CHAPTER 10

DIFFERENTIATION

10.1 INTRODUCTION

In the previous chapter, we have learnt about differentiability of a function at a point. The same was extended to the domain of a function. In case, a function is differentiable at every point of its domain, then each point in its domain can be associated to the derivative of the function at that point. Such a correspondence between points in the domain and the set of values of derivatives at those points defines a new function which is known as the derivative or differentiation of the given function. In the previous class, we have studied that the derivative of a function $f(x)$ is given by

$$\frac{d}{dx}\left(f\ (x)\right) = \lim_{h \to 0}\ \frac{f\ (x+h)-f\ (x)}{h} \quad \text{or,} \quad \frac{d}{dx}\left(f\ (x)\right) = \lim_{h \to 0}\ \frac{f\ (x-h)-f\ (x)}{-h}$$

This is also called the derivative or differentiation with respect to x and is also denoted by $f\,'(x)$ or, $D\,f\,(x)$. Sometime the derivative or differentiation of a function $f\,(x)$ is also called the differential coefficient of $f\,(x)$. The process of finding the derivative of a function by using the above definition is called the differentiation from first principles or by *ab*-initio method or by delta method.

Following are derivatives of some standard functions which have been derived in Class XI from first principles.

(i) $\frac{d}{dx}(x^n) = n\,x^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$ (iii) $\frac{d}{dx}(a^x) = a^x \log_e a\,,\ a > 0$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ (v) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}\,,\ a > 0,\ a \neq 1$

(vi) $\frac{d}{dx}(\sin x) = \cos x$ (vii) $\frac{d}{dx}(\cos x) = -\sin x$ (viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ (x) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Let us now have a brief recall of what else we have studied in Class XI.

10.2 RECAPITULATION

In the previous class, we have learnt about the following fundamental rules for differentiation.

(i) Differentiation of a constant functions zero i.e., $\frac{d}{dx}(c) = 0$

(ii) Let $f\,(x)$ be a differentiable function and let c be a constant. Then, $c\,f\,(x)$ is also differentiable such that

$$\frac{d}{dx}\left(c\ f(x)\right) = c\,\frac{d}{dx}\left(f\ (x)\right)$$

i.e. the derivative of a constant times a function is the constant times the derivative of the function.

(iii) *Product rule*: If $f(x)$ and $g(x)$ are differentiable functions, then $f(x)\,g(x)$ is also differentiable function such that

$$\frac{d}{dx}\{f(x)\,g(x)\} = \frac{d}{dx}\Big(f(x)\Big)g(x) + f(x)\,.\,\frac{d}{dx}\Big(g(x)\Big)$$

If $f(x)$, $g(x)$ and $h(x)$ are differentiable functions, then

$$\frac{d}{dx}\Big(f(x)\,g(x)\,h(x)\Big) = \frac{d}{dx}\Big(f(x)\Big)g(x)\,h(x) + f(x)\,\frac{d}{dx}\Big(g(x)\Big)h(x) + f(x)\,g(x)\,\frac{d}{dx}\Big(h(x)\Big)$$

(iv) *Quotient rule:* If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\,\frac{d}{dx}\Big(f(x)\Big) - f(x)\,\frac{d}{dx}\Big(g(x)\Big)}{\{g(x)\}^2}$$

BASED ON BASIC CONCEPTS (BASIC)

ILLUSTRATION 1 *Differentiate the following functions with respect to x:*

(i) $\dfrac{2^x \cot x}{\sqrt{x}}$ (ii) $e^x \log\sqrt{x}\tan x$

SOLUTION (i) We have,

$$\frac{d}{dx}\left\{\frac{2^x\cot x}{\sqrt{x}}\right\}$$

$$= \frac{d}{dx}\left\{2^x \cot x\; x^{-1/2}\right\}$$

$$= \left\{\frac{d}{dx}(2^x)\right\}(\cot x)\;x^{-1/2} + 2^x\left\{\frac{d}{dx}(\cot x)\right\}x^{-1/2} + 2^x\cot x\left\{\frac{d}{dx}(x^{-1/2})\right\}$$

$$= 2^x \log_e 2 \cot x\, x^{-1/2} + 2^x(-\operatorname{cosec}^2 x)\,x^{-1/2} + 2^x \cot x \times -\frac{1}{2}x^{-3/2}$$

$$= \frac{2^x\log_e 2\cot x}{\sqrt{x}} - \frac{2^x \operatorname{cosec}^2 x}{\sqrt{x}} - \frac{2^{x-1}\cot x}{x\sqrt{x}}$$

(ii) $$\frac{d}{dx}\{e^x\log\sqrt{x}\tan x\}$$

$$= \frac{d}{dx}\left\{e^x \times \frac{1}{2}\log x \times \tan x\right\}$$

$$= \frac{1}{2}\frac{d}{dx}\left\{e^x \log x \tan x\right\}$$

$$= \frac{1}{2}\left[\left\{\frac{d}{dx}(e^x)\right\}\log x\tan x + e^x\left\{\frac{d}{dx}(\log x)\right\}\tan x + e^x\log x\left\{\frac{d}{dx}(\tan x)\right\}\right]$$

$$= \frac{1}{2}\left\{e^x\log x\tan x + \frac{e^x\tan x}{x} + e^x\log x\sec^2 x\right\}$$

$$= \frac{1}{2}e^x\left\{\log x\tan x + \frac{\tan x}{x} + \log x\sec^2 x\right\}$$

ILLUSTRATION 2 *Differentiate the following functions with respect to x:*

(i) $\dfrac{e^x + \sin x}{1 + \log x}$ (ii) $\dfrac{\sin x - x\cos x}{x\sin x + \cos x}$

SOLUTION (i) We have,

$$\frac{d}{dx}\left\{\frac{e^x + \sin x}{1 + \log x}\right\}$$

$$= \frac{(1 + \log x)\dfrac{d}{dx}(e^x + \sin x) - (e^x + \sin x)\dfrac{d}{dx}(1 + \log x)}{(1 + \log x)^2}$$

$$= \frac{(1 + \log x)(e^x + \cos x) - (e^x + \sin x)\left(0 + \dfrac{1}{x}\right)}{(1 + \log x)^2} = \frac{(1 + \log x)(e^x + \cos x) - \dfrac{e^x + \sin x}{x}}{(1 + \log x)^2}$$

(ii) $$\frac{d}{dx}\left\{\frac{\sin x - x\cos x}{x\sin x + \cos x}\right\}$$

$$= \frac{(x\sin x + \cos x)\dfrac{d}{dx}(\sin x - x\cos x) - (\sin x - x\cos x)\dfrac{d}{dx}(x\sin x + \cos x)}{(x\sin x + \cos x)^2}$$

$$= \frac{(x\sin x + \cos x)(\cos x - \cos x + x\sin x) - (\sin x - x\cos x)(\sin x + x\cos x - \sin x)}{(x\sin x + \cos x)^2}$$

$$= \frac{(x\sin x + \cos x)(x\sin x) - (\sin x - x\cos x)(x\cos x)}{(x\sin x + \cos x)^2}$$

$$= \frac{(x^2\sin^2 x + x\sin x\cos x) - (x\sin x\cos x - x^2\cos^2 x)}{(x\sin x + \cos x)^2}$$

$$= \frac{x^2(\sin^2 x + \cos^2 x)}{(x\sin x + \cos x)^2} = \frac{x^2}{(x\sin x + \cos x)^2}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

ILLUSTRATION 3 *If* $y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\dots(1 + x^{2^n})$, *find* $\dfrac{dy}{dx}$.

SOLUTION We have,

$$y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\dots(1 + x^{2^n})$$

$$\Rightarrow \quad y = \frac{(1 - x)(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\dots(1 + x^{2^n})}{1 - x}$$

$$\Rightarrow \quad y = \frac{(1 - x^2)(1 + x^2)(1 + x^4)(1 + x^8)\dots(1 + x^{2^n})}{1 - x}$$

$$\Rightarrow \quad y = \frac{1 - x^{2^{n+1}}}{1 - x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1 - x)\dfrac{d}{dx}(1 - x^{2^{n+1}}) - (1 - x^{2^{n+1}})\dfrac{d}{dx}(1 - x)}{(1 - x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{(1-x)\left(-2^{n+1}x^{2^{n+1}-1}\right)+\left(1-x^{2^{n+1}}\right)}{(1-x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2^{n+1}x^{2^{n+1}-1}+2^{n+1}x^{2^{n+1}}+1-x^{2^{n+1}}}{(1-x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{-2^{n+1}x^{2^{n+1}-1}+1+x^{2^{n+1}}\left(2^{n+1}-1\right)}{(1-x)^2}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

ILLUSTRATION 4 *If $f(x)=|\cos x|$, find $f'\left(\frac{\pi}{4}\right)$ and $f'\left(\frac{3\pi}{4}\right)$.* [NCERT EXEMPLAR]

SOLUTION We have,

$$f(x)=|\cos x|=\begin{cases}\cos x, \text{ if } 0<x\le\frac{\pi}{2}\\ -\cos x, \text{ if } \frac{\pi}{2}<x<\pi\end{cases} \Rightarrow f'(x)=\begin{cases}-\sin x, \text{ if } 0<x<\frac{\pi}{2}\\ \sin x, \text{ if } \frac{\pi}{2}<x<\pi\end{cases}$$

Note that $f(x)$ is not differentiable at $x=\frac{\pi}{2}$.

$$\therefore \quad f'\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}} \text{ and, } f'\left(\frac{3\pi}{4}\right)=\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$$

ILLUSTRATION 5 *If $f(x)=|\cos x-\sin x|$, find $f'\left(\frac{\pi}{6}\right)$ and $f'\left(\frac{\pi}{3}\right)$.* [NCERT EXEMPLAR]

SOLUTION We have,

$$f(x)=|\cos x-\sin x|=\begin{cases}\cos x-\sin x, \text{ if } 0<x<\frac{\pi}{4}\\ -(\cos x-\sin x), \text{ if } \frac{\pi}{4}<x<\frac{\pi}{2}\end{cases}$$

$$\Rightarrow \quad f(x)=\begin{cases}\cos x-\sin x, \text{ if } 0<x<\frac{\pi}{4}\\ \sin x-\cos x, \text{ if } \frac{\pi}{4}<x<\frac{\pi}{2}\end{cases} \Rightarrow f'(x)=\begin{cases}-\sin x-\cos x, \text{ if } 0<x<\frac{\pi}{4}\\ \cos x+\sin x, \text{ if } \frac{\pi}{4}<x<\frac{\pi}{2}\end{cases}$$

$$\Rightarrow \quad f'\left(\frac{\pi}{6}\right)=-\sin\frac{\pi}{6}-\cos\frac{\pi}{6}=-\frac{\sqrt{3}+1}{2} \text{ and } f'\left(\frac{\pi}{3}\right)=\cos\frac{\pi}{3}+\sin\frac{\pi}{3}=\frac{\sqrt{3}+1}{2}$$

ILLUSTRATION 6 *If $f(x)=|\log x|$, $x>0$, find $f'(1/e)$ and $f'(e)$.* [NCERT EXEMPLAR]

SOLUTION We have,

$$f(x)=|\log x|=\begin{cases}-\log x, \text{ if } 0<x<1\\ \log x, \text{ if } x\ge 1\end{cases} \Rightarrow f'(x)=\begin{cases}-1/x, \text{ if } 0<x<1\\ 1/x, \text{ if } x>1\end{cases} \Rightarrow f'(e)=1/e \text{ and } f'(1/e)=-e$$

10.3 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS FROM FIRST PRINCIPLES

In the previous class, we have learnt that the derivative of a function $f(x)$ is given by

$$\frac{d}{dx}\left\{f(x)\right\}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or,} \quad \frac{d}{dx}\left\{f(x)\right\}=\lim_{h\to 0}\frac{f(x-h)-f(x)}{-h}$$

The process of finding the derivative of a function by using the above definition is called the differentiation from first principles or, by *ab*-initio method or, by delta method.

In this section, we will find the derivatives or differentiations or differential coefficients of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\text{cosec}^{-1} x$ and $\cot^{-1} x$ from first principles.

Following results will be very useful to find the same:

(i) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(ii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right\}$

(iii) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left\{\dfrac{x \pm y}{1 \mp xy}\right\}$ (iv) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n\, a^{n-1}$

(v) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$ (vi) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$, $\lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$

(vii) $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1$, $\lim\limits_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1$ (viii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a$, $a > 0, a \neq 1$

(ix) $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1$ (x) $\lim\limits_{x \to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$.

THEOREM 1 *If $x \in (-1, 1)$, then the differentiation of $\sin^{-1} x$ with respect to x is $\dfrac{1}{\sqrt{1-x^2}}$.*

i.e., $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$

PROOF Let $f(x) = \sin^{-1} x$. Then, $f(x+h) = \sin^{-1}(x+h)$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}(x+h) - \sin^{-1} x}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}}{\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}} \times \frac{\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{(x+h)\sqrt{1-x^2} - x\sqrt{1-(x+h)^2}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^2(1-x^2) - x^2\{1-(x+h)^2\}}{h} \times \frac{1}{\left\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\right\}}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \times \frac{1}{\left\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} (2x+h) \times \frac{1}{\left\{(x+h)\sqrt{1-x^2} + x\sqrt{1-(x+h)^2}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{2x}{2x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

Hence, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, where $-1 < x < 1$. **Q.E.D.**

THEOREM 2 *If $x \in (-1, 1)$, then the differentiation of $\cos^{-1} x$ with respect to x is $-\frac{1}{\sqrt{1-x^2}}$.*

i.e., $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

PROOF Let $f(x) = \cos^{-1} x$. Then, $f(x+h) = \cos^{-1}(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cos^{-1}(x+h) - \cos^{-1} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(x+h)\right\} - \left\{\frac{\pi}{2} - \sin^{-1} x\right\}}{h} \quad \left[\because \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin^{-1}(x+h) - \sin^{-1} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{1}{\sqrt{1-x^2}}$$ [See Theorem 1]

Hence, $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$ **Q.E.D.**

THEOREM 3 *The differentiation of $\tan^{-1} x$ with respect to x is $\frac{1}{1+x^2}$* i.e., $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

PROOF Let $f(x) = \tan^{-1} x$. Then, $f(x+h) = \tan^{-1}(x+h)$

Now, $\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} = \lim_{h \to 0} \frac{\tan^{-1}\left\{\frac{x+h-x}{1+x(x+h)}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{\frac{\tan^{-1}\left(\frac{h}{1+x^2+hx}\right)}{\left(\frac{h}{1+x^2+hx}\right)}\right\} \times \frac{1}{(1+x^2+hx)} = 1 \times \frac{1}{1+x^2} = \frac{1}{1+x^2}$$

Hence, $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ for all $x \in R$. **Q.E.D.**

THEOREM 4 *The differentiation of $\cot^{-1} x$ with respect to x is $-\frac{1}{1+x^2}$* i.e., $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

PROOF Let $f(x)=\cot^{-1}x$. Then, $f(x)=\frac{\pi}{2}-\tan^{-1}x$ and so $f(x+h)=\frac{\pi}{2}-\tan^{-1}(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\frac{\pi}{2}-\tan^{-1}(x+h)\right\}-\left\{\frac{\pi}{2}-\tan^{-1}x\right\}}{h} = \lim_{h\to 0}\frac{\tan^{-1}x-\tan^{-1}(x+h)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\tan^{-1}\left\{\frac{x-(x+h)}{1+x(x+h)}\right\}}{h} = \lim_{h\to 0}\frac{\tan^{-1}\left\{\frac{-h}{1+x^2+xh}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\left\{\frac{\tan^{-1}\left(\frac{-h}{1+x^2+hx}\right)}{\left(\frac{-h}{1+x^2+hx}\right)}\right\}\times\frac{1}{1+x^2+hx} = 1\times\frac{-1}{1+x^2} = \frac{-1}{1+x^2}$$

Hence, $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$ **Q.E.D.**

THEOREM 5 *If $x\in R-[-1,1]$, then the differentiation of $\sec^{-1}x$ with respect to x is* $\frac{1}{|x|\sqrt{x^2-1}}$.

i.e., $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$

PROOF Let $f(x) = \sec^{-1}x$. Then, $f(x) = \begin{cases}\tan^{-1}\sqrt{x^2-1}, & \text{if } x\geq 1\\ \pi-\tan^{-1}\sqrt{x^2-1}, & \text{if } x\leq -1\end{cases}$

Case I *When* $x>1$.

We have, $f(x) = \tan^{-1}\sqrt{x^2-1}$ and $f(x+h) = \tan^{-1}\sqrt{(x+h)^2-1}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\tan^{-1}\sqrt{(x+h)^2-1}-\tan^{-1}\sqrt{x^2-1}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{1}{h}\tan^{-1}\left\{\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{1+\sqrt{(x+h)^2-1}\times\sqrt{x^2-1}}\right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\left\{\frac{\tan^{-1}\left\{\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{1+\sqrt{(x+h)^2-1}\times\sqrt{x^2-1}}\right\}}{\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{1+\sqrt{(x+h)^2-1}\times\sqrt{x^2-1}}}\right\}\times\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{h\left\{1+\sqrt{(x+h)^2-1}\times\sqrt{x^2-1}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{1}{h}\left\{\frac{(x+h)^2 - 1 - (x^2-1)}{1+\sqrt{(x+h)^2-1} \times \sqrt{x^2-1}}\right\} \times \frac{1}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{1}{h}\left\{\frac{2hx+h^2}{1+\sqrt{(x+h)^2-1} \times \sqrt{x^2-1}}\right\} \times \frac{1}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{\frac{2x+h}{1+\sqrt{(x+h)^2-1} \times \sqrt{x^2-1}}\right\} \times \frac{1}{\sqrt{(x+h)^2-1}+\sqrt{x^2-1}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{2x}{1+x^2-1} \times \frac{1}{\sqrt{x^2-1}+\sqrt{x^2-1}} = \frac{1}{x\sqrt{x^2-1}}$$

Case II *When* $x < -1$: Proceeding as in Case I, we obtain: $\frac{d}{dx}(\sec^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

Thus, we obtain : $\frac{d}{dx}(\sec^{-1} x) = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{for } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{for } x < -1 \end{cases}$

Hence, $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ for all $x \neq \pm 1$.

Q.E.D.

THEOREM 6 *If* $x \in R - [-1, 1]$, *then the differentiation of* $\operatorname{cosec}^{-1} x$ *with respect to* x *is* $\frac{-1}{|x|\sqrt{x^2-1}}$.

i.e., $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$ for all $x \neq \pm 1$.

PROOF Let $f(x) = \operatorname{cosec}^{-1} x$. Then, $f(x+h) = \operatorname{cosec}^{-1}(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\operatorname{cosec}^{-1}(x+h) - \operatorname{cosec}^{-1} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\frac{\pi}{2} - \sec^{-1}(x+h)\right\} - \left\{\frac{\pi}{2} - \sec^{-1} x\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sec^{-1}(x+h) - \sec^{-1} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{1}{|x|\sqrt{x^2-1}}$$

[See Theorem 5]

Hence, $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$ for all $x \neq \pm 1$.

Q.E.D.

The above results and derivatives of other standard functions are listed below for ready reference.

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$ (iii) $\frac{d}{dx}(a^x) = a^x \log_e a$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ (v) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ (vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$ (viii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (ix) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (xi) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(xii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ (xiii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

(xiv) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ (xv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

(xvi) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$ (xvii) $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Following examples will illustrate some more applications of differentiation by first principles.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Differentiate the following functions with respect to x from first-principles:*

(i) e^{x^2} (ii) e^{2x} (iii) $e^{\sqrt{x}}$ [CBSE 2003] (iv) $e^{\sin x}$

SOLUTION (i) Let $f(x) = e^{x^2}$. Then, $f(x+h) = e^{(x+h)^2}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{(x+h)^2} - e^{x^2}}{h} = \lim_{h \to 0} \frac{e^{x^2} e^{2hx + h^2} - e^{x^2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} e^{x^2} \left\{ \frac{e^{2hx + h^2} - 1}{2hx + h^2} \right\} \times \left(\frac{2hx + h^2}{h} \right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{x^2} \lim_{h \to 0} \left\{ \frac{e^{2hx + h^2} - 1}{2hx + h^2} \right\} \times \lim_{h \to 0} (2x + h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{x^2} \lim_{\theta \to 0} \left(\frac{e^{\theta} - 1}{\theta} \right) \times \lim_{h \to 0} (2x + h), \text{ where } \theta = 2hx + h^2$$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{x^2} \times 1 \times 2x = 2x\, e^{x^2}$

$\therefore \quad \frac{d}{dx}(e^{x^2}) = 2x\, e^{x^2}$

(ii) Let $f(x) = e^{2x}$. Then, $f(x+h) = e^{2(x+h)}$

$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{2(x+h)} - e^{2x}}{h} = \lim_{h \to 0} \frac{e^{2x} . e^{2h} - e^{2x}}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2e^{2x} \lim_{h \to 0} \left(\frac{e^{2h} - 1}{2h}\right) = 2e^{2x} \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right)$, where $y = 2h$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2e^{2x} \times 1 = 2e^{2x}$ $\left[\because \lim_{y \to 0} \frac{e^y - 1}{y} = 1\right]$

$\therefore \quad \frac{d}{dx}(e^{2x}) = 2e^{2x}$

(iii) Let $f(x) = e^{\sqrt{x}}$. Then, $f(x+h) = e^{\sqrt{x+h}}$

$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{\sqrt{x+h}} - e^{\sqrt{x}}}{h} = e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{h}\right)$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{\sqrt{x+h} - \sqrt{x}}\right) \times \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{x}} \lim_{h \to 0} \left(\frac{e^{\sqrt{x+h} - \sqrt{x}} - 1}{\sqrt{x+h} - \sqrt{x}}\right) \times \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{x}} \lim_{y \to 0} \left(\frac{e^y - 1}{y}\right) \times \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$, where $y = \sqrt{x+h} - \sqrt{x}$

$[\because$ when $h \to 0, y \to 0]$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{x}} \times 1 \times \left(\frac{1}{\sqrt{x} + \sqrt{x}}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$ $\left[\because \lim_{y \to 0} \frac{e^y - 1}{y} = 1\right]$

(iv) Let $f(x) = e^{\sin x}$. Then, $f(x+h) = e^{\sin(x+h)}$

$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{\sin(x+h)} - e^{\sin x}}{h} = e^{\sin x} \lim_{h \to 0} \frac{e^{\sin(x+h) - \sin x} - 1}{h}$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sin x} \lim_{h\to 0} \left\{ \frac{e^{\sin(x+h)-\sin x}-1}{\sin(x+h)-\sin x} \right\} \times \left\{ \frac{\sin(x+h)-\sin x}{h} \right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sin x} \lim_{h\to 0} \left\{ \frac{e^{\sin(x+h)-\sin x}-1}{\sin(x+h)-\sin x} \right\} \times \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sin x} \lim_{y\to 0} \left(\frac{e^{y}-1}{y} \right) \times \lim_{h\to 0} \frac{2\sin(h/2)\cos(x+h/2)}{2(h/2)}, \text{ where } y=\sin(x+h)-\sin x \quad [\because \text{ when } h\to 0, y\to 0]$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sin x} \lim_{y\to 0} \left(\frac{e^{y}-1}{y} \right) \times \lim_{h\to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h\to 0} \cos(x+h/2)$$

$$\Rightarrow \frac{d}{dx}(f(x)) = e^{\sin x}(1)\times(1)\times(\cos x) = e^{\sin x}\times\cos x$$

EXAMPLE 2 *Differentiate xe^{x} from first principles.*

SOLUTION Let $f(x) = xe^{x}$. Then, $f(x+h) = (x+h)e^{(x+h)}$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(x+h)e^{x+h}-xe^{x}}{h} = \lim_{h\to 0} \frac{(xe^{x+h}-xe^{x})+he^{x+h}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \left\{ xe^{x}\left(\frac{e^{h}-1}{h}\right) + e^{x+h} \right\} = xe^{x} \lim_{h\to 0} \left(\frac{e^{h}-1}{h}\right) + \lim_{h\to 0} e^{x+h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = xe^{x}+e^{x} = (x+1)e^{x}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Differentiate $\log \sin x$ by first principles.*

SOLUTION Let $f(x) = \log \sin x$. Then, $f(x+h) = \log \sin(x+h)$.

