Linear Algebra

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Chapter - I : Matrices and Linear Equations

a) Important theorems :

- Every matrix has a unique Row Canonical Form
- Let A be a square matrix. Then the following are equivalent :
  (i) A can be reduced to I by a sequence of elementary row operations
  (ii) A is a product of elementary matrices.
  (iii) A is invertible
  (iv) The system $Ax = 0$ has only the trivial solution $x = 0$.

b) Important Applications :

- Gauss Elimination Method (GEM) to solve $Ax = b$:
  We can ~~do~~ easily transform any matrix to the Row - Echelon Form (REF) by the following 3 elementary operations (EROs) :
  (i) Interchange of 2 rows $\boxed{T_{ij}}$
  (ii) Addition of a scalar multiple of a row to another row $\boxed{T_{ij}(m)}$
  (iii) Multiplication of a row by a non-zero scalar. $\boxed{T_i(m)}$

Note : We transform the augmented matrix $[A \mid b]$

A matrix $A \in \mathbb{R}^{m \times n}$ is said to be in REF if and only if :
(i) Non - zero rows precede zero rows.
(ii) If there are r non - zero rows and K_i be the pivotal column of i^{th} row, ~~the K_i's are in~~ $K_1 < K_2 < \dots < K_r$.

Pivot : The first non- zero entry from the left.

Let $x = [x_1 \quad x_2 \quad \dots \quad x_n]^T$

$x_{K_1}, x_{K_2} \dots x_{K_r}$ are the pivotal variables and other are free variables.

To solve $Ax=b$ if A is in REF:

First let the free variable to be all zero and find a particular solution (if it exists) x_0 by back substitution. Then find $n-r$ basic solutions (s_i) of $Ax=0$ by setting one free variable $=1$ and others 0.

The solution to our linear equation system is:
(i) None if x_0 does not exist.
(ii) $x_0 + \sum_{i=1}^{n-r} \alpha_i s_i$ if x_0 exists.

• Gauss-Jordan Method for Inverse of a Matrix:

Consider the augmented matrix $[A \mid I]$.
Transform it to $[I \mid C]$ by E.R.O.s. Then $\boxed{C=A^{-1}}$

~~Then~~ Chapter-2 : Determinants

a) Important theorems:

- Let f be an alternating multilinear function of order n and d be a determinant function of order n.
  $\forall$ ~~Ae~~ $n\times n$ ~~B~~ matrix $A=(A_1, A_2, \dots, A_n)$
  $$f(A_1, A_2, \dots A_n) = d(A_1, A_2 \dots A_n)\, f(e_1, e_2, \dots e_n).$$
  In particular if f is also a determinant function then
  $f(A_1, A_2, \dots A_n) = d(A_1, A_2, \dots A_n)$ $\boxed{\text{Determinant is unique}}$
- Let A be an $n\times n$-matrix. Then the function
  $f(A) = a_{11}\det A_{11} - a_{12}\det A_{12} + \dots + (-1)^{n+1} a_{1n}\det A_{1n}$ is the determinant function on $n\times n$ matrices. Hence, by induction on n, the ~~exction~~ $\boxed{\text{existence of determinant is proved}}$
- (i) Let U be an upper triangular or a lower triangular matrix. Then $\boxed{\det U = \text{product of diagonal entries}}$
  (ii) If $\boxed{E = T_{ij}(m)}$ for some $i \neq j$, $\boxed{\det E = 1}$
  (iii) If $\boxed{E = T_{ij}}$ for some $i \neq j$, $\boxed{\det E = -1}$
  (iv) If $\boxed{E = T_i(m)}$ then $\boxed{\det E = m}$.
- If A, B are 2 $n\times n$ matrices. $\boxed{\det(AB) = \det A \times \det B}$.

