Deriving Rotational Inertia Formulas

Physics 2210, University of Utah

Derive the formula for the rotational inertia of each object using integral calculus.
The rotational inertia of a point mass m at distance r from the axis is mr^2.

<u>One Dimension</u>

1. A thin rod of mass M and length L, about an axis perpendicular to the rod and through its center
2. A thin rod of mass M and length L, about an axis perpendicular to the rod and through one end
3. A thin hoop of mass M and radius R, about its central axis
4. A thin hoop of mass M and radius R, about a diameter

<u>Two Dimensions</u>

5. A solid disk of mass M and radius R, about its central axis
6. A solid disk of mass M and radius R, about a diameter
7. A washer of mass M, inner radius R_1, and outer radius R_2, about its central axis
8. A thin rectangular slab of length L and width W, about an edge of length W
9. A thin rectangular slab of length L and width W, about an axis through its center and parallel to the edge of length W
10. A thin rectangular slab of length L and width W, about an axis through its center and perpendicular to the slab

<u>Three Dimensions</u>

11. A hollow cylinder of mass M, radius R, and length L, about its central axis
12. A solid cylinder of mass M, radius R, and length L, about its central axis
13. A solid cylinder of mass M, radius R, and length L, about a diameter
14. A hollow sphere of mass M and radius R, about an axis through its center
15. A solid sphere of mass M and radius R, about an axis through its center
16. A hollow circular cone of mass M, base radius R, and height H, about its central axis
17. A solid circular cone of mass M, base radius R, and height H, about its central axis
18. A solid square pyramid of mass M, base side length L, and height H, about an axis perpendicular to the base and through its center

Steps to Deriving Rotational Inertia Formulas:

1. Divide the object into an infinite number of pieces of similar shape. Make sure that you already know the formula for the rotational inertia of an arbitrary piece.
2. Define the integration variable(s). Make sure to distinguish these from the constant dimensions of the object. The integration variables are the things that change as you go from piece to piece.
3. Write the formula for the rotational inertia of an arbitrary piece in differential form by changing I to dI and m to dm. For example, if each piece is a point mass with inertia $I = mr^2$, then the differential form is $dI = dm\, r^2$.
4. Write everything in terms of the variables from Step 2. This includes writing dm (the mass of the piece) in terms of the mass density of the object and the dimensions of the piece:
    - 1D: $dm = \lambda\, dL = M/L\, dL$
    - 2D: $dm = \sigma\, dA = M/A\, dA$
    - 3D: $dm = \rho\, dV = M/V\, dV$
5. Add up the inertias for all the pieces by integrating both sides. Make sure to include the correct limits of integration.

Answers

1. $\frac{1}{12}ML^2$
2. $\frac{1}{3}ML^2$
3. MR^2
4. $\frac{1}{2}MR^2$
5. $\frac{1}{2}MR^2$
6. $\frac{1}{4}MR^2$
7. $\frac{1}{2}M({R_1}^2 + {R_2}^2)$
8. $\frac{1}{3}ML^2$
9. $\frac{1}{12}ML^2$
10. $\frac{1}{12}M(L^2 + W^2)$
11. MR^2
12. $\frac{1}{2}MR^2$
13. $\frac{1}{4}MR^2 + \frac{1}{12}ML^2$
14. $\frac{2}{3}MR^2$
15. $\frac{2}{5}MR^2$
16. $\frac{1}{2}MR^2$
17. $\frac{3}{10}MR^2$
18. $\frac{1}{10}ML^2$