CALCULUS II: REVIEW OF CALCULUS I

Contents

1. Motivation

This section provides an introduction to the topics that we will learn in our Calculus II course and how they relate and build upon the topics that we learned in Calculus I, as well as discussing how we will complexify and expand upon these topics in Calculus III and Multivariable Calculus.

Our course begins with a simple question: given a curve $y = f(x)$ in the cartesian plane between two x-values a and b where $a \leqslant b$, what is the area of the shape under the curve and above the x-axis? This is known as the *area problem*. For example, observe the figure to the right. Figure 1 shows the graph of a particular function $y = f(x)$ including a blue shaded region below $f(x)$ and above the x-axis betweeen $x = 2$ and $x = 6$. A primary motivation behind this class is: given a function $f(x)$ and values a and b, when can we find the area under the curve and what is the area?

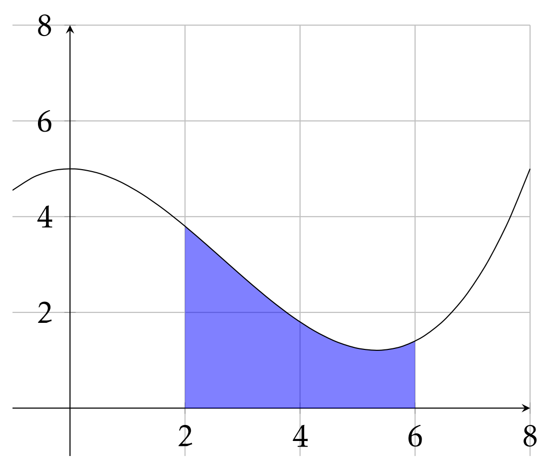

Figure 1. Area under $y = f(x)$.

We will strive to answer this question for the first several weeks of the class and, about halfway through, we will formulate a way to calculate this area. In the process we will discover connections to derivatives and limits from Calculus I and explore the mathematical concepts that appear in our investigation of solving the are problem.

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2. Back to Calculus I

Recall that the two main concepts from Calculus I were *limits* and *derivatives*.

2.1. **Limits.** Limits are an essential tool in mathematics. They are the basic tool that we use to find derivatives. They will also be a very important in helping us solve the area problem.

Definition of Limit

The *limit* of the function $f(x)$ as x approaches a is L if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$. If the limit of $f(x)$ as x approaches a is L, we write

$$\lim_{x \to a} f(x) = L.$$

This is the rigorous mathematical definition of a limit in mathematics using universal and existental quantifiers. It is a fancy way of saying "as x gets really close to the value a, the value of the function $f(x)$ gets really close to L." You may ask then, why do we go through the trouble of this definition. The answer is that all mathematics must be painstakingly formulated so there is no room for ambiguity. For example, "really close" could mean many things, and the exact definition tells us what it means in the context of a limit. This is the realm of a course called *Real Analysis* that goes into the foundations of calculus that we will not discuss here.

Before we move on to derivatives, let's look at a few examples of how to calculate limits of functions. This first example will reintroduce continuity and the helpfulness that it can play when we calculate limits. The second example will be a bit more difficult and remind us that, sometimes, we have to do a bit of algebra before we can calculate the limit.

Example

Determine whether $\lim_{x \to 5} \frac{1}{4}\left(x^2 - 4x + 2\right)$ exists. If so, find the limit.

Solution. There are a few ways to solve this problem. Firstly, we can graph this function and find values as x gets closer and closer to 5. A video will be provided and linked for this method. Another way is that we can recall that this function

$$f(x) = \frac{1}{4}\left(x^2 - 4x + 2\right)$$

is a polynomial therefore it is continuous. This means that we can simply plug $x = 5$ into the function $f(x)$ to find the limit, so the limit exists and

$$\lim_{x\to 5} \frac{1}{4}\left(x^2 - 4x + 2\right) = \frac{1}{4}\left((5)^2 - 4(5) + 2\right) = \frac{7}{4}.$$

□

This problem reminds us of another concept from Calculus I, *continuity*.

Definition of Continuous

A function $f(x)$ is *continuous* at $x = a$ if (1) the limit of $f(x)$ as x approaches a exists, (2) the value of $f(x)$ at a exists, and

$$(3) \quad \lim_{x\to a} f(x) = f(a).$$

Recall from Calculus I that all polynomials are continuous as well as other common functions such as $\cos(x)$, $\sin(x)$, e^x, $\log(x)$, $|x|$ as well as any linear combinations, products, or compositions of these functions. Let's look at another limits example.

