NAME: ____ UID: ________________

CS 2100: Discrete Structures

Quiz 4

Spring 2020

This is a take-home exam.

Pleases read these instructions carefully and put your name and UID on the first page:

- This exam is released online via Canvas at 8:00 a.m. on Tuesday, March 24, it is due at 7:59 a.m. on Wednesday, March 25th, giving you 24 hours to complete it.
- Please submit the exam via Gradescope on the due date and time.
- The exam takes, on average, 80 minutes to complete. You can take as long as you want to work on the exam within the given time window (i.e., 24 hours).
- Students with CDS document can turn in exam after 48 hours, on or before Thursday, March 26, at 7:59 a.m. by emailing it as a PDF to TA Don Wang at comidon@outlook.com.
- The exam is take-home, open-book and open-notes. However, **please do not use the internet to search for solutions**.
- The exam is to be done independently. Please do not discuss solutions with anyone.
- Please ask any clarifying question regarding the exam via Piazza under "quiz 4". Please do not post solutions via Piazza.
- To work on the exam, there are a few options (similar to solving homework problems):
  - You can write your answers by hand, using WORD, or using LATEX, then submit a scanned image or a PDF. We only need your solution to each problem.
  - You can also download the PDF and use Adobe or Preview (or similar tools) for editing your solutions directly on the PDF file.
  - If you are solving by hand (please use a black pen), pleases submit a scanned version in PDF or a photoed version.
  - **Please do not put more than one problem on each page**.
  - If you need more space to provide a solution to a problem, use a blank piece of paper. Do not use a single paper for more than one problem.
- **Important submission information**: the exam should be submitted via Gradescope as a PDF (preferred) or an image file. The submission process is very similar to the submission of a homework problem:
  - Please match each problem to the outline specified on Gradescope: the page number containing the answer to each individual question should be specified during submission.
  - If no pages are specified for each problem, then 20% is deducted from the final score.
  - **The submission should be legible; any problem that is not legible will not be graded and will not receive any credit.**
- **To be eligible for partial credit, you must show your work.**
- It is the student's responsibility to ensure the successful and timely submission of the exam via Gradescope— start early and follow the instructions carefully. Corrupted or missing files are not grounds for extensions — double-check your submissions and save a digital copy of all of your work in your CADE account.

1. **(10 points)** Consider the function $f(x) = 2x - 1$. Find a formula for the composition functions,

   (a). $h = f \circ f$.

   (b). $g = f \circ f \circ f$.

   Solution:

2. **(10 points)** Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.
The function $f : A \to B$ is defined by

$$f(x) = \frac{4!}{(4-x)!x!}.$$

(a). Determine if f is one-to-one.

(b). Determine if f is onto.

For each question, either give a proof or a counterexample.
For partial/full credits, please show your work.

Solution:

3. **(10 points)** Let d be any positive integer greater than 1. Let $\mathbb{Z}$ denote the set of integers.

(a) Prove that the "multiplication by d" function $f : \mathbb{Z} \to \mathbb{Z}$ defined below is one-to-one:

$$f(n) = d \cdot n, \forall n \in \mathbb{Z}.$$

(b) Prove that the "mod d" function $g : \mathbb{Z} \to \{0, 1, \cdots, d-1\}$ defined below is onto and not one-to-one,

$$\forall n \in \mathbb{Z}, g(n) = (n \mod d).$$

Recall the modulo operation ($x \mod y$) finds the remainder after division of x by y.

Solution:

4. **(10 points)** Let X be a set $X = \{1,2,3,4,5,6,7,8,9\}$.
Let $Q(X)$ be the collection of all subsets of X with at least three elements.
For example $\{1,2,3\} \in Q(X)$ but $\{2,3\} \notin Q(X)$.
Each of the following defines a relation on $Q(X)$:

(a) $R : A \subseteq B$; (b) $S : A$ is disjoint from B; and (c) $T : A \cup B = X$.

In other words,

(a). $R \subseteq Q(X) \times Q(X)$ is defined by $(A, B) \in R$ if $A \subseteq B$, for $A, B \in Q(X)$.
In other words, $R = \{(A, B) \mid A \subseteq B, \text{for } A, B \in Q(X)\}$

(b). $S \subseteq Q(X) \times Q(X)$ is defined by
$S = \{(A, B) \mid A \cap B = \emptyset, \text{for } A, B \in Q(X)\}$

(c). $T \subseteq Q(X) \times Q(X)$ is defined by
$T = \{(A, B) \mid A \cup B = X, \text{for } A, B \in Q(X)\}$

Determine which of the above relations are reflexive.
For partial/full credits, please show your work and provide justifications.

Solution:

5. **(15 points)**

(a). Suppose Q is an antisymmetric relation on a set A, show that $Q \cap S$ is antisymmetric for **any** relation S on A.

(b). Show, by giving a counterexample, that R and T may be transitive relations on A, but $R \cup T$ need not be transitive.

Solution:

6. **(15 points)** Let $\mathbb{Z}$ denote the set of integers. Define a relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$, where $(a, b) \in R$ (for $a, b \in \mathbb{Z}$) if for some positive integer r, $b = a^r$. For example, $(2, 8) \in R$ since $8 = 2^3$.

   Show that R is a partial order on $\mathbb{Z}$, that is, show that R is

   (a). Reflexive;

   (b). Antisymmetric;

   (c). Transitive.

   Solution:

7. **(10 points)** Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Determine whether or not each of the following is a partition of S:

   (a). $\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}$

   (b). $\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}$

   (c). $\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}$

   (d). $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   Please provide justification for your decision.

   Solution:

8. **(10 points)** Let $A = \{1, 2, 3, 4, 5, 6\}$ and let R be the equivalence relation on A defined by:

$$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}.$$

Find the partition of A induced by R, i.e., find the equivalence classes of R.
Hint: one solution is obtained by using arrow diagrams of visualize the relation R and find the independent "islands".

Solution:

9. **(10 points)** Recall the definition of a total ordering (Page 309 of textbook) as follows: A relation R is a total ordering on A if R is a reflexive, transitive, and antisymmetric relation on A that also satisfies the "comparability" property, that is,

$$\forall a, b \in A, \text{ if } a \neq b, \text{ either } (a, b) \in R \text{ or } (b, a) \in R.$$

In other words, "comparability" means that any pair of elements in the set of the relation are comparable under the relation.

Suppose $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of (positive) natural numbers.
Suppose $\mathbb{N}$ is ordered by divisibility relation $R \subseteq \mathbb{N} \times \mathbb{N}$.
For example $(2, 6) \in R$ because 2 divides 6, but $(3, 5) \notin R$ because 3 does not divide 5.

Determine whether the following subsets of $\mathbb{N}$ are totally ordered (under the "divisibility" relation):

(a). $\{24, 2, 6\}$

(b). $\{3, 15, 5\}$

(c). $\mathbb{N} = \{1, 2, 3, \cdots\}$

(d). $\{2, 8, 32, 4\}$

(e). $\{7\}$

(f). $\{15, 5, 30\}$

Solution: