Homework 3

Qianlang Chen

Section 3.1 (p193)

1. Exercise 2

Part (a)

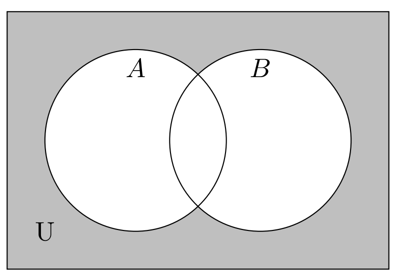
4, 8, 12, 16, 20.

Part (c)

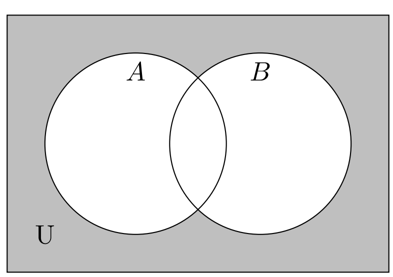
4, 8, 12, 16, 20.

2. Exercise 16

Part (b)



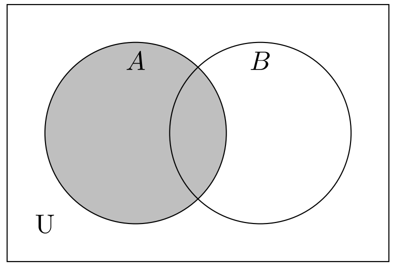
The Left-hand side: $(A \cup B)'$



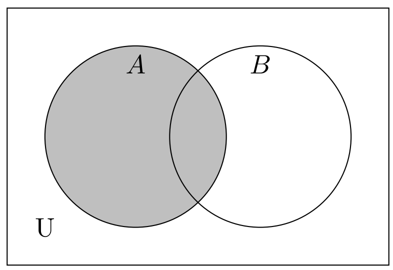
The Right-hand side: $A' \cap B'$

Therefore, since the Venn diagrams for both sides are the same, $(A \cup B)' = A' \cap B'$.

Part (d)



The Left-hand side: $A \cap (A \cup B)$

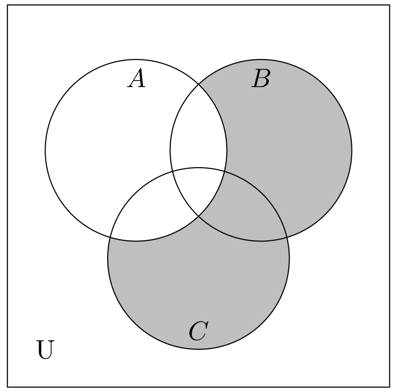


The Right-hand side: A

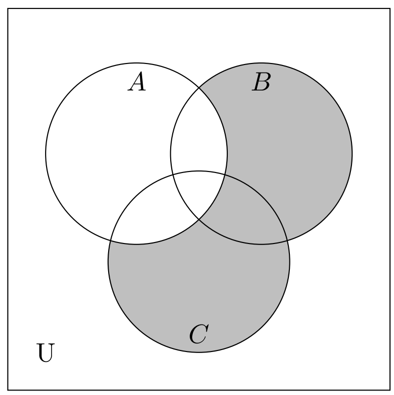
Therefore, since the Venn diagrams for both sides are the same, $A \cap (A \cup B) = A$.

3. Exercise 17

Part (b)



