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In-Class Questions Q 1.2

$\boxed{1}$

We use proof by contradiction. Suppose, the claim is false. Then a and b ~~must~~ can be $> \sqrt{n}$ and $a \cdot b = n$.

$$a > \sqrt{n} \text{ so}$$
$$ab > \sqrt{n}\, b > \sqrt{n}\sqrt{n} = n$$
$$\text{Hence, } ab > n$$

This is a contradiction. ~~If~~ Therefore, either a or b must be $\leq \sqrt{n}$.

$\boxed{2}$

The proof is same ~~as~~ for $\sqrt{3}$ with two modifications: replace 2 by 3 and "is even" by "divisible by 3".

Some generalizations:

More general

- For any $m, k > 0$ if m^k is divisible by a prime number, then m must be divisible by p.
- For any positive integer n, k, $\sqrt[k]{n}$ must be either integer or an irrational.

Lemma

- ~~If~~ The real roots of a non-zero polynomial with integer coefficients are either integral or irrational. (Problem 1.15) → solution below

(a) ~~If~~ If $\sqrt[m]{k}$ is not an m^{th} power of an integer, ~~so~~ it is irrational, because it is a root of $x^m - k$, and is not an integer (by definition). Hence, must be irrational.

(b) We use proof by contradiction. ~~A~~ Suppose, a non-integral rational root exists. Then it can be written as $\frac{n}{d}$ where n and d

have no common divisors except 1 and -1, $d \neq 0, 1$, $n \neq 0$. ~~and $d > 0$~~ and $d > 0$

$$a_0 + a_1\left(\frac{n}{d}\right) + a_2\left(\frac{n}{d}\right)^2 + \dots a_{m-1}\left(\frac{n}{d}\right)^{m-1} + \left(\frac{n}{d}\right)^m = 0$$

Multiplying by d^m on both sides

$$\boxed{a_0 d^m + a_1 n d^{m-1} + a_2 n^2 d^{m-2} + \dots + a_{m-1} n^{m-1} d + n^m = 0}$$

We divide it into two cases.

Case 1: $n = 0$. Then $a_0 = 0$. No solution and hence contradiction ($a_0 \neq 0$)

Case 2: $n \neq 0$. Since $d \neq 0, 1$ some prime p divides it. ~~Since p~~ Hence, it divides all powers of d. Since 0 is a multiple of all integers, p divides 0.

Then, p divides $(a_0 d^m + a_1 n d^{m-1} + \dots + a_{m-1} n^{m-1} d + n^m)$

Since p divides $a_0 d^m, a_1 n d^{m-1}, \dots a_{m-1} n^{m-1} d$, p divides n^m. Hence, p divides n by the following lemma:

If a prime p is a factor of some power of an integer, then it is a factor of the integer.

This is a contradiction to the fact that n and d have no common divisors except ± 1. Hence, any real root is either integral or irrational.

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The proof is by case analysis. There are 2 cases:

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Consider $a = \sqrt{2}$, $b = \sqrt{2}$. Then a^b is rational while a and b are both irrational. Hence, we have found the required numbers to prove the hypothesis.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Consider $a = \sqrt{2}^{\sqrt{2}}$, $b = \sqrt{2}$. Then $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$ is rational. Hence, the hypothesis is proved in this case too.

So in any case, there exist $a, b \in \mathbb{R} \setminus \mathbb{Q}$ such that $a^b \in \mathbb{Q}$.

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We use proof by contradiction. Suppose $2\log_2 3$ is rational. Then $2\log_2 3 = \frac{n}{d}$ where $n, d \in \mathbb{Z}, d \neq 0$

and in its lowest terms.

Then $\log_2 3 = \frac{n}{2d} \Rightarrow 3 = 2^{\frac{n}{2d}} \Rightarrow \boxed{3^{2d} = 2^n}$

This is impossible as LHS is divisible by 3, but RHS is not.
This is a contradiction. Therefore, $2\log_2 3$ is irrational.