CHAPTER 27

HYPERBOLA

27.1 INTRODUCTION

We have discussed in earlier chapters that a hyperbola is the particular case of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$. The analytical definition of a hyperbola is as follows:

HYPERBOLA *A hyperbola is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (called directrix) is always constant which is always greater than unity.*

The constant ratio is generally denoted by e and is known as the *eccentricity* of the hyperbola.

If S is the focus, $Z\ Z'$ is the directrix and P is any point on the hyperbola, then by definition

$$\frac{SP}{PM} = e \Rightarrow SP = e \cdot PM$$

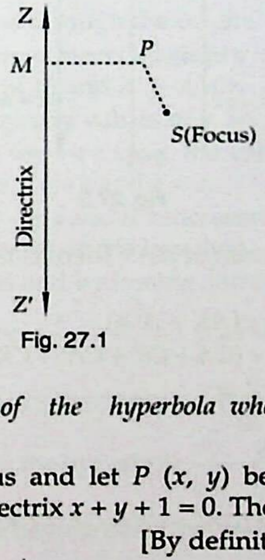

Fig. 27.1

ILLUSTRATION *Find the equation of the hyperbola whose focus is (1, 2), directrix the line $x + y + 1 = 0$ and eccentricity 3/2.*

SOLUTION Let $S(1, 2)$ be the focus and let $P\ (x, y)$ be a point on the hyperbola. Draw perpendicular PM from P on the directrix $x + y + 1 = 0$. Then,

$SP = e\ PM$ [By definition]

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2} = \frac{3}{2}\left|\frac{x+y+1}{\sqrt{1^2+1^2}}\right|$$

$$\Rightarrow (x-1)^2 + (y-2)^2 = \frac{9}{4}\left\{\frac{(x+y+1)^2}{2}\right\}$$

$$\Rightarrow 8\left\{(x-1)^2 + (y-2)^2\right\} = 9\,(x+y+1)^2$$

$$\Rightarrow 8x^2 + 8y^2 - 16x - 32y + 40 = 9x^2 + 9y^2 + 9 + 18xy + 18x + 18y$$

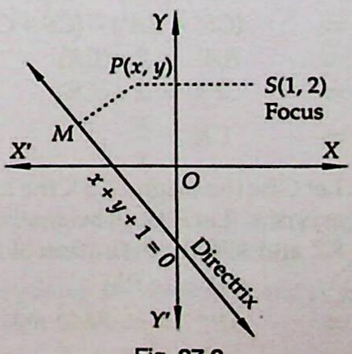

Fig. 27.2

$\Rightarrow \quad x^2 + y^2 + 18xy + 34x + 50y - 31 = 0$, which is the required equation of the hyperbola.

27.2 EQUATION OF THE HYPERBOLA IN STANDARD FORM

Let S be the focus, ZK be the directrix and e be the eccentricity of the hyperbola whose equation is required. Draw SK perpendicular from S on the directrix ZK and divide SK internally and externally at A and A' (on SK produced) respectively in the ratio $e : 1$. Then,

$$SA = e\,AK \quad \text{...(i)}$$

and, $$SA' = e\,A'K \quad \text{...(ii)}$$

Since A and A' are such points that their distances from the focus bear constant ratio e (> 1) to their respective distances from the directrix. Therefore, these points lie on the hyperbola.

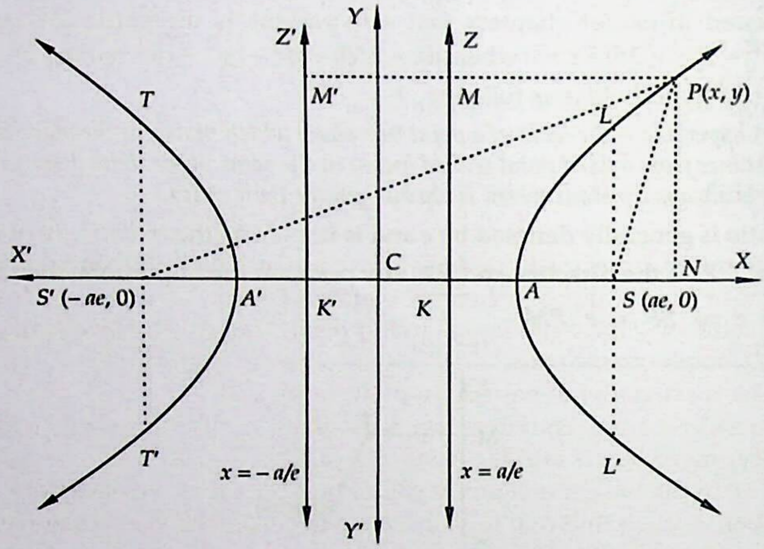

Fig. 27.3

Let $AA' = 2a$ and C be the middle point of AA'. Then, $CA = CA' = a$.

Adding (i) and (ii), we get

$$SA + SA' = e\,(AK + A'K)$$

$$\Rightarrow \quad CS - CA + CS + CA' = e\,(CA - CK + CA' + CK)$$

$$\Rightarrow \quad 2\,CS = 2\,ae$$

$$\Rightarrow \quad CS = ae$$

Subtracting (i) from (ii), we get

$$SA' - SA = e\,(A'K - AK)$$

$$\Rightarrow \quad (CS' + SA') - (CS + CA) = e\,(CA' + CK - CA + CK)$$

$$\Rightarrow \quad AA' = 2\,e\,(CK)$$

$$\Rightarrow \quad 2\,a = 2\,e\,(CK)$$

$$\Rightarrow \quad CK = \frac{a}{e}$$

Let C be the origin, CSX the axis of x and a straight line CY through C perpendicular to CX as the axis of Y. Let $P\,(x, y)$ be any point on the hyperbola and PM, PN be the perpendiculars from P on KZ and KX. By definition of hyperbola

$$SP = e\,PM$$

$$\Rightarrow \quad SP^2 = e^2\,PM^2$$

$$\Rightarrow \quad SP^2 = e^2\,KN^2$$

$$\Rightarrow \quad SP^2 = e^2 (CN - CK)^2$$

$$\Rightarrow \quad (x - ae)^2 + y^2 = e^2 \left(x - \frac{a}{e}\right)^2$$

$$\Rightarrow \quad x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2 (e^2 - 1)$$

This is the equation of the hyperbola in the standard form.

27.2.1 TRACING OF HYPERBOLA

The equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

$$\therefore \quad y = \pm \frac{b}{a} \sqrt{x^2 - a^2} \quad \text{...(ii)} \qquad \text{and, } x = \pm \frac{a}{b} \sqrt{y^2 + b^2} \quad \text{...(iii)}$$

In order to trace the graph of the hyperbola (i), we observe the following points:

(a) *Symmetry:* For every value of x there are equal and opposite values of y [sec (ii)]. Similarly, for every value of y there are equal and opposite values of x [See (iii)]. So, the curve is symmetric about both the axes.

(b) *Origin*: The curve does not pass through the origin.

(c) *Intersection with the axes :* The curve meets x-axis at $y = 0$. Putting $y = 0$ in (iii), we get $x = \pm a$. So ,the curve meets x-axis at $A\ (a, 0)$ and $A'\ (-a,\ 0)$.
Putting $x = 0$ in (ii), we get imaginary values of y. So ,the curve does not meet y-axis.

