QNT 402 Quantum Computation
Homework 1

1. [45] *Depolarizing quantum channel.* Suppose a qubit is simultaneously subject to σ_x, σ_y, and σ_z errors, each with probability p_x, p_y, and p_z, respectively. Such a channel can be described by the Kraus operators:

$$\hat{K}_0 = \sqrt{1-p}\hat{I}$$
$$\hat{K}_1 = \sqrt{p_x}\hat{\sigma}_x$$
$$\hat{K}_2 = \sqrt{p_y}\hat{\sigma}_y$$
$$\hat{K}_3 = \sqrt{p_z}\hat{\sigma}_z$$

with $p = p_x + p_y + p_z$.

   a.) [10] Assuming the system initially started in the state $|0\rangle$ find the density matrix after the operation is complete.

   b.) [10] Assuming the system initially started in the state $|X\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, find the density matrix after the operation is complete.

   c.) [15] As we learned last quarter, any qubit density matrix can be described as:

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\cdot\vec{\sigma}\right),$$

   where $\vec{\sigma} = \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}$ and $\vec{r} = r_x\,\hat{x} + r_y\,\hat{y} + r_z\,\hat{z}$. We saw that for a pure state $\vec{r}$ was analogous to the statevector on the Bloch sphere. Calculate the effect of the depolarizing channel on a density matrix described this way and show that it can be written as

$$\hat{\rho} = \frac{1}{2}\left(\hat{I} + \vec{r}\,'\cdot\vec{\sigma}\right),$$

   where $\vec{r}\,'$is the vector after the channel and determine $\vec{r}\,'$.

   d.) [10] Find the value of $|\vec{r}\,'|/|\vec{r}|$ for $p_x = p_y = p_z = p/3$. Why do you think this is called the depolarizing channel?

2. [30] *When is an error an error?* A qubit in the state $|\psi_o\rangle = 1/\sqrt{2}\,(|0\rangle + |1\rangle)$ is subject to a phase-flip channel with probability p.

   a.) [15] Calculate the fidelity of the final state with the initial state.

   b.) [15] Now assume a qubit in $|\psi_o\rangle$ is subject to a bit-flip channel with probability p. Calculate the fidelity of the final state with the initial state.

3. [30] *Entropy measures* Suppose that two qubits are initially in the $|\Psi^+\rangle$ Bell state. Calculate the following:

   a.) [5] The von Neuman entropy

   b.) [5] The entanglement entropy of the first qubit

   c.) [20] Suppose the two qubits are subjected to a two-qubit dephasing channel, where the probability of a single qubit dephasing is p. The Kraus operators can be written as:

   $\hat{K}_o = (\sqrt{1-p}\,\hat{I}_1) \otimes (\sqrt{1-p}\,\hat{I}_2) = (1-p)\,\hat{I}_1 \otimes \hat{I}_2$ – no dephasing

$\hat{K}_1 = \sqrt{p}\hat{\sigma}_z^{(1)} \otimes \hat{I}_2$ – qubit 1 dephasing
$\hat{K}_2 = \sqrt{p}\hat{I}_1 \otimes \hat{\sigma}_z^{(2)}$—qubit 2 dephasing
$\hat{K}_3 = p\ \hat{\sigma}_z^{(1)} \otimes \hat{\sigma}_z^{(2)}$ – qubit 1 and qubit 2 dephasing

Calculate the von Neumann entropy and the entanglement entropy of the first qubit after the quantum channel.

4. [45] *Decoherence in QuTip.* Assume we have a qubit being driven by near resonant radiation, such that the Hamiltonian in the rotating frame after the RWA is the usual:

$$\hat{H} = -\frac{\delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x$$

a.) [5] Use QuTip's built-in master equation solver mesolve to find the evolution of the density matrix and plot ρ_{22} as a function of time. Assume at t = 0 the system starts in the ground state. Pick $\delta = 0$ and $\Omega = 2\pi$.

b.) [20] Redo part (a) but add a collapse operator (c_ops) for spontaneous emission. The appropriate operator is $\sqrt{\Gamma}\hat{\sigma}_-$, where Γ is the spontaneous emission rate. Plot the evolution you find for ρ_{22} as a function of time for $\Gamma = 0.1\Omega$, $\Gamma = \Omega$, and $\Gamma = 3\Omega$.

c.) [20] Redo part (b) but use mcsolve instead of mesolve. For this step plot the result for using 1, 10, 100 trajectories. For more information, on mcsolve and changing the trajectory number see: https://qutip.org/docs/latest/guide/dynamics/dynamics-monte.html