UNIT 2   FRACTALS AND ROUGHNESS OF REALITY

Contents

2.0   OBJECTIVES

This unit is conceived to give a general introduction to the concept of fractals that can represent and analyse the roughness of reality. Rather than going into the scientific intricacies, the philosophical and scientific significance associated with fractals are emphasized. In comparison with Euclidean geometry, which remained unchallenged for centuries, we shall see how fractal geometry comes in as a worthy alternative. We shall also analyse how Benoit Mandelbrot, the father of fractal geometry, successfully represented the irregular and rough texture of nature with fractal dimensions while the Euclidean geometry unsuccessfully tried to fit in the complexities of nature into predefined shapes. A discussion on fractals like Koch curve, Mandelbrot set and others will give a basic understanding on some of the famous fractals. The significance of fractals can be seen in their practical application in myriad fields initiating a tectonic shift in scientific and philosophical outlook. Thus by the end of this Unit you should be able to:

- have a basic understanding of fractals

- contrast Euclidean geometry and Fractal Geometry

- understand how fractals represents the roughness of reality

- analyse the practical applications of fractals

- creatively ponder on the philosophical implications of fractals

2.1   INTRODUCTION

The fractal Geometry better represents the complexity of nature which was once bracketed off by the Euclidean Geometry and also reveals the micro-macro relationship. Rather than singularly approaching and analysing nature from the perspective of Platonic perfection, the introduction of fractals has initiated a paradigm shift that very well incorporates and analyses the irregularity and roughness of the nature.

Benoit Mandelbrot was the man behind the discovery of fractal dimension. The fractal dimension was initially called the Hausdorff-Besicovitch dimensions after the mathematicians Felix Hausdorff and A.S. Besicovitch who invented and developed it. The concept of fractal dimension was considered by mathematicians to be an abstract mathematical idea. They used to go behind the fractal dimensions purely for their own entertainment. So the fractal dimension which was once ignored by most mathematicians was given a fresh lease of life by Mandelbrot. Thus today, these fractal dimensions finds application in real life as it helps in analyzing the complexity of fractal shapes, describing a wide range of natural phenomena.

Mandelbrot wrote in his book, *The Fractal Geometry of Nature*, "clouds are not spheres, mountains are not cones; coastlines are not circles, and bark is not smooth, nor does lightning travel in straight line." He moved beyond the Euclidean geometry where space has three dimension, a plane has two, a line has one and a point has zero dimension to the fractional dimensions. Mandelbrot evolved a new type of mathematics, capable of describing and analyzing the structured irregularity of the natural world, and coined a new name for the geometric forms involved: fractals. Fractional dimension became a way of measuring qualities that otherwise have no clear definition: the degree of roughness or brokenness or irregularities of the object. Fractals are ubiquitous in nature; they are seen in clouds and coastlines, ferns and trees. The human circulatory system is a series of self similar fractal from 'aorta to capillary.'

Mandelbrot made a claim which the world stood up and listened. The claim was that the degree of irregularity remains constant over different scales. Surprisingly, often, the claim turns out to be true. Over and over again, the world displays a regular irregularity. Rather than applying Euclidean geometry which bypassed the irregular structures, the fractal geometry made sure that the irregularities of the nature were incorporated. Above all it finds order in the apparent disorder which is seen in the nature. It is a movement inspired from nature rather than dominating nature with our own set of rules.

2.2 FROM EUCLIDEAN TO FRACTAL GEOMETRY

Euclid collected, studied the works of his predecessors on geometry and systematically presented in his book *Elements* which contains treatises on maths and geometry. Until the arrival Non-Euclidean Geometry in the second half of nineteenth century, Euclidean Geometry stood unchallenged. Geometric statements in Greece were derived by logic alone from self evident geometric assumptions called axioms. Euclid chose five axioms and from them he formed the body of theorems known as Euclidean geometry. The Euclidean postulates can be given in modern terms as: 1. A line joining two points is a straight line. 2. This line can be extended indefinitely. 3. Given any straight line, a circle can be drawn having the segment as radius and one point as the centre. 4. All right angles are congruent. 5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles then two lines inevitably must intersect each other on that side if extended far enough. As the fifth postulate stands as basis for the uniqueness of parallel lines, it is called the parallel postulate.

The classical Euclidean geometry is predominantly influenced by the Platonic philosophy. The Platonic philosophy gave utmost importance to the world of forms or ideas which were considered as eternal and absolute. The temporal and changing world which we live in was considered only to be the shadow of the unperceived real world that is permanent and unchanging. As a necessary after effect of Platonic philosophy, irregularities or complexity of nature was not taken up into Euclidean geometry. Thus we have the shapes of classical geometry restricted to lines and planes, circles and spheres, triangles and cones. Whatever be the irregular features that were present in the reality, they were all powerfully abstracted into these perfect shapes.

