CHAPTER 20

AREAS OF BOUNDED REGIONS

20.1 INTRODUCTION

Integration has a large number of applications in Science and Engineering. In this chapter, we shall use integration for finding the areas of bounded regions. The first step in finding the areas of bounded regions is to identify the region whose area is to be computed. For this, we first draw the rough sketches of the various curves which enclose the region. In order to draw the rough sketches of the curves, readers are advised to go through the appendix prior to this chapter.

20.2 AREA AS A DEFINITE INTEGRAL

THEOREM *Let $f(x)$ be a continuous function defined on $[a, b]$. Then, the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is given by*

$$\int_a^b f(x)\, dx \text{ or, } \int_a^b y\, dx$$

PROOF Let AD be the curve $y = f(x)$ between the ordinates BA $(x = a)$ and CD $(x = b)$. Then, the required area is the area of region $ABCD$. Let $P(x, y)$ be any point on the curve and $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point on it. Draw ordinates PL and QM. Then, $PL = y$, $QM = y + \Delta y$ and $LM = \Delta x$.

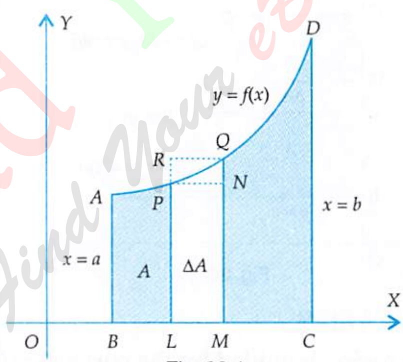

Fig. 20.1

Let A denote the Area $BLPA$, and let $A + \Delta A$ be the Area $BMQA$. Then, ΔA = Area $LMQP$.

Also, Area $LMNP = y\,\Delta x$, and Area $LMQR = (y + \Delta y)\,\Delta x$.

Clearly, Area of rectangle $LMNP \leq$ Area $LMQP \leq$ Area of rectangle $LMQR$

$$\Rightarrow \quad y\,\Delta x \leq \Delta A \leq (y + \Delta y)\,\Delta x$$

$$\Rightarrow \quad y \leq \frac{\Delta A}{\Delta x} \leq y + \Delta y$$

$$\Rightarrow \quad \lim_{\Delta x \to 0} y \leq \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} \leq \lim_{\Delta y \to 0} (y + \Delta y)$$

$$\Rightarrow \quad y \leq \frac{dA}{dx} \leq y$$

$$\Rightarrow \quad \frac{dA}{dx} = y$$

$$\Rightarrow \quad \int_a^b \frac{dA}{dx}\, dx = \int_a^b y\, dx \qquad \text{[Integrating between the limits } a \text{ and } b\text{]}$$

$$\Rightarrow \quad \Big[A\Big]_{x=a}^{x=b} = \int_a^b y\, dx$$

$$\Rightarrow \quad (\text{Area } A \text{ when } x = b) - (\text{Area } A \text{ when } x = a) = \int_a^b y\, dx$$

$$\Rightarrow \quad \text{Area } ABCD - 0 = \int_a^b y\, dx \qquad \left[\begin{array}{r}\text{When } x = a,\ PL \text{ coincides with } AB \\ \text{So, area } ABLP = 0 \text{ when } x = a\end{array}\right].$$

$$\Rightarrow \quad \text{Area } ABCD = \int_a^b y\, dx = \int_a^b f(x)\, dx$$

REMARK 1 *If the curve $y = f(x)$ lies below x-axis, then the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$ is negative. So, area is given by $\int_a^b |f(x)|\, dx$ or, $\int_a^b |y|\, dx$.*

REMARK 2 *The area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = c$ and $y = d$ is given by*

$$\int_c^d |f(y)|\, dy \quad \text{or,} \quad \int_c^d |x|\, dy$$

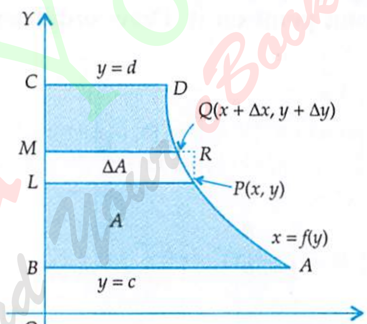

Fig. 20.2

20.3 AREA USING VERTICAL STRIPS

In order to find the area of the region bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$ and $x = b$, we may use the following algorithm.

ALGORITHM

Step I *Make a sketch of the curve and identify the region whose area is to be found.*

Step II *Slice the region into vertical strips. Take an arbitrary point $P(x, y)$ on the curve and construct a representative strip of width dx having two ends of its base on x-axis at points $\left(x - \frac{dx}{2}, 0\right)$ and $\left(x + \frac{dx}{2}, 0\right)$ and $(x, 0)$ as the mid-point of its base.*

Step III *Construct an approximating rectangle whose base is same as that of the representative strip and height equal to $|y| = |f(x)|$.*

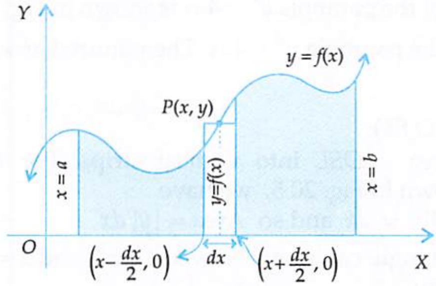

Fig. 20.3

Step IV *Find the area of the approximating rectangle as* $|y|\,dx = |f(x)|\,dx$.

Step V *Find the values of x, say,* $x = a$ *and* $x = b$ *within which the approximating rectangle can move horizontally in the given region and form the integral* $\int_a^b |f(x)|\,dx$ *or,* $\int_a^b |y|\,dx$.

Step VI *Evaluate the integral obtained in step V. The value of integral so obtained is the required area.*

REMARK 1 *If the curve* $y = f(x)$ *lies above x-axis on interval* $[a, b]$, *then the area of the region bounded by the curve* $y = f(x)$, *x-axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$\int_a^b |f(x)|\,dx = \int_a^b f(x)\,dx = \int_a^b y\,dx \qquad \left[\because f(x) \geq 0 \text{ for all } x \in [a, b] \ \therefore |f(x)| = f(x)\right]$$

REMARK 2 *If the curve* $y = f(x)$ *lies below x-axis on interval* $[a, b]$, *then the area of the region bounded by the curve* $y = f(x)$, *x-axis and the ordinates* $x = a$ *and* $x = b$ *is given by*

$$\int_a^b |f(x)|\,dx = -\int_a^b f(x)\,dx = -\int_a^b y\,dx \qquad \left[\because f(x) \leq 0 \text{ for all } x \in [a, b] \ \therefore |f(x)| = -f(x)\right]$$

REMARK 3 *If* $f(x)$ *is a continuous function defined on* $[a, b]$ *and* $c \in (a, b)$ *such that the curve* $y = f(x)$ *lies above x-axis on* $[a, c]$ *below x-axis on* $[c, b]$ *as shown in Fig. 20.4. Then, area A of the region bounded by* $y = f(x)$, *x-axis,* $x = a$ *and* $x = b$ *is given by*

$$A = \int_a^b |f(x)|\,dx = \int_a^c |f(x)|\,dx + \int_c^b |f(x)|\,dx = \int_a^c f(x)\,dx + \int_c^b -f(x)\,dx$$

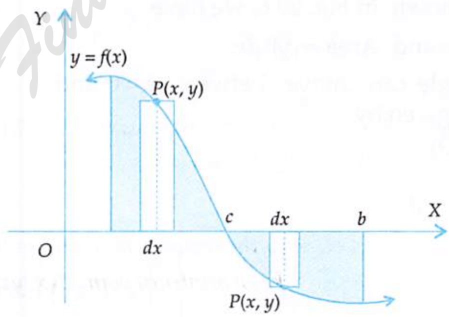

Fig. 20.4

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the area bounded by the parabola* $y^2 = 4ax$ *and its latus rectum.* [NCERT]

SOLUTION A rough sketch of the parabola $y^2 = 4ax$ is shown in Fig. 20.5. Let S$(a, 0)$ be the focus and $L\ SL'$ be the directrix of the parabola $y^2 = 4ax$. The required area is $LO\ L'\ L$. Since the curve is symmetrical about x-axis.

So, required area = 2 (Area $LO\ SL$).

Here, we slice the area $LOSL$ into vertical strips. For the approximating rectangle shown in Fig. 20.5, we have

Length $= |y|$, Width $= dx$ and so Area $= |y|\ dx$

Since the approximating rectangle can move between $x = 0$ and $x = a$..

So, required area A is given by

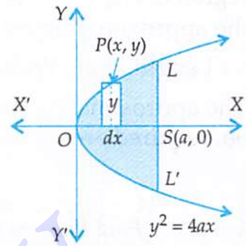

Fig. 20.5

$$A = 2\,(\text{Area } LOSL)$$

$$A = 2\int_0^a |y|\ dx$$

$$\Rightarrow \quad A = 2\int_0^a y\ dx \qquad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad A = 2\int_0^a \sqrt{4ax}\ dx \qquad \left[\because P(x,y) \text{ lies on } y^2 = 4ax\ \therefore y = \sqrt{4ax}\right]$$

$$\Rightarrow \quad A = 4\sqrt{a}\int_0^a \sqrt{x}\ dx = 4\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^a = 4\sqrt{a} \times \frac{2}{3}\,(a^{3/2} - 0) = \frac{8}{3}\,a^2 \text{ sq. units}$$

EXAMPLE 2 *Using integration, find the area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$.*

SOLUTION The equation $y^2 = 16x$ represents a parabola with vertex at the origin and axis of symmetry along the positive direction of x-axis as shown in Fig. 20.6. Clearly, $x = 4$ is a line parallel to y-axis. The region is the shaded portion shown in Fig. 20.6. Since $y^2 = 16x$ is symmetrical about x-axis.

$\therefore$ Required area = 2 (Area $OCAO$)

Here, we slice the area above x-axis into vertical strips. For the approximating rectangle shown in Fig. 20.6, we have

Length $= |y|$, Width $= dx$ and Area $= |y|\ dx$.

The approximating rectangle can move between $x = 0$ and $x = 4$. So, required area A is given by

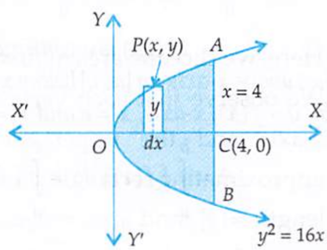

Fig. 20.6

$$A = 2\ (\text{Area } OCAO)$$

$$\Rightarrow \quad A = 2\int_0^4 |y|\ dx = 2\int_0^4 y\ dx \qquad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad A = 2\int_0^4 \sqrt{16x}\ dx \qquad \left[\because\ P(x, y) \text{ lies on } y^2 = 16x\ \therefore\ y = \sqrt{16x}\right]$$

$$\Rightarrow \quad A = 8\int_0^4 \sqrt{x}\ dx = 8\left[\frac{x^{3/2}}{3/2}\right]_0^4 = \frac{16}{3}\,(4^{3/2} - 0^{3/2}) = \frac{16}{3} \times 8 = \frac{128}{3} \text{ sq. units}$$

EXAMPLE 3 *Sketch the region bounded by $y = 2x - x^2$ and x-axis and find its area using integration.*

SOLUTION The equation of the curve is $y = 2x - x^2$. Clearly, it represents a parabola opening downward which cuts x-axis at (0, 0) and (2, 0). The sketch of the curve is as shown in Fig. 20.7. The required region is the shaded region in Fig. 20.7. Here, we slice this region into vertical strips. For the approximating rectangle shown in Fig. 20.7, we have

Length = $|y|$, Width = dx and, Area = $|y|\ dx$

The approximating rectangle can move from $x = 0$ to $x = 2$.
So, required area A is given by

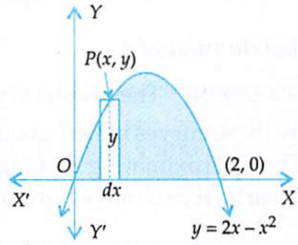

Fig. 20.7

$$\therefore \quad A = \int_0^2 |y|\, dx = \int_0^2 y\, dx \quad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad A = \int_0^2 (2x - x^2)\, dx \qquad \left[\because\ P(x, y) \text{ lies on } y = 2x - x^2\right]$$

$$\Rightarrow \quad A = \left[x^2 - \frac{x^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}$$

EXAMPLE 4 *Find the area of the region bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. [NCERT]

SOLUTION The equation of the curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In this equation all powers of x and y both are even. So, it is symmetrical about both the axes as shown in Fig. 20.8.

$\therefore$ Area enclosed by the ellipse= 4 [Area enclosed by the ellipse and the coordinate axes in the first quadrant].

Here, we slice the area in first quadrant into vertical strips. We observe that each vertical strip has its lower end on x-axis and the upper end on the ellipse. So, the approximating rectangle shown in Fig. 20.8 has width = dx, length = $|y|$, and area = $|y|\ dx$. The approximating rectangle can move between $x = 0$ and $x = a$. So, required area A is given by

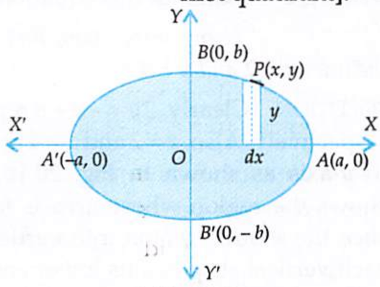

Fig. 20.8

$$A = 4\int_0^a |y|\, dx = 4\int_0^a y\, dx \qquad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx \qquad \left[\because\ P(x, y) \text{ lies on } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ \therefore\ y = \frac{b}{a}\sqrt{a^2 - x^2}\right]$$

$$\Rightarrow \quad A = \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx = \frac{4b}{a}\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_0^a$$

$$\Rightarrow \quad A = \frac{4b}{a}\left\{0 + \frac{1}{2}a^2 \sin^{-1}(1)\right\} = \frac{4b}{a} \times \frac{1}{2}a^2\left(\frac{\pi}{2}\right) = \pi ab \text{ sq. units}$$

EXAMPLE 5 *If the area above x-axis, bounded by the curves $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $\dfrac{3}{\log_e 2}$, then find the value of k.*

SOLUTION The graphs of $y = 2^{kx}$, $x = 0$ and $x = 2$ are shown in Fig. 20.9 and the region bounded by these curves in first quadrant is shaded in Fig. 20.9. Let us slice this region into vertical strips. The approximating rectangle shown in Fig. 20.9, has length $= |y|$, width $= dx$ and area $= |y|\, dx$. Clearly, it can move horizontally between $x = 0$ and $x = 2$.

$\therefore$ Area of the shaded region $= \displaystyle\int_0^2 |y|\, dx = \int_0^2 y\, dx$ $\quad [\because\ y \geq 0\ \therefore |y| = y]$

But, it is given that the area of the shaded region is $\dfrac{3}{\log_e 2}$.

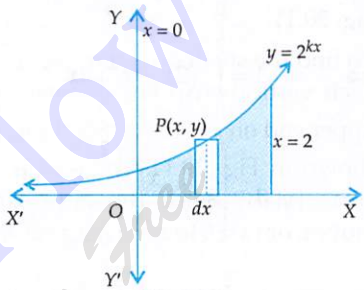

Fig. 20.9

$$\therefore \quad \int_0^2 |y|\, dx = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \int_0^2 y\, dx = \frac{3}{\log_e 2} \qquad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad \int_0^2 2^{kx}\, dx = \frac{3}{\log_e 2}$$

$$\Rightarrow \quad \left[\frac{2^{kx}}{k \log_e 2}\right]_0^2 = \frac{3}{\log_e 2} \Rightarrow \frac{2^{2k}}{k \log_e 2} - \frac{1}{k \log_e 2} = \frac{3}{\log_e 2} \Rightarrow \frac{4^k - 1}{k} = 3$$

Clearly, $k = 1$ satisfies this equation. Hence, $k = 1$.

EXAMPLE 6 *Using integration, find the area of the region bounded by the line $2y = -x + 8$, x-axis and the lines $x = 2$ and $x = 4$.*

SOLUTION Clearly, $2y = -x + 8$ represents a straight line cutting x and y-axes at (8, 0) and (0, 4) respectively. Also, $x = 2$ and $x = 4$ are straight lines parallel to y-axis as shown in Fig. 20.10. The shaded portion shows the region whose area is to be found. When we slice the shaded region into vertical strips, we find that each vertical strip has its lower end on x-axis and upper end on the line $2y = -x + 8$. So, the approximating rectangle shown in Fig. 20.10 has, length $= |y|$, width $= dx$ and area $= |y|\, dx$. The approximating rectangle can move from $x = 2$ to $x = 4$. So, required area A is given by

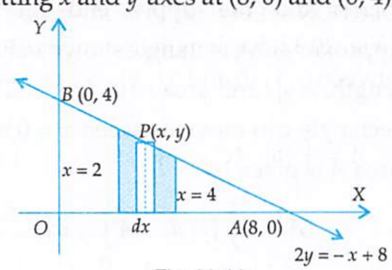

Fig. 20.10

$$A = \int_2^4 |y|\, dx = \int_2^4 y\, dx \qquad [\because\ y \geq 0\ \therefore |y| = y]$$

$$\Rightarrow \quad A = \int_2^4 \left(\frac{-x + 8}{2}\right) dx \qquad \left[\because\ P(x, y) \text{ lies on } 2y = -x + 8\ \therefore\ y = \frac{-x + 8}{2}\right]$$

$$\Rightarrow \quad A = \frac{1}{2}\left[-\frac{x^2}{2} + 8x\right]_2^4 = \frac{1}{2}\left\{\left(-\frac{16}{2} + 32\right) - \left(-\frac{4}{2} + 16\right)\right\} = 5 \text{ sq. units}$$

EXAMPLE 7 *Draw a rough sketch of the curve $y = \cos^2 x$ in $[0, \pi]$ and find the area enclosed by the curve, the lines $x = 0$, $x = \pi$ and the x-axis.*

SOLUTION We prepare a table for the values of $\cos^2 x$ at different points between 0 and π as given below.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\cos^2 x$	1	0.75	0.5	0.25	0	0.25	0.5	0.75	1

By plotting these points and joining them with a free hand, we obtain a rough sketch of $y = \cos^2 x$ as shown in Fig. 20.11. The required region is the shaded region in Fig. 20.11.

To find the shaded region, we slice it into vertical strips. Each vertical strip has its lower end on x-axis and the upper end on $y = \cos^2 x$ So, the approximating rectangle shown in Fig. 20.11 has, length $= |y|$, width $= dx$ and area $= |y|\, dx$. Since the approximating rectangle can move from $x = 0$ to $x = \pi$. So, required area A is given by

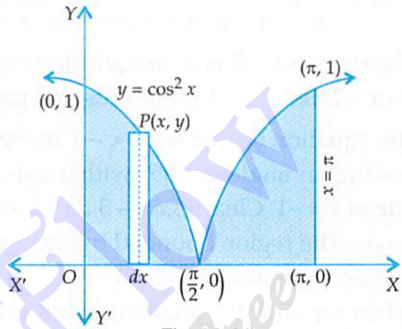

Fig. 20.11

$$A = \int_0^\pi |y|\, dx = \int_0^\pi y\, dx \qquad [\because\ y \geq 0 \ \therefore |y| = y]$$

$$\Rightarrow \quad A = \int_0^\pi \cos^2 x\, dx \qquad [\because\ P(x, y) \text{ lies on } y = \cos^2 x]$$

$$\Rightarrow \quad A = \frac{1}{2}\int_0^\pi (1 + \cos 2x)\, dx = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^\pi = \frac{1}{2}\left\{\left(\pi + \frac{\sin 2\pi}{2}\right) - 0\right\} = \frac{\pi}{2} \text{ sq. units}$$

EXAMPLE 8 *Find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -1$ and $x = 1$.* **[NCERT]**

SOLUTION Let $P(x, y)$ be any point on the line $y = 3x + 2$. The approximating rectangle shown in Fig. 20.12, has length $|y|$ and width dx. Clearly, it can move between $(-1, 0)$ and $(1, 0)$. So, required area A is given by

$$A = \int_{-1}^{1} |y|\, dx = \int_{-1}^{-2/3} |y|\, dx + \int_{-2/3}^{1} |y|\, dx$$

$$\Rightarrow A = \int_{-1}^{-2/3} -y\, dx + \int_{-2/3}^{1} y\, dx \qquad \left[\begin{array}{l}\because y < 0 \text{ for } -1 < x < -\dfrac{2}{3} \\ \text{and } y > 0 \text{ for } -\dfrac{2}{3} < x < 1\end{array}\right]$$

$$\Rightarrow A = -\int_{-1}^{-2/3} (3x + 2)\, dx + \int_{-2/3}^{1} (3x + 2)\, dx$$

$$\Rightarrow A = -\left[\frac{3}{2}x^2 + 2x\right]_{-1}^{-2/3} + \left[\frac{3}{2}x^2 + 2x\right]_{2/3}^{1} = \frac{1}{6} + \frac{25}{6} = \frac{13}{3} \text{ sq. units.}$$

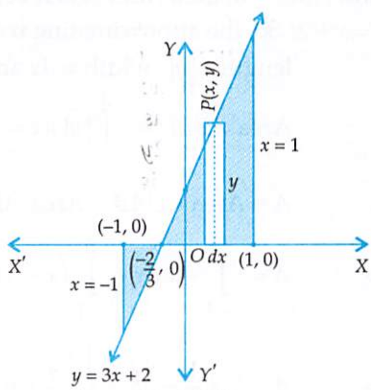

Fig. 20.12

EXAMPLE 9 *Using integration, find the area of the region bounded by the following curves, after making a rough sketch:* $y = 1 + |x+1|$, $x = -3, x = 3, y = 0$. **[NCERT EXEMPLAR, CBSE 2014]**

SOLUTION We have,

$$y = 1 + |x+1| \Rightarrow y = \begin{cases} 1 + x + 1 \ , & \text{if } x + 1 \geq 0 \\ 1 - (x+1), & \text{if } x + 1 < 0 \end{cases} = \begin{cases} x + 2, & \text{if } x \geq -1 \\ -x \ , & \text{if } x < -1 \end{cases}$$

Thus, the equations of the given curves are

$$y = x + 2 \text{ for } x \geq -1, \ y = -x \text{ for } x < -1, \ x = -3, x = 3 \text{ and } y = 0$$

Clearly, $y = x + 2$ is a straight line cutting x and y-axes at $(-2, 0)$ and $(0, 2)$ respectively. So, $y = x + 2$, for $x > -1$ represents that part of the line which is on the right side of $x = -1$.

The equation $y = -x$ for $x < -1$ represents that part of the line passing through the origin and making an angle of 135° with x-axis which is on the left side of $x = -1$. Clearly, $x = -3$ and $x = 3$ are lines parallel to y-axis. The region bounded by the given curves is shaded portion shown in Fig. 20.13.

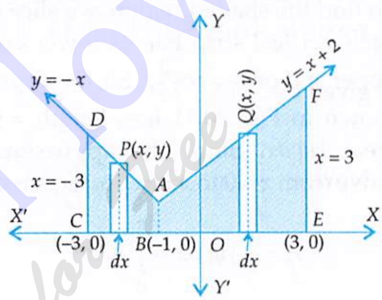

Fig. 20.13

When we slice the shaded region into vertical strips, we observe that vertical strips change their character at point A.

So, required area = Area $CDAB$ + Area $ABEF$.

Area CDAB: In area $CDAB$, each vertical strip has its upper end on $y = -x$ and lower end on x-axis. So, the approximating rectangle shown in Fig. 20.13 has, length $= |y|$, width $= dx$ and area $= |y|\, dx$. Since the approximating rectangle can move from $x = -3$ to $x = -1$.

$\therefore$ $$\text{Area } CDAB = \int_{-3}^{-1} |y|\, dx = \int_{-3}^{-1} y\, dx = \int_{-3}^{-1} -x\, dx \qquad [\because P(x, y) \text{ lies on } y = -x]$$

Let A denote the required area. Then,

Area ABEF: In area $ABEF$, each vertical strip has its lower end on x-axis and the upper end on $y = x + 2$. So, the approximating rectangle has,

length $= |y|$, width $= dx$ and area $= |y|\, dx$. As it can move from $x = -1$ to $x = 3$.

$\therefore$ $$\text{Area } ABEF = \int_{-1}^{3} |y|\, dx = \int_{-1}^{3} y\, dx = \int_{-1}^{3} (x+2)\, dx \qquad [\because Q(x, y) \text{ lies on } y = x + 2]$$

$\therefore$ A = Area $CDAB$ + Area $ABEF$

$$\Rightarrow \quad A = \int_{-3}^{-1} -x\, dx + \int_{-1}^{3} (x+2)\, dx = -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$\Rightarrow \quad A = -\left\{\frac{1}{2} - \frac{9}{2}\right\} + \left\{\left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)\right\} = 16 \text{ sq. units}$$

EXAMPLE 10 *Sketch the graph* $y = |x+1|$. *Evaluate* $\int_{-3}^{1} |x+1|\, dx$. *What does this value represent on the graph?*

SOLUTION The equation of the given curve is

$$y = |x+1| = \begin{cases} x+1, & \text{if } x+1 \geq 0 \\ -(x+1), & \text{if } x+1 < 0 \end{cases} \quad \text{or,} \quad y = |x+1| = \begin{cases} x+1, & \text{if } x \geq -1 \\ -x-1, & \text{if } x < -1 \end{cases}$$

Clearly, $y = x+1$ is a straight line cutting x and y-axes at $(-1, 0)$ and $(0, 1)$ respectively. So, $y = x+1, x \geq -1$ represents that portion of the line which lies on the right side of $x = -1$. Similarly, $y = -x-1, x < -1$ represents that part of the line $y = -x-1$ which is on the left side of $x = -1$.

A rough sketch of $y = |x+1|$ is shown in Fig. 20.14. Clearly, $x = -3$ and $x = 1$ are lines parallel to y-axis. The region bounded by $y = |x+1|$, $x = -3$, $x = 1$ and x-axis is shaded in Fig. 20.14. The area of this region is given by $\int_{-3}^{1} |x+1| \, dx$.

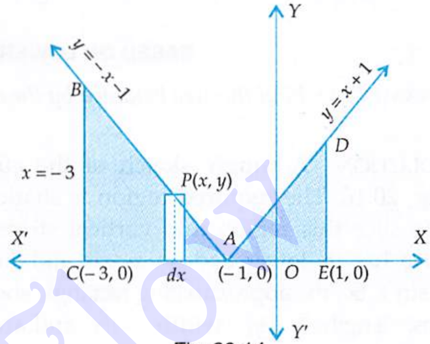

Fig. 20.14

Now, $$\int_{-3}^{1} |x+1| \, dx = \int_{-3}^{-1} |x+1| \, dx + \int_{-1}^{1} |x+1| \, dx = \int_{-3}^{-1} -(x+1) \, dx + \int_{-1}^{1} (x+1) \, dx$$

$$= -\left[\frac{(x+1)^2}{2}\right]_{-3}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{1} = -\left[0 - \frac{4}{2}\right] + \left[\frac{4}{2} - 0\right] = 4$$

This value represents the area of the shaded region shown in Fig. 20.14.

EXAMPLE 11 *Using integration find the area of the triangle formed by positive x-axis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.* **[CBSE 2015]**

SOLUTION The equation of the circle is $x^2 + y^2 = 4$. Differentiating with respect to x, we obtain

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(1, \sqrt{3})} = -\frac{1}{\sqrt{3}}$$

So, the equations of tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$ are

$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x-1) \text{ and } y - \sqrt{3} = \sqrt{3}(x-1)$$

respectively or, $x + \sqrt{3}y = 4$ and $y = \sqrt{3}x$.

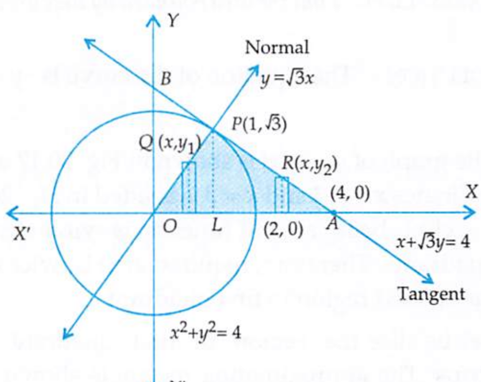

Fig. 20.15

The triangular region bounded by the positive x-axis, tangent and normal to the circle is the shaded region in Fig. 20.15. Let A be its area. Then,

$$A = \text{Area of the region } OPLO + \text{Area of the region } LPAL = \int_{0}^{1} y_1 \, dx + \int_{1}^{4} y_2 \, dx$$

Points $Q(x, y_1)$ and $R(x, y_2)$ lie on $y=\sqrt{3}x$ and $x+\sqrt{3}y=4$ respectively. Therefore, $y_1=\sqrt{3}x$ and $x+\sqrt{3}\,y_2=4$.

$$\therefore \quad A=\int_0^1 \sqrt{3}\,x\,dx+\int_1^4 \frac{4-x}{\sqrt{3}}\,dx=\sqrt{3}\left[\frac{x^2}{2}\right]_0^1+\frac{1}{\sqrt{3}}\left[4x-\frac{x^2}{2}\right]_1^4=\left(\frac{\sqrt{3}}{2}+\frac{3\sqrt{3}}{2}\right)=2\sqrt{3} \text{ sq. units}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 12 *Find the area bounded by the curve $y=\sin x$ between $x=0$ and $x=2\pi$.*

[NCERT EXEMPLAR]

SOLUTION A rough sketch of the curve $y=\sin x$ between $x=0$ and $x=2\pi$ is shown in Fig. 20.16. The required region is shaded in Fig. 20.16. We slice this region into vertical strips. Each vertical strip has its lower end on x-axis and the upper end on $y=\sin x$. So, the approximating rectangle shown in Fig. 20.16 has, length $=|y|$. width $=dx$, and area $=|y|\,dx$. The approximating rectangle can move horizontally between (0, 0) and $(2\pi, 0)$.

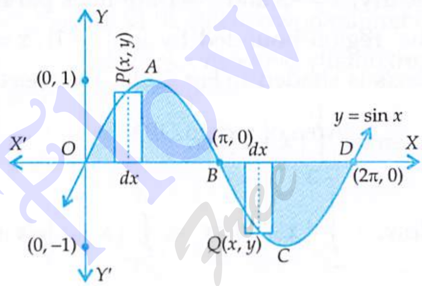

Fig. 20.16

So, required area A is given by

$$A=\int_0^{2\pi}|y|\,dx=\int_0^{\pi}|y|\,dx+\int_{\pi}^{2\pi}|y|\,dx$$

$$\Rightarrow \quad A=\int_0^{\pi} y\,dx+\int_{\pi}^{2\pi} -y\,dx \qquad \left[\begin{array}{l}\because\ y\geq 0 \text{ for } 0\leq x\leq\pi \text{ and } y<0 \text{ for } \pi<x<2\pi\\ \therefore\ |y|=y \text{ for } 0\leq x\leq\pi \text{ and } |y|=-y \text{ for } \pi<x<2\pi\end{array}\right]$$

$$\Rightarrow \quad A=\int_0^{\pi}\sin x\,dx+\int_{\pi}^{2\pi}-\sin x\,dx \qquad [\because P(x, y) \text{ lies on } y=\sin x]$$

$$\Rightarrow \quad A=\Big[-\cos x\Big]_0^{\pi}+\Big[\cos x\Big]_{\pi}^{2\pi}=2+2=4 \text{ sq. units}$$

EXAMPLE 13 *Find the area bounded by the curve $y=x\,|x|$, x-axis and the ordinates $x=-3$ and $x=3$.*

SOLUTION The equation of the curve is $y=x|x|=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$

The graph of $y=x|x|$ is shown in Fig. 20.17 and the region bounded by $y=x|x|$, x-axis and the ordinates $x=-3$ and $x=3$ is shaded in Fig. 20.17. Clearly, $y=x\,|x|$, being an odd function is symmetric in opposite quadrants. Therefore, required area is twice of the area of the shaded region in first quadrant.

Let us slice the region in first quadrant into vertical strips. The approximating rectangle shown in Fig. 20.17 has length $=|y_1|$, width $=dx$ and area $=|y_1|\,dx$. As it can move between $x=0$ and $x=3$. Therefore, area of the shaded region in first quadrant

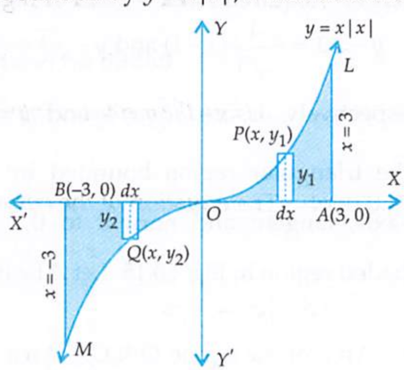

Fig. 20.17

$$A=\int_0^3|y_1|\,dx=\int_0^3 y_1\,dx \qquad [\because y_1\geq 0\ \therefore |y_1|=y_1]$$

$$\Rightarrow \quad A = \int_0^3 x^2\, dx = \left[\frac{x^3}{3}\right]_0^3 = 9 \text{ sq. units.} \qquad \left[\because P(x, y_1) \text{ lies on } y = x^2 \therefore\ y_1 = x^2\right]$$

Hence, required area $= 2A = 2 \times 9 = 18$ sq. units.

ALITER The graph of $y = x|x|$ is shown in Fig. 20.17. We have to find the area of the shaded region. Clearly,

Required area = Area of region $OBMO$ + Area of region $OALO$.

Area of region OBMO: Let us slice the region $OBMO$ into vertical strips. The approximating rectangle shown in Fig. 20.17 has length $= |y_2|$, width $= dx$ and area $= |y_2|\, dx$. Clearly, it can move horizontally between $x = -3$ and $x = 0$.

$$\therefore \quad \text{Area of region } OBMO = \int_{-3}^{0} |y_2|\, dx = \int_{-3}^{0} -y_2\, dx \qquad \left[\because y_2 < 0 \ \therefore |y_2| = -y_2\right]$$

$$= \int_{-3}^{0} -(-x^2)\, dx \qquad \left[\because Q(x, y_2) \text{ lies on } y = -x^2 \ \therefore\ y_2 = -x^2\right]$$

$$= \int_{-3}^{0} x^2\, dx = \left[\frac{x^3}{3}\right]_{-3}^{0} = 0 - (-9) = 9 \text{ sq. units.}$$

Area of region OALO: Let us slice the region $OALO$ into the vertical strips. The approximating rectangle shown in Fig. 20.17 has length $= |y_1|$, width $= dx$ and area $= |y_1|\, dx$. Clearly, it can move between $x = 0$ and $x = 3$.

$$\therefore \quad \text{Area of region } OALO = \int_0^3 |y_1|\, dx = \int_0^3 y_1\, dx S \qquad \left[\because y_1 > 0 \ \therefore\ |y_1| = y_1\right]$$

$$= \int_0^3 x^2\, dx = \left[\frac{x^3}{3}\right]_0^3 = 9 \text{ sq. units} \qquad \left[\because P(x, y_1) \text{ lies on } y = x^2 \therefore y_1 = x^2\right]$$

Hence, Required Area = Area of region $OBMO$ + Area of region $OALO = (9 + 9) = 18$ sq. units.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 14 *Using the method of integration find the area-bounded by the curve $|x| + |y| = 1$.* [NCERT]

SOLUTION The equation of the curve is

$$|x| + |y| = 1 \Leftrightarrow \begin{cases} x + y = 1 & \text{when } x \geq 0, y \geq 0 \\ -x + y = 1 & \text{when } x < 0, y \geq 0 \\ x - y = 1 & \text{when } x \geq 0, y < 0 \\ -x - y = 1 & \text{when } x < 0, y < 0 \end{cases}$$

Lines $x+y=1, -x+y=1, x-y=1$ and $-x-y=1$ enclose a square shown in Fig. 20.18. The area A bounded by the square is given by

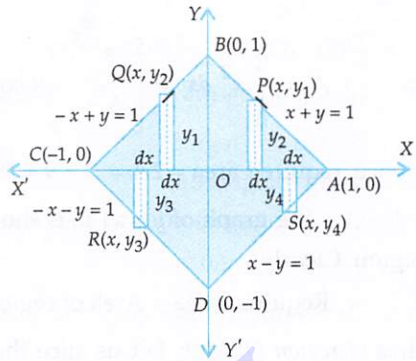

Fig. 20.18

$$A = \text{Area } COBO + \text{Area } OABO + \text{Area } CODC + \text{Area } ODAO$$

$$\Rightarrow A = \int_{-1}^{0} |y_1|\, dx + \int_{0}^{1} |y_2|\, dx + \int_{-1}^{0} |y_3|\, dx + \int_{0}^{1} |y_4|\, dx$$

$$\Rightarrow A = \int_{-1}^{0} |1+x|\, dx + \int_{0}^{1} |1-x|\, dx + \int_{-1}^{0} |-x-1|\, dx + \int_{0}^{1} |x-1|\, dx$$

$$\Rightarrow A = \int_{-1}^{0} (1+x)\, dx + \int_{0}^{1} (1-x)\, dx + \int_{-1}^{0} (x+1)\, dx + \int_{0}^{1} (1-x)\, dx$$

$$\Rightarrow A = 2\int_{-1}^{0} (x+1)\, dx + 2\int_{0}^{1} (1-x)\, dx = 2\left[\frac{x^2}{2}+x\right]_{-1}^{0} + 2\left[x-\frac{x^2}{2}\right]_{0}^{1}$$

$$\Rightarrow A = 2\left\{0-\left(\frac{1}{2}-1\right)\right\} + 2\left\{\left(1-\frac{1}{2}\right)-0\right\} = 1+1 = 2 \text{ sq. units.}$$

EXERCISE 20.1

BASIC

1. Using integration, find the area of the region bounded between the line $x=2$ and the parabola $y^2 = 8x$.
2. Using integration, find the area of the region bounded by the line $y-1=x$, the x-axis and the ordinates $x=-2$ and $x=3$. [CBSE 2002, 2022]
3. If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x=4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a, where $a>0$. [CBSE 2022]
4. Find the area lying above the x-axis and under the parabola $y = 4x - x^2$.
5. Draw a rough sketch to indicate the region bounded between the curve $y^2 = 4x$ and the lines $x=0$ and $x=3$. Also, find the area of this region. [CBSE 2022]
6. Make a rough sketch of the graph of the function $y = 4 - x^2$, $0 \le x \le 2$ and determine the area enclosed by the curve, the x-axis and the lines $x=0$ and $x=2$.
7. Sketch the graph of $y=\sqrt{x+1}$ in $[0, 4]$ and determine the area of the region enclosed by the curve, the x-axis and the lines $x=0$, $x=4$.
8. Find the area under the curve $y=\sqrt{6x+4}$ above x-axis from $x=0$ to $x=2$. Draw a sketch of curve also.
9. Draw the rough sketch of $y^2+1=x$, $x \le 2$. Find the area enclosed by the curve and the line $x=2$.

10. Draw a rough sketch of the graph of the curve $\frac{x^2}{4}+\frac{y^2}{9}=1$ and evaluate the area of the region under the curve and above the x-axis. [NCERT]
11. Sketch the region $\{(x, y): 9x^2+4y^2=36\}$ and find the area of the region enclosed by it, using integration.
12. Using integration, find the area of the region enclosed by the line $y=\sqrt{3}x$, semi-circle $y=\sqrt{4-x^2}$ and x-axis in first quadrant. [CBSE 2022]
13. Determine the area under the curve $y=\sqrt{a^2-x^2}$ included between the lines $x=0$ and $x=a$. [NCERT EXEMPLAR]
14. Using integration, find the area of the region bounded by the line $2y=5x+7$, x-axis and the lines $x=2$ and $x=8$.
15. Using definite integrals, find the area of the circle $x^2+y^2=a^2$.
16. Using integration, find the area of the region bounded by the following curves, after making a rough sketch: $y=1+|x+1|$, $x=-2$, $x=3$, $y=0$.
17. Sketch the graph $y=|x-5|$. Evaluate $\int_0^1 |x-5|\,dx$. What does this value of the integral represent on the graph. [NCERT]
18. Sketch the graph $y=|x+3|$. Evaluate $\int_{-6}^{0} |x+3|\,dx$. What does this integral represent on the graph? [NCERT, CBSE 2011]
19. Sketch the graph $y=|x+1|$. Evaluate $\int_{-4}^{2} |x+1|\,dx$. What does the value of this integral represent on the graph?
20. Find the area of the region bounded by the curve $xy-3x-2y-10=0$, x-axis and the lines $x=3$, $x=4$.
21. Using integration, find the area of the region enclosed by the curve $y=x^2$, the x-axis and the ordinates $x=-2$ and $x=1$. [CBSE 2022]
22. Draw a rough sketch of the curve $y=\frac{x}{\pi}+2\sin^2 x$ and find the area between the x-axis, the curve and the ordinates $x=0$ and $x=\pi$.

