Homework 5

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Problem 1

Part (a)

Since a person can only either have a heart attack or not have one,

$$\begin{aligned}
P(S=0) &= P(S=0 \cap H=0) + P(S=0 \cap H=1) \\
&= 0.50 + 0.03 \\
&= \boxed{0.53}
\end{aligned}$$

Part (b)

Similarly, since a person is either a smoker or a non-smoker,

$$\begin{aligned}
P(H=0) &= P(H=0 \cap S=0) + P(H=0 \cap S=1) \\
&= 0.50 + 0.44 \\
&= \boxed{0.94}
\end{aligned}$$

Part (c)

$$\begin{aligned}
P(H=1|S=0) &= \frac{P(H=1 \cap S=0)}{P(S=0)} \\
&= \frac{0.03}{0.53} \\
&\approx \boxed{0.057}
\end{aligned}$$

$$\begin{aligned}
P(H=1|S=1) &= \frac{P(H=1 \cap S=1)}{P(S=1)} \\
&= \frac{0.03}{1-0.53} \\
&\approx \boxed{0.064}
\end{aligned}$$

These two numbers indicate the probability of drawing a non-smoker having heart attack and drawing a smoker having heart attack, respectively. The probability of the latter is higher, which matches our intuition.

Part (d)

No. According to the results of the calculations done above, $P(H = 1) \neq P(H = 1|S = 1)$. By definition of independence, the variables H and S are not independent.

Problem 2

Part (a)

$$p_{X,Y}(s,t) = \begin{cases} \dfrac{1}{\pi}, & s^2 + t^2 \leq 1 \\ 0, & s^2 + t^2 > 1 \end{cases}$$

Part (b)

Intuitively, $p_X(0) > p_X(1)$. The reason is, if $X = 1$, then the only value Y can be is 0 for the point to still be inside the unit disk; the probability of $Y = 0$ for a continuous random variable Y is infinitely small. On the other hand, when $X = 0$, Y can be anything between -1 and 1, which happens 100% of the time. Therefore, $p_X(0)$ should be greater.

Part (c)

$\mathbb{E}[X + Y] = 0$ since X and Y are both symmetric about the origin.

Part (d)

This is not a valid way to sample uniformly. For example, as described in *Part (b)*, we have $p_X(0) > p_X(1)$, meaning that X is "denser" around 0 and it is more likely to have an X value close to 0 than one close to 1. Therefore, X cannot be sampled uniformly between [-1, 1] in the first step.

Problem 3

Part (a)

By definition of a binomial distribution, the probability of i heads occurring out of n tosses is

$$P(Y = i) = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

where p is the probability of heads occurring for a single toss.

Therefore, by definition of an expected value,

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_i i \cdot P(Y = i) \\
&= \boxed{\sum_{i=0}^{n} i \cdot \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}}
\end{aligned}$$

Part (b)

Since each toss can be viewed as independent to other tosses, the probability of heads occurring on any toss is p. Therefore, by definition of an expected value,

$$\begin{aligned}
\mathbb{E}[X_i] &= \sum_i i \cdot P(X_i = i) \\
&= 1 \cdot p + 0 \cdot (1-p) \\
&= \boxed{p}
\end{aligned}$$

Part (c)

Since the total number heads is equal to the sum of "whether each toss is heads":

$$Y = \sum_{i=0}^{n} X_i$$

we have

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[\sum_{i=0}^{n} X_i] \\
&= \sum_{i=0}^{n} \mathbb{E}[X_i] \\
&= \sum_{i=0}^{n} p \\
&= \boxed{np}
\end{aligned}$$