Week 2 Homework

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Exercise 1

Part 1

(a)

$$
\begin{aligned}
&\gcd(1084, 412) & 1084 &= 2 \times 412 + 260 \\
=\ &\gcd(412, 260) & 412 &= 1 \times 260 + 152 \\
=\ &\gcd(260, 152) & 260 &= 1 \times 152 + 108 \\
=\ &\gcd(152, 108) & 152 &= 1 \times 108 + 44 \\
=\ &\gcd(108, 44) & 108 &= 2 \times 44 + 20 \\
=\ &\gcd(44, 20) & 44 &= 2 \times 20 + 4 \\
=\ &\gcd(20, 4) & 20 &= 5 \times 4 + 0 \\
=\ &\gcd(4, 0) \\
=\ &\boxed{4}
\end{aligned}
$$

(b)

$$
\begin{aligned}
&\gcd(1979, 531) && 1979 = 3 \times 531 + 386 \\
&= \gcd(531, 386) && 531 = 1 \times 386 + 145 \\
&= \gcd(386, 145) && 386 = 2 \times 145 + 96 \\
&= \gcd(145, 96) && 145 = 1 \times 96 + 49 \\
&= \gcd(96, 49) && 96 = 1 \times 49 + 47 \\
&= \gcd(49, 47) && 49 = 1 \times 47 + 2 \\
&= \gcd(47, 2) && 47 = 23 \times 2 + 1 \\
&= \gcd(2, 1) && 2 = 2 \times 1 + 0 \\
&= \gcd(1, 0) \\
&= \boxed{1}
\end{aligned}
$$

(c)

$$
\begin{aligned}
&\gcd(305, 185) && 305 = 1 \times 185 + 120 \\
&= \gcd(185, 120) && 185 = 1 \times 120 + 65 \\
&= \gcd(120, 65) && 120 = 1 \times 65 + 55 \\
&= \gcd(65, 55) && 65 = 1 \times 55 + 10 \\
&= \gcd(55, 10) && 55 = 5 \times 10 + 5 \\
&= \gcd(10, 5) && 10 = 2 \times 5 + 0 \\
&= \gcd(5, 0) \\
&= \boxed{5}
\end{aligned}
$$

Part 2

(a)

$$\frac{1084}{412} = [2; 1, 1, 1, 2, 2, 5] = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{5}}}}}}$$

(b)

$$\frac{1979}{531} = [3; 1, 2, 1, 1, 1, 23, 2] = 3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{23 + \cfrac{1}{2}}}}}}}$$

(c)

$$\frac{305}{185} = [1; 1, 1, 1, 5, 2] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{2}}}}}$$

Part 3

(a)

$\frac{305}{185} = [1; 1, 1, 1, 5, 2]$, and $[1; 1, 1, 1, 5] = \frac{28}{17}$. Note that $x = 17, y = -28$ is one solution.

The other integer solutions are in the form of

$$\begin{aligned} x &= 17 + \frac{185}{\gcd(305, 185)} k = \boxed{17 - 37k}, \\ y &= -28 + \frac{305}{\gcd(305, 185)} k = \boxed{-28 + 61k} \end{aligned}$$

for $k \in \mathbb{Z}$.

(b)

$\frac{1979}{531} = [3; 1, 2, 1, 1, 1, 23, 2]$, and $[3; 1, 2, 1, 1, 1, 23] = \frac{969}{260}$. Note that $x = 260, y = -969$ is one solution.

The other integer solutions are in the form of

$$\begin{aligned} x &= 260 + \frac{1979}{\gcd(1979, 531)} k = \boxed{260 - 531k}, \\ y &= -969 + \frac{531}{\gcd(1979, 531)} k = \boxed{-969 + 1979k} \end{aligned}$$

for $k \in \mathbb{Z}$.

(c)

First, let's use the Euclidean Algorithm to find $\gcd(15750, 9150)$:

$$\begin{aligned}
&\gcd(15750, 9150) & 15750 &= 1 \times 9150 + 6600 \\
&= \gcd(9150, 6600) & 9150 &= 1 \times 6600 + 2550 \\
&= \gcd(6600, 2550) & 6600 &= 2 \times 2550 + 1500 \\
&= \gcd(2550, 1500) & 2550 &= 1 \times 1500 + 1050 \\
&= \gcd(1500, 1050) & 1500 &= 1 \times 1050 + 450 \\
&= \gcd(1050, 450) & 1050 &= 2 \times 450 + 150 \\
&= \gcd(450, 150) & 450 &= 3 \times 150 + 0 \\
&= \gcd(150, 0) & & \\
&= \boxed{150} & &
\end{aligned}$$

$\dfrac{15750}{9150} = [1; 1, 2, 1, 1, 2, 3]$, and $[1; 1, 2, 1, 1, 2] = \dfrac{31}{18}$. Note that $x = -18, y = -31$ is one solution.

