Functional Analysis

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Banach and Hilbert Spaces

A *norm space* $(X, \|.\|)$ is a vector space X together with a norm $\|.\|$. Such a space is called a *Banach space* if X is *complete* under the metric topology induced by $d(x, y) = \|x - y\|$. Note that every norm space has a *completion* that is Banach. Also note that in a finite dimensional space, all norms induce equivalent topologies since for every pair of norms $\|.\|_1$ and $\|.\|_2$ and all points $x \in X$, there exists a uniform constant $1 < c < \infty$ such that $\frac{1}{c}\|x\|_2 \leq \|x\|_1 \leq c\|x\|_2$.

A *linear functional* ℓ on $(X, \|.\|)$ is a linear function $\ell : X \to \mathbb{R}$ or $\mathbb{C}$. ℓ is said to be *bounded* if $|\ell(x)| \leq c\|x\|$ for all $x \in X$ and $c < \infty$. Equivalently, ℓ is bounded if and only if ℓ is continuous. Every norm space $(X, \|.\|_X)$ has a *dual* $(X', \|.\|_{X'})$ given by the vector space X' of all bounded linear functionals ℓ on X, together with the norm

$$\|\ell\|_{X'} = \sup_{\substack{x \neq 0 \\ x \in X}} \frac{|\ell(x)|}{\|x\|_X} = \sup_{\|x\|_X = 1} |\ell(x)|.$$

In the special case of L^p (or ℓ^p), we have for $1 \leq p < \infty$, $(L^p(X))' = L^q(X)$ where $\frac{1}{p} + \frac{1}{q} = 1$ (Riesz Representation). It immediately follows that L^p is *reflexive* (meaning $(L^p)'' = L^p$) for $1 < p < \infty$, and $(L^2)' = L^2$. Furthermore, we have the Hölder Inequality: If $\frac{1}{r} = \frac{1}{q} + \frac{1}{p}$ and $a \in \ell^p$, $b \in \ell^q$, then $\|ab\|_r \leq \|a\|_p \|b\|_q$.

An *inner product* or *scalar product* space $(X, \langle \cdot, \cdot \rangle)$ is a vector space X together with a scalar product $\langle \cdot, \cdot \rangle : X \times X \to F$ (where $F = \mathbb{R}$ or $\mathbb{C}$) that satisfies

- Bilinearity: If $F = \mathbb{R}$, then $\langle \cdot, \cdot \rangle$ is linear in the first and second arguments. If $F = \mathbb{C}$, then $\langle \cdot, \cdot \rangle$ is linear in the first argument and *sesquilinear* in the second ($\langle x, \alpha y \rangle = \bar{\alpha}\langle x, y \rangle$).
- Symmetry/Skew Symmetry: If $F = \mathbb{R}$, then $\langle x, y \rangle = \langle y, x \rangle$. If $F = \mathbb{C}$, then $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- Positivity: $\langle x, x \rangle > 0$ if $x \neq 0$ (and $\langle 0, 0 \rangle = 0$ by linearity).

Every scalar product induces a norm $\|x\|^2 = \langle x, x \rangle$. If X is complete under the topology induced by this norm (i.e., $(X, \|.\|)$ is Banach), then $(X, \langle \cdot, \cdot \rangle)$ is called a *Hilbert space*. Note that since $(L^2)' = L^2$, L^2 is a Hilbert space under $\langle f, g \rangle = \int f\bar{g}$. In fact, every Hilbert space is isomorphic to its dual.

Every inner product $\langle \cdot, \cdot \rangle$ that induces a norm $\|.\|$ satisfies the following identities:

- Cauchy-Schwarz: $|\langle x, y \rangle| \leq \|x\|\|y\|$.

- Parallelogram Law: $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$.
- Polarization ID: $\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2$.

Furthermore, given a norm space $(X, \|.\|)$, there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\langle x, x\rangle = \|x\|^2$ if and only if $\|.\|$ satisfies the parallelogram law. Furthermore, that inner product is given by the polarization identity.

Given a set of vectors S in a Hilbert space H, the span of S is given by the set of all finite linear combinations of elements in S. I.e., span S is the smallest subspace of H containing S. Note that in the infinite-dimensional case, span S need not be closed (as it is in the finite dimensional case). A collection of vectors S in H is said to be *orthonormal* if their *projections* $\langle y_\alpha, y_\beta\rangle = 0$ ($y_\alpha \perp y_\beta$) for all $\alpha \neq \beta$ and $\langle y_\alpha, y_\alpha\rangle = 1$. An orthonormal set is called an orthonormal *basis* if the closure of its span satisfies $\overline{\text{span}}\ S = H$. It follows that for any orthonormal set S:

- (Parseval's ID) If S is basis for H, then $\|x\|^2 = \sum_{e_\alpha \in S} |\langle x, e_\alpha\rangle|^2$ for all $x \in H$.
- (Bessel's Ineq) In general, $\|x\|^2 \geq \sum_{e_\alpha \in S} |\langle x, e_\alpha\rangle|^2$ for all $x \in H$.

Every Hilbert space has a basis, which can be orthogonalized using the Gram-Schmidt process. So, by identifying basis vectors, we see that two Hilbert spaces are isomorphic if and only if their bases have equal cardinalities. It follows that every Hilbert space of countably infinite dimension is isomorphic to L^2.

Important Theorems

Zorn's Lemma

Let S be a partially ordered set (poset) under a relation ($\leq$) such that every totally ordered subset has an upper bound. Then S has a maximal element under the relation "$\leq$."

Notes on Zorn's Lemma:
In the infinite-dimensional case, Zorn's lemma is often used in place of induction in a process called "Zornification."

Hahn-Banach Theorem

Let X be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$ and let p be a positive real-valued function on X that satisfies

- Positive homogeneity: for $a \in \mathbb{R}_+$ (where in the complex case, $\mathbb{R}_+$ denotes the positive half-plane) and $x \in X$, $p(ax) = |a|p(x)$.
- Sub-additivity: for $x, y \in X$, $p(x+y) \leq p(x) + p(y)$.

Let Y be a subspace of X and let $\ell : Y \to F$ be a linear functional on Y such that $|\ell(y)| \leq p(y)$ for all $y \in Y$ (p dominates ℓ). Then ℓ can be extended to a linear functional on X that is still dominated by p.

Clarkson's Theorem

Let X be a norm space that is uniformly convex. Also let $z \in X$ and let K be a closed convex subset of X. Then there exists a unique point $y_0 \in K$ that solves:

$$\|x - y_0\| = \inf_{y \in K} \|z - y\| = dist(z, k).$$

Riesz's Lemma

Let Y be a closed proper subspace of a norm space X. Then there exists a unit vector $z \in X$ such that for all $y \in Y$, $\|z - y\| \geq \frac{1}{2}$.

Notes on Riesz's Lemma:
It follows from Riesz's Lemma that in an infinite-dimensional space, the closed unit ball is not compact (as it is in the finite-dimensional case). This inspires the *weak topology*, the weakest topology in which all $\ell \in X'$ are continuous. I.e., $x_n \to x$ weakly if $|\ell(x_n) - \ell(x)| \to 0$ for all $\ell \in X'$. Under the weak topology, the closed unit ball is indeed compact. Furthermore, if $x_n \to x$ in the weak topology, then $\{x_n\}$ either does not converge in the norm topology, or $\|x_n - x\| \to 0$.

Baire Category Theorem

Let X be a complete metric space and let $X = \cup_{n=1}^{\infty} S_n$ (where $\{S_n\}$ is countable). Then at least one S_n is **not** nowhere dense.

