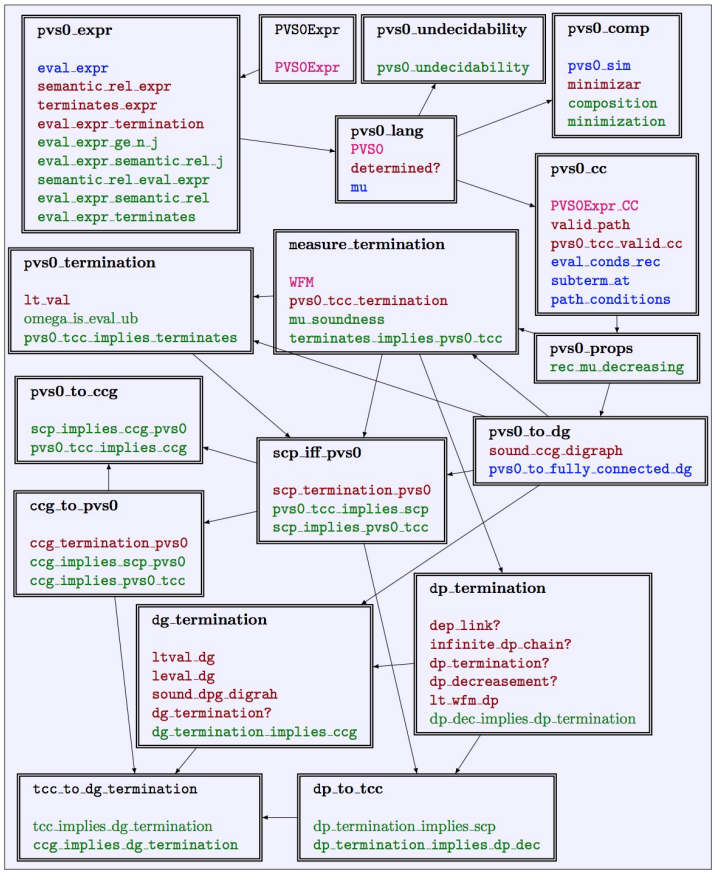
- Formalization of the Computational Theory of a functional language - Thiago Mendonça Ferreira Ramos (PhD Inf UnB Student), Mariano Moscato & César Muñoz (NIA / NASA LaRC FM)

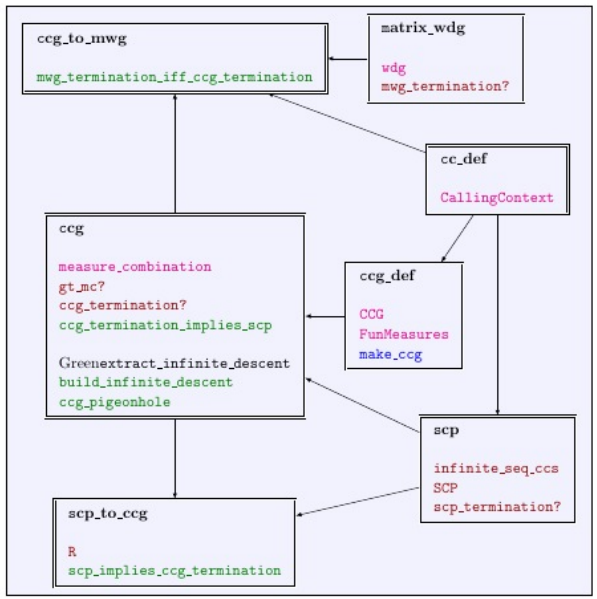
  
  (2018) *"Formalization of the Undecidability of the Halting Problem for a Functional Language"*

- TRS Termination by Dependency Pairs Criteria Theorem — Ariane Alves Almeida (PhD Inf UnB Student)

  
  (submitted, 2020) *"Formalizing the Dependency Pair Criterion for Innermost Termination"*

PVS PVS0 and CCG Theories (Around 404 and 348 theorems, resp.)





