Assignment: Statistical Principles

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Problem 1

```python
# The size of the domain in question.
n = 3000
```

Part A

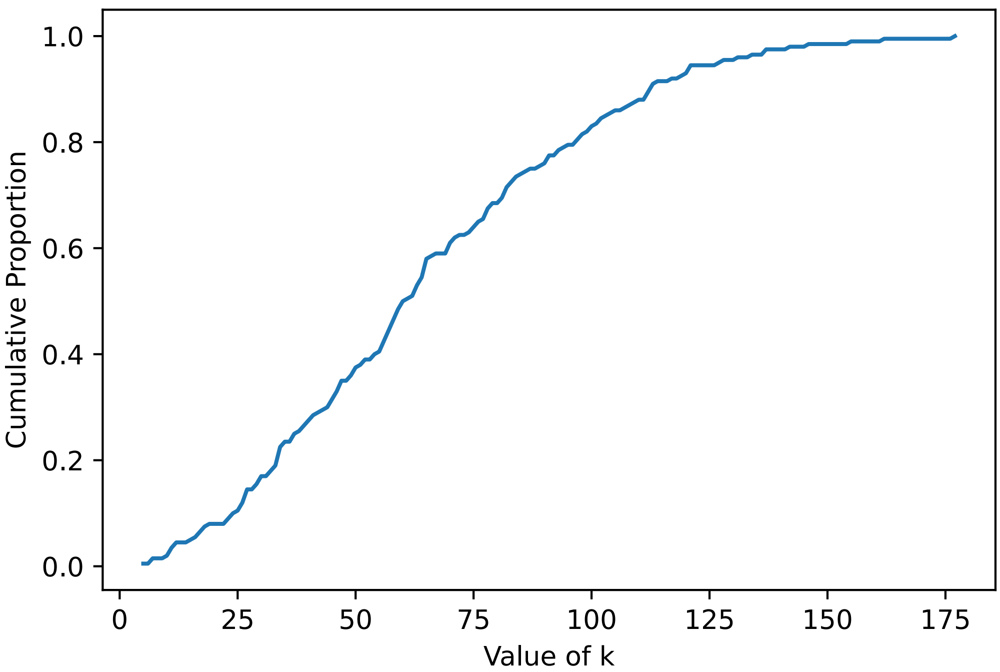
```python
import random

# Runs the experiment of generating random numbers in the range [0, n)
# until two numbers generated equal. Returns the value `k` where the
# k-th number equals to one of the numbers generated before.
def experiment(n):
    generated = set()
    for k in range(1, n + 1):
        x = random.randint(0, n - 1)
        if x in generated: return k
        generated.add(x)

# Run the experiment once and report the k-value.
k = experiment(n)
print(f'Running the experiment once took {k} random trials')
```

```
Running the experiment once took 82 random trials
```

Part B

```python
import itertools
from matplotlib import pyplot

# The number of experiments to run.
m = 200

# Run the experiment `m` times and record the frequency for each value
# of `k`.
k_freq = {}
for _ in range(m):
    k = experiment(n)
    k_freq[k] = k_freq.get(k, 0) + 1

# Plot the k-frequencies.
X = list(range(min(k_freq.keys()), max(k_freq.keys()) + 1))
Y = list(itertools.accumulate(k_freq.get(x, 0) / m for x in X))
pyplot.xlabel('Value of k')
pyplot.ylabel('Cumulative Proportion')
pyplot.plot(X, Y)
```

```
[<matplotlib.lines.Line2D at 0x23ee24ac088>]
```



Part C

```
k_bar = sum(k * x for k, x in k_freq.items()) / m
print(f'Empirical estimate of the expected value of `k`: {k_bar}')
```

```
Empirical estimate of the expected value of `k`: 65.37
```

Part D

In the implementation of the experiment:

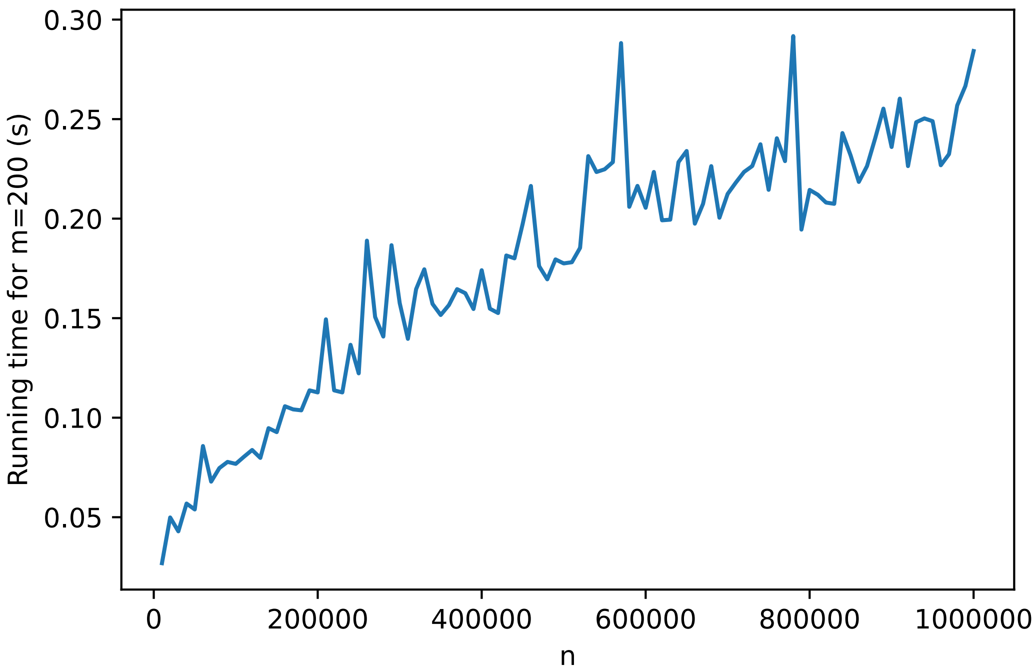
- I used a hash-set to keep track of the numbers that had appeared (been randomly generated);
- I kept generating random numbers until the first time a number generated exists in that hash-set;
- When the above happened, I immediately terminated the experiment and reported the number of trails it took.

```python
import time

# Returns the time in seconds that running an experiment with some
# values of `n` and `m` takes.
def time_experiment(n, m):
    start_time = time.time()
    while time.time() - start_time < .25: pass # warm-up loop
    start_time = time.time()
    for _ in range(m): experiment(n)
    return time.time() - start_time

t = time_experiment(n, m)
print(f'The experiment with n={n} and m={m} took {t:.3f} s')
```

```
The experiment with n=3000 and m=200 took 0.015 s
```

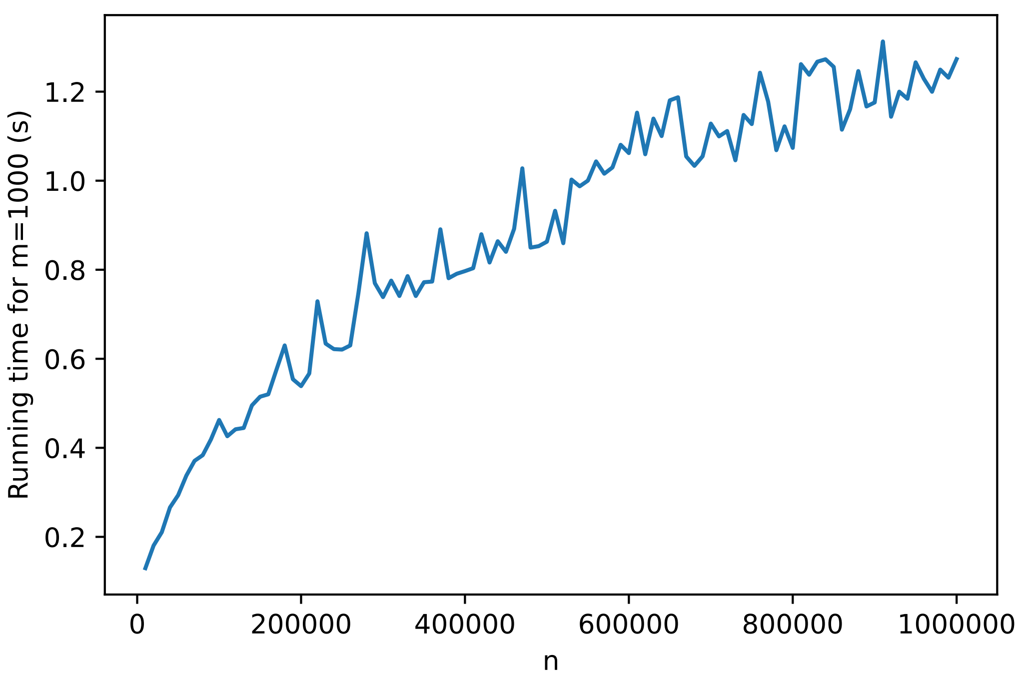
Now, for more values of n and m:

```python
def time_m(m):
    X = list(range(10**4, 10**6 + 1, 10**4))
    Y = list(time_experiment(x, m) for x in X)
    pyplot.ticklabel_format(useOffset=False, style='plain')
    pyplot.xlabel('n')
    pyplot.ylabel(f'Running time for m={m} (s)')
    pyplot.plot(X, Y)

time_m(200)
```



