Q1 $\quad u \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x} \right) + \varphi$

$L = 1$ $\quad \phi(0) = 1$

$\phi(L) = 0$

No. of grid points $= n$

$$\boxed{dx = \frac{L}{n-1}}$$

a. i. Break down the derivates in terms of finite diff slopes.

$$\left.\frac{\partial \phi}{\partial x}\right|_{\phi_i} = \frac{\phi_i - \phi_{i-1}}{h} \quad \text{(Backward Diff)}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{\phi_i} = \frac{\phi_{i+1} - \phi_i}{h} \quad \text{(Forward Diff)}$$

$$\left.\frac{\partial \phi}{\partial x}\right|_{\phi_i} = \frac{\phi_{i+1} - \phi_{i-1}}{2h} \quad \text{(Central Diff)}$$

$$\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{\phi_i} = \frac{\phi_{i+1} + \phi_{i-1} - 2\phi_i}{h^2} \quad \text{(Central Diff Method)}$$

ii. Replace those finite diff. terms in the governing physical eqn

You obtain eq^n of the form:

$$a \cdot \phi_{i+1} + b \cdot \phi_i + c \cdot \phi_{i-1} = S_i$$

iii. For $n-1$ divisions $\rightarrow n$ nodes $\rightarrow$ 2 BCs. $\Rightarrow$ $n-2$ unknown ϕ_i

$n-2$ eqns.

iv. Form a matrix eqⁿ to represent the $n-2$ eqⁿ.

$$A\Phi = B$$

$A : (n-2) \times (n-2)$

$\Phi : (n-2) \times 1$

$B : (n-2) \times 1$

v. Solve for ϕ :

In general $\Phi = A^{-1}B$.

For special forms of A such as TDM $\Rightarrow$ Apply TDMA.

vi. TDMA : • Obtain UTM from A & correspondingly modified B.
• Perform Back substitution to obtain solⁿ matrix Φ.

6. $u=1 \qquad \tau=0.1 \qquad Q=0.$

$$\Rightarrow \frac{\partial \phi}{\partial x} = 0.1\,\frac{\partial^2 \phi}{\partial x^2}$$

$$\Rightarrow \frac{\phi_{i+1}-\phi_{i-1}}{2h} = 0.1\left(\frac{\phi_{i+1}+\phi_{i-1}-2\phi_i}{h^2}\right)$$

$$\Rightarrow \phi_{i+1}\left(\frac{0.1}{h^2}-\frac{1}{2h}\right)+\phi_i\left(-\frac{2}{h^2}\right)+\phi_{i-1}\left(\frac{0.1}{h^2}+\frac{1}{2h}\right)$$

$i:\ 1 \to n-1$

$$a\phi_{i+1}+b\phi_i+c\phi_{i-1}=0$$

$i=1 \quad a\phi_2+b\phi_1+c\phi_0=0 \Rightarrow a\phi_2+b\phi_1=-c\phi_0$

$i=2 \quad a\phi_3+b\phi_2+c\phi_1=0$

$i=3 \quad a\phi_4+b\phi_3+c\phi_2=0$

$\vdots \qquad \vdots$

$i=n-2 \quad a\phi_{n-1}+b\phi_{n-2}+c\phi_{n-3}=0$

$i=n-1 \quad a\phi_n+b\phi_{n-1}+c\phi_{n-2}=0 \Rightarrow b\phi_{n-1}+c\phi_{n-2}=-a\phi_n$

$$\underset{A}{\begin{bmatrix} b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ 0 & c & b & a & 0 \\ 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 0 & c & b & a \\ 0 & 0 & 0 & c & b \end{bmatrix}} \underset{\phi}{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \phi_{n-2} \\ \phi_{n-1} \end{bmatrix}} = \underset{B}{\begin{bmatrix} -c\phi_0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -a\phi_n \end{bmatrix}}$$

A is a Tri-Diagonal Matrix

- For A to UTM: $k = \dfrac{A[i][i-1]}{A[i-1][i-1]}$

$i: 1 \to n-1$ $A[i][i] = A[i][i] - k \cdot A[i-1][i]$

- Change in B:

$i: 1 \to n-1$ $B[i][0] = B[i][0] - k \cdot B[i-1][0]$

- You obtain UTM U & modified B. B'

$U\phi = B'$

$\phi = w$ from Back substitution

c) $0.1 \frac{\partial}{\partial x}\left[(1+\phi)\frac{\partial \phi}{\partial x}\right] = -Q$

$$\Rightarrow \frac{0.1}{\Delta x}\left[\left((1+\phi)\frac{\partial \phi}{\partial x}\right)_{i+1/2} - \left((1+\phi)\frac{\partial \phi}{\partial x}\right)_{i-1/2}\right] = -Q$$

$$\Rightarrow \frac{0.1}{\Delta x}\left[\left(1+\underbrace{\phi_{i+\frac{1}{2}}}_{\frac{\phi_i+\phi_{i+1}}{2}}\right)\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}\right) - \left(1+\underbrace{\phi_{i-\frac{1}{2}}}_{\frac{\phi_i+\phi_{i-1}}{2}}\right)\left(\frac{\phi_i-\phi_{i-1}}{\Delta x}\right)\right] = -Q_i$$

$$\Rightarrow \frac{0.1}{\Delta x}\left[\left(1+\frac{\phi_i+\phi_{i+1}}{2}\right)\left(\frac{\phi_{i+1}-\phi_i}{\Delta x}\right) - \left(1+\frac{\phi_i+\phi_{i-1}}{2}\right)\left(\frac{\phi_i-\phi_{i-1}}{\Delta x}\right)\right]$$

$$= -Q_i$$

$\Rightarrow \quad a_i \phi_{i-1} + b_i \phi_i + c_i \phi_{i+1} = d_i$

$$a_i = \frac{-1}{\Delta x^2}\left(1+\frac{\phi_i+\phi_{i-1}}{2}\right)$$

$$c_i = \frac{-1}{\Delta x^2}\left(1+\frac{\phi_i+\phi_{i+1}}{2}\right)$$

$$b_i = \frac{1}{\Delta x^2}\left(2+\frac{\phi_{i+1}+\phi_{i-1}+2\phi_i}{2}\right)$$

$$d_i = -Q_i \Delta x^2$$

$x=0, \ \phi_0 = 1$
$x=L, \ \phi_{N-1} = 0$

Solution Scheme:

i) We have nonlinear system.
We apply Picard's Iteration for the solution

ii) Initial guess for ϕ.

↳ Linear interpolation $\phi = 1 - x$

iii) Calculate a_i, b_i, c_i based on $\phi^{(k)}$

iv) Solve for the TDM using TDMA. $\Rightarrow \phi^{(k+1)}_{TDMA}$

v) Add under-relaxation to stabilise solution

$$\phi^{(k+1)} = \phi^{k}(1-w) + \phi^{(k+1)}_{TDMA} \cdot w \qquad w < 1$$