Quiz 4 Solutions

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Problem 1

Part (a)

$$h(x) = f(f(x)) = f(2x - 1) = 2 \cdot (2x - 1) - 1 = \boxed{4x - 3}$$

Part (b)

$$g(x) = f(f(f(x))) = f(4x - 2) = 2 \cdot (4x - 2) - 1 = \boxed{8x - 7}$$

Problem 2

Part (a)

No, the function f is not one-to-one because

$$f(1) = \frac{4!}{(4-1)! \times 1!} = \frac{24}{6 \times 1} = 4$$

and

$$f(3) = \frac{4!}{(4-3)! \times 3!} = \frac{24}{1 \times 6} = 4 = f(1)$$

but $1 \neq 3$.

Part (b)

No, the function f is not onto because there does not exist an x in the domain where $f(x) = 2$ (but 2 is in the codomain):

$$\begin{aligned}
f(1) &= \frac{4!}{(4-1)! \times 1!} = \frac{24}{6 \times 1} = 4, \\
f(2) &= \frac{4!}{(4-2)! \times 2!} = \frac{24}{2 \times 2} = 6, \\
f(3) &= \frac{4!}{(4-3)! \times 3!} = \frac{24}{1 \times 6} = 4, \\
f(4) &= \frac{4!}{(4-4)! \times 4!} = \frac{24}{1 \times 24} = 1
\end{aligned}$$

Problem 3

Part (a)

Let $a, b \in \mathbb{Z}$ be given such that $f(a) = f(b)$. By definition of the function f, we have

$$d \cdot a = d \cdot b$$

Dividing by d on both sides, we have

$$a = b$$

By definition of one-to-one, the function f is one-to-one. $\square$

Part (b)

First, let us show that the function g is onto.

Let $y \in \mathbb{Z}$ be given.

Let $x = d + y$. By definition of the function g and the definition of modulo,

$$g(x) = x \bmod d = (d + y) \bmod d = y$$

By definition of onto, the function g is onto.

Now, to show that the function g is not one-to-one, we have

$$g(1) = 1 \bmod d = 1$$

and

$$g(d + 1) = (d + 1) \bmod d = 1 = g(1)$$

but $1 \neq d + 1$ since $d > 1$. Therefore, by definition of one-to-one, the function g is not one-to-one. $\square$

Problem 4

Part (a)

Let an element $A \in Q(X)$ be given.

By definition of a subset, $A \subseteq A$.

By definition of the relation R, $(A, A) \in R$.

Therefore, by definition of reflexive, the relation R is reflexive. $\square$

Part (b)

Let $A = \{1, 2, 3\}$. Since $A \subseteq Q$ and $|A| \geq 3$, $A \in Q(X)$.

However, $A \cap A = \{1, 2, 3\} \neq \emptyset$, meaning that $(A, A) \notin S$, by definition of the relation S.

Therefore, by definition of reflexive, the relation S is not reflexive. $\square$

Part (c)

Let $A = \{1, 2, 3\}$. Since $A \subseteq Q$ and $|A| \geq 3$, $A \in Q(X)$.

However, $A \cup A = \{1, 2, 3\} \neq X$, meaning that $(A, A) \notin T$, by definition of the relation T.

Therefore, by definition of reflexive, the relation T is not reflexive. $\square$

Problem 5

Part (a)

Let $(a, b) \in Q \cap S$ be given, where $a \neq b$. By definition of an intersection, $(a, b) \in Q$.

Since Q is antisymmetric, $(b, a) \notin Q$.

However, by definition of an intersection, if some element $x \notin Q$, then $x \notin Q \cap S$.

Therefore, $(b, a) \notin Q \cap S$. Now, by definition of antisymmetric, the relation $Q \cap S$ is antisymmetric. $\square$

Part (b)

Let $A = \{1, 2, 3\}$, $R = \{(1, 2)\}$, and $T = \{(2, 3)\}$. By definition of transitive, the relations R and T are both transitive.

However, $R \cup T = \{(1, 2), (2, 3)\}$ is not transitive since $(1, 3) \notin R \cup T$.

Problem 6

First, let us show that the relation R is reflexive.

Let $a \in \mathbb{Z}$ be given. Since $a = a^1$, by definition of the relation R, $(a, a) \in R$.

By definition of reflexive, the relation R is reflexive.

Next, let us show that the relation R is antisymmetric.

Let $(a, b) \in R$ be given such that $(b, a) \in R$. By definition of the relation R, we have $b = a^r, a = b^s$ for some positive integers r, s.

Since $r, s > 0$, the only way this can happen is when $r = s = 1$. The reason is, if $b = a^r$, then $a = b^{1/r}$, meaning that $s = \frac{1}{r}$, but r and s are both integers.

Now, since $s = 1$, we have that $a = b^s = b^1 = b$.

By definition of antisymmetric, the relation R is antisymmetric.

Finally, let us show that the relation R is transitive.

Let $(a, b), (b, c) \in R$ be given. By definition of the relation R, we have $b = a^r, c = b^s$ for some positive integers r, s.

Now, we have $c = b^s = (a^r)^s = a^{rs} = a^t$ where $t = rs$. By closure under multiplication, $t \in \mathbb{Z}$, and since $r, s > 0$, $t > 0$.

By definition of the relation R, $(a, c) \in R$, and by definition of transitive, the relation R is transitive.

