Homework: Deriving Rotational Inertias

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Problem 2

Divide up the rod so that each segment is a single point on the rod (with negligible thickness). Each segment has mass ($\frac{M}{L}dx$) and rotational inertia ($x^2\frac{M}{L}dx$). The total rotational inertia of the rod is

$$\begin{aligned} I &= \int_0^L x^2\left(\frac{M}{L}\ dx\right) \\ &= \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L \\ &= \frac{M}{L}\left(\frac{L^3}{3}\right) \\ &= \boxed{\frac{ML^2}{3}} \end{aligned}$$

Problem 4

Divide up the hoop so that each segment is a single point on the hoop (with negligible thickness). Each segment has mass $(\frac{M}{2\pi}d\theta)$ and therefore has rotational inertia $((R\sin\theta)^2\frac{M}{2\pi}d\theta)$. The total rotational inertia of the hoop is

$$\begin{aligned}
I &= \int_0^{2\pi} (R\sin\theta)^2 \left(\frac{M}{2\pi}\ d\theta\right) \\
&= \frac{MR^2}{2\pi} \int_0^{2\pi} \sin^2\theta\ d\theta \\
&= \frac{MR^2}{2\pi} \int_0^{2\pi} \frac{1-\cos(2\theta)}{2}\ d\theta \\
&= \frac{MR^2}{2\pi} \left[\frac{\theta}{2} + \sin(2\theta)\right]_0^{2\pi} \\
&= \frac{MR^2}{2\pi}(\pi) \\
&= \boxed{\frac{MR^2}{2}}
\end{aligned}$$

Problem 6

Divide up the disk so that each segment is a thin hoop with some radius r from the center of the disk. Each segment has mass

$$dm = \frac{M}{A} dA = \frac{M}{\pi R^2} 2\pi r \ dr = \frac{2M}{R^2} r \ dr$$

According to the results from *Problem 4*, each segment has rotational inertia

$$dI = \frac{1}{2} r^2 dm = \frac{1}{2} r^2 \left(\frac{2M}{R^2} r \ dr \right) = \frac{M}{R^2} r^3 dr$$

Therefore, the total rotational inertia of the disk is

$$\begin{aligned} I &= \int_0^R \frac{M}{R} r^3 dr \\ &= \frac{M}{R^2} \left[\frac{r^4}{4} \right]_0^R \\ &= \frac{M}{R^2} \left(\frac{R^4}{4} \right) \\ &= \boxed{\frac{MR^2}{4}} \end{aligned}$$

Problem 10

Divide up the slab so that each segment is a thin rod with length L some distance x to the right of the center. Each segment has mass

$$dm = \frac{M}{A}dA = \frac{M}{WL}L\ dx = \frac{M}{W}dx$$

By the Parallel Axis Theorem and according to the results from *Problem 2*, each segment has rotational inertia

$$dI = \frac{1}{12}L^2 dm + x^2 dm = \left(x^2 + \frac{L^2}{12}\right) dm = \left(x^2 + \frac{L^2}{12}\right)\left(\frac{M}{W}dx\right)$$

Therefore, the total rotational inertia of the slab is

$$\begin{aligned} I &= \int_{-\frac{W}{2}}^{\frac{W}{2}} \left(x^2 + \frac{L^2}{12}\right)\left(\frac{M}{W}dx\right) \\ &= \frac{M}{W}\left[\frac{x^3}{3} + \frac{L^2 x}{12}\right]_{-\frac{W}{2}}^{\frac{W}{2}} \\ &= \frac{M}{W}\left(\frac{W^3}{12} + \frac{WL^2}{12}\right) \\ &= \boxed{\frac{M\ (W^2 + L^2)}{12}} \end{aligned}$$