Homework 4

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Problem 1

According to the properties of probability mass functions, the total probabality of all possible cases combined should equal to 1. We can form the following equation using this feature:

$$\begin{aligned}
\sum_{i=1}^{4} p_i &= 1 \\
\sum_{i=1}^{4} \alpha \cdot (i+1) &= 1 \\
\alpha \cdot (1+1) + \alpha \cdot (2+1) + \alpha \cdot (3+1) + \alpha \cdot (4+1) &= 1 \\
14\alpha &= 1 \\
\alpha &= \boxed{\frac{1}{14}}
\end{aligned}$$

Problem 2

First, let us find the probability density function of X:

$$f(x) = \begin{cases} 0, & x < 0 \text{ or } x > 2 \\ \dfrac{1}{b-a} = \dfrac{1}{2-0} = \dfrac{1}{2}, & 0 \leq x \leq 2 \end{cases}$$

According to the *Change-of-variable formula*, we have

$$\begin{aligned} \mathbb{E}(g(X)) &= \int_{-\infty}^{\infty} g(x)\ f(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{0} (x^3 + x^2) \cdot 0\ \mathrm{d}x + \int_{0}^{2} (x^3 + x^2) \cdot \frac{1}{2}\ \mathrm{d}x + \int_{2}^{\infty} (x^3 + x^2) \cdot 0\ \mathrm{d}x \\ &= 0 + \frac{10}{3} + 0 \\ &= \boxed{\frac{10}{3}} \end{aligned}$$

Problem 3

Part (a)

$$P(X = k) = \begin{cases} 0, & k \leq 0 \\ (1 - \frac{1}{2})^{k-1} \cdot (\frac{1}{2})^1 = \frac{1}{2^k}, & k > 0 \end{cases}$$

Part (b)

$$\begin{aligned} \mathbb{E}(X) &= \sum_k k \cdot P(X = k) \\ &= \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k} \\ &= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots \\ &= \boxed{2} \end{aligned}$$

Problem 4

Probability Density Functions

PDF of X:

$$f(x) = \begin{cases} 0, & x < -1 \text{ or } x > 1 \\ \dfrac{1}{b-a} = \dfrac{1}{1-(-1)} = \dfrac{1}{2}, & -1 \le x \le 1 \end{cases}$$

PDF of Y:

$$f(y) = \begin{cases} 0, & y < -4 \text{ or } y > 4 \\ \dfrac{1}{b-a} = \dfrac{1}{4-(-4)} = \dfrac{1}{8}, & -4 \le y \le 4 \end{cases}$$

PDF of Z:

$$f(z) = \begin{cases} 0, & z < 4 \text{ or } z > 6 \\ \dfrac{1}{b-a} = \dfrac{1}{6-4} = \dfrac{1}{2}, & 4 \le z \le 6 \end{cases}$$

Expected Values

Expected Value of X:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, \mathrm{d}x = \int_{-1}^{1} x \cdot \frac{1}{2} \, \mathrm{d}x = 0$$

Expected Value of Y:

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} y \cdot f(y) \, \mathrm{d}y = \int_{-4}^{4} y \cdot \frac{1}{8} \, \mathrm{d}y = 0$$

Expected Value of Z:

$$\mathbb{E}(Z) = \int_{-\infty}^{\infty} z \cdot f(z) \, \mathrm{d}z = \int_{4}^{6} z \cdot \frac{1}{2} \, \mathrm{d}z = 5$$

Variances

Let us use the shortcut formula described in the textbook:

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$$

Variance of X:

$$\begin{aligned} \mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}^2(X) \\ &= \int_{-1}^{1} x^2 \cdot \frac{1}{2} \, \mathrm{d}x - 0^2 \\ &= \frac{1}{3} - 0 \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

Variance of Y:

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathbb{E}(Y^2) - \mathbb{E}^2(Y) \\
&= \int_{-4}^{4} y^2 \cdot \frac{1}{8}\ \mathrm{d}y - 0^2 \\
&= \frac{16}{3} - 0 \\
&= \boxed{\frac{16}{3}}
\end{aligned}$$

Variance of Z:

$$\begin{aligned}
\mathrm{Var}(Z) &= \mathbb{E}(Z^2) - \mathbb{E}^2(Z) \\
&= \int_{4}^{6} z^2 \cdot \frac{1}{2}\ \mathrm{d}z - 5^2 \\
&= \frac{76}{3} - 25 \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

Observations

The variances of X and Z are the same because their random variables come from regions with the same size (2) and are both uniform.

Problem 5

not feeling like doing this.