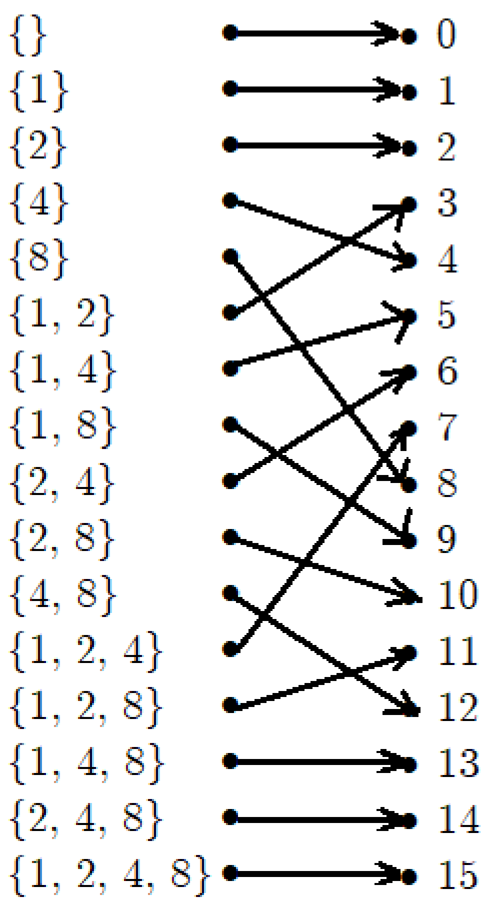
Homework 4

Qianlang Chen

Section 4.1 (p266)

1. Exercise 9.c



2. Exercise 13

(a).

3. Exercise 18.b

The inverse function $f^{-1}(x) = g(x) = \frac{x}{3} - \frac{5}{3}$.

Proof: let an element $x \in \mathbb{R}$ be given.

$$\begin{aligned} g(f(x)) &= g(3x+5) \\ &= \frac{3x+5}{3} - \frac{5}{3} \\ &= x - \frac{5}{3} + \frac{5}{3} \\ &= x \end{aligned}$$

By definition of an inverse function, $g(x)$ is the inverse of $f(x)$.

4. For each of the following Java methods, state whether or not it meets the definition of a function with domain and codomain `Color`. If not, state the reason why.

Part (a)

Yes.

Part (b)

No. Colors with `red` values different than `green` values is not covered by the output, since the outputting colors always have the same `red` and `green` values.

Part (c)

Yes.

5. Implement a method that is the inverse of the *grayscale* method above, i.e. *grayscale*$^{-1}$, or state why *grayscale* is not invertible:

This function is not invertible because its input space is bigger than its output space, meaning that the function is not one-to-one. The input space is all colors with RGB values that can each be distinct, whereas the function only outputs colors with the same values for RGB.

Section 4.2 (p277)

