# Fodor's Lemma in Lean

## Or, I formalized some nice facts about club sets

Theofanis Chatzidiamantis Christoforidis

January 2024

# Basic Definitions

Let $o$ be a limit ordinal and $C \subseteq o$ a set.

## Definition

- $C$ is called **unbounded** in $o$ if $\sup(C \cap o) = o$.

## Basic Definitions

Let $o$ be a limit ordinal and $C \subseteq o$ a set.

### Definition

- $C$ is called **unbounded** in $o$ if $\sup(C \cap o) = o$.
- $C$ is called **closed** in $o$ if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.

## Basic Definitions

Let $o$ be a limit ordinal and $C \subseteq o$ a set.

### Definition

- $C$ is called **unbounded** in $o$ if $\sup(C \cap o) = o$.
- $C$ is called **closed** in $o$ if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.
- $C$ is called **club** in $o$ if it is closed and unbounded in $o$.

## Basic Definitions

Let $o$ be a limit ordinal and $C \subseteq o$ a set.

### Definition

- $C$ is called **unbounded** in $o$ if $\sup(C \cap o) = o$.
- $C$ is called **closed** in $o$ if for every $\alpha < o$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.
- $C$ is called **club** in $o$ if it is closed and unbounded in $o$.

### Remark

Equivalently, $C$ is unbounded in $o$ if for every $\alpha < o$, there is a $\beta \in C$ such that $\alpha < \beta$ and $\beta < o$.

# Basic Definitions

## Definition

Let $o$ be a limit ordinal and $S \subseteq o$ a set. $S$ is called **stationary** in $o$ if for every club set $C \subseteq o$, $S \cap C \neq \varnothing$.

# Basic Definitions

### Definition

Let $o$ be a limit ordinal and $S \subseteq o$ a set. $S$ is called **stationary** in $o$ if for every club set $C \subseteq o$, $S \cap C \neq \varnothing$.

### Definition

Let $f : C \to D$ be a function, where $C$ and $D$ are sets of ordinals. $f$ is callted **regressive** if for every $0 < \alpha \in C$, $f(\alpha) < \alpha$.

# Basic Definitions

## Definition

Let $\kappa$ be a cardinal.

- The **cofinality** $cof(\kappa)$ of $\kappa$ is the smallest possible cardinality of an unbounded set in $\kappa$

## Basic Definitions

### Definition

Let $\kappa$ be a cardinal.

- The **cofinality** $cof(\kappa)$ of $\kappa$ is the smallest possible cardinality of an unbounded set in $\kappa$
- A cardinal is **regular** if $cof(\kappa) = \kappa$

## Basic Definitions

### Definition

Let $\kappa$ be a cardinal.

- The **cofinality** $cof(\kappa)$ of $\kappa$ is the smallest possible cardinality of an unbounded set in $\kappa$
- A cardinal is **regular** if $cof(\kappa) = \kappa$

### Remark

Cofinality generalizes to ordinals and this is what Lean actually does.

## The Theorem

Our goal is to prove the following:

**Theorem (Fodor's Lemma)**

Let $\kappa$ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in $\kappa$.

# The Theorem

Our goal is to prove the following:

**Theorem (Fodor's Lemma)**

Let $\kappa$ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in $\kappa$.

In other words, a regressive function on a stationary set is constant on some stationary subset of its domain.

# Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.

## Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.
- With this in mind, our main objects of study here are of type ***Set Ordinal***.

## Sets and Ordinals are different types.

- In set theory, everyting is a set, including cardinals and ordinals. This is not true in the type-theoretic framework we are working with.
- With this in mind, our main objects of study here are of type ***Set Ordinal***.
- This makes sense: An ordinal contains every ordinal stricty smaller than itself.

## Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- `strict_Ordinal_res C o` $:= \{c \in C | c < o\}$, corresponding to $C \cap o$.

# Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- `strict_Ordinal_res C o` $:= \{c \in C | c < o\}$, corresponding to $C \cap o$.
- `Ordinal_res C o` $:= \{c \in C | c \leq o\}$

# Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

- `strict_Ordinal_res C o` $:= \{c \in C | c < o\}$, corresponding to $C \cap o$.
- `Ordinal_res C o` $:= \{c \in C | c \leq o\}$

"strict" is the correct notion, but Ordinal_res has nicer properties and we use it whenever they are equal (i.e., when $o \notin C$)

To prove Fodor's Lemma we need one last definition :

To prove Fodor's Lemma we need one last definition :

### Definition

Let $\kappa$ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of $\kappa$. The **diagonal intersection** of $(C_\alpha)_{\alpha<\kappa}$ is defined to be

$$\Delta_{\alpha<\kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \ \forall \theta < \beta\}$$

# How to prove Fodor's Lemma

- First we prove that for $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club

# How to prove Fodor's Lemma

- First we prove that for $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club (This is about 95% of the lines of code).

# How to prove Fodor's Lemma

- First we prove that for $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.

# How to prove Fodor's Lemma

- First we prove that for $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa}C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.
- Then for each $o < \kappa$, choose a club set that does not intersect its preimage.
- Take the diagonal intersection of the sequence you just created. this is club and thus intersects $S$.

# How to prove Fodor's Lemma

- First we prove that for $(C_\alpha)_{\alpha<\kappa}$ a sequence of club sets, $\Delta_{\alpha<\kappa} C_\alpha$ is club (This is about 95% of the lines of code).
- Assume that no $f^{-1}(\theta)$ is stationary.
- Then for each $o < \kappa$, choose a club set that does not intersect its preimage.
- Take the diagonal intersection of the sequence you just created. this is club and thus intersects $S$.
- Use the definition of $\Delta$ to prove that $f$ is not regressive, contradiction.

## General Structure:

Basic properties of supremums + Definitions + regularity of $\kappa$

$\rightsquigarrow$

The intersection of two club sets is club

$\rightsquigarrow$ induction

The intersection of less than $\kappa$ club sets is club

$\rightsquigarrow$

The diagonal intersection of $\kappa$ club sets is club

$\rightsquigarrow$

Fodor's Lemma

# Proving Closedness:

This came down to manipulating supremums.

## Proving Closedness:

This came down to manipulating supremums.

All their basic properties are in mathlib, but I formalized some of my own for the previously defined restrictions.

# Proving Unboundedness:

For each of the main theorems about club sets, we construct a "good" increasing sequence recursively.

# Proving Unboundedness:

For each of the main theorems about club sets, we construct a "good" increasing sequence recursively.

We then use our closedness assumptions to prove that its limit is in our intersection.

These arguments use both closedness and unboundedness assumptions.

## On Cofinality

Why is it useful to work with sequences of length $\omega$ whenever possible?

# On Cofinality

Why is it useful to work with sequences of length $\omega$ whenever possible?

If $\kappa$ is uncountable and regular, the limit of such an ascending sequence will always be less than $\kappa$ by regularity of $\kappa$, since $\mathbb{N}$ is countable.

## On Cofinality

Why is it useful to work with sequences of length $\omega$ whenever possible?

If $\kappa$ is uncountable and regular, the limit of such an ascending sequence will always be less than $\kappa$ by regularity of $\kappa$, since $\mathbb{N}$ is countable.

With this in mind, a very easy generalisation for some of the properties of club sets is apparent:

### Remark

We can replace our uncountable regular cardinal with any ordinal $o$ satisfying $cof(o) > \aleph_0$.