

# Fodor's Lemma in Lean

Or, I formalized some nice facts about club sets

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## Basic Definitions

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### Remark

Equivalently,  $C$  is unbounded in  $o$  if for every  $\alpha < o$ , there is a  $\beta \in C$  such that  $\alpha < \beta$  and  $\beta < o$ .

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### Definition

Let  $f : C \rightarrow D$  be a function, where  $C$  and  $D$  are sets of ordinals.  $f$  is called **regressive** if for every  $0 < \alpha \in C$ ,  $f(\alpha) < \alpha$ .

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## Remark

Cofinality generalizes to ordinals and this is what Lean actually does.

# The Theorem

Our goal is to prove the following:

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Let  $\kappa$  be an uncountable regular cardinal,  $S \subseteq \kappa$  a stationary set and let  $f : S \rightarrow \kappa$  be a regressive function. Then there is an ordinal  $\theta < \kappa$  such that  $f^{-1}(\{\theta\})$  is stationary in  $\kappa$ .

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In other words, a regressive function on a stationary set is constant on some stationary subset of its domain.

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- ▶ With this in mind, our main objects of study here are of type ***Set*** ***Ordinal***.
- ▶ This makes sense: An ordinal contains every ordinal strictly smaller than itself.

## Sets and Ordinals are different types.

In Lean, for  $C : \text{Set } \text{Ordinal}$  and  $o : \text{Ordinal}$ , we define:

- ▶  $\text{strict\_Ordinal\_res } C \ o := \{c \in C \mid c < o\}$ ,  
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"strict" is the correct notion, but `Ordinal_res` has nicer properties and we use it whenever they are equal (i.e., when  $o \notin C$ )

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### Definition

Let  $\kappa$  be a cardinal and let  $(C_\alpha)_{\alpha < \kappa}$  be a sequence of subsets of  $\kappa$ .  
The **diagonal intersection** of  $(C_\alpha)_{\alpha < \kappa}$  is defined to be

$$\Delta_{\alpha < \kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \ \forall \theta < \beta\}$$

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- ▶ Then for each  $\alpha < \kappa$ , choose a club set that does not intersect its preimage.
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- ▶ Use the definition of  $\Delta$  to prove that  $f$  is not regressive, contradiction.

## General Structure:

Basic properties of supremums + Definitions + regularity of  $\kappa$



The intersection of two club sets is club



The intersection of less than  $\kappa$  club sets is club



The diagonal intersection of  $\kappa$  club sets is club



Fodor's Lemma

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All their basic properties are in mathlib, but I formalized some of my own for the previously defined restrictions.

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For each of the main theorems about club sets, we construct a "good" increasing sequence  $\mathbb{N} \rightarrow \text{Ordinal}$  recursively.

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We then use our closedness assumptions to prove that its limit is in our intersection.

These arguments use both closedness and unboundedness assumptions.

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With this in mind, a very easy generalisation for the properties of club sets is apparent:

### Remark

We can replace our uncountable regular cardinal with any ordinal  $\alpha$  satisfying  $\text{cof}(\alpha) > \aleph_0$ .