CHAPTER 12

DERIVATIVE AS A RATE MEASURER

12.1 DERIVATIVE AS A RATE MEASURER

Let $y = f(x)$ be a function of x. Let Δy be the change in y corresponding to a small change Δx in x. Then, $\frac{\Delta y}{\Delta x}$ represents the change in y due to a unit change in x. In other words, $\frac{\Delta y}{\Delta x}$ represents the average rate of change of y with respect to x as x changes from x to $x + \Delta x$.

As $\Delta x \to 0$, the limiting value of this average rate of change of y with respect to x in the interval $[x, x + \Delta x]$ becomes the instantaneous rate of change of y with respect to x.

Thus,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{Instantaneous rate of change of } y \text{ with respect to } x$$

$$\Rightarrow \quad \frac{dy}{dx} = \text{Rate of change of } y \text{ with respect to } x \qquad \left[\because \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}\right]$$

The word "instantaneous" is often dropped.

Hence, $\frac{dy}{dx}$ represents the rate of change of y with respect to x for a definite value of x.

REMARK 1 *The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change of y with respect to x at $x = x_0$.*

REMARK 2 *If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, provided that $\frac{dx}{dt} \neq 0$.*

Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

REMARK 3 *Throughout this chapter, the term "rate of change" will mean the instantaneous rate of change unless stated otherwise.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with respect to its radius when the radius is 7 cm.*

SOLUTION Let x be the radius and y be the volume of the balloon. Then,

$$y = \frac{4}{3}\pi x^3 \Rightarrow \frac{dy}{dx} = 4\pi x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{x=7} = 4\pi (7)^2 = 196\pi \text{ cm}^2$$

Hence, the volume is increasing with respect to its radius at the rate of $196\pi \text{ cm}^2$, when the radius is 7 cm.

EXAMPLE 2 *Find the rate of change of the area of a circle with respect to its radius. How fast is the area changing with respect to the radius when the radius is 3 cm?* **[NCERT]**

SOLUTION Let A be the area of the circle. Then,

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$$

Thus, the rate of change of the area of the circle with respect to its radius r is $2\pi r$.
When $r = 3$ cm, we obtain

$$\frac{dA}{dr} = (2\pi \times 3) \text{ cm} = 6\pi \text{ cm}.$$

EXAMPLE 3 *A balloon, which always remains spherical, has a variable diameter* $\frac{3}{2}(2x+3)$. *Determine the rate of change of volume with respect to x.*

SOLUTION Let V be the volume of the balloon. Then,

$$V = \frac{4\pi}{3}\left\{\frac{3}{4}(2x+3)\right\}^3 = \frac{9\pi}{16}(2x+3)^3$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{9\pi}{16} \times 3(2x+3)^2 \frac{d}{dx}(2x+3) = \frac{27\pi}{8}(2x+3)^2$$

EXAMPLE 4 *The total cost C (x) associated with the production of x units of an item is given by*

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$$

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. **[CBSE 2018, NCERT]**

SOLUTION Since the marginal cost is the rate of change of total cost with respect to the output.

$$\therefore \quad \text{Marginal cost (MC)} = \frac{d}{dx}(C(x)) = \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 5000)$$
$$= 0.005(3x^2) - 0.02(2x) + 30$$

When $x = 3$, we get

$$\text{Marginal cost (MC)} = 0.005 \times 3 \times 3^2 - 0.02 \times 2 \times 3 + 30 = 0.135 - 0.12 + 30 = 30.015$$

Hence, the required marginal cost is ₹ 30.02 (nearly).

EXAMPLE 5 *The total revenue received from the sale of x units of a product is given by* $R(x) = 3x^2 + 36x + 5$. *Find the marginal revenue when* $x = 5$, *where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant.* **[NCERT]**

SOLUTION Since the marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$$\therefore \quad \text{Marginal revenue (MR)} = \frac{dR}{dx} = \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36$$

When $x = 5$, we obtain

$$\text{Marginal revenue} = 6 \times 5 + 36 = 66$$

Hence, the required marginal revenue is ₹ 66.

EXAMPLE 6 *A car starts from a point P at time t = 0 second and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by* $x = t^2\left(2 - \frac{t}{3}\right)$. *Find the time taken by it to reach at Q and also find distance between P and Q.* **[NCERT]**

SOLUTION We have,

$$x = t^2\left(2 - \frac{t}{3}\right) \Rightarrow x = 2t^2 - \frac{t^3}{3} \Rightarrow \frac{dx}{dt} = 4t - t^2.$$ This gives velocity of the car at any time t.

Suppose the car stops at Q after t_1 second. Then, at $t = t_1$

$$\frac{dx}{dt} = 0$$

or, $\left(\frac{dx}{dt}\right)_{t=t_1} = 0$

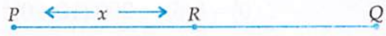

Fig. 12.1

$\Rightarrow \quad 4t_1 - t_1^{\,2} = 0 \Rightarrow t_1(4-t_1) = 0 \Rightarrow t_1 = 4$ $\quad$ [$\because$ $t_1 = 0$ is for point P]

Thus, the car takes 4 seconds to reach at Q.
The distance between P and Q is the value of x at $t = t_1$ i.e. at $t = 4$.

$\therefore \quad PQ = (\text{Value of } x \text{ at } t = 4) = 2 \times 4^2 - \frac{4^3}{3} = 32 - \frac{64}{3} = \frac{32}{3}$ m

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Find the rate of change of volume of a sphere with respect to its surface area when the radius is 2 cm.*

SOLUTION Let r be the radius, V the volume and S be the surface area of the sphere. Then,

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

We have, to find $\frac{dV}{dS}$ when $r = 2$.

Now, $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2 \Rightarrow \frac{dV}{dr} = 4\pi r^2$ and $\frac{dS}{dr} = 8\pi r$

$$\therefore \quad \frac{dV}{dS} = \frac{\frac{dV}{dr}}{\frac{dS}{dr}} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2} \Rightarrow \left(\frac{dV}{dS}\right)_{r=2} = \frac{2}{2} = 1.$$

EXAMPLE 8 *If x and y are the sides of two squares such that $y = x - x^2$. Find the change of the area of second square with respect to the area of the first square.* **[NCERT EXEMPLAR]**

SOLUTION Let A_1 and A_2 denote the areas of squares of sides x and y respectively. Then,

$$A_1 = x^2 \text{ and } A_2 = y^2$$

$\Rightarrow \quad A_1 = x^2$ and $A_2 = (x - x^2)^2$ $\quad$ [$\because$ $y = x - x^2$ (given)]

$$\Rightarrow \quad \frac{dA_1}{dx} = 2x \text{ and } \frac{dA_2}{dx} = 2(x - x^2)(1 - 2x)$$

$$\therefore \quad \frac{dA_2}{dA_1} = \frac{dA_2/dx}{dA_1/dx} = \frac{2(x-x^2)(1-2x)}{2x} = (1-x)(1-2x) = 1 - 3x + 2x^2.$$

EXAMPLE 9 *A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200\,(10-t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?* **[NCERT EXEMPALR]**

SOLUTION We have to find $\frac{dL}{dt}$ at $t = 5$.

Now, $L = 200\,(10-t)^2 \Rightarrow \frac{dL}{dt} = -400\,(10-t) \Rightarrow \left(\frac{dL}{dt}\right)_{t=5} = -400\,(10-5) = -2000$

Thus, the water is running out at the rate of 2000 litres per second at the end of 5 seconds.
The average rate at which the water flows out during the first 5 seconds is given by

$$\frac{L(0)-L(5)}{5}=\frac{200\,(10-0)^2-200\,(10-5)^2}{5}=\frac{20000-5000}{5}=3000 \text{ litres/sec.}$$

EXERCISE 12.1

BASIC

1. Find the rate of change of the total surface area of a cylinder of radius r and height h, when the radius varies.
2. Find the rate of change of the volume of a sphere with respect to its diameter.
3. Find the rate of change of the volume of a cone with respect to the radius of its base.
4. Find the rate of change of the area of a circle with respect to its radius r when $r = 5$ cm.
5. Find the rate of change of the volume of a ball with respect to its radius r. How fast is the volume changing with respect to the radius when the radius is 2 cm?
6. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced. [NCERT]
7. The total revenue received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$. [NCERT]

BASED ON LOTS

8. Find the rate of change of the volume of a sphere with respect to its surface area when the radius is 2 cm.
9. Find the rate of change of the area of a circular disc with respect to its circumference when the radius is 3 cm.
10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (Marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$, and write which value does the question indicate. [CBSE 2013]

ANSWERS

1. $4\pi r + 2\pi h$ 2. $\frac{\pi r^2}{2}$, r is the diameter 3. $\frac{2}{3}\pi r h$ 4. $10\pi \text{ cm}^2/\text{cm}$
5. $4\pi r^2$, $16\pi \text{ m}^3/\text{m}$ 6. ₹ 20.967 7. ₹ 208 8. 1 cm 9. 3 cm
10. MR = ₹ 66. It indicates the extra money spent when number of employees increase from 5 to 6.

HINTS TO SELECTED PROBLEMS

6. We have, $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000 \Rightarrow \frac{d}{dx}(C(x)) = 0.021x^2 - 0.006x + 15$

$$\therefore \left(\frac{d}{dx}C(x)\right)_{x=17} = 0.021\times 17^2 - 0.006\times 17 + 15. \text{ Hence, marginal cost} = ₹\,20.967$$

7. We have, $R(x) = 13x^2 + 26x + 15$

$$\therefore \frac{d}{dx}(R(x)) = 26x + 26 \Rightarrow \left(\frac{d}{dx}(R(x))\right)_{x=7} = 26\times 7 + 26 = 208$$

8. We have,

$$V = \frac{4}{3}\pi r^3 \text{ and, } S = 4\pi r^2 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \text{ and, } \frac{dS}{dr} = 8\pi r$$

$$\therefore \frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2} \Rightarrow \left(\frac{dV}{dS}\right)_{r=2} = \frac{2}{2} = 1 \text{ cm}$$

9. We have, $A = \pi r^2$ and, $C = 2\pi r \Rightarrow \dfrac{dA}{dr} = 2\pi r$ and, $\dfrac{dC}{dr} = 2\pi$

$$\therefore \quad \frac{dA}{dC} = \frac{dA/dr}{dC/dr} = r \Rightarrow \left(\frac{dA}{dC}\right)_{r=3} = 3 \text{ cm}$$

12.2 RELATED RATES

Generally we come across with the problems in which the rate of change of one of the quantities involved is required corresponding to the given rate of change of another quantity. For example, suppose the rate of change of volume of a spherical balloon is required when the rate of change of its radius is given. In such type of problems, we must find a relation connecting such quantities and differentiate this relation w.r. to time. The procedure is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *An edge of a variable cube is increasing at the rate of 10 cm/sec. How fast the volume of the cube is increasing when the edge is 5 cm long?*

SOLUTION Let x be the length of the edge of the cube and V be its volume at any time t. Then,

$$V = x^3 \text{ and } \frac{dx}{dt} = 10 \text{ cm/sec} \qquad \text{[Given]}$$

Now, $V = x^3$

$$\Rightarrow \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt} = (3x^2)(10) = 30x^2 \qquad \left[\because \frac{dx}{dt} = 10\right]$$

$$\Rightarrow \quad \left(\frac{dV}{dt}\right)_{x=5} = 30(5)^2 = 750 \text{ cm}^3\text{sec.}$$

Thus, the volume of the cube is increasing at the rate of 750 cm^3/sec when the edge is 5 cm long.

EXAMPLE 2 *The radius of a circle is increasing uniformly at the rate of 4 cm/sec. Find the rate at which the area of the circle is increasing when the radius is 8 cm.*

SOLUTION Let r be the radius and A be the area of a circle at any time t. Then,

$$A = \pi r^2 \text{ and } \frac{dr}{dt} = 4 \text{ cm/sec} \qquad \text{[Given]}$$

Now, $A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r \dfrac{dr}{dt} \Rightarrow \left(\dfrac{dA}{dt}\right)_{r=8} = 2\pi \times 8 \times 4 \text{ cm}^2/\text{sec} = 64\pi \text{ cm}^2/\text{sec}.$

EXAMPLE 3 *Find an angle* θ, $0 < \theta < \dfrac{\pi}{2}$, *which increases twice as fast as it sine.*

[NCERT EXEMPLAR]

SOLUTION It is given that

$$\frac{d\theta}{dt} = 2\frac{d}{dt}(\sin\theta) \Rightarrow \frac{d\theta}{dt} = 2\cos\theta \frac{d\theta}{dt} \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ \left[\because \frac{d\theta}{dt} \neq 0\right]$$

Hence, the measure of angle is 60°.

EXAMPLE 4 *The side of an equilateral triangle is increasing at the rate of 2 cm/sec. At what rate is its area increasing when the side of the triangle is 20 cm?* [CBSE 2015]

SOLUTION At any time t, let x cm be the length of a side of an equilateral triangle and A be its area. Then,

$$A=\frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt}=\frac{\sqrt{3}}{4}\times 2x\frac{dx}{dt}=\frac{\sqrt{3}}{2}x\frac{dx}{dt}$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{x=20}=\frac{\sqrt{3}}{2}\times 20\times 2=20\sqrt{3}\ \text{cm}^2/\text{sec} \qquad \left[\because \frac{dx}{dt}=2\ \text{cm/sec (given)}\right]$$

Hence, the area is increasing at the rate of $20\sqrt{3}\ \text{cm}^2/\text{sec}$.

EXAMPLE 5 *The radius of a balloon is increasing at the rate of 10 cm/sec. At what rate is the surface area of the balloon increasing when the radius is 15 cm?*

SOLUTION Let r be the radius and S be the surface area of the balloon at any time t. Then,

$$S=4\pi r^2 \text{ and } \frac{dr}{dt}=10\ \text{cm/sec}$$

Now, $S = 4\pi r^2$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 80\pi r \qquad \left[\because \frac{dr}{dt}=10\ \text{cm/sec.}\right]$$

$$\Rightarrow \quad \left(\frac{dS}{dt}\right)_{r=15} = 80\pi(15) = 1200\pi\ \text{cm}^2/\text{sec.}$$

EXAMPLE 6 *A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?* [NCERT]

SOLUTION Let r be the radius and A be the area of the circular wave at any time t. Then,

$$A = \pi r^2 \text{ and } \frac{dr}{dt} = 3\cdot 5\ \text{cm/sec.} \qquad \text{[Given]}$$

Now, $A = \pi r^2$

$$\Rightarrow \quad \frac{dA}{dt} = \pi\left(2r\frac{dr}{dt}\right) = 2\pi r\frac{dr}{dt} = 2\pi r(3\cdot 5) = 7\pi r \qquad \left[\because \frac{dr}{dt}=3.5\ \text{cm/sec}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{r=7\cdot 5} = 7\pi(7\cdot 5) = 52\cdot 5\pi\ \text{cm}^2/\text{sec.}$$

EXAMPLE 7 *For the curve $y=5x-2x^3$, if x increases at the rate of 2 units/sec, then how fast is the slope of the curve changing when $x = 3$?* [NCERT EXEMPLAR]

SOLUTION Let m be the slope of the curve at an arbitrary point (x, y) on it. Then,

$$m=\frac{dy}{dx} \Rightarrow m=5-6x^2 \qquad \left[\because y=5x-2x^3 \therefore \frac{dy}{dx}=5-6x^2\right]$$

It is given that $\frac{dx}{dt}=2$ units/sec and we have to find $\frac{dm}{dt}$ when $x = 3$.

