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In - Class Questions 1.3

$\boxed{1}$

- $S(n)$ and ~~r~~ NOT $(3|n)$
- $3|0$, so it ~~is~~ ~~no~~ ~~does~~ ~~not~~ $0 \notin C$
- Since $m>0$, atleast one stamp must have been used. Discarding any one stamp leaves with a valid postage. Since only 6 or 15 cent stamps are there, $S(m-6)$ or $S(m-15)$ holds.
- $m-6<m$ and if NOT $(3|m-6)$, then $m-6 \in C$ contradicting minimality of m.
- ~~C is empty~~. Since $3|m-6$ and $3|m$, $3|(m-6)+6$. Hence $3|m$
- C must be empty. So, there are no counterexamples

$\boxed{2}$

Let C be the set of counterexamples to the proposition. ~~i.e~~ i.e.

$$C := \{n \mid \sum_{k=0} k^2 \neq \frac{n(n+1)(2n+1)}{6}\}$$

Assume C is non-empty for proof by contradiction. By WOP, there is a smallest element $m \in C$. ~~B~~ This $m>0$ because

$$\sum_{k=0}^{0} k^2 = 0^2 = 0 = \frac{0 \times (0+1) \times (2 \times 0+1)}{6}$$

and hence $0 \notin C$.
Since m is the smallest element of C $m-1 \notin C$ and $m-1 \notin \mathbb{N}$ as $m>0$.

$$\Rightarrow \sum_{k=0}^{m-1} k^2 = \frac{(m-1)(m-1+1)(2m-2+1)}{6}$$

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Now, $\sum_{k=0}^{m} k^2 = \sum_{k=0}^{m-1} k^2 + m^2$

$$= \frac{(m-1)(m)(2m-1)}{6} + m^2$$

$$= m\left[\frac{2m^2-3m+1}{6} + m\right]$$

$$= \frac{m(2m^2+3m+1)}{6}$$

$$\sum_{k=0}^{m} m^2 = \frac{m(m+1)(2m+1)}{6}$$

This is a contradiction to $m \in C$. Hence C must be empty, proving the proposition.

$\boxed{3}$

Let C be the set of all a's ~~such that~~ from all (a,b,c,d) such that the equation is satisfied.

$$C := \{a \mid 8a^4 + 4b^4 + 2c^4 = d^4 \text{ for some } b,c,d \in \mathbb{N}^+\}$$

Assume C is non-empty for proof by contradiction.
By WOP, there is a smallest element $p \in C$

$$\Rightarrow \boxed{8p^4 + 4b^4 + 2c^4 = d^4} \text{ for some } b,c,d \in \mathbb{N}^+$$

Since $2 \mid$ LHS, $2 \mid d^4$ and hence $2 \mid d$

$$\Rightarrow d = 2s \text{ for some } s \in \mathbb{N}^+$$

$$\Rightarrow 8p^4 + 4b^4 + 2c^4 = 16s^4$$

Dividing by 2,

$$\boxed{4p^4 + 2b^4 + c^4 = 8s^4}$$

Again $2 \mid$ RHS $\Rightarrow 2 \mid \text{LHS} = 4p^4 + 2b^4 + c^4 \Rightarrow 2 \mid c^4 \Rightarrow 2 \mid c$

$$\Rightarrow c = 2r \text{ for some } r \in \mathbb{N}^+$$

$$\Rightarrow 4p^4 + 2b^4 + 16r^4 = 8s^4$$

Dividing by 2

$$\boxed{2p^4 + b^4 + 8r^4 = 4s^4}$$

Similar to previous arguments $2 | b$, and hence $b = 2q$ for some $q \in \mathbb{N}^+$

$$\Rightarrow 2p^4 + 16q^4 + 8r^4 = 4s^4$$

Dividing by 2,

$$\boxed{p^4 + 8q^4 + 4r^4 = 2s^4}$$

Again, $2 | p$ and hence $p = 2t$ for some $t \in \mathbb{N}^+$

$$\Rightarrow 16t^4 + 8q^4 + 4r^4 = 2s^4$$

Divide by 2 ~~$\Rightarrow 8t^4 + 4q$~~

$$\Rightarrow \boxed{8t^4 + 4q^4 + 2r^4 = s^4}$$

Hence, (t, q, r, s) is a solution of the equation.
But $t = \frac{p}{2} < p$ and $t \in C$.

This contradicts the minimality of p and hence C must be empty proving that there are no solutions $\in \mathbb{N}^+$.

$\boxed{4}$

Let $C := \{m \mid \text{Property } m \text{ does not hold}\}$.
Assume C is non-empty for proof by contradiction.
By WOP, there is a smallest element $n \in C$.
$n \neq 0$ because ~~on~~ $2^{0+1} = 2$ and both $0, 1 < 2$ can be formed with selecting the 1 dollar envelope and ~~not~~- not- ~~of~~ selecting any envelope.
Hence, Property $(n-1)$ holds as n is the minimum element of C. Also since $n > 0$ $n-1 \in \mathbb{N}$.

We use case-analysis

Case 1: The target dollars $0 \le t < 2^n$.
Since Property $(n-1)$ holds, the selection can be made from $1, 2, 4, \ldots 2^{n-1}$ envelopes and not selecting 2^n envelope.

Case 2: Target $2^n \le t < 2^{n+1}$
Let us select the 2^n envelope.
Then remaining $t' = t - 2^n$ are constrained by $0 \le t' < 2^n$ which can be made from $1, 2, 4, \ldots 2^{n-1}$ envelope.

Hence property n holds contradicting $n \in C$.
Hence C must be empty, proving that Property m holds $\forall m \in \mathbb{N}$.

$\boxed{5}$

Let $P(n) :=$ n can be represented as the sum of non-negative integer multiple of 6, 10, 15

Let $C := \{ n \geq 30 \mid \text{NOT}(P(n)) \}$
Assume C is non-empty for proof by contradiction.
By WOP, there is a smallest $m \in C$.

Now, $P(n)$ is true for $n \in \{30, 31, \ldots, 35\}$, as

$30 = 5 \times 6 + 0 \times 10 + 0 \times 15$

$31 = 1 \times 6 + 1 \times 10 + 1 \times 15$

$32 = 2 \times 6 + 2 \times 10 + 0 \times 15$

$33 = 3 \times 6 + 0 \times 10 + 1 \times 15$

$34 = 4 \times 6 + 1 \times 10 + 0 \times 15$

$35 = 0 \times 6 + 2 \times 10 + 1 \times 15$

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Hence, $m \geq 36$. Also $P(m-6)$ is true because $m-6 \geq 30$ and m is minimum element of C.

$\Rightarrow m-6 = 6a+10b+15c \quad a,b,c \in \mathbb{N}$

$\Rightarrow \boxed{m = 6(a+1)+10b+15c}$

As $a+1, b, c \in \mathbb{N}$ $P(m)$ be true contradicting $m \in C$.
Therefore, C is empty. and hence $P(n)$ is true $\forall\ n \geq 30$.