Homework 2

Qianlang Chen

Section 2.1

1. Exercise 11 (p97)

Part (a)

Let a, b be the two given odd integers. By definition of odd, we have

$$a = 2m + 1, b = 2n + 1 : m, n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1, \text{ where } k = 2mn + m + n \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of odd, we have that ab is an odd number.

Part (b)

Let a be the given odd integer and b be the given even integer. By definition of odd and even, we have

$$a = 2m + 1, b = 2n : m, n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= (2m+1)2n \\ &= 4mn + 2n \\ &= 2(2mn + n) \\ &= 2k, \text{ where } k = 2mn + n \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of even, we have that ab is an even number.

Part (c)

Let a be the given even integer and b be the given integer that is divisible by 3.

By definition of even, we have

$$a = 2m : m \in \mathbb{Z}$$

By definition of being divisible by 3, we have

$$b = 3n : n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= 2m \cdot 3n \\ &= 6mn \\ &= 6k, \text{ where } k = mn \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 6, we have that ab is divisible by 6.

2. Exercise 12.d (p97)

This pair of statements are *not* contrapositives of one another.

- Counterexample of case (i): a person that likes computers but does not like computer science.
- Counterexample of case (ii): a person that does not like computers but likes computer science.

3. Exercise 13.a (p97)

Contrapositive: "if $m = 0$ and $n = 0$, then $m^2 + n^2 = 0$."

Proof of the contrapositive: Since $m = 0$ and $n = 0$, by substitution, we have

$$m^2 + n^2 = 0^2 + 0^2 = 0$$

Since the contrapositive is true, the original statement "if $m^2 + n^2 \neq 0$, then $m \neq 0$ or $n \neq 0$" is true.

4. Exercise 14.c (p98)

Let n be a number preceding a perfect cube. Since any perfect cube is in the form of x^3 where x is an integer, we have that $n = x^3 - 1$.

The expression $x^3 - 1$ is equivalent to $(x - 1)(x^2 + x + 1)$:

$$\begin{aligned}(x-1)(x^2+x+1) &= x(x^2+x+1) - 1(x^2+x+1) \\ &= (x \cdot x^2 + x \cdot x + x \cdot 1) - (1 \cdot x^2 + 1 \cdot x + 1 \cdot 1) \\ &= x^3 + x^2 + x - x^2 - x - 1 \\ &= x^3 - 1\end{aligned}$$

Now, since $n = x^3 - 1$, n can always be factorized into $(x - 1)$ and $(x^2 + x + 1)$.

- When x is zero or negative, n is negative, which can never be prime.
- When x is 1, $n = 1^3 - 1 = 0$, which is not prime.
- When x is 2, $n = 2^3 - 1 = 7$, which is prime.
- When x is 3 or higher, since n can always be factorized into $(x - 1)$ and $(x^2 + x + 1)$, but $(x - 1)$ is not 1 or n, n is divisible by some other integer that is not 1 or n. Therefore, n is not prime.
- When x is not an integer, n is not an integer, which is not prime.

Since the above cases cover all possibilities of x, 7 is indeed the only prime preceding a perfect cube.

Section 2.2

5. Exercise 7.e (p108)

Let m be the given $10^{n-1} - 1$ that is divisible by 9. Since m is divisible by 9, $m = 9k : k \in \mathbb{Z}$.

Since $10^n = 10 \cdot 10^{n-1}$, we have

$$\begin{aligned}
10^n - 1 &= 10 \cdot 10^{n-1} - 1 \\
&= 10 \cdot (10^{n-1} - 1 + 1) - 1 \\
&= 10 \cdot (m + 1) - 1 && \text{Substituting } m = 10^{n-1} - 1 \\
&= 10 \cdot (9k + 1) - 1 && \text{Substituting } 9k = m \\
&= 90k + 10 - 1 && \text{Distributing } 10 \\
&= 90k + 9 \\
&= 9 \cdot (10k + 1) && \text{Factoring out } 9 \\
&= 9c && \text{Where } c = 10k + 1
\end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, $10^n - 1$ is divisible by 9.

6. Exercise 16 (p109)

Since n is not divisible by 3, by the division theorem, $n = 3q + r : q, r \in \mathbb{Z}, 0 < r < 3$. In other words, one of the following cases about r must be true:

Case I, when r = 1, meaning that n = 3q + 1:

$$\begin{aligned}
n^2 + 2 &= (3q+1)^2 + 2 && \text{Substituting } 3q+1 = n \\
&= 9q^2 + 6q + 1 + 2 \\
&= 9q^2 + 6q + 3 \\
&= 3 \cdot (3q^2 + 2q + 1) && \text{Factoring out 3} \\
&= 3k && \text{Where } k = 3q^2 + 2q + 1
\end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 3, $n^2 + 2$ is divisible by 3.

Case II, when r = 2, meaning that n = 3q + 2:

$$\begin{aligned}
n^2 + 2 &= (3q+2)^2 + 2 && \text{Substituting } 3q+2 = n \\
&= 9q^2 + 12q + 4 + 2 \\
&= 9q^2 + 12q + 6 \\
&= 3 \cdot (3q^2 + 4q + 2) && \text{Factoring out 3} \\
&= 3k && \text{Where } k = 3q^2 + 4q + 2
\end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 3, $n^2 + 2$ is divisible by 3.

Therefore, $n^2 + 2$ is divisible by 3 in every possible case.

7. Exercise 20 (p109)

Let n be any integer. Let $s = (n-1)^3 + n^3 + (n+1)^3$, which is the sum of three consecutive perfect cubes mentioned in the problem. After simplifying the expression of s, we have

$$\begin{aligned} s &= (n-1)^3 + n^3 + (n+1)^3 \\ &= (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) \\ &= 3n^3 + 6n \\ &= 3n \cdot (n^2 + 2) && \text{Factoring out } 3n \end{aligned}$$

Now, one of the following cases about n must be true:

Case I, when n is divisible by 3:

Since n is divisible by 3, meaning that $n = 3k : k \in \mathbb{Z}$, we have

$$\begin{aligned} s &= 3n \cdot (n^2 + 2) \\ &= 3 \cdot 3k \cdot (n^2 + 2) && \text{Substituting } 3k = n \\ &= 9k \cdot (n^2 + 2) \\ &= 9c && \text{Where } c = k \cdot (n^2 + 2) \end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, s is divisible by 9.

Case II, when n is not divisible by 3:

Since n is not divisible by 3, according to what is proven by *Exercise 16,* $(n^2 + 2)$ is divisible by 3. This means that $n^2 + 2 = 3k : k \in \mathbb{Z}$, and we have

$$\begin{aligned} s &= 3n \cdot (n^2 + 2) \\ &= 3n \cdot 3k && \text{Substituting } 3k = n^2 + 2 \\ &= 9nk \\ &= 9c && \text{Where } c = nk \end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, s is divisible by 9.

Therefore, s is divisible by 9 is every possible case.

8. ***Show that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.***

Let n be the given integer that is not divisible by 5. By the division theorem, since n is not divisible by 5, $n = 5q + r : q, r \in \mathbb{Z}, 0 < r < 5$. In other words, one of the following cases about r must be true:

Case I, when r = 1, meaning that $n = 5q + 1$:

$$\begin{aligned} n^2 &= (5q+1)^2 && \text{Substituting } 5q+1 = n \\ &= (25q^2 + 10q) + 1 \\ &= 5 \cdot (5q^2 + 2q) + 1 && \text{Factoring out 5} \\ &= 5k + 1 && \text{Where } k = 5q^2 + 2q \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 1 when divided by 5.

Case II, when r = 2, meaning that $n = 5q + 2$:

$$\begin{aligned} n^2 &= (5q+2)^2 && \text{Substituting } 5q+2 = n \\ &= (25q^2 + 20q) + 4 \\ &= 5 \cdot (5q^2 + 4q) + 4 && \text{Factoring out 5} \\ &= 5k + 4 && \text{Where } k = 5q^2 + 4q \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 4 when divided by 5.