https://github.com/nasa/pvslib

Formalized Mathematics by GTC members — Functional and Rewriting Termination

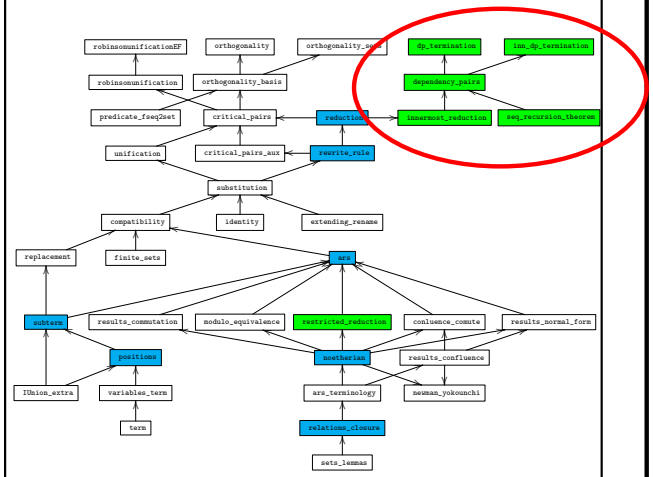
PVS0 Functional Programs

PVS0/CCG Theories

CCG termination

Term Rewriting Systems

TRS Theory



Innermost DP termination

Formalized Mathematics by GTC members — Nominal Equational Reasoning

equality check: $s = t$? matching: $\exists\sigma : s\sigma = t$? unification: $\exists\sigma : s\sigma = t\sigma$?

- Formalization of Functional Nominal Unification — Ana Cristina Rocha Oliveira (PhD Inf UnB 2016)

   (2015) *"Completeness in PVS of a Nominal Unification Algorithm"*

- Formalization of Rule-Inference Nominal Unification and Matching Modulo C — Washington de Carvalho Segundo (PhD Inf UnB 2019)

   (2017) *"Nominal C-Unification"*

- Formalization of Functional Nominal Equality Check Modulo AC — W. de Carvalho

   (2019) *"A formalisation of nominal α-equivalence with A, C, and AC function symbols"*

- Formalization of Functional Nominal Unification and Matching Modulo C — W. de Carvalho and Gabriel Ferreira Silva (PhD student MAT UnB)

    (2020 and to be submitted) *"Functional Formalisation of Nominal C-Unification and Matching with Protected Variables"*

Formalized Mathematics by GTC members — Nominal Equational Reasoning

Specification and formalization of algorithms in PVS and Coq. The PVS theory consists of around theorems.

`nominal.cic.unb.br`

Among several novel important theoretical associated results:

- Permutational Nominal Approach for dealing with Freshness and Fixed Points — Maribel Fernández (King's College) & Daniele Nantes (UnB)

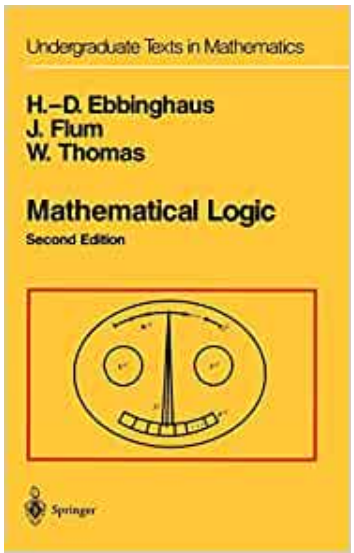
   (2020) *"On Nominal Syntax and Permutation Fixed Points"*

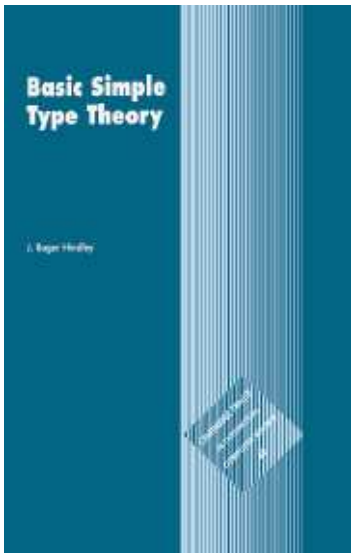
- Intersection Types for Nominal Logical Systems — Ana Cristina Rocha Oliveira (PhD Inf UnB 2016), Maribel Fernández (King's College) & Daniel Ventura

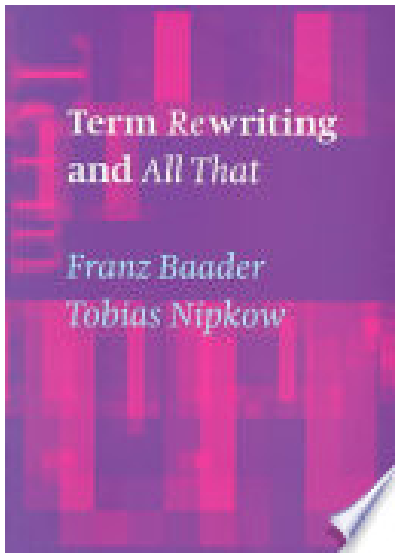
   (2018) *"Nominal essential intersection types"*

Formalized Mathematics by GTC members

You are welcome!







Gentzen Calculus

Sequents:

$$\Gamma \quad \Rightarrow \quad \Delta$$

Γ: antecedent

Δ: succedent

Gentzen Calculus

Table: Rules of deduction *à la* Gentzen for predicate logic

Left rules	Right rules
Axioms:	
$\Gamma, \varphi \Rightarrow \varphi, \Delta \ \ (Ax)$	$\bot, \Gamma \Rightarrow \Delta \ \ (L_{\bot})$
Structural rules:	
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \ (LWeakening)$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \ (RWeakening)$
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \ (LContraction)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \ (RContraction)$

Gentzen Calculus

Table: Rules of deduction *à la* Gentzen for predicate logic

Left rules	Right rules
Logical rules:	
$\dfrac{\varphi_{i\in\{1,2\}}, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L_\wedge)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (R_\wedge)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (L_\vee)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i\in\{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (R_\vee)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (L_\to)$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (R_\to)$
$\dfrac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\forall)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/y]}{\Gamma \Rightarrow \Delta, \forall_x \varphi}\ (R_\forall), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$
$\dfrac{\varphi[x/y], \Gamma \Rightarrow \Delta}{\exists_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\exists), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists_x \varphi}\ (R_\exists)$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$\varphi \Rightarrow \varphi \ \ (Ax)$$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$(RW)\ \frac{\varphi \Rightarrow \varphi \ \ (Ax)}{\varphi \Rightarrow \varphi, \psi}$$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$(RW)\ \frac{\varphi \Rightarrow \varphi \ \ (Ax)}{\varphi \Rightarrow \varphi, \psi}$$

$$(R_{\to})\ \frac{\varphi \Rightarrow \varphi, \psi}{\Rightarrow \varphi, \varphi \to \psi}$$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$\dfrac{\dfrac{\varphi \Rightarrow \varphi \ \ (Ax)}{\varphi \Rightarrow \varphi, \psi}\,(RW)}{\Rightarrow \varphi, \varphi \to \psi}\,(R_{\to}) \qquad \varphi \Rightarrow \varphi \ \ (Ax)$$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \ \ (Ax)}{\varphi \Rightarrow \varphi, \psi}\,(RW)}{\Rightarrow \varphi, \varphi \to \psi}\,(R_{\to}) \qquad \varphi \Rightarrow \varphi \ \ (Ax)}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi}\,(L_{\to})$$

Gentzen Calculus

Derivation of the Peirce's law: $\vdash ((\varphi \to \psi) \to \psi) \to \psi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \ \ (Ax)}{\varphi \Rightarrow \varphi, \psi}\,(RW)}{\Rightarrow \varphi, \varphi \to \psi}\,(R_{\to}) \qquad \varphi \Rightarrow \varphi \ \ (Ax)}{(\varphi \to \psi) \to \varphi \Rightarrow \varphi}\,(L_{\to})}{\Rightarrow ((\varphi \to \psi) \to \varphi) \to \varphi}\,(L_{\to})$$

Gentzen Calculus

Cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma\Gamma' \Rightarrow \Delta\Delta'} \ (Cut)$$

Gentzen Calculus - dealing with negation: c-equivalence

$$\varphi, \Gamma \Rightarrow \Delta \text{ one-step c-equivalent } \Gamma \Rightarrow \Delta, \neg\varphi$$

$$\Gamma \Rightarrow \Delta, \varphi \text{ one-step c-equivalent } \neg\varphi, \Gamma \Rightarrow \Delta$$

The c-equivalence is the equivalence closure of this relation.

Lemma 1 (One-step c-equivalence)

(i) $\vdash_G \varphi, \Gamma \Rightarrow \Delta$, *iff* $\vdash_G \Gamma \Rightarrow \Delta, \neg\varphi$;

(ii) $\vdash_G \neg\varphi, \Gamma \Rightarrow \Delta$, *iff* $\vdash_G \Gamma \Rightarrow \Delta, \varphi$.

Gentzen Calculus - dealing with negation

Proof.

