Discrete Mathematics (II) Spring 2020

Lecture 4: Proposition, Connectives and Truth Tables

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1 Overview

In last lecture, we give a brief introduction to mathematical logic and then redefine order and tree, which has minor differences with the structures presented in last semester.

In this lecture, we discuss sentential logic. Topics focus first on the syntax of proposition. Semantics would be discussed shallowly. And the *Adequacy Theorem* is a very important theorem.

2 Propositions

What is a proposition? It is the base of proposition logic. We should define clearly the object discussed in our class. Rethinking about every day language, especially the English, we need define some elementary objects to construct some complicated one before starting to learn the language.

Let's consider the following statements:

1. I am a student.
2. I am not a student.
3. I am a student and I study computer science.
4. I am a boy or I am a girl.
5. If I am a student, I have a class in a week.
6. I am student if and only if I am a member of some university.

They are the statements in our life. ”I am a student” is just declarative. And ”I am not a student” just denies the former one.

However, we do not consider the statements like the following:

1. Are you a student?
2. Sit down please.
3. What are you doing?

They are interrogative, imperative, and interrogative sentences respectively. It is very different with the sentences in previous examples.Why we do not handle the sentence like ”are you a student?”. Because it is not determined or it is not declarative. All these statements are not declarative.

We first define a *propositional letter*, which represents a statement and claims one thing. For example, ”I am a student” is a sentence. But ”I am a boy or I am a girl” is hard to say because it is not as simple as "I am a student". A propositional letter is something like a word in a sentence. It can't be divide into more basic one.

A sentence is composed of some words linked by some function word such as verbal word, just like "I am a student and I study computer science". To describe a complicated case, we need some word such as *and*, *or*, and *if...not ...*, which are called conjunctions. Furthermore, if some more complicated scenario is to be depicted, we may need several sentences, a paragraph, which is separated by a series of punctuation.

We also need define some symbols in propositional logic. Sometimes we call it *language*, all complicated strings are constructed based on these symbols with some rules.

Definition 1. *The* language *of propositional logic consists of the following symbols:*

1. *Connectives:* $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$
2. *Parentheses:), (*
3. *Propositional Letters:* $A, A_1, A_2, \cdots, B, B_1, B_2, \cdots$.

where we suppose the set of proposition letters is countable, which is reasonable for human can hardly grasp uncountable set.

There are many words. But we need the grammar to write a sentence, a paragraph, even a book. We also define an inductive approach to construct a proposition.

Definition 2 (Proposition)**.** *A* proposition *is a sequence of many symbols which can be constructed in the following approach:*

1. *Propositional letters are propositions.*
2. *if* α *and* β *are propositions, then* $(\alpha \vee \beta), (\alpha \wedge \beta), (\neg\alpha), (\alpha \rightarrow \beta)$ *and* $(\alpha \leftrightarrow \beta)$ *are propositions.*
3. *A string of symbols is a proposition if and only if it can be obtained by starting with propositional letters (1) and repeatedly applying (2).*

It is obvious that infinite propositions can be generated even if there are only finite proposition letters.

Actually, there are more strings (a sequence of symbols defined in Definition 1) than those generated according to Definition 2.

Example 1. *Given the following strings, check whether it is a proposition.*

1. $(A \vee B), ((A \wedge B) \rightarrow C)$.
2. $A \vee \neg, (A \wedge B$

In practice, we also admit $A \vee B$ as a valid proposition. But it is not generated according to definition of proposition. So we give a definition of those propositions generated according to definition.

Definition 3. *The proposition constructed according to Definition 2 is* well-defined *or* well-formed*.*

We call them well-formed because of their good properties, which are good for us to design some procedure to recognize a proposition. This is a topic in the next lecture.

3 Truth table

We only consider two value logic. It means for any proposition it is whether *true* or *false*. So it is clear for propositional letters. For connectives is concerned, the semantics of a proposition other than propositional letter is not self-evident. We define the following truth tables for a given connectives.

We first define the truth table of unary connective $\neg$. It is easy to understand and to find the

α	$\neg\alpha$
T	F
F	T

Figure 1: Truth table of unary connective $\neg$

corresponding example in our life.

Conjunction and disjunction both are easy to be understood. They have the following truth table. Actually, we can find that $\vee$ and $\wedge$ obey *De Morgan Law*.

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

(a) Disjunction $\vee$

α	β	$\alpha \wedge \beta$
T	T	T
T	F	F
F	T	F
F	F	F

(b) Conjunction $\wedge$

Figure 2: Truth table of binary connectives

However, the truth table of $\to$ is not well understandable. But the truth table of bi-condition (equality) is intuitive to be understood. In table, α is called *premise* and β is called *consequence*.

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Conditional $\to$

α	β	$\alpha \leftrightarrow \beta$
T	T	T
T	F	F
F	T	F
F	F	T

(b) Bi-conditional $\leftrightarrow$

Figure 3: Truth table of binary connectives

Truth tables of these connectives are the base of propositional logic system. They define the functional behavior of a connective, which will be formally represented as a function. They indeed reflect the human's naive reasoning system.

Condition can be represented as "if ... then ...". If condition is satisfied, it is natural that we can obtain a correct consequence. The difficult is that why we let $\alpha \to \beta$ true when α is false?

As we said in previous lecture, logic is a discipline to study deduction. $\alpha \to \beta$ should be taken as one satement, which characterizes a way of deduction. In another word, is the assertion reasonable?

Example 2. *Consider the proposition, if $n > 2$, then $n^2 > 4$. Let $n = 3, 1, -3$. It is always true when $n \in R$.*

Remark 1. *We have TT, FF, FT, which all sounds reasonable. However, when premise is true consequence is false. It means we can reach wrong result from right basis step by step without fault. It is ridiculous in our life.*

To impress the understanding, consider the following examples:

Example 3. *With the following assumption:*

1. *If man can fly like a bird!*
2. *Everything in a folk world.*

We can make many irrational assertion.

We hope they could help you. If you have any good examples, please sent me an email.

4 Connectives

From the point of view of function, every connective can be taken as a function with form $f : \{0,1\}^k \to \{0,1\}$.

Definition 4 (Truth function)**.** *An n-ary connective is truth functional if the truth value for $\sigma(A_1, \ldots, A_n)$ is uniquely determined by the truth value of $A_1, \ldots, A_n$.*

The truth value T, F sometimes represented as $0, 1$, which is called as Boolean value. So the truth function is also named as Boolean function.

Definition 5. *A k-place* Boolean function *is a function from $\{F, T\}^k$ to $\{T, F\}$. We let F and T themselves to be 0-place Boolean functions.*

Example 4. *We can define $\to$ as a Boolean function in Figure 4.*

x_1	x_2	$x_1 \to x_2$	$f_\to(x_1, x_2)$
T	T	T	$f_\to(T,T) = T$
T	F	F	$f_\to(T,F) = F$
F	T	T	$f_\to(F,T) = T$
F	F	T	$f_\to(F,F) = T$

Figure 4: Boolean Function

Let $I_i(x_1, x_2, \ldots, x_n) = x_i$, which is a projection function of i-th parameter. We have the following properties on truth function.

1. For each n, there are 2^{2^n} n-place Boolean functions. It can be easily calculated only if you just think about the truth table.
2. 0-ary connectives: T and F.

3. Unary connectives: $\neg$, I , T and F.

4. Binary connectives: 10 of 16 are real binary functions.

Definition 6 (Adequate connectives)**.** *A set S of truth functional connectives is adequate if, given any truth function connective σ, we can find a proposition built up from the connectives in S with the same abbreviated truth table as σ.*

Let's consider the following example, which actually shows two different propositions with the same brief truth table.

Example 5. *Given the following two truth tables.*

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Conditional $\to$

α	β	$\neg\alpha \vee \beta$
T	T	T
T	F	T
F	T	F
F	F	T

(b) $\neg\alpha \vee \beta$

Figure 5: Equivalent truth table

It means $(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$. In another word, $\to$ can be represented by $\{\neg, \vee\}$. In general, we have the following *Adequacy* theorem.

Theorem 7 (Adequacy)**.** $\{\neg, \vee, \wedge\}$ *is adequate(complete).*

It means that a few connectives is enough. Before formal proof. We first consider whether we can use $\{\neg, \vee, \wedge\}$ to represent $\to$.

It is a little tricky to find $(\alpha \to \beta) \equiv ((\neg\alpha) \vee \beta)$, which can be observed from the abbreviated truth table of $\to$ as shown in Figure 3a.

We want to develop a systematic approach to general cases. Lets consider a row with the last column with T. Other than the last column, we use a new literal to represent each one, say x_i, as following:

$$x_i' = \begin{cases} x_i & \text{if } T \\ \neg x_i & \text{o.w.} \end{cases}$$

For the first row, we have $(\alpha \wedge \beta)$. We can verify that it is true only if the input is same as (T, T). Considering all rows with the last column as T, the proposition just constructed only represent one case. So we have $(\alpha \to \beta) = (\alpha \wedge \beta) \vee (\neg\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta) = \sigma$. They have the same brief truth table, shown as Figure 6.

We first just consider a very special case, all A_i are true and $\sigma(A_1, \ldots, A_k)$ is also true. Consider the conjunction $A_1 \wedge A_2 \wedge \cdots \wedge A_k$, you would find that the configuration of other row would make it false.

Generally, we have the following proof.

