CS 2100: Discrete Structures

Final Exam
Spring 2020

Take Home Final Exam Rules

Any violation of these rules and additional rules stated in the the Academic Misconduct Policy of CS 2100 will result in an Academic Misconduct filing.

- Do not communicate with anyone else (especially other students in the course) about the exam. The only exception are questions to the instructor/TAs via Piazza about how to interpret the exam questions. All general clarifications will be posted on Piazza.
- Do not search or solicit solutions to exam questions online or elsewhere.
- Do not submit as one's own, work that is copied from another student or an outside source (including any online source other than the course Canvas webpage).
- Do not violate any rule as detailed within the Academic Misconduct Policy of CS 2100.
- If you are aware of any efforts to violate these rules (including others communicating with you), it is your duty to report those violations to the instructor. A failure to communicate to the instructor a violation of these rules is itself a first class Academic Misconduct.

Allowable sources include materials provided by the instructor for CS 2100 Spring 2020 at the University of Utah, including:

1. Resources directly linked off of the course Canvas website.
2. Your personal, handwritten notes directly related to resources from item 1.
3. The course textbook: Discrete Mathematics: Mathematical Reasoning and Proof with Puzzles, Patterns, and Games by Douglas E. Ensley and J. Winston Crawley.
4. To solve problems in this exam, nothing should be required beyond the class textbook and resources directly linked off of the course Canvas website.
5. When in doubt, please ask the instructor via Pizza BEFORE using a source other than the ones described in items 1 and 3 above.

Any resource that is not part of the allowable sources is considered an outside source and therefore is not allowed.

By signing below, I pledge that (a) I did not violate any of the above take home final rules, and (b) I fully understand the Academic Misconduct Policy of CS 2100 and I did not violate any additional rules stated within the Academic Misconduct Policy.

Sign Here: ______________________________ UID: ______________

Pleases read these instructions regarding the final exam carefully.
Please make sure that you sign your name and put your UID on the first page.
Please submit your solutions together with the signed pledge page (the 1st page). You submission will not be graded if you do not sign and submit the pledge page. This is a take-home exam.

- The exam is a take-home exam. You are allowed to use *allowable resources* as described on the first page during the exam.
- The exam is to be done independently. Submitting as one's own, work that is copied from another student or an outside source (including an online source other than the course Canvas webpage) is considered academic misconduct. **For academic misconduct in CS 2100, the sanction is to fail the course.**
- This exam is released online via Canvas at 8:00 a.m. on Thursday, April 23, it is due at 7:59 a.m. on Friday, April 24, giving you 24 hours to complete it.
- Please submit the exam via Gradescope on the due date and time.
- The exam takes, on average, 80 to 120 minutes to complete. You can take as long as you want to work on the exam within the given time window (i.e., 24 hours).
- Students with CDS accommodations can turn in exam after 48 hours, on or before Saturday, April 25, at 7:59 a.m. by emailing it as a PDF to TA Don Wang at comidon@outlook.com.
- Please ask any clarifying question regarding the exam via Piazza under "final exam". Please do not post solutions via Piazza.
- To work on the exam, there are a few options (similar to solving homework problems):
  - You can write your answers by hand, using WORD, or using LATEX, then submit a scanned image or a PDF. **We only need your solution to each problem TOGETHER with the pledge page**.
  - You can also download the PDF and use Adobe or Preview (or similar tools) for editing your solutions directly on the PDF file.
  - If you are solving by hand (please use a black pen), pleases submit a scanned version in PDF or a photoed version.
  - **Please do not put more than one problem on each page**.
  - If you need more space to provide a solution to a problem, use a blank piece of paper. Do not use a single paper for more than one problem.
- **Important submission information**: the exam should be submitted via Gradescope as a PDF (preferred) or an image file. The submission process is very similar to the submission of a homework problem:
  - Please match each problem to the outline specified on Gradescope: the page number containing the answer to each individual question should be specified during submission.
  - If no pages are specified for each problem, then 20% is deducted from the final score.
  - **The submission should be legible; any problem that is not legible will not be graded and will not receive any credit.**
- **To be eligible for partial credit, you must show your work.**
- It is the student's responsibility to ensure the successful and timely submission of the exam via Gradescope— start early and follow the instructions carefully. Corrupted or missing files are not grounds for extensions — double-check your submissions and save a digital copy of all of your work in your CADE account.

- For all problems, express your solutions in terms of factorials, exponents, multiplication, division, addition, and/or subtraction. You need not simplify to the actual numerical solution unless specifically required by the problem. However, you may not give final solutions in terms of $P(n,r)$ and/or $C(n,r)$, please provide the actual mathematical formulation.

- For example, in stead of writing $P(10,8)$ as a final answer, write $P(10,2) = 10!/8! = 90$ (when the math is straightforward); instead of writing $P(100,95)$, write at least $P(100,95) = \frac{100!}{(100-95)!} = \frac{100!}{5!}$ (when the exact answer is hard to get without a calculator).

- For mathematical formulation you choose, please provide justifications to be eligible for partial/full credit.

1. **(5 points)** We study a sequence defined as follows:

   1. Start with any positive integer.
   2. Each term is obtained from the previous term as follows:

      2.a If the previous term is even, the next term is one half of the previous term.

      2.b If the previous term is odd, the next term is 3 times the previous term plus 1.

   A conjecture in mathematics states that no matter what value of n, the sequence will always reach 1. So far, this conjecture remains open. A partial proof came out in 2019 that shows that the conjecture is *almost* true for *almost* all numbers.

   Here are some examples of sequences generated following the above conjecture:

   - Starting with 12, you will get the sequence: $12, 6, 3, 10, 5, 16, 8, 4, 2, 1$.
   - Starting with 15, you will get the sequence: 15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

   Solve the following questions:

   (a). **(3 points)** Starting with 7, please give a sequence following the above conjecture until it reaches 1.

   (b). **(2 points)** Let x denote a positive integer. Let f be a function that maps x to the next number in the sequence as described above. For example, if $x = 12$, then $f(x) = 6$; if $x = 15$, then $f(15) = 46$. Describe $f(x)$ mathematically.

   Solution:

2. **(10 points)** We work with a sequence with a recursive formula is as follows,

$$x_0 = x_1 = x_2 = 1;$$

$$x_n = x_{n-2} + x_{n-3}, \quad \forall n \geq 3.$$

The sequence therefore looks like:

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \cdots$$

For example, $x_3 = x_1 + x_0 = 1 + 1 = 2$, $x_4 = x_2 + x_1 = 2$, and $x_5 = x_3 + x_2 = 3$, $x_6 = x_4 + x_3 = 4$, $x_7 = x_5 + x_4 = 5$, etc.
Prove by induction the following statement:

$$\sum_{i=0}^{n} x_i = x_{n+5} - 2, \quad \forall n \geq 0$$

Solution:

3. **(10 points)** Using the Properties of Boolean Algebra, prove the following Boolean equality:

$$(b' + a) \cdot (b + c) \cdot (a + c') = a \cdot (b + c).$$

Please specify the rules used during each step.

Solution:

4. **(10 points)**

   (a). **(5 points)** Let P, Q and R be sets, P', Q' and R' denote their complements respectively. Prove that $P' \times Q' \times R' \subseteq (P \times Q \times R)'$.

   (b). **(5 points)** Let A be a set, A' be its complement, and $\mathcal{P}(A)$ be its power set. Prove that $\mathcal{P}(A') - \{\emptyset\} \subseteq (\mathcal{P}(A))'$.

   Solution:

5. **(10 points)** Functions and relations.

(a). **(3 points)** A constant function is a function whose (output) value is the same for every input value. For example, $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = 5$ is a constant function; $f = \{(1,1), (2,1), (3,1)\}$ is also a constant function.
Please give an example of a constant function which is both a one-to-one function and an onto function. If such an example does not exist, please explain why.

(b). **(3 points)** Function f is describe by a set of pairs formed between its domain and codomain $f = \{(1,2), (2,1), (3,1), (4,4)\}$. Find the composition function $f \circ f \circ f$.

(c). **(4 points)** Recall the transitive closure of a binary relation R on a set A is the smallest relation on A that contains R and is transitive. A relation Q on $A = \{1,2,3,4\}$ is given as $Q = \{(1,3), (1,4), (3,3), (3,2), (3,4)\}$. Find Q's transitive closure, denoted as Q^*.

Solution:

6. **(5 points)** A baseball team contains 9 players. We need to form a baseball team from 20 candidates. How many different ways can we form a team of 9 players if:

   (a). **(1 point)** We put no restriction on the selection process.

   (b). **(2 point)** A particular candidate has to be on the team.

   (c). **(2 point)** A particular candidate is never on the team.

   Please provide justifications to your solution.

   Solution:

7. **(10 points)** Probabilities with a deck of cards. There are 52 cards in a standard deck of cards. There are 4 suits (Clubs, Hearts, Diamonds, and Spades) and there are 13 cards in each suit. Clubs/Spades are black, Hearts/Diamonds are red. There are 12 face cards. Face cards are those with a Jack (J), King (K), or Queen (Q) on them. For this question, we will consider the Ace (A) card to be a number card (i.e., number 1). Then for each suit, there are 10 number cards.

   Recall that without replacement means the card **IS NOT** put back into the deck. With replacement means the card **IS** put back into the deck.

   (a). **(5 points)** What is the probability of drawing a red King, then a black Jack, followed by a red number card without replacement?

   (b). **(5 points)** In a poker game, a *royal flush* is a hand consisting of the cards A, K, Q, J, 10 of the same suit. Two players A and B are playing poker with a deck of cards. First, A is dealt a hand (i.e. 5 cards) from the deck without replacement. Then B is dealt a hand (i.e., 5 cards) from the remaining deck without replacement. What is the conditional probability that B also gets a royal flush given that A has a royal flush?

   Solution:

8. **(10 points)**

(a). **(5 points)** We play the following game: throw a pair of dice 20 times, what is the probability of getting at least one double 6?

