Categorical Semantics of Type Theory

Kunhong Du

In the classical settings of model theory, logical systems are interpreted in set-based structures. It turns out to work well for first order logic, but in some more elaborate cases, traditional, set-theoretic models may lack generality or at least cause much inconvenience. That's why we introduce the idea of *structure valued in a category.* For a particular logical concept, we seek properties and extra structures of a category to intepret the concepts (for example, contexts, types and terms in a type theory) such that the interpretation respects the logical rules and axioms.

1 Formal type theory

Since we need to interpret the rules and axioms of type theory, we need a more careful treatment of the formal system of type theory.

By a context, we mean a finite (possibly empty) list of distinct variables with their respective typings:

$$x_1 : A_1, ..., x_n : A_n$$

Contexts are often denoted by Γ, Δ, etc.

We have five forms of judgments:

1. $\Gamma\ \mathsf{ctx}$
2. $\Gamma \vdash A\ \mathsf{type}$
3. $\Gamma \vdash a : A$
4. $\Gamma \vdash A \equiv B\ \mathsf{type}$
5. $\Gamma \vdash a \equiv b : A$

The derivibility of those judements are defined inductively by the *inference rules* of type theory.

Let B be a term / type / context, write $B[a/x]$ for the *substitution* of a term a for free occurrences of the variable x in the term B. Moreover, let $B[a_1, ..., a_n/x_1, ..., x_n]$ denote the simultaneous substitution.

2 Models of type theory

2.1 A naîve interpretation

We want to 'interpret' every judgement by something in a category, such that *the interpretation respects the logical rules and axioms.*

Let $\mathcal{C}$ be a category.

Type-theoretic judgments	Interpretation in a category $\mathcal{C}$
$\Gamma\ \mathsf{ctx}$	$[\![\Gamma]\!] \in \mathrm{Ob}\,\mathcal{C}$
$\Gamma \vdash A\ \mathsf{type}$	A morphism with codomain $[\![\Gamma]\!]$
$\Gamma \vdash a : A$	A section of $[\![\Gamma \vdash A\ \mathsf{type}]\!]$
Definitional equalities	Strict equalities between morphisms

What do we mean by a sound interpretation?

With the semantics of rules given, a partial function from judgements to categorical terms (objects and morphisms) is obtained by structural induction. We say the interpretation is sound if all derivable judgements are defined and all equality judgements are validated with respect to the actual equality in the category.

Structural rules

We want to examine what properties of the interpretation and the category are needed for it to respect the inference rules and axioms.

For the first two rules about contexts, things are easy.

$$\frac{}{\mathsf{ctx}}\ \text{ctx-EMP} \quad \Big|\ \mathcal{C} \text{ has a terminal object 1, and } [\![]\!] := 1$$

$$\frac{\Gamma \vdash A\ \ \mathsf{type}}{\Gamma.A\ \mathsf{ctx}}\ \text{ctx-EXT} \quad \Big|\ \text{Have a map } \gamma \text{ with codomain } \Gamma. \text{ Let } \Gamma.A \text{ be the domain of } \gamma.$$

VBLE

$$\frac{\Gamma \vdash A\ \ \mathsf{type}}{\Gamma, x : A \vdash x : A}\ \text{VBLE}$$

$$\begin{tikzcd}
\Gamma.A \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& \Gamma.A \times_\Gamma \Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\
& \Gamma.A \arrow[r] & \Gamma
\end{tikzcd}$$

SUBST

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash B\ \mathsf{type}}{\Gamma, \Delta[a/x] \vdash B[a/x]\ \mathsf{type}}\ \text{SUBST}$$

$[\![\Gamma.\Delta[a/x]]\!]$ is the pullback

$$\begin{tikzcd}
[\![\Gamma.\Delta[a/x]]\!] \arrow[r] \arrow[d] & [\![\Gamma.A.\Delta]\!] \arrow[d, "pr"] \\
[\![\Gamma]\!] \arrow[r, "a"] & [\![\Gamma.A]\!]
\end{tikzcd}$$

And the conclusion of the rule is the dashed arrow.

$$
\begin{tikzcd}
\llbracket A.\Delta[a/x].B[a/x] \rrbracket \arrow[r] \arrow[d, dashed] & \llbracket \Gamma.A.\Delta.B \rrbracket \arrow[d, "pr"'] \\
\llbracket \Gamma.\Delta[a/x] \rrbracket \arrow[r] \arrow[d] & \llbracket \Gamma.A.\Delta \rrbracket \arrow[d, "pr"] \\
\llbracket \Gamma \rrbracket \arrow[r, "a"] & \llbracket \Gamma.A \rrbracket
\end{tikzcd}
$$

