Homework 2

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Problem 1

Part (a)

Part (b)

Part (c)

Problem 2

Part (a)

Part (b)

Part (c)

Part (d)

Problem 3

Part (a)

Let $x = -1$. Now, the vectors p and q are linearly dependent because they are scaled versions of each other:

$$\begin{bmatrix} 1 \\ -2 \\ 4 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix}$$

Part (b)

Let $x = 21$. Now, the vectors p and q are orthogonal because their dot-product is zero:

$$\begin{aligned}
\begin{bmatrix} 1 \\ -2 \\ 4 \\ x \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} &= \begin{bmatrix} 1 & -2 & 4 & x \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -2 & 4 & 21 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 8 \\ -2 \end{bmatrix} \\
&= 1 \times 2 + 2 \times 4 + 4 \times 8 - 21 \times 2 \\
&= 0
\end{aligned}$$

Part (c)

$$||\mathbf{q}||_1 = \left\| \begin{matrix} 2 \\ -4 \\ 8 \\ -2 \end{matrix} \right\|_1 = |2| + |-4| + 8 + |-2| = \boxed{16}$$

Part (d)

$$||\mathbf{q}||_2^2 = \left\| \begin{matrix} 2 \\ -4 \\ 8 \\ -2 \end{matrix} \right\|_2^2 = 2^2 + (-4)^2 + 8^2 + (-2)^2 = \boxed{88}$$

Problem 4

Part (a)

$$
\begin{aligned}
A^T B &= \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 3 \\ -1 & -1 & -2 \\ 4 & 0 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2\times 0+0\times 1+3\times 0 & 2\times 0+0\times 0+3\times 2 & 2\times 1+0\times 0+3\times 0 \\ -1\times 0-1\times 1-2\times 0 & -1\times 0-1\times 0-2\times 2 & -1\times 1-1\times 0-2\times 0 \\ 4\times 0+0\times 1+6\times 0 & 4\times 0+0\times 0+6\times 2 & 4\times 1+0\times 0+6\times 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 0 & 6 & 2 \\ -1 & -4 & -1 \\ 0 & 12 & 4 \end{bmatrix}}
\end{aligned}
$$

Part (b)

$$
\begin{aligned}
AB &= \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 2\times 0-1\times 1+4\times 0 & 2\times 0-1\times 0+4\times 2 & 2\times 1-1\times 0+4\times 0 \\ 0\times 0-1\times 1+0\times 0 & 0\times 0-1\times 0+0\times 2 & 0\times 1-1\times 0-0\times 0 \\ 3\times 0-2\times 1+6\times 0 & 3\times 0-2\times 0+6\times 2 & 3\times 1-2\times 0+6\times 0 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} -1 & 8 & 2 \\ -1 & 0 & 0 \\ -2 & 12 & 3 \end{bmatrix}}
\end{aligned}
$$

Part (c)

$$
\begin{aligned}
BA &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 2 + 0 \times 0 + 1 \times 3 & 0 \times -1 - 0 \times 1 - 1 \times 2 & 0 \times 4 + 0 \times 0 + 1 \times 6 \\ 1 \times 2 + 0 \times 0 + 0 \times 3 & 1 \times -1 - 0 \times 1 - 0 \times 2 & 1 \times 4 + 0 \times 0 + 0 \times 6 \\ 0 \times 2 + 2 \times 0 + 0 \times 3 & 0 \times -1 - 2 \times 1 - 0 \times 2 & 0 \times 4 + 2 \times 0 + 0 \times 6 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 3 & -2 & 6 \\ 2 & -1 & 4 \\ 0 & -2 & 0 \end{bmatrix}}
\end{aligned}
$$

Part (d)

$$
\begin{aligned}
B + A &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 0+2 & 0-1 & 1+4 \\ 1+0 & 0-1 & 0+0 \\ 0+3 & 2-2 & 0+6 \end{bmatrix} \\
&= \boxed{\begin{bmatrix} 2 & -1 & 5 \\ 1 & -1 & 0 \\ 3 & 0 & 6 \end{bmatrix}}
\end{aligned}
$$

Part (e)

$$
\begin{aligned}
B^T &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}^T \\
&= \boxed{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}}
\end{aligned}
$$

Part (f)

Matrix A is *not* invertable because its columns are not linearly independent (one column can be written as the linear-combination between the others):

$$\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = 0 \times \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix} + \frac{1}{2} \times \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

On the other hand, matrix B *is* invertable. To find its inverse, let us augment it with the identity matrix and perform row-reductions:

$$\left[\begin{array}{ccc:ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Which tells us that the inverse of matrix B is:

$$B^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$