Vector Calculus

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Definitions

- *Scalar field*: a function $f : V \to F$ where V is a vector space (usually $\mathbb{R}^2$ or $\mathbb{R}^3$) and F is a field (usually $\mathbb{R}$).
  - For $V = \mathbb{R}^3$, this is often denoted: $f(x, y, z)$.
- *Vector field*: a function $F : V \to V$ where V is a vector space (usually $\mathbb{R}^2$ or $\mathbb{R}^3$).
  - For $V = \mathbb{R}^3$, this is often denoted: $F(x, y, z) = P(x, y, z)\hat{\imath} + Q(x, y, z)\hat{\jmath} + R(x, y, z)\hat{k}$, where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ denote the unit basis vectors in the x, y, and z directions.
- A curve C in $\mathbb{R}^2$ or a surface S in $\mathbb{R}^3$ is *parameterized* by a function $\gamma(t)$ or $\gamma(u, v)$ if γ traces C or S while t or (u, v) are in the domain of γ.
  - A curve C parameterized by $\gamma(t)$, $t \in [a, b]$ is *closed* if $\gamma(a) = \gamma(b)$.
  - A curve C is *simple* if it does not cross itself anywhere.
  - A set $D \subset \mathbb{R}^n$ is *connected* if every pair of points in D can be connected by a curve in D, and *simply connected* if D is connected and every simple closed curve in D encloses only points in D.
- A curve C is *positively oriented* if it is traced in the counterclockwise direction by $\gamma(t)$.
- A surface S with exactly two faces is *positively/outward/upward* oriented if all of its normal vectors point outward from the surface.
- For a scalar field f on $\mathbb{R}^3$, the *gradient vector field* is denoted by: $\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$, where f_x, f_y, and f_z denote the partial derivatives of f w.r.t. x, y, and z respectively.
- A vector field F is said to be *conservative* if $F = \nabla f$ for some scalar field f (called the *potential function*).
  - In a conservative vector field, the work done moving a particle around a closed curve is 0. I.e., for a force F and a closed curve C parameterized by γ, the work done is given by $\int_C F \cdot d\gamma = 0$.
  - In a conservative vector field, the work done in moving a particle from point a to b is independent of the path $\gamma(t)$.

- For a vector field $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, the *divergence* of F is given by
$$div(F) := \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$
  - Physical intuition: Let F represent fluid flow in $\mathbb{R}^3$ and let p be a point in $\mathbb{R}^3$. Then if $div(F(p)) > 0$, the net flow of the fluid (flux) is outward at p; and if $div(F(p)) < 0$, the net flow of fluid is inward at p.

- For a scalar field f, the *Laplacian* of f is given by
$$\triangle f := div(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

- For a vector field $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$, the *curl* of F is given by
$$curl(F) := \nabla \times F = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle.$$

Surface and Line Integrals

In general, line and surface integrals are computed using the measure γ defined by $d\gamma = |\gamma'|dt$, where γ parameterizes the line/surface.

- The *line integral* or *curve integral* of a scalar field $f : \mathbb{R}^2 \to \mathbb{R}$ over a curve C in $\mathbb{R}^2$ parameterized by $\gamma(t)$, $t \in [a,b]$ is given by:
$$\int_C f d\gamma := \int_a^b f(\gamma(t))\|\gamma'(t)\|dt.$$
  - Note: for a curve C, the negative orientation of C is denoted $-C$. Let $\gamma(t)$ parameterize C for $t \in [a,b]$. Then $\phi(t) = \gamma(a+b-t)$ traces $-C$ for $t \in [a,b]$, and $\phi \int_{-C} f(x,y)d\phi = \int_C f(x,y)d\gamma$.
  - Physical intuition: if $\rho(x,y)$ gives the density of a wire at all points (x,y) defined over some curve $C \subset \mathbb{R}^2$, then $m = \int_C \rho d\gamma$ gives the *mass* of the wire, and the *center of mass* is given by $(\bar{x}, \bar{y})$ where $\bar{x} = \frac{1}{m}\int_C x\rho(x,y)d\gamma$ and $\bar{y} = \frac{1}{m}\int_C y\rho(x,y)d\gamma$.
  - Note that $M_y := x\rho(x,y)$ is called the *moment* about the y-axis, and $M_x := y\rho(x,y)$ is called the moment about the x-axis.
  - All the above still hold for a curve $C \in \mathbb{R}^3$ and a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$.

- The line integral of a vector field $F(x,y,z) = P(x,y,z)\hat{i} + Q(x,y,z)\hat{j} + R(x,y,z)\hat{k}$ over a curve C in $\mathbb{R}^3$ parameterized by $\gamma(t)$, for $(t) \in [a,b]$, is given by
$$\int_C F \cdot d\gamma := \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt.$$

- Note: Let $\gamma(t)$ parameterize C for $t \in [a,b]$. Then $\int_{-C} F \cdot d\gamma = -\int_C F \cdot d\gamma$.
- Physical intuition: if $F(x,y,z)$ represents a force exerted on a particle at position $(x,y,z) \in \mathbb{R}^3$ and $\gamma(t)$ parameterizes the path travelled by that particle through space, then the work done in moving the particle from $\gamma(a)$ to $\gamma(b)$ is given by $\int_C F \cdot d\gamma$.

- The *surface integral* of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ over a surface S parameterized by $\gamma(u,v)$, $(u,v) \in D$ is given by

$$\int\int_S f(x,y,z)d\gamma := \int\int_D f\left(\gamma(u,v)\right) \|\gamma_u \times \gamma_v\| dA,$$

where γ_u and γ_v denote the partial derivatives of γ w.r.t. u and v respectively, and dA denotes the standard double integral over the region $D \subseteq \mathbb{R}^2$.

  - Note: $\int\int_D \|\gamma_u \times \gamma_v\| dA$ gives the surface area of S.
  - Physical intuition: If $\rho(x,y,z)$ is a density function defined over a surface S, then the mass of S is given by $m = \int\int_S \rho(x,y,z)d\gamma$, and the center of mass is given by $(\bar{x}, \bar{y}, \bar{z})$ where $\bar{x}$, $\bar{y}$, and $\bar{z}$ are defined similarly as in the case of the wire.

- The *surface integral* of a vector field $F(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ over a surface S parameterized by $\gamma(u,v)$, $(u,v) \in D$ is given by

$$\int\int_S F \cdot d\gamma := \int\int_D f\left(\gamma(u,v)\right) \cdot \left(\gamma_u \times \gamma_v\right) dA,$$

where γ_u and γ_v denote the partial derivatives of γ w.r.t. u and v respectively, and dA denotes the standard double integral over the region $D \subseteq \mathbb{R}^2$.

  - Physical intuition: if $F(x,y,z)$ represents the rate of fluid flow at a point (x,y,z), then $\int\int_S F(x,y,z) \cdot d\gamma$ gives the *flux* of F over S, i.e., the rate at which fluid is flowing out of S (flux > 0) or into S (flux < 0).

Major Theorems

The following two theorems establish the fundamental properties of conservative vector fields as analogues to continuous functions.

The Fundamental Theorem of Calculus for Line Integrals

Given a once continuously differentiable scalar field f ($f \in C^1$) defined over a curve C parameterized $\gamma(t)$ (for $t \in [a,b]$), $\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$.

Note that as a corollary, it follows that every conservative vector field F satisfies:

- Conservation of energy: For a closed curve C, $\int_C F \cdot d\gamma = 0$.
- Independence of paths: For any two paths C_1 and C_2 satisfying $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$, $\int_{C_1} F \cdot d\gamma_1 = \int_{C_2} F(\gamma(t)) \cdot d\gamma_2$.

Conservative Vector Fields

Let $F(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ be a a vector field, for which all first order partials exist and are continuous (C^1) in some simply connected domain D. Then, F is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ for all $(x,y) \in D$.

The remaining theorems establish a fundamental relationship between the value of an integral and its boundary, establishing a link between PDEs and boundary value problems.

Green's Theorem

Let C be a simple closed curve enclosing some region D in $\mathbb{R}^2$. Then for a vector field $F(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$,

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Stoke's Theorem

Let S be a positively oriented surface in $\mathbb{R}^3$ (parameterized by $\phi(u,v)$) enclosed by a curve $C := \partial S$ (parameterized by $\gamma(t)$). Let $F(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ be a vector field. Then

$$\int_C F \cdot d\gamma = \int\int_S curl(F) \cdot d\phi.$$

Gauss's (Divergence) Theorem

Let E be a solid in $\mathbb{R}^3$ (of type 1, 2, or 3), and let S be the surface bounding E (with positive orientation) parameterized by γ. Let $F(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$ be a vector field with continuous first order partial derivatives. Then

$$\int\int_S F \cdot d\gamma = \int\int\int_E div(F)dV.$$