Formalization of the Proof of Fodor's Lemma

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Remark

Equivalently, C is unbounded in o if for every $\alpha < o$, there is a $\beta \in C$ such that $\alpha < \beta$.

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Let $f : C \to D$ be a function, where C and D are sets of ordinals. f is callted **regressive** if for every $0 < \alpha \in C$, $f(\alpha) < \alpha$.

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Remark

Cofinality generalizes to ordinals and this is what Lean actually does.

The Theorem

Our goal is to prove the following:

Theorem (Fodor's Lemma)

Let κ be an uncountable regular cardinal, $S \subseteq \kappa$ a stationary set and let $f : S \to \kappa$ be a regressive function. Then there is an ordinal $\theta < \kappa$ such that $f^{-1}(\{\theta\})$ is stationary in κ.

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- With this in mind, our main objects of study here are of type ***Set Ordinal***.
- This makes sense: recall that in ZFC, an ordinal contains every ordinal strictly smaller than itself.

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Working with supremums is hard.

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General Structure:

$$\text{Basic properties of supremums + Definitions + regularity of } \kappa$$
$$\rightsquigarrow$$
$$\text{The intersection of two club sets is club}$$
$$\rightsquigarrow \text{induction}$$
$$\text{The intersection of less than } \kappa \text{ club sets is club}$$
$$\rightsquigarrow$$
$$\text{The diagonal intersection of } \kappa \text{ club sets is club}$$
$$\rightsquigarrow$$
$$\text{Fodor's Lemma}$$

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With this in mind, a very easy generalisation for the properties of club sets is apparent:

Remark

We can replace our uncountable regular cardinal with any ordinal *o* satisfying $cof(o) > \aleph_0$.