- For any $n\times n$ matrix A, $\boxed{\det A = \det A^t}$
- Let A be an $n\times n$ matrix and let $1 \le k \le n$. Then
$$\boxed{\det A = \sum_{i=1}^{n} (-1)^{k+i} a_{ik} \det A_{ik}}$$
- For any $n\times n$ matrix A, $\boxed{A(\cot A)^t = (\det A)I = (\cot A)^t A}$

$\cot A$ is the cofactor matrix $= (\cot a_{ij})$ $\cot a_{ij} = (-1)^{i+j} \det A_{ij}$

Here A_{ij} is the matrix obtained by deleting i^{th} row, j^{th} column.

b) Important applications:

- Computation of determinant by Gauss ~~Jordan~~ Elimination:
Let r row exchange have been used to transform A to its REF, U which is upper triangular. Then $\boxed{\det A = (-1)^r \det U}$ ($\det U$ is easy, is product of diagonal)
- Computation of inverse by cofactor matrix:
If $\det A \neq 0$, $\boxed{A^{-1} = \frac{(\cot A)^t}{\det A}}$ otherwise inverse does not exist.
- Cramer's Rule:
Let $Ax = b$ has to solved and A is invertible.
Let C_j be the matrix obtained by replacing j^{th} column of A by b. Let $x = [x_1, x_2, \dots x_n]^T$
Then $\forall j \in 1, 2, \dots n$, $\boxed{x_j = \frac{\det C_j}{\det A}}$

Chapter 3: Vector Spaces

a) Important theorems:

- Any two bases of a finite dimensional vector space have same number of elements, called its $\boxed{\text{dimension}}$
- Let A be an $m\times n$ matrix. Then $\boxed{\dim R(A) = \dim C(A)}$
- Rank-Nullity Theorem: Let $A \in \mathbb{F}^{m\times n}$ Then
$\boxed{\text{rank } A + \text{nullity } A = n}$. Here ~~nullity~~
$\boxed{\text{rank } A = \dim R(A) = \dim C(A)}$ $\boxed{\text{nullity } A = \dim N(A)}$

- A matrix A has rank $r \geq 1 \iff \det M \neq 0$ for some order r minor M of A and $\det N = 0$ for all order $r+1$ minors of A.

Minor : $r \times r$ submatrix of A.

b) Important Applications:

- Finding a basis of column space, null space and row space by GEM:

Transform the matrix A to its ~~REF~~ REF U by GEM.

Then Basis $[C(A)]$ = set of pivotal columns of A

Basis $[N(A)]$ = set of basic solutions (i.e. $Ax = 0$)

Basis $[R(A)]$ = set of ~~pivotal rows~~ non-zero rows.

Chapter-4: Linear Transformations

a) Important theorems:

- Rank-Nullity Theorem: Let V and W be vector spaces where V is finite dimensional. Let $T: V \to W$ be a linear transformation. Then $\boxed{\text{rank}(T) + \text{nullity}(T) = \dim V}$

Here $\text{rank}(T) = \dim[Im(T)]$ where $Im(T) = \{T(v) \mid v \in V\}$

$\text{nullity}(T) = \dim[N(T)]$ where $N(T) = \{v \in V \mid T(v) = 0\}$

- $\boxed{M_B^B(T) = (M_C^B)^{-1} M_C^C(T) M_C^B}$

where $M_F^E(T) = [[T(e_1)]_F \ [T(e_2)]_F \ \ldots \ [T(e_n)]_F]$

where $T: V \to W$ is a linear transformation and $E = (e_1, e_2, \ldots e_n)$ is an ordered basis of V and $F = (f_1, f_2, \ldots f_m)$ is an ordered basis of W.

Also, Let $v \in S$ where S is a vector space with the ordered basis $\{s_1, s_2, \ldots s_n\}$

Then $v = a_1 s_1 + a_2 s_2 + \ldots + a_n s_n$

$[v]_{B_S} = [a_1 \ a_2 \ . \ a_n]^T$

$M_C^B = M_C^B(I)$ where $I(v) = v$ and $M_C^B = (M_B^C)^{-1}$

- Let V, W be subspace of a finite dimensional vector space U. Then
$$\boxed{\dim V + \dim W = \dim(V \cap W) + \dim(V + W)}$$
where $V + W = L(V \cup W)$ (linear span of $V \cup W$)

