XXIX Semana do IME/UFG e VI Seminário de Pesquisa e Pós-Graduação do IME/UFG

Formalização de Teoremas em Assistentes de Prova

Section 2: Caso de estudo - Teoria de Grupos

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Talk's Plan

Closure in a group

```
G: VAR set[T]

closed?(G): bool = FORALL (x,y:(G)): member(x*y,G)

group?(G): bool = closed?(G) AND
                  associative?[(G)](*) AND
                  member(e,G) AND identity?[(G)](*)(e) AND
                  inv_exists?(G)
```

Conjecture power_closed in pred_algebra.pvs

For all group G, $y \in G$ and $n \in \mathbb{N}$ one can prove that $y^n = \underbrace{y * \ldots * y}_{n-times} \in G$.

A recursive function in PVS

$$\wedge(y, n) = \prod_{i=1}^{n} y$$

In PVS:

```
^(y : T, n : nat) : RECURSIVE T =
                    IF n = 0 THEN e
                    ELSE y * ^(y, n-1) ENDIF
                    MEASURE n
```

Type Correctness Conditions (TCCs)

The specification provides two conditions to be verified:

- **A TCC about the type of the argument in the recursive call**

```
% Subtype TCC generated (at line 52, column 22) for  n - 1
    % expected type  nat
caret_TCC1: OBLIGATION FORALL (n: nat): NOT n = 0 IMPLIES n - 1 >= 0;
```

- **A TCC that guarantees the termination of the recursive call**

```
% Termination TCC generated (at line 52, column 17) for  ^(y, n - 1)
caret_TCC2: OBLIGATION FORALL (n: nat): NOT n = 0 IMPLIES n - 1 < n;
```

Induction scheme: weak induction on naturals

power_closed:

|---

$[1]$ $\mathtt{FORALL}(\mathtt{G} : (\mathtt{group?}), \mathtt{y} : (\mathtt{G}), \mathtt{n} : \mathtt{nat}) : \mathtt{member}(\wedge(\mathtt{y}, \mathtt{n}), \mathtt{G})$

Rule? (induct"n")

- Base case: power_closed.1

  |---

  $[1]$ $\mathtt{FORALL}(\mathtt{G} : (\mathtt{group?}), \mathtt{y} : (\mathtt{G})) : \mathtt{member}(\wedge(\mathtt{y}, 0), \mathtt{G})$

- Inductive Step: power_closed.2

  |---

  $[1]$ $\mathtt{FORALL}\, \mathtt{j} :$

  $(\mathtt{FORALL}(\mathtt{G} : (\mathtt{group?}), \mathtt{y} : (\mathtt{G})) : \mathtt{member}((\mathtt{y} \wedge \mathtt{j}), \mathtt{G}))$ IMPLIES

  $(\mathtt{FORALL}(\mathtt{G} : (\mathtt{group?}), \mathtt{y} : (\mathtt{G})) : \mathtt{member}((\mathtt{y} \wedge (\mathtt{j} + 1)), \mathtt{G}))$

Strong induction on naturals

Fibonacci Sequence

```
fibonacci(n:nat): RECURSIVE nat =
                    IF n <= 1 THEN n ELSE
                    fibonacci(n-1) + fibonacci(n-2)
                    ENDIF
                    MEASURE n
```

Conjecture `fibonacci_exp_lim` in `fibonacci.pvs`

$\texttt{fibonacci}(\texttt{n}) \leq 1.7^n$, for all $n \in \mathbb{N}$.

Induction scheme: strong induction on naturals

fibonacci_exp_lim:

$$|\texttt{---}$$

$$[1]\ \texttt{FORALL}(\texttt{n}:\texttt{nat}):\texttt{fibonacci}(\texttt{n}) <= \texttt{expt}(1.7,\texttt{n})$$

$$\texttt{Rule}?\ (\texttt{measure}-\texttt{induct}+\ \text{``}\texttt{n}\text{''}\ \text{``}\texttt{n}\text{''})$$

$$\downarrow$$

$$[-1]\ \texttt{FORALL}(\texttt{y}:\texttt{nat}):\texttt{y} < \texttt{x!1}\ \ \texttt{IMPLIES}\ \ \texttt{fibonacci}(\texttt{y}) <= \texttt{expt}(1.7,\texttt{y})$$

$$|\texttt{---}$$

$$[1]\ \texttt{fibonacci}(\texttt{x!1}) <= \texttt{expt}(1.7,\texttt{x!1})$$

Induction scheme: strong induction on naturals

Base cases:

```
fibonacci_exp_lim:

    [-1] FORALL(y : nat) : y < x!1  IMPLIES  fibonacci(y) <= expt(1.7, y)
    |---
    [1] fibonacci(x!1) <= expt(1.7, x!1)
    Rule? (case - replace "x!1 = 0")

fibonacci_exp_lim.1:

    [-1] x!1 = 0
    [-2] FORALL(y : nat) : y < x!1  IMPLIES  fibonacci(y) <= expt(1.7, y)
    |---
    [1] fibonacci(0) <= expt(1.7, 0)
    Rule? (grind)

This completes the proof of fibonacci_exp_lim.1.
```

Induction scheme: strong induction on naturals

Base cases:

`fibonacci_exp_lim.2:`

```
[-1] FORALL(y : nat) : y < x!1  IMPLIES  fibonacci(y) <= expt(1.7, y)
|---
[1] x!1 = 0
[2] fibonacci(x!1) <= expt(1.7, x!1)
Rule? (case - replace "x!1 = 1")
```

`fibonacci_exp_lim.2.1:`

```
[-1] x!1 = 1
[-2] FORALL(y : nat) : y < x!1  IMPLIES  fibonacci(y) <= expt(1.7, y)
|---
[1] 1 = 0
[2] fibonacci(1) <= expt(1.7, 1)
Rule? (grind)
```

This completes the proof of fibonacci_exp_lim.2.1.

Induction scheme: strong induction on naturals

Inductive Step:

fibonacci_exp_lim.2.2:

$[-1]$ FORALL(y : nat) : y < x!1 IMPLIES fibonacci(y) <= expt(1.7, y)

|---

$[1]$ $x!1 = 1$

$[2]$ $x!1 = 0$

$[3]$ fibonacci(x!1) <= expt(1.7, x!1)

Rule? (expand "fibonacci" 3) (assert)

$[-1]$ FORALL(y : nat) : y < x!1 IMPLIES fibonacci(y) <= expt(1.7, y)

|---

$[1]$ $x!1 = 1$

$[2]$ $x!1 = 0$

$[3]$ fibonacci(x!1 − 1) + fibonacci(x!1 − 2) <= expt(17/10, x!1)

Rule? (inst − cp − 1 "x!1 − 1")

Induction scheme: strong induction on naturals

Inductive Step:

$[-1]$ `FORALL(y : nat) : y < x!1 IMPLIES fibonacci(y) <= expt(1.7, y)`

$[-2]$ `x!1 - 1 < x!1 IMPLIES fibonacci(x!1 - 1) <= expt(17/10, x!1 - 1)`

`|---`

$[1]$ $x!1 = 1$

$[2]$ $x!1 = 0$

$[3]$ `fibonacci(x!1 - 1) + fibonacci(x!1 - 2) <= expt(17/10, x!1)`

`Rule? (inst -1 "x!1 - 2")`

$[-1]$ `x!1 - 2 < x!1 IMPLIES fibonacci(x!1 - 2) <= expt(17/10, x!1 - 2)`

$[-2]$ `x!1 - 1 < x!1 IMPLIES fibonacci(x!1 - 1) <= expt(17/10, x!1 - 1)`

`|---`

$[1]$ $x!1 = 1$

$[2]$ $x!1 = 0$

$[3]$ `fibonacci(x!1 - 1) + fibonacci(x!1 - 2) <= expt(17/10, x!1)`

`Rule? (assert)`

Induction scheme: strong induction on naturals

Inductive Step:

$[-1]$ $\texttt{fibonacci}(\texttt{x!1}-2) <= \texttt{expt}(17/10, \texttt{x!1}-2)$

$[-2]$ $\texttt{fibonacci}(\texttt{x!1}-1) <= \texttt{expt}(17/10, \texttt{x!1}-1)$

|---

$[1]$ $x!1 = 1$

$[2]$ $x!1 = 0$

$[3]$ $\texttt{fibonacci}(\texttt{x!1}-1) + \texttt{fibonacci}(\texttt{x!1}-2) <= \texttt{expt}(17/10, \texttt{x!1})$

Exercises - A case study on Group Theory

See the file `pred_algebra.pvs` in Exercises directory