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\sin(x+h)-\log\sin x}{h} = \lim_{h\to 0} \frac{\log\left\{\frac{\sin(x+h)}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)}{\sin x}-1\right\}}{h} = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \times \left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1 + \frac{\sin(x+h) - \sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h) - \sin x}{\sin x}\right\}} \times \frac{\sin(x+h) - \sin x}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1 + \frac{\sin(x+h) - \sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h) - \sin x}{\sin x}\right\}} \times \lim_{h \to 0} \frac{2 \sin \frac{h}{2} \cos\left(x + \frac{h}{2}\right)}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1 + \frac{\sin(x+h) - \sin x}{\sin x}\right\}}{\left\{\frac{\sin(x+h) - \sin x}{\sin x}\right\}} \times \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right) \cos\left(x + \frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \times \frac{1}{\sin x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \cos x \times \frac{1}{\sin x} = \cot x.$$

EXAMPLE 4 *Differentiate* log sec x *from first principles.*

SOLUTION Let $f(x) = \log \sec x$. Then, $f(x+h) = \log \sec (x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log \sec (x+h) - \log \sec x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{\frac{\sec(x+h)}{\sec x}\right\}}{h} = \lim_{h \to 0} \frac{\log\left\{1 + \left(\frac{\cos x}{\cos(x+h)} - 1\right)\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1 + \frac{\cos x - \cos(x+h)}{\cos(x+h)}\right\}}{h\left\{\frac{\cos x - \cos(x+h)}{\cos(x+h)}\right\}} \times \frac{\cos x - \cos(x+h)}{\cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log\left\{1 + \frac{\cos x - \cos(x+h)}{\cos(x+h)}\right\}}{\left\{\frac{\cos x - \cos(x+h)}{\cos(x+h)}\right\}} \times \lim_{h \to 0} \frac{2 \sin\left(x + \frac{h}{2}\right) \sin \frac{h}{2}}{h \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \lim_{h \to 0} \frac{\sin\left(x + \frac{h}{2}\right)}{\cos(x+h)} \times \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) = 1 \times \frac{\sin x}{\cos x} \times 1 = \tan x.$$

EXAMPLE 5 *If* $f(x) = x \tan^{-1} x$, *find* $f'(\sqrt{3})$ *by first principles.*

SOLUTION We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore \quad f'(\sqrt{3}) = \lim_{h \to 0} \frac{f(\sqrt{3}+h) - f(\sqrt{3})}{h} = \lim_{h \to 0} \frac{(\sqrt{3}+h)\tan^{-1}(\sqrt{3}+h) - \sqrt{3}\tan^{-1}\sqrt{3}}{h}$$

$$\Rightarrow \quad f'(\sqrt{3}) = \lim_{h \to 0} \frac{\sqrt{3}\left\{\tan^{-1}(\sqrt{3}+h) - \tan^{-1}\sqrt{3}\right\} + h\tan^{-1}(\sqrt{3}+h)}{h}$$

$$\Rightarrow \quad f'(\sqrt{3}) = \lim_{h \to 0} \frac{\sqrt{3}}{h}\tan^{-1}\left(\frac{\sqrt{3}+h-\sqrt{3}}{1+\sqrt{3}(\sqrt{3}+h)}\right) + \lim_{h \to 0} \tan^{-1}(\sqrt{3}+h)$$

$$\Rightarrow \quad f'(\sqrt{3}) = \sqrt{3}\lim_{h \to 0}\left\{\frac{\tan^{-1}\left(\frac{h}{4+\sqrt{3}h}\right)}{\frac{h}{4+\sqrt{3}h}}\right\} \times \frac{1}{4+\sqrt{3}h} + \lim_{h \to 0} \tan^{-1}(\sqrt{3}+h)$$

$$\Rightarrow \quad f'(\sqrt{3}) = \sqrt{3} \times 1 \times \frac{1}{4} + \tan^{-1}\sqrt{3} = \frac{\sqrt{3}}{4} + \tan^{-1}\sqrt{3}.$$

EXAMPLE 6 *Differentiate* $\cos^{-1}(2x+3)$ *from first principles.*

SOLUTION Let $f(x) = \cos^{-1}(2x+3)$. Then, $f(x+h) = \cos^{-1}(2x+3+2h)$.

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cos^{-1}(2x+3+2h) - \cos^{-1}(2x+3)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\frac{\pi}{2} - \sin^{-1}(2x+3+2h)\right\} - \left\{\frac{\pi}{2} - \sin^{-1}(2x+3)\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}(2x+3) - \sin^{-1}(2x+3+2h)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}\left\{(2x+3)\sqrt{1-(2x+3+2h)^2} - (2x+3+2h)\sqrt{1-(2x+3)^2}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1} Z}{Z} \times \frac{Z}{h},$$

where $Z = (2x+3)\sqrt{1-(2x+3+2h)^2} - (2x+3+2h)\sqrt{1-(2x+3)^2}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{Z}{h} \qquad \left[\because \lim_{h \to 0} \frac{\sin^{-1} Z}{Z} = 1\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2x+3)\sqrt{1-(2x+3+2h)^2} - (2x+3+2h)\sqrt{1-(2x+3)^2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2x+3)^2 \{1-(2x+3+2h)^2\} - (2x+3+2h)^2 \{1-(2x+3)^2\}}{h\left\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2x+3)^2 - (2x+3+2h)^2}{h\left\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-4h(2x+3) - 4h^2}{h\left\{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-4(2x+3) - 4h}{(2x+3)\sqrt{1-(2x+3+2h)^2} + (2x+3+2h)\sqrt{1-(2x+3)^2}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{4(2x+3)}{2(2x+3)\sqrt{1-(2x+3)^2}} = \frac{-2}{\sqrt{1-(2x+3)^2}}.$$

EXAMPLE 7 *Differentiate $e^{\sqrt{\tan x}}$ from first principle.*

SOLUTION Let $f(x) = e^{\sqrt{\tan x}}$. Then, $f(x+h) = e^{\sqrt{\tan(x+h)}}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{\sqrt{\tan(x+h)}} - e^{\sqrt{\tan x}}}{h} = \lim_{h \to 0} e^{\sqrt{\tan x}} \left\{\frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{h}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \lim_{h \to 0} \left\{\frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \times \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \left\{\frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}}\right\} \times \lim_{h \to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times 1 \times \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \frac{\sin h}{h \cos(x+h) \cos x} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \times \frac{1}{2\sqrt{\tan x}} = \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x.$$

EXAMPLE 8 *Differentiate $x \tan^{-1} x$ from first principles.*

SOLUTION Let $f(x) = x \tan^{-1} x$. Then, $f(x+h) = (x+h)\tan^{-1}(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)\tan^{-1}(x+h) - x \tan^{-1} x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left[x \left\{ \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} \right\} + \frac{h \tan^{-1}(x+h)}{h} \right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{ \frac{x \tan^{-1}\left(\dfrac{x+h-x}{1+x(x+h)} \right)}{h} \right\} + \lim_{h \to 0} \tan^{-1}(x+h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = x \lim_{h \to 0} \left\{ \frac{\tan^{-1}\left(\dfrac{h}{1+x(x+h)} \right)}{\dfrac{h}{1+x(x+h)}} \times \frac{1}{\{1+x(x+h)\}} \right\} + \tan^{-1} x = \frac{x}{1+x^2} + \tan^{-1} x$$

EXAMPLE 9 *Differentiate* $\sin^{-1}\sqrt{x}\ (0 < x < 1)$ *from first principles.*

SOLUTION Let $f(x) = \sin^{-1}\sqrt{x}$. Then, $f(x+h) = \sin^{-1}\sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}\sqrt{x+h} - \sin^{-1}\sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1}\left\{ \sqrt{x+h}\sqrt{1-x} - \sqrt{x}\sqrt{1-x-h} \right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin^{-1} Z}{Z} \times \frac{Z}{h}, \text{ where } Z = \sqrt{x+h}\sqrt{1-x} - \sqrt{x}\sqrt{1-x-h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{Z}{h} \qquad \left[\because \lim_{h \to 0} \frac{\sin^{-1} Z}{Z} = \lim_{Z \to 0} \frac{\sin^{-1} Z}{Z} = 1 \right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)(1-x) - x(1-x-h)}{h} \times \frac{1}{\sqrt{x+h}\sqrt{1-x} + \sqrt{x}\sqrt{1-x-h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h(1-x+x)}{h} \times \frac{1}{\sqrt{x+h}\sqrt{1-x} + \sqrt{x}\sqrt{1-x-h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{1}{\sqrt{x+h}\sqrt{1-x} + \sqrt{x}\sqrt{1-x-h}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

REMARK *It should be noted that* $\frac{d}{dx}$ *is an operator such that when it is applied on* $y = f(x)$ *gives us* $\frac{d}{dx}(f(x)) = \frac{dy}{dx}$. *Also,* $\frac{dy}{dx}$ *is not simply a fraction obtained by dividing dy by dx. For example, if* $\frac{d}{dx}$ *is applied on* $\sin x$ *it gives us* $\cos x$ *i.e.,* $\frac{d}{dx}(\sin x) = \cos x$. *The operator* $\frac{d}{dx}$ *is called the differential operator.*

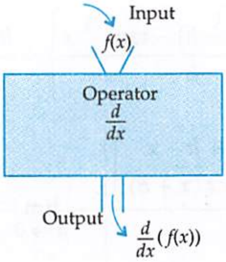

Fig. 10.1

EXERCISE 10.1

BASIC

Differentiate the following functions from first principles:

1. e^{-x} 2. e^{3x} 3. e^{ax+b} 4. $e^{\cos x}$ 5. $e^{\sqrt{2x}}$

BASED ON HOTS

Differentiate each of the following functions from first principal:

6. $\log \cos x$ 7. $e^{\sqrt{\cot x}}$ 8. $x^2 e^x$ 9. $\log \operatorname{cosec} x$ 10. $\sin^{-1}(2x+3)$

ANSWERS

1. $-e^{-x}$ 2. $3e^{3x}$ 3. ae^{ax+b} 4. $-e^{\cos x} \sin x$ 5. $\dfrac{e^{\sqrt{2x}}}{\sqrt{2x}}$ 6. $-\tan x$

7. $-\dfrac{e^{\sqrt{\cot x}}}{2\sqrt{\cot x}} \operatorname{cosec}^2 x$ 8. $(x^2+2x)e^x$ 9. $-\cot x$ 10. $\dfrac{2}{\sqrt{1-(2x+3)^2}}$

10.4 DIFFERENTIATION OF A FUNCTION OF A FUNCTION

In this section, we will study about the differentiation of composition of two or more functions.

THEOREM *(Chain Rule) If f(x) and g(x) are differentiable functions, then fog is also differentiable and*

$$(fog)'(x) = f'(g(x))\, g'(x)$$

or, $$\frac{d}{dx}\left\{(fog)(x)\right\} = \frac{d}{d\,g(x)}\{f(g(x))\}\frac{d}{dx}\left(g(x)\right).$$

PROOF Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \text{ and, } \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \quad \ldots(i)$$

Now, $$\frac{d}{dx}\left\{(fog)(x)\right\} = \lim_{h \to 0} \frac{fog(x+h)-fog(x)}{h} = \lim_{h \to 0} \frac{f\{g(x+h)\}-f\{g(x)\}}{h}$$

$$\Rightarrow \frac{d}{dx}\left\{(fog)(x)\right\} = \lim_{h \to 0} \frac{f\{g(x+h)\}-f\{g(x)\}}{g(x+h)-g(x)} \times \frac{g(x+h)-g(x)}{h}$$

$$\Rightarrow \frac{d}{dx}\left\{(fog)(x)\right\} = \lim_{h \to 0} \frac{f\{g(x+h)\}-f\{g(x)\}}{g(x+h)-g(x)} \times \lim_{h \to 0} \frac{g(x+h)-g(x)}{h}$$

Given that $g(x)$ is differentiable and therefore it is continuous and hence $\lim\limits_{h \to 0} g(x+h) = g(x)$.

$$\therefore \quad \frac{d}{dx}\left\{(fog)\ (x)\right\} = \lim_{g(x+h)\to g(x)} \frac{f\{g(x+h)\} - f\{g(x)\}}{g(x+h)-g(x)} \times \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}\left\{(fog)\ (x)\right\} = \frac{d}{dg(x)}\left\{(fog)\ (x)\right\} \times \frac{d}{dx}\left\{g\ (x)\right\}.$$

REMARK 1 *The above rule can also be restated as if* $z = f(y)$ and $y = g(x)$, *then* $\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$

OR

Derivative of z with respect to x = (Derivative of z with respect to y) × (Derivative of y with respect to x)

REMARK 2 *This chain rule can be extended further.*

Derivative of z with respect to x = (Derivative of z with respect to u) × (Derivative of u with respect to v) × (Derivative of v with respect to x)

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\sin(x^2+1)$ (ii) $e^{\sin x}$ (iii) $\log \sin x$

SOLUTION (i) Let $y = \sin(x^2+1)$. Putting $u = x^2+1$, we get

$$y = \sin u \text{ and } u = x^2+1 \Rightarrow \frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = 2x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = (\cos u)\, 2x = 2x\cos(x^2+1) \qquad [\because u = x^2+1]$$

Hence, $\frac{d}{dx}\{\sin(x^2+1)\} = 2x\cos(x^2+1)$

ALITER We have,

$$\frac{d}{dx}\{\sin(x^2+1)\} = \frac{d}{d(x^2+1)}\{\sin(x^2+1)\}\times\frac{d}{dx}(x^2+1)$$

$$\Rightarrow \quad \frac{d}{dx}\{\sin(x^2+1)\} = \{\cos(x^2+1)\}\times 2x \qquad \left[\because \frac{d}{d(x^2+1)}\{\sin(x^2+1)\} = \cos(x^2+1)\right]$$

$$\Rightarrow \quad \frac{d}{dx}\{\sin(x^2+1)\} = 2x\cos(x^2+1).$$

(ii) Let $y = e^{\sin x}$. Putting $u = \sin x$, we get

$$y = e^u \text{ and } u = \sin x \Rightarrow \frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = \cos x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = e^u\cos x = \cos x\, e^{\sin x} \qquad [\because u = \sin x]$$

Hence, $\frac{d}{dx}\{e^{\sin x}\} = e^{\sin x}\cos x.$

ALITER $\frac{d}{dx}\{e^{\sin x}\} = \frac{d}{d(\sin x)}\{e^{\sin x}\}\times\frac{d}{dx}\{\sin x\} = e^{\sin x}\times\cos x.$

(iii) Let $y = \log\sin x$. Putting $u = \sin x$, we get

$$y = \log u \text{ and } u = \sin x \Rightarrow \frac{dy}{du} = \frac{1}{u} \text{ and } \frac{du}{dx} = \cos x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \cos x = \frac{1}{\sin x} \times \cos x = \cot x.$$

Hence, $\frac{d}{dx}\{\log \sin x\} = \cot x$.

ALITER $\frac{d}{dx}\{\log \sin x\} = \frac{d}{d(\sin x)}\{\log \sin x\} \times \frac{d}{dx}(\sin x) = \frac{1}{\sin x} \times \cos x = \cot x$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $\log \sin x^2$ (ii) $e^{\sin x^2}$ (iii) $\sin (e^{x^2})$

SOLUTION (i) Let $y = \log \sin x^2$. Putting $v = x^2$ and $u = \sin x^2 = \sin v$, we get

$$y = \log u, \ u = \sin v \text{ and } v = x^2 \Rightarrow \frac{dy}{du} = \frac{1}{u}, \ \frac{du}{dv} = \cos v \text{ and } \frac{dv}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = \frac{1}{u} \times \cos v \times 2x = \frac{1}{\sin v} \cos v \times 2x \qquad [\because u = \sin v]$$

$$\Rightarrow \quad \frac{dy}{dx} = (\cot v)\, 2x = 2x \cot x^2 \qquad [\because v = x^2]$$

Hence, $\frac{d}{dx}(\log \sin x^2) = 2x \cot x^2$

ALITER $\frac{d}{dx}\{\log \sin x^2\} = \frac{d}{d(\sin x^2)}\{\log \sin x^2\} \times \frac{d}{d(x^2)}(\sin x^2) \times \frac{d}{dx}(x^2)$

$$= \frac{1}{\sin x^2} \times \cos x^2 \times 2x = 2x \cot x^2.$$

(ii) Let $y = e^{\sin x^2}$. Putting $x^2 = v$ and $u = \sin x^2 = \sin v$, we get

$$y = e^u, \ u = \sin v \text{ and } v = x^2 \Rightarrow \frac{dy}{du} = e^u, \ \frac{du}{dv} = \cos v \text{ and } \frac{dv}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = e^u \times \cos v \times 2x = e^{\sin v} \times \cos v \times 2x \qquad [\because u = \sin v]$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sin x^2} \times \cos x^2 \times 2x \qquad [\because v = x^2]$$

ALITER $\frac{d}{dx}\left(e^{\sin x^2}\right) = \frac{d}{d(\sin x^2)}\left(e^{\sin x^2}\right) \times \frac{d}{dx}\left(\sin x^2\right) \times \frac{d}{dx}(x^2) = e^{\sin x^2} \times \cos x^2 \times 2x$

(iii) Let $y = \sin (e^{x^2})$. Putting $x^2 = v$ and $u = e^{x^2} = e^v$, we get

$$y = \sin u, \text{ where } u = e^v \text{ and } v = x^2 \Rightarrow \frac{dy}{du} = \cos u, \ \frac{du}{dv} = e^v \text{ and } \frac{dv}{dx} = 2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = \cos u \times e^v \times 2x = \cos (e^v) \times e^v \times 2x \qquad [\because u = e^v]$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos (e^{x^2}) \times e^{x^2} \times 2x \qquad [\because v = x^2]$$

ALITER $\frac{d}{dx}\{\sin e^{x^2}\} = \frac{d}{d(e^{x^2})}(\sin e^{x^2}) \times \frac{d}{d(x^2)}(e^{x^2}) \times \frac{d}{dx}(x^2) = \cos(e^{x^2}) \times e^{x^2} \times 2x$

EXAMPLE 3 *Differentiate the following functions with respect to x:*

(i) $(x^2 + x + 1)^4$ (ii) $\sqrt{x^2 + x + 1}$ (iii) $\sin^3 x$ (iv) $\frac{1}{\sqrt{a^2 - x^2}}$

SOLUTION (i) Let $y = (x^2 + x + 1)^4$. Putting $x^2 + x + 1 = u$, we get

$$y = u^4 \text{ and } u = x^2 + x + 1 \Rightarrow \frac{dy}{du} = 4u^3 \text{ and } \frac{du}{dx} = 2x + 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3(2x+1) = 4(x^2 + x + 1)^3(2x+1).$$

ALITER We have,

$$\frac{d}{dx}\{(x^2 + x + 1)^4\} = \frac{d}{d(x^2 + x + 1)}\{(x^2 + x + 1)^4\} \times \frac{d}{dx}(x^2 + x + 1) = 4(x^2 + x + 1)^3(2x+1)$$

(ii) Let $y = \sqrt{x^2 + x + 1}$. Putting $x^2 + x + 1 = u$, we get

$$y = \sqrt{u} \text{ and } u = x^2 + x + 1 \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2\sqrt{u}} \text{ and } \frac{du}{dx} = 2x + 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2\sqrt{u}} \times (2x+1) = \frac{1}{2\sqrt{x^2 + x + 1}} \times (2x+1) \qquad [\because u = x^2 + x + 1]$$

ALITER We have,

$$\frac{d}{dx}\left\{\sqrt{x^2 + x + 1}\right\} = \frac{d}{d(x^2 + x + 1)}\left\{(x^2 + x + 1)^{1/2}\right\} \times \frac{d}{dx}(x^2 + x + 1)$$

$$= \frac{1}{2}(x^2 + x + 1)^{-1/2}(2x+1)$$

(iii) Let $y = \sin^3 x$. Putting $u = \sin x$, we get

$$y = u^3 \text{ and } u = \sin x \Rightarrow \frac{dy}{du} = 3u^2 \text{ and } \frac{du}{dx} = \cos x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3u^2 \times \cos x = 3(\sin x)^2 \times \cos x = 3\sin^2 x \cos x. \qquad [\because u = \sin x]$$

ALITER $\frac{d}{dx}(\sin^3 x) = \frac{d}{dx}\{(\sin x)^3\} = \frac{d}{d(\sin x)}\{(\sin x)^3\} \times \frac{d}{dx}(\sin x)$

$$= 3(\sin x)^{3-1} \times \cos x = 3\sin^2 x \cos x.$$

(iv) Let $y = \frac{1}{\sqrt{a^2 - x^2}}$. Putting $u = a^2 - x^2$, we get

$$y = \frac{1}{\sqrt{u}} = u^{-1/2} \text{ and } u = a^2 - x^2 \Rightarrow \frac{dy}{du} = -\frac{1}{2}u^{-3/2} \text{ and } \frac{du}{dx} = -2x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{2}u^{-3/2} \times (-2x) = -\frac{1}{2u^{3/2}} \times (-2x) = \frac{x}{(a^2 - x^2)^{3/2}} \qquad [\because u = a^2 - x^2]$$

ALITER $\frac{d}{dx}\left\{(a^2-x^2)^{-1/2}\right\} = \frac{d}{d(a^2-x^2)}\left\{(a^2-x^2)^{-1/2}\right\} \times \frac{d}{dx}\left(a^2-x^2\right)$

$$= -\frac{1}{2}\left(a^2-x^2\right)^{-3/2}(0-2x) = \frac{x}{(a^2-x^2)^{3/2}}$$

EXAMPLE 4 *Differentiate the following functions with respect to x:*

(i) $\log(\sec x + \tan x)$ (ii) $e^{x \sin x}$ (iii) $\sin^{-1}(x^3)$ (iv) $\sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right), b > a$

SOLUTION (i) Let $y = \log(\sec x + \tan x)$. Putting $u = \sec x + \tan x$, we get

$$y = \log u \text{ and } u = \sec x + \tan x \Rightarrow \frac{dy}{du} = \frac{1}{u} \text{ and } \frac{du}{dx} = \sec x \tan x + \sec^2 x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times (\sec x \tan x + \sec^2 x) = \frac{1}{\sec x + \tan x} \sec x(\tan x + \sec x) = \sec x.$$