(d) *Region:* From (ii), we find that for $-a < x < a$, the values of y are imaginary. So, the curve does not exist between the lines $x = -a$ and $x = a$.
From (ii), we find that $y = 0$ at $x = \pm a$ and if x increases and is greater than a, the values of y also increase. Similarly, if decreases and is less than $-a$, y also increases.
With the help of the above facts and by joining some convenient points on the hyperbola the general shape of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is as shown in Fig. 27.3.

27.2.2 SECOND FOCUS AND SECOND DIRECTRIX OF THE HYPERBOLA

Similar to ellipse it can be shown that the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, b^2 = a^2 (e^2 - 1)$ has second focus $S'\ (-ae, 0)$ and second directrix $Z'K'$ having equation $x = -\frac{a}{e}$.

27.2.3 VARIOUS ELEMENTS OF HYPERBOLA

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have following points:

VERTICES *In Fig. 27.3, the points A and A', where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola. The coordinates of A and A' are (a, 0) and (– a, 0) respectively.*

TRANSVERSE AND CONJUGATE AXES *In Fig. 27.3, the straight line joining the vertices A and A' is called the transverse axis of the hyperbola. Its length AA' is generally taken to be 2a.*

The straight line through the centre which is perpendicular to the transverse axis does not meet the hyperbola in real points. But if B, B' be the points on this line such that $CB = CB' = b$, the line BB' is called the conjugate axis such that $BB' = 2b$.

FOCI *In Fig. 27.3, the points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola.*

DIRECTRICES *In Fig. 27.3, ZK and Z′K′ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively.*

CENTRE *In Fig. 27.3, the middle point C of AA′ bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.*

27.2.4 ECCENTRICITY

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$b^2 = a^2(e^2 - 1)$$

$$\Rightarrow \quad e^2 = \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2}$$

$$\Rightarrow \quad e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{(2b)^2}{(2a)^2}} \Rightarrow e = \sqrt{1 + \frac{(\text{conjugate axis})^2}{(\text{transverse axis})^2}}$$

27.2.5 LENGTH OF THE LATUS-RECTUM

In Fig. 27.3, LSL' is the latus-rectum and LS is called the semi latus-rectum. $TS'T'$ is also a latus-rectum.

The coordinates of L are (ae, SL). As L lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the coordinates of L will satisfy the equation of the hyperbola.

$$\therefore \quad \frac{(ae)^2}{a^2} - \frac{(SL)^2}{b^2} = 1$$

$$\Rightarrow \quad (SL)^2 = b^2(e^2 - 1)$$

$$\Rightarrow \quad (SL)^2 = b^2\left(\frac{b^2}{a^2}\right) \qquad \left[\because b^2 = a^2(e^2 - 1) \Rightarrow e^2 - 1 = \frac{b^2}{a^2}\right]$$

$$\Rightarrow \quad SL = \frac{b^2}{a}$$

$$\therefore \quad SL = SL' = \frac{b^2}{a}$$

Hence, length of the latus-rectum $= 2(SL) = \frac{2b^2}{a} = 2a(e^2 - 1)$.

27.2.6 FOCAL DISTANCES OF A POINT

The distances of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ from its foci are known as the focal distances of that point.

THEOREM *The difference of the focal distances of any point on a hyperbola is constant and equal to the length of the transverse axis of the hyperbola.*

PROOF Let $P\ (x,\ y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (See Fig. 27.3). Then, by definition, we have

$$SP = ePM \text{ and } S'P = e\,PM'.$$

Now, $SP = e\,PM \;\Rightarrow\; SP = e\,(NK) = e\,(CN - CK) = e\left(x - \frac{a}{e}\right) = ex - a.$

and, $S'P = e\,PM' \Rightarrow S'P = e\,(NK') = e\,(CN + CK') = e\left(x + \frac{a}{e}\right) = e\,x + a$

$\therefore \quad S'P - SP = (ex + a) - (ex - a) = 2\,a =$ Transverse axis.

Hence, the difference of the focal distances of a point on the hyperbola is constant and is equal to the length of the transverse axis of the hyperbola. **Q.E.D**

On account of this property, a second definition of the hyperbola may be given as follows:

A hyperbola is the locus of a point which moves in such a way that the difference of its distances from two fixed points (foci) is always constant.

27.2.7 CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Its shape is as shown in Fig. 27.4.

The eccentricity of the conjugate hyperbola is given by $a^2 = b^2(e^2 - 1)$ and the length of the latus-rectum is $\frac{2\,a^2}{b}$.

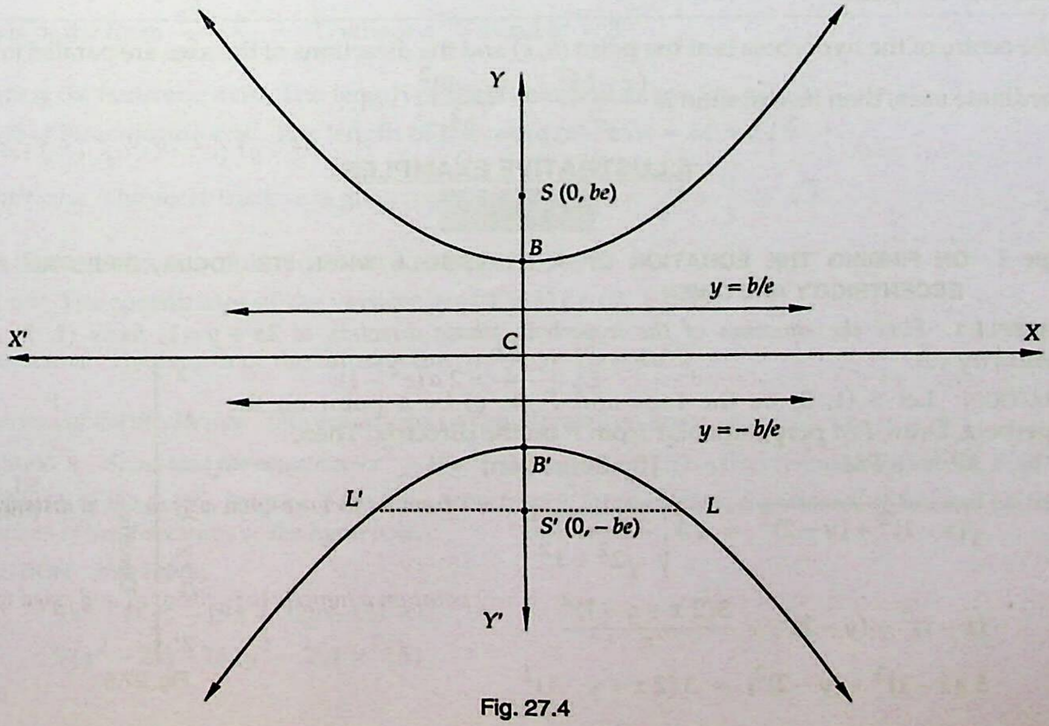

Fig. 27.4

Various results related to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given in the following table for ready reference.

