Homework 2

Qianlang Chen

Section 2.1

1. Exercise 11 (p97)

Part (a)

Let a, b be the two given odd integers. By definition of odd, we have

$$a = 2m + 1, b = 2n + 1 : m, n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k + 1, \text{ where } k = 2mn + m + n \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of odd, we have that ab is an odd number.

Part (b)

Let a be the given odd integer and b be the given even integer. By definition of odd and even, we have

$$a = 2m + 1, b = 2n : m, n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= (2m + 1)2n \\ &= 4mn + 2n \\ &= 2(2mn + n) \\ &= 2k, \text{ where } k = 2mn + n \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of even, we have that ab is an even number.

Part (c)

Let a be the given even integer and b be the given integer that is divisible by 3.

By definition of even, we have

$$a = 2m : m \in \mathbb{Z}$$

By definition of being divisible by 3, we have

$$b = 3n : n \in \mathbb{Z}$$

By substitution,

$$\begin{aligned} ab &= 2m \cdot 3n \\ &= 6mn \\ &= 6k, \text{ where } k = mn \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 6, we have that ab is divisible by 6.

2. Exercise 12.d (p97)

This pair of statements are *not* contrapositives of one another.

- Counterexample of case (i): a person that likes computers but does not like computer science.
- Counterexample of case (ii): a person that does not like computers but likes computer science.

3. Exercise 13.a (p97)

Contrapositive: “if $m = 0$ and $n = 0$, then $m^2 + n^2 = 0$.”

Proof of the contrapositive: Since $m = 0$ and $n = 0$, by substitution, we have

$$m^2 + n^2 = 0^2 + 0^2 = 0$$

Since the contrapositive is true, the original statement “if $m^2 + n^2 \neq 0$, then $m \neq 0$ or $n \neq 0$” is true.

4. Exercise 14.c (p98)

Let n be a number preceding a perfect cube. Since any perfect cube is in the form of x^3 where x is an integer, we have that $n = x^3 - 1$.

The expression $x^3 - 1$ is equivalent to $(x - 1)(x^2 + x + 1)$:

$$\begin{aligned}(x-1)(x^2+x+1) &= x(x^2+x+1) - 1(x^2+x+1) \\ &= (x \cdot x^2 + x \cdot x + x \cdot 1) - (1 \cdot x^2 + 1 \cdot x + 1 \cdot 1) \\ &= x^3 + x^2 + x - x^2 - x - 1 \\ &= x^3 - 1\end{aligned}$$

Now, since $n = x^3 - 1$, n can always be factorized into $(x - 1)$ and $(x^2 + x + 1)$.

- When x is zero or negative, n is negative, which can never be prime.
- When x is 1, $n = 1^3 - 1 = 0$, which is not prime.
- When x is 2, $n = 2^3 - 1 = 7$, which is prime.
- When x is 3 or higher, since n can always be factorized into $(x - 1)$ and $(x^2 + x + 1)$, but $(x - 1)$ is not 1 or n, n is divisible by some other integer that is not 1 or n. Therefore, n is not prime.
- When x is not an integer, n is not an integer, which is not prime.

Since the above cases cover all possibilities of x, 7 is indeed the only prime preceding a perfect cube.

Section 2.2

5. Exercise 7.e (p108)

Let m be the given $10^{n-1} - 1$ that is divisible by 9. Since m is divisible by 9, $m = 9k : k \in \mathbb{Z}$.

Since $10^n = 10 \cdot 10^{n-1}$, we have

$$\begin{aligned}
10^n - 1 &= 10 \cdot 10^{n-1} - 1 \\
&= 10 \cdot (10^{n-1} - 1 + 1) - 1 \\
&= 10 \cdot (m + 1) - 1 && \text{Substituting } m = 10^{n-1} - 1 \\
&= 10 \cdot (9k + 1) - 1 && \text{Substituting } 9k = m \\
&= 90k + 10 - 1 && \text{Distributing 10} \\
&= 90k + 9 \\
&= 9 \cdot (10k + 1) && \text{Factoring out 9} \\
&= 9c && \text{Where } c = 10k + 1
\end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, $10^n - 1$ is divisible by 9.

