COMPUTING THE UMBRELLA NEIGHBOURHOOD OF A VERTEX IN THE DELAUNAY TRIANGULATION AND A SINGLE VORONOI CELL IN ARBITRARY DIMENSION

Tyler Chang[a], Layne Watson[abc], Thomas Lux[a], Sharath Raghvendra[a], Bo Li[a], Li Xu[d], Ali Butt[a], Kirk Cameron[a], and Yili Hong[d]

Departments of Computer Science[a], Mathematics[b], Aerospace & Ocean Engineering[c], and Statistics[d]

Virginia Polytechnic Institute and State University
Blacksburg, VA 24061-0106 USA

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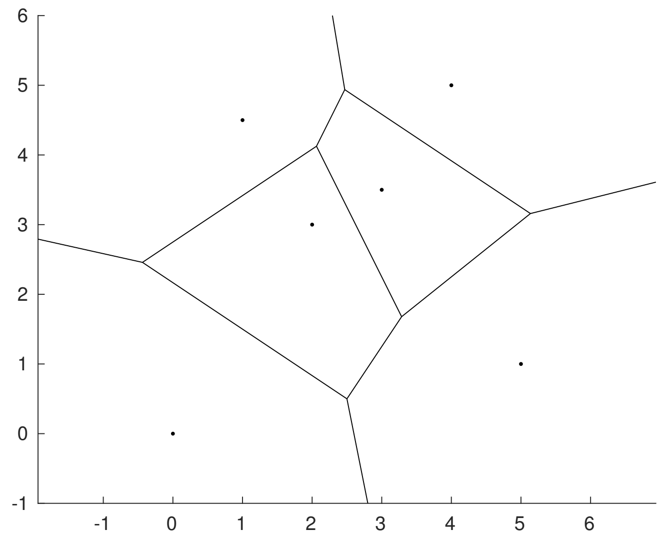
What is a Tessellation?

A d-dimensional *tessellation* of some convex region $X \subseteq \mathbb{R}^d$ is a division of X into closed (typically convex) sets called cells such that:

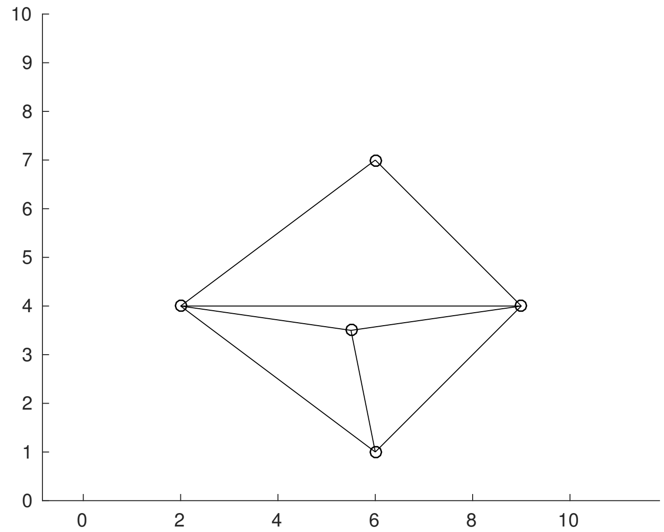
- They are disjoint except along their shared boundaries.
- Their union is X.



What is a Triangulation?

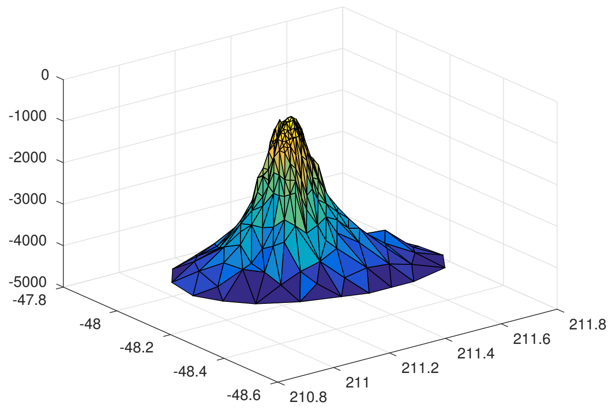
A d-dimensional *triangulation* T of a (finite) set of points P in $\mathbb{R}^d$ is a tessellation of the *convex hull* of P, such that:

- Each cell in T is a d-simplex.
- The vertices of each simplex are points in P.



Who Cares?

- Discretizing space.
- Interpolation and mesh generation.
- Computer vision/graphics.
- Topological data anlaysis (α-shapes and k-skeletons).
- Notions of "nearness" and "neighbors" in high-dimensions.



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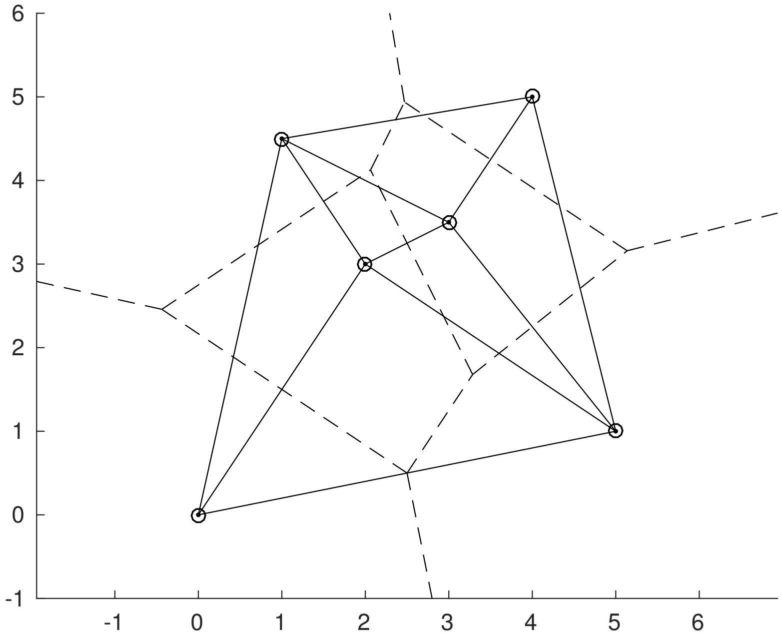
The Voronoi Diagram and Delaunay Triangulation

Given points $P \in \mathbb{R}^d$, the Voronoi diagram is a “nearest neighbor” tessellation.

The Delaunay triangulation is its *geometric dual*.



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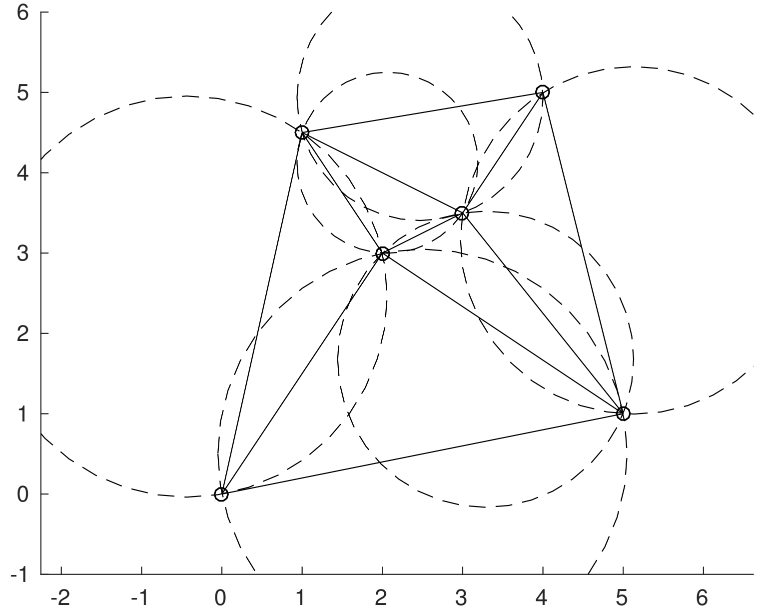
The Empty Circumsphere Property

More usefully, the Delaunay triangulation can be defined in terms of its *empty circumsphere property*:



Note that the center of each circumsphere is a Voronoi vertex.

Existence and Uniqueness

- Trivially, the Voronoi diagram always exists and is unique.
- Since the Delaunay triangulation is defined in terms of the Voronoi diagram, it exists so long as not all the points in P lie in an *affine subspace*.
- Unique if no $d + 2$ or more points lie on the same circumsphere.
- Since the probability of degeneracy is zero, assume the points are in *general position* and the Delaunay triangulation exists and is unique.

The Curse of Dimensionality

Both can be reasonably computed in two or three dimensions

Unfortunately both the Delaunay triangulation and the Voronoi diagram grow exponentially in *size* with respect to the dimension.

In the worst case, the number of Voronoi vertices/Delaunay simplices for a set of n points in $\mathbb{R}^d$ is

$$\mathcal{O}(n^{\lceil d/2 \rceil}).$$

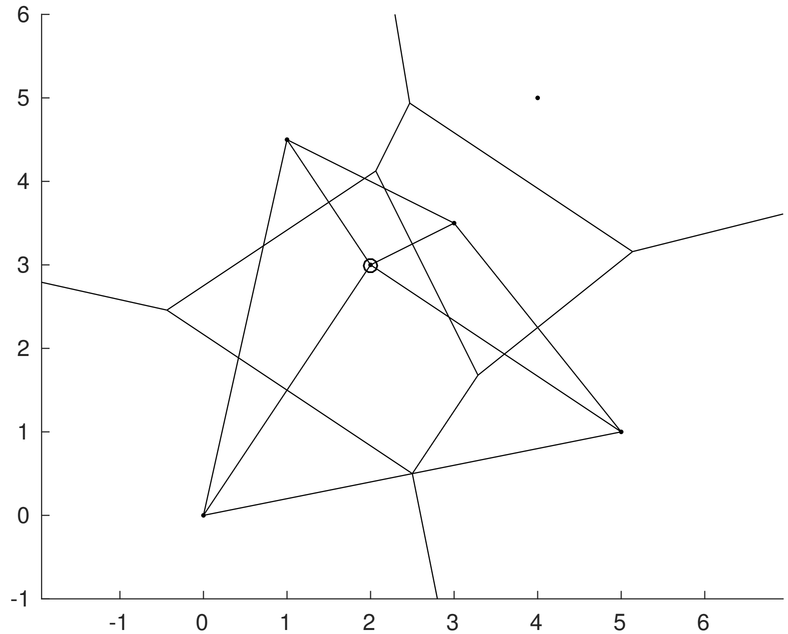
That means that the computational time *and* space complexity *must* grow *exponentially*!

Umbrella Neighborhoods and Single Voronoi Cells

The Delaunay *umbrella neighborhood* of a point $p \in P$ is the set of simplices "incident" at P.

I.e., the set of simplices with P as a vertex.

Since the circumcenter of each simplex is a Voronoi vertex and vice versa, computing the umbrella neighborhood of p is equivalent to computing the *Voronoi cell* of p.



Computing the Umbrella Neighborhood

Applications that only require the umbrella neighborhood/a single Voronoi cell:

- Multiobjective optimization: Well-spacedness in parameter space (Deshpande, Watson, and Canfield, 2016).
- Modelling of porosity in a pebble-bed nuclear reactor: How much space between each pebble? (Rycroft et al., 2006).

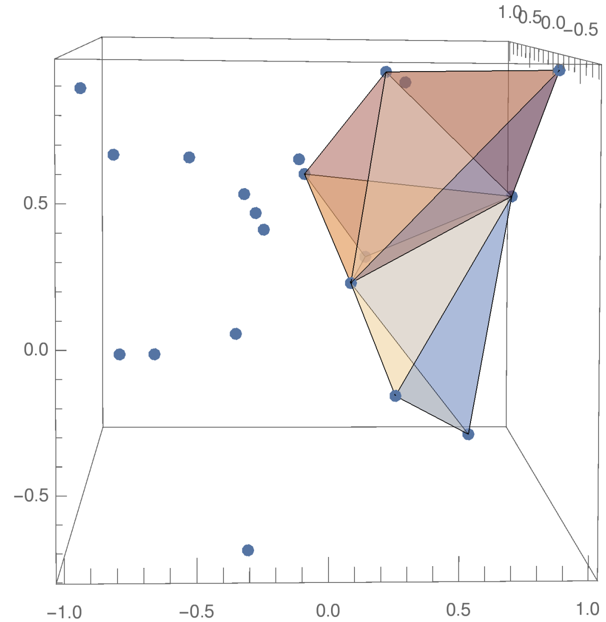
Research Questions

Can one compute just an umbrella neighborhood/single cell without computing the whole triangulation/diagram?

How much time will that save?

New Algorithm for Computing the Umbrella Neighbourhood

- Grow an initial simplex from p (through a sequence of least squares problems).
- Flip accross each facet containing p as a vertex to "close" them.
- Keep track of all "open" facets containing p as a vertex in a data structure (called the "AFL").
- Iterate until all open facets have been closed.



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Algorithm Analysis

- Takes $\mathcal{O}\left(nd^4\right)$ time to compute the first simplex.
- Takes $\mathcal{O}\left(nd^3\right)$ time to perform each "flip."
- If there are k simplices in the umbrella neighbourhood, must perform k "flips."
- If $k << |DT(P)|$, this will save a significant amount of compute time/space!

Results: Empirical Analysis

Average run times in seconds for sizes n and dimensions d (with a sample size of 20, for randomly generated data)

	$n = 2K$	$n = 8K$	$n = 16K$	$n = 32K$
$d = 2$	0.1 s	1.6 s	6.3 s	25.0 s
$d = 3$	0.1 s	1.8 s	7.0 s	27.8 s
$d = 4$	0.2 s	2.0 s	7.6 s	30.0 s
$d = 5$	0.3 s	2.6 s	9.2 s	34.1 s

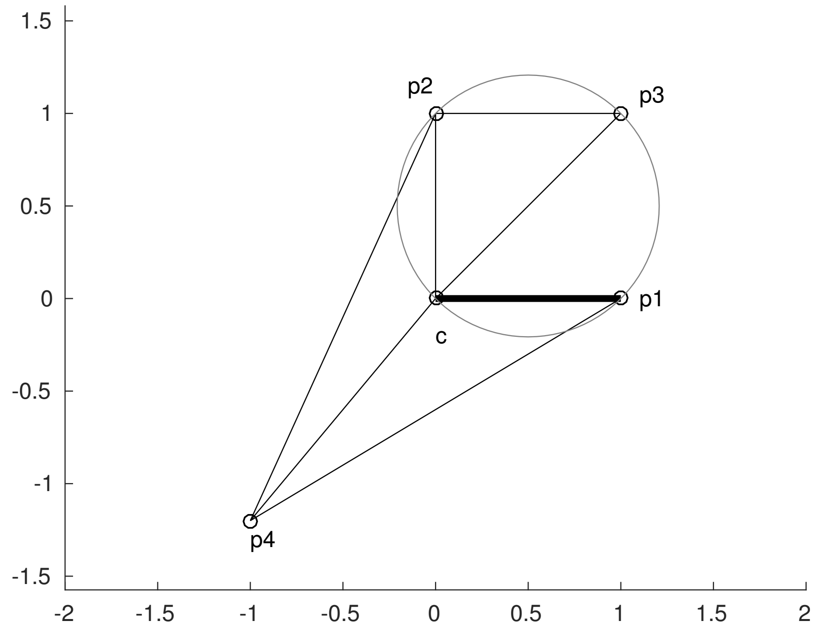
Average number of simplices in the umbrella neighbourhood

	$n = 2K$	$n = 8K$	$n = 16K$	$n = 32K$
$d = 2$	6.30	5.90	6.25	6.65
$d = 3$	28.10	28.90	29.50	29.10
$d = 4$	151.90	170.55	173.60	161.00
$d = 5$	1122.90	1115.70	1111.00	1038.70

Issues in Stability

Thus far have only considered points in *general position*. In the real world, degeneracies happen, and this can be disastrous for the algorithm.

- The only known solution is perturbation.



Conclusion

A new scalable algorithm for computing Delaunay umbrella neighbourhoods / Voronoi diagrams was introduced.

Empirically, it is able to save a significant amount of time by computing a small subset of the complete triangulation.

Unfortunately, it has trouble handling degenerate data.

Future Work

Investigate perturbation strategies and their stability guarantees.

Compare to **Voro++** library for computing a single Voronoi cell in 3-dimensions.