CHAPTER 19

ARITHMETIC PROGRESSIONS

19.1 SEQUENCE

A sequence is a function whose domain is the set N of natural numbers.

It is customary to denote a sequence by a letter 'a' and the image $a(n)$ of $n \in N$ under a by a_n. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. The images of 1, 2, 3, ..., n, ... under a sequence 'a' are generally denoted by $a_1, a_2, a_3, \dots a_n, \dots$ respectively. $a_1, a_2, a_3, \dots a_n, \dots$ are known as first term, second term ..., nth term,... respectively of the sequence. If a_n is the nth term of a sequence, 'a' then we write $a = < a_n >$.

REAL SEQUENCE *A sequence whose range is a subset of R is called a real sequence.*

In other words, a real sequence is a function with domain N and the range a subset of the set R of real numbers.

REPRESENTATION OF A SEQUENCE *There are several ways of representing a real sequence.*

One way to represent a real sequence is to list its first few terms till the rule for writing down other terms becomes clear. For example, 1, 3, 5, ... is a sequence whose nth term is $(2n - 1)$.

Another way to represent a real sequence is to give a rule of writing the nth term of the sequence. For example, the sequence 1, 3, 5, 7, ... can be written as $a_n = 2n - 1$.

Sometimes we represent a real sequence by using a recursive relation. For example, the Fibonacci sequence is given by

$$a_1 = 1,\ a_2 = 1 \text{ and } a_{n+1} = a_n + a_{n-1},\ n \geq 2$$

The terms of this sequence are 1, 1, 2, 3, 5, 8, ...

ILLUSTRATION 1 *Give first 3 terms of the sequence defined by* $a_n = \dfrac{n}{n^2 + 1}$.

SOLUTION Putting $n = 1, 2, 3$ in $a_n = \dfrac{n}{n^2 + 1}$, we get

$$a_1 = \frac{1}{1^2 + 1} = \frac{1}{2},\quad a_2 = \frac{2}{2^2 + 1} = \frac{2}{5} \text{ and } a_3 = \frac{3}{3^2 + 1} = \frac{3}{10}.$$

ILLUSTRATION 2 *Find the first four terms of the sequence whose first term is 1 and whose (n + 1)th term is obtained by subtracting n from its nth term.*

SOLUTION We are given that $a_1 = 1$ and $a_{n+1} = a_n - n$.

Putting $n = 1$, we obtain

$$a_2 = a_1 - 1 \Rightarrow a_2 = 1 - 1 = 0 \qquad [\because\ a_1 = 1]$$

Putting $n = 2$, we obtain

$$a_3 = a_2 - 2 \Rightarrow a_3 = 0 - 2 = -2$$

Similarly, by putting $n = 3$, we obtain

$$a_4 = a_3 - 3 = -2 - 3 = -5$$

SERIES *If $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + a_4 + \ldots + a_n + \ldots$ is a series.*

A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

PROGRESSIONS *It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the nth term. Those sequences whose terms follow certain patterns are called progressions.*

In this chapter, we shall study arithmetical progressions as defined below.

19.2 ARITHMETIC PROGRESSION (A.P.)

A sequence is called an arithmetic progression if the difference of a term and the previous term is always same.

i.e. $a_{n+1} - a_n = \text{constant } (= d) \text{ for all } n \in N$

The constant difference, generally denoted by d is called the common difference.

ILLUSTRATION 1 *1, 4, 7, 10, ... is an A.P. whose first term is 1 and the common difference is equal to* $4 - 1 = 3$.

ILLUSTRATION 2 *11, 7, 3, –1, ... is an A.P. whose first term is 11 and the common difference is equal to* $7 - 11 = -4$.

In order to determine whether a sequence is an A.P. or not when its nth term is given, we may use the following algorithm.

ALGORITHM

<u>STEP I</u> *Obtain a_n.*

<u>STEP II</u> *Replace n by n + 1 in a_n to get a_{n+1}.*

<u>STEP III</u> *Calculate $a_{n+1} - a_n$.*

<u>STEP IV</u> *If $a_{n+1} - a_n$ is independent of n, the given sequence is an A.P. Otherwise it is not an A.P.*

Following examples illustrate the procedure:

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that the sequence defined by $a_n = 4n + 5$ is an A.P. Also, find its common difference.*

SOLUTION We have, $a_n = 4n + 5$

Replacing n by $(n + 1)$, we get

$$a_{n+1} = 4(n+1) + 5 = 4n + 9$$

$$\therefore \quad a_{n+1} - a_n = (4n+9) - (4n+5) = 4$$

Clearly, $a_{n+1} - a_n$ is independent of n and is equal to 4.

So, the given sequence is an A.P. with common difference 4.

EXAMPLE 2 *Show that the sequence defined by $a_n = 2n^2 + 1$ is not an A.P.*

SOLUTION We have, $a_n = 2n^2 + 1$

Replacing n by $(n + 1)$ in a_n, we obtain

$$a_{n+1} = 2(n+1)^2 + 1 = 2n^2 + 4n + 3$$

$$\therefore \quad a_{n+1} - a_n = (2n^2 + 4n + 3) - (2n^2 + 1) = 4n + 2$$

Clearly, $a_{n+1} - a_n$ is not independent of n and is therefore not constant. So, the given sequence is not an A.P.

EXAMPLE 3 *Show that the sequence* $\log a, \log\left(\frac{a^2}{b}\right), \log\left(\frac{a^3}{b^2}\right), \log\left(\frac{a^4}{b^3}\right), \ldots\ldots$ *forms an A.P.*

SOLUTION We have,

$$\log\left(\frac{a^2}{b}\right) - \log a = \log\left(\frac{a^2}{b} \times \frac{1}{a}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^3}{b^2}\right) - \log\left(\frac{a^2}{b}\right) = \log\left(\frac{a^3}{b^2} \times \frac{b}{a^2}\right) = \log\left(\frac{a}{b}\right)$$

$$\log\left(\frac{a^4}{b^3}\right) - \log\left(\frac{a^3}{b^2}\right) = \log\left(\frac{a^4}{b^3} \times \frac{b^2}{a^3}\right) = \log\left(\frac{a}{b}\right)$$

This shows that the difference of a term and the preceding term is always same. Hence, the given sequence forms an A.P.

ALITER From the symmetry, we obtain

$$a_n = \log\left(\frac{a^n}{b^{n-1}}\right)$$

$$\Rightarrow \quad a_{n+1} = \log\left(\frac{a^{n+1}}{b^n}\right)$$

$$\therefore \quad a_{n+1} - a_n = \log\left(\frac{a^{n+1}}{b^n}\right) - \log\left(\frac{a^n}{b^{n-1}}\right) = \log\left(\frac{a^{n+1}}{b^n} \times \frac{b^{n-1}}{a^n}\right) = \log\left(\frac{a}{b}\right)$$

Clearly, $a_{n+1} - a_n$ is constant for all values of n.

So, the given sequence is an A.P. with common difference $\log\left(\frac{a}{b}\right)$.

LEVEL-2

EXAMPLE 4 *Show that a sequence is an A.P. if its nth term is a linear expression in n and in such a case the common difference is equal to the coefficient of n.*

SOLUTION Let a_n be the n^{th} term of a sequence. Let a_n be a linear expression in n.

i.e. $\quad a_n = An + B$, where A, B are constants.

$$\Rightarrow \quad a_{n+1} = A(n+1) + B$$

$$\therefore \quad a_{n+1} - a_n = \{A(n+1) + B\} - \{An + B\} = A$$

Clearly, $a_{n+1} - a_n$ is independent of n and is therefore a constant.

Hence, the sequence is an A.P. with common difference A.

NOTE *Students are advised to use the statement of the above example as a standard result.*

EXAMPLE 5 *The* n^{th} *term of a sequence is* $3n - 2$. *Is the sequence an A.P. ? If so, find its 10th term.*

SOLUTION Here, $a_n = 3n - 2$.

Clearly, a_n is a linear expression in n. So, the given sequence is an A.P. with common difference 3.

Putting $n = 10$, we get: $a_{10} = 3 \times 10 - 2 = 28$

REMARK *It is evident from the above examples that a sequence is not an A.P. if its nth term is not a linear expression in n.*

EXERCISE 19.1

1. If the n^{th} term a_n of a sequence is given by $a_n = n^2 - n + 1$, write down its first five terms.
2. A sequence is defined by $a_n = n^3 - 6n^2 + 11n - 6, n \in N$. Show that the first three terms of the sequence are zero and all other terms are positive.
3. Find the first four terms of the sequence defined by $a_1 = 3$ and, $a_n = 3a_{n-1} + 2$, for all $n > 1$. [NCERT]
4. Write the first five terms in each of the following sequences:
   (i) $a_1 = 1, a_n = a_{n-1} + 2, n > 1$
   (ii) $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$
   (iii) $a_1 = a_2 = 2, a_n = a_{n-1} - 1, n > 2$ [NCERT]
5. The Fibonacci sequence is defined by $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}$ for $n > 2$.
   Find $\dfrac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$. [NCERT]
6. Show that each of the following sequences is an A.P. Also, find the common difference and write 3 more terms in each case.
   (i) $3, -1, -5, -9 \ldots$ (ii) $-1, 1/4, 3/2, 11/4, \ldots$
   (iii) $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, 7\sqrt{2}, \ldots$ (iv) $9, 7, 5, 3, \ldots$
7. The n^{th} term of a sequence is given by $a_n = 2n + 7$. Show that it is an A.P. Also, find its 7th term.
8. The n^{th} term of a sequence is given by $a_n = 2n^2 + n + 1$. Show that it is not an A.P.

ANSWERS

1. $a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 13,\ a_5 = 21$ 3. $a_1 = 3, a_2 = 11, a_3 = 35, a_4 = 107$
4. (i) $a_1 = 1, a_2 = 3, a_3 = 5, a_4 = 7, a_5 = 9$ (ii) $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, a_5 = 5$
   (iii) $a_1 = 2, a_2 = 2, a_3 = 1, a_4 = 0, a_5 = -1$
6. (i) -4 (ii) $\dfrac{5}{4}$ (iii) $2\sqrt{2}$ (iv) -2 7. 21

HINTS TO NCERT & SELECTED PROBLEMS

3. We have, $a_1 = 3$ and $a_n = 3a_{n-1} + 2$ for $n > 1$
   $\therefore \quad a_2 = 3a_1 + 2 = 3 \times 3 + 2 = 11,\ a_3 = 3a_2 + 2 = 3 \times 11 + 2 = 35$
   and, $a_4 = 3a_3 + 2 = 3 \times 35 + 2 = 107$.
4. (iii) We have, $a_1 = a_2 = 2$ and $a_n = a_{n-1} - 1$ for $n > 2$.
   $\therefore \quad a_3 = a_2 - 1 = 2 - 1 = 1,\ a_4 = a_3 - 1 = 1 - 1 = 0$ and, $a_5 = a_4 - 1 = 0 - 1 = -1$
5. We have, $a_1 = 1 = a_2$ and $a_n = a_{n-1} + a_{n-2}$ for $n > 2$.
   $\therefore \quad a_3 = a_2 + a_1 = 1 + 1 = 2,\ a_4 = a_3 + a_2 = 2 + 1 = 3$
   $a_5 = a_4 + a_3 = 3 + 2 = 5,\ a_6 = a_5 + a_4 = 5 + 3 = 8$
   We have to find $\dfrac{a_{n+1}}{a_n}$ for $n = 1, 2, 3, 4, 5$. i.e. $\dfrac{a_2}{a_1}, \dfrac{a_3}{a_2}, \dfrac{a_4}{a_3}, \dfrac{a_5}{a_4}$, and $\dfrac{a_6}{a_5}$

Clearly, $\frac{a_2}{a_1} = \frac{1}{1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}, \frac{a_5}{a_4} = \frac{5}{3}$ and $\frac{a_6}{a_5} = \frac{8}{5}$

7. We have,

$a_n = 2n + 7 \Rightarrow a_{n+1} = 2(n+1) + 7 = 2n + 9.$

$\therefore \quad a_{n+1} - a_n = (2n+9) - (2n+7) = 2$ (a constant). So, the sequence is an A.P.

8. Show that $a_{n+1} - a_n$ is not independent of n.

19.3 GENERAL TERM OF AN A.P.

THEOREM *Let a be the first term and d be the common difference of an A.P. Then, its nth term is $a + (n-1)\,d$ i.e. $a_n = a + (n-1)\,d$.*

PROOF *Let $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ be the given A.P. Then,*

$a_1 =$ First term $= a$

$\Rightarrow \quad a_1 = a + (1-1)\,d.$

By the definition, we have

$$a_2 - a_1 = d \Rightarrow a_2 = a_1 + d \Rightarrow a_2 = a + d \Rightarrow a_2 = a + (2-1)\,d$$
$$a_3 - a_2 = d \Rightarrow a_3 = a_2 + d \Rightarrow a_3 = (a+d) + d \Rightarrow a_3 = a + 2d \Rightarrow a_3 = a + (3-1)\,d$$
$$a_4 - a_3 = d \Rightarrow a_4 = a_3 + d \Rightarrow a_4 = (a+2d) + d \Rightarrow a_4 = a + 3d \Rightarrow a_4 = a + (4-1)\,d$$

Similarly, $a_5 = a + (5-1)\,d$, $a_6 = a + (6-1)\,d$, ..., $a_n = a + (n-1)\,d$.

Hence, nth term of an A.P. with first term a and common difference d is $a_n = a + (n-1)\,d$.

Q.E.D.

19.3.1 *n*th TERM OF AN A.P. FROM THE END

Let a be the first term and d be the common difference of an A.P. having m terms. Then, n^{th} term from the end is $(m-n+1)^{\text{th}}$ term from the beginning.

$\therefore \quad n^{\text{th}}$ term from the end $= a_{m-n+1} = a + (m-n+1-1)\,d = a + (m-n)\,d$

For finding the n^{th} term from the end, we may take a_m as the first term and $-d$ as the common difference.

Taking a_m as the first term and common difference equal to '$-d$', we find that

n^{th} term from the end $= a_m + (n-1)(-d)$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE INDICATED TERM OF AN A.P.**

EXAMPLE 1 *Show that the sequence 9, 12, 15, 18, ... is an A.P. Find its 16th term and the general term.*

SOLUTION Clearly, $(12-9) = (15-12) = (18-15) = 3$, so the given sequence is an A.P. with common difference $d = 3$ and first term $a = 9$.

$\therefore \quad$ 16th term $= a_{16} = a + (16-1)\,d = a + 15d = 9 + 15 \times 3 = 54 \quad [\because a_n = a + (n-1)\,d]$

and, General term $=$ *nth* term $= a_n = a + (n-1)\,d = 9 + (n-1) \times 3 = 3n + 6$

EXAMPLE 2 *Show that the sequence log a, log (ab), log (ab^2), log (ab^3), ... is an A.P. Find its nth term.*

SOLUTION We have,

$$\log(ab) - \log a = \log\left(\frac{ab}{a}\right) = \log b, \quad \log(ab^2) - \log(ab) = \log\left(\frac{ab^2}{ab}\right) = \log b$$

$$\log(ab^3) - \log(ab^2) = \log\left(\frac{ab^3}{ab^2}\right) = \log b$$

It follows from the above results that the difference of a term and the preceding term is always same. So, the given sequence is an A.P. with common difference log b.

$\therefore \quad a_n = a + (n-1)\, d$

$\Rightarrow \quad a_n = \log a + (n-1) \log b = \log a + \log b^{n-1} = \log (ab^{n-1})$

EXAMPLE 3 *Which term of the sequence 72, 70, 68, 66, ... is 40?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 72$ and common difference $d = -2$. Let its nth term be 40.

i.e. $\quad a_n = 40$

$\Rightarrow \quad a + (n-1) d = 40$

$\Rightarrow \quad 72 + (n-1)(-2) = 40 \qquad [\because\ a_n = a + (n-1)\, d]$

$\Rightarrow \quad 72 - 2n + 2 = 40 \Rightarrow 2n = 34 \Rightarrow n = 17$

Hence, 17th term of the given sequence is 40.

EXAMPLE 4 *Which term of the sequence 4, 9, 14, 19, ... is 124?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 4$ and common difference $d = 5$. Let 124 be the nth term of the given sequence. Then,

$$a_n = 124 \Rightarrow a + (n-1)\, d = 124 \Rightarrow 4 + (n-1) \times 5 = 124 \Rightarrow n = 25$$

Hence, 25th term of the given sequence is 124.

EXAMPLE 5 *How many terms are there in the sequence 3, 6, 9, 12, ..., 111?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 3$. Let there be n terms in the given sequence. Then,

$$n^{\text{th}} \text{ term} = 111 \Rightarrow a + (n-1)\, d = 111 \Rightarrow 3 + (n-1) \times 3 = 111 \Rightarrow n = 37$$

Thus, the given sequence contains 37 terms.

EXAMPLE 6 *Is 184 a term of the sequence 3, 7, 11, ... ?*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 3$ and common difference $d = 4$. Let the nth term of the given sequence be 184. Then,

$$a_n = 184 \Rightarrow a + (n-1)\, d = 184 \Rightarrow 3 + (n-1) \times 4 = 184 \Rightarrow 4n = 185 \Rightarrow n = 46\frac{1}{4}.$$

Since n is not a natural number. So, 184 is not a term of the given sequence.

EXAMPLE 7 *Which term of the sequence* $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots$ *is the first negative term?*

SOLUTION The given sequence is an A.P. in which first term $a = 20$ and common difference $d = -\frac{3}{4}$. Let the nth term of the given A.P. be the first negative term. Then,

$\quad a_n < 0$

$\Rightarrow \quad a + (n-1)\, d < 0$

$\Rightarrow \quad 20 + (n-1) \times (-3/4) < 0 \Rightarrow \frac{83}{4} - \frac{3n}{4} < 0 \Rightarrow 83 - 3n < 0 \Rightarrow 3n > 83 \Rightarrow n > 27\frac{2}{3}$

Since 28 is the natural number just greater than $27\frac{2}{3}$. So, $n = 28$. Thus, 28th term of the given sequence is the first negative term.

EXAMPLE 8 *Which term of the sequence* $8 - 6i, 7 - 4i, 6 - 2i, \ldots$ *is (i) purely real (ii) purely imaginary?*

SOLUTION The given sequence is clearly an A.P. with first term $a = 8 - 6i$ and common difference $d = -1 + 2i$. The nth term of the given A.P. is given by

$$a_n = a+(n-1)\,d = 8-6i+(n-1)(-1+2i) = (9-n)+i\,(2n-8)$$

(i) Let the nth term of the given sequence be purely real. Then, a_n is purely real.

$\therefore \quad (9-n)+i(2n-8)$ is purely real $\Rightarrow 2n-8 = 0 \Rightarrow n = 4$

So, 4th term of the given sequence is purely real.

(ii) Let the nth term of the given sequence be purely imaginary. Then, a_n is purely imaginary

$\therefore \quad (9-n)+i(2n-8)$ is purely imaginary

$\Rightarrow \quad 9-n = 0 \Rightarrow n = 9$

Thus, 9th term of the given sequence is purely imaginary.

Type II **PROBLEMS BASED UPON** $a_n = a+(n-1)\,d$

EXAMPLE 9 *If pth, qth and rth terms of an A.P. are a, b, c respectively, then show that:*

(i) $a(q-r)+b(r-p)+c(p-q)=0$ *(ii)* $(a-b)\,r+(b-c)\,p+(c-a)\,q=0$ **[NCERT]**

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = p\text{th term} \Rightarrow a = A+(p-1)\,D \quad \text{...(i)}$$

$$b = q\text{th term} \Rightarrow b = A+(q-1)\,D \quad \text{...(ii)}$$

$$c = r\text{th term} \Rightarrow c = A+(r-1)\,D \quad \text{...(iii)}$$

(i) Substituting these values of a, b, c, in $a\,(q-r)+b\,(r-p)+c\,(p-q)$, we obtain

$$\begin{aligned}
& a\,(q-r)+b\,(r-p)+c\,(p-q) \\
&= \{A+(p-1)\,D\}\,(q-r)+\{A+(q-1)\,D\}\,(r-p)+\{A+(r-1)\,D\}\,(p-q) \\
&= A\{(q-r)+(r-p)+(p-q)\}+D\{(p-1)\,(q-r)+(q-1)\,(r-p)+(r-1)\,(p-q)\} \\
&= A\,.\,0+D\,\{p\,(q-r)+q\,(r-p)+r\,(p-q)-(q-r)-(r-p)-(p-q)\} \\
&= A\,.\,0+D\,.\,0 = 0
\end{aligned}$$

(ii) On subtracting (ii) from (i); (iii) from (ii) and (i) from (iii), we get

$$a-b = (p-q)\,D \text{ ...(iv)} \quad (b-c) = (q-r)\,D \text{ ...(v)} \quad c-a = (r-p)\,D \quad \text{...(vi)}$$

$$\begin{aligned}
\therefore \quad (a-b)\,r+(b-c)\,p+(c-a)\,q &= (p-q)\,D\,r+(q-r)\,D\,p+(r-p)\,D\,q \\
&= D[(p-q)\,r+(q-r)\,p+(r-p)\,q] = D\times 0 = 0
\end{aligned}$$

EXAMPLE 10 *Show that the sum of* $(m+n)^{th}$ *and* $(m-n)^{th}$ *term of an A.P. is equal to twice the* m^{th} *term.* **[NCERT]**

SOLUTION Let a be the first term and d be the common different of the AP. Then,

$$a_{m+n} = (m+n)^{th} \text{ term } = a+(m+n-1)\,d \text{ and, } a_{m-n} = (m-n)^{th} \text{ term } = a+(m-n-1)\,d$$

$$\begin{aligned}
\therefore \quad a_{m+n}+a_{m-n} &= \{a+(m+n-1)\,d\}+\{a+(m-n-1)\,d\} \\
&= 2a+(m+n-1+m-n-1)\,d \\
&= 2a+2\,(m-1)\,d \\
&= 2\,\{a+(m-1)\,d\} \\
&= 2\,a_m.
\end{aligned}$$

EXAMPLE 11 *If m times the mth term of an A.P. is equal to n times its nth term, show that the* $(m+n)$*th term of the A.P. is zero.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

m times m^{th} term $= n$ times nth term

$\Rightarrow \quad m\,a_m = n\,a_n$

$\Rightarrow \quad m\{a+(m-1)\,d\} = n\{a+(n-1)\,d\}$

$\Rightarrow \quad m\{a+(m-1)d\}-n\{a+(n-1)d\}=0$

$\Rightarrow \quad a(m-n)+\{m(m-1)-n(n-1)\}d=0$

$\Rightarrow \quad a(m-n)+\{(m^2-n^2)-(m-n)\}d=0$

$\Rightarrow \quad a(m-n)+(m-n)(m+n-1)d=0$

$\Rightarrow \quad (m-n)\{a+(m+n-1)d\}=0$

$\Rightarrow \quad a+(m+n-1)d=0$ $\quad [\because m \neq n]$

$\Rightarrow \quad a_{m+n}=0$

Hence, the $(m+n)^{th}$ term of the given A.P. is zero.

EXAMPLE 12 *If the pth term of an A.P. is q and the q^{th} term is p, prove that its nth term is $(p+q-n)$.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

pth term $= q \Rightarrow a+(p-1)d = q$ ...(i)

qth term $= p \Rightarrow a+(q-1)d = p$ ...(ii)

Subtracting (ii) from (i), we get

$(p-q)d = (q-p) \Rightarrow d=-1$

Putting $d=-1$ in (i), we get: $a = (p+q-1)$

$\therefore \quad n$th term $= a+(n-1)d = (p+q-1)+(n-1)\times(-1) = (p+q-n)$

EXAMPLE 13 *If the mth term of an A.P. be 1/n, and nth term be 1/m then show that its (mn)th term is 1.*

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Then,

$\frac{1}{n} = m\text{th term} \Rightarrow \frac{1}{n} = a+(m-1)d$ ...(i)

$\frac{1}{m} = n\text{th term} \Rightarrow \frac{1}{m} = a+(n-1)d$ ...(ii)

On subtracting (ii) from (i), we get

$\frac{1}{n}-\frac{1}{m} = (m-n)d \Rightarrow \frac{m-n}{mn} = (m-n)d \Rightarrow d = \frac{1}{mn}$

Putting $d = \frac{1}{mn}$ in (i), we get

$\frac{1}{n} = a+\frac{(m-1)}{mn} \Rightarrow \frac{1}{n} = a+\frac{1}{n}-\frac{1}{mn} \Rightarrow a = \frac{1}{mn}$

$\therefore \quad (mn)\text{th term} = a+(mn-1)d = \frac{1}{mn}+(mn-1)\frac{1}{mn} = 1$

EXAMPLE 14 *Determine the number of terms in the A.P. 3, 7, 11, ... 407. Also, find its 20th term from the end.*

SOLUTION Clearly, the given sequence is an A.P. with first term 3 and the common difference 4. Let there be n terms in the given A.P. Then,

$407 = n\text{th term} \Rightarrow 407 = 3+(n-1)\times 4 \Rightarrow 4n = 408 \Rightarrow n = 102$

Now,

20th term from the end = $[102-20+1]$th term from the beginning

= 83rd term from the beginning $= 3+(83-1)\times 4 = 331$

<u>ALITER</u> To find 20th term from the end, we consider the given sequence as an A.P. with first term = 407 and common difference − 4.

$\therefore$ 20th term from the end $= 407+(20-1)\times(-4) = 331$.

EXAMPLE 15 *How many numbers of two digits are divisible by 7 ?*

SOLUTION First two digit number divisible by 7 is 14 and last two digit number divisible by 7 is 98. So, we have to determine the number of terms in the sequence 14, 21, 28, ..., 98. Let there be n terms in this sequence. Then,

$$98 = n\text{th term} \Rightarrow 98 = 14 + (n-1) \times 7 \Rightarrow 7n = 91 \Rightarrow n = 13$$

LEVEL-2

EXAMPLE 16 *Show that there is no A.P. which consists of only distinct prime numbers.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be an A.P. consisting only of prime numbers. Let d be the common difference of the A.P. Since the difference of two consecutive prime numbers is greater than or equal to 1. Therefore, $d > 1$.

Now,

$$(a_1 + 1)^{\text{th}} \text{ term of this A.P.} = a_1 + (a_1 + 1 - 1)\,d = a_1(1 + d)$$

$$\Rightarrow \quad (a_1 + 1)^{\text{th}} \text{ term is not a prime number}$$

This is a contradiction that the A.P. consists of only prime numbers as its terms.
Hence, there cannot be an A.P. which consists only of distinct prime numbers.

EXAMPLE 17 *Show that in an A.P. the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.* **[NCERT EXEMPLAR]**

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference 'd'. Then,

$$k\text{th term from the beginning} = a_k = a_1 + (k-1)\,d$$

and, $\quad k\text{th term from the end} = (n-k+1)\text{th term from the beginning}$

$$= a_{n-k+1}$$

$$= a_1 + (n-k+1-1)\,d = a_1 + (n-k)\,d$$

$\therefore \quad (k\text{th term from the beginning}) + (k\text{th term from the end})$

$$= a_k + a_{n-k+1}$$

$$= \{a_1 + (k-1)\,d\} + \{a_1 + (n-k)\,d\} = 2a_1 + (n-1)\,d = a_1 + \{a_1 + (n-1)\,d\} = a_1 + a_n$$

Thus, $\quad a_k + a_{n-k+1} = a_1 + a_n \quad \text{for all } k = 1, 2, \ldots n$

$$\Rightarrow \quad a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \ldots = a_1 + a_n$$

Hence, the sum of the terms equidistant from the beginning and end is always same and equal to the sum of first and last terms.

NOTE *The statement of the above example may be treated as a standard result.*

EXAMPLE 18 *In the arithmetic progressions 2, 5, 8, ... upto 50 terms, and 3, 5, 7, 9, ... upto 60 terms, find how many terms are identical.*

SOLUTION Let the mth term of the first A.P. be equal to the nth term of the second A.P. Then,

$$2 + (m-1) \times 3 = 3 + (n-1) \times 2$$

$$\Rightarrow \quad 3m - 1 = 2n + 1$$

$$\Rightarrow \quad 3m = 2n + 2$$

$$\Rightarrow \quad \frac{m}{2} = \frac{n+1}{3} = k \text{ (say)}$$

$$\Rightarrow \quad m = 2k \text{ and } n = 3k - 1$$

$$\Rightarrow \quad 2k \le 50 \text{ and } 3k - 1 \le 60 \qquad [\because m \le 50 \text{ and } n \le 60]$$

$$\Rightarrow \quad k \le 25 \text{ and } k \le 20\frac{1}{3}$$

$\Rightarrow \quad k \leq 20$ [$\because$ k is a natural number]

$\Rightarrow \quad k = 1, 2, 3, \ldots 20$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 20 identical terms in the two A.P.'s.

EXAMPLE 19 *Find the number of terms common to the two A.P.'s: 3, 7, 11, ... 407 and 2, 9, 16, ..., 709.*

SOLUTION Let the number of terms in two A.P.'s be m and n respectively. Then,

$$407 = m\text{th term of first A.P. and, } 709 = n\text{th term of second A.P.}$$

$$\Rightarrow \quad 407 = 3 + (m-1) \times 4 \quad \text{and} \quad 709 = 2 + (n-1) \times 7$$

$$\Rightarrow \quad m = 102 \quad \text{and} \quad n = 102$$

So, each A.P. consists of 102 terms.

Let pth term of first A.P. be identical to qth term of the second A.P. Then,

$$3 + (p-1) \times 4 = 2 + (q-1) \times 7$$

$$\Rightarrow \quad 4p - 1 = 7q - 5$$

$$\Rightarrow \quad 4p + 4 = 7q$$

$$\Rightarrow \quad 4(p+1) = 7q$$

$$\Rightarrow \quad \frac{p+1}{7} = \frac{q}{4} = k \text{ (say)}$$

$$\Rightarrow \quad p = 7k - 1 \text{ and } q = 4k$$

Since each A.P. consists of 102 terms.

$$\therefore \quad p \leq 102 \quad \text{and} \quad q \leq 102$$

$$\Rightarrow \quad 7k - 1 \leq 102 \quad \text{and} \quad 4k \leq 102$$

$$\Rightarrow \quad k \leq 14\frac{5}{7} \quad \text{and } k \leq 25\frac{1}{2}$$

$$\Rightarrow \quad k \leq 14 \Rightarrow k = 1, 2, 3, \ldots, 14$$

Corresponding to each value of k, we get a pair of identical terms.

Hence, there are 14 identical terms in two A.P.'s.

EXAMPLE 20 *If $a_1, a_2, a_3, \ldots, a_n$ are in A.P., where $a_i > 0$ for all i, show that*

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}.$$

[NCERT EXEMPLAR]

SOLUTION Let d be the common difference of the given A.P. Then

$$a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d \qquad \ldots(i)$$

Now, $$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \frac{1}{\sqrt{a_3} + \sqrt{a_4}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{(\sqrt{a_2} + \sqrt{a_1})(\sqrt{a_2} - \sqrt{a_1})} + \frac{\sqrt{a_3} - \sqrt{a_2}}{(\sqrt{a_3} + \sqrt{a_2})(\sqrt{a_3} - \sqrt{a_2})} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{(\sqrt{a_n} + \sqrt{a_{n-1}})(\sqrt{a_n} - \sqrt{a_{n-1}})}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{a_2 - a_1} + \frac{\sqrt{a_3} - \sqrt{a_2}}{a_3 - a_2} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{a_n - a_{n-1}}$$

$$= \frac{\sqrt{a_2} - \sqrt{a_1}}{d} + \frac{\sqrt{a_3} - \sqrt{a_2}}{d} + \ldots + \frac{\sqrt{a_n} - \sqrt{a_{n-1}}}{d} \qquad \text{[Using (i)]}$$

$$= \frac{1}{d}\left\{(\sqrt{a_2} - \sqrt{a_1}) + (\sqrt{a_3} - \sqrt{a_2}) + \ldots + (\sqrt{a_n} - \sqrt{a_{n-1}})\right\}$$

$$= \frac{1}{d}\left\{\sqrt{a_n} - \sqrt{a_1}\right\}$$

$$= \frac{(\sqrt{a_n} - \sqrt{a_1})(\sqrt{a_n} + \sqrt{a_1})}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{a_n - a_1}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{a_1 + (n-1)d - a_1}{d(\sqrt{a_n} + \sqrt{a_1})} = \frac{n-1}{\sqrt{a_n} + \sqrt{a_1}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$

EXAMPLE 21 *If $a_1, a_2, a_3, \dots, a_n$ be an A.P. of non-zero terms, prove that*

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_{n-1} a_n} = \frac{n-1}{a_1 a_n}.$$

SOLUTION Let 'd' be the common difference of the given A.P. Then,

$$a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = d \text{ (say)}. \quad \text{...(i)}$$

Now,

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \frac{1}{a_3 a_4} + \dots + \frac{1}{a_{n-1} a_n}$$

$$= \frac{1}{d}\left\{\frac{d}{a_1 a_2} + \frac{d}{a_2 a_3} + \frac{d}{a_3 a_4} + \dots + \frac{d}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\frac{(a_2 - a_1)}{a_1 a_2} + \frac{(a_3 - a_2)}{a_2 a_3} + \frac{(a_4 - a_3)}{a_3 a_4} + \dots + \frac{(a_n - a_{n-1})}{a_{n-1} a_n}\right\}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \left(\frac{1}{a_3} - \frac{1}{a_4}\right) + \dots + \left(\frac{1}{a_{n-1}} - \frac{1}{a_n}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a_1} - \frac{1}{a_n}\right\} = \frac{1}{d}\left\{\frac{a_n - a_1}{a_1 a_n}\right\} = \frac{1}{d}\left\{\frac{a_1 + (n-1)d - a_1}{a_1 a_n}\right\} = \frac{n-1}{a_1 a_n}$$

EXAMPLE 22 *If $a_1, a_2, a_3, \dots, a_n$ are in AP with common difference d (where $d \neq 0$), then the sum of series.*

$\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \dots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$ is equal to $\cot a_1 - \cot a_n$.

[NCERT EXEMPALR]

SOLUTION We have,

$\sin d(\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_2 \operatorname{cosec} a_3 + \dots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n)$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \frac{\sin d}{\sin a_3 \sin a_4} + \dots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3 - a_2)}{\sin a_2 \sin a_3} + \frac{\sin(a_4 - a_3)}{\sin a_3 \sin a_4} + \dots + \frac{\sin(a_n - a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_2 - \cos a_1 \sin a_2}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \dots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= (\cot a_1 - \cot a_2) + (\cot a_2 - \cot a_3) + \dots + (\cot a_{n-1} - \cot a_n)$$

$$= \cot a_1 - \cot a_n$$

EXERCISE 19.2

LEVEL-1

1. Find:
   (i) 10th term of the A.P. 1, 4, 7, 10, ... (ii) 18th term of the A.P. $\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}, \dots$
   (iii) *nth* term of the A.P. 13, 8, 3, –2, ...

2. If the sequence $< a_n >$ is an A.P., show that $a_{m+n} + a_{m-n} = 2a_m$.

3. (i) Which term of the A.P. 3, 8, 13, ... is 248 ?

   (ii) Which term of the A.P. 84, 80, 76, ... is 0 ?

   (iii) Which term of the A.P. 4, 9, 14, ... is 254 ?

4. (i) Is 68 a term of the A.P. 7, 10, 13, ... ?

   (ii) Is 302 a term of the A.P. 3, 8, 13, ... ?

5. (i) Which term of the sequence 24, $23\frac{1}{4}, 22\frac{1}{2}, 21\frac{3}{4}$... is the first negative term ?

   (ii) Which term of the sequence $12 + 8i, 11 + 6i, 10 + 4i$, ... is (a) purely real (b) purely imaginary ?

6. (i) How many terms are there in the A.P. 7, 10, 13, ... 43 ?

   (ii) How many terms are there in the A.P. $-1, -\frac{5}{6}, -\frac{2}{3}, -\frac{1}{2}, \ldots, \frac{10}{3}$?

7. The first term of an A.P. is 5, the common difference is 3 and the last term is 80; find the number of terms.

8. The 6th and 17th terms of an A.P. are 19 and 41 respectively, find the 40th term.

9. If 9th term of an A.P. is zero, prove that its 29th term is double the 19th term.

10. If 10 times the 10th term of an A.P. is equal to 15 times the 15th term, show that 25th term of the A.P. is zero.

11. The 10th and 18th terms of an A.P. are 41 and 73 respectively. Find 26th term.

12. In a certain A.P. the 24th term is twice the 10th term. Prove that the 72nd term is twice the 34th term.

13. If $(m + 1)$th term of an A.P. is twice the $(n + 1)$th term, prove that $(3m + 1)$th term is twice the $(m + n + 1)$th term.

14. If the nth term of the A.P. 9, 7, 5, ... is same as the nth term of the A.P. 15, 12, 9, ... find n.

15. Find the 12th term from the end of the following arithmetic progressions:

    (i) 3, 5, 7, 9, ... 201 (ii) 3, 8, 13, ... , 253 (iii) 1, 4, 7, 10, ..., 88

16. The 4th term of an A.P. is three times the first and the 7th term exceeds twice the third term by 1. Find the first term and the common difference.

17. Find the second term and *nth* term of an A.P. whose 6th term is 12 and the 8th term is 22.

18. How many numbers of two digit are divisible by 3 ?

19. An A.P. consists of 60 terms. If the first and the last terms be 7 and 125 respectively, find 32nd term.

20. The sum of 4th and 8th terms of an A.P. is 24 and the sum of the 6th and 10th terms is 34. Find the first term and the common difference of the A.P.

LEVEL-2

21. The first and the last terms of an A.P. are a and l respectively. Show that the sum of *nth* term from the beginning and *nth* term from the end is $a + l$.

22. If an A.P. is such that $\frac{a_4}{a_7} = \frac{2}{3}$, find $\frac{a_6}{a_8}$.

23. If $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ are in AP, whose common difference is d, show that
    $\sec\theta_1 \sec\theta_2 + \sec\theta_2 \sec\theta_3 + \ldots + \sec\theta_{n-1} \sec\theta_n = \frac{\tan\theta_n - \tan\theta_1}{\sin d}$ **[NCERT EXEMPALR]**

ANSWERS

1. (i) 28 (ii) $35\sqrt{2}$ (iii) $-5n+18$ 3. (i) 50 (ii) 22 (iii) 51
4. (i) No (ii) No 5. (i) 34th (ii) (a) 5 (b) 13
6. (i) 13 (ii) 27 7. 26 8. 87 11. 105 14. 7
15. (i) 179 (ii) 198 (iii) 55 16. First term = 3, common difference = 2
17. $a_2=-8$, $a_n=5n-18$ 18. 30 19. 69 20. $-\frac{1}{2}, \frac{5}{2}$ 22. $\frac{4}{5}$

19.4 SELECTION OF TERMS IN AN A.P.

Sometimes we require certain number of terms in A.P. The following ways of selecting terms are generally very convenient.

Number of terms	Terms	Common difference
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

It should be noted that in case of an odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a-d$, $a+d$ and the common differences is $2d$.

The following examples will illustrate the use of such representations.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *The sum of three numbers in A.P. is – 3, and their product is 8. Find the numbers.*

SOLUTION Let the numbers be $(a-d)$, a, $(a+d)$. Then,

$$\text{Sum} = -3 \Rightarrow (a-d)+a+(a+d) = -3 \Rightarrow 3a = -3 \Rightarrow a = -1$$

and, Product = 8

$$\Rightarrow (a-d)(a)(a+d) = 8$$

$$\Rightarrow a(a^2-d^2) = 8$$

$$\Rightarrow (-1)(1-d^2) = 8 \qquad [\because a=-1]$$

$$\Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

When $a=-1$ and $d=3$, the numbers are $-4, -1, 2$. When $a=-1$ and $d=-3$, the numbers are $2, -1, -4$. So, the numbers are $-4, -1, 2$, or $2, -1, -4$.

EXAMPLE 2 *Find four numbers in A.P. whose sum is 20 and the sum of whose squares is 120.*

SOLUTION Let the numbers be $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$. Then,

$$\text{Sum} = 20 \Rightarrow (a-3d)+(a-d)+(a+d)+(a+3d) = 20 \Rightarrow 4a = 20 \Rightarrow a = 5$$

and, Sum of the squares = 120

$$\Rightarrow (a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2 = 120$$

$$\Rightarrow \quad 4a^2 + 20d^2 = 120$$

$$\Rightarrow \quad a^2 + 5d^2 = 30$$

$$\Rightarrow \quad 25 + 5d^2 = 30 \qquad [\because a = 5]$$

$$\Rightarrow \quad 5d^2 = 5 \Rightarrow d = \pm 1$$

If $d = 1$, and $a = 5$, then the numbers are 2, 4, 6, 8. If $d = -1$, and $a = 5$, then the numbers are 8, 6, 4, 2.

Thus, the numbers are 2, 4, 6, 8 or 8, 6, 4, 2.

EXAMPLE 3 *Divide 32 into four parts which are in A.P. such that the product of extremes is to the product of means is 7 : 15.*

SOLUTION Let the four parts be $(a - 3d)$, $(a - d)$, $(a + d)$ and $(a + 3d)$. Then,

$$\text{Sum} = 32 \Rightarrow (a - 3d) + (a - d) + (a + d) + (a + 3d) = 32 \Rightarrow 4a = 32 \Rightarrow a = 8$$

It is given that the product of extremes is to the product of means is 7 : 15.

$$\therefore \quad \frac{(a - 3d)(a + 3d)}{(a - d)(a + d)} = \frac{7}{15}$$

$$\Rightarrow \quad \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15} \Rightarrow \frac{64 - 9d^2}{64 - d^2} = \frac{7}{15} \Rightarrow 128d^2 = 512 \Rightarrow d^2 = 4 \Rightarrow d = \pm 2$$

Thus, the four of 32 parts are 2, 6, 10, 14.

When $a = 8$ and $d = 2$ four parts are: 2, 6, 10, and 14. When $a = 8$ and $d = -2$ four parts are 14, 10, 6 and 2.

LEVEL-2

EXAMPLE 4 *The product of three numbers in A.P. is 224, and the largest number is 7 times the smallest. Find the numbers.* **[NCERT EXEMPLAR]**

SOLUTION Let the three numbers in A.P. be $a - d$, a, $a + d$, where $d > 0$. Clearly, $a + d$ is the largest number and $a - d$ is the smallest number.

It is given that :

Product of numbers = 224 and, The largest number = 7 (The smallest numbers)

$$\Rightarrow \quad (a - d)\, a\, (a + d) = 224 \text{ and, } a + d = 7(a - d)$$

$$\Rightarrow \quad a(a^2 - d^2) = 224 \text{ and, } 6a = 8d$$

$$\Rightarrow \quad a(a^2 - d^2) = 224 \text{ and, } d = \frac{3a}{4}$$

$$\Rightarrow \quad a\left(a^2 - \frac{9}{16}a^2\right) = 224 \qquad [\text{On eliminating } d]$$

$$\Rightarrow \quad \frac{7a^3}{16} = 224$$

$$\Rightarrow \quad a^3 = 512 = 8^3$$

$$\Rightarrow \quad a = 8.$$

Putting $a = 8$ in $d = \frac{3a}{4}$, we obtain $d = 6$.

Hence, three numbers are 2, 8, 14.

EXAMPLE 5 *If the fourth power of the common difference of an A.P. with integer entries is added to the product of any four consecutive terms of it. Prove that the resulting sum is the square of an integer.*

SOLUTION Let $a-3d, a-d, a+d, a+3d$ be four consecutive terms of an A.P. with integer terms. Clearly, the common difference is $2d$. Since the terms are integers, therefore a and d are also integers.

Now, Given sum $= (a-3d)(a-d)(a+d)(a+3d)+(2d)^4$

$$= (a^2-9d^2)(a^2-d^2)+16d^4$$

$$= a^4-10a^2d^2+9d^4+16d^4$$

$$= a^4-10a^2d^2+25d^4$$

$= (a^2-5d^2)^2$, which is square of an integer as a and d are integers.

EXERCISE 19.3

LEVEL-1

1. The sum of three terms of an A.P. is 21 and the product of the first and the third terms exceeds the second term by 6, find three terms.
2. Three numbers are in A.P. If the sum of these numbers be 27 and the product 648, find the numbers.
3. Find the four numbers in A.P., whose sum is 50 and in which the greatest number is 4 times the least.

LEVEL-2

4. The sum of three numbers in A.P. is 12 and the sum of their cubes is 288. Find the numbers.
5. If the sum of three numbers in A.P. is 24 and their product is 440, find the numbers. [NCERT]
6. The angles of a quadrilateral are in A.P. whose common difference is 10°. Find the angles.

ANSWERS

1. 1, 7, 13	2. 6, 9, 12	3. 5, 10, 15, 20	4. 2, 4, 6 or 6, 4, 2
5. 5, 8, 11	6. 75°, 85°, 95°, 105°		

HINTS TO NCERT & SELECTED PROBLEMS

5. Let the three numbers be $a-d, a, a+d$. It is given that the sum and product of these numbers are 24 and 440 respectively. Therefore,

$$a-d+a+a+d=24 \text{ and } (a-d)\,a\,(a+d)=440$$

$$\Rightarrow \quad 3a=24 \text{ and } a(a^2-d^2)=440$$

$$\Rightarrow \quad a=8 \text{ and } a(a^2-d^2)=440$$

Now,

$$a(a^2-d^2)=440 \Rightarrow 8(64-d^2)=440 \Rightarrow 64-d^2=55 \Rightarrow d^2=9 \Rightarrow d=\pm 3$$

Hence, three numbers are 5, 8, 11 or 11, 8, 5.

19.5 SUM TO n TERMS OF AN A.P.

THEOREM *Show that the sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is*

$$S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\}$$

or, $$S_n = \frac{n}{2}(a + l),\ \textit{where } l = \textit{last term} = a + (n-1)\, d$$

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with first term a and common difference d. Then

$$a_1 = a,\ a_2 = a + d,\ a_3 = a + 2d,\ a_4 = a + 3d, \ldots, a_n = a + (n-1)\, d$$

Now,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

$$\Rightarrow \quad S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-2)\, d) + \{a + (n-1)\, d\} \quad \ldots\text{(i)}$$

Writing the above series in a reverse order, we get

$$S_n = \{a + (n-1)\, d\} + \{a + (n-2)\, d\} + \ldots + (a + d) + a \quad \ldots\text{(ii)}$$

Adding the corresponding terms of (i) and (ii), we get

$$2S_n = \{2a + (n-1)\, d\} + \{2a + (n-1)\, d\} + \ldots + \{2a + (n-1)\, d\}$$

$$\Rightarrow \quad 2S_n = n\{2a + (n-1)\, d\} \quad [\because 2a + (n-1)\, d \text{ repeats } n \text{ times}]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\}$$

Now, l = last term $= a + (n-1)\, d$

$$\therefore \quad S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\} = \frac{n}{2}\left[a + \left\{a + (n-1)\, d\right\}\right] = \frac{n}{2}\left(a + l\right)$$

ALITER We have,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_{n-2} + a_{n-1} + a_n \quad \ldots\text{(i)}$$

or, $$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_3 + a_2 + a_1 \quad \ldots\text{(ii)}$$

Adding corresponding terms in (i) and (ii), we get

$$2S_n = (a_1 + a_n) + (a_2 + a_{n-1}) + (a_3 + a_{n-2}) + \ldots + (a_{n-1} + a_2) + (a_n + a_1)$$

$$\Rightarrow \quad 2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \ldots + (a_1 + a_n) + (a_1 + a_n)$$

$$\Rightarrow \quad 2S_n = n\,(a_1 + a_n) \quad [\because a_1 + a_n = a_k + a_{n-k+1} \text{ for } k = 2, 3, \ldots, n]$$

$$\Rightarrow \quad S_n = \frac{n}{2}(a_1 + a_n)$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{a_1 + a_1 + (n-1)\, d\right\} \quad [\because a_n = a_1 + (n-1)\, d]$$

$$\Rightarrow \quad S_n = \frac{n}{2}\left\{2a_1 + n - 1)\, d\right\}$$

NOTE 1 *In the formula $S_n = \frac{n}{2}\left\{2a + (n-1)\, d\right\}$, there are four quantities viz. S_n, a, n and d. If any three of these are known, the fourth can be determined. Sometimes two of these quantities are given, in such cases remaining two quantities are provided by some other relations.*

NOTE 2 *If the sum S_n of n terms of a sequence is given, then nth term a_n of the sequence can be determined by the using formula: $a_n = S_n - S_{n-1}$*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SUM OF GIVEN NUMBER OF TERMS OF AN A.P.**

Formula: $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$ or, $S_n = \frac{n}{2}\left\{a + l\right\}$.

EXAMPLE 1 *Find the sum of 20 terms of the A.P. 1, 4, 7, 10, ...*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Clearly, $a = 1$, $d = 3$. We have to find the sum of 20 terms of the given A.P. Putting $a = 1$, $d = 3$, $n = 20$ in $S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$, we get

$$S_{20} = \frac{20}{2}\left\{2 \times 1 + (20-1) \times 3\right\} = 10 \times 59 = 590$$

EXAMPLE 2 *Find the sum of the series : 5 + 13 + 21 + ... + 181.*

SOLUTION The terms of the given series form an A.P. with first term $a = 5$ and common difference $d = 8$. Let there be n terms in the given series. Clearly,

$$a_n = 181 \Rightarrow a + (n-1)\,d = 181 \Rightarrow 5 + (n-1) \times 8 = 181 \Rightarrow 8n = 184 \Rightarrow n = 23$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{23}{2}(5 + 181) = 2139.$$

EXAMPLE 3 *Find the sum of all three digit natural numbers, which are divisible by 7.*

SOLUTION The smallest and the largest numbers of three digits, which are divisible by 7 are 105 and 994 respectively. So, the sequence of three digit numbers which are divisible by 7 are 105, 112, 119, ..., 994. Clearly, these numbers are in A.P. with first term $a = 105$ and common difference $d = 7$.

Let there be n terms in this sequence. Then,

$$a_n = 994 \Rightarrow a + (n-1)\,d = 994 \Rightarrow 105 + (n-1) \times 7 = 994 \Rightarrow n = 128$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = \frac{128}{2}\left\{2 \times 105 + (128-1) \times 7\right\} = 70336$$

EXAMPLE 4 *Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.*

SOLUTION Clearly, the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258, ..., 999. These numbers are in A.P. with first term $a = 252$, common difference = 3 and last term = 999. Let there be n terms in this A.P. Then,

$$a_n = 999 \Rightarrow a + (n-1)\,d = 999 \Rightarrow 252 + (n-1) \times 3 = 999 \Rightarrow n = 250$$

$$\therefore \quad \text{Required sum} = S_n = \frac{n}{2}(a + l) = \frac{250}{2}(252 + 999) = 156375$$

EXAMPLE 5 *Find the sum of all odd integers between 2 and 100 divisible by 3.*

SOLUTION The odd integers between 2 and 100 which are divisible by 3 are 3, 9, 15, 21, ... , 99. Clearly, these numbers are in A.P. with first term $a = 3$ and common difference $d = 6$. Let there be n terms in this sequence. Then,

$$a_n = 99 \Rightarrow a + (n-1)\,d \Rightarrow 3 + (n-1) \times 6 = 99 \Rightarrow n = 17$$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + l) = \frac{17}{2}(3 + 99) = 867.$$

EXAMPLE 6 *Find the sum of first 20 terms of an A.P., in which 3rd term is 7 and 7th term is two more than thrice of its 3rd term.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_3 = 7 \quad \text{and} \quad a_7 = 3a_3 + 2$$

$$\Rightarrow \quad a + 2d = 7 \quad \text{and} \quad a + 6d = 3(a + 2d) + 2$$

$\Rightarrow \quad a + 2d = 7$ and $a = -1$

$\Rightarrow \quad a = -1, d = 4$

$\therefore \quad S_{20} = \frac{20}{2}\left\{2 \times -1 + (20-1) \times 4\right\}$ $\left[\text{Using: } S_n = \frac{n}{2}\left\{2a + (n-1)d\right\}\right]$

$\Rightarrow \quad S_{20} = \frac{20}{2}(-2+76) = 740$

EXAMPLE 7 *The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.* **[NCERT]**

SOLUTION Let there be n terms in the A.P. with first term $a = 11$ and common difference d. It is given that

Sum of first four terms $= 56$

$\Rightarrow \quad \frac{4}{2}\left\{2 \times 11 + (4-1)d\right\} = 56$

$\Rightarrow \quad 22 + 3d = 28 \Rightarrow 3d = 6 \Rightarrow d = 2$

It is also given that

Sum of last four terms $= 112$

$\Rightarrow \quad a_n + a_{n-1} + a_{n-2} + a_{n-3} = 112$

$\Rightarrow \quad \frac{4}{2}(a_n + a_{n-3}) = 112$ $\left[\text{Using: } S_n = \frac{n}{2}(a+l)\right]$

$\Rightarrow \quad a_n + a_{n-3} = 56$

$\Rightarrow \quad \{11 + (n-1) \times 2\} + \{11 + (n-4) \times 2\} = 56$

$\Rightarrow \quad 22 + 2(2n-5) = 56$

$\Rightarrow \quad 4n = 44 \Rightarrow n = 11.$

Hence, there are 11 terms in the A.P.

EXAMPLE 8 *If the sum of n terms of an A.P. is $pn + qn^2$, where p and q are constants, find the common difference.* **[NCERT]**

SOLUTION Let S_n denote the sum of n terms and a_n denote the nth term of the A.P. Then,

$S_n = pn + qn^2$

$\Rightarrow \quad S_{n-1} = p(n-1) + q(n-1)^2$ [On replacing n by $(n-1)$ in S_n]

Now, $\quad a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = \{pn + qn^2\} - \{p(n-1) + q(n-1)^2\}$

$\Rightarrow \quad a_n = pn - p(n-1) + qn^2 - q(n-1)^2$

$\Rightarrow \quad a_n = p\{n - (n-1)\} + q\{n^2 - (n-1)^2\}$

$\Rightarrow \quad a_n = p + q(2n-1)$

$\therefore \quad a_{n-1} = p + q\{2(n-1) - 1\}$ [Replacing n by $(n-1)$ in a_n]

Let d be the common difference of the A.P. Then,

$d = a_n - a_{n-1}$

$\Rightarrow \quad d = \{p + q(2n-1)\} - [p + q\{2(n-1) - 1\}]$

$\Rightarrow \quad d = \{p + q(2n-1)\} - \{p + q(2n-3)\}$

$\Rightarrow \quad d = q(2n - 1 - 2n + 3) = 2q$

Hence, the common difference $= 2q$.

EXAMPLE 9 *If the sum of n terms of an A.P. is $3n^2+5n$ and its mth term is 164, find the value of m.* **[NCERT]**

SOLUTION Let S_n denote the sum of n terms and a_n be the nth term of the given A.P. Then,

$$S_n = 3n^2 + 5n$$

$\Rightarrow \quad S_{n-1} = 3(n-1)^2 + 5(n-1) = 3n^2 - n - 2$ [On replacing n by $(n-1)$ in S_n]

Now, $a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = (3n^2+5n) - (3n^2 - n - 2)$

$\Rightarrow \quad a_n = 6n + 2$

Now, $a_m = 164$ [Given]

$\Rightarrow \quad 6m + 2 = 164 \Rightarrow 6m = 162 \Rightarrow m = 27$

EXAMPLE 10 *Find the sum to n terms of the sequence given by $a_n = 5 - 6n, n \in N$.*

SOLUTION We have, $a_n = 5 - 6n$

$\therefore \quad a_{n+1} = 5 - 6(n+1) = -1 - 6n$

$\therefore \quad a_{n+1} - a_n = (-1-6n) - (5-6n) = -6$, for all $n \in N$

Since $a_{n+1} - a_n$ is constant for all $n \in N$. So, the given sequence is an A.P. with common difference -6.

Putting $n = 1$, in $a_n = 5 - 6n$, we get: $a_1 = -1$.

So, the sum S_n to n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(-1 + 5 - 6n) = n(2 - 3n)$$

EXAMPLE 11 *If the m^{th} term of an A.P. is $\frac{1}{n}$ and the n^{th} term is $\frac{1}{m}$, show that the sum of mn terms is $\frac{1}{2}(mn+1)$, where $m \neq n$.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. It is given that

$$a_m = \frac{1}{n} \text{ and } a_n = \frac{1}{m}$$

Now, $a_m = \frac{1}{n} \Rightarrow a + (m-1)\,d = \frac{1}{n}$ ...(i)

and, $a_n = \frac{1}{m} \Rightarrow a + (n-1)\,d = \frac{1}{m}$ ...(ii)

Subtracting (ii) from (i), we get

$$(m-n)\,d = \frac{1}{n} - \frac{1}{m} \Rightarrow (m-n)\,d = \frac{m-n}{mn} \Rightarrow d = \frac{1}{mn}$$

Putting $d = \frac{1}{mn}$ in (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n} \Rightarrow a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

$$\therefore \quad S_{mn} = \frac{mn}{2}\left\{2a + (mn-1)\,d\right\} = \frac{mn}{2}\left\{\frac{2}{mn} + (mn-1) \times \frac{1}{mn}\right\} = \frac{1}{2}(mn+1)$$

Type II **FINDING THE NUMBER OF TERMS IN AN A.P. WHEN THE SUM OF ITS n TERMS IS GIVEN**

EXAMPLE 12 *How many terms of the series 54, 51, 48, ... be taken so that their sum is 513 ? Explain the double answer.*

SOLUTION Clearly, the given sequence is an A.P. with first term $a = 54$ and common difference $d = -3$. Let the sum of n terms be 513. Then,

$$S_n = 513$$

$$\Rightarrow \frac{n}{2}\left\{2a + (n-1)\, d\right\} = 513$$

$$\Rightarrow \frac{n}{2}\left\{108 + (n-1) \times -3\right\} = 513 \Rightarrow n^2 - 37n + 342 = 0 \Rightarrow (n-18)(n-19) = 0 \Rightarrow n = 18 \text{ or, } 19$$

Here, the common difference is negative. So, the terms are in decreasing order and the value of 19^{th} term is $54 + (19-1) \times -3 = 0$. Thus, the sum of 18 terms as well as that of 19 terms is 513.

