

# SS: Harmonic Analysis

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## Fourier Series and Transforms on the Circle

In general, the Fourier series for a function  $f$  is obtained by projecting  $f$  onto the sine and cosine *basis functions* of the form:  $\cos(nx)$  and  $\sin(nx)$ .

Let  $S^1$  denote the unit circle and suppose  $f$  is real-valued. Note that in the  $L^2(S^1)$  *inner product space*, the trigonometric polynomials are *orthogonal*:

$$\begin{aligned}\int_0^{2\pi} \cos(nx) \cos(mx) &= 0 && \text{if } n \neq m \\ \int_0^{2\pi} \sin(nx) \sin(mx) &= 0 && \text{if } n \neq m \\ \int_0^{2\pi} \cos(nx) \sin(mx) &= 0 && \text{for all } n, m.\end{aligned}$$

It follows that when normalized by  $\frac{1}{\sqrt{\pi}}$ , the series:

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(x), \frac{1}{\sqrt{\pi}} \sin(x), \frac{1}{\sqrt{\pi}} \cos(2x), \frac{1}{\sqrt{\pi}} \sin(2x), \dots$$

forms an *orthonormal basis* for some subspace of  $L^2(S^1)$ .

The best  $L^2$  approximation to  $f$  in the span of these basis functions is therefore given by the *projection* onto the sine and cosine polynomials. Since they are orthonormal, this projection is easy to compute:

$$f(x) \approx a_0 \frac{1}{\sqrt{2\pi}} + a_1 \frac{1}{\sqrt{\pi}} \cos(x) + b_1 \frac{1}{\sqrt{\pi}} \sin(x) + a_2 \frac{1}{\sqrt{\pi}} \cos(2x) + b_2 \frac{1}{\sqrt{\pi}} \sin(2x) + \dots$$

where

$$\begin{aligned}a_0 &= \langle f, \frac{1}{\sqrt{2\pi}} \rangle &&= \int_0^{2\pi} f(x) \frac{1}{\sqrt{2\pi}} dx, \\ a_n &= \langle f, \frac{1}{\sqrt{\pi}} \cos(nx) \rangle &&= \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \cos(nx) dx, \\ b_m &= \langle f, \frac{1}{\sqrt{\pi}} \sin(mx) \rangle &&= \int_0^{2\pi} f(x) \frac{1}{\sqrt{\pi}} \sin(mx) dx.\end{aligned}$$

More generally, if  $f$  is a complex-valued function, and we allow  $S^1$  to have any length  $L$ , we can perform the same analysis with the basis functions

$$\frac{1}{\sqrt{L}}, \frac{\sqrt{2}}{\sqrt{L}} \cos(2\pi x/L), \frac{\sqrt{2}}{\sqrt{L}} i \sin(2\pi x/L), \frac{\sqrt{2}}{\sqrt{L}} \cos(4\pi x/L), \frac{\sqrt{2}}{\sqrt{L}} i \sin(4\pi x/L), \dots$$

where the sine polynomials now carry an  $i$  out front and each function is normalized by  $L$ .

Using *Euler's identity* and folding the normalizing terms  $\frac{1}{L}$  into the coefficients  $\hat{f}(n)$ , the Fourier series for any function on the circle  $S^1$  (or equivalently, any function that is periodic over an interval  $[a, b]$  of length  $L$ ) can be written:

$$f(x) \approx \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x / L} \quad \text{where} \quad \hat{f}(n) = \frac{1}{L} \langle f, e^{2\pi i n x / L} \rangle = \frac{1}{L} \int_a^b f(x) e^{-2\pi i n x / L} dx.$$

The function  $\hat{f}(n)$  that gives Fourier coefficients is known as the *Fourier transform* of  $f$ .

As a consequence of the *Weierstrass Approximation Theorem* and the fact that Fourier series is the  $L^2$  projection of  $f$ , we obtain the *inverse theorem*:

$$\lim_{N \rightarrow \infty} S_N := \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x / L} \rightarrow f(x).$$

Note that the above convergence is w.r.t.  $\|\cdot\|_{L^2}$  and may not be uniform.