Case III, when r = 3, meaning that $n = 5q + 3$:

$$\begin{aligned} n^2 &= (5q+3)^2 && \text{Substituting } 5q+3 = n \\ &= 25q^2 + 30q + 9 \\ &= (25q^2 + 30q + 5) + 4 \\ &= 5 \cdot (5q^2 + 6q + 1) + 4 && \text{Factoring out 5} \\ &= 5k + 4 && \text{Where } k = 5q^2 + 6q + 1 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 4 when divided by 5.

Case IV, when r = 4, meaning that $n = 5q + 4$:

$$\begin{aligned} n^2 &= (5q+4)^2 && \text{Substituting } 5q+4 = n \\ &= 25q^2 + 40q + 16 \\ &= (25q^2 + 40q + 15) + 1 \\ &= 5 \cdot (5q^2 + 8q + 3) + 1 && \text{Factoring out 5} \\ &= 5k + 1 && \text{Where } k = 5q^2 + 8q + 3 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 1 when divided by 5.

Therefore, the n^2 leaves a remainder of 1 or 4 when divided by 5 in every possible case.

9. Exercise 26.c (p109)

First, let us prove that a perfect square k^2 is divisible by 4 if and only if k is even.

Let n be a perfect square, and by definition, $n = k^2 : k \in \mathbb{Z}$. Since k must be either even or odd, one of the following cases must be true:

Case I, when k is even:

Since k is even, $k = 2c : c \in \mathbb{Z}$. By substitution, we have

$$n = k^2 = (2c)^2 = 4c^2 = 4m \text{ where } m = c^2$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By definition of being divisible by 4, n is divisible by 4.

Case II, when k is odd:

Since k is odd, $k = 2c + 1 : c \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} n = k^2 &= (2c+1)^2 \\ &= 4c^2 + 4c + 1 \\ &= 4 \cdot (c^2 + c) + 1 \\ &= 4m + 1 \text{ where } m = c^2 + c \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, n leaves a remainder of 1 when divided by 4, implying that n is not divisible by 4.

Therefore, *every perfect square k^2 is divisible by 4 if and only if k is even.*

Now, let us prove the contrapositive of the original problem statement: "for every three integers a, b, and c where $c^2 = a^2 + b^2$, if not both a and b are even, then c is not even."

Since every integer is either even or odd, at least one of $\{a, b\}$ must be odd for the above condition to be true. Therefore, we can let j be one of $\{a, b\}$ that is odd and k be the other (which can be either even or odd). Since j is odd, $j = 2p + 1 : p \in \mathbb{Z}$, and this leads us into one of the following:

Case I, when k is even:

By definition of even, $k = 2q : q \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} c^2 &= j^2 + k^2 \\ &= (2p+1)^2 + (2q)^2 && \text{Substituting } 2p+1 = j \text{ and } 2q = k \\ &= 4p^2 + 4p + 1 + 4p^2 \\ &= 4 \cdot (p^2 + p + q^2) + 1 && \text{Factoring out 4} \\ &= 4m + 1 && \text{Where } m = p^2 + p + q^2 \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, c^2 is not divisible by 4, which means that c is not even, according to what we have proved earlier.

Case II, when k is odd:

By definition of odd, $k = 2q + 1 : q \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} c^2 &= j^2 + k^2 \\ &= (2p+1)^2 + (2q+1)^2 && \text{Substituting } 2p+1 = j \text{ and } 2q+1 = k \\ &= 4p^2 + 4p + 1 + 4q^2 + 4q + 1 \\ &= 4p^2 + 4p + 4q^2 + 4q + 2 \\ &= 4 \cdot (p^2 + p + q^2 + q) + 2 && \text{Factoring out 4} \\ &= 4m + 2 && \text{Where } m = p^2 + p + q^2 + q \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, c^2 is not divisible by 4, which means that c is not even, according to what we have proved earlier.

Therefore, c is not even in every possible case. Also, since the contrapositive is true, the original statement is true.