For typing judgement:

$$
\frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] : B[a/x]} \text{SUBST}
$$

is then the section given by the universal property of pullback

$$
\begin{tikzcd}
\llbracket A.\Delta[a/x].B[a/x] \rrbracket \arrow[r] \arrow[d] & \llbracket \Gamma.A.\Delta.B \rrbracket \arrow[d] \\
\llbracket \Gamma.\Delta[a/x] \rrbracket \arrow[r] \arrow[u, dashed, bend left] & \llbracket \Gamma.A.\Delta \rrbracket \arrow[u, bend left, "b"]
\end{tikzcd}
$$

The equality judement is then the strict equalities of the pullbacks or sections. The slogan is ”substitution is pullback”.

WKG

$$
\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, \Delta \vdash B \ \mathsf{type}}{\Gamma, x : A, \Delta \vdash B \ \mathsf{type}} \text{WKG}
$$

is interpreted as the dashed arrow of

$$
\begin{tikzcd}
\llbracket \Gamma.A.\Delta.B \rrbracket \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma.\Delta.B \rrbracket \arrow[d] \\
\llbracket \Gamma.A.\Delta \rrbracket \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma.\Delta \rrbracket \arrow[d] \\
\llbracket \Gamma.A \rrbracket \arrow[r] & \llbracket \Gamma \rrbracket
\end{tikzcd}
$$

The typing judgement

$$
\frac{\Gamma \vdash A \ \mathsf{type} \qquad \Gamma, \Delta \vdash b : B}{\Gamma, x : A, \Delta \vdash b : B} \text{WKG}
$$

is then again given by the universal property of pullback:

$$
\begin{tikzcd}
\llbracket \Gamma.A.\Delta.B \rrbracket \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma.\Delta.B \rrbracket \arrow[d] \\
\llbracket \Gamma.A.\Delta \rrbracket \arrow[r] \arrow[d] \arrow[u, dashed, bend left] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma.\Delta \rrbracket \arrow[d] \arrow[u, bend left, "b"] \\
\llbracket \Gamma.A \rrbracket \arrow[r] & \llbracket \Gamma \rrbracket
\end{tikzcd}
$$

Logical rules

First we notice, for $f : B \to A$, let $f^* : \mathcal{C}/A \to \mathcal{C}/B$ be the pullback along f. Then the composition $\Sigma_f \dashv f^*$. What about its right adjoint? We say a category is locally cartesian closed if it has pullback, a terminal object and that every f^* admits a right adjoint denoted by Π_f. As the notion implies, they are essentially how we interpret the Σ-types and Π-types.

Π-types

$$\frac{\Gamma, x : A \vdash B(x)\ \ \mathsf{type}}{\Gamma \vdash \Pi_{x:A} B(x)\ \ \mathsf{type}}\ \Pi\text{-}\textsc{form}
\qquad\Bigg|\qquad
\begin{tikzcd}
\llbracket \Gamma.A.B \rrbracket \arrow[d, "g"] & \Pi_f \llbracket \Gamma.A.B \rrbracket \arrow[d, dashed, "\Pi_f(g)"] \\
\llbracket \Gamma.A \rrbracket \arrow[r, "f"] & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

The Intro rule is given by the adjointness.

Σ-types

$$\frac{\Gamma, x : A \vdash B(x)\ \ \mathsf{type}}{\Gamma \vdash \Sigma_{x:A} B(x)\ \ \mathsf{type}}\ \Sigma\text{-}\textsc{form}
\qquad\Bigg|\qquad
\begin{tikzcd}
\llbracket \Gamma.A.B \rrbracket \arrow[d, "g"'] \arrow[dr, "f \circ g = \Sigma_f(g)"] & \\
\llbracket \Gamma.A \rrbracket \arrow[r, "f"'] & \llbracket \Gamma \rrbracket
\end{tikzcd}$$

$$\frac{\Gamma, x : A \vdash B(x)\ \ \mathsf{type}}{\Gamma, x : A, y : B(x) \vdash (x, y) : \Sigma_{x:A} B(x)\ \ \mathsf{type}}\ \Sigma\text{-}\textsc{form}$$

$$\begin{tikzcd}
\llbracket \Gamma.A.B.\Sigma \rrbracket \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \llbracket \Gamma.\Sigma \rrbracket = \llbracket \Gamma.A.B \rrbracket \arrow[d] \arrow[dr] & \\
\llbracket \Gamma.A.B \rrbracket \arrow[r] \arrow[u, dashed, bend left, "\Delta"] & \llbracket \Gamma.A \rrbracket \arrow[r] & \llbracket \Gamma \rrbracket & \llbracket \Gamma.A \rrbracket \arrow[l]
\end{tikzcd}$$

Id-types

$$\frac{\Gamma \vdash A\ \ \mathsf{type}}{\Gamma, x : A, y : A \vdash \mathsf{Id}_A(x, y)\ \ \mathsf{type}}\ \mathsf{Id}\text{-}\textsc{form}
\qquad\Bigg|\qquad
\begin{tikzcd}
\Gamma.A = \mathsf{Id}_A \arrow[dr, dashed, "\Delta"] & & \\
& \Gamma.A \times_\Gamma \Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d] \\
& \Gamma.A \arrow[r] & \Gamma
\end{tikzcd}$$

In a more natural or "parameterized" form:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathsf{Id}_A(a, b)\ \ \mathsf{type}}\ \mathsf{Id}\text{-}\textsc{form}
\qquad\Bigg|\qquad
\mathsf{Id}_A(a, b) \xrightarrow{\text{equalizer}} \Gamma \underset{b}{\overset{a}{\rightrightarrows}} \Gamma.A$$

This is obtained by using substitution twice:

$$
\begin{tikzcd}
\mathsf{Id}_A(a,b) \arrow[rr] \arrow[d, red] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathsf{Id}_A \arrow[d] & \\
\Gamma \arrow[r, "b"] \arrow[rr, bend left, "\langle a,b\rangle"] & A \arrow[r, "\langle a\circ\pi,\mathrm{id}\rangle"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A\times_\Gamma A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "\pi"'] \\
& \Gamma \arrow[r, "a"] & A \arrow[r, "\pi"] & \Gamma
\end{tikzcd}
$$

$$
\frac{\Gamma \vdash A \ \ \mathsf{type}}{\Gamma, x : A \vdash \mathsf{refl}_x : \mathsf{Id}_A(x,x)}\ \mathsf{Id}\text{-}\textsc{intro}
\quad\Bigg|\quad
\begin{tikzcd}
\Delta^*\,\mathsf{Id}_A = A \arrow[r] \arrow[d] & \mathsf{Id}_A = A \arrow[d, "\Delta"'] \\
A \arrow[u, dashed, bend left, "\mathsf{refl}_A"] \arrow[r, "\Delta"] & A\times_\Gamma A
\end{tikzcd}
$$

$$
\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}_a : \mathsf{Id}_A(a,a)}\ \mathsf{Id}\text{-}\textsc{intro}
\quad\Bigg|\quad
\begin{tikzcd}
\mathsf{Id}_A(a,a) \arrow[r, "\text{equalizer}"] & \Gamma \arrow[r, shift left, "a"] \arrow[r, shift right, "a"'] \arrow[l, dashed, bend left, "\text{unique section}"] & \Gamma.A
\end{tikzcd}
$$

Finally, the trickiest one.

$$
\frac{\Gamma, x, y : A, u : \mathsf{Id}_A(x,y) \vdash C(x,y,u) \qquad \Gamma, z : A \vdash d(z) : C(z,z,\mathsf{refl}_A(z))}{\Gamma, x, y : A, u : \mathsf{Id}_A(x,y) \vdash J_{z,d}(x,y,u) : C(x,y,u)}\ \mathsf{Id}\text{-}\textsc{elim}
$$

First a careful analysis of the axiom on the right, which is derived by performing substitution twice:

$$
\frac{\Gamma, x : A \vdash \mathsf{refl}_A(x) : \mathsf{Id}_A(x,x) \qquad \dfrac{\Gamma, x : A \vdash x : A \qquad \Gamma, x, y : A, u : \mathsf{Id}_A(x,y) \vdash C(x,y,u)}{\Gamma, x : A, u : \mathsf{Id}_A(x,x) \vdash C(x,x,u)}}{\Gamma, x : A \vdash d(x) : C(x,x,\mathsf{refl}_A(x))}
$$

which is interpreted as

$$
\begin{tikzcd}
\mathsf{refl}_A^*(\Delta^* C) \arrow[r] \arrow[d] & \Delta^* C \arrow[r] \arrow[d] & C \arrow[d] \\
A \arrow[u, bend left, "d"] \arrow[r, "\mathsf{refl}_A"] & \Delta^*\,\mathsf{Id}_A \arrow[r] \arrow[d] & \mathsf{Id}_A \arrow[u, dashed, bend left, "J"] \arrow[d] \\
& A \arrow[r, "\Delta"] & A\times_\Gamma A
\end{tikzcd}
$$

The rule is really saying: given a section d, we have a section J.