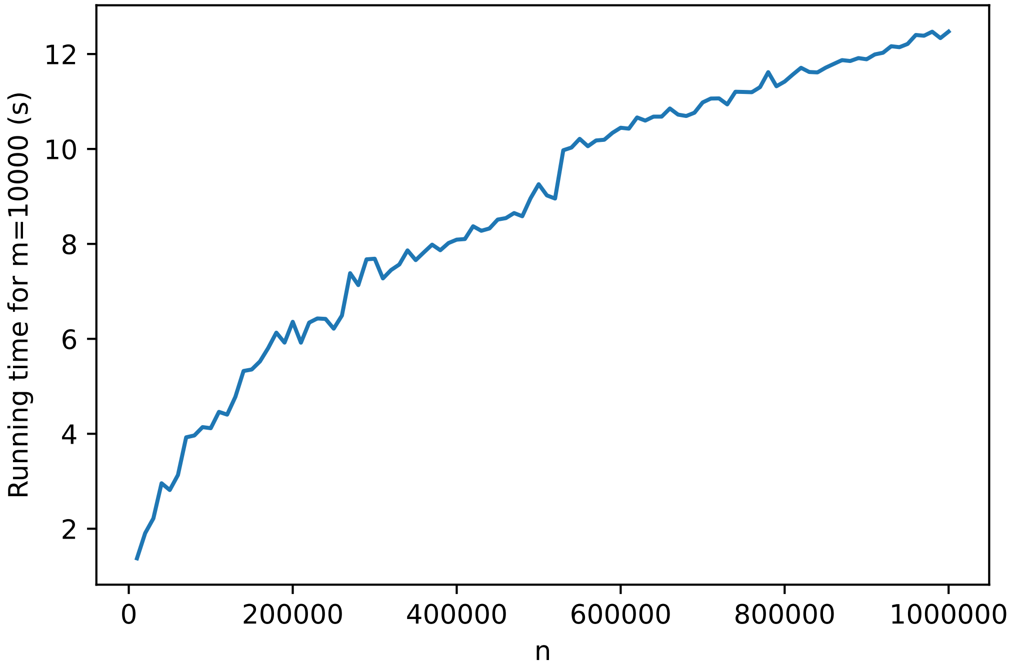
0.30
0.25
0.20
0.15
0.10
0.05
Running time for m=200 (s)
0
200000
400000
600000
800000
1000000
n


```
time_m(10**3)
```



1.2
1.0
0.8
0.6
0.4
0.2
Running time for m=1000 (s)
0
200000
400000
600000
800000
1000000
n

```
time_m(10**4)
```



Running time for m=10000 (s)
2
4
6
8
10
12
0
200000
400000
600000
800000
1000000
n

Problem 2

```python
# The size of the domain in question.
n = 200
```

Part A

```python
import itertools, random

# Runs the experiment of generating random numbers in the range [0, n)
# until every possible number has been generated. Returns the value `k`
# where as of the k-th generation, all possible numbers have been
# generated.
def experiment(n):
    not_generated = {x for x in range(n)}
    for k in itertools.count(1):
        x = random.randint(0, n - 1)
        not_generated.discard(x)
        if not not_generated: return k

# Run the experiment once and report the k-value.
k = experiment(n)
print(f'Running the experiment once took {k} random trials')
```

