HHL Algorithm

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The HHL Algorithm is a quantum algorithm that can be used to solve linear systems of equations. The algorithm is based on the quantum Fourier transform and the quantum phase estimation algorithm. The algorithm is based on the following equation:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

To start, we crate a matrix A which is a hermition matrix. Lets do a 2 qubit example:

$$A = \begin{bmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}$$

This matrix A has the following eigenvalues:

$$\lambda_1 = \frac{3}{4} \lambda_2 = \frac{1}{4}$$

And the following eigenvectors:

$$v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We then create a vector b which is a vector of length 2^n where n is the number of qubitss:

$$b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since A is a hermition matrix, it has the following spectral decomposition:

$$A = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle v_i|$$

We can also write A^{-1} as:

$$A^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i} |v_i\rangle\langle v_i|$$

A is both invertible and Hermition, so we can write b as:

$$b = \sum_{i=1}^{n} b_i |v_i\rangle$$

Therefore our solution to x is:

$$|x\rangle = \sum_{i=1}^{n} \frac{b_i}{\lambda_i} |v_i\rangle$$

To start our algorithm, we load b into the last registers and run QPE on A^{-1} to get the eigenvalues of A^{-1}. QPE for the 2 qubit case will have a binary approximation of $\frac{\lambda_i t}{2\pi}$. This will give a 2 qubit approximation of: