Fodor's Lemma in Lean

Or, I formalized some nice facts about club sets

Theofanis Chatzidiamantis Christoforidis

January 2024

Basic Definitions

Let o be a limit ordinal and $C \subseteq o$ a set.

Definition

- C is called **unbounded** in o if $\sup(C \cap o) = o$.

Basic Definitions

Let o be a limit ordinal and $C \subseteq o$ a set.

Definition

- C is called **unbounded** in o if $\sup(C \cap o) = o$.
- C is called **closed** in o if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.

Basic Definitions

Let o be a limit ordinal and $C \subseteq o$ a set.

Definition

- C is called **unbounded** in o if $\sup(C \cap o) = o$.
- C is called **closed** in o if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.
- C is called **club** in o if it is closed and unbounded in o.

Basic Definitions

Let o be a limit ordinal and $C \subseteq o$ a set.

Definition

- C is called **unbounded** in o if $\sup(C \cap o) = o$.
- C is called **closed** in o if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.
- C is called **club** in o if it is closed and unbounded in o.

Remark

Equivalently, C is unbounded in o if for every $\alpha < o$, there is a $\beta \in C$ such that $\alpha < \beta$ and $\beta < o$.

Basic Definitions

Definition

Let o be a limit ordinal and $S \subseteq o$ a set. S is called **stationary** in o if for every club set $C \subseteq o$, $S \cap C \neq \varnothing$.

Basic Definitions

Definition

Let o be a limit ordinal and $S \subseteq o$ a set. S is called **stationary** in o if for every club set $C \subseteq o$, $S \cap C \neq \emptyset$.

Definition

Let $f : C \to D$ be a function, where C and D are sets of ordinals. f is callted **regressive** if for every $0 < \alpha \in C$, $f(\alpha) < \alpha$.

Basic Definitions

Definition

Let κ be a cardinal.

- The **cofinality** $cof(\kappa)$ of κ is the smallest possible cardinality of an unbounded set in κ

Basic Definitions

Definition

Let κ be a cardinal.

- The **cofinality** $cof(\kappa)$ of κ is the smallest possible cardinality of an unbounded set in κ
- A cardinal is **regular** if $cof(\kappa) = \kappa$

Basic Definitions

Definition

Let κ be a cardinal.

- The **cofinality** $cof(\kappa)$ of κ is the smallest possible cardinality of an unbounded set in κ
- A cardinal is **regular** if $cof(\kappa) = \kappa$

Remark

Cofinality generalizes to ordinals and this is what Lean actually does.

The Theorem

Our goal is to prove the following:

Theorem (Fodor's Lemma)

Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in κ.

The Theorem

Our goal is to prove the following:

Theorem (Fodor's Lemma)

Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in κ.

In other words, a regressive function on a stationary set is constant on some stationary subset of its domain.

Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.

Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.
- With this in mind, our main objects of study here are of type ***Set Ordinal***.

Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.
- With this in mind, our main objects of study here are of type ***Set Ordinal***.
- This makes sense: recall that in ZFC, an ordinal contains every ordinal strictly smaller than itself.

Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- strict_Ordinal_res C o := $\{c \in C | c < o\}$, corresponding to $C \cap o$.

Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- strict_Ordinal_res C o := $\{c \in C | c < o\}$, corresponding to $C \cap o$.
- Ordinal_res C o := $\{c \in C | c \leq o\}$

Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- strict_Ordinal_res C o := $\{c \in C | c < o\}$, corresponding to $C \cap o$.
- Ordinal_res C o := $\{c \in C | c \leq o\}$

"strict" is the correct notion, but Ordinal_res has nicer properties and we use it whenever they are equal (i.e., when $o \notin C$)

To prove Fodor's Lemma we need one last definition :

To prove Fodor's Lemma we need one last definition :

Definition

Let κ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of κ. The **diagonal intersection** of $(C_\alpha)_{\alpha<\kappa}$ is defined to be

$$\Delta_{\alpha<\kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \ \forall \theta < \beta\}$$

How to prove Fodor's Lemma

- First we prove that $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club

How to prove Fodor's Lemma

- First we prove that $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa}C_\alpha$ is club (This is about 95% of the lines of code).

How to prove Fodor's Lemma

- First we prove that $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.

How to prove Fodor's Lemma

- First we prove that $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa}C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.
- Then for each $o < \kappa$, choose a club set that does not intersect its preimage.
- Take the diagonal intersection of the sequence you just created. this is club and thus intersects S.

How to prove Fodor's Lemma

- First we prove that $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.
- Then for each $o < \kappa$, choose a club set that does not intersect its preimage.
- Take the diagonal intersection of the sequence you just created. this is club and thus intersects S.
- Use the definition of Δ to prove that f is not regressive, contradiction.

General Structure:

Basic properties of supremums + Definitions + regularity of κ

$\rightsquigarrow$

The intersection of two club sets is club

$\rightsquigarrow$ induction

The intersection of less than κ club sets is club

$\rightsquigarrow$

The diagonal intersection of κ club sets is club

$\rightsquigarrow$

Fodor's Lemma

Proving Closedness:

This came down to manipulating supremums.

Proving Closedness:

This came down to manipulating supremums.

All their basic properties are in mathlib, but I formalized some of my own for the previously defined restrictions.

Proving Unboundedness:

For each of the main theorems about club sets, we construct a "good" increasing sequence $\mathbb{N} \to \mathit{Ordinal}$ recursively.

Proving Unboundedness:

For each of the main theorems about club sets, we construct a "good" increasing sequence $\mathbb{N} \to \mathit{Ordinal}$ recursively.

We then use our closedness assumptions to prove that its limit is in our intersection.

These arguments use both closedness and unboundedness assumptions.

On Cofinality

Why is it useful to work with sequences of length ω?

On Cofinality

Why is it useful to work with sequences of length ω?

If κ is uncountable and regular, the limit of such an ascending sequence will always be less than κ by regularity of κ, since $\mathbb{N}$ is countable.

On Cofinality

> Why is it useful to work with sequences of length ω?
>
> If κ is uncountable and regular, the limit of such an ascending sequence will always be less than κ by regularity of κ, since $\mathbb{N}$ is countable.

With this in mind, a very easy generalisation for the properties of club sets is apparent:

Remark

We can replace our uncountable regular cardinal with any ordinal o satisfying $cof(o) > \aleph_0$.