(ii) Let $y = e^{x \sin x}$. Putting $u = x \sin x$, we get

$$y = e^u \text{ and } u = x \sin x \Rightarrow \frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = x\cos x + \sin x.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u(x\cos x + \sin x) = e^{x \sin x}(x\cos x + \sin x)$$

ALITER $\frac{d}{dx}(e^{x \sin x}) = \frac{d}{d(x \sin x)}\{e^{x \sin x}\} \times \frac{d}{dx}(x \sin x) = e^{x \sin x} \times (x\cos x + \sin x).$

(iii) Let $y = \sin^{-1} x^3$. Putting $u = x^3$, we get

$$y = \sin^{-1} u \text{ and } u = x^3 \Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \text{ and } \frac{du}{dx} = 3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times 3x^2 = \frac{1}{\sqrt{1-x^6}} \times 3x^2$$

ALITER $\frac{d}{dx}(\sin^{-1} x^3) = \frac{d}{d(x^3)}(\sin^{-1} x^3) \times \frac{d}{dx}(x^3) = \frac{1}{\sqrt{1-x^6}} \times 3x^2$

(iv) Let $y = \sin^{-1}\left(\frac{a+b\cos x}{b+a\cos x}\right)$. Putting $u = \frac{a+b\cos x}{b+a\cos x}$, we get

$$y = \sin^{-1} u \text{ and } u = \frac{a+b\cos x}{b+a\cos x}$$

$$\Rightarrow \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \text{ and, } \frac{du}{dx} = \frac{(b+a\cos x)(0-b\sin x)-(a+b\cos x)(0-a\sin x)}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}} \text{ and, } \frac{du}{dx} = \frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times \frac{(a^2-b^2)\sin x}{(b+a\cos x)^2} = \frac{1}{\sqrt{1-\left(\frac{a+b\cos x}{b+a\cos x}\right)^2}} \times \frac{(a^2-b^2)\sin x}{(b+a\cos x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(b + a\cos x)}{\sqrt{(b + a\cos x)^2 - (a + b\cos x)^2}} \times \frac{(a^2 - b^2)\sin x}{(b + a\cos x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(a^2 - b^2)\sin x}{\sqrt{b^2(1 - \cos^2 x) - a^2(1 - \cos^2 x)}} \times \frac{1}{(b + a\cos x)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(a^2 - b^2)\sin x}{\sqrt{(b^2 - a^2)\sin^2 x}} \times \frac{1}{b + a\cos x} = -\frac{(b^2 - a^2)\sin x}{\sqrt{b^2 - a^2}\,\sin x} \times \frac{1}{b + a\cos x} = \frac{-\sqrt{b^2 - a^2}}{b + a\cos x}.$$

EXAMPLE 5 *Differentiate the following functions with respect to x:*

(i) e^{e^x} (ii) $\log_7(\log_7 x)$ (iii) $\log_x 2$

SOLUTION (i) Let $y = e^{e^x}$. Putting $e^x = u$, we get

$$y = e^u \text{ and } u = e^x \Rightarrow \frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = e^x$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u \times e^x = e^{e^x} \times e^x \qquad [\because u = e^x]$$

ALITER $\frac{d}{dx}\left(e^{e^x}\right) = \frac{d}{d(e^x)}\left(e^{e^x}\right) \times \frac{d}{dx}(e^x) = e^{e^x} \times e^x$

(ii) Let $y = \log_7(\log_7 x)$. Putting $u = \log_7 x$, we get

$$y = \log_7 u \text{ and } u = \log_7 x \Rightarrow \frac{dy}{du} = \frac{1}{u\log_e 7} \text{ and } \frac{du}{dx} = \frac{1}{x\log_e 7}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u\log_e 7} \times \frac{1}{x\log_e 7}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log_7 x \times \log_e 7 \times x\log_e 7} = \frac{1}{x(\log_7 x)(\log_e 7)^2} \qquad [\because u = \log_7 x]$$

ALITER $\frac{d}{dx}\{\log_7(\log_7 x)\} = \frac{d}{d(\log_7 x)}\left(\log_7(\log_7 x)\right) \times \frac{d}{dx}(\log_7 x) = \frac{1}{(\log_7 x)\log_e 7} \times \frac{1}{x\log_e 7}$

$$= \frac{1}{x(\log_7 x)(\log_e 7)^2}$$

(iii) Let $y = \log_x 2$. Then, $y = \frac{1}{\log_2 x}$. Putting $u = \log_2 x$, we get $\left[\because \log_b a = \frac{1}{\log_a b}\right]$

$$y = \frac{1}{u} \text{ and } u = \log_2 x \Rightarrow \frac{dy}{du} = -\frac{1}{u^2} \text{ and } \frac{du}{dx} = \frac{1}{x\log_e 2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = -\frac{1}{u^2} \times \frac{1}{x\log_e 2} = -\frac{1}{(\log_2 x)^2} \times \frac{1}{x\log_e 2} \qquad [\because u = \log_2 x]$$

ALITER Using chain rule, we obtain

$$\frac{d}{dx}(\log_x 2) = \frac{d}{dx}\left(\frac{1}{\log_2 x}\right) = \frac{d}{d(\log_2 x)}\left(\frac{1}{\log_2 x}\right) \times \frac{d}{dx}(\log_2 x) = -\frac{1}{(\log_2 x)^2} \times \frac{1}{x\log_e 2}$$

EXAMPLE 6 *Differentiate the following functions with respect to x:*

(i) $\sec(\log x^n)$ (ii) $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ [CBSE 2002] (iii) $\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$.

SOLUTION (i) Let $y=\sec(\log x^n)$. Putting $x^n=v$, $u=\log x^n=\log v$, we get

$$y=\sec u,\ u=\log v \text{ and } v=x^n \Rightarrow \frac{dy}{du}=\sec u\tan u,\ \frac{du}{dv}=\frac{1}{v} \text{ and } \frac{dv}{dx}=nx^{n-1}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}=\sec u\tan u\times\frac{1}{v}\times nx^{n-1}$$

$$\Rightarrow \quad \frac{dy}{dx}=\sec(\log x^n)\tan(\log x^n)\times\frac{1}{x^n}\times nx^{n-1}=\frac{n}{x}\times\sec(\log x^n)\tan(\log x^n)$$

(ii) Let $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$. Putting $\frac{\pi}{4}+\frac{x}{2}=v$, $\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)=\tan v=u$, we get

$$y=\log u,\ u=\tan v \text{ and } v=\frac{\pi}{4}+\frac{x}{2} \Rightarrow \frac{dy}{du}=\frac{1}{u},\ \frac{du}{dv}=\sec^2 v \text{ and } \frac{du}{dx}=\frac{1}{2}.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dv}\times\frac{dv}{dx}=\frac{1}{u}\times\sec^2 v\times\frac{1}{2}=\frac{1}{\tan v}\sec^2 v\times\frac{1}{2} \qquad [\because u=\tan v]$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1}{2\sin v\cos v}=\frac{1}{\sin 2v}=\frac{1}{\sin\left(\frac{\pi}{2}+x\right)}=\frac{1}{\cos x}=\sec x. \qquad \left[\because v=\frac{\pi}{4}+\frac{x}{2}\right]$$

(iii) Let $y=\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$

Putting $\frac{x^2}{3}-1=v$, $\sin\left(\frac{x^2}{3}-1\right)=\sin v=u$ and $\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}=\log u=z$, we get

$$y=\sqrt{z},\ z=\log u,\ u=\sin v \text{ and } v=\frac{x^2}{3}-1 \Rightarrow \frac{dy}{dz}=\frac{1}{2\sqrt{z}},\ \frac{dz}{du}=\frac{1}{u},\ \frac{du}{dv}=\cos v \text{ and } \frac{dv}{dx}=\frac{2x}{3}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{du}\times\frac{du}{dv}\times\frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{1}{2\sqrt{z}}\right)\left(\frac{1}{u}\right)(\cos v)\left(\frac{2x}{3}\right)=\frac{x}{3}\times\frac{\cos v}{u\sqrt{\log u}} \qquad [\because z=\log u]$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{3}\times\frac{\cos\left(\frac{x^2}{3}-1\right)}{\sin\left(\frac{x^2}{3}-1\right)\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}=\frac{x\cot\left(\frac{x^2}{3}-1\right)}{3\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}.$$

EXAMPLE 7 *Differentiate the following functions with respect to x:*

(i) $\log\left(x+\sqrt{a^2+x^2}\right)$ **[CBSE 2003, NCERT EXEMPLAR]** (ii) $\log\left\{\dfrac{a+b\sin x}{a-b\sin x}\right\}$

(iii) $\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$ (iv) $\log\sqrt{\dfrac{1+\sin x}{1-\sin x}}$

SOLUTION (i) Let $y=\log\left(x+\sqrt{a^2+x^2}\right)$. Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\log\left(x+\sqrt{a^x+x^2}\right)\right\}=\frac{1}{x+\sqrt{a^2+x^2}}\times\frac{d}{dx}\left\{x+\sqrt{a^2+x^2}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{1}{2}(a^2+x^2)^{-1/2}\times\frac{d}{dx}(a^2+x^2)\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{x+\sqrt{a^2+x^2}}\left\{1+\frac{1}{2\sqrt{a^2+x^2}}\times 2x\right\}=\frac{1}{x+\sqrt{a^2+x^2}}\times\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}}=\frac{1}{\sqrt{a^2+x^2}}$$

(ii) Let $y=\log\left\{\dfrac{a+b\sin x}{a-b\sin x}\right\}$. Then,

$$y=\log(a+b\sin x)-\log(a-b\sin x)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{d}{dx}\{\log(a+b\sin x)\}-\frac{d}{dx}\{\log(a-b\sin x)\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{a+b\sin x}\times\frac{d}{dx}(a+b\sin x)-\frac{1}{a-b\sin x}\times\frac{d}{dx}(a-b\sin x)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{a+b\sin x}\times\frac{d}{dx}(a+b\sin x)-\frac{1}{a-b\sin x}\times\frac{d}{dx}(a-b\sin x)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{1}{a+b\sin x}(0+b\cos x)-\frac{1}{a-b\sin x}(0-b\cos x)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{b\cos x}{a+b\sin x}+\frac{b\cos x}{a-b\sin x}=b\cos x\left\{\frac{1}{a+b\sin x}+\frac{1}{a-b\sin x}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=b\cos x\left\{\frac{a-b\sin x+a+b\sin x}{(a+b\sin x)(a-b\sin x)}\right\}=\frac{2ab\cos x}{a^2-b^2\sin^2 x}$$

(iii) Let $y=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$. Then,

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\frac{e^x+e^{-x}}{e^x-e^{-x}}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{(e^x-e^{-x})\frac{d}{dx}(e^x+e^{-x})-(e^x+e^{-x})\frac{d}{dx}(e^x-e^{-x})}{(e^x-e^{-x})^2}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}\quad\left[\because \frac{d}{dx}(e^{-x})=e^{-x}\frac{d}{dx}(-x)=-e^{-x}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$$

(iv) Let $y = \log \sqrt{\dfrac{1+\sin x}{1-\sin x}}$. Then,

$$y = \log \left\{ \frac{1+\sin x}{1-\sin x} \right\}^{1/2} = \frac{1}{2} \log \left\{ \frac{1+\sin x}{1-\sin x} \right\} = \frac{1}{2} \{\log (1+\sin x) - \log (1-\sin x)\}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2} \left\{ \frac{d}{dx} \{\log (1+\sin x)\} - \frac{d}{dx} \{\log (1-\sin x)\} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \left\{ \frac{1}{1+\sin x} \times \frac{d}{dx}(1+\sin x) - \frac{1}{1-\sin x} \times \frac{d}{dx}(1-\sin x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \left\{ \frac{1}{1+\sin x}(0+\cos x) - \frac{1}{1-\sin x}(0-\cos x) \right\} = \frac{1}{2} \left\{ \frac{\cos x}{1+\sin x} + \frac{\cos x}{1-\sin x} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2} \cos x \left\{ \frac{1-\sin x + 1 + \sin x}{1-\sin^2 x} \right\} = \frac{1}{2} \cos x \left(\frac{2}{1-\sin^2 x} \right) = \frac{\cos x}{\cos^2 x} = \sec x.$$

EXAMPLE 8 *Find* $\dfrac{dy}{dx}$, *when*

(i) $y = e^{ax} \cos (bx+c)$ (ii) $y = \dfrac{e^x + \log x}{\sin 3x}$ (iii) $y = e^x \log (1+x^2)$ (iv) $y = \dfrac{\sin x + x^2}{\cot 2x}$

SOLUTION (i) Using product rule, we get

$$\frac{dy}{dx} = e^{ax} \times \frac{d}{dx}\{\cos (bx+c)\} + \cos (bx+c) \times \frac{d}{dx}(e^{ax})$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax} \times - \sin (bx+c) \times \frac{d}{dx}(bx+c) + \cos (bx+c) \times e^{ax} \times \frac{d}{dx}(ax)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax} \times \{-\sin (bx+c)\} \times b + \cos (bx+c) \times e^{ax} \times a$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{ax} \{-b \sin (bx+c) + a \cos (bx+c)\}.$$

(ii) Using quotient rule, we get

$$\frac{dy}{dx} = \frac{\sin 3x \times \frac{d}{dx}(e^x + \log x) - (e^x + \log x) \times \frac{d}{dx}(\sin 3x)}{(\sin 3x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin 3x \left(e^x + \frac{1}{x} \right) - (e^x + \log x)(\cos 3x) \frac{d}{dx}(3x)}{\sin^2 3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sin 3x \left(e^x + \frac{1}{x} \right) - (e^x + \log x)(\cos 3x) \times 3}{\sin^2 3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(e^x + 1/x) \sin 3x - 3(e^x + \log x) \cos 3x}{\sin^2 3x}$$

(iii) Using product rule, we get

$$\frac{dy}{dx} = e^x \times \frac{d}{dx}\{\log(1+x^2)\} + \log(1+x^2) \times \frac{d}{dx}(e^x)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x \times \frac{1}{1+x^2} \times \frac{d}{dx}(1+x^2) + \log(1+x^2) \times e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x}{1+x^2} \times 2x + e^x \times \log(1+x^2) = e^x\left\{\frac{2x}{1+x^2} + \log(1+x^2)\right\}$$

(iv) We have, $y = \dfrac{\sin x + x^2}{\cot 2x} = (\sin x + x^2)\tan 2x$

Using product rule, we get

$$\frac{dy}{dx} = (\sin x + x^2)\frac{d}{dx}(\tan 2x) + \tan 2x \frac{d}{dx}(\sin x + x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x + x^2)(\sec^2 2x)\frac{d}{dx}(2x) + (\tan 2x)(\cos x + 2x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x + x^2)(\sec^2 2x) \times 2 + (\tan 2x)(\cos x + 2x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 2(\sin x + x^2)\sec^2 2x + (\cos x + 2x)\tan 2x.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *If* $y = \left\{x + \sqrt{x^2 + a^2}\right\}^n$, *then prove that* $\dfrac{dy}{dx} = \dfrac{ny}{\sqrt{x^2 + a^2}}$.

[CBSE 2005]

SOLUTION We have, $y = \left\{x + \sqrt{x^2 + a^2}\right\}^n$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left\{x + \sqrt{x^2 + a^2}\right\}^n = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \frac{d}{dx}\left\{x + \sqrt{x^2 + a^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{\frac{d}{dx}(x) + \frac{d}{dx}\sqrt{x^2 + a^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{1 + \frac{1}{2}(x^2 + a^2)^{-1/2} \times \frac{d}{dx}(x^2 + a^2)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \times \left\{1 + \frac{1}{2\sqrt{x^2 + a^2}} \times 2x\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \left\{1 + \frac{x}{\sqrt{x^2 + a^2}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = n\left\{x + \sqrt{x^2 + a^2}\right\}^{n-1} \left\{\frac{\sqrt{x^2 + a^2} + x}{\sqrt{x^2 + a^2}}\right\} = \frac{n\left\{x + \sqrt{x^2 + a^2}\right\}^n}{\sqrt{x^2 + a^2}} = \frac{ny}{\sqrt{x^2 + a^2}}.$$

EXAMPLE 10 *If* $y = \dfrac{x\sin^{-1}x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$, *then prove that* $\dfrac{dy}{dx} = \dfrac{\sin^{-1}x}{(1-x^2)^{3/2}}$.

SOLUTION We have, $y = x\sin^{-1}x\,(1-x^2)^{-1/2} + \dfrac{1}{2}\log(1-x^2)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{x\sin^{-1}x\,(1-x^2)^{-1/2}\right\} + \frac{1}{2}\frac{d}{dx}\left\{\log(1-x^2)\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1}x\,(1-x^2)^{-1/2}\frac{d}{dx}(x) + x\frac{d}{dx}(\sin^{-1}x)\,(1-x^2)^{-1/2} + x\sin^{-1}x\frac{d}{dx}(1-x^2)^{-1/2} + \frac{1}{2}\times\frac{1}{1-x^2}\times\frac{d}{dx}(1-x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}\times 1 + x\times\frac{1}{\sqrt{1-x^2}}\times\frac{1}{\sqrt{1-x^2}} + x\sin^{-1}x\times\left(-\frac{1}{2}\right)(1-x^2)^{-3/2}\frac{d}{dx}(1-x^2) + \frac{1}{2(1-x^2)}(0-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{x}{1-x^2} - \frac{x}{2}\frac{(\sin^{-1}x)}{(1-x^2)^{3/2}}(0-2x) - \frac{x}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + \frac{x^2\sin^{-1}x}{(1-x^2)^{3/2}} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}\left\{1+\frac{x^2}{1-x^2}\right\} = \frac{\sin^{-1}x}{\sqrt{1-x^2}}\times\frac{1}{1-x^2} = \frac{\sin^{-1}x}{(1-x^2)^{3/2}}.$$

EXAMPLE 11 *Differentiate the following functions with respect to x:*

(i) $\sin(m\sin^{-1}x)$ (ii) $a^{(\sin^{-1}x)^2}$ (iii) $e^{\cos^{-1}\left(\sqrt{1-x^2}\right)}$

SOLUTION (i) We have, $y = \sin(m\sin^{-1}x)$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left\{\sin(m\sin^{-1}x)\right\} = \cos(m\sin^{-1}x)\times\frac{d}{dx}(m\sin^{-1}x) = \cos(m\sin^{-1}x)\times m\frac{d}{dx}(\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = \cos(m\sin^{-1}x)\times m\times\frac{1}{\sqrt{1-x^2}} = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x)$$

(ii) We have, $y = a^{(\sin^{-1}x)^2}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left\{a^{(\sin^{-1}x)^2}\right\} = a^{(\sin^{-1}x)^2}\log a\times\frac{d}{dx}\left\{(\sin^{-1}x)^2\right\}$$

$$\Rightarrow \frac{dy}{dx} = a^{(\sin^{-1}x)^2}\log a\times 2(\sin^{-1}x)^{2-1}\times\frac{d}{dx}(\sin^{-1}x)$$

$$\Rightarrow \frac{dy}{dx} = a^{(\sin^{-1}x)^2}\log a\times 2\sin^{-1}x\times\frac{1}{\sqrt{1-x^2}} = \frac{2\log a\cdot\sin^{-1}x}{\sqrt{1-x^2}}\times a^{(\sin^{-1}x)^2}$$

(iii) We have, $y = e^{\cos^{-1}\sqrt{1-x^2}}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left\{e^{\cos^{-1}\sqrt{1-x^2}}\right\} = e^{\cos^{-1}\sqrt{1-x^2}} \times \frac{d}{dx}\left\{\cos^{-1}\sqrt{1-x^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\cos^{-1}\sqrt{1-x^2}} \times \frac{-1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\cos^{-1}\sqrt{1-x^2}} \times \frac{-1}{\sqrt{1-(1-x^2)}} \times \frac{1}{2}(1-x^2)^{\frac{1}{2}-1} \times \frac{d}{dx}(1-x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\cos^{-1}\sqrt{1-x^2}} \times \left(-\frac{1}{x}\right) \times \frac{1}{2\sqrt{1-x^2}}(-2x) = e^{\cos^{-1}\sqrt{1-x^2}} \times \frac{1}{\sqrt{1-x^2}}.$$

EXAMPLE 12 *Differentiate the following functions with respect to x:*

(i) $\log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$ (ii) $5^{3-x^2} + (3-x^2)^5$

SOLUTION (i) Let $y = \log_{10} x + \log_x 10 + \log_x x + \log_{10} 10$. Then,

$$y = \log_{10} x + \frac{1}{\log_{10} x} + 1 + 1 = \log_{10} x + (\log_{10} x)^{-1} + 2.$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{x \log_e 10} + (-1)(\log_{10} x)^{-2} \times \frac{d}{dx}(\log_{10} x) + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x \log_e 10} - (\log_{10} x)^{-2} \times \frac{1}{x \log_e 10}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x \log_e 10} - \frac{1}{x(\log_{10} x)^2} \times \frac{1}{\log_e 10}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x \log_e 10} - \frac{1}{x(\log_{10} x \cdot \log_e 10)^2} \log_e 10 = \frac{1}{x \log_e 10} - \frac{1}{x(\log_e x)^2} \log_e 10.$$

(ii) Let $y = 5^{3-x^2} + (3-x^2)^5$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(5^{3-x^2}\right) + \frac{d}{dx}\left\{(3-x^2)^5\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = 5^{3-x^2} \log_e 5 \times \frac{d}{dx}(3-x^2) + 5(3-x^2)^{5-1} \times \frac{d}{dx}(3-x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = 5^{3-x^2} \log_e 5 \times (0-2x) + 5(3-x^2)^4 \times (0-2x) = -2x\left\{5^{3-x^2} \log_e 5 + 5(3-x^2)^4\right\}.$$

EXAMPLE 13 *If* $y = \dfrac{\sqrt{a^2+x^2} + \sqrt{a^2-x^2}}{\sqrt{a^2+x^2} - \sqrt{a^2-x^2}}$, *show that* $\dfrac{dy}{dx} = -\dfrac{2a^2}{x^3}\left\{1 + \dfrac{a^2}{\sqrt{a^4-x^4}}\right\}$.

SOLUTION We have,

$$y = \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}} = \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}} \times \frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$$

$$\Rightarrow \quad y = \frac{\left\{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right\}^2}{(a^2+x^2)-(a^2-x^2)} = \frac{a^2+x^2+a^2-x^2+2\sqrt{a^2+x^2}\sqrt{a^2-x^2}}{2x^2}$$

$$\Rightarrow \quad y = \frac{2a^2+2\sqrt{a^4-x^4}}{2x^2} = \frac{a^2}{x^2}+\frac{\sqrt{a^4-x^4}}{x^2} = a^2x^{-2}+\sqrt{a^4-x^4}\,x^{-2}$$

$$\therefore \quad \frac{dy}{dx} = a^2\frac{d}{dx}(x^{-2})+\frac{d}{dx}\left\{\sqrt{a^4-x^4}\,x^{-2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -2a^2x^{-3}+(-2)\,x^{-3}\sqrt{a^4-x^4}+(x^{-2})\,\frac{1}{2}(a^4-x^4)^{-1/2}\times\frac{d}{dx}(a^4-x^4)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2a^2}{x^3}-\frac{2}{x^3}\sqrt{a^4-x^4}+\frac{1}{2x^2\sqrt{a^4-x^4}}(-4x^3)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2a^2}{x^3}-\frac{2}{x^3}\sqrt{a^4-x^4}-\frac{2x}{\sqrt{a^4-x^4}} = -\frac{2a^2}{x^3}-2\left\{\frac{\sqrt{a^4-x^4}}{x^3}+\frac{x}{\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2a^2}{x^3}-2\left\{\frac{a^4-x^4+x^4}{x^3\sqrt{a^4-x^4}}\right\} = -\frac{2a^2}{x^3}-\frac{2a^4}{x^3\sqrt{a^4-x^4}} = -\frac{2a^2}{x^3}\left\{1+\frac{a^2}{\sqrt{a^4-x^4}}\right\}$$

EXAMPLE 14 *If* $y=\sqrt{\frac{1-x}{1+x}}$, *prove that* $(1-x^2)\frac{dy}{dx}+y=0$. **[CBSE 2004]**

SOLUTION We have, $y=\sqrt{\frac{1-x}{1+x}}$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2-1}\times\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\times\frac{(1+x)\frac{d}{dx}(1-x)-(1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\times\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}\times\frac{-1-x-1+x}{(1+x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{1+x}{1-x}}\times\frac{1}{(1+x)^2}$$

$$\Rightarrow \quad (1-x^2)\frac{dy}{dx} = -\sqrt{\frac{1+x}{1-x}}\,\frac{1}{(1+x)^2}(1-x^2) \qquad \text{[Multiplying both sides by } (1-x^2)\text{]}$$

$$\Rightarrow \quad (1-x^2)\frac{dy}{dx} = -\sqrt{\frac{1-x}{1+x}} \Rightarrow (1-x^2)\frac{dy}{dx} = -y \Rightarrow (1-x^2)\frac{dy}{dx} + y = 0$$

EXAMPLE 15 *If* $y = \sqrt{\frac{1+e^x}{1-e^x}}$, *show that* $\frac{dy}{dx} = \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$.

SOLUTION We have, $y = \sqrt{\frac{1+e^x}{1-e^x}}$. Differentiating both sides with respect to x, we obtain.

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+e^x}{1-e^x}\right)^{1/2-1} \times \frac{d}{dx}\left(\frac{1+e^x}{1-e^x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-e^x}{1+e^x}} \times \frac{(1-e^x)\frac{d}{dx}(1+e^x) - (1+e^x)\frac{d}{dx}(1-e^x)}{(1-e^x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-e^x}{1+e^x}} \times \frac{(1-e^x)e^x + (1+e^x)e^x}{(1-e^x)^2} = \frac{1}{2}\sqrt{\frac{1-e^x}{1+e^x}} \times \frac{2e^x}{(1-e^x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x\sqrt{\frac{1-e^x}{1+e^x}} \times \frac{1}{(1-e^x)^2} = \frac{e^x}{\sqrt{1+e^x}\,(1-e^x)^{3/2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{e^x}{\sqrt{1+e^x}\sqrt{1-e^x}\,(1-e^x)} = \frac{e^x}{(1-e^x)\sqrt{1-e^{2x}}}$$

EXAMPLE 16 *If* $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$, *using derivatives prove that*

(i) $C_1 + 2C_2 + \ldots + nC_n = n.2^{n-1}$ (ii) $C_1 - 2C_2 + 3C_3 + \ldots + (-1)^{n-1}nC_n = 0$

SOLUTION We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Differentiating both sides with respect to x, we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

Putting $x = 1$ and -1 successively, we get

$$C_1 + 2C_2 + 3C_3 + \ldots + nC_n = n.2^{n-1} \text{ and, } C_1 - 2C_2 + 3C_3 + \ldots + (-1)^{n-1}nC_n = 0$$

EXAMPLE 17 *Using the fact:* $\sin(A+B) = \sin A\cos B + \cos A\sin B$ *and the technique of differentiation, obtain the sum formula for* cosines.

SOLUTION We have,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Taking B as a constant, A as a variable and differentiating both sides with respect to A, we get

$$\frac{d}{dA}(\sin(A+B)) = \cos B\frac{d}{dA}(\sin A) + \sin B\frac{d}{dA}(\cos A)$$

$$\Rightarrow \quad \cos(A+B) = \cos B\cos A - \sin B\sin A \text{ or, } \cos(A+B) = \cos A\cos B - \sin A\sin B.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 18 *If* $f(x) = \sqrt{x^2+1}$, $g(x) = \frac{x+1}{x^2+1}$ *and* $h(x) = 2x-3$, *then find* $f'(h'(g'(x)))$. **[CBSE 2015]**

SOLUTION We have, $f(x) = \sqrt{x^2+1}$, $g(x) = \frac{x+1}{x^2+1}$ and $h(x) = 2x-3$

$\therefore \quad f'(x) = \dfrac{x}{\sqrt{x^2+1}},\ g'(x) = \dfrac{1-2x-x^2}{(x^2+1)^2}$ and $h'(x) = 2$ for all $x \in R$

Now,

$$h'(x) = 2 \text{ for all } x \in R$$

$$\Rightarrow \quad h'\left(g'(x)\right) = 2 \text{ for all } x \in R$$

$$\Rightarrow \quad f'\left(h'\left(g'(x)\right)\right) = f'(2) \text{ for all } x \in R$$

$$\Rightarrow \quad f'\left(h'\left(g'(x)\right)\right) = \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}} \text{ for all } x \in R \qquad \left[\because f'(x) = \frac{x}{\sqrt{x^2+1}} \therefore f'(2) = \frac{2}{\sqrt{5}}\right]$$

EXAMPLE 19 *If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, find $\dfrac{dy}{dx}$.* **[CBSE 2020]**

SOLUTION We have, $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(f(x^2)) = f'(x^2)\frac{d}{dx}(x^2) = f'(x^2)\, 2x = e^{|x|}\, 2x = 2x e^{|x|} \qquad [\because f'(x) = e^{\sqrt{x}} \therefore f'(x^2) = e^{|x|}]$$

10.4.1 DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS BY CHAIN RULE

In section 10.3, we have obtained the derivative of inverse trigonometric functions from first principles. In this section, we will obtain the same by using chain rule.

THEOREM 1 *If $x \in (-1, 1)$, then the differentiation of $\sin^{-1} x$ with respect to x is $\dfrac{1}{\sqrt{1-x^2}}$.*

i.e., $\quad \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$.

PROOF Let $y = \sin^{-1} x$. Then, $\sin(\sin^{-1} x) = x \Rightarrow \sin y = x$

Differentiating both sides with respect to x, we get

$$1 = \frac{d}{dx}(\sin y)$$

$$\Rightarrow \quad 1 = \frac{d}{dy}(\sin y) \times \frac{dy}{dx} \qquad \text{[By chain rule]}$$

$$\Rightarrow \quad 1 = \cos y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}},\ -1 < x < 1$$

or, $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

THEOREM 2 *If $x \in (-1, 1)$, then the differentiation of $\cos^{-1} x$ with respect to x is $\dfrac{-1}{\sqrt{1-x^2}}$.*

i.e., $\quad \dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1-x^2}}$, for $x \in (-1, 1)$.

PROOF Let $y = \cos^{-1} x$. Then,

$$\cos(\cos^{-1} x) = x$$

$$\Rightarrow \quad \cos y = x$$

$$\Rightarrow \quad \frac{d}{dx}(\cos y) = \frac{d}{dx}(x) \qquad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{d}{dy}(\cos y) \times \frac{dy}{dx} = 1 \qquad \text{[By chain rule]}$$

$$\Rightarrow \quad -\sin y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\sin y} = \frac{-1}{\sqrt{1-\cos^2 y}} = \frac{-1}{\sqrt{1-x^2}}, -1 < x < 1 \qquad [\because x = \cos y]$$

$$\text{or,} \quad \frac{d}{dx}(\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

THEOREM 3 *The differentiation of* $\tan^{-1} x$ *with respect to* x *is* $\frac{1}{1+x^2}$.

i.e., $$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

PROOF Let $y = \tan^{-1} x$. Then,

$$\tan(\tan^{-1} x) = x$$

$$\Rightarrow \quad \tan y = x$$

$$\Rightarrow \quad \frac{d}{dx}(\tan y) = \frac{d}{dx}(x) \qquad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{d}{dy}(\tan y) \times \frac{dy}{dx} = 1 \qquad \text{[By chain rule]}$$

$$\Rightarrow \quad \sec^2 y \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\text{or,} \quad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad [\because y = \tan^{-1} x \text{ and } \tan y = x]$$

THEOREM 4 *The differentiation of* $\cot^{-1} x$ *with respect to* x *is* $\frac{-1}{1+x^2}$.

i.e., $$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

PROOF Let $y = \cot^{-1} x$. Then,

$$\cot(\cot^{-1} x) = x$$

$$\Rightarrow \quad \cot y = x$$

$$\Rightarrow \quad \frac{d}{dx}(\cot y) = \frac{d}{dx}(x) \qquad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{d}{dy}(\cot y) \times \frac{dy}{dx} = 1 \qquad \text{[Using chain rule]}$$

$$\Rightarrow \quad -\operatorname{cosec}^2 y \times \frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{\operatorname{cosec}^2 y} = \frac{-1}{1+\cot^2 y} = \frac{-1}{1+x^2}$$

$$\text{or,} \quad \frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2} \qquad [\because y = \cot^{-1} x \text{ and } x = \cot y]$$

THEOREM 5 *If* $x \in R - [-1, 1]$, *then the differentiation of* $\sec^{-1} x$ *with respect to* x *is* $\frac{1}{|x|\sqrt{x^2-1}}$.

i.e., $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$, $x \in R - [-1, 1]$.

PROOF Let $y = \sec^{-1} x$. Then,

$$\sec(\sec^{-1} x) = x$$

$\Rightarrow \quad \sec y = x$

$\Rightarrow \quad \frac{d}{dx}(\sec y) = \frac{d}{dx}(x)$ [Differentiating both sides with respect to x]

$\Rightarrow \quad \frac{d}{dy}(\sec y) \times \frac{dy}{dx} = 1$ [Using chain rule]

$\Rightarrow \quad \sec y \tan y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$

If $x > 1$, then

$y \in (0, \pi/2) \Rightarrow \sec y > 0, \tan y > 0 \Rightarrow |\sec y|\ |\tan y| = \sec y \tan y$.

If $x < -1$, then

$y \in (\pi/2, \pi) \Rightarrow \sec y < 0, \tan y < 0 \Rightarrow |\sec y|\ |\tan y| = (-\sec y)(-\tan y) = \sec y \tan y$

$\therefore \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} \Rightarrow \frac{dy}{dx} = \frac{1}{|\sec y|\ |\tan y|}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{|\sec y|\sqrt{\tan^2 y}} = \frac{1}{|\sec y|\sqrt{\sec^2 y - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}$

or, $\quad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$

THEOREM 6 *If $x \in R - [-1, 1]$, then the differentiation of $\operatorname{cosec}^{-1} x$ with respect to x is $\frac{-1}{|x|\sqrt{x^2 - 1}}$.*

i.e., $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{1}{|x|\sqrt{x^2 - 1}}$

PROOF Let $y = \operatorname{cosec}^{-1} x$. Then,

$$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$$

$\Rightarrow \quad \operatorname{cosec} y = x$

$\Rightarrow \quad \frac{d}{dx}(\operatorname{cosec} y) = \frac{d}{dx}(x)$ [Differentiating both sides with respect to x]

$\Rightarrow \quad \frac{d}{dy}(\operatorname{cosec} y) \times \frac{dy}{dx} = 1$ [Using chain rule]

$\Rightarrow \quad -\operatorname{cosec} y \cot y \times \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y}$

If $x > 1$, then $y\ (0, \pi/2) \Rightarrow \operatorname{cosec} y > 0, \cot y > 0 \Rightarrow |\operatorname{cosec} y||\cot y| = \operatorname{cosec} y \cot y$

If $x < -1$, then

$y \in (-\pi/2, 0) \Rightarrow \operatorname{cosec} y < 0$ and $\cot y < 0 \Rightarrow |\operatorname{cosec} y||\cot y| = (-\operatorname{cosec} y)(-\cot y)$

$\therefore \quad \frac{dy}{dx} = \frac{-1}{\operatorname{cosec} y \cot y} \Rightarrow \frac{dy}{dx} = \frac{-1}{|\operatorname{cosec} y||\cot y|} = \frac{-1}{|\operatorname{cosec} y|\sqrt{\operatorname{cosec}^2 y - 1}} = \frac{-1}{|x|\sqrt{x^2 - 1}}$

or, $\quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2 - 1}}$

EXERCISE 10.2

BASIC

Differentiate the following functions with respect to x (1-57) :

1. $\sin(3x+5)$
2. $\tan^2 x$
3. $\tan(x^\circ + 45^\circ)$
4. $\sin(\log x)$
5. $e^{\sin\sqrt{x}}$
6. $e^{\tan x}$
7. $\sin^2(2x+1)$
8. $\log_7(2x-3)$
9. $\tan 5x^\circ$
10. 2^{x^3}
11. 3^{e^x}
12. $\log_x 3$
13. 3^{x^2+2x}
14. $\sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$
15. $3^{x\log x}$
16. $\sqrt{\dfrac{1+\sin x}{1-\sin x}}$
17. $\sqrt{\dfrac{1-x^2}{1+x^2}}$
18. $(\log\sin x)^2$
19. $\sqrt{\dfrac{1+x}{1-x}}$
20. $\sin\left(\dfrac{1+x^2}{1-x^2}\right)$
21. $e^{3x}\cos 2x$
22. $\sin(\log\sin x)$
23. $e^{\tan 3x}$
24. $e^{\sqrt{\cot x}}$
25. $\log\left(\dfrac{\sin x}{1+\cos x}\right)$
26. $\log\sqrt{\dfrac{1-\cos x}{1+\cos x}}$ [CBSE 2003]
27. $\tan(e^{\sin x})$
28. $\log(x+\sqrt{x^2+1})$
29. $\dfrac{e^x\log x}{x^2}$
30. $\dfrac{3x^2\sin x}{\sqrt{7-x^2}}$
31. $\dfrac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}$
32. $\log\left(\dfrac{x^2+x+1}{x^2-x+1}\right)$
33. $\tan^{-1}(e^x)$
34. $e^{\sin^{-1}2x}$
35. $\sin(2\sin^{-1}x)$
36. $e^{\tan^{-1}\sqrt{x}}$
37. $\sqrt{\tan^{-1}\left(\dfrac{x}{2}\right)}$
38. $\log(\tan^{-1}x)$
39. $\dfrac{2^x\cos x}{(x^2+3)^2}$
40. $\sin^2\{\log(2x+3)\}$
41. $e^x\log\sin 2x$
42. $(\sin^{-1}x^4)^4$
43. $\sin^{-1}\left(\dfrac{x}{\sqrt{x^2+a^2}}\right)$
44. $\dfrac{e^x\sin x}{(x^2+2)^3}$
45. $3e^{-3x}\log(1+x)$
46. $\dfrac{x^2+2}{\sqrt{\cos x}}$
47. $\dfrac{x^2(1-x^2)^3}{\cos 2x}$
48. $\log\left\{\cot\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right\}$
49. $e^{ax}\sec x\tan 2x$
50. $\log(\cos x^2)$
51. $\cos(\log x)^2$
52. $\log\sqrt{\dfrac{x-1}{x+1}}$
53. $\log(\operatorname{cosec} x-\cot x)$
54. $x\sin 2x+5^x+k^k+(\tan^2 x)^3$
55. $\log(3x+2)-x^2\log(2x-1)$
56. $\dfrac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}}$
57. $\log\left\{x+2+\sqrt{x^2+4x+1}\right\}$
58. If $y=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$, prove that $\dfrac{dy}{dx}=1-y^2$
59. If $y=\sqrt{x^2+a^2}$, prove that $y\dfrac{dy}{dx}-x=0$
60. If $y=e^x+e^{-x}$, prove that $\dfrac{dy}{dx}=\sqrt{y^2-4}$ [CBSE 2020]

61. If $y = \sqrt{a^2 - x^2}$, prove that $y \dfrac{dy}{dx} + x = 0$

62. If $xy = 4$, prove that $x\left(\dfrac{dy}{dx} + y^2\right) = 3y$

63. If $f(x) = \sqrt{\dfrac{\sec x - 1}{\sec x + 1}}$, find $f'\left(\dfrac{\pi}{3}\right)$. [CBSE 2020]

64. If $f(x) = \sqrt{\tan \sqrt{x}}$, then find $f'\left(\dfrac{\pi^2}{16}\right)$. [CBSE 2020]

65. If $y = \dfrac{x}{x+2}$, prove that $x \dfrac{dy}{dx} = (1 - y)\, y$

66. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, prove that $2x \dfrac{dy}{dx} = \sqrt{x} - \dfrac{1}{\sqrt{x}}$

67. If $y = \log \sqrt{\dfrac{1 + \tan x}{1 - \tan x}}$, prove that $\dfrac{dy}{dx} = \sec 2x$. [CBSE 2011]

68. If $y = \dfrac{x \sin^{-1} x}{\sqrt{1 - x^2}}$, prove that $(1 - x^2) \dfrac{dy}{dx} = x + \dfrac{y}{x}$.

69. If $y = x \sin^{-1} x + \sqrt{1 - x^2}$, prove that $\dfrac{dy}{dx} = \sin^{-1} x$.

70. If $y = (1 - x) \log (x - 1) - (x + 1) \log (x + 1)$, prove that $\dfrac{dy}{dx} - \log\left(\dfrac{x - 1}{1 + x}\right)$

71. If $y = \log\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)$, prove that $\dfrac{dy}{dx} = \dfrac{x - 1}{2x\,(x + 1)}$

72. If $y = \sqrt{x + 1} + \sqrt{x - 1}$, prove that $\sqrt{x^2 - 1}\, \dfrac{dy}{dx} = \dfrac{1}{2} y$

73. If $y = \dfrac{1}{2} \log\left(\dfrac{1 - \cos 2x}{1 + \cos 2x}\right)$, prove that $\dfrac{dy}{dx} = 2 \operatorname{cosec} 2x$

74. Prove that $\dfrac{d}{dx}\left\{\dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2} \sin^{-1} \dfrac{x}{a}\right\} = \sqrt{a^2 - x^2}$ [CBSE 2004]

75. If $y = \log\left\{\sqrt{x - 1} - \sqrt{x + 1}\right\}$, show that $\dfrac{dy}{dx} = \dfrac{-1}{2\sqrt{x^2 - 1}}$

ANSWERS

1. $3 \cos (3x + 5)$
2. $2 \tan x \sec^2 x$
3. $\dfrac{\pi}{180} \sec^2 (x^\circ + 45^\circ)$
4. $\dfrac{1}{x} \cos (\log x)$
5. $\dfrac{\cos \sqrt{x}\, e^{\sin \sqrt{x}}}{2\sqrt{x}}$
6. $e^{\tan x} \sec^2 x$
7. $2 \sin (4x + 2)$
8. $\dfrac{2}{(2x - 3) \log_e 7}$
9. $\dfrac{5\pi}{180} \sec^2 (5x^\circ)$
10. $3x^2 \cdot 2^{x^3} \log 2$
11. $3^{e^x} \log 3 \cdot e^x$
12. $-\dfrac{1}{x \log_e 3\, (\log_3 x)^2}$
13. $(3^{x^2 + 2x} \log 3)(2x + 2)$
14. $\dfrac{-2a^2 x}{\sqrt{a^2 - x^2}\, (a^2 + x^2)^{3/2}}$
15. $3^{x \log x} (\log 3)(1 + \log x)$

16. $\sec x(\tan x+\sec x)$

17. $\dfrac{-2x}{\sqrt{1-x^2}\,(1+x^2)^{3/2}}$

18. $2(\log\sin x)\cot x$

19. $\dfrac{1}{\sqrt{1+x}\,(1-x)^{3/2}}$

20. $\dfrac{4x}{(1-x^2)^2}\cos\left(\dfrac{1+x^2}{1-x^2}\right)$

21. $e^{3x}(3\cos 2x-2\sin 2x)$

22. $\cos(\log\sin x)\cdot\cot x$

23. $3\,e^{\tan 3x}\cdot\sec^2 3x$

24. $-\dfrac{1}{2}\,\dfrac{e^{\sqrt{\cot x}}}{\sqrt{\cot x}}\times\operatorname{cosec}^2 x$

25. $\operatorname{cosec} x$

26. $\operatorname{cosec} x$

27. $\sec^2(e^{\sin x})\;e^{\sin x}\cos x$

28. $\dfrac{1}{\sqrt{x^2+1}}$

29. $e^x x^{-2}\left(\log x+\dfrac{1}{x}-\dfrac{2}{x}\log x\right)$

30. $\dfrac{6x\sin x+3x^2\cos x}{\sqrt{7-x^2}}+\dfrac{3x^3\sin x}{(7-x^2)^{3/2}}$

31. $\dfrac{-8}{(e^{2x}-e^{-2x})^2}$

32. $-\dfrac{2(x^2-1)}{x^4+x^2+1}$

33. $\dfrac{e^x}{1+e^{2x}}$

34. $\dfrac{2}{\sqrt{1-4x^2}}\;e^{\sin^{-1}2x}$

35. $\cos(2\sin^{-1}x)\cdot\dfrac{2}{\sqrt{1-x^2}}$

36. $\dfrac{e^{\tan^{-1}\sqrt{x}}}{2\sqrt{x}\,(1+x)}$

37. $\dfrac{1}{(4+x^2)\sqrt{\tan^{-1}\left(\dfrac{x}{2}\right)}}$

38. $\dfrac{1}{(1+x^2)\tan^{-1}x}$

39. $\dfrac{2^x}{(x^2+3)^2}\left\{\cos x\cdot\log_e 2-\sin x-\dfrac{4x\cos x}{x^2+3}\right\}$

40. $\sin\{2\log(2x+3)\}\cdot\left(\dfrac{2}{2x+3}\right)$

41. $2\,e^x\cot 2x+e^x\log\sin 2x$

42. $\dfrac{16x^3(\sin^{-1}x^4)^3}{\sqrt{1-x^8}}$

43. $\dfrac{a}{a^2+x^2}$

44. $\dfrac{e^x\sin x+e^x\cos x}{(x^2+2)^3}-\dfrac{6x\,e^x\sin x}{(x^2+2)^4}$

45. $3\,e^{-3x}\left\{\dfrac{1}{x+1}-3\log(x+1)\right\}$

46. $\dfrac{1}{\sqrt{\cos x}}\left\{2x+\left(\dfrac{x^2+2}{2}\right)\tan x\right\}$

47. $2x(1-x^2)^2\sec 2x\,\{1-4x^2+x(1-x^2)\tan 2x\}$

48. $-\sec x$

49. $e^{ax}\sec x\,\{a\tan 2x+\tan x\tan 2x+2\sec^2 2x\}$

50. $-2x\tan x^2$

51. $\dfrac{-2\log x\sin(\log x)^2}{x}$

52. $\dfrac{1}{x^2-1}$

53. $\operatorname{cosec} x$

54. $\sin 2x+2x\cos 2x+5^x\log 5+6\tan^5 x\sec^2 x$

55. $\dfrac{3}{3x+2}-\dfrac{2x^2}{2x-1}-2x\log(2x-1)$

56. $2x+\dfrac{2x^3}{\sqrt{x^4-1}}$

57. $\dfrac{1}{\sqrt{x^2+4x+1}}$

63. $\dfrac{2}{3}$

64. $\dfrac{2}{\pi}$

10.5 DIFFERENTIATION BY USING TRIGONOMETRICAL SUBSTITUTIONS

Sometimes, it becomes very easy to differentiate a function by using trigonometrical transformations. Usually this is done in case of inverse trigonometrical functions. Some important results on trigonometrical and inverse trigonometrical functions are given below for ready reference.

