Homework 3

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Problem 1

Part (a)

- **Upper bound:** when all four coins land with heads up: 4 heads − 0 tails = $\boxed{4}$;
- **Lower bound:** when all four coins land with tails up: 0 heads − 4 tails = $\boxed{-4}$.

Part (b)

$$\text{pmf}(X = k) = \begin{cases} \binom{4}{0} \cdot (1 - \frac{1}{2})^4 = \frac{1}{16}, k = -4 \\ \binom{4}{1} \cdot \frac{1}{2} \cdot (1 - \frac{1}{2})^3 = \frac{1}{4}, k = -2 \\ \binom{4}{2} \cdot (\frac{1}{2})^2 \cdot (1 - \frac{1}{2})^2 = \frac{3}{8}, k = 0 \\ \binom{4}{3} \cdot (\frac{1}{2})^3 \cdot (1 - \frac{1}{2}) = \frac{1}{4}, k = 2 \\ \binom{4}{4} \cdot (\frac{1}{2})^4 = \frac{1}{16}, k = 4 \\ 0, k \notin \{-4, -2, 0, 2, 4\} \end{cases}$$

Part (c)

$$\text{cdf}(X \leq k) = \begin{cases} 0, k < -4 \\ \text{pmf}(X = -4) = \frac{1}{16}, -4 \leq k < -2 \\ \text{pmf}(X = -4) + \text{pmf}(X = -2) = \frac{5}{16}, -2 \leq k < 0 \\ \text{pmf}(X = -4) + \text{pmf}(X = -2) + \text{pmf}(X = 0) = \frac{11}{16}, 0 \leq k < 2 \\ \text{pmf}(X = -4) + \text{pmf}(X = -2) + \text{pmf}(X = 0) + \text{pmf}(X = 2) = \frac{15}{16}, 2 \leq k < 4 \\ 1, k \geq 4 \end{cases}$$

Problem 2

```
## returns the outcome/random variable X for performing 4 coin tosses.
simulate.coin.tosses = function()
{
  ## the final random variable X.
  x = 0

  for(i in 1:4)
  {
    ## 1/2 of the time the random var X increases and 1/2 decreases.
    r = sample(0:1, 1)
    if (r == 0)
      x = x + 1
    else
      x = x - 1
  }

  return(x)
}

# initialize counting variables
x.less = 0
x.neg.4 = 0
x.neg.3 = 0
x.neg.2 = 0
x.neg.1 = 0
x.0 = 0
x.1 = 0
x.2 = 0
x.3 = 0
x.4 = 0
x.more = 0

# run the simulation 1,000 times and record the outcomes

n = 1000
for (i in 1:n)
{
  r = simulate.coin.tosses()

  if (r < -4)
    x.less = x.less + 1
  else if (r == -4)
    x.neg.4 = x.neg.4 + 1
  else if (r == -3)
    x.neg.3 = x.neg.3 + 1
  else if (r == -2)
    x.neg.2 = x.neg.2 + 1
  else if (r == -1)
    x.neg.1 = x.neg.1 + 1
  else if (r == 0)
    x.0 = x.0 + 1
```

```r
  else if (r == 1)
    x.1 = x.1 + 1
  else if (r == 2)
    x.2 = x.2 + 1
  else if (r == 3)
    x.3 = x.3 + 1
  else if (r == 4)
    x.4 = x.4 + 1
  else if (r > 4)
    x.more = x.more + 1
}

# display the results
sprintf('P(X < -4): %f', x.less / n)
```

```
## [1] "P(X < -4): 0.000000"
```

```r
sprintf('P(X = -4): %f', x.neg.4 / n)
```

```
## [1] "P(X = -4): 0.064000"
```

```r
sprintf('P(X = -3): %f', x.neg.3 / n)
```

```
## [1] "P(X = -3): 0.000000"
```

```r
sprintf('P(X = -2): %f', x.neg.2 / n)
```

```
## [1] "P(X = -2): 0.244000"
```

```r
sprintf('P(X = -1): %f', x.neg.1 / n)
```

```
## [1] "P(X = -1): 0.000000"
```

```r
sprintf('P(X = 0): %f', x.0 / n)
```

```
## [1] "P(X = 0): 0.368000"
```

```r
sprintf('P(X = 1): %f', x.1 / n)
```

```
## [1] "P(X = 1): 0.000000"
```

```r
sprintf('P(X = 2): %f', x.2 / n)
```

```
## [1] "P(X = 2): 0.254000"
```

```r
sprintf('P(X = 3): %f', x.3 / n)
```

```
## [1] "P(X = 3): 0.000000"
```

```r
sprintf('P(X = 4): %f', x.4 / n)
```

```
## [1] "P(X = 4): 0.070000"
```

```r
sprintf('P(X > 4): %f', x.more / n)
```

```
## [1] "P(X > 4): 0.000000"
```

Problem 3

Part (a)

- **Upper bound:** all n coins land with heads up: $\boxed{n}$;
- **Lower bound:** all n coins land with tails up; tossing them all again, all tails up again: $\boxed{0}$.

Part (b)

Let us consider the outcome X for just a single coin. There are two possibilities:

- The coin lands with heads up. This can happen in one of two ways: landing heads on the first toss (probability $\frac{1}{2}$), or landing tails on the first toss but heads on the second toss (probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$). Total probability for heads: $\dfrac{1}{2} + \dfrac{1}{4} = \boxed{\dfrac{3}{4}}$.
- The coin lands with tails up. This can only happen in one way: landing tails on both the first and second tosses. Total probability for tails: $\dfrac{1}{2} \cdot \dfrac{1}{2} = \boxed{\dfrac{1}{4}}$.

Now, when there are n coins, since each coin has a certain chance of producing heads – incrementing the random variable X, we have

$$\begin{aligned} \text{pmf}(X = k) &= \text{Prob of } k \text{ heads and } (n-k) \text{ tails} \times \text{Number of ways this can happen} \\ &= (\frac{3}{4})^k \cdot (\frac{1}{4})^{(n-k)} \cdot \binom{n}{k} \end{aligned}$$

This fits the form of a **binomial distribution.**

Problem 4

Since the recognition system can only either pass or fail and the pass rate is $\frac{9}{10}$, it means that the probability of failing is $1 - \frac{9}{10} = \frac{1}{10}$.

Mathematical Expressions:

a. $\binom{40}{6} \cdot (\frac{1}{10})^6 \cdot (\frac{9}{10})^{(40-6)} \approx 11\%$

b. $\sum_{n=0}^{2} \binom{40}{n} \cdot (\frac{1}{10})^n \cdot (\frac{9}{10})^{(40-n)} \approx 22\%$

c. $\sum_{n=8}^{40} \binom{40}{n} \cdot (\frac{1}{10})^n \cdot (\frac{9}{10})^{(40-n)} \approx 4.2\%$

R Expressions:

a. `dbinom(6, 40, 1/10)`

b. `pbinom(2, 40, 1/10)`

c. `pbinom(8, 40, 1/10, FALSE) + dbinom(8, 40, 1/10)`