BASED ON LOTS

23. Find the area bounded by the curve $y=\cos x$, x-axis and the ordinates $x=0$ and $x=2\pi$. [NCERT]
24. Show that the areas under the curves $y=\sin x$ and $y=\sin 2x$ between $x=0$ and $x=\frac{\pi}{3}$ are in the ratio 2 : 3.
25. Compare the areas under the curves $y=\cos^2 x$ and $y=\sin^2 x$ between $x=0$ and $x=\pi$.
26. Find the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the ordinates $x=ae$ and $x=0$, where $b^2=a^2(1-e^2)$ and $e<1$. [NCERT]
27. Find the area of the minor segment of the circle $x^2+y^2=a^2$ cut off by the line $x=\frac{a}{2}$.
28. Find the area of the region bounded by the curve $x=at^2$, $y=2at$ between the ordinates corresponding $t=1$ and $t=2$. [NCERT EXEMPLAR]

29. Find the area enclosed by the curve $x = 3\cos t,\ y = 2\sin t$. [NCERT EXEMPLAR]

30. If the area between the curves $x = y^2$ and $x = 4$ is divided into two equal parts by the line $x = a$, then find the value of a by using integration. [CBSE 2020]

ANSWERS

1. $\frac{32}{3}$ sq. units
2. $\frac{17}{2}$ sq. units
3. $2\sqrt{2}$
4. $\frac{32}{3}$ sq. units
5. $8\sqrt{3}$ sq. units
6. $\frac{16}{3}$ sq. units
7. $\frac{2}{3}(5^{3/2} - 1)$
8. $\frac{56}{9}$ sq. units
9. $\frac{4}{3}$ sq. units
10. 3π sq. units
11. 6π sq. units
12. $\frac{2\pi}{3}$ sq. units
13. $\frac{\pi a^2}{4}$ sq. units
14. 96 sq. units
15. πa^2 sq. units
16. $\frac{27}{2}$ sq. units
17. $\frac{9}{2}$ sq. units
18. 9 sq. units
19. 9 sq. units
20. $3 + 16\log 2$ sq. units
21. 3 sq. units
22. $\frac{3\pi}{2}$ sq. units
23. 4 sq. units
25. Each equal to $\frac{\pi}{2}$ sq. unit.
26. $ab\left\{e\sqrt{1-e^2} + \sin^{-1} e\right\}$
27. $\frac{a^2}{12}(4\pi - 3\sqrt{3}))$ sq. units
28. $\frac{56}{3}a^2$ sq. units
29. 6π sq. units
30. $16^{1/3}$

HINTS TO SELECTED PROBLEMS

11. A rough sketch of the graph of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is shown in Fig. 20.19.

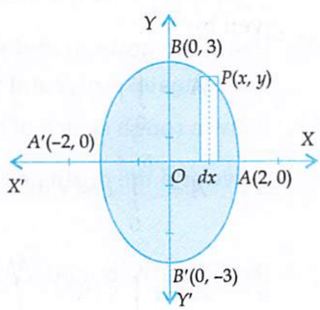

Fig. 20.19

Required area $= 2\int_{-2}^{2} |y|\, dx = 2\int_{-2}^{2} y\, dx \quad [\because y \geq 0]$

$= 2 \times \frac{3}{2}\int_{-2}^{2}\sqrt{4 - x^2}\, dx = 3\left[\frac{1}{2}x\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$

$= 3\left\{(0 + 2\sin^{-1} 1) - (2\sin^{-1}(-1)\right\} = 6\pi$ sq. units.

12. We have, $y = 2\sqrt{1 - x^2} \Rightarrow y^2 = 4 - 4x^2 \Rightarrow \frac{x^2}{1} + \frac{y^2}{4} = 1.$

This is the equation of an ellipse. So, $y = 2\sqrt{1 - x^2},\ x \in [0, 1]$ represents the portion of the ellipse lying in the first quadrant. So, required area A is given by

$$A = \int_0^1 y\, dx = \int_0^1 2\sqrt{1 - x^2}\, dx = \frac{\pi}{2} \text{ sq. units}$$

17. $\int_0^1 |x - 5|\, dx = \int_0^1 -(x - 5)\, dx$

$$= \left[\frac{-x^2}{2} + 5x\right]_0^2 = \left(-\frac{1}{2} + 5\right) \text{sq. units}$$

$$= \frac{9}{2} \text{ sq. units}$$

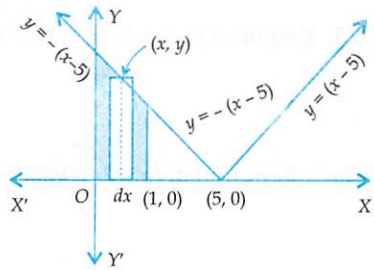

Fig. 20.20

As is evident from Fig. 20.20, $\int_0^1 |x-5|\,dx$ represents the area of the shaded region.

18. $\int_{-6}^{0} |x+3|\,dx = \int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx$

$$= \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

$$= \left[-\frac{(x+3)^2}{2}\right]_{-6}^{-3} + \left[\frac{(x+3)^2}{2}\right]_{-3}^{0}$$

$$= 0 + \frac{9}{2} + \frac{9}{2} - 0 = 9 \text{ sq. units}$$

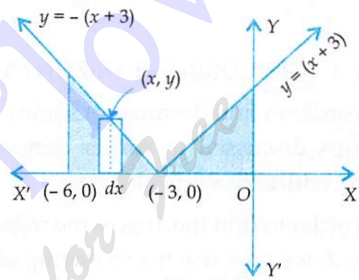

Fig. 20.21

The given integral represents the area bounded by $y = |x+3|$, x-axis and the ordinates $x = -3$ and $x = 0$.

23. The region bounded by $y = \cos x$, x-axis and the ordinates $x = 0$ and $x = 2\pi$ is the shaded region in Fig. 20.22. Its area A is given by

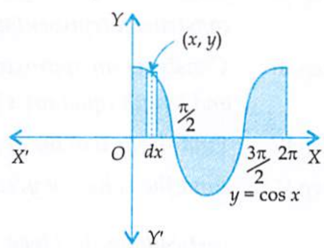

Fig. 20.22

$$A = \int_0^{2\pi} |y|\,dx = \int_0^{\pi/2} |y|\,dx + \int_{\pi/2}^{3\pi/2} |y|\,dx + \int_{3\pi/2}^{2\pi} |y|\,dx$$

$$\Rightarrow \quad A = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$$

$$\Rightarrow \quad A = \int_0^{\pi/2} y\,dx + \int_{\pi/2}^{3\pi/2} -y\,dx + \int_{3\pi/2}^{2\pi} y\,dx = \Big[\sin x\Big]_0^{\pi/2} + \Big[-\sin x\Big]_{\pi/2}^{3\pi/2} + \Big[\sin x\Big]_{3\pi/2}^{2\pi}$$

$$\Rightarrow \quad A = 1 + (1+1) + 0 - (-1) = 4 \text{ sq. units}$$

26. Required area A is given by

$A = 2$ (Area of shaded region in first quadrant)

$$\Rightarrow A = 2\int_0^{ae} |y|\,dx = 2\int_0^{ae} y\,dx \quad [\because y \geq 0 \;\therefore |y| = y]$$

$$\Rightarrow A = 2\,\frac{b}{a}\left[\int_0^{ae} \sqrt{a^2 - x^2}\,dx\right]$$

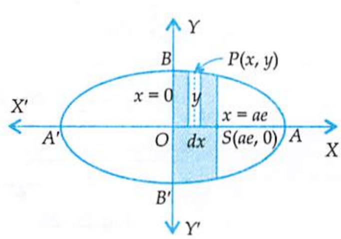

Fig. 20.23

27. Required area $= 2\int_{a/2}^{a} \sqrt{a^2 - x^2}\, dx$

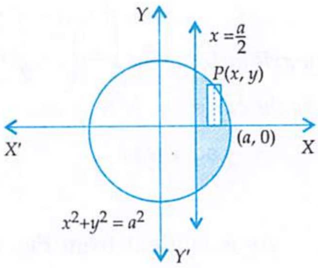

Fig. 20.24

28. Required area $= 2\int_{a}^{4a} y\, dx = 2\int_{a}^{4a} \sqrt{4\,ax}\, dx$

OR

Required area $= 2\int_{1}^{2} y \frac{dx}{dt}\, dt = 2\int_{1}^{2} (2at)(2at)\, dt = 8a^2 \int_{1}^{2} t^2\, dt \quad [\because\ x = at^2,\ y = 2at \Rightarrow y^2 = 4ax]$

20.4 AREA USING HORIZONTAL STRIPS

In order to find the areas of some regions, it is easier to form horizontal strips rather than vertical strips discussed in earlier section. The procedure for horizontal strips is analogous to the procedure for vertical strips.

In order to find the area of the region bounded by the curve $x = f(y)$, y-axis and the lines $y = c$ and $y = d$, we may use the following algorithm.

ALGORITHM

Step I *Make sketch of the curve and identify the region whose area is to be found.*

Step II *Slice the region into horizontal strips. Take an arbitrary point $P(x, y)$ on the curve and construct a representative strip of width dy and $(y, 0)$ as the mid-point of its base.*

Step III *Construct an approximating rectangle whose base is same as that of the representative strip and length equal to $|x| = |f(y)|$.*

Step IV *Find the area of the approximating rectangle as $|x|\, dy = |f(y)|\, dy$.*

Step V *Find the values of y, say $y = c$ and $y = d$ within which the approximating rectangle can move vertically in the given region and form the integral $\int_c^d |x|\, dy = \int_c^d |f(y)|\, dy$.*

Step VI *Evaluate the integral obtained in step V. The value so obtained is the required area.*

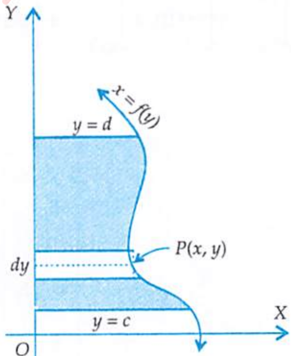

Fig. 20.25

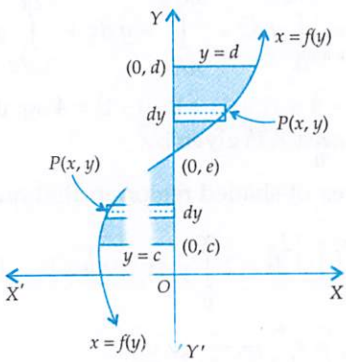

Fig. 20.26

REMARK 1 *If the function $f(y)$ is defined on $[c, d]$ on y-axis such that the curve $x = f(y)$ lies on the left of y-axis on $[c, e]$ and on the right of y-axis on $[e, d]$ as shown in Fig. 20.26, then area of the region bounded by the curve $x = f(y)$, y-axis and lines $y = c$ and $y = d$ is given by*

$$A = \int_c^d |x|\, dy = \int_c^e |x|\, dy + \int_e^d |x|\, dy$$

or, $$A = \int_c^d |f(y)|\, dy = \int_c^e |f(y)|\, dy + \int_e^d |f(y)|\, dy$$

or, $$A = \int_c^e -f(y)\, dy + \int_e^d f(y)\, dy \quad [\because\ x = f(y) < 0 \text{ for } c < y < e \text{ and } x = f(y) > 0 \text{ for } e < y < d]$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.*

SOLUTION The equation of the given curve is

$$y^2 = 2y - x \Rightarrow y^2 - 2y = -x \Rightarrow y^2 - 2y + 1 = -x + 1 \Rightarrow (y-1)^2 = -(x-1)$$

Clearly, this equation represents a parabola with vertex at (1, 1) and opens on the left. Putting $x = 0$ in $y^2 = 2y - x$, we get: $y^2 - 2y = 0 \Rightarrow y = 0, 2$.

So, the curve meets y-axis at (0, 0) and (0, 2). A rough sketch of the curve is as shown in Fig. 20.27 and the area of the region bounded by the curve and y-axis is shaded in Fig. 20.27. Here, we slice the shaded region into horizontal strips. For the approximating rectangle shown in Fig. 20.27, we have width $= dy$, length $= |x|$, and area $= |x|\, dy$. The approximating rectangle can move vertically from $y = 0$ to $y = 2$. So, required area A is given by

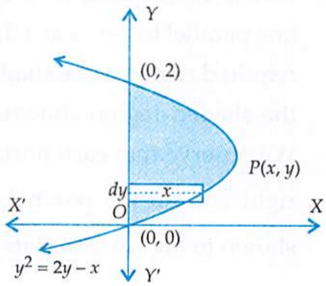

Fig. 20.27

$$\therefore \quad A = \int_0^2 |x|\, dy = \int_0^2 x\, dy \qquad [\because x \geq 0 \ \therefore |x| = x]$$

$$\Rightarrow \quad A = \int_0^2 (2y - y^2)\, dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 \qquad [\because\ P(x, y) \text{ lies on } y^2 = 2y - x \ \therefore\ x = 2y - y^2]$$

$$\Rightarrow \quad A = 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.}$$

EXAMPLE 2 *Sketch the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Find the area of the region using integration.*

SOLUTION The equation $y = 9x^2$ represents an upward opening parabola with axis as y-axis and vertex at the origin. Clearly, the shaded region is the region lying in the first quadrant and bounded by $y = 9x^2$, $x = 0$, $y = 1$ and $y = 4$. Let us slice this region into horizontal strips. The approximating rectangle shown in Fig. 20.28 has length $= |x|$ width $= dy$ and area $= |x|\, dy$. Clearly, it can move vertically between $y = 1$ and $y = 4$. So, required area A is given by

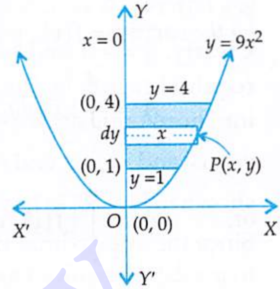

Fig. 20.28

$$A = \int_1^4 |x|\, dy = \int_1^4 x\, dy \qquad [\because x \geq 0 \therefore |x| = x]$$

$$\Rightarrow A = \int_1^4 \sqrt{\frac{y}{9}}\, dy = \frac{1}{3}\int_1^4 \sqrt{y}\, dy = \frac{1}{3} \times \frac{2}{3}\left[y^{3/2}\right]_1^4 \qquad \left[\because P(x, y) \text{ lies on } y = 9x^2 \therefore x = \sqrt{\frac{y}{9}}\right]$$

$$\Rightarrow A = \frac{2}{9}(8 - 1) = \frac{14}{9} \text{ sq. units}$$

EXAMPLE 3 *Find the area bounded by the curve $y^2 = 4ax$ and the lines $y = 2a$ and y-axis.*

SOLUTION Clearly, the equation $y^2 = 4ax$ represents a parabola with vertex (0, 0) and axis as x-axis. The equation $y = 2a$ represents a straight line parallel to x-axis at a distance $2a$ from it as shown in Fig. 20.29. The required region is the shaded portion in Fig. 20.29. To find the area of the shaded region shown in Fig. 20.29, we slice it into horizontal strips. We observe that each horizontal strip has its left end on y-axis and the right end on the parabola $y^2 = 4ax$. So, the approximating rectangle shown in Fig. 20.29 has its length $= |x|$, width $= dy$ and area $= |x|\, dy$.

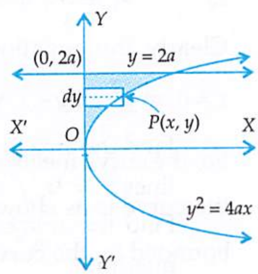

Fig. 20.29

Since the approximating rectangle can move vertically from $y = 0$ to $y = 2a$. So, required area A is given by

$$A = \int_0^{2a} |x|\, dy = \int_0^{2a} x\, dy \qquad [\because x \geq 0 \therefore |x| = x]$$

$$\Rightarrow \quad A = \int_0^{2a} \frac{y^2}{4a}\, dy = \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{2a} \qquad \left[\because P(x, y) \text{ lies on } y^2 = 4ax \therefore x = \frac{y^2}{4a}\right]$$

$$\Rightarrow \quad A = \frac{1}{4a}\left(\frac{8a^3}{3} - 0\right) = \frac{2a^2}{3} \text{ sq. units}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the area of the region bounded by* $y=-1, y=2, x=y^3$ and $x=0$.

SOLUTION A rough sketch of the curve $x=y^3$ as shown in Fig. 20.30.

Clearly, $y=-1$ and $y=2$ are straight lines parallel to x-axis. The required region is shaded in Fig. 20.30. When we slice this region into horizontal strips, we observe that each strip has its one end on y-axis and other end on $x=y^3$. So, the approximating rectangle shown in Fig. 20.30 has, length $=|x|$, width $=dy$ and area $=|x|\,dy$. Since the approximating rectangle can move vertically from $y=-1$ to $y=2$. So, required area A is given by

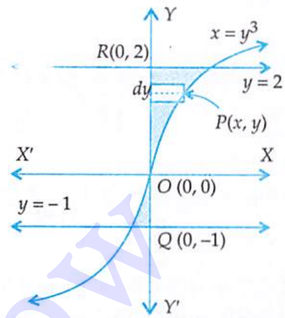

Fig. 20.30

$$A=\int_{-1}^{2}|x|\,dy=\int_{-1}^{0}|x|\,dy+\int_{0}^{2}|x|\,dy$$

$$\Rightarrow \quad A=\int_{-1}^{0}-x\,dy+\int_{0}^{2}x\,dy \qquad [\because x<0 \text{ for } -1\le y<0 \text{ and } x>0 \text{ for } 0\le y\le 2]$$

$$\Rightarrow \quad A=\int_{-1}^{0}-y^3\,dy+\int_{0}^{2}y^3\,dy=-\left[\frac{y^4}{4}\right]_{-1}^{0}+\left[\frac{y^4}{4}\right]_{0}^{2} \qquad [\because P(x,y) \text{ lies on } x=y^3 \therefore x=y^3]$$

$$\Rightarrow \quad A=\frac{1}{4}+4=\frac{17}{4} \text{ sq. units}$$

EXERCISE 20.2

BASIC

1. Find the area of the region in the first quadrant bounded by the parabola $y=4x^2$ and the lines $x=0, y=1$ and $y=4$.
2. Find the area of the region bounded by $x^2=16y$, $y=1$, $y=4$ and the y-axis in the first quadrant.
3. Find the area of the region bounded by $x^2=4ay$ and its latusrectum.
4. Find the area of the region bounded by $x^2+16y=0$ and its latusrectum.

BASIC ON LOTS

5. Find the area of the region bounded by the curve $ay^2=x^3$, the y-axis and the lines $y=a$ and $y=2a$.

ANSWERS

1. $\frac{7}{3}$ sq. units 2. $\frac{56}{3}$ sq. units 3. $\frac{8a^2}{3}$ sq. units 4. $\frac{128}{3}$ sq. units
5. $\frac{3}{5}(2^{5/3}-1)\,a^2$ sq. units

HINTS TO SELECTED PROBLEMS

5. Required area $=\int_{a}^{2a}x\,dy$

$$= \int_{a}^{2a} (ay^2)^{1/3}\, dy = a^{1/3} \int_{a}^{2a} y^{2/3}\, dy$$

$$= a^{1/3}\left[\frac{3}{5} y^{5/3}\right]_0^{2a}$$

$$= \frac{3}{5} a^{1/3} \left\{(2a)^{5/3} - a^{5/3}\right\} \text{ sq. units}$$

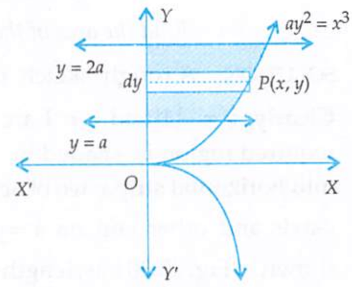

Fig. 20.31

20.5 AREA BETWEEN TWO CURVES BY USING VERTICAL STRIPS

Uptill now we have been discussing problems either on finding the areas of the regions bounded by a curve $y = f(x)$, x-axis and on the left and right by the vertical lines $x = a$ and $x = b$ respectively or on finding the areas of the regions bounded by a curve $x = f(y)$, y-axis and between the horizontal lines $y = c$ and $y = d$ on bottom and top respectively. In some practical problems, we may be interested in computing the area between two curves one below the other or one on the left of other. If two curves are such that one is below the other and we wish to find the area of the region bounded by them and on the left and right by vertical lines. In such cases, we may use the following algorithm.

ALGORITHM

Step I *Draw given curves $y = f(x)$ and $y = g(x)$ and vertical lines $x = a$ and $x = b$.*

Step II *Identify the region included between the curves and vertical lines drawn in step I.*

Step III *Take an arbitrary point $P(x, y_1)$ on one of the curves, say $y = f(x)$, and draw a vertical line through P to meet the other curves $y = g(x)$, at $Q(x, y_2)$. Clearly, $y_1 = f(x)$ and $y_2 = g(x)$.*

Step IV *Draw a vertical approximately rectangle of width $= dx$, height (length) $= |y_1 - y_2| = |f(x) - g(x)|$ such that $P(x, y_1)$ and $Q(x, y_2)$ are mid-points of horizontal sides AB and CD as shown in Fig. 20.32.*

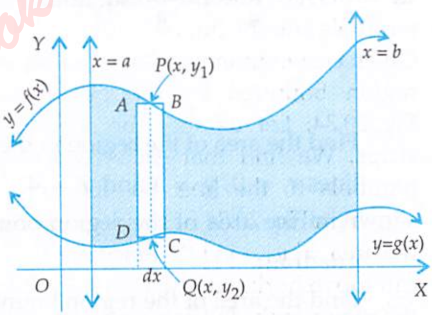

Fig. 20.32

Step V *Find the area of the approximating rectangle drawn in step IV. Let ΔA be the area. Then, $\Delta A = |f(x) - g(x)|\, dx$.*

Step VI *Use the formula $A = \int_a^b |f(x) - g(x)|\, dx$ to find the area of the region between $y = f(x)$, $y = g(x)$ and on the left and right by the vertical lines $x = a$ and $x = b$ respectively.*

If two function $f(x)$ and $g(x)$ are defined on $[a, b]$ which can be divided into sub-intervals $[a, c]$ and $[c, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, c]$ and $f(x) \leq g(x)$ for all $x \in [c, b]$. That is the curve $y = f(x)$ is on top of the curve $y = g(x)$ over the sub-interval $[a, c]$ and the curve $y = g(x)$ is on top of the curve $y = f(x)$ over the sub-interval $[c, b]$ as shown in Fig. 20.33. Then, the area A between the two curves on the interval $[a, b]$ is given by

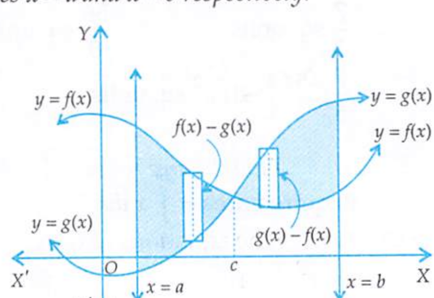

Fig. 20.33

$$A = \int_a^b |f(x) - g(x)|\, dx = \int_a^c |f(x) - g(x)|\, dx + \int_c^b |f(x) - g(x)|\, dx$$

$$\Rightarrow \quad A = \int_a^c \{f(x) - g(x)\}\, dx + \int_c^b \{g(x) - f(x)\}\, dx$$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the area of the region included between the parabola* $y = \frac{3x^2}{4}$ *and the line* $3x - 2y + 12 = 0$. [NCERT, NCERT EXEMPLAR]

SOLUTION The equations of the given curves are

$$y = \frac{3x^2}{4} \quad \text{...(i)} \qquad \text{and,} \qquad 3x - 2y + 12 = 0 \quad \text{...(ii)}$$

Equation (i) represents a parabola having vertex at the origin, axis along the positive direction of y-axis and opens upwards. A free hand sketch of the parabola $y = \frac{3x^2}{4}$ is shown in Fig. 20.34.

Equation $3x - 2y + 12 = 0$ represents a straight line. The straight line given by (ii) meets x-axis at $(-4, 0)$ and y-axis at $(0, 6)$.

In order to find the points of intersection of the given parabola and the line, we solve (i) and (ii) simultaneously. Given curves intersect at the points $(-2, 3)$ and $(4, 12)$. The region bounded by the given curves in shaded in Fig. 20.34. Let us slice the shaded region into vertical strips. We find that each vertical strip runs from the parabola to the line. So, the approximating rectangle shown in Fig. 20.34 has width $= dx$, length $= |y_2 - y_1|$ and the area $= |y_2 - y_1|\, dx$. Since the approximating rectangle can move horizontally from $x = -2$ to $x = 4$. So, area A of the region included between curves (i) and (ii) is given by

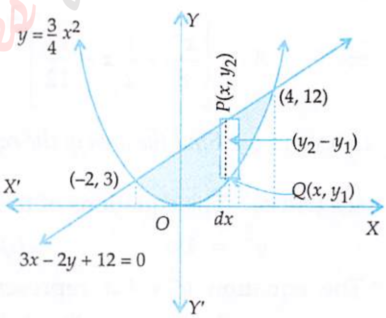

Fig. 20.34

$$A = \int_{-2}^{4} |y_2 - y_1|\, dx = \int_{-2}^{4} (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_{-2}^{4} \left(\frac{3x + 12}{2} - \frac{3}{4}x^2 \right) dx \qquad \left[\begin{array}{l} \because (x, y_1) \text{ and } (x, y_2) \text{ lie on (i) and (ii) respectively} \\ \therefore y_1 = \frac{3x + 12}{2} \text{ and } y_2 = \frac{3x^2}{4} \end{array} \right]$$

$$\Rightarrow \quad A = \left[\frac{3}{4}x^2 + 6x - \frac{x^3}{4} \right]_{-2}^{4} = \left(\frac{3}{4} \times 16 + 6 \times 4 - \frac{64}{4} \right) - \left(\frac{3}{4} \times 4 + 6(-2) + \frac{8}{4} \right) = 27 \text{ sq. units}$$

EXAMPLE 2 *Find the area bounded by the curve* $x^2 = 4y$ *and the straight line* $x = 4y - 2$. [NCERT, CBSE 2004, 2005, 2010, 2013, 2014]

SOLUTION The equations of the given curves are

$$x^2 = 4y \quad \text{...(i)} \qquad \text{and,} \qquad x = 4y - 2 \quad \text{...(ii)}$$

Equation (i) represents a parabola with vertex at the origin and axis along positive direction of y-axis. Equation (ii) represents a straight line which meets the coordinate axes at $(-2, 0)$ and $(0, 1/2)$ respectively. To find the points of intersection of the given parabola and the line, we solve (i) and (ii) simultaneously. Solving the two equations simultaneously we obtain that the points of intersection of the given parabola and the line are $(2, 1)$ and $(-1, 1/4)$. The region whose area is to be found out is shaded in Fig. 20.35.

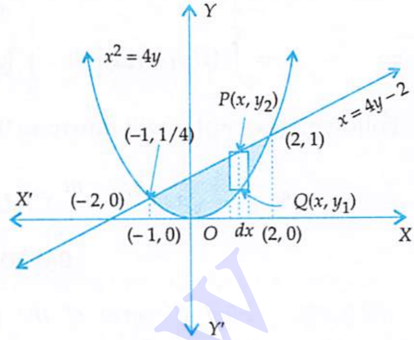

Fig. 20.35

Let us slice the shaded region into vertical strips. We find that each vertical strip runs from parabola (i) to the line (ii). So, the approximating rectangle shown in Fig. 20.35 has width $= dx$, length $= |y_2 - y_1|$, and area $= |y_2 - y_1|\, dx$. Since the approximating rectangle can move horizontally from $x = -1$ to $x = 2$. So, area A of the shaded region is given by

$$A = \int_{-1}^{2} |y_2 - y_1|\, dx = \int_{-1}^{2} (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \ \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx \qquad \left[\begin{array}{l} \because P(x, y_2) \text{ and } Q(x, y_1) \text{ lie on (ii) and (i) respec.} \\ \therefore y_2 = \dfrac{x+2}{4} \text{ and } y_1 = \dfrac{x^2}{4} \end{array} \right]$$

$$\Rightarrow \quad A = \left[\frac{x^2}{8} + \frac{1}{2}x - \frac{x^3}{12} \right]_{-1}^{2} = \left(\frac{4}{8} + \frac{2}{2} - \frac{8}{12} \right) - \left(\frac{1}{8} - \frac{1}{2} + \frac{1}{12} \right) = \frac{9}{8} \text{ sq. units}$$

EXAMPLE 3 *Find the area of the region enclosed by the parabola $y^2 = 4ax$ and the line $y = mx$.* [NCERT]

SOLUTION The equations of the given curves are

$$y^2 = 4ax \quad \text{...(i)} \qquad \text{and,} \qquad y = mx \quad \text{...(ii)}$$

The equation $y^2 = 4ax$ represents a parabola in standard form and the equation $y = mx$ represents a line passing through the origin having slope m.

In order to find the points of intersection of (i) and (ii), we solve them simultaneously.

Putting $y = mx$ from (ii) in (i), we get

$$m^2x^2 = 4ax \Rightarrow x(m^2x - 4a) = 0 \Rightarrow x = 0,\ x = \frac{4a}{m^2}$$

From (ii), we find that

$$x = 0 \Rightarrow y = 0 \quad \text{and} \quad x = \frac{4a}{m^2} \Rightarrow y = \frac{4a}{m}$$

So, the points of intersection of the given curves are $(0, 0)$ and $(4a/m^2,\ 4a/m)$.

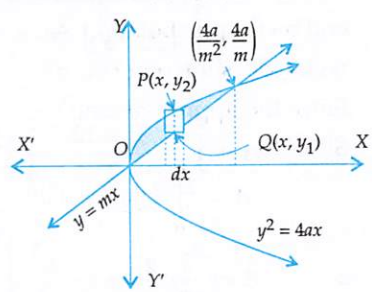

Fig. 20.36

The rough sketch of the two curves is shown in Fig. 20.36 and the shaded portion is the region enclosed by the parabola and the line. Let us slice the shaded region into vertical strips. We observe that each vertical strip has lower end on the line $y = mx$ and the upper end on the parabola $y^2 = 4ax$. So, the approximating rectangle shown in Fig. 20.36 has width $= dx$, length $= |y_2 - y_1|$ and area $= |y_2 - y_1|\, dx$. Since the approximating rectangle can move from $x = 0$ to $x = 4a/m^2$. So, required area A is given by

$$A = \int_0^{4a/m^2} |y_2 - y_1|\, dx = \int_0^{4a/m^2} (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_0^{4a/m^2} (2\sqrt{ax} - mx)\, dx \qquad \left[\begin{array}{l} \because P(x, y_2) \text{ and } Q(x, y_1) \text{ lie on (i) and (ii) respectively} \\ \therefore y_2^{\,2} = 4ax \text{ and } y_1 = mx \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{4}{3}\sqrt{a}\; x^{3/2} - \frac{m}{2} x^2\right]_0^{4a/m^2} = \frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{m}{2}\left(\frac{4a}{m^2}\right)^2 = \frac{8a^2}{3m^3} \text{ sq. units}$$

EXAMPLE 4 *Find the area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, where $a > 0$.* **[CBSE 2003, 2004]**

SOLUTION The equations of the given curves are

$$y^2 = 4ax \quad ...(i) \qquad \text{and,} \qquad x^2 = 4ay \quad ...(ii)$$

Clearly, (i) and (ii) represent parabolas in standard forms. The rough sketch of these parabolas can easily be drawn as shown in Fig. 20.37. In order to find the points of intersection of the curves (i) and (ii), we solve them simultaneously.

Putting $y = \dfrac{x^2}{4a}$ from (ii) into (i), we get

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow x^4 = 64a^3 x \Rightarrow x(x^3 - 64a^3) = 0 \Rightarrow x = 0 \text{ or, } x = 4a$$

From (ii), we find that

$$x = 0 \Rightarrow y = 0 \text{ and } x = 4a \Rightarrow y = 4a.$$

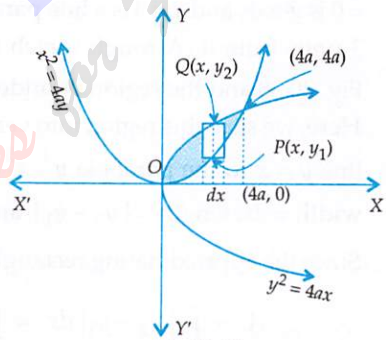

Fig. 20.37

So, the two curves intersect at (0, 0) and (4*a*, 4*a*). The region enclosed by the two parabolas is shaded in Fig. 20.37.

Let us we slice this region into vertical strips. We observe that each vertical strip has its lower end on the parabola $x^2 = 4ay$ and the upper end on the parabola $y^2 = 4ax$. For the approximating rectangle shown in Fig. 20.37, we have width $= dx$, length $= |y_2 - y_1|$ and area $= |y_2 - y_1|\, dx$.

Since the approximating rectangle can move between $x = 0$ and $x = 4a$. So, required area A is given by

$$A = \int_0^{4a} |y_2 - y_1|\, dx = \int_0^{4a} (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_0^{4a} \left\{2\sqrt{ax} - \frac{x^2}{4a}\right\} dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } (x, y_2) \text{ lie on (ii) and (i) respectively} \\ \therefore x^2 = 4ay_1 \text{ and } y_2^2 = 4ax \Rightarrow y_2 = \sqrt{4ax} \text{ and } y_1 = \dfrac{x^2}{4a} \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{4\sqrt{a}}{3} x^{3/2} - \frac{x^3}{12a}\right]_0^{4a} = \frac{4\sqrt{a}}{3}(4a)^{3/2} - \frac{(4a)^3}{12a} = \frac{32a^2}{3} - \frac{16a^2}{3} = \frac{16a^2}{3} \text{ sq. units}$$

NOTE *The above area can also be obtained by horizontal slicing. In that case,*

$$\text{Required area} = \int_0^{4a} (x_2 - x_1)\, dy = \int_0^{4a} \left\{ \frac{y^2}{4a} - \sqrt{4ay} \right\} dy$$

EXAMPLE 5 *Find the area of the region bounded by the curves $y = x^2 + 2, y = x, x = 0$ and $x = 3$.*

SOLUTION The equations of the given curves are

$y = x^2 + 2$ ...(i)

$y = x$ ...(ii)

$x = 0$ ...(iii)

and, $x = 3$ ...(iv)

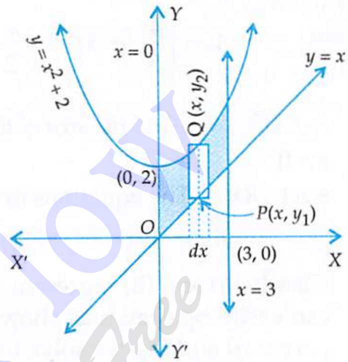

Fig. 20.38

Clearly, $y = x^2 + 2$ represents a parabola with vertex at (0, 2) and axis along positive direction of y-axis. This parabola opens upwards as shown in Fig. 20.38. Clearly, $y = x$ is a line passing through the origin and makes 45° angle with x-axis, $x = 0$ is y-axis and $x = 3$ is a line parallel to y-axis at a distance of 3 units from it. A rough sketch of these curves is shown in Fig. 20.38 and the region bounded by these curves is shaded.

Here, we slice this region into vertical strips. We observe that each vertical strip runs from the line $y = x$ to the parabola $y = x^2 + 2$. So, the approximating rectangle shown in Fig. 20.38 has, width $= dx$, length $= |y_2 - y_1|$ and area $= |y_2 - y_1|\, dx$.

Since the approximating rectangle can move from $x = 0$ to $x = 3$. So, required area A is given by

$$A = \int_0^3 |y_2 - y_1|\, dx = \int_0^3 (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_0^3 \left\{ (x^2 + 2) - x \right\} dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) and (i) respectively} \\ \therefore y_1 = x \text{ and } y_2 = x^2 + 2 \end{array} \right]$$

$$\Rightarrow \quad A = \left[\frac{x^3}{3} + 2x - \frac{x^2}{2} \right]_0^3 = 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq. units}$$

EXAMPLE 6 *Find the area of the region $\{(x, y) : x^2 \le y \le x\}$.* **[CBSE 2005, 2011, 2013]**

SOLUTION Let $R = \{(x, y) : x^2 \le y \le x\}$. Then, $R = \{(x, y) : x^2 \le y\} \cap \{(x, y) : y \le x\} = R_1 \cap R_2$

where $R_1 = \{(x, y) : x^2 \le y\}$ and $R_2 = \{(x, y) : y \le x\}$

Region R_1: Clearly, $x^2 = y$ represents a parabola with vertex at (0, 0), positive direction of y-axis as its axis and it opens upwards. The region R_1 represented by $x^2 \le y$ is the interior of the parabola.

Region R_2: Clearly, $y = x$ is a line passing through the origin and making an angle of 45° with the x-axis. So, region R_2 represented by $y \le x$ is the region lying below the line $y = x$.

Hence, the required region R is the shaded region shown in Fig. 20.39.

Solving $y = x^2$ and $y = x$, we obtain $O\ (0, 0)$ and $A\ (1, 1)$ as their points of intersection.

Here, we slice this region R into vertical strips. We observe that each vertical strip has its lower end on the parabola $y = x^2$ and upper end on $y = x$. So, the approximating rectangle shown in Fig. 20.39 has, length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\, dx$.

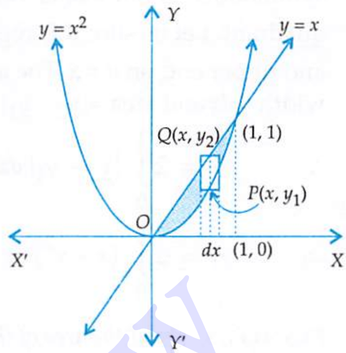

Fig. 20.39

Since the approximating rectangle can move from $x = 0$ to $x = 1$.. So, required area A is given by

$$A = \int_0^1 |y_2 - y_1|\, dx$$

$$\Rightarrow \quad A = \int_0^1 (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \ \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_0^1 (x - x^2)\, dx \qquad \left[\begin{array}{l} \because P\ (x, y_1) \text{ and } Q\ (x, y_2) \text{ lie on } y = x^2 \text{ and} \\ y = x \text{ respectively. So, } y_1 = x^2 \text{ and } y_2 = x \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. unit}$$

EXAMPLE 7 *Find the area of the region* $\left\{(x, y) : x^2 \le y \le |x|\right\}$. **[NCERT, CBSE 2002, 2012]**

SOLUTION Let $R = \left\{(x, y) : x^2 \le y \le |x|\right\}$. Then,

$$R = \left\{(x, y) : x^2 \le y\right\} \cap \left\{(x, y) : y \le |x|\right\}$$

$$\Rightarrow \quad R = \left\{(x, y) : x^2 \le y\right\} \cap \left[\left\{(x, y) : y \le x,\ x \ge 0\right\} \cup \left\{(x, y) : y \le -x,\ x < 0\right\}\right]$$

$$\Rightarrow \quad R = R_1 \cap (R_2 \cup R_3), \text{ where}$$

$$R_1 = \left\{(x, y) : x^2 \le y\right\},\ R_2 = \left\{(x, y) : y \le x,\ x \ge 0\right\} \text{ and } R_3 = \left\{(x, y) : y \le -x,\ x < 0\right\}$$

Region R_1: Clearly, $x^2 = y$ represents a parabola with vertex at (0, 0), positive direction of y-axis as its axis and it opens upwards. The interior of the parabola is the region R_1.

Region R_2: Clearly, $y = x$, $x \ge 0$ is a line passing through the origin and making an angle of 45° with the positive direction of x-axis. So, R_2 is the region lying below the line $y = x$.

Region R_3: Clearly, $y = -x$, $x < 0$ is a line passing through the origin and making an angle of 135° with the positive direction of x-axis. So, R_3 is the region lying below the line $y = -x$.

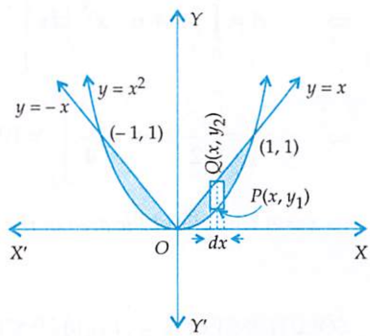

Fig. 20.40

The required region R is the shaded region shown in Fig. 20.40. Since both the curves are symmetrical about y-axis. So, required area A is twice the area of the shaded region in first quadrant. Let us slice this region into vertical strips. Each vertical strip has lower end on $y = x^2$ and upper end on $y = x$. The approximating rectangle shown in Fig. 20.40 has, length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\, dx$. Clearly, it can move horizontally between O (0, 0) and A (1, 1).

$$\therefore \quad A = 2\int_0^1 |y_2 - y_1|\, dx = 2\int_0^1 (y_2 - y_1)\, dx \qquad [\because\ y_2 > y_1\ \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = 2\int_0^1 (x - x^2)\, dx = 2 \times \frac{1}{6} = \frac{1}{3} \text{ sq. units} \qquad \text{[See Example 6]}$$

EXAMPLE 8 *Find the area of the region bounded by the curve $y = x^3$ and the lines $y = x + 6$ and $x=0$.*

[NCERT EXEMPALR]

SOLUTION The x-coordinate of the point of intersection of $y = x^3$ and $y = x + 6$ is a root of the equation

$$x^3 = x + 6 \qquad \text{[On eliminating } y \text{ between } y = x^3 \text{ and } y = x + 6]$$

$$\Rightarrow \quad x^3 - x - 6 = 0$$

$$\Rightarrow \quad (x-2)(x^2 + 2x + 3) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2 \qquad [\because\ x^2 + 2x + 3 \neq 0]$$

Putting $x = 2$ in $y = x^3$, we obtain $y = 8$. Thus, the line $y = x + 6$ cuts $y = x^3$ at (2, 8).

The region bounded by the curve $y = x^3$ and the lines $y = x + 6$ and $x = 0$ is shaded region in Fig. 20.41. We slice this region into vertical strips. Each vertical strip has its lower end on $y = x^3$ and upper end on $y = x + 6$. So, the approximating rectangle shown in Fig. 20.41 has length $= (y_2 - y_1)$, width $= dx$ and area $= (y_2 - y_1)\, dx$. Clearly, it can move horizontally between $x = 0$ and $x = 2$. So, the required area A is given by

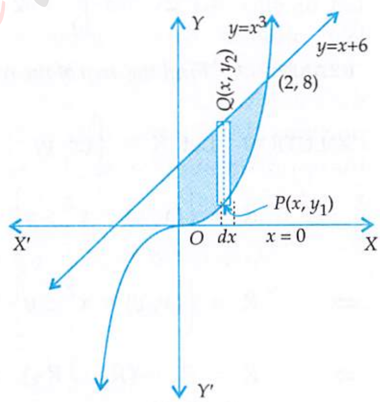

Fig. 20.41

$$A = \int_0^2 (y_2 - y_1)\, dx$$

$$\Rightarrow \quad A = \int_0^2 \left(x + 6 - x^3\right) dx \quad \left[\begin{array}{l} \because\ P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = x^3 \text{ and } y = x + 6 \text{ respectively.} \\ \therefore\ y_2 = x + 6 \text{ and } y_1 = x^3 \end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2 = 10 \text{ sq. units.}$$

EXAMPLE 9 *Find the area of the region $\left\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\right\}$.*

[NCERT, CBSE 2001C]

SOLUTION Let $R = \left\{(x, y) : 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1, 0 \le x \le 2\right\}$. Then,

$$R = \left\{(x, y) : 0 \le y \le x^2 + 1\right\} \cap \left\{(x, y) : 0 \le y \le x + 1\right\} \cap \left\{(x, y) : 0 \le x \le 2\right\}$$

$\Rightarrow$ $R = R_1 \cap R_2 \cap R_3$, where

$$R_1 = \left\{(x, y) : 0 \le y \le x^2 + 1\right\}, R_2 = \left\{(x, y) : 0 \le y \le x + 1\right\} \text{ and } R_3 = \left\{(x, y) : 0 \le x \le 2\right\}$$

Region R_1: Clearly, $y = x^2 + 1$ represents a parabola with vertex at (0, 1), axis along the positive direction of y-axis and it opens upwards.

Now, $0 \le y \le x^2 + 1 \Leftrightarrow y \ge 0$ and $y \le x^2 + 1$.

So, R_1 is the region lying outside the parabola $y = x^2 + 1$ and above x-axis.

Region R_2: Clearly, $y = x + 1$ represents a straight line cutting x and y axis at (–1, 0) and (0, 1) respectively. Since $0 \le y \le x + 1 \Leftrightarrow y \ge 0$ and $y \le x + 1$. Therefore, R_2 is the region lying above x-axis and below the line $y = x + 1$.

Region R_3: Clearly, R_3 is the region lying between the lines $x = 0$ i.e. y-axis and $x = 2$.

Hence, the required region R is the shaded region as shown in Fig. 20.42.

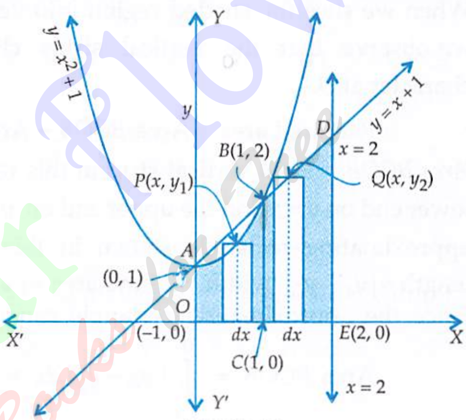

Fig. 20.42

Solving $y = x^2 + 1$ and $y = x + 1$, we find that these two intersect at A (0, 1) and B (1, 2).

Let us slice the shaded region into vertical strips. We observe that vertical strips change their character at the point B. Draw a line BC parallel to the y-axis which divides the area $OABDEO$ into two portions $OABCO$ and $BDECB$. For the area $OABCO$, the approximating rectangle has length $= |y_1|$, width $= dx$ and area $= |y_1|\, dx$. As it can move from $x = 0$ to $x = 1$.

$$\therefore \quad \text{Area } OABCO = \int_0^1 |y_1|\, dx = \int_0^1 y_1\, dx \qquad [\because\ y_1 \ge 0\ \therefore |y_1| = y_1]$$

$$= \int_0^1 (x^2 + 1)\, dx \qquad [\because\ P(x, y_1) \text{ lies on } y = x^2 + 1\ \therefore y_1 = x^2 + 1]$$

For the area $BDECB$, the approximating rectangle shown in Fig. 20.42 has length $= |y_2|$, width $= dx$ and area $= |y_2|\, dx$. Since it can move from $x = 1$ to $x = 2$.

$$\therefore \quad \text{Area } BDECB = \int_1^2 |y_2|\, dx = \int_1^2 y_2\, dx \qquad [\because\ y_2 \ge 0\ \therefore |y_2| = y_2]$$

$$= \int_1^2 (x + 1)\, dx \qquad [\because\ Q(x, y_2) \text{ lies on } y = x + 1\ \therefore y_2 = x + 1]$$

$$\therefore \quad \text{Required area} = \text{Area } OABCO + \text{Area } BDECB = \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2 = \frac{4}{3} + \frac{5}{2} = \frac{23}{6} \text{ sq. units}$$

EXAMPLE 10 *Find the area bounded by the curves $y = x$ and $y = x^3$.*

SOLUTION The equations of the given curves are

$$y = x \quad ...(i) \qquad \text{and,} \qquad y = x^3 \quad ...(ii)$$

The sketch of the curve $y = x^3$ is shown in Fig. 20.43. Clearly, $y = x$ is a line passing through the origin and making an angle of 45° with x-axis. The shaded portion shown in Fig. 20.43 is the region bounded by the curves $y = x$ and $y = x^3$. Solving $y = x$ and $y = x^3$ simultaneously, find that the two curves intersect at $O(0, 0)$, $A(1, 1)$ and $B(-1, -1)$.

When we slice the shaded region into vertical strips, we observe that the vertical strips change their character at O.

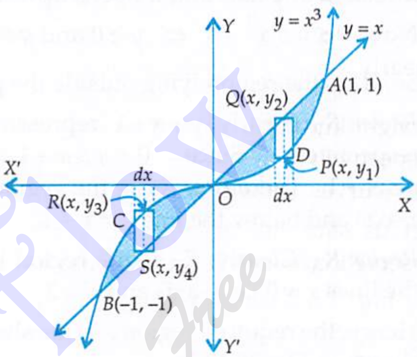

Fig. 20.43

$\therefore$ Required area = Area $BCOB$ + Area $ODAO$

Area BCOB: Each vertical strip in this region has its lower end on $y = x$ and the upper end on $y = x^3$. So, the approximating rectangle shown in this region has length $= |y_4 - y_3|$, width $= dx$ and area $= |y_4 - y_3|\, dx$. Since the approximating rectangle can move from $x = -1$ to $x = 0$.

$$\therefore \quad \text{Area } BCOB = \int_{-1}^{0} |y_4 - y_3|\, dx = \int_{-1}^{0} -(y_4 - y_3)\, dx \qquad [\because y_4 < y_3 \therefore y_4 - y_3 < 0]$$

$$= \int_{-1}^{0} -(x - x^3)\, dx \qquad \left[\begin{array}{l}\because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on (ii) and (i)} \\ \text{respectively} \therefore y_3 = x^3 \text{ and } y_4 = x\end{array}\right]$$

$$= \int_{-1}^{0} (x^3 - x)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4} \text{ sq. units}$$

Area ODAO: Each vertical strip in this region has its two ends on (ii) and (i) respectively. So, the approximating rectangle shown in this region has length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\, dx$. Since it can move from $x = 0$ to $x = 1$

$$\therefore \quad \text{Area } ODAO = \int_{0}^{1} |y_2 - y_1|\, dx = \int_{0}^{1} (y_2 - y_1)\, dx \qquad [\because y_2 > y_1 \therefore y_2 - y_1 > 0]$$

$$= \int_{0}^{1} (x - x^3)\, dx \qquad \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) and (i)} \\ \text{respectively} \therefore y_1 = x^3 \text{ and } y_2 = x\end{array}\right]$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ sq. units}$$

$$\therefore \quad \text{Required area} = \text{Area } BCOB + \text{Area } ODAO = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \text{ sq. units.}$$

EXAMPLE 11 *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the straight line* $\frac{x}{a}+\frac{y}{b}=1$. **[NCERT]**

SOLUTION The equations of the given curves are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \ldots(i) \qquad \text{and,} \qquad \frac{x}{a}+\frac{y}{b}=1 \quad \ldots(ii)$$

Clearly, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ represents an ellipse as shown in Fig. 20.44 and $\frac{x}{a}+\frac{y}{b}=1$ is the equation of a straight line cutting x and y-axes at $(a, 0)$ and $(0, b)$ respectively. The smaller region bounded by these two curves is the shaded region in Fig. 20.44.

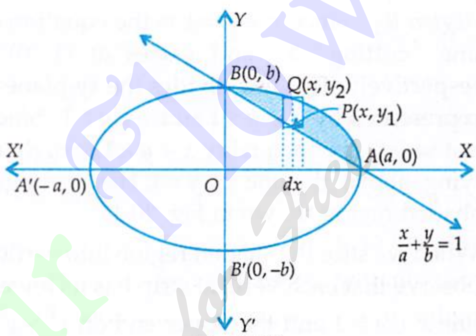

Fig. 20.44

Let us slice this region into vertical strips. We observe that each vertical strip has its lower end on the line $\frac{x}{a}+\frac{y}{b}=1$ and upper end on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. So, the approximating rectangle shown in Fig. 20.44 has length $=|y_2-y_1|$, width $=dx$ and area $=|y_2-y_1|\,dx$.

Since the approximating rectangle can move from $x=0$ to $x=a$. So, required area A is given by

$$A=\int_0^a |y_2-y_1|\,dx=\int_0^a (y_2-y_1)\,dx \qquad [\because y_2>y_1 \;\therefore |y_2-y_1|=y_2-y_1] \qquad \ldots(i)$$

Since point $P(x, y_1)$ lies on $\frac{x}{a}+\frac{y}{b}=1$ and point $Q(x, y_2)$ lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\therefore \qquad \frac{x}{a}+\frac{y_1}{b}=1 \text{ and } \frac{x^2}{a^2}+\frac{y_2^2}{b^2}=1 \Rightarrow y_1=\frac{b}{a}(a-x) \text{ and } y_2=\frac{b}{a}\sqrt{a^2-x^2}$$

Substituting these values in (i), the required area A is given by

$$A=\int_0^a\left\{\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}(a-x)\right\}dx=\frac{b}{a}\left[\int_0^a\sqrt{a^2-x^2}\,dx-\int_0^a(a-x)\,dx\right]$$

$$\Rightarrow \qquad A=\frac{b}{a}\left[\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a-\left[ax-\frac{x^2}{2}\right]_0^a\right]$$

$$\Rightarrow \qquad A=\frac{b}{a}\left[\left\{0+\frac{1}{2}a^2\sin^{-1}(1)\right\}-\left\{a^2-\frac{a^2}{2}\right\}\right]=\frac{b}{a}\left\{\frac{1}{2}a^2\times\frac{\pi}{2}-\frac{a^2}{2}\right\}=\frac{1}{2}\left(\frac{\pi}{2}-1\right)ab \text{ sq. units}$$

EXAMPLE 12 *Find the area of the region* $\left\{(x, y): x^2+y^2\le 1\le x+y\right\}$. **[CBSE 2010]**

SOLUTION Let $R=\left\{(x, y): x^2+y^2 \leq 1 \leq x+y\right\}$. Then,

$$R=\left\{(x, y): x^2+y^2 \leq 1 \leq x+y\right\}=\left\{(x, y): x^2+y^2 \leq 1\right\} \cap\left\{(x, y): 1 \leq x+y\right\}$$

$$\Rightarrow \quad R=R_1 \cap R_2, \text{ where } R_1=\left\{(x, y): x^2+y^2 \leq 1\right\} \text{ and, } R_2=\left\{(x, y): 1 \leq x+y\right\}$$

Region R_1: Clearly $x^2+y^2=1$ represents a circle with centre at (0, 0) and radius unity. Since $x^2+y^2 \leq 1$, so region R_1 represents the interior of circle $x^2+y^2=1$.

Region R_2: Since $x+y=1$ is the equation of a straight line cutting x and y-axes at (1, 0) and (0, 1) respectively. This line divides the xy-plane in two parts represented by $x+y \leq 1$ and $x+y \geq 1$. Since (0, 0) does not satisfy the inequality $x+y \geq 1$. So, R_2 is the region lying above the line $x+y=1$. Hence, region R is the shaded region shown in Fig. 20.45.

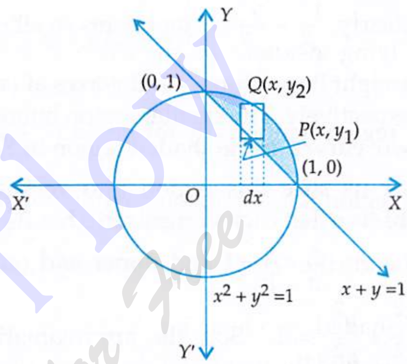

Fig. 20.45

When we slice the shaded region into vertical strips, we observe that each vertical strip has its lower end on the line $x+y=1$ and the upper end on $x^2+y^2=1$. So, the approximating rectangle shown in Fig. 20.45 has, length $=|y_2-y_1|$, width $=dx$ and area $=|y_2-y_1|\,dx$. Since the vertical approximating rectangle can move horizontally from $x=0$ to $x=1$. So, required area A is given by

$$A=\int_0^1 |y_2-y_1|\,dx=\int_0^1 (y_2-y_1)\,dx \qquad [\because y_2>y_1 \because |y_2-y_1|=y_2-y_1]$$

$$\Rightarrow A=\int_0^1\left\{\sqrt{1-x^2}-(1-x)\right\} dx \qquad \left[\begin{array}{l}\because P(x, y_1), \,\&\, Q(x, y_2) \text{ lie on } x+y=1 \,\&\, x^2+y^2=1 \text{ resp.} \\ \therefore x+y_1=1 \,\&\, x^2+y_2^2=1. \text{ So, } y_1=1-x \,\&\, y_2=\sqrt{1-x^2}\end{array}\right]$$

$$\Rightarrow A=\left[\frac{1}{2} x \sqrt{1-x^2}+\frac{1}{2} \sin^{-1}\left(\frac{x}{1}\right)-x+\frac{x^2}{2}\right]_0^1=\left\{0+\frac{1}{2} \sin^{-1} 1-1+\frac{1}{2}\right\}-0$$

$$\Rightarrow A=\left(\frac{\pi}{4}-\frac{1}{2}\right) \text{ sq. units}$$

EXAMPLE 13 *Find the area of the region* $\left\{(x, y): y^2 \leq 4x, 4x^2+4y^2 \leq 9\right\}$. **[CBSE 2017, NCERT]**

or

Find the area of the circle $4x^2+4y^2=9$ which is interior to the parabola $y^2=4x$. **[CBSE 2010, 13, 19]**

SOLUTION Let $R=\left\{(x, y): y^2 \leq 4x, 4x^2+4y^2 \leq 9\right\}$. Then,

$$R=\left\{(x, y): y^2 \leq 4x\right\} \cap\left\{(x, y): 4x^2+4y^2 \leq 9\right\}$$

$$\Rightarrow \quad R=R_1 \cap R_2, \text{ where } R_1=\left\{(x, y): y^2 \leq 4x\right\} \text{ and } R_2=\left\{(x, y): 4x^2+4y^2 \leq 9\right\}.$$

Region R_1 : Clearly, $y^2 = 4x$ is the equation of the parabola with vertex at the origin and axis along x-axis. Clearly, $y^2 \leq 4x$ represents the region lying inside or on the parabola $y^2 = 4x$. So, R_1 is the region lying inside the parabola $y^2 = 4x$.

Region R_2 : The equation $4x^2 + 4y^2 = 9$ or, $x^2 + y^2 = \left(\frac{3}{2}\right)^2$ represents a circle with centre at the origin and radius $\frac{3}{2}$. Therefore, $x^2 + y^2 \leq \frac{9}{4}$ is the region lying inside or on the circle $x^2 + y^2 = \frac{9}{4}$. So, R_2 is the region lying inside the circle $x^2 + y^2 = \left(\frac{3}{2}\right)^2$. Thus, the region R is the region bounded by the parabola $y^2 = 4x$ and the circle $x^2 + y^2 = \left(\frac{3}{2}\right)^2$, as shown by the shaded region in Fig. 20.46.

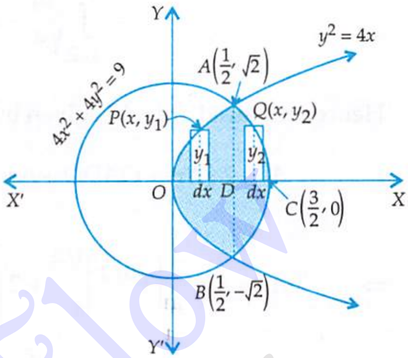

Fig. 20.46

To find the points of intersection of the given curves, we solve their equations simultaneously.

We have, $y^2 = 4x$ ...(i) and, $4x^2 + 4y^2 = 9$ ..(ii)

Putting $y^2 = 4x$ from (i) into (ii), we get

$$4x^2 + 16x = 9 \Rightarrow 4x^2 + 16x - 9 = 0 \Rightarrow (2x + 9)(2x - 1) = 0 \Rightarrow x = \frac{1}{2} \text{ or, } x = -\frac{9}{2}$$

From (i), we find that

$$x = \frac{1}{2} \Rightarrow y = \pm\sqrt{2} \text{ and } x = -\frac{9}{2} \text{ gives imaginary values of } y.$$

So, the two curves intersect at $(1/2, \sqrt{2})$ and $(1/2, -\sqrt{2})$. Clearly, both the curves are symmetrical about x-axis.

$\therefore$ Required area = 2 (Area of the shaded region lying above x-axis).

To find this area, we slice it into vertical strips. We observe that vertical strips change their character at the point A. Draw a line AD parallel to y-axis which divides the shaded region lying above x-axis into two portions $OADO$ and $ADCA$.

For the area $OADO$, the approximating rectangle has, length $= |y_1|$, width $= dx$ and area $= |y_1|\, dx$. As it can move from $x = 0$ to $x = \frac{1}{2}$.

$$\therefore \quad \text{Area } OADO = \int_0^{1/2} |y_1|\, dx = \int_0^{1/2} y_1\, dx \qquad [\because y_1 > 0 \therefore |y_1| = y_1]$$

$$= \int_0^{1/2} 2\sqrt{x}\, dx \qquad \left[\because P(x, y_1) \text{ lies on } y^2 = 4x \therefore y_1^2 = 4x \Rightarrow y_1 = 2\sqrt{x}\right]$$

For the area $ADCA$, the approximating rectangle shown in Fig. 20.46 has, length $= |y_2|$, width $= dx$ and area $= |y_2|\, dx$. As it can move from $x = \frac{1}{2}$ to $x = \frac{3}{2}$.

$$\therefore \quad \text{Area } ADCA = \int_{1/2}^{3/2} |y_2|\, dx = \int_{1/2}^{3/2} y_2\, dx \qquad [\because y_2 > 0 \therefore |y_2| = y_2]$$

$$= \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx \qquad \left[\begin{array}{l} \because Q(x, y_2) \text{ lies on } 4x^2 + 4y^2 = 9 \\ \therefore 4x^2 + 4y_2^2 = 9 \Rightarrow y_2 = \sqrt{\frac{9}{4} - x^2} \end{array}\right]$$

Hence, required area A is given by

$$A = 2\,[\text{Area } OADO + \text{Area } ADCA] = 2\int_0^{1/2} 2\sqrt{x}\, dx + 2\int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\, dx$$

$$\Rightarrow \quad A = 4 \times \frac{2}{3}\left[x^{3/2}\right]_0^{1/2} + 2\left[\frac{1}{2}x\sqrt{\frac{9}{4} - x^2} + \frac{1}{2} \times \frac{9}{4}\sin^{-1}\frac{2x}{3}\right]_{1/2}^{3/2}$$

$$\Rightarrow \quad A = \frac{8}{3}\left(\frac{1}{2\sqrt{2}} - 0\right) + 2\left[\left\{\frac{9}{8}\sin^{-1}(1)\right\} - \left\{\frac{1}{2\sqrt{2}} + \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$\Rightarrow \quad A = \frac{2\sqrt{2}}{3} + \left\{\frac{9}{8}\pi - \frac{1}{\sqrt{2}} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\} = \left\{\frac{\sqrt{2}}{6} + \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)\right\} \text{ sq. units}$$

EXAMPLE 14 *Find the area of the region* $\left\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\right\}$.

[NCERT EXEMPLAR, CBSE 2016]

SOLUTION Let $R = \left\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\right\}$. Then,

$$R = \left\{(x, y) : x^2 + y^2 \le 2ax\right\} \cap \left\{(x, y) : y^2 \ge ax\right\} \cap \left\{(x, y) : x \ge 0\ y \ge 0\right\}$$

$$\Rightarrow \quad R = R_1 \cap R_2 \cap R_3, \text{ where}$$

$$R_1 = \left\{(x, y) : x^2 + y^2 \le 2ax\right\},\ R_2 = \left\{(x, y) : y^2 \ge ax\right\} \text{ and } R_3 = \left\{(x, y) : x \ge 0,\ y \ge 0\right\}$$

Region R_1: Clearly, $x^2 + y^2 = 2ax$ or, $(x - a)^2 + (y - 0)^2 = a^2$ represents a circle with centre at $(a, 0)$ and radius a as shown in Fig. 20.47.

We have, $(x - a)^2 + (y - 0)^2 \le a^2$. So, R_1 is the region lying inside the circle shown in Fig. 20.47.

Region R_2: Clearly, $y^2 = ax$ represents a parabola with vertex at (0, 0) and its axis along x-axis. It is given that $y^2 \ge ax$. So, region R_2 is the region lying outside the parabola $y^2 = ax$.

