A (hopefully formalization-friendly) Proof of Fodor's Lemma

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1 Definition. Let λ be a limit ordinal and $C \subseteq \lambda$ a set.

i. C is called **unbounded** in λ if $\sup(C \cap \lambda) = \lambda$.
ii. C is called **closed** in λ if for every $\alpha < \lambda$, if $C \cap \alpha \neq \varnothing$, then $\sup(C \cap \alpha) \in C$.
iii. C is called **club** in λ if it is closed and unbounded in λ.

1.1 Remark. Equivalently, $C \subseteq \lambda$ is unbounded in λ if for every $\alpha < \lambda$, there is a $\beta \in C$ such that $\alpha < \beta$.

2 Definition. Let λ be a limit ordinal and $S \subseteq \lambda$ a set. S is called **stationary** in λ if for every club set $C \subseteq \lambda$, $S \cap C \neq \varnothing$.

Our goal is to give a self-contained (assuming the contents of Mathlib) proof of the following:

3 Theorem (Fodor's Lemma)**.** *[regressive_on_stationary]*
Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be an ordinal function such that $f(\alpha) < \alpha$ for all $\alpha \in S, \alpha > 0$. Then there is an ordinal $\theta < \kappa$ such that the preimage $f^{-1}(\{\theta\})$ is stationary in κ.

We mostly follow the structure of [1, Chapter 8].

4 Lemma. The intersection of two club sets is club.

Proof. Let $C, D \subseteq \lambda$ be club in λ.

- To show that $C \cap D$ is closed in λ, assume $\alpha < \lambda$ and that $C \cap D \cap \alpha \neq \varnothing$. Let $\beta = \sup(C \cap D \cap \alpha)$. Then $\sup(C \cap \beta) = \sup(C \cap \sup(C \cap D \cap \alpha)) = \sup(C \cap D \cap \alpha) = \beta$, and thus $\beta \in C$ as C is club. Similarly $\sup(D \cap \beta) = \beta$ and $\beta \in D$, meaning $\beta \in C \cap D$.
- We now show that $C \cap D$ is unbounded in λ.

□

5 Proposition. *[int_lt_card_club]* Let κ be an uncountable regular cardinal and let $(C_\alpha)_{\alpha<\lambda}$ be a sequence of subsets of κ where $\lambda < \kappa$. If every C_α is club in κ, then the intersection $\cap_{\alpha<\lambda} C_\alpha$ is club in κ.

Proof. □

6 Definition. [*diag_int*] Let κ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of κ. The **diagonal intersection** of $(C_\alpha)_{\alpha<\kappa}$ is defined to be

$$\Delta_{\alpha<\kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \ \forall \theta < \beta\}$$

7 Lemma. [*diag_int_club*] Let κ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of κ. If every C_α is club in κ, then $\Delta_{\alpha<\kappa} C_\alpha$ is club in κ.

Proof. □

Proof of Fodor's Lemma. Assume, towards a contradiction, that there exists no such θ, i.e.,

$$\{\alpha \in S : f(\alpha) = \beta\}$$

is not stationary for any $\beta < \kappa$. Then, for every $\beta < \kappa$, we can choose a club set C_β satisfying $f(\alpha) \neq \beta$ for every $\alpha \in S \cap C_\beta$. By (7), $C := \Delta_{\beta<\kappa} C_\beta$ is club in κ. Pick an $\alpha \in C \cap S$, which is nonempty since C is club and S is stationary in κ. By definition of the diagonal intersection, this means that $f(\alpha) \neq \beta$ for every $\beta < \alpha$. This implies $f(\alpha) \geq \alpha$, contradicting the assumption that f is regressive on S. □

References

[1] Jech, Thomas. Set Theory: The Third Millennium Edition, Springer, 2003.

[2] Schimmerling, Ernest. A Course on Set Theory, Cambridge University Press, 2011.