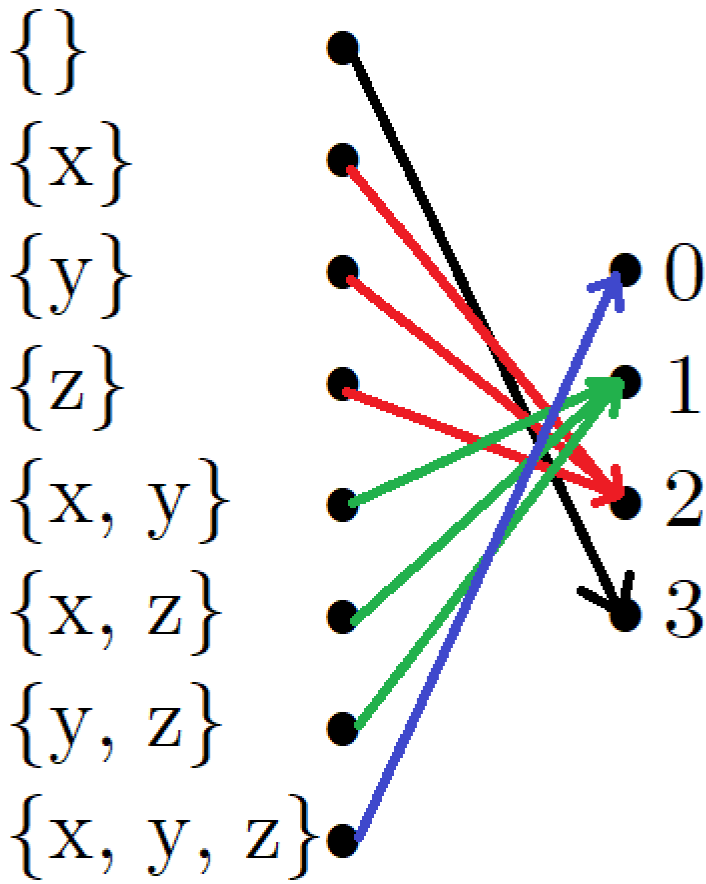
6. Exercise 2.a

$$\begin{aligned}(g \circ f)(z) &= g(f(z)) \\ &= g(2z+1) \\ &= \boxed{(2z+1)^2 - 1}\end{aligned}$$

7. Exercise 4

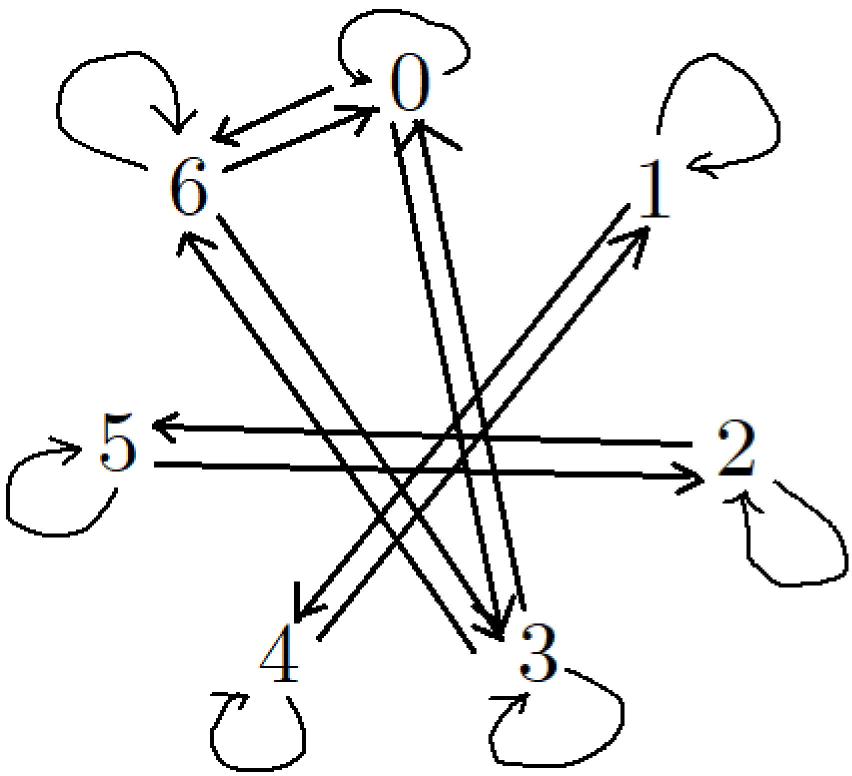
$n \circ c$ is defined.

- **Domain:** $\wp(\{x, y, z\})$;
- **Codomain:** $\{0, 1, 2, 3\}$;
- **Arrow diagram:**

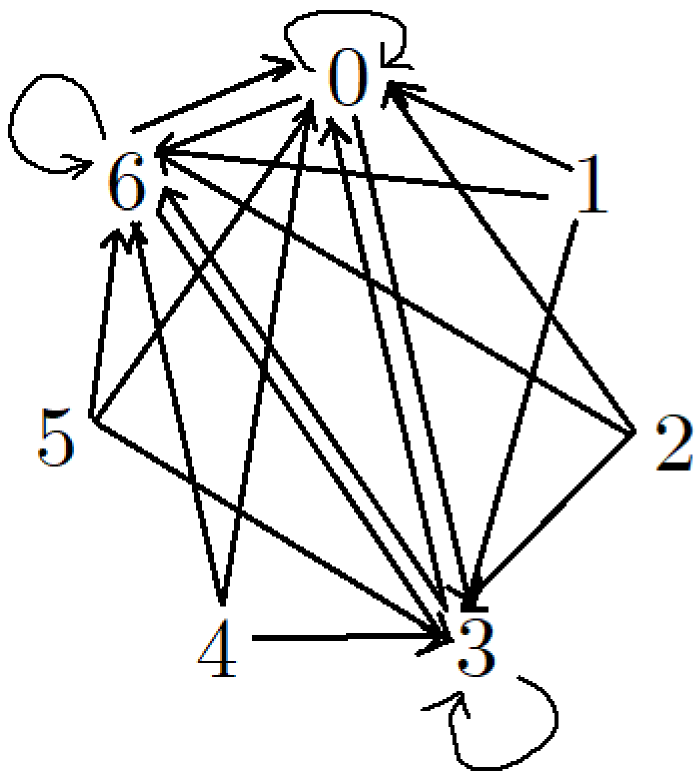


8. Exercise 23

- **Arrow diagram for R:**



- **Arrow diagram for $R \circ R$:**



- **Rule for $R \circ R$:** $x(R \circ R)y$ if $x - (x - y)$ is divisible by 3.

9. Implement a method that is the *grayscale* method composed with the negative method, i.e. *grayscale* ∘ *negative*, or state why it is not possible.

```
/** Returns a new Color that is the grayscale of the negative of the given Color. */
public static Color grayscaleComposedWithNegative(Color c)
{
    Color neg = new Color(255 - c.red, 255 - c.green, 255 - c.blue);

    int average = (neg.red + neg.green + neg.blue) / 3;
    return new Color(average, average, average);
}
```

Section 4.3 (p298)

10. Exercise 4.b

This function is neither one-to-one, onto, nor invertible.

11. Exercise 9

Let $c \in C$ be given. Since the function g is onto, there must exist a $b \in B$ such that $g(b) = c$. Since the function f is also onto, there must exist an $a \in A$ such that $f(a) = b$. If we take $a \in A$ such that $f(a) = b$, it follows that

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Therefore, the function $g \circ f$ is also onto. $\square$

12. Exercise 13

First, let us show that the function g is onto.

Let $n \in \mathbb{N}$ be given. Since every integer is either even or odd, one of the following cases about n must be true:

Case I, when n is even, meaning that $\exists k \in \mathbb{Z} : n = 2k$. It follows that $n = 2k = -2 \cdot (-k) = -2z$, where $z = -k$. By closure under addition and multiplication, $z \in \mathbb{Z}$. Also, since $n \in \mathbb{N}$, meaning that $n \geq 0$, we have that $-2z \geq 0 \implies z \leq 0$.

Case II, when n is odd, meaningThat $\exists k \in \mathbb{Z} : n = 2k + 1$. It follows that $n = 2k + 1 = 2k + 2 - 1 = 2 \cdot (k + 1) - 1 = 2z - 1$, where $z = k + 1$. By closure under addition and multiplication, $z \in \mathbb{Z}$. Also, since $n \in \mathbb{N}$ and n is odd, meaning that $n > 0$, we have that $2z - 1 > 0 \implies z > \frac{1}{2} \implies z > 0$.

Therefore, there exists a $z \in \mathbb{Z} : g(z) = n$ in every possible case, meaning that the function g is onto.

Now, to show that g is one-to-one, let $z, y \in \mathbb{Z}$ be given, where $g(z) = g(y)$. By properties of integers, one of the following cases about z and y must be true:

Case I, when $z, y \leq 0$. By definition of the function g, since $g(z) = g(y)$, we have that $-2z = -2y \implies z = y$.

Case II, when $z, y > 0$. By definition of the function g, since $g(z) = g(y)$, we have that $2z - 1 = 2y - 1 \implies z = y$.

