CHAPTER 26

ELLIPSE

26.1 INTRODUCTION

In previous chapter, we have discussed that an ellipse is a particular case of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ when $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 < ab$. The analytical definition of an ellipse is as follows.

ELLIPSE *An ellipse is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.*

The constant ratio is generally denoted by e and is known as the *eccentricity* of the ellipse.

If S is the focus, ZZ' is the directrix and P is any point on the ellipse, then by definition

$$\frac{SP}{PM} = e \Rightarrow SP = e.PM$$

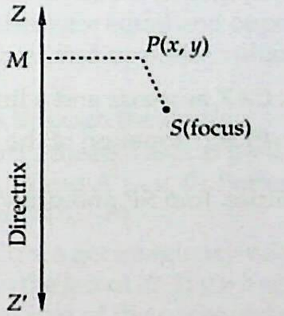

Fig. 26.1

ILLUSTRATION 1 *Find the equation of the ellipse whose focus is (1, 0), the directrix is $x + y + 1 = 0$ and eccentricity is equal to $1/\sqrt{2}$.*

SOLUTION Let S (1, 0) be the focus and ZZ' be the directrix. Let P (x, y) be any point on the ellipse and PM be perpendicular from P on the directrix. Then, by definition

$$SP = e \cdot PM, \quad \text{where } e = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad SP^2 = e^2 PM^2$$

$$\Rightarrow \quad (x-1)^2 + (y-0)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 \left|\frac{x+y+1}{\sqrt{1+1}}\right|^2$$

$$\Rightarrow \quad 4[(x-1)^2 + y^2] = (x+y+1)^2$$

$$\Rightarrow \quad 4x^2 + 4y^2 - 8x + 4 = x^2 + y^2 + 1 + 2xy + 2x + 2y$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 2xy - 10x - 2y + 3 = 0$$

This is the equation of the required ellipse.

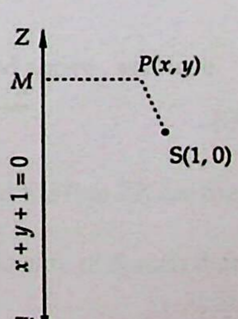

Fig. 26.2

26.2 EQUATION OF THE ELLIPSE IN STANDARD FORM $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let S be the focus, ZK the directrix and e the eccentricity of the ellipse whose equation is required. Draw SK perpendicular from S on the directrix. Divide SK internally and externally at A and A' (on KS produced) respectively in the ratio $e:1$.

$$\therefore \quad \frac{SA}{AK} = \frac{e}{1} \Rightarrow SA = e \cdot AK \qquad \text{...(i)}$$

$$\text{and,} \quad \frac{SA'}{A'K} = \frac{e}{1} \Rightarrow SA' = e\, A'K \qquad \text{...(ii)}$$

Since A and A' are such points that their distances from the focus bear constant ratio e (<1) to their respective distances from the directrix. Therefore these points lie on the ellipse.

Let $AA' = 2a$ and C be the mid-point of AA'. Then, $CA = CA' = a$

Adding (i) and (ii), we get

$$SA + SA' = e\,(AK + A'K)$$

$$\Rightarrow \quad 2a = e\,(CK - CA + A'C + CK)$$

$$\Rightarrow \quad 2a = 2e\,CK \qquad [\because CA = A'C = a]$$

$$\Rightarrow \quad CK = \frac{a}{e} \qquad \text{...(iii)}$$

Subtracting (i) from (ii), we get

$$SA' - SA = e\,(A'K - AK)$$

$$\Rightarrow \quad (SC + CA') - (CA - CS) = e\,(AA')$$

$$\Rightarrow \quad 2CS = 2ae$$

$$\Rightarrow \quad CS = ae \qquad \text{...(iv)}$$

Now let us choose C as the origin. CAX as x-axis and a line CY perpendicular to AA' as y-axis. Therefore, coordinates of S are $(ae, 0)$ and equation of the directrix ZK is $x = \frac{a}{e}$.

Let $P(x, y)$ be any point on the ellipse. Join SP and draw $PM \perp ZK$. Then, by definition of the ellipse

$$SP = e\,PM$$

$$\Rightarrow \quad SP^2 = e^2\,PM^2$$

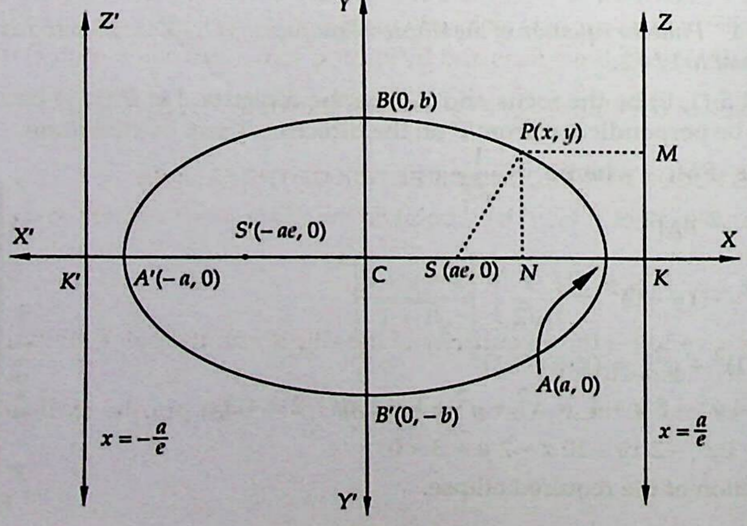

Fig. 26.3

$$\Rightarrow \quad SP^2 = e^2 (NK)^2$$

$$\Rightarrow \quad SP^2 = e^2 (CK - CN)^2$$

$$\Rightarrow \quad (x - ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$$\Rightarrow \quad x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1$$

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2 (1 - e^2)$$

This is the standard equation of the ellipse.

<u>NOTE</u> *We have,* $e < 1$*. Therefore,* $1 - e^2 < 1 \Rightarrow a^2 (1 - e^2) < a^2 \Rightarrow b^2 < a^2$.

26.2.1 TRACING OF THE ELLIPSE

We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$ ...(i)

$$\therefore \quad y = \pm \frac{b}{a} \sqrt{a^2 - x^2} \quad \text{...(ii)} \qquad \text{and, } x = \pm \frac{a}{b} \sqrt{b^2 - y^2} \quad \text{...(iii)}$$

In order to trace the ellipse (i), we observe the following points:

(a) *Symmetry*: For every value of x there are equal and opposite values of y [see (ii)]. Similarly, for every value of y there are equal and opposite values of x [see (iii)]. Thus, the curve is symmetric about both the axes.

(b) *Origin*: The curve does not pass through the origin.

(c) *Intersection with the axes*: The curve meets x axis at $y = 0$. Putting $y = 0$ in (iii), we get $x = \pm a$. So the curve meets x-axis at $A\ (a, 0)$ and $A'\ (-a, 0)$. Putting $x = 0$ in (ii), we get $y = \pm b$. So, the curve meets y-axis at $B\ (0, b)$ and $B'\ (0, -b)$.

(d) *Region*: If $x > a$ or $x < -a$, from (ii) we get imaginary values of y. Therefore, there is no part of the curve to the right of A or to the left of A'. If $y > b$ or $y < -b$, from (iii) we get imaginary values of x. Therefore, there is no part of the curve above $B\ (0, b)$ or below $B'\ (0, -b)$.
From (ii), we find that at $x = 0$, $y = \pm b$ and as x increases the values of y decrease and $y = 0$ at $x = a$. Therefore, the curve is a closed curve.
With the help of the above facts and by joining some convenient points on the ellipse, the general shape of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is as shown in Fig. 26.3.

26.2.2 SECOND FOCUS AND SECOND DIRECTRIX OF THE ELLIPSE

In Fig. 26.3 of an ellipse let $P\ (x, y)$ be a point on the curve. Then as discussed above, we have

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1 \quad \text{...(i)}$$

where $CA = CA' = a$ and e is the eccentricity of the ellipse and the point S and the line ZK are the focus and directrix respectively.

Let S' and K' be points on the x-axis on the side of C which is opposite to the side of S such that $CS' = ae$ and $CK' = \frac{a}{e}$.

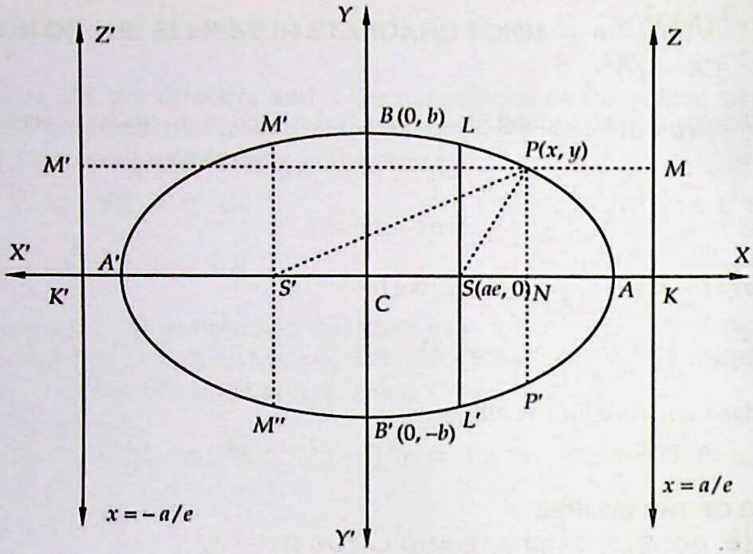

Fig. 26.4

Let $Z'K' \perp CK'$, $PM' \perp Z'K'$ as shown in Fig. 26.4. Join P and S'. Clearly $PM' = NK' = x + \frac{a}{e}$.

Now, equation (i) can be written as

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\Rightarrow \quad x^2 + y^2 + a^2e^2 = a^2 + e^2x^2$$

$$\Rightarrow \quad (x^2 + 2aex + a^2e^2) + y^2 = a^2 + 2aex + e^2x^2$$

$$\Rightarrow \quad (x+ae)^2 + y^2 = (a+ex)^2$$

$$\Rightarrow \quad (x+ae)^2 + (y-0)^2 = e^2\left(x + \frac{a}{e}\right)^2$$

$$\Rightarrow \quad S'P^2 = e^2 PM'^2$$

$$\Rightarrow \quad S'P = e\, PM'$$

$\Rightarrow$ Distance of P from $S' = e$ (Distance of P from $Z'K'$)

Hence, we would have obtained the same curve had we started with S' as focus and $Z'K'$ as directrix. This shows that the ellipse has a second focus $S'(-ae, 0)$ and a second directrix $x = -\frac{a}{e}$.

26.2.3 VERTICES, MAJOR AND MINOR AXES, FOCI, DIRECTRICES AND CENTRE

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, we have the following definitions of some terms.

VERTICES *The points A and A′ in Fig. 26.4 where the curve meets the line joining the foci S and S′, are called the vertices of the ellipse. The coordinates of A and A′ are (a, 0) and (– a, 0) respectively.*

MAJOR AND MINOR AXES *In Fig. 26.4 the distances* $AA' = 2a$ *and,* $BB' = 2b$ *are called the major and minor axes of the ellipse.*

Since $e < 1$ and $b^2 = a^2(1-e^2)$. Therefore, $a > b \Rightarrow 2a > 2b \Rightarrow AA' > BB'$.

FOCI In Fig. 26.4, the points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the ellipse.

DIRECTRICES *ZK and Z′ K′ are two directrices of the ellipse and their equations are* $x = \frac{a}{e}$ *and* $x = -\frac{a}{e}$ *respectively.*

CENTRE *Since the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *every chord, passing through C is bisected at C (0, 0). Therefore, C is the centre of the ellipse in Fig. 26.4 and it is the mid-point of AA′.*

ECCENTRICITY For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$b^2 = a^2(1 - e^2) \Rightarrow e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = 1 - \frac{4b^2}{4a^2} = 1 - \left(\frac{2b}{2a}\right)^2$$

$$\Rightarrow \quad e = \sqrt{1 - \left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$$

26.2.4 ORDINATE, DOUBLE ORDINATE AND LATUS-RECTUM

We have the following terms associated to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$:

ORDINATE AND DOUBLE ORDINATE Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that PN produced meets the ellipse at P'. Then, PN is called the ordinate of P and PNP' the double ordinate of P.

LATUS-RECTUM It is a double ordinate passing through the focus.

In Fig. 26.4, LSL' is the latus-rectum and LS is called the semi-latus-rectum. $MS'M'$ is also a latus-rectum. The coordinates of L are $(ae,\ SL)$. As L lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the coordinates of L will satisfy the equation of the ellipse.

$$\therefore \quad \frac{(ae)^2}{a^2} + \frac{(SL)^2}{b^2} = 1$$

$$\Rightarrow \quad (SL)^2 = b^2(1 - e^2)$$

$$\Rightarrow \quad (SL)^2 = b^2 \times \frac{b^2}{a^2} \qquad \left[\because b^2 = a^2(1 - e^2) \Rightarrow 1 - e^2 = \frac{b^2}{a^2}\right]$$

$$\Rightarrow \quad SL = \frac{b^2}{a}$$

$$\therefore \quad SL = SL' = \frac{b^2}{a}$$

Hence, Length of the latus-rectum $LL' = 2(SL) = \frac{2b^2}{a} = 2a(1 - e^2)$

26.2.5 FOCAL DISTANCES OF A POINT ON THE ELLIPSE

The distances of any point on the ellipse from its foci are known as its focal distances.

THEOREM *The sum of the focal distances of any point on an ellipse is constant and equal to the length of the major axis of the ellipse.*

PROOF Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (see Fig. 26.4). Then,

$$SP = e\,PM = e\,(NK) = e\,(CK - CN) = e\left(\frac{a}{e} - x\right) = a - ex \quad \text{...(i)}$$

and, $$S'P = e\,PM' = e\,(NK') = e\,(CK' + CN) = e\left(\frac{a}{e} + x\right) = a + ex \quad \text{...(ii)}$$

$\therefore$ $$SP + S'P = a - ex + a + ex = 2a = \text{Major axis } (= \text{Constant})$$

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

REMARK *On account of this property, a second definition of the ellipse may be given as follows: An ellipse is the locus of a point which moves in such a way that the sum of its distances from two fixed points (foci) is always constant.*

26.3 EQUATION OF ELLIPSE IN OTHER FORMS

In the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$ or $a^2 > b^2$ (denominator of x^2 is greater than that of y^2), then the major and minor axes lie along x-axis and y-axis respectively as shown in Fig. 26.4. But, if $a < b$ or $a^2 < b^2$ (denominator of x^2 is less than that of y^2), then the major axis of the ellipse lies along the y-axis and is of length $2b$ and the minor axis along the x-axis and is of length $2a$. The coordinates of foci S and S' are $(0, be)$ and $(0, -be)$ respectively. The equations of the directrices ZK and $Z'K'$ are $y = \frac{b}{e}$ and $y = -\frac{b}{e}$ respectively. The eccentricity e is given by the formula

$$a^2 = b^2(1 - e^2) \Rightarrow e = \sqrt{1 - \frac{a^2}{b^2}}$$

The shape of the ellipse is shown in Fig. 26.5.

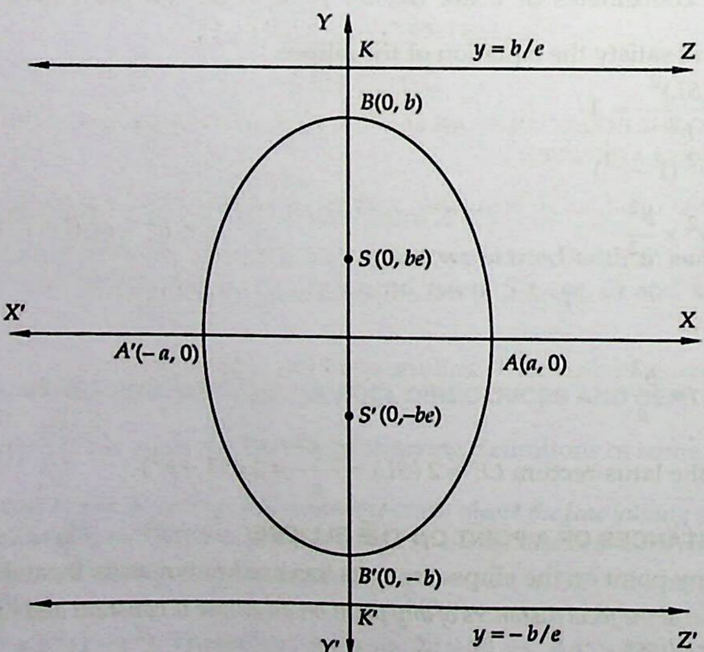

Fig. 26.5

Various results related to the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a<b)$ are given in the following table for ready reference.

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a>b$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a<b$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of the major axis	$2a$	$2b$
Length of the minor axis	$2b$	$2a$
Equation of the major axis	$y=0$	$x=0$
Equation of the minor axis	$x=0$	$y=0$
Equations of the directrices	$x=\frac{a}{e}$ and $x=-\frac{a}{e}$	$y=\frac{b}{e}$ and $y=-\frac{b}{e}$
Eccentricity	$e=\sqrt{1-\frac{b^2}{a^2}}$	$e=\sqrt{1-\frac{a^2}{b^2}}$
Length of the latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$

SPECIAL FORM *If the centre of the ellipse is at point (h, k) and the directions of the axes are parallel to the coordinate axes, then its equation is* $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF AN ELLIPSE WHEN ITS FOCUS, DIRECTRIX AND ECCENTRICITY ARE GIVEN**

EXAMPLE 1 *Find the equation of the ellipse with focus at $(1, 1)$ and eccentricity $\frac{1}{2}$ and directrix $x-y+3=0$. Also, find the equation of its major axis.*

SOLUTION Let $P(x, y)$ be a point on the ellipse. Then, by definition

$$SP = e\,PM$$

Here $e=\frac{1}{2}$, coordinates of S are $(1, 1)$ and the equation of the directrix is $x-y+3=0$.

$$\therefore \quad SP = \frac{1}{2}PM$$

$$\Rightarrow \quad \sqrt{(x-1)^2+(y-1)^2} = \frac{1}{2}\left|\frac{x-y+3}{\sqrt{1^2+(-1)^2}}\right|$$

$$\Rightarrow \quad 8[(x-1)^2+(y-1)^2] = (x-y+3)^2$$

$$\Rightarrow \quad 7x^2+7y^2+2xy-22x-10y+7 = 0$$

This is the required equation of the ellipse.

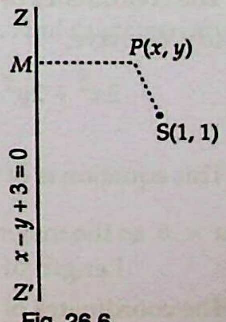

Fig. 26.6

The major axis is a line perpendicular to the directrix and passing through the focus. Therefore the equation of the major axis is $y-1=-1(x-1)$
or, $x+y-2=0$.

EXAMPLE 2 *Find the equation of the ellipse whose eccentricity is 1/2, the focus is $(-1, 1)$ and the directrix is $x-y+3=0$.*

SOLUTION Let $P(x, y)$ be any point on the ellipse whose focus is $S(-1, 1)$ and eccentricity $e=1/2$. Let PM be perpendicular from P on the directrix. Then,

$$SP = e\,PM$$

$$\Rightarrow \quad SP = \frac{1}{2}(PM)$$

$$\Rightarrow \quad 4(SP)^2 = PM^2$$

$$\Rightarrow \quad 4\left\{(x+1)^2+(y-1)^2\right\} = \left|\frac{x-y+3}{\sqrt{1^2+(-1)^2}}\right|^2$$

$$\Rightarrow \quad 8(x^2+y^2+2x-2y+2) = (x-y+3)^2$$

$$\Rightarrow \quad 7x^2+7y^2+10x-10y+2xy+7 = 0$$

This is the required equation of the ellipse.

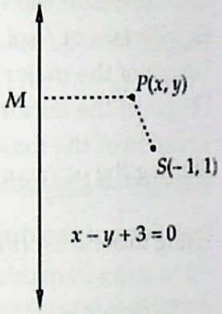

Fig. 26.7

Type II **ON FINDING THE VARIOUS ELEMENTS OF AN ELLIPSE WHEN ITS EQUATIONS IS GIVEN**

EXAMPLE 3 *For the following ellipses find the lengths of major and minor axes, coordinates of foci, vertices and the eccentricity:*

(i) $16x^2+25y^2 = 400$ (ii) $3x^2+2y^2 = 6$ (iii) $x^2+4y^2-2x = 0$

SOLUTION (i) We have,

$$16x^2+25y^2=400 \Rightarrow \frac{x^2}{25}+\frac{y^2}{16}=1 \quad \text{...(i)}$$

This is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a^2=25$ and $b^2=16$ i.e. $a=5$ and $b=4$. Clearly, $a>b$, therefore the major and minor axes of the ellipse (i) are along x and y axes respectively.

$\therefore$ Length of major axis $=2a=10$, Length of minor axis $=2b=8$.

The coordinates of the vertices are $(a, 0)$ and $(-a, 0)$ i.e. $(5, 0)$ and $(-5, 0)$.

Let e be the eccentricity of the ellipse. Then,

$$e=\sqrt{1-\frac{b^2}{a^2}}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$$

The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$ i.e. $(3, 0)$ and $(-3, 0)$.

(ii) We have,

$$3x^2+2y^2 = 6 \Rightarrow \frac{x^2}{2}+\frac{y^2}{3} = 1 \quad \text{...(i)}$$

This equation is of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, where $a^2=2$ and $b^2=3$ i.e. $a=\sqrt{2}$ and $b=\sqrt{3}$. Clearly, $a<b$, so the major and minor axes of the given ellipse are along y and x-axes respectively.

$\therefore$ Length of the major axis $=2b=2\sqrt{3}$, Length of the minor axis $=2=2\sqrt{2}$

The coordinates of the vertices are $(0, b)$ and $(0, -b)$ i.e. $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

The eccentricity e of the ellipse is given by

$$e = \sqrt{1-\frac{a^2}{b^2}} = \sqrt{1-\frac{2}{3}} = \frac{1}{\sqrt{3}}.$$

The coordinates of the foci are $(0, be)$ and $(0, -be)$ i.e. $(0, 1)$ and $(0, -1)$.

(iii) We have,

$$x^2 + 4y^2 - 2x = 0$$

$$\Rightarrow \quad (x^2 - 2x + 1) + 4y^2 = 0 + 1$$

$$\Rightarrow \quad (x-1)^2 + 4(y-0)^2 = 1$$

$$\Rightarrow \quad \frac{(x-1)^2}{1^2} + \frac{(y-0)^2}{(1/2)^2} = 1 \qquad \text{...(i)}$$

Shifting the origin at $(1, 0)$ without rotating the coordinate axes, we have

$$x = X + 1 \quad \text{and} \quad y = Y + 0 \qquad \text{...(ii)}$$

Using these relations in (i), it reduces to

$$\frac{X^2}{1^2} + \frac{Y^2}{(1/2)^2} = 1 \qquad \text{...(iii)}$$

Clearly, this equation is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a^2 = 1$ and $b^2 = 1/4$ i.e. $a = 1$ and $b = 1/2$.

We find that $a > b$. So, the major and minor axes of the ellipse (iii) are along X and Y axes respectively.

$\therefore$ Length of the major axis $= 2a = 2$; Length of the minor axis $= 2b = 1$.

The eccentricity e is given by $e = \sqrt{1-\frac{b^2}{a^2}} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

The coordinates of the vertices with respect to the new axes are $(X = 1, Y = 0)$ and $(X = -1, Y = 0)$. So, the coordinates of the vertices with respect to the old axes are $(2, 0)$ and $(0, 0)$ [Putting $X = 1, Y = 0$ and $X = -1, Y = 0$ separately in (ii)]

The coordinates of the foci with respect to the new axes are

$$\left(X = \frac{\sqrt{3}}{\sqrt{2}}, Y = 0\right) \text{ and } \left(X = -\frac{\sqrt{3}}{\sqrt{2}}, Y = 0\right) \qquad [\text{Coordinates of foci are } (\pm\, ae, 0)]$$

So, the coordinates of the foci with respect to the old axes are

$$\left(\sqrt{\frac{3}{2}} + 1, 0\right) \text{ and } \left(1 - \sqrt{\frac{3}{2}}, 0\right). \qquad \left[\text{Putting } X = \pm\frac{\sqrt{3}}{\sqrt{2}}, Y = 0 \text{ in (ii)}\right]$$

EXAMPLE 4 *Show that $x^2 + 4y^2 + 2x + 16y + 13 = 0$ is the equation of an ellipse. Find its eccentricity, vertices, foci, directrices and, the length and the equation of the latus-rectum.*

SOLUTION We have,

$$x^2 + 4y^2 + 2x + 16y + 13 = 0$$

$$\Rightarrow \quad (x^2 + 2x + 1) + 4(y^2 + 4y + 4) = 4$$

$$\Rightarrow \quad (x+1)^2 + 4(y+2)^2 = 4$$

$$\Rightarrow \quad \frac{(x+1)^2}{2^2} + \frac{(y+2)^2}{1^2} = 1 \qquad \text{...(i)}$$

Shifting the origin at (–1, –2) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we have

$$x = X - 1 \quad \text{and} \quad y = Y - 2 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a = 2$ and $b = 1$.

Thus, the given equation represents an ellipse. Clearly, $a > b$. So, the given equation represents an ellipse whose major and minor axes are along X and Y axes respectively.

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}$$

Vertices: The vertices of the ellipse with respect to the new axes are $(X = \pm a,\ Y = 0)$ i.e. $(X = \pm 2,\ Y = 0)$. So, the vertices with respect to the old axes are given by

$$(\pm 2 - 1, -2) \text{ i.e. } (-3, -2) \text{ and } (1, -2) \qquad [\text{Putting } x = \pm 2,\ y = 0 \text{ in (ii)}]$$

Foci: The coordinates of the foci with respect to the new axes are given by $(X = \pm ae,\ Y = 0)$ i.e. $(X = \pm\sqrt{3},\ Y = 0)$. So, the coordinates of foci with respect to the old axes are given by

$$(\pm\sqrt{3} - 1,\ -2) \qquad [\text{Putting } X = \pm\sqrt{3},\ Y = 0 \text{ in (ii)}]$$

Directrices: The equations of the directrices with respect to the new axes are $X = \pm\frac{a}{e}$ i.e. $X = \pm\frac{4}{\sqrt{3}}$. So, the equations of the directrices with respect to the old axes are

$$x = \pm\frac{4}{\sqrt{3}} - 1 \text{ i.e. } x = \frac{4}{\sqrt{3}} - 1 \text{ and } x = -\frac{4}{\sqrt{3}} - 1 \qquad \left[\text{Putting } X = \pm\frac{4}{\sqrt{3}} \text{ in (ii)}\right]$$

Length of the latus-rectum: The length of the latusrectum $= \frac{2b^2}{a} = \frac{2}{2} = 1.$

Equations of Latus-recta: The equations of the latusrecta with respect to the new axes are $X = \pm ae$ *i.e.* $X = \pm\sqrt{3}$. So, the equations of the latus-recta with respect to the old axes are

$$x = \pm\sqrt{3} - 1 \text{ i.e. } x = \sqrt{3} - 1 \text{ and } x = -\sqrt{3} - 1. \qquad [\text{Putting } X = \pm\sqrt{3} \text{ in (ii)}]$$

EXAMPLE 5 *Find the eccentricity, centre, vertices, foci, minor axis, major axis, directrices and latus-rectum of the ellipse* $25x^2 + 9y^2 - 150x - 90y + 225 = 0$.

SOLUTION The equation of the ellipse is

$$25x^2 + 9y^2 - 150x - 90y + 225 = 0$$

$$\Rightarrow \quad 25x^2 - 150x + 9y^2 - 90y = -225$$

$$\Rightarrow \quad 25(x^2 - 6x) + 9(y^2 - 10y) = -225$$

$$\Rightarrow \quad 25(x^2 - 6x + 9) + 9(y^2 - 10y + 25) = -225 + 225 + 225$$

$$\Rightarrow \quad 25(x - 3)^2 + 9(y - 5)^2 = 225$$

$$\Rightarrow \quad \frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1 \qquad \text{...(i)}$$

Shifting the origin at (3, 5) without rotating the coordinate axes and denoting the new coordinates with respect to the new axes by X and Y, we have

$$x = X + 3 \quad \text{and} \quad y = Y + 5 \qquad \text{...(ii)}$$

Using these relations, equation (i) reduces to

$$\frac{X^2}{3^2} + \frac{Y^2}{5^2} = 1 \qquad \text{...(iii)}$$

This is of the form $\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$, where $a^2 = 3^2$ and $b^2 = 5^2$. Clearly, $a < b$. So, equation (iii) represents an ellipse whose major and minor axes along Y and X axes respectively.

Eccentricity: The eccentricity e is given by

$$e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

Centre: The coordinates of the centre with respect to new axes are $(X = 0,\ Y = 0)$. So, the coordinates of the centre with respect to old axes are $(3, 5)$.

Vertices: The vertices of the ellipse with respect to the new axes are $(X = 0, Y = \pm b)$ i.e. $(X = 0, Y = \pm 5)$. So, the vertices with respect to the old axes are

$$(3, 5 \pm 5) \ \textit{i.e.}\ (3, 0) \ \text{and} \ (3, 10) \qquad [\text{Putting } X = 0,\ Y = \pm 5 \text{ in (ii)}]$$

Foci: The coordinates of the foci with respect to the old axes are $(X = 0, Y = \pm be)$ i.e. $(X = 0, Y = \pm 4)$. So, the coordinates of the foci with respect to the old axes are

$$(3, \pm 4 + 5) \ \textit{i.e.}\ (3, 1) \ \text{and} \ (3, 9) \qquad [\text{Putting } X = 0,\ Y = \pm 4 \text{ in (ii)}]$$

Directrices: The equations of the directrices with respect to the new axes are $Y = \pm \frac{b}{e}$ i.e. $Y = \pm \frac{25}{4}$.

So, the equations of the directrices with respect to the old axes are

$$y = \pm \frac{25}{4} + 5 \ \textit{i.e.}\ y = -\frac{5}{4} \ \text{and} \ y = \frac{45}{4}. \qquad \left[\text{Putting } Y = \pm \frac{25}{4} \text{ in (ii)}\right]$$

Axes: Lengths of the major and minor axes are: Major axis $= 2b = 10$, Minor axis $= 2a = 6$.

Equation of the major axis with respect to the new axes is $X = 0$. So, the equation of the major axis with respect to the old axes is $x = 3$. [Putting $X = 0$ in (ii)]

The equation of the minor axis with respect to the new axes is $Y = 0$. So, the equation of the minor axis with respect to the old axes is $y = 5$. [Putting $Y = 0$ in (ii)]

Latus-rectum: The length of the latus-rectum $= \frac{2a^2}{b} = \frac{2 \times 9}{5} = \frac{18}{5}$.

The equations of the latus-recta with respect to the new axes are $Y = \pm ae$ *i.e.* $Y = \pm 4$. So, the equations of the latus-recta with respect to the old axes are

$$y = \pm 4 + 5 \ \textit{i.e.}\ y = 1 \ \text{and} \ y = 9. \qquad [\text{Putting } Y = \pm 4 \text{ in (ii)}]$$

EXAMPLE 6 *Find the eccentricity, foci and the length of the latusrectum of the ellipse* $x^2 + 4y^2 + 8y - 2x + 1 = 0$.

SOLUTION The given equation of the ellipse is

$$x^2 + 4y^2 + 8y - 2x + 1 = 0$$

$$\Rightarrow \quad x^2 - 2x + 4y^2 + 8y = -1$$

$$\Rightarrow \quad (x^2 - 2x + 1) + 4(y^2 + 2y + 1) = -1 + 1 + 4$$

$$\Rightarrow \quad (x - 1)^2 + 4(y + 1)^2 = 4$$

$$\Rightarrow \quad \frac{(x-1)^2}{2^2} + \frac{(y+1)^2}{1^2} = 1 \qquad \text{...(i)}$$

Shifting the origin to $(1, -1)$ without rotating the axes and denoting the new coordinates with respect to these axes by X and Y, we obtain

$$x = X + 1,\ y = Y - 1 \qquad \text{...(ii)}$$

Using these relations equation (i) reduces to

$$\frac{X^2}{2^2} + \frac{Y^2}{1^2} = 1$$

This is of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$. On compairing, we get

$$a^2 = 2^2 \text{ and } b^2 = 1 \Rightarrow a = 2 \text{ and } b = 1.$$

Let e be the eccentricity of the ellipse. Then,

$$b^2 = a^2(1-e^2) \Rightarrow 1 = 4(1-e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$$

The coordinates of foci with respect to new axes are $(X = \pm\, ae,\ Y = 0)$ i.e., $(X = \pm\sqrt{3},\ Y = 0)$.
So, coordinates of foci with respect to old axes are $(1 \pm \sqrt{3}, -1)$ [Putting $X = \pm\sqrt{3}, Y = 0$ in (ii)]

$$\text{Length of the latus-rectum} = \frac{2b^2}{a} = \frac{2(1)^2}{2} = 1.$$

EXAMPLE 7 *Find the distance between the directrices the ellipse* $\frac{x^2}{36} + \frac{y^2}{20} = 1.$

[NCERT EXEMPLAR]

SOLUTION Comparing the given equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we obtain $a^2 = 36$ and $b^2 = 20$.

Let e be the eccentricity of the ellipse. Then,

$$b^2 = a^2(1-e^2) \Rightarrow 20 = 36(1-e^2) \Rightarrow 36e^2 = 16 \Rightarrow e = \frac{2}{3}$$

$\therefore$ Distance between the directrices $= \frac{2a}{e} = \frac{2\times 6}{2/3} = 18.$

Type III ON FINDING SOME ELEMENTS OF AN ELLIPSE FROM GIVEN ELEMENTS

EXAMPLE 8 *If the eccentricity of an ellipse is* $\frac{5}{8}$ *and the distance between its foci is 10, then find the latusrectum of the ellipse.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the required ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and let e its eccentricity.

We have, $e = \frac{5}{8}$ and $2ae = 10$

$$\Rightarrow \quad e = \frac{5}{8} \text{ and } ae = 5$$

$$\Rightarrow \quad e = \frac{5}{8} \text{ and } a = 8$$

$$\therefore \quad b^2 = a^2(1-e^2) \Rightarrow b^2 = 64\left(1 - \frac{25}{64}\right) = 39$$

Hence, length of the latusrectum $= \frac{2b^2}{a} = 2 \times \frac{39}{8} = \frac{39}{4}$

EXAMPLE 9 *If the latusrectum of an ellipse is equal to half of minor axis, find its eccentricity.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let e be its eccentricity.

It is given that

$$\text{Latusrectum} = \frac{1}{2}\ \text{(Minor axis)}$$

$$\Rightarrow \quad \frac{2b^2}{a} = \frac{1}{2}(2b)$$

$$\Rightarrow \quad 2b = a$$

$$\Rightarrow \quad 4b^2 = a^2$$

$$\Rightarrow \quad 4a^2(1-e^2) = a^2 \Rightarrow 4-4e^2 = 1 \Rightarrow 4e^3 = 3 \Rightarrow e = \frac{\sqrt{3}}{2}$$

Hence, the eccentricity is $\frac{\sqrt{3}}{2}$.

Type IV **ON FINDING THE EQUATION OF AN ELLIPSE WHEN SOME OF ITS ELEMENTS ARE GIVEN**

EXAMPLE 10 *Find the equation of the ellipse whose axes are along the coordinate axes, vertices are* $(\pm 5, 0)$ *and foci at* $(\pm 4, 0)$.

SOLUTION Let the equation of the required ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

The coordinates of its vertices and foci are $(\pm a, 0)$ and $(\pm ae, 0)$ respectively. But, the coordinates of vertices and foci are given as $(\pm 5, 0)$ and $(\pm 4, 0)$.

$$\therefore \quad a = 5 \text{ and } ae = 4 \Rightarrow e = \frac{4}{5}$$

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 25\left(1-\frac{16}{25}\right) = 9.$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{25}+\frac{y^2}{9}=1$, which is the equation of the required ellipse.

EXAMPLE 11 *Find the equation of the ellipse whose axes are along the coordinate axes, vertices are* $(0, \pm 10)$ *and eccentricity* $e = 4/5$.

SOLUTION Let the equation of the required ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

Since the vertices of the ellipse are on y-axis. So, the coordinates of the vertices are $(0, \pm b)$. But, the coordinates of vertices are given to be $(0, \pm 10)$.

$$\therefore \quad b = 10.$$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = 100\left(1-\frac{16}{25}\right) = 36$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{36}+\frac{y^2}{100}=1$ as the equation of the required ellipse.

EXAMPLE 12 *Find the equation of the ellipse whose axes are along the coordinate axes, foci at $(0, \pm 4)$ and eccentricity 4/5.*

SOLUTION Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(i)}$$

The coordinates of the foci are $(0, \pm 4)$. This means that the major and minor axes of the ellipse are along y and x axes respectively and the coordinates of foci are $(0, \pm be)$.

$\therefore \quad be = 4$

$\Rightarrow \quad b\,(4/5) = 4 \qquad [\because\ e = 4/5]$

$\Rightarrow \quad b = 5.$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = 25\left(1-\frac{16}{25}\right) = 9$

Substituting the values of a^2 and b^2 in (i), we obtain $\frac{x^2}{9}+\frac{y^2}{25}=1$ as the equation of the required ellipse.

EXAMPLE 13 *The foci of an ellipse are $(\pm 2, 0)$ and its eccentricity is $1/2$, find its equation if it is given that its centre is at the origin and axes are along the coordinates axes.*

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The, coordinates of its foci are $(\pm ae, 0)$.

But, the coordinates of foci are given as $(\pm 2, 0)$.

$\therefore \quad ae = 2$

$\Rightarrow \quad a \times \frac{1}{2} = 2 \qquad [\because\ e = 1/2]$

$\Rightarrow \quad a = 4.$

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 16\left(1-\frac{1}{4}\right) = 12.$

Substituting $a=4$ and $b^2=12$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we obtain $\frac{x^2}{16}+\frac{y^2}{12}=1$ as the equation of the ellipse.

EXAMPLE 14 *Find the equation of the ellipse with foci at $(\pm 5, 0)$ and $x=\frac{36}{5}$ as one of the directrices.*

[NCERT EXEMPALR]

SOLUTION Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and let e be its eccentricity. The coordinates of its foci and the equations of the directrices are $(\pm ae, 0)$ and $x = \pm a/e$ respectively. But, it is given that the coordinates of foci are $(\pm 5, 0)$ and the equations of one of the directrices is $x = 36/5$.

$\therefore \quad ae = 5$ and $\frac{a}{e}=\frac{36}{5}$

$\Rightarrow \quad ae \times \frac{a}{e} = 5 \times \frac{36}{5}$

$\Rightarrow \quad a^2 = 36$

$\Rightarrow \quad a = 6$

Now, $\quad b^2 = a^2(1-e^2)$

$\Rightarrow \quad b^2 = a^2-(ae)^2 = 36-25 = 11$

$\Rightarrow \quad b = \sqrt{11}$

Substituting $a=6$ and $b=\sqrt{11}$ in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we obtain $\frac{x^2}{36}+\frac{y^2}{11}=1$ as the required equation.

EXAMPLE 15 *Find the equation of the ellipse whose axes are parallel to the coordinate axes having its centre at the point (2, – 3) one focus at (3, – 3) and one vertex at (4, – 3).*

SOLUTION Let $2a$ and $2b$ be the major and minor axes of the ellipse. Then, its equation is

$$\frac{(x-2)^2}{a^2} + \frac{(y+3)^2}{b^2} = 1 \quad \text{...(i)}$$

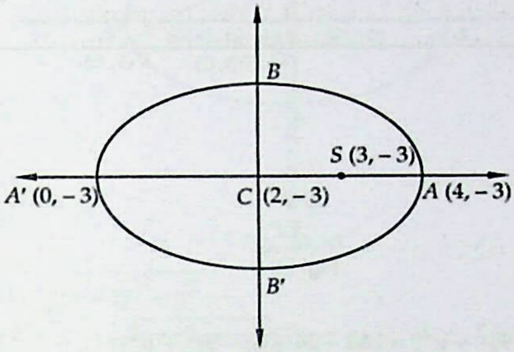

Fig. 26.8

Clearly,

$CA = a$ (= Semi-Major axis)

$\Rightarrow \quad \sqrt{(4-2)^2 + (-3+3)^2} = a$

$\Rightarrow \quad a = 2.$ ...(ii)

Since the distance between the focus and centre of an ellipse is equal to ae, where e is the eccentricity.

$\therefore \quad CS = ae$

$\Rightarrow \quad \sqrt{(2-3)^2 + (-3+3)^2} = ae$

$\Rightarrow \quad ae = 1$ ...(iii)

From (ii) and (iii), we get : $e = \frac{1}{2}$.

Now, $b^2 = a^2(1-e^2) \Rightarrow b^2 = 4\left(1-\frac{1}{4}\right) = 3.$

Substituting the values of a and b in (i), we obtain

$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{3} = 1$, as the required equation of the ellipse.

LEVEL-2

EXAMPLE 16 *Find the equation of the set of all points the sum of whose distances from the points* (3, 0) *and* (9, 0) *is* 12.

