Homework 2

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Problem 1

Part (a)

We *can* only use Chebyshev's Inequality to calculate the upper-bound of such probability because we know the values of $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, which are all what the inequality needs.

We *cannot* use Markov's Inequality because we are not sure if $X \geq 0$ at all times, which is the requirement of using the Markov's Inequality. Even though the expect value of X is 100 (quite big), there is no guarantee that X never goes below zero.

We also *cannot* use the Chernoff-Hoeffding Inequality because we don't know the bounds of the variable X. X needs to have a defined bound to use the C-H Inequality.

Part (b)

Since $\mathrm{E}(X) = 100$ and $\mathrm{Var}(X) = 144$, by Chebyshev's Inequality,

$$\begin{aligned}
\Pr(X < 75) &= \Pr(X < 100 - 25) \\
&= \Pr(X < \mathrm{E}(X) - 25) \\
&= \Pr(|X - \mathrm{E}(X)| > 25) \qquad (\implies \varepsilon = 25) \\
&\leq \frac{\mathrm{Var}(X)}{\varepsilon^2} = \frac{144}{25^2} \approx \boxed{23\%}
\end{aligned}$$

Part (c)

An example of that could be a normal (Gaussian) distribution. A normal distribution does not have a definite bound, which is required in order to use either Markov's or Chernoff-Hoeffding Inequality.

Problem 2

Part (a)

First of all, by linearity of expectation, we have

$$\begin{aligned}
\mathrm{E}(\bar{X}) &= \mathrm{E}(\frac{1}{n} \cdot \sum_{i=1}^{n} X_i) \\
&= \frac{1}{n} \cdot \sum_{i=1}^{n} \mathrm{E}(X_i) \\
&= \frac{1}{n} \cdot n \cdot 7 = 7
\end{aligned}$$

Moreover, since $n = 2$, we have

$$\mathrm{Var}(\bar{X}) = \frac{\mathrm{Var}(X_i)}{n} = \frac{2}{2} = 1$$

Now, by Chebyshev's Inequality,

$$\begin{aligned}
\Pr(\bar{X} > 12) &= \Pr(\bar{X} > 7 + 5) \\
&= \Pr(\bar{X} > \mathrm{E}(\bar{X}) + 5) \\
&= \Pr(|\bar{X} - \mathrm{E}(\bar{X})| > 5) \quad (\implies \varepsilon = 5) \\
&\leq \frac{\mathrm{Var}(\bar{X})}{\varepsilon^2} = \frac{1}{5^2} = \boxed{4\%}
\end{aligned}$$

Part (b)

Since $\Delta = t - b = 13 - 1 = 12$, by the Chernoff-Hoeffding Inequality (reusing values from *Part (a)*),

$$\begin{aligned}
\Pr(\bar{X} > 12) &= \Pr(|\bar{X} - \mathrm{E}(\bar{X})| > 5) \qquad\qquad (\implies \varepsilon = 5) \\
&\leq 2 \cdot \exp\left(\frac{-2\varepsilon^2 n}{\Delta^2}\right) \\
&\leq 2 \cdot \exp\left(\frac{-2 \times 5^2 \times 2}{12^2}\right) \approx \boxed{99.9\%}
\end{aligned}$$

Part (c)

Similar to *Part (a)*, by Chebyshev's Inequality,

$$
\begin{aligned}
\Pr(\bar{X} > 12) &= \Pr(\bar{X} > 7 + 5) \\
&= \Pr(\bar{X} > \mathrm{E}(\bar{X}) + 5) \\
&= \Pr(|\bar{X} - \mathrm{E}(\bar{X})| > 5) && (\implies \varepsilon = 5) \\
&\leq \frac{\mathrm{Var}(\bar{X})}{\varepsilon^2} \\
&\leq \frac{\mathrm{Var}(X_i)}{n\varepsilon^2} = \frac{2}{20 \times 5^2} = \boxed{0.4\%}
\end{aligned}
$$

Part (d)

Similar to *Part (b)*, by the Chernoff-Hoeffding Inequality (reusing values from *Part (c)*),

$$
\begin{aligned}
\Pr(\bar{X} > 12) &= \Pr(|\bar{X} - \mathrm{E}(\bar{X})| > 5) && (\implies \varepsilon = 5) \\
&\leq 2 \cdot \exp\left(\frac{-2\varepsilon^2 n}{\Delta^2}\right) \\
&\leq 2 \cdot \exp\left(\frac{-2 \times 5^2 \times 20}{12^2}\right) \approx \boxed{0.2\%}
\end{aligned}
$$