6. Exercise 16 (p109)

Since n is not divisible by 3, by the division theorem, $n = 3q + r : q, r \in \mathbb{Z}, 0 < r < 3$. In other words, one of the following cases about r must be true:

Case I, when r = 1, meaning that n = 3q + 1:

$$\begin{aligned} n^2 + 2 &= (3q+1)^2 + 2 && \text{Substituting } 3q+1 = n \\ &= 9q^2 + 6q + 1 + 2 \\ &= 9q^2 + 6q + 3 \\ &= 3 \cdot (3q^2 + 2q + 1) && \text{Factoring out 3} \\ &= 3k && \text{Where } k = 3q^2 + 2q + 1 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 3, $n^2 + 2$ is divisible by 3.

Case II, when r = 2, meaning that n = 3q + 2:

$$\begin{aligned} n^2 + 2 &= (3q+2)^2 + 2 && \text{Substituting } 3q+2 = n \\ &= 9q^2 + 12q + 4 + 2 \\ &= 9q^2 + 12q + 6 \\ &= 3 \cdot (3q^2 + 4q + 2) && \text{Factoring out 3} \\ &= 3k && \text{Where } k = 3q^2 + 4q + 2 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By definition of being divisible by 3, $n^2 + 2$ is divisible by 3.

Therefore, $n^2 + 2$ is divisible by 3 in every possible case.

7. Exercise 20 (p109)

Let n be any integer. Let $s = (n-1)^3 + n^3 + (n+1)^3$, which is the sum of three consecutive perfect cubes mentioned in the problem. After simplifying the expression of s, we have

$$\begin{aligned} s &= (n-1)^3 + n^3 + (n+1)^3 \\ &= (n^3 - 3n^2 + 3n - 1) + n^3 + (n^3 + 3n^2 + 3n + 1) \\ &= 3n^3 + 6n \\ &= 3n \cdot (n^2 + 2) && \text{Factoring out } 3n \end{aligned}$$

Now, one of the following cases about n must be true:

Case I, when n is divisible by 3:

Since n is divisible by 3, meaning that $n = 3k : k \in \mathbb{Z}$, we have

$$\begin{aligned} s &= 3n \cdot (n^2 + 2) \\ &= 3 \cdot 3k \cdot (n^2 + 2) && \text{Substituting } 3k = n \\ &= 9k \cdot (n^2 + 2) \\ &= 9c && \text{Where } c = k \cdot (n^2 + 2) \end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, s is divisible by 9.

Case II, when n is not divisible by 3:

Since n is not divisible by 3, according to what is proven by *Exercise 16,* $(n^2 + 2)$ is divisible by 3. This means that $n^2 + 2 = 3k : k \in \mathbb{Z}$, and we have

$$\begin{aligned} s &= 3n \cdot (n^2 + 2) \\ &= 3n \cdot 3k && \text{Substituting } 3k = n^2 + 2 \\ &= 9nk \\ &= 9c && \text{Where } c = nk \end{aligned}$$

By closure under addition and multiplication, $c \in \mathbb{Z}$. By definition of being divisible by 9, s is divisible by 9.

Therefore, s is divisible by 9 is every possible case.

8. *Show that the square of an integer not divisible by 5 leaves a remainder of 1 or 4 when divided by 5.*

Let n be the given integer that is not divisible by 5. By the division theorem, since n is not divisible by 5, $n = 5q + r : q, r \in \mathbb{Z}, 0 < r < 5$. In other words, one of the following cases about r must be true:

Case I, when r = 1, meaning that $n = 5q + 1$:

$$\begin{aligned} n^2 &= (5q+1)^2 && \text{Substituting } 5q+1 = n \\ &= (25q^2 + 10q) + 1 \\ &= 5 \cdot (5q^2 + 2q) + 1 && \text{Factoring out 5} \\ &= 5k + 1 && \text{Where } k = 5q^2 + 2q \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 1 when divided by 5.

Case II, when r = 2, meaning that $n = 5q + 2$:

$$\begin{aligned} n^2 &= (5q+2)^2 && \text{Substituting } 5q+2 = n \\ &= (25q^2 + 20q) + 4 \\ &= 5 \cdot (5q^2 + 4q) + 4 && \text{Factoring out 5} \\ &= 5k + 4 && \text{Where } k = 5q^2 + 4q \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 4 when divided by 5.

Case III, when r = 3, meaning that $n = 5q + 3$:

$$\begin{aligned} n^2 &= (5q+3)^2 && \text{Substituting } 5q+3 = n \\ &= 25q^2 + 30q + 9 \\ &= (25q^2 + 30q + 5) + 4 \\ &= 5 \cdot (5q^2 + 6q + 1) + 4 && \text{Factoring out 5} \\ &= 5k + 4 && \text{Where } k = 5q^2 + 6q + 1 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 4 when divided by 5.