Therefore, since the relation R is reflexive, antisymmetric, and transitive, by definition of a partial order, the relation R is a partial order on $\mathbb{Z}$. $\square$

Problem 7

Part (a)

No, this is not a partition of S because $\{1,3,5\} \cup \{2,6\} \cup \{4,8,9\} = \{1,2,3,4,5,6,8,9\} \neq S$.

Part (b)

No, this is not a partition of S because $\{1,3,5\} \cap \{5,7,9\} = \{5\} \neq \emptyset$.

Part (c)

Yes, this is a partition of S because

- $\{1,3,5\}, \{2,4,6,8\}$ and $\{7,9\}$ are all not empty;
- $\{1,3,5\}, \{2,4,6,8\}$ and $\{7,9\}$ are all disjoint to each other; and
- $\{1,3,5\} \cup \{2,4,6,8\} \cup \{7,9\} = S$.

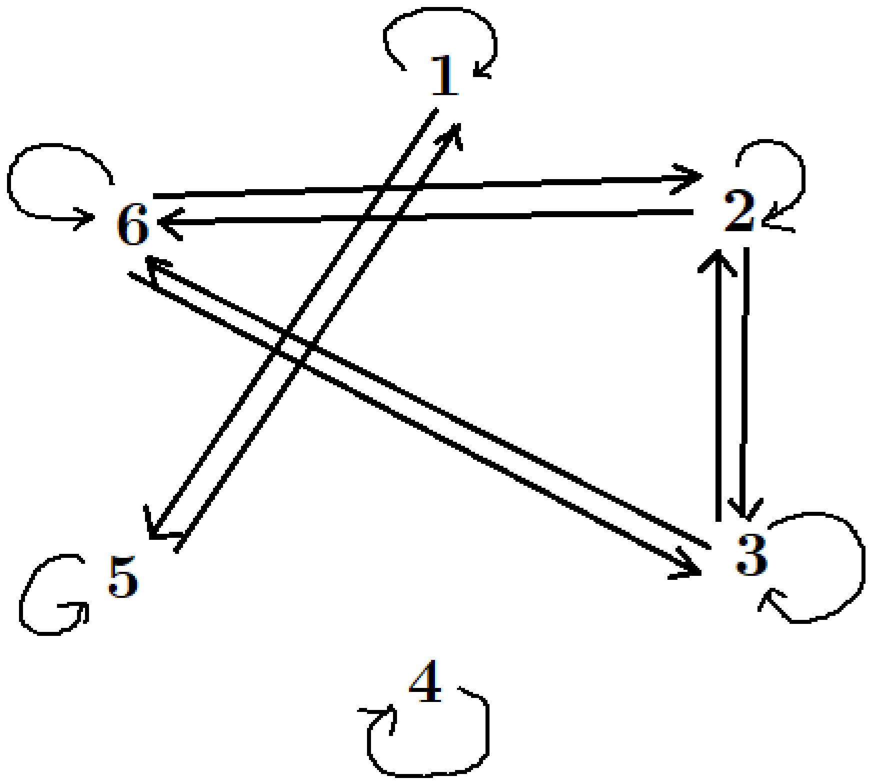
Part (d)

Yes, this is a partition of S because

- $\{1,2,3,4,5,6,7,8,9\}$ is not empty;
- $\{1,2,3,4,5,6,7,8,9\}$ is the only set in the set *(d)* (so there is not a potential set to be not disjoint to this set); and
- $\{1,2,3,4,5,6,7,8,9\} = S$.

Problem 8

Generating an arrow diagram for the relation R:



As seen from the above diagram, there are three clear "independent islands": $\{1, 5\}, \{2, 3, 6\}$, and $\{4\}$. Therefore, the partition of A induced by the relation R is:

$$\{\{1, 5\}, \{2, 3, 6\}, \{4\}\}$$

Problem 9

Part (a)

$R = \{(2, 2), (2, 6), (2, 24), (6, 6), (6, 24), (24, 24)\}$.

Yes, this set is totally ordered because the relation R is reflexive, antisymmetric, transitive and satisfies the "comparable" property.

Part (b)

$R = \{(3, 3), (3, 15), (5, 5), (5, 15), (15, 15)\}$.

No, this set is not totally ordered because it does not satisfy the "comparable" property. (The relation R does not contain $(3, 5)$ or $(5, 3)$.)

Part (c)

No, this set is not totally ordered because it does not satisfy the "comparable" property. (For example, the relation R does not contain $(2, 3)$ or $(3, 2)$ since 2 and 3 are coprimes.)

Part (d)

$R = \{(2, 2), (2, 4), (2, 8), (2, 32), (4, 4), (4, 8), (4, 32), (8, 8), (8, 32), (32, 32)\}$.

Yes, this set is totally ordered because the relation R is reflexive, antisymmetric, transitive and satisfies the "comparable" property.

Part (e)

$R = \{(7, 7)\}$.

Yes, this set is totally ordered because the relation R is reflexive, antisymmetric, transitive and satisfies the "comparable" property.

Part (f)

$R = \{(5, 5), (5, 15), (5, 30), (15, 15), (15, 30), (30, 30)\}$.

Yes, this set is totally ordered because the relation R is reflexive, antisymmetric, transitive and satisfies the "comparable" property.