Problem 3

Part (a)

Let $x = -1$. Now, the vectors $\vec{p}$ and $\vec{q}$ are linearly dependent because they are scaled versions of each other:

$$\begin{bmatrix} 1 \\ -2 \\ 4 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix}$$

Part (b)

Let $x = 21$. Now, the vectors $\vec{p}$ and $\vec{q}$ are orthogonal because their dot-product is zero:

$$\begin{aligned} \begin{bmatrix} 1 \\ -2 \\ 4 \\ x \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} &= \begin{bmatrix} 1 & -2 & 4 & x \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 4 & 21 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} \\ &= 1 \times 2 + 2 \times 4 + 4 \times 8 - 21 \times 2 \\ &= 0 \end{aligned}$$

Part (c)

$$||\vec{q}||_1 = \left\| \begin{matrix} 2 \\ -4 \\ 8 \\ -2 \end{matrix} \right\|_1 = |2| + |-4| + 8 + |-2| = \boxed{16}$$

Part (d)

$$||\vec{q}||_2^2 = \left\| \begin{array}{r} 2 \\ -4 \\ 8 \\ -2 \end{array} \right\|_2^2 = 2^2 + (-4)^2 + 8^2 + (-2)^2 = \boxed{88}$$

Problem 4

Part (a)

$$
\begin{aligned}
A^T B &= \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 3 \\ -1 & -1 & -2 \\ 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 0 + 0 \times 1 + 3 \times 0 & 2 \times 0 + 0 \times 0 + 3 \times 2 & 2 \times 1 + 0 \times 0 + 3 \times 0 \\ -1 \times 0 - 1 \times 1 - 2 \times 0 & -1 \times 0 - 1 \times 0 - 2 \times 2 & -1 \times 1 - 1 \times 0 - 2 \times 0 \\ 4 \times 0 + 0 \times 1 + 6 \times 0 & 4 \times 0 + 0 \times 0 + 6 \times 2 & 4 \times 1 + 0 \times 0 + 6 \times 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0 & 6 & 2 \\ -1 & -4 & -1 \\ 0 & 12 & 4 \end{bmatrix}}
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
AB &= \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \times 0 - 1 \times 1 + 4 \times 0 & 2 \times 0 - 1 \times 0 + 4 \times 2 & 2 \times 1 - 1 \times 0 + 4 \times 0 \\ 0 \times 0 - 1 \times 1 + 0 \times 0 & 0 \times 0 - 1 \times 0 + 0 \times 2 & 0 \times 1 - 1 \times 0 - 0 \times 0 \\ 3 \times 0 - 2 \times 1 + 6 \times 0 & 3 \times 0 - 2 \times 0 + 6 \times 2 & 3 \times 1 - 2 \times 0 + 6 \times 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} -1 & 8 & 2 \\ -1 & 0 & 0 \\ -2 & 12 & 3 \end{bmatrix}}
\end{aligned}
$$

Part (c)

$$BA = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \times 2 + 0 \times 0 + 1 \times 3 & 0 \times -1 - 0 \times 1 - 1 \times 2 & 0 \times 4 + 0 \times 0 + 1 \times 6 \\ 1 \times 2 + 0 \times 0 + 0 \times 3 & 1 \times -1 - 0 \times 1 - 0 \times 2 & 1 \times 4 + 0 \times 0 + 0 \times 6 \\ 0 \times 2 + 2 \times 0 + 0 \times 3 & 0 \times -1 - 2 \times 1 - 0 \times 2 & 0 \times 4 + 2 \times 0 + 0 \times 6 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 3 & -2 & 6 \\ 2 & -1 & 4 \\ 0 & -2 & 0 \end{bmatrix}}$$

Part (d)

$$B + A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0+2 & 0-1 & 1+4 \\ 1+0 & 0-1 & 0+0 \\ 0+3 & 2-2 & 0+6 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2 & -1 & 5 \\ 1 & -1 & 0 \\ 3 & 0 & 6 \end{bmatrix}}$$

Part (e)

$$B^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}^T$$

$$= \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}}$$

Part (f)

Matrix A is *not* invertable because its columns are not linearly independent (one column can be written as a linear-combination of the others):

$$\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 0 \times \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

On the other hand, matrix B *is* invertable. To find its inverse, let us augment it with the identity matrix and perform row-reductions:

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Which tells us that the inverse of matrix B is:

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$