Fodor's Lemma in Lean

Or, I formalized some nice facts about club sets

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Basic Definitions

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Remark

Equivalently, C is unbounded in o if for every $\alpha < o$, there is a $\beta \in C$ such that $\alpha < \beta$ and $\beta < o$.

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Let $f : C \to D$ be a function, where C and D are sets of ordinals. f is callted **regressive** if for every $0 < \alpha \in C$, $f(\alpha) < \alpha$.

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Remark

Cofinality generalizes to ordinals and this is what Lean actually does.

The Theorem

Our goal is to prove the following:

Theorem (Fodor's Lemma)

Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in κ.

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In other words, a regressive function on a stationary set is constant on some stationary subset of its domain.

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- With this in mind, our main objects of study here are of type ***Set Ordinal***.
- This makes sense: An ordinal contains every ordinal stricly smaller than itself.

Sets and Ordinals are different types.

In Lean, for C : *Set Ordinal* and o : *Ordinal*, we define:

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"strict" is the correct notion, but Ordinal_res has nicer properties and we use it whenever they are equal (i.e., when $o \notin C$)

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Definition

Let κ be a cardinal and let $(C_\alpha)_{\alpha<\kappa}$ be a sequence of subsets of κ. The **diagonal intersection** of $(C_\alpha)_{\alpha<\kappa}$ is defined to be

$$\Delta_{\alpha<\kappa} C_\alpha := \{\beta < \kappa \mid \beta \in C_\theta \; \forall \theta < \beta\}$$

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- Assume that no $f^{-1}(\theta)$ is stationary.
- Then for each $o < \kappa$, choose a club set that does not intersect its preimage.
- Take the diagonal intersection of the sequence you just created. this is club and thus intersects S.

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- Use the definition of Δ to prove that f is not regressive, contradiction.

General Structure:

Basic properties of supremums + Definitions + regularity of κ

$\rightsquigarrow$

The intersection of two club sets is club

$\rightsquigarrow$ induction

The intersection of less than κ club sets is club

$\rightsquigarrow$

The diagonal intersection of κ club sets is club

$\rightsquigarrow$

Fodor's Lemma

Proving Closedness:

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All their basic properties are in mathlib, but I formalized some of my own for the previously defined restrictions.

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For each of the main theorems about club sets, we construct a "good" increasing sequence $\mathbb{N} \to$ *Ordinal* recursively.

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We then use our closedness assumptions to prove that its limit is in our intersection.

These arguments use both closedness and unboundedness assumptions.

On Cofinality

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With this in mind, a very easy generalisation for the properties of club sets is apparent:

Remark

We can replace our uncountable regular cardinal with any ordinal o satisfying $cof(o) > \aleph_0$.