Synthetic Algebraic Geometry
Talk II

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Recall: Propositions are certain types! The terms correspond to the proofs, that the proposition holds.

Notation. Let X be a type.

- We write $X \sqcup X'$ for the sum type.

- we have two equivalent ways to think about subtypes of X and we use the following notation to jump between the two

$$(x : X \ \vdash \ Ux : \mathrm{Prop}) \longmapsto \{x : X \mid Ux\} \equiv (\textstyle\sum_{x:X} Ux) \subset X$$

$$(x : X \ \vdash \ x \in U : \mathrm{Prop}) \longleftarrow U \subset X$$

- If U is a proposition indexed by terms $x : X$ then we write $\forall x : X, Ux \equiv \prod_{x:X} Ux$
- The propositional truncation of X is some $\eta_X : X \to \|X\|$ such that [1]

$$\begin{tikzcd} X \arrow[r, "\forall"] \arrow[d, "\eta_X"'] & P : \mathrm{Prop} \\ \|X\| \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Axiom 1 (Loc)**.** R is a local ring.

Let A denote a finitely presented ($R-$)algebra, i.e. its merely of the form

$$A = R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)$$

Definition 0.1. The spectrum is type of R-algebra homomorphisms

$$\mathrm{Spec}\, A = \mathrm{Hom}_R(A, R)$$

A type is a affine, or , an affine scheme, if it is merely of the form $\mathrm{Spec}\, A$ for some finitely presenteted algebra.

Lemma 0.2 (The Fundamental Theorem of AG)**.** *Picking a presentation gives an equivalence*

$$\begin{aligned} \mathrm{Spec}(R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)) &\to \{(x_1, \ldots, x_n) \in R^n \mid f_i(x_1, \ldots, x_n) = 0 \ \forall i\} \\ \phi &\mapsto (\phi(X_1), \ldots, \phi(X_n)) \end{aligned}$$

[1] I emphasize its important that P is a proposition. If P is more generally a set, one has to show, that the map $A \to P$ actually does not depend on the term $a : A$.This is called Lemma von Kraus

Axiom 2 (Duality)**.** The **duality map**

$$\begin{aligned} A &\to R^{\operatorname{Spec} A} \\ f &\mapsto (x \mapsto x(f)) \end{aligned}$$

is an R-algebra isomorphism between A and the algebra of R-valued functions on $\operatorname{Spec} A$. We may use this as a coercion, i.e for $f : A, x : \operatorname{Spec} A$ we may write $f(x) \equiv x(f)$.

Lemma 0.3 (Weak field)**.** *For any $x : R$, $x \neq 0$ iff x is invertible. More generally, a vector is non-zero iff one of its entries is non-zero.*

1 Affines have no choice

Recall the type theoretic principle of choice

Proposition 1.1 (Type theoretic principle of choice)**.** *Given a relation R depending on $x : A$ and $y : B$*

$$\left(\prod_{x:A} \{y : B \mid xRy\}\right) \simeq \{f : A \to B \mid \forall x : A, xR(fx)\}$$

Definition 1.2. A type A has choice, if for any type family B_x over $x : A$ we have

$$(\forall x : A, \|B_x\|) \to \|\prod_{x:A} B_x\|$$

Example 1.3. *If A has choice then any surjection $g : B \to A$ merely has a section $f : A \to B$. This follows by applying choice to the type family of fibers of g*

$$B_x \equiv \{y : B \mid \underbrace{xRy}_{\equiv gy=x}\}$$

and then use 1.1.

In synthetic algebraic geometry, we dont expect affine schemes to have choice:

Example 1.4. *We want to see that the local ring R does not have choice. For any $x : R$, consider the proposition*

$$(x \neq 0) \vee (1 - x \neq 0) \equiv \|(x \neq 0) \sqcup (x \neq 1)\|$$

this mereley holds by 0.3 and (Loc). A section would correspond to a term in

$$p : \prod_{x:R} (x \neq 0) \sqcup (x \neq 1)$$

We want to show that such a term is impossible Consider the function[2]

$$\begin{aligned} f \colon R &\to R \\ x &\mapsto \begin{cases} 1 & , \textit{ if } px \ \mathsf{isIn}\ x \neq 0 \\ 0 & , \textit{ if } px \ \mathsf{isIn}\ x \neq 1 \end{cases} \end{aligned}$$

which is an idempotent element in the algebra R^R. By Duality and 0.2 this is an idempotent polynomial $f : R[X]$, which is nonzero, e.g. at 1 by Loc. Moreover $f(0) = 0$ but every idempotent polynomial with this property is zero: We can factor $f = X \cdot g$ and then $f = f^n = X^n g^n$ so all coefficients vanish.

Remark 1. Secretly we have shown in the last example, that R is connected, i.e. every function $R \to 1 \sqcup 1$ is constant.

[2] If A, B are types , $p : A \sqcup B$, we write p isIn A for the decidable proposition $\sum_{a:A} \mathsf{inl}(a) = p$

2 The last axiom

Definition 2.1. for $f\colon A$, define the *principal open subsets of* $\operatorname{Spec} A$

$$D(f) = \{x\colon \operatorname{Spec} A \mid \operatorname{isInv}(f(x))\}$$

Lemma 2.2. *For $f : A$ we have $D(f) \simeq \operatorname{Spec}(A_f)$*

Proof.

$$\operatorname{Spec} A_f = \operatorname{Hom}_R(A_f, R) = \{x : \operatorname{Hom}_R(A, R) | \operatorname{isInv}(x(f))\} = D(f)$$

by the universal property of the localization. □

Axiom 3. for any family B of merely inhabited types over $\operatorname{Spec} A$ we merely find a principal open cover $\operatorname{Spec} A = \bigcup_i D(f_i)$ and local sections $\prod_{x:D(f_i)} Bx$:

$$(\forall x : X, \|Bx\|) \to \left\| \underbrace{\sum_{n:\mathbb{N}} \sum_{(f_1,\ldots,f_n):A^n} \left(\bigcup_{i=1}^{n} D(f_i) = \operatorname{Spec} A\right)}_{\text{principal open cover}} \times \prod_{i=1}^{n} \underbrace{\prod_{x:D(f_i)} Bx}_{\text{local sections}} \right\|$$

There is this connection of principal open covers to algebra:

Proposition 2.3 (Unimodularity)**.** *Given $f_1, \ldots, f_n : A$ we have $(f_1, \ldots, f_n) = A$ iff $\bigcup_i D(f_i) = \operatorname{Spec} A$*

We prove this later. Hence the axiom can be restated by just using commutative algebra and not principal open covers.

Lemma 2.4. *If $(f_1, \ldots, f_n) = (1)$ then $(f_1^k, \ldots, f_n^k) = (1)$ for all $k \geq 1$.*

Proof. Express the goal through 2.3. and then use

$$D(f_i) = \{x : X \mid \underbrace{f_i(x) \neq 0}_{\leftrightarrow f_i(x)^k \neq 0}\} = D(f_i^k)$$

□

Proposition 2.5 (Rewrite along equivalence)**.** *If $f : A' \simeq A$ and B is a family of propositions over A, then*

$$\{a' : A' \mid B(fa')\} \simeq \{a : A \mid Ba\}$$
$$(a', x) \mapsto (fa', x)$$

Example 2.6. *For some A and $g, g' : A$ define*

$$g \mid_A g' \equiv \|\{h : A \mid hg =_A g'\}\|$$

Claim: For any $g, g' : A$, we have

$$g|_A g' \leftrightarrow \forall x : \operatorname{Spec} A, gx|_R g'x$$

Proof. $\to$ is obvious using that the duality map is an algebra isomorphism.
$\leftarrow$. For any $x : \operatorname{Spec} A$ we merely find some $h : R$ with $h \cdot g(x) = g'(x)$, i.e. we define our family of inhabited types as

