Classical Mechanics

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June 22, 2018

Basic Definitions

- In classical mechanics, a *closed system* consists of states and laws of motion. Given any state, called an *initial condition* the laws of motion should allow one to perfectly predict the future and past.
  - Every state must map to exactly one new state under the laws of motion.
  - That is, the laws must be *deterministic* and *reversible*.
  - No information can be lost in such a system, i.e., there can be no singularities.
- A quantity that does not change with time is said to be *conserved*.
- Newton's laws govern most physical systems:
  1. In the absense of a force, velocity is constant (special case of 2).
  2. $F = m\ddot{x} = \dot{p}$, where m is mass, x is position, and $p := m\dot{x}$ is *momentum*.
  3. $\sum_{ij} F_{ij} = 0$, where F_{ij} is the force between particles i and j. It follows, $\frac{d}{dt}p_{total} = F_{total} = 0$, so momentum is conserved.
- For a force F dependent on position, the *potential energy* V is given by the potential function $F =: -\nabla V$.
- The *kinetic energy* is given by $T(\dot{x}) := \frac{m}{2}\dot{x}^2$.
- The total *energy* of a system is given by $E := T + V$, and is a conserved quantity ($\frac{d}{dt}E = 0$).
- An *inertial reference frame* is a reference frame in which Newton's laws hold.
- The *phase space* of a system with n coordinates is the $2n$ dimensional space obtained by considering both position and momentum.

Newton's laws of motion and the corresponding notions of energy and momentum are only one of many equally legitimate sets of laws and definitions. In the following sections, we will derive several frameworks in which all classical mechanics can be cast.

The Principle of Stationary Action and Lagrangian

The *action* of a system is given by the functional

$$S[q(t)] := \int_{t_1}^{t_f} L[q(t), \dot{q}(t), t]dt,$$

where L is called the *Lagrangian* and the integral is taken over some path through phase space. *The principle of stationary action* states that the laws of motion are given by the path through phase space for which the action is *stationary* (either a minimum, maximum, or saddle point) for fixed start and end points $q(t_1)$ and $q(t_f)$.

Given an arbitrary Lagrangian, one can solve for the resulting path using *calculus of variations*. The solution is given by the *Euler-Lagrange equations*:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}.$$

Of course, for an arbitrary Lagrangian, the notion of momentum may not be well-defined, so we define the momentum by

$$p := \frac{\partial L}{\partial \dot{x}} \qquad \text{and it follows that} \qquad \dot{p} = \frac{\partial L}{\partial x}.$$

A continuous transformation is given by any transformation of the form

$$\delta x_i = f_i(x)\varepsilon$$

where δx_i is the change in the ith component of position and f_i is any continuous function of position. Such a tranformation is called a *symmetry* if it doesn't affect the Lagrangian ($\delta L = 0$). Given a symmetry, there exists a quantity $Q = \sum_i p_i f_i$ that is conserved ($\frac{d}{dt}Q = 0$). For example, if the Lagrangian is rotationally invariant (rotation is a symmetry), then angular momentum is a conserved quantity.

To recover Newton's equations, one can use the following Lagrangian

$$L(x, \dot{x}) = T(\dot{x}) - V(x)$$

where $T(\dot{x})$ and $V(x)$ are the kinetic and potential energies as defined in Newton's formula. More generally, for an arbitrary Lagrangian, if L can be expressed as the difference between a quadratic function of $\dot{x}$ and a function of x, we can define the kinetic energy to be the function of $\dot{x}$ and the potential energy to be the function of x.

The Hamiltonian

The Hamiltonian is given by

$$H(x, \dot{x}, t) := \langle p, \dot{x} \rangle - L(x, \dot{x}, t)$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product. In systems where there exists a notion of energy, the energy is defined to be the Hamiltonian. If the Lagrangian has no explicit dependence on time ($L(x, \dot{x})$) then the Hamiltonian/energy is conserved.

Hamilton's equations of motion are then given by

$$\dot{p} = \frac{\partial H}{\partial x} \qquad \text{and} \qquad \dot{x} = \frac{\partial H}{\partial p}.$$

Recall that a flow is said to be *incompressible* if the volume of inflow is equal to the volume of outflow for all regions, or equivalently, if the flow maps regions to regions of equal volume. This is the analogue to determinism and reversibility in a continuous state space.

Liouville's Theorem: The flow in phase space (for any Hamilton equations) is incompressible, i.e., $div(V) = 0$, where V is the vector field describing the laws of motion in phase space for some Hamiltonian.

One final important piece of notation is the *Poisson bracket*. The brackets are defined for two functions F and G in phase space:

$$\{F, G\} := \sum_i \left(\frac{\partial F}{\partial x_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial x_i} \right).$$

If $F(p, x)$ is a force in some system governed by a Hamiltonian H, then $\dot{F} = \{F, H\}$. Also, if G is a conserved quantity, then $\{G, H\} = 0$.

Electrostatics

Magnetic fields (in the absense of monopoles) are incompressible vector fields that describe a force on charged particles. The *Lorentz equation* describes the forces on a charged particle in the presence of a stationary electromagnetic field:

$$F = \frac{e}{c} \left(-\nabla V + (\dot{x} \times B) \right)$$

where $E = -\nabla V$ is the electric field, B is the magnetic field, e is the charge of the particle, and c is the speed of light.

Note that since B is incompressible, B can be expressed as the curl of some other field $B = \nabla \times A$, where A is called the *vector potential*. Of course, since $\nabla \times \nabla S = 0$ for any scalar field S (called the Gauge), A is not unique and can be *Gauge transformed* without affecting the magnetic field B. However, the choice of Gauge does not affect the equations of motion so this is not an issue.

The Lagrangian that yields Lorentz's equations of motion is

$$L(x, \dot{x}) = \frac{m}{2} \dot{x}^2 + \frac{e}{c} (A \cdot \dot{x}).$$