Case III, when $z \leq 0$ and $y > 0$. By definition of the function g, since $g(z) = g(y)$, we have that

$$\begin{aligned}
-2z &= 2y - 1 \\
2 \cdot (-z) &= 2y - 2 + 1 \\
2 \cdot (-z) &= 2 \cdot (y - 1) + 1 \\
2m &= 2n + 1 \qquad \text{where } m = -z \text{ and } n = y - 1
\end{aligned}$$

By closure under addition and multiplication, $m, n \in \mathbb{Z}$. According to the definition of even and odd, the above experssion implies that an even number is equal to an odd number, which can never happen. Therefore, this case is not possible in the first place.

Case IV, when $z > 0$ and $y \leq 0$. By definition of the function g, since $g(z) = g(y)$, we have that

$$\begin{aligned}
2z - 1 &= -2y \\
2z - 2 + 1 &= 2 \cdot (-y) \\
2 \cdot (z - 1) + 1 &= 2 \cdot (-y) \\
2m + 1 &= 2n \qquad \text{where } m = z - 1 \text{ and } n = -y
\end{aligned}$$

By closure under addition and multiplication, $m, n \in \mathbb{Z}$. According to the definition of even and odd, the above experssion implies that an odd number is equal to even number, which can never happen. Therefore, this case is not possible in the first place.

Therefore, $z = y$ in every possible case, meaning that the function g is one-to-one. $\square$

13. Exercise 20

Part (a)

Let elements $a, b, c, d \in A$ be given such that $f(a, b) = f(c, d)$, meaning that $2^a \cdot 3^b$ and $2^c \cdot 3^d$ are the same factor of 144. According to the Fundamental Theorem of Arithmetic, $a = c$ and $b = d$. Therefore, the function f is one-to-one.

Part (b)

Let an element $n \in B$ be given, meaning that n is a factor of 144. Since $144 = 2^4 \cdot 3^2$, according to the Fundamental Theorem of Arithmetic, that is the only way to write 144 as a product of primes. Therefore, there must exist $a \in \{0, 1, 2, 3, 4\}$ and $b \in \{0, 1, 2\}$ such that $n = 2^a \cdot 3^b$, meaning that the function f is onto.

Part (c)

Theorem 7: "Let $f : A \to B$ be a fucntion, where A and B are finite sets of sizes m a n, respectively. If f is a one-to-one correspondence (both one-to-one and onto), then $m = n$."

Part (d)

Set A's cardinality is easy to determine. Since 144 has a prime factorization of $2^4 \cdot 3^2$, the set A in this case has a cardinality of $(4 + 1) \cdot (2 + 1) = 15$. More generally, if a positive integer has a prime factorization of $p_1^{k_1} \cdot p_2^{k_2} \cdots p_n^{k_n}$ where p_1 through p_n are unique primes and k_1 through k_n are their powers, then the set A for that particular integer has a cardinality of $(k_1 + 1) \cdot (k_2 + 1) \cdots (k_n + 1)$.

However, while set A's cardinality is easier to explain, since set B always has the same cardinality as A, its cardinality is just as easy to determine.

Section 4.4 (p311)

14. Exercise 6

Part (a)

Irreflexive and antisymmetric.

Part (b)

Irreflexive, transitive, and antisymmetric.

15. Exercise 14

Part (a)

$$R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$$

Part (b)

$$R = \{(1,1),(1,2),(2,2),(2,3),(3,3)\}$$

Part (c)

$$R = \{(1,2),(1,3),(2,3)\}$$

16. Exercise 18

Part (a)

True.

Since R is reflexive, $(a,a) \in R$ for all $a \in A$. By definition of an inverse relation, the inverse of every $(a,a) \in R$ is still (a,a), for all $a \in A$. Therefore, the inverse is also a reflexive relation.

Part (b)

True.

Since R is antisymmetric, for all $a,b \in A$ where $(a,b) \in R$, we have that $(b,a) \notin R$. By definition of an inverse relation, the inverse of every $(a,b) \in R$ is (b,a). However, since $(b,a) \notin R$, $(a,b) \notin R^{-1}$. Therefore, the inverse is also a antisymmetric relation.

Part (c)

True.

Since R is transitive, for all $a,b,c \in A$ where $(a,b),(b,c) \in R$, we have that $(a,c) \in R$. By definition of an inverse relation, the inverse of every $(a,b),(b,c),(a,c) \in R$ is $(b,a),(c,b),(c,a) \in R^{-1}$. Since for every $(c,b),(b,a) \in R^{-1}$, we have $(c,a) \in R^{-1}$, the inverse is also a transitive relation.

17. Exercise 22

Let elements $a,c \in A$ be given such that $(a,c) \in (R \circ R)$. By definition of a composed relation, $(a,c) \in (R \circ R)$ because (a,b) and (b,c) exist in R for some $b \in A$. Since R is transitive, (a,c) must also exist in R. Therefore, by definition of a subset, $(R \circ R)$ is a subset of R. $\square$

Section 4.5 (p322)

18. Exercise 9.b

By definition of being transitive that if $(a,b),(b,c) \in R$ then $(a,c) \in R$, in the context of the problem, it means "for all $a,b,c \in \mathbb{Z}$, if a^2-b^2 and b^2-c^2 are both divisible by 5, then a^2-c^2 is also divisible by 5."

To show that the above statement is true, let integers a,b,c be given such that a^2-b^2 and b^2-c^2 are both divisible by 5. Since a^2-b^2 is divisible by 5, by definition, $a^2-b^2=5m$ for some $m \in \mathbb{Z}$. Since b^2-c^2 is divisible by 5, by definition, $b^2-c^2=5n$ for some $n \in \mathbb{Z}$. Now, we have

$$\begin{aligned} a^2-c^2 &= a^2-b^2+b^2-c^2 \\ &= 5m+5n \\ &= 5\cdot(m+n) \\ &= 5q && \text{where } q=m+n \end{aligned}$$

By closure under addition and multiplication, $q \in \mathbb{Z}$, and by definition of being divisible by 5, a^2-c^2 is divisible by 5. Now, since we showed that if $(a,b),(b,c) \in T_2$ then $(a,c) \in T_2$, by definition of being transitive, the relation T_2 is transitive. $\square$

19. Exercise 11

Part (a)

Let an element $x \in A$ be given. Since $x-x=0$ which is divisible by 3, $(x,x) \in R$. By definition of being reflexive, R is reflexive.

Part (b)

Let elements $x,y \in A$ be given such that $(x,y) \in R$, meaning that $x-y$ is divisible by 3. By definition of being divisible by 3, $\exists k \in \mathbb{Z} : 3k = x-y$. Now, $y-x=-(x-y)=-3k=3\cdot(-k)=3n$ for some n. By closure under addition and multiplication, $n \in \mathbb{Z}$, and by definition of being divisible by 3, $y-x$ is divisible by 3. Since $(y,x) \in R$, by definition of being symmetric, R is symmetric.

Part (c)

Let elements $x,y,z \in A$ be given such that $(x,y),(y,z) \in R$, meaning that $x-y$ and $y-z$ are both divisible by 3. Similar to the logic used in *Exercise 9.b*, we can show that if $x-y$ and $y-z$ are both divisible by 3, then $x-z$ is divisible by 3. This means that $(x,z) \in R$, and by definition of being transitive, R is transitive.

Part (d)

$$\{\{0,3,6\},\{1,4\},\{2,5\}\}$$

20. Exercise 21

Part I, show that $R \subseteq R^{-1}$.

Let elements $a,b \in A$ be given such that $(a,b) \in R$. Since R is symmetric, we have that $(b,a) \in R$. Now, because of this, by definition of an inverse relation, $(a,b) \in R^{-1}$. Therefore, by definition of a subset, $R \subseteq R^{-1}$.

Part II, show that $R^{-1} \subseteq R$.

Let elements $a,b \in A$ be given such that $(a,b) \in R^{-1}$. By definition of an inverse relation, $(b,a) \in R$. Since R is symmetric, we have that $(a,b) \in R$.Therefore, by definition of a subset, $R^{-1} \subseteq R$.

Therefore, since $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$, we have that $R = R^{-1}$. $\square$