Now,

$$m=5-6x^2 \Rightarrow \frac{dm}{dt}=-12x\frac{dx}{dt} \Rightarrow \left(\frac{dm}{dt}\right)_{x=3}=-12\times 3\times 2=-72\ \text{units/sec} \quad \left[\because x=3 \text{ and } \frac{dx}{dt}=2\right]$$

Thus, the slope of the curve is decreasing at the rate of 72 units/sec when x is increasing at the rate of 2 units/sec.

EXAMPLE 8 *The volume of a cube is increasing at a rate of 7 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?*

SOLUTION Let x be the length of an edge of the cube, V be the volume and S be the surface area at any time t. Then, $V = x^3$ and $S=6x^2$. It is given that

$$\frac{dV}{dt} = 7\,\text{cm}^3/\text{sec} \Rightarrow \frac{d}{dt}(x^3) = 7 \Rightarrow 3x^2\frac{dx}{dt} = 7 \Rightarrow \frac{dx}{dt} = \frac{7}{3x^2}$$

Now, $S = 6x^2$

$$\Rightarrow \frac{dS}{dt} = 12x\frac{dx}{dt} = 12x \times \frac{7}{3x^2} = \frac{28}{x} \qquad \left[\because \frac{dx}{dt} = \frac{7}{3x^2}\right]$$

$$\Rightarrow \left(\frac{dS}{dt}\right)_{x=12} = \frac{28}{12}\,\text{cm}^2/\text{sec} = \frac{7}{3}\,\text{cm}^2/\text{sec}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *If the area of circle increases at a uniform rate, then prove that the perimeter varies inversely as the radius.* **[NCERT EXEMPLAR]**

SOLUTION Let r be the radius P be the perimeter and A be the area of the circle at any time t. Then, $A = \pi r^2$ and $P = 2\pi r$. It is given that $\frac{dA}{dt}$ = constant (k), where $k > 0$.

Now,

$$A = \pi r^2 \quad \text{and} \quad P = 2\pi r$$

$$\Rightarrow A = \pi\left(\frac{P}{2\pi}\right)^2 \qquad \left[\because P = 2\pi r \Rightarrow r = \frac{P}{2\pi}\right]$$

$$\Rightarrow A = \frac{1}{4\pi}P^2$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{4\pi} \times 2P\frac{dP}{dt} \Rightarrow k = \frac{1}{2\pi}P\frac{dP}{dt} \Rightarrow k = \frac{1}{2\pi}(2\pi r)\frac{dP}{dt} \qquad [\because P = 2\pi r]$$

$$\Rightarrow \frac{dP}{dt} = \frac{k}{r} \Rightarrow P \text{ varies inversely as the radius } r.$$

ALITER We have,

$$A = \pi r^2 \text{ and } P = 2\pi r$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt} \text{ and } \frac{dP}{dt} = 2\pi\frac{dr}{dt} \Rightarrow k = 2\pi r\frac{dr}{dt} \text{ and } \frac{dP}{dt} = 2\pi\frac{dr}{dt} \qquad \left[\because \frac{dA}{dt} = k\right]$$

$$\Rightarrow \frac{dP}{dt} = 2\pi\left(\frac{k}{2\pi r}\right) \qquad \left[\text{On eliminating } \frac{dr}{dt}\right]$$

$$\Rightarrow \frac{dP}{dt} = \frac{k}{r} \Rightarrow P \text{ varies inversely as the radius } r.$$

EXAMPLE 10 *A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.* **[NCERT EXEMPLAR]**

SOLUTION Let V, S and r denote respectively the volume, surface area and radius of the salt ball at any instant t. Then,

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

It is given that the rate of decrease of the volume V is proportional to the surface area S.

i.e. $\frac{dV}{dt} \propto S$

$$\Rightarrow \frac{dV}{dt} = -kS, \text{ where } k > 0 \text{ is the constant of proportionality}$$

It is given that V is decreasing with time, so that is why negative sign is taken.

Now,

$$\frac{dV}{dt} = -kS \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = -k\,(4\pi r^2) \Rightarrow 4\pi r^2 \frac{dr}{dt} = -4\pi k r^2 \Rightarrow \frac{dr}{dt} = -k$$

$\therefore$ r decrease with a constant rate

Hence, the radius is decreasing at a constant rate.

EXAMPLE 11 *A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.* **[NCERT]**

SOLUTION Let the required point be $P(x, y)$. It is given that

Rate of change of y coordinate = 8 (Rate of change of x-coordinate) i.e. $\frac{dy}{dt} = 8\,\frac{dx}{dt}$ ...(i)

Now, $6y = x^3 + 2$

$\Rightarrow$ $6\,\frac{dy}{dt} = 3x^2\,\frac{dx}{dt}$ [Differentiating both sides with respect to t]

$\Rightarrow$ $6\left(8\,\frac{dx}{dt}\right) = 3x^2\,\frac{dx}{dt}$ [Using (i)]

$\Rightarrow$ $3x^2 = 48 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Now, $x = 4 \Rightarrow 6y = 4^3 + 2 = 66 \Rightarrow y = 11$ and, $x = -4 \Rightarrow 6y = (-4)^3 + 2 = -62$ $y = -\frac{31}{3}$

So, the required points are $(-4, -31/3)$ and $(4, 11)$.

EXAMPLE 12 *The volume of a cube is increasing at a constant rate. Prove that the increase in surface area varies inversely as the length of the edge of the cube.* **[NCERT EXEMPLAR]**

SOLUTION Let x be the length of each edge of the cube, S be its surface area and V be its volume at any time t. Then, $S = 6x^2$ and $V = x^3$. It is given that $\frac{dV}{dt} = k$ (constant).

Now, $V = x^3 \Rightarrow \frac{dV}{dt} = 3x^2\,\frac{dx}{dt} \Rightarrow k = 3x^2\,\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{k}{3x^2}$ ...(i)

and, $S = 6x^2$

$\Rightarrow$ $\frac{dS}{dt} = 12\,x\,\frac{dx}{dt} = 12\,x\left(\frac{k}{3x^2}\right) = \frac{4k}{x}$ [Using (i)]

$\Rightarrow$ $\frac{dS}{dt} \propto \frac{1}{x}$.

Hence, the rate of increase in surface area varies inversely as the length of the edge of the cube.

EXAMPLE 13 *Two men M_1 and M_2 start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, find the rate at which they are separated.* **[NCERT EXEMPLAR]**

SOLUTION Let O be the junction and OA and OB be two roads inclined at an angle of 45°. Let men M_1 and M_2 travel by roads OA and OB respectively and let at any time P and Q be their positions such that $OP = OQ = x$ (both men travel with same speed v). Then,

$$\frac{dx}{dt} = v$$

Let $PQ = y$. We have to find $\frac{dy}{dt}$.

Using cosine formula in ΔOPQ, we obtain

$$PQ^2 = OP^2 + OQ^2 - 2\,OP \cdot OQ \cos 45°$$

$\Rightarrow$ $y^2 = x^2 + x^2 - 2x^2 \times \frac{1}{\sqrt{2}}$

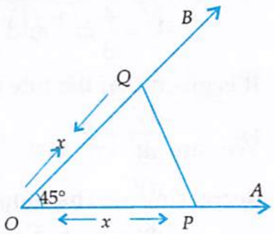

Fig. 12.2

$\Rightarrow \quad y = \sqrt{2-\sqrt{2}}\; x$

$\Rightarrow \quad \frac{dy}{dt} = \sqrt{2-\sqrt{2}}\; \frac{dx}{dt} = \sqrt{2-\sqrt{2}}\; v \qquad \left[\because \frac{dx}{dt} = v\right]$

Hence, two men M_1 and M_2 are separated at the rate $\left(\sqrt{2-\sqrt{2}}\right) v$.

EXAMPLE 14 *The length x of a rectangle is decreasing at the rate of 2 cm/sec and the width y is increasing at the rate of 2 cm/sec. When x = 12 cm and y = 5 cm, find the rate of change of (i) the perimeter and (ii) the area of the rectangle.* **[NCERT]**

SOLUTION Let P be the perimeter and A be the area of the rectangle at any time t. Then,

$P = 2(x+y)$ and $A = xy$

It is given that $\frac{dx}{dt} = -2$ cm/sec and $\frac{dy}{dt} = 2$ cm/sec.

(i) We have, $P = 2(x+y)$

$\Rightarrow \frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = 2(-2+2) = 0$ cm/sec i.e. the perimeter remains constant.

(ii) We have, $A = xy$

$\Rightarrow \frac{dA}{dt} = \left(\frac{dx}{dt}\right) y + x\left(\frac{dy}{dt}\right) = -2\times 5 + 12\times 2 = 14\ \text{cm}^2/\text{sec}. \qquad [\because x = 12 \text{ cm and } y = 5 \text{ cm (given)}]$

EXAMPLE 15 *A man 2 metres high, walks at a uniform speed of 6 metres per minute away from a lamp post, 5 metres high. Find the rate at which the length of his shadow increases.* **[NCERT]**

SOLUTION Let AB be the lamp-post. Let at any time t, the man CD be at a distance x metres from the lamp-post and y metres be the length of his shadow CE. Then,

$\frac{dx}{dt} = 6$ metres/minute [Given] ...(i)

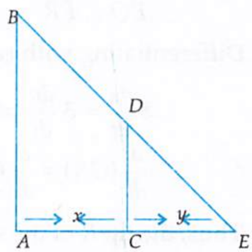

Fig. 12.3

Clearly, triangles ABE and CDE are similar.

$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE}$

$\Rightarrow \quad \frac{5}{2} = \frac{x+y}{y}$

$\Rightarrow \quad 3y = 2x \Rightarrow 3\frac{dy}{dt} = 2\frac{dx}{dt} \Rightarrow 3\frac{dy}{dt} = 2(6)$ [Using (i)]

$\Rightarrow \quad \frac{dy}{dt} = 4$

Thus, the shadow increases at the rate of 4 metres/minute.

EXAMPLE 16 *A man is walking at the rate of 6.5 km/hr towards the foot of a tower 120 m high. At what rate is he approaching the top of the tower when he is 50 m away from the tower?*

SOLUTION Let at any time t, the man be at distances of x and y metres from the foot and top of the tower respectively. Then,

$y^2 = x^2 + (120)^2$ ...(i)

$\Rightarrow \quad 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

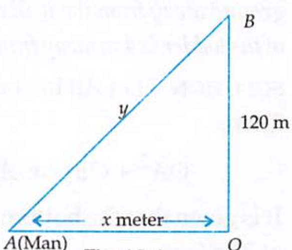

Fig. 12.4

We are given that $\frac{dx}{dt} = -6\cdot 5$ km/hr (negative sign due to decreasing x). Therefore,

$\frac{dy}{dt} = -\frac{6\cdot 5x}{y}$ ...(ii)

Putting $x = 50$ in (i), we get: $y = \sqrt{50^2 + 120^2} = 130.$

Putting $x = 50$, $y = 130$ in (ii), we get : $\frac{dy}{dt} = -\frac{6\cdot5 \times 50}{130} = -2\cdot5.$

Thus, the man is approaching the top of the tower at the rate of $2\cdot5$ km/hr.

EXAMPLE 17 *A man 2 m tall, walks at the rate of $1\frac{2}{3}$ m/sec towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?* **[NCERT EXEMPLAR]**

SOLUTION Let OA be the street light of height $5\frac{1}{3}$ m. At any time t, let PQ be the position of the man and let PR be the length of his shadow such that $PR = x$ and $OP = y$.

It is given that the man is walking at the rate of $\frac{5}{3}$ m/sec towards the street light.

$$\therefore \qquad \frac{dy}{dt} = -\frac{5}{3} \text{ m/sec} \qquad \ldots\text{(i)}$$

We have to find the rate at which the tip of the shadow is moving i.e. we have to find $\frac{d}{dt}(x+y)$.

For this we require the value of $\frac{dx}{dt}$. So, let us first find $\frac{dx}{dt}$.

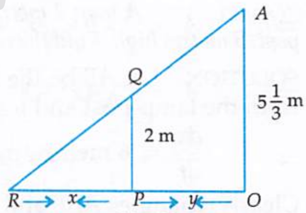

Fig. 12.5

Δ's AOR and QPR are similar triangles.

$$\therefore \qquad \frac{AO}{PQ} = \frac{OR}{PR} \Rightarrow \frac{16/3}{2} = \frac{x+y}{x} \Rightarrow 8x = 3x + 3y \Rightarrow 5x = 3y$$

Differentiating with respect to t, we obtain

$$5\frac{dx}{dt} = 3\frac{dy}{dt} \Rightarrow 5\frac{dx}{dt} = 3 \times -\frac{5}{3} \Rightarrow \frac{dx}{dt} = -1 \text{ m/sec} \qquad \ldots\text{(ii)}$$

$$\therefore \qquad \frac{d}{dt}(OR) = \frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt} = -1 - \frac{5}{3} = -\frac{8}{3} \qquad \text{[Using (i) and (ii)]}$$

Thus, the tip R of the shadow PR is moving towards the base of the street light at the rate of $8/3$ m/sec.

From (ii), we obtain : $\frac{dx}{dt} = -1$ for all x, y.

Thus, the length of the shadow is reducing at the rate of 1 m/sec when the man is $3\frac{1}{3}$ m from the base of light.

EXAMPLE 18 *A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/sec. How fast its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?* **[CBSE 2012, NCERT, CBSE 2019]**

SOLUTION Let AB be the position of the ladder at any time t such that $OA = x$ and $OB = y$. Then,

$$OA^2 + OB^2 = AB^2 \Rightarrow x^2 + y^2 = 5^2 \qquad \ldots\text{(i)}$$

It is given that the bottom of the ladder is pulled along the ground away from the wall at the rate of 2 m/sec.

$$\therefore \quad \frac{dx}{dt} = 2 \text{ m/sec.}$$

Now, $x^2 + y^2 = 5^2$

$$\Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2x(2) + 2y\frac{dy}{dt} = 0 \qquad \left[\because \frac{dx}{dt} = 2\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{2x}{y} \qquad \text{...(ii)}$$

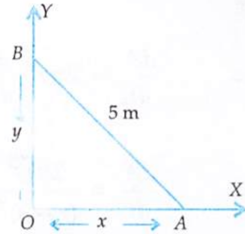

Fig. 12.6

Putting $x = 4$ in (i), we get: $y = \sqrt{25-16} = 3$. Putting $x = 4$ and $y = 3$ in (ii), we get:

$$\frac{dy}{dt} = -\frac{8}{3} \text{ m/sec.}$$

Hence, the rate of decrease in the height of the ladder on the wall is $\frac{8}{3}$ m/sec.