SOLUTION Let $P(x, y)$ be a point such that the sum of its distances from $S(3, 0)$ and $S'(9, 0)$ is 12.

i.e. $\quad PS + PS' = 12$

$\Rightarrow \quad \sqrt{(x-3)^2 + (y-0)^2} + \sqrt{(x-9)^2 + (y-0)^2} = 12$

$\Rightarrow \quad \sqrt{(x-3)^2 + y^2} = 12 - \sqrt{(x-9)^2 + y^2}$

$\Rightarrow \quad (x-3)^2 + y^2 = 144 - 24\sqrt{(x-9)^2 + y^2} + \{(x-9)^2 + y^2\}$

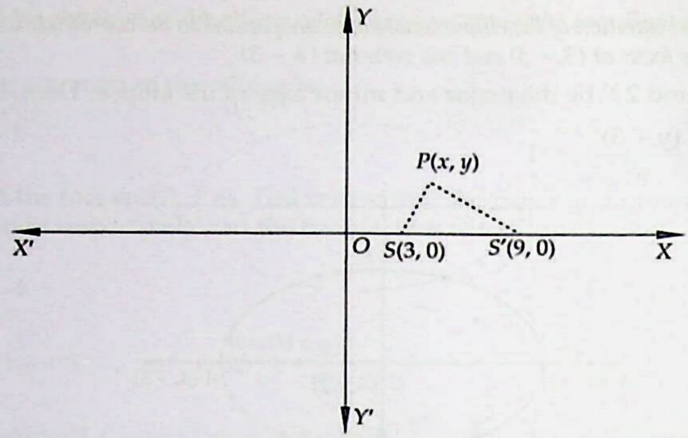

Fig. 26.9

$\Rightarrow \quad \{(x-3)^2+y^2\}-\{(x-9)^2+y^2\}=144-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad 12x-72=144-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad 12x-216=-24\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad x-18=-2\sqrt{(x-9)^2+y^2}$

$\Rightarrow \quad (x-18)^2=4\{(x-9)^2+y^2\}$

$\Rightarrow \quad x^2-36x+324=4x^2-72x+324+4y^2$

$\Rightarrow \quad 3x^2-36x+4y^2=0$, which is the required equation.

<u>ALITER</u> We know that the sum of the focal distances of a point on the ellipse is constant equal to major axis. Therefore, the curve is an ellipse having its foci at $S(3, 0)$ and $S'(9, 0)$ and major axis $2a=12$. The distance between the foci S and S' is 6.

$\therefore \quad 2ae=6 \Rightarrow 12e=6 \Rightarrow e=\frac{1}{2}$

Now, $\quad b^2=a^2(1-e^2) \Rightarrow b^2=36\left(1-\frac{1}{4}\right)=27$

The centre of the ellipse is the mid-point of segment SS'. So, the coordinates of centre are $(6, 0)$. Hence, the equation of the ellipse is

$$\frac{(x-6)^2}{6^2}+\frac{(y-0)^2}{27}=1 \text{ or, } 3x^2+4y^2-36x=0$$

EXAMPLE 17 *Find the equation of the ellipse whose centre is at the origin, foci are $(1, 0)$ and $(-1, 0)$ and eccentricity is $1/2$.*

SOLUTION Here coordinates of two foci S and S' are $(1, 0)$ and $(-1, 0)$ respectively. Therefore, $SS'=2$. Let $2a$ and $2b$ be the lengths of the major and minor axes of the required ellipse and e be the eccentricity. Then, $SS'=2ae \Rightarrow 2ae=2 \Rightarrow ae=1 \Rightarrow a\left(\frac{1}{2}\right)=1 \Rightarrow a=2$.

Let $P(x, y)$ be any point on the ellipse. Then,

$SP+S'P=2a$ [See section 26.2.5]

$\Rightarrow \quad SP+S'P=4$ [$\because a=2$]

$\Rightarrow \quad \sqrt{(x-1)^2+(y-0)^2}+\sqrt{(x+1)^2+(y-0)^2}=4$

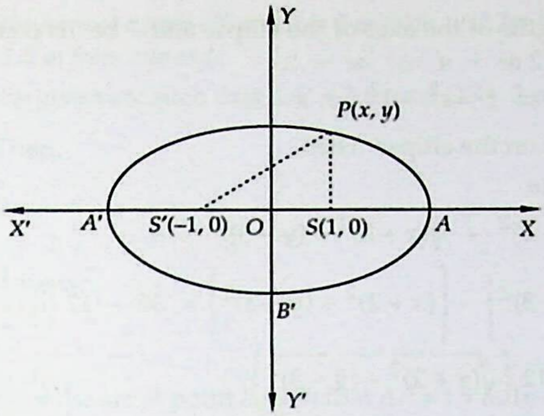

Fig. 26.10

$$\Rightarrow \quad \sqrt{(x-1)^2+y^2} = 4-\sqrt{(x+1)^2+y^2}$$

$$\Rightarrow \quad \left\{\sqrt{(x-1)^2+y^2}\right\}^2 = \left\{4-\sqrt{(x+1)^2+y^2}\right\}^2$$

$$\Rightarrow \quad (x-1)^2+y^2 = 16-8\sqrt{(x+1)^2+y^2}+(x+1)^2+y^2$$

$$\Rightarrow \quad \left\{(x-1)^2+y^2\right\}-\left\{(x+1)^2+y^2\right\} = 16-8\left\{\sqrt{(x+1)^2+y^2}\right\}$$

$$\Rightarrow \quad -4x = 16-8\left\{\sqrt{(x+1)^2+y^2}\right\}$$

$$\Rightarrow \quad x+4 = 2\left\{\sqrt{(x+1)^2+y^2}\right\}$$

$$\Rightarrow \quad (x+4)^2 = 4\left\{(x+1)^2+y^2\right\}$$

$\Rightarrow \quad 3x^2+4y^2-12 = 0$, which is the required equation of the ellipse.

ALITER Let $S(1, 0)$ and $S'(-1, 0)$ be the foci of the ellipse and e be its eccentricity. The centre of the ellipse is the mid-point of segment SS'. So, the coordinates of the centre are $(0, 0)$. Let $2a$ and $2b$ be the lengths of major and minor axes of the ellipse.

Now, $\quad SS' = \sqrt{(-1-1)^2+(0-0)^2} = 2$

$\Rightarrow \quad 2ae = 2$

$\Rightarrow \quad ae = 1$

$\Rightarrow \quad a \times \frac{1}{2} = 1 \qquad \left[\because\ e = \frac{1}{2} \text{ (given)}\right]$

$\Rightarrow \quad a = 2.$

Now, $\quad b^2 = a^2(1-e^2) = 4\left(1-\frac{1}{4}\right) = 3$

Hence, the equation of the ellipse is $\frac{x^2}{4}+\frac{4^2}{3} = 1.$

EXAMPLE 18 *Find the equation of the ellipse whose foci are $(2, 3)$, $(-2, 3)$ and whose semi-minor axis is $\sqrt{5}$.*

SOLUTION Let S and S' be two foci of the required ellipse. Then, the coordinates of S and S' are $(2, 3)$ and $(-2, 3)$ respectively. Therefore, $SS' = 4$

Let $2a$ and $2b$ be the lengths of the axes of the ellipse and e be its eccentricity. Then,

$$SS' = 2ae. \Rightarrow 2ae = 4 \Rightarrow ae = 2.$$

Now, $b^2 = a^2(1 - e^2) \Rightarrow 5 = a^2 - 2^2 \Rightarrow a = 3.$

Let $P(x, y)$ be any point on the ellipse. Then,

$$SP + S'P = 2a \qquad \text{[See section 26.2.5]}$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} + \sqrt{(x+2)^2 + (y-3)^2} = 6$$

$$\Rightarrow \left\{(x-2)^2 + (y-3)^2\right\} - \left\{(x+2)^2 + (y-3)^2\right\} = 36 - 12\left\{\sqrt{(x+2)^2 + (y-3)^2}\right\}$$

$$\Rightarrow -8x = 36 - 12\left\{\sqrt{(x+2)^2 + (y-3)^2}\right\}$$

$$\Rightarrow (2x+9)^2 = 9\{(x+2)^2 + (y-3)^2\}$$

$$\Rightarrow 5x^2 + 9y^2 - 54y + 36 = 0,$$ which is the required equation of the ellipse.

EXAMPLE 19 *A rod AB of length 15 cm rests in between two coordinate axes in such a way that the end point A lies on x-axis and end point B lies on y-axis. A point is taken on the rod in such a way that AP = 6 cm. Show that the locus of P is an ellipse. Also, find its eccentricity.* [NCERT]

SOLUTION Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. Let the coordinates of P be (h, k).

We have, $AP = 6$ cm and $AB = 15$ cm.

$\therefore \quad BP = 9$ cm.

Since $P(h, k)$ divides AB in the ratio $6 : 9$. Therefore,

$$h = \frac{9a}{15} \text{ and } k = \frac{6b}{15}$$

$$\Rightarrow a = \frac{15h}{9} \text{ and } b = \frac{15k}{6}$$

$$\Rightarrow a = \frac{5h}{3} \text{ and } b = \frac{5k}{2}$$

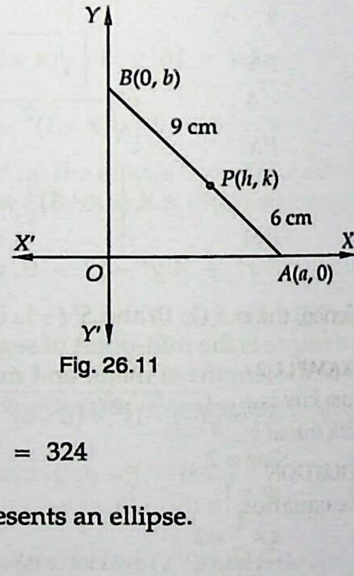

Fig. 26.11

In ΔOAB, we have,

$$OA^2 + OB^2 = AB^2$$

$$\Rightarrow a^2 + b^2 = 15^2 \Rightarrow \frac{25h^2}{9} + \frac{25k^2}{4} = 15^2 \Rightarrow 4h^2 + 9k^2 = 324$$

Hence, the locus of $P(h, k)$ is $4x^2 + 9y^2 = 324$. Clearly, it represents an ellipse.

Now, $4x^2 + 9y^2 = 324 \Rightarrow \frac{x^2}{81} + \frac{y^2}{36} = 1$

Comparing this equation with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we obtain

$$a^2 = 81 \text{ and } b^2 = 36 \Rightarrow a = 9 \text{ and } b = 6$$

Let e be the eccentricity of the ellipse. Then,

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{36}{81}} = \frac{\sqrt{5}}{3}.$$

EXAMPLE 20 *An arc is in the form of a semi-ellipse. It is 8 m wide and 2m high at the centre. Find the height of the arch at a point 1.5 m from one end.*

SOLUTION Let ABA' be the given arc such that $AA' = 8$ m and $OB = 2$ m. Let the arc be a part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then,

$$AA' = 8 \text{ m} \Rightarrow 2a = 8 \Rightarrow a = 4$$

and, $OB = 2 \text{ m} \Rightarrow b = 2.$

So, the equation of the ellipse is

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \text{...(i)}$$

We have, to find the height of the arc at point P such that $AP = 1.5$ m. In other words, we have to find the y-coordinate at P.

$\because \quad OA = 4 \text{ m} \text{ and } AP = 1.5 \text{ m}$

$\therefore \quad OP = OA - AP = (4 - 1.5) \text{ m} = 2.5 \text{ m}.$

Thus, the coordinates of M are $\left(\frac{5}{2}, PM\right)$.

Since M lies on the ellipse (i). Therefore,

$$\frac{25}{4 \times 16} + \frac{PM^2}{4} = 1$$

$$\Rightarrow \quad \frac{PM^2}{4} = 1 - \frac{25}{64}$$

$$\Rightarrow \quad \frac{PM^2}{4} = \frac{39}{64}$$

$$\Rightarrow \quad PM = \sqrt{\frac{39}{16}} \text{ m} = \frac{\sqrt{39}}{4} \text{ m}$$

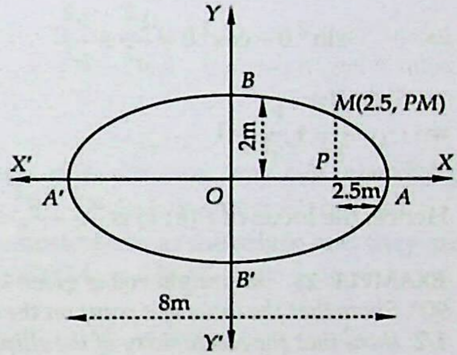

Fig. 26.12

Hence, the height of the arc at a point 1.5 m from one end is $\frac{\sqrt{39}}{4}$ m.

EXAMPLE 21 *A man running a race-course notes that the sum of the distances from the two flag posts from him is always 10 metres and the distance between the flag posts is 8 metres. Find the equation of the path traced by the man.* **[NCERT]**

SOLUTION Clearly, the path traced by the man is an ellipse having its foci at two flag posts. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b^2 = a^2(1 - e^2)$$

It is given that the sum of the distances of the man from the two flag posts is 10 metres. This means that the sum of the focal distnaces of a point on the ellipse is 10 m.

$\therefore \quad 2a = 10 \Rightarrow a = 5 \quad$ [$\because$ Sum of the focal distances of a point $= 2a$]

It is also given that the distance between the flag posts is 8 metres.

$\therefore \quad 2ae = 8 \Rightarrow ae = 4 \quad$ [$\because$ Distance between two foci $= 2ae$]

Now, $\quad b^2 = a^2(1 - e^2)$

$\Rightarrow \quad b^2 = a^2 - a^2e^2 = 25 - 16 = 9$

Hence, the equation of the path is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

EXAMPLE 22 *A bar of given length moves with its extremities on two fixed straight lines at right angles. Show that any point on the bar describes an ellipse.* **[NCERT EXEMPLAR]**

SOLUTION Let AB be a bar of length l which slides between the coordinate axes and let $P(h, k)$ be a point on the bar such that $PA = a$ and $PB = b$.

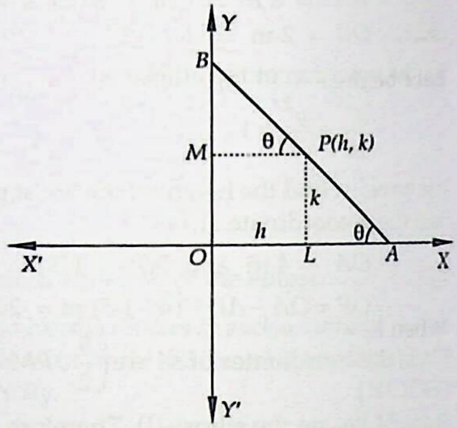

Fig. 26.13

Let $\angle OAB = \theta$. Then, $\angle MPB = \theta$.

In Δ's ALP and PMB, we have

$$\sin\theta = \frac{PL}{AP} \text{ and } \cos\theta = \frac{PM}{BP}$$

$$\Rightarrow \quad \sin\theta = \frac{k}{a} \text{ and } \cos\theta = \frac{h}{b}$$

$$\Rightarrow \quad \sin^2\theta + \cos^2\theta = \frac{k^2}{a^2} + \frac{h^2}{b^2}$$

$$\Rightarrow \quad \frac{h^2}{b^2} + \frac{k^2}{a^2} = 1$$

Hence, the locus of $P(h, k)$ is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, which is an ellipse.

EXAMPLE 23 *A straight rod of given length slides between two fixed bars which include an angle of 90°. Show that the locus of a point on the rod which divides it in a given ratio is an ellipse. If this ratio be 1/2, show that the eccentricity of the ellipse is $\sqrt{2}/3$.*

SOLUTION Let the two lines be along the coordinate axes. Let PQ be the rod of length a such that $\angle OPQ = \theta$. Then, the coordinates of P and Q are $(a\cos\theta, 0)$ and $(0, a\sin\theta)$ respectively. Let $R(h, k)$ be the point dividing PQ in the ratio $\lambda : 1$. Then,

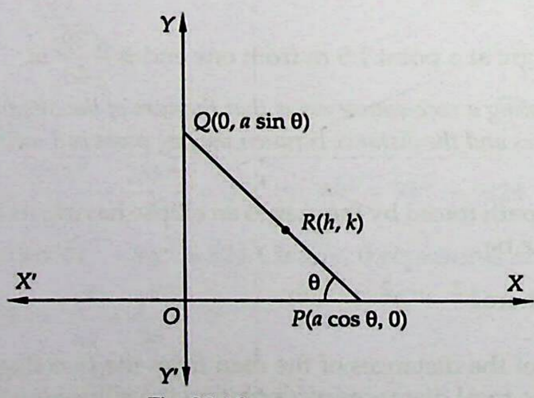

Fig. 26.14

$$h = \frac{a\cos\theta}{\lambda+1} \quad \text{and} \quad k = \frac{\lambda a \sin\theta}{\lambda+1}$$

$$\Rightarrow \quad \cos\theta = \frac{h}{a}(\lambda+1) \text{ and } \sin\theta = \frac{k}{a\lambda}(\lambda+1)$$

$$\Rightarrow \quad \cos^2\theta + \sin^2\theta = \frac{h^2}{a^2}(\lambda+1)^2 + \frac{k^2}{a^2\lambda^2}(\lambda+1)^2$$

$$\Rightarrow \quad \frac{h^2}{\left(\frac{a}{\lambda+1}\right)^2} + \frac{k^2}{\left(\frac{a\lambda}{\lambda+1}\right)^2} = 1$$

Hence, the locus of (h, k) is $\dfrac{x^2}{\left(\frac{a}{\lambda+1}\right)^2} + \dfrac{y^2}{\left(\frac{a\lambda}{\lambda+1}\right)^2} = 1$, which is an ellipse.

Let e be the eccentricity of this ellipse. Then,

$$e = \sqrt{1 - \frac{\left(\frac{a\lambda}{\lambda+1}\right)^2}{\left(\frac{a}{\lambda+1}\right)^2}} \qquad \left[\because\ e = \sqrt{1-\frac{b^2}{a^2}}\right]$$

$$\Rightarrow \quad e = \sqrt{1-\lambda^2}$$

When $\lambda = \frac{1}{2}$, we obtain

$$e = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

EXAMPLE 24 *A point moves so that the sum of the squares of its distances from two intersecting straight lines is constant. Prove that its locus is an ellipse.*

SOLUTION Let us assume that the two intersecting lines intersects at the origin and they are equally inclined with the positive direction of x-axis i.e. $\angle XOA = \angle XOC = \theta$.

The equations OA and OB are respectively

$$y = x \tan\theta \quad \text{and} \quad y = -x\tan\theta$$

or, $\quad x\sin\theta - y\cos\theta = 0$ and $x\sin\theta + y\cos\theta = 0$

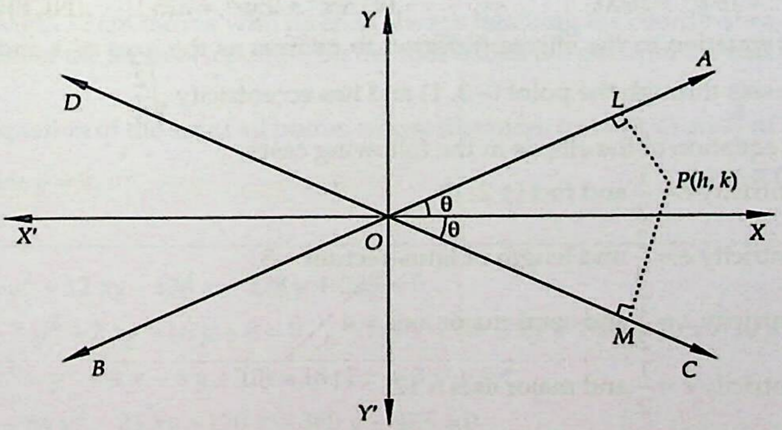

Fig. 26.15

Let $P(h, k)$ be a variable point such that the sum of the squares of its distances from OA and OB is constant.

i.e. $\quad PL^2 + PM^2 = \lambda^2$ (constant)

$$\Rightarrow \quad \left(\frac{h\sin\theta - k\cos\theta}{\sqrt{\sin^2\theta+\cos^2\theta}}\right)^2 + \left(\frac{h\sin\theta + k\cos\theta}{\sqrt{\sin^2\theta+\cos^2\theta}}\right)^2 = \lambda^2$$

$$\Rightarrow \quad (h \sin \theta - k \cos \theta)^2 + (h \sin \theta + k \cos \theta)^2 = 2\lambda^2$$

$$\Rightarrow \quad h^2 \sin^2 \theta + k^2 \cos^2 \theta = \lambda^2$$

$$\Rightarrow \quad \frac{h^2}{\lambda^2 \operatorname{cosec}^2 \theta} + \frac{k^2}{\lambda^2 \sec^2 \theta} = 1$$

Hence, the locus of (h, k) is

$\dfrac{x^2}{(\lambda \operatorname{cosec} \theta)^2} + \dfrac{y^2}{(\lambda \sec \theta)^2} = 1$, which is an ellipse having its centre at the intersection point of the given lines.

EXERCISE 26.1

LEVEL-1

1. Find the equation of the ellipse whose focus is $(1, -2)$, the directrix $3x - 2y + 5 = 0$ and eccentricity equal to $1/2$.
2. Find the equation of the ellipse in the following cases:
   (i) focus is $(0, 1)$, directrix is $x + y = 0$ and $e = \frac{1}{2}$.
   (ii) focus is $(-1, 1)$, directrix is $x - y + 3 = 0$ and $e = \frac{1}{2}$.
   (iii) focus is $(-2, 3)$, directrix is $2x + 3y + 4 = 0$ and $e = \frac{4}{5}$.
   (iv) focus is $(1, 2)$, directrix is $3x + 4y - 5 = 0$ and $e = \frac{1}{2}$.
3. Find the eccentricity, coordinates of foci, length of the latus-rectum of the following ellipse:
   (i) $4x^2 + 9y^2 = 1$ (ii) $5x^2 + 4y^2 = 1$ (iii) $4x^2 + 3y^2 = 1$
   (iv) $25x^2 + 16y^2 = 1600$. (v) $9x^2 + 25y^2 = 225$ [NCERT EXEMPLAR]
4. Find the equation to the ellipse (referred to its axes as the axes of x and y respectively) which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$.
5. Find the equation of the ellipse in the following cases:
   (i) eccentricity $e = \frac{1}{2}$ and foci $(\pm 2, 0)$
   (ii) eccentricity $e = \frac{2}{3}$ and length of latus-rectum $= 5$
   (iii) eccentricity $e = \frac{1}{2}$ and semi-major axis $= 4$
   (iv) eccentricity $e = \frac{1}{2}$ and major axis $= 12$
   (v) The ellipse passes through $(1, 4)$ and $(-6, 1)$.
   (vi) Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$ [NCERT]
   (vii) Vertices $(0, \pm 13)$, foci $(0, \pm 5)$ [NCERT]
   (viii) Vertices $(\pm 6, 0)$, foci $(\pm 4, 0)$ [NCERT]
   (ix) Ends of major axis $(\pm 3, 0)$, ends of minor axis $(0, \pm 2)$ [NCERT]
   (x) Ends of major axis $(0, \pm \sqrt{5})$, ends of minor axis $(\pm 1, 0)$ [NCERT]
   (xi) Length of major axis 26, foci $(\pm 5, 0)$ [NCERT]
   (xii) Length of minor axis 16 foci $(0, \pm 6)$ [NCERT]

(xiii) Foci$(\pm 3, 0)$, $a = 4$ [NCERT]

6. Find the equation of the ellipse whose foci are $(4, 0)$ and $(-4, 0)$, eccentricity $= 1/3$.
7. Find the equation of the ellipse in the standard form whose minor axis is equal to the distance between foci and whose latus-rectum is 10.
8. Find the equation of the ellipse whose centre is $(-2, 3)$ and whose semi-axis are 3 and 2 when major axis is (i) parallel to x-axis (ii) parallel to y-axis.
9. Find the eccentricity of an ellipse whose latus-rectum is
   (i) half of its minor axis (ii) half of its major axis.
10. Find the centre, the lengths of the axes, eccentricity, foci of the following ellipse:
    (i) $x^2 + 2y^2 - 2x + 12y + 10 = 0$ (ii) $x^2 + 4y^2 - 4x + 24y + 31 = 0$
    (iii) $4x^2 + y^2 - 8x + 2y + 1 = 0$ (iv) $3x^2 + 4y^2 - 12x - 8y + 4 = 0$
    (v) $4x^2 + 16y^2 - 24x - 32y - 12 = 0$ (vi) $x^2 + 4y^2 - 2x = 0$
11. Find the equation of an ellipse whose foci are at $(\pm 3, 0)$ and which passes through $(4, 1)$.
12. Find the equation of an ellipse whose eccentricity is $2/3$, the latus-rectum is 5 and the centre is at the origin.
13. Find the equation of an ellipse with its foci on y-axis, eccentricity $3/4$, centre at the origin and passing through $(6, 4)$.
14. Find the equation of an ellipse whose axes lie along coordinate axes and which passes through $(4, 3)$ and $(-1, 4)$.
15. Find the equation of an ellipse whose axes lie along the coordinate axes, which passes through the point $(-3, 1)$ and has eccentricity equal to $\sqrt{2/5}$.
16. Find the equation of an ellipse, the distance between the foci is 8 units and the distance between the directrices is 18 units.
17. Find the equation of an ellipse whose vertices are $(0, \pm 10)$ and eccentricity $e = \dfrac{4}{5}$.

LEVEL-2

18. A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with x-axis.
19. Find the equation of the set of all points whose distances from $(0, 4)$ are $\dfrac{2}{3}$ of their distances from the line $y = 9$. [NCERT EXEMPLAR]

ANSWERS

1. $43x^2 + 48y^2 + 12xy - 134x + 228y + 235 = 0$
2. (i) $7x^2 + 7y^2 - 2xy - 16y + 8 = 0$ (ii) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$
   (iii) $325(x^2 + y^2 + 4x - 6y + 13) = 16(2x + 3y + 4)^2$
   (iv) $91x^2 + 84y^2 - 24xy - 170x - 360y + 475 = 0$
3. (i) $e = \dfrac{\sqrt{5}}{3}; \left(\pm\dfrac{\sqrt{5}}{6}, 0\right); \dfrac{4}{9}$ (ii) $e = \dfrac{1}{\sqrt{5}}; \left(0, \pm\dfrac{1}{2\sqrt{5}}\right); \dfrac{4}{5}$
   (iii) $e = \dfrac{1}{2}; \left(0, \pm\dfrac{1}{2\sqrt{3}}\right); \dfrac{\sqrt{3}}{2}$ (iv) $e = \dfrac{3}{5}, (0, \pm 6); \dfrac{64}{5}$
   (v) $e = \dfrac{4}{5}; (\pm 4, 0); \dfrac{18}{5}$
4. $3x^2 + 5y^2 = 32$

5. (i) $3x^2 + 4y^2 = 48$ (ii) $20x^2 + 36y^2 = 405$ (iii) $3x^2 + 4y^2 = 48$

(iv) $3x^2 + 4y^2 = 108$ (v) $3x^2 + 7y^2 = 115$ (vi) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

(vii) $\frac{x^2}{144} + \frac{y^2}{169} = 1$ (viii) $\frac{x^2}{36} + \frac{y^2}{20} = 1$ (ix) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

(x) $\frac{x^2}{1} + \frac{y^2}{5} = 1$ (xi) $\frac{x^2}{169} + \frac{y^2}{144} = 1$ (xii) $\frac{x^2}{64} + \frac{y^2}{100} = 1$

(xiii) $\frac{x^2}{16} + \frac{y^2}{7} = 1$ 6. $\frac{x^2}{9} + \frac{y^2}{8} = 16$ 7. $x^2 + 2y^2 = 100$

8. (i) $4x^2 + 9y^2 + 16x - 54y + 61 = 0$ (ii) $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

9. (i) $e = \frac{\sqrt{3}}{2}$ (ii) $e = \frac{1}{\sqrt{2}}$

10.

	Centre	Major axis	Minor axis	Eccentricity	Foci
(i)	$(1, -3)$	6	$3\sqrt{2}$	$\frac{1}{\sqrt{2}}$	$\left(1 \pm \frac{3}{\sqrt{2}}, -3\right)$
(ii)	$(2, -3)$	6	3	$\frac{\sqrt{3}}{2}$	$\left(2 \pm \frac{3\sqrt{3}}{2}, -3\right)$
(iii)	$(1, -1)$	4	2	$\frac{\sqrt{3}}{2}$	(1, <196 >1 ± $\sqrt{3}$)
(iv)	$(2, 1)$	4	$2\sqrt{3}$	$\frac{1}{2}$	$(2 \pm 1, 1)$
(v)	$(3, 1)$	8	4	$\sqrt{3}/2$	$(3 \pm 2\sqrt{3}, 1)$
(vi)	$(1, 0)$	2	1	$\sqrt{3}/2$	$(1 \pm \sqrt{3}/2), 0)$

11. $\frac{x^2}{18} + \frac{y^2}{9} = 1$ 12. $\frac{4x^2}{81} + \frac{4y^2}{45} = 1$ 13. $\frac{x^2}{43} + \frac{7y^2}{688} = 1$ 14. $\frac{7x^2}{247} + \frac{15y^2}{247} = 1$

15. $3x^2 + 5y^2 = 32$ 16. $\frac{x^2}{36} + \frac{y^2}{20} = 1$ 17. $100x^2 + 36y^2 = 3600$

.18. $x^2 + 9y^2 = 81$ 19. $9x^2 + 5y^2 = 180$

HINTS TO NCERT & SELECTED PROBLEMS

5. (vi) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

It is given that vertices are at $(\pm 5, 0)$ and foci are at $(\pm 4, 0)$. Therefore, $a = 5$ and $ae = 4$. Now,

$$b^2 = a^2(1 - e^2) \Rightarrow b^2 = a^2 - (ae)^2 = 25 - 16 = 9$$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{25} + \frac{y^2}{9} = 1$ as the equation of the ellipse.

(vii) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Its vertices are at $(0, \pm 13)$ and foci are at $(0, \pm 5)$.

$\therefore \quad b = 13$ and $be = 5$

Now, $a^2 = b^2(1-e^2) \Rightarrow a^2 = b^2 - (be)^2 = 169 - 25 = 144$

Substituting the values of a and b in (i), we obtain $\frac{x^2}{144} + \frac{y^2}{169}$ as the equation of the ellipse.

(viii) Proceed as in 5 (vi).

(ix) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

It is given that ends of major and minor axes are at $(\pm 3, 0)$ and $(0, \pm 2)$. But, coordinates of end points of major and minor axes are $(\pm a, 0)$ and $0, \pm b)$.

$\therefore \quad a = 3$ and $b = 2$.

Hence, the equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

(x) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad ...(i)$$

where $b > a$.

The coordinates of end points of its major and minor axes are $(0, \pm b)$ and $(\pm a, 0)$.

$\therefore \quad b = \sqrt{5}$ and $a = 1$

Hence, the equation of the ellipse is $\frac{x^2}{5} + \frac{y^2}{1} = 1$.

(xi) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that $2a = 26$ and $ae = 5$

$\Rightarrow \quad a = 13$ and $a^2e^2 = 25$

$\Rightarrow \quad a = 13$ and $a^2 - b^2 = 25 \qquad \left[\because b^2 = a^2(1-e^2)\right]$

$\Rightarrow \quad a = 13$ and $b^2 = 169 - 25 = 144$

Hence, the equation of the ellipse is $\frac{x^2}{169} + \frac{y^2}{144} = 1$.

(xii) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, b > a$. It is given that

$2a = 16$ and $be = 6$. Therefore, $a = 8$ and $be = 6$.

Now,

$be = 6 \Rightarrow b^2e^2 = 36 \Rightarrow b^2 - a^2 = 36 \qquad \left[\because a^2 = b^2(1-e^2)\right]$

$\Rightarrow \quad b^2 = 64 + 36 = 100$

Hence, the equation of the ellipse is $\frac{x^2}{64} + \frac{y^2}{100} = 1$.

(xiii) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that $a = 4$ and $ae = 3$.

$\therefore \quad b^2 = a^2(1-e^2) \Rightarrow b^2 = 16 - 9 = 7$

Hence, the equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{7}=1$.

11. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then, $ae = 3$ and $\frac{16}{a^2}+\frac{1}{b^2} = 1$

$\Rightarrow \quad \frac{16}{a^2}+\frac{1}{a^2-a^2e^2}=1 \Rightarrow \frac{16}{a^2}+\frac{1}{a^2-9}=1$

$\Rightarrow \quad a^4-26a^2+144=0 \Rightarrow (a^2-18)(a^2-8)=0 \Rightarrow a^2=18, a^2=8.$

$\therefore \quad b^2 = a^2(1-e^2)$ and $ae = 3 \Rightarrow b^2=a^2-9.$

Now, $a^2=18 \Rightarrow b^2=18-9 = 9$

$a^2=8 \Rightarrow b^2=8-9=-1$, which is not possible.

Hence, the equation of the ellipse is $\frac{x^2}{18}+\frac{y^2}{9}=1$.

12. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. We have, $e=\frac{2}{3}$, and $\frac{2b^2}{a}=5$.

Now, $\frac{2b^2}{a}=5 \Rightarrow 2b^2=5a \Rightarrow 2a^2(1-e^2)=5a \Rightarrow 2a\left(1-\frac{4}{9}\right)=5 \Rightarrow a=\frac{9}{2}$.\ff

$\therefore \quad \frac{2b^2}{a} = 5 \Rightarrow b^2=\frac{45}{4}.$

Hence, the equation of the ellipse is $\frac{4x^2}{81}+\frac{4y^2}{45}=1$.

13. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where b > a. We have, $e=\frac{3}{4}$

$\therefore \quad a^2=b^2(1-e^2) \Rightarrow a^2=\frac{7}{16}b^2.$

So, the equation of the ellipse becomes $\frac{x^2}{a^2}+\frac{7y^2}{16a^2}=1$. It passes through (6, 4).

$\therefore \quad \frac{36}{a^2}+\frac{112}{16a^2}=1 \Rightarrow a^2=43.$

$\therefore \quad b^2 = \frac{16a^2}{7} \Rightarrow b^2=\frac{16\times 43}{7}=\frac{688}{7}$

Hence, the equation of the ellipse is $\frac{x^2}{43}+\frac{7y^2}{688}=1$.

14. Let the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. It passes through (4, 3) and (–1, 4).

$\therefore \quad \frac{16}{a^2}+\frac{9}{b^2}=1$ and $\frac{1}{a^2}+\frac{16}{b^2}=1$

$\Rightarrow \quad 16\alpha+9\beta=1$ and $\alpha+16\beta=1$, where $\alpha=\frac{1}{a^2}$ and $\beta=\frac{1}{b^2}$.

Solving these two equations, we get $\alpha=\frac{7}{247}$ and $\beta=\frac{15}{247}$.

Therefore, $a^2=\frac{247}{7}$ and $b^2=\frac{247}{15}$.

Hence, the equation of the ellipse is $7x+15y^2=247$.

16. Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

We have,

$$2ae = 8 \text{ and } 2a/e = 18 \Rightarrow 2ae \times \frac{2a}{e} = 8\times 18 \Rightarrow 4a^2 = 8\times 18 \Rightarrow a = 6.$$

Now, 2ae = 8 and $a = 6 \Rightarrow e = 2/3$.

$$\therefore \quad b^2 = a^2(1-e^2) = 36\left(1-\frac{4}{9}\right) = 20.$$

Hence, the equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If the lengths of semi-major and semi-minor axes of an ellipse are 2 and $\sqrt{3}$ and their corresponding equations are $y-5=0$ and $x+3=0$, then write the equation of the ellipse.
2. Write the eccentricity of the ellipse $9x^2+5y^2-18x-2y-16=0$.
3. Write the centre and eccentricity of the ellipse $3x^2+4y^2-6x+8y-5=0$.
4. PSQ is a focal chord of the ellipse $4x^2+9y^2=36$ such that $SP=4$. If S' is the another focus, write the value of $S'Q$.
5. Write the eccentricity of an ellipse whose latus-rectum is one half of the minor axis.
6. If the distance between the foci of an ellipse is equal to the length of the latus-rectum, write the eccentricity of the ellipse.
7. If S and S' are two foci of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and B is an end of the minor axis such that $\Delta BSS'$ is equilateral, then write the eccentricity of the ellipse.
8. If the minor axis of an ellipse subtends an equilateral triangle with vertex at one end of major axis, then write the eccentricity of the ellipse.
9. If a latus-rectum of an ellipse subtends a right angle at the centre of the ellipse, then write the eccentricity of the ellipse.

ANSWERS

1. $3x^2+4y^2+18x-40y+115=0$ 2. $\frac{1}{2}$ 3. $(1,-1), \frac{1}{2}$ 4. $\frac{26}{5}$

5. $\frac{\sqrt{3}}{2}$ 6. $\frac{\sqrt{5}-1}{2}$ 7. $\frac{1}{2}$ 8. $\sqrt{\frac{2}{3}}$ 9. $\frac{\sqrt{5}-1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. For the ellipse $12x^2+4y^2+24x-16y+25=0$

(a) centreis $(-1, 2)$ (b) lengths of the axes are $\sqrt{3}$ and 1

(c) eccentricity $=\sqrt{\frac{2}{3}}$ (d) all of these

2. The equation of the ellipse with focus $(-1, 1)$, directrix $x - y + 3 = 0$ and eccentricity $1/2$ is

(a) $7x^2 + 2xy + 7y^2 + 10x + 10y + 7 = 0$ (b) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

(c) $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$ (d) none of these

3. The equation of the circle drawn with the two foci of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as the end-points of a diameter is

(a) $x^2 + y^2 = a^2 + b^2$ (b) $x^2 + y^2 = a^2$

(c) $x^2 + y^2 = 2a^2$ (d) $x^2 + y^2 = a^2 - b^2$

4. The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if its latus-rectum is equal to one half of its minor axis, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) none of these

5. The eccentricity of the ellipse, if the distance between the foci is equal to the length of the latus-rectum, is

(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$ (c) $\frac{\sqrt{5}-1}{4}$ (d) none of these

6. The eccentricity of the ellipse, if the minor axis is equal to the distance between the foci, is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{\sqrt{2}}{3}$

7. The difference between the lengths of the major axis and the latus-rectum of an ellipse is

(a) ae (b) $2ae$ (c) ae^2 (d) $2ae^2$

8. The eccentricity of the conic $9x^2 + 25y^2 = 225$ is

(a) 2/5 (b) 4/5 (c) 1/3 (d) 1/5 (e) 3/5

9. The latus-rectum of the conic $3x^2 + 4y^2 - 6x + 8y - 5 = 0$ is

(a) 3 (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) none of these

10. The equations of the tangents to the ellipse $9x^2 + 16y^2 = 144$ from the point (2, 3) are

(a) $y = 3, x = 5$ (b) $x = 2, y = 3$ (c) $x = 3, y = 2$ (d) $x + y = 5, y = 3$

11. The eccentricity of the ellipse $4x^2 + 9y^2 + 8x + 36y + 4 = 0$ is

(a) $\frac{5}{6}$ (b) $\frac{3}{5}$ (c) $\frac{\sqrt{2}}{3}$ (d) $\frac{\sqrt{5}}{3}$

12. The eccentricity of the ellipse $4x^2 + 9y^2 = 36$ is

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{\sqrt{5}}{6}$

13. The eccentricity of the ellipse $5x^2 + 9y^2 = 1$ is

(a) 2/3 (b) 3/4 (c) 4/5 (d) 1/2

14. For the ellipse $x^2 + 4y^2 = 9$

(a) the eccentricity is 1/2 (b) the latus-rectum is 3/2

(c) a focus is $(3\sqrt{3}, 0)$ (d) a directrix is $x = -2\sqrt{3}$

15. If the latus-rectum of an ellipse is one half of its minor axis, then its eccentricity is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{4}$

16. An ellipse has its centre at $(1, -1)$ and semi-major axis $= 8$ and it passes through the point $(1, 3)$. The equation of the ellipse is

(a) $\frac{(x+1)^2}{64} + \frac{(y+1)^2}{16} = 1$ (b) $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{16} = 1$

(c) $\frac{(x-1)^2}{16} + \frac{(y+1)^2}{64} = 1$ (d) $\frac{(x+1)^2}{64} + \frac{(y-1)^2}{16} = 1$

17. The sum of the focal distances of any point on the ellipse $9x^2 + 16y^2 = 144$ is

(a) 32 (b) 18 (c) 16 (d) 8

18. If $(2, 4)$ and $(10, 10)$ are the ends of a latus-rectum of an ellipse with eccentricity $1/2$, then the length of semi-major axis is

(a) $20/3$ (b) $15/3$ (c) $40/3$ (d) none of these

19. The equation $\frac{x^2}{2-\lambda} + \frac{y^2}{\lambda - 5} + 1 = 0$ represents an ellipse, if

(a) $\lambda < 5$ (b) $\lambda < 2$ (c) $2 < \lambda < 5$ (d) $\lambda < 2$ or $\lambda > 5$

20. The eccentricity of the ellipse $9x^2 + 25y^2 - 18x - 100y - 116 = 0$, is

(a) $25/16$ (b) $4/5$ (c) $16/25$ (d) $5/4$

21. If the major axis of an ellipse is three times the minor axis, then its eccentricity is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\frac{2\sqrt{2}}{3}$ (e) $\frac{2}{3\sqrt{2}}$

22. The eccentricity of the ellipse $25x^2 + 16y^2 = 400$ is

(a) $3/5$ (b) $1/3$ (c) $2/5$ (d) $1/5$

23. The eccentricity of the ellipse $5x^2 + 9y^2 = 1$ is

(a) $2/3$ (b) $3/4$ (c) $4/5$ (d) $1/2$

24. The eccentricity of the ellipse $4x^2 + 9y^2 = 36$ is

(a) $\frac{1}{2\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{5}}{3}$ (d) $\frac{\sqrt{5}}{6}$

ANSWERS

1. (d)	2. (b)	3. (d)	4. (b)	5. (a)	6. (c)	7. (d)	8. (b)
9. (a)	10. (d)	11. (d)	12. (c)	13. (a)	14. (b)	15. (c)	16. (b)
17. (d)	18. (a)	19. (c)	20. (b)	21. (d)	22. (a)	23. (a)	24. (c)

SUMMARY

1. An ellipse is the locus of a point in a plane which moves in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.
The constant ratio of generally denoted by e and is known as the eccentricity of the ellipse.
If S is the focus, ZZ' is the directrix and P is any point on the ellipse, such that M is the foot of perpendicular from P on ZZ', then $SP = e \cdot PM$.
The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents an ellipse, if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0 \text{ and } h^2 < ab.$$

2. The equation of the ellipse whose axes are parallel to the coordinate axes and whose centre is at the origin, is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the following properties:

	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, -b)$ and $(0, -b)$
Coordinates of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of the major axis	$2a$	$2b$
Length of the minor axis	$2b$	$2a$
Equation of the major axis	$y = 0$	$x = 0$
Equation of the minor axis	$x = 0$	$y = 0$
Equations of the directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$y = \frac{b}{e}$ and $y = -\frac{b}{e}$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Length of the latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$