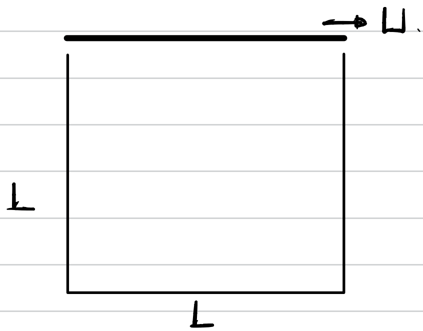
Question 3



$Re = 100.$

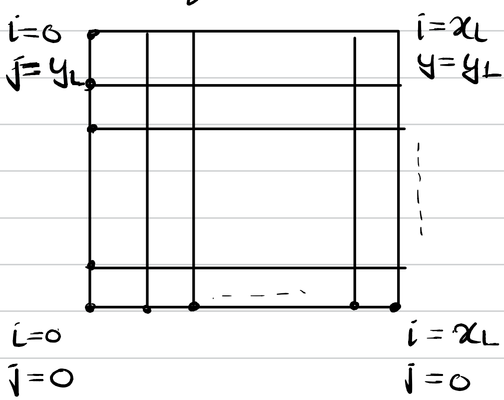
a. Main Governing Eq^n:

Continuity : $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

NS Eq^n : x : $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\mu$

y : $\rho\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\mu$

Discretisation of Grid :



No. of nodes = $n+1$

No. of faces = n

Say here $n = 10$

Each div = L/n

$dx = dy = L/n$

Discretisation of Equations

Objective: To solve steady state N.S. & Continuity Eqn.

Unknowns: u, v, P

$u^{n}_{i,j}$ → Time march

→ Spatial march

To obtain steady state $\left| u^{n+1}_{i,j} - u^{n}_{i,j} \right| < tol$

1. Time Stepping :

$$u^{n+1}, v^{n+1} \quad \text{from} \quad u^{n}, v^{n}$$

2. Space Stepping

* why do we use backward diff rather that central difference in convective term

Convective Term: $\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right)_{i,j}$ (apply finite diff. method)

$$\simeq u_{i,j}\left(\frac{u_{i,j} - u_{i-1,j}}{\Delta x}\right) + v_{i,j}\left(\frac{u_{i,j} - u_{i,j-1}}{\Delta y}\right)$$

Diffusion Term: $\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j}$

$$= \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{(\Delta x)^2} + \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{(\Delta y)^2}$$

Pressure Source: $\left(\frac{\partial P}{\partial x}\right)_{i,j} = \frac{P_{i+1,j} - P_{i-1,j}}{2\Delta x}$

The Eqns become :

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t\left[-\bar{V}(\nabla u)_{i,j}^{n} - \frac{1}{\rho}\nabla\cdot P + \nu(\nabla^2 u)_{i,j}^{n} \right]$$

$$v_{i,j}^{n+1} = \text{——}\,\|\,\text{——}$$

i) Get Approx u^*, v^* ignoring the Pressure source term

$$u^*_{i,j} = u_{i,j}^{n} + \Delta t\left[-\bar{V}(\nabla u)_{i,j}^{n} + \nu(\nabla^2 u)_{i,j}^{n} \right]$$

$$v^*_{i,j} = \text{—}\,\|\,\text{—}$$

ii) Now Add the correction term :

$$u_{i,j}^{n+1} = u^*_{i,j} + \Delta t\left[-\frac{1}{\rho}\nabla P_{i,j}^{n+1} \right]$$

iii) Impose continuity eqn on the time-marched velocity field

$$\Rightarrow \nabla\cdot\bar{V}_{i,j}^{n+1} = 0$$

$$\Rightarrow \nabla\bar{V}^*_{i,j} + \Delta t\left[-\frac{1}{\rho}\nabla\cdot(\nabla P_{i,j}^{n+1}) \right] = 0$$

iv) $$\boxed{\nabla^2 P_{i,j}^{n+1} = \frac{\rho}{\Delta t}\nabla\cdot\bar{V}^*_{i,j}}$$ Pressure-Poisson Eqn

Steps ii) - iv) give direct rise to the Pressure-Poisson Eqn.

After step i) , directly jump to:

v) $$\nabla^2 P^{n+1}_{i,j} = \frac{\rho}{\Delta t} \nabla \cdot \bar{V}^{*}_{i,j}$$

$$= \frac{\rho}{\Delta t}\left(\frac{u^{*}_{i+1,j} - u^{*}_{i-1,j}}{2\Delta x} + \frac{v^{*}_{i,j+1} - v^{*}_{i,j-1}}{2\Delta y}\right)$$

vi) $$u^{n+1}_{i,j} = u^{*}_{i,j} - \frac{\Delta t}{\rho} \nabla \cdot P^{n+1}_{i,j}$$

$$= u^{*}_{i,j} - \frac{\Delta t}{\rho}\left(\frac{P^{n+1}_{i+1,j} - P^{n+1}_{i-1,j}}{2\Delta x}\right)$$

$$\Rightarrow \nabla^2 p_{i,j} = b_{i,j}$$

$$\Rightarrow \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b_{i,j}$$

$$\Rightarrow \frac{p_{i+1,j} + p_{i-1,j} - 2p_{i,j}}{(\Delta x)^2} + \frac{p_{i,j+1} + p_{i,j-1} - 2p_{i,j}}{(\Delta y)^2} = b_{i,j}$$

$$\Rightarrow \left(-\frac{2}{\Delta x^2 + \Delta y^2}\right) p_{i,j} + \frac{1}{\Delta x^2}(p_{i+1,j} + p_{i-1,j}) + \frac{1}{\Delta y^2}(p_{i,j+1} + p_{i,j-1}) = b_{i,j}$$

$$\Rightarrow p_{i,j} = \frac{(p_{i+1,j} + p_{i-1,j})\Delta y^2 + (p_{i,j+1} + p_{i,j-1})\Delta x^2 - b_{i,j}\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

Gauss Siedel Iterative Solution Method