Necessity:

$$\dfrac{\dfrac{\varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta, \bot}\,(\mathrm{RW})}{\Gamma \Rightarrow \Delta, \neg\varphi}\,(\mathrm{R}_{\rightarrow})$$

Sufficiency:

$$\dfrac{(\mathrm{LW})\ \dfrac{\Gamma \Rightarrow \Delta, \neg\varphi}{\varphi, \Gamma \Rightarrow \Delta, \neg\varphi} \qquad \dfrac{(Ax)\ \varphi, \Gamma \Rightarrow \Delta, \varphi \qquad \bot, \varphi, \Gamma \Rightarrow \Delta\ (L_{\bot})}{\neg\varphi, \varphi, \Gamma \Rightarrow \Delta}\,(\mathrm{L}_{\rightarrow})}{\varphi, \Gamma \Rightarrow \Delta}\,(\textsc{Cut})$$

Gentzen Calculus - dealing with negation

Necessity:

$$\dfrac{\dfrac{\dfrac{(Ax)\ \varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi, \bot}{\Gamma \Rightarrow \Delta, \varphi, \varphi, \neg\varphi\ (\mathrm{R}_{\rightarrow})} \qquad \bot, \Gamma \Rightarrow \Delta, \varphi, \varphi\ (L_{\bot})}{\neg\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \varphi\ (\mathrm{L}_{\rightarrow})}}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi \rightarrow \varphi\ (\mathrm{R}_{\rightarrow})} \qquad \dfrac{\dfrac{\dfrac{\neg\varphi, \Gamma \Rightarrow \Delta}{\neg\varphi, \Gamma \Rightarrow \Delta, \varphi, \bot}\ (\mathrm{RW})}{\Gamma \Rightarrow \Delta, \varphi, \neg\neg\varphi}\ (\mathrm{R}_{\rightarrow}) \qquad \varphi, \Gamma \Rightarrow \Delta, \varphi\ (Ax)}{\neg\neg\varphi \rightarrow \varphi, \Gamma \Rightarrow \Delta, \varphi}\ (\mathrm{L}_{\rightarrow})}{\Gamma \Rightarrow \Delta, \varphi}\ (\textsc{Cut})$$

Sufficiency:

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \bot, \Gamma \Rightarrow \Delta}{\neg\varphi, \Gamma \Rightarrow \Delta}\ (\mathrm{L}_{\rightarrow})$$

The Prototype Verification System (PVS)

PVS is a verification system, developed by the SRI International Computer Science Laboratory, which consists of

1. a *specification language*:
   - based on *higher-order logic*;
   - a type system based on Church's simple theory of types augmented with *subtypes* and *dependent types*.
2. an *interactive theorem prover*:
   - based on **sequent calculus**; that is, goals in PVS are sequents of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite sequences of formulae, with the usual Gentzen semantics.

The Prototype Verification System (PVS) — Libraries

- The `prelude` library
  - It is a collection of basic *theories* containing specifications about:
    - functions;
    - sets;
    - predicates;
    - logic; among others.
  - The theories in the prelude library are visible in all PVS contexts;
  - It provides the infrastructure for the PVS typechecker and prover, as well as much of the basic mathematics needed to support specification and verification of systems.

The Prototype Verification System (PVS) — Libraries

- NASA LaRC PVS library (`nasalib`)
  - It includes the *theories*
    - ⋆ structures, analysis, algebra, graphs, digraphs,
    - ⋆ real arithmetic, floating point arithmetic, groups, interval arithmetic,
    - ⋆ linear algebra, measure integration, metric spaces,
    - ⋆ orders, probability, series, sets, topology,
    - ⋆ term rewriting systems, unification, etc. etc.
  - The `nasalib` is maintaned by the NASA LaRC formal methods group;
  - The `nasalib` is result of research developed by the NASA LaRC formal methods group and the cientific comunity in general.

Sequent Calculus in PVS

A sequent of the form $\Gamma \vdash \Delta$ (or $A_1, A_2, ..., A_n \vdash B_1, B_2, ..., B_m$, since Γ and Δ are finite sequences of formulae) is:

- interpreted as:
  $A_1 \wedge A_2 \wedge ... \wedge A_n \vdash B_1 \vee B_2 \vee ... \vee B_m$,
  that is, from the conjunction of the antecedent formulae one obtains the disjunction of the succedent formulae.

- represented in PVS as:

$$
\begin{array}{l}
\texttt{[-1]}\ A_1 \\
\vdots \\
\texttt{[-n]}\ A_n \\
\texttt{|----------} \\
\texttt{[1]}\ B_1 \\
\vdots \\
\texttt{[m]}\ B_m
\end{array}
$$

Sequent Calculus in PVS

- Inference rules
  - Premises and conclusions are simultaneously constructed:

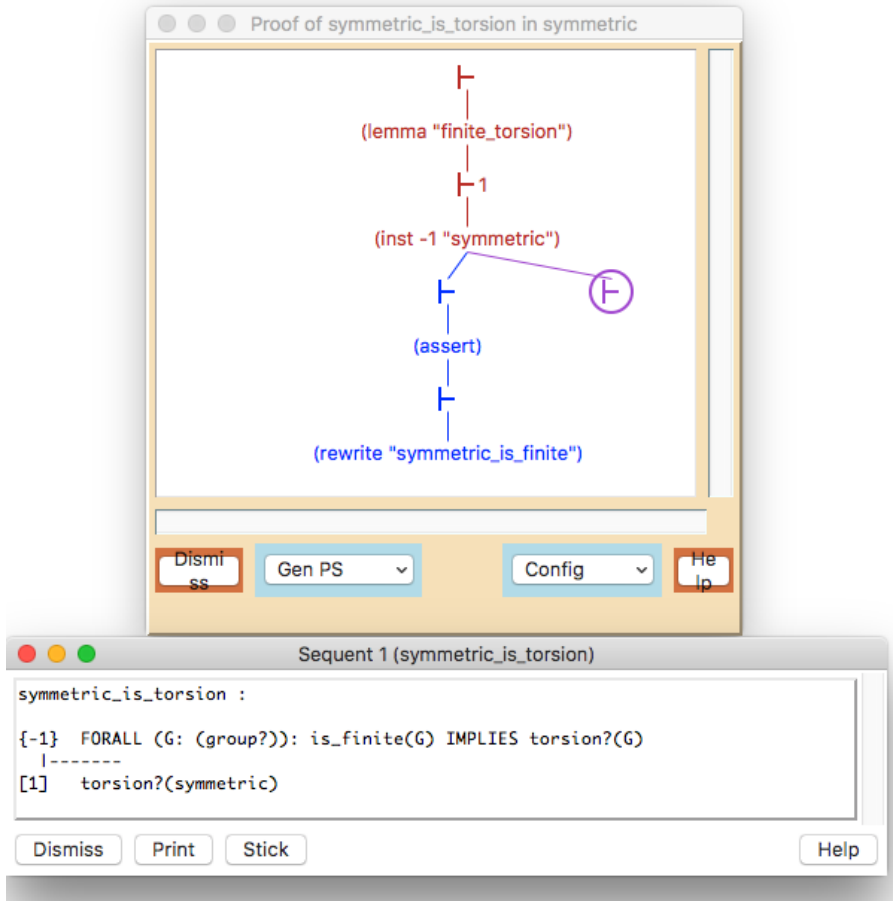
$$\frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}$$

  - A PVS proof command corresponds to the application of an inference rule. In general:

$$\frac{\Gamma \vdash \Delta}{\Gamma_1 \vdash \Delta_1 \ldots \Gamma_n \vdash \Delta_n} \textbf{ (Rule Name)}$$

- Goal: $\vdash \Delta$.

- Proof tree: each node is labelled by a sequent



Some inference rules in PVS

- Structural:

Deduction rule	PVS command
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LContraction)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\varphi, \varphi, \Gamma \vdash \Delta}\ (copy)$

$$\begin{array}{c} \vdots \\ [-i]\ A \wedge \neg B \\ \vdots \\ |------ \\ \vdots \\ [j]\ \neg C \to D \\ \vdots \end{array} \quad (\texttt{copy} - \texttt{i}) \quad \rightsquigarrow \quad \begin{array}{c} [-1]\ A \wedge \neg B \\ \vdots \\ [-(i+1)]\ A \wedge \neg B \\ \vdots \\ |------ \\ \vdots \\ [j]\ \neg C \to D \\ \vdots \end{array}$$

Some inference rules in PVS

- Structural:

Deduction rule	PVS command
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LW\,eakening)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (hide)$

$$
\begin{array}{l}
[-1]\ A \wedge \neg B \\
\quad\vdots \\
[-(i+1)]\ A \wedge \neg B \\
\quad\vdots \\
\vdash\!\!- - - - - - - \\
\quad\vdots \\
[j]\ \neg C \to D \\
\quad\vdots
\end{array}
\qquad (\texttt{hide} - (\texttt{i} + \texttt{1})) \qquad \rightsquigarrow \qquad
\begin{array}{l}
[-1]\ A \wedge \neg B \\
\quad\vdots \\
\vdash\!\!- - - - - - - \\
\quad\vdots \\
[j]\ \neg C \to D \\
\quad\vdots
\end{array}
$$