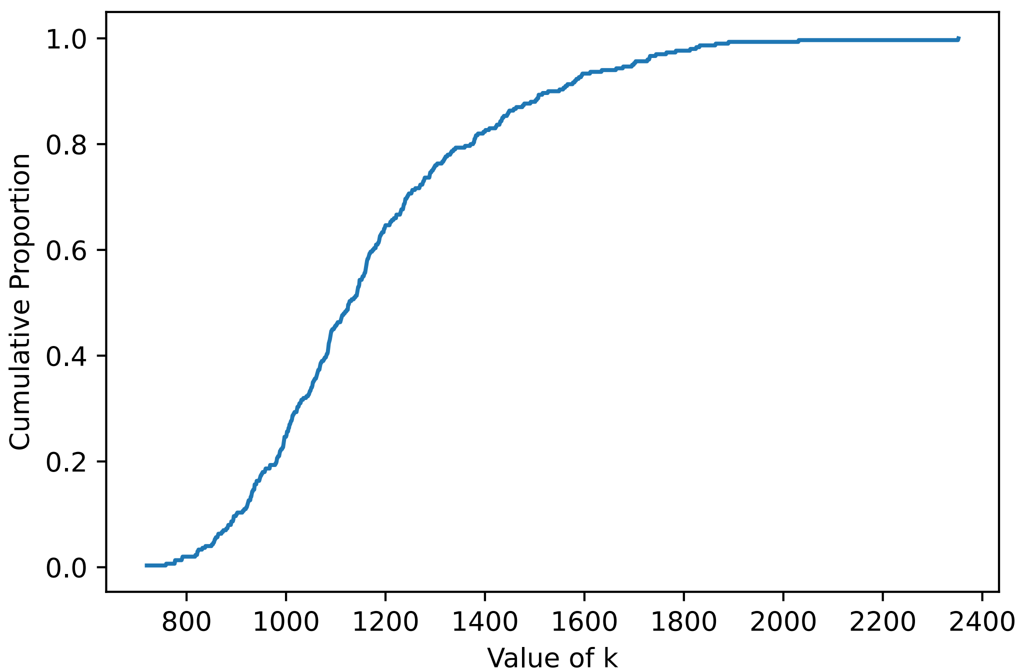
```
Running the experiment once took 1126 random trials
```

Part B

```python
from matplotlib import pyplot

# The number of experiments to run.
m = 300

# Run the experiment `m` times and record the frequency for each value
# of `k`.
k_freq = {}
```

```python
for _ in range(m):
    k = experiment(n)
    k_freq[k] = k_freq.get(k, 0) + 1

# Plot the k-frequencies.
X = list(range(min(k_freq.keys()), max(k_freq.keys()) + 1))
Y = list(itertools.accumulate(k_freq.get(x, 0) / m for x in X))
pyplot.xlabel('Value of k')
pyplot.ylabel('Cumulative Proportion')
pyplot.plot(X, Y)
```

```
[<matplotlib.lines.Line2D at 0x23ee2728d88>]
```



Part C

```
k_bar = sum(k * x for k, x in k_freq.items()) / m
print(f'Empirical estimate of the expected value of `k`: {k_bar}')
```

```
Empirical estimate of the expected value of `k`: 1176.1733333333334
```

Part D

In the implementation of the experiment:

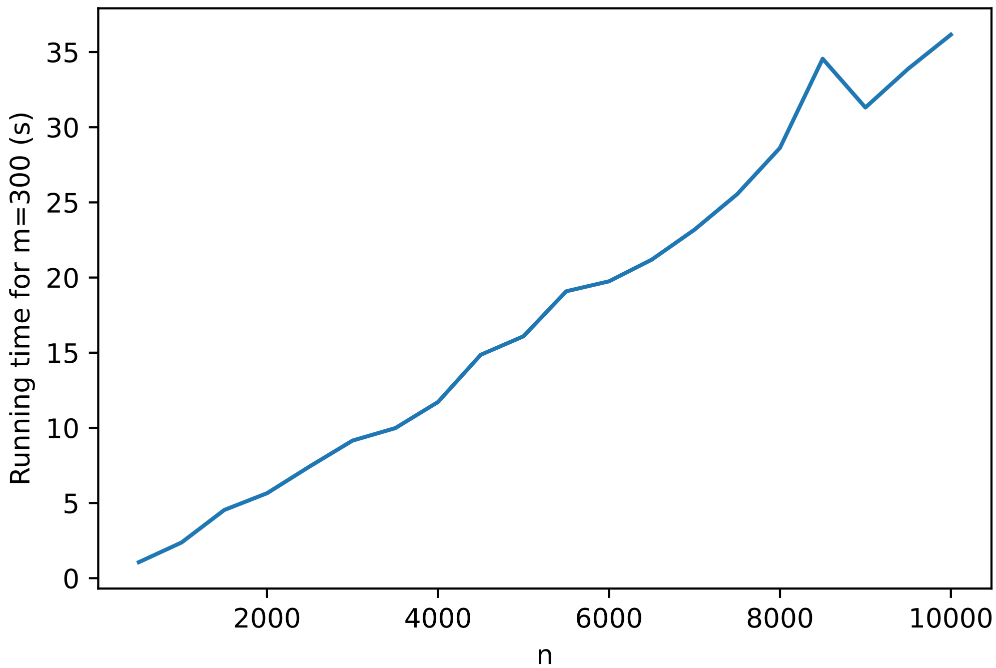
- I used a hash-set to keep track of the numbers that had *not yet* appeared (*not* been randomly generated);
- I kept generating random numbers until that hash-set became empty, meaning that all possible numbers had been generated;
- When the above happened, I immediately terminated the experiment and reported the number of trails it took.

```
import time

# Returns the time in seconds that running an experiment with some
# values of `n` and `m` takes.
def time_experiment(n, m):
    start_time = time.time()
    while time.time() - start_time < .25: pass # warm-up loop
    start_time = time.time()
    for _ in range(m): experiment(n)
    return time.time() - start_time

t = time_experiment(n, m)
print(f'The experiment with n={n} and m={m} took {t:.3f} s')
```

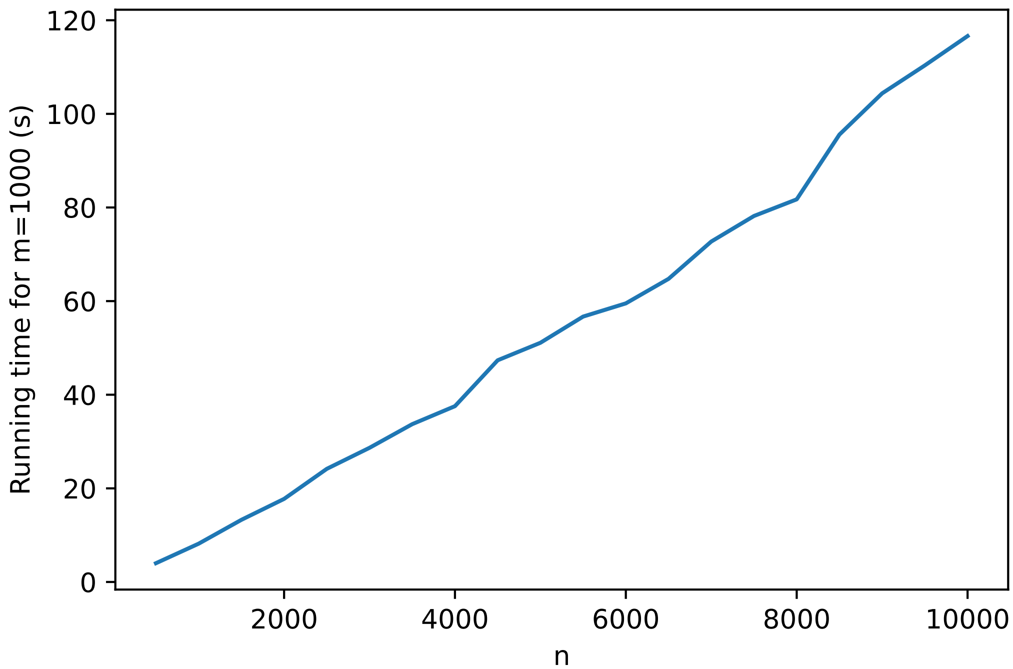
```
The experiment with n=200 and m=300 took 0.319 s
```

Now, for more values of n and m:

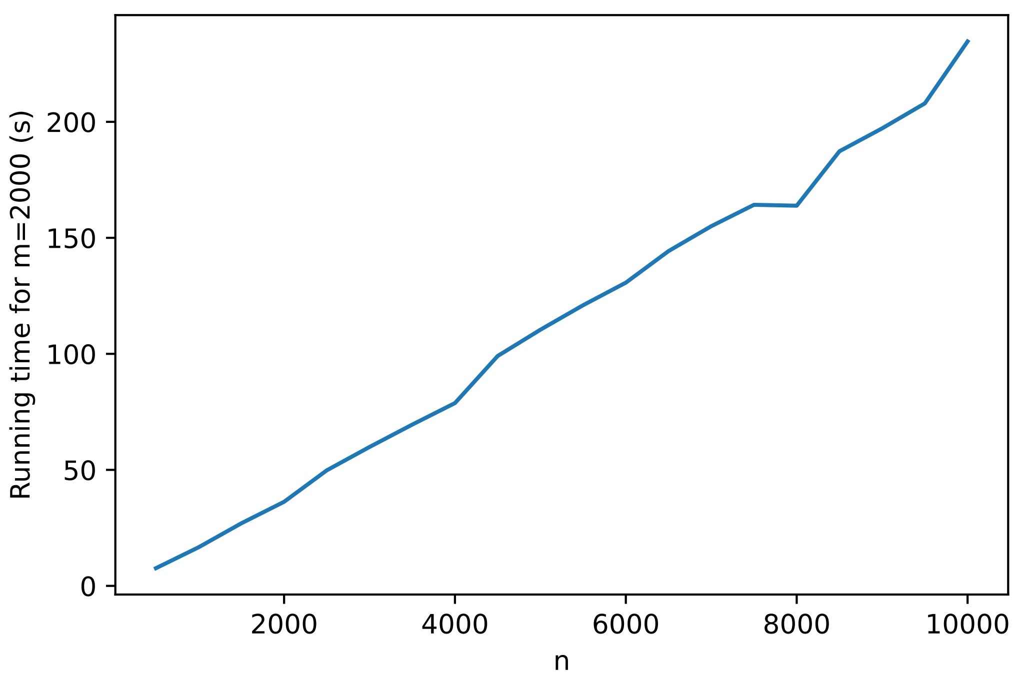
```python
def time_m(m):
    X = list(range(500, 10**4 + 1, 500))
    Y = list(time_experiment(x, m) for x in X)
    pyplot.ticklabel_format(useOffset=False, style='plain')
    pyplot.xlabel('n')
    pyplot.ylabel(f'Running time for m={m} (s)')
    pyplot.plot(X, Y)

time_m(300)
```




```python
time_m(1000)
```


Running time for m=1000 (s)
0
20
40
60
80
100
120
2000
4000
6000
8000
10000
n

```
time_m(2000)
```


Running time for m=2000 (s)
0
50
100
150
200
2000
4000
6000
8000
10000
n

Problem 3

Part A

I will use Method 1 from the lecture to estimate the actual expected value of k:

$$\begin{aligned}
\Pr(\text{Collision with } k \text{ objects in a domain sized } n) &\approx 1 - \left(1 - \frac{1}{n}\right)^{k^2/2} \\
\frac{1}{2} &\approx 1 - \left(1 - \frac{1}{3000}\right)^{k^2/2} \\
\frac{k^2}{2} &\approx \log\left(\frac{1}{2}, \frac{2999}{3000}\right) \\
k &\approx \sqrt{2\log\left(\frac{1}{2}, \frac{2999}{3000}\right)} \\
&\approx \boxed{64.48}
\end{aligned}$$

Therefore, after generating about 65 random numbers, we can expect the probability that two of the random numbers equal to be more than 50%. This result is reasonably close to the estimate from *Problem 1-C* (65.37).

Part B

I will use the (only) method from the lecture to estimate the actual expected value of k:

$$\begin{aligned}
\mathrm{E}[k] &= n \cdot \sum_{i=1}^{n} \frac{1}{i} \\
&= 200 \cdot \sum_{i=1}^{200} \frac{1}{i} \\
&= 200 \cdot \left(1 + \frac{1}{2} + \cdots + \frac{1}{200}\right) \\
&\approx \boxed{1176}
\end{aligned}$$

Therefore, we can expect to see every possiblility being generated after generating about 1176 random numbers, which is reasonably close to the estimate from *Problem 2-C* (1176).

Problem 4

Assume that the random numbers are generated from the domain $\{1, \cdots, n\}$, which has size n. Now, f_i represents the number of trials that generate the number i, where $1 \leq i \leq n$.

$$\begin{aligned}
\Pr\left[\left|\mu - \frac{1}{n}\right| \geq \varepsilon\right] &= \Pr\left[\exists\, i \in \{1, \cdots, n\}, \left|\frac{f_i}{k} - \frac{1}{n}\right| \geq \varepsilon\right] \\
&= \Pr\left[\left(\left|\frac{f_1}{k} - \frac{1}{n}\right| \geq \varepsilon\right) \cup \cdots \cup \left(\left|\frac{f_n}{k} - \frac{1}{n}\right| \geq \varepsilon\right)\right] \\
&\leq \Pr\left[\left|\frac{f_1}{k} - \frac{1}{n}\right| \geq \varepsilon\right] + \cdots + \Pr\left[\left|\frac{f_n}{k} - \frac{1}{n}\right| \geq \varepsilon\right]
\end{aligned}$$

The above is true because the events $|f_i/k - 1/n| \geq \varepsilon$ for any i are not disjoint, meaning that there could be more than one values of i where such an event is true. In other words, there could be more than one numbers in $\{1, \cdots, n\}$ that occur more often than $\varepsilon + 1/n$.

Suppose that $\Pr[|f_i/k - 1/n| \geq \varepsilon] \leq .02/n$ for all i. This would directly imply that

$$\Pr\left[\left|\frac{f_1}{k} - \frac{1}{n}\right| \geq \varepsilon\right] + \cdots + \Pr\left[\left|\frac{f_n}{k} - \frac{1}{n}\right| \geq \varepsilon\right] \leq n \cdot \frac{.02}{n} = .02$$

And thus, $\Pr[|\mu - 1/n| \geq \varepsilon] \leq .02$.

Now, in order for $\Pr[|f_i/k - 1/n| \geq \varepsilon] \leq .02/n$ to be true for any i, first, notice that

$$\Pr\left[\left|\frac{f_i}{k} - \frac{1}{n}\right| \geq \varepsilon\right] = \Pr\left[\left|f_i - \frac{k}{n}\right| \geq k\varepsilon\right]$$

Consider a Chebyshev's Inequality for the variable $X = f_i$; specifically,

$$\begin{aligned}
\mathrm{E}[X] &= \mathrm{E}[f_i] = \frac{k}{n}, \\
\mathrm{Var}[X] &= \mathrm{Var}[f_i] = \mathrm{Var}\left[\mathrm{Binom}\left(k, \frac{1}{n}\right)\right] = k \cdot \frac{1}{n} \cdot \frac{n-1}{n} = \frac{k(n-1)}{n^2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Pr[|X - \mathrm{E}[X]| \geq \varepsilon] &\leq \frac{\mathrm{Var}[X]}{\varepsilon^2} \\
\Pr\left[\left|f_i - \frac{k}{n}\right| \geq k\varepsilon\right] &\leq \frac{k(n-1)}{n^2(k\varepsilon)^2} \leq \frac{.02}{n} \\
\frac{n-1}{nk\varepsilon^2} &\leq .02 \\
k &\geq \boxed{\frac{n-1}{.02n\varepsilon^2}}
\end{aligned}$$

By similar reasoning,

$$\Pr\left[\left|\mu - \frac{1}{n}\right| \geq \varepsilon\right] \leq .002 \implies k \geq \frac{n-1}{.002n\varepsilon^2}$$