This does hold simply because so many identities here:

$$
\begin{tikzcd}
C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & C \arrow[d] \\
A \arrow[u, bend left, "d"] \arrow[r, "\mathsf{refl}_A=\mathrm{id}"] & A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[u, dashed, bend left, "J"] \arrow[d, "\Delta"] \\
& A \arrow[r, "\Delta"] & A\times_\Gamma A
\end{tikzcd}
$$

Now it's easy to see $\Gamma, x : A \vdash J_{z,d}(x, x, \mathsf{refl}_A(x)) : C(x, x, \mathsf{refl}_A(x))$ is derived from performing substitution twice. Hence the computation rules says that $J'' = d$ as morphisms.

$$
\begin{tikzcd}
\mathsf{refl}_A^*(\Delta^* C) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Delta^* C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d] \\
A \arrow[r, "\mathsf{refl}_A"] \arrow[u, dashed, bend left, "J''"] \arrow[u, bend right, "d"'] & \Delta^* \mathsf{Id}_A \arrow[r] \arrow[d] \arrow[u, dashed, bend left, "J'"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathsf{Id}_A \arrow[d] \arrow[u, bend left, "J"] \\
& A \arrow[r, "\Delta"] & A \times_\Gamma A
\end{tikzcd}
$$

3 Problems of the naîve model

3.1 Coherence

Type-theoretically substitution is a strictly functorial operation, whereas its categorical counterpart, pullback, without making any choices, is functorial only up to isomorphism.

Consider

$$\vdash a : A \quad x : A \vdash b : B \quad x : A, y : B \vdash T \ \ \mathsf{type}\,.$$

Then we can do substitution twice and obtain

$$\vdash (T[b/y])[a/x] \ \ \mathsf{type}\,.$$

This process has semantics as performing pullback twice:

$$
\begin{tikzcd}
(T[b/y])[a/x] \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & T[b/y] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\
1 \arrow[r, "a"] & A \arrow[r, "b"] & B
\end{tikzcd}
$$

On the other hand, we have, syntatically,

$$(T[b/y])[a/x] = (T[a/x])[(b[a/x])/y].$$

And the right formula has semantics:

$$
\begin{tikzcd}
(T[a/x])[(b[a/x])/y] \arrow[r] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & T[a/x] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\
1 \arrow[r, "{b[a/x]}"] & B[a/x] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
& 1 \arrow[r, "a"] & A
\end{tikzcd}
$$

One can show that the resulting maps are isomorphic but not neccessarily equal if the pullbacks are chosen arbitrarily.

One possible solution is to make pullbacks a definite data of a category. (For example, a category with attributes.) In this way, making substitution is no longer arbitrary but under control.

3.2 Intensional vs. extensional

A type theory with Id-type is said to be extensional if the *reflection* rule is derivable:

$$\frac{\Gamma \vdash p : \mathsf{Id}_A(a, b)}{\Gamma \vdash a \equiv b : A}\ \mathsf{Id}\text{-}\textsc{refl}$$

The interpretation above does validate Id-REFL because if the following equalizer admits a section, then $a = b$.

$$\mathsf{Id}_A(a, b) \xrightarrow{\text{equalizer}} \Gamma \underset{b}{\overset{a}{\rightrightarrows}} \Gamma.A$$

Hence Id-REFL is indeed consistent with MLTT, but is it derivable? The question is answered negetively by a counter model given Hofmann and Streicher [1].

3.3 Universe and univalence

Don't forget the axiom making the HoTT distinguished. It's a fact that it is consistent with MLTT [2] but constructing a model is way more complicate and will be left for my further talks.

References

[1] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. *Twenty-five years of constructive type theory (Venice, 1995)*, 36:83–111, 1998.

[2] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after voevodsky). *Journal of the European Mathematical Society*, 23(6):2071–2126, 2021.