Some inference rules in PVS

- Propositional:

$$|---------$$

$$[1]\ A \wedge B \rightarrow (C \vee D \rightarrow C \vee (A \wedge C))$$

$$\downarrow \texttt{(flatten)}$$

$$[-1]\ A$$

$$[-2]\ B$$

$$[-3]\ C \vee D$$

$$|---------$$

$$[1]\ C$$

$$[2]\ A \wedge C$$

Deduction rule	PVS command $(flatten)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ (R_{\rightarrow})$	$\dfrac{\Gamma \vdash \Delta, \varphi \rightarrow \psi}{\varphi, \Gamma \vdash \Delta, \psi}$
$\dfrac{\varphi_1, \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L_{\wedge})$	$\dfrac{\varphi_1 \wedge \varphi_2, \Gamma \vdash \Delta}{\varphi_{i \in \{1,2\}}, \Gamma \vdash \Delta}$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i \in \{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (R_{\vee})$	$\dfrac{\Gamma \vdash \Delta, \varphi_1 \vee \varphi_2}{\Gamma \vdash \Delta, \varphi_1, \varphi_2}$

Some inference rules in PVS

- Propositional:

Deduction rule	PVS command
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (L_{\to})$	$\dfrac{\varphi \to \psi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \varphi \quad \psi, \Gamma \vdash \Delta}\ (split)$

$$[-1]\ (A \to B) \to A$$

$$|-------\quad (\texttt{split}\ -1)$$

$$[1]\ A$$

$$\swarrow \qquad\qquad \searrow$$

$$[-1]\ A \qquad\qquad |------$$

$$|------ \qquad\qquad [1]\ A \to B$$

$$[1]\ A \qquad\qquad [2]\ A$$

Some inference rules in PVS

- Propositional:

$$
\begin{array}{l}
\vdash \; - - - - - - - \quad (\texttt{case "m} \geq \texttt{n"}) \\
[1] \;\; \gcd(m,n) = \gcd(n,m)
\end{array}
\rightsquigarrow
\begin{array}{l}
[-1] \; m \geq n \\
\vdash \; - - - - - - - \\
[1] \;\; \gcd(m,n) = \gcd(n,m)
\end{array}
\qquad
\begin{array}{l}
\vdash \; - - - - - - - \\
[1] \; m \geq n \\
[2] \;\; \gcd(m,n) = \gcd(n,m)
\end{array}
$$

Some inference rules in PVS

- Propositional - semantics of PVS instructions:

$$\dfrac{\dfrac{a, \Gamma \vdash \Delta, b}{\Gamma \vdash \Delta, a \to b}\ \texttt{(flatten)} \qquad \dfrac{\Gamma \vdash \Delta, a, c}{\Gamma \vdash \Delta, \neg a \to c}\ \texttt{(flatten)}}{\Gamma \vdash \Delta, \textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ endif}}\ \texttt{(split)}$$

$$\dfrac{\dfrac{a, b, \Gamma \vdash \Delta}{a \wedge b, \Gamma \vdash \Delta}\ \texttt{(flatten)} \qquad \dfrac{c, \Gamma \vdash \Delta, a}{\neg a \wedge c, \Gamma \vdash \Delta}\ \texttt{(flatten)}}{\textbf{if } a \textbf{ then } b \textbf{ else } c \textbf{ endif}, \Gamma \vdash \Delta}\ \texttt{(split)}$$

Some inference rules in PVS

- Propositional (propax):

$$\frac{\Gamma, A \vdash A, \Delta}{}\ \textbf{(Ax)} \qquad \frac{\Gamma, FALSE \vdash \Delta}{}\ \textbf{(FALSE}\vdash\textbf{)}$$

$$\frac{\Gamma \vdash TRUE, \Delta}{}\ (\vdash \textbf{TRUE})$$

Exercises - infinity of Primes

See the file `preliminaries.pvs` in Exercises directory

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Structural Left Rules vs Proof Commands

Structural left rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LWeakening)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (hide)$
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LContraction)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\varphi, \varphi, \Gamma \vdash \Delta}\ (copy)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Structural Right Rules vs Proof Commands

Structural right rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (RWeakening)$	$\dfrac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta}\ (hide)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (RContraction)$	$\dfrac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \varphi, \varphi}\ (copy)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Logical Left Rules vs Proof Commands