EXAMPLE 13 *Find the number of terms in the series* $20, 19\frac{1}{3}, 18\frac{2}{3}, \ldots$ *of which the sum is 300, explain the double answer.*

SOLUTION The given sequence is an A.P. with first term $a = 20$ and the common difference $d = -\frac{2}{3}$. Let the sum of n terms be 300. Then,

$$S_n = 300$$

$$\Rightarrow \quad \frac{n}{2}\left\{2a + (n-1)\, d\right\} = 300$$

$$\Rightarrow \quad \frac{n}{2}\left\{2 \times 20 + (n-1)\left(-\frac{2}{3}\right)\right\} = 300$$

$$\Rightarrow \quad n^2 - 61n + 900 = 0 \Rightarrow (n-25)(n-36) = 0 \Rightarrow n = 25 \text{ or, } 36$$

$\therefore$ Sum of 25 terms = Sum of 36 terms = 300.

Here, the common difference is negative therefore terms go on diminishing and 31*st* term becomes zero. All terms following 31*st* term are negative. These negative terms when added to positive terms from 26*th* term to 30*th term*, they cancel out each other and the sum remains same. Hence, the sum of 25 terms as well as that of 36 terms is 300.

EXAMPLE 14 *Solve* $1 + 6 + 11 + 16 + \ldots + x = 148$.

SOLUTION Clearly, terms of the given series form an A.P. with first term $a = 1$ and common difference $d = 5$. Let there be n terms in this series. Then,

$$1 + 6 + 11 + 16 + \ldots + x = 148$$

$$\Rightarrow \quad \text{Sum of } n \text{ terms} = 148$$

$$\Rightarrow \quad \frac{n}{2}\left\{2a + (n-1)\, d\right\} = 148$$

$$\Rightarrow \quad \frac{n}{2}\left\{2 + (n-1) \times 5\right\} = 148 \Rightarrow 5n^2 - 3n - 296 = 0 \Rightarrow (n-8)(5n+37) = 0 \Rightarrow n = 8$$

Clearly, $x = n^{th}$ term

$$\Rightarrow \quad x = a + (n-1)\, d = 1 + (8-1) \times 5 = 36 \qquad [\because a = 1, d = 5, n = 8]$$

Type III PROVING RESULTS RELATED TO THE SUM OF *n* TERMS OF AN A.P.

EXAMPLE 15 *The sum of the first p, q, r terms of an A.P. are a, b, c respectively. Show that*

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$ **[NCERT]**

SOLUTION Let A be the first term and D be the common difference of the given A.P. Then,

$$a = \text{Sum of } p \text{ terms} \Rightarrow a = \frac{p}{2}\left\{2A+(q-1)D\right\} \Rightarrow \frac{2a}{p} = \left\{2A+(p-1)D\right\} \quad \text{...(i)}$$

$$b = \text{Sum of } q \text{ terms} \Rightarrow b = \frac{q}{2}\left\{2A+(q-1)D\right\} \Rightarrow \frac{2b}{q} = \left\{2A+(q-1)D\right\} \quad \text{...(ii)}$$

and, $$c = \text{Sum of } r \text{ terms} \Rightarrow c = \frac{r}{2}\left\{2A+(r-1)D\right\} \Rightarrow \frac{2c}{r} = \left\{2A+(r-1)D\right\} \quad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by $(q-r)$, $(r-p)$ and $(p-q)$ respectively and adding, we get

$$\frac{2a}{p}(q-r)+\frac{2b}{q}(r-p)+\frac{2c}{r}(p-q)$$

$$= \{2A+(p-1)D\}(q-r)+\{2A+(q-1)D\}(r-p)+\{2A+(r-1)D\}(p-q)$$

$$= 2A(q-r+r-p+p-q)+D\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\}$$

$$= 2A \times 0 + D \times 0 = 0$$

EXAMPLE 16 *The sum of n, 2n, 3n terms of an A.P. are S_1, S_2, S_3 respectively. Prove that: $S_3 = 3(S_2 - S_1)$.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_1 = \text{Sum of } n \text{ terms} \Rightarrow S_1 = \frac{n}{2}\left\{2a+(n-1)d\right\} \quad \text{...(i)}$$

$$S_2 = \text{Sum of } 2n \text{ terms} \Rightarrow S_2 = \frac{2n}{2}\left\{2a+(2n-1)d\right\} \quad \text{...(ii)}$$

and, $$S_3 = \text{Sum of } 3n \text{ terms} \Rightarrow S_3 = \frac{3n}{2}\left\{2a+(3n-1)d\right\} \quad \text{...(iii)}$$

Now, $$S_2 - S_1 = \frac{2n}{2}\left\{2a+(2n-1)d\right\} - \frac{n}{2}\left\{2a+(n-1)d\right\}$$

$$\Rightarrow \quad S_2 - S_1 = \frac{n}{2}\left[2\left\{2a+(2n-1)d\right\} - \left\{2a+(n-1)d\right\}\right] = \frac{n}{2}\left\{2a+(3n-1)d\right\}$$

$$\therefore \quad 3(S_2 - S_1) = \frac{3n}{2}\left\{2a+(3n-1)d\right\} \quad \text{...(iv)}$$

From (iii) and (iv), we get

$$3(S_2 - S_1) = S_3$$

EXAMPLE 17 *The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3. The first term of each is unity and the common differences are 1, 2 and 3 respectively. Prove that $S_1 + S_3 = 2S_2$.*

SOLUTION We have,

S_1 = Sum of n terms of an A.P. with first term 1 and common difference 1

$$\Rightarrow \quad S_1 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 1\right\} = \frac{n}{2}(n+1)$$

S_2 = Sum of n terms of an A.P. with first term 1 and common difference 2

$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 2\right\} = n^2$

S_3 = Sum of n terms of and A.P. with first term 1 and common difference 3

$\Rightarrow \quad S_2 = \frac{n}{2}\left\{2 \times 1 + (n-1) \times 3\right\} = \frac{n}{2}(3n-1)$

$\therefore \quad S_1 + S_3 = \frac{n}{2}(n+1) + \frac{n}{2}(3n-1) = 2n^2$

Hence, $S_1 + S_3 = 2S_2$ $\quad [\because S_2 = n^2]$

EXAMPLE 18 *If in an A.P. the sum of m terms is equal to n and the sum of n terms is equal to m, then prove that the sum of (m + n) terms is – (m + n). Also, find the sum of first (m – n) terms (m > n).*

[NCERT EXEMPLAR]

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = n \Rightarrow \frac{m}{2}\left\{2a + (m-1)d\right\} = n \Rightarrow 2am + m(m-1)d = 2n \quad \text{...(i)}$$

and, $$S_n = m \Rightarrow \frac{n}{2}\left\{2a + (n-1)d\right\} = m \Rightarrow 2an + n(n-1)d = 2m \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$2a(m-n) + \{m(m-1) - n(n-1)\}d = 2n - 2m$$

$$\Rightarrow \quad 2a(m-n) + \{(m^2 - n^2) - (m-n)\}d = -2(m-n)$$

$$\Rightarrow \quad 2a + (m+n-1)d = -2 \quad \text{[On dividing both sides by } (m-n)] \quad \text{...(iii)}$$

Now, $$S_{m+n} = \frac{m+n}{2}\left\{2a + (m+n-1)d\right\}$$

$$\Rightarrow \quad S_{m+n} = \frac{(m+n)}{2}(-2) \quad \text{[Using (iii)]}$$

$$\therefore \quad S_{m+n} = -(m+n)$$

From (iii), we obtain

$$2a = -2 - (m+n-1)d \quad \text{...(iv)}$$

Substituting this value of $2a$ in (i), we obtain

$$-2m - m(m+n-1)d + m(m-1)d = 2n$$

$$\Rightarrow \quad d = -2\left(\frac{m+n}{mn}\right) \quad \text{...(v)}$$

Putting $d = -2\left(\frac{m+n}{mn}\right)$ in (iv), we obtain

$$2a = -2 + \frac{2}{mn}(m+n-1)(m+n) \quad \text{...(vi)}$$

Now,

$$S_{m-n} = \frac{m-n}{2}\left\{2a + (m-n-1)d\right\}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2 + \frac{2}{mn}(m+n-1)(m+n) - \frac{2}{mn}(m-n-1)(m+n)\right\} \quad \text{[Using (v) and (vi)]}$$

$$\Rightarrow \quad S_{m-n} = \frac{m-n}{2}\left\{-2 + \frac{4n}{mn}(m+n)\right\} = \frac{1}{m}(m-n)(m+2n)$$

EXAMPLE 19 *If the sum of first m terms of an A.P. is the same as the sum of its first n terms, show that the sum of its (m + n) terms is zero.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then,

$$S_m = S_n$$

$$\Rightarrow \quad \frac{m}{2}\left\{2a + (m-1)\,d\right\} = \frac{n}{2}\left\{2a + (n-1)\,d\right\}$$

$$\Rightarrow \quad 2a\,(m-n) + \{m\,(m-1) - n\,(n-1)\}\,d = 0$$

$$\Rightarrow \quad 2a\,(m-n) + \{(m^2 - n^2) - (m-n)\}\,d = 0$$

$$\Rightarrow \quad (m-n)\,\{2a + (m+n-1)\,d\} = 0$$

$$\Rightarrow \quad 2a + (m+n-1)\,d = 0 \qquad [\because m - n \neq 0] \quad \ldots \text{(i)}$$

$$\therefore \quad S_{m+n} = \frac{m+n}{2}\left\{2a + (m+n-1)\,d\right\} = \frac{m+n}{2} \times 0 = 0 \qquad \text{[Using (i)]}$$

EXAMPLE 20 *The ratio of the sums of m and n terms of an A.P. is* $m^2 : n^2$. *Show that the ratio of the mth and nth terms is* $(2m-1):(2n-1)$. **[NCERT]**

SOLUTION Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}\left\{2a + (m-1)\,d\right\} \quad \text{and,} \quad S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} \text{ respectively.}$$

It is given that

$$\frac{S_m}{S_n} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{\frac{m}{2}\left\{2a + (m-1)\,d\right\}}{\frac{n}{2}\left\{2a + (n-1)\,d\right\}} = \frac{m^2}{n^2}$$

$$\Rightarrow \quad \frac{2a + (m-1)\,d}{2a + (n-1)\,d} = \frac{m}{n}$$

$$\Rightarrow \quad \{2a + (m-1)\,d\}\,n = \{2a + (n-1)\,d\}\,m$$

$$\Rightarrow \quad 2a\,(n-m) = d\,\{(n-1)\,m - (m-1)\,n\}$$

$$\Rightarrow \quad 2a\,(n-m) = d\,(n-m)$$

$$\Rightarrow \quad d = 2a$$

$$\therefore \quad \frac{T_m}{T_n} = \frac{a + (m-1)\,d}{a + (n-1)\,d} = \frac{a + (m-1)\,2a}{a + (n-1)\,2a} = \frac{2m-1}{2n-1}$$

EXAMPLE 21 *The interior angles of a polygon are in A.P. The smallest angle is* $120°$ *and the common difference is* $5°$. *Find the number of sides of the polygon.* **[NCERT]**

SOLUTION Let there be n sides of the polygon. Then, the sum of its interior angles is given by

$$S_n = (2n-4) \text{ right angles} = (n-2) \times 180° \qquad \ldots \text{(i)}$$

Thus, the interior angles form an A.P. with first term $a = 120°$ and common difference $d = 5°$.

$$\therefore \quad S_n = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\} \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$(n-2) \times 180° = \frac{n}{2}\left\{2 \times 120° + (n-1) \times 5°\right\}$$

$$\Rightarrow \quad (n-2) \times 360 = n\,(5n + 235)$$

$$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n-16)\,(n-9) = 0 \Rightarrow n = 16 \text{ or, } n = 9$$

For $n=16$, we obtain
Last angle $= a_n = a + (n-1)\,d = 120° + (16-1) \times 5 = 195°$, which is not possible.
Hence, $n = 9$.

EXAMPLE 22 *The first, second and the last terms of an A.P. are a, b, c respectively. Prove that the sum is* $\dfrac{(a+c)(b+c-2a)}{2(b-a)}$. **[NCERT EXEMPLAR]**

SOLUTION Let d be the common difference of the given A.P. Then, $d = b - a$. Let there be n terms in the given A.P. Then,

$$c = n\text{th term}$$
$$\Rightarrow \quad c = a + (n-1)\,d$$
$$\Rightarrow \quad c = a + (n-1)(b-a) \qquad [\because d = b - a]$$
$$\Rightarrow \quad n - 1 = \frac{c-a}{b-a} \Rightarrow n = \frac{c-a}{b-a} + 1 \Rightarrow n = \frac{b+c-2a}{b-a}$$
$$\therefore \quad \text{Sum of the A.P.} = \text{Sum of its } n \text{ terms}$$
$$= \frac{n}{2}(a+c) \qquad \left[\text{Using}: S_n = \frac{n}{2}(a+l)\right]$$
$$= \frac{(a+c)(b+c-2a)}{2(b-a)}.$$

EXAMPLE 23 *Let S_n denote the sum of the first n terms of an A.P. If $S_{2n} = 3\,S_n$, then prove that* $\dfrac{S_{3n}}{S_n} = 6$.

SOLUTION Let a be the first term and d the common difference of the given A.P. Then,

$$S_{2n} = 3\,S_n$$
$$\Rightarrow \quad \frac{2n}{2}\left\{a + (2n-1)\,d\right\} = \frac{3n}{2}\left\{a + (n-1)\,d\right\}$$
$$\Rightarrow \quad 2\,\{a + (2n-1)\,d\} = 3\,\{a + (n-1)\,d\}$$
$$\Rightarrow \quad 2a - (3n - 3 - 4n + 2)\,d = 0$$
$$\Rightarrow \quad 2a - (n+1)\,d = 0$$
$$\Rightarrow \quad 2a = (n+1)\,d \qquad \text{...(i)}$$

Now,
$$\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}\left\{2a + (3n-1)\,d\right\}}{\frac{n}{2}\left\{2a + (n-1)\,d\right\}}$$
$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{3\left\{(n+1)\,d + (3n-1)\,d\right\}}{\left\{(n+1)\,d + (n-1)\,d\right\}} \qquad \text{[Using (i)]}$$
$$\Rightarrow \quad \frac{S_{3n}}{S_n} = \frac{12\,n\,d}{2nd} = 6.$$

LEVEL-2

Type IV ON SUM OF TERMS OF AN A.P.

EXAMPLE 24 *Prove that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants.*

SOLUTION Let S_n be the sum of n terms of an A.P. with first term a and common difference d. Then,

$$S_n = \frac{n}{2}\left\{2a + (n-1)\,d\right\} = an + \frac{n^2}{2}d - \frac{n}{2}d = \left(\frac{d}{2}\right)n^2 + \left(a - \frac{d}{2}\right)n$$

$\Rightarrow \quad S_n = An^2 + Bn$, where $A = \dfrac{d}{2}$ and $B = a - \dfrac{d}{2}$

Thus, the sum of n terms of an A.P. is of the form $An^2 + Bn$.

Conversely, let the sum S_n of n terms of a sequence $a_1, a_2, a_3, \dots, a_n, \dots$ be of the form $An^2 + Bn$. Then, we have to show that the sequence is an A.P.

We have, $S_n = An^2 + Bn$

$\Rightarrow \quad S_{n-1} = A(n-1)^2 + B(n-1)$ [On replacing n by $n-1$]

Now, $a_n = S_n - S_{n-1}$

$\Rightarrow \quad a_n = \{An^2 + Bn\} - \{A(n-1)^2 + B(n-1)\} = 2An + (B-A)$

$\Rightarrow \quad a_{n+1} = 2A(n+1) + (B-A)$ [On replacing n by $n+1$]

$\therefore \quad a_{n+1} - a_n = \{2A(n+1) + B - A\} - \{2An + (B-A)\} = 2A$

Clearly, $a_{n+1} - a_n = 2A$ for all $n \in N$. So, the sequence is an A.P. with common difference $2A$.

REMARK *It follows from this example that a sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2. For example, if $S_n = 3n^2 + 2n$, one can easily say that it is the sum of n terms of an A.P. with common difference 6. Similarly, $S_n = nP + \frac{1}{2}n(n-1)Q = \frac{Q}{2}n^2 + \left(P - \frac{Q}{2}\right)n$ is the sum of n terms of an A.P, with common difference Q.*

EXAMPLE 25 *Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \dots$, if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.* **[NCERT EXEMPLAR]**

SOLUTION We know that in an A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term i.e. $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \dots$ So, if an A.P. consists of 24 terms, then $a_1 + a_{24} = a_5 + a_{20} = a_{10} + a_{15}$.

Now, $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow \quad (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow \quad 3(a_1 + a_{24}) = 225$

$\Rightarrow \quad a_1 + a_{24} = \dfrac{225}{3} = 75$ ...(i)

$\therefore \quad S_{24} = \dfrac{24}{2}(a_1 + a_{24})$ $\left[\text{Using } S_n = \dfrac{n}{2}(a_1 + a_n)\right]$

$\Rightarrow \quad S_{24} = 12(75) = 900$ [Using (i)]

EXAMPLE 26 *The first term of an A.P. is a and the sum of first p terms is zero, show that the sum of its next q terms is $-\dfrac{a(p+q)q}{p-1}$.* **[NCERT EXEMPLAR]**

SOLUTION Let d be the common difference of the A.P. It is gtiven that the sum of first p terms is zero

i.e. $\quad S_p = 0 \Rightarrow \dfrac{p}{2}\left\{2a + (p-1)d\right\} = 0 \Rightarrow d = -\dfrac{2a}{p-1}$.

Let S be the required sum. Then,

$S = a_{p+1} + a_{p+2} + \dots + a_{p+q}$

$\Rightarrow \quad S = (a_1 + a_2 + \dots + a_p + a_{p+1} + \dots + a_{p+q}) - (a_1 + a_2 + \dots + a_p)$

$\Rightarrow \quad S = S_{p+q} - S_p$

$\Rightarrow \quad S = S_{p+q} - 0 \qquad [\because S_p = 0 \text{ (given)}]$

$$\Rightarrow \quad S = \frac{p+q}{2}\left\{2a + (p+q-1)d\right\}$$

$$\Rightarrow \quad S = \frac{p+q}{2}\left\{2a + (p+q-1)\left(-\frac{2a}{p-1}\right)\right\}$$

$$\Rightarrow \quad S = (p+q)\, a\left\{1 - \left(\frac{p+q-1}{p-1}\right)\right\} = (p+q)\, a\left(\frac{p-1-p-q+1}{p-1}\right) = -\frac{(p+q)\, a}{p-1}$$

EXAMPLE 27 *If the first term of an A.P. is 2 and the sum of first five terms is equal to one-fourth of the sum of the next five terms, find the sum of first 30 terms.*

SOLUTION Let $a_1, a_2, a_3, \ldots$ be given A.P. with common difference d. It is given that $a_1 = 2$ and the sum of first five terms is equal to one fourth of the sum of next five terms.

i.e. $$a_1 + a_2 + a_3 + a_4 + a_5 = \frac{1}{4}(a_6 + a_7 + a_8 + a_9 + a_{10})$$

$$\Rightarrow \quad 4(a_1 + a_2 + a_3 + a_4 + a_5) = (a_6 + a_7 + a_8 + a_9 + a_{10})$$

$$\Rightarrow \quad 5(a_1 + a_2 + a_3 + a_4 + a_5) = (a_1 + a_2 + \ldots + a_{10})$$

$$\Rightarrow \quad 5S_5 = S_{10}$$

$$\Rightarrow \quad 5\left[\frac{5}{2}\left\{2 \times 2 + (5-1)d\right\}\right] = \frac{10}{2}\left\{2 \times 2 + (10-1)d\right\}$$

$$\Rightarrow \quad 50(1+d) = 20 + 45d$$

$$\Rightarrow \quad d = -6$$

Thus, we have $a = 2$ and $d = -6$.

$$\therefore \quad \text{Required sum } = S_{30} = \frac{30}{2}\left\{2 \times 2 + (30-1) \times -6\right\} = -2550.$$

EXAMPLE 28 *The p^{th} term of an A.P. is a and qth term is b. Prove that the sum of its $(p+q)$ terms is* $\frac{p+q}{2}\left\{a + b + \frac{a-b}{p-q}\right\}$. **[NCERT EXEMPLAR]**

SOLUTION Let A and D be the first term and common difference respectively of the given A.P. Then,

$$a = p\text{th term} \Rightarrow a = A + (p-1)D \qquad \text{...(i)}$$

$$b = q\text{th term} \Rightarrow b = A + (q-1)D \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we get: $D = \frac{a-b}{p-q}$

Adding (i) and (ii), we get $a + b = 2A + (p+q-2)D$

$$\Rightarrow \quad a + b = 2A + (p+q-1)D - D$$

$$\Rightarrow \quad (a+b) + D = 2A + (p+q-1)D$$

$$\Rightarrow \quad (a+b) + \frac{a-b}{p-q} = 2A + (p+q-1)D \qquad \text{...(iii)}$$

Now, S_{p+q} = Sum of $(p+q)$ terms

$$\Rightarrow \quad S_{p+q} = \frac{p+q}{2}\left\{2A + (p+q-1)D\right\} = \frac{p+q}{2}\left\{a + b + \frac{a-b}{p-q}\right\} \quad \text{[Using (iii)]}$$

EXAMPLE 29 *The ratio of the sum of n terms of two A.P.'s is $(7n+1):(4n+27)$. Find the ratio of their mth terms.*

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums S_n and S_n' of their n terms are given by

$$S_n = \frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}, \text{ and } S_n' = \frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}$$

$$\therefore \quad \frac{S_n}{S_n'} = \frac{\frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}}{\frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27} \quad \ldots \text{(i)}$$

We have to find the ratio to m^{th} terms of two A.P.'s i.e., $\frac{a_1 + (m-1)d_1}{a_1 + (m-1)d_2}$. Clearly, this can be obtained by replacing $\frac{n-1}{2}$ by $(m-1)$ on the LHS of (i). Replacing $\frac{n-1}{2}$ by $m-1$ i.e. n by $(2m-1)$ on both side of (i), we get

$$\frac{a_1 + (m-1)d_1}{a_2 + (m-1)d_2} = \frac{7(2m-1)+1}{4(2m-1)+27} = \frac{14m-6}{8m+23}$$

Hence, the ratio of the mth terms of the two A.P.'s is $(14m-6):(8m+23)$.

<u>REMARK</u> *It is evident from the above example that if we are given the ratio of the sums of n terms of two A.P.'s then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$.*

EXAMPLE 30 *The sum of n terms of two arithmetic progressions are in the ratio $(3n+8):(7n+15)$. Find the ratio of their 12th terms.* **[NCERT]**

SOLUTION Let a_1, a_2 be the first terms and d_1, d_2 the common differences of the two given A.P.'s. Then, the sums of their n terms are given by

$$S_n = \frac{n}{2}\{2a_1 + (n-1)d_1\} \text{ and, } S_n' = \frac{n}{2}\{2a_2 + (n-1)d_2\}$$

It is given that

$$\frac{S_n}{S_n'} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\left\{2a_1 + (n-1)d_1\right\}}{\frac{n}{2}\left\{2a_2 + (n-1)d_2\right\}} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{3n+8}{7n+15}$$

$$\Rightarrow \quad \frac{a_1+\left(\frac{n-1}{2}\right)d_1}{a_2+\left(\frac{n-1}{2}\right)d_2}=\frac{3n+8}{7n+15}$$

Replacing $\frac{n-1}{2}$ by 11 i.e. n by 23 on both sides, we get

$$\frac{a_1+11\,d_1}{a_2+11\,d_2}=\frac{3\times 23+8}{7\times 23+15}=\frac{77}{176}=\frac{7}{16}$$

Hence, the required ratio is 7 : 16.

ALITER If the ratio of the sums of n terms of two A.P.s is given, then the ratio of their m^{th} terms is obtained by replacing n by $(2m-1)$ in the given ratio. So, required ratio is obtained by replacing n by $2\times 12-1=23$ in $(3n+8):(7n+15)$.

Hence, required ratio $=(69+8):(161+15)=7:16$.

EXAMPLE 31 *If there are $(2n+1)$ terms in A.P., then prove that the ratio of the sum of odd terms and the sum of even terms is $(n+1):n$.* **[NCERT EXEMPLAR]**

SOLUTION Let a and d be the first term and common difference respectively of the given A.P. Let a_k denote the k^{th} terms of the given A.P. Then, $a_k=a+(k-1)\,d$.

Now, $\quad S_1=$ Sum of odd terms $=a_1+a_3+a_5+\ldots+a_{2n+1}$

$$\Rightarrow \quad S_1=\frac{n+1}{2}\left(a_1+a_{2n+1}\right)=\frac{n+1}{2}\left\{a+a+(2n+1-1)\,d\right\}=(n+1)\,(a+nd)$$

and, $\quad S_2=$ Sum of even terms $=a_2+a_4+a_6+\ldots+a_{2n}$

$$\Rightarrow \quad S_2=\frac{n}{2}\left(a_2+a_{2n}\right)=\frac{n}{2}\left[(a+d)+\left\{a+(2n-1)\,d\right\}\right]=n\,(a+nd)$$

$$\therefore \quad S_1:S_2=(n+1)\,(a+nd):n\,(a+nd)=(n+1):n$$

EXAMPLE 32 *Let S_k be the sum of first k terms of an A.P. What must this progression be for the ratio $\frac{S_{kx}}{S_x}$ to be independent of x ?*

SOLUTION Let a be the first term and d common difference of the given progression. Then,

$$\frac{S_{kx}}{S_x}=\frac{\frac{kx}{2}\left\{2a+(kx-1)\,d\right\}}{\frac{x}{2}\left\{2a+(x-1)\,d\right\}}=\frac{k\left\{kxd+(2a-d)\right\}}{\left\{xd+(2a-d)\right\}}$$

Clearly, the RHS of the above relation will be independent of x iff $2a-d=0$ i.e. $d=2a$.

Hence, the progression is $a,\ 3a,\ 5a,\ 7a,\ \ldots,$ where a is any non-zero real number.

EXAMPLE 33 *Let S_n be the sum of first n terms of an A.P. with non-zero common difference. Find the ratio of first term and common difference if $\frac{S_{n_1 n_2}}{S_{n_1}}$ is independent of n_1.*

SOLUTION Let the first term and common difference of the A.P. be a and d respectively. Then,

$$S_{n_1 n_2}=\frac{n_1 n_2}{2}\left\{2a+(n_1 n_2-1)\,d\right\} \text{ and, } S_{n_1}=\frac{n_1}{2}\left\{2a+(n_1-1)\,d\right\}$$

$$\therefore \quad \frac{S_{n_1 n_2}}{S_{n_1}}=\frac{n_2\left\{2a+(n_1 n_2-1)\,d\right\}}{\left\{2a+(n_1-1)\,d\right\}}=\frac{n_2\left\{(2a-d)+n_1 n_2\,d\right\}}{\left\{(2a-d)+n_1\,d\right\}}$$

Clearly, RHS will be independent of n_1 iff $2a - d = 0$ i.e. $d = 2a$.

Hence, $\frac{a}{d} = \frac{1}{2}$.

EXAMPLE 34 *If $S_1, S_2, S_3, \ldots S_m$ are the sums of n terms of m A.P.'s whose first terms are 1, 2, 3, ..., m and common differences are 1, 3, 5, ..., (2 m − 1) respectively. Show that*

$$S_1 + S_2 + \ldots + S_m = \frac{mn}{2}(mn+1)$$

SOLUTION The first terms, common difference and the sums of their n terms are as under:

First terms	*Common differences*	*Sums of n terms*
1	1	$S_1 = \frac{n}{2}\left\{2\times 1+(n-1)\times 1\right\}$
2	3	$S_2 = \frac{n}{2}\left\{2\times 2+(n-1)\times 3\right\}$
3	5	$S_3 = \frac{n}{2}\left\{2\times 3+(n-1)\times 5\right\}$
$\vdots$	$\vdots$	
m	$2m-1$	$S_m = \frac{n}{2}\left\{2m+(n-1)(2m-1)\right\}$

$\therefore$

$$\begin{aligned}
& S_1 + S_2 + \ldots + S_m \\
&= \frac{n}{2}\left[2\times 1+(n-1)\times 1\right] + \frac{n}{2}\left[2\times 2+(n-1)\times 3\right] + \ldots + \frac{n}{2}\left[2m+(n-1)(2m-1)\right] \\
&= \frac{n}{2}\left[2\times(1+2+3+\ldots+m)+(n-1)(1+3+5+\ldots+(2m-1)\right] \\
&= \frac{n}{2}\left[2\times\frac{m}{2}(1+m)+(n-1)\frac{m}{2}\{1+(2m-1)\}\right] \\
&= \frac{n}{2}\left[m(m+1)+m^2(n-1)\right] = \frac{mn}{2}(mn+1)
\end{aligned}$$

EXAMPLE 35 *If the sum of m terms of an A.P. is equal to the sum of either the next n terms or the next p terms, then prove that*

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right).$$

[NCERT EXEMPLAR]

SOLUTION Let a denote the first term and d the common difference of the A.P. Further, let a_k denote the k^{th} term of the A.P. Then,

Sum of m terms = Sum of next n terms

$$\begin{aligned}
&\Rightarrow\ a_1 + a_2 + a_3 + \ldots + a_m = a_{m+1} + a_{m+2} + \ldots + a_{m+n} \\
&\Rightarrow\ 2(a_1 + a_2 + \ldots + a_m) = a_1 + a_2 + \ldots + a_m + a_{m+1} + a_{m+2} + \ldots + a_{m+n} \\
&\Rightarrow\ 2S_m = S_{m+n} \\
&\Rightarrow\ 2\,\frac{m}{2}\left\{2a+(m-1)d\right\} = \frac{m+n}{2}\left\{2a+(m+n-1)d\right\} \\
&\Rightarrow\ \frac{2m}{m+n} = \frac{2a+(m+n-1)d}{2a+(m-1)d} \\
&\Rightarrow\ \frac{2m}{m+n} - 1 = \frac{2a+(m+n-1)d}{2a+(m-1)d} - 1 \\
&\Rightarrow\ \frac{m-n}{m+n} = \frac{nd}{2a+(m-1)d} \qquad \ldots\text{(i)}
\end{aligned}$$

Similarly,

Sum of m terms = Sum of next p terms

$$\Rightarrow \quad \frac{m-p}{m+p} = \frac{pd}{2a+(m-1)d} \quad \text{...(ii)}$$

Dividing (i) by (ii), we get

$$\frac{m-n}{m+n} \cdot \frac{m+p}{m-p} = \frac{n}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{n} = \frac{(m+n)(m-p)}{p}$$

$$\Rightarrow \quad \frac{(m-n)(m+p)}{nm} = \frac{(m+n)(m-p)}{mp}$$

$$\Rightarrow \quad (m+p)\left(\frac{m-n}{mn}\right) = (m+n)\left(\frac{m-p}{mp}\right)$$

$$\Rightarrow \quad (m+n)\left(\frac{1}{p}-\frac{1}{m}\right) = (m+p)\left(\frac{1}{n}-\frac{1}{m}\right)$$

$$\Rightarrow \quad (m+n)\left(\frac{1}{m}-\frac{1}{p}\right) = (m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

EXERCISE 19.4

LEVEL-1

1. Find the sum of the following arithmetic progressions:
   (i) 50, 46, 42, ... to 10 terms
   (ii) 1, 3, 5, 7, ... to 12 terms
   (iii) 3, 9/2, 6, 15/2, ... to 25 terms
   (iv) 41, 36, 31, ... to 12 terms
   (v) $a+b, a-b, a-3b, \ldots$ to 22 terms
   (vi) $(x-y)^2, (x^2+y^2), (x+y)^2, \ldots$ to n terms
   (vii) $\frac{x-y}{x+y}, \frac{3x-2y}{x+y}, \frac{5x-3y}{x+y}, \ldots$ to n terms
2. Find the sum of the following series:
   (i) $2+5+8+\ldots+182$
   (ii) $101+99+97+\ldots+47$
   (iii) $(a-b)^2+(a^2+b^2)+(a+b)^2+\ldots+[(a+b)^2+6ab]$
3. Find the sum of first n natural numbers.
4. Find the sum of all natural numbers between 1 and 100, which are divisible by 2 or 5. **[NCERT]**
5. Find the sum of first n odd natural numbers.
6. Find the sum of all odd numbers between 100 and 200.
7. Show that the sum of all odd integers between 1 and 1000 which are divisible by 3 is 83667.
8. Find the sum of all integers between 84 and 719, which are multiples of 5.

9. Find the sum of all integers between 50 and 500 which are divisible by 7.

10. Find the sum of all even integers between 101 and 999.

11. Find the sum of all integers between 100 and 550, which are divisible by 9.

12. Find the sum of the series: $3 + 5 + 7 + 6 + 9 + 12 + 9 + 13 + 17 + \ldots$ to $3n$ terms.

13. Find the sum of all those integers between 100 and 800 each of which on division by 16 leaves the remainder 7.

14. Solve: (i) $25 + 22 + 19 + 16 + \ldots + x = 115$ (ii) $1 + 4 + 7 + 10 + \ldots + x = 590$.

15. Find the rth term of an A.P., the sum of whose first n terms is $3n^2 + 2n$.

[NCERT EXEMPLAR]

16. How many terms are there in the A.P. whose first and fifth terms are -14 and 2 respectively and the sum of the terms is 40 ?

17. The sum of first 7 terms of an A.P. is 10 and that of next 7 terms is 17. Find the progression.

18. The third term of an A.P. is 7 and the seventh term exceeds three times the third term by 2. Find the first term, the common difference and the sum of first 20 terms.

19. The first term of an A.P. is 2 and the last term is 50. The sum of all these terms is 442. Find the common difference.

20. The number of terms of an A.P. is even; the sum of odd terms is 24, of the even terms is 30, and the last term exceeds the first by 10 1/2, find the number of terms and the series.

21. If $S_n = n^2 p$ and $S_m = m^2 p$, $m \neq n$, in an A.P., prove that $S_p = p^3$.

22. If 12th term of an A.P. is -13 and the sum of the first four terms is 24, what is the sum of first 10 terms ?

23. If the 5th and 12th terms of an A.P. are 30 and 65 respectively, what is the sum of first 20 terms ?

24. Find the sum of n terms of the A.P. whose kth terms is $5k + 1$. **[NCERT]**

25. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder. **[NCERT]**

26. If the sum of a certain number of terms of the AP 25, 22, 19, ... is 116. Find the last term. **[NCERT]**

27. Find the sum of odd integers from 1 to 2001. **[NCERT]**

28. How many terms of the A.P. $-6, -\frac{11}{2}, -5, \ldots$ are needed to give the sum -25?

29. In an A.P. the first term is 2 and the sum of the first five terms is one fourth of the next five terms. Show that 20th term is -112. **[NCERT]**

LEVEL-2

30. If S_1 be the sum of $(2n + 1)$ terms of an A.P. and S_2 be the sum of its odd terms, then prove that: $S_1 : S_2 = (2n + 1) : (n + 1)$.

31. Find an A.P. in which the sum of any number of terms is always three times the squared number of these terms.

32. If the sum of n terms of an A.P. is $nP + \frac{1}{2} n(n-1) Q$, where P and Q are constants, find the common difference. **[NCERT]**

33. The sums of n terms of two arithmetic progressions are in the ratio $5n + 4 : 9n + 6$. Find the ratio of their 18th terms. **[NCERT]**

34. The sums of first n terms of two A.P.'s are in the ratio $(7n + 2) : (n + 4)$. Find the ratio of their 5th terms.

ANSWERS

1. (i) 320 (ii) 144 (iii) 525 (iv) 162 (v) $22a - 440b$ (vi) $n\{(x-y)^2 + (n-1)\,xy\}$

(vii) $\dfrac{n}{2(x+y)}\left\{n(2x-y)-y\right\}$ 2. (i) 5612 (ii) 2072 (iii) $6(a^2+b^2+3ab)$

3. $\dfrac{n(n+1)}{2}$ 4. 3050 5. n^2 6. 7500 8. 50800 9. 17696 10. 246950

11. 16425 12. $3n(2n+3)$ 13. 19668 14. (i) -2 (ii) 58 15. $6r-1$

16. 10 17. $a=1,\ d=1/7$ 18. $-1, 4, 740$ 19. 3

20. 8 terms, $1\frac{1}{2}, 3, 4\frac{1}{2}, \ldots$ 22. 0 23. 1150 24. $\dfrac{n}{2}(5n+7)$ 25. 1210

26. 4 27. 1002001 28. 5 or 20 31. 3, 9, 15, 21 32. Q

33. 179 : 321 34. 5 : 1

HINTS TO NCERT & SELECTED PROBLEMS

3. Required sum $= 1+2+3+\ldots+n = \dfrac{n}{2}(1+n)$

4. Required sum = Sum of natural numbers between1 and 100 which are divisible by 2
+ Sum of natural numbers between 1 and 100 which are divisible by 5
− Sum of natural numbers between 1 and 100 which are divisible by 2 and 5 both i.e. by 10

$$= (2+4+\ldots+100)+(5+10+15+\ldots+100)-(10+20+\ldots+100)$$
$$= \frac{50}{2}(2+100)+\frac{20}{2}(5+100)-\frac{10}{2}(10+100)$$
$$= 2550+1050-550 = 3050$$

5. Required sum $= 1+3+5+\ldots+(2n-1) = \dfrac{n}{2}\left\{1+(2n-1)\right\} = n^2$

6. Required sum $= 101+103+\ldots+199 = \dfrac{50}{2}(101+199) = 7500$

8. Required sum $= 85+90+\ldots+715 = \dfrac{127}{2}(85+715) = 50800$

9. Required sum $= 56+63+\ldots+497$

10. Required sum $= 102+104+\ldots+998$

11. Required sum $= 108+117+\ldots+549$

12. Required sum $= (3+6+9+\ldots$ to n terms$) + (5+9+13+\ldots$ to n terms$) + (7+12+17+\ldots$ to n terms$)$

13. Required sum $= 103+119+135+\ldots+791$

15. $a_r = S_r - S_{r-1} = (3r^2+2r) - \left\{3(r-1)^2 + 2(r-1)\right\} = 6r-1$

17. We have, $S_7 = 10$ and $S_{14} = 10+17 = 27$

24. We have,

$$a_k = 5k+1 \Rightarrow a_1 = 6 \text{ and } a_n = 5n+1$$
$$\therefore \quad S_n = \frac{n}{2}\left(a_1 + a_n\right)$$
$$\Rightarrow \quad S_n = \frac{n}{2}\left(6+5n+1\right) = \frac{n}{2}\left(5n+7\right)$$

25. We have to find the sum of all two digit numbers of the form $4k+1$, $k \in N$. Clearly, such numbers are 13, 17, 21, 25, ..., 97 and are forming an A.P. with common difference 4. Let such numbers be n in number. Then,

$$97 = n^{\text{th}} \text{ term of AP with first term 13 and common difference 4}$$

$$\Rightarrow \quad 97 = 13 + (n-1) \times 4$$

$$\Rightarrow \quad n - 1 = 21$$

$$\Rightarrow \quad n = 22$$

Let S be the sum of such numbers. Then,

$$S = \frac{n}{2}\left(a_1 + a_n\right)$$

$$\Rightarrow \quad S = \frac{22}{2}\left(13 + 97\right) = 1210$$

26. Let the sum of n terms of the A.P. 25, 22, 19, ... be 116. Then,

$$116 = \frac{n}{2}\left\{2 \times 25 + (n-1) \times (-3)\right\}$$

$$\Rightarrow \quad 232 = n(-3n + 53)$$

$$\Rightarrow \quad 3n^2 - 53n + 232 = 0$$

$$\Rightarrow \quad 3n^2 - 24n - 29n + 232 = 0$$

$$\Rightarrow \quad (n-8)(3n-29) = 0$$

$$\Rightarrow \quad n = 8 \qquad \left[\because n \neq \frac{29}{3}\right]$$

$$\therefore \quad a_8 = 25 + 7 \times (-3) = 4$$

27. The odd integers from 1 to 2001 are 1, 3, 5, 7, ..., 2001. Let the number of such integers be n. Then,

$$2001 = 1 + (n-1) \times 2 \Rightarrow n = 1001$$

$$\therefore \quad \text{Required sum} = \frac{1001}{2}(1 + 2001) = 1001 \times 1001 = 1002001.$$

<u>ALITER</u> The sum of first n odd integers is n^2. So, the sum of odd integers 1, 3, 5, 7, ..., 2001 is $(1001)^2 = 1002001$.