In fact, if  $f$  is just  $\alpha$ -Hölder continuous, then the convergence will be uniform, and the rate of convergence for the partial sums  $S_N \rightarrow f(x)$  will be proportional to the order of differentiability. If  $f$  is discontinuous but still almost everywhere  $\alpha$ -Hölder smooth, the above still holds except in some neighborhood of the discontinuities, where the convergence can be highly non-uniform. This is known as the *Gibbs phenomenon*.

Two other important theorems are

- Parseval's Identity:  $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{\sqrt{L}} \|f\|_2^2$ , which can be thought of as a generalization of the Pythagorean theorem, and
- General Parseval:  $\sum_{n=-\infty}^{\infty} \hat{f}(n) \text{conj}(\hat{g}(n)) = \frac{1}{\sqrt{L}} \langle f, g \rangle$ , where the *conj*() denotes the complex conjugate.

## Wavelets

An alternative interpretation of the Fourier series is the *wavelet* approach. Note that  $\hat{f}(n)$  intuitively tells us how well  $f$  aligns with a pure wavelet of frequency  $n$ . So, by checking how well  $f$  aligned with every wavelet, we can reconstruct  $f$ . By this interpretation, the partial sums  $S_N = \sum_{n=-N}^N \hat{f}(n) e^{i n x}$  can be interpreted as *convolutions* with the *Dirichlet kernel*:  $D_N(x) = \sum_{n=-N}^N e^{i n x} = \frac{\sin((N+1/2)x)}{\sin(x/2)}$ .

$$S_N = (f * D_N)(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) D_N(x - y) dy.$$

Any kernel  $K_N$  which satisfies the following three properties is a “good kernel,” meaning  $\lim_{N \rightarrow \infty} (f * K_N)$  converges uniformly to  $f$ :

1.  $\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} K_N(x) dx = 1$  for all  $N \geq 1$ ,
2.  $\int_{-\pi}^{\pi} |K_N(x)| dx \leq M$  for some fixed  $M$  for all  $N \geq 1$ , and
3.  $\int_{\delta \leq |x| \leq \pi} |K_N(x)| dx \rightarrow 0$  as  $N \rightarrow \infty$

Unfortunately,  $D_N$  fails property (3) since its radius of influence shrinks too slowly. However, if  $f$  is “nice enough” (e.g.,  $f$  has some level of smoothness), the uniform convergence is still recovered since  $f$  makes up for the slow convergence of  $D_N$ .

## Fourier Analysis on the Real Line

Now we will extend the previous analysis to the entire real line, as opposed to just the unit circle  $S^1$ . When we do this, the discrete transform  $\hat{f}(n)$  becomes a continuous function  $\hat{f}(\xi)$ . Then the continuous analogue of the discrete series is the integral:

$$f(x) \approx \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

where the continuous transform  $\hat{f}$  of a frequency  $\xi \in \mathbb{R}$  is given by:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} dx.$$

Note that the previous theorems (inverse theorem and parseval) have continuous analogues, given that  $f$  is a “nice” function. To denote the space of nice functions, we will say that  $f$  is *Schwartz function* ( $f \in \mathcal{S}(\mathbb{R})$ ) if  $f$  is infinitely-differentiable and  $\sup_{x \in \mathbb{R}} |x|^k |f^{(\ell)}(x)|$  is bounded for all  $k, \ell \geq 0$ . The analogues of the above theorems are given by:

- Inverse theorem: for  $f \in \mathcal{S}(\mathbb{R})$ ,  $f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$ .
- Plancharel’s Theorem: If  $f \in \mathcal{S}(\mathbb{R})$ , then  $\|\hat{f}\|_{L^2} = \|f\|_{L^2}$  (can be thought of as the limit of Parseval).
- Good Kernels: Same three properties, but instead denoted  $K_\delta$  where  $\delta$  is a continuous variable describing the radius of  $K_\delta$ . Consequently, for property (3) we will take  $\delta \rightarrow 0$  in contrast to the bounded case where we took  $N \rightarrow \infty$ .