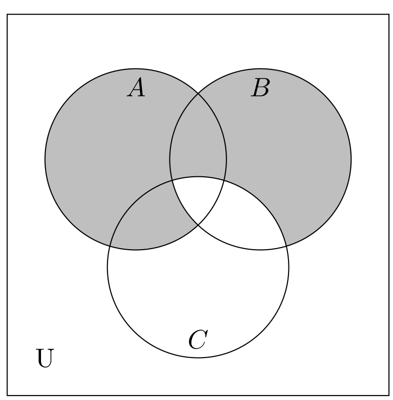
The Left-hand side: $(B \cup C) - A$



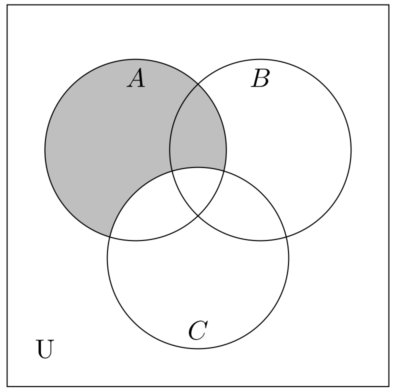
The Right-hand side: $(B - A) \cup (C - A)$

Therefore, since the Venn diagrams for both sides are the same, $(B \cup C) - A = (B - A) \cup (C - A)$.

Part (d)

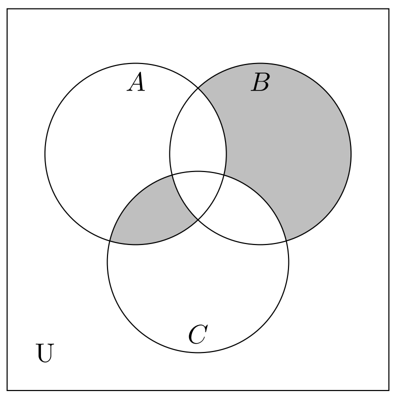


The Left-hand side: $(A - B) \cup (B - C)$



The Right-hand side: $A - C$

As seen from the Venn diagrams, there are some areas covered by the left-hand side but not the right-hand side. Specifically, these are the mentioned areas:



Therefore, the left-hand side may not be a subset of the right-hand side. $(A - B) \cup (B - C) \nsubseteq A - C$. As an example, if

$$\begin{aligned} A &= \{1, 2, 3\}, \\ B &= \{4, 5, 6\}, \\ C &= \{7, 8, 9\}, \end{aligned}$$

then

$$\begin{aligned} (A - B) \cup (B - C) &= \{1, 2, 3, 4, 5, 6\}, \\ A - C &= \{1, 2, 3\}; \\ \{1, 2, 3, 4, 5, 6\} &\nsubseteq \{1, 2, 3\} \end{aligned}$$

4. Exercise 18

Part (b)

$$A - B = \{k : k \in \mathbb{Z}, k \bmod 6 \neq 0\}$$

Part (e)

$$\mathbb{Z} - A = \{2k + 1 : k \in \mathbb{Z}\}$$

Section 3.2 (p208)

5. Exercise 1

Part (b)

$$\begin{aligned}(A \times B) - (A \times A) &= \{2,4\} \times \{1,2,8\} - \{2,4\} \times \{2,4\} \\ &= \{(2,1),(2,2),(2,8),(4,1),(4,2),(4,8)\} - \{(2,2),(2,4),(4,2),(4,4)\} \\ &= \boxed{\{(2,1),(2,8),(4,1),(4,8)\}}\end{aligned}$$

Part (d)

$$\begin{aligned}\wp(B \cap C) &= \wp(\{1,2,8\} \cap \{1,2,5,6,10\}) \\ &= \wp(\{1,2\}) \\ &= \boxed{\{\ \{\}, \{1\}, \{2\}, \{1,2\}\ \}}\end{aligned}$$

6. Exercise 9

According to the division theorem, every integer falls into one of the following categories:

- $4q$,
- $4q+1$,
- $4q+2$,
- $4q+3$,

where $q \in \mathbb{Z}$. In other words, $\mathbb{Z} = B \cup C \cup S \cup T$, where

- $B = \{4q+1 : q \in \mathbb{Z}\}$ (given by the problem statement),
- $C = \{4q+3 : q \in \mathbb{Z}\}$ (given by the problem statement),
- $S = \{4q : q \in \mathbb{Z}\}$,
- $T = \{4q+2 : q \in \mathbb{Z}\}$.