(i) $\sin 2x = 2\sin x\cos x$

(ii) $1 + \cos 2x = 2\cos^2 x$ or, $\cos 2x = 2\cos^2 x - 1$

(iii) $1 - \cos 2x = 2\sin^2 x$ or, $\cos 2x = 1 - 2\sin^2 x$

(iv) $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$

(v) $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x}$

(vi) $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

(vii) $\sin 3x = 3\sin x - 4\sin^3 x$

(viii) $\cos 3x = 4\cos^3 x - 3\cos x$

(ix) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1-3\tan^2 x}$

(x) $\sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

$$= \begin{cases} \sin^{-1} x + \sin^{-1} y & \text{, if } -1 \le x, y < 1 \text{ and } x^2+y^2 \le 1 \text{ or, if } xy < 0 \text{ and } x^2+y^2 > 1 \\ \pi - (\sin^{-1} x + \sin^{-1} y) & \text{, if } 0 < x, y \le 1 \text{ and } x^2+y^2 > 1 \\ -\pi - (\sin^{-1} x + \sin^{-1} y) & \text{, if } -1 \le x, y < 0 \text{ and } x^2+y^2 > 1 \end{cases}$$

$\sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}$

$$= \begin{cases} \sin^{-1} x - \sin^{-1} y & \text{, if } -1 \le x, y \le 1 \text{ and } x^2+y^2 \le 1 \text{ or, if } xy > 0 \,\&\, x^2+y^2 > 1 \\ \pi - (\sin^{-1} x - \sin^{-1} y) & \text{, if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2+y^2 > 1 \\ -\pi - (\sin^{-1} x - \sin^{-1} y) & \text{, if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2+y^2 \ge 1 \end{cases}$$

(xi) $$\cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \begin{cases} \cos^{-1} x + \cos^{-1} y, & \text{if } -1 \le x, y \le 1 \text{ and } x+y \ge 0 \\ 2\pi - (\cos^{-1} x + \cos^{-1} y), & \text{if } -1 \le x, y \le 1 \text{ and } x+y \le 0 \end{cases}$$

$$\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \begin{cases} \cos^{-1} x - \cos^{-1} y, & \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -(\cos^{-1} x - \cos^{-1} y), & \text{if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

(xii) $$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \begin{cases} \tan^{-1} x + \tan^{-1} y, & \text{if } xy < 1 \\ \pi - (\tan^{-1} x + \tan^{-1} y), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ \pi + (\tan^{-1} x + \tan^{-1} y), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(xiii) $$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \begin{cases} \tan^{-1} x - \tan^{-1} y, & \text{if } xy > -1 \\ \pi - (\tan^{-1} x - \tan^{-1} y), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi - (\tan^{-1} x - \tan^{-1} y), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

(xiv) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, if $-1 \le x \le 1$ (xv) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ for all $x \in R$

(xvi) $\sec^{-1} x + \text{cosec}^{-1} x = \frac{\pi}{2}$, if $x \in (-\infty, -1] \cup [1, \infty)$ (xvii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$ for $x \in [-1, 1]$

(xviii) $\tan^{-1}(-x) = -\tan^{-1} x$ for $x \in R$ (xix) $\sin^{-1}(-x) = -\sin^{-1} x$ for $x \in [-1,1]$

(xx) $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$, if $x \in (-\infty, -1] \cup [1, \infty)$

(xxi) $\cos^{-1} x = \sec^{-1} \frac{1}{x}$, if $x \in (-\infty, -1] \cup [1, \infty)$

(xxii) $\tan^{-1} x = \begin{cases} \cot^{-1}\left(\frac{1}{x}\right), & \text{if } x > 0 \\ -\pi + \cot^{-1}\left(\frac{1}{x}\right), & \text{if } x < 0 \end{cases}$ or, $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, x > 0 \\ -\pi + \cot^{-1} x, x < 0 \end{cases}$

(xxiii) $\sin^{-1}(\sin \theta) = \theta$, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$; $\cos^{-1}(\cos \theta) = \theta$, if $0 \le \theta \le \pi$

$\tan^{-1}(\tan \theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$; $\text{cosec}^{-1}(\text{cosec}\, \theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}, \theta \ne 0$

$\sec^{-1}(\sec \theta) = \theta$, if $0 < \theta < \pi, \theta \ne \frac{\pi}{2}$; $\cot^{-1}(\cot \theta) = \theta$, if $0 < \theta < \pi$

Following are some substitutions useful in finding derivatives:

Expression	*Substitution*	*Expression*	*Substitution*
(i) $a^2 + x^2$	$x = a \tan \theta$ or, $a \cot \theta$	(iv) $\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(ii) $a^2 - x^2$	$x = a \sin \theta$ or, $a \cos \theta$	(v) $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or, $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2 = a^2 \cos 2\theta$
(iii) $x^2 - a^2$	$x = a \sec \theta$ or, $a\, \text{cosec}\, \theta$		

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\sin^{-1}(\sin x), x \in [0, 2\pi]$ (ii) $\cos^{-1}(\cos x), x \in [0, 2\pi]$

(iii) $\tan^{-1}(\tan x), x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

SOLUTION (i) Let $y = \sin^{-1}(\sin x)$. Then,

$$y = \sin^{-1}(\sin x) = \begin{cases} x, & \text{if } x \in \left[0, \frac{\pi}{2}\right] \\ \pi - x, & \text{if } x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi + x, & \text{if } x \in \left[\frac{3\pi}{2}, 2\pi\right] \end{cases}$$

We observe that

$$\left(\text{LHD of } y \text{ at } x = \frac{\pi}{2}\right) = 1 \text{ and}, \left(\text{RHD of } y \text{ at } x = \frac{\pi}{2}\right) = -1$$

$$\left(\text{LHD of } y \text{ at } x = \frac{3\pi}{2}\right) = -1 \text{ and}, \left(\text{RHD of } y \text{ at } x = \frac{3\pi}{2}\right) = 1$$

So, $y = \sin^{-1}(\sin x)$ is not differentiable at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

$$\therefore \quad \frac{dy}{dx} = \begin{cases} 1 \ , & \text{if } x \in \left[0, \frac{\pi}{2}\right) \\ -1 \ , & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ 1 \ , & \text{if } x \in \left(\frac{3\pi}{2}, 2\pi\right] \end{cases}$$

(ii) Let $y = \cos^{-1}(\cos x)$. Then,

$$y = \cos^{-1}(\cos x) = \begin{cases} -x \ , & \text{if } x \in [-\pi, 0] \\ x \ , & \text{if } x \in [0, \pi] \\ 2\pi - x \ , & \text{if } x \in [\pi, 2\pi] \\ -2\pi + x \ , & \text{if } x \in [2\pi, 3\pi] \text{ and so on.} \end{cases}$$

Clearly,

(LHD at $x = 0$) $= -1$ and (RHD at $x = 0$) $= 1$; (LHD at $x = \pi$) $= 1$ and (RHD at $x = \pi$) $= -1$

(LHD at $x = 2\pi$) $= -1$ and (RHD at $x = 2\pi$) $= 1$

So, $y = \cos^{-1}(\cos x)$ is not differentiable at $x = 0, \pi, 2\pi$. Hence, $\frac{dy}{dx} = \begin{cases} 1 \ , & \text{if } x \in (0, \pi) \\ -1 \ , & \text{if } x \in (\pi, 2\pi) \end{cases}$

(iii) Let $y = \tan^{-1}(\tan x)$. Then,

$$y = \tan^{-1}(\tan x) = \begin{cases} x \ , & \text{if } x \in \left[0, \frac{\pi}{2}\right) \\ x - \pi \ , & \text{if } x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ x - 2\pi \ , & \text{if } x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \end{cases} \quad \text{and so on}$$

$$\therefore \quad \frac{dy}{dx} = 1, \text{ if } x \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$$

EXAMPLE 2 *Differentiate* $\sin^{-1}\left(2x\sqrt{1 - x^2}\right)$ *with respect to x, if*

(i) $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ (ii) $\frac{1}{\sqrt{2}} < x < 1$ (iii) $-1 < x < -\frac{1}{\sqrt{2}}$

SOLUTION Let $y=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$. Putting $x=\sin\theta$, we get

$$y = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$$

(i) If $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$, then

$$x = \sin\theta \Rightarrow -\frac{1}{\sqrt{2}}<\sin\theta<\frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4}<\theta<\frac{\pi}{4} \Rightarrow -\frac{\pi}{2}<2\theta<\frac{\pi}{2}$$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) = 2\theta \qquad \left[\because -\frac{\pi}{2}<2\theta<\frac{\pi}{2}\right]$

$\Rightarrow \quad y = 2\sin^{-1}x \qquad [\because \sin\theta = x \Rightarrow \theta=\sin^{-1}x]$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

(ii) If $\frac{1}{\sqrt{2}}<x<1$, then

$$x = \sin\theta \Rightarrow \frac{1}{\sqrt{2}}<\sin\theta<1 \Rightarrow \frac{\pi}{4}<\theta<\frac{\pi}{2} \Rightarrow \frac{\pi}{2}<2\theta<\pi$$

$\therefore \quad y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow \quad y = \sin^{-1}(\sin(\pi-2\theta)) = \pi-2\theta \qquad \left[\because \frac{\pi}{2}<2\theta<\pi \Rightarrow 0<\pi-2\theta<\frac{\pi}{2}\right]$

$$\Rightarrow \quad y = \pi-2\sin^{-1}x \Rightarrow \frac{dy}{dx} = 0-\frac{2}{\sqrt{1-x^2}} = -\frac{2}{\sqrt{1-x^2}}$$

(iii) If $-1<x<-\frac{1}{\sqrt{2}}$, then

$$x = \sin\theta \Rightarrow -1<\sin\theta<-\frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{2}<\theta<-\frac{\pi}{4} \Rightarrow -\pi<2\theta<-\frac{\pi}{2}$$

$\therefore \quad y = \sin^{-1}(\sin 2\theta)$

$\Rightarrow \quad y = \sin^{-1}(-\sin(\pi+2\theta))$

$\Rightarrow \quad y = \sin^{-1}(\sin(-\pi-2\theta)) = -\pi-2\theta \qquad \left[\because -\pi<2\theta<-\frac{\pi}{2} \Rightarrow -\frac{\pi}{2}<\pi+2\theta<0\right]$

$$\Rightarrow \quad y = -\pi-2\sin^{-1}x \Rightarrow \frac{dy}{dx} = -0-\frac{2}{\sqrt{1-x^2}} = -\frac{2}{\sqrt{1-x^2}}.$$

EXAMPLE 3 *Differentiate* $\sin^{-1}(3x-4x^3)$ *with respect to x, if*

(i) $-\frac{1}{2}<x<\frac{1}{2}$ (ii) $\frac{1}{2}<x<1$ (iii) $-1<x<-\frac{1}{2}$

SOLUTION Let $y=\sin^{-1}(3x-4x^3)$. Putting $x=\sin\theta$, we get

$$y = \sin^{-1}(3\sin\theta-4\sin^3\theta) = \sin^{-1}(\sin 3\theta)$$

(i) If $-\frac{1}{2}<x<\frac{1}{2}$, then

$$x = \sin\theta \Rightarrow -\frac{1}{2}<\sin\theta<\frac{1}{2} \Rightarrow -\frac{\pi}{6}<\theta<\frac{\pi}{6} \Rightarrow -\frac{\pi}{2}<3\theta<\frac{\pi}{2}$$

$\therefore \quad y = \sin^{-1}(\sin 3\theta) = 3\theta \qquad \left[\because -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = 3\sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}}$

(ii) If $\frac{1}{2} < x < 1$, then

$x = \sin\theta \Rightarrow \frac{1}{2} < \sin\theta < 1 \Rightarrow \frac{\pi}{6} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3\theta < \frac{3\pi}{2}$

$\therefore \quad y = \sin^{-1}(\sin 3\theta)$

$\Rightarrow \quad y = \sin^{-1}\{\sin(\pi - 3\theta)\} = \pi - 3\theta \qquad \left[\because \frac{\pi}{2} < 3\theta < \frac{3\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi - 3\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = \pi - 3\sin^{-1} x \qquad [\because x = \sin\theta \Rightarrow \theta = \sin^{-1} x]$

$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{3}{\sqrt{1-x^2}} = -\frac{3}{\sqrt{1-x^2}}$

(iii) If $-1 < x < -\frac{1}{2}$, then

$x = \sin\theta \Rightarrow -1 < \sin\theta < -\frac{1}{2} \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2}$

$\therefore \quad y = \sin^{-1}(\sin 3\theta)$

$\Rightarrow \quad y = \sin^{-1}\{\sin(-\pi - 3\theta)\} = -\pi - 3\theta \qquad \left[\because -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi - 3\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = -\pi - 3\sin^{-1} x \Rightarrow \frac{dy}{dx} = -0 - \frac{3}{\sqrt{1-x^2}} = -\frac{3}{\sqrt{1-x^2}}$

EXAMPLE 4 *Differentiate* $\cos^{-1}(2x^2 - 1)$ *with respect to x, if* (i) $0 < x < 1$ (ii) $-1 < x < 0$

SOLUTION Let $y = \cos^{-1}(2x^2 - 1)$. Putting $x = \cos\theta$, we get

$y = \cos^{-1}(2\cos^2\theta - 1) = \cos^{-1}(\cos 2\theta)$

(i) If $0 < x < 1$, then

$x = \cos\theta \Rightarrow 0 < \cos\theta < 1 \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$

$\therefore \quad y = \cos^{-1}(\cos 2\theta) = 2\theta \qquad [\because 0 < 2\theta < \pi$

$\Rightarrow \quad y = 2\cos^{-1} x \qquad [\because x = \cos\theta \Rightarrow \theta = \cos^{-1} x]$

$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

(ii) If $-1 < x < 0$, then

$x = \cos\theta \Rightarrow -1 < \cos\theta < 0 \Rightarrow \frac{\pi}{2} < \theta < \pi \Rightarrow \pi < 2\theta < 2\pi$

$\therefore \quad y = \cos^{-1}(\cos 2\theta)$

$\Rightarrow \quad y = \cos^{-1}\{\cos(2\pi - 2\theta)\} = 2\pi - 2\theta \qquad [\because \pi < 2\theta < 2\pi \Rightarrow 0 < 2\pi - 2\theta < \pi]$

$\Rightarrow \quad y = 2\pi - 2\cos^{-1} x \Rightarrow \frac{dy}{dx} = 0 + \frac{2}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}$

EXAMPLE 5 *Differentiate* $\cos^{-1}(1-2x^2)$ *with respect to x, if*

(i) $0 < x < 1$ (ii) $-1 < x < 0$

SOLUTION Let $y = \cos^{-1}(1-2x^2)$. Putting $x = \sin\theta$, we get

$$y = \cos^{-1}(1-2\sin^2\theta) = \cos^{-1}(\cos 2\theta)$$

(i) If $0 < x < 1$, then

$$x = \sin\theta \Rightarrow 0 < \sin\theta < 1 \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$$

$$\therefore \quad y = \cos^{-1}(\cos 2\theta) = 2\theta = 2\sin^{-1}x \qquad [\because x = \sin\theta \Rightarrow \theta = \sin^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

(ii) If $-1 < x < 0$, then

$$x = \sin\theta \Rightarrow -1 < \sin\theta < 0 \Rightarrow -\frac{\pi}{2} < \theta < 0 \Rightarrow -\pi < 2\theta < 0$$

$$\therefore \quad y = \cos^{-1}(\cos 2\theta)$$

$$\Rightarrow \quad y = \cos^{-1}(\cos(-2\theta)) = -2\theta = -2\sin^{-1}x \qquad [\because -\pi < 2\theta < 0 \Rightarrow 0 < -2\theta < \pi]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

EXAMPLE 6 *Differentiate* $\cos^{-1}(4x^3-3x)$ *with respect to x, if*

(i) $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ (ii) $x \in \left(\frac{1}{2}, 1\right)$ (iii) $x \in \left(-1, -\frac{1}{2}\right)$

SOLUTION Let $y = \cos^{-1}(4x^3-3x)$. Putting $x = \cos\theta$, we get

$$y = \cos^{-1}(4\cos^3\theta - 3\cos\theta) = \cos^{-1}(\cos 3\theta)$$

(i) If $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, then

$$x = \cos\theta \Rightarrow -\frac{1}{2} < \cos\theta < \frac{1}{2} \Rightarrow \frac{\pi}{3} < \theta < \frac{2\pi}{3} \Rightarrow \pi < 3\theta < 2\pi$$

$$\therefore \quad y = \cos^{-1}(\cos 3\theta)$$

$$\Rightarrow \quad y = \cos^{-1}\{\cos(2\pi - 3\theta)\} = 2\pi - 3\theta \qquad [\because \pi < 3\theta < 2\pi \Rightarrow 0 < 2\pi - 3\theta < \pi]$$

$$\Rightarrow \quad y = 2\pi - 3\cos^{-1}x \qquad [\because x = \cos\theta \Rightarrow \theta = \cos^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - 3 \times -\frac{1}{\sqrt{1-x^2}} = \frac{3}{\sqrt{1-x^2}}$$

(ii) If $x \in \left(\frac{1}{2}, 1\right)$, then

$$x = \cos\theta \Rightarrow \frac{1}{2} < \cos\theta < 1 \Rightarrow 0 < \theta < \frac{\pi}{3} \Rightarrow 0 < 3\theta < \pi$$

$$\therefore \quad y = \cos^{-1}(\cos 3\theta) = 3\theta \qquad [\because 0 < 3\theta < \pi]$$

$$\Rightarrow \quad y = 3\cos^{-1}x \qquad [\because x = \cos\theta \Rightarrow \theta = \cos^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{3}{\sqrt{1-x^2}}$$

(iii) If $-1 < x < -\frac{1}{2}$, then

$$x = \cos\theta \Rightarrow -1 < \cos\theta < -\frac{1}{2} \Rightarrow \frac{2\pi}{3} < \theta < \pi \Rightarrow 2\pi < 3\theta < 3\pi$$

$$\therefore \quad y = \cos^{-1}(\cos 3\theta)$$

$$\Rightarrow \quad y = \cos^{-1}\{\cos(2\pi - 3\theta)\}$$

$$\Rightarrow \quad y = \cos^{-1}\{\cos(3\theta - 2\pi)\} = 3\theta - 2\pi \qquad [\because 2\pi < 3\theta < 3\pi \Rightarrow 0 < 3\theta - 2\pi < \pi]$$

$$\Rightarrow \quad y = 3\cos^{-1}x - 2\pi \Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} - 0 = \frac{-3}{\sqrt{1-x^2}}$$

EXAMPLE 7 *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to x, if*

(i) $x \in (-1, 1)$ (ii) $x \in (-\infty, -1)$ (iii) $x \in (1, \infty)$.

SOLUTION Let $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$. Putting $x = \tan\theta$, we get : $y = \tan^{-1}(\tan 2\theta)$

(i) If $-1 < x < 1$, then

$$x = \tan\theta \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

$$\therefore \quad y = \tan^{-1}(\tan 2\theta) = 2\theta \qquad \left[\because -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = 2\tan^{-1}x \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+x^2}$$

(ii) If $-\infty < x < -1$, then

$$x = \tan\theta \Rightarrow -\infty < \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2}$$

$$\therefore \quad y = \tan^{-1}(\tan 2\theta)$$

$$\Rightarrow \quad y = \tan^{-1}\{\tan(\pi + 2\theta)\} = \pi + 2\theta \qquad \left[\because -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi + 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \pi + 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = 0 + \frac{2}{1+x^2} = \frac{2}{1+x^2}$$

(iii) If $x \in (1, \infty)$, then

$$x = \tan\theta \Rightarrow 1 < \tan\theta < \infty \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$$\therefore \quad y = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{-\tan(\pi - 2\theta)\} = \tan^{-1}\{\tan(2\theta - \pi)\}$$

$$\Rightarrow \quad y = 2\theta - \pi \qquad \left[\because \frac{\pi}{2} < 2\theta < \pi \Rightarrow -\frac{\pi}{2} < 2\theta - \pi < 0\right]$$

$$\Rightarrow \quad y = 2\tan^{-1}x - \pi \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2} - 0 = \frac{2}{1+x^2}$$

EXAMPLE 8 *Differentiate* $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, *if*

(i) $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$ [NCERT] (ii) $x > \frac{1}{\sqrt{3}}$ (iii) $x < -\frac{1}{\sqrt{3}}$

SOLUTION Let $y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$. Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

(i) If $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, then

$$x = \tan\theta \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$$

$\therefore \quad y = \tan^{-1}(\tan 3\theta) = 3\theta \qquad \left[\because -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = 3\tan^{-1}x \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{1 + x^2}$

(ii) If $x > \frac{1}{\sqrt{3}}$, then

$$x = \tan\theta \Rightarrow \tan\theta > \frac{1}{\sqrt{3}} \Rightarrow \frac{\pi}{6} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3\theta < \frac{3\pi}{2}$$

$\therefore \quad y = \tan^{-1}(\tan 3\theta) = \tan^{-1}\{-\tan(\pi - 3\theta)\} = \tan^{-1}\{\tan(3\theta - \pi)\}$

$\Rightarrow \quad y = 3\theta - \pi \qquad \left[\because \frac{\pi}{2} < 3\theta < \frac{3\pi}{2} \Rightarrow -\frac{\pi}{2} < 3\theta - \pi < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = 3\tan^{-1}x - \pi \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{1 + x^2} - 0 = \frac{3}{1 + x^2}$

(iii) If $x < -\frac{1}{\sqrt{3}}$, then

$$x = \tan\theta \Rightarrow -\infty < \tan\theta < -\frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2}$$

$\therefore \quad y = \tan^{-1}(\tan 3\theta) = \tan^{-1}\{\tan(\pi + 3\theta)\}$

$\Rightarrow \quad y = \pi + 3\theta = \pi + 3\tan^{-1}x \qquad \left[\because -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi + 3\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad \frac{dy}{dx} = 0 + \frac{3}{1 + x^2} = \frac{3}{1 + x^2}$

EXAMPLE 9 *Differentiate* $\sin^{-1}\left(\frac{2x}{1 + x^2}\right)$ *with respect to x, when*

(i) $x \in (-1, 1)$ (ii) $x \in (1, \infty)$ (iii) $x \in (-\infty, -1)$

SOLUTION Let $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$. Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

(i) If $x \in (-1, 1)$, then

$$x = \tan\theta \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

$$\therefore \quad y = \sin^{-1}(\sin 2\theta) = 2\theta \qquad \left[\because -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = 2\tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{2}{1+x^2}$$

