Real Analysis 2

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Differentiation Theory

To define the derivative of a function F, consider *Dini's derivatives* which exist for all x:

$$(\overline{D}^{+}F)(x) := \limsup_{h\to 0^+} \tfrac{F(x+h)-F(x)}{h} \qquad (\underline{D}^{+}F)(x) := \liminf_{h\to 0^+} \tfrac{F(x+h)-F(x)}{h}$$

$$(\overline{D}^{-}F)(x) := \limsup_{h\to 0^+} \tfrac{F(x)-F(x-h)}{h} \qquad (\underline{D}^{-}F)(x) := \liminf_{h\to 0^+} \tfrac{F(x)-F(x-h)}{h}.$$

If all four Dini's derivatives are equal, then we say that the *derivative* $F'(x)$ exists and

$$F'(x) := \lim_{y\to x} \frac{F(y)-F(x)}{|y-x|}.$$

If $F : [a,b] \to \mathbb{R}$ is differentiable for all $x \in [a,b]$, then we say that F is *differentiable*. Similarly, if F is differentiable except on a null set, we say F is *almost everywhere differentiable*.

First, consider which classes of functions are a.e. differentiable? Measurability, boundedness, and even continuity are not enough to ensure differentiability. As a counterexample, the Weierstrass function is bounded and continuous, but nowhere differentiable:

$$F(x) = \sum_{n=1}^{\infty} a^{-n}\cos(b^n \pi x) \qquad \text{for } a > 1 \text{ and } \frac{b}{a} > 1 + \frac{3}{2}\pi.$$

It is the oscillations of the cosine function that break differentiability. In fact, all measurable monotone functions are a.e. differentiable (*Monotone differentiation theorem*). However, a stronger statement can be made.

Let the *total variation* of F be given by:

$$\|F\|_{TV(\mathbb{R})} := \sup_{x_0 < x_1 < \ldots < x_n} \sum_{i=1}^{n} |F(x_i) - F(x_{i-1})|$$

where the supremum is taken over all finite increasing sequences. A function F is said to have *bounded variation* ($F \in BV(\mathbb{R})$) if $\|F\|_{BV} := \|F\|_{TV} = M < \infty$. Every function of bounded variation can be decomposed into the difference between two bounded monotone nondecreasing functions: $F = F^{+} - F^{-}$. It follows that every $f \in BV(R)$ is a.e. differentiable. Note: every Lipschitz continuous function f satisfies $f \in BV(\mathbb{R})$.

We would like to extend the two *fundamental theorems of calculus* to the a.e. differentiable case using Lebesgue integrals instead of Riemann integrals. However, the theorems are harder to prove in the Lebesgue case, given that for F a.e. differentiable, both *Rolle's theorem* and the *Mean Value Theorem* in general fail.

The analogue of the first fundamental theorem is:

The Lebesgue Differentiation Theorem: Let $f : \mathbb{R} \to \mathbb{C}$ be L^1 and let $F : \mathbb{R} \to \mathbb{C}$ be the Lebesgue integral: $F(x) = \int_{-\infty}^{x} f(t)dt$. Then F is continuous and a.e. differentiable, and $F'(x) = f(x)$ for a.e. $x \in \mathbb{R}$.

The 2nd FTC is much harder to show. Indeed, it fails in general for both monotone and continuous functions. Instead, we must introduce a new notion that is stronger than both *continuity* and *uniform continuity*:

A function $F : \mathbb{R} \to \mathbb{R}$ is *absolutely continuous* ($F \in AC(\mathbb{R})$) if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all collections of disjoint intervals $\{(a_j, b_j)\}$ that satisfy $\sum_{j=1}^{n} |b_j - a_j| \leq \delta$, $\sum_{j=1}^{n} |F(b_j) - F(a_j)| \leq \varepsilon$.

2nd FTC for AC Functions: Let $F \in AC([a, b])$ for all compact $[a, b] \subset \mathbb{R}$. Then F' exists and $\int_a^b F' = F(b) - F(a)$. Conversely, for $f \in L^1([a, b])$, there exists an $F \in AC([a, b])$ such that $F' = f$ a.e. and $F(x) = \int_a^x f$.

L^p Spaces

Extending the notion of the L^1 class, define the class of L^p functions as follows for any p:

$$f \in L^p(X) \iff \int_X |f(x)|^p d\mu < \infty.$$

For $p > 1$, the exponent p increases the rate at which f decays, making this a broader class than L^1. Taking the limit as $p \to \infty$, we say that $f \in L^\infty$ if $|f|$ is bounded by some $M < \infty$ almost everywhere. For $p \geq 1$, L^p is a *norm space* under the norm

$$\|f\|_{L^p} = \left(\int |f|^p \right)^{1/p}.$$

If $p < 1$, then L^p is only a *quasi-norm space*.

Under the metric induced by $\|.\|_{L^p}$, both the addition $+ : L^p \times L^p \to L^p$ and multiplication $\cdot : \mathbb{C} \times L^p \to L^p$ operations are jointly continuous (under the product topology). Therefore, we say that L^p is a *toplogical vector space*. Furthermore, L^p is actually an infinite-dimensional *Banach space*, i.e., a *complete metric space*, and satisfies the *Hausdorff property*. Put in practical terms, this means that for every *Cauchy sequence* in an L^p space, the limit exists and is unique.

The *Riesz Representation Theorem* tells us about the *dual space* $(L^p)^*$. For $\frac{1}{p} + \frac{1}{q} = 1$, $(L^p)^* = L^q$. That is, for any *linear functional* $\ell \in (L^p)^*$, there exists a unique $g \in L^q$ such that $\ell(f) = \int f\overline{g}$. For $1 < p < \infty$, L^p space is *reflexive*, meaning $(L^p)^{**} = L^p$. Furthermore, in the special case where $p = 2$, $(L^2)^* = L^2$. Therefore, L^2 space is a *Hilbert space* (complete inner product space) under the inner product

$$\langle f, g \rangle := \int f\overline{g} \qquad \text{for all } f, g \in L^2.$$

Basic Functional Analysis

One of the fundamental theorems of functional analysis is:

The Baire Category Theorem (BCT): Let (X, d) be a complete metric space, and let $\{E_n\}_{n=1}^{\infty}$ be a countable sequence of subsets of X. If $\cup_{n=1}^{\infty} E_n \supset B$ for any ball B in (X, d), then there exists a nonempty sub-ball $B' \subset B$ in which at least one E_n is dense.

Using BCT, we categorize topological sets. Note, that the analogies are informal:

Name of set	Definition	Analogy in a measure space
Meager (1st category)	An at most countable union of nowhere dense sets	null sets
2nd category	Any set that is not 1st category	sets of positive measure
Co-meager (residual)	E is co-meager if E^C is meager	full-measure sets
Baire (almost open)	E is Baire if $E \triangle U$ is meager for some open set U	measurable sets

BCT has several important corollaries involving *linear operators* (linear maps $T : X \to Y$ between norm spaces). If T satisfies $\|T\| := \sup_{x\in X} \frac{\|Tx\|_Y}{\|x\|_X} = M < \infty$, then T is called a *bounded linear operator*.

The Uniform Boundedness Principle (UBP): Let X be a Banach space and Y be a general norm space, and let $\{T_\alpha\}_{\alpha\in A} : X \to Y$ be a family of bounded linear operators. Then the following are equivalent:

- (Pointwise boundedness) For all $x \in X$, the collection in Y given by $\{T_\alpha x : \alpha \in A\}$ is bouned in Y-norm.
- (Uniform bounedness) The operator norms $\{\|T_\alpha\| : \alpha \in A\}$ are uniformly bounded (i.e., they are all bounded by some finite constant).

For a general function $f : X \to Y$ where both X and Y are topological spaces, f is an *open map* if for all open $U \in X$, $f(U)$ is open in Y. That is, f maps open sets to open sets. Contrast this with a *continuous map*, where $f^{-1}(U)$ is open in X for all U open in Y.