Section 2.3

10. Exercise 2 (p121)

Part (a)

- $R(1)$: $a_1 = 2^{1-1} + 3 = 4$
- $R(2)$: $a_2 = 2^{2-1} + 3 = 5 = 2a_1 - 3$
- $R(3)$: $a_3 = 2^{3-1} + 3 = 7 = 2a_2 - 3$
- $R(4)$: $a_4 = 2^{4-1} + 3 = 11 = 2a_3 - 3$

According to the above, $R(1), R(2), R(3)$, and $R(4)$ are all true.

Part (b)

- $R(8)$: $a_8 = 2^{8-1} + 3 = 131$
- $R(9)$: $a_9 = 2^{9-1} + 3 = 259$

Part (c)

Since $R(8)$ is verified to be true, meaning that $a_n = a_k$ for $n = k = 8$, we have

$$\begin{aligned}
a_{n+1} &= 2^{(n-1)+1} + 3 \\
&= 2 \cdot 2^{n-1} + 3 \\
&= 2 \cdot (2^{n-1} + 3 - 3) + 3 \\
&= 2 \cdot (2^{n-1} + 3) - 6 + 3 \\
&= 2a_n - 3 \\
&= 2a_k - 3 \\
&= a_{k+1}
\end{aligned}$$

Since $a_{n+1} = a_{k+1}$ and $n + 1 = k + 1 = 9$, $R(9)$ is now verified to be true.

11. Exercise 3.f (p121)

Statement: show that $a_n = a_k$, where $a_n = 2^{n-1} + 3$ and $a_k = 2a_{k-1} - 3$.

Base case: when $n = k = 1$, $a_n = 2^{1-1} + 3 = 4 = a_k$.

Induction hypothesis: assume that $a_n = a_k$ when $n = k = m$, $m \in \mathbb{Z} : m \geq 1$.

Induction step: when $n = k = m + 1$,

$$\begin{aligned}
a_n &= 2^{(m+1)-1} + 3 \\
&= 2 \cdot 2^{m-1} + 3 \\
&= 2 \cdot (2^{m-1} + 3 - 3) + 3 \\
&= 2 \cdot (2^{m-1} + 3) - 6 + 3 \\
&= 2a_m - 3 \\
&= a_{m+1} \\
&= a_k
\end{aligned}$$

Verifying the first four terms:

- $a_1 = 2^{1-1} + 3 = 4$
- $a_2 = 2^{2-1} + 3 = 5 = 2a_1 - 3$
- $a_3 = 2^{3-1} + 3 = 7 = 2a_2 - 3$
- $a_4 = 2^{4-1} + 3 = 11 = 2a_3 - 3$

12. Exercise 4.e (p122)

Statement: show that $b_n = b_k$, where $b_n = 3 \cdot 2^n - n - 2$ and $b_k = 2b_{k-1} + k$.

Base case: when $n = k = 1$, $b_n = 3 \cdot 2^1 - 1 - 2 = 3 = b_k$.

Induction hypothesis: assume that $a_n = a_k$ when $n = k = m - 1$, $m \in \mathbb{Z} : m > 1$.

Induction step: when $n = k = m$,

$$\begin{aligned}
a_n &= 3 \cdot 2^m - m - 2 \\
&= 3 \cdot (2 \cdot 2^{m-1}) - m + m - m - 2 \\
&= 2 \cdot (3 \cdot 2^{m-1}) - 2m - 2 + m \\
&= 2 \cdot (3 \cdot 2^{m-1} - m - 1) + m \\
&= 2 \cdot (3 \cdot 2^{m-1} - m + 1 - 1 - 1) + m \\
&= 2 \cdot (3 \cdot 2^{m-1} - (m - 1) - 2) + m \\
&= 2a_{m-1} + m \\
&= 2a_{k-1} + k \\
&= a_k
\end{aligned}$$

13. Exercise 8.f (p122)

Statement: show that $a_n = b_n$, where $a_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$ and $b_n = \frac{n}{n+1}$.

Base case:

$$\begin{aligned} a_1 &= \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \\ &= \frac{1}{1+1} = b_1 \end{aligned}$$

Induction hypothesis: assume that $a_{k-1} = b_{k-1}$, $k \in \mathbb{Z} : k > 1$.