Chapter 5: Inner product Spaces

a) Important theorems:

- Let $v, w \in V$ and $\boxed{v \perp w \ (\langle v, w\rangle = 0)}$, then
(Pythagoras) $\boxed{\|v + w\|^2 = \|v\|^2 + \|w\|^2}$ $\boxed{\|v\| = \sqrt{\langle v, v\rangle}}$
- Cauchy-Schwarz inequality: $\boxed{|\langle w, v\rangle| \leq \|v\| \|w\|}$
- Triangle inequality: $\boxed{\|v + w\| \leq \|v\| + \|w\|}$
- Let V be a finite dimensional inner product space and W be its subspace with orthogonal basis $\{w_1, w_2, \ldots w_m\}$. If $W \neq V$ then $\exists \{w_{m+1}, w_{m+2}, \ldots w_n\} \in V$ such that $\{w_1, w_2, \ldots w_n\}$ is an orthogonal basis of V.
Taking $W = \{0\}$ we see that for any inner-product space we can find an orthogonal, and hence orthonormal basis.
- Every $v \in V$ can be written uniquely as $v = x + y$, where $x \in W$ and $y \in W^\perp$. Moreover, ~~$\dim V = \dim W$~~ $\dim V = \dim W + \dim W^\perp$ (W is a subspace of V).
- $P_W(v)$ is the best approximation to v by vectors in W. i.e. for any $w \in W$, $\|v - P_W(v)\| \leq \|v - w\|$
$P_W(v)$ is ~~projection of~~ v orthogonal projection of v onto W. i.e. x in above theorem.

b) Important applications:

- Gram-Schmidth Orthogonalization Process:
Let $\{v_1, v_2, v_3, \ldots v_n\}$ be a basis of V,
Let $w_1 = v_1$
Then $w_2 = v_2 - P_{w_1}(v_2)$
$\vdots$
$w_n = v_n - P_{w_1}(v_n) - P_{w_2}(v_n) \cdots - P_{w_{n-1}}(v_n)$

Then $\{w_1, w_2, \ldots w_n\}$ is an orthogonal basis of V.

- Best approximation of a vector in $C(A)$:

Let A be an $n \times m$ $(m \le n)$ matrix and let $b \in \mathbb{R}^n$. Let ~~the~~ best approximate of b in $C(A)$ be $\boxed{p = Ax}$

x is ~~the~~ any solution of $\boxed{A^t A x = A^t b}$ [Normal equations]

- Least squares approximation:

Let $y(x) = s_0 + s_1 x + s_2 x^2 + \ldots + s_m x^m$ be the best fit polynomial of degree m to the data points (x_i, y_i), $i = 1, \ldots n$ so as to minimize

$$\sum_{i=1}^{n} \sqrt{(y_i - s_0 - s_1 x_i - s_2 x_i^2 - \ldots - s_m x_i^m)^2}$$. Then the coefficient $s_0, s_1, \ldots s_m$ can be found by solving $\boxed{A^t A x = A^t b}$ where

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^m \\ 1 & x_2 & x_2^2 & \cdots & x_2^m \\ \vdots & \vdots & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^m \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad x = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_m \end{bmatrix}$$

Chapter-6: Eigenvalues and eigenvectors

a) Important theorems:

- Let $T: V \to V$ be a linear operator. Let $\lambda_1, \lambda_2, \ldots \lambda_n \in \mathbb{F}$ be distinct eigenvalues of T and let $v_1, v_2, \ldots v_n$ be corresponding eigenvectors. Then $v_1, v_2, \ldots v_n$ are linearly independent.
- Let V be a finite dimensional vector space over $\mathbb{F}$. Then the geometric multiplicity of an eigenvalue $\mu \in \mathbb{F}$ of T is less than or equal to ~~that~~ its algebraic multiplicity.
- T is diagonalizable $\Leftrightarrow$ $\boxed{\sum_\lambda \dim V_\lambda = \dim V}$ where V_λ are the eigenspaces corresponding to eigenvalue λ. If $\mathbb{F} = \mathbb{C}$, then T is diagonalizable $\Leftrightarrow$ algebraic multiplicity $=$ geometric multiplicity $\forall \lambda$

- (a) If $A \in \mathbb{R}^{n \times n}$ and $\mathbb{R}^n$ has an orthonormal basis of eigenvectors of A, then A is symmetric.
  (b) If $A \in \mathbb{C}^{n \times n}$ and $\mathbb{C}^n$ has an orthonormal basis of eigenvectors of A, then A is normal.
- The eigenvalues of a Hermitian matrix are real.
- ~~If~~ T is self-adjoint $\iff M_B^B(T)$ is self-adjoint for every ordered orthonormal basis B of V
- If T is self-adjoint, then there exists an orthonormal basis of V consisting of eigenvectors of T. (Spectral theorem for Self-Adjoint Operators).
- Let T be self-adjoint. Let $\lambda_1, \lambda_2 \dots \lambda_r$ be distinct eigenvalues of T. Then $\boxed{V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_r}}$ and $\boxed{\dim V = \sum_{i=1} \dim V_{\lambda_i}}$ [$\oplus$ $\Rightarrow$ $A \oplus B \equiv A + B$, but $A \cap B = \{0\}$]
- <u>Spectral Theorem for Real Symmetric matrices</u>: Let A be an $n \times n$ real symmetric matrix with (real) eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$. Let $D = \operatorname{diag}(\lambda_1, \dots \lambda_n)$. Then there exists an $n \times n$ real orthogonal matrix S such that $\boxed{S^t A S = D}$
- Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1, \dots \lambda_n \in \mathbb{R}$. Let $D = \operatorname{diag}(\lambda_1, \dots \lambda_n)$. Then there exists an $n \times n$ unitary matrix U such that $\boxed{U^* A U = D}$.
- Let ~~V be an~~ A, B be two commuting self-adjoint operators on V. Then there exists an orthonormal basis $(v_1, \dots v_n)$ of V such that each v_i is an eigenvector of both A and B.
- <u>Spectral Theorem for Normal matrices</u>: A complex normal matrix has an orthonormal basis of $\mathbb{C}^n$, consisting of eigenvectors of A.
- Let A be real symmetric and U be orthogonal and $U^t A U = D = \operatorname{diag}(\lambda_1, \lambda_2, \dots \lambda_n)$. $X = UY = U(y_1, y_2, \dots y_n)^t$ then ~~$Q(x) = x^t$~~
  $Q(x) = x^t A x = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$

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b) Important Applications:

- Finding eigenvalues and ~~basis of~~ eigenspaces of a matrix:

Eigenvalues: $\det(A - \lambda I) = 0$ → Roots of this equation.

Basis ~~of~~ Eigenspace: $N(A - \lambda I) = V_\lambda$

- Matrix exponentiation by diagonalization:

$$A = PDP^{-1} \Rightarrow \boxed{A^n = PD^nP^{-1}}$$

- ~~Let ax^2+~~ Identification of conic sections:

Let $ax^2 + bxy + cy^2 + dx + ey + f$ be a conic section in $\mathbb{R}^2$.

Set $x = x_1$, $y = x_2$

$$\Rightarrow [x_1 \; x_2] \underbrace{\begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + \underbrace{[d \; e]}_{B} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} + f = 0$$

$$\Rightarrow X^t A X + BX + C = 0, \quad [f] = C$$

Let $U = [U_1, U_2]$ be an orthogonal matrix with U_1, U_2 as eigenvectors of A and eigenvalues λ_1, λ_2. Put $X = UY$, $Y = [y_1 \; y_2]^T$

$\Rightarrow$ ~~$\lambda_1 y_1^2 + \lambda_2 y_2^2 + BUY + f = 0$~~

$$\Rightarrow \lambda_1 y_1^2 + \lambda_2 y_2^2 + [d \; e][U_1 \; U_2]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + f = 0.$$

and hence the conic can be easily identified.