Also, to show that $A = \{2k : k \in \mathbb{Z}\} = S \cup T$, since every integer is either even or odd,

- when k is even, $2k = 2(2q) = 4q$, where q is an integer by definition of even;
- when k is odd, $2k = 2(2q+1) = 4q+2$, where q is an integer by definition of odd.

Therefore, we showed that $A = S \cup T$. Now, we have $\mathbb{Z} = B \cup C \cup S \cup T = A \cup B \cup C$.

According to the definition of a partition, and

- $A \neq B \neq C \neq \emptyset$,
- $A \cap B = A \cap C = B \cap C = \emptyset$,
- $A \cup B \cup C = \mathbb{Z}$,

$\{A, B, C\}$ is a partition of $\mathbb{Z}$.

7. Exercise 11

Part (a)

True.

Part (b)

False. One counter-example: if $A = \{1\}$ and $B = \{2\}$,

$$\begin{aligned}
(A \cup B) \times (A - B) &= \{1, 2\} \times \{1\} \\
&= \{(1, 1), (2, 1)\}; \\
A^2 - B^2 &= \{(1, 1)\} - \{(2, 2)\} \\
&= \{(1, 1)\}; \\
(A \cup B) \times (A - B) &\neq A^2 - B^2
\end{aligned}$$

Part (c)

False. One counter-example: if $A = \{1\}, B = \{2\}$ and $C = \{3\}$,

$$\begin{aligned}
A \times (B \times C) &= \{1\} \times (\{2\} \times \{3\}) \\
&= \{1\} \times \{(2, 3)\} \\
&= \{(1, (2, 3))\}; \\
(A \times B) \times C &= (\{1\} \times \{2\}) \times \{3\} \\
&= \{(1, 2)\} \times \{3\} \\
&= \{((1, 2), 3)\}; \\
A \times (B \times C) &\neq (A \times B) \times C
\end{aligned}$$

8. Exercise 22

According to *Theorem 1*, $|A \times B| = |A|\ |B|$. Therefore,

$$\begin{aligned}
|S_1 \times S_2 \times \cdots \times S_{k-1} \times S_k| &= |S_1|\ |S_2|\ \cdots\ |S_{k-1}|\ |S_k| \\
&= (|S_1|\ |S_2|\ \cdots\ |S_{k-1}|)\ |S_k| \\
&= |S_1 \times S_2 \times \cdots \times S_{k-1}|\ |S_k| \\
&= \boxed{|(S_1 \times S_2 \times \cdots \times S_{k-1}) \times S_k|}
\end{aligned}$$

Section 3.3 (p219)

9. Exercise 2.e

Let an element $x : x \in (\{2n+1 : n \in \mathbb{Z}\} \cap \{5m+4 : m \in \mathbb{Z}\})$ be given. By definition of set union, we have that $\exists n \in \mathbb{Z} : x = 2n+1$ and that $\exists m \in \mathbb{Z} : x = 5m+4$. Since every integer is either even or odd, one of the following cases about m must be true:

Case I, when m is even: by definition of even, $\exists k \in \mathbb{Z} : 2k = m$. By substitution, $x = 5m+4 = 5 \cdot (2k)+4 = 10k+4 = 2 \cdot (5k+2) = 2c$ for some c. Since x has to be in the form of $2n+1$, this is not a possibility for x.

Case II, when m is odd: by definition of odd, $\exists k \in \mathbb{Z} : 2k+1 = m$. By substitution, $x = 5m+4 = 5 \cdot (2k+1)+4 = 10k+9 = 2 \cdot (5k+4)+1 = 2c+1$ for some c. By closure under addition, $c \in \mathbb{Z}$. Since this fits the definition of x and x can be written in the form of $10k+9$, by definition of a subset, $(\{2n+1 : n \in \mathbb{Z}\} \cap \{5m+4 : m \in \mathbb{Z}\}) \subseteq \{10k+9 : k \in \mathbb{Z}\}$.