## Fourier Analysis in $\mathbb{R}^d$

The idea is that we will define our function in polar coordinates then extend all definitions and theorems similarly as before. The Schwartz space extends to  $\mathcal{S}(\mathbb{R}^d)$  where  $f \in \mathcal{S}(\mathbb{R}^d)$  if  $\sup_{x \in \mathbb{R}^d} |x^\alpha (\frac{\partial}{\partial x})^\beta f(x)|$  is finite for all *multi-indices*  $\alpha, \beta$ . The Fourier transform of  $f \in \mathcal{S}(\mathbb{R}^d)$  is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} dx.$$

The inverse theorem and plancharel both still hold for  $f \in \mathcal{S}(\mathbb{R}^d)$ .

The same definition of good kernels extends to kernels  $K \in \mathcal{S}(\mathbb{R}^d)$ . In a quick connection to probability theory, it is known that the Gaussian kernels  $K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\pi x^2/\delta}$  are a family of good kernels under this extended definition. This means that any function  $f$  can be reconstructed by testing how well it aligns with every shift of  $K_\delta$  as  $\delta \rightarrow 0$ .

## Finite Fourier Analysis

If we take  $f$  to instead be a function over some finite Abelian group  $G$ , then we can define  $f$  similarly and repeat all the Fourier analysis. However, for a group  $G$  of cardinality  $N$ , the first  $N$  basis functions form a complete orthonormal basis for all of space. These basis functions are the  $N$  roots of unity:  $e^{-2\pi i k/N}$ . The *Fast Fourier Transform* (FFT) is a recursive algorithm for computing all  $N$  projections (discrete Fourier transforms) in  $\mathcal{O}(N \log N)$  time.

## Some Applications

The Fourier series was originally proposed as the solution to the standing wave equation:  $u(x, t) = \phi(x)\psi(t)$ , and to the heat equation:  $\frac{\partial u}{\partial t} = \Delta u$  where  $\Delta u$  denotes the Laplacian of  $u$ .

In the realm of number theory, the discrete Fourier series is also used in the proof of Dirichlet's Theorem. In engineering domains, Fourier series are used for signal compression and reconstruction, solving inverse problems such as the Radon transform. In physics, Fourier transforms can be used to derive the Heisenberg uncertainty principle and Huygen's principle.

## Fractals and the Hausdorff Dimension

The *Hausdorff dimension* of a set  $E$ , is the natural dimension in which  $E$  has nontrivial measure. This can be computed by defining the Hausdorff measure:

$$H_\alpha^\varepsilon(E) = \inf_{B_j} \sum_{j=1}^n r_j^\alpha$$

where  $B_j$  are balls of radius  $r_j < \varepsilon$  that cover  $E$ . Then let  $H_\alpha := \lim_{\varepsilon \rightarrow 0^+} H_\alpha^\varepsilon(E)$ .

Let  $\alpha_0$  satisfy  $H_\alpha(E) = \infty$  for  $\alpha < \alpha_0$  and  $H_\alpha(E) = 0$  for  $\alpha > \alpha_0$ . Then the Hausdorff dimension of  $E$  is given by:

$$\dim_H(E) := \alpha_0.$$

That is,  $\dim_H(E)$  is the smallest dimension in which  $E$  has finite measure and the largest dimension in which  $E$  has nonzero measure. For sets, where  $\dim(E)$  is obvious,  $\dim_H(E)$  agrees with our intuition. However, this definition also allows for  $\dim_H(E)$  to take non-integer values (i.e., consider the Cantor set). Such sets are called *fractals*.

*Frostman's lemma* states that a compact set  $A$  has Hausdorff dimension of at least  $s$  if and only if there exists a probability measure  $\mu$  supported on  $A$  and a constant  $C$  such that  $\mu(B(x, r)) \leq Cr^s$  for all balls  $B(x, r)$  of radius  $r$ . For each such  $\mu$ , we can further derive a finite energy integral:

$$I_s(\mu) = \int \int |x - y|^{-s} d\mu(x) d\mu(y) = M < \infty.$$

Then we can use a Fourier transform of the measure  $\mu$  to derive the inverse statement.