Analysis of Stress Distribution in Long Fixed-End Cylindrical Torsional Shaft

Semester Course Project - ES221: Mechanics of Solids
Group 1

Supervisor: Prof. Harmeet Singh

1 Objectives

The primary objectives of this project are:

- Assuming **small deformation theories**, perform a detailed **Stress Analysis** by applying individual loading cases: Twisting Moment, Bending Moment, Transverse Loading and Combination of all using superposition principle under certain assumptions.
- Develop **Shear Force and Bending Moment Diagrams** of a fixed-end, slender circular shaft using Mathematica software for multiple transverse loading.
- Employ **Mohr's Circle** method to find principal stress in the shaft geometry.
- Create **graphical visualizations of stress** distribution in the shaft geometry using **Heatmaps**.
- Validate theoretical results with finite element method software simulations (ANSYS) and available experimental data.

2 Problem Statement

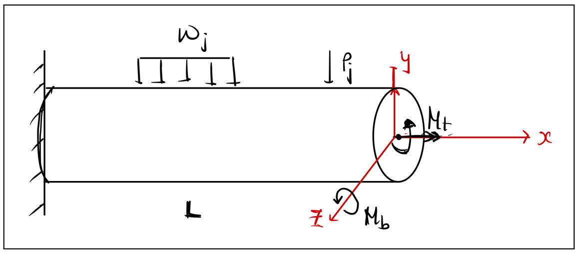

Figure 1: Initial Setup

Consider a **linearly elastic, homogenous, slender cylindrical rod of Length 5 m and radius 0.25 m (Slenderness ration = r/L = 0.05) made of material obeying the Hooke's Law** fixed at one end: Let the shaft be subjected to the following loading conditions: Twisting moment, Bending moment, Transverse loading

- We wish to write a **Mathematica** code that can be used to find the **Shear Force, Bending Moment and Twisting Moment diagrams** for the loadings independently. Thereupon, we will generate the **stress tensor** for the complete Shaft Geometry based on the **Torsional Twisitng Theory, Euler-Beam Bending Theory and Engineering Theory of Beams**.
- The Stress Tensor would then be used to Post-process various results. This includes determining **internal stress states, visualizing stress distributions, and identifying failure zones** based on material properties.
- The generated theoretical stress analysis of the shaft would be verified with **FEM Simulations** on similar geometry.

- In the end, Von-Mises Stress needs to be plotted for each point in the 3D geometry and for the plane of symmetry.

3 Methodology

3.1 Determining the Shear Force and Bending Moment Diagram

The Shear Force and Bending Moment Diagram were constructed by sectioning the beam and then performing Force balance and Moment balance for the sections.

3.2 Determining the Stress Distribution

To find stress distribution because of these Forces and Moments, we used the following analytical results derived in class and available in the Solid Mechanics book by Crandall and Dahl.

1. **Pure Torsion:** Pure Torsion induces shear stress in the θx plane leading to $\tau_{\theta x}$. Thus, $\tau_{\theta x}$ and $\tau_{x\theta}$ are the only non-zero components in this stress matrix. The formula of $\tau_{\theta x}$ has been provided in section 4.2. The coordinate system was then changed to **Cartesian coordinate** using Transformation Matrix

2. **Bending and Transverse Loading:** We apply the assumptions of the Engineering Analysis of Beam under Loading to obtain the stress under Bending and Transverse Loading. We have Shear Stress τ_{xz} and Normal Stress σ_x induced in the beam while the other components are zero. The formula of τ_{xz} and σ_x have been provided in section 4.2.

3.3 Principle of Superposition

After getting results from the above steps, we **applied the Principle of Superposition** under assumptions of **Material Linearity, Geometric Linearity and small deformations** to get combined loading scenarios. According to the principle of superposition, the stress tensor matrices for independent loading were added to obtain the final stress state of the geometry.

3.4 Mohr's Circle

For a 3D Stress Tensor, the Mohr's Circle method takes the form of Eigenvalue Calculation for determining the principal stresses. This is implemented in Python supplemented with the Numpy library.

3.5 FEM Verification

ANSYS was used to conduct the verification of Theoretical and Code Calculations with FEM Simulation. The model was created in the ANSYS Static Structural software. Geometry was kept congruent to the geometry used in Mathematica and loading was given the the torsional bar in the form of Transverse Point Load and Torsional Moment acting at the free end. Cylindrical coordinate system was used for the Torisional case to visualise $\tau_{\theta x}$ and Cartesian coordinate system was used for the Transverse point load to visualise σ_x and τ_{xz} .

3.6 3D Visualization through Heatmaps and Failure Prediction in the Shaft using Von-Mises Yield Criteria

The Analytical Formula for Total Stress Tensor for all Loading was used to determine Principle Stress by Finding Eigenvalues using Numerical Code and Used to Visualize (Von-Mises Stress) at each point and predict the most probable Failure Point

$$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

4 Numerical Implementation

4.1 Shear Force and Bending Moment Diagram

For an arbitrary section at position x along a cantilever beam, the internal shear force $V(x)$ and the bending moment $M(x)$ are computed by simple equilibrium equations taking in account forces and moment that act before the section.

Note: We are going to follow the sign convention of the book *An Introduction to the Mechanics of Solids also the distributed loads are not a function of x; that is, they are constant and uniform across the length of application.*

Shear Force, $V(x)$

The shear force at a section x from the fixed point of the cantilever beam is given by:

$$V(x) = -R_{\text{vertical}} + \sum_{x_j \leq x} P_j + \sum_{\text{distributed loads}} \begin{cases} w\,(x-a), & \text{if } a \leq x < b, \\ w\,(b-a), & \text{if } x \geq b, \end{cases}$$

where:

- R_{vertical} is the vertical reaction force at the fixed end.
- P_j denotes the magnitude of the j-th point load applied at position x_j.
- For a distributed load acting from $x = a$ to $x = b$ with constant intensity w:
  - If $a \leq x < b$: the load applied on the beam up to position x is $w\,(x-a)$.
  - If $x \geq b$: the full load contribution is $w\,(b-a)$.

Bending Moment, $M(x)$

The bending moment at a section x is computed as:

$$M(x) = -M_{\text{reaction}} + R_{\text{vertical}}\,x - \sum_{x_j \leq x} \left[P_j\,(x-x_j)\right] - \sum_{\text{distributed loads}} \begin{cases} \dfrac{w\,(x-a)^2}{2}, & \text{if } a \leq x < b, \\ w\,(b-a)\,\left[x - \left(\frac{a+b}{2}\right)\right], & \text{if } x \geq b. \end{cases}$$

Here:

- M_{reaction} is the moment reaction at the fixed end.
- The term $R_{\text{vertical}}\,x$ accounts for the moment induced by the vertical reaction force.
- $\sum_{x_j \leq x} \left[P_j\,(x-x_j)\right]$ sums the moment contributions from all point loads to the left of (or at) x. Each point load contributes a moment equal to the force P_j times its lever arm $(x - x_j)$.
- For distributed load, there can be two cases:
  - If the current section x is within the load span $(a \leq x < b)$, the moment contribution is:

$$[w(x-a)]\frac{(x-a)}{2} = \frac{w\,(x-a)^2}{2},$$

  - If x is beyond the load span $(x \geq b)$, the entire distributed load $w(b-a)$ acts at its centroid $\left(\frac{a+b}{2}\right)$. Thus, the moment contribution is:

$$w\,(b-a)\,\left[x - \left(\frac{a+b}{2}\right)\right].$$

4.2 Stress Matrix Determination

4.2.1 Pure Torsion

Following is the form of the stress matrix in the case of pure torsion.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rx} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta x} \\ \sigma_{rx} & \sigma_{\theta x} & \sigma_{xx} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{M_t r}{I_{xx}} \\ 0 & \frac{M_t r}{I_{xx}} & 0 \end{bmatrix}$$

where:

- M_t = Applied Twisting torque
- r = Radial distance from the shaft axis
- I_{xx} = Polar moment of inertia for cylinder $J = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}$

The Transformation Matrix for converting it to cartesian coordinate is

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\sigma}_{\text{Torsion}} = \boldsymbol{\sigma}_{\text{Cartesian}} = \mathbf{T}\,\boldsymbol{\sigma}_{\text{Cylindrical}}\,\mathbf{T}^T$$

Then we would replace θ and r as follows:

$$r = \sqrt{y^2 + z^2}, \quad \theta = \tan^{-1}\left(\frac{y}{z}\right)$$

4.2.2 Bending due to pure Bending Moment or Transverse Load

For Pure Bending or Bending due to Transverse Load , The Stress Tensor takes the form

$$\boldsymbol{\sigma}_{\text{Bending}} = \begin{bmatrix} \sigma_{xx} & 0 & \tau_{xz} \\ 0 & 0 & 0 \\ \tau_{xz} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{M \cdot y}{I_{zz}} & 0 & \frac{V \cdot Q}{I_{zz} . t} \\ 0 & 0 & 0 \\ \frac{V \cdot Q}{I_{zz} . t} & 0 & 0 \end{bmatrix}$$

- V = Shear Force
- M = Bending moment
- y = Distance from the neutral axis
- I_{zz} = 2nd Moment of area across z - axis
- Q = 1st Moment of area above the point of interest in y-direction
- t = Thickness/width of Cross-section along z direction at the point of interest

For Cylindrical Shaft the Shear Stress takes the form:

$$\tau_{xy}(y) = \frac{4V}{3\pi R^4}(R^2 - y^2)$$

4.2.3 Application of Superposition Principle

Under Assumption of **Material Linearity, Geometric Linearity and small deformations**, on application of superpostion principle we can get the total stress Tensor by simply adding the Stress Tensors obtained by individual Loading Scenarios

$$\boldsymbol{\sigma}_{\text{total}} = \boldsymbol{\sigma}_{\text{Bending}} + \boldsymbol{\sigma}_{\text{Torsion}}$$

4.3 Von-Mises Stress

As we have obtained a stress Tensor for any point in the shaft , we need some criteria or formula to determine or assign a single value of stress at a point.

Thus we used Von-Mises Stress formula which is the following

$$\sigma_{\mathrm{VM}} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}$$

Where $\sigma_1, \sigma_2, \sigma_3$ are found by determining Eigenvalues of Total Stress Tensor at each point

4.4 ANSYS Model

The ANSYS Static Structural Solver is used to model the Torsion and Transverse Loading of the Torsional Bar. The Boundary conditions such as fixed support and Point Forces were carefully setup on the geometry. The mesh for the geometry had an element size of 1e-3. The material used during the simulation was Structural Steel with an added modification of Posisson's Ratio as 0.

5 Results and Discussion

- The combined bending and torsional loading resulted in a nice and well defined analytical formula for Total Stress Tensor for a particular value of the Loading
- The Stress Heatmaps along the length as well as of the Cross-section provided a better intuitive understanding and Visualization for the state of stress at various points
- Mohr's circle analysis allowed for identifying the Principle Stresses.
- The Heatmaps provided the zones having maximum state of stress thus help in determining failue criteria .
- Computational results visually matched with the theoretical results however the magnitude of stresses were different as we have made certain assumption to simplify our problem.

5.1 Shear Force and Bending Moment

The mathematica code has capabilities to provide Point Load, Distributed Load and Bending Moment to the Torsional Bar. It was able to generate SFD and BMD for any given set of loading adjusted by the user through sliders. The results agreed with sanity checking points such as linearly increasing Shear force for distributed loads, Jump Discontinuities in SFD for Point Loads and the Shear force and Bending Moment are maximum at the fixed end and 0 at the free end.

5.2 Stress Distribution for Loading

5.2.1 Pure Torsion

- The Shear Stress for maximum at the outer cylindrical layers.
- The Shear Stress was axis-symmetric with respect to the x-axis (axial).
- The Maximum Shear Stress values matched closely for Mathematica Code (4.074e7 Pa) and ANSYS Simulation (4.0856e7 Pa) for a congruent geometry and torsional loading.
- The Shear Stress was uniform throughout the length of the rod at a particular r, θ.

5.2.2 Transverse Loading and Bending

- The Normal Stress was maximum at the outer cylindrical layers for particular x. Also overall, the Normal Stress was maximum near the wall as the Bending Moment was maximum
- The Shear Stress was maximum near the Neutral Axis for particular x. Also overall, the Shear Stress was maximum near wall as the shear force is maximum there. y.
- The Value of Stresses were not matching with that obtained through ANSYS simulations , however the heatmaps Visually were similar and Congruent.

5.2.3 Von-Mises Stress

- With the help of Total stress Tensor using superposition principle we were able to determine the Most Probable Failure Points , which is the outermost point near the wall (x=0; y = -R,R ; Z = 0)

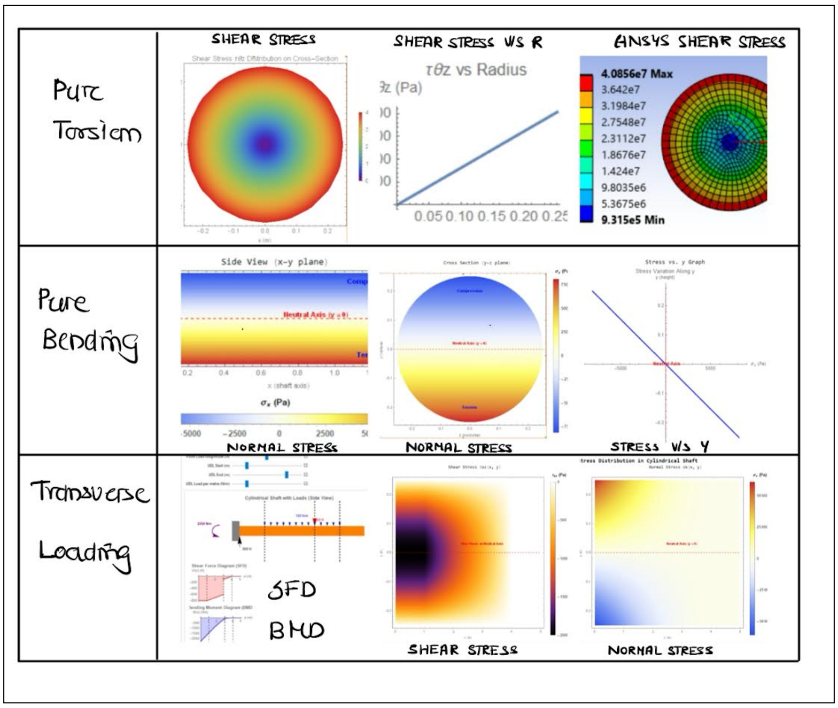

Figure 2: Compiled Results

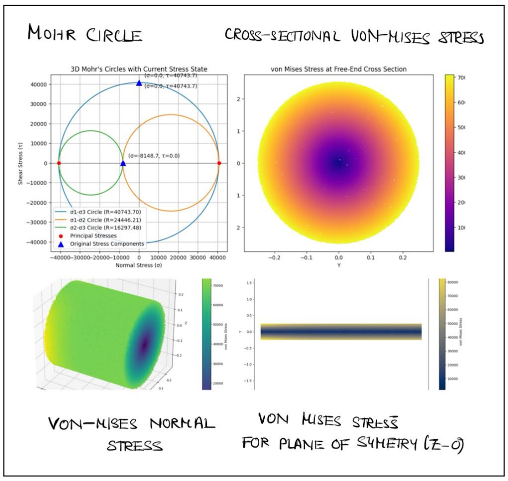

Figure 3: Compiled Von Mises Stress Results

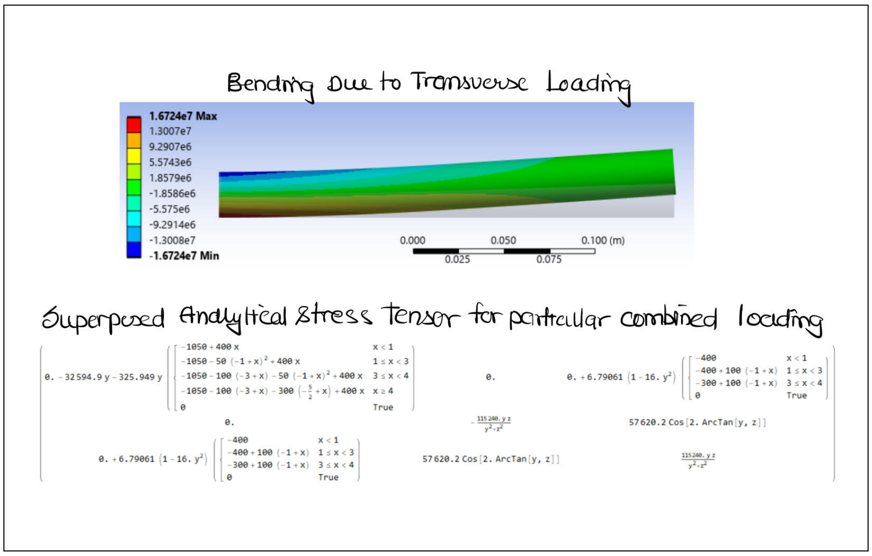

Figure 4: Superposed Stress Tensor

6 Learning Outcomes

Through this project, the team achieved the following:

- Implementation of Stress-Analysis Theories for a complete system of Loading .
- Applied and Visualized the concept of stress for Torsion and Bending of the Cylindrical Shaft with the help of Heat Maps.
- were able to comprehend on how different assumptions be it Materialistic or Geometric and affect the Result by Comparing solution through theories and FEM
- Assigning a quantitative value to state of stress using Von Mises Stress formula and predict the most probable point of failure
- At last gained some hands-on experience in Coding in Mathematica/Python and FEM software like ANSYS

References

[1] Crandall, S. H., Dahl, N. C., & Lardner, T. J. (2012). *An Introduction to the Mechanics of Solids* (3rd ed., SI Units). Tata McGraw-Hill Education. Available on Amazon

[2] Hibbeler, R. C. (2017). *Mechanics of Materials* (10th ed.). Pearson Education. Available on Amazon

[3] Static Structural Analysis of a Cantilever Beam – A practical demonstration of cantilever beam analysis using finite element methods.

Link for Result Images and Code