Assignment: Dimensionality Reduction

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```
import numpy

A = numpy.loadtxt('./data/A.csv', delimiter=',')
print(A.shape)
```

```
(3500, 20)
```

Problem 1

Part A

```
from scipy import linalg

U, S, Vt = linalg.svd(A)
S = numpy.identity(len(S)) * S

def Ak(k):
    Uk = U[:,:k]
    Sk = S[:k,:k]
    Vtk = Vt[:k,:]
    return Uk @ Sk @ Vtk

K = tuple(range(1, 11))
```

```python
for k in K:
    print(f'k = {k}: {linalg.norm(A - Ak(k), 2)}')
```

```
k = 1: 100.00246446709691
k = 2: 92.11139537014573
k = 3: 87.43842799887248
k = 4: 70.84270143876726
k = 5: 58.89263665883072
k = 6: 57.77458152758134
k = 7: 25.86135552441823
k = 8: 24.871609868369216
k = 9: 24.395000417106157
k = 10: 22.997273663616927
```

Part B

```python
norm_A = linalg.norm(A, 2)

K = tuple(range(1, 21))
for k in K:
    if linalg.norm(A - Ak(k), 2) < .2 * norm_A:
        print(k)
        break
```

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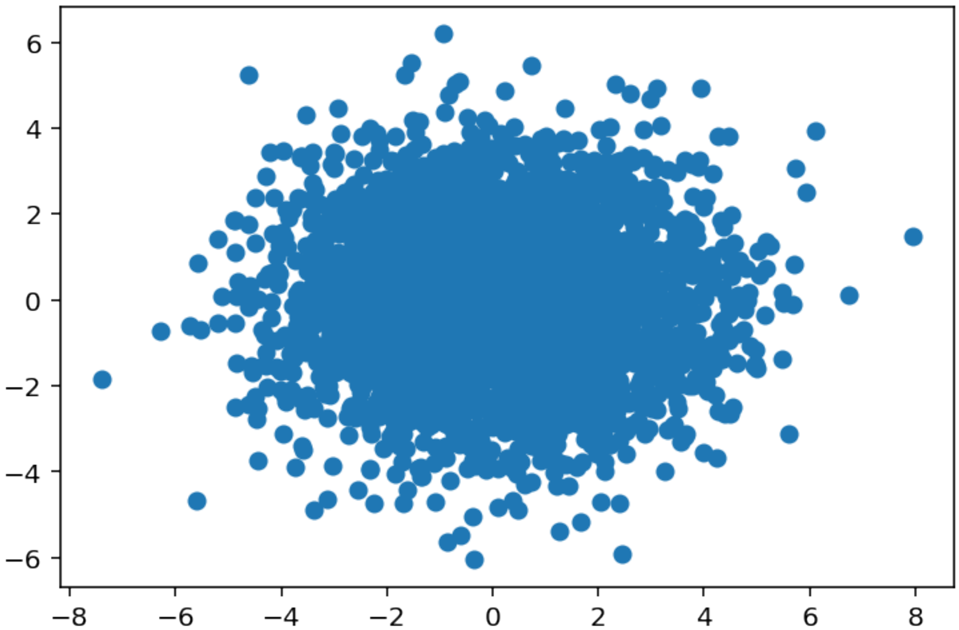
```

Part C

```python
%config InlineBackend.figure_format = 'retina'

from matplotlib import pyplot
from sklearn import decomposition

A2 = decomposition.PCA(2).fit_transform(A)
_ = pyplot.scatter(A2[:, 0], A2[:, 1])
```



I applied the PCA method from the lecture that found a subspace B minimizing $\mathsf{SSE}(A, B)$. The algorithm first centered the data points and then performed SVD. I made use of an existing implementation for this method from SciKit-Learn (https://scikit-learn.org/stable/modules/generated/sklearn.decomposition.PCA.html).

Problem 2

Part A

- How large does l need to be for the above error to be at most $||A||_F^2/20$?

```
from lib import FD

def find_min_l(A, err_bound):
    n, d = A.shape
    for l in range(1, n // 2):
        B = FD.freq_dir(A, l)
        err = linalg.norm(A.T @ A - B.T @ B)
        if err <= err_bound:
            return l, err

l, err = find_min_l(A, linalg.norm(A)**2 / 20)
print(l)
```

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```

- How does this compare to the theoretical bound (e.g. for $k = 0$)?

```
print(f'||Ax||^2 - ||Bx||^2:\t\t\t\t{err}')
for k in range(l):
    print(f'||A - Ak||_F^2 / (l - k) [with l = {l}, k = {k}]:'
          f'\t{linalg.norm(A - Ak(k))**2 / (l - k)}')
```

```
||Ax||^2 - ||Bx||^2:                              2467.226247591852
||A - Ak||_F^2 / (l - k) [with l = 7, k = 0]:    8075.627550160504
||A - Ak||_F^2 / (l - k) [with l = 7, k = 1]:    7175.904154353963
||A - Ak||_F^2 / (l - k) [with l = 7, k = 2]:    6610.98640532616
||A - Ak||_F^2 / (l - k) [with l = 7, k = 3]:    6142.605717398873
||A - Ak||_F^2 / (l - k) [with l = 7, k = 4]:    5641.648059560493
||A - Ak||_F^2 / (l - k) [with l = 7, k = 5]:    5953.127915769583
||A - Ak||_F^2 / (l - k) [with l = 7, k = 6]:    8437.913178910114
```

The tightest bound occurred when $k = 4$ (5642), but the bounding expression

$$||Ax||^2 - ||Bx||^2 \leq ||A - A_k||_F^2/(l - k)$$

still holds true.

- How large does l need to be for the above error to be at most $||A - A_k||_F^2/20$ (for $k = 2$)?

```
k = 2
l, err = find_min_l(A, linalg.norm(A - Ak(k))**2 / 20)
print(l)
```

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```

Appendix

lib/FD.py

```
import numpy
from scipy import linalg

def freq_dir(A, l):
    n, d = A.shape
    B = numpy.zeros((2 * l, d))
    zero_index = 0
    for i in range(n):
        B[zero_index] = A[i]
        zero_index += 1
        if zero_index == 2 * l: # full
            U, S, Vt = linalg.svd(B)
            delta = S[-1]**2
            S_prime = numpy.zeros((2 * l, d))
            numpy.fill_diagonal(S_prime,
                                [(sigma**2 - delta)**.5 for sigma in S])
            B = S_prime @ Vt
            for zero_index in range(2 * l):
                if not B[zero_index].any(): break # the first empty row
    return B
```