Region R_3: It is given that $x \ge 0$ and $y \ge 0$. So, region R_3 is the region in the first quadrant.

Thus, $R = R_1 \cap R_2 \cap R_3$ is the shaded region shown in Fig. 20.47.

The equations of the given curves are

$$x^2 + y^2 = 2ax \qquad \text{...(i)}$$

and, $\quad y^2 = ax \qquad$ ...(ii)

To find the points of intersection of these two curves we solve (i) and (ii) simultaneously. Solving these two equations simultaneously, we find that the two curves intersect at $O\ (0, 0)$ and $A(a, a)$.

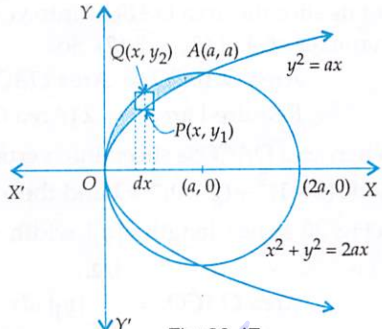

Fig. 20.47

To find the area of the shaded region shown in Fig. 20.47, we slice it into vertical strips. We observe that each vertical strip has its lower end on the parabola $y^2 = ax$ and upper end on $x^2 + y^2 = 2ax$. So, the approximating rectangle shown in Fig. 20.47 has length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\ dx$. Since the approximating rectangle can move from $x = 0$ to $x = a$.

So, required area A is given by

$$A = \int_0^a |y_2 - y_1|\ dx = \int_0^a (y_2 - y_1)\ dx \qquad [\because y_2 > y_1 \ \therefore |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad A = \int_0^a \left\{\sqrt{a^2 - (x-a)^2} - \sqrt{ax}\right\} dx \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (ii) \& (i) respectively} \\ \therefore (x-a)^2 + y_2^2 = a^2 \ \& \ y_1^2 = ax\end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{1}{2}(x-a)\sqrt{a^2 - (x-a)^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x-a}{a}\right) - \frac{2\sqrt{a}}{3}\ x^{3/2}\right]_0^a$$

$$\Rightarrow \quad A = \left[\left\{0 + \frac{1}{2}a^2 \sin^{-1}(0) - \frac{2}{3}a^2\right\} - \left\{0 + \frac{1}{2}a^2 \sin^{-1}(-1) - 0\right\}\right]$$

$$\Rightarrow \quad A = -\frac{2}{3}a^2 - \frac{1}{2}a^2\left(-\frac{\pi}{2}\right) = \left(\frac{\pi}{4} - \frac{2}{3}\right)a^2 \text{ sq. units}$$

EXAMPLE 15 *Find the area of the region enclosed between the two circles* $x^2 + y^2 = 1$ *and* $(x-1)^2 + y^2 = 1$. **[CBSE 2007, 2013]**

SOLUTION The equations of the given circles are

$$x^2 + y^2 = 1 \qquad \text{...(i)}$$

and, $$(x-1)^2 + (y-0)^2 = 1 \qquad \text{...(ii)}$$

Clearly, $x^2 + y^2 = 1$ represents a circle with centre at $(0, 0)$ and radius unity. Also, $(x-1)^2 + (y-0)^2 = 1$ represents a circle with centre at $(1, 0)$ and radius unity. To find the points of intersection of the given curves, we solve (i) and (ii) simultaneously. Solving (i) and (ii) simultaneously, we find that the two curves intersect at $A\ (1/2, \sqrt{3}/2)$ and $D\ (1/2, -\sqrt{3}/2)$.

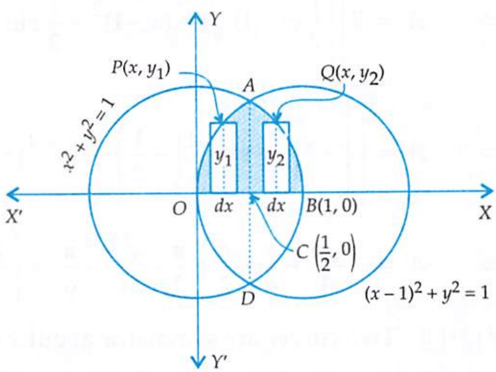

Fig. 20.48

Since both the curves are symmetrical about x-axis.

$\therefore$ Required area $= 2\,(\text{Area } OABCO)$

Let us slice the area $OABCO$ into vertical strips. We observe that the vertical strips change their character at $A\,(1/2, \sqrt{3}/2)$. So,

$$\text{Area } OABCO = \text{Area } OACO + \text{Area } CABC.$$

$$\therefore \quad \text{Required area} = 2\,[\text{Area } OACO + \text{Area } CABC]$$

When area $OACO$ is sliced into vertical strips, we find that each strip has its upper end on the circle $(x-1)^2 + (y-0)^2 = 1$ and the lower end on x-axis. So, the approximating rectangle shown in Fig. 20.48 has, length $= |y_1|$, width $= dx$ and area $= |y_1|\, dx$. As it can move from $x = 0$ to $x = 1/2$.

$$\therefore \quad \text{Area } OACO = \int_0^{1/2} |y_1|\, dx = \int_0^{1/2} y_1\, dx \qquad [\because\ y_1 > 0 \ \therefore |y_1| = y_1]$$

$$= \int_0^{1/2} \sqrt{1-(x-1)^2}\, dx \qquad \left[\begin{array}{l} \because\ P\,(x, y_1) \text{ lies on } (x-1)^2 + y^2 = 1 \\ \therefore (x-1)^2 + y_1^2 = 1 \Rightarrow y_1 = \sqrt{1-(x-1)^2} \end{array}\right]$$

Similarly, approximating rectangle in the region $CABC$ has, length $= |y_2|$, width $= dx$ and area $= |y_2|\, dx$. As it can move from $x = \dfrac{1}{2}$ to $x = 1$.

$$\therefore \quad \text{Area } CABC = \int_{1/2}^{1} |y_2|\, dx = \int_{1/2}^{1} y_2\, dx$$

$$= \int_{1/2}^{1} \sqrt{1-x^2}\, dx \qquad \left[\because Q\,(x, y_2) \text{ lies on } x^2 + y^2 = 1 \ \therefore x^2 + y_2^2 = 1 \Rightarrow y_2 = \sqrt{1-x^2}\right]$$

Hence, required area A is given by

$$A = 2\,[\text{Area } OACO + \text{Area } CABC]$$

$$A = 2\left[\int_0^{1/2} \sqrt{1-(x-1)^2}\, dx + \int_{1/2}^{1} \sqrt{1-x^2}\, dx\right]$$

$$\Rightarrow \quad A = 2\left[\left[\frac{1}{2}(x-1)\sqrt{1-(x-1)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-1}{1}\right)\right]_0^{1/2} + \left[\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{1/2}^{1}\right]$$

$$\Rightarrow \quad A = \left[\left\{-\frac{\sqrt{3}}{4} + \sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right\} + \left\{\sin^{-1}(1) - \frac{\sqrt{3}}{4} - \sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$\Rightarrow \quad A = -\frac{\sqrt{3}}{4} - \frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

ALITER Two curves are symmetric about $x = 1/2$. So, required area $= 4$ (Area $OACO$).

EXAMPLE 16 *Using integration, find the area of triangle ABC whose vertices have coordinates $A\,(2, 5)$, $B\,(4, 7)$ and $C\,(6, 2)$.* **[CBSE 2001, 2010, 2011]**

SOLUTION Let us first find the equations of the sides of triangle ABC by using the formula

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The equation of AB is $y-5 = \dfrac{7-5}{4-2}(x-2) \Rightarrow x-y+3=0$ ...(i)

The equation of BC is $y-7 = \dfrac{2-7}{6-4}(x-4)$

$\Rightarrow \quad 5x+2y-34=0$ ...(ii)

The equation of side AC is $y-5 = \dfrac{2-5}{6-2}(x-2)$

$\Rightarrow \quad 3x+4y-26=0$ ...(iii)

Clearly, Area of $\Delta\, ABC$ = Area ADB + Area BDC.

Area ADB : To find area ADB, we slice it into vertical strips. We observe that each vertical strip has its lower end on side AC and the upper end on AB. So, the approximating rectangle has length $=|y_2-y_1|$, width $= dx$ and area $=|y_2-y_1|\,dx$. Since the approximating rectangle can move from $x=2$ to $x=4$.

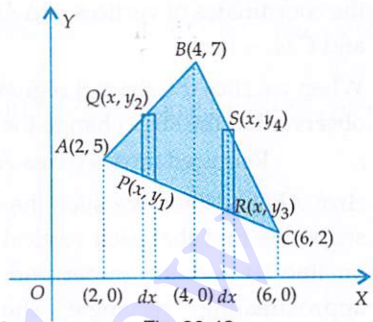

Fig. 20.49

$$\therefore \quad \text{Area } ADB = \int_2^4 |y_2-y_1|\,dx = \int_2^4 (y_2-y_1)\,dx \qquad [\because\ y_2>y_1 \ \therefore |y_2-y_1| = y_2-y_1]$$

$$= \int_2^4 \left\{(x+3)-\left(\frac{26-3x}{4}\right)\right\} dx \qquad \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on (iii) \& (i)} \\ \text{respectively} \therefore 3x+4y_1-26=0 \ \&\ y_2=x+3\end{array}\right]$$

$$= \frac{1}{4}\int_2^4 (7x-14)\,dx$$

Similarly, we obtain

$$\text{Area } BDC = \int_4^6 |y_4-y_3|\,dx = \int_4^6 (y_4-y_3)\,dx \qquad [\because\ y_4>y_3 \ \therefore |y_4-y_3| = y_4-y_3]$$

$$= \int_4^6 \left\{\left(\frac{34-5x}{2}\right)-\left(\frac{26-3x}{4}\right)\right\} dx \qquad \left[\begin{array}{l}\because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on (iii) and} \\ \text{(ii) respectively. } \therefore 3x+4y_3-26=0 \\ \text{and } 5x+2y_4-34=0\end{array}\right]$$

$$= \frac{1}{4}\int_4^6 (42-7x)\,dx$$

$$\therefore \quad \text{Area of } \Delta\, ABC = \frac{1}{4}\int_2^4 (7x-14)\,dx + \frac{1}{4}\int_4^6 (42-7x)\,dx = \frac{1}{4}\left[\left[\frac{7x^2}{2}-14x\right]_2^4 + \left[42x-\frac{7x^2}{2}\right]_4^6\right]$$

$$= \frac{1}{4}\left[\{(56-56)-(14-28)\}+\{(252-126)-(168-56)\}\right] = 7 \text{ sq. units}$$

EXAMPLE 17 *Compute the area bounded by the lines $x+2y=2, y-x=1$ and $2x+y=7$.* **[NCERT EXEMPLAR]**

SOLUTION The equations of the given lines are

$x+2y=2$ ...(i)

$y-x=1$ ...(ii)

$2x+y=7$ ... (iii)

The line $x+2y=2$ meets x and y-axes at $(2, 0)$ and $(0, 1)$ respectively. By joining these two points we obtain the graph of $x+2y = 2$. Similarly, graphs of other two lines are drawn as shown in Fig. 20.50.

Solving equations (i), (ii) and (iii) in pairs, we obtain that the coordinates of vertices of ΔABC are: $A\ (0, 1)$, $B\ (2, 3)$ and $C\ (4, -1)$.

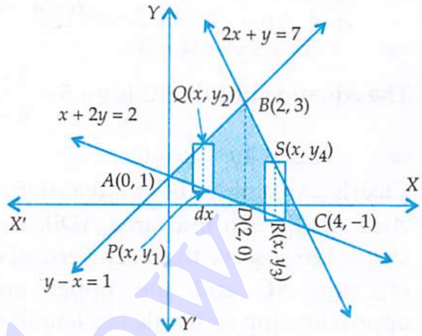

Fig. 20.50

When we slice the shaded region into vertical strips, we observe that the strip change their character at B.

$\therefore$ Required area = (Area ABD) + (Area BDC)

Area ABD : When we slice the area ABD into vertical strips, we find that each vertical strip has its lower end on line (i) and the upper end on line (ii). So, the approximating rectangle shown in Fig. 20.50 has, length $=|y_2-y_1|$, width $= dx$ and area $=|y_2-y_1|\,dx$.

Since the approximating rectangle can move from $x=0$ to $x=2$.

$$\therefore \quad \text{Area } ABD = \int_0^2 |y_2-y_1|\,dy = \int_0^2 (y_2-y_1)\,dx \qquad [\because y_2>y_1 \therefore |y_2-y_1| = y_2-y_1]$$

$$\Rightarrow \quad \text{Area } ABD = \int_0^2 \left\{(x+1)-\left(\frac{2-x}{2}\right)\right\} dx \qquad \left[\begin{array}{l}\because P\ (x, y_1) \text{ and } Q, (x, y_2) \text{ lie on (i) and (ii)} \\ \text{respectively} \therefore y_1-x=1 \text{ and } x+2y_2=2\end{array}\right]$$

Area BDC : When area BDC is sliced into vertical strips, then each strip has its lower end on the line (i) and the upper end on the line (iii). So, the approximating rectangle shown in Fig. 20.50 has, length $=|y_4-y_3|$, width $= dx$ and area $=|y_4-y_3|\,dx$. Since the approximating rectangle can move from $x=2$ to $x=4$.

$$\therefore \quad \text{Area } BDC = \int_2^4 |y_4-y_3|\,dx = \int_2^4 (y_4-y_3)\,dx \qquad [\because y_4>y_3 \therefore |y_4-y_3| = y_4-y_3]$$

$$\Rightarrow \quad \text{Area } BDC = \int_2^4 \left\{(7-2x)-\left(\frac{2-x}{2}\right)\right\} dx \qquad \left[\begin{array}{l}\because R\ (x, y_3) \text{ and } S\ (x, y_4) \text{ lie on (i) and (ii)} \\ \text{respectively.} \\ \therefore x+2y_3=2 \text{ and } 2x+y_4=7\end{array}\right]$$

$\therefore$ Required area = Area ABD + Area BDC

$$= \int_0^2 \left\{(x+1)-\left(\frac{2-x}{2}\right)\right\} dx + \int_2^4 \left\{(7-2x)-\left(\frac{2-x}{2}\right)\right\} dx$$

$$= \int_0^2 \frac{3x}{2}\,dx + \int_2^4 \left(6-\frac{3x}{2}\right) dx = \left[\frac{3x^2}{4}\right]_0^2 + \left[6x-\frac{3x^2}{4}\right]_2^4 = 6 \text{ sq. units}$$

EXAMPLE 18 *Draw a rough sketch of the curves $y=\sin x$ and $y=\cos x$ as x varies from 0 to $\frac{\pi}{2}$ and find the area of the region enclosed by them and x-axis.*

SOLUTION The graphs of $y=\sin x$ and $y=\cos x$ are shown in Fig. 20.51. These two curves intersect at point A. At the point of intersection, we have

$$\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$$

The required region is the shaded region shown in Fig. 20.51. To find the area of this shaded region, we slice it into vertical strips. We observe that vertical strips change their character at A. So, required area A is given by

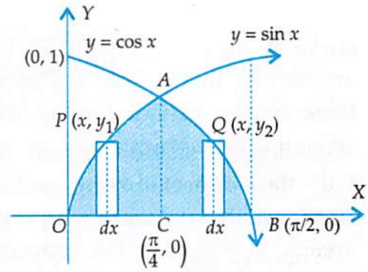

Fig. 20.51

$$A = \text{Area } OAC + \text{Area } ACB$$

$$\Rightarrow A = \int_0^{\pi/4} |y_1|\, dx + \int_{\pi/4}^{\pi/2} |y_2|\, dx$$

$$\Rightarrow A = \int_0^{\pi/4} y_1\, dx + \int_{\pi/4}^{\pi/2} y_2\, dx \qquad [\because y_1, y_2 > 0 \therefore |y_1| = y, |y_2| = y_2]$$

$$\Rightarrow A = \int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = \sin x \text{ and} \\ y = \cos x \text{ respectively } \therefore y_1 = \sin x \text{ and } y_2 = \cos x \end{array}\right]$$

$$\Rightarrow A = \Big[-\cos x\Big]_0^{\pi/4} + \Big[\sin x\Big]_{\pi/4}^{\pi/2} = \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 19 *Find the area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.*

SOLUTION The curve $y = 2x - x^2$ represents a parabola opening downward and crossing x-axis at (0, 0) and (2, 0). Clearly, $y = -x$ represents a line passing through the origin and making 135° with x-axis. A rough sketch of the two curves is shown in Fig. 20.52. The region whose area is to be found is shaded in Fig. 20.52.

Here, we slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 20.52, we have width $= dx$, length $= |y_1 - y_2|$ and area $= |y_1 - y_2|\, dx$. The approximating rectangle can move horizontal between $x = 0$ and $x = 3$.

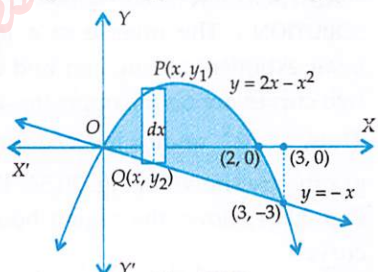

Fig. 20.52

So, required area A is given by

$$A = \int_0^3 |y_1 - y_2|\, dx = \int_0^3 (y_1 - y_2)\, dx$$

$$[\because y_1 > y_2 \therefore |y_1 - y_2| = y_1 - y_2]$$

$$\Rightarrow \quad A = \int_0^3 \left\{(2x - x^2) - (-x)\right\} dx \qquad \left[\begin{array}{l} \because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2x - x^2 \text{ and} \\ y = -x \text{ respectively } \therefore y_1 = 2x - x^2 \text{ and } y_2 = -x \end{array}\right]$$

$$\Rightarrow \quad A = \int_0^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2} \text{ sq. units}$$

EXAMPLE 20 *Compute the area of the figure bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$.*

SOLUTION The equation $y = 2x - x^2$ represents a parabola opening downwards having vertex at $(1, 1)$ and crossing x-axis at (0, 0) and (2, 0). The equation $y = 2^x$ represents the exponential curve as shown in Fig. 20.53. Lines $x = 0$ and $x = 2$ are shown in Fig. 20.53. The region bounded by these curves is shaded in Fig. 20.53.

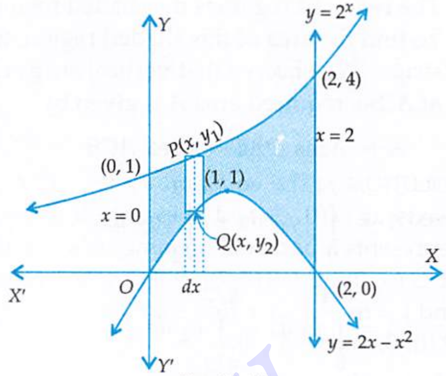

Fig. 20.53

We slice the shaded region into vertical strips. For the approximating rectangle shown in Fig. 20.53, we have length $= |y_1 - y_2|$, width $= dx$ and, area $= |y_1 - y_2|\, dx$. The approximating rectangle can move horizontally between $x = 0$ and $x = 2$. So, required area A is given by

$$A = \int_0^2 |y_1 - y_2|\, dx = \int_0^2 (y_1 - y_2)\, dx$$

$$[\because y_1 > y_2 \therefore |y_1 - y_2| = y_1 - y_2]$$

$$\Rightarrow \quad A = \int_0^2 (2^x - 2x + x^2)\, dx \qquad \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lies on } y = 2^x \text{ and } y = 2x - x^2 \\ \text{respectively. } \therefore y_1 = 2^x \text{ and } y_2 = 2x - x^2\end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3}\right]_0^2 = \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2} = \left(\frac{3}{\log 2} - \frac{4}{3}\right) \text{ sq. units}$$

EXAMPLE 21 *Sketch the curves and identify the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$. Find the area of this region.*

SOLUTION The inverse of a logarithmic function is an exponential function and vice- versa and these two curves are on the opposite sides of the line $y = x$. Thus, $y = 2^x$ and $y = \log x$ do not intersect. Their graphs are shown in Fig. 20.54. The shaded region in Fig. 20.54 shows the region bounded by the given curves.

