Week 1 Homework

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Math 4400 Spring 2022

Exercise 1

First, create a list of integers from 2 to 100 (as we know 1 is not prime):

$$\begin{gathered}
2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \\
21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, \\
41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, \\
61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, \\
81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100
\end{gathered}$$

Circle the first non-circled number from the list as our next prime, then cross out all multiples of that number:

$$\begin{gathered}
\boxed{2}, 3, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, 9, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, 15, \cancel{16}, 17, \cancel{18}, 19, \cancel{20}, \\
21, \cancel{22}, 23, \cancel{24}, 25, \cancel{26}, 27, \cancel{28}, 29, \cancel{30}, 31, \cancel{32}, 33, \cancel{34}, 35, \cancel{36}, 37, \cancel{38}, 39, \cancel{40}, \\
41, \cancel{42}, 43, \cancel{44}, 45, \cancel{46}, 47, \cancel{48}, 49, \cancel{50}, 51, \cancel{52}, 53, \cancel{54}, 55, \cancel{56}, 57, \cancel{58}, 59, \cancel{60}, \\
61, \cancel{62}, 63, \cancel{64}, 65, \cancel{66}, 67, \cancel{68}, 69, \cancel{70}, 71, \cancel{72}, 73, \cancel{74}, 75, \cancel{76}, 77, \cancel{78}, 79, \cancel{80}, \\
81, \cancel{82}, 83, \cancel{84}, 85, \cancel{86}, 87, \cancel{88}, 89, \cancel{90}, 91, \cancel{92}, 93, \cancel{94}, 95, \cancel{96}, 97, \cancel{98}, 99, \cancel{100}
\end{gathered}$$

Repeat this process:

$$\begin{gathered}
\boxed{2}, \boxed{3}, \cancel{4}, 5, \cancel{6}, 7, \cancel{8}, \cancel{9}, \cancel{10}, 11, \cancel{12}, 13, \cancel{14}, \cancel{15}, \cancel{16}, 17, \cancel{18}, 19, \cancel{20}, \\
\cancel{21}, \cancel{22}, 23, \cancel{24}, 25, \cancel{26}, \cancel{27}, \cancel{28}, 29, \cancel{30}, 31, \cancel{32}, \cancel{33}, \cancel{34}, 35, \cancel{36}, 37, \cancel{38}, \cancel{39}, \cancel{40}, \\
41, \cancel{42}, 43, \cancel{44}, \cancel{45}, \cancel{46}, 47, \cancel{48}, 49, \cancel{50}, \cancel{51}, \cancel{52}, 53, \cancel{54}, 55, \cancel{56}, \cancel{57}, \cancel{58}, 59, \cancel{60}, \\
61, \cancel{62}, \cancel{63}, \cancel{64}, 65, \cancel{66}, 67, \cancel{68}, \cancel{69}, \cancel{70}, 71, \cancel{72}, 73, \cancel{74}, \cancel{75}, \cancel{76}, 77, \cancel{78}, 79, \cancel{80}, \\
\cancel{81}, \cancel{82}, 83, \cancel{84}, 85, \cancel{86}, \cancel{87}, \cancel{88}, 89, \cancel{90}, 91, \cancel{92}, \cancel{93}, \cancel{94}, 95, \cancel{96}, 97, \cancel{98}, \cancel{99}, \cancel{100}
\end{gathered}$$

...Until all numbers in the list are either circled or crossed out:

$$\boxed{2}, \boxed{3}, \cancel{4}, \boxed{5}, \cancel{6}, \boxed{7}, \cancel{8}, \cancel{9}, \cancel{10}, \boxed{11}, \cancel{12}, \boxed{13}, \cancel{14}, \cancel{15}, \cancel{16}, \boxed{17}, \cancel{18}, \boxed{19}, \cancel{20},$$
$$\cancel{21}, \cancel{22}, \boxed{23}, \cancel{24}, \cancel{25}, \cancel{26}, \cancel{27}, \cancel{28}, \boxed{29}, \cancel{30}, \boxed{31}, \cancel{32}, \cancel{33}, \cancel{34}, \cancel{35}, \cancel{36}, \boxed{37}, \cancel{38}, \cancel{39}, \cancel{40},$$
$$\boxed{41}, \cancel{42}, \boxed{43}, \cancel{44}, \cancel{45}, \cancel{46}, \boxed{47}, \cancel{48}, \cancel{49}, \cancel{50}, \cancel{51}, \cancel{52}, \boxed{53}, \cancel{54}, \cancel{55}, \cancel{56}, \cancel{57}, \cancel{58}, \boxed{59}, \cancel{60},$$
$$\boxed{61}, \cancel{62}, \cancel{63}, \cancel{64}, \cancel{65}, \cancel{66}, \boxed{67}, \cancel{68}, \cancel{69}, \cancel{70}, \boxed{71}, \cancel{72}, \boxed{73}, \cancel{74}, \cancel{75}, \cancel{76}, \cancel{77}, \cancel{78}, \boxed{79}, \cancel{80},$$
$$\cancel{81}, \cancel{82}, \boxed{83}, \cancel{84}, \cancel{85}, \cancel{86}, \cancel{87}, \cancel{88}, \boxed{89}, \cancel{90}, \cancel{91}, \cancel{92}, \cancel{93}, \cancel{94}, \cancel{95}, \cancel{96}, \boxed{97}, \cancel{98}, \cancel{99}, \cancel{100}$$

Now, the circled numbers above are the primes between 1 and 100:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

Exercise 2

Suppose $\sqrt{2}$ is rational and $\sqrt{2} = \frac{a}{b}$, where $a, b \in \mathbb{Z}^+$.

By rearranging, we have $a^2 = 2b^2$.

Factorize a so that $a = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$. It follows that $a^2 = p_1^{2k_1} p_2^{2k_2} \cdots p_n^{2k_n}$, meaning that the prime factorization of a^2 must have an even number of 2s (or any other prime) in it.

By similar logic, the prime factorization of b^2 must also have an even number of 2s in it.

However, we said that $a^2 = 2 \cdot b^2$, meaning that the prime factorization of a^2 must now have an odd number of 2s in it, which is a contradiction. Therefore, $\sqrt{2}$ cannot be rational.