The other integer solutions are in the form of

$$\begin{aligned} x &= -18 + \frac{9150}{\gcd(15750, 9150)}k = \boxed{-18 - 61k}, \\ y &= 31 + \frac{15750}{\gcd(15750, 9150)}k = \boxed{31 + 105k} \end{aligned}$$

for $k \in \mathbb{Z}$.

(d)

First, let's use the Euclidean Algorithm to find $\gcd(427, 259)$:

$$\begin{aligned} &\gcd(427, 259) & 427 &= 1 \times 259 + 168 \\ =&\gcd(259, 168) & 259 &= 1 \times 168 + 91 \\ =&\gcd(168, 91) & 168 &= 1 \times 91 + 77 \\ =&\gcd(91, 77) & 91 &= 1 \times 77 + 14 \\ =&\gcd(77, 14) & 77 &= 5 \times 14 + 7 \\ =&\gcd(14, 7) & 14 &= 2 \times 7 + 0 \\ =&\gcd(7, 0) \\ =&\boxed{7} \end{aligned}$$

Since the right-hand side 13 is not divisible by 7, this equation has no integer solutions.

Exercise 2

Part 1

Let's run the Continued Fraction Algorithm to see what $\sqrt{3}$ looks like as a continued fraction. Set $\alpha_0 = \sqrt{3}$.

- $a_0 = \lfloor\sqrt{3}\rfloor = 1, \beta_0 = \sqrt{3} - 1, \alpha_1 = \frac{1}{\sqrt{3}-1} = \frac{\sqrt{3}+1}{2}$
- $a_1 = \lfloor\frac{\sqrt{3}+1}{2}\rfloor = 1, \beta_1 = \frac{\sqrt{3}+1}{2} - 1 = \frac{\sqrt{3}-1}{2}, \alpha_2 = \frac{2}{\sqrt{3}-1} = \sqrt{3} + 1$
- $a_2 = \lfloor\sqrt{3} + 1\rfloor = 2, \beta_2 = \sqrt{3} + 1 - 2 = \sqrt{3} - 1$

We've found a repeat as now $\beta_2 = \beta_0$. Therefore, this algorithm can never finish, meaning that $\sqrt{3}$ does not have a terminating continued fraction. So, $\sqrt{3}$ is not rational.

$\sqrt{3} \approx 1.732$, and the fourth convergent is $[1; 1, 2, 1] = \frac{7}{4} = 1.75$, which is correct for 2 digits (1 fractional digit).

Part 2

Let's run the Continued Fraction Algorithm to see what $\sqrt{7}$ looks like as a continued fraction. Set $\alpha_0 = \sqrt{7}$.

- $a_0 = \lfloor\sqrt{7}\rfloor = 2, \beta_0 = \sqrt{7} - 2, \alpha_1 = \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}$
- $a_1 = \lfloor\frac{\sqrt{7}+2}{3}\rfloor = 1, \beta_1 = \frac{\sqrt{7}+2}{3} - 1 = \frac{\sqrt{7}-1}{3}, \alpha_2 = \frac{3}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{2}$
- $a_2 = \lfloor\frac{\sqrt{7}+1}{2}\rfloor = 1, \beta_2 = \frac{\sqrt{7}+1}{2} - 1 = \frac{\sqrt{7}-1}{2}, \alpha_3 = \frac{2}{\sqrt{7}-1} = \frac{\sqrt{7}+1}{3}$
- $a_3 = \lfloor\frac{\sqrt{7}+1}{3}\rfloor = 1, \beta_3 = \frac{\sqrt{7}+1}{3} - 1 = \frac{\sqrt{7}-2}{3}, \alpha_4 = \frac{3}{\sqrt{7}-2} = \sqrt{7} + 2$
- $a_4 = \lfloor\sqrt{7} + 2\rfloor = 4, \beta_4 = \sqrt{7} + 2 - 4 = \sqrt{7} - 2$

We've found a repeat as now $\beta_4 = \beta_0$. Therefore, this algorithm can never finish, meaning that $\sqrt{7}$ does not have a terminating continued fraction. So, $\sqrt{7}$ is not rational.

$\sqrt{7} \approx 2.646$, and the fourth convergent is $[2; 1, 1, 1] = \frac{8}{3} \approx 2.667$, which is correct for 2 digits (1 fractional digit).

Part 3

The definition of α has a self-containing feature. In other words,

$$\alpha = 1 + \frac{1}{1 + \frac{1}{1 + \ddots}} = 1 + \frac{1}{\alpha}$$

By rearranging and solving,

$$\begin{aligned} \alpha^2 &= \alpha + 1 \\ \alpha^2 - \alpha - 1 &= 0 \\ \implies \alpha &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

However, from the original equation it looks like $\alpha > 0$. Therefore, $\alpha = \dfrac{1 + \sqrt{5}}{2}$.