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Cubical Sperner Lemmas as Applications of Generalized Complementary Pivoting

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Two cubical versions of Sperner's lemma, due to Kuhn and Fan, are proved constructively without resorting to a simplicial decomposition of the cube, presenting examples of generalized complementary pivoting discussed by Todd (*Math. Programming* **6** (1974)). The first version is essentially equivalent to Sperner's lemma in that it implies Brouwer's fixed point theorem, thereby answering a question raised by Kuhn (*IBM J. Res. Develop.* **4** (1960)). The second has the property that although the structure is that of generalized complementarity, there is a uniquely defined path or algorithm associated with it. The basic structure used is a cubical decomposition of the cube, a special case of a cubical pseudomanifold, presented by Fan (*Arch. Math.* **11** (1960)). Given the existence of a constructive algorithm for Sperner's lemma (see Cohen, *J. Combinatorial Theory* **2** (1967)) and its generalization by Fan *J. Combinatorial Theory* **2** (1967)) allied to the large amount of recent progress in complementary pivot theory, resulting in particular from the works of Lemke (*Manage. Sci.* **11** (1965)) and Scarf ("The Computation of Economic Equilibria") the computational attractions of a simplicial decomposition have become apparent. However, a cubical decomposition leads to certain advantages when a search for more than one "completely labeled" region is required, and no simplicial construction for the Fan lemma is known.

INTRODUCTION

Throughout this article we shall consider a cube in n-space $\Gamma_p = \{x \in R^n \mid 0 \leqslant x_i \leqslant p\}$, where p is a positive integer, divided up into p^n unit cubes, having the set Δ_p of $(p+1)^n$ points with integer coefficients between 0 and p as vertices.

We shall define two types of labelings:

(i) An *integer labeling* is a function $l\colon \Delta_p \to \{1, 2,..., n+1\}$. It is a *proper* integer labeling if for $z \in \Delta_p$: $l(z) \leqslant i$ if $z_i = 0$ and $l(z) \neq i$ if $z_i = p$.

(ii) A 0–1 *labeling* is a function $l\colon \Delta_p \to \Delta_1$. It is a *proper* 0–1 labeling if for $z \in \Delta_p$: $l_i(z) = 0$ if $z_i = 0$, $l_i(z) = 1$ if $z_i = p$. It has the *neighborhood* *property* if z and z' being neighbors in Δ_p implies that $l(z)$ and $l(z')$ are neighbors or are identical in Δ_1, i.e., $\sum_{i=1}^{n} | l_i(z) - l_i(z')| \leqslant 1$ whenever $\sum_{i=1}^{n} | z_i - z_i' | = 1$.

Now the two weak cubical versions of Sperner's lemma can be given. Below stronger versions of both will be proved constructively.

LEMMA 1 [4]. *Given a proper integer labeling of Γ_p, there exists a unit cube having all the labels $\{1, 2,..., n + 1\}$ among the labels of its vertices.*

LEMMA 2. *Given a proper 0–1 labeling of Γ_p with the neighborhood property, there exists a unit cube whose vertices are mapped by l onto Δ_1. (The stronger form of this result, see Lemma 2', is due to Fan* [2].)

Below we shall use the fact that the $2n$ faces of an n-cube are $(n - 1)$-cubes; a 0-cube is a point, a 1-cube is a line segment with two 0-cubes as endpoints (faces), etc.

Given a unit k-cube, we shall denote by L the set of 2^k not necessarily distinct labels associated with its vertices.

1. INTEGER LABELING

DEFINITION 1. Using an integer labeling, a unit 0-cube is *completely labeled* (CL) if $L = \{1\}$. A unit k-cube is *completely labeled* if $L = \{1, 2,..., k + 1\}$ and it has an odd number of CL $(k - 1)$-faces. A k-cube is *almost completely labeled* (ACL) if $L \subsetneq \{1, 2,..., k + 1\}$ and it has a CL $(k - 1)$-face.

LEMMA 1'. *Given a proper integer labeling of $\Gamma_p \in R^n$, there exist an odd number of completely labeled n-cubes.*

As a CL n-cube necessarily has all the labels $\{1, 2,..., n + 1\}$ we see that Lemma 1' is stronger than Lemma 1.

The fundamental result in the proof of Lemma 1' is the following proposition.

PROPOSITION 1. *Given a proper integer labeling an ACL n-cube that is not CL contains an even number of CL $(n - 1)$-faces.*

Proof. By Definition 1 an ACL n-cube that has $n + 1$ among its labels, and is not CL, has an even number of CL $(n - 1)$-faces. Therefore we only need consider the case where $n + 1 \notin L$. This will be proved by induction on the dimension $q = n$.

For $q = 1$, the set of possible labels is $\{1, 2\}$. An ACL 1-cube that is not CL necessarily has a label 1 on both endpoints, and hence contains exactly two CL 0-faces.

Assume the proposition holds for $q = k - 1$. Consider a k-cube Q which is ACL but not CL. We shall show that all its CL($k - 1$)-faces can be paired off by constructing disjoint paths of CL ($k - 2$)-faces between them. The number of such faces is then clearly even.

Remark 1. Given an ACL ($k - 1$)-face of Q containing some CL ($k - 2$)-face there is exactly one other ($k - 1$)-face of Q containing the CL ($k - 2$)-face. By definition this ($k - 1$)-face is also ACL.

Remark 2. The algorithm below will generate a path $\{S^0, T^0, S^1, T^1, \ldots, S^i, T^i, S^*\}$ consisting of an alternating sequence of ACL ($k - 1$)-faces $\{S^i\}$ and distinct CL ($k - 2$)-faces $\{T^i\}$ beginning and ending with distinct CL ($k - 1$)-faces.

The k-cube Q is ACL and hence contains at least one CL ($k - 1$)-face. Choose any such face, denoted S^0.

Step 0. Let T^0 be some CL ($k - 2$)-face of S^0 (of which there are an odd number by the induction hypothesis). Set $i = 0$.

Step 1. Let $S^{i+1} \neq S^i$ be the ACL ($k - 1$)-face of Q containing T^i (Remark 1).

Step 2. If S^{i+1} does not contain a CL ($k - 2$)-face distinct from $\{T^0, T^1, \ldots, T^i\}$, go to Step 4.

Step 3. Let T^{i+1} be some CL ($k - 2$)-face of S^{i+1} distinct from $\{T^0, T^1, \ldots, T^i\}$. Set $i \to i + 1$, and go to Step 1.

Step 4. Let $S^* = S^{i+1}$. Termination in this state must occur, as the T^j are distinct and finite in number. We claim that S^* is a CL ($k - 1$)-face distinct from S^0.

Consider the other possibilities.

(i) If $S^* = S^0$, $\{T^0, T^1, \ldots, T^i\}$ contains an even number of distinct CL ($k - 2$)-faces lying in S^0 (T^0, T^i, and two for every other visit). The total number of such faces is odd as S^0 is CL, contradicting the move at Step 2.

(ii) If S^* is ACL but not CL, $\{T^0, T^1, \ldots, T^i\}$ contains an odd number of distinct CL ($k - 2$)-faces lying in S^* (T^i, and two for each previous visit to S^*). However, from the induction hypothesis S^* contains an even number of such faces, again contradicting the move at Step 2.

Step 5. Suppose Q contains other CL ($k - 1$)-faces distinct from S^0 and S^*. Take any of these faces S^{00} and repeat the above procedure using CL ($k - 2$)-faces distinct from the path $\{T^0, T^1, \ldots, T^i\}$ already generated. Again this procedure must terminate at a CL ($k - 1$)-face $S^{**} \neq S^{00}$. Also $S^{**} \neq S^0$ or S^*, because by construction S^0 has an even number of CL ($k - 2$)-faces distinct from $\{T^0, T^1, \ldots, T^i\}$, and S^* has none.

This continues till all the CL $(k - 1)$-faces have been paired off, and Q has therefore an even number of CL $(k - 1)$-faces.

Proof of Lemma 1′. Again by induction on the dimension. For $q = 1$ the lemma is equivalent to Sperner's lemma on a simplex.

Assume true for $q = n - 1$. We note that the only face of Γ_p containing CL $(n - 1)$-faces is the face $x_n = 0$. This face is an $(n - 1)$-cube with a proper integer labeling, and hence by the induction hypothesis contains an odd number of CL $(n - 1)$-cubes.

Starting from one such CL $(n - 1)$-cube T^0 a path alternating between distinct CL $(n - 1)$-cubes, and ACL n-cubes can be constructed, terminating either in some other CL $(n - 1)$-cube T^j on $x_n = 0$, or in a CL n-cube (as a consequence of Proposition 1).

The only other possible paths must begin and end in distinct CL n-cubes, and hence the number of CL n-cubes is necessarily odd.

EXAMPLE. Cubical Sperner Lemma 1. $n = 2$.
Figure 1 indicates the steps of the algorithm.

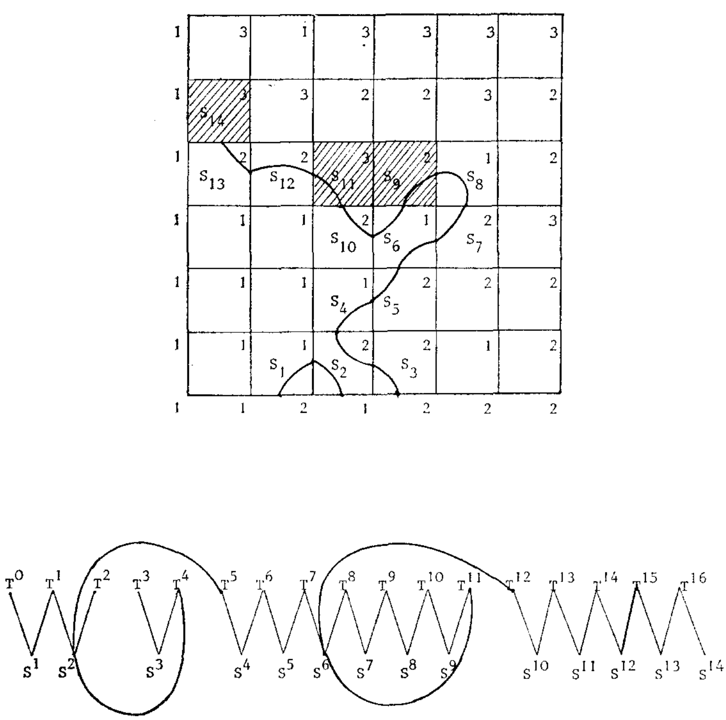

FIGURE 1

Note that the two shaded cubes S^9 and S^{11} have all three labels, but are not completely labeled. Hence the algorithm can pick up more than one cube satisfying the conditions of Lemma 1. In higher dimensions the path can even continue on through completely labeled cubes if they contain more than one completely labeled face. (In comparison with [3, Figs. 3 and 4] where each room has zero, one, or two doors, we see that S_2, S_6, S_8 have four doors.)

2. 0–1 Labeling

We can describe a k-face of the unit cube $\Delta_1 \in R^n$ by: $\Delta_1(j_1, \delta_1;...;j_{n-k}, \delta_{n-k}) = \{z \in \Delta_1 \mid z_{j_1} = \delta_1, ..., z_{j_{n-k}} = \delta_{n-k}\}$ where the δ_i's are either 0 or 1.

Definition 2. Using 0–1 labeling a k-cube is *completely labeled* (CL) if its vertices are mapped by l *onto* a k-face of Δ_1 of the form: $\Delta_1(k+1, \delta_{k+1};...;n, \delta_n)$.

A k-cube is *almost completely labeled* (ACL) if it has a CL $(k-1)$-face with the kth coefficients equal to zero (i.e., a face $\Delta_1(k, 0; k+1, \delta_{k+1};...;n, \delta_n)$).

Lemma 2′. *Given a proper* 0–1 *labeling with the neighborhood property of* $\Gamma_p \in R^n$, *there exists an odd number of completely labeled n-cubes.*

Remark 3. Given a 0–1 labeling with the neighborhood property, a k-cube which is CL contains exactly two CL $(k-1)$-faces, one having all $l_k = 0$, the other all $l_k = 1$.