Induction step:

$$\begin{aligned} a_k &= \sum_{i=1}^{k} \frac{1}{i(i+1)} \\ &= (\sum_{i=1}^{k-1} \frac{1}{i(i+1)}) + \frac{1}{k(k+1)} \\ &= a_{k-1} + \frac{1}{k(k+1)} \\ &= b_{k-1} + \frac{1}{k(k+1)} \\ &= \frac{k-1}{(k-1)+1} + \frac{1}{k(k+1)} \\ &= \frac{(k-1)(k+1)}{k(k+1)} + \frac{1}{k(k+1)} \\ &= \frac{k^2-1+1}{k(k+1)} \\ &= \frac{k}{k+1} \\ &= b_k \end{aligned}$$

Verifying the first four terms:

$$\begin{aligned} a_1 &= \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2} = \frac{1}{1+1} = b_1; \\ a_2 &= \sum_{i=1}^{2} \frac{1}{i(i+1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{2+1} = b_2; \\ a_3 &= \sum_{i=1}^{2} \frac{1}{i(i+1)} = \frac{2}{3} + \frac{1}{12} = \frac{3}{4} = \frac{3}{3+1} = b_3; \\ a_4 &= \sum_{i=1}^{2} \frac{1}{i(i+1)} = \frac{3}{4} + \frac{1}{20} = \frac{4}{5} = \frac{4}{4+1} = b_4; \end{aligned}$$

14. Exercise 13 (p122)

Statement: show that $a_n = b_n$, where

$$\begin{aligned} a_n &= \sum_{i=0}^{n} x^i, \\ b_n &= \frac{x^{n+1}-1}{x-1}, \\ x &\in \mathbb{R} : x \neq 1 \end{aligned}$$

Base case:

$$\begin{aligned} a_0 &= \sum_{i=0}^{0} x^i = x^0 = 1 \\ &= \frac{x-1}{x-1} = \frac{x^{0+1}-1}{x-1} = b_0 \end{aligned}$$

Induction hypothesis: assume that $a_{k-1} = b_{k-1}$, $k \in \mathbb{Z} : k > 0$.

Induction step:

$$\begin{aligned} a_k &= \sum_{i=0}^{k} x^i \\ &= (\sum_{i=0}^{k-1} x^i) + x^k \\ &= a_{k-1} + x^k \\ &= b_{k-1} + x^k \\ &= \frac{x^{(k-1)+1}-1}{x-1} + x^k \\ &= \frac{x^k-1}{x-1} + \frac{x^k(x-1)}{x-1} \\ &= \frac{x^k - 1 + x^k \cdot x - x^k}{x-1} \\ &= \frac{x^{k+1}-1}{x-1} \\ &= b_k \end{aligned}$$

Section 2.4

15. Exercise 1.d (p130)

$$d_n = d_{n-1} + \frac{1}{(2n-1)(2n+1)}, d_1 = \frac{1}{3}$$

16. Exercise 2.d (p130)

Statement: show that $a_n = d_n$, where $a_n = a_{n-1} + \frac{1}{(2n-1)(2n+1)}, a_1 = \frac{1}{3}$, and $d_n = \frac{n}{2n+1}$.

Base case:

$$a_1 = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1} = d_1$$

Induction hypothesis: assume that $a_{k-1} = d_{k-1}$, $k \in \mathbb{Z} : k > 1$.

Induction step:

$$\begin{aligned}
a_k &= a_{k-1} + \frac{1}{(2k-1)(2k+1)} \\
&= d_{k-1} + \frac{1}{(2k-1)(2k+1)} \\
&= \frac{k-1}{2(k-1)+1} + \frac{1}{(2k-1)(2k+1)} \\
&= \frac{k-1}{2k-1} + \frac{1}{(2k-1)(2k+1)} \\
&= \frac{(k-1)(2k+1)+1}{(2k-1)(2k+1)} \\
&= \frac{2k^2+k-2k-1+1}{(2k-1)(2k+1)} \\
&= \frac{2k^2-k}{(2k-1)(2k+1)} \\
&= \frac{k(2k-1)}{(2k-1)(2k+1)} \\
&= \frac{k}{2k+1} \\
&= d_k
\end{aligned}$$

17. Exercise 4.b (p130)

Statement: let $D(n)$ be the statement "$n^3 - n$ is divisible by 3." Show that $D(n)$ is true for each integer n, where $n \geq 1$.

