A (hopefully formalization-friendly) Proof of Fodor's Lemma

Theofanis Chatzidiamantis Christoforidis

1 Definition. Let λ be a limit ordinal and $C \subseteq \lambda$ a set.

i. C is called **unbounded** in λ if $sup(C \cap \lambda) = \lambda$.

ii. C is called **closed** in λ if for every $\alpha < \lambda$, if $C \cap \alpha \neq \varnothing$, then $sup(C \cap \alpha) = \alpha$.

iii. C is called **club** in λ if it is closed and unbounded in λ.

2 Definition. Let λ be a limit ordinal and $S \subseteq \lambda$ a set. S is called **stationary** in λ if for every club set $C \subseteq \lambda$, $S \cap C \neq \varnothing$.

Our goal is to give a self-contained (assuming the contents of Mathlib) proof of the following:

3 Theorem (Fodor's Lemma)**.** Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ be a stationary set and let $f : S \to \kappa$ be an ordinal function such that $f(\alpha) < \alpha$ for all $\alpha \in S, \alpha > 0$. Then there is a stationary set $T \subset S$ and some $\theta < \kappa$ such that $f(\alpha) = \theta$ for all $\alpha \in T$.

References

[1] Jech, Thomas. Set Theory: The Third Millennium Edition, Springer, 2003.

[2] Schimmerling, Ernest. A Course on Set Theory, Cambridge University Press, 2011.