Complex Analysis

Tyler Chang
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Basic Definitions

Let $z = re^{i\theta} = x + iy$ be a complex number.

- $|z| = r = \sqrt{x^2 + y^2}$ is called the *modulus* of z.
- arg $z = \theta$ is called the *argument* of z.
  - Note that arg z is not well-defined since $re^{i\theta} = re^{i(\theta+2\pi m)}$ for all $m \in \mathbb{Z}$.
- Arg $z = \theta$, where $-\pi < \theta \leq \pi$ is called the *principle argument* of z.
  - Note that Arg z is well-defined, but not continuous since there is a "slit" on $\mathbb{R}_-$ (the negative real-axis), where Arg z "jumps" in value from $-\pi$ to π or vice versa.

Let $f = u + iv$ be a complex-valued function.

- If f is one-to-one on $\mathbb{C} \setminus \mathbb{R}_-$, the *inverse* is not well-defined unless it is independent of arg z. For example, the inverse of $e^{i\theta}$ is given by $\log z = \log|z| + i\text{arg } z$, and therefore not well-defined. We call the range of arg z the *branch* of $\log z$. Log $z = \log|z| + i\text{Arg } z$ is then called the *principle branch* of the function, and is well-defined, but typically not continuous.
- A complex-valued function is said to be a *linear fractional transformation* (LFT) if it is of the form $w(z) = \frac{ax+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.
  - Every LFT is a composition of complex-valued translations, dilations, and inversions.
  - 3 pairs $(z_1, w(z_1))$, $(z_2, w(z_2))$, and $(z_3, w(z_3))$ uniquely determine an LFT.
  - The image of a circle or line under an LFT is either a circle or a line.
- A complex-valued function f is said to be differentiable at a point z_0 if $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists (for every direction, determined by arg z). If the limit exists, it is called $f'(z)$.

Let $\gamma(t)$ where $a \leq t \leq b$ be a *path* from A to B. Let $P = u_1(x, y) + iv_1(x, y)$, $Q = u_2(x, y) + iv_2(x, y)$ be a complex-valued functions (in the complex plane with $z = x + iy$).

- $P\ dx + Q\ dy$ is called a *differential.*
- $\int_\gamma P\ dx + Q\ dy = \int_\gamma P dx + \int_\gamma Q dy$ is called the *line integral* of that differential.

- $P\ dx + Q\ dy$ is called *exact* if there exists $dh = P\ dx + Q\ dy$.
- The line integral $\int_\gamma P\ dx + Q\ dy$ is *independent of path* if it has the same value for all paths with the same endpoints, i.e., the differential $P\ dx + Q\ dy$ is somehow *conservative* in a sense, depending only on its endpoints.
- $P\ dx + Q\ dy$ is exact if and only if $\int_\gamma P\ dx + Q\ dy$ is independent of path.
- A differential $P\ dx + Q\ dy$ is *closed* if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.
- Exact always implies closed, but closed only implies exact in a *star-shaped region* D, meaning every point in D is "visible" from some central point.
- *Green's Theorem* tells us that if P and Q are smooth on a domain D with piecewise smooth boundary, and the partials of P and Q exist and are continuous to ∂D, then

$$\int_{\partial D} P\ dx + Q\ dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\ dy.$$

Holomorphic/Analytic Functions

A complex function $f = u + iv$ is said to be *holomorphic* or *analytic* on an open domain D if f is once continuously differentiable everywhere in D. Some equivalent conditions follow:

- Both partial derivatives of u and v exist in D and *the Cauchy-Riemann equations* hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

- $f\ dz$ is closed. Note, this implies that $f\ dz$ is exact if D is star-shaped. This is important because it implies $\int_\gamma f\ dz$ is independent of paths.
- f is continuous and $\int_{\partial R} f\ dz = 0$ for all rectangles R in D.
- If $f'(z)$ exists for all $z \in D$, then f' must be continuous, so f is automatically analytic if it is just differentiable everywhere in D.

For f analytic in D, the *Fundamental Theorem of Calculus* holds, i.e., the integral $\int_\gamma f' dz = f(B) - f(A)$ is independent of paths. Furthermore, in a star-shaped region, there exists an analytic F such that $F' = f$ and $F(z) = \int_{z_0}^z f(\xi) d\xi$.

If D is star-shaped and bounded with piecewise smooth boundary and f is continuous on $\overline{D}$, then by Green's theorem $\int_{\partial D} f\ dz = 0$ (*Cauchy's thoerem*). The *Cauchy Integral Formula* follows:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} dw.$$

It follows by interchanging the order of integration that f is infinitely differentiable and

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{m+1}} dw.$$

In fact, the closed path ∂D can be replaced by any loop in D enclosing z.

Furthermore, for f analytic in a ball $B(r, z_0)$ centered at z_0 with radius r, then the *Taylor series*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(m)}(z_0)}{m!} (z - z_0)^k$$

converges uniformly. The largest radius R for which f is analytic in $B(R, z_0)$ is called the *radius of convergence*. As a corollary, if f is analytic in a star-shaped region, then f is determined globally by its local behavior, since f's Taylor series at each point can be extrapolated to determine f's behavior in the entire star-shaped region. A function f that is analytic on the entire complex plane is called *entire*, and *Liouville's theorem* tells us that every bounded entire function is identically equal to a constant.

If f is not analytic in a ball around z_0, but is instead analytic in the annulus $r_1 < |z - z_0| < r_2$, then f has a *Laurent series*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

where $a_k = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{k+1}} dz$, with the line integral evaluated counterclockwise. Now consider any *singularity*, i.e., point s where f is not analytic. If s is isolated, then in some annulus $0 < |z - s| < \varepsilon$, f is analytic with a Laurent series $f(z) = \sum_{k=-\infty}^{\infty} a_k (z - s)^k$.

- If all $a_k = 0$ for all $k < 0$, then f is a *removable singularity*, meaning f is actually analytic in the ball $B(\varepsilon, s)$.
- If $a_k \neq 0$ for only finitely many $k < 0$, then s is a *pole*, meaning f behaves like an analytic function near a_k, but diverges such that $f(s) \to \infty$.
- If $a_k = 0$ for infinitely many $k < 0$, then s is an *essential singularity*, meaning $\lim\limits_{z \to s} f(z)$ does not exist. In fact, the limit of a sequence $f(z_n)$ where $z_n \to s$ attains nearly every value, depending on the path taken by z_n.

A function f that is analytic except at isolated singularities, each of which is a pole, is called *meromorphic*. A function f is meromorphic in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ if and only if f is *rational*.

If f is analytic and one-to-one in D, then f is said to be *univalent* on D. Every analytic function f is univalent except in a neighborhood of any *critical points* (points where $f'(z) = 0$). Furthermore, if f is nonconstant, then by the *open mapping theorem*, f maps open sets to open sets. If follows that f^{-1} exists and is analytic except in a neighborhood of f's critical points. Furthermore, if f is univalent, then f is *conformal*, meaning f preserves angles.

Harmonic Functions

A function $u(x, y)$ on a two-dimensional domain D is said to be *harmonic* if it satisfies Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. If $f = u + iv$ is an analytic function, then u and v are real-valued harmonic functions. It follows that every analytic function is harmonic. Furthermore, for all $z_0 \in D$, every harmonic function u has a local *harmonic conjugate*, i.e., a harmonic function v such that $f = u + iv$ is analytic in some neighborhood of z_0. If D is star-shaped, then v is a harmonic conjugate to u on all of D.

Every harmonic function also satisfies the *mean value property*,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

If h is a bounded complex-valued harmonic function on D that extends to the boundary, then h must attain its maximum on ∂D. Furthermore, if $|h(z)| \leq M$ for all $z \in D$ and $|h(z_0)| = M$ for some $z_0 \in D$, then $h \equiv C$, where C is a constant. These properties are often referred to as the *maximum principle.*

Dirichlet showed that if $h(\theta)$ is continuous on a circle S, then there exists a unique harmonic function $\tilde{h}(re^{i\theta})$ defined on the disk $\mathbb{D}$ with boundary S such that $\tilde{h} = h$ on S. His solution is given by

$$\tilde{h}(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{i(\theta - \phi)}) P_r(\phi) d\phi$$

where $P_r(\phi) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\phi}$ is the *Poisson kernel.*

