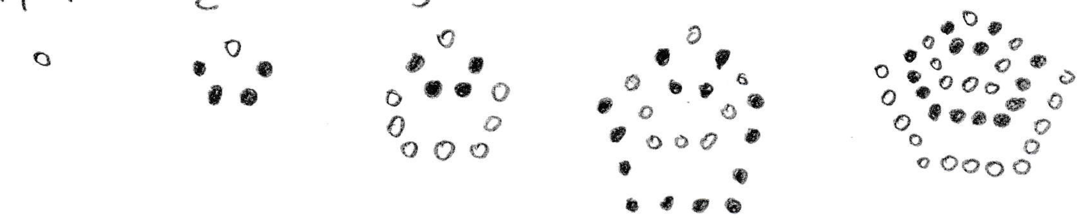
EXERCISE 1

(1) $P_1 = 1$ $\quad P_2 = 5$ $\quad P_3 = 12$ $\quad P_4 = 22$ $\quad P_5 = 35$



(2) Looks like $P_{n+1} - P_n = 3n+1$

(3) $P_n = 1 + (3\cdot1+1) + (3\cdot2+1) + \cdots + (3\cdot(n-1)+1)$

$$= n + 3\,\frac{n(n-1)}{2} = n + \frac{3n^2}{2} - \frac{3n}{2} = \boxed{\frac{3n^2-n}{2}}$$

(4) $m^2 = \dfrac{3n^2-n}{2}$

$2m^2 = 3n^2 - n$

$24m^2 = 36n^2 - 12n$

$6(4m^2) = (36n^2 - 12n + 1) - 1$

$6\underbrace{(2m)}_{y}{}^2 = \underbrace{(6n-1)}_{x}{}^2 - 1$

$6y^2 = x^2 - 1$

$\boxed{x^2 - 6y^2 = 1}$

(5) $m^2 = 1 \Rightarrow m = 1$

$\dfrac{3n^2-2}{2} = 1 \Rightarrow n = 1$

$x = 6n - 1 = 6\cdot1 - 1 = \boxed{5}$

$y = 2m = 2\cdot1 = \boxed{2}$

$x=5, y=2$ is the smallest solution to $x^2 - 6y^2 = 1$ because $6y^2$ for $y=1$ is not one less than a perfect square.

(6) $K=1 \Rightarrow 5+2\sqrt{6} \Rightarrow x=5, y=2 \Rightarrow n=1, m=1$

$K=2 \Rightarrow 49+20\sqrt{6} \Rightarrow x=49, y=20 \Rightarrow n=25/3, m=10$

$K=3 \Rightarrow 485+198\sqrt{6} \Rightarrow x=485, y=198 \Rightarrow \boxed{n=81, m=99}$

$m^2 = \boxed{9801}$ is a square-pentagonal number

EXERCISE 2

(1) Choose $x=22 < 10\pi$ and $y=7 < 10$, we have

$$22 - 7\pi < \frac{1}{10} \Rightarrow \boxed{\frac{22}{7}} - \pi < \frac{1}{7^2}$$

(2) Create buckets named $0, 1, \ldots, 690$.

Take the numbers $1, 11, 111, \ldots, \underbrace{111\cdots111}_{\text{length } 692}$ and put each number into the bucket with name equal to the number mod 691.

Since there are 692 numbers and 691 buckets, at least two numbers must share the same bucket.

Call these two numbers a and b where $a>b$.

Now, $a-b$ is a multiple of 691 because $a \equiv b \bmod 691$.

Also, $a-b$ is made up with zero and one only because both a and b are only made up with ones.