UNIT 3 INDIRECT PROOF

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3.0. OBJECTIVES

The central theme of this unit is to provide an exposition of Indirect Proof. This will bring us to the last Rule with which we are concerned in our analysis of arguments which comprise of compound and simple propositions. As mentioned in the previous units, this is another tool devised to test arguments in as simple manner as possible.

3.1. INTRODUCTION

Indirect Proof, as its name suggests, is a proof procedure that establishes the validity of an argument indirectly when it is either very difficult to prove directly or just impossible. Modern logic adapted this method from geometry when arguments with which it is concerned were of varying complexities. This addition rendered the task of logician much easier.

3.2. THE MEANING OF INDIRECT PROOF

The method of indirect proof is often called Reductio ad Absurdum (R A A), a method quite common in the construction of proof of geometrical theorems. This method is characterized by a special feature. In order to prove a certain statement, its contradiction is assumed to be true from which the conclusion is logically deduced which in turn contradicts our assumption. Suppose that A is derived from certain premises. If A contradicts B, then either A must be false or B must be false. A cannot be false because it is logically deduced from what is purported to be true. Therefore B must be false, which means that A must be true. This is how a theorem in geometry or an argument in logic is, sometimes, proved.

This method has a distinct advantage. Sometimes the length of proof is too long. In logic it is important that we use least number of steps. Second requirement is clarity. Combination of these two is what is most desired and also desirable. In such circumstances this method is most useful. The use of this method consists in beginning with the contradiction of what is to be proved. A point to be noted here is that, the contradiction of what has to be proved is marked by writing I.P. or R. A. on the right hand side (R.H.S.) just adjacent to the assumption. In this unit both abbreviations are used for the sake of familiarity. Here, the expression R.A. and R.P. stand for 'Reductio assumption' and 'Reductio proof' respectively. The denial of conclusion is named as Reduction Proof (R.P.).

In C.P. also we begin with an assumption. The difference is that in I. P. what is assumed is the contradiction of the conclusion whereas in the case of C. P. it is the antecedent of the conclusion. Consider this argument.

1)

1.	A => (B ^ C)		
2.	(B v D) => E		
3.	D v A	/ ∴ E	
4.	¬ E		I.P. (R. A.)
5.	¬ (B v D)	2, 4,	M.T.
6.	¬B ^ ¬ D	5,	De. M.
7.	¬ D	3,	Simp.
8.	A	3, 7,	D.S.
9.	B ^ C	1, 8,	M.P.
10.	B	9,	Simp.
11.	B v D	10,	Add.
12.	E	2, 11,	M.P.
13.	E ^ ¬ E	12, 4,	Conj.

From 10th Step onwards the problem can be reworked in this manner.

11.	¬B	6,	Simp.
12.	B ^ ¬ B	10, 11,	Conj
13.	∴ E		R. P.

Whether we get E ? E or B ? B, the result remains the same. In both the cases there are two steps in the argument whose conjunction leads to contradiction. Whenever there is contradiction one conjunct must be false so that the other one has to be true.

Check Your Progress I.

Note: Use the space provided for your answer.

1) Explain the scope of Indirect Proof.

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2) Give the meaning of R.A and R.P?

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3.3. APPLICATION OF INDIRECT PROOF

We learnt the application of Rules on earlier occasions when we considered arguments with varied structure. On similar lines, Indirect Proof is applied to arguments and thereby validity of arguments is tested.

2)	1.	(A v C)	
	2.	(A => C) / ∴ C	
	3.	¬C	R.A
	4.	¬A	2, 3, M.T.
	5.	C v A	1, Com.
	6.	A	5, 3, D.S.
	7.	A ^ ¬A	6, 4, Conj.
	8.	C	R.P.

7th step involves contradiction. The final step in which the conclusion is repeated is redundant, but it is permitted for the sake of comprehensiveness. The names 'reductio assumption' and 'reduction proof' are not very frequent and we have several other usages. Yet the names introduced here will serve the purpose. Accomplishing an explicit contradiction itself is more than adequate to show that the preferred conclusion is derivable, because (A?~A) C is tautology. The unique conclusion can be derived after accomplishing an explicit contradiction by taking the negation of the conclusion in question as a conditional assumption.

3)	1.	A v (B ^ C)			
	2.	A => C	/ ∴ C		
	3.	¬C	I.P.		
	4.	¬A		2, 3,	M.T.
	5.	B ^ C		1, 4,	D.S.
	6.	C		5,	Simp.
	7.	C ^ ¬C		6, 3,	Conj.
	8.	∴C		R. P.	

7th step involves contradiction; therefore ¬C is false which means that C is true.

4)	1.	(D v E) => (F => G)		
	2.	(¬G v H) => (D ^ F) /∴G		
	3.	¬ G	I.P.	
	4.	¬G v H	3,	Add.
	5.	D ^ F	2, 4,	M.P.
	6.	D	5,	Simp.

7.	D v E	6,	Add.
8.	F => G	1, 7,	M.P.
9.	¬F	8, 3,	M.T.
10.	F	5,	Simp.
11.	F ^ ¬F	10, 9,	Conj.
12.	∴G	R. P.	

11th step is contradiction. Therefore G is false; which means that G is true

5)

1.	(H => I) ^ (J => K)		
2.	(I v K) => L		
3.	¬L	/ ∴ ¬(H v J)	
4.	H v J	I.P.	
5.	I v K	1, 4,	C.D.
6.	L	2, 5,	M.P.
7.	L ^ ¬L	6, 3,	Conj.
8.	∴ ¬(H v J)	R. P.	

7th step involves contradiction. Therefore 4 is false; which means that ¬(H v J) is true.

6)

1.	(M v N) => (O ^ P)		
2.	(O v Q) => ¬ R ^ S		
3.	(R v T) => (M v N) / ∴ ¬R		
4.	R		I.P.
5.	R v T	4,	Add.
6.	M v N	3, 5,	M.P.
7.	O ^ P	1, 6,	M.P.
8.	O	7,	Simp.
9.	O v Q	8,	Add.
10.	¬ R ^ S	2, 9,	M.P.
11.	¬ R	10,	Simp.
12.	R ^ ¬ R	4, 11,	Conj.
13.	∴ ¬ R	R. P.	

12th step involves contradiction. Therefore R is false which means that R is true.

7)

1.	(V => ¬ W) ^ (X => Y)		
2.	(¬ W => Z) ^ (Y=> ¬ A)		
3.	(Z => ¬ B) ^ (¬A => C)		
4.	V ^ X / ∴ ¬ B ^ C		
5.	¬ (¬ B ^ C)		I.P.
6.	B v ¬ C	5,	De. M.
7.	¬ Z v A	3, 6,	D.D.
8.	W v ¬Y	2, 7,	D.D.
9.	¬V v ¬X	1, 8,	D.D.
10.	(V ^ X) ^ (¬ V v ¬ X)	4, 9,	Conj.
11.	∴ ¬ B ^ C	R. P.	

10th Step involves contradiction. According to de Morgan's law (V ^ X) and (¬V v ¬X) are contradictories. Therefore ¬ (¬B ^ C) is false, which means that (¬B ^ C) is true. We can also prove these arguments using formal proof of validity. Consider the 5th argument.

8)

1.	(H => I) ^ (J => K)		
2.	(I v K) => L		
3.	¬ L	/ ∴ ¬ (H v J)	
4.	¬(I v K)		2, 3 MT.
5.	¬ I ^ ¬ K		4, De. M.
6.	¬ I		5, Simp.
7.	¬ I v ¬ K		6, Add.
8.	∴ ¬ H v ¬ J		1, 7, D. D.

When the 5th argument was solved using IP method, it involved thirty three words and five steps, whereas formal proof required forty words and nine steps. Therefore the former is shorter and preferable.

Now consider the seventh argument.

9)

1.	(V => ¬ W) ^ (X => Y)			
2.	(¬ W => Z) ^ (Y => ¬ A)			
3.	(Z=> ¬B) ^ (¬A=> C)			
4.	V ^ X	/ ∴ ¬ B ^ C		
5.	V => ¬ W		1,	Simp.
6.	V		4,	Simp.
7.	¬ W		5, 6,	M.P.
8.	X => Y		1,	Simp.
9.	X		4,	Simp.
10.	Y		8, 9,	M.P.
11.	¬ W=> Z		2,	Simp.
12.	Z		11, 7,	M.P.
13.	Y => ¬ A		2,	Simp.
14.	¬ A		13, 10,	M.P.
15.	Z => ¬ B		3,	Simp.
16.	¬ B		15, 12,	M.P.
17.	¬ A => C		3,	Simp.
18.	C		17, 14,	M.P.
19.	∴ ¬ B ^ C		16, 18,	Conj.

When the 7th argument was solved using I.P. method, it involved fifty seven words and seven steps, whereas formal proof required ninety words and nineteen steps. Therefore the former is shorter and preferable.

We learnt in the earlier unit that sometimes C. P. is shorter than formal proof and sometimes it is longer than formal proof. Same situation prevails in the case of I. P. also. Consider the following example.

1	H => (I => J) `	
2	K => (I => J)	
3	(¬ H ^ ¬ K) => (¬ L v ¬ M)	
4	(¬ L => ¬ N) ^ (¬ M => ¬ O)	
5	(P => N) ^ (Q => O)	
6	¬ (I => J)	/∴ ¬ P v ¬ Q
7	¬ H	1,6, M.T.
8	¬ K	2,6, M.T.
9	¬ H ^ ¬ K	7,8, Conj.
10	¬ L v ¬ M	3,9, M.P.
11	¬ N v ¬ O	4,10, C.D.
12	∴ ¬ P v ¬ Q	5,11, D.D.

This proof construction consists of forty two words, six lines and five Rules used six times. Let us use Indirect Proof method to know which method is shorter and simpler.

1	H => (I => J) `	
2	K => (I => J)	
3	(¬ H ^ ¬ K) => (¬ L v ¬ M)	
4	(¬ L => ¬ N) ^ (¬ M => ¬ O)	
5	(P => N) ^ (Q => O)	
6	¬ (I => J)	/∴ ¬ P v ¬ Q
7	¬ (¬ P v ¬ Q)	I.P.
8	P ^ Q	7, De. M.
9	P	8, Simp.
10	P v Q	9, Add.
11	N v O	5, 10, C.D.
12	L v M	4, 11, D.D.
13	¬ H	1, 6, M. T.
14	¬ K	2, 6, M.T.
15	¬ H ^ ¬ K	13, 14, Conj.
16	¬ L v ¬ M	3, 15, M. P.

It must be more than obvious that even after ten steps and sixty nine words and the application of nine Rules I. P. did not yield the expected results. It must be noted that L v M does not negate the twelfth step i.e. L v M is not the contradictory of the sixteenth step i.e., L v M. Therefore even if subsequent steps yield the desired result, it is, surely, not profitable to follow the I. P. method in this case.

3.4 EXAMPLES

1)

1 (P v Q)	
2 (P = > Q)	/ ∴ Q
3 ¬ Q	R.A
4 ¬ P	2, 3, M.T.
5 Q ∨ P	1, Com.
6 P	5, 3, D.S.
7 P ^ ¬ P	6 , 4, Conj.
8 Q	R.P.

2)

1 (A$\vee$ B)	
2 (A = > B)	/$\therefore$ B
3 $\neg$ B	R.A
4 $\neg$ A	2, 3,M.T.
5 B $\vee$ A	1, Com.
6 A	5, 3, D.S.
7 A ^ $\neg$ A	6 , 4, Conj.
8 B	R.P.

3)

1. P => (Q => R)	
2. S $\vee$ (P $\vee$ R)	
3. P => Q	/$\therefore$ (S $\vee$ R)
4. $\neg$ (.S$\vee$ R)	R.A.
5. $\neg$ S ^ $\neg$ R	4, De .M.
6. $\neg$ S	5, Simp.
7. P $\vee$ R	2, 6, D.S
8. (P ^ Q) => R	1, Exp.
9. P => (P ^ Q)	3, Abs.
10 P => R	9, 8, H.S.
11 $\neg$ R ^ $\neg$ S	5, Com.
12. $\neg$ R	11, Simp.
13. $\neg$ P	10, 12, M.T.
14. R $\vee$ P	7, Com.
15. P	14, 12, D.S .
16. P ^ $\neg$ P	15, 13, Conj.
17. $\therefore$S $\vee$ R	R.P.

3.5 EXERCISES ON INDIRECT PROOF

Evaluate the relative advantages and disadvantages of formal proof and I. P. methods with the help of following arguments.

1.
(B v N) => (K ^ L)
$\neg$ K
$\neg$ M /$\therefore$ $\neg$ B ^ $\neg$ M

2.
(M v N) => (P ^ Q)
N /$\therefore$ P

3. A=> (B ^ C)
$\neg$ B / $\therefore$ $\neg$ A

4. P $\vee$ Q
P $\vee$ $\neg$ Q /$\therefore$ P

5. A => B
A $\vee$ B /$\therefore$B

1. (B v N) => (K ^ L)
– K

6. $(M \vee \neg M) \Rightarrow \neg(\neg N \wedge \neg O)$
$(N \vee O) \Rightarrow \neg P \quad / \therefore \neg P$

7. $[(W \vee X) \Rightarrow (Y \wedge W)]$
$(X \Rightarrow Y)$
$[\neg Z \Rightarrow (W \vee X)] \quad / \therefore (Z \vee W)$

8. $P \Rightarrow (Q \wedge R)$
$\neg Q \quad / \therefore \neg P$

9. $\neg(\neg P \wedge \neg Q)$
$\neg P \Rightarrow \neg Q \quad / \therefore P$

10. $\neg B \Rightarrow \neg A$
$\neg(\neg A \wedge \neg B) \quad / \therefore B$

11. $(A \vee \neg A) \Rightarrow \neg(\neg B \wedge \neg O)$
$(B \vee O) \Rightarrow \neg Q \quad / \therefore \neg Q$

12. $[(X \vee W) \Rightarrow (W \wedge Y)]$
$(\neg Y \Rightarrow \neg X)$
$[\neg Z \Rightarrow (X \vee W)] \quad / \therefore (W \vee Z)$

13. $\neg A \vee (C \wedge X)$
$\neg X \quad / \therefore \neg A$

14. $B \vee A$
$\neg B \vee A \quad / \therefore A$

15. $\neg A \vee D$
$A \vee D \quad / \therefore D$

16. $(P \vee \neg P) \Rightarrow \neg(\neg N \wedge \neg O)$
$(N \vee O) \Rightarrow \neg Q \quad / \therefore \neg Q$

17. $[\neg(\neg W \wedge \neg X) \Rightarrow \neg(\neg Y \vee \neg W)]$
$(\neg Y \Rightarrow \neg X)$
$[\neg Z \Rightarrow \neg(\neg W \wedge \neg X)] \quad / \therefore \neg(\neg Z \wedge \neg W)$

18. $A \text{ v } (B \wedge C)$
$A \Rightarrow C \ / \therefore C$

19. $(D \text{ v } E) \Rightarrow (F \Rightarrow G)$
$(\neg G \text{ v } H) \Rightarrow (D \wedge F) \ / \therefore G$

20. $(G \Rightarrow H) \Rightarrow (I \text{ v } J)$
$K \text{ v } \neg(L \Rightarrow M)$
$(G \Rightarrow H) \text{ v } \neg K$
$N \Rightarrow (L \Rightarrow M)$
$\neg(I \text{ v } J) \ / \therefore \neg N$

3.6 INDIRECT PROOF AND PROOF OF TAUTOLOGY

Just as arguments are classified as valid and invalid, statements are classified as tautologous and nontautologous. Under the latter category there is further classification into contingent and contradiction. All conditional arguments can be transformed into statement forms. If an argument is valid, then its corresponding statement form is tautologous and if a statement form is tautologous, then its corresponding conditional argument is necessarily valid. Such statement form also is conditional whose premise is the antecedent and conclusion is the consequent of the original argument. We must remember that disjunctive form also is conditional.

Consider the simplest conditional argument with two variables; p and q.

$$p \Rightarrow q$$
$$p$$
$$\therefore q$$

Since this argument is valid we should first determine the truth-value of the premises and conclusion in order to ensure that false conclusion is not derived from true premises. This can be achieved with the help of truth-table.

p	q	¬ p	¬ q	p	=>	q
1	1	0	0	1	1	1
1	0	0	1	1	0	0
0	1	1	0	0	1	1
0	0	1	1	0	1	1

There are four rows in which the truth-value of p => q needs to be determined. If there are three variables, then we will have eight rows. It means that the number of rows is expressed in the form of formula

$$2^n$$

where n stands for the number of variables. Against this background, we shall consider statement form for the conditional argument of the form mentioned above.

(For the sake of simplicity we can omit negations of p and q since they are not required). We obtain the statement form by conjoining premises to which the conclusion is connected using implication again.

p	q	p => q	[{(p => q) ^ p} => q]
1	1	1 1 1	1 1 1 1 1
1	0	1 0 0	0 0 1 1 0
0	1	0 1 1	1 0 0 1 1
0	0	0 1 1	1 0 0 1 0

It should be noticed that in the last but one column the truth-value obtained is 1 in all instances. Therefore we say that the statement form is tautologous. If the value happened to be 0 in all instances, then, the statement form becomes contradictory. On the other hand, if the statement form takes 1 and 0 in different instances, then the statement form is said to be contingent. Let us restrict ourselves to tautology. When there are many variables truth-table method becomes quite complex and not viable on practical grounds. In such circumstances I.P. method becomes useful. Consider an example.

1)

1	(A => B) v (A => ¬ B)	
2	¬ {(A => B) v (A => ¬ B)}	I. P.
3	¬ (A => B) ^ ¬ (A => ¬ B)	2, De. M.
4	¬ (¬ A v B) ^ ¬ (¬ A v ¬ B)	3, Impl.
5	(A ^ ¬ B) ^ (A ^ B)	4, De. M.
6	A ^ ¬ B	5, Simp.
7	¬ B	6, Simp.
8	A ^ B	5, Simp.
9	B	8, Simp.
10	B ^ ¬ B	9, 7, Conj.

11 In tenth step there is contradiction. Therefore ¬ {(A => B) v (A => ¬ B)} is false which shows that (A => B) v (A => B) is tautologous.

2)

1	(A => B) v (B => A)	
2	¬ [(A => B) v (B => A)]	I. P.
2	¬ (A => B) ^ ¬ (B => A)	2, De. M.
3	¬ (¬ A v B) ^ ¬ (¬ B v A)	3, Impl.
4	(A ^ ¬ B) ^ (B ^ ¬ A)	4, De. M.
5	A ^ ¬ B	5,Simp.
6	A	6, Simp.
7	B ^ ¬ A	5, Simp.
8	¬ A	8, Simp.
9	A ^ ¬ A	8, 9, Conj.

Result is similar to the first argument.

