General Topology

Tyler Chang
March 19, 2018

Basic Definitions

A topology is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a set of subsets of X such that $\mathcal{T}$ satisfies three properties:

- $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.
- $\mathcal{T}$ is closed under arbitrary unions. I.e., if some (possibly uncountable) family of sets $\{U_\alpha\} \in \mathcal{T}$, then $\cup U_\alpha \in \mathcal{T}$.
- $\mathcal{T}$ is closed under finite intersections. I.e., if $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* under the topology $(X, \mathcal{T})$, and the complements of open sets are called *closed sets* under $(X, \mathcal{T})$. A set is *clopen* if it is both open and closed (note, $\emptyset$ and X are always clopen by definition).

An open set containing some $x \in X$ is called an open neighborhood of x. The largest open set contained in a set E is called the *interior* of E and is denoted E°. The smallest closed set containing E is called the *closure* of E and is denoted $\bar{E}$. The *boundary* of a set E, denoted ∂E, is the set difference between the closure of E and the interior of E. A point is on the boundary of E if and only if every open neighborhood of x contains points both in E and in E^c. A set E is dense in X if $\bar{E} = X$, and nowhere dense if $\left(\bar{E}\right)^\circ = \emptyset$.

A sequence of points $\{x_n\}$ in X is said to *converge* to a point $x \in X$ if for every open neighborhood U of x, there exists some value N such that $x_n \in U$ for all $n > N$. Then x is said to be a *limit point* of $\{x_n\}$. In general, the limit point of a sequence need not be unique. However, in a Hausdorff space (described later) every limit is unique. Using this terminology, the following are equivalent definitions of closedness:

- A set C is closed (by definition) if it is the complement of an open set.
- A set C is closed if and only if every convergent sequence of points in C converges to a point in C (i.e., C contains all its limit points).
- A set C is closed if and only if it contains all its boundary points.

A topology $\mathcal{T}$ is said to be generated by a basis $\mathcal{B}$ if all the elements of $\mathcal{T}$ can be written as unions of elements in $\mathcal{B}$. The standard topology on $\mathbb{R}^d$ is the topology generated by open balls $B(x, r)$ where $B(x, r) = \{y : d(y, x) < r\}$ where $d(x, y) = \|x - y\|_2$. In general, any topology generated by open balls with respect to some metric d is called a metric space. A metric space X is said to be *complete* if every Cauchy sequence: ($\{x_n\}$ such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$) converges to a point in X.

A set K is compact if every open cover of K (i.e., set of open sets whose union contains K) has a finite subcover. The *Heine-Borel Theorem* states that in a finite-dimensional metric space, this is equivalent to stating that K is closed and bounded. In an infinite-dimensional metric space, we need the further condition that K is closed and bounded, and K lies within some the ε-neighborhood of some finite-dimensional metric space where $\varepsilon > 0$. An entire space X is compact if as a set, X is compact under its own topology.

Continuity and Homeomorphisms

A function $f : X \to Y$ is said to be *continuous* with respect to $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ if $f^{-1}(U) \in \mathcal{T}_X$ for all in $U \in \mathcal{T}_Y$.

Note, that if we were to rename the elements of X while maintaining the same topology on the renamed set, then the change would be superficial and, from a topological perspective, we wouldn't be able to distinguish between the spaces. For example,

$$X = \{a, b, c\}, \qquad \mathcal{T}_X = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$$

is indistiguishable from

$$Y = \{1, 2, 3\}, \qquad \mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}.$$

We capture this notion of sameness by saying X and Y are *homeomorphic*. If a continuous bijective function h exists between X and Y, then clearly they are homeomorphic, and we call h a *homeomorphism* between X and Y. Properties of a topology that are preserved under homeomorphisms are called *topological invariants*. For example, the existance of clopen sets other than X and $\emptyset$ is a topological invariant.

One important topological invariant is the *Hausdorff* property, which states that for every pair of points $x_1, x_2 \in X$, there exists a neighborhood U_1 of x_1 and U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. Another important topological invariant is *separability*, defined as the existence of a countable dense subset of X. For example, $\mathbb{R}$ is separable under the standard topology since the rationals are countable and desne in $\mathbb{R}$.

Manifolds

A n-dimensional *manifold* is a topological space that "looks like" n-dimensional Euclidean space when we zoom in close enough. To put this formally, every point x in a manifold has an open neighborhood U_x that is homeomorphic to $\mathbb{E}^n$ (that is, if we restrict the topology on X to its intersection with U_x). For example, the surface of the Earth is a 2-dimensional manifold because to a person standing on its surface, it looks like a flat plane.