Group action

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1 Action

Action of an algebraic object on some other is a general construct of which group action is one. The simplest concept of an action involves one set, X, acting on another set Y and such an action is given by a function A, from $X \times Y$ to Y,

$$A : X \times Y \to Y$$

For fixed $x \in X$, this produces an endofunction $A_x : Y \to Y$, and hence called "action by X" (i.e., every $x \in X$ correspond to a map from Y to Y). In this way the whole of X acts on Y. The image of the action, therefore is the union of the image of all the A_x s.

$$\text{i.e.,} \quad \mathrm{Im}(A) = \bigcup_{x \in X} \mathrm{Im}(A_x)$$

Hence one can talk about the map $x \mapsto A_x$, given by the induced function (say $\hat{A}$),

$$\hat{A} : X \to Y^Y$$

where Y^Y is the set of all endofunctions on Y (X^Y is the set of all functions from Y to X).

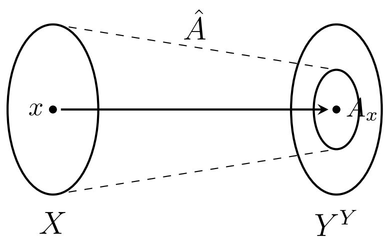

Fig:1.1

If the map $\hat{A}$ is injective, the corresponding action A is called "faithful", and unfaithful otherwise.

For a 'group action', X needs to be a group G, and $\hat{A}$, a group homomorphism.

Note. *The idea here is to carry the algebraic structure of the actor to the object being acted upon.*

Analogous structures of Y^Y (i.e., the codomain of $\hat{A}$, here called the "target space") in different categories of X (actor) and Y (being acted upon):

Remark. *Target space structure results from the additional structure we impose on $\hat{A}$. Oftentimes that, $\hat{A}$ is a morphism of the category of the actor. The operation(s) under which the target spaces have these structures are often defined pointwise using the operation(s) of Y.*

Category of X	**Category of Y**	**Target Space**	**Structure**	**Formal Name**
Set[1]	**Set**	Y^Y	Monoid	Action
Grp[2]	**Set**	$\mathrm{Sym}(Y)$	Group	Group Action
Grp	**Vect**[3]	$\mathrm{GL}(V)$	Group	Group Representation
Ring[4]	**Ab**[5]	$\mathrm{End}(M)$	Ring	Ring Representation
Alg[6]	**Vect**	$\mathrm{End}(V)$	Algebra	Algebra Representation

Note. *In the III row, GL(Y) = GL(V), for Y is a vector space V.*
In the IV row, End(Y) = End(M), for Y is an abelian group thought of as a $\mathbb{Z}-$module M.
In the V row, End(Y) = End(V), for Y is a vector space V.

Note: Y^Y is a superset of all other target spaces.

Irrelevant info: Notice how action goes along the process of experimentation. A test subject Y, being acted upon by a known structure X to comment of the unknown structure of Y based on the interaction (for eg. a scattering experiment).

2 Group action

In case of group actions, we add an extra condition on the induced function of the action A (i.e., $\hat{A}$), that it must be a group homomorphism. Let's call this new, more restricted $\hat{A}$ as $\hat{A_G}$. Let's rename X (the actor) as G, for it's a group, and Y (the set being acted upon) as X, for Y always succeeds X and there's no X to preceed it, so a group action A_G is given by,

$$A_G : G \times X \to X.$$

Now, the associated induced function

$$\hat{A_G} : G \to X^X$$

needs to be a group homomorphism.

For a function $\rho : G \to H$ to be a group homomorphism where both G and H are groups, satisfying, $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$, is enough. Because from this condition alone, $\rho(e_G) = e_H$ is implied,

[1] category w/ sets as objects and functions as morphisms
[2] category of groups and group homomorphisms
[3] category of vector spaces and linear maps
[4] category of rings and ring homomorphisms
[5] category of abelian groups and group homomorphisms
[6] category of associative algebras and algebra homomorphism

thanks to the invertibility of H. The proof goes, $\rho(e_G e_G) = \rho(e_G)\rho(e_G) = \rho(e_G) \implies \rho(e_G) = e_H$ solely because $\rho(e_G)$ is an element of H, therefore invertible (e_G and e_H are the identities of G and H respectively).

But here, X^X (say, M) is just a monoid under composition (i.e., elements are not necessarily invertible) which forces us to add the condition $\rho(e_G) = e_M$ (e_M is the identity of the monoid M) to ensure invertibility in the image of the map ρ (reasoning in section 2.1), rendering ρ into a group homomorphism.

i.e., given $\rho(e_G) = e_M$, $\rho(gg^{-1}) = \rho(g)\rho(g^{-1}) = e_M$ and hence $\rho(g)$ is invertible with inverse $\rho(g^{-1})$for all $g \in G$.

2.1 Basis of homomorphism conditions

As the name suggests, homomorphisms are the morphisms that morphs objects into the same category, i.e., if the domain is a group, so is the image of a group homomorphism. Let's look at how the conditions $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$ and $\rho(e_G) = e_M$ combinedly ensure this when the codomain is a monoid.

$\rho(gh) = \rho(g)\rho(h)$ ensures closure of the image. G is closed $\implies gh \in G$ for all $g, h \in G$ and $\rho(g) \in \mathrm{Im}(\rho)$ for all $g \in G \implies \rho(gh) = \rho(g)\rho(h) \in \mathrm{Im}(\rho)$ for all $\rho(g), \rho(h) \in \mathrm{Im}(\rho) \implies \mathrm{Im}(\rho)$ is closed.

Associativity is inherited for any non-empty subset of a monoid, since a monoid is associative.

Now comes the crucial part. $\mathrm{Im}(\rho)$ being invertible. i.e., for each $\rho(g) \in \mathrm{Im}(\rho)$, there exists $(\rho(g))^{-1}$ such that, $\rho(g)(\rho(g))^{-1} = e_M$. Assume $\rho(e_G) = m \in M$ and $m \neq e_M \implies \rho(e_G e_G) = \rho(e_G)\rho(e_G) \implies \rho(e_G) = m^2 \implies m = m^2 \implies m$ is an idempotent element.

Therefore the identity, in itself, has this property of getting mapped to an idempotent elment under the condition $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$.

Now, we also need m to be invertible, for we need $\mathrm{Im}(\rho)$ to be a group. i.e., there exists $m^{-1} \in Im(\rho)$ such that $mm^{-1} = e_M \implies mmm^{-1} = me_M \implies m^2m^{-1} = m \implies mm^{-1} = m \implies m = e_M \implies$ $\boxed{\rho(e_G) = e_M}$

i.e., the only invertible idempotent element in a monoid is the identity.

The conditions, $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in G$ and $\rho(e_G) = e_M$ ensure that $\mathrm{Im}(\rho)$ is a group, the associated map ρ, a group homomorphism, which is what we wanted the induced map $\hat{A_G}$ of our group action A to be (i.e., a group homomorphism).

Now, eventhough the codomain M is a monoid, the image of $\hat{A_G}$ is a group, a subgroup of the monoid M. And for any monoid M, the subset of units, M^* is a group, and is the biggest subgroup of the monoid to which every other subgroup is a subgroup of. Therefore $Im(\rho) < M^*$,[7] and the the group of units M^* of our monoid ($M = X^X$), is the set of all invertible endofunctions on X, a.k.a. bijections from X to X, which is nothing but the symmetric group S_X[8].

Thus the induced map of a group action $A_G : G \times X \to X$ is

$$\hat{A_G} : G \to S_X,$$

which is a group homomorphism.

[7] whenever I use, $\subset$ or $<$, the former is an improper subset, the latter, an improper substructure (i.e., subspace of a vector space, subsgroup of a group etc.,) otherwise would be explicitly mentioned, a subgroup of a monoid for example, would be explicitly mentioned. For proper couterparts, i use $\subsetneq$ and $\lneq$ respectively.

[8] S_X is the symmetric group of X, $\mathrm{Sym}(X)$

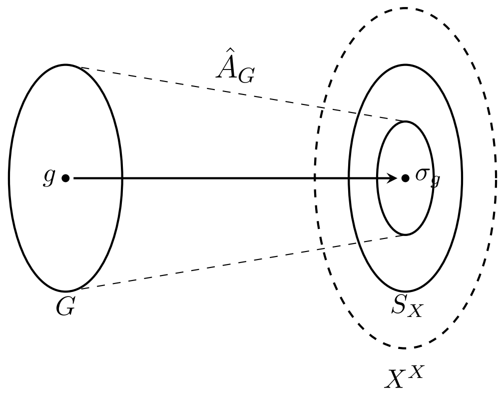

Fig:1.2

Note. *σ_g is a bijection (as any element of a symmetric group) that g is getting mapped to.*

Now one can draw analogies from Fig:1.1, that when the actor gets changed from a set to a group, the mapping gets more restrictive, codomain shrinks from Y^Y to S_X (ofcourse, it's because we wanted $\hat{A_G}$ to be a group homomorphism). Since $\hat{A_G}$ preserves group structure, unlike $\hat{A}$ in the case of action by a set.

Let's see how this naturally leads us to the axioms of a group action.

For $\hat{A}_G$ to be a group homomorphism, the following must be satisfied:

$$\hat{A_G}(gh) = \hat{A_G}(g)\hat{(}A_G(h)) \text{ for all } g, h \in G$$
$$\hat{A_G}(e_G) = e_M$$

Now, $\hat{A}_G$ is a function from G to S_X, for 2 functions to be deemed 'same' in S_X, we need them to be same for every $x \in X$.

$$\text{i.e.,} \quad \hat{A}_G(gh)(x) = \hat{A}_G(g)(\hat{A}_G(h)(x)) \text{ for all } g, h \in G \text{ for each } x \in X$$
$$\hat{A}_G(e_G)(x) = e_M(x) = x \text{ for all } x \in X$$

Since the identity of the monoid X^X or the group S_X is nothing but the identity function (i.e., $e_M(x) = x$ for all $x \in X$).

Note: $A_G(g, x) = \hat{A_G}(g)(x) = \sigma_g(x)$

Therefore, the above reads,

$$\boxed{\begin{aligned} &A_G(gh, x) = A_G(g, A_G(h, x)) \text{ for all } g, h \in G \text{ for each } x \in X \\ &A_G(e_G, x) = x \text{ for all } x \in X \end{aligned}}$$

which exactly are the axioms of a group action.

Note. *For notational convenience, $A_G(g, x) = g \cdot x$*

Theorem 2.1 (Cayley's theorem)**.** *Every group is embedded in a symmetric group.*

Proof. The goal here is to show that for every group G, there exist a set X on which G acts faithfully. Since for a faithful group action, $\hat{A_G}$ is injective (which already is a group homomorphism), resulting in an embedding of G into S_X (i.e., $\mathrm{Im}(\hat{A_G})$ is isomorphic to G). The standard proof involves the set X being the group G itself ($X = G$), and the action being left multiplication.

$$A_G : G \times G \to G \text{ given by } A_G(g, h) = g \cdot h = gh$$

Note. *Don't confuse the condition as A_G being an injection. It's $\hat{A_G}$ that needs to be an injection, for A_G to be faithful.*

the endofunction (which is a bijection) defined by a specific $g \in G$ is

$$A_G(g, h) = \sigma_g(h) \text{ for all } h \in G.$$

The induced map is

$$\hat{A}_G : G \to S_G \text{ given by } \hat{A}_G(g) = \sigma_g.$$

The goal is to show this induced map is injective. In other words, every $g \in G$ correspond to a unique bijection from G to G.

For this map to be a non-injection, $\sigma_{g_1}(h) = \sigma_{g_2}(h)$ for all $h \in G$ for some $g_1, g_2 \in G$. But here, $\sigma_g(e_G) = ge_G = g$ for each $g \in G$. Therefore, $\sigma_{g_1} = \sigma_{g_2} \implies g_1 = g_2$. Hence the map is $\hat{A_G}$ is injective.

Now, is $\hat{A_G}$ a group homomorphism? Yes, because

$$\begin{aligned}
&\hat{A}_G(gh)(h) = \sigma_{gh}(h) = (gh) \cdot h = g \cdot (h \cdot h) = \sigma_g(\sigma_h(h)) = \hat{A}_G(g)(\hat{A}_G(h)(h)), \\
&\hat{A}_G(e_G)(h) = \sigma_{e_G}(h) = e_G \cdot h = h.
\end{aligned}$$

Therefore G is embedded in S_G.

Note. *Alternatively, one could show that A_G satisfies the axioms of a group action directly instead of showing $\hat{A_G}$ is a group homomorphism.*

Remark. *To give an alternate perspective to look at this action, look at G (being acted upon) as a set X (i.e., just forget that G had a binary operation). Now, there exist a trvial bijection from G to X, say, $\phi : G \to X$, given by $\phi(g) = x_g$ for all $g \in G$ (where x_g is just the element g thought of as an element of the set X). Now, define a the group action A_G as follows,*

$$A_G : G \times X \to X \text{ given by } A_G(g, x_h) = \phi(g \cdot \phi^{-1}(x_h)) = \phi(gh)$$

This gives a sense that, however the left multiplication is definitive of this action (for it fixes the permutation correspondence of g), it's not limited to it. The group elements act by permuting the elements of a set X (i.e., bijection from X to X).

2.2 Orbits and Stabilizers

Definition 2.1. Once the group action is defined, we have the following constructs,

- **Orbit**: For $x \in X$, the orbit of x under the action of G is

$$\mathrm{Orb}(x) = \{g \cdot x \mid g \in G\} \subset X$$

- **Stabilizer**: For $x \in X$, the stabilizer of x under the action of G is

$$\mathrm{Stab}(x) = \{g \in G \mid g \cdot x = x\} \subseteq G$$