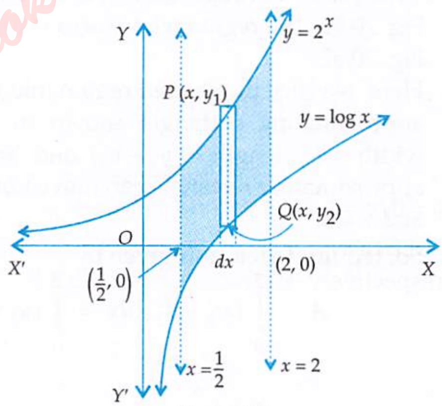

Fig. 20.54

Let us slice this region into vertical strips as shown in Fig. 20.54. For the approximating rectangle shown in Fig. 20.54, we have length $= |y_1 - y_2|$, width $= dx$ and, area $= |y_1 - y_2|\, dx$. As the approximating rectangle can move horizontally between $x = \frac{1}{2}$ and $x = 2$. So, required area A is given by

$$A = \int_{1/2}^2 |y_1 - y_2|\, dx = \int_{1/2}^2 (y_1 - y_2)\, dx \qquad [\because y_1 > y_2 \therefore |y_1 - y_2| = y_1 - y_2]$$

$$\Rightarrow \quad A = \int_{1/2}^2 (2^x - \log x)\, dx \qquad \left[\begin{array}{l}\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2^x \text{ and } y = \log x \\ \text{respectively } \therefore y_1 = 2^x \text{ and } y_2 = \log x\end{array}\right]$$

$$\Rightarrow \quad A = \left[\frac{2^x}{\log 2} - x \log x + x\right]_{1/2}^2$$

$$\Rightarrow \quad A = \left\{ \frac{4}{\log 2} - 2 \log 2 + 2 \right\} - \left\{ \frac{\sqrt{2}}{\log 2} + \frac{1}{2} \log 2 + \frac{1}{2} \right\} = \left\{ \frac{(4 - \sqrt{2})}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \right\} \text{ sq. units}$$

EXAMPLE 22 *Find the area bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.*

SOLUTION The equation $y = 6x - x^2$ represents a parabola opening downward and cutting x-axis at $O(0, 0)$ and $A(6, 0)$. Similarly, $y = x^2 - 2x$ also represents a parabola opening upward and crossing x-axis at $O(0, 0)$ and $B(2, 0)$. Solving the equations $y = x^2 - 2x$ and $y = 6x - x^2$, we find that the two parabolas intersect at $O(0, 0)$ and $C(4, 8)$.

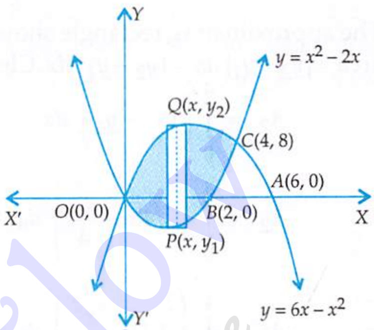

Fig. 20.55

The rough sketches of the two parabolas are shown in Fig. 20.55 and the shaded region is the region enclosed the two curves. We slice this region into vertical strips as shown in Fig. 20.55. The approximating rectangle shown in Fig. 20.55 has length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\, dx$. Clearly, it can move horizontally between $O(0, 0)$ and $C(4, 8)$. So, required area A is given by

$$A = \int_0^4 |y_2 - y_1|\, dx = \int_0^4 (y_2 - y_1)\, dx \qquad [\because\ y_2 > y_1 \ \therefore |y_2 - y_1| = (y_2 - y_1)]$$

$$\Rightarrow \quad A = \int_0^4 \left\{ (6x - x^2) - (x^2 - 2x) \right\} dx \qquad \left[\begin{array}{l} \because\ (x, y_1) \text{ lies on } y = x^2 - 2x \ \therefore\ y_1 = x^2 - 2 \\ (x, y_2) \text{ lies on } y = 6x - x^2 \ \therefore\ y_2 = 6x - x^2 \end{array} \right]$$

$$\Rightarrow \quad A = \int_0^4 (8x - 2x^2)\, dx = \left[4x^2 - \frac{2}{3} x^3 \right]_0^4 = 64 - \frac{128}{3} = \frac{64}{3} \text{ sq. units}$$

EXAMPLE 23 *Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $y = 0$, $x = 4$ and $y = 4$ into three equal parts.* **[NCERT, CBSE 2009, 2015, 2016, 2019]**

SOLUTION Let A_1, A_2 and A_3 denote areas of the regions $OAPLO$, $OAPBO$ and $OBPMO$ respectively. We have to prove that $A_1 = A_2 = A_3$.

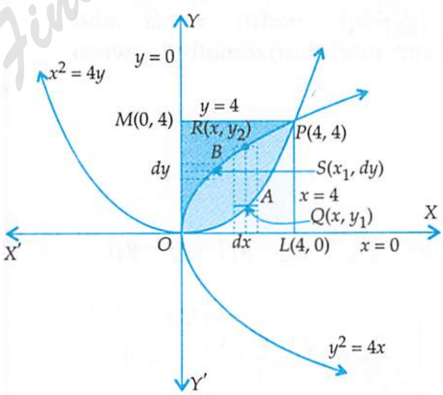

Fig. 20.56

$$\text{Now,} \quad A_1 = \int_0^4 |y_1|\, dx = \int_0^4 y_1\, dx \qquad [\because\ y_1 > 0 \ \therefore |y_1| = y_1]$$

$$\Rightarrow \quad A_1 = \int_0^4 \frac{x^2}{4}\,dx \qquad \left[\because (x, y_1) \text{ lies on } x^2 = 4y \;\therefore\; x^2 = 4y_1 \Rightarrow y_1 = \frac{x^2}{4}\right]$$

$$\Rightarrow \quad A_1 = \frac{1}{4}\int_0^4 x^2\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{1}{4}\times\frac{64}{3} = \frac{16}{3} \text{ sq. units}$$

The approximating rectangle shown in the region *OAPBO* has length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\,dx = (y_2 - y_1)\,dx$. Clearly, it can move horizontally between $x = 0$ and $x = 4$.

$$\therefore \quad A_2 = \int_0^4 \left(y_2 - y_1\right) dx$$

$$\Rightarrow \quad A_2 = \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4}\right) dx \qquad \left[\begin{array}{l}\because Q(x, y_1) \text{ lie on } x^2 = 4y \;\therefore\; x^2 = 4y_1 \\ \because R(x, y_2) \text{ lies on } y_2 = 4x \;\therefore\; y_2^2 = 4x\end{array}\right]$$

$$\Rightarrow \quad A_2 = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \left(\frac{4}{3}\times 8 - \frac{64}{12}\right) = \frac{16}{3} \text{ sq. units}$$

The approximating rectangle shown in region *OBPMO* has length $= |x_1|$, width $= dy$ and area $= |x_1|\,dy$. Clearly, it can move vertically between $y = 0$ and $y = 4$.

$$\text{and,} \quad A_3 = \text{Area } OBPMO = \int_0^4 |x_1|\,dy = \int_0^4 x_1\,dy \qquad [\because x_1 > 0 \;\therefore\; |x_1| = x_1]$$

$$\Rightarrow \quad A_3 = \int_0^4 \frac{y^2}{4}\,dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^4 = \frac{1}{4}\times\frac{64}{3} = \frac{16}{3} \qquad [\because S(x_1, y) \text{ lies on } y^2 = 4x \;\therefore y^2 = 4x_1]$$

Clearly, $A_1 = A_2 = A_3 = \dfrac{16}{3}$ sq. units

EXAMPLE 24 *If the area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 square unit, then find the value of a.*

SOLUTION Clearly, $y = ax^2$ and $x = ay^2$ intersect at $O(0, 0)$ and $(1/a, 1/a)$. The shaded region in Fig. 20.57 is the region enclosed by the two parabolas. The approximating rectangle shown in Fig. 20.57 has length $= |y_2 - y_1|$, width $= dx$ and area $= |y_2 - y_1|\,dx$. Clearly, it can more horizontally between $x = 0$ and $x = 1/a$.

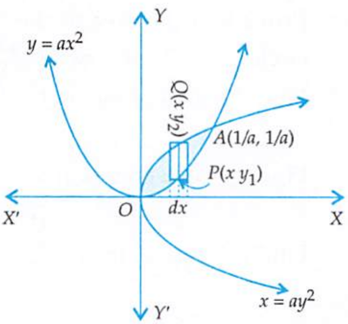

Fig. 20.57

$$\therefore \quad \int_0^{1/a} |y_2 - y_1|\,dx = 1$$

$$\Rightarrow \quad \int_0^{1/a} \left(y_2 - y_1\right) dx = 1 \qquad [\because y_2 > y_1 \;\therefore\; |y_2 - y_1| = y_2 - y_1]$$

$$\Rightarrow \quad \int_0^{1/a} \left\{\sqrt{\frac{x}{a}} - ax^2\right\} dx = 1 \Rightarrow \left[\frac{2}{3\sqrt{a}}x^{3/2} - \frac{a}{3}x^3\right]_0^{1/a} = 1$$

$$\Rightarrow \frac{2}{3\sqrt{a}}\times\frac{1}{a^{3/2}} - \frac{a}{3}\times\frac{1}{a^3} = 1 \Rightarrow \frac{2}{3a^2} - \frac{1}{3a^2} = 1 \Rightarrow \frac{1}{3a^2} = 1 \Rightarrow a^2 = \frac{1}{3} \Rightarrow a = \frac{1}{\sqrt{3}} \; [\because a > 0]$$

EXERCISE 20.3

BASIC

1. Calculate the area of the region bounded by the parabolas $y^2 = 6x$ and $x^2 = 6y$. [NCERT EXEMPLAR]
2. Find the area of the region common to the parabolas $4y^2 = 9x$ and $3x^2 = 16y$.
3. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.
4. Find the area bounded by the curve $y = 4 - x^2$ and the lines $y = 0, y = 3$.
5. Find the area of the region $\left\{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \le \frac{x}{a} + \frac{y}{b}\right\}$.
6. Using integration, find the area of the region bounded by the triangle whose vertices are $(2, 1)$, $(3, 4)$ and $(5, 2)$.
7. Using integration, find the area of the region bounded by the triangle ABC whose vertices A, B, C are $(-1, 1)$, $(0, 5)$ and $(3, 2)$ respectively. [NCERT EXEMPLAR, CBSE 2008]
8. Using integration, find the area of the triangular region, the equations of whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. [CBSE 2011, 2012]
9. Find the area of the region $\{(x, y) : y^2 \le 8x, x^2 + y^2 \le 9\}$.
10. Find the area of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 6x$.
11. Find the area of the region between the circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. [NCERT, CBSE 2010, 2012, 2013]
12. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.
13. Draw a rough sketch of the region $\{(x, y) : y^2 \le 3x,\ 3x^2 + 3y^2 \le 16\}$ and find the area enclosed by the region using method of integration.
14. Draw a rough sketch of the region $\{(x, y) : y^2 \le 5x, 5x^2 + 5y^2 \le 36\}$ and find the area enclosed by the region using method of integration.
15. Using integration, find the area of the region enclosed by the parabola $y = 3x^2$ and the line $3x - y + 6 = 0$ [CBSE 2020]
16. Find the area included between the parabolas $y^2 = 4ax$ and $x^2 = 4by$.
17. Prove that the area in the first quadrant enclosed by the x-axis, the line $x = \sqrt{3}y$ and the circle $x^2 + y^2 = 4$ is $\pi/3$. [CBSE 2012, 2017]
18. Find the area of the region bounded by $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis. [NCERT EXEMPLAR]
19. Find the area common to the circle $x^2 + y^2 = 16\,a^2$ and the parabola $y^2 = 6\,ax$.

    OR [CBSE 2004, 2012, 2013]

    Find the area of the region $\{(x, y) : y^2 \le 6ax\}$ and $\{(x, y) : x^2 + y^2 \le 16a^2\}$. [NCERT EXEMPLAR]
20. Find the area, lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and the parabola $y^2 = 4x$. [CBSE 2008, 2019, 2020]
21. Using integration, find the area of the smaller region enclosed by the curve $4x^2 + 4y^2 = 9$ and the line $2x + 2y = 3$. [CBSE 2022]

22. Prove that the area common to the two parabolas $y = 2x^2$ and $y = x^2 + 4$ is $\frac{32}{3}$ sq. units.

23. Using integration, find the area of the region bounded by the triangle whose vertices are
(i) $(-1, 2), (1, 5)$ and $(3, 4)$ **[CBSE 2014]** (ii) $(-2, 1), (0, 4)$ and $(2, 3)$ **[CBSE 2017]**
(iii) $(2, 5), (4, 7)$ and $(6, 2)$ **[CBSE 2019]**

24. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$. **[NCERT EXEMPLAR]**

25. Find the area of the region in the first quadrant enclosed by x-axis, the line $y = \sqrt{3}\ x$ and the circle $x^2 + y^2 = 16$. **[CBSE 2012, 2017]**

26. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y - 1 = 0$.

27. Find the area of the region bounded by the curves $y = x - 1$ and $(y - 1)^2 = 4(x + 1)$.

28. Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$. **[NCERT EXEMPLAR]**

29. Find the area bounded by the parabola $y = 2 - x^2$ and the straight line $y + x = 0$.

30. Using the method of integration, find the area of the region bounded by the following lines:
(i) $3x - y - 3 = 0,\ 2x + y - 12 = 0,\ x - 2y - 1 = 0.$ **[CBSE 2012]**
(ii) $3x - 2y + 1 = 0, 2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$ **[CBSE 2019]**

31. Sketch the region bounded by the curves $y = x^2 + 2,\ y = x,\ x = 0$ and $x = 1$. Also, find the area of this region.

32. Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

33. Using integration, find the area of the triangle ABC coordinates of whose vertices are $A(4, 1), B(6, 6)$ and $C(8, 4)$.

34. Using integration find the area of the region: $\left\{(x, y) : |x - 1| \le y \le \sqrt{5 - x^2}\right\}$. **[CBSE 2010]**

35. Find the area of the region bounded by $y = |x - 1|$ and $y = 1$.

36. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[CBSE 2014, 2018]**

37. Find the area of the circle $x^2 + y^2 = 16$ which is exterior to the parabola $y^2 = 6x$. **[CBSE 2007]**

38. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$. **[NCERT, CBSE 2005]**

39. Make a sketch of the region $\left\{(x, y) : 0 \le y \le x^2 + 3;\ 0 \le y \le 2x + 3;\ 0 \le x \le 3\right\}$ and find its area using integration.

40. Find the area of the region bounded by the curve $y = \sqrt{1 - x^2}$, line $y = x$ and the positive x-axis. **[CBSE 2005]**

41. Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$. **[CBSE 2005]**

42. Find the area of the region enclosed between the two curves $x^2 + y^2 = 9$ and $(x - 3)^2 + y^2 = 9$. **[CBSE 2009, 2020]**

43. Find the area of the region $\left\{(x, y) : x^2 + y^2 \le 9, x + y \ge 3\right\}$. **[CBSE 2012, 2022]**

44. Using integration, find the area of the following region: $\left\{(x, y): \dfrac{x^2}{9}+\dfrac{y^2}{4} \leq 1 \leq \dfrac{x}{3}+\dfrac{y}{2}\right\}$.

[CBSE 2010, 2014, 2019, 2022]

45. Using integration find the area of the region bounded by the curve $y=\sqrt{4-x^2}$, $x^2+y^2-4x=0$ and the x-axis. [CBSE 2016]

BASED ON LOTS

46. Find the area enclosed by the curves $y=|x-1|$ and $y=-|x-1|+1$.

47. Find the area enclosed by the curves $3x^2+5y=32$ and $y=|x-2|$.

48. Find the area enclosed by the parabolas $y=4x-x^2$ and $y=x^2-x$.

49. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y=4x-x^2$ and $y=x^2-x$?

50. Find the area of the figure bounded by the curves $y=|x-1|$ and $y=3-|x|$.

51. If the area bounded by the parabola $y^2=4ax$ and the line $y=mx$ is $\dfrac{a^2}{12}$ sq. units, then using integration, find the value of m.

52. If the area enclosed by the parabolas $y^2=16ax$ and $x^2=16ay$, $a>0$ is $\dfrac{1024}{3}$ square units, find the value of a.

ANSWERS

1. 12 sq. units
2. 4 sq. units
3. $\dfrac{1}{6}$ sq. units
4. $\dfrac{28}{3}$ sq. unit
5. $(\pi-2)\dfrac{ab}{4}$ sq. units
6. 4 sq. units
7. $\dfrac{15}{2}$ sq. units
8. 8 sq. unit
9. $2\left(\dfrac{\sqrt{2}}{3}+\dfrac{9\pi}{4}-\dfrac{9}{2}\sin^{-1}\dfrac{1}{3}\right)$ sq. units
10. $\dfrac{4}{3}(4\pi+\sqrt{3})$ sq. units
11. $\left(\dfrac{8\pi}{3}-2\sqrt{3}\right)$ sq. units
12. $\dfrac{7}{6}$ sq. units
13. $\dfrac{4}{\sqrt{3}}a^{3/2}+\dfrac{8\pi}{3}-a\sqrt{163-a^2}-\dfrac{16}{3}\sin^{-1}\left(\dfrac{\sqrt{3}a}{4}\right)$, where $a=\dfrac{-9+\sqrt{273}}{6}$
14. $\dfrac{4\sqrt{5}}{3}a^{3/2}+\dfrac{18\pi}{5}-a\sqrt{\dfrac{36}{5}-a^2}-\dfrac{36}{5}\sin^{-1}\left(\dfrac{a\sqrt{5}}{6}\right)$, $a=\dfrac{-25+\sqrt{1345}}{10}$
15. $\dfrac{27}{2}$ sq. units.
16. $\dfrac{16}{3}ab$ sq. units
18. 9 sq. units
19. $\dfrac{4a^2}{3}(4\pi+\sqrt{3})$ sq. units
20. $\left(4\pi-\dfrac{32}{3}\right)$ sq. units
21. $\dfrac{9}{8}\left(\dfrac{\pi}{2}-1\right)$ sq. units
23. (i) 14 sq. units (ii) 4 sq. units

24. 1/6 sq. units 25. $8\pi/3$ sq. units 26. $\frac{16}{3}$ sq. units 27. $\frac{64}{3}$ sq. units

28. $\frac{9}{2}$ sq. units 29. $\frac{9}{2}$ sq. units 30. 11 sq. units 31. $\frac{11}{6}$ sq. units

32. 4 sq. units 33. 7 sq. units 34. $\left\{\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{5}}\right)-\frac{3}{2}\right\}$ sq. units

35. 1 sq. units 36. 4π sq. units 37. $\left(\frac{32}{3}\pi-\frac{4\sqrt{3}}{3}\right)$ sq. units

38. $\frac{9}{2}$ sq. units 39. $\frac{50}{3}$ sq. units 40. $\frac{\pi}{8}$ sq. units 41. $\frac{15}{2}$ sq. units

42. $6\pi-\frac{9\sqrt{3}}{2}$ sq. units 43. $\frac{9}{4}(\pi-2)$ sq. units 44. $\left(\frac{3\pi}{2}-3\right)$ sq. units

45. $\left(\frac{4\pi}{3}-\sqrt{3}\right)$ sq. units 46. $\frac{1}{2}$ sq. units 47. $\frac{33}{2}$ sq. units

48. $\frac{125}{24}$ sq. units 49. 121:4 50. 4 sq. units 51. $m=2$

52. $2\sqrt{3}$

HINTS TO SELECTED PROBLEMS

11. Clearly, given circles intersect at $(1, \sqrt{3})$ and $(1, -\sqrt{3})$.

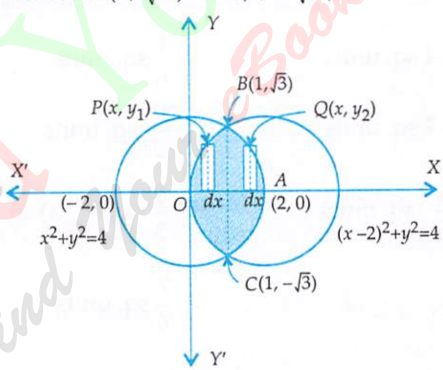

Fig. 20.58

$\therefore$ Required area $= 2\,(\text{Area } OABO)$

$$= 2\left(\int_0^1 y_1\,dx+\int_1^2 y_2\,dx\right) = 2\left\{\int_1^2 \sqrt{4-x^2}\,dx+\int_0^1 \sqrt{4-(x-2)^2}\,dx\right\}$$

$$= \left[x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}\right]_1^2+\left[(x-2)\sqrt{4-(x-2)^2}+4\sin^{-1}\frac{x-2}{2}\right]_0^1$$

$$= 4\sin^{-1}1-\left(\sqrt{3}+4\times\frac{\pi}{6}\right)+\left\{-\sqrt{3}+4\sin^{-1}\left(-\frac{1}{2}\right)\right\}-\left\{0+4\sin^{-1}(-1)\right\}$$

$$= 4 \times \frac{\pi}{2} - \left(\sqrt{3} + \frac{2\pi}{3}\right) + \left(-\sqrt{3} - \frac{4\pi}{6}\right) - \left(-\frac{4\pi}{2}\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq. units}$$

28. Required area $= \int_{-1}^{2} |y_2 - y_1|\, dx$

$$= \int_{-1}^{2} (y_1 - y_2)\, dx \quad \left[\begin{array}{l} \because\ y_2 < y_1 < 0 \Rightarrow y_2 - y_1 < 0 \\ \Rightarrow\ |y_2 - y_1| = y_1 - y_2 \end{array}\right]$$

$$= \int_{-1}^{2} (-x^2 + x + 2)\, dx = \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_{-1}^{2}$$

$$= \left(-\frac{8}{3} + 6\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{9}{2} \text{ sq.units}$$

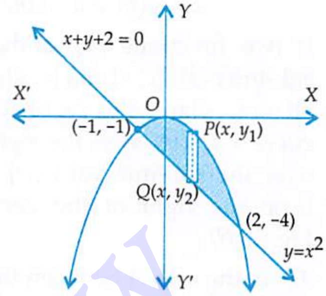

Fig. 20.59

38. Required area $= \int_{-1}^{2} (y_2 - y_1)\, dx$

$$= \int_{-1}^{2} \left\{(x+2) - x^2\right\} dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{10}{3} + \frac{7}{6}\right) = \frac{9}{2} \text{ sq. units}$$

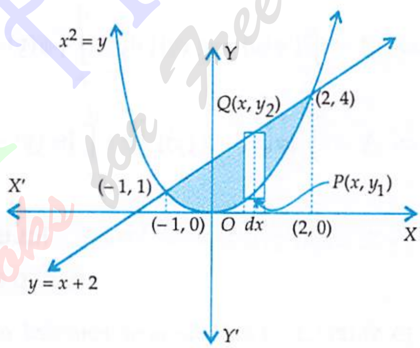

Fig. 20.60

20.6 AREAS BETWEEN TWO CURVES BY USING HORIZONTAL STRIPS

To find the area between two curves of the form $x = \phi(y)$ and $x = \psi(y)$ on the interval $[c, d]$ on the y-axis, we form horizontal strips. The procedure is explained in the following algorithm.

ALGORITHM

Step I *Draw given curves $x = \phi(y)$ and $x = \psi(y)$ and horizontal lines $y = c$ and $y = d$.*

Step II *Identify the region included between the curves and horizontal lines drawn in step I.*

Step III *Take an arbitrary point $P(x_1, y)$ on one of the curves say $x = \phi(y)$, and draw a horizontal line through P to meet the other curve $x = \psi(y)$ at $Q(x_2, y)$. Clearly, $x_1 = \phi(y)$ and $x_2 = \psi(y)$.*

Step IV *Draw horizontal approximating rectangle of width $= \Delta y$, length $= |x_2 - x_1| = |\psi(y) - \phi(y)|$ such that $P(x_1, y)$ and $Q(x_2, y)$ are mid-point of horizontal sides AB and CD as shown in Fig. 20.61.*

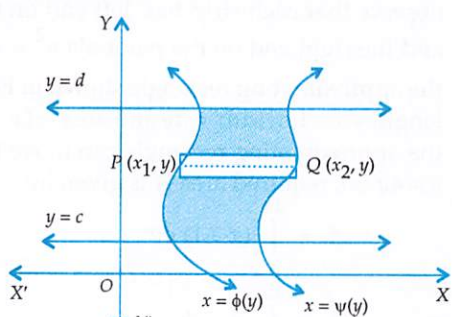

Fig. 20.61

Step V *Find the area of the approximating rectangle drawn in step IV. Let ΔA be its area. Then, $\Delta A = |\psi(y) - \phi(y)|\, \Delta y$.*

Step VI *Use the formula* $A = \int_c^d |\psi(y) - \phi(y)|\, dy$ *to find the area of the region between* $x = \phi(y)$, $x = \psi(y)$ *and on the bottom and top by the horizontal lines* $y = c$ *and* $y = d$ *respectively.*

If two functions $\phi(y)$ and $\psi(y)$ are defined on $[c, d]$ on y-axis which can be divided into sub-intervals $[c, e]$ and $[e, d]$ such that $\phi(y) \ge \psi(y)$ for all $y \in [c, e]$ and $\phi(y) \le \psi(y)$ for all $y \in [e, d]$. That is the curve $x = \phi(y)$ is on the right of the curve $x = \psi(y)$ over the sub-interval $[c, e]$ and the curve $x = \psi(y)$ is on the right of the curve $x = \phi(y)$ as shown in Fig. 20.62.

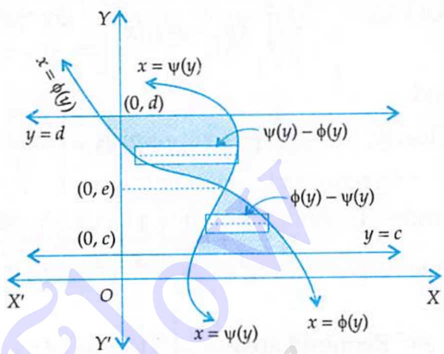

Fig. 20.62

Then, the area A between the curves on the interval $[c, d]$ on y-axis is given by

$$A = \int_c^d |\phi(y) - \psi(y)|\, dy$$

$$\Rightarrow A = \int_c^e |\phi(y) - \psi(y)|\, dy + \int_e^d |\phi(y) - \psi(y)|\, dy$$

$$\Rightarrow A = \int_c^e \{\phi(y) - \psi(y)\}\, dy + \int_e^d \{\psi(y) - \phi(y)\}\, dy$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the area bounded by the curve* $y^2 = 4a^2(x-1)$ *and the lines* $x = 1$ *and* $y = 4a$.

SOLUTION The equation of the given curve is $y^2 = 4a^2(x-1)$ or, $(y-0)^2 = 4a^2(x-1)$.

Clearly, this equation represents a parabola with vertex at (1, 0) as shown in Fig. 20.63. The region enclosed by $y^2 = 4a^2(x-1)$, $x = 1$ and $y = 4a$ is the area of shaded portion in Fig. 20.63. When we slice the area of the shaded portion in horizontal strips, we observe that each strip has left end on the line $x = 1$ and the right end on the parabola $y^2 = 4a^2(x-1)$. So, the approximating rectangle shown in Fig. 20.63 has, length $= x - 1$, width $= dy$ and area $= (x-1)\, dy$. Since, the approximating rectangle can move from $y = 0$ to $y = 4a$. So, required area A is given by

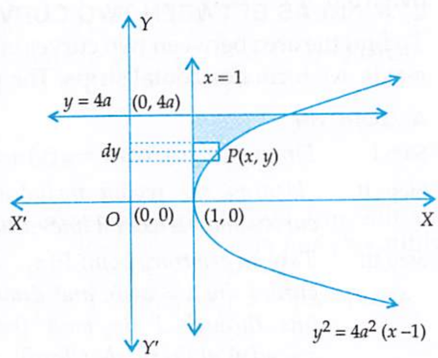

Fig. 20.63

$$A = \int_0^{4a} (x-1)\, dy$$

$$\Rightarrow \quad A = \int_0^{4a} \frac{y^2}{4a^2}\, dy \qquad [\because P(x, y) \text{ lies on } y^2 = 4a^2(x-1) \therefore x - 1 = y^2/4a^2]$$

$$\Rightarrow \quad A = \frac{1}{4a^2}\left[\frac{y^3}{3}\right]_0^{4a} = \frac{1}{4a^2}\left(\frac{64a^3}{3}\right) = \frac{16a}{3} \text{ sq. units}$$

EXAMPLE 2 *Sketch the region common to the circle $x^2 + y^2 = 16$ and the parabola $x^2 = 6y$. Also, find the area of the region using integration.*

OR

Using integration, find the area of the region $\left\{(x, y) : x^2 + y^2 \le 16 \text{ and}, x^2 \le 6y\right\}$. **[CBSE 2010]**

SOLUTION The equations of the given curves are

$$x^2 + y^2 = 16 \quad \text{...(i)}$$

and, $$x^2 = 6y \quad \text{...(ii)}$$

Clearly, $x^2 + y^2 = 16$ represents a circle having centre at the origin and radius four units and $x^2 = 6y$ represents a parabola opening with axis along y-axis upward and having its vertex at the origin. To find the points of intersection of these two curves, we solve (i) and (ii) simultaneously. Putting $y = \frac{x^2}{6}$, obtained from (ii), in (i), we get

$$x^2 + \frac{x^4}{36} = 16 \Rightarrow x^4 + 36x^2 - 576 = 0 \Rightarrow (x^2 + 48)(x^2 - 12) = 0 \Rightarrow x^2 - 12 = 0 \Rightarrow x = \pm 2\sqrt{3}$$

Putting $x = \pm 2\sqrt{3}$ in (ii), we get $y = 2$. Thus, the two curves intersect at $A(2\sqrt{3}, 2)$ and $C(-2\sqrt{3}, 2)$. Clearly, both the curves are symmetrical about y-axis.

$\therefore$ Required area = 2 (Area of the shaded region lying in first quadrant)

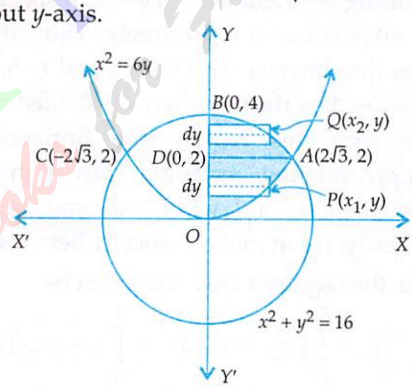

Fig. 20.64

To find this area, we slice it into horizontal strips. We observe that the horizontal strips change their character at the point A. Draw a line AD parallel to x-axis which divides the region $OABO$ into two portions $OADO$ and $ADBA$.

For the region $OADO$, the approximating rectangle has, length $= |x_1|$, width $= dy$ and area $= |x_1|\, dy$. As it can move from $y = 0$ to $y = 2$.

$$\therefore \text{ Area } OADO = \int_0^2 |x_1|\, dy = \int_0^2 x_1\, dy \quad \left[\begin{array}{l} \because x_1 > 0 \\ \therefore |x_1| = x_1 \end{array}\right]$$

$$= \int_0^2 \sqrt{6y}\, dy \quad \left[\because (x_1, y_1) \text{ lies on } x^2 = 6y \;\therefore x_1^2 = 6y \Rightarrow x_1 = \sqrt{6y}\right]$$

For the area $DABD$, the approximating rectangle as shown in Fig. 20.35 has, length $= x_2$, width $= dy$ and area $= x_2\, dy$. As it can move from $y = 2$ to $y = 4$.

$$\therefore \text{ Area } DABD = \int_2^4 |x_2|\, dy = \int_2^4 x_2\, dy \quad [\because x_2 > 0 \;\therefore |x_2| = x_2]$$

$$= \int_2^4 \sqrt{16 - y^2}\, dy \left[\because (x_2, y) \text{ lies on } x^2 + y^2 = 16 \therefore x_2^2 + y^2 = 16 \Rightarrow x_2 = \sqrt{16 - y^2}\right]$$

$$\therefore \text{ Required area } = 2\,[\text{Area } OADO + \text{Area } DABD] = 2\left[\int_0^2 \sqrt{6y}\, dy + \int_2^4 \sqrt{16 - y^2}\, dy\right]$$

$$= 2\left[\frac{2}{3}\sqrt{6}\left[y^{3/2}\right]_0^2 + \left[\frac{1}{2}y\sqrt{16 - y^2} + \frac{16}{2}\sin^{-1}\frac{y}{4}\right]_2^4\right]$$

$$= 2\left[\frac{2\sqrt{6}}{3}\times(2\sqrt{2}-0)+\left\{8\sin^{-1}1-\sqrt{16-4}-8\sin^{-1}\frac{1}{2}\right\}\right]$$

$$= 2\left\{\frac{4}{3}\sqrt{12}+8\times\frac{\pi}{2}-\sqrt{12}-8\times\frac{\pi}{6}\right\} = 2\left(\frac{8}{3}\sqrt{3}+4\pi-2\sqrt{3}-\frac{4\pi}{3}\right) = \left(\frac{4\sqrt{3}}{3}+\frac{16\pi}{3}\right) \text{ sq. units}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Find the area of the region between the parabola $x = y^2 - 6y$ and the line $x = -y$.*

SOLUTION The equation $x = y^2 - 6y$ can be written as $(y-3)^2 = x + 9$. Clearly, it represents the parabola having vertex at $(-9, 3)$ and opens rightward. The sketch of the parabola is shown in Fig. 20.65. The equation $y = -x$ represents a line passing through the origin making 135° angle with x-axis. To find the points of intersection of these two curves, we solve the equations $x = y^2 - 6y$ and $x = -y$ simultaneously.

Putting $x = -y$ in $x = y^2 - 6y$, we get

$$y^2 - 6y = -y \Rightarrow y(y-5) = 0 \Rightarrow y = 0 \text{ and } y = 5$$

Putting $y = 0$ and $y = 5$ in $y = -x$ respectively, we obtain $x = 0$ and $x = -5$ respectively. Thus, the parabola and the line intersect at $O(0, 0)$ and $(-5, 5)$. The region enclosed by the two curves is shaded in Fig. 20.65. Let us slice this region into horizontal strips. The approximating rectangle shown in Fig. 20.65 has length $= |x_2 - x_1|$, width $= dy$ and, area $= |x_2 - x_1|\, dy$. Clearly, it can move vertically between $y = 0$ and $y = 5$.

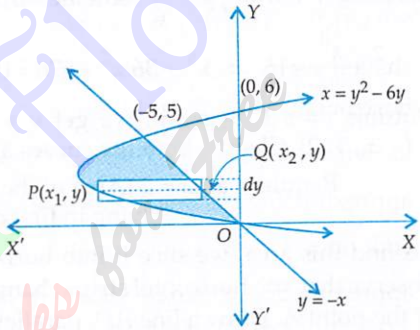

Fig. 20.65

So, the required area A is given by

$$A = \int_0^5 |x_2 - x_1|\, dy = \int_0^5 (x_2 - x_1)\, dy \qquad [\because x_2 > x_1 \therefore |x_2 - x_1| = x_2 - x_1]$$

$$\Rightarrow A = \int_0^5 \left\{-y-(y^2-6y)\right\} dy \qquad \left[\begin{array}{l}\because Q(x_2, y) \text{ and } P(x_1, y) \text{ lie on } y = -x \ \& \ x = y^2 - 6y \text{ respec.}\\ \therefore x_2 = -y \text{ and } x_1 = y^2 - 6y\end{array}\right]$$

$$\Rightarrow A = \int_0^5 (5y - y^2)\, dy = \left[\frac{5}{2}y^2 - \frac{y^3}{3}\right]_0^5 = \frac{125}{6} \text{ sq. units}$$

EXAMPLE 4 *Find the area of the region bounded by the parabola $y^2 = 2x$ and straight line $x - y = 4$.*

[NCERT EXEMPLAR]

SOLUTION The region bounded by the parabola $y^2 = 2x$ and the line $x - y = 4$ is the shaded region in Fig. 20.66. We slice this region into horizontal strips. We observe that the left end of each horizontal strip is on the parabola $y^2 = 2x$ and the right end is on the line $x - y = 4$. The approximating rectangle shown in Fig. 20.66 is of length $(x_2 - x_1)$, width $= dy$ and area $= (x_2 - x_1)\, dy$. Clearly, it can move vertically from $y = -2$ to $y = 4$. So, the required area A is given by

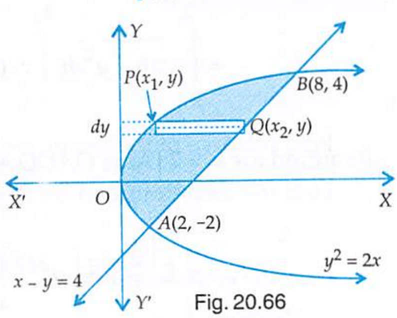

Fig. 20.66

$$A = \int_{-2}^{4} (x_2 - x_1)\, dy$$

$$\Rightarrow A = \int_{-2}^{4} \left\{ (y+4) - \frac{y^2}{2} \right\} dy \qquad \left[\begin{array}{l} \because P(x_1, y) \text{ and } Q(x_2, y) \text{ lie on } y^2 = 2x \text{ and } x - y = 4 \text{ respectively.} \\ \therefore y^2 = 2x_1 \text{ and } x_2 - y = 4 \end{array} \right]$$

$$\Rightarrow A = \frac{1}{2} \int_{-2}^{4} (2y + 8 - y^2)\, dy = \frac{1}{2} \left[y^2 + 8y - \frac{y^3}{3} \right]_{-2}^{4} = \frac{1}{2} \left\{ \left(48 - \frac{64}{3} \right) - \left(-12 + \frac{8}{3} \right) \right\} = 18 \text{ sq units.}$$

EXAMPLE 5 *Find the area of the region bounded by the curve $y = x^3$ and the lines $y = x + 6$ and $y = 0$.* **[NCERT EXEMPLAR]**

SOLUTION The region bounded by the curve $y = x^3$ and the lines $y = x + 6$ and $y = 0$ is the shaded region in Fig. 20.67. Let us slice this region into horizontal strips. We find that each horizontal strip has its left end on $y = x + 6$ and right end on $y = x^3$. The approximating rectangle shown in Fig. 20.67 has its left end on $y = x + 6$, right and on $y = x^3$. Therefore, the approximating rectangle shown in Fig. 20.67 has length $= (x_2 - x_1)$, width $= dy$ and area $= (x_2 - x_1)\, dy$.

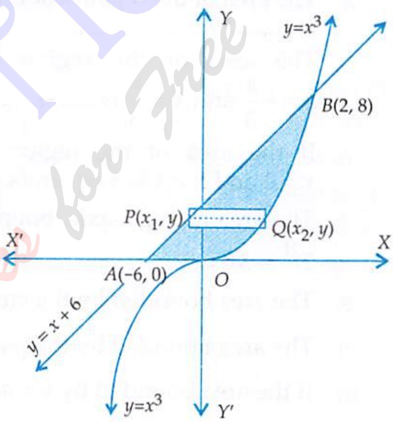

Fig. 20.67

Clearly, the approximating rectangle can move vertically between $y = 0$ and $y = 8$. So, the area A of the shaded region is given by

$$A = \int_{0}^{8} (x_2 - x_1)\, dy$$

$$\Rightarrow A = \int_{0}^{8} \left\{ y^{1/3} - (y - 6) \right\} dy \qquad \left[\begin{array}{l} \because P(x_1, y) \text{ and } Q(x_2, y) \text{ lie on } y = x + 6 \text{ and } y = x^3 \text{ respectively.} \\ \therefore\ y = x_1 + 6,\ y = x_2^{\,3} \end{array} \right]$$

$$\Rightarrow A = \left[\frac{3}{4} y^{4/3} - \frac{y^2}{2} + 6y \right]_{0}^{8} = \left\{ (12 - 32 + 48) - 0 \right\} = 28 \text{ sq. units}$$

EXERCISE 20.4

BASED ON LOTS

1. Find the area of the region between the parabola $x = 4y - y^2$ and the line $x = 2y - 3$.
2. Find the area bounded by the parabola $x = 8 + 2y - y^2$; the y-axis and the lines $y = -1$ and $y = 3$.
3. Find the area bounded by the parabola $y^2 = 4x$ and the line $y = 2x - 4$.
   (i) By using horizontal strips (ii) By using vertical strips.

4. Find the area of the region bounded by the parabola $y^2 = 2x$ and the straight line $x - y = 4$. [NCERT EXEMPLAR]

ANSWERS

1. $\frac{32}{3}$ sq. units 2. $\frac{92}{3}$ sq. units 3. 9 sq. units 4. 18 sq. units

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The area of the region bounded by the curve $x = y^2$, y-axis and the lines $y = 3$ and $y = 4$ is ....
2. The area of the region bounded by the curve $y = x^2 + x$, x-axis and the lines $x = 2$ and $x = 5$ is equal to ....
3. The area of the region bounded by the curve $xy = c$, x-axis and between the lines $x = 1$ and $x = 4$, is ....
4. The area of the region bounded by the curve $y = \sin x$, x-axis and between $x = 0$ and $x = 2\pi$ is ....
5. The area of the region bounded by the curve $y = \tan x$, x-axis and the lines $x = -\frac{\pi}{3}$ and $x = \frac{\pi}{3}$ is ....
6. If the area of the region bounded by the curve $y = a\sqrt{x} + bx$, x-axis and the lines $x = 0$ and $x = 4$ is 8 sq. units, then the value of $2a + 3b$ is ....
7. The area above x-axis, bounded by the curve $y = 2^{kx}$ and $x = 0$ and $x = 2$ is $3\log_2 e$, then $2^{2k} - 3k =$ ....
8. The area bounded by the curve $y^2 = x$, line $y = 4$ and y-axis is ....
9. The area bounded by the parabola $y^2 = 4ax$ and its latusrectum is ....
10. If the area bounded by $y = ax^2$ and $x = ay^2$, $a > 0$, is 1 sq. units, then $a =$ ....
11. The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the x-axis is .... [NCERT EXEMPLAR]
12. Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is .... [NCERT EXEMPLAR]
13. The area of the region bounded by the circle $x^2 + y^2 = 1$ is .... [NCERT EXEMPLAR]
14. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is.......... . [NCERT EXEMPLAR]
15. The area of the region bounded by the curve $y = x + 1$, x-axis and the lines $x = 2$ and $x = 3$ is................................ .

ANSWERS

1. $\frac{37}{3}$ sq.units 2. $\frac{297}{6}$ sq.units 3. $2c\log_e 2$ sq. units 4. 4 sq. units
5. $2\log 2$ sq. units 6. 3 7. 1 8. $\frac{64}{3}$ sq. units
9. $\frac{8a^2}{3}$ sq.units 10. $\frac{1}{\sqrt{3}}$ 11. 1 sq. unit 12. 2 sq. unit
13. π sq. units 14. 20π sq. units 15. $\frac{7}{2}$ sq. units