Base Case: when $n = 1$, $D(n)$ states that $1^3 - 1 = 0$ is divisible by 3, which is true.

Induction hypothesis: assume that $D(n)$ is true for $n = k - 1$, $k \in \mathbb{Z} : k > 1$.

Induction step: since $D(k-1)$ is true, meaning that $(k-1)^3 - (k-1)$ is divisible by 3, we have $(k-1)^3 - (k-1) = 3q : q \in \mathbb{Z}$, by definition of being divisible by 3.

Now, to show that $D(k)$ is true, we have

$$\begin{aligned}
k^3 - k &= (k^3 - 3k^2 + 2k) + 3k^2 - 3k \\
&= (k^3 - 3k^2 - 3k - 1) - (k-1) + 3k^2 - 3k \\
&= (k-1)^3 - (k-1) + 3k^2 - 3k \\
&= 3q + 3k^2 - 3k && \text{Substituting } 3q = (k-1)^3 - (k-1) \\
&= 3(q + k^2 - k) \\
&= 3c && \text{Where } c = q + k^2 - k
\end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 3, $k^3 - k$ is divisible by 3, meaning that $D(k)$ is true.

18. Exercise 6 (p131)

Statement: let $D(n)$ be the statement "$2^{3n} - 1$ is divisible by 7." Show that $D(n)$ is true for each integer n, where $n \geq 2$.

Base Case: when $n = 2$, $D(n)$ states that $2^{3 \cdot 2} - 1 = 63 = 7 \cdot 9$ is divisible by 7, which is true.

Induction hypothesis: assume that $D(n)$ is true for $n = k - 1$, $k \in \mathbb{Z} : k > 1$.

Induction step: since $D(k-1)$ is true, meaning that $2^{3(k-1)} - 1$ is divisible by 7, we have $2^{3(k-1)} - 1 = 7q : q \in \mathbb{Z}$, by definition of being divisible by 7.

Now, to show that $D(k)$ is true, we have

$$\begin{aligned}
2^{3k} - 1 &= 2^3 \cdot 2^{3(k-1)} - 1 \\
&= 8 \cdot 2^{3(k-1)} - 8 + 7 \\
&= 8 \cdot (2^{3(k-1)} - 1) + 7 \\
&= 8 \cdot 7q + 7 && \text{Substituting } 7q = 2^{3(k-1)} - 1 \\
&= 7 \cdot (8q + 1) \\
&= 7c && \text{Where } c = 8q + 1
\end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 7, $2^{3k} - 1$ is divisible by 7, meaning that $D(k)$ is true. Now, since $2^{3n} - 1 > 7$ when $n \geq 2$, but $2^{3n} - 1$ is divisible by 7, $2^{3n} - 1$ is not prime for all $n \geq 2$.

Section 2.5

19. Exercise 12 (p147)

Let us assume that $a+b$ is rational. By definition of rational, $a+b=\frac{m}{n}: m,n \in \mathbb{Z}, n \neq 0$.

Since $a \in \mathbb{Q}$, by definitional of rational, $a=\frac{p}{q}: p,q \in \mathbb{Z}, q \neq 0$.

By substitution,

$$\begin{aligned} b &= (a+b)-a \\ &= \frac{m}{n}-\frac{p}{q} \\ &= \frac{mq-np}{nq} \\ &= \frac{j}{k} \text{ where } j=mq-np \text{ and } k=nq \end{aligned}$$

By closure under addition and multiplication, $j,k \in \mathbb{Z}$. Also, since $n \neq 0$ and $q \neq 0$, $k=nq \neq 0$.

Since we showed that $b=\frac{j}{k}$, meaning that b must be rational, but the problem states that b is irrational, we have arrived at a contradiction. This indicates that our assumption in the beginning must be incorrect, meaning that $a+b$ cannot be rational. Therefore, $a+b$ is irrational.

