HW 6

Thursday, March 2, 2023 9:23 AM

1) A) $E[t] = \int t \cdot P(t)\, dt = \langle t \rangle$

$$= \int t \cdot \frac{1}{\tau} e^{-\frac{t}{\tau}} dt$$

$$= \frac{1}{\tau} \int t \cdot e^{-\frac{t}{\tau}} dt$$

$$\int u\, dv = uv - \int du\, v$$

let $u = t$ $\quad du = 1$

$dv = e^{-\frac{t}{\tau}}$ $\quad v = -\tau e^{-\frac{t}{\tau}}$

$$= (t)\left(-\tau e^{-\frac{t}{\tau}}\right) - \int e^{-\frac{t}{\tau}} dt$$

$$= -t\tau e^{-\frac{t}{\tau}} - \left(-\tau e^{-t/\tau}\right) + C$$

$$\boxed{= -e^{-\frac{t}{\tau}}(t + \tau) + C}$$

B) $E[x^2 - \langle x \rangle] = \int x^2 P(x)\, dx - \langle x \rangle^2$

$$= \int t^2 \frac{1}{\tau} e^{-\frac{t}{\tau}} dt - \left(-e^{-\frac{t}{\tau}}(t + \tau) + C\right)^2$$

$$\frac{1}{\tau} \int t^2 e^{-\frac{t}{\tau}} dt$$

$u = t^2$ $\quad du = 2t\, dt$

$v = -\tau e^{-\frac{t}{\tau}}$ $\quad dv = e^{-\frac{t}{\tau}} dt$

$$\int u\, dv = uv - \int v\, du$$

$$= (t^2)\left(-\tau e^{-\frac{t}{\tau}}\right) - \left[\int -\tau e^{-\frac{t}{\tau}} \cdot 2t\, dt\right]$$

...

$$= -e^{-t/\tau}(t^2 + 2\tau^2 + 2\tau t) + D$$

$\therefore$ our variance is

$$\boxed{\left[-e^{-t/\tau}(t^2 + 2\tau^2 + 2\tau t) + D\right] - \left[-e^{-\frac{t}{\tau}}(t + \tau) + C\right]^2}$$

C) Our Likelyhood is given by

$$L(\theta) = P(x_1) f(x_2) \cdots f(x_n) = \prod_{i=1}^{N} P(x_i, \theta)$$

And our MLE for θ is:

$$\frac{dL}{d\theta} = 0 \longrightarrow \theta_e$$

$\therefore$ our Likeliness function is

$\theta = \{\tau\}$

$$L(t, \theta) = \prod_{i=1}^{n} \frac{1}{\tau} e^{-\frac{t_i}{\tau}}$$

find where $\frac{d \ln L(t,\theta)}{dt} = 0$

$$\ln L(t, \theta) = \prod_{i=1}^{n} \ln\left(\frac{1}{\tau}\right)\left(-\frac{t_i}{\tau}\right) = 0$$

$$\frac{d \ln L(t,\theta)}{dt} = \frac{d}{dt} \sum_{i=1}^{n} \ln\left(\frac{1}{\tau}\right) - \frac{t_i}{\tau} = 0$$

$$= \frac{d}{dt}\left[n \ln\left(\frac{1}{\tau}\right) - \sum_{i=1}^{n} \frac{t_i}{\tau}\right] = 0$$

$\therefore$

$$\boxed{\tau_e = \tau}$$

4) $f(x, y) = x + y$

where our Covariance matrix is

$$V = \begin{bmatrix} V_{00} & V_{10} \\ V_{01} & V_{11} \end{bmatrix} = \begin{bmatrix} cov(x,x) & cov(x,y) \\ cov(y,x) & cov(y,y) \end{bmatrix} = \boxed{\frac{1}{N}\begin{bmatrix} \sigma_x^2 & \sigma_x \sigma_y \\ \sigma_y \sigma_x & \sigma_y^2 \end{bmatrix}}$$

- where the variance of $f(x,y)$ is given by

$$\sigma_f^2 = \sum_{i,j} \frac{df}{d\theta_i} \frac{df}{d\theta_j} \cdot cov(\theta_i, \theta_j) \longrightarrow$$ sum over our matrix elements

↳ for $f(x,y)$ $\frac{df}{dx} = 1$ $\quad \frac{df}{dy} = 1$

$$\boxed{\sigma_f^2 = \sigma_x^2 + 2\sigma_x\sigma_y + \sigma_y^2}$$