The Open Mapping Theorem (OMT): Let $T : X \to Y$ be a bounded linear operator between Banach spaces X and Y. Then the following are equivalent:

- T is onto (surjective).
- T is open.
- (Qualitative solvability) For all $y \in Y$, there exists $x \in X$ such that $Tx = y$.
- (Quantitative solvability) For all $y \in Y$, there exists $x \in X$ such that $Tx = y$ and $\|x\|_X \leq M\|y\|_Y$ for some constant M.
- (Quantitative solvability on a dense subclass) There exists a constant $M > 0$ such that for any dense subset E of Y, there exists $x \in X$ such that $Tx = y$ for all $y \in E$ and $\|x\|_X \leq M\|y\|_Y$.

The Closed Graph Theorem (CGT): Let $T : X \to Y$ be a linear operator between Banach spaces. Then the following are equivalent:

- T is continuous.
- T is closed: I.e., the graph $\Gamma := \{(x, Tx) : x \in X\}$ is closed under the product topology $X \times Y$.
- T is weakly continuous: I.e., there exists some topology $\mathcal{F}$ on Y, weaker than the norm topology but still Hausdorff, such that $T : X \to (Y, \mathcal{F})$ is continuous.

Infinite Dimensional Spaces and Weak Topologies

In an infinite dimesional space (such as $L^p(\mathbb{R})$), it is difficult to formulate compactness arguments due to the failure of the *Heine-Borel Theorem*, which tells us that in a finite dimensional metric space, the following definitions of compactness are equivalent. X is compact if:

- (Compactness) Every open cover of X has a finite subcover.
- (Sequential Compactness) Every sequence $\{x_n\}$ in X has a convergent subsequence.
- (Closed and Boundedness) X is closed and bounded.

To see what can go wrong in an infinite dimensional space, consider the closed unit ball B. Intuitively, since B is closed and bounded, it should be compact. However, in an infinite-dimesional space, one can construct an infinite sequencei $\{e_j\}$, each of whose elements is the unit basis vector in a new dimension. Then no subsequence of $\{e_j\}$ converges.

To recover a friendlier topology in infinite-dimensional space, consider the following weaker topology: Let $(X, \|.\|)$ be a normed space. For each $\ell \in X^*$ (bounded), define the seminorm $\rho_\ell(x) := |\ell(x)|$. Then the *weak topology* is the topology generated by all open strips/balls of the form: $B_r^\ell(x) := \{y \in X : \rho_\ell(x - y) < r\}$. Similarly, the weak* topology on X^* is generated by all seminorms of the form $\rho_x(\ell) := |\ell(x)|$. Under the weak topology on a reflexive separable Banach space X, the closed unit ball is indeed sequentially compact.

We have already been briefly introduced to the *norm operator topology* (NOT). Let X, Y be Banach spaces and let $B(X, Y)$ be the collection of all bounded linear operators $T : X \to Y$. Then $B(X, Y)$ is a norm space under:

$$\|T\| := \sup_{x \in X,\, x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \qquad \text{for all } T \in B(X, Y).$$

Similarly as above, we also introduce the *strong operator topology* (SOT) and *weak operator topology* (WOT). SOT is the topology induced by seminorms $\rho_x(T) := \|Tx\|$ for all $x \in X$, and WOT is the topology induced by seminorms $\rho_{x,\ell}(T) := |\ell(Tx)|$ for all $x \in X$ and $\ell \in Y^*$.

SOT and WOT are generally used in arguments of convergence:

$$T_n \to T \text{ in SOT iff: } \|T_n x - Tx\| \to 0 \text{ for all } x \in X.$$
$$T_n \to T \text{ in WOT iff: } |\ell(T_n x) - \ell(Tx)| \to 0 \text{ for all } x \in X,\ \ell \in Y^*.$$

In order of strength, convergence in NOT is stronger than in SOT is stronger than in WOT.