Principle of Uniform Boundedness

Let X be a complete metric space and let $\mathcal{F}$ be a collection of real-valued continuous functionals on X. If $\mathcal{F}$ is *pointwise bounded* ($|f(x)| \leq M(x)$ for all $x \in X$ and $f \in \mathcal{F}$), then there exists an open set $U \subset X$ in which $\mathcal{F}$ is *uniformly bounded* ($|f(x)| \leq M < \infty$ for all $x \in U$ and $f \in \mathcal{F}$). In the special case where $\mathcal{F}$ are sub-additive and absolutely homogeneous (most commonly, when $\mathcal{F}$ are linear) and X is Banach with norm $\|.\|$, $\mathcal{F}$ is uniformly bounded on the entire space X: $|f(x)| \leq c\|x\|$ for $c < \infty$.

Open Mapping Theorem

Every bounded linear map $M : Y \to Y$ is *open* (If U is open, so is $M(U)$).

Closed Graph Theorem

Let X and Y be Banach and let $M : X \to Y$ be a linear operator. We call $\Gamma(M) = \{(x, Mx) : x \in X\}$ the *graph* of M. We say M is *closed* if its graph is closed in $X \times Y$ under the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. The closed graph theorem says that if M is closed, then M is bounded.

Bounded Linear Operators

Let T be a linear operator $T : X \to Y$, where X, Y are Banach. Then T is bounded if

$$\|T\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y \tag{1}$$

is bounded by some constant M. Furthermore, we consider the Banach space $\mathcal{L}(X, Y)$ of all bounded linear functionals from X to Y under the norm defined in (1). Other useful topologies on $\mathcal{L}(X, Y)$ include the *strong operator topology* (the weakest topology in which $T : X \to Y$ is continuous for all $T \in \mathcal{L}(X, Y)$) and the *weak operator topology* (the weakest topology in which $T : X \times Y' \to F$ is continuous for all $T \in \mathcal{L}(X, Y)$).

If $V \subset X$ is a subspace of X then

- The *annihilator* $V^{\perp}$ of V is given by the set of all linear functionals $\ell \in X'$ that vanish on V. I.e., for all $\ell \in V^{\perp}$ and $y \in V$, $\ell(y) = 0$.
- X/V denotes the *quotient space*, i.e., the space of equivalence classes $[x]$ such that $[x] = [z]$ for $x, z \in X$ and $x - z \in V$.

If X, Y are Hilbert spaces, then we can define the *transpose* of a linear operator T as the operator $T' \in \mathcal{L}(Y', X')$ such that $\langle x, T'\ell \rangle = \langle Tx, \ell \rangle$ for all $x \in X$ and $\ell \in Y'$. Similarly, the *Hermitian adjoint* of T is the operator $T^* \in \mathcal{L}(Y, X)$ such that $\langle T^*x, y \rangle = \langle x, Ty \rangle$ for all $x \in X$ and $y \in Y$. Such an operator $T \in \mathcal{L}(X, X)$ is said to be *symmetric* or *self-adjoint* if $T^* = T$.

In the special case where $X = Y$, we denote $\mathcal{L}(X, X)$ by $\mathcal{L}(X)$. A few notable linear operators in $\mathcal{L}(L^2(\mathbb{R}))$ and $\mathcal{L}(L^2(\mathbb{R}_+))$ are the Fourier and Laplace transforms respectively. If an operator $T \in \mathcal{L}(X)$ is bijective, then it follows by the open mapping theorem that its inverse T^{-1} exists. The space $\mathcal{GL}(X)$ of all invertible operators on X is itself a subspace of $\mathcal{L}(X)$, and furthermore, $\mathcal{GL}(X)$ is a two-sided ideal under multiplication. Let $K \in \mathcal{L}(X)$ such that $K = I + E$, where $\|E\| < 1$ and I denotes the identity on X. Then $K \in \mathcal{GL}(X)$ and its inverse is given by the *Neumann series* $K^{-1} = \sum_{n=0}^{\infty} (-1)^n E^n$, which converges geometrically.