(ii) If $x \in (1, \infty)$, then

$$x = \tan\theta \Rightarrow 1 < \tan\theta < \infty \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$$\therefore \quad y = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{\sin(\pi - 2\theta)\} \qquad [\because \sin(\pi - 2\theta) = \sin 2\theta]$$

$$\Rightarrow \quad y = \pi - 2\theta \qquad \left[\because \frac{\pi}{2} < 2\theta < \pi \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \pi - 2\tan^{-1}x \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

(iii) If $x \in (-\infty, -1)$, then

$$x = \tan\theta \Rightarrow -\infty < \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2}$$

$$\therefore \quad y = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{-\sin(\pi + 2\theta)\} = \sin^{-1}\{\sin(-\pi - 2\theta)\}$$

$$\Rightarrow \quad y = -\pi - 2\theta \qquad \left[\because -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi - 2\theta < 0\right]$$

$$\Rightarrow \quad y = -\pi - 2\tan^{-1}x \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1}x]$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$$

EXAMPLE 10 *Differentiate* $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *with respect to x, when*

(i) $x \in (0, \infty)$ (ii) $x \in (-\infty, 0)$

SOLUTION Let $y = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

(i) If $x \in (0, \infty)$, then

$$x = \tan\theta \Rightarrow 0 < \tan\theta < \infty \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$$

$\therefore \quad y = \cos^{-1}(\cos 2\theta) = 2\theta \qquad [\because\ 0 < 2\theta < \pi]$

$\Rightarrow \quad y = 2\tan^{-1} x \qquad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

(ii) If $x \in (-\infty, 0)$, then

$$x = \tan\theta \Rightarrow -\infty < \tan\theta < 0 \Rightarrow -\frac{\pi}{2} < \theta < 0 \Rightarrow -\pi < 2\theta < 0$$

$\therefore \quad y = \cos^{-1}(\cos 2\theta) = \cos^{-1}\{\cos(-2\theta)\} = -2\theta \qquad [\because\ -\pi < 2\theta < 0 \Rightarrow 0 < -2\theta < \pi]$

$\Rightarrow \quad y = -2\tan^{-1} x \Rightarrow \frac{dy}{dx} = -\frac{2}{1+x^2}$

EXAMPLE 11 *Differentiate each of the following functions with respect to x*

(i) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$ **[NCERT]** (ii) $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$ **[NCERT]**

(iii) $\cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$ **[NCERT]** (iv) $\sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$ **[NCERT]**

SOLUTION (i) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $0 < x < 1$. Putting $x = \tan\theta$, we obtain

$$y = \cos^{-1}\left(\frac{1-\tan^2}{1+\tan^2\theta}\right) = \cos^{-1}(\cos 2\theta)$$

$\Rightarrow \quad y = 2\theta \qquad \left[\because\ 0 < x < 1 \Rightarrow 0 < \tan\theta < 1 \Rightarrow 0 < \theta < \frac{\pi}{4} \Rightarrow 0 < 2\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = 2\tan^{-1} x \qquad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+x^2}$

(ii) Let $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, where $0 < x < 1$. Putting $x = \tan\theta$, we get

$$y = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\cos 2\theta) = \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$\Rightarrow \quad y = \frac{\pi}{2} - 2\theta \qquad \left[\because 0 < x < 1 \Rightarrow 0 < \tan\theta < 1 \Rightarrow 0 < \theta < \frac{\pi}{4} \Rightarrow 0 < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - 2\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1} x \qquad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$

$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{1+x^2} = -\frac{2}{1+x^2}$

(iii) Let $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, where $-1 < x < 1$. Putting $x = \tan\theta$, we get

$$y = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos\left(\frac{\pi}{2}-2\theta\right)\right\}$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\theta \left[\because -1 < x < 1 \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - 2\theta < \pi\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{-2}{1+x^2}$$

(iv) Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$, where $0 < x < \frac{1}{\sqrt{2}}$. Putting $x = \cos\theta$, we obtain

$$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right) = \cos^{-1}(2\cos^2\theta - 1) \qquad \left[\because \sec^{-1}\frac{1}{x} = \cos^{-1} x\right]$$

$$\Rightarrow \quad y = \cos^{-1}(\cos 2\theta)$$

$$\Rightarrow \quad y = 2\theta \qquad \left[\because 0 < x < \frac{1}{\sqrt{2}} \Rightarrow 0 < \cos\theta < \frac{1}{\sqrt{2}} \Rightarrow 0 < \theta < \frac{\pi}{4} \Rightarrow 0 < 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = 2\cos^{-1} x \qquad [\because x = \cos\theta \Rightarrow \theta = \cos^{-1} x]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

EXAMPLE 12 *Differentiate each of the following functions with respect to x:*

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ (ii) $\cos^{-1}\left(2x\sqrt{1-x^2}\right), -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

SOLUTION (i) Let $y = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$, where $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Putting $x = \sin\theta$, we get

$$y = \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow \quad y = 2\theta \left[\because -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} < \sin\theta < \frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = 2\sin^{-1} x \qquad [\because x = \sin\theta \Rightarrow \theta = \sin^{-1} x]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

(ii) Let $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$, where $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. Putting $x = \sin\theta$, we get

$$y = \cos^{-1}(2\sin\theta\cos\theta) = \cos^{-1}(\sin 2\theta) = \cos^{-1}\left\{\cos\left(\frac{\pi}{2} - 2\theta\right)\right\}$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\theta \qquad \left[\begin{array}{l}\because -\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \Rightarrow -\frac{1}{\sqrt{2}} < \sin\theta < \frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \\ \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - 2\theta < \pi\end{array}\right]$$

$$\Rightarrow \quad y = \frac{\pi}{2} - 2\sin^{-1} x \qquad [\because x = \sin\theta \Rightarrow \theta = \sin^{-1} x]$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - \frac{2}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$$

EXAMPLE 13 *Differentiate the following functions with respect to x:*

(i) $\tan^{-1}\left\{\frac{1-\cos x}{\sin x}\right\}, -\pi < x < \pi$ (ii) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, -\pi < x < \pi$

[NCERT EXEMPLAR]

(iii) $\tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\}, 0 < x < \pi$ (iv) $\tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\}, 0 < x < \pi$

(v) $\tan^{-1}\left\{\sqrt{\frac{1+\sin x}{1-\sin x}}\right\}, -\frac{\pi}{2} < x < \frac{\pi}{2}$ (vi) $\tan^{-1}(\sec x + \tan x), -\frac{\pi}{2} < x < \frac{\pi}{2}$

[NCERT EXEMPLAR]

SOLUTION (i) Let $y = \tan^{-1}\left\{\frac{1-\cos x}{\sin x}\right\}$. Then,

$$y = \tan^{-1}\left\{\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right\} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2} \qquad \left[\because -\pi < x < \pi \Rightarrow -\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}$$

(ii) Let $y = \tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}$. Then,

$$y = \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x/2}{2\cos^2 x/2}}\right\} = \tan^{-1}\left|\tan\frac{x}{2}\right| = \begin{cases} \tan^{-1}\left(\tan\frac{x}{2}\right), & \text{if } \tan\frac{x}{2} \ge 0 \\ \tan^{-1}\left(-\tan\frac{x}{2}\right), & \text{if } \tan\frac{x}{2} < 0 \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} \frac{x}{2}, & \text{if } 0 \le x < \pi \\ -\frac{x}{2}, & \text{if } -\pi < x < 0 \end{cases} \Rightarrow \frac{dy}{dx} = \begin{cases} \frac{1}{2}, & \text{if } 0 < x < \pi \\ -\frac{1}{2}, & \text{if } -\pi < x < 0 \end{cases}$$

(iii) Let $y = \tan^{-1}\left\{\sqrt{\frac{1+\cos x}{1-\cos x}}\right\}$. Then,

$$y = \tan^{-1}\left\{\sqrt{\frac{2\cos^2 x/2}{2\sin^2 x/2}}\right\} = \tan^{-1}\left|\cot\frac{x}{2}\right| = \tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \frac{x}{2}\right)\right\} = \frac{\pi}{2} - \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0 - \frac{1}{2} = -\frac{1}{2}.$$

(iv) Let $y = \tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$. Then,

$$y = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\} = \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = 0-\frac{1}{2} = -\frac{1}{2}.$$

(v) Let $y = \tan^{-1}\left\{\sqrt{\dfrac{1+\sin x}{1-\sin x}}\right\}$. Then,

$$\Rightarrow \quad y = \tan^{-1}\left\{\sqrt{\frac{1-\cos(\pi/2+x)}{1+\cos(\pi/2+x)}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\sin^2(\pi/4+x/2)}{2\cos^2(\pi/4+x/2)}}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2} \qquad \left[\because -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow 0<\frac{x}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}$$

(vi) Let $y = \tan^{-1}(\sec x + \tan x)$. Then,

$$y = \tan^{-1}\left\{\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right\} = \tan^{-1}\left\{\frac{1-\cos\left(\frac{\pi}{2}+x\right)}{\sin\left(\frac{\pi}{2}+x\right)}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 14 *Differentiate the following functions with respect to x:*

(i) $\tan^{-1}\left\{\sqrt{1+x^2}+x\right\}$, $x \in R$

(ii) $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\}$, $x \in R$

(iii) $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-1}{x}\right\}$, $x \neq 0$ [CBSE 2004, 2012]

(iv) $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}+1}{x}\right\}$, $x \neq 0$

(v) $\cot^{-1}\left\{\sqrt{1+x^2}+x\right\}$ (vi) $\tan^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}, 0<x<\pi.$

[CBSE 2004]

SOLUTION (i) Let $y=\tan^{-1}(\sqrt{1+x^2}+x)$. Putting $x=\cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta+\cot\theta) = \tan^{-1}\left(\frac{1}{\sin\theta}+\frac{\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{2}-\frac{1}{2}\theta$$

$$\Rightarrow \quad y = \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x \qquad [\because x=\cot\theta \;\therefore\; \theta=\cot^{-1}x]$$

$$\therefore \quad \frac{dy}{dx} = 0-\frac{1}{2}\left(-\frac{1}{1+x^2}\right) = \frac{1}{2(1+x^2)}$$

(ii) Let $y=\tan^{-1}(\sqrt{1+x^2}-x)$. Putting $x=\cot\theta$, we get

$$y = \tan^{-1}(\operatorname{cosec}\theta-\cot\theta) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(-\frac{1}{1+x^2}\right) = -\frac{1}{2(1+x^2)}$$

(iii) Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$. Putting $x=\tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\Rightarrow \quad y = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{1+x^2}\right)$$

(iv) Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+1}{x}\right)$. Putting $x=\tan\theta$, we get

$$y = \tan^{-1}\left(\frac{\sec\theta+1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{2}-\frac{1}{2}\theta = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = 0-\frac{1}{2}\times\frac{1}{1+x^2} = -\frac{1}{2(1+x^2)}$$

(v) Let $y = \cot^{-1}(\sqrt{1+x^2}+x)$. Putting $x = \cot\theta$, we get

$$y = \cot^{-1}(\operatorname{cosec}\theta+\cot\theta) = \cot^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right) = \cot^{-1}\left(\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \cot^{-1}\left(\cot\frac{\theta}{2}\right)$$

$$\Rightarrow \quad y = \frac{1}{2}\theta = \frac{1}{2}\cot^{-1}x$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2(1+x^2)}$$

(vi) Let $y = \tan^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$. We know that:

$$\sqrt{1+\sin x} = \sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}} = \sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} = \left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|$$

$$\Rightarrow \quad \sqrt{1+\sin x} = \cos\frac{x}{2}+\sin\frac{x}{2}, \text{ for } 0 < x < \pi.$$

and, $$\sqrt{1-\sin x} = \sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}} = \sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|$$

$$\Rightarrow \quad \sqrt{1-\sin x} = \begin{cases} \cos\frac{x}{2}-\sin\frac{x}{2}, & \text{if } 0 < x < \frac{\pi}{2} \\ -\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right), & \text{if } \frac{\pi}{2} < x < \pi \end{cases}$$

Thus, we have following cases:

Case I *When* $0 < x < \frac{\pi}{2}$: In this case, we obtain

$$y = \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} = \tan^{-1}\left(\cot\frac{x}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\}$$

$$\Rightarrow y = \frac{\pi}{2}-\frac{x}{2} \qquad \left[\because\ 0 < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{4} < \frac{\pi}{2}-\frac{x}{2} < \frac{\pi}{2}\right]$$

$$\therefore \frac{dy}{dx} = 0-\frac{1}{2} = -\frac{1}{2}$$

Case II *When* $\frac{\pi}{2} < x < \pi$: In this case, we obtain

$$y = \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)}\right\} = \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}.$$

EXAMPLE 15 *Differentiate the following functions with respect to x:*

(i) $\tan^{-1}\left(\dfrac{a+x}{1-ax}\right)$ (ii) $\tan^{-1}\sqrt{\dfrac{a-x}{a+x}}, -a<x<a$

(iii) $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), -\dfrac{1}{\sqrt{3}}<\dfrac{x}{a}<\dfrac{1}{\sqrt{3}}$ **[NCERT EXEMPLAR]**

(iv) $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ *and* $\dfrac{a}{b}\tan x>-1$ **[NCERT EXEMPLAR]**

SOLUTION (i) $\dfrac{d}{dx}\left\{\tan^{-1}\left(\dfrac{a+x}{1-ax}\right)\right\} = \dfrac{d}{dx}\{\tan^{-1}a+\tan^{-1}x\} = \dfrac{d}{dx}(\tan^{-1}a)+\dfrac{d}{dx}(\tan^{-1}x)$

$$= 0+\frac{1}{1+x^2} = \frac{1}{1+x^2}$$

(ii) Let $y=\tan^{-1}\left\{\sqrt{\dfrac{a-x}{a+x}}\right\}$, where $-a<x<a$. Substituting $x=a\cos\theta$, we get

$$y = \tan^{-1}\left\{\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}\right\} = \tan^{-1}\left\{\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}\right\} = \tan^{-1}\left|\tan\frac{\theta}{2}\right|$$

Now,

$$-a<x<a \text{ and } x=a\cos\theta \Rightarrow -a<a\cos\theta<a \Rightarrow -1<\cos\theta<1 \Rightarrow \theta\in(0,\pi) \Rightarrow \frac{\theta}{2}\in\left(0,\frac{\pi}{2}\right)$$

$$\therefore \quad y = \tan^{-1}\left|\tan\frac{\theta}{2}\right| = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{2}\times\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\times\frac{d}{dx}\left(\frac{x}{a}\right) = -\frac{1}{2\sqrt{a^2-x^2}}$$

(iii) Let $y=\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right)$. Then,

$$y = \tan^{-1}\left(\frac{\frac{3x}{a}-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right) \qquad \text{[Dividing numerator and denominator by } a^3\text{]}$$

Putting $\frac{x}{a} = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$\Rightarrow \quad y = 3\theta \qquad \left[\because -\frac{1}{\sqrt{3}} < \frac{x}{a} < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = 3\tan^{-1}\frac{x}{a}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3}{1 + \frac{x^2}{a^2}} \times \frac{d}{dx}\left(\frac{x}{a}\right) = \frac{3a^2}{a^2 + x^2} \times \frac{1}{a} = \frac{3a}{a^2 + x^2}$$

(iv) Let $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$. Then,

$$y = \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right) \qquad \text{[Dividing numerator and denominator by } b\cos x]$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{a}{b}\right) - \tan^{-1}(\tan x) = \tan^{-1}\left(\frac{a}{b}\right) - x \qquad \left[\because -\frac{\pi}{2} < x < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 - 1 = -1$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 16 *If* $y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}$ *and* $0 < x < 1$, *then find* $\frac{dy}{dx}$.

[CBSE 2010]

SOLUTION We have,

$$y = \sin^{-1}\left\{x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right\}, \text{ where } 0 < x < 1$$

$$\Rightarrow \quad y = \sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right\}$$

$$\Rightarrow \quad y = \sin^{-1}x - \sin^{-1}\sqrt{x} \qquad \left[\text{Using}: \sin^{-1}x - \sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}\right]$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

EXAMPLE 17 *If* $y = \cos^{-1}\left\{x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right\}$ *and* $0 < x < 1$, *find* $\frac{dy}{dx}$.

SOLUTION We have,

$$y = \cos^{-1}\left\{x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2}\right\} = \cos^{-1}\left\{x\sqrt{1-(\sqrt{x})^2} + \sqrt{x}\sqrt{1-x^2}\right\}$$

$$\Rightarrow \quad y=\frac{\pi}{2}-\sin^{-1}\left\{x\sqrt{1-(\sqrt{x})^2}+\sqrt{x}\sqrt{1-x^2}\right\}=\frac{\pi}{2}-\left\{\sin^{-1}x+\sin^{-1}\sqrt{x}\right\}$$

$$\Rightarrow \quad y=\left(\frac{\pi}{2}-\sin^{-1}x\right)-\sin^{-1}\sqrt{x} = \cos^{-1}x-\sin^{-1}\sqrt{x}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x})=-\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x}\sqrt{1-x^2}}$$

EXAMPLE 18 *If* $y=\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\}, -1<x<1, x\neq 0$ *find* $\dfrac{dy}{dx}$.

[CBSE 2015]

SOLUTION Putting $x^2=\cos 2\theta$, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right) = \tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$\Rightarrow \quad y = \tan^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\theta \quad \left[\because 0<x^2<1 \Rightarrow 0<\cos 2\theta<1 \Rightarrow 0<2\theta<\frac{\pi}{2} \Rightarrow 0<\theta<\frac{\pi}{4} \Rightarrow \frac{\pi}{4}<\frac{\pi}{4}+\theta<\frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2 \quad \left[\because \cos 2\theta = x^2 \Rightarrow \theta=\frac{1}{2}\cos^{-1}x^2\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}\right)+\frac{1}{2}\frac{d}{dx}(\cos^{-1}x^2) = 0+\frac{1}{2}\frac{(-1)}{\sqrt{1-x^4}}\frac{d}{dx}(x^2) = -\frac{1}{2}\times\frac{2x}{\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

EXAMPLE 19 *Differentiate* $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. *with respect to x, if*

(i) $x\in(0,1)$ (ii) $x\in(-1,0)$ (iii) $x\in(1,\infty)$ (iv) $x\in(-\infty,-1)$

SOLUTION Let $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. Putting $x=\tan\theta$, we get

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)+\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\theta)$$

(i) *When* $0<x<1$: We have,

$$x=\tan\theta \text{ and } 0<x<1 \Rightarrow 0<\tan\theta<1 \Rightarrow 0<\theta<\frac{\pi}{4} \Rightarrow 0<2\theta<\frac{\pi}{2}$$

$$\therefore \quad y = \sin^{-1}(\sin 2\theta)+\cos^{-1}(\cos 2\theta) = 2\theta+2\theta=4\theta=4\tan^{-1}x \quad [\because x=\tan\theta \Rightarrow \theta=\tan^{-1}x]$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{1+x^2}$$

(ii) *When* $-1<x<0$: We have,

$$x=\tan\theta \text{ and } -1<x<0 \Rightarrow -1<\tan\theta<0 \Rightarrow -\frac{\pi}{4}<\theta<0 \Rightarrow -\frac{\pi}{2}<2\theta<0$$

$\Rightarrow \quad \sin^{-1}(\sin 2\theta) = 2\theta$ and $\cos^{-1}(\cos 2\theta) = \cos^{-1}\{\cos(-2\theta)\} = -2\theta$.

$\therefore \quad y = \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta) = 2\theta + (-2\theta) = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = 0.$

(iii) *When* $x \in (1, \infty)$ We have,

$x = \tan\theta$ and $1 < x < \infty \Rightarrow 1 < \tan\theta < \infty \Rightarrow \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < 2\theta < \pi$

$\Rightarrow \quad \cos^{-1}(\cos 2\theta) = 2\theta$ and $\sin^{-1}(\sin 2\theta) = \sin^{-1}\{\sin(\pi - 2\theta)\} = \pi - 2\theta$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta) = \pi - 2\theta + 2\theta = \pi$

$\Rightarrow \quad \dfrac{dy}{dx} = 0.$

(iv) *When* $x \in (-\infty, -1)$: We have,

$x = \tan\theta$ and $-\infty < x < -1 \Rightarrow -\infty < \tan\theta < -1 \Rightarrow -\dfrac{\pi}{2} < \theta < -\dfrac{\pi}{4} \Rightarrow -\pi < 2\theta < -\dfrac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}(\sin 2\theta) = \sin^{-1}\{-\sin(\pi + 2\theta)\} = \sin^{-1}\{\sin(-\pi - 2\theta)\} = -\pi - 2\theta$

and, $\cos^{-1}(\cos 2\theta) = \cos^{-1}\{\cos(-2\theta)\} = -2\theta$

$\therefore \quad y = \sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta) = -\pi - 2\theta - 2\theta = -\pi - 4\tan^{-1} x$

$[\because\ x = \tan\theta \therefore \theta = \tan^{-1} x]$

$\Rightarrow \quad \dfrac{dy}{dx} = 0 - \dfrac{4}{1+x^2} = -\dfrac{4}{1+x^2}$

EXAMPLE 20 *Differentiate* $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right) + \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *with respect to x, when*

(i) $x \in (0, 1)$ (ii) $x \in (1, \infty)$ (iii) $x \in (-1, 0)$ (iv) $x \in (-\infty, -1)$

SOLUTION Let $y = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) + \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$. Putting $x = \tan\theta$, we get

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) + \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta) + \cos^{-1}(\cos 2\theta)$$

(i) *When* $x \in (0, 1)$: We have,

$0 < x < 1$ and $x = \tan\theta \Rightarrow 0 < \tan\theta < 1 \Rightarrow 0 < \theta < \dfrac{\pi}{4} \Rightarrow 0 < 2\theta < \dfrac{\pi}{2}$

$\Rightarrow \tan^{-1}(\tan 2\theta) = 2\theta$ and $\cos^{-1}(\cos 2\theta) = 2\theta$

$\therefore\ y = \tan^{-1}(\tan 2\theta) + \cos^{-1}(\cos 2\theta) = 2\theta + 2\theta = 4\theta = 4\tan^{-1} x \quad [\because\ x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{4}{1+x^2}$

(ii) *When* $x \in (1, \infty)$: We have, $x > 1$ and $x = \tan\theta$

$\Rightarrow \tan\theta > 1 \Rightarrow \dfrac{\pi}{4} < \theta < \dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < 2\theta < \pi \Rightarrow \dfrac{\pi}{2} < 2\theta < \pi$ and $-\dfrac{\pi}{2} < 2\theta - \pi < 0$

$\therefore\ \cos^{-1}(\cos 2\theta) = 2\theta$

and, $\tan^{-1}(\tan 2\theta) = \tan^{-1}\left(-\tan(\pi - 2\theta)\right) = -\tan^{-1}\{\tan(\pi - 2\theta)\} = -(\pi - 2\theta) = 2\theta - \pi$.