20. Exercise 20 (p148)

Contrapositive: "for all real numbers x and y, if $x=0$ and $y=0$, then $x^2+y^2=0$."

Proof of the contrapositive: Since $x=0$ and $y=0$, by substitution, we have

$$x^2+y^2=0^2+0^2=0$$

Since the contrapositive is true, the original statement "for all real numbers x and y, if $x^2+y^2 \neq 0$, then $x \neq 0$ or $y \neq 0$" is true.

21. Exercise 32.b (p149)

Let five integers be given. Place these numbers into boxes labeled 0, 1, 2, and 3 according to the rule: a number x goes into the box labeled i if i or $(7-i)$ is the remainder when x is divided by 7. By the basic pigeonhole principle, some box (call its label r) contains at least two numbers. Call these numbers a and b. Since a and b are in the box labeled r, by the division theorem, one of the following cases must be true:

Case I, when $a = 7p + r$ and $b = 7q + r$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a - b &= (7p + r) - (7q + r) \\ &= 7p - 7q \\ &= 7 \cdot (p - q) \\ &= 7k \text{ where } k = p - q \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 7, $a - b$ is divisible by 7.

Case II, when $a = 7p + (7 - r)$ and $b = 7q + r$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a + b &= (7p + 7 - r) + (7q + r) \\ &= 7p + 7q + 7 \\ &= 7 \cdot (p + q + 1) \\ &= 7k \text{ where } k = p + q + 1 \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 7, $a + b$ is divisible by 7.

Case III, when $a = 7p + r$ and $b = 7q + (7 - r)$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a + b &= (7p + r) + (7q + 7 - r) \\ &= 7p + 7q + 7 \\ &= 7 \cdot (p + q + 1) \\ &= 7k \text{ where } k = p + q + 1 \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 7, $a + b$ is divisible by 7.

Case IV, when $a = 7p + (7 - r)$ and $b = 7q + (7 - r)$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a - b &= (7p + 7 - r) - (7q + 7 - r) \\ &= 7p - 7q \\ &= 7 \cdot (p - q) \\ &= 7k \text{ where } k = p - q \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 7, $a - b$ is divisible by 7.

Therefore, $a + b$ or $a - b$ is divisible by 7 in every possible case.

22. Exercise 34.b (p149)

Let fifty-two integers be given. Place these numbers into boxes labeled using integers [0..50] according to the rule: a number x goes into the box labeled i if i or $(100 - i)$ is the remainder when x is divided by 100. By the basic pigeonhole principle, some box (call its label r) contains at least two numbers. Call these numbers a and b. Since a and b are in the box labeled r, by the division theorem, one of the following cases must be true:

Case I, when $a = 100p + r$ and $b = 100q + r$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a - b &= (100p + r) - (100q + r) \\ &= 100p - 100q \\ &= 100 \cdot (p - q) \\ &= 100k \text{ where } k = p - q \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 100, $a - b$ is divisible by 100.

Case II, when $a = 100p + (100 - r)$ and $b = 100q + r$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a + b &= (100p + 100 - r) + (100q + r) \\ &= 100p + 100q + 100 \\ &= 100 \cdot (p + q + 1) \\ &= 100k \text{ where } k = p + q + 1 \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 100, $a + b$ is divisible by 100.

Case III, when $a = 100p + r$ and $b = 100q + (100 - r)$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a + b &= (100p + r) + (100q + 100 - r) \\ &= 100p + 100q + 100 \\ &= 100 \cdot (p + q + 1) \\ &= 100k \text{ where } k = p + q + 1 \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 100, $a + b$ is divisible by 100.

Case IV, when $a = 100p + (100 - r)$ and $b = 100q + (100 - r)$, $p, q \in \mathbb{Z}$:

$$\begin{aligned} a - b &= (100p + 100 - r) - (100q + 100 - r) \\ &= 100p - 100q \\ &= 100 \cdot (p - q) \\ &= 100k \text{ where } k = p - q \end{aligned}$$

By closure under addition, $k \in \mathbb{Z}$. By definition of being divisible by 100, $a - b$ is divisible by 100.

Therefore, $a + b$ or $a - b$ is divisible by 100 in every possible case.