A (hopefully formalization-friendly) Proof of Fodor's Lemma

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1 Definition. Let λ be a limit ordinal and $C \subseteq \lambda$ a set.

i. C is called **unbounded** in λ if $sup(C \cap \lambda) = \lambda$.

ii. C is called **closed** in λ if for every $\alpha < \lambda$, if $C \cap \alpha \neq \varnothing$, then $sup(C \cap \alpha) = \alpha$.

iii. C is called **club** in λ if it is closed and unbounded in λ.

1.1 Remark. Equivalently, $C \subseteq \lambda$ is unbounded in λ if for all $\alpha < \lambda$, there is a $\beta \in C$ such that $\alpha < \beta$.

This may be an easier definition to work with.

2 Definition. Let λ be a limit ordinal and $S \subseteq \lambda$ a set. S is called **stationary** in λ if for every club set $C \subseteq \lambda$, $S \cap C \neq \varnothing$.

Our goal is to give a self-contained (assuming the contents of Mathlib) proof of the following:

3 Theorem (Fodor's Lemma)**.** Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ be a stationary set and let $f : S \to \kappa$ be an ordinal function such that $f(\alpha) < \alpha$ for all $\alpha \in S, \alpha > 0$. Then there is a stationary set $T \subset S$ and some $\theta < \kappa$ such that $f(\alpha) = \theta$ for all $\alpha \in T$.

4 Lemma. The intersection of two club sets is club.

Proof. Let $C, D \subseteq \lambda$ be club in λ.

- To show $C \cap D$ is closed,

$\square$

5 Definition. Let κ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of κ. The **diagonal intersection** of $(C_\alpha)_{\alpha<\kappa}$ is defined to be

$$\Delta_{\alpha<\kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \ \forall \theta < \beta\}$$

References

[1] Jech, Thomas. Set Theory: The Third Millennium Edition, Springer, 2003.

[2] Schimmerling, Ernest. A Course on Set Theory, Cambridge University Press, 2011.