$$Bx = \{h : R \mid h \cdot g(x) = g'(x)\}$$

By zariski local choice we merely find some principal open cover $\operatorname{Spec} A = \bigcup_{i=1}^n D(f_i)$ and local sections

$$\begin{aligned} &\prod_{x:D(f_i)} \{h_i : R \mid h_i \cdot g(x) = g'(x)\} \\ &\overset{1.1}{\simeq} \{h_i : D(f_i) \to R \mid (h_i x) \cdot g(x) = g'(x)\} \\ &\overset{Duality}{\simeq} \left\{h_i : A_{f_i} \mid h_i \cdot \frac{g}{1} =_{A_{f_i}} \frac{g'}{1}\right\} \end{aligned}$$

We can multiply h_i by high enough powers of f_i to obtain some $h_i : A$ with $h_i \cdot g = g' \cdot f_i^n$ for some $n : \mathbb{N}$. we may assume that n does not depend on $i = 1, \ldots, n$ by taking the maximum and multiplying the h_i again with enough powers of f_i. Now use 2.4 to write $1 = \sum_{i=1}^n \ell_i f_i^n$ for some $\ell_i : A$ and then

$$(\sum_i \ell_i h_i) \cdot g = \sum_i \ell_i f_i^n g' = 1g' = g'$$

□

3 Topology

Definition 3.1.

- A proposition P is open iff there merely exists $f_1, \ldots, f_n \colon R$ such that
$$P = (f_1 \neq 0) \vee \ldots \vee (f_n \neq 0)$$
.
- A proposition P is closed iff there merely exists $f_1, \ldots, f_n \colon R$ such that
$$P = (f_1 = 0) \wedge \ldots \wedge (f_n = 0).$$
- A subtype $U \subset X$ is open (closed) iff the proposition $x \in U$ is open (closed) for all $x : X$.

Definition 3.2. Let A be an fin pres. algebra.

- For $f_1, \ldots, f_n \colon A$ define the closed subset, the *vanishing locus*
$$V(f_1, \ldots, f_n) = \{x : \operatorname{Spec} A \mid \bigwedge_{i=1}^n f_i(x) = 0\} \overset{0.2}{=} \operatorname{Spec}(A/(f_1, \ldots, f_n))$$
- a subset of $\operatorname{Spec} A$ is called Zariski open iff its merely of the form
$$D(f_1, \ldots, f_n) = \{x : \operatorname{Spec} A \mid \bigvee_{i=1}^n f_i(x) \neq 0\} = \bigcup_{i=1}^n D(f_i)$$

Example 3.3. *For any open subtype $U \subset Y$ and any map $f : X \to Y$ the preimage $\{x \colon X \mid f(x) \in U\}$ is open.*

Lemma 3.4. *The Open subsets of X contain $\varnothing, X$ (e.g. $D(1) = X$) and are stable under finite unions and intersections (0.3). Furthermore*

3.1 Proof of the unimodularity

Lemma 3.5 (Complements of closeds)**.** *for any $f_1, \dots, f_n$ we have $V(f_1, \dots, f_n)^c = D(f_1, \dots, f_n)$.*

Proof. The vector $(f_1(x), \dots, f_n(x))$ is non-zero if $f_i \neq 0$ for some i by 0.3 ∎

Unimodularity follows from setting $g = 1$ in

Lemma 3.6. *$D(g) \subset \bigcup_i D(f_i)$ iff $g \in \sqrt{(f_1, \dots, f_n)}$.*

Proof.

$$\begin{aligned}
&D(g) \subset D(f_1, \dots, f_n) \overset{3.5}{=} V(f_1, \dots, f_n)^c \\
&\leftrightarrow \varnothing = V(f_1, \dots, f_n) \cap D(g) = \mathrm{Spec}((A/(f_1, \dots, f_n)_g)) \qquad |\text{nullstellensatz} \\
&\leftrightarrow (A/(f_1, \dots, f_n))_g = 0 \\
&\leftrightarrow g \in \sqrt{(f_1, \dots, f_n)}.
\end{aligned}$$

∎

4 Subsets of affines

We want to equip the set $\mathrm{Spec}\, A$ with some kind of topology. On one hand, there is the pointwise definition of open. On the other hand, there is also the *Zariski topology* on , i.e. where the principal opens form a basis of the topology, i.e. the zariski open subsets. We need the third axioms to show that those two notions of topology coincide.

Lemma 4.1. *The map that evaluates every coeffcient*

$$\begin{aligned}
A[T] &\to (\mathrm{Spec}\, A \to R[T]) \\
g = \sum_{j=0}^{k} g_j T^j &\mapsto g^* = \left(x \mapsto \sum_{j=0}^{k} g_j(x) T^k \right)
\end{aligned}$$

is an equivalence.

Proof. Observing $\mathrm{Spec}\, A[T] \simeq \mathrm{Spec}\, A \times \mathrm{Spec}\, R[T]$ (by the universal property of the polynomial ring) gives

$$R^{\mathrm{Spec}\, A[T]} \simeq R^{\mathrm{Spec}\, A \times \mathrm{Spec}\, R[T]} \simeq (\mathrm{Spec}\, A \to R^{\mathrm{Spec}\, R[T]})$$

Conclude by rewriting along duality at $A[T]$ and $R[T]$. ∎

4.1 Closed subsets

Lemma 4.2. *A proposition is closed iff its merely of the form $g = 0$ for some $g : R[T]$*

Proof. A polynomial is zero iff all its coefficients are zero. ∎

Lemma 4.3. *We have $V(f_1, \dots, f_n) \subset V(g_1, \dots, g_m)$ iff $(g_1, \dots, g_m) \subset (f_1, \dots, f_n)$.*

Proof. Such an inclusion corresponds to a map $\mathrm{Spec}(A/(f_1, \dots, f_n)) \to \mathrm{Spec}(A/(g_1, \dots, g_m))$ over $\mathrm{Spec}\, A$. We conclude by fully faithfullness of Spec. ∎

Example 4.4. *Classically, Consider an inclusion of subsets* $V(x^2) \subset V(x)$ *inside* $\operatorname{Spec} R[x]$*. Now it does hold* $(x) \not\subset (x^2)$*. Instead, classically we have an inclusion of subsets of* $\operatorname{Spec} A$ $V(I) \subset V(J)$ *iff* $J \subset \sqrt{I}$*. So only if we consider them as closed subschemes we get the condition* $J \subset I$*. Hence internally the vanishing locus carries more structure than just the underlying set!*

Proposition 4.5. *Let* $C \subset \operatorname{Spec} A$ *be closed. Then there exists a unique finitely generated ideal* $I \subset A$ *such that* $C = \operatorname{Spec}(A/I)$

Proof. For any $x : X$ we merely find some $g : R[X]$ such that $x \in C$ iff $g = 0$. By Zariski Local choice we find a principal open cover $\operatorname{Spec} A = \bigcup D(f_i)$ and a term in

$$\begin{aligned}
&\prod_{x:D(f_i)} \{g : R[T] \mid x \in C \leftrightarrow g = 0\} \\
&\overset{1.1}{\simeq} \{g : D(f_i) \to R[T] \mid \forall x : D(f_i), x \in C \leftrightarrow gx = 0\} \\
&\overset{4.1}{\simeq} \{g : A_{f_i}[T] \mid \forall x : D(f_i), x \in C \leftrightarrow g^* x = 0\} \\
&\simeq \{g : A_{f_i}[X] \mid \{x : D(f_i) \mid x \in C\} = V((g_j)_j)\}
\end{aligned}$$

The finitely generated Ideals $I_i = (g_j)_j \subset A_{f_i}$ satisfy $I_i \cdot A_{f_i f_j} = I_j \cdot A_{f_i f_j}$ because under duality they both determine the functions $D(f_i f_j) \to R$ that vanish on $C \cap D(f_i f_j)$. By commutative algebra the ideals $I_i \subset A_{f_i}$ glue uniquely to some finitely generated ideal I such that $I \cdot A_{f_i} = I_i$. □

4.2 Opens subsets

Lemma 4.6. *Let* $U \subset \operatorname{Spec} A$*.* U *is Zariski-open iff there merely exists some* $g : A[T]$ *such that* $x \in U \leftrightarrow g^* x \neq 0$ *for all* $x : \operatorname{Spec} A$*.*

Proof. For any $x : \operatorname{Spec} A$, the proposition $g_0(x) \neq 0 \vee \ldots g_n(x) \neq 0$ is equivalent to $\sum_{i=0}^{n} g_i(x) T^i \neq 0$ in $R[T]$ by 0.3. □

Example 4.7. *A proposition (a subtype of* $\operatorname{Spec} R \simeq 1$*) is open iff its merely of the form* $g \neq 0$ *for some* $g : R[T]$*.*

Theorem 4.8. *A subtype* $U \subset \operatorname{Spec} A$ *is open iff it is Zariski-open.*

Proof. The backwards direction is easy: If $U = D(f_1, \ldots, f_n)$, then $x \in U$ iff $f_1(x) \neq 0 \vee \ldots \vee f_n(x) \neq 0)$.
Conversely, let U be an open subtype. For any $x : \operatorname{Spec} A$ $x \in U$ is merely of the form $g \neq 0$ for some $g : R[T]$. By Zariski-Local choice we find a principal open cover $\operatorname{Spec} A = \bigcup D(f_i)$ and a term in

$$\begin{aligned}
&\prod_{x:D(f_i)} \{g : R[T] \mid x \in U \leftrightarrow g \neq 0\} \\
&\overset{1.1}{\simeq} \{g : D(f_i) \to R[T] \mid \forall x : D(f_i), x \in U \leftrightarrow gx \neq 0\} \\
&\overset{4.1}{\simeq} \{g : A_{f_i}[T] \mid \forall x : D(f_i), x \in U \leftrightarrow g^* x \neq 0 \\
&\simeq \{g : A_{f_i}[T] \mid \{x : D(f_i) \mid x \in U\} = D((g_j)_j)\}
\end{aligned}$$

i.e. for all i, $U \cap D(f_i)$ is Zariski open subset of $D(f_i)$. By the next lemma this is enough. □

Lemma 4.9. *Let* $f : A$*. Every Zariski-open* $U \subset D(f)$ *of a principal open is Zariski open in* $\operatorname{Spec} A$

Proof. Its enough to check that principal opens of principal opens are principal open. For this use that localization of a localization is a localization. □

5 Line bundles

Definition 5.1.
- An n-dimensional $(R-)$vectorspace is an $R-$module V such that its merely equal to R^n

$$\mathsf{RVect}_n \equiv \sum_{V:\mathsf{RMod}} \|V =_{\mathsf{RMod}} R^n\|$$

- Let $X = \operatorname{Spec} A$. The type of rank n vectorbundle on X is

$$X \to \mathsf{RVect}_n$$

- Let $\mathcal{L} : X \to \mathsf{RVect}_n$ be a vectorbundle. The type of global section of $\mathcal{L}$ is

$$\Gamma(X, \mathcal{L}) \equiv \prod_{x:X} \mathcal{L}_x$$

which becomes an A-module by setting

$$(a \cdot s)_x \equiv a(x) \cdot s_x$$

Example 5.2. *The trivial rank n vector bundle is*

$$\mathcal{O}_X^{\oplus n} \equiv (x : X \mapsto (R^n : \mathsf{RVect}_n))$$

Lemma 5.3 (using Zariski Local Choice)**.** *Every vectorbundle $\mathcal{L} : X \to \mathsf{RVect}_n$ is locally trivial, i.e. there exists an open cover $X = \bigcup_{i=1}^n U_i$ such that $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus n}$, i.e.*

$$\prod_{x:U_i} (\mathcal{L}_x \simeq R^n)$$

6 Schemes

Now we are finally able to state the definition of a scheme.

Definition 6.1. A type X is a scheme if there merely exists a finite cover of open subtypes $(U_i)_i$ such that each U_i is affine.