29. We have,

$$a_1 = 2 \text{ and } a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$$

Now,

$$a_1 + a_2 + \ldots + a_5 = \frac{1}{4}(a_6 + a_7 + \ldots + a_{10})$$

$$\Rightarrow \quad 4(a_1 + a_2 + \ldots + a_5) = a_6 + a_7 + \ldots + a_{10}$$

$$\Rightarrow \quad 4S_5 = S_{10} - S_5$$

$$\Rightarrow \quad 5S_5 = S_{10}$$

$$\Rightarrow \quad 5\left[\frac{5}{2}\{2 \times 2 + (5-1)d\}\right] = \frac{10}{2}\left[2 \times 2 + (10-1)d\right]$$

$$\Rightarrow \quad \frac{25}{2}(4 + 4d) = \frac{10}{2}(9d + 4)$$

$$\Rightarrow \quad 20(1+d) = 2(9d+4) \Rightarrow 10 + 10d = 9d + 4 \Rightarrow d = -6$$

$\therefore \quad a_{20} = a_1 + 19d = 2 + 19 \times (-6) = -112$

31. Use $S_n = 3n^2$ and $a_n = S_n - S_{n-1}$.

32. We have,

$$S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_{n-1} = (n-1)P + \frac{1}{2}(n-1)(n-2)Q$$

Let a_n be the n^{th} term. Then,

$$a_n = S_n - S_{n-1}$$

$$\Rightarrow \quad a_n = \left\{nP + \frac{1}{2}n(n-1)Q\right\} - \left\{(n-1)P + \frac{1}{2}(n-1)(n-2)Q\right\}$$

$$\Rightarrow \quad a_n = P + \frac{1}{2}(n-1)\{n-(n-2)\}Q$$

$$\Rightarrow \quad a_n = P + (n-1)Q$$

$$\Rightarrow \quad a_{n-1} = P + (n-2)Q$$

Let d be the common difference. Then,

$$d = a_n - a_{n-1} = \{P + (n-1)Q\} - \{P + (n-2)Q\} = Q$$

<u>ALITER</u> We have,

$$S_n = nP + \frac{1}{2}n(n-1)Q \Rightarrow S_n = \frac{1}{2}n^2 Q + \left(P - \frac{Q}{2}\right)n$$

Clearly, S_n is of the form $An^2 + Bn$. Hence, the sequence is an A.P. with common difference $2A = Q$.

33. Let S_n and S'_n be the sums of n terms of two arithmetic progressions. Then,

$$\frac{S_n}{S_n'} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{\frac{n}{2}\{2a_1 + (n-1)d_1\}}{\frac{n}{2}\{2a_2 + (n-1)d_2\}} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{5n+4}{9n+6}$$

$$\Rightarrow \quad \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{5n+4}{9n+6}$$

Replacing $\frac{n-1}{2}$ by 17 i.e. n by 35, we get $\frac{a_1 + 17d_1}{a_2 + 17d_2} = \frac{179}{321}$

<u>ALITER</u> If the ratio of the sums of n terms is given, then to find the ratio of their n^{th} terms, we replace n by $(2n-1)$. So, to find the ratio of 18th terms, we replace n by $2 \times 18 - 1 = 35$ in the ratio $5n + 4 : 9n + 6$

Hence, required ratio is $(5 \times 35 + 4) : (9 \times 35 + 6)$ i.e. $179 : 321$.

19.6 PROPERTIES OF ARITHMETIC PROGRESSIONS

In this section, we shall discuss some properties of arithmetical progressions which will be frequently used in this chapter and in the subsequent chapters.

PROPERTY 1 *If a constant is added to or subtracted from each term of an A.P., then the resulting sequence is also an A.P. with the same common difference.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d, and let k be a fixed constant which is added to each term of this A.P. Then, the resulting sequence is $a_1 + k, a_2 + k, a_3 + k_1 \ldots$

Let $b_n = a_n + k, n = 1, 2, \ldots$ Then, the new sequence is $b_1, b_2, b_3, \ldots$

Now, $b_{n+1} - b_n = (a_{n+1} + k) - (a_n + k) = a_{n+1} - a_n = d$ for all $n \in N$

Thus, the new sequence is also an A.P. with common difference d.

PROPERTY 2 *If each term of a given A.P. is multiplied or divided by a non-zero constant k, then the resulting sequence is also an A.P. with common difference kd or d/k, where d is the common difference of the given A.P.*

PROOF Let $a_1, a_2, a_3, \ldots$ be an A.P. with common difference d and let k be a non-zero constant. Let $b_1, b_2, b_3, \ldots$ be sequence obtained by multiplying each term of the given A.P. by k. Then,

$$b_1 = a_1 k, b_2 = a_2 k, \ldots, b_n = a_n k, \ldots$$

Now, $b_{n+1} - b_n = a_{n+1} k - a_n k = (a_{n+1} - a_n) k = dk$ for all $n \in N$ $[\because a_{n+1} - a_n = d$ for all $n \in N]$

This shows that the new sequence is an A.P. with common difference dk.

Similarly, it can be proved that on dividing each term of a given A.P. by a non-zero constant, we obtain a sequence which is also an A.P.

PROPERTY 3 *In a finite A.P. the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term.*

i.e. $a_k + a_{n-(k-1)} = a_1 + a_n$ for all $k = 1, 2, 3, \ldots, n-1$.

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be an A.P. with common difference d. We have to show that

$$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = a_4 + a_{n-3} = \ldots$$

i.e. $a_1 + a_n = a_k + a_{n-(k-1)}$ for all $k = 1, 2, 3, \ldots, n-1$

For any $k = 1, 2, \ldots, n-1$

$$\begin{aligned} a_k + a_{n-(k-1)} &= a_k + a_{n+1-k} \\ &= [a_1 + (k-1)d] + [a_1 + (n+1-k-1)d] \\ &= 2a_1 + (k-1+n+1-k-1)d \\ &= 2a_1 + (n-1)d = a_1 + [a_1 + (n-1)d] = a_1 + a_n. \end{aligned}$$

PROPERTY 4 *Three numbers a, b, c are in A.P. iff $2b = a + c$.*

PROOF First, let a, b, c be in A.P. Then,

$b - a =$ Common difference and, $c - b =$ Common difference

$\Rightarrow \quad b - a = c - b$

$\Rightarrow \quad 2b = a + c$

Conversely, let a, b, c be three numbers such that $2b = a + c$. Then, we have to show that a, b, c are in A.P.

We have, $2b = a + c \Rightarrow b - a = c - b \Rightarrow a, b, c$ are in A.P.

ILLUSTRATION *If $\frac{2}{3}, k, \frac{5}{8}$ are in A.P., find the value of k.*

SOLUTION It is given that,

$$\frac{2}{3}, k, \frac{5}{8} \text{ are in A.P.} \Rightarrow 2k = \frac{2}{3} + \frac{5}{8} \Rightarrow 2k = \frac{31}{24} \Rightarrow k = \frac{31}{48}.$$

PROPERTY 5 *A sequence is an A.P. iff its nth term is a linear expression in n i.e. $a_n = An + B$, where A, B are constants. In such a case the coefficient of n in a_n is the common difference of the A.P.*

PROOF See example 3 on page 19.3.

PROPERTY 6 *A sequence is an A.P. iff the sum of its first n terms is of the form $A n^2 + Bn$, where A, B are constants independent of n. In such a case the common difference is 2A i.e. 2 times the coefficient of n^2.*

PROOF See example 6 on page 19.17.

PROPERTY 7 *If the terms of an A.P. are chosen at regular intervals, then they form an A.P.*

PROPERTY 8 *If a_n, a_{n+1} and a_{n+2} are three consecutive terms of an A.P., then $2a_{n+1} = a_n + a_{n+2}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **TO PROVE THAT THREE NUMBERS ARE IN A.P. WHEN THREE GIVEN NUMBERS ARE IN A.P.**

EXAMPLE 1 *If a, b, c are in A.P., prove that the following are also in A.P.*

(i) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (ii) $b+c, c+a, a+b$

(iii) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ [NCERT] (iv) $a^2(b+c), b^2(c+a), c^2(a+b)$

(v) $\left\{(b+c)^2-a^2\right\}, \left\{(c+a)^2-b^2\right\}, \left\{(a+b)^2-c^2\right\}$ (vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

SOLUTION (i) a, b, c are in A.P.

$\Rightarrow \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

(ii) It is given that

a, b, c are in A.P.

$\Rightarrow a-(a+b+c), b-(a+b+c), c-(a+b+c)$ are in A.P. [Subtracting $a+b+c$ from each term]

$\Rightarrow -(b+c), -(c+a), -(a+b)$ are in A.P. [Multiplying each term by – 1]

$\Rightarrow b+c, c+a, a+b$ are in A.P.

(iii) a, b, c are in A.P.

$\Rightarrow \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. [On dividing each term by abc and using Property 2]

$\Rightarrow \frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

[On multiplying each term by $ab+bc+ca$ and using Property 2]

$\Rightarrow \frac{ab+bc+ca}{bc}-1, \frac{ab+bc+ca}{ca}-1, \frac{ab+bc+ca}{ab}-1$ are in A.P.

[On adding –1 to each term and using Property 1]

$\Rightarrow \frac{ab+ac}{bc}, \frac{ab+bc}{ca}, \frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

(iv) $a^2(b+c), b^2(c+a), c^2(a+b)$ will be in A.P.

if $b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$

i.e. if $c(b^2 - a^2) + ab(b-a) = a(c^2 - b^2) + bc(c-b)$

i.e. if $(b-a)(ab+bc+ca) = (c-b)(ab+bc+ca)$

i.e. if $b - a = c - b$

i.e. if $2b = a + c$

i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

ALITER It is given that

a, b, c are in A.P.

$\Rightarrow$ $\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P.

$\Rightarrow$ $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ are in A.P.

$\Rightarrow$ $\frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ca}, \frac{ab+bc+ca}{ab}$ are in A.P.

$\Rightarrow$ $1+\frac{ab+ca}{bc}, 1+\frac{ab+bc}{ca}, 1+\frac{bc+ca}{ab}$ are in A.P.

$\Rightarrow$ $\frac{a(b+c)}{bc}, \frac{b(a+c)}{ca}, \frac{c(a+b)}{ab}$ are in A.P. [Subtracting 1 from each term]

$\Rightarrow$ $\frac{a^2(b+c)}{abc}, \frac{b^2(a+c)}{abc}, \frac{c^2(a+b)}{abc}$ are in A.P.

$\Rightarrow$ $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P. [Multiplying each term by abc]

(v) It is given that

a, b, c are in A.P.

$\Rightarrow$ $-2a, -2b, -2c$ are in A.P. [Multiplying each term by -2]

$\Rightarrow$ $a+b+c-2a, a+b+c-2b, a+b+c-2c$ are in A.P. [Adding $a+b+c$ to each term]

$\Rightarrow$ $b+c-a, c+a-b, a+b-c$ are in A.P.

$\Rightarrow$ $(a+b+c)(b+c-a), (a+b+c)(c+a-b), (a+b+c)(a+b-c)$ are in A.P.

[Multiplying each terming $a+b+c$]

$\Rightarrow$ $(b+c)^2 - a^2, (c+a^2) - b^2, (a+b)^2 - c^2$ are in A.P.

(vi) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ will be in A.P.

if $\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{(\sqrt{c}+\sqrt{a})(\sqrt{b}+\sqrt{c})} = \frac{(\sqrt{c}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a})}$

i.e. if $\frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}} = \frac{\sqrt{c}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$

i.e. if $b - a = c - b$

i.e. if $2b = a + c$

i.e. if a, b, c are in A.P.

Thus, a, b, c are in A.P. $\Rightarrow \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

<u>ALITER</u> It is given that

a, b, c are in A.P.

$\Rightarrow \quad b-a=c-b$

$\Rightarrow \quad \left(\sqrt{b}+\sqrt{a}\right)\left(\sqrt{b}-\sqrt{a}\right)=\left(\sqrt{c}-\sqrt{b}\right)\left(\sqrt{c}+\sqrt{b}\right)$

$\Rightarrow \quad \frac{\sqrt{b}-\sqrt{a}}{\sqrt{b}+\sqrt{c}}=\frac{\sqrt{c}-\sqrt{b}}{\sqrt{b}+\sqrt{a}}$

$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{a}+\sqrt{c}\right)}{\sqrt{b}+\sqrt{c}}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{b}+\sqrt{a}\right)}{\sqrt{b}+\sqrt{a}}$

$\Rightarrow \quad \frac{\left(\sqrt{b}+\sqrt{c}\right)-\left(\sqrt{c}+\sqrt{a}\right)}{\left(\sqrt{b}+\sqrt{c}\right)\left(\sqrt{c}+\sqrt{a}\right)}=\frac{\left(\sqrt{c}+\sqrt{a}\right)-\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{c}+\sqrt{a}\right)}$

$\Rightarrow \quad \frac{1}{\sqrt{c}+\sqrt{a}}-\frac{1}{\sqrt{b}+\sqrt{c}}=\frac{1}{\sqrt{a}+\sqrt{b}}-\frac{1}{\sqrt{c}+\sqrt{a}}$

$\Rightarrow \quad \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

EXAMPLE 2 *If a^2, b^2, c^2 are in A.P, then prove that the following are also in A.P.*

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$.

SOLUTION (i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ will be in A.P.

if $\quad \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$

i.e. if $\quad \frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$

i.e. if $\quad \frac{b-a}{b+c}=\frac{c-b}{a+b}$

i.e. if $\quad b^2-a^2=c^2-b^2$

i.e. if $\quad 2b^2=a^2+c^2$

i.e. if $\quad a^2, b^2, c^2$ are in A.P.

Thus, a^2, b^2, c^2 are in A.P. $\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

<u>ALITER</u> (i) It is given that

a^2, b^2, c^2 are in A.P.

$\Rightarrow \quad b^2-a^2=c^2-b^2$

$\Rightarrow \quad (b-a)(b+a)=(c-b)(c+b)$

$\Rightarrow \quad \dfrac{b-a}{b+c}=\dfrac{c-b}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{b+c}=\dfrac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{(a+c)(b+c)}=\dfrac{(c+a)-(b+a)}{(a+b)(a+c)}$ [Multiplying both side by $\dfrac{1}{a+b}$]

$\Rightarrow \quad \dfrac{1}{a+c}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{a+c}$

$\Rightarrow \quad \dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$, are in A.P.

(ii) $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ will be in A.P.

if $\dfrac{a}{b+c}+1, \dfrac{b}{c+a}+1, \dfrac{c}{a+b}+1$ are in A.P. [On adding 1 to each term]

i.e. if $\dfrac{a+b+c}{b+c}, \dfrac{a+b+c}{c+a}, \dfrac{a+b+c}{c+a}$ are in A.P.

i.e. if $\dfrac{1}{b+c}, \dfrac{1}{c+a}, \dfrac{1}{a+b}$ are in A.P. [On dividing each term by $a+b+c$]

i.e. if $\dfrac{1}{c+a}-\dfrac{1}{b+c} = \dfrac{1}{a+b}-\dfrac{1}{c+a}$

i.e. if $\dfrac{b-a}{(c+a)(b+c)} = \dfrac{c-b}{(a+b)(c+a)}$

i.e. if $\dfrac{b-a}{b+c} = \dfrac{c-b}{a+b}$

i.e. if $b^2-a^2 = c^2-b^2$

i.e. if $2b^2 = a^2+c^2$

i.e. if a^2, b^2, c^2 are in A.P.

Thus, a^2, b^2, c^2 are in A.P. $\Rightarrow \dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P.

<u>ALITER</u> It is given that

a^2, b^2, c^2 are in A.P.

$\Rightarrow \quad b^2-a^2=c^2-b^2.$

$\Rightarrow \quad (b+a)(b-a)=(c+b)(c-b)$

$\Rightarrow \quad \dfrac{b-a}{b+c}=\dfrac{c-b}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{b+c}=\dfrac{(c+a)-(b+a)}{a+b}$

$\Rightarrow \quad \dfrac{(b+c)-(a+c)}{(a+c)(b+c)}=\dfrac{(c+a)-(b+a)}{(a+b)(a+c)}$

$\Rightarrow \quad \dfrac{1}{a+c}-\dfrac{1}{b+c}=\dfrac{1}{a+b}-\dfrac{1}{a+c}$

$\Rightarrow \frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A.P.

$\Rightarrow \frac{a+b+c}{b+c}, \frac{a+b+c}{c+a}, \frac{a+b+c}{a+b}$ are in A.P.

$\Rightarrow 1+\frac{a}{b+c}, 1+\frac{b}{c+a}, 1+\frac{c}{a+b}$ are in A.P.

$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

EXAMPLE 3 *If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.*

SOLUTION $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \left\{\frac{b+c-a}{a}+2\right\}, \left\{\frac{c+a-b}{b}+2\right\}, \left\{\frac{a+b-c}{c}+2\right\}$ are in A.P. [Adding 2 to each term]

$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P [Dividing each term by $a+b+c$]

EXAMPLE 4 *If $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P., show that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $a^2+2bc, b^2+2ac, c^2+2ab$ are in A.P.

$\Rightarrow (a^2+2bc)-(ab+bc+ca), (b^2+2ac)-(ab+bc+ca), (c^2+2ab)-(ab+bc+ca)$ are in A.P.

[On subtracting $(ab+bc+ca)$ from each term]

$\Rightarrow a^2+bc-ab-ca, b^2+ca-ab-bc, c^2+ab-bc-ca$ are in A.P.

$\Rightarrow (a-b)(a-c), (b-c)(b-a), (c-a)(c-b)$ are in A.P.

$\Rightarrow \frac{-1}{b-c}, \frac{-1}{c-a}, \frac{-1}{a-b}$ are in A.P. [On dividing each term by $(a-b)(b-c)(c-a)$]

$\Rightarrow \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P. [On multiplying each term by -1]

EXAMPLE 5 *If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., prove that $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.*

SOLUTION $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P.

$\Rightarrow (c-a)^2-(b-c)^2 = (a-b)^2-(c-a)^2$

$\Rightarrow (b-a)(2c-a-b) = (c-b)(2a-b-c)$ ...(i)

$\Rightarrow (b-a)\{(c-a)+(c-b)\} = (c-b)\{(a-b)+(a-c)\}$

$\Rightarrow (b-a)(c-a)+(b-a)(c-b) = (c-b)(a-b)+(a-c)(c-b)$

$\Rightarrow -(a-b)(c-a)+(a-b)(b-c) = -(a-b)(b-c)+(b-c)(c-a)$

$\Rightarrow -\frac{1}{b-c}+\frac{1}{c-a} = -\frac{1}{c-a}+\frac{1}{a-b}$ [Dividing throughout by $(a-b)(b-c)(c-a)$]

$\Rightarrow \frac{2}{c-a} = \frac{1}{a-b}+\frac{1}{b-c}$

$\Rightarrow \quad \frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in AP.

Thus, if $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., then $\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b}$ are in A.P.

EXAMPLE 6 *If a, b, c are in A.P., then prove that:*

(i) $(a-c)^2 = 4(b^2 - ac)$ (ii) $a^3 + 4b^3 + c^3 = 3b(a^2 + c^2)$

SOLUTION (i) It is given that a, b, c are in A.P.

$\therefore \quad 2b = a + c \Rightarrow b = \frac{a+c}{2}$

Putting $b = \frac{a+c}{2}$ on RHS, we obtain

$$\text{RHS} = 4(b^2 - ac) = 4\left\{\left(\frac{a+c}{2}\right)^2 - ac\right\} = 4\left\{\frac{(a+c)^2 - 4ac}{4}\right\} = (a+c)^2 - 4ac = (a-c)^2 = \text{LHS}$$

(ii) It is given that a, b, c are in A.P.

$\therefore \quad 2b = a + c \Rightarrow b = \frac{a+c}{2}$

$$\text{LHS} = a^3 + 4b^3 + c^3$$

$$= a^3 + 4\left(\frac{a+c}{2}\right)^3 + c^3 = (a^3 + c^3) + \frac{1}{2}(a+c)^3$$

$$= \frac{1}{2}\left\{2(a^3 + c^3) + (a+c)^3\right\} = \frac{1}{2}\left\{2(a+c)(a^2 - ac + c^2) + (a+c)^3\right\}$$

$$= \frac{1}{2}(a+c)\left\{2(a^2 - ac + c^2) + (a+c)^2\right\}$$

$$= \frac{1}{2}(a+c)\,3(a^2 + c^2) = 3\left(\frac{a+c}{2}\right)(a^2 + c^2) = 3b(a^2 + c^2) = \text{RHS}$$

ALITER LHS $= a^3 + 4b^3 + c^3$

$$= (a^3 + c^3) + 4b^3$$

$$= (a+c)^3 - 3ac(a+c) + 4b^3$$

$$= (2b)^3 - 3ac(2b) + 4b^3$$

$$= 12b^3 - 6abc$$

$$= 3b(4b^2 - 2ac) = 3b\left\{(2b)^2 - 2ac\right\} = 3b\left\{(a+c)^2 - 2ac\right\} = 3b(a^2 + c^2) = \text{RHS}$$

EXAMPLE 7 *If $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P., show that either a, b, c are in A.P. or $ab + bc + ca = 0$.*

SOLUTION $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P.

$$\Rightarrow \quad b^2(c+a) - a^2(b+c) = c^2(a+b) - b^2(c+a)$$

$$\Rightarrow \quad (b^2a - a^2b) + (b^2c - a^2c) = (c^2b - b^2c) + (c^2a - b^2a)$$

$$\Rightarrow \quad (b-a)(ab + bc + ca) = (c-b)(ab + bc + ca)$$

$$\Rightarrow \quad (ab + bc + ca)(2b - a - c) = 0$$

$$\Rightarrow \quad ab + bc + ca = 0 \text{ or } 2b - a - c = 0$$

$$\Rightarrow \quad ab + bc + ca = 0 \text{ or } a, b, c \text{ are in A.P.}$$

EXERCISE 19.5

LEVEL-1

1. If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P., prove that:

   (i) $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P. (ii) $a(b+c), b(c+a), c(a+b)$ are in A.P.

2. If a^2, b^2, c^2 are in A.P., prove that $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in A.P.

3. If a, b, c are in A.P., then show that:

   (i) $a^2(b+c), b^2(c+a), c^2(a+b)$ are also in A.P.

   (ii) $b+c-a, c+a-b, a+b-c$ are in A.P.

   (iii) $bc-a^2, ca-b^2, ab-c^2$ are in A.P.

4. If $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P., prove that:

   (i) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. (ii) bc, ca, ab are in A.P.

5. If a, b, c are in A.P., prove that:

   (i) $(a-c)^2 = 4(a-b)(b-c)$ (ii) $a^2 + c^2 + 4ac = 2(ab+bc+ca)$

   (iii) $a^3 + c^3 + 6abc = 8b^3$

6. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

7. Show that x^2+xy+y^2, z^2+zx+x^2 and y^2+yz+z^2 are consecutive terms of an A.P., if x, y and z are in A.P. [NCERT EXEMPLAR]

HINTS TO NCERT & SELECTED PROBLEMS

5. (i) Put $b = \frac{a+c}{2}$ on RHS (ii) Put $b = \frac{a+c}{2}$ on RHS (iii) Put $b = \frac{a+c}{2}$ on RHS and LHS

6. $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right)+1, b\left(\frac{1}{c}+\frac{1}{a}\right)+1, c\left(\frac{1}{a}+\frac{1}{b}\right)+1$ are A.P. [$\because$ Adding 1 throughout]

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{a}{a}, b\left(\frac{1}{c}+\frac{1}{a}\right)+\frac{b}{b}, c\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{c}{c}$ are in A.P.

$\Rightarrow$ $a\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{a}\right), b\left(\frac{1}{c}+\frac{1}{a}+\frac{1}{b}\right), c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ are in A.P.

$\Rightarrow$ a, b, c are in A.P. $\left[\text{Dividing each term by } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$

19.7 INSERTION OF ARITHMETIC MEANS

If between two given quantities a and b we have to insert n quantities $A_1, A_2, \ldots, A_n$ such that $a, A_1, A_2, \ldots A_n, b$ form an A.P., then we say that $A_1, A_2, \ldots, A_n$ are arithmetic means between a and b.

ILLUSTRATION *Since 15, 11, 7, 3, –1, –5 are in A.P., it follows that 11, 7, 3, –1 are four arithmetic means between 15 and –5.*

If a, A, b are in A.P., we say that A is the arithmetic mean of a and b.

19.7.1 INSERTION OF ARITHMETIC MEANS

Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between two quantities a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. Let d be the common difference of this A.P. Clearly, it contains $(n+2)$ terms.

$$\therefore \quad b = (n+2)^{\text{th}} \text{ term} \Rightarrow b = a + (n+1)\,d \Rightarrow d = \frac{b-a}{n+1}$$

Now,
$$A_1 = a + d = a + \frac{b-a}{n+1}$$
$$A_2 = a + 2d = a + \frac{2(b-a)}{n+1}$$
$$A_3 = a + 3d = a + \frac{3(b-a)}{n+1}$$
$$\vdots \qquad \vdots \quad \vdots$$
$$A_n = a + nd = a + \frac{n(b-a)}{n+1}$$

These are the required arithmetic means between a and b.

19.7.2 INSERTION OF A SINGLE ARITHMETIC MEAN BETWEEN TWO NUMBERS

Let a and b be two numbers and A be the single arithmetic mean between them. Then,

$$a, A, b \text{ are in A.P.}$$
$$\Rightarrow \quad A - a = b - A$$
$$\Rightarrow \quad 2A = a + b \Rightarrow A = \frac{a+b}{2}$$

Thus, the arithmetic mean of a and b is $\frac{a+b}{2}$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Insert three arithmetic means between 3 and 19.*

SOLUTION Let A_1, A_2, A_3 be 3 A.M.'s between 3 and 19. Then $3, A_1, A_2, A_3, 19$ are in A.P. whose common difference d is given by $d = \frac{19-3}{3+1} = 4$.

$$\therefore \quad A_1 = 3 + d = 3 + 4 = 7,\ A_2 = 3 + 2d = 3 + 2\times 4 = 11,\ A_3 = 3 + 3d = 3 + 3\times 4 = 15.$$

Hence, the required A.M.'s are 7, 11, 15.

EXAMPLE 2 *For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b ?* **[NCERT]**

SOLUTION The A.M. of a and b is $\frac{a+b}{2}$. Therefore, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ will be the A.M. of a and b, if

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$
$$\Rightarrow \quad 2(a^{n+1}+b^{n+1}) = (a^n+b^n)(a+b)$$
$$\Rightarrow \quad 2a^{n+1}+2b^{n+1} = a^{n+1}+a^n b+b^n a+b^{n+1}$$
$$\Rightarrow \quad a^{n+1}+b^{n+1} = a^n b + b^n a$$
$$\Rightarrow \quad a^n(a-b) = b^n(a-b)$$
$$\Rightarrow \quad a^n = b^n \Rightarrow \frac{a^n}{b^n} = 1 \Rightarrow \left(\frac{a}{b}\right)^n = 1 \Rightarrow \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \Rightarrow n = 0.$$

EXAMPLE 3 *If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1 : 3, then find the value of n.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between 20 and 80 and let d be the common difference of the A.P. 20, $A_1, A_2, \ldots A_n$, 80. Then,

$$d = \frac{80-20}{n+1} = \frac{60}{n+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_1 = 20 + d \Rightarrow A_1 = 20 + \dfrac{60}{n+1} = 20\left(\dfrac{n+4}{n+1}\right)$

And, $A_n = 20 + nd \Rightarrow A_n = 20 + \dfrac{60n}{n+1} = 20\left(\dfrac{4n+1}{n+1}\right)$

It is given that

$$\frac{A_1}{A_n} = \frac{1}{3} \Rightarrow \frac{\frac{20(n+4)}{n+1}}{\frac{20(4n+1)}{n+1}} = \frac{1}{3} \Rightarrow \frac{n+4}{4n+1} = \frac{1}{3} \Rightarrow 4n+1 = 3n+12 \Rightarrow n = 11$$

EXAMPLE 4 *Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and (m − 1)th means is 5 : 9. Find the value of m.* **[NCERT]**

SOLUTION Let $A_1, A_2, \ldots, A_m$ be m arithmetic means between 1 and 31. Then 1, $A_1, A_2, \ldots, A_m$, 31 is an A.P. with common difference d given by

$$d = \frac{31-1}{m+1} = \frac{30}{m+1} \qquad \left[\text{Using: } d = \frac{b-a}{n+1}\right]$$

Now, $A_7 = 1 + 7d = 1 + \dfrac{7 \times 30}{m+1} = \dfrac{m+211}{m+1}$

and, $A_{m-1} = 1 + (m-1)d = 1 + \dfrac{30(m-1)}{m+1} = \dfrac{31m-29}{m+1}$

It is given that

$$\frac{A_7}{A_{m-1}} = \frac{5}{9} \Rightarrow \frac{m+211}{31m-29} = \frac{5}{9} \Rightarrow 9m + 1899 = 155m - 145 \Rightarrow 146m = 2044 \Rightarrow m = 14$$

LEVEL-2

EXAMPLE 5 *Prove that the sum of n arithmetic means between two numbers is n times the single A.M. between them.*

SOLUTION Let $A_1, A_2, \ldots, A_n$ be n arithmetic means between a and b. Then, $a, A_1, A_2, \ldots, A_n, b$ is an A.P. with common difference d given by $d = \dfrac{b-a}{n+1}$.

Now, $A_1 + A_2 + \ldots + A_n = \dfrac{n}{2}\left(A_1 + A_n\right)$ $\qquad \left[\because S_n = \dfrac{n}{2}(a+l)\right]$

$= \dfrac{n}{2}(a+b)$ $\qquad [\because a, A_1, A_2, \ldots, A_n, b$ is an A.P. $\therefore a + b = A_1 + A_n]$

$= n\left(\dfrac{a+b}{2}\right) = n \times$ (A.M. between a and b)

EXAMPLE 6 *The sum of two numbers is $\frac{13}{6}$. An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1. Find the number of means inserted.*

SOLUTION Let a and b be two numbers such that $a + b = \frac{13}{6}$ ...(i)

Let $A_1, A_2, \dots, A_{2n}$ be $2n$ arithmetic means between a and b. Then,

$$A_1 + A_2 + \dots + A_{2n} = 2n\left(\frac{a+b}{2}\right) \quad \text{[Using result of Example 5]}$$

$$\Rightarrow \quad A_1 + A_2 + \dots + A_{2n} = n(a+b) = \frac{13}{6}n \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 2n+1 = \frac{13}{6}n \quad [\because A_1 + A_2 + \dots + A_{2n} = 2n+1 \text{ (given)}]$$

$$\Rightarrow \quad 12n + 6 = 13$$

$$\Rightarrow \quad n = 6$$

EXAMPLE 7 *If the A.M. between pth and qth terms of an A.P. be equal to the A.M. between rth and sth terms of the A.P., then show that $p + q = r + s$.*

SOLUTION Let a be the first term and d be the common difference of the given A.P. Then

$$a_p = p\text{th term} = a + (p-1)d\,;\; a_q = q\text{th term} = a + (q-1)d$$

$$a_r = r\text{th term} = a + (r-1)d \text{ and, } a_s = s\text{th term} = a + (s-1)d$$

It is given that

$$\text{A.M. between } a_p \text{ and } a_q = \text{A.M. between } a_r \text{ and } a_s$$

$$\Rightarrow \quad \frac{1}{2}(a_p + a_q) = \frac{1}{2}(a_r + a_s)$$

$$\Rightarrow \quad a_p + a_q = a_r + a_s$$

$$\Rightarrow \quad \{a + (p-1)d\} + \{a + (q-1)d\} = \{a + (r-1)d\} + \{a + (s-1)d\}$$

$$\Rightarrow \quad (p + q - 2)d = (r + s - 2)d$$

$$\Rightarrow \quad p + q = r + s$$

EXAMPLE 8 *Suppose x and y are two real numbers such that the rth mean between x and 2y is equal to the rth mean between 2x and y when n arithmetic means are inserted between them in both the cases. Show that $\frac{n+1}{r} - \frac{y}{x} = 1$.*

SOLUTION Let $A_1, A_2, \dots, A_n$ be n arithmetic means between x and $2y$. Then, $x, A_1, A_2, \dots, A_n, 2y$ are in AP with common difference d_1 given by $d_1 = \frac{2y - x}{n+1}$.

$$\therefore \quad r^{\text{th}} \text{ mean} = A_r = x + r\,d_1 = x + r\left(\frac{2y - x}{n+1}\right)$$

Let $A_1', A_2', \dots, A_n'$ be n arithmetic means between $2x$ and y. Then, $2x, A_1', A_2', \dots, A_n', y$ are in A.P. with common difference d_2 given by $d_2 = \frac{y - 2x}{n+1}$.

$$\therefore \quad r^{\text{th}} \text{ mean} = A_r' = 2x + r\,d_2 = 2x + r\left(\frac{y - 2x}{n+1}\right)$$

It is given that :

$$A_r = A_r'$$

$$\Rightarrow \quad x + r\left(\frac{2y - x}{n+1}\right) = 2x + r\left(\frac{y - 2x}{n+1}\right)$$

$\Rightarrow \quad (n+1)\,x + r\,(2y - x) = (n+1)\,2x + r\,(y - 2x)$

$\Rightarrow \quad (n+1)\,x - ry = rx$

$\Rightarrow \quad \dfrac{n+1}{r} - \dfrac{y}{x} = 1$

EXERCISE 19.6

LEVEL-1

1. Find the A.M. between:
   (i) 7 and 13 (ii) 12 and –8 (iii) $(x - y)$ and $(x + y)$.
2. Insert 4 A.M.s between 4 and 19.
3. Insert 7 A.M.s between 2 and 17.
4. Insert six A.M.s between 15 and –13.
5. There are n A.M.s between 3 and 17. The ratio of the last mean to the first mean is 3 : 1. Find the value of n.
6. Insert A.M.s between 7 and 71 in such a way that the 5th A.M. is 27. Find the number of A.M.s.
7. If n A.M.s are inserted between two numbers, prove that the sum of the means equidistant from the beginning and the end is constant.
8. If x, y, z are in A.P. and A_1 is the A.M. of x and y and A_2 is the A.M. of y and z, then prove that the A.M. of A_1 and A_2 is y.
9. Insert five numbers between 8 and 26 such that the resulting sequence is an A.P. [NCERT]

ANSWERS

1. (i) 10 (ii) 2 (iii) x 2. 7, 10, 13, 16

3. $\dfrac{31}{8}, \dfrac{23}{4}, \dfrac{61}{8}, \dfrac{19}{2}, \dfrac{91}{8}, \dfrac{53}{4}, \dfrac{121}{8}$

4. 11, 7, 3, –1, –5, –9 5. 6 6. 15 9. 11, 14, 17, 20, 23

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9. Let a_1, a_2, a_3, a_4, a_5 be five natural numbers between 8 and 26 such that $8, a_1, a_2, a_3, a_4, a_5, 26$ is an A.P. Let d be the common difference. Then,

$$d = \frac{26 - 8}{5 + 1} = 3 \qquad \left[\because d = \frac{b - a}{n + 1}\right]$$

$\therefore \quad a_1 = 8 + 3 = 11,\ a_2 = a_1 + 3 = 14,\ a_3 = 17,\ a_4 = 20$ and $a_5 = 23$

Hence, five numbers are 11, 14, 17, 20 and 23.

19.8 APPLICATIONS OF A.P.

In this section, we shall discuss some problems based upon the applications of arithmetic progressions.

LEVEL-1

EXAMPLE 1 *The digits of a positive integer, having three digits, are in A.P. and their sum is 15. The number obtained by reversing the digits is 594 less than the original number. Find the number.*

SOLUTION Let the digits at ones, tens and hundreds place be $(a - d)$, a and $(a + d)$ respectively. Then the number is

$$(a + d) \times 100 + a \times 10 + (a - d) = 111a + 99d$$

The number obtained by reversing the digits is

$$(a-d)\times 100 + a\times 10 + (a+d) = 111a - 99d$$

It is given that $(a-d)+a+(a+d)=15$

and, $111a-99d = 111a + 99d - 594$

$\Rightarrow$ $3a = 15$ and $198d = 594 \Rightarrow a = 5$ and $d = 3$

So, the number is $111a + 99d = 111\times 5 + 99\times 3 = 852$.

EXAMPLE 2 *Two cars start together in the same direction from the same place. The first goes with uniform speed of 10 km/h. The second goes at a speed of 8 km/h in the first hour and increases the speed by 1/2 km each succeeding hour. After how many hours will the second car overtake the first car if both cars go non-stop ?*

SOLUTION Suppose the second car overtakes the first car after t hours. Then the two cars travel the same distance in t hours.

Distance travelled by the first car in t hours = $10\,t$ km.

Distance travelled by the second car in t hours

= Sum of t terms of an A.P. with first term 8 and common difference 1/2.

$$= \frac{t}{2}\left\{2\times 8 + (t-1)\times\frac{1}{2}\right\} = \frac{t\,(t+31)}{4}$$

When the second car overtakes the first car. The distance travelled by both cars is same.

$$\therefore \quad 10t = \frac{t\,(t+31)}{4} \Rightarrow t\,(t-9) = 0 \Rightarrow t = 9 \qquad [\because t \neq 0]$$

Thus, the second car will overtake the first car in 9 hours.

EXAMPLE 3 *A man repays a loan of ₹ 3250 by paying ₹ 20 in the first month and then increases the payment by ₹ 15 every month. How long will it take him to clear the loan ?*

SOLUTION Suppose the loan is cleared in n months. Clearly, the amounts form an A.P. with first term 20 and the common difference 15.

$\therefore$ Sum of the amounts = 3250

$$\Rightarrow \quad \frac{n}{2}\left\{2\times 20 + (n-1)\times 15\right\} = 3250$$

$$\Rightarrow \quad 3n^2 + 5n - 1300 = 0 \Rightarrow (n-20)\,(3n+65) = 0 \Rightarrow n = 20 \qquad \left[\because\ 3n+65 \neq 0\right]$$

Thus, the loan is cleared in 20 months.

LEVEL-2

EXAMPLE 4 *150 workers were engaged to finish a piece of work in a certain number of days. Four workers dropped the second day, four more workers dropped the third day and so on. It takes 8 more days to finish the work now. Find the number of days in which the work was completed.*

SOLUTION Suppose the work is completed in n days when the workers started dropping. Since 4 workers are dropped on every day except the first day. Therefore, the total number of workers who worked all the n days is the sum of n terms of an A.P. with first term 150 and common difference -4.

$$\text{i.e.} \quad \frac{n}{2}\left\{2\times 150 + (n-1)\times -4\right\} = n\,(152-2n)$$

Had the workers not dropped then the work would have finished in $(n-8)$ days with 150 workers working on each day. Therefore, the total number of workers who would have worked all the n days is $150\,(n-8)$.

$$\therefore \quad n\,(152-2n) = 150\,(n-8) \Rightarrow n^2 - n - 600 = 0 \Rightarrow (n-25)\,(n+24) = 0 \Rightarrow n = 25.$$

Thus, the work is completed in 25 days.

EXAMPLE 5 *Along a road lie an odd number of stones placed at intervals of 10 metres. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried the job with one of the end stones by carrying them in succession. In carrying all the stones he covered a distance of 3 km. Find the number of stones.*

SOLUTION Let there be $(2n + 1)$ stones. Clearly, one stone lies in the middle and n stones on each side of it in a row. Let P be the mid-stone and let A and B be the end stones on the left and right of P respectively. Clearly, there are n intervals each of length 10 metres on both the sides of P. Now, suppose the man starts from A. He picks up the end stone on the left of mid-stone and goes to the mid-stone, drops it and goes to $(n - 1)$th stone on left, picks it up, goes to the mid-stone and drops it. This process is repeated till he collects all stones on the left of the mid-stone at the mid-stone. So, distance covered in collecting stones on the left of the mid-stones

$$= 10 \times n + 2\,[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1].$$

After collecting all stones on left of the mid-stone the man goes to the stone B on the right side of the mid-stone, picks it up, goes to the mid-stone and drops it. Then he goes to $(n-1)^{\text{th}}$ stone on the right and the process is repeated till he collects all stones at the mid-stone.

Distance covered in collecting the stones on the right side of the mid-stone

$$= 2\,[10 \times n + 10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1]$$

$\therefore$ Total distance covered

$$= 10 \times n + 2\,[10 \times (n-1) + 10 \times (n-2) + \ldots + 10 \times 2 + 10 \times 1] + 2\,[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1]$$

$$= 4\,[10 \times n + 10 \times (n-1) + \ldots + 10 \times 2 + 10 \times 1] - 10 \times n$$

$$= 40\left[1 + 2 + 3 + \ldots + n\right] - 10n = 40\left\{\frac{n}{2}(1+n)\right\} - 10n = 20n^2 + 10n.$$

But, the total distance covered is 3 km = 3000 m.

$\therefore$ $20n^2 + 10n = 3000 \Rightarrow 2n^2 + n - 300 = 0 \Rightarrow (n-12)(2n+25) = 0 \Rightarrow n = 12$

Hence, the number of stones $= 2n + 1 = 25$.

EXERCISE 19.7

LEVEL-1

1. A man saved ₹ 16500 in ten years. In each year after the first he saved ₹ 100 more than he did in the receding year. How much did he save in the first year ?
2. A man saves ₹ 32 during the first year, ₹ 36 in the second year and in this way he increases his savings by ₹ 4 every year. Find in what time his saving will be ₹ 200.
3. A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which form an arithmetic series. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid, find the value of the first instalment.
4. A manufacturer of radio sets produced 600 units in the third year and 700 units in the seventh year. Assuming that the product increases uniformly by a fixed number every year, find (i) the production in the first year (ii) the total product in 7 years and (iii) the product in the 10th year.
5. There are 25 trees at equal distances of 5 metres in a line with a well, the distance of the well from the nearest tree being 10 metres. A gardener waters all the trees separately starting from the well and he returns to the well after watering each tree to get water for the next. Find the total distance the gardener will cover in order to water all the trees.
6. A man is employed to count ₹ 10710. He counts at the rate of ₹ 180 per minute for half an hour. After this he counts at the rate of ₹ 3 less every minute than the preceding minute. Find the time taken by him to count the entire amount.

7. A piece of equipment cost a certain factory ₹ 600,000. If it depreciates in value, 15% the first, 13.5% the next year, 12% the third year, and so on. What will be its value at the end of 10 years, all percentages applying to the original cost ?
8. A farmer buys a used tractor for ₹ 12000. He pays ₹ 6000 cash and agrees to pay the balance in annual instalments of ₹ 500 plus 12% interest on the unpaid amount. How much the tractor cost him?
9. Shamshad Ali buys a scooter for ₹ 22000. He pays ₹ 4000 cash and agrees to pay the balance in annual instalments of ₹ 1000 plus 10% interest on the unpaid amount. How much the scooter will cost him.
10. The income of a person is ₹ 300,000 in the first year and he receives an increase of ₹ 10000 to his income per year for the next 19 years. Find the total amount, he received in 20 years. [NCERT]
11. A man starts repaying a loan as first instalment of ₹ 100. If he increases the instalments by ₹ 5 every month, what amount he will pay in the 30th instalment? [NCERT]
12. A carpenter was hired to build 192 window frames. The first day he made five frames and each day thereafter he made two more frames than he made the day before. How many days did it take him to finish the job? [NCERT EXEMPLAR]
13. We know that the sum of the interior angles of a triangle is 180°. Show that the sums of the interior angles of polygons with 3, 4, 5, 6,... sides form an arithmetic progression. Find the sum of the interior angles for a 21 sided polygon. [NCERT EXEMPLAR]
14. In a potato race 20 potatoes are placed in a line at intervals of 4 meters with the first potato 24 metres from the starting point. A contestant is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing back all the potatoes? [NCERT EXEMPLAR]
15. A man accepts a position with an initial salary of ₹ 5200 per month. It is understood that he will receive an automatic increase of ₹320 in the very next month and each month thereafter.
    (i) Find his salary for the tenth month.
    (ii) What is his total earnings during the first year?
16. A man saved ₹66000 in 20 years. In each succeeding year after the first year he saved ₹ 200 more than what he saved in the previous year. How much did he save in the first year?
17. In a cricket team tournament 16 teams participated. A sum of ₹ 8000 is to be awarded among themselves as prize money. If the last place team is awarded ₹ 275 in prize money and the award increases by the same amount for successive finishing places, how much amount will the first place team receive?

ANSWERS

1. ₹ 1200	2. 5 yrs 3. ₹ 51	4. (i) 550 (ii) 4375 (iii) 775	
5. 3500 m	6. 89 minutes	7. ₹ 105000	8. ₹ 16680
9. ₹ 39100	10. ₹7900,000	11. ₹ 245	12. 12 days
13. 3420°	14. 2480 m	15. (i) ₹ 8080 (ii) ₹ 83520	
16. ₹1400	17. ₹ 725		

HINTS TO NCERT & SELECTED PROBLEMS

10. Here, $a = 300{,}000$, $d = 10{,}000$ and $n = 20$. Let S be the total amount received in 20 years. Then,

$$S = ₹\ \frac{20}{2}\{2 \times 300{,}000 + (20-1) \times 10{,}000\} = ₹10\,(600{,}000 + 190{,}000) = ₹\ 7900{,}000$$

11. Here, $a = 100$, $d = 5$ and $n = 30$.

$\therefore$ Amount to be paid in 30th instalment $= a_{30} = a + 29d = 100 + 29 \times 5 = 245$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the common difference of an A.P. whose nth term is $xn + y$.
2. Write the common difference of an A.P. the sum of whose first n terms is $\frac{P}{2}n^2 + Qn$.
3. If the sum of n terms of an AP is $2n^2 + 3n$, then write its nth term.
4. If $\log 2$, $\log(2^x - 1)$ and $\log(2^x + 3)$ are in A.P., write the value of x.
5. If the sums of n terms of two arithmetic progressions are in the ratio $2n + 5 : 3n + 4$, then write the ratio of their mth terms.
6. Write the sum of first n odd natural numbers.
7. Write the sum of first n even natural numbers.
8. Write the value of n for which nth terms of the A.P.s 3, 10, 17, ... and 63, 65, 67,.... are equal.
9. If $\frac{3+5+7+\ldots+\text{upto } n \text{ terms}}{5+8+11+\ldots\text{ upto 10 terms}} = 7$, then find the value of n.
10. If mth term of an A.P. is n and nth term is m, then write its pth term.
11. If the sums of n terms of two A.P.'s are in the ratio $(3n+2) : (2n+3)$, find the ratio of their 12th terms.

ANSWERS

1. x	2. P	3. $4n + 1$	4. $\log_2 5$
5. $(4m + 3) : (6m + 1)$		6. n^2	7. $n(n + 1)$
8. 13	9. 35	10. $m + n - p$	11. 71 : 49

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If 7th and 13th terms of an A.P. be 34 and 64 respectively, then its 18th term is
   (a) 87 (b) 88 (c) 89 (d) 90
2. If the sum of p terms of an A.P. is q and the sum of q terms is p, then the sum of $p + q$ terms will be
   (a) 0 (b) $p - q$ (c) $p + q$ (d) $-(p + q)$
3. If the sum of n terms of an A.P. be $3n^2 - n$ and its common difference is 6, then its first term is
   (a) 2 (b) 3 (c) 1 (d) 4
4. Sum of all two digit numbers which when divided by 4 yield unity as remainder is
   (a) 1200 (b) 1210 (c) 1250 (d) none of these.
5. In A.M.'s are introduced between 3 and 17 such that the ratio of the last mean to the first mean is 3 : 1, then the value of n is
   (a) 6 (b) 8 (c) 4 (d) none of these.
6. If S_n denotes the sum of first n terms of an A.P. $< a_n >$ such that $\frac{S_m}{S_n} = \frac{m^2}{n^2}$, then $\frac{a_m}{a_n} =$
   (a) $\frac{2m+1}{2n+1}$ (b) $\frac{2m-1}{2n-1}$ (c) $\frac{m-1}{n-1}$ (d) $\frac{m+1}{n+1}$.

7. The first and last terms of an A.P. are 1 and 11. If the sum of its terms is 36, then the number of terms will be
(a) 5 (b) 6 (c) 7 (d) 8

8. If the sum of n terms of an A.P., is $3n^2 + 5n$ then which of its terms is 164?
(a) 26th (b) 27th (c) 28th (d) none of these.

9. If the sum of n terms of an A.P. is $2n^2 + 5n$, then its nth term is
(a) $4n - 3$ (b) $3n - 4$ (c) $4n + 3$ (d) $3n + 4$

10. If $a_1, a_2, a_3, \ldots a_n$ are in A.P. with common difference d, then the sum of the series $\sin d$ $[\operatorname{cosec} a_1 \operatorname{cosec} a_2 + \operatorname{cosec} a_1 \operatorname{cosec} a_3 + \ldots + \operatorname{cosec} a_{n-1} \operatorname{cosec} a_n]$ is
(a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$
(c) $\cot a_1 - \cot a_n$ (d) $\tan a_1 - \tan a_n$

11. In the arithmetic progression whose common difference is non-zero, the sum of first $3n$ terms is equal to the sum of next n terms. Then the ratio of the sum of the first $2n$ terms to the next $2n$ terms is
(a) 1/5 (b) 2/3 (c) 3/4 (d) none of these

12. If $a_1, a_2, a_3, \ldots a_n$ are in A.P. with common difference d, then the sum of the series
$$\sin d\, [\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \ldots + \sec a_{n-1} \sec a_n], \text{ is}$$
(a) $\sec a_1 - \sec a_n$ (b) $\operatorname{cosec} a_1 - \operatorname{cosec} a_n$
(c) $\cot a_1 - \cot a_n$ (d) $\tan a_n - \tan a_1$

13. If four numbers in A.P. are such that their sum is 50 and the greatest number is 4 times the least, then the numbers are
(a) 5, 10, 15, 20 (b) 4, 10, 16, 22 (c) 3, 7, 11, 15 (d) none of these

14. If n arithmetic means are inserted between 1 and 31 such that the ratio of the first mean and nth mean is 3 : 29, then the value of n is
(a) 10 (b) 12 (c) 13 (d) 14

15. Let S_n denote the sum of n terms of an A.P. whose first term is a. If the common difference d is given by $d = S_n - k S_{n-1} + S_{n-2}$, then $k =$
(a) 1 (b) 2 (c) 3 (d) none of these

16. The first and last term of an A. P. are a and l respectively. If S is the sum of all the terms of the A.P. and the common difference is given by $\dfrac{l^2 - a^2}{k - (l + a)}$, then $k =$
(a) S (b) $2S$ (c) $3S$ (d) none of these

17. If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then $k =$
(a) $\dfrac{1}{n}$ (b) $\dfrac{n-1}{n}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

18. If the first, second and last term of an A.P are a, b and $2a$ respectively, then its sum is
(a) $\dfrac{ab}{2(b-a)}$ (b) $\dfrac{ab}{b-a}$ (c) $\dfrac{3ab}{2(b-a)}$ (d) none of these

19. If, S_1 is the sum of an arithmetic progression of 'n' odd number of terms and S_2 the sum of the terms of the series in odd places, then $\dfrac{S_1}{S_2} =$
(a) $\dfrac{2n}{n+1}$ (b) $\dfrac{n}{n+1}$ (c) $\dfrac{n+1}{2n}$ (d) $\dfrac{n+1}{n}$

20. If in an A.P., $S_n = n^2 p$ and $S_m = m^2 p$, where S_r denotes the sum of r terms of the A.P., then S_p is equal to
   (a) $\frac{1}{2}p^3$ (b) $mn\,p$ (c) p^3 (d) $(m+n)\,p^2$
21. If in an A.P., the pth term is q and $(p+q)^{th}$ term is zero, then the q^{th} term is
   (a) $-p$ (b) p (c) $p+q$ (d) $p-q$
22. The 10^{th} common term between the A.P.s 3, 7, 11, 15, ... and 1, 6, 11, 16, ... is
   (a) 191 (b) 193 (c) 211 (d) none of these
23. If in an A.P. $S_n = n^2 q$ and $S_m = m^2 q$, where S_r denotes the sum of r terms of the A.P., then S_q equals
   (a) $\frac{q^3}{2}$ (b) mnq (c) q^3 (d) $(m^2+n^2)\,q$
24. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3\,S_n$, then $S_{3n} : S_n$ is equal to
   (a) 4 (b) 6 (c) 8 (d) 10

ANSWERS

1. (c)	2. (d)	3. (a)	4. (b)	5. (a)	6. (b)	7. (b)	8. (b)
9. (c)	10. (c)	11. (a)	12. (d)	13. (a)	14. (d)	15. (b)	16. (b)
17. (d)	18. (c)	19. (a)	20. (c)	21. (b)	22. (a)	23. (c)	24. (b)

SUMMARY

1. A sequence is a function whose domain is the set N of all natural numbers or some subsets of the type $\{1, 2, 3, ..., n\}$.
   A sequence containing a finite number of terms is called a finite sequence.
   A sequence is called an infinite sequence if it is not a finite sequence.
2. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a sequence, then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a series.
   A series is called a finite series if it has got finite number of terms, otherwise, it is called an infinite series.
3. Those sequences whose terms follow certain patterns are called progressions.
4. A sequence is called an arithmetic progression if the difference of a term and the previous term is always same, i.e. $a_{n+1} - a_n = \text{constant} (= d)$ for all $n \in N$
   The constant difference 'd' is called the common difference.
5. A sequence is an arithmetic progression if and only if its nth terms is a linear expression in n and in such a case the common difference is equal to the coefficient of n.
6. If a is the first term and d is the common difference of an A.P., then its nth term is given by
   $$a_n = a + (n-1)\,d$$
7. If an A.P. consists of m terms, then nth term from the end is equal to $(m-n+1)^{th}$ term from the beginning.
8. The following ways of selecting terms of an A.P. are generally very convenient:

Number of terms	*Terms*	*Common difference*
3	$a-d, a, a+d$	d
4	$a-3d, a-d, a+d, a+3d$	$2d$
5	$a-2d, a-d, a, a+d, a+2d$	d
6	$a-5d, a-3d, a-d, a+d, a+3d, a+5d$	$2d$

9. The sum S_n of n terms of an A.P. with first term 'a' and common difference 'd' is given by
$$S_n = \frac{n}{2}\{2a + (n-1)\,d\} \text{ or, } S_n = \frac{n}{2}\,(a + l), \text{ where } l = \text{last term} = a + (n-1)\,d.$$
10. If the sum S_n of n terms of a sequence is given, then n^{th} term a_n of the sequence can be determined by using the formula $a_n = S_n - S_{n-1}$
11. A sequence is an A.P. iff the sum of its n terms is of the form $An^2 + Bn$ i.e. a quadratic expression in n and in such a case the common difference is twice the coefficient of n^2.
12. If the ratio of the sums of n terms of two A.P.'s is given, then the ratio of their n^{th} terms is obtained by replacing n by $(2n-1)$ in the given ratio
13. Three numbers a, b, c are in A.P. iff $2b = a + c$. In such a case b is called the arithmetic mean of a and c.
14. The arithmetic mean of a and b is $\frac{a+b}{2}$.
15. If n numbers $A_1, A_2, \ldots, A_n$ are inserted between two given numbers a and b such that $a, A_1, A_2, \ldots, A_n, b$ is an arithmetic progression, then $A_1, A_2, \ldots, A_n$ are known as n arithmetic means between a and b and the common difference of the A.P. is $d = \frac{b-a}{n+1}$.

    Also, $A_1 + A_2 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$.
16. In an A.P. the sum of the terms equidistant from the beginning and the end is always same and is equal to the sum of first and last term.