	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	*Conjugate hyperbola* $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordiantes of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	$2a$	$2b$
Length of the conjugate axis	$2b$	$2a$
Equations of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$ or, $b^2 = a^2(e^2 - 1)$	$e = \sqrt{\frac{b^2 + a^2}{b^2}}$ or, $a^2 = b^2(e^2 - 1)$
Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$
Focal distances	$ex \pm a$	$ey \pm b$
Difference of the focal distances of a point	$2a$	$2b$

If the centre of the hyperbola is at the point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF A HYPERBOLA WHEN ITS FOCUS, DIRECTRIX AND ECCENTRICITY ARE GIVEN**

EXAMPLE 1 *Find the equation of the hyperbola whose directrix is* $2x + y = 1$, *focus* $(1, 2)$ *and eccentricity* $\sqrt{3}$.

SOLUTION Let $S\ (1, 2)$ be the fous and $P\ (x, y)$ be a point on the hyperbola. Draw PM perpendicular from P on the directrix. Then,

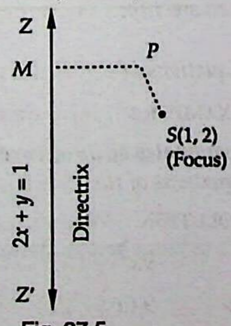

Fig. 27.5

$$SP = e\,PM \quad \text{[By definition]}$$

$$\Rightarrow \quad \sqrt{(x-1)^2 + (y-2)^2} = \sqrt{3}\left|\frac{2x + y - 1}{\sqrt{2^2 + 1^2}}\right|$$

$$\Rightarrow \quad (x-1)^2 + (y-2)^2 = \frac{3(2x + y - 1)^2}{5}$$

$$\Rightarrow \quad 5\{(x-1)^2 + (y-2)^2\} = 3(2x + y - 1)^2$$

$\Rightarrow \quad 5x^2 + 5y^2 - 10x - 20y + 25 = 3(4x^2 + y^2 + 1 + 4xy - 4x - 2y)$

$\Rightarrow \quad 7x^2 - 2y^2 + 12xy - 2x + 14y - 22 = 0$, which is the required equation of the hyperbola.

Type II **ON FINDING THE CENTRE, LENGTHS OF TRANSVERSE AND CONJUGATE AXES, ECCENTRICITY, FOCI, VERTICES, LATUS-RECTUM, DIRECTRICES etc. OF A GIVEN HYPERBOLA**

EXAMPLE 2 *For the following hyperbolas find the lengths of transverse and conjugate axes, eccentricity and coordinates of foci and vertices; length of the latus-rectum, equations of the directrices:*

(i) $16x^2 - 9y^2 = 144$ (ii) $3x^2 - 6y^2 = -18$

SOLUTION (i) The equation $16x^2 - 9y^2 = 144$ can be written as $\frac{x^2}{9} - \frac{y^2}{16} = 1$.

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 9$ and $b^2 = 16$.

Length of the transverse axis: The length of the transverse axis $= 2a = 6$

Length of the conjugate axis: The length of the conjugate axis $= 2b = 8$

Eccentricity: The eccentricity e is given by $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

Foci: The coordinates of the foci are $(\pm ae, 0)$ *i.e.* $(\pm 5, 0)$

Vertices: The coordinates of the vertices are $(\pm a, 0)$ *i.e.* $(\pm 3, 0)$.

Latus-rectum: The length of the latus-rectum $= \frac{2b^2}{a} = \frac{32}{3}$

Equations of the directrices: The equations of the directrices are $x = \pm \frac{a}{e}$ *i.e.* $x = \pm \frac{9}{5}$.

(ii) The equation $3x^2 - 6y^2 = -18$ can be written as $\frac{x^2}{6} - \frac{y^2}{3} = -1$.

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, where $a^2 = 6$ and $b^2 = 3$.

Length of the transverse axis: The length of the transverse axis $= 2b = 2\sqrt{3}$.

Length of the conjugate axis: The length of the conjugate axis $= 2a = 2\sqrt{6}$.

Eccentricity: The eccentricity e is given by $e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{6}{3}} = \sqrt{3}$

Foci: The coordinates of the foci are $(0, \pm be)$ i.e. $(0, \pm 3)$

Vertices: The coordinates of the vertices are $(0, \pm b)$ *i.e.* $(0, \pm \sqrt{3})$

Latusrectum: The length of the latusrectum $= \frac{2a^2}{b} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$.

Equations of the directrices: The equations of the directrices are $y = \pm b/e$ *i.e.* $y = \pm 1$

EXAMPLE 3 *Show that the equation* $9x^2 - 16y^2 - 18x + 32y - 151 = 0$ *represents a hyperbola. Find the coordinates of the centre, lengths of the axes, eccentricity, latus-rectum, coordinates of foci and vertices, equations of the directrices of the hyperbola.*

SOLUTION We have,

$$9x^2 - 16y^2 - 18x + 32y - 151 = 0$$

$$\Rightarrow \quad 9(x^2 - 2x) - 16(y^2 - 2y) = 151$$

$$\Rightarrow \quad 9(x^2-2x+1)-16(y^2-2y+1)=151+9-16$$

$$\Rightarrow \quad 9(x-1)^2-16(y-1)^2 = 144$$

$$\Rightarrow \quad \frac{(x-1)^2}{16}-\frac{(y-1)^2}{9} = 1 \qquad \text{...(i)}$$

Shifting the origin at (1, 1) without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X+1 \quad \text{and} \quad y = Y+1 \qquad \text{...(ii)}$$

Using these relations, equations (i) reduces to

$$\frac{X^2}{16}-\frac{Y^2}{9} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$, where $a^2=16$ and $b^2=9$.

Centre: The coordinates of the centre with respect to the new axes are $(X=0, Y=0)$.

So, the coordinates of the centre with respect to the old axes are

$(1, 1)$ [Putting $X=0, Y=0$ in (ii)]

Transverse axis: Length of the transverse axis $= 2a = 8$

Conjugate axis: Length of the conjugate axis $= 2b = 6$

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\frac{9}{16}} = \frac{5}{4}$$

Latusrectum: Length of the latusrectum $=\frac{2b^2}{a}=\frac{18}{4}=\frac{9}{2}$.

Foci: The coordinates of foci with respect to the new axes are $(X=\pm ae, Y=0)$ i.e. $(X=\pm 5,\ Y=0)$. So, the coordinates of foci with respect to the old axes are

$(1\pm 5, 1)$ *i.e.* $(6, 1)$ and $(-4, 1)$ [Putting $X=\pm 5,\ Y=0$ in (ii)]

Vertices: The coordinates of the vertices with respect to the new axes are $(X=\pm a,\ Y=0)$ i.e. $(X=\pm 4,\ Y=0)$. So, the coordinates of the vertices with respect to the old axes are

$(\pm 4+1, 1)$ i.e., $(5, 1)$ and $(-3, 1)$ [Putting $X=\pm 4,\ Y=0$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are $X=\pm\frac{a}{e}$ i.e. $X=\pm\frac{16}{5}$.