10. Exercise 3.b

Let an element $x : x \in (\{n^2-1 : n \in \mathbb{Z}\} \cap \{2k : k \in \mathbb{Z}\})$ be given. By definition of set union, we have that $\exists n \in \mathbb{Z} : x = n^2-1$ and that $\exists k \in \mathbb{Z} : k = 2k$. Since every integer is either even or odd, one of the following cases about n must be true:

Case I, when n is even: by definition of even, $\exists m \in \mathbb{Z} : 2m = n$. By substitution, $x = n^2-1 = (2m)^2-1 = 4m^2-1 = 2 \cdot (2m^2-1)+1 = 2c+1$ for some c. Since x has to be in the form of $2k$, this is not a possibility for x.

Case II, when n is odd: by definition of odd, $\exists m \in \mathbb{Z} : 2m+1 = n$. By substitution, $x = n^2-1 = (2m+1)^2-1 = 4m^2+4m+1-1 = 2 \cdot (2m^2+2m) = 2c$ for some c, and by closure under addition, $c \in \mathbb{Z}$. Also, $4m^2+4m = 4(m^2+m) = 4d$ for some d, and by closure under addition, $d \in \mathbb{Z}$. Since this fits the definition of x and x can be written in the form of $4m$, by definition of a subset, $(\{n^2-1 : n \in \mathbb{Z}\} \cap \{2k : k \in \mathbb{Z}\}) \subseteq \{4m : m \in \mathbb{Z}\}$.

11. Exercise 11.d

First, we have that $C \subseteq (A \cup C)$ because $C \subseteq C$ already.

Now, let us show that $(A \cup C) \subseteq C$.

Let an element $x : x \in (A \cup C)$ be given. By definition of set union, one of the following cases about x must be true:

Case I, when $x \in A$: since $A \cup B = B$, we have that $x \in B$. Since $B \cup C = C$, we have that $x \in C$.

Case II, when $x \in C$ already.

Therefore, $x \in C$ in every possible case. By definition of a subset, $(A \cup C) \subseteq C$.

Finally, since $C \subseteq (A \cup C)$ and $(A \cup C) \subseteq C$, we have that $(A \cup C) = C$.

12. Exercise 13.c

First, let us show that $(A \cup (B - A)) \subseteq B$. Let an element $x : x \in (A \cup (B - A))$ be given. By definition of set union, one of the following cases about x must be true:

Case I, when $x \in A$: since $A \subseteq B$, $x \in B$.

Case II, when $x \in (B - A)$: by definition of set difference, we have that $x \in B$.

Therefore, $x \in B$ in every possible case. By definition of a subset, $(A \cup (B - A)) \subseteq B$.

Now, to show that $B \subseteq (A \cup (B - A))$, let an element $x : x \in B$ be given. Since $A \subseteq B$, one of the following cases about x must be true:

Case I, when $x \in A$ already, meaning that $x \in (A \cup (B - A))$.

Case II, when $x \in B$ but $x \notin A$: by definition of set difference, $x \in (B - A)$, also meaning that $x \in (A \cup (B - A))$.

Therefore, $x \in (A \cup (B - A))$ in every possible case. By definition of a subset, $B \subseteq (A \cup (B - A))$.

Finally, since $(A \cup (B - A)) \subseteq B$ and $x \in (A \cup (B - A))$, we have that $(A \cup (B - A)) = B$.

13. Exercises 14.e and 15.e

Exercise 14.e

$$\begin{aligned} (A \cup B) \cap (A' \cap C)' &= (A \cup B) \cap (A \cup C') && \text{De Morgan's} \\ &= A \cup (B \cap C') && \text{Inverse Distributive} \end{aligned}$$

Exercise 15.e

$$(A \cap B) \cup (A' \cup C)' = (A \cap (B \cup C'))$$

14. Exercise 19.b

$$\begin{aligned} A \cup (B \cap A') &= (A \cup B) \cap (A \cup A') && \text{Distributive} \\ &= B \cap (A \cup A') && \text{Substituting } B = A \cup B \\ &= B \cap U && \text{Negation} \\ &= B && \text{Identity} \end{aligned}$$

15. Exercises 22.a

First, let us show that $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.