Case IV, when r = 4, meaning that $n = 5q + 4$:

$$\begin{aligned} n^2 &= (5q+4)^2 && \text{Substituting } 5q+4 = n \\ &= 25q^2 + 40q + 16 \\ &= (25q^2 + 40q + 15) + 1 \\ &= 5 \cdot (5q^2 + 8q + 3) + 1 && \text{Factoring out 5} \\ &= 5k + 1 && \text{Where } k = 5q^2 + 8q + 3 \end{aligned}$$

By closure under addition and multiplication, $k \in \mathbb{Z}$. By the division theorem, n^2 leaves a remainder of 1 when divided by 5.

Therefore, the n^2 leaves a remainder of 1 or 4 when divided by 5 in every possible case.

9. Exercise 26.c (p109)

First, let us prove that a perfect square k^2 is divisible by 4 if and only if k is even.

Let n be a perfect square, and by definition, $n = k^2 : k \in \mathbb{Z}$. Since k must be either even or odd, one of the following cases must be true:

Case I, when k is even:

Since k is even, $k = 2c : c \in \mathbb{Z}$. By substitution, we have

$$n = k^2 = (2c)^2 = 4c^2 = 4m \text{ where } m = c^2$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By definition of being divisible by 4, n is divisible by 4.

Case II, when k is odd:

Since k is odd, $k = 2c + 1 : c \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} n = k^2 &= (2c+1)^2 \\ &= 4c^2 + 4c + 1 \\ &= 4 \cdot (c^2 + c) + 1 \\ &= 4m + 1 \text{ where } m = c^2 + c \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, n leaves a remainder of 1 when divided by 4, implying that n is not divisible by 4.

Therefore, *every perfect square k^2 is divisible by 4 if and only if k is even.*

Now, let us prove the contrapositive of the original problem statement: "for every three integers a, b, and c where $c^2 = a^2 + b^2$, if not both a and b are even, then c is not even."

Since every integer is either even or odd, at least one of $\{a, b\}$ must be odd for the above condition to be true. Therefore, we can let j be one of $\{a, b\}$ that is odd and k be the other (which can be either even or odd). Since j is odd, $j = 2p + 1 : p \in \mathbb{Z}$, and this leads us into one of the following:

Case I, when k is even:

By definition of even, $k = 2q : q \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} c^2 &= j^2 + k^2 \\ &= (2p+1)^2 + (2q)^2 && \text{Substituting } 2p+1 = j \text{ and } 2q = k \\ &= 4p^2 + 4p + 1 + 4p^2 \\ &= 4 \cdot (p^2 + p + q^2) + 1 && \text{Factoring out 4} \\ &= 4m + 1 && \text{Where } m = p^2 + p + q^2 \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, c^2 is not divisible by 4, which means that c is not even, according to what we have proved earlier.

Case II, when k is odd:

By definition of odd, $k = 2q + 1 : q \in \mathbb{Z}$. By substitution, we have

$$\begin{aligned} c^2 &= j^2 + k^2 \\ &= (2p+1)^2 + (2q+1)^2 && \text{Substituting } 2p+1 = j \text{ and } 2q+1 = k \\ &= 4p^2 + 4p + 1 + 4q^2 + 4q + 1 \\ &= 4p^2 + 4p + 4q^2 + 4q + 2 \\ &= 4 \cdot (p^2 + p + q^2 + q) + 2 && \text{Factoring out 4} \\ &= 4m + 2 && \text{Where } m = p^2 + p + q^2 + q \end{aligned}$$

By closure under addition and multiplication, $m \in \mathbb{Z}$. By the division theorem, c^2 is not divisible by 4, which means that c is not even, according to what we have proved earlier.

Therefore, c is not even in every possible case. Also, since the contrapositive is true, the original statement is true.

Section 2.3

10. Exercise 2 (p121)

Part (a)

Part (b)

Part (c)

Part (d)

11. Exercise 3.f (p121)

12. Exercise 4.e (p122)

13. Exercise 8.f (p122)

14. Exercise 13 (p122)

Section 2.4

15. Exercise 1.d (p130)

16. Exercise 2.d (p130)

17. Exercise 4.b (p130)

18. Exercise 6 (p131)

Section 2.5

19. Exercise 12 (p147)

20. Exercise 20 (p148)

21. Exercise 32.b (p149)

22. Exercise 34.b (p149)