Proof. Construct the truth table of any connective $\sigma(A_1, \ldots, A_k)$. Consider the row with function value 1. For details, please refer to our textbook. □

α	β	$\alpha \to \beta$
T	T	T
T	F	F
F	T	T
F	F	T

(a) Conditional $\to$

α	β	$\alpha \wedge \beta$	$\neg\alpha \wedge \beta$	$\neg\alpha \wedge \neg\beta$	σ
T	T	T	F	F	T
T	F	F	F	F	F
F	T	F	T	F	T
F	F	F	F	T	T

(b) $(\alpha \wedge \beta) \vee (\neg\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$

Figure 6: Equivalent truth table

Furthermore, we can remove connective $\wedge$ such that the remainders are still adequate.

Corollary 8. $\{\neg, \vee\}$ *is adequate.*

You just find a way to represent $\wedge$ by $\{\neg, \vee\}$. The proof is very simple, which is left as an exercise.

We may wonder whether there is a adequate set with only one connective. The answer is positive. You can refer to exercise 4. The propositional logic with only one connective is very elegant. However, it first serve people. In practice, we also hope that the system is friend. With tradeoff, most of text book will adopt the set $\{\neg, \wedge, \vee\}$. $\to$ and $\leftrightarrow$ are two one which characterize human's thinking style. So they are also kept in the symbol set.

In class, we have not mentioned the following two definitions. Any truth function is represented as a proposition with form of disjunction of conjunction. It is called as following:

Definition 9 (DNF)**.** α *is called* disjunctive normal form *(abbreviated DNF). If* α *is a disjunction*

$$\alpha = \gamma_1 \vee \cdots \vee \gamma_k,$$

where each γ_i *is a conjunction*

$$\gamma_i = \beta_{i1} \wedge \cdots \wedge \beta_{in_i}$$

and each β_{ij} *is a proposition letter or the negation of a proposition letter.*

Example 6. $\alpha = (A_1 \wedge A_2 \wedge A_3) \vee (\neg A_1 \wedge A_2 \wedge A_3) \vee (A_1 \wedge A_2 \wedge \neg A_3)$ *is a DNF.*

For symmetry, we can also exchange the position of conjunction and disjunction in DNF and get the following:

Definition 10 (CNF)**.** α *is called* conjunctive normal form *(abbreviated CNF). If* α *is a conjunction*

$$\alpha = \gamma_1 \wedge \cdots \wedge \gamma_k,$$

where each γ_i *is a disjunction*

$$\gamma_i = \beta_{i1} \vee \cdots \vee \beta_{in_i}$$

and each β_{ij} *is a proposition letter or the negation of a proposition letter.*

Example 7. $\alpha = (A_1 \vee A_2 \vee A_3) \wedge (\neg A_1 \vee A_2 \vee A_3) \wedge (A_1 \vee A_2 \vee \neg A_3)$ *is a CNF.*

According to proof of *Adequacy* theorem, we have the following theorem.

Theorem 11. *Any proposition can be reformed as a DNF and a CNF.*

Proof. According to adequacy theorem. □

Exercies

1. Which of the following expressions are well-formed?

   (a) $((\neg(A \vee B)) \wedge C)$

   (b) $(A \wedge B) \vee C$

   (c) $(((C \vee B) \wedge A) \leftrightarrow D)$

2. Prove that set of propositions generated according to definition is countable.

3. Prove that $\{\neg, \rightarrow\}$ is an adequate set of connectives.

4. Prove that the binary connective $(\alpha|\beta)$ ("not both ...and") called the *Sheffer stroke* whose truth table is given in the following table is adequate.

α	β	$\alpha\|\beta$
T	T	F
T	F	T
F	T	T
F	F	T

5. Prove that $\{\wedge, \vee\}$ is not adequate.

6. Let $+^3$ be the ternary connective such that $+^3\alpha\beta\gamma$ is equivalent to $\alpha + \beta + \gamma$.

   (a) Show that $\{F, T, \wedge, +^3\}$ is complete.

   (b) Show that no proper subset is complete.

   Remark: $+^3$ is the ternary *parity* connective; $+\alpha_1\alpha_2\alpha_3$ is true if and only if odd number of $\alpha_1, \alpha_2, \alpha_3$ is T.

7. Prove that any proposition(Boolean function) can be represented as a CNF.

8. Determine whether the following proposition is a normal form. If it is not, transform it into a normal form.

   (a) $(A_1 \vee A_2 \vee A_3) \wedge (B_1 \vee (\neg B_2) \vee B_3) \wedge (C_1 \vee C_2)$

   (b) $(A_1 \wedge A_2) \vee (A_1 \wedge (\neg B_1)) \vee (A_2 \wedge B_1)$

9. Design an algorithm to transform a proposition into a DNF. You can suppose there are n different proposition letters.