An operator $C \in \mathcal{L}(X, Y)$ is said to be *compact* if $C(B_1(0))$ (where $B_1(0)$ denotes the unit ball in X) is precompact in Y. The space $\mathcal{C}(X, Y)$ of compact operators between X and Y is itself a closed proper subspace of $\mathcal{L}(X, Y)$. Let $C \in \mathcal{C}(X)$ and let $T = I - C$, where I denotes the identity on X. Then the T is invertible and its Neumann series converges. Furthermore, if N_T denotes the nullspace of T and R_T denotes the range of T, then

- The sequence N_{T^n} $k = 1, \ldots, \infty$ saturates at some finite value $N_{T^N} = \lim_{n\to\infty} N_{T^n}$.
- The dimension of N_T is given by dim N_T = dim (X/R_T).

These statements hold trivially in the finite-dimensional case (the second statement is equivalent to the statement that dim $X = R_T + N_T$). However, in the infinite-dimensional case they generally fail. For a general operator $K \in \mathcal{L}(X)$, the discrepancy dim N_T- dim(X/R_T) is called the *Fredholm index* of K.

Spectral Theory

For a general operator $T \in \mathcal{L}(X)$, the *resolvent set* is given by $\rho(T) = \{\lambda \in \mathcal{C} : (\lambda I - T) \in \mathcal{GL}(X)\}$. The *spectrum* of T is then given by $\sigma(T) = \mathbb{C} \setminus \rho(T)$. For every operator, the spectrum satisfies $\sigma(T) \subseteq \overline{B_{\|T\|}(0)}$ where $\overline{B_{\|T\|}(0)}$ denotes the closed ball of radius $\|T\|$ centered at the origin. Furthermore, $\sigma(T)$ is always closed and nonempty. Unfortunately, for a general operator $T \in \mathcal{L}(X)$, $\sigma(T)$ can be uncountable and may contain elements that are **not** eigenvalues for T. This makes the spectrum difficult to work with for a general bounded linear operator.

However, for a compact operator $C \in \mathcal{C}(X)$, every $\lambda_j \neq 0$ in $\sigma(C)$ **is** an eigenvalue of C, the spectrum of C is at most countably infinite, and the only possible accumulation point for $\lambda_j \in \sigma(C)$ is 0. Furthermore, if C is self-adjoint and compact, we can conclude that C is diagonally dominated in the sense that $\|T\| = \max(\lambda_+, -\lambda_-)$ where $\lambda_+ = \underset{\|u\|=1}{\sup} \langle Tu, u\rangle$ and $\lambda_- = \underset{\|u\|=1}{\inf} \langle Tu, u\rangle$. Furthermore, the sup and inf are both attained, and $\lambda_+ = \max(\sigma(T))$, $\lambda_- = \min(\sigma(T))$. It follows that for $C \in \mathcal{C}(X)$ where X is a separable Hilbert space and C is self-adjoint, C has an *eigenbasis*, i.e., there exists an orthonormal basis $\{x_n\}_{n=1}^{\infty}$ for X such that $Cx_n = \lambda_n x_n$. For example, the Laplace transform is a compact self-adjoint operator in $\mathcal{C}(L^2(\mathbb{R}_+))$, with $\lambda_+ = \sqrt{\pi}$ where $(\sqrt{\pi}, t^{-1/2})$ is an Eigenpair.

More generally, for $C \in \mathcal{C}(X)$ where X is a Hilbert space and $C = C^*$, we have the *max-min* and *min-max principles*:
Let S_n denote any n-dimensional subspaces of X. Then if

$$\alpha_n = \max_{S_n} \min_{\substack{u \in S_n \\ \|u\|=1}} \langle Tu, u\rangle, \qquad \textbf{or} \qquad \alpha_n = \min_{\substack{u \in S_n \\ \|u\|=1}} \max_{S_n} \langle Tu, u\rangle,$$

then $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, and all non-zero α_j are eigenvalues of T.