Contour Integration and the Residue Theorem

The *residue theorem* generalizes Cauchy's theorem. Let z_0 be an isolated singularity of $f(z)$. Then if $f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n$, $0 < |z - z_0| < \rho$, then the *residue* of f at z_0 is given

$$\mathrm{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \int_{|z - z_0| = r < \rho} f(z) dz.$$

Furthermore, if D is a star-shaped bounded domain in $\mathbb{C}$ with p.w. smooth boundary (as in Cauchy's theorem), and f is meromorphic in D (with singularities z_i) and smooth in $\overline{D}$, then

$$\int_{\partial D} f(z) dz = 2\pi i \sum_{i=1}^{\infty} \mathrm{Res}(f(z), z_i).$$

Contour integration is a process for integrating meromorphic (or equivalently, rational) functions f using the residue theorem. Suppose we wish to integrate a rational function f over some path γ (most commonly, a subset of the real axis, i.e., a definite real integral). Suppose f cannot be integrated over γ, but can be integrated over another path ϕ in D, with the same endpoints as γ. Then using the residue theorem, $\int_\gamma f = \sum_{i=1}^{n} \mathrm{Res}(f(z), z_i) - \int_\phi f$ where z_i

are singularities contained in the loop $\gamma \cup \phi$, and each integral is taken in the counterclockwise direction.

To make residues easy to compute, there are three rules:

- If z_0 is a *simple pole* (only a_{-1} is nonzero) then $a_{-1} = \lim_{z \to z_0} (z - z_0) f(z)$.
- If z_0 is a pole of order n, then $a_{-1} = \lim_{z \to z_0} \frac{d^{(n+1)}}{dz^{(n+1)}} ((z - z_0)^n f(z))$.
- If f, g are analytic at z_0 and g has a simple zero at z_0, then $\frac{f}{g}$ has a simple pole if and only if $f(z_0) \neq 0$.

Furthermore, when constructing loops, one must avoid singularities. The *fractional residue theorem* tells us that for a simple pole z_0, for C_ε an arc of the circle $|z - z_0| = \varepsilon$ with an angle of $\alpha > 0$ oriented counterclockwise, $\lim_{\varepsilon \to 0} \int_{C_\varepsilon} f(z)dz = \alpha i \text{Res}(f, z_0)$.

Homotopy, Winding Numbers, and Simply Connected Regions

The *logarithmic integral* of an analytic function f over a closed curve γ in D is given by $\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} dz = \frac{1}{2\pi} \int_\gamma d(\text{arg} f(z))dz$. The logarithmic integral is always a nonnegative integer, and the change in arg f around γ is given by 2π times the logarithmic integral. If D is a bounded domain, with p.w. smooth boundary and f is meromorphic in $\overline{D}$ with no zeros or poles on ∂D, then the logarithmic integral of f around the boundary of D is equal to $N_0 - N_\infty$ where N_0 denotes the number of zeros of f in D, and N_∞ denotes the number of poles (with multiplicity). This property is called the *argument principle* and can be used to perform root-finding on f. Furthermore, if g is also analytic in $\overline{D}$ and $|f|$ dominates $|g|$, then f and $f + g$ have the same number of zeros in D.

Furthermore, if $f(z_0) = w_0$, then there exists a neighborhood $N_\delta = |w - w_0| < \delta$ in which f attains all $w \in N_\delta$ the same number of times. Let U_ε be a sufficiently small neighborhood of z_0. Then the number of times f hits $w \in N_\delta$ is given by the *winding number* of the image $f(\partial U_\varepsilon)$ around w_0.

The *winding number* of a closed curve $\gamma(t)$ $(a \leq t \leq b)$ about a point z_0 is given by

$$W(\gamma, z_0) = \frac{h(b) - h(a)}{2\pi}$$

and tells the integer number of times γ "wraps around" z_0 (i.e., the change in $\arg(z - z_0)$ around γ divided by 2π). Two curves $\gamma_0(t)$ and $\gamma_1(t)$ defined in a domain D are *homotopic* if there exists a curve $\gamma(s,t)$, $0 \leq s \leq t$, such that $\gamma(0,t) = \gamma_0(t)$, $\gamma(1,t) = \gamma_1(t)$, and $\gamma(s,t)$ is jointly continuous in both s and t. If γ_0 is a point, then $\gamma_1(t)$ is said to be homotopic to a point. Equivalently, this implies that $W(\gamma_1(t), \xi) = 0$ for all $\xi \notin D$. A *simply connected* domain is defined as a domain in which every curve γ is homotopic to a point.

This terminology allows us to generalize Cauchy's theorem, so that if f is analytic in D with p.w. smooth boundary, and $\gamma \subset D$ is a closed curve with $W(\gamma, \xi) = 0$ for all $\xi \notin D$, then

$\int_\gamma f(z)dz = 0$. Note, this implies that an analytic f is infinitely differentiable in any *simply connected domain*. Furthermore, the condition that ∂D is p.w. smooth can be dropped if we are willing to "back off" slightly from the edge of D.

Analytic Continuation and Reflection

Consider now a function f defined by its Taylor series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$. In some ball $B(R, z_0)$. The goal of this section is to extend f to its entire *natural domain*.

In fact, if γ is any (not necessarily closed) curve on which f has no singularities, and we can define a sequence of partially overlapping balls $B(R_n, z_n)$ such that each $z_n \in \gamma$, then f can be extended along the entire path γ by power series centered at z_n. This process is called *analytic continuation*, and the resulting function f will be analytic in $\cup_n B(R_n, z_n)$ if and only if this sequence of balls does not "wrap around" any singularities (in which case, f may not agree where $B(R_n, z_n)$ wrap back on themselves). Furthermore, if f can be continued along two paths γ_0 and γ_1 which are homotopic by a sequence γ_s of paths along which f can also be analytically continued, then the continuations of f along γ_0 and γ_1 agrees (the *Monodromy theorem*). It follows that f can be analytically continued along any path in a simply connected domain.

Alternatively, if D is a domain that is symmetric about the real-axis, and f^+ is analytic on the top-half of D (D^+), then f^+ can be reflected across the real-axis to the bottom half D_- by $f^-(z) = \overline{f}(\overline{z})$ for all $z \in D_-$. This reflecton defines an analytic function f on all of D if $\lim_{z\to\mathbb{R}} f^+(z) \in \mathbb{R}$ for all limit points in $D \cap \mathbb{R}$. It follows that by applying a LFT, to any domain E whose boundary shares an arc or line in $\mathbb{C}^*$, a similar reflection can be performed, if E's image under the LFT satisfies the properties above.

Hyperbolic and Spherical Geometries

Let $f : \mathbb{D} \to \mathbb{D}$ be analytic (where $\mathbb{D}$ denotes the open unit disk). Then *Schwarz's lemma* tells us that if $f(0) = 0$ and $|f(z)| \le |z|$, then f is a rotation if any $z_0 \neq 0$ satisfies $|f(z_0)| = |z_0|$, and a contraction otherwise. This generalizes to *Pick's lemma*: If f $f : \mathbb{D} \to \mathbb{D}$ is analytic then

$$|f'(z)| \le \frac{1 - |f(z)|^2}{1 - |z|^2},$$

and if equality holds for any $z_0 \in D$, then f is a conformal self-map of D.

Let $f : \mathbb{D} \to \mathbb{D}$ be conformal. The *hyperbolic length* of a curve γ is defined by

$$\ell(\gamma) = 2\int_\gamma \frac{|dz|}{1 - |z|^2},$$

and $\ell(\gamma) = \ell(f \circ \gamma)$. The *hyperbolic distance* between two points is then defined by

$$\rho(z_1, z_2) = \inf_\gamma \ell(\gamma),$$

where the inf is taken over all curves γ connecting z_1 and z_2. Note that this inf is obtained for some unique path γ, called the *hyperbolic geodesic*. Note that in such a *hyperbolic geometry*, the sum of all angles in a triangle is strictly less than $180°$.

Similarly, the *chordal metric* is defined for two points p and q on $\mathbb{C}^*$ by taking the inverse stereoscopic projection of $\mathbb{C}^*$ onto the sphere, and computing the length of the line segment joining p and q on the sphere. The *chordal distance* is given by twice the length of the shortest such line segment (which corresponds to an arc of a great circle on the sphere). Note that in general, the *chordal geodesic* is not necessarilly unique, and the sum of angles in a triangle is strictly greater than $180°$.