Left rules	PVS commands
$\dfrac{\varphi_1, \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L_\wedge)$	$\dfrac{\varphi_1 \wedge \varphi_2, \Gamma \vdash \Delta}{\varphi_{i\in\{1,2\}}, \Gamma \vdash \Delta}\ (flatten)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (L_\vee)$	$\dfrac{\varphi \vee \psi, \Gamma \vdash \Delta}{\varphi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}\ (split)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}\ (L_\rightarrow)$	$\dfrac{\varphi \rightarrow \psi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \varphi \quad \psi, \Gamma \vdash \Delta}\ (split)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Logical Right Rules vs Proof Commands

Right rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (R_\wedge)$	$\dfrac{\Gamma \vdash \Delta, \varphi \wedge \psi}{\Gamma \vdash \Delta, \varphi \quad \Gamma \vdash \Delta, \psi}\ (split)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i \in \{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (R_\vee)$	$\dfrac{\Gamma \vdash \Delta, \varphi_1 \vee \varphi_2}{\Gamma \vdash \Delta, \varphi_1, \varphi_2}\ (flatten)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ (R_\rightarrow)$	$\dfrac{\Gamma \vdash \Delta, \varphi \rightarrow \psi}{\varphi, \Gamma \vdash \Delta, \psi}\ (flatten)$

Gentzen Calculus

Table: Rules of deduction *à la* Gentzen for predicate logic

Left rules	Right rules
Axioms:	
$\Gamma, \varphi \Rightarrow \varphi, \Delta \ \ (Ax)$	$\bot, \Gamma \Rightarrow \Delta \ \ (L_{\bot})$
Structural rules:	
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \ (LWeakening)$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi} \ (RWeakening)$
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \ (LContraction)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi} \ (RContraction)$

Gentzen Calculus

Table: Rules of deduction *à la* Gentzen for predicate logic

Left rules	Right rules
Logical rules:	
$\dfrac{\varphi_{i\in\{1,2\}}, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L_\wedge)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (R_\wedge)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (L_\vee)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i\in\{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (R_\vee)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (L_\to)$	$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (R_\to)$
$\dfrac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\forall)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/y]}{\Gamma \Rightarrow \Delta, \forall_x \varphi}\ (R_\forall), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$
$\dfrac{\varphi[x/y], \Gamma \Rightarrow \Delta}{\exists_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\exists), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists_x \varphi}\ (R_\exists)$

Gentzen Calculus

Derivation of: $\vdash \exists_x \neg\varphi \Rightarrow \neg\forall_x \varphi$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi[x/t] \Rightarrow \varphi[x/t]}{\forall_x \varphi \Rightarrow \varphi[x/t]}\,(L_\forall)}{\neg\varphi[x/t], \forall_x \varphi \Rightarrow}\,(\text{C-EQUIV})}{\neg\varphi[x/t] \Rightarrow \neg\forall_x \varphi}\,(\text{C-EQUIV})}{\exists_x \neg\varphi \Rightarrow \neg\forall_x \varphi}\,(L_\exists)$$

Some inference rules in PVS

- Predicate:

Deduction rule	PVS command
$\dfrac{\varphi[x/y], \Gamma \Rightarrow \Delta}{\exists_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\exists), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\exists_x \varphi, \Gamma \vdash \Delta}{\varphi[x/y], \Gamma \vdash \Delta}\ (skolem), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$
$\dfrac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\forall)$	$\dfrac{\forall_x \varphi, \Gamma \vdash \Delta}{\varphi[x/t], \Gamma \vdash \Delta}\ (inst)$

$$
\begin{array}{l}
[-1]\ \forall_{x:T} : P(x) \\
[-2]\ \exists_{x:T} : \neg P(x) \quad (\mathtt{skolem}\ -2\ \text{“}\mathtt{z}\text{”}) \quad \rightsquigarrow \\
\mathtt{|---} \\
\end{array}
\qquad
\begin{array}{l}
[-1]\ \forall_{x:T} : P(x) \\
\mathtt{|---} \\
[1]\ P(z)
\end{array}
$$

$$
\begin{array}{l}
[-1]\ \forall_{x:T} : P(x) \\
\mathtt{|---} \\
[1]\ P(z)
\end{array}
\quad (\mathtt{inst}\ -1\ \text{“}\mathtt{z}\text{”}) \quad \rightsquigarrow \quad
\left(
\begin{array}{c}
[-1]\ P(z) \\
\mathtt{|---} \\
[1]\ P(z)
\end{array}
\right)
\quad \mathtt{Q.E.D.}
$$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Structural Left Rules vs Proof Commands

