Homework 1

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Problem 1

Part (a)

$$\begin{aligned}\Pr(Y \neq 1) &= \Pr(Y = 0)\\ &= \Pr(Y = 0 \cap X = 0) + \Pr(Y = 0 \cap X = 1)\\ &= 0.4 + 0.1\\ &= \boxed{0.5}\end{aligned}$$

Part (b)

$$\Pr(X = 1 \cap Y = 0) = \boxed{0.1}$$

Part (c)

$$\begin{aligned}\Pr(X = 1 | Y = 0) &= \Pr(X = 1 \cap Y = 0) \div \Pr(Y = 0)\\ &= 0.1 \div 0.5\\ &= \boxed{0.2}\end{aligned}$$

Part (d)

Yes since all of these equalities hold:

$$\begin{aligned}\Pr(X = 0 \cap Y = 0) = 0.4 = (0.4 + 0.4) \times (0.4 + 0.1) = \Pr(X = 0) \times \Pr(Y = 0)\\ \Pr(X = 1 \cap Y = 0) = 0.1 = (0.1 + 0.1) \times (0.4 + 0.1) = \Pr(X = 1) \times \Pr(Y = 0)\\ \Pr(X = 0 \cap Y = 1) = 0.4 = (0.4 + 0.4) \times (0.4 + 0.1) = \Pr(X = 0) \times \Pr(Y = 1)\\ \Pr(X = 1 \cap Y = 1) = 0.1 = (0.1 + 0.1) \times (0.4 + 0.1) = \Pr(X = 1) \times \Pr(Y = 1)\end{aligned}$$

Problem 2

Part (a)

For $i \in \{1, 2, \cdots, 8\}$, by linearity of expectation, we have

$$\begin{aligned}
\mathrm{E}(C_i) &= \mathrm{E}[A_i \times \mathrm{E}(M_{ji})] \\
&= \mathrm{E}(A_i) \times \mathrm{E}(M_{ji}) \\
&= [2 \times \Pr(A_i = 2) + 1 \times \Pr(A_i = 1)] \times [1 \times \Pr(M_{ji} = 1) + 0 \times \Pr(M_{ji} = 0)] \\
&= (2 \times 0.5 + 1 \times 0.5) \times (1 \times 0.7) \\
&= \boxed{1.05}
\end{aligned}$$

Part (b)

Since C_i is the same for every $i \in \{1, 2, \cdots, 8\}$, the answer is

$$8 \times \mathrm{E}(C_i) = 8 \times 1.05 = \boxed{8.4}$$

Part (c)

For $i \in \{9, 10, 11, 12\}$, by linearity of expectation, we have

$$\begin{aligned}
\mathrm{E}(C_i) &= \mathrm{E}[A_i \times \mathrm{E}(M_{ji})] \\
&= \mathrm{E}(A_i) \times \mathrm{E}(M_{ji}) \\
&= [2 \times \Pr(A_i = 2) + 1 \times \Pr(A_i = 1)] \times [1 \times \Pr(M_{ji} = 1) + 0 \times \Pr(M_j i = 0)] \\
&= (2 \times 0.5 + 1 \times 0.5) \times (1 \times 0.6) \\
&= \boxed{0.9}
\end{aligned}$$

Part (d)

Since C_i is the same for every $i \in \{9, 10, 11, 12\}$, the answer is

$$4 \times \mathrm{E}(C_i) = 4 \times 0.9 = \boxed{3.6}$$

Part (e)

$$C_1 + C_2 + ... + C_{12} = 8.4 + 3.6 = \boxed{12}$$

Part (f)

By linearity of expectation,

$$\begin{aligned} \text{E(Points)} &= \text{E}(3 \times \text{Field goals made}) \\ &= 3 \times \text{E(Field goals made)} \\ &= 3 \times 12 \\ &= \boxed{36} \end{aligned}$$

Problem 3

Part (a)

For this particular case, a reasonable prior distribution could be one that is uniform between 0 and 1:

$$\pi(p) = \begin{cases} 1, 0 \leq p \leq 1 \\ 0, \text{otherwise} \end{cases}$$

Part (b)

Let x be the number of heads (the number of 1's in the data). By Bayes' Theorem,

$$\begin{aligned}
\Pr(p|X) &\propto \; \Pr(X|p)\ \pi(p) \\
\text{argmax}_p \Pr(p|X) &= \text{argmax}_p \Pr(X|p)\ \pi(p) \\
&= \text{argmax}_p\ p^x(1-p)^{10-x} \cdot 1 \\
\text{argmax}_p\ \ln[\Pr(p|X)] &= \text{argmax}_p\ \ln[p^x(1-p)^{10-x}] \\
&= \text{argmax}_p\ x\ln(p) + (10-x)\ln(1-p)
\end{aligned}$$

The maximum value occurs when the right-hand side's derivative equals 0:

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}p}[x\ln(p) + (10-x)\ln(1-p)] \\
&= \frac{x}{p} - \frac{10-x}{1-p} \implies \boxed{p = \frac{x}{10}}
\end{aligned}$$

Therefore, the best value of p for the most likely model is the number of heads divided by 10, which is $\frac{8}{10} = 0.8$.

Problem 4

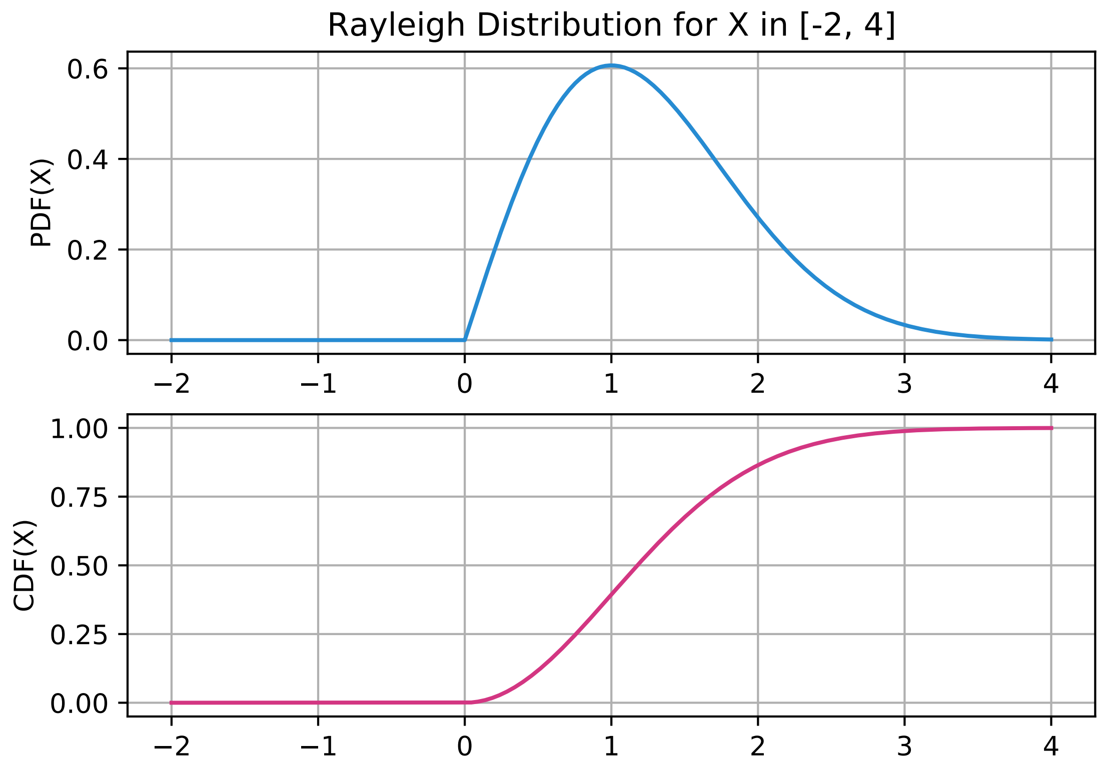
```
import numpy
from scipy.stats import rayleigh
from matplotlib import pyplot

# create a large number of data points evenly distributed within [-2, 4]
x = numpy.linspace(-2, 4, 1729)

# calculate pdf(x) and cdf(x) with the rayleigh library
pdf_x = rayleigh.pdf(x)
cdf_x = rayleigh.cdf(x)

# generate plots
figure, plots = pyplot.subplots(2)
plots[0].set_title("Rayleigh Distribution for X in [-2, 4]")
plots[0].set_ylabel("PDF(X)")
plots[0].grid()
plots[0].plot(x, pdf_x, "#268BD2")
plots[1].set_ylabel("CDF(X)")
plots[1].grid()
plots[1].plot(x, cdf_x, "#D33682")

# results spilled to the next page :)
```



Rayleigh Distribution for X in [-2, 4]
PDF(X)
0.6
0.4
0.2
0.0
−2
−1
0
1
2
3
4
CDF(X)
1.00
0.75
0.50
0.25
0.00
−2
−1
0
1
2
3
4