Let a tuple $(x, c) \in (A \cup B) \times C$ be given. By definition of a Cartisian Product and set union, one of the following cases about x must be true:

Case I, when $x \in A$, meaning that $(x, c) \in (A \times C)$. This also means that $(x, c) \in (A \times C) \cup (B \times C)$.

Case II, when $x \in B$, meaning that $(x, c) \in (B \times C)$. This also means that $(x, c) \in (A \times C) \cup (B \times C)$.

Therefore, $(x, c) \in (A \times C) \cup (B \times C)$ in every possible case. By definition of a subset, $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$.

Next, let us show that $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$.

Let a tuple $(x, c) \in (A \times C) \cup (B \times C)$ be given. By definition of a Cartisian Produt and set union, one of the following cases about x must be true:

Case I, when $(x, c) \in (A \times C)$, meaning that $x \in A$.

Case II, when $(x, c) \in (B \times C)$, meaning that $x \in B$.

Therefore, $x \in A$ or $x \in B$ in every possible case, and by definition of set union, we have that $x \in (A \cup B)$, meaning that $(x, c) \in (A \cup B) \times C$. Also, by definition of a subset, $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$.

Finally, since $(A \cup B) \times C \subseteq (A \times C) \cup (B \times C)$ and $(A \times C) \cup (B \times C) \subseteq (A \cup B) \times C$, we have that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

Section 3.4 (p227)

16. Exercise 1.d

$$(p + p'q)' = p'q'$$

17. Exercise 2.f

$$(ab) \cdot (bc)' = (ab) \cdot c'$$

18. Exercise 6.b

$$\begin{aligned}
a' + b &= (ab)' + b && \text{Substituting } ab = a \\
&= (a' + b') + b && \text{De Morgan's} \\
&= a' + (b' + b) && \text{Associative} \\
&= a' + 1 && \text{Negation} \\
&= 1 && \text{Universal Bound}
\end{aligned}$$

19. Exercise 14

$$\begin{aligned}
ab + (a' + b') &= (ab + a') + b' && \text{Associative} \\
&= (a' + ab) + b' && \text{Commutative} \\
&= (a' + a)(a' + b) + b' && \text{Distributive} \\
&= (a + a')(a' + b) + b' && \text{Commutative} \\
&= 1 \cdot (a' + b) + b' && \text{Negation} \\
&= (a' + b) \cdot 1 + b' && \text{Commutative} \\
&= (a' + b) + b' && \text{Identity} \\
&= a' + (b + b') && \text{Associative} \\
&= a' + 1 && \text{Negation} \\
&= 1 && \text{Universal Bound}
\end{aligned}$$

$$\begin{aligned}
ab \cdot (a' + b') &= (ab) \cdot a' + (ab) \cdot b' && \text{Distributive} \\
&= a' \cdot (ab) + (ab) \cdot b' && \text{Commutative} \\
&= (a'a) \cdot b + (ab) \cdot b' && \text{Associative} \\
&= (aa') \cdot b + (ab) \cdot b' && \text{Commutative} \\
&= 0 \cdot b + (ab) \cdot b' && \text{Negation} \\
&= b \cdot 0 + (ab) \cdot b' && \text{Commutative} \\
&= 0 + (ab) \cdot b' && \text{Universal Bound} \\
&= 0 + a \cdot (bb') && \text{Associative} \\
&= 0 + a \cdot 0 && \text{Negation} \\
&= 0 + 0 && \text{Universal Bound} \\
&= 0 && \text{Idempotent}
\end{aligned}$$