Homework 2

Qianlang Chen

Problem 1

Part (a)

Rolling a six-sided die leaves us with the following sample space for the outcomes:

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Also, according to the problem description, we have

$$\begin{aligned} A &= \{1, 2, 3, 4\}; \\ B &= \{2, 4, 6\}; \\ A \cap B &= \{2, 4\} \end{aligned}$$

Since these outcomes are equally likely to happen,

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{2}{6} = \frac{1}{3}$$

However,

$$P(A) \times P(B) = \frac{|A|}{|\Omega|} \times \frac{|B|}{|\Omega|} = \frac{4}{6} \times \frac{3}{6} = \frac{1}{3} = P(A \cap B)$$

Therefore, by definition of independence, the events A and B are independent.

Part (b)

Flipping two fair coins leaves us with the following sample space for the outcomes:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

Also, according to the problem description, we have

$$\begin{aligned} A &= \{(H, H), (T, T)\}; \\ B &= \{(H, T), (T, H), (T, T)\}; \\ A \cap B &= \{(T, T)\} \end{aligned}$$

Since these outcomes are equally likely to happen,

$$P(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{1}{4}$$

However,

$$P(A) \times P(B) = \frac{|A|}{|\Omega|} \times \frac{|B|}{|\Omega|} = \frac{2}{4} \times \frac{3}{4} = \frac{3}{8} \neq P(A \cap B)$$

Therefore, by definition of independence, the events A and B are not independent.

Problem 2

Let D be the event that the patient has heart disease and T be the event that the test turns out positive. According to the problem statement, we have

$$\begin{aligned} P(D) &= \frac{1}{5}; \\ P(T|D^c) &= \frac{1}{10}; \\ P(T|D) &= \frac{9}{10} \end{aligned}$$

By the Law of Total Probability, we have

$$\begin{aligned} P(T) &= P(T|D) \times P(D) + P(T|D^c) \times P(D^c) \\ &= \frac{9}{10} \times \frac{1}{5} + \frac{1}{10} \times (1 - \frac{1}{5}) \\ &= \frac{13}{50} \end{aligned}$$

By Bayes' Rule, we have

$$\begin{aligned} P(D|T) &= P(T|D) \times \frac{P(D)}{P(T)} \\ &= \frac{9}{10} \times \frac{1}{5} \div \frac{13}{50} \\ &= \boxed{\frac{9}{13}} \end{aligned}$$

Problem 3

Part (a)

Since A and B are independent events, by definition, $P(A) = P(A|B)$ and also $P(A \cap B) = P(A) \times P(B)$.

Now, according to the Probability Rules, we have

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
&= P(A) + P(B) - P(A) \times P(B) \\
&= P(A|B) + P(B) - P(A|B) \times P(B) \\
&= \frac{1}{2} + x - \frac{1}{2}x \\
&= \boxed{\frac{1+x}{2}}
\end{aligned}$$

Part (b)

First of all, since $B \subseteq (A \cup B)$, we have $B \cap (A \cup B) = B$.

Now, we have,

$$\begin{aligned}
P(B|A \cup B) &= \frac{P(B \cap (A \cup B))}{P(A \cup B)} \\
&= \frac{P(B)}{P(A \cup B)}
\end{aligned}$$

By rearranging, we have

$$\begin{aligned}
P(B) &= P(B|A \cup B) \times P(A \cup B) \\
x &= \frac{2}{3} \times \frac{1+x}{2} \\
\implies x &= \boxed{\frac{1}{2}}
\end{aligned}$$

Problem 4

Part (a)

By definition of independence, $P(A \cap B) = P(A) \times P(B)$ if A and B are independent events. Since A and B are disjoint, $P(A \cap B) = 0$. However, the problem states that $P(A) \neq 0$ and $P(B) \neq 0$. Therefore, A and B cannot be disjoint and independent at the same time.

Part (b)

Same logic as *Part (a)*, two events A and B cannot be disjoint and independent at the same time given that $P(A) \neq 0$ and $P(B) \neq 0$.

Part (c)

Since $A \subset B$, $P(A \cap B) = P(A)$. If A and B are independent, it would mean that $P(A \cap B) = P(A) \times P(B) = P(A) \times 1$, but the problem states that $P(B) \neq 1$. Therefore, if $A \subset B$, A and B cannot be independent.

Part (d)

Let events A and $A \cup B$ be independent. By definition of independence, we have $P(A|(A \cup B)^c) = P(A)$.

However, we also have

$$\begin{aligned}
P(A|(A \cup B)^c) &= \frac{P(A \cap (A \cup B)^c)}{P((A \cup B)^c)} \\
&= \frac{P(A \cap A^c \cap B^c)}{P((A \cup B)^c)} \\
&= \frac{0}{P((A \cup B)^c)} \\
&= 0
\end{aligned}$$

This implies $P(A) = 0$, but the problem states that $P(A) \neq 0$. Therefore, the events A and $A \cup B$ cannot be independent.

Problem 5

Part (a)

First, given how the events A, B, and C are defined, we have

$$A \cap B \cap C = \{1\}$$

Since all 8 outcomes from the sample space is equally likely,

$$P(A) \times P(B) \times P(C) = \frac{4}{8} \times \frac{4}{8} \times \frac{4}{8} = \frac{1}{8} = P(A \cap B \cap C)$$

Part (b)

$$P(A) \times P(C) = \frac{4}{8} \times \frac{4}{8} = \frac{1}{4}$$

However,

$$P(A \cap C) = \frac{|\{1\}|}{|\Omega|} = \frac{1}{8} \neq P(A) \times P(C)$$

Therefore, by definition of independence, the events A and C are not independent.