3)

1	(A => B) v (¬ A => B)	
2	¬ [(A => B) v (¬ A => B)]	I. P.
3	¬ (A => B) ^ ¬ (¬ A => B)	2, De.M.
4	¬ (¬ A v B) % ¬ (A v B)	3, Impl.
5	(A ^ ¬ B) ^ (¬ A ^ ¬ B)	4, De. M.
6	A ^ ¬ B	5, Simp.
7	A	6,Simp.
8	¬ A ^ ¬ B	5, Simp.
9	¬ A	8, Simp.
10	A ^ ¬ A	7, 9, Conj.

In this proof system also we have obtained the same result. It means that the second and third statements are tautologous.

Combination of *Reductio* ad absurdum and truth- table methods is another technique of testing arguments. While doing so we have to make two assumptions. In the first place we must assume that all premises are true. Secondly, the conclusion must be assumed to be false. If this combination can be achieved, then the given argument is invalid. Otherwise, the argument must be valid. Examine this argument.

1)

1 (A v B) => (C ^ D)

2 (D v E) => F/ ∴ A => F

The conclusion is false only if A is true and F is false. The second premise can be true provided D and E are false because F is false. If the first premise must be true then false consequent should not be implied by true antecedent. In the first premise the consequent is false because D is false and conjunction of which D is a component is false. Since the consequent is false the antecedent must be false so that the implication is true. But the antecedent is true because one of the components of the antecedent is true. Therefore antecedent fails to satisfy the requirement. This shows that the consequent cannot be false.

Now consider an argument with conjunctive conclusion.

2)

(B v N) => (K ^ L)

¬K

¬M / ∴ ¬ B ^ ¬M

Assume that the conclusion is false. Then at least one component must be false. Let us assume that B is false. Then B must be true. Therefore (K Λ L) must be true. This is possible when both K and L are true. But when K is true ¬K is necessarily false. We have assumed that every premise must be true. The obtained result contradicts our assumption. Therefore the conclusion must be true. Therefore the argument is valid.

Now consider an argument with simple conclusion.

3)

(M v N) => (O ^ P)

(O v Q) => ¬R ^ S

(R v T) => (M v N) / ∴ R

Let R be false. Then ¬R Λ S is false. Therefore the second premise can be true only if (O v Q) is false. (O v Q) can be false only when both the components are false. Since O is false, (O ^ P) is also false. Therefore the first premise can be true only if (M v N) is false. Now we shall examine the last premise. Since the consequent is false, the antecedent also must be false if the premise must be true. However, R is true since R is false. (R v T) is true since R is true. When we derive false conclusion from true antecedent the premise becomes false which contradicts our assumption which states that all premises must be true. Therefore the conclusion must be true.

If verbal explanation is replaced by assigning of truth-values to all the variables and sentential connectives, then the determination of validity becomes far simpler than one can imagine. This is left for the reader as an exercise.

Check your progress II.

Note: Use the space provided for your answer.

Use I. P. and formal methods to test the following arguments.

1) A v B

¬ (A ^ B) / ∴B

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2) ¬ (P ^ ¬ Q)

¬ (¬Q ^ ¬P) / ∴ Q

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3.7 LET US SUM UP

In this unit we have provided an exposition of indirect proof, by giving definition and principles of this method. The conclusion ought to be negated and assumed as an additional premise (R.A.). Once it is reduced to absurdity, we restated the conclusion and named it as Reduction proof (R.P.). With the help of this method we have learnt that logicians use indirect method to prove an argument with least effort.

3.8 KEY WORDS

Theorem: A logical truth.

Valid argument form: An argument form which has no invalid substitution instance.

Contradiction: An obvious contradiction that is a substitution instance of the statement form (A ^ ¬ A)

Reductio Ad Absurdum (R.A.A): Proof technique of reducing an assumption to absurdity by deriving explicit contradiction from it.

3.9 FURTHER READINGS AND REFERENCES

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