On the other hand it follows immediately from the neighborhood property that a k-cube which is ACL but not CL does not contain any CL $(k-1)$-face with $l_k = 1$. Hence a k-cube which is ACL and contains a second CL $(k-1)$-face with $l_k = 1$ is CL.

Remark 4. Every $(k-1)$-face of an n-cube lies in exactly two k-faces of the n-cube. If the $(k-1)$-face is CL with $l_k = 0$, the two k-faces are ACL.

Proposition 2. *Given a* 0–1 *labeling with the neighborhood property an ACL n-cube that is not CL contains an even number of CL* $(n-1)$*-faces with the* nth *coefficient* l_n *equal to zero.*

Proof. See Fan [2]. Here we give a constructive proof using induction on the dimension $q = n$.

For $q = 1$, an ACL 1-cube necessarily has one vertex with label $(0, l_2, l_3, ..., l_n)$ where $l_i = 0$ or 1. The other vertex either has label $(1, l_2, l_3, ..., l_n)$, in which case the 1-cube is CL, or it is of the form $(0, l_2', l_3', ..., l_n')$ with $\sum_{i=2}^{n} | l_i - l_i' | \leqslant 1$, and it contains two of the required faces.

Assume the proposition holds for $q = k - 1$.

Consider a k-cube Q which is ACL but not CL. Let S^0 be a CL $(k-1)$-face of Q with $l_k = 0$.

Step 0. Let T^0 be the unique CL $(k-2)$-face of S^0 with $l_{k-1} = 0$ (Remark 1). Set $i = 0$.

Step 1. Let $S^{i+1} \neq S^i$ be the other ACL $(k-1)$-face of Q containing T^i (Remark 2).

Step 2. Either (a) S^{i+1} is a CL $(k-1)$-face of Q with $l_k = 0$ ($S^{i+1} \neq S^0$ by the uniqueness of T^0) (go to Step 4); or (b) $S^{i+1} \neq S^i$ is ACL, but not CL, and hence (by induction) contains an odd number of CL $(k-2)$-faces with $l_{k-1} = 0$ distinct from $\{T^0, T^1,..., T^i\}$ (go to Step 3).

Step 3. Let T^{i+1} be some CL $(k-2)$-face of S^{i+1} with $l_{k-1} = 0$ distinct from the sequence $\{T^0, T^1,..., T^i\}$. Set $i \to i + 1$ and return to Step 1.

Step 4. Termination in this state must occur as the T^j are distinct and finite in number. S^0 and $S^* = S^{i+1}$ are two distinct CL $(k-1)$-faces of Q with $l_k = 0$. (Note that neither S^0 nor S^* contains any CL $(k-2)$-faces with $l_{k-1} = 0$ distinct from $\{T^0, T^1,..., T^i\}$ (see Remark 1).)

Step 5. Suppose that Q contains other CL $(k-1)$-faces with $l_{k-1} = 0$. Starting from one such face, construct a new path of CL $(k-2)$-faces with $l_{k-1} = 0$ disjoint from the previous paths. Each such path will produce a pair of distinct CL $(k-1)$-faces of Q.

Proof of Lemma 2′. Again by induction on the dimension. For $q = 1$ the lemma is equivalent to the simplicial Sperner lemma.

Assume the lemma is true for $q = n - 1$. The only face of Γ_p containing CL $(n-1)$-faces with $l_n = 0$ is the face $x_n = 0$ This face is an $(n-1)$-cube with a proper 0–1 labeling, and hence by the induction hypothesis contains an odd number of CL $(n-1)$-cubes with $l_n = 0$.

Starting from one such CL $(n-1)$-cube T^0 with $l_n = 0$, a path alternating between distinct CL $(n-1)$-cubes with $l_n = 0$, and ACL n-cubes can be constructed, terminating either in some other CL $(n-1)$-cube T^j with $l_n = 0$ on $x_n = 0$, or in a CL n-cube (by Proposition 2).

Therefore the number of CL n-cubes is necessarily odd.

Remark 5. A surprising feature of Lemma 2′ is that even though this is a case of generalized complementary pivoting (i.e., the choice of T^{i+1} at Step 3 is not always unique), there is a well-defined algorithm which produces a *uniquely defined* path for arriving at a CL n-cube. This is due to the fact that there is an imbedding of the cubical structure for dimensions 0 through n.

PROPOSITION 3. *Given a* 0–1 *labeling with the neighborhood property of* $\Gamma_p \in R^n$, *there is a well-defined algorithm which starts from a CL*$(n-1)$*-face* *with* $l_n = 0$ *of an ACL n-cube, and either finds a second such face, or demonstrates that the cube is CL.*

Proof. By induction on the dimension of the cubes. For $q = 1$, the given CL $(n - 1)$-face has label $(0, l_2, ..., l_n)$.

ALGORITHM. Look at the label of the other vertex of the ACL 1-cube. Either we have $(0, l_2', ..., l_n')$ with $\sum_{i=2}^{n} | l_i - l_i' | \leqslant 1$ and the second face is found, or we have $(1, l_1, l_2, ..., l_n)$ and the 1-cube is CL. END.

Suppose the induction hypothesis holds for $q = k - 1$. Now consider the case $q = k$.

Let Q be an ACL k-cube having a CL $(k - 1)$-face S^0 with $l_k = 0$. (We essentially repeat the steps of Proposition 2).

Step 0. (Unchanged.)

Step 1. (Unchanged.)

Step 2. S^{i+1} is an ACL $(k - 1)$-face of Q containing T^i, a CL $(k - 2)$-face with $l_{k-1} = 0$. By the induction hypothesis there exists a well-defined algorithm (path) which

(a) finds a CL $(k - 2)$-face with $l_{k-1} = 0$, i.e., T^{i+1} (continue as in Step 3), or

(b) shows that S^{i+1} is CL.

(i) Either S^{i+1} is a CL $(k - 1)$-face with $l_k = 0$. Q is not CL. A face of the required type has been found. Stop. Or,

(ii) S^{i+1} is a CL $(k - 1)$-face with $l_k = 1$. Now Q is shown to be a CL k-cube (Remark 1). Stop.

The result is shown for $q = k$.

Remark 6. As an immediate consequence of Proposition 2 there is a well-defined algorithm for finding a CL n-cube.

EXAMPLE. Cubical Sperner Lemma 2, $n = 2$, $p = 3$. We denote by iterations the step from one CL $(n - 1)$-cube to another.

Iteration 0. Beginning at A we require that a CL 1-cube with $l_2 = 0$ on AD be found. Start from $S^0 = A$ a CL 0-cube with $l_1 = 0$ within the ACL 1-cube AB. Call algorithm $k = 0$ to find a second such CL 0-cube, or show AB is CL. AB is a CL 1-cube.

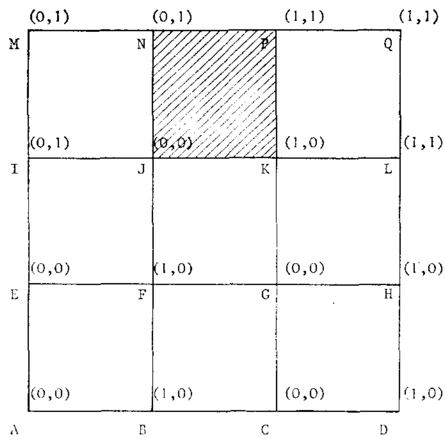

FIGURE 2

Iteration 1. Start from $S^0 = AB$ a CL 1-cube with $l_2 = 0$ within the ACL 2-cube $ABEF$. Call algorithm $k = 1$ to find a second such CL 1-cube or show $ABEF$ is CL.

$$T^0 = A \text{ is a CL 0-cube with } l_1 = 0,$$
$$S^1 = AE \text{ is a CL 1-cube containing } T^0.$$

Call algorithm $k = 0$.

$$T^1 = E \text{ is a CL 0-cube with } l_1 = 0 \text{ in } S^1,$$
$$S^2 = EF \text{ is an ACL 1-cube containing } T^1.$$

Call algorithm $k = 0$.

$$S^2 = EF \text{ is a CL 1-cube.}$$

As $l_2 = 0$, $ABEF$ is not CL, and we have:

Iteration 2. $S^0 = EF$ is a CL 1-cube with $l_2 = 0$ within the ACL 2-cube $EFIJ$. Call algorithm $k = 1$ to find:

$$S^1 = EI \text{ is an ACL 1-cube containing } T^0 = E.$$

Call algorithm $k = 0$:

$$T^1 = I,$$
$$S^2 = IJ.$$

Call algorithm $k = 0$.

$$T^2 = J,$$
$$S^3 = JF.$$

Call algorithm $k = 0$:

$$S^3 = JF \text{ is a CL 1-cube.}$$

As $l_2 = 0$, $EFIJ$ is not CL.

Iteration 3. $S^0 = JF$ is a CL 1-cube with $l_2 = 0$ within the ACL 2-cube $FGJK$. Call algorithm $k = 1$ to find:

$$S^1 = JK \text{ is an ACL 1-cube containing } T^0 = J.$$

Call algorithm $k = 0$:

$$S^1 = JK \text{ is a CL 1-cube.}$$

As $l_2 = 0$, $FGJK$ is not CL.

Iteration 4. $S^0 = JK$ is a CL 1-cube with $l_2 = 0$ within the ACL 2-cube $JKNP$. Call algorithm $k = 1$ to find:

$$S^1 = JN \text{ is an ACL 1-cube containing } T^0 = J.$$

Call algorithm $k = 0$:

$$T^1 = N,$$
$$S^2 = NP.$$

Call algorithm $k = 0$:

$$S^2 = NP \text{ is a CL 1-cube.}$$

As $l_2 = 1$, $JKNP$ is CL. END.

We note that a new vertex must be inspected for its label each time the algorithm with $k = 0$ is called.

3. Conclusions

Todd [7, p. 258] has recently defined generalizations of complementary pivoting, suggesting that no practical applications of these structures were known. The structure presented in Sections 1 and 2 would represent the standard case if, in the respective propositions, one could replace the word 'even" by "two." However, the examples show that this is clearly impossible. Alternatively in the language of Fan [3, p. 596], rooms have an even or odd number of doors rather than 0, 1, 2.

Evidently the question arises as to the relation between the cubical and simplicial versions of the above theorems. The following relations are easily shown, and are natural after the fact.

LEMMA 3. (Integer Labeling). *Take a simplicial decomposition of an n-cube into $n!$ simplexes (i.e., [4]). A CL n-cube contains an odd number of simplexes with labels $\{1, 2, \ldots, n+1\}$. An ACL n-cube that is not CL contains an even number of such simplexes.*

LEMMA 4. (0–1 *Labeling with the Neighborhood Property). Given a simplicial decomposition of Γ_p with Δ_p as the vertices, there exists at least one simplex whose vertices $\{I_0 \leqslant I_1 \leqslant \cdots \leqslant I_n = I_0 + e\}$ are mapped in such a fashion that $l(I_0) = e - l(I_n)$, where $e = (1, 1, \ldots, 1)^T$.*

Computationally the advantages of cubical (over simplicial) decomposition for Sperner's Lemma 1 appear to be negative, as at some stage 2^n rather than $n + 1$ labels should be examined, and bookkeeping may be needed to indicate which ACL faces have already been used. The unique path approach of the theorem of Section 2 may break down due to the existence of ACL k-cubes with labels $\{1, 2, \ldots, k + 1\}$ which are not CL. However, as the example shows the paths can pick up several cubes of the required type.

For Sperner's Lemma 2 there is no exact simplicial analog, but given a function f used to define the labels l, it is not clear what conditions on the function f imply the neighborhood property on the label l. The theorem does suggest that in other cases of generalized complementary pivoting one may well find associated well-defined algorithms.

More generally, other applications of generalized complementary pivoting, perhaps as a way of systematically exploring parameter space for the linear complementary problem, appear to merit further investigation.

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