Synthetic Algebraic Geometry
Constructive working over a field with nonzero nilpotents

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May 2024

1 HoTT revisited

1.1 Curry Howard Correspondence

$$
\begin{tikzcd}
MLTT \arrow[rr, bend left, "\|\cdot\|"] \arrow[dr, shift left, "interpretation"] & & Constructive\ Logic \arrow[dl, shift left, "\{proofs\ of\ A\} \leftarrow\!\shortmid A"] \\
& Constructive\ Set\ Theory \arrow[ul, shift left, "\supseteq"] \arrow[ur, shift left, "A \mapsto {}''A \neq \varnothing''"] &
\end{tikzcd}
$$

Martin Löff Type theory	Sets as Types	Propositions as Types
A Type	A is Set	A is Prop
$a : A$	$a \in A$	a is a proof that A holds
$A \to B$	functions $A \to B$	A implies B
$\neg A \equiv A \to \bot$	A is empty	A is false
$A \simeq B$	{ Bijections $A \cong B$ }	$A \leftrightarrow B$
$A \times B$	$A \times B$	$A \wedge B$
$A + B$	$A \sqcup B$	$A \vee B$
$\sum_{x:X} Bx$	$\bigsqcup_{x \in A} Bx$	$\exists x \in X, Bx$ (X a Set)
$\prod_{x:X} Bx$	Sections of $(\bigsqcup_{x \in A} Bx) \to X$	$\forall x \in X, Bx$ (X a Set)
$x =_A y$	{ proofs of the proposition that $x = y$ holds }	automatic

Here:

$$A \simeq B \equiv \sum_{f:A \to B} \mathrm{isEquiv}(f),$$

where

$$\mathrm{isEquiv}(f) \equiv \mathrm{sec}(f) \times \mathrm{ret}(f)$$

We have an internal notion, when a type A is a proposition:

$$\begin{aligned} \mathrm{isProp}(A) &\equiv \prod_{x,y:A} x =_A y \\ &\simeq (A \to \mathrm{isContr}(A)) \end{aligned}$$

where

$$\mathrm{isContr}(A) \equiv \sum_{x:A} \prod_{y:A} x =_A y.$$

Slogan: Once a proposition is true I dont care anymore, why its true.

Proposition 1.1. $\mathrm{isProp}(A), \mathrm{isContr}(A), \mathrm{isEquiv}(f)$ *are propositions.*

One realizes, that propositions are not stable under $+$ and Σ:

Example. • *If* 1 *denotes the unit type, then* $1+1$ *is not a proposition!*

- $\sum_{x:\,A} Bx$ *is the type of all terms* x *such that* Bx *holds, so different* x *such that* Bx *give different terms!*

We see in action where Martin Löff Type theory is not rich enough, when we try to define the image of a map:

Example. *Let* $f\colon A \to B$. *If one defines* $\operatorname{im} f \equiv \sum_{y:B}\sum_{x:A} fx =_B y$ *then we would get*

$$\begin{aligned}\operatorname{im} f &\equiv \sum_{y:B}\sum_{x:A} fx =_B y\\ &\simeq \sum_{x:A}\underbrace{\sum_{y:B} fx =_B y}_{contractible}\\ &\simeq \sum_{x:A} 1\\ &\simeq A\end{aligned}$$

Similar calculations would show that for the naive definition of surjectivity

$$\operatorname{isSurj}(f) \equiv \prod_{y:B}\sum_{x:A} fx = y \simeq \sum_{g:B\to A}\prod_{y:B} f(gy) = y \equiv \sec(f).$$

1.2 Propositional Truncation

We need to add another type former to Martin-Löff-Typetheory, called the propositional truncation

Definition 1.2. Given a type A, we add a type $\|A\|$, called the propositional truncation, equipped with a map $\eta_A : A \to \|A\|$, such that

- $\|A\|$ is a proposition
- for any proposition P the map

$$\begin{aligned}(\|A\| \to P) &\to (A \to P)\\ f &\mapsto f \circ \eta_A\end{aligned}$$

  is an equivalence.

$$\begin{tikzcd} A \arrow[r, "\forall"] \arrow[d, "\eta_A"'] & P \\ \|A\| \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

So $\|A\|$ is the proposition, that says that A is inhabited. To emphasize I only have a term of type $\|A\|$, we say A **has merely a term**. Now we can fix the issues we add earlier:

Definition 1.3. • For propositions A, B we define

$$A \vee B \equiv \|A + B\|$$

- For family of propositions Bx indexed over a type $x\colon A$ we define

$$\exists x : A, Bx \equiv \|\sum_{x:A} Bx\|$$

In words : " There merely exists an $x : A$ such that Bx " , and

$$\forall x : A, Bx \equiv \prod_{x\colon A} Bx$$

Suggestively: Assuming $\exists x : A, Bx$, only if we want to show a proposition, we are allowed to extract a witness $x\colon A$ that satisfies Bx. That the dependent product of propositions is a proposition, has to be added as an axiom[1] to Martin-Löff-Type theory.