So, the equations of the directrices with respect to the old axes are

$$x = \pm\frac{16}{5}+1 \qquad \left[\text{Putting } X=\pm\frac{16}{5} \text{ in (ii)}\right]$$

or, $\quad x = \frac{21}{5}$ and $x = -\frac{11}{5}$

EXAMPLE 4 *Show that the equation $x^2-2y^2-2x+8y-1=0$ represents a hyperbola. Find the coordinates of the centre, lengths of the axes, eccentricity, latusrectum, coordinates of foci and vertices and equations of directrices of the hyperbola.*

SOLUTION We have,

$$x^2-2y^2-2x+8y-1 = 0 \Rightarrow (x^2-2x)-2(y^2-4y) = 1$$

$$\Rightarrow \quad (x^2-2x+1)-2(y^2-4y+4) = -6$$

$$\Rightarrow \quad (x-1)^2-2(y-2)^2 = -6$$

$$\Rightarrow \quad \frac{(x-1)^2}{(\sqrt{6})^2} - \frac{(y-2)^2}{(\sqrt{3})} = -1 \qquad \text{...(i)}$$

Shifting the origin at (1, 2) without rotating the coordinate axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X+1 \text{ and } y = Y+2 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{(\sqrt{6})^2} - \frac{Y^2}{(\sqrt{3})^2} = -1 \qquad \text{...(iii)}$$

This equation is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = -1$, where $a^2 = (\sqrt{6})^2$ and $b^2 = (\sqrt{3})^2$.

Centre: The coordinates of the centre with respect to the new axes are ($X = 0$, $Y = 0$). So, the coordinates of the centre with respect to the old axes are

$(1, 2)$ [Putting $X = 0, Y = 0$ in (ii)]

Lengths of the axes: Since the transverse axis of the hyperbola is along new Y-axis.

$\therefore$ Transverse axis $= 2b = 2\sqrt{3}$ and, Conjugate axis $= 2a = 2\sqrt{6}$.

Eccentricity: The eccentricity e is a given by

$$e = \sqrt{1+\frac{a^2}{b^2}} = \sqrt{1+\frac{6}{3}} = \sqrt{3}$$

Latusrectum: Length of the latusrectum $= \frac{2a^2}{b} = \frac{12}{\sqrt{3}} = 4\sqrt{3}$

Foci: The coordinates of foci with respect to the new axes are ($X = 0$, $Y = \pm be$) i.e. ($X = 0$, $Y = \pm 3$). So, the coordinates of foci with respect to the old axes are

$(1, 2 \pm 3)$ *i.e.* $(1, 5)$ and $(1, -1)$ [Putting $X = 0, Y = \pm 3$ in (ii)]

Vertices: The coordinates of the vertices with respect to the new axes are $X = 0, Y = \pm b$) i.e. ($X = 0, Y = \pm\sqrt{3}$)

So, the coordinates of the vertices with respect to the old axes are

$(1, 2 \pm \sqrt{3})$ i.e. $(1, 2+\sqrt{3})$ and $(1, 2-\sqrt{3})$ [Putting $X = 0, Y = \pm\sqrt{3}$ in (ii)]

Directrices: The equations of the directrices with respect to the new axes are $Y = \pm b/e$ i.e. $Y = \pm 1$. So, the equations of the directrices with respect to the old axes are

$y = 2 \pm 1$ i.e. $y = 1$ and $y = 3$ [Putting $Y = \pm 2$ in (ii)]

Type III **ON FINDING THE EQUATION OF A HYPERBOLA WHEN SOME OF ITS PARTS ARE GIVEN**

EXAMPLE 5 *Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:*

(i) *Vertices at* $(\pm 5, 0)$, *Foci at* $(\pm 7, 0)$ (ii) *Vertices at* $(0, \pm 7)$, $e = \frac{4}{3}$

SOLUTION (i) Since the vertices lie on x-axis, so let the equation of the required hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

The coordinates of its vertices and foci are $(\pm a, 0)$ and $(\pm ae, 0)$ respectively. But, the coordinates of vertices and foci are given as $(\pm 5, 0)$ and $(\pm 7, 0)$ respectively.

$$\therefore \quad a = 5 \text{ and } ae = 7 \Rightarrow e = \frac{7}{5}$$

Now, $b^2 = a^2(e^2-1) \Rightarrow b^2 = 25\left(\frac{49}{25}-1\right) = 24.$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{25} - \frac{y^2}{24} = 1$ as the equation of the required hyperbola.

(ii) Since the vertices of the required hyperbola lie on y-axis. So, let its equation be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

The coordinates of vertices of this hyperbola are $(0, \pm b)$ and the coordinates of vertices are given as $(\pm 7, 0)$. So, $b = 7$.

Now, $a^2 = b^2(e^2 - 1) \Rightarrow a^2 = 49\left(\frac{16}{9} - 1\right) \Rightarrow a^2 = 49 \times \frac{7}{9} = \frac{343}{9}$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{9x^2}{343} - \frac{y^2}{49} = -1$ as the equation of the desired hyperbola.

EXAMPLE 6 *Referred to the principal axes as the axes of coordinates find the equation of the hyperbola whose foci are at $(0, \pm\sqrt{10})$ and which passes through the point $(2, 3)$.*

SOLUTION Since the vertices are on y-axis, so let the equation of the required hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

It passes through $(2, 3)$.

$$\therefore \quad \frac{4}{a^2} - \frac{9}{b^2} = -1$$

$$\Rightarrow \quad \frac{4}{b^2(e^2 - 1)} - \frac{9}{b^2} = -1 \quad [\because a^2 = b^2(e^2 - 1)]$$

$$\Rightarrow \quad \frac{4}{b^2 e^2 - b^2} - \frac{9}{b^2} = -1 \quad \text{...(ii)}$$

The coordinates of foci are given to be $(0, \pm\sqrt{10})$.

$$\therefore \quad be = \sqrt{10} \Rightarrow b^2 e^2 = 10 \quad \text{...(iii)}$$

From (ii) and (iii), we get

$$\frac{4}{10 - b^2} - \frac{9}{b^2} = -1$$

$$\Rightarrow \quad 4b^2 - 9(10 - b^2) = -b^2(10 - b^2)$$

$$\Rightarrow \quad 13b^2 - 90 = -10b^2 + b^4$$

$$\Rightarrow \quad b^4 - 23b^2 + 90 = 0 \Rightarrow (b^2 - 18)(b^2 - 5) = 0 \Rightarrow b^2 = 18 \text{ or, } b^2 = 5.$$

Now, $a^2 = b^2(e^2 - 1) \Rightarrow a^2 = (be)^2 - b^2 \Rightarrow a^2 = 10 - b^2$ $\quad [\because be = \sqrt{10}]$

If $b^2 = 18$, then $a^2 = 10 - b^2 \Rightarrow a^2 = 10 - 18 = -8$, which is not possible.

$\therefore \quad b^2 = 5$ and hence $a^2 = 10 - b^2 \Rightarrow a^2 = 10 - 5 = 5$.

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{5} - \frac{y^2}{5} = -1$ i.e. $x^2 - y^2 = -5$ as the equation of the required hyperbola.

EXAMPLE 7 *Find the equation of the hyperbola, the length of whose latusrectum is 8 and eccentricity is $3/\sqrt{5}$.*

SOLUTION Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

The length of its latusrectum is $\frac{2b^2}{a}$. It is given that the length of its latusrectum is 8.

$$\therefore \quad \frac{2b^2}{a} = 8$$

$$\Rightarrow \quad b^2 = 4a \Rightarrow a^2(e^2-1) = 4a \qquad \left[\because b^2 = a^2(e^2-1)\right]$$

$$\Rightarrow \quad a(e^2-1) = 4 \Rightarrow a\left(\frac{9}{5}-1\right) = 4 \Rightarrow a = 5$$

Putting $a = 5$ in $b^2 = 4a$, we get $b^2 = 20$.

Substituting the values of a and b in (i), we obtain $\frac{x^2}{25} - \frac{y^2}{20} = 1$ as the required equation of the hyperbola

EXAMPLE 8 *The foci of a hyperbola coincide with the foci of the ellipse* $\frac{x^2}{25} + \frac{y^2}{9} = 1$. *Find the equation of the hyperbola, if its eccentricity is 2.*

SOLUTION The equation of the ellipse is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a^2 = 25$ and $b^2 = 9$.

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

So, the coordinates of foci are $(\pm ae, 0)$ i.e. $(\pm 4, 0)$.

It is given that the foci of the hyperbola coincide with the foci of the ellipse. So, the coordinates of foci of the hyperbola are $(\pm 4, 0)$.

Let e' be the eccentricity of the required hyperbola and its equation be

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1 \qquad \text{...(i)}$$

The coordinates of its foci are $(\pm a'e', 0)$.

$$\therefore \quad a'e' = 4 \Rightarrow 2a' = 4 \Rightarrow a' = 2 \qquad [\because e = 2]$$

Also, $b'^2 = a'^2(e'^2 - 1) \Rightarrow b'^2 = 4(4-1) = 12$.

Substituting the values of a' and b' in (i), we obtain $\frac{x^2}{4} - \frac{y^2}{12} = 1$ as the equation of the required hyperbola.

EXAMPLE 9 *Find the equation of the hyperbola whose conjugate axis is 5 and the distance between the foci is 13.*

SOLUTION Let $2a$ and $2b$ be the transverse and conjugate axes and e be the eccentricity. Let the centre be the origin and the transverse and the conjugate axes the coordinate axes. Then, the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{...(i)}$$

We have, $2b = 5$ and $2ae = 13$.

Now, $b^2 = a^2(e^2 - 1)$

$$\Rightarrow \quad b^2 = a^2e^2 - a^2$$

$\Rightarrow \quad \dfrac{25}{4} = \dfrac{169}{4} - a^2 \Rightarrow a^2 = \dfrac{144}{4} \Rightarrow a = 6.$

Substituting the values of a and b in (i), the equation of the hyperbola is

$$\frac{x^2}{36} - \frac{y^2}{25/4} = 1 \Rightarrow 25x^2 - 144y^2 = 900.$$

EXAMPLE 10 *Find the equation of the hyperbola whose foci are (8, 3) and (0, 3) and eccentricity* $= \dfrac{4}{3}$.

SOLUTION The centre of the hyperbola is the mid-point of the line joining the two foci. So, the coordinates of the centre are $\left(\dfrac{8+0}{2}, \dfrac{3+3}{2}\right)$ i.e. $(4, 3)$.

Let $2a$ and $2b$ be the length of transverse and conjugate axes and let e be the eccentricity. Then, the equation of the hyperbola is

$$\frac{(x-4)^2}{a^2} - \frac{(y-3)^2}{b^2} = 1 \qquad \text{...(i)}$$

The coordinates of two foci are (8, 3) and (0, 3).

$\therefore$ Distance between two foci $= \sqrt{(8-0)^2 + (3-3)^2} = 8.$

But, the distance between the two foci is equal to $2ae$.

$\therefore \quad 2ae = 8 \Rightarrow ae = 4 \Rightarrow \dfrac{4a}{3} = 4 \Rightarrow a = 3 \qquad \left[\because e = \dfrac{4}{3}\right]$

Now, $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 9\left(-1 + \dfrac{16}{9}\right) = 7$

Thus, the equation of the hyperbola is

$$\frac{(x-4)^2}{9} - \frac{(y-3)^2}{7} = 1 \qquad \text{[Putting the values of } a \text{ and } b \text{ in (i)]}$$

or, $\quad 7x^2 - 9y^2 - 56x + 54y - 32 = 0.$

LEVEL-2

Type IV MISCELLANEOUS PROBLEMS ON HYPERBOLA

EXAMPLE 11 *If e and e' be the eccentricities of a hyperbola and its conjugate, prove that* $\dfrac{1}{e^2} + \dfrac{1}{e'^2} = 1.$

SOLUTION Let the equation of the hyperbola be $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \qquad$ ...(i)

Then, the equation of the hyperbola conjugate to (i) is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1 \qquad$ ...(ii)

Now, $\quad e = \text{Eccentricity of (i)} = \sqrt{1 + \left(\dfrac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

$\Rightarrow \quad e = \sqrt{1 + \left(\dfrac{2b}{2a}\right)^2}$

$\Rightarrow \quad e^2 = 1 + \dfrac{b^2}{a^2} \Rightarrow e^2 = \dfrac{a^2 + b^2}{a^2} \qquad$ ...(iii)

and, $\quad e' = \text{Eccentricity of (ii)} = \sqrt{1 + \left(\dfrac{\text{Conjugate axis}}{\text{Transverse axis}}\right)^2}$

$$\Rightarrow \quad e' = \sqrt{1 + \left(\frac{2a}{2b}\right)^2}$$

$$\Rightarrow \quad e'^2 = 1 + \frac{a^2}{b^2} \Rightarrow e'^2 = \frac{a^2 + b^2}{b^2} \qquad \text{...(iv)}$$

From (iii) and (iv), we have

$$\frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = \frac{a^2 + b^2}{a^2 + b^2} \Rightarrow \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

EXAMPLE 12 *Find the locus of the point of intersection of the lines* $\sqrt{3}x - y - 4\sqrt{3}\lambda = 0$ *and* $\sqrt{3}\lambda x + \lambda y - 4\sqrt{3} = 0$ *for different values of* λ.

SOLUTION Let (h, k) be the point of intersection of the given lines. Then,

$$\sqrt{3}h - k - 4\sqrt{3}\lambda = 0 \text{ and } \sqrt{3}\lambda h + \lambda k - 4\sqrt{3} = 0$$

$$\Rightarrow \quad \sqrt{3}h - k = 4\sqrt{3}\lambda \text{ and } \lambda(\sqrt{3}h + k) = 4\sqrt{3}$$

$$\Rightarrow \quad (\sqrt{3}h - k)\,\lambda(\sqrt{3}h + k) = (4\sqrt{3}\lambda)(4\sqrt{3})$$

$$\Rightarrow \quad 3h^2 - k^2 = 48$$

Hence, the locus of (h, k) is $3x^2 - y^2 = 48$.

EXERCISE 27.1

LEVEL-1

1. The equation of the directrix of a hyperbola is $x - y + 3 = 0$. Its focus is $(-1, 1)$ and eccentricity 3. Find the equation of the hyperbola.
2. Find the equation of the hyperbola whose
   (i) focus is $(0, 3)$, directrix is $x + y - 1 = 0$ and eccentricity $= 2$
   (ii) focus is $(1, 1)$, directrix is $3x + 4y + 8 = 0$ and eccentricity $= 2$
   (iii) focus is $(1, 1)$ directrix is $2x + y = 1$ and eccentricity $= \sqrt{3}$
   (iv) focus is $(2, -1)$, directrix is $2x + 3y = 1$ and eccentricity $= 2$
   (v) focus is $(a, 0)$, directrix is $2x - y + a = 0$ and eccentricity $= \frac{4}{3}$
   (vi) focus is $(2, 2)$, directrix is $x + y = 9$ and eccentricity $= 2$.
3. Find the eccentricity, coordinates of the foci, equations of directrices and length of the latus-rectum of the hyperbola
   (i) $9x^2 - 16y^2 = 144$ [NCERT EXEMPLAR] (ii) $16x^2 - 9y^2 = -144$
   (iii) $4x^2 - 3y^2 = 36$ (iv) $3x^2 - y^2 = 4$
   (v) $2x^2 - 3y^2 = 5$.
4. Find the axes, eccentricity, latus-rectum and the coordinates of the foci of the hyperbola $25x^2 - 36y^2 = 225$.
5. Find the centre, eccentricity, foci and directrices of the hyperbola
   (i) $16x^2 - 9y^2 + 32x + 36y - 164 = 0$ (ii) $x^2 - y^2 + 4x = 0$
   (iii) $x^2 - 3y^2 - 2x = 8$.
6. Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:
   (i) the distance between the foci $= 16$ and eccentricity $= \sqrt{2}$
   (ii) conjugate axis is 5 and the distance between foci $= 13$
   (iii) conjugate axis is 7 and passes through the point $(3, -2)$.

7. Find the equation of the hyperbola whose
   (i) foci are (6, 4) and (– 4, 4) and eccentricity is 2.
   (ii) vertices are (– 8, – 1) and (16, – 1) and focus is (17, – 1)
   (iii) foci are (4, 2) and (8, 2) and eccentricity is 2.
   (iv) vertices are at (0 ± 7) and foci at $\left(0, \pm \frac{28}{3}\right)$.
   (v) vertices are at $(\pm 6, 0)$ and one of the directrices is $x = 4$. [NCERT EXEMPLAR]
   (vi) foci at $(\pm 2, 0)$ and eccentricity is $3/2$. [NCERT EXEMPLAR]
8. Find the eccentricity of the hyperbola, the length of whose conjugate axis is $\frac{3}{4}$ of the length of transverse axis.
9. Find the equation of the hyperboala whose
   (i) focus is at (5, 2), vertex at (4, 2)and centre at (3, 2)
   (ii) focus is at (4, 2), centre at (6, 2) and $e = 2$.
10. If P is any point on the hyperbola whose axis are equal, prove that $SP \cdot S'P = CP^2$.
11. In each of the following find the equations of the hyperbola satisfying the given conditions:
   (i) vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$ [NCERT]
   (ii) vertices $(0, \pm 5)$, foci $(0, \pm 8)$ [NCERT]
   (iii) vertices $(0, \pm 3)$, foci $(0, \pm 5)$ [NCERT]
   (iv) foci $(\pm 5, 0)$, transverse axis = 8 [NCERT]
   (v) foci $(0, \pm 13)$, conjugate axis = 24 [NCERT]
   (vi) foci $(\pm 3\sqrt{5}, 0)$, the latus-rectum = 8 [NCERT]
   (vii) foci $(\pm 4, 0)$, the latus-rectum = 12 [NCERT]
   (viii) vertices $(0, \pm 6)$, $e = \frac{5}{3}$ [NCERT EXEMPLAR]
   (ix) foci $(0, \pm \sqrt{10})$, passing through (2, 3) [NCERT]
   (x) foci $(0, \pm 12)$, latus-rectum = 36 [NCERT]
12. If the distance between the foci of a hyperbola is 16 and its ecentricity is $\sqrt{2}$, then obtain its equation. [NCERT EXEMPLAR]
13. Show that the set of all points such that the difference of their distances from (4, 0) and (– 4, 0) is always equal to 2 represents a hyperbola. [NCERT EXEMPLAR]

ANSWERS

1. $7(x^2 + y^2) - 18xy + 50x - 50y + 77 = 0$.
2. (i) $x^2 + y^2 + 4xy - 4x + 2y - 7 = 0$
   (ii) $11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$
   (iii) $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$
   (iv) $3x^2 + 23y^2 + 48xy + 36x - 50y - 61 = 0$
   (v) $19x^2 - 64xy - 29y^2 + 154ax - 32ay - 29a^2 = 0$
   (vi) $x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$.
3.

	Eccentricity	Foci	Directrices	L.R.
(i)	$\frac{5}{4}$	$(\pm 5, 0)$	$5x \mp 16 = 0$	$\frac{9}{2}$
(ii)	$\frac{5}{4}$	$(0, \pm 5)$	$5y \mp 16 = 0$	$\frac{9}{2}$

(iii)	$\frac{\sqrt{13}}{3}$	$(\pm\sqrt{13}, 0)$	$\sqrt{13}\,x \mp 3\sqrt{3} = 0$	$\frac{8}{\sqrt{3}}$
(iv)	2	$\left(\pm\frac{4}{\sqrt{3}}, 0\right)$	$\sqrt{3}\,x \mp 1 = 0$	$4\sqrt{3}$
(v)	$\sqrt{\frac{5}{3}}$	$\left(\pm\frac{5}{\sqrt{6}}, 0\right)$	$\sqrt{2}\,x \mp \sqrt{3} = 0$	$\frac{10}{3}\sqrt{\frac{2}{5}}$

4. Transverse axis = 6, conjugate axis = 5, $e = \frac{\sqrt{61}}{6}$, L.R. $= \frac{25}{6}$, foci $\left(\pm\frac{\sqrt{61}}{2}, 0\right)$.

5.	*Centre*	*Eccentricity*	*Foci*	*Directrices*
(i)	$(-1, 2)$	$\frac{5}{3}$	$(4, 2), (-6, 2)$	$5x = 4, 5x + 14 = 0$
(ii)	$(-2, 0)$	$\sqrt{2}$	$(-2 \pm 2\sqrt{2}, 0)$	$x + 2 = \pm\sqrt{2}$
(iii)	$(1, 0)$	$\frac{2\sqrt{3}}{3}$	$(1 \pm 2\sqrt{3}, 0)$	$x = 1 \pm \frac{9}{2\sqrt{3}}$

6. (i) $x^2 - y^2 = 32$ (ii) $25x^2 - 144y^2 = 900$ (iii) $65x^2 - 36y^2 = 441$.

7. (i) $12x^2 - 4y^2 - 24x + 32y - 127 = 0$ (ii) $25x^2 - 144y^2 - 200x - 288y - 3344 = 0$

(iii) $3x^2 - y^2 - 36x + 4y + 101 = 0$ (iv) $\frac{9x^2}{343} - \frac{y^2}{49} = -1$

(v) $\frac{x^2}{36} - \frac{y^2}{45} = 1$ (vi) $\frac{x^2}{4} - \frac{y^2}{5} = \frac{4}{9}$

8. 5/4 9. (i) $3(x-3)^2 - (y-2)^2 = 3$ (ii) $3(x-6)^2 - (y-2)^2 = 3$

11. (i) $\frac{x^2}{4} - \frac{y^2}{5} = 1$ (ii) $\frac{x^2}{39} - \frac{y^2}{25} = -1$ (iii) $\frac{x^2}{16} - \frac{y^2}{9} = -1$ (iv) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

(v) $\frac{x^2}{144} - \frac{y^2}{25} = -1$ (vi) $\frac{x^2}{25} - \frac{y^2}{20} = 1$ (vii) $\frac{x^2}{4} - \frac{y^2}{12} = 1$ (viii) $\frac{x^2}{49} - \frac{9y^2}{343} = 1$

(ix) $\frac{x^2}{5} - \frac{y^2}{5} = -1$ (x) $3y^2 - x^2 = 108$ 12. $x^2 - y^2 = 32$.

HINTS TO NCERT & SELECTED PROBLEMS

11. (i) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

It is given that its vertices are at $(\pm 2, 0)$ and foci are at $(\pm 3, 0)$.

$\therefore \quad a = 2$ and $ae = 3$

Now,

$$b^2 = a^2(e^2 - 1) \Rightarrow b^2 = (ae)^2 - a^2 = 9 - 4 = 5.$$

Hence, the equation of the hyperbola is

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

[Substituting $a = 2, b^2 = 5$ in (i)]

(ii) Let the equation of the hyperbola be

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Its vertices are at $(0, \pm 5)$ and foci are at $(0, \pm 8)$.

$\therefore \quad b = 5$ and $be = 8$.

Now,

$$a^2 = b^2(e^2-1) \Rightarrow a^2 = (be)^2 - b^2 = 64 - 25 = 39$$

Substituting the values of a and b in (i), we get

$-\dfrac{x^2}{39} + \dfrac{y^2}{25} = 1$ as the equation of the hyperbola.

(iii) Proceed as in (ii)

(iv) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Its foci are at $(\pm 5, 0)$ and transverse axis is 8.

$\therefore \quad ae = 5$ and $2a = 8$

$\Rightarrow \quad a^2e^2 = 25$ and $a = 4$

$\Rightarrow \quad a^2 + b^2 = 25$ and $a = 4$ $\quad \left[\because b^2 = a^2(e^2-1) \Rightarrow a^2e^2 = a^2 + b^2\right]$

$\Rightarrow \quad a = 4, b = 3$

Substituting the values of a and b in (i), we obtain that the equation of the hyperbola is $\dfrac{x^2}{16} - \dfrac{y^2}{9} = 1$.

(v) The foci of the given hyperbola are on y-axis. So, let its equation be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

Its foci are at $(0, \pm 13)$ and conjugate axis is 24.

$\therefore \quad be = 13$ and $2a = 24$

Now, $a^2 = b^2(e^2-1) \Rightarrow a^2 = (be)^2 - b^2 \Rightarrow 144 = 169 - b^2 \Rightarrow b^2 = 25$

Substituting $a^2 = 144$ and $b^2 = 25$ in (i), we obtain $\dfrac{x^2}{144} - \dfrac{y^2}{25} = -1$ as the equation of the hyperbola.

(vi) Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

It is given that its foci are at $(\pm 3\sqrt{5}, 0)$ and latus-rectum = 8.

$\therefore \quad ae = 3\sqrt{5}$ and $\dfrac{2b^2}{a} = 8$

Now, $\dfrac{2b^2}{a} = 8$

$\Rightarrow \quad b^2 = 4a$

$\Rightarrow \quad a^2(e^2-1) = 4a$

$\Rightarrow \quad 45 - a^2 = 4a \Rightarrow a^2 + 4a - 45 = 0 \Rightarrow (a+9)(a-5) = 0 \Rightarrow a = 5$

Again, $\dfrac{2b^2}{a} = 8 \Rightarrow b^2 = 4a \Rightarrow b^2 = 20$ $\quad [\because a = 5]$

Substituting the values of a and b in (i), we obtain $\dfrac{x^2}{25} - \dfrac{y^2}{20} = 1$ as the equation of the hyperbola.

(vii) Proceed as in (vi)

(viii) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

Its vertices are at $(0, \pm 6)$ and $e=\frac{5}{3}$.

$\therefore \quad b=6$ and $e = \frac{5}{3}$

Now, $a^2=b^2(e^2-1) \Rightarrow a^2 = 36\left(\frac{25}{9}-1\right)=64$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{64}-\frac{9y^2}{36}=-1$ as the equation of hyperbola.

(ix) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1 \quad \text{...(i)}$$

Its foci are at $(0, \pm\sqrt{10})$ and passes through $(2, 3)$.

$\therefore \quad be=\sqrt{10}$ and $\frac{4}{a^2}-\frac{9}{b^2}=-1$

$\Rightarrow \quad b^2e^2 = 10$ and $\frac{4}{b^2(e^2-1)}-\frac{9}{b^2} = -1$ $\quad \left[\because a^2=b^2(e^2-1)\right]$

$\Rightarrow \quad \frac{4}{10-b^2}-\frac{9}{b^2} = -1$

$\Rightarrow \quad 4b^2-9(10-b^2) = -b^2(10-b^2)$

$\Rightarrow \quad b^4-23b^2+90 = 0$

$\Rightarrow \quad (b^2-5)(b^2-18) = 0 \Rightarrow b^2 = 5, 18$

Now, $a^2=b^2(e^2-1)$ and $be=\sqrt{10} \Rightarrow a^2 = 10-b^2$

For $b^2=5$, $a^2=5$ and for $b^2=18$, $a^2=-8$, which is absurd as $a^2>0$.

$\therefore \quad a^2=5$ and $b^2=5$.

So, the equation of the hyperbola is $\frac{x^2}{5}-\frac{y^2}{5}=-1$. $\quad$ [Substituting $a^2=5, b^2=5$ in (i)]

(x) Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = -1 \quad \text{...(i)}$$

Its foci are at $(0, \pm 2)$ and latusrectum = 36

$\therefore \quad be = 12$ and $\frac{2a^2}{b} = 36 \Rightarrow be = 12$ and $a^2 = 18b$

Now, $a^2=18b$

$\Rightarrow \quad b^2(e^2-1) = 18b$ $\quad \left[\because a^2=b^2(e^2-1)\right]$

$\Rightarrow \quad 144-b^2 = 18b$ $\quad [\because be=12]$

$\Rightarrow \quad b^2+18b-144 = 0$

$\Rightarrow \quad (b+24)(b-6) = 0$

$\Rightarrow \quad b = 6$

$\therefore \quad a^2 = 18b \Rightarrow a^2 = 18 \times 6 = 108$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{108} - \frac{y^2}{36} = -1$ as the equation of the hyperbola.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.
2. Write the eccentricity of the hyperbola whose latus-rectum is half of its transverse axis.
3. Write the coordinates of the foci of the hyperbola $9x^2 - 16y^2 = 144$.
4. Write the equation of the hyperbola of eccentricity $\sqrt{2}$, if it is known that the distance between its foci is 16.
5. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, write the value of b^2.
6. Write the length of the latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$.
7. If the latus-rectum through one focus of a hyperbola subtends a right angle at the farther vertex, then write the eccentricity of the hyperbola.
8. Write the distance between the directrices of the hyperbola
   $x = 8 \sec \theta, y = 8 \tan \theta$.
9. Write the equation of the hyperbola whose vertices are $(\pm 3, 0)$ and foci at $(\pm 5, 0)$.
10. If e_1 and e_2 are respectively the eccentricities of the ellipse $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then write the value of $2e_1^2 + e_2^2$.

ANSWERS

1. $\frac{5}{4}$ 2. $\frac{1}{\sqrt{2}}$ 3. $(\pm 5, 0)$ 4. $x^2 - y^2 = 32$ 5. 7 6. $\frac{4}{3}$

7. 2 8. $8\sqrt{2}$ 9. $16x^2 - 9y^2 = 144$ 10. 3

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Equation of the hyperbola whose vertices are $(\pm 3, 0)$ and foci at $(\pm 5, 0)$, is
   (a) $16x^2 - 9y^2 = 144$ (b) $9x^2 - 16y^2 = 144$
   (c) $25x^2 - 9y^2 = 225$ (d) $9x^2 - 25y^2 = 81$
2. If e_1 and e_2 are respectively the eccentricities of the ellipse $\frac{x^2}{18} + \frac{y^2}{4} = 1$ and the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$, then the relation between e_1 and e_2 is
   (a) $3e_1^2 + e_2^2 = 2$ (b) $e_1^2 + 2e_2^2 = 3$
   (c) $2e_1^2 + e_2^2 = 3$ (d) $e_1^2 + 3e_2^2 = 2$

3. The distance between the directrices of the hyperbola $x = 8 \sec \theta, y = 8 \tan \theta$, is
(a) $8\sqrt{2}$ (b) $16\sqrt{2}$ (c) $4\sqrt{2}$ (d) $6\sqrt{2}$

4. The equation of the conic with focus at $(1, -1)$ directrix along $x - y + 1 = 0$ and eccentricity $\sqrt{2}$ is
(a) $xy = 1$ (b) $2xy + 4x - 4y - 1 = 0$
(c) $x^2 - y^2 = 1$ (d) $2xy - 4x + 4y + 1 = 0$

5. The eccentricity of the conic $9x^2 - 16y^2 = 144$ is
(a) $\frac{5}{4}$ (b) $\frac{4}{3}$ (c) $\frac{4}{5}$ (d) $\sqrt{7}$

6. A point moves in a plane so that its distances PA and PB from two fixed points A and B in the plane satisfy the relation $PA - PB = k\ (k \neq 0)$, then the locus of P is
(a) a hyperbola (b) a branch of the hyperbola
(c) a parabola (d) an ellipse

7. The eccentricity of the hyperbola whose latus-rectum is half of its transverse axis, is
(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\sqrt{\frac{3}{2}}$ (d) none of these

8. The eccentricity of the hyperbola $x^2 - 4y^2 = 1$ is
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{5}}$

9. The difference of the focal distances of any point on the hyperbola is equal to
(a) length of the conjugate axis (b) eccentricity
(c) length of the transverse axis (d) Latus-rectum

10. The foci of the hyperbola $9x^2 - 16y^2 = 144$ are
(a) $(\pm 4, 0)$ (b) $(0, \pm 4)$ (c) $(\pm 5, 0)$ (d) $(0, \pm 5)$

11. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$, then equation of the hyperbola is
(a) $x^2 + y^2 = 32$ (b) $x^2 - y^2 = 16$ (c) $x^2 + y^2 = 16$ (d) $x^2 - y^2 = 32$

12. If e_1 is the eccentricity of the conic $9x^2 + 4y^2 = 36$ and e_2 is the eccentricity of the conic $9x^2 - 4y^2 = 36$, then
(a) $e_1^2 - e_2^2 = 2$ (b) $2 < e_2^2 - e_1^2 < 3$
(c) $e_2^2 - e_1^2 = 2$ (d) $e_2^2 - e_1^2 > 3$

13. If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then $\alpha =$
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

14. The equation of the hyperbola whose foci are $(6, 4)$ and $(-4, 4)$ and eccentricity 2, is
(a) $\frac{(x-1)^2}{25/4} - \frac{(y-4)^2}{75/4} = 1$ (b) $\frac{(x+1)^2}{25/4} - \frac{(y+4)^2}{75/4} = 1$
(c) $\frac{(x-1)^2}{75/4} - \frac{(y-4)^2}{25/4} = 1$ (d) none of these

15. The length of the straight line $x - 3y = 1$ intercepted by the hyperbola $x^2 - 4y^2 = 1$ is
(a) $\frac{6}{\sqrt{5}}$ (b) $3\sqrt{\frac{2}{5}}$ (c) $6\sqrt{\frac{2}{5}}$ (d) none of these

16. The latus-rectum of the hyperbola $16x^2 - 9y^2 = 144$ is

(a) $16/3$ (b) $32/3$ (c) $8/3$ (d) $4/3$

17. The foci of the hyperbola $2x^2 - 3y^2 = 5$ are

(a) $(\pm 5/\sqrt{6}, 0)$ (b) $(\pm 5/6, 0)$ (c) $(\pm \sqrt{5}/6, 0)$ (d) none of these

18. The eccentricity the hyperbola $x = \frac{a}{2}\left(t + \frac{1}{t}\right), y = \frac{a}{2}\left(t - \frac{1}{t}\right)$ is

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{3}$ (d) $3\sqrt{2}$

19. The equation of the hyperbola whose centre is (6, 2) one focus is (4, 2) and of eccentricity 2 is

(a) $3(x-6)^2 - (y-2)^2 = 3$ (b) $(x-6)^2 - 3(y-2)^2 = 1$

(c) $(x-6)^2 - 2(y-2)^2 = 1$ (d) $2(x-6)^2 - (y-2)^2 = 1$

20. The locus of the point of intersection of the lines $\sqrt{3}\,x - y - 4\sqrt{3}\,\lambda = 0$ and $\sqrt{3}\,\lambda x + \lambda y - 4\sqrt{3} = 0$ is a hyperbola of eccentricity

(a) 1 (b) 2 (c) 3 (d) 4

ANSWERS

1. (a)	2. (c)	3. (a)	4. (d)	5. (a)	6. (a)	7. (c)	8. (b)
9. (c)	10. (c)	11. (d)	12. (b)	13. (b)	14. (a)	15. (c)	16. (d)
17. (a)	18. (a)	19. (a)	20. (b)				

SUMMARY

1. A hyperbola is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed line (called directrix) is always constant which is always greater than unity.
   The fixed point is called the focus, the fixed line is called the directrix and the constant ratio, generally denoted by e, is known as the eccentricity of the hyperbola.
   The general equation of the hyperbola is of the form
   $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$.
2. The equation of the hyperbola having its centre at the origin and axes along the coordinate axes is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the following property:

	Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	*Conjugate hyperbola* $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordiantes of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(\pm ae, 0)$	$(0, \pm be)$
Length of the transverse axis	$2a$	$2b$
Length of the conjugate axis	$2b$	$2a$
Equations of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$ or, $b^2 = a^2 (e^2 - 1)$	$e = \sqrt{\frac{b^2 + a^2}{b^2}}$ or, $a^2 = b^2 (e^2 - 1)$
Length of the latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of the transverse axis	$y = 0$	$x = 0$
Equation of the conjugate axis	$x = 0$	$y = 0$

3. A hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.
4. If the centre of the hyperbola is at the point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$