Structural left rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LWeakening)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}\ (hide)$
$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ (LContraction)$	$\dfrac{\varphi, \Gamma \vdash \Delta}{\varphi, \varphi, \Gamma \vdash \Delta}\ (copy)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Structural Right Rules vs Proof Commands

Structural right rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ (RWeakening)$	$\dfrac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta}\ (hide)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ (RContraction)$	$\dfrac{\Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta, \varphi, \varphi}\ (copy)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Logical Left Rules vs Proof Commands

Left rules	PVS commands
$\dfrac{\varphi_1, \varphi_2, \Gamma \Rightarrow \Delta}{\varphi_1 \wedge \varphi_2, \Gamma \Rightarrow \Delta}\ (L_\wedge)$	$\dfrac{\varphi_1 \wedge \varphi_2, \Gamma \vdash \Delta}{\varphi_{i\in\{1,2\}}, \Gamma \vdash \Delta}\ (flatten)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \vee \psi, \Gamma \Rightarrow \Delta}\ (L_\vee)$	$\dfrac{\varphi \vee \psi, \Gamma \vdash \Delta}{\varphi, \Gamma \vdash \Delta \quad \psi, \Gamma \vdash \Delta}\ (split)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}\ (L_\rightarrow)$	$\dfrac{\varphi \rightarrow \psi, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \varphi \quad \psi, \Gamma \vdash \Delta}\ (split)$
$\dfrac{\varphi[x/t], \Gamma \Rightarrow \Delta}{\forall_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\forall)$	$\dfrac{\forall_x \varphi, \Gamma \vdash \Delta}{\varphi[x/t], \Gamma \vdash \Delta}\ (inst)$
$\dfrac{\varphi[x/y], \Gamma \Rightarrow \Delta}{\exists_x \varphi, \Gamma \Rightarrow \Delta}\ (L_\exists), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\exists_x \varphi, \Gamma \vdash \Delta}{\varphi[x/y], \Gamma \vdash \Delta}\ (skolem), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$

Summary - Gentzen Deductive Rules vs Proof Commads

Table: Logical Right Rules vs Proof Commands

Right rules	PVS commands
$\dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}\ (R_\wedge)$	$\dfrac{\Gamma \vdash \Delta, \varphi \wedge \psi}{\Gamma \vdash \Delta, \varphi \quad \Gamma \vdash \Delta, \psi}\ (split)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi_{i \in \{1,2\}}}{\Gamma \Rightarrow \Delta, \varphi_1 \vee \varphi_2}\ (R_\vee)$	$\dfrac{\Gamma \vdash \Delta, \varphi_1 \vee \varphi_2}{\Gamma \vdash \Delta, \varphi_1, \varphi_2}\ (flatten)$
$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}\ (R_\rightarrow)$	$\dfrac{\Gamma \vdash \Delta, \varphi \rightarrow \psi}{\varphi, \Gamma \vdash \Delta, \psi}\ (flatten)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/y]}{\Gamma \Rightarrow \Delta, \forall_x \varphi}\ (R_\forall), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$	$\dfrac{\Gamma \vdash \Delta, \forall_x \varphi}{\Gamma \vdash \Delta, \varphi[x/y]}\ (skolem), \quad y \notin \mathtt{fv}(\Gamma, \Delta)$
$\dfrac{\Gamma \Rightarrow \Delta, \varphi[x/t]}{\Gamma \Rightarrow \Delta, \exists_x \varphi}\ (R_\exists)$	$\dfrac{\Gamma \vdash \Delta, \exists_x \varphi}{\Gamma \vdash \Delta, \varphi[x/t]}\ (inst)$

Summary - Completing the GC vs PVS rules

	(hide)	(copy)	(flatten)	(split)	(skolem)	(inst)	(lemma) (case) ×
(LW)	×						
(LC)		×					
$(L_\wedge)$			×				
$(L_\vee)$				×			×
$(L_\rightarrow)$				×			
$(L_\forall)$						×	
$(L_\exists)$					×		
(RW)	×						
(RC)		×					
$(R_\wedge)$				×			
$(R_\vee)$			×				
$(R_\rightarrow)$			×				
$(R_\forall)$					×		
$(R_\exists)$						×	
(Cut)							×