$\therefore \quad y = \tan^{-1}(\tan 2\theta) + \cos^{-1}(\cos 2\theta) = 2\theta - \pi + 2\theta = 4\theta - \pi = 4\tan^{-1} x - \pi$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4}{1+x^2} - 0 = \dfrac{4}{1+x^2}$

(iii) *When* $x \in (-1, 0)$: We have,

$-1 < x < 0$ and $x = \tan\theta \Rightarrow -1 < \tan\theta < 0 \Rightarrow -\dfrac{\pi}{4} < \theta < 0 \Rightarrow -\dfrac{\pi}{2} < 2\theta < 0$

$\Rightarrow \quad \tan^{-1}(\tan 2\theta) = 2\theta$ and $\cos^{-1}(\cos 2\theta) = \cos^{-1}\{\cos(-2\theta)\} = -2\theta$

$\therefore \quad y = \tan^{-1}(\tan 2\theta) + \cos^{-1}(\cos 2\theta) = 2\theta + (-2\theta) = 0$

$\Rightarrow \quad \dfrac{dy}{dx} = 0$

(iv) *When* $x \in (-\infty, -1)$: We have, $-\infty < x < -1$ and $x = \tan\theta$

$\Rightarrow \quad -\infty < \tan\theta < -1 \Rightarrow -\dfrac{\pi}{2} < \theta < -\dfrac{\pi}{4} \Rightarrow -\pi < 2\theta < -\dfrac{\pi}{2} \Rightarrow \dfrac{\pi}{2} < -2\theta < \pi$, and $0 < \pi + 2\theta < \dfrac{\pi}{2}$

$\Rightarrow \cos^{-1}(\cos 2\theta) = \cos^{-1}\{\cos(-2\theta)\} = -2\theta$

and, $\tan^{-1}(\tan 2\theta) = \tan^{-1}\{\tan(\pi + 2\theta)\} = \pi + 2\theta$

$\therefore \quad y = \tan^{-1}(\tan 2\theta) + \cos^{-1}(\cos 2\theta) = \pi + 2\theta - 2\theta = \pi$

$\Rightarrow \quad \dfrac{dy}{dx} = 0.$

EXAMPLE 21 *If* $y = \sin^{-1} x + \sin^{-1}\sqrt{1-x^2}$, *find* $\dfrac{dy}{dx}$ *in each of the following cases:*

(i) $x \in (0, 1)$ (ii) $x \in (-1, 0)$ **[NCERT]**

SOLUTION Putting $x = \sin\theta$, we obtain : $y = \sin^{-1}(\sin\theta) + \sin^{-1}(\cos\theta)$

(i) We have,

$x \in (0, 1)$ and $x = \sin\theta \Rightarrow 0 < \sin\theta < 1 \Rightarrow 0 < \theta < \dfrac{\pi}{2} \Rightarrow 0 < \dfrac{\pi}{2} - \theta < \dfrac{\pi}{2}$

$\therefore \quad y = \sin^{-1}(\sin\theta) + \sin^{-1}(\cos\theta) = \sin^{-1}(\sin\theta) + \sin\left\{\sin\left(\dfrac{\pi}{2} - \theta\right)\right\} = \theta + \dfrac{\pi}{2} - \theta = \dfrac{\pi}{2}$

$\therefore \quad \dfrac{dy}{dx} = 0$

(ii) We have, $x \in (-1, 0)$ and $x = \sin\theta \Rightarrow -1 < \sin\theta < 0 \Rightarrow -\dfrac{\pi}{2} < \theta < 0$

$\Rightarrow \quad \sin^{-1}(\sin\theta) = \theta$ and $\sin^{-1}(\cos\theta) = \sin^{-1}\left\{\sin\left(\dfrac{\pi}{2} + \theta\right)\right\} = \dfrac{\pi}{2} + \theta$

$\therefore \quad y = \sin^{-1}(\sin\theta) + \sin^{-1}(\cos\theta) = \theta + \dfrac{\pi}{2} + \theta = \dfrac{\pi}{2} + 2\sin^{-1} x$

$\Rightarrow \quad \dfrac{dy}{dx} = 0 + \dfrac{2}{\sqrt{1-x^2}} = \dfrac{2}{\sqrt{1-x^2}}$

EXERCISE 10.3

BASIC

Differentiate the following functions with respect to x:

1. $\cos^{-1}\left\{2x\sqrt{1-x^2}\right\}$, $\dfrac{1}{\sqrt{2}} < x < 1$

2. $\cos^{-1}\left\{\sqrt{\dfrac{1+x}{2}}\right\}$, $-1 < x < 1$

3. $\sin^{-1}\left\{\sqrt{\frac{1-x}{2}}\right\}, 0<x<1$

4. $\sin^{-1}\left\{\sqrt{1-x^2}\right\}, 0<x<1$

5. $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a<x<a$

6. $\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

7. $\sin^{-1}(2x^2-1), 0<x<1$

8. $\sin^{-1}(1-2x^2), 0<x<1$

9. $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

10. $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right), -\pi<x<\pi$ [NCERT]

11. $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$

12. $\cos^{-1}\left(\frac{1-x^{2n}}{1+x^{2n}}\right), 0<x<\infty$

13. $\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\sec^{-1}\left(\frac{1+x^2}{1-x^2}\right), x\in R$

14. $\tan^{-1}\left(\frac{a+x}{1-ax}\right)$

15. $\tan^{-1}\left(\frac{\sqrt{x}+\sqrt{a}}{1-\sqrt{xa}}\right)$

16. $\tan^{-1}\left(\frac{a+b\tan x}{b-a\tan x}\right)$

17. $\tan^{-1}\left(\frac{a+bx}{b-ax}\right)$

18. $\tan^{-1}\left(\frac{x-a}{x+a}\right)$

19. $\tan^{-1}\left(\frac{x}{1+6x^2}\right)$

20. $\tan^{-1}\left\{\frac{5x}{1-6x^2}\right\}, -\frac{1}{\sqrt{6}}<x<\frac{1}{\sqrt{6}}$

21. $\tan^{-1}\left\{\frac{\cos x+\sin x}{\cos x-\sin x}\right\}, -\frac{\pi}{4}<x<\frac{\pi}{4}$

22. $\tan^{-1}\left\{\frac{x^{1/3}+a^{1/3}}{1-(a\,x)^{1/3}}\right\}$

23. $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ [CBSE 2013]

24. $\tan^{-1}\left\{\frac{4x}{1-4x^2}\right\}, -\frac{1}{2}<x<\frac{1}{2}$

25. $\tan^{-1}\left(\frac{2^{x+1}}{1-4^x}\right), -\infty<x<0$

26. $\tan^{-1}\left(\frac{2a^x}{1-a^{2x}}\right), a>1, -\infty<x<0$

27. $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ [CBSE 2018]

BASED ON LOTS

28. $\sin^{-1}\left\{\frac{\sin x+\cos x}{\sqrt{2}}\right\}, -\frac{3\pi}{4}<x<\frac{\pi}{4}$

29. $\cos^{-1}\left\{\frac{\cos x+\sin x}{\sqrt{2}}\right\}, -\frac{\pi}{4}<x<\frac{\pi}{4}$

[NCERT EXEMPLAR]

30. $\tan^{-1}\left\{\frac{x}{1+\sqrt{1-x^2}}\right\}, -1<x<1$

31. $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}, -a<x<a$

32. $\sin^{-1}\left\{\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}, -\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$

33. $\cos^{-1}\left\{\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}, -1<x<1$

34. $\sin^{-1}\left\{\dfrac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}, 0<x<1$

35. $\tan^{-1}\left\{\dfrac{\sqrt{1+a^2x^2}-1}{ax}\right\}, x\neq 0$

BASED ON HOTS

36. If $y=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right), 0<x<1$, prove that $\dfrac{dy}{dx}=\dfrac{4}{1+x^2}$.

37. If $y = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)+\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right), 0<x<\infty$, prove that $\dfrac{dy}{dx}=\dfrac{2}{1+x^2}$.

38. Differentiate the following with respect to x:

(i) $\cos^{-1}(\sin x)$ (ii) $\cot^{-1}\left(\dfrac{1-x}{1+x}\right)$ [NCERT, CBSE 2004]

39. If $y=\cot^{-1}\left\{\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}$, show that $\dfrac{dy}{dx}$ is independent of x. [NCERT]

40. If $y=\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)+\sec^{-1}\left(\dfrac{1+x^2}{1-x^2}\right), x>0$, prove that $\dfrac{dy}{dx}=\dfrac{4}{1+x^2}$.

41. If $y=\sec^{-1}\left(\dfrac{x+1}{x-1}\right)+\sin^{-1}\left(\dfrac{x-1}{x+1}\right), x>0$. Find $\dfrac{dy}{dx}$.

42. If $y=\sin\left[2\tan^{-1}\left\{\sqrt{\dfrac{1-x}{1+x}}\right\}\right]$, find $\dfrac{dy}{dx}$.

43. If $y=\cos^{-1}(2x)+2\cos^{-1}\sqrt{1-4x^2}, 0<x<\dfrac{1}{2}$, find $\dfrac{dy}{dx}$.

44. If the derivative of $\tan^{-1}(a+bx)$ takes the value 1 at $x=0$, prove that $1+a^2=b$.

45. If $y=\cos^{-1}(2x)+2\cos^{-1}\sqrt{1-4x^2}, -\dfrac{1}{2}<x<0$, find $\dfrac{dy}{dx}$.

46. If $y=\tan^{-1}\left\{\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}$, find $\dfrac{dy}{dx}$. [CBSE 2003, 2008]

47. If $y=\cos^{-1}\left\{\dfrac{2x-3\sqrt{1-x^2}}{\sqrt{13}}\right\}$, find $\dfrac{dy}{dx}$. [CBSE 2010]

48. Differentiate $\sin^{-1}\left\{\dfrac{2^{x+1}\cdot 3^x}{1+(36)^x}\right\}$ with respect to x. [CBSE 2013]

49. If $y=\sin^{-1}\left(6x\sqrt{1-9x^2}\right), -\dfrac{1}{3\sqrt{2}}<x<\dfrac{1}{3\sqrt{2}}$, then find $\dfrac{dy}{dx}$. [CBSE 2017]

ANSWERS

1. $\dfrac{2}{\sqrt{1-x^2}}$ 2. $-\dfrac{1}{2\sqrt{1-x^2}}$ 3. $-\dfrac{1}{2\sqrt{1-x^2}}$ 4. $-\dfrac{1}{\sqrt{1-x^2}}$

5. $\dfrac{1}{\sqrt{a^2-x^2}}$ 6. $\dfrac{a}{a^2+x^2}$ 7. $\dfrac{2}{\sqrt{1-x^2}}$ 8. $\dfrac{-2}{\sqrt{1-x^2}}$

9. $-\dfrac{a}{a^2+x^2}$ 10. $\dfrac{1}{2}$ 11. $-\dfrac{1}{1+x^2}$ 12. $\dfrac{2nx^{n-1}}{1+x^{2n}}$

13. 0 14. $\dfrac{1}{1+x^2}$ 15. $\dfrac{1}{2\sqrt{x}(1+x)}$ 16. 1

17. $\dfrac{1}{1+x^2}$ 18. $\dfrac{a}{a^2+x^2}$ 19. $\dfrac{3}{1+9x^2}-\dfrac{2}{1+4x^2}$

20. $\dfrac{3}{1+9x^2}+\dfrac{2}{1+4x^2}$ 21. 1 22. $\dfrac{1}{3}\cdot\dfrac{x^{-2/3}}{1+x^{2/3}}$

23. $\dfrac{2^{x+1}}{1+4^x}\log 2$ 24. $\dfrac{4}{1+4x^2}$ 25. $\dfrac{2^{x+1}\log_e 2}{1+4^x}$ 26. $\dfrac{2\cdot a^x\log a}{1+a^{2x}}$

27. $-\dfrac{1}{2}$ 28. 1 29. -1 30. $\dfrac{1}{2\sqrt{1-x^2}}$

31. $\dfrac{1}{2\sqrt{a^2-x^2}}$ 32. $\dfrac{1}{\sqrt{1-x^2}}$ 33. $-\dfrac{1}{\sqrt{1-x^2}}$ 34. $-\dfrac{1}{2\sqrt{1-x^2}}$

35. $\dfrac{1}{2}\left(\dfrac{a}{1+a^2x^2}\right)$ 38. (i) -1 (ii) $\dfrac{1}{1+x^2}$ 41. 0

42. $\dfrac{-x}{\sqrt{1-x^2}}$ 43. $\dfrac{2}{\sqrt{1-4x^2}}$ 45. $-\dfrac{6}{\sqrt{1-4x^2}}$ 46. $\dfrac{1}{2\sqrt{1-x^2}}$

47. $\dfrac{-1}{\sqrt{1-x^2}}$ 48. $\dfrac{2(\log 6)\,6^x}{1+36^x}$ 49. $\dfrac{6}{\sqrt{1-9x^2}}$

HINTS TO SELECTED PROBLEMS

15. Given function $= \tan^{-1}\sqrt{x}+\tan^{-1}\sqrt{a}$

16. Given function $= \tan^{-1}\left\{\dfrac{(a/b)+\tan x}{1-(a/b)\tan x}\right\} = \tan^{-1}(a/b)+\tan^{-1}(\tan x)$

17. Given function $I = \tan^{-1}\left\{\dfrac{(a/b)+x}{1-(a/b)\,x}\right\} = \tan^{-1}(a/b)+\tan^{-1}x$

18. Given function $= \tan^{-1}\left\{\dfrac{1-(a/x)}{1+(a/x)}\right\} = \tan^{-1}(1)-\tan^{-1}(a/x)$

19. Given function $= \tan^{-1}\left\{\dfrac{3x-2x}{1+(3x)(2x)}\right\} = \tan^{-1}3x-\tan^{-1}2x$

20. Given function $= \tan^{-1}3x+\tan^{-1}2x$

21. Given function $= \tan^{-1}\left(\dfrac{1+\tan x}{1-\tan x}\right) = \tan^{-1}(1) + \tan^{-1}(\tan x) = \dfrac{\pi}{4} + x$

22. Given function $= \tan^{-1} x^{1/3} + \tan^{-1} a^{1/3}$

36. Putting $x = \tan\theta$, we get $y = 2\tan^{-1} x + 2\tan^{-1} x = 4\tan^{-1} x$

39. We have,

$$y = \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \quad y = \cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$\Rightarrow \quad y = \cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\}$$

$$\Rightarrow \quad y = \begin{cases} \cot^{-1}\left\{\dfrac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\}, & 0 < x \le \dfrac{\pi}{4} \\[2ex] \cot^{-1}\left\{\dfrac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\}, & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} \cot^{-1}\left(\cot\dfrac{x}{2}\right), & 0 < x\ x \le \dfrac{\pi}{4} \\[2ex] \cot^{-1}\left(\tan\dfrac{x}{2}\right), & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases} \Rightarrow y = \begin{cases} \cot^{-1}\left(\cot\dfrac{x}{2}\right), & 0 < x \le \dfrac{\pi}{4} \\[2ex] \cot^{-1}\left\{\cot\left(\dfrac{\pi}{2}-\dfrac{x}{2}\right)\right\}, & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases}$$

$$\Rightarrow \quad y = \begin{cases} \dfrac{x}{2}, & 0 < x \le \dfrac{\pi}{4} \\[2ex] \dfrac{\pi}{2}-\dfrac{x}{2}, & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases} \Rightarrow \frac{dy}{dx} = \begin{cases} \dfrac{1}{2}, & 0 < x < \dfrac{\pi}{4} \\[2ex] -\dfrac{1}{2}, & \dfrac{\pi}{4} < x < \dfrac{\pi}{2} \end{cases}$$

Hence, $\dfrac{dy}{dx}$ is independent of x.

41. Use: $\sec^{-1}\left(\dfrac{x+1}{x-1}\right) = \cos^{-1}\dfrac{x-1}{x+1}$ and $\cos^{-1}\theta + \sin^{-1}\theta = \dfrac{\pi}{2}$

43. Putting $2x = \cos\theta$, we get

$$y = \theta + 2\cos^{-1}(\sin\theta) = \theta + 2\cos^{-1}\{\cos(\pi/2-\theta)\} = \theta + 2(\pi/2-\theta) = \pi - \theta = \pi - \cos^{-1}(2x)$$

10.6 RELATION BETWEEN $\frac{dy}{dx}$ AND $\frac{dx}{dy}$

Let x and y be two variables connected by a relation of the form $f(x, y) = 0$. Let Δx be a small change in x and let Δy be the corresponding change in y. Then,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \text{ and, } \frac{dx}{dy} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta y}$$

Now, $\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} = 1$

$$\Rightarrow \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \times \frac{\Delta x}{\Delta y} \right) = 1$$

$$\Rightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \times \lim_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = 1 \qquad [\because \Delta x \to 0 \Leftrightarrow \Delta y \to 0]$$

$$\Rightarrow \frac{dy}{dx} \times \frac{dx}{dy} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{dx/dy}$$

10.7 DIFFERENTIATION OF IMPLICIT FUNCTIONS

Up till now we have discussed derivatives of functions of the form $y = f(x)$. If the variables x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function x in the form $y = \phi(x)$, then y is said to be an implicit function of x. To find $\frac{dy}{dx}$ in such a case, we differentiate both sides of the given relation with respect to x, keeping in mind that the derivative of $\phi(y)$ with respect to x is $\frac{d\phi}{dy} \cdot \frac{dy}{dx}$.

For example, $\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$, $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.

It should be noted that $\frac{d}{dy}(\sin y) = \cos y$ but $\frac{d}{dx}(\sin y) = \cos y \cdot \frac{dy}{dx}$.

Similarly, $\frac{d}{dy}(y^3) = 3y^2$ whereas $\frac{d}{dx}(y^3) = 3y^2 \frac{dy}{dx}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, find $\frac{dy}{dx}$ and $\frac{dx}{dy}$. Also, show that $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$.*

SOLUTION We have, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ...(i)

Differentiating both sides of this with respect to x, we get

$$\frac{d}{dx}(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) + \frac{d}{dx}(2gx) + \frac{d}{dx}(2fy) + \frac{d}{dx}(c) = \frac{d}{dx}(0)$$

$$\Rightarrow a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) + 2g\frac{d}{dx}(x) + 2f\frac{d}{dx}(y) + 0 = 0$$

$$\Rightarrow 2ax + 2h\left(x\frac{dy}{dx} + y\right) + b\,2y\frac{dy}{dx} + 2g \times 1 + 2f \times \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx}(2hx + 2by + 2f) + 2ax + 2hy + 2g = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2(ax+hy+g)}{2(hx+by+f)} = -\left(\frac{ax+hy+g}{hx+by+f}\right) \quad \text{...(ii)}$$

Differentiating both sides of (i) with respect to y, we obtain

$$\frac{d}{dy}(ax^2) + \frac{d}{dy}(2hxy) + \frac{d}{dy}(by^2) + \frac{d}{dy}(2gx) + \frac{d}{dy}(2fy) + \frac{d}{dy}(c) = \frac{d}{dy}(0)$$

$$\Rightarrow \quad a\frac{d}{dy}(x^2) + 2h\frac{d}{dy}(xy) + b\frac{d}{dy}(y^2) + 2g\frac{d}{dy}(x) + 2f\frac{d}{dy}(y) + \frac{d}{dy}(c) = 0$$

$$\Rightarrow \quad a\left(2x\frac{dx}{dy}\right) + 2h\left(y\frac{dx}{dy} + x\right) + b(2y) + 2g\frac{dx}{dy} + 2f \times 1 + 0 = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = -\frac{2(hx+by+f)}{2(ax+hy+g)} = -\left(\frac{hx+by+f}{ax+hy+g}\right) \quad \text{...(iii)}$$

From (ii) and (iii), we obtain

$$\frac{dy}{dx}\cdot\frac{dx}{dy} = -\left(\frac{ax+hy+g}{hx+by+f}\right) \times -\left(\frac{hx+by+f}{ax+hy+g}\right) = 1$$

EXAMPLE 2 *If* $x^2 + 2xy + y^3 = 42$, *find* $\frac{dy}{dx}$

SOLUTION We have, $x^2 + 2xy + y^3 = 42$. Differentiating both sides of this with respect to x, we get

$$\frac{d}{dx}(x^2) + 2\frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = \frac{d}{dx}(42)$$

$$\Rightarrow \quad 2x + 2\left(x\frac{dy}{dx} + y\right) + 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2x + 2y + \frac{dy}{dx}(2x + 3y^2) = 0 \Rightarrow \frac{dy}{dx}(2x + 3y^2) = -2(x+y) \Rightarrow \frac{dy}{dx} = -\frac{2(x+y)}{(2x+3y^2)}.$$

EXAMPLE 3 *If* $x^3 + y^3 = 3axy$, *find* $\frac{dy}{dx}$

SOLUTION Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3a\frac{d}{dx}(xy)$$

$$\Rightarrow \quad 3x^2 + 3y^2\frac{dy}{dx} = 3a\left\{x\frac{dy}{dx} + y\right\}$$

$$\Rightarrow \quad (3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2 \Rightarrow 3(y^2 - ax)\frac{dy}{dx} = 3(ay - x^2) \Rightarrow \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

EXAMPLE 4 *If* $\log(x^2 + y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, *show that* $\frac{dy}{dx} = \frac{x+y}{x-y}$.

SOLUTION Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}\left\{\log(x^2 + y^2)\right\} = 2\frac{d}{dx}\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\}$$

$$\Rightarrow \quad \frac{1}{x^2+y^2} \times \frac{d}{dx}(x^2 + y^2) = 2 \times \frac{1}{1 + (y/x)^2} \times \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{1}{x^2+y^2}\left\{\frac{d}{dx}(x^2)+\frac{d}{dx}(y^2)\right\} = 2\times\frac{x^2}{x^2+y^2}\left\{\frac{x\frac{dy}{dx}-y\times 1}{x^2}\right\}$$

$$\Rightarrow \quad \frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = \frac{2}{x^2+y^2}\left\{x\frac{dy}{dx}-y\right\}$$

$$\Rightarrow \quad 2\left\{x+y\frac{dy}{dx}\right\} = 2\left\{x\frac{dy}{dx}-y\right\}$$

$$\Rightarrow \quad x+y\frac{dy}{dx} = x\frac{dy}{dx}-y \Rightarrow \frac{dy}{dx}(y-x) = -(x+y) \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

EXAMPLE 5 *If* $x\sqrt{1+y}+y\sqrt{1+x}=0$ *and* $x \neq y$, *prove that* $\frac{dy}{dx}=-\frac{1}{(x+1)^2}$. **[CBSE 2012, NCERT]**

SOLUTION We have,

$$x\sqrt{1+y}+y\sqrt{1+x} = 0$$

$$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x}$$

$$\Rightarrow \quad x^2(1+y) = y^2(1+x) \qquad \text{[On squaring both sides]}$$

$$\Rightarrow \quad x^2-y^2 = y^2x-x^2y$$

$$\Rightarrow \quad (x+y)(x-y) = -xy(x-y)$$

$$\Rightarrow \quad x+y = -xy \Rightarrow x = -y-xy \Rightarrow y(1+x) = -x \Rightarrow y = -\frac{x}{1+x} \qquad [\because x \neq y]$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left\{\frac{(1+x)\times 1 - x(0+1)}{(1+x)^2}\right\} \Rightarrow \frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *If* $\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \tan^{-1} a$, *prove that* $\frac{dy}{dx}=\frac{y}{x}$.

SOLUTION We have,

$$\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right) = \tan^{-1} a \Rightarrow \frac{x^2-y^2}{x^2+y^2} = \cos(\tan^{-1} a) = \lambda, \text{ say}$$

$$\Rightarrow \quad \frac{2x^2}{-2y^2} = \frac{\lambda+1}{\lambda-1} \qquad \text{[Applying Componendo and dividendo]}$$

$$\Rightarrow \quad \frac{x^2}{y^2} = \frac{1+\lambda}{1-\lambda}$$

Differentiating both sides with respect to x, we get

$$\Rightarrow \quad \frac{y^2\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(y^2)}{(y^2)^2} = 0 \Rightarrow \frac{y^2\times 2x - x^2\times 2y\frac{dy}{dx}}{y^4} = 0$$

$$\Rightarrow \quad 2xy^2-2x^2y\frac{dy}{dx} = 0 \Rightarrow 2x^2y\frac{dy}{dx} = 2xy^2 \Rightarrow \frac{dy}{dx} = \frac{2xy^2}{2x^2y} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

ALITER We have,

$$\cos^{-1}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\tan^{-1}a \Rightarrow \cos^{-1}\left\{\frac{1-(y/x)^2}{1+(y/x)^2}\right\}=\tan^{-1}a$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}a \Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{2}\tan^{-1}a \Rightarrow \frac{y}{x}=\tan\left(\frac{1}{2}\tan^{-1}a\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{y}{x}\right)=0 \Rightarrow \frac{x\frac{dy}{dx}-y\times 1}{x^2}=0 \Rightarrow x\frac{dy}{dx}-y=0 \Rightarrow \frac{dy}{dx}=\frac{y}{x}$$

EXAMPLE 7 *If* $\sin y = x\sin(a+y)$, *prove that* $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$. **[CBSE 2009, 2011, 2012]**

SOLUTION Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}(\sin y) = \frac{d}{dx}\left\{x\sin(a+y)\right\}$$

$$\Rightarrow \quad \cos y\frac{dy}{dx} = 1\times\sin(a+y)+x\cos(a+y)\frac{d}{dx}(a+y)$$

$$\Rightarrow \quad \cos y\frac{dy}{dx} = \sin(a+y)+x\cos(a+y)\frac{dy}{dx}$$

$$\Rightarrow \quad \cos y\frac{dy}{dx}-x\cos(a+y)\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \quad \left\{\cos y - x\cos(a+y)\right\}\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \quad \left\{\cos y-\frac{\sin y}{\sin(a+y)}\cos(a+y)\right\}\frac{dy}{dx}=\sin(a+y) \qquad \left[\begin{array}{l}\because\ \sin y = x\sin(a+y)\\ \therefore\ x=\dfrac{\sin y}{\sin(a+y)}\end{array}\right]$$

$$\Rightarrow \quad \left\{\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin(a+y)}\right\}\frac{dy}{dx}=\sin(a+y)$$

$$\Rightarrow \quad \frac{\sin(a+y-y)}{\sin(a+y)}\times\frac{dy}{dx}=\sin(a+y) \Rightarrow \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

ALITER 1 We have,

$$\sin y = x\sin(a+y) \Rightarrow x=\frac{\sin y}{\sin(a+y)}$$

Differentiating both sides with respect to y, we get

$$\frac{dx}{dy}=\frac{\sin(a+y)\cos y-\sin y\cos(a+y)}{\sin^2(a+y)}=\frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{dx/dy}=\frac{\sin^2(a+y)}{\sin a}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *If* $\sqrt{1-x^6}+\sqrt{1-y^6}=a(x^3-y^3)$, *prove that* $\dfrac{dy}{dx}=\dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$, *where* $-1\ \ x<1$ *and* $-1<y<1$.

SOLUTION Putting $x^3=\sin A$ and $y^3=\sin B$ in the given relation, we get

$$\sqrt{1-\sin^2 A}+\sqrt{1-\sin^2 B}=a(\sin A-\sin B)$$

$$\Rightarrow \quad \cos A + \cos B = a(\sin A - \sin B)$$

$$\Rightarrow \quad 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \Rightarrow \cot\left(\frac{A-B}{2}\right) = a$$

$$\Rightarrow \quad \frac{A-B}{2} = \cot^{-1}(a) \Rightarrow A - B = 2\cot^{-1}(a) \Rightarrow \sin^{-1}x^3 - \sin^{-1}y^3 = 2\cot^{-1}(a).$$

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-x^6}} \times \frac{d}{dx}(x^3) - \frac{1}{\sqrt{1-y^6}} \times \frac{d}{dx}(y^3) = 0 \Rightarrow \frac{1}{\sqrt{1-x^6}} \times 3x^2 - \frac{1}{\sqrt{1-y^6}} \times 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

ALITER 1 We have, $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$ ...(i)

Differentiating both sides of the given relation with respect to x, we get

$$\frac{1}{2\sqrt{1-x^6}}\frac{d}{dx}(1-x^6) + \frac{1}{2\sqrt{1-y^6}}\frac{d}{dx}(1-y^6) = a\frac{d}{dx}(x^3 - y^3)$$

$$\Rightarrow \quad \frac{1}{2\sqrt{1-x^6}} \times -6x^5 + \frac{1}{2\sqrt{1-y^6}} \times -6y^5\frac{dy}{dx} = a\left(3x^2 - 3y^2\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{-3x^5}{\sqrt{1-x^6}} - \frac{3y^5}{\sqrt{1-y^6}}\frac{dy}{dx} = 3ax^2 - 3ay^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left\{ay^2 - \frac{y^5}{\sqrt{1-y^6}}\right\} = ax^2 + \frac{x^5}{\sqrt{1-x^6}}$$

$$\Rightarrow \quad y^2\frac{dy}{dx}\frac{\left\{a\sqrt{1-y^6} - y^3\right\}}{\sqrt{1-y^6}} = x^2\frac{\left\{a\sqrt{1-x^6} + x^3\right\}}{\sqrt{1-x^6}} \Rightarrow \frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}\left\{\frac{a\sqrt{1-x^6} + x^3}{a\sqrt{1-y^6} - y^3}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}\left\{\frac{\dfrac{\left\{\sqrt{1-x^6} + \sqrt{1-y^6}\right\}\sqrt{1-x^6}}{x^3 - y^3} + x^3}{\dfrac{\left\{\sqrt{1-x^6} + \sqrt{1-y^6}\right\}\sqrt{1-y^6}}{x^3 - y^3} - y^3}\right\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}\left\{\frac{1 - x^6 + \sqrt{1-x^6}\sqrt{1-y^6} + x^6 - x^3y^3}{\sqrt{1-x^6}\sqrt{1-y^6} + 1 - y^6 - x^3y^3 + y^6}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}\left\{\frac{1 - x^3y^3 + \sqrt{1-x^6}\sqrt{1-y^6}}{1 - x^3y^3 + \sqrt{1-x^6}\sqrt{1-y^6}}\right\} \Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

EXAMPLE 9 *If* $x^2 + y^2 = t - \frac{1}{t}$ *and* $x^4 + y^4 = t^2 + \frac{1}{t^2}$, *then prove that* $\frac{dy}{dx} = \frac{1}{x^3y}$.

SOLUTION We have, $x^2+y^2=t-\frac{1}{t}$

$$\Rightarrow \quad (x^2+y^2)^2 = \left(t-\frac{1}{t}\right)^2 \Rightarrow x^4+y^4+2x^2y^2=t^2+\frac{1}{t^2}-2$$

$$\Rightarrow \quad x^4+y^4+2x^2y^2 = x^4+y^4-2 \qquad \left[\because x^4+y^4=t^2+\frac{1}{t^2}\right]$$

$$\Rightarrow \quad 2x^2y^2 = -2 \Rightarrow x^2y^2 = -1 \Rightarrow y^2 = -\frac{1}{x^2} \Rightarrow y^2 = -x^{-2}$$

Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = -(-2)\,x^{-3} \Rightarrow y\frac{dy}{dx} = \frac{1}{x^3} \Rightarrow \frac{dy}{dx} = \frac{1}{x^3\,y}$$

EXAMPLE 10 *If* $y=b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right)$, *find* $\frac{dy}{dx}$.

SOLUTION We have,

$$y = b\tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right) \Rightarrow \frac{y}{b} = \tan^{-1}\left(\frac{x}{a}+\tan^{-1}\frac{y}{x}\right) \Rightarrow \tan\frac{y}{b} = \frac{x}{a}+\tan^{-1}\frac{y}{x}$$

Differentiating both sides with respect to x, we get

$$\frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a}+\frac{1}{1+\left(\frac{y}{x}\right)^2}\times\frac{x\frac{dy}{dx}-y}{x^2} \Rightarrow \frac{1}{b}\sec^2\left(\frac{y}{b}\right)\frac{dy}{dx} = \frac{1}{a}+\frac{x\frac{dy}{dx}-y}{x^2+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}\right\} = \frac{1}{a}-\frac{y}{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{\frac{1}{a}-\frac{y}{x^2+y^2}}{\frac{1}{b}\sec^2\left(\frac{y}{b}\right)-\frac{x}{x^2+y^2}}$$

EXAMPLE 11 *If* $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$, *then show that* $\frac{dy}{dx}=\sqrt{\frac{3}{\cos x\cos 3x}}$.

SOLUTION We have, $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$

$$\Rightarrow \quad \cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}} \qquad \ldots\text{(i)}$$

$$\Rightarrow \quad \cos y = \sqrt{\frac{4\cos^3 x-3\cos x}{\cos^3 x}} \Rightarrow \cos y = \sqrt{4-3\sec^2 x} \Rightarrow \cos^2 y = 4-3(1+\tan^2 x)$$

$$\Rightarrow \quad 1-\cos^2 y = 3\tan^2 x \Rightarrow \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$$

Differentiating both sides with respect to x, we get

$$\cos y\frac{dy}{dx} = \sqrt{3}\sec^2 x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{3}}{\cos y \cos^2 x} = \frac{\sqrt{3}}{\cos^2 x} \times \sqrt{\frac{\cos^3 x}{\cos 3x}} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cos 3x}}$$

EXERCISE 10.4

BASIC

Find $\frac{dy}{dx}$ in each of the following (1-11):

1. $xy = c^2$
2. $y^3 - 3xy^2 = x^3 + 3x^2 y$
3. $x^{2/3} + y^{2/3} = a^{2/3}$
4. $4x + 3y = \log(4x - 3y)$
5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
6. $x^5 + y^5 = 5xy$
7. $(x + y)^2 = 2axy$
8. $(x^2 + y^2)^2 = xy$ [CBSE 2009, 2018]
9. $\tan^{-1}(x^2 + y^2) = a$
10. $e^{x - y} = \log\left(\frac{x}{y}\right)$
11. $\sin xy + \cos(x + y) = 1$
12. If $xy = 1$, prove that $\frac{dy}{dx} + y^2 = 0$.
13. If $xy^2 = 1$, prove that $2\frac{dy}{dx} + y^3 = 0$.

BASED ON LOTS

14. If $\sqrt{1 - x^2} + \sqrt{1 - y^2} = a(x - y)$, prove that $\frac{dy}{dx} = \sqrt{\frac{1 - y^2}{1 - x^2}}$. [NCERT EXEMPLAR]
15. If $y\sqrt{1 - x^2} + x\sqrt{1 - y^2} = 1$, prove that $\frac{dy}{dx} = -\sqrt{\frac{1 - y^2}{1 - x^2}}$.
16. If $x\sqrt{1 + y} + y\sqrt{1 + x} = 0$, prove that $(1 + x)^2 \frac{dy}{dx} + 1 = 0$. [CBSE 2011]
17. If $\log\sqrt{x^2 + y^2} = \tan^{-1}\left(\frac{y}{x}\right)$, prove that $\frac{dy}{dx} = \frac{x + y}{x - y}$. [CBSE 2019, 2020]
18. If $\sec\left(\frac{x + y}{x - y}\right) = a$, prove that $\frac{dy}{dx} = \frac{y}{x}$.
19. If $\tan^{-1}\left(\frac{x^2 - y^2}{x^2 + y^2}\right) = a$, prove that $\frac{dy}{dx} = \frac{x}{y}\frac{(1 - \tan a)}{(1 + \tan a)}$.
20. If $xy\log(x + y) = 1$, prove that $\frac{dy}{dx} = -\frac{y(x^2 y + x + y)}{x(xy^2 + x + y)}$.
21. If $y = x\sin(a + y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin(a + y) - y\cos(a + y)}$.
22. If $x\sin(a + y) + \sin a\cos(a + y) = 0$, prove that $\frac{dy}{dx} = \frac{\sin^2(a + y)}{\sin a}$. [NCERT EXEMPLAR]
23. If $y = x\sin y$, prove that $\frac{dy}{dx} = \frac{\sin y}{(1 - x\cos y)}$.

24. If $y\sqrt{x^2+1} = \log\left(\sqrt{x^2+1} - x\right)$, show that $(x^2+1)\dfrac{dy}{dx} + xy + 1 = 0$.

25. If $\sin(xy) + \dfrac{y}{x} = x^2 - y^2$, find $\dfrac{dy}{dx}$.

26. If $\tan(x+y) + \tan(x-y) = 1$, find $\dfrac{dy}{dx}$.

27. If $e^x + e^y = e^{x+y}$, prove that $\dfrac{dy}{dx} = -\dfrac{e^x(e^y-1)}{e^y(e^x-1)}$ or, $\dfrac{dy}{dx} + e^{y-x} = 0$ [CBSE 2014]

28. If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$. [NCERT]

29. If $\sin^2 y + \cos xy = k$, find $\dfrac{dy}{dx}$ at $x = 1$, $y = \dfrac{\pi}{4}$. [CBSE 2017]

BASED ON HOTS

30. If $y = \{\log_{\cos x} \sin x\}\{\log_{\sin x} \cos x\}^{-1} + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, find $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$.

31. If $\sqrt{y+x} + \sqrt{y-x} = c$, show that $\dfrac{dy}{dx} = \dfrac{y}{x} - \sqrt{\dfrac{y^2}{x^2} - 1}$.

ANSWERS

1. $-\dfrac{y}{x}$

2. $\dfrac{dy}{dx} = \dfrac{(x+y)^2}{y^2 - 2xy - x^2}$

3. $\left(-\dfrac{y}{x}\right)^{1/3}$

4. $\dfrac{4(1-4x+3y)}{3(4x-3y+1)}$

5. $-\dfrac{b^2x}{a^2y}$

6. $\dfrac{y-x^4}{y^4-x}$

7. $\dfrac{ay-x-y}{x+y-ax}$

8. $\dfrac{4x(x^2+y^2)-y}{x-4y(x^2+y^2)}$

9. $-\dfrac{x}{y}$

10. $\dfrac{y}{x}\dfrac{(xe^{x-y}-1)}{(ye^{x-y}-1)}$

11. $\dfrac{\sin(x+y) - y\cos(xy)}{x\cos(xy) - \sin(x+y)}$

25. $\dfrac{2x^3 + y - x^2 y\cos(xy)}{x\{x^2\cos xy + 1 + 2xy\}}$

26. $\dfrac{\sec^2(x-y) + \sec^2(x+y)}{\sec^2(x-y) - \sec^2(x+y)}$

29. $\dfrac{\pi}{4}(\sqrt{2}+1)$

30. $8\left\{\dfrac{4}{\pi^2+16} - \dfrac{1}{\log 2}\right\}$

HINTS TO SELECTED PROBLEMS

12. We have,

$$xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2} \text{ Therefore, } \frac{dy}{dx} + y^2 = -\frac{1}{x^2} + \frac{1}{x^2} = 0$$

13. We have,

$$xy^2 = 1 \Rightarrow x = \frac{1}{y^2} \Rightarrow \frac{dx}{dy} = -\frac{2}{y^3} \Rightarrow \frac{dy}{dx} = -\frac{y^3}{2} \Rightarrow 2\frac{dy}{dx} + y^3 = 0$$

14. Put $x = \sin A$, $y = \sin B$ and proceed as in Ex. 8.

15. Put $x = \sin A$ and $y = \sin B$

28. We have,

$$\cos y = x \cos (a+y)$$

$$\Rightarrow \quad x = \frac{\cos y}{\cos (a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-\cos (a+y) \sin y - \cos y \times -\sin (a+y)}{\{\cos (a+y)\}^2}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\sin (a+y) \cos y - \cos (a+y) \sin y}{\cos^2 (a+y)} = \frac{\sin (a+y-y)}{\cos^2 (a+y)} = \frac{\sin a}{\cos^2 (a+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2 (a+y)}{\sin a}$$

10.8 LOGARITHMIC DIFFERENTIATION

We have learnt about the derivatives of the functions of the form $[f(x)]^n, n^{f(x)}$ and n^n, where $f(x)$ is a function of x and n is a constant. In this section, we will be mainly discussing derivatives of the functions of the form $[f(x)]^{g(x)}$ where $f(x)$ and $g(x)$ are functions of x. To find the derivative of this type of functions we proceed as follows:

Let $y = [f(x)]^{g(x)}$. Taking logarithm of both the sides, we get

$$\log y = g(x) \cdot \log \{f(x)\}$$

Differentiating with respect to x, we get

$$\frac{1}{y} \frac{dy}{dx} = g(x) \times \frac{1}{f(x)} \frac{d}{dx}\Big(f(x)\Big) + \log \{f(x)\} \cdot \frac{d}{dx}\Big(g(x)\Big)$$

$$\therefore \quad \frac{dy}{dx} = y \left\{ \frac{g(x)}{f(x)} \cdot \frac{d}{dx}\Big(f(x)\Big) + \log \{f(x)\} \cdot \frac{d}{dx}\Big(g(x)\Big) \right\}$$

Alternatively, we may write

$$y = [f(x)]^{g(x)} = e^{g(x) \log \{f(x)\}}$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = e^{g(x) \log \{f(x)\}} \left\{ g(x) \cdot \frac{1}{f(x)} \frac{d}{dx}\Big(f(x)\Big) + \log \{f(x)\} \cdot \frac{d}{dx}\Big(g(x)\Big) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = [f(x)]^{g(x)} \left\{ \frac{g(x)}{f(x)} \cdot \frac{d}{dx}\Big(f(x)\Big) + \log \{f(x)\} \cdot \frac{d}{dx}\Big(g(x)\Big) \right\}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) x^x (ii) $x^{\sin x}$ [NCERT] (iii) $(\sin x)^{\log x}$

SOLUTION (i) Let $y = x^x$. Then, $y = e^{x \cdot \log x}$ $\qquad [\because a^b = e^{\log a^b} = e^{b \log a}]$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x \log x} \frac{d}{dx}(x \log x) = x^x \left\{ \log x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\log x) \right\} \qquad [\because e^{x \log x} = x^x]$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x \left(\log x + x \times \frac{1}{x}\right) = x^x \ (1 + \log x)$$

(ii) Let $y = x^{\sin x}$. Then, $y = e^{\sin x \log x}$ $\qquad [\because a^b = e^{\log a^b} = e^{b \log a}]$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\sin x \,.\, \log x} \frac{d}{dx}(\sin x \,.\, \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sin x} \left\{\log x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\log x)\right\} \qquad [\because e^{\sin x \log x} = x^{\sin x}]$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sin x} \left\{\cos x \,.\, \log x + \frac{\sin x}{x}\right\}$$

(iii) Let $y = (\sin x)^{\log x}$. Then, $y = e^{\log x \,.\, \log \sin x}$ $\qquad [\because a^b = e^{b \log a}]$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\log x \,.\, \log \sin x} \frac{d}{dx}\{\log x \,.\, \log \sin x\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\log x} \left\{\log \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\log \sin x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\log x} \left\{\frac{\log \sin x}{x} + \log x \times \frac{1}{\sin x} \times \cos x\right\} = (\sin x)^{\log x} \left\{\frac{\log \sin x}{x} + \cot x \,.\, \log x\right\}$$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $(\cos x)^x$ (ii) $x^{\sqrt{x}}$ (iii) $(\log x)^{\sin x}$ (iv) $(\sin x)^{\cos x}$

SOLUTION Let $y = (\cos x)^x$. Then, $y = e^{x \log \cos x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x \log \cos x} \frac{d}{dx}(x \log \cos x)$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x \left\{\log \cos x \frac{d}{dx}(x) + x \frac{d}{dx}(\log \cos x)\right\} = (\cos x)^x \left\{\log \cos x + x \frac{1}{\cos x}(-\sin x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\cos x)^x \ (\log \cos x - x \tan x)$$

(ii) Let $y = x^{\sqrt{x}}$. Then, $y = e^{\sqrt{x} \log x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\sqrt{x} \log x} \frac{d}{dx}(\sqrt{x} \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sqrt{x}} \left\{\left(\log x\right) \frac{d}{dx}(\sqrt{x}) + \sqrt{x} \frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sqrt{x}} \left\{\left(\log x\right) \frac{1}{2\sqrt{x}} + \sqrt{x} \times \frac{1}{x}\right\} = x^{\sqrt{x}} \left(\frac{\log x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

(iii) Let $y = (\log x)^{\sin x}$. Then, $y = e^{\sin x \log (\log x)}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\sin x \,.\, \log (\log x)} \frac{d}{dx}\{\sin x \,.\, \log (\log x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^{\sin x} \left\{ \log(\log x) \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\log(\log x)) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^{\sin x} \left\{ \log(\log x) \,.\, \cos x + \sin x \times \frac{1}{\log x} \times \frac{1}{x} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^{\sin x} \left\{ \log(\log x) \,.\, \cos x + \frac{\sin x}{x \log x} \right\}$$

(iv) Let $y = (\sin x)^{\cos x}$. Then, $y = e^{\cos x \,.\, \log \sin x}$.

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\cos x \,.\, \log \sin x} \frac{d}{dx}(\cos x \,.\, \log \sin x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x} \left\{ \log \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\log \sin x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x} \left\{ -\sin x \log \sin x + \cos x \times \frac{1}{\sin x} \times \cos x \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos x} \left\{ -\sin x \log \sin x + \frac{\cos^2 x}{\sin x} \right\}$$

EXAMPLE 3 *Differentiate the following functions with respect to x :*

(i) $x^{\cos^{-1} x}$ (ii) $(\sin x)^{\cos^{-1} x}$

SOLUTION Let $y = x^{\cos^{-1} x}$. Then, $y = e^{\cos^{-1} x \,.\, \log x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\cos^{-1} x \,.\, \log x} \frac{d}{dx}(\cos^{-1} x \,.\, \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\cos^{-1} x} \left\{ \log x \frac{d}{dx}(\cos^{-1} x) + \cos^{-1} x \frac{d}{dx}(\log x) \right\} = x^{\cos^{-1} x} \left\{ \frac{-\log x}{\sqrt{1-x^2}} + \frac{\cos^{-1} x}{x} \right\}$$

(ii) Let $y = (\sin x)^{\cos^{-1} x}$. Then, $y = e^{\cos^{-1} x \,.\, \log \sin x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{\cos^{-1} x \,.\, \log \sin x} \frac{d}{dx}(\cos^{-1} x \,.\, \log \sin x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos^{-1} x} \left\{ \cos^{-1} x \frac{d}{dx}(\log \sin x) + \log \sin x \frac{d}{dx}(\cos^{-1} x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos^{-1} x} \left\{ \cos^{-1} x \times \frac{1}{\sin x} \times \cos x + (\log \sin x) \times \frac{-1}{\sqrt{1-x^2}} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\cos^{-1} x} \left\{ \cos^{-1} x \cot x - \frac{\