The absoluteness of Euclidean geometry was challenged by mathematicians like Janos Bolayi, Nikolai Lobachevski, and Karl Friedrich Gauss in the nineteenth century. The Non Euclidean geometry came up with the realization that the Euclid's parallel postulate was independent of the other four axioms and could never be derived from them. They also understood that a consistent system of geometry can be built if one add the negation of the parallel postulate to other axioms. The two main types of non Euclidean geometry are hyperbolic geometry and elliptical geometry, having negative and positive curvature respectively. The properties of the Non-Euclidean geometry such as curvature were far beyond the every day human experience which Einstein could apply to his theory of relativity.

The strong influence which Euclidean geometry wielded can be well understood from Immanuel Kant who claimed that people could think about space only in Euclidean terms. The arrival of non Euclidean geometry showed that there are other conceivable descriptions of space. Non Euclidean geometry was the first step forward from the rigid notion of viewing the reality from a linear outlook with excessive emphasis on perfection and leaving out imperfect and irregular feature. The arrival of fractal geometry brought in the once ignored realm of complexity found in nature.

2.3 FRACTAL GEOMETRY AND THE THEORY OF ROUGHNESS

Euclid was preoccupied by the vision of unrealistically ordered universe, and could not provide a geometry that could picture the roughness and irregularity of the world Galileo stated that the Great book of Nature is written in the language of mathematics with characters being circles, triangles, and other such shapes.. With Fractal geometry, Mandelbrot was knitting together a theory of roughness. This changed the notion that pure mathematics was incompatible in representing the irregularity of the substances in the world through the fractal geometry of nature.

Emphasizing the roughness of the universe, Mandelbrot states that everything in the world is rough except for the circles. Geometry and physics have ignored the roughness of reality as they were too complicated. Beyond the regular geometrical structures like circle, cone, sphere, straight line and the like, Mandelbrot wanted to show that the odd shapes that is found in the nature carry meaning. He heralded the arrival of a new geometry which provided a language to describe the nature with its imperfections and irregularities.

The fractal geometry measures the degree of roughness or brokenness or irregularity of an object. The term fractal was coined by Mandelbrot in 1975 from the Latin term *fractus* which describes a broken stone – broken up and irregular. According to Mandelbrot, "Fractals are geometrical shapes that, contrary to those of Euclid, are not regular at all. First, they are irregular all over. Secondly, they have the same degree of irregularity on all scales. A fractal object looks the same when examined from far away or nearby – it is self similar." Fractal geometry is called the geometry of nature because unlike the classical or Euclidean geometry it helps us model things that we see in every day life.

Self similarity, a hall mark of fractals, means the object is similar to a part of itself exactly or in an approximate way. Many examples of fractals can be seen in nature like ferns, cauliflower and many other plants where the branch and twig are very like the whole. There are three types of self similarity namely exact self similarity, quasi self similarity and statistical self similarity. Objects displaying the exact self similarity appear identical at different scales. In quasi self similarity, the fractal would be approximately self similar across scales. Statistical self similarity means a numerical or statistical measure of the fractal is preserved across scales.

Yet another feature of fractal geometry is the fractional or non integer dimension. In 1967, Mandelbrot wrote a seminal paper titled, "How long is the coast of Britain? Statistical self-similarity and fractional dimensionality," in *Science* magazine. Though it seemed to be simple question, the answer would depend on the way we measure the coastline. If we measure it with a meter stick, the answer will be approximate as it would be overlooking the little nooks and crannies. If measured with a smaller scale then we would get a greater length for the coastline as it would go into smaller spaces of the coastline and count them. Thus the length of the coastline would depend on the one's choice of measuring stick. This problem according to Mandelbrot can be solved only by moving away from the ordinary three dimensions to fractal dimensions.

Going beyond the classical Euclidean geometry that works with objects existing in integer dimension, Mandelbrot moved to the non integer dimension. Non integer dimension helps in measuring the qualities of an object which exhibits roughness or brokenness or irregularity. He specified the ways of calculating the fractal dimension of real objects and stated that irregularity remains constant over different scales. The fractal used to model coastlines have the dimension between 1 and 2, while those used to model mountain surfaces have dimension between 2 and 3. The fractal dimension of earth's surface as indicated by NASA photos is 2.1 while that of Mars is 2.4.

Even though the term fractal has been coined recently by Mandelbrot; mathematicians knew well about the complex fractal structures for a century or more. For pure mathematicians these fractal sets that they derived from complex numbers were a means of entertainment. During those times geometry was considered inferior to pure mathematics. Mathematicians remained aloof from the real world confining themselves to the world of numbers. With the arrival of Mandelbrot, mathematics again combined hands with reality. Thus, fractal sets which were once the object of bizarre fancies of mathematicians were found to be useful for describing diverse natural phenomena.

Check Your Progress I

Note: Use the space provided for your answers.

1) Why was Euclidean geometry unable to represent the roughness of reality?

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2) Explain the fractional dimension and self similarity associated with fractal geometry.

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2.4 SOME FAMOUS FRACTALS

In order to model the fractal shapes found in nature, geometric figures that exhibit precise self-similarity are constructed. Iteration is the method by which mathematical fractals are constructed. It means continuously repeating a certain mathematical operation. There are three types of iterations namely generator iteration, Iterated Function System (IFS) Iteration and Formula Iteration. In generator iteration, certain geometric shapes are repeatedly substituted by other shapes to create a fractal. Iterated Function System iteration applies geometric transformation repeatedly in order to create a fractal. In the formula iteration a certain mathematical formula or several formulas are repeated to create a fractal.

Koch Curve

Koch curve or the Snow flake curve is one of the simplest fractal shapes generated by iteration. It was discovered by Swedish Mathematician Helge von Koch in 1906. The process starts with a line segment. It is divided into three equal parts and the central section is replaced by two sides of an equilateral triangle. This process is repeated for the four segments repeated after the first iteration and the process is repeated to get the Koch curve.

The Cantor Set

Cantor set is named after the nineteenth-century mathematician George Cantor. The generation of this set involves iteration of a single operation on a line of unit length. The middle third from each line segment of the previous set is removed with each iteration. With the increasing iterations the number of separate line segments tends to infinity while the length of each segment approaches zero. Magnification reveals self similarity as the structure is essentially distinguishable from the whole.

The Sierpinski Triangle

Named after the polish mathematician Waclaw Sierpinski, Sierpiniski triangle is made by making infinite removals. Each triangle is divided into four smaller, upside down triangles. Then we remove the centre of the four triangles. Iterating this process, an infinite number of times the total area of the set tends to infinity as the size of each new triangle goes to zero. Upon zooming into the completed Sierpinski triangle to any depth would reveal the exact replica of the entire Sierpinski triangle. It provides one of the basic examples of a self similar set.

The Mandelbrot Set

The Mandelbrot set, one of the famous fractals in existence, is the modern development of a theory developed independently in 1918 by Gaston Julia and Plerru Fatou. This set is named after Benoit Mandelbrot. It is born out of the quadratic equation $z = z^2+c$, where z and c are complex umbers. Mandelbrot set is the set of all complex c such that iterating $z = z^2+c$ does not diverge. One of the interesting features of Mandelbrot set is that it retains its highly complicated structure even while zooming at higher levels of magnification.

The Julia Set

The Julia set was discovered by Gaston Julia and Pierre Fatou. The Julia shares a very close relation with Mandelbrot set. It is the difference in function iteration that separates Julia set and Mandelbrot set. The Mandelbrot set iterates $z = z^2+c$ with z always starting at 0 and varying the c value. The Julia set iterates $z = z^2+c$ c for a fixed c value and varying z values. While the Julia set is in the dynamical space or the z plane, the Mandelbrot set is in the parameter space, or c plane.

Check Your Progress II

Note: Use the space provided for your answers.

1) What is iteration used for and what are its different types?

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2) How is a Mandelbrot set created?

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2.5   PRACTICAL APPLICATIONS OF FRACTALS

The fractal concept and the fractal objects find immense use in diverse fields. From image compression to finance, artists and scientists are experiencing the value of fractals. Fractal Geometry is used to describe many complex phenomena. From physics to astrophysics, biology to chemistry and even in market fluctuations fractals have found various applications.

Yet another important application of fractals is in the non linear dynamics, especially with regard to chaos theory. Both chaos theory and fractals appeared unrelated in the nineteen seventies during their infancy even though they are mathematical cousins. The predictability in a chaotic system is ruled out due to sensitive dependence on initial condition and makes one wonder about the scope of finding any order in the system. Here fractal structure of the strange attractor comes to the rescue. Fractals help us decode the language of chaos and present us with a meaningful picture. The strange attractors of chaotic systems are often fractal. This does not mean that all fractals are examples of chaos. Chaos naturally produces fractals. As nearby trajectories expand they must eventually fold over close to one another for the motion to remain finite. This is repeated again and again, generating folds within folds, up to infinity. The beautiful microscopic structure of chaotic attractors results from this process. The advantage of fractal image is that the extraordinary detail present in fractal images can be generated by very simple recipes. The fractal intricacy of an attractor gives the best solution to the modelling problems that involves a structure with infinite intricacy and having the characteristic of chaos.