EXAMPLE 19 *The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm/sec. How fast is the area decreasing when the two equal sides are equal to the base?* [NCERT]

SOLUTION Let at any time t, the length of each equal side be x cm and area of the triangle be A. Then,

$$A = \frac{1}{2}(BC \times AD)$$

$$\Rightarrow \quad A = \frac{1}{2} \times b \times \sqrt{x^2 - \frac{b^2}{4}}$$

$$\Rightarrow \quad A = \frac{b}{4}\sqrt{4x^2 - b^2}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}} \frac{d}{dt}(4x^2 - b^2)$$

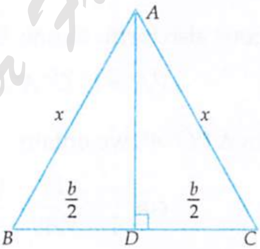

Fig. 12.7

$$\Rightarrow \quad \frac{dA}{dt} = \frac{b}{8\sqrt{4x^2 - b^2}} \times 8x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{bx}{\sqrt{4x^2 - b^2}}\frac{dx}{dt} = \frac{3bx}{\sqrt{4x^2 - b^2}} \qquad \left[\because \frac{dx}{dt} = 3 \text{ cm/sec (given)}\right]$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{x=b} = \frac{3b^2}{\sqrt{4b^2 - b^2}} = \sqrt{3}\, b \text{ cm}^2\text{/sec.}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 20 *An airforce plane is ascending vertically at the rate of 100 km/h. If the radius of the earth is r km, how fast is the area of the earth, visible from the plane, increasing at 3 minutes after it started ascending? Given that the visible area A at height h is given by* $A = 2\pi r^2 \frac{h}{r+h}$.

SOLUTION It is given that the plane is ascending vertically at the constant rate of 100 km/h.

$$\therefore \quad \frac{dh}{dt} = 100 \text{ km/h} \qquad \text{...(i)}$$

$\Rightarrow$ Height of the plane after 3 minutes $= 100 \times \frac{3}{60} = 5$ km. [Using $h = vt$]

Now, $A = 2\pi r^2 \dfrac{h}{r+h}$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r^2 \frac{d}{dt}\left(\frac{h}{r+h}\right) = 2\pi r^2 \left\{\frac{(r+h)\frac{dh}{dt} - h\frac{d}{dt}(r+h)}{(r+h)^2}\right\} = 2\pi r^2 \left\{\frac{(r+h)\frac{dh}{dt} - h\frac{dh}{dt}}{(r+h)^2}\right\}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi r^3}{(r+h)^2}\frac{dh}{dt} \Rightarrow \frac{dA}{dt} = \frac{2\pi r^3}{(r+h)^2} \times 100 = \frac{200\pi r^3}{(r+h)^2} \qquad \left[\because \frac{dh}{dt} = 100 \text{ km/h}\right]$$

We have to find $\frac{dA}{dt}$ when $t = 3$ minutes and at $t = 3$, we have $h = 5$ km.

$$\therefore \quad \left(\frac{dA}{dt}\right)_{t=3} = \frac{200\pi r^3}{(r+5)^2}.$$

EXAMPLE 21 *Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at the uniform rate of 2 cm^2/sec in its surface area through a tiny hole at the vertex in the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.* **[NCERT EXEMPLAR]**

SOLUTION Let VAB be a conical funnel of semi-vertical angle $\frac{\pi}{4}$. At any time t the water in the cone also forms a cone. Let r be its radius, l be the slant height and S be the surface area. Then,

$$VA' = l, \; O'A' = r \text{ and } \angle A'VO' = \frac{\pi}{4}.$$

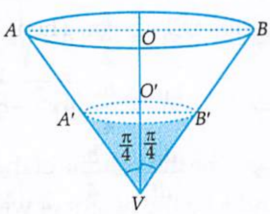

Fig. 12.8

In $\Delta VO'A'$, we obtain

$$\cos\frac{\pi}{4} = \frac{VO'}{VA'} = \frac{VO'}{l} \text{ and, } \sin\frac{\pi}{4} = \frac{O'A'}{VA'} = \frac{O'A'}{l}.$$

$$\Rightarrow \quad VO' = l\cos\frac{\pi}{4} \text{ and, } O'A' = l\sin\frac{\pi}{4}.$$

The surface area S of the conical funnel is given by

$$S = \pi(O'A')(VA') \qquad [\text{Using: } S = \pi r l]$$

$$\Rightarrow \quad S = \pi\left(l\sin\frac{\pi}{4}\right) l = \pi l^2 \sin\frac{\pi}{4} = \frac{\pi l^2}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt}$$

$$\Rightarrow \quad -2 = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt} \qquad \left[\because \frac{dS}{dt} = -2 \text{ cm}^2/\text{sec}\right]$$

$$\Rightarrow \quad \frac{dl}{dt} = -\frac{\sqrt{2}}{\pi l} \Rightarrow \left(\frac{dl}{dt}\right)_{l=4} = -\frac{\sqrt{2}}{4\pi} \text{ cm/sec.}$$

Thus, the rate of decrease of the slant height is $\frac{\sqrt{2}}{4\pi}$ cm/sec.

EXAMPLE 22 *Sand is pouring from a pipe at the rate of 12 cm^3/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4 cm?* **[NCERT, CBSE 2011]**

SOLUTION Let r be the radius, h be the height and V be the volume of the sand-cone at any time t. Then,

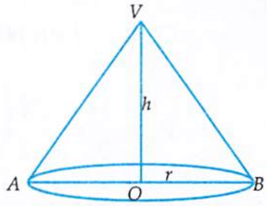

Fig. 12.9

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(36h^2)h = 12\pi h^3 \qquad [\because r = 6h]$$

$$\Rightarrow \quad \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt} \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{3\pi h^2} \qquad \left[\because \frac{dV}{dt} = 12 \text{ (Given)}\right]$$

$$\Rightarrow \quad \left(\frac{dh}{dt}\right)_{h=4} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}$$

Thus, the height of the sand-cone is increasing at the rate of $\frac{1}{48\pi}$ cm/sec.

EXAMPLE 23 *An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of 3/2 c.c. per minute. Find the rate at which the level of water in the cone is rising when the depth is 4 cm.*

SOLUTION Let α be the semi-vertical angle of the cone VAB whose height VO is 10 cm and radius $OB = 5$ cm. Then,

$$\tan\alpha = \frac{5}{10} = \frac{1}{2}$$

Let V be the volume of the water in the cone i.e. the volume of the cone $VA'B'$ after time t minutes and h be the height of water. Then,

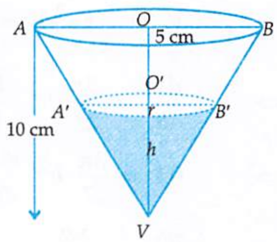

Fig. 12.10

$$V = \frac{1}{3}\pi(OB')^2(VO')$$

$$\Rightarrow V = \frac{1}{3}\pi h^3 \tan^2\alpha \quad \left[\because \tan\alpha = \frac{O'B'}{VO'} = \frac{O'B'}{h} \Rightarrow O'B' = h\tan\alpha\right]$$

$$\Rightarrow V = \frac{\pi}{12}h^3 \quad \left[\because \tan\alpha = \frac{1}{2}\right]$$

$$\Rightarrow \frac{dV}{dt} = \frac{\pi}{12}3h^2\frac{dh}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}$$

$$\Rightarrow \frac{3}{2} = \frac{\pi h^2}{4}\frac{dh}{dt} \qquad \left[\because \frac{dV}{dt} = \frac{3}{2} \text{ cm}^3\text{/minute (given)}\right]$$

$$\Rightarrow \frac{dh}{dt} = \frac{6}{\pi h^2} \Rightarrow \left(\frac{dh}{dt}\right)_{h=4} = \frac{6}{\pi(4)^2} = \frac{3}{8\pi} \text{ cm/min.}$$

EXAMPLE 24 *Water is dripping out from a conical funnel at a uniform rate of 4 cm^3/sec through a tiny hole at the vertex in the bottom. When the slant height of the water is 3 cm, find the rate of decrease of the slant height of the water-cone. Given that the vertical angle of the funnel is 120°.* **[NCERT EXEMPLAR]**

SOLUTION Let at any time t, V be the volume of the water in the cone i.e., the volume of the water-cone $VA'B'$, and let l be the slant height. Then,

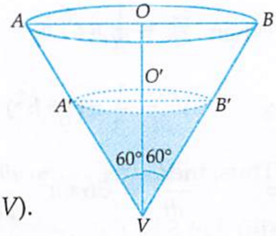

Fig. 12.11

$$O'A' = l\sin 60° = \frac{\sqrt{3}\,l}{2} \text{ and } VO' = l\cos 60° = \frac{l}{2}.$$

$$\therefore \quad V = \frac{1}{3}\pi\left(\frac{\sqrt{3}\,l}{2}\right)^2\left(\frac{l}{2}\right) = \frac{\pi l^3}{8} \Rightarrow \frac{dV}{dt} = \frac{3\pi l^2}{8}\frac{dl}{dt} \quad \text{...(i)}$$

We are given that $\frac{dV}{dt} = -4\ \text{cm}^3/\text{sec}$ (negative sign due to decreasing V).

$$\therefore \quad -4 = \frac{3\pi}{8} l^2 \frac{dl}{dt} \quad \left[\text{Putting } \frac{dV}{dt} = -4 \text{ in (i)}\right]$$

$$\Rightarrow \quad \frac{dl}{dt} = -\frac{32}{3\pi l^2} \Rightarrow \left(\frac{dl}{dt}\right)_{l=3} = -\frac{32}{3\pi(3)^2} = -\frac{32}{27\pi} \text{ cm/sec}$$

Thus, the slant height of the water-cone is decreasing at the rate of $\frac{32}{27}$ cm/sec.

EXAMPLE 25 *Water is running into a conical vessel, 15 cm deep and 5 cm in radius, at the rate of 0.1 cm^3/sec. When the water is 6 cm deep, find at what rate is*

(i) *the water level rising?* (ii) *the water-surface area increasing?*

(iii) *the wetted surface of the vessel increasing?*

SOLUTION Let V be the volume of the water in the cone i.e. the volume of the water-cone $VA'B'$ at any time t. Let $VO' = h$, $O'A' = r$ and $VA' = l$. Let α be the semi-vertical angle of the cone. Then,

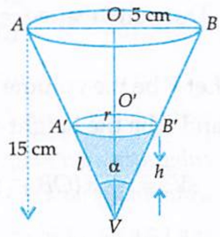

Fig. 12.12

$$\tan\alpha = \frac{OA}{VO} = \frac{5}{15} = \frac{1}{3}. \text{ Also, } \tan\alpha = \frac{O'A'}{VO'} = \frac{r}{h}$$

$$\therefore \quad \frac{1}{3} = \frac{r}{h} \Rightarrow 3r = h$$

(i) We have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3 \quad [\because 3r = h]$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{3\pi}{27}h^2\frac{dh}{dt}$$

$$\Rightarrow \quad 0.1 = \frac{3\pi}{27}h^2\frac{dh}{dt} \quad \left[\because \frac{dV}{dt} = 0.1\ \text{cm}^3/\text{sec (Given)}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{2.7}{3\pi h^2} \Rightarrow \left(\frac{dh}{dt}\right)_{h=6} = \frac{2.7}{3\pi(36)} = \frac{1}{40\pi}$$

Thus, the water level is rising at the rate of $\frac{1}{40\pi}$ cm/sec.

(ii) Let A be the water surface area at any time t. Then,

$$A = \pi r^2$$

$$\Rightarrow \quad A = \pi\frac{h^2}{9} \quad [\because 3r = h]$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2\pi h}{9}\frac{dh}{dt}$$

When $h = 6$, $\dfrac{dh}{dt} = \dfrac{1}{40\pi}$, we obtain:

$$\frac{dA}{dt} = \frac{2\pi \times 6}{9} \times \frac{1}{40\pi} = \frac{1}{30} \text{ cm}^2/\text{sec}$$

Thus, the water-surface area is increasing at the rate of $\dfrac{1}{30}$ cm^2/sec.

(iii) Let S be the wetted surface area of the vessel at any time t. Then, $S = \pi r l$.

From Fig. 12.12, we obtain

$$l^2 = VA'^2 = VO'^2 + O'A'^2$$

$$\Rightarrow \quad l^2 = h^2 + r^2$$

$$\Rightarrow \quad l^2 = h^2 + \frac{h^2}{9} \qquad [\because\ 3r = h]$$

$$\Rightarrow \quad l = \frac{\sqrt{10}\, h}{3}$$

$$\therefore \quad S = \pi r l \Rightarrow S = \pi\left(\frac{h}{3}\right)\left(\frac{\sqrt{10}\, h}{3}\right) \Rightarrow S = \frac{\pi}{9}\sqrt{10}\, h^2 \Rightarrow \frac{dS}{dt} = \frac{2\pi\sqrt{10}\, h}{9}\frac{dh}{dt}$$

Since $h = 6$ and, $\dfrac{dh}{dt} = \dfrac{1}{40\pi}$. Therefore, $\dfrac{dS}{dt} = \dfrac{2\pi\sqrt{10}}{9} \times 6 \times \dfrac{1}{40\pi} = \dfrac{\sqrt{10}}{30}$ cm^2/sec.

Thus, the wetted surface area of the vessel is increasing at the rate of $\dfrac{\sqrt{10}}{30}$ cm^2/sec.

EXAMPLE 26 *A water tank has the slope of an inverted right circular cone with its axis vertical and vertex lower most. Its semi-vertical angle is* $\tan^{-1}(0.5)$. *Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m.* **[NCERT]**

SOLUTION Let α be the semi-vertical angle of the water tank in the form of cone. Then,

$$\tan\alpha = 0.5 = \frac{1}{2} \Rightarrow \frac{r}{h} = \frac{1}{2} \Rightarrow r = \frac{h}{2}$$

Let $V\ A'\ B'$ be the water cone of volume V. Then,

$$\frac{dV}{dt} = 5 \text{ m}^3/\text{hr} \qquad \text{[Given]}$$

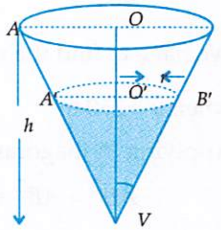

Fig. 12.13

We have to find $\dfrac{dh}{dt}$ when $h = 4$ m.

Now,

$$V = \frac{1}{3} r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12} h^3$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \Rightarrow 5 = \frac{\pi}{4} \times 4^2 \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{5}{4\pi} = \frac{5}{4} \times \frac{7}{22} \text{ m/h} = \frac{35}{88} \text{ m/h}$$

Thus, the rate of change of water level is $\dfrac{35}{88}$ m/h.

EXAMPLE 27 *A man is moving away from a tower 41.6 m high at the rate of 2 m/sec. Find the rate at which the angle of elevation of the top of tower is changing, when he is at a distance of 30 m from the foot of the tower. Assume that the eye level of the man is 1.6 m from the ground.*

SOLUTION Let AB be the tower. Let at any time t, the man be at a distance of x metres from the tower AB and let θ be the angle of elevation at that time.Then,

$$\tan\theta = \frac{BC}{PC} \Rightarrow \tan\theta = \frac{40}{x} \Rightarrow x = 40\cot\theta \qquad \text{...(i)}$$

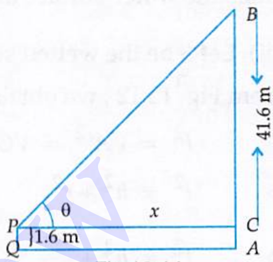

Fig.12.14

$$\therefore \quad \frac{dx}{dt} = -40\,\text{cosec}^2\theta\,\frac{d\theta}{dt}$$

We are given that $\frac{dx}{dt} = 2$ m/sec.

$$\therefore \quad 2 = -40\,\text{cosec}^2\theta\,\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{20\,\text{cosec}^2\theta} \qquad \text{...(ii)}$$

When $x = 30$, we get

$$\cot\theta = \frac{30}{40} = \frac{3}{4} \qquad \text{[Putting } x = 30 \text{ in (i)]}$$

$$\therefore \quad \text{cosec}^2\theta = 1 + \cot^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$$

Substituting $\text{cosec}^2\theta = \frac{25}{16}$ in (ii), we get: $\frac{d\theta}{dt} = -\frac{1}{20 \times \frac{25}{16}} = -\frac{4}{125}$ radians/sec

Thus, the angle of elevation of the top of tower is decreasing at the rate of 4/125 radians/sec.

EXAMPLE 28 *A kite is moving horizontally at the height of 151.5 meters. If the speed of kite is 10 m/sec, how fast is the string being let out; when the kite is 250 m away from the boy who is flying the kite? The height of the boy is 1.5 m.* **[NCERT EXEMPLAR]**

SOLUTION Let OA be the boy of height 1.5 m and kite be flying at a height $OB = 151.5$ m from the horizon OX. Therefore, $AB = OB - OA = (151.5 - 1.5)$ m $= 150$m.

Let at any time t, kite be at P such that $BP = x$ and $AP = y$. It is given that the kite is moving horizontally at the speed of 10 m/sec.

$$\therefore \quad \frac{dx}{dt} = 10 \text{ m/sec}$$

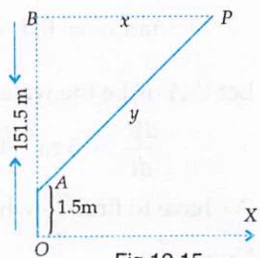

Fig.12.15

We have to find the rate at which the string is being let out i.e. $\frac{dy}{dt}$ when $y = 250$ m.

Applying Pythagoras theorem in ΔABP, we obtain

$$AP^2 = AB^2 + BP^2 \Rightarrow y^2 = 150^2 + x^2 \qquad \text{...(i)}$$

Differentiating (i) with respect to t, we obtain

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{x}{y} \times 10 \Rightarrow \frac{dy}{dt} = \frac{10x}{y} \qquad \text{...(ii)}$$

Putting $y = 250$ in (i), we obtain

$$250^2 = 150^2 + x^2 \Rightarrow x^2 = 40000 \Rightarrow x = 200$$

Putting $x = 200$ and $y = 250$ in (ii), we obtain: $\frac{dy}{dt} = 10 \times \frac{200}{250} = 8$.

Hence, the string is being let out at the rate of 8 m/sec.

EXERCISE 12.2

BASIC

1. The side of a square sheet is increasing at the rate of 4 cm per minute. At what rate is the area increasing when the side is 8 cm long?
2. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long?
3. The side of a square is increasing at the rate of 0.2 cm/sec. Find the rate of increase of the perimeter of the square.
4. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference? [NCERT]
5. The radius of a spherical soap bubble is increasing at the rate of 0.2 cm/sec. Find the rate of increase of its surface area, when the radius is 7 cm.
6. A balloon which always remains spherical, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm. [NCERT]
7. The radius of an air bubble is increasing at the rate of 0.5 cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?
8. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm/sec. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing? [NCERT]
9. A particle moves along the curve $y = x^3$. Find the points on the curve at which the y-coordinate changes three times more rapidly than the x-coordinate.
10. Find an angle θ
    (i) which increases twice as fast as its cosine.
    (ii) whose rate of increase twice is twice the rate of decrease of its consine.

BASED ON LOTS

11. A man 2 metres high walks at a uniform speed of 5 km/hr away from a lamp-post 6 metres high. Find the rate at which the length of his shadow increases.
12. A ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5 m/sec. How fast is the angle θ between the ladder and the ground is changing when the foot of the ladder is 12 m away from the wall.
13. A particle moves along the curve $y = x^2 + 2x$. At what point(s) on the curve are the x and y coordinates of the particle changing at the same rate?
14. If $y = 7x - x^3$ and x increases at the rate of 4 units per second, how fast is the slope of the curve changing when $x = 2$?
15. The top of a ladder 6 metres long is resting against a vertical wall on a level pavement, when the ladder begins to slide outwards. At the moment when the foot of the ladder is 4 metres from the wall, it is sliding away from the wall at the rate of 0.5 m/sec. How fast is the top-sliding downwards at this instance?
    How far is the foot from the wall when it and the top are moving at the same rate?
16. A balloon in the form of a right circular cone surmounted by a hemisphere, having a diametre equal to the height of the cone, is being inflated. How fast is its volume changing with respect to its total height h, when $h = 9$ cm.
17. Water is running into an inverted cone at the rate of π cubic metres per minute. The height of the cone is 10 metres, and the radius of its base is 5 m. How fast the water level is rising when the water stands 7.5 m below the base.

18. The surface area of a spherical bubble is increasing at the rate of 2 cm^2/s. When the radius of the bubble is 6 cm, at what rate is the volume of the bubble increasing? [CBSE 2005]
19. The radius of a cylinder is increasing at the rate 2 cm/sec. and its altitude is decreasing at the rate of 3 cm/sec. Find the rate of change of volume when radius is 3 cm and altitude 5 cm. [CBSE 2017]
20. The volume of metal in a hollow sphere is constant. If the inner radius is increasing at the rate of 1 cm/sec, find the rate of increase of the outer radius when the radii are 4 cm and 8 cm respectively.
21. A particle moves along the curve $y = \left(\frac{2}{3}\right) x^3 + 1$. Find the points on the curve at which the y-coordinate is changing twice as fast as the x-coordinate.
22. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate. [CBSE 2002C]
23. The volume of a cube is increasing at the rate of 9 cm^3/sec. How fast is the surface area increasing when the length of an edge is 10 cm?
24. The volume of a spherical balloon is increasing at the rate of 25 cm^3/sec. Find the rate of change of its surface area at the instant when radius is 5 cm. [CBSE 2004, 2017]
25. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (i) the perimeter (ii) the area of the rectangle. [CBSE 2009]
26. A circular disc of radius 3 cm is being heated. Due to expansion, its radius increases at the rate of 0.05 cm/sec. Find the rate at which its area is increasing when radius is 3.2 cm. [NCERT]

BASED ON HOTS

27. Sand is being poured onto a conical pile at the constant rate of 50 cm^3/minute such that the height of the cone is always one half of the radius of its base. How fast is the height of the pile increasing when the sand is 5 cm deep.
28. A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 52 m/sec, find the rate at which the string is being paid out.

ANSWERS

1. 64 cm^2/minute
2. 900 cm^3/sec
3. 0.8 cm/sec
4. 1.4 π cm/sec
5. 11.2π cm^2/sec
6. $1/\pi$ cm/sec
7. 2π cm^3/sec
8. 80π cm^2/sec
9. (1, 1), (−1, −1)
10. (i) $7\pi/6$ (ii) $\pi/6$
11. 5/2 km/h
12. 0.3 radian/sec
13. (−1/2, −3/4)
14. 48
15. $\frac{1}{\sqrt{5}}$ m/sec, $3\sqrt{2}$m
16. 12π cm^3/sec
17. 0.64 metre/minute
18. 6 cm^3/sec
19. 33π cm^3/sec
20. 1/4 cm/sec
21. (1, 5/3) and (−1, 1/3)
22. (2, 4)
23. 3.6 cm^2/sec
24. 10 cm^2/sec
25. (i) −2 cm/minute (ii) 2 cm/minute
26. 0.320 π cm^2/sec
27. $1/2\pi$ cm/minute
28. 20 m/sec.