Remark 1. One can realize propositional truncation as a higher inductive type, but they will not play a role today.

1.3 Subtypes

We fix a universe $\mathcal{U}$.
We write $\mathrm{Prop} \equiv \sum_{A:\mathcal{U}} \mathrm{isProp}(A)$

Definition 1.4 (Subtypes)**.** Given a type A, a subtype of A is a map

$$U\colon A \to \mathrm{Prop}$$

written $U \subset A$. For $x\colon A$, we write the respective proposition as

$$x \in U \equiv U(x)$$

And we may use the notation

$$U \equiv (x\colon A \mapsto Ux) \equiv \{x\colon A \mid Ux\}$$

For subtypes U, V we write $U \subseteq V \equiv \forall x\colon A, x \in U \to x \in V$.

Example. *The empty subtype is defined as*

$$\varnothing = \{x\colon A \mid \bot\}$$

Definition 1.5. Let $(U_i)_i : I \to (A \to \mathrm{Prop})$ be a family of subtypes of A. Then

- $$\bigcup_{i\in I} U_i \equiv \{x\colon A \mid \exists i\colon I, x \in U_i\}$$

- $$\bigcap_{i\in I} U_i \equiv \{x\colon A \mid \forall i\colon I, x \in U_i\}$$

Definition 1.6. A map $f : A \to B$ is an embedding if for all $x, y\colon A$ the map

$$\mathrm{ap}_f : x =_A y \to fx =_B fy$$

induced by path-induction, is an equivalence. We write $A \hookrightarrow B$ for the type of maps $A \to B$ that are embeddings.

[1] up to equivalence its called function extensionality

Lemma 1.7. *A map $f : A \to B$ is an embedding iff for any $y \colon B$, the fiber $\mathrm{fib}_f(y) \equiv \sum_{x:A} fx =_B y$ is a proposition.*

Proposition 1.8. *Assuming the univalence axiom, there are mutual inverse equivalences*

$$(A \to \mathrm{Prop}) \simeq \sum_{B:\mathcal{U}} B \hookrightarrow A$$
$$U \mapsto (\sum_{x\colon A} x \in U, \mathrm{proj}_1)$$
$$\{x \colon A \mid \mathrm{fib}_f(x)\} \leftarrow\!\shortmid (f : B \hookrightarrow A)$$

2 Synthetic Algebraic Geometry

2.1 Algebra

Definition 2.1. A type A is a set if $\prod_{x,y\colon A} \mathrm{isProp}(x =_A y)$. We write Set for the subtype of $\mathcal{U}$ that consists of the sets.

We can do algebra in HoTT. The type of (commutative unital) Rings can be written as

$$\mathsf{Ring} \equiv \sum_{R:\mathsf{Set}} \sum_{+,\cdot:R\times R\to R} \sum_{0,1\colon R} \mathrm{isAbelianGroup}(R,+,0)\times\mathrm{isAbelianMonoid(R},\cdot,1)\times\mathrm{isDistributive}(R,+,\cdot)$$

where we omit the definitions of the algebraic-flavoured propositions.

Remark 2 (Lean)**.** In Lean, this can be formalized by structures. Conversely a structure in Lean can be encoded as an iterated Sigma-type in HoTT.

Definition 2.2. Let $(R,+,\cdot,0,1)\colon \mathsf{Ring}$. Given $x\colon R$,define

$$\mathrm{isInv}(x) \equiv \sum_{y:R} x \cdot y =_R 1$$

One can show, that this is indeed a proposition, because inverses are unique if they exist.

Definition 2.3.

$$\mathrm{isLocal(R)} \equiv 1 \neq 0 \ \wedge \ \forall x, y \colon R, (\mathrm{isInv}(x+y) \to \mathrm{isInv}(x) \vee \mathrm{isInv}(y))$$

In synthetic algebraic geometry, there is always the datum of a commutative ring R in the context. The first axiom says:

Axiom 1 (Loc)**.** R is a local ring.

Definition 2.4.

- The type of algebras is

$$\mathrm{Alg} \equiv \sum_{A\colon \mathsf{Ring}} \mathrm{Hom}_{\mathsf{Ring}}(R, A)$$

  $\mathrm{Hom}_R(A,B)$ is the set of homomorphisms of R-algebras $A \to B$.

- An algebra A is finitely presented, if

$$\| \sum_{m,n:\mathbb{N}} \sum_{f_1,\ldots,f_m:R[X_1,\ldots,X_n]} A =_{\mathrm{Alg}} R[X_1,\ldots,X_n]/(f_1,\ldots,f_m)\|$$

Remark 3. By the univalence axiom, the latter equality in Alg can be replaced by asking for an isomorphism of R-algebras.

Definition 2.5. For A a (finitely presented) R-algebra, we define the spectrum as the set

$$\operatorname{Spec} A \equiv \operatorname{Hom}_R(A, R)$$

The fundamental observation of algebraic geometry is, that once we pick a representation of A through polynomials $f_1, \ldots, f_m$, the spectrum corresponds to the set of common zeros over R of those polynomials.

Lemma 2.6 (The Fundamental Theorem of AG)**.** *We have an equivalence*

$$\begin{aligned} \operatorname{Spec}(R[X_1, \ldots, X_n]/(f_1, \ldots, f_m)) &\to \{(x_1, \ldots, x_n) \in R^n \mid f_i(x_1, \ldots, x_n) = 0 \forall i\} \\ \phi &\mapsto (\phi(X_1), \ldots, \phi(X_n)) \end{aligned}$$

Proof. This is a direct consequence of the universal property of the quotient ring and the universal property of the polynomial algebra. ☐

Remark 4 (Classically!)**.** The above equivalence holds if R is an algebraically closed field and one replace Spec by MaxSpec, the set of maximal ideals.

Definition 2.7. Given a type X, we have the algebra R^X of R-valued functions on X, i.e. Its underlying set is the function type $X \to R$ and it inherits an R-algebra structure from the target by doing everything pointwise. Its not finitely presented in general.

With the next axiom, the spectrum gets new interesting geometric structure:

Axiom 2 (Duality)**.** For any finitely presented algebra A the **duality map**

$$\begin{aligned} A &\to R^{\operatorname{Spec} A} \\ f &\mapsto (x \mapsto x(f)) \end{aligned}$$

is an equivalence.
We may use this as a coercion, i.e for $f : A, x : \operatorname{Spec} A$ we may write $f(x) \equiv x(f)$.

What does this tell us?

Example. *Every function $R \to R$ is a polynomial!*

Proof. Indeed, $\operatorname{Spec} R[X] = \operatorname{Hom}_R(R[X], R) \simeq R$ as a set by The Fundamental Theorem of AG, hence we only have to apply Duality to $A = R[X]$. ☐

Let A denote a finitely presented algebra.

Proposition 2.8 (Weak Nullstellensatz)**.** $A = 0$ *iff* $\operatorname{Spec} A = \varnothing$.

Proof. Both sides are propositions, where we rephrase the left hand side as $1 =_A 0$. If $1 =_A 0$, then there are no R-algebra homomorphisms $A \to R$ because $0, 1$ have to get preserved but they are not equal in R by Loc.
If $\operatorname{Spec} A = \varnothing$ then

$$A \simeq R^{\operatorname{Spec} A} = R^{\varnothing}$$

is contractible as a set, hence $1 =_A 0$. ☐

We now see some consequences of this:
The next proposition shows, that R is a field in a weak way (from a constructive point of view) and $\neg\neg(x = 0)$ is the same as x beeing nilpotent.

Proposition 2.9.

- *A vector in R^n is non zero iff one of its entries is invertible.*

- *Let $x : R$. x is not invertible iff x is nilpotent*

Proof.

- $(x_1, \dots, x_n) \neq 0$ iff $\mathrm{Hom}_R(R/(x_1, \dots, x_n), R) = \mathrm{Spec}(R/(x_1, \dots, x_n))$ is empty iff $R/(x_1, \dots, x_n) = 0$ iff $(x_1, \dots, x_n) = R$ iff one of the x_i is invertible by Loc.
- x is not invertible iff $\mathrm{Spec}(R_x)$ is empty iff $R_x = 0$ iff $1 = 0$ in R_x iff x is nilpotent.

$\square$

Classically, the next example does not work in a field, but constructively with our notion of a field ($x \neq 0 \to \mathrm{isInv}(x)$ it does.

Example. *In R not every nilpotent is zero.*

Proof. We prove something stronger, namely than not every element that squares to zero is itself zero. If it would then $\mathrm{Spec}(R[x]/x^2)$ would be a signleton by The Fundamental Theorem of AG. But the R algebra $R[x]/x^2$ is not isomorphic to R^1 as an R-algebra. $\square$

2.2 Spec is right adjoint and fully faithfull when restricted to FP-algebras

Proposition 2.10 (Adjunction). *For any A : Alg, we have a natural equivalence*

$$(X \to \mathrm{Spec}\, A) \xrightarrow{\sim} \mathrm{Hom}_R(A, R^X)$$
$$\mathrm{id}_{\mathrm{Spec}\, A} \mapsto \text{duality map of } A$$

Proof. We have the natural flip map

$$(X \to (A \to R)) \xrightarrow{\simeq} (A \to (X \to R))$$

which is an equivalence. I claim that this map restricts to a map $(X \to \mathrm{Spec}\, A) \simeq \mathrm{Hom}_R(A, R^X)$. One now has to check, that an element on the left hand side has image in algebra homomorphisms $\mathrm{Hom}_R(A, R)$ if and only if the corresponding function on the right hand side will be an algebra homomorphism. $\square$

Now invoking, that by Duality the counit of the above adjunction is an equivalence, we deduce the following two purely formal statements

Corollary (Fullyfaithfullness of Spec)**.** Spec *enhances to a fully faithfull functor*

$$\mathrm{Spec} : \mathrm{Alg}_{FP} \to \mathcal{U}$$

That means: For A, B finitely presented A-algebras, the map

$$\mathrm{Hom}_R(A, B) \to (\mathrm{Spec}\, B \to \mathrm{Spec}\, A)$$
$$\varphi \mapsto (x \mapsto x \circ \varphi)$$

is an equivalence.

Corollary. *A type X is equivalent to* $\mathrm{Spec}\, A$ *for some finitely presented A iff the unit map $X \to \mathrm{Spec}(R^X)$ is an equivalence. In this case we call X* **affine***.*

2.3 Topology

A subtype $U : X \to \mathrm{Prop}$ is open if all the propositions $U(x)$ are merely of the form $(f_1 = 0) \wedge \ldots \wedge (f_n = 0)$ for some $f_i \colon R$.

Definition 2.11.

- A proposition P is open iff there merely exists $f_1, \ldots, f_n \colon R$ such that
$$P = \mathrm{isInv}(f_1) \vee \ldots \vee \mathrm{isInv}(f_n)$$
.
- A proposition P is closed iff there merely exists $f_1, \ldots, f_n \colon R$ such that
$$P = (f_1 = 0) \wedge \ldots \wedge (f_n = 0).$$
- A subtype $U \subset X$ is open (closed) iff the proposition $x \in U$ is open (closed) for all $x : X$.
- for $f \colon A$, define the *principal open subsets of* $\operatorname{Spec} A$
$$D(f) = \{x \colon \operatorname{Spec} A \mid \mathrm{isInv}(f(x))\}$$
- For $f_1, \ldots, f_n \colon A$ define the closed subset, the *vanishing locus*
$$V(f_1, \ldots, f_n) = \{x : \operatorname{Spec} A \mid f_i(x) = 0 \ \forall i\}$$

Example. For any open subtype $U \subset Y$ and any map $f : X \to Y$ the preimage $\{x \colon X \mid f(x) \in U\}$ is open.

Example. *We have* $D((X - a_1) \ldots (X - a_n)) = \operatorname{Spec} R[X] \setminus \{a_1, \ldots, a_n\}$

On the other hand one could define for $X = \operatorname{Spec} A$ a subtype $U \subset X$ to be open if its merely of the form $U = \bigcup_{i=1}^n D(f_i)$ for some $f_1, \ldots, f_n \in A$. To show that those two topologies coincide we need one last axiom.

2.4 The last axiom

On the other hand, there is also the *Zariski topology* on $\operatorname{Spec} A$, i.e. where the principal opens form a basis of the topology. To show, that this coincide with the pointwise topology above, we need one last axiom.

Axiom 3. for any family B of inhabited types over $\operatorname{Spec} A$ we merely find a principal open cover $\operatorname{Spec} A = \bigcup_i D(f_i)$ and local sections $\prod_{x : D(f_i)} Bx$:

$$\left\| \underbrace{\sum_{n:\mathbb{N}} \sum_{(f_1,\ldots,f_n):A^n} \left(\bigcup_{i=1}^n D(f_i) = \operatorname{Spec} A \right)}_{\text{principal open cover}} \times \prod_{i=1}^n \underbrace{\prod_{x:D(f_i)} Bx}_{\text{local sections}} \right\|$$

With this we also understand how closed subtypes of $\operatorname{Spec} A$ look like. Also computing cohomology seems not possible without this axiom.

There is this connection of principal opens to algebra:

Lemma 2.12. *Given* $f_1, \ldots, f_n : A$ *we have* $(f_1, \ldots, f_n) = A$ *iff* $\bigcup_i D(f_i) = \operatorname{Spec} A$

Proof. We prove more generally, that $D(g) \subset \bigcup_i D(f_i)$ iff $g \in \sqrt{(f_1, \dots, f_n)}$. First observe that open subsets satisfy the law of excluded middle by 2.9. Hence we may understand when $(\bigcup D(f_i))^c \cap D(g) = \varnothing$, but by the same proposition

$$\Big(\bigcup D(f_i))^c \cap D(g) = V(f_1, \dots, f_n\Big) \cap D(g) = \operatorname{Spec}(A/(f_1, \dots, f_n)_g) = \varnothing$$

iff $(A/(f_1, \dots, f_n))_g = 0$ by the nullstellensatz iff $g \in \sqrt{(f_1, \dots, f_n)}$.

□

Definition 2.13. The type of unimodular functions is defined as:

$$\operatorname{Um}(A) = \sum_{n:\mathbb{N}} \sum_{f_1,\dots,f_n:A} (f_1, \dots f_n) = (1)$$

In words, For any finitely presented algebra A and any family $B : \operatorname{Spec} A \to \mathcal{U}$ we have a map

$$(\forall x : X, \|Bx\|) \to \left\| \sum_{(n,f_1,\dots,f_n):\operatorname{Um}(A)} \prod_{i=1}^{n} \prod_{x:D(f_i)} Bx \right\|$$

Now we are finally able to state the definition of a scheme.

Definition 2.14. A type X is a scheme if there merely exists a cover of open subtypes $(U_i)_i$ such that each U_i is affine.

2.5 More on topology

Lemma 2.15. *The Open subsets of X contain $\varnothing, X$ and are stable under finite unions and intersections. Furthermore*

Lemma 2.16. *For $f_1, \dots, f_n : A$, There is an equivalence*

$$V(f_1, \dots, f_n) \to \operatorname{Spec}(A/(f_1, \dots, f_n))$$

and $D(f) \simeq \operatorname{Spec}(A_f)$

Proof. By the universal property, An algebra homomorphism $A/(f_1, \dots, f_n) \to R$ is an algebra homomorphism $x : A \to R$ such that $f_i(x) = x(f_i) = 0$ for $i = 1, \dots, n$. □

Proof. Indeed $(x - a_1) \dots (x - a_n)$ is invertible iff one of the $(x - a_i)$ is invertible, i.e. $x \neq a_i$ by 2.9 □

Lemma 2.17. *We have $V(f_1, \dots, f_n) \subset V(g_1, \dots, g_m)$ iff $(g_1, \dots, g_m) \subset (f_1, \dots, f_n)$.*

Proof. Such an inclusion corresponds to a map $\operatorname{Spec}(A/(f_1, \dots, f_n)) \to \operatorname{Spec}(A/(g_1, \dots, g_m))$ over $\operatorname{Spec} A$. We conclude by fully faithfullness of Spec. □

Example. Classically, Consider an inclusion of subsets $V(x^2) \subset V(x)$ inside $\operatorname{Spec} R[x]$. Now it does hold $(x) \not\subset (x^2)$. Instead, classically we have an inclusion of subsets of $\operatorname{Spec} A$ $V(I) \subset V(J)$ iff $J \subset \sqrt{I}$ and only if we consider them as closed subschemes we get the condition $J \subset I$. Hence internally the vanishing locus carries more structure than just the underlying set!