Mandelbrot was involved in the study of financial prices during the early and toward the end of his career. He stated that the price fluctuations, as assumed by economists are not smooth but often discontinuous and irregular and the most important ones are concentrated in time in such a way that the wealth acquired in stock markets is confined to a very small number of favourable periods. Great attention is received for Mandelbrot's notions of fractals and multi-fractal forms of economic concentration. His study of the financial markets in 1963 resulted in a paper titled, "The Variation of Certain Speculative Prices." and in 2004 he co-authored a book with journalist Richard Hudson titled *The (Mis) Behaviour of Markets: A Fractal View of Risk, Ruin, and Reward*.

Fractals aid in modelling self similar natural forms. It helps in mimicking the large scale real world objects. Fractals comes in handy for an environmentalist who want to estimate the effect of a disaster like a large scale oil spill or the doctor who needs to calculate the surface area covered by bronchial tubes within a human lung. In both the cases fractals are used to an approximate the structure of a real object for the better understanding and implementation.

Fractals create amazing visual effects which has been successfully tried and tested in many films like Star Wars and Star Trek. They not only have the aesthetic value but also trick the mind. The Fractal images are aesthetically and economically far better alternative to the outdoor costly and elaborate shooting sets.

Fractals are used in architecture. According to Mandelbrot, "some cultures have a very strong fractal aspect; Persian, Indian and Mughal architecture often show the contours of small domes within large ones." Identifying the fractal structure of a building helps one to adjust the structure so as to improve the strength and integrity.

Fractals also find use in psychology and counselling. Theoretical biologists have began finding fractal organization that control structures all through the body a decade after Mandelbrot published his physiological speculations. Geologists have found that the distribution of earth quakes fitted into a mathematical pattern which is fractal. The fractal patters also helps in making weather predictions.

2.6   SIGNIFICANCE OF FRACTALS

Fractals have stimulated new and deeper investigations in diverse fields. It unravels a new regime of nature which is subjected to mathematical modelling. The formless and irregular patterns receive a new meaning with fractals. Fractals convincingly explain and explore the innumerable complex natural phenomena which were conveniently bypassed for the want of scientific explanation.

Fractal geometry was successful in representing the reality with its irregularities and imperfections which classical and Euclidean geometry could not do. The idea of fractal dimension was a worthy replacement for the Euclidean measurements that failed to capture the essence of irregular shapes. Thus, rather than bracketing out the roughness of the reality as an imperfection, fractals helps in incorporating them with others thus resulting in a holistic understanding of the reality.

It is impossible to quantitatively measure the individual details of a fractal. We have a shift from quantity to quality. The fractal dimension gives the numerical measure of the degree of roughness of a fractal. The understanding of fractals displaces the general notion that the 'complexity of structure is a result of complicated interwoven processes.' This would mean acknowledging the greater significance of simple process, which could result in complex patterns. The fractal patterns arising out of the iteration of simple equations underline this fact.

The fractal dimension has reignited the interest in mathematics by creating a renewed interest both for teaching and studying. Fractals are extremely useful in teaching mathematics in combination with physics and some aspects of art. Mandelbrot finds the tremendous interest seen in fractal geometry outside the mathematics community as a healthy development. This would impart mathematical literacy to the otherwise highly educated people. As Ian Stewart states, "So today, fractals appear in science in two different ways. They may occur as the primary object, a descriptive tool for studying irregular process and forms. Or they may be a mathematical deduction from an underlying chaotic dynamic."

Check Your Progress III

Note: Use the space provided for your answers.

1) How is chaos theory and fractals related?

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2) Explain how fractals have stimulated new and deeper investigations in diverse fields.

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2.7 LET US SUM UP

In this unit we have initiated a discussion on the concept of fractals. In comparison to Euclidean geometry that strived for Platonic perfection and lost out in representing the roughness of reality, fractal geometry is presented as means to picture the roughness of reality in the best possible way. We have seen how Benoit Mandelbrot related the fractals, which once remained an exclusive property of mathematicians in successfully explaining the complexity of nature. Some of the famous fractals and the way in which they were modelled also have been discussed in brief. After the discussion on the practical application of fractals, we conclude by outlining the far reaching significance fractals have in store for mankind.

2.8 KEY WORDS

Benoit B. Mandelbrot: The mathematician who coined the term Fractal. It was Mandelbrot that the fractal sets which were once the object of bizarre fancies of mathematicians were found to be useful for describing diverse natural phenomena.

Fractals: A fractal is a rough geometric shape that can be split into parts, each of which is exactly, approximately or statistically a reduced-size copy of the whole, a property called self-similarity.

Fractal Dimension: Fractal dimension is the fractional or non integer dimension which helps in measuring the qualities of an object which exhibits roughness, brokenness or irregularity in a better way.

2.9   FURTHER READINGS AND REFERENCES

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