HINTS TO SELECTED PROBLEMS

10. (i) We have,

$$\frac{d\theta}{dt} = 2\frac{d}{dt}(\cos\theta) \Rightarrow \frac{d\theta}{dt} = -2\sin\theta\frac{d\theta}{dt} \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = \frac{7\pi}{6}$$

(ii) It is given that

$$\frac{d\theta}{dt} = -2\frac{d}{d\theta}(\cos\theta) \Rightarrow \frac{d\theta}{dt} = 2\sin\theta\frac{d\theta}{dt} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

12. Let the bottom of the ladder be at a distance x m from the wall and the top be at a height y from the ground. Then,

$$x^2 + y^2 = 13^2 \text{ and } \tan\theta = \frac{y}{x}$$

$$\therefore \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

$$\Rightarrow \quad 3x + 2y\frac{dy}{dt} = 0 \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - \frac{3y}{2}}{x^2} \qquad \left[\because \frac{dx}{dt} = 1.5\right]$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{3x}{2y} \text{ and } \sec^2\theta\frac{d\theta}{dt} = \frac{x \times -\frac{3x}{2y} - \frac{3y}{2}}{x^2}$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y\sec^2\theta} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y(1+\tan^2\theta)} = -\frac{3}{2}\frac{(x^2+y^2)}{x^2 y\left(1+\frac{y^2}{x^2}\right)} = -\frac{3}{2y}$$

When $x = 12,\ x^2 + y^2 = 13^2 \Rightarrow y = 5. \quad \therefore \quad \frac{d\theta}{dt} = -\frac{3}{10}$

13. We have, $y = x^2 + 2x \quad \Rightarrow \frac{dy}{dt} = (2x+2)\frac{dx}{dt} \Rightarrow 1 = 2x + 2 \Rightarrow x = -\frac{1}{2}$

14. We have , m = Slope of the curve $= \frac{dy}{dx} = 7 - 3x^2$.

Now, $\quad m = 7 - 3x^2$

$$\Rightarrow \quad \frac{dm}{dt} = -6x\frac{dx}{dt} = -6x(4) = -24x \qquad \left[\because \frac{dx}{dt} = 4 \text{ (given)}\right]$$

$$\Rightarrow \quad \left(\frac{dm}{dt}\right)_{x=2} = -48$$

18. We have, $S = 4\pi r^2 \quad \Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{2}{8\pi r}$

Now,

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi r^2 \times \frac{2}{8\pi r} = r \qquad \text{[Using (i)]}$$

Hence, $\frac{dV}{dt} = 6$ when $r = 6$.

19. We have, $V = \pi r^2 h,\ \frac{dr}{dt} = 2$ and $\frac{dh}{dt} = -3$

$$\therefore \quad V = \pi r^2 h \quad \Rightarrow \frac{dV}{dt} = \pi\left\{2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right\} \Rightarrow \frac{dV}{dt} = \pi(4rh - 3r^2)$$

When $r = 3, h = 5$, we obtain: $\frac{dV}{dt} = \pi(60 - 27) = 33\pi$

26. Let r be the radius and A be the area of the disc at any time t. Then, $A = \pi r^2$. It is given that $\frac{dr}{dt} = 0.05$ cm/sec.

Now, $A = \pi r^2$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow \left(\frac{dA}{dt}\right)_{r=3.2} = 2\pi \times 3.2 \times 0.05 = 0.320\,\pi \text{ cm}^2\text{/sec.}$$

28. We have, $y^2 = x^2 + (120)^2$

$$\Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = 52\frac{x}{y} \quad \left[\because \frac{dx}{dt} = 52\right]$$

Putting $y = 130$ in $y^2 = x^2 + (120)^2$, we get $x = 50$.

$$\therefore \frac{dy}{dt} = \frac{52 \times 50}{130} = 20.$$

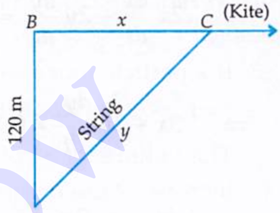

Fig.12.14

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The rate of change of $\sqrt{x^2+16}$ with respect to $\frac{x}{x-1}$ at $x = 3$ is ....
2. The rate of change of the surface are of a sphere of radius r when the radius is increasing at the rate of 2 cm/sec is.............................. .
3. The diagonal of a square is changing at the rate of $\frac{1}{2}$ cm/sec. Then the rate of change of area, when the area is 400 cm^2, is equal to ....
4. The rate of change of volume of a sphere with respect to its surface area, when the radius is 2 cm, is ....
5. The angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine, is ....
6. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when the side is 10 cm, is ....
7. Gas is being pumped into a spherical balloon at the rate of 30 cm^3/min. The rate at which the radius increases when it reaches the value 15 cm, is ....
8. The distance s described by a particle in t seconds is given by $s = ae^t + \frac{b}{e^t}$. Then the acceleration of the particle at time t is equal to ....
9. The volume V and depth x of water in a vessel are connected by the relation $V = 5x - \frac{x^2}{6}$ and the volume of water is increasing the rate of 5 cm^3/sec, when $x = 2$ cm. The rate of which the depth of water is increasing is equal to ....
10. Water is flowing into a vertical cylindrical tank of radius 2 ft at the rate of 8 cubic/minute. The rate at which the water level is rising, is ....
11. If the radius of a circle is increasing at the rate of 0.5 cm/sec, then the rate of increase of its circumference is .... **[CBSE 2020]**
12. The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of .... cm^2/sec. **[CBSE 2020]**

ANSWERS

1. $\frac{-12}{5}$ 2. $16\pi r$ 3. $10\sqrt{2}$ cm^2/sec 4. 1 5. $\frac{\pi}{6}$ 6. $10\sqrt{3}$ sq.unit/sec
7. $\frac{1}{30\pi}$ cm/min 8. s 9. $\frac{15}{13}$ cm/sec 10. $\frac{2}{\pi}$ ft/minute 11. π cm/sec 12. 12π

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If a particle moves in a straight line such that the distance travelled in time t is given by $s = t^3 - 6t^2 + 9t + 8$. Find the initial velocity of the particle.
2. The volume of a sphere is increasing at 3 cubic centimeter per second. Find the rate of increase of the radius, when the radius is 2 cms.
3. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. How far is the area increasing when the side is 10 cms? **[NCERT EXEMPLAR]**
4. The side of a square is increasing at the rate of 0.1 cm/sec. Find the rate of increase of its perimeter.
5. The radius of a circle is increasing at the rate of 0.5 cm/sec. Find the rate of increase of its circumference.
6. The side of an equilateral triangle is increasing at the rate of $\frac{1}{3}$ cm/sec. Find the rate of increase of its perimeter.
7. Find the surface area of a sphere when its volume is changing at the same rate as its radius.
8. If the rate of change of volume of a sphere is equal to the rate of change of its radius, find the radius of the sphere.
9. The amount of pollution content added in air in a city due to x diesel vehicles is given by $P(x) = 0.005x^3 + 0.02x^2 + 30x$. Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above questions. **[CBSE 2013]**
10. A ladder, 5 meter long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides down wards at the rate of 10 cm/sec, then find the rate at which the angle between the floor and ladder is decreasing when lower end of ladder is 2 metres from the wall. **[NCERT EXEMPLAR]**

ANSWERS

1. 9 units/unit time 2. $3/16\pi$ cm/sec 3. $10\sqrt{3}$ cm^2/sec 4. 0.4 cm/sec
5. π cm/sec 6. 1 cm/sec 7. 1 square unit 8. $1/2\sqrt{\pi}$ units
9. 30.255 units, Pollution level due to x diesel vehicles. 10. 1/20 radian/second