(b). **(5 points)** A group of 10 friends are playing a game of guessing the outcome of a tournament played among 8 teams (labeled team 1 to team 8) with a fixed schedule as follows.

In round 1, team 1 plays against teams 2, team 3 plays against team 4, team 5 plays against team 6, and team 7 plays against team 8. In round 2, the winner of (1,2) plays against the winner of (3,4), the winner of (5,6) plays against the winner of (7,8). In round 3, the winners of round 2 plays against each other.

The person who correctly predicts the outcome of all 3 rounds wins the game. The idea is that each person will put \$2 forward (with a total of \$20). What is the expected amount of money a person will lose or gain by playing this game?

Solution:

9. **(10 points)**

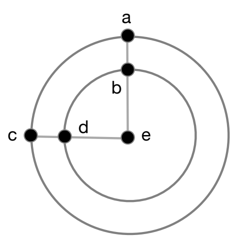
(a). Is graph G_1 an Eulerian graph (Yes/No)? And Why?

(b). Is there an Eulerian trail in G_1 (Yes/No)? And Why?

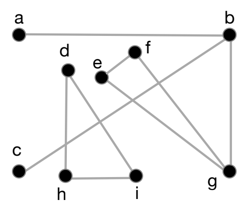
(c). Find the connected components of the graph G_2. Describe each connected component by its vertex set.

(d). Is graph G_3 planar (Yes/No)? If yes, draw it on the plane so that none of its edges cross. If no, explain why.

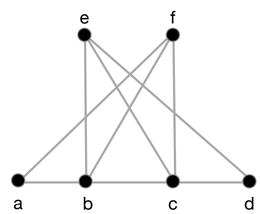
(e). Kuratowski's theorem states that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or of $K_{3,3}$ (complete bipartite graph on six vertices, three of which connect to each of the other three). Use Kuratowski's theorem to decide: is G_4 planar (Yes/No)? And why?



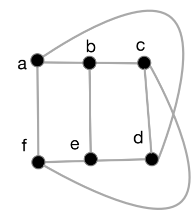
G_1



G_2



G_3



G_4

Solution:

10. **(10 points)** Nancy is learning about graph theory. She tried to prove the following claim using induction.

Claim: Every simple, connected graph H with at least 3 vertices and whose vertices all have degree at least 2 contains a cycle of length 3.

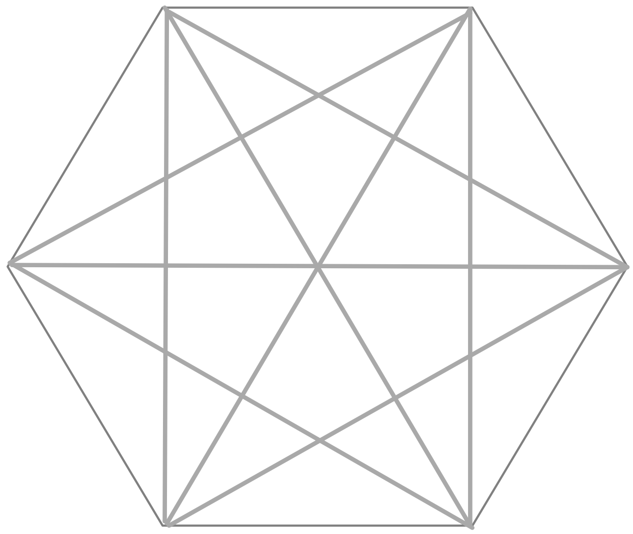
Nancy's induction proof is as follows:
Problem statement: Let $P(n)$ denote the statement that the above claim is true for any graph H with n vertices where $n \geq 3$ and every vertex in H has degree ≥ 2.
Base case: We need to show $P(3)$ holds, that is, when $n = 3$, the claim is true. By going over two possible cases of simple, connected graphs containing 3 vertices, we can easily see that any simple connected graph with 3 vertices with all vertices with degree ≥ 2 must be a cycle of length 3.
Induction hypothesis: Assume $P(m-1)$ is true. That is, the claim holds for any simple, connected graph with $m-1$ vertices.
Induction step: To prove $P(m)$ is true, let H be a graph on $m-1$ vertices for which the claim holds. Now we can construct a new graph H' by adding one new vertex u to H, such that u is connected to at least 2 vertices in H. Since H contains a cycle of length 3, the newly constructed graph H' also contains a cycle of length 3. This concludes the proof.

(a). **(5 points)** There is a mistake in Nancy's proof, making it invalid. Please detect and describe her mistake.

(b). **(5 points)** Is the claim generally true (Yes/No)? If no, please provide a counterexample. If yes, please provide a new inductive proof.

Solution:

11. **(10 points)** Given the following graph G, we want to color each edge either red or green. Prove that no matter how we color the edges, there must be a triangle of red edges, a triangle of green edges, or both.
Hint: consider that every vertex in G is connected to 5 other vertices via 5 edges, and how many of these edges will have the same color. Consider applying Pigeonhole principles (PHP).



Solution: