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Exercise 1

(1) False

(2) True

(3) False

(4) False

(5) True

(6) False

(7) True

(8) True

(9) False

(10) False

(11) False

(12) False

(13) True

(14) True

(15) False

Exercise 2

(1) $\gcd(71, 24)$ $\quad 71 = 24 \times 2 + 23$

$= \gcd(24, 23)$ $\quad 24 = 23 \times 1 + 1$ $\quad \Rightarrow \frac{71}{24} = [2; 1, 23]$

$= \gcd(23, 1)$ $\quad 23 = 1 \times 23 + 0$

$= \gcd(1, 0) = 1$

$[2; 1] = 3/1$ and $24 \times \boxed{3} = 1 \mod 71$

$\Rightarrow$ 3 is 24's multiplicative inverse in $\mathbb{F}_{71}$

(2) $\gcd(x^5 + x^3 + x^2 + 1,\ x^3 + 2x^2 + x + 2)$

$x^5 + x^3 + x^2 + 1$
$= (x^3 + 2x^2 + x + 2)(x^2 - 2x + 4)$
$\cancel{+ (-7x^2 - 7)}$
$+ 0$

$= \gcd(x^3 + 2x^2 + x + 2,\ 0)$

$= \boxed{x^3 + 2x^2 + x + 2}$

(3) $\gcd(30 + i, 53)$ $\quad 53 = 2(30 + i) + (-7 - 2i)$

$= \gcd(30 + i, -7 - 2i)$ $\quad 30 + i = (-4 + i)(-7 - 2i) + 0$

$= \gcd(-7 - 2i, 0) = \boxed{-7 - 2i}$

Exercise 3

(1) $\left(\frac{17}{589}\right) = \left(\frac{589}{17}\right) = \left(\frac{19}{17}\right)\left(\frac{31}{17}\right) = \left(\frac{2}{17}\right)\left(\frac{14}{17}\right)$

$= 1 \times \left(\frac{2}{17}\right)\left(\frac{7}{17}\right) = 1 \times \left(\frac{17}{7}\right) = \left(\frac{3}{7}\right) = -\left(\frac{7}{3}\right)$

$= -\left(\frac{1}{3}\right) = -1 \Rightarrow \boxed{\text{No}}$

(2) $x^2 - 4 + 10 = 0$

$x^2 - 4 + 10 - 6 = -6$ mod 131

$x^2 - 4 + 4 = 125$

$(x-2)^2 = 125 \leftarrow$ is 125 a square mod 131?

$\left(\frac{125}{131}\right) = \left(\frac{5}{131}\right)^3 = \left(\frac{131}{5}\right)^3 = \left(\frac{1}{5}\right)^3 = 1 \Rightarrow \boxed{\text{yes}}$

Exercise 4

$$g = 2 \Rightarrow g^{\frac{p-1}{4}} = 2^{\frac{52}{4}} = 2^{13} \Rightarrow 2^{13} \bmod 53 = 30$$

$$\Rightarrow 30^2 + 1^2 = 901 = 17 \times 53$$

$$u = 30 \bmod 17 = -4$$

$$v = 1 \bmod 17 = 1$$

$$(u - vi)(a + bi) = (-4 - i)(30 + i) = -119 - 34i$$

$$\frac{(-119 - 34i)}{17} = -7 - 2i$$

$$\Rightarrow (-7)^2 + (-2)^2 = 53$$

$$\boxed{x = -7, \ y = -2}$$

Exercise 5

$x^2 - 5y^2 = 1 \Rightarrow (x=9, y=4)$ is the smallest solution.

$k=1 \Rightarrow 9 + 4\sqrt{5} \Rightarrow x=9, \ y=4$

$k=2 \Rightarrow (9+4\sqrt{5})^2 = 161 + 72\sqrt{5} \Rightarrow x=161, \ y=72$

$$\Rightarrow \boxed{\begin{cases} x=9, \ y=4 \\ x=161, \ y=72 \end{cases}}$$