Example

Determine whether $\displaystyle\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2}$ exists. If so, find the limit.

Solution. In this problem, if we try to simply plug $x = 2$ into the function $f(x)$, we will get $\frac{0}{0}$, which is an indeterminate form. This means that we have to do some algebra before trying to plug in the value. Thus we proceed as

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2} = \lim_{x\to 2}(x+3) = 5.$$

We can plug in $x = 2$ at the end of the last line since $x + 3$ is a continuous function.

□

Exercise

Compute the limit

$$\lim_{x\to 0} \frac{x}{3 - \sqrt{x+9}}.$$

2.2. **Derivatives.** We begin by recalling the definition of the derivative of a function at a point.

Definition of Derivative

The *derivative* of $f(x)$ at $x = a$ exists if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

In this case, we denote the derivative of $f(x)$ at $x = a$ as $f'(a)$ and

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

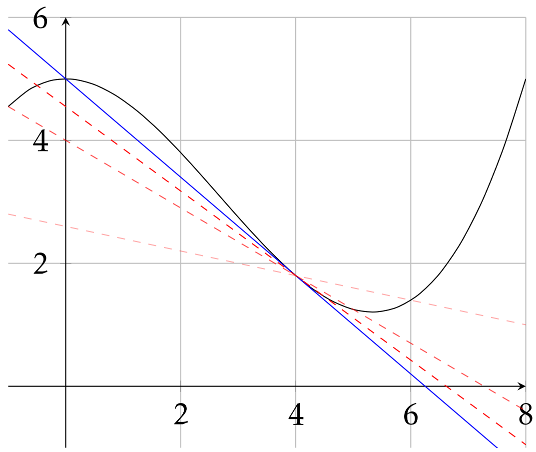

FIGURE 2. Secant lines.

The physical interpretation of the derivative of $f(x)$ at $x = a$ is the slope of the tangent line to $f(x)$ at $x = a$ and is the limit of the slopes of the secant lines. Consider the figure to the right. Figure 2 shows a function $y = f(x)$ with a blue line representing the tangent line to the curve at $x = 4$. The red dashed lines are secant lines passing through $x = 4$ and $x = 6, 5$, and 4.5, respectively. The slopes of these lines are -0.2, -0.55, and -0.6875, slowly approaching the true value of the slope of the tangent line which is -0.8

Remark 1. The derivative can also be defined as a different, but equivalent limit. The derivative of $f(x)$ at $x = a$ can also be calculated as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Let's now look at an example of calculating the derivative of a particular function using the definition.

Example

Find the derivative of $f(x) = x^2 - x$ at $x = a$ using the definition of the derivative.

Solution. Using Section 2.2, we will find a formula for $f'(a)$

$$\begin{aligned} f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} &= \lim_{x\to a}\frac{(x^2-x)-(a^2-a)}{x-a} \\ &= \lim_{x\to a}\frac{x^2-a^2-x+a}{x-a} \\ &= \lim_{x\to a}\frac{(x-a)(x+a)-(x-a)}{x-a} \\ &= \lim_{x\to a}\frac{(x-a)((x+a)-1)}{x-a} \\ &= \lim_{x\to a} x+a-1 \\ &= 2a-1. \end{aligned}$$

Therefore, we find that $f'(a) = 2a-1$. $\square$

Exercise

Find the derivative of $f(x) = x^2 - x$ at $x = a$ using the definition from Remark 1.

After we learned how to calculate derivatives using limits, we found out that we can calculate derivatives much more easily using a few rules.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
k, constant	0
x^n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

Figure 3. Table of Common Derivatives.

Function	Derivative
$y = f(x) + g(x)$	$\frac{dy}{dx} = f'(x) + g'(x)$
$y = kf(x)$	$\frac{dy}{dx} = kf'(x)$
$y = f(x)g(x)$	$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$
$y = \frac{f(x)}{g(x)}$	$\frac{dy}{dx} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$
$y = f(g(x))$	$\frac{dy}{dx} = f'(g(x))g'(x)$

Figure 4. Derivative Rules.

These rules make the example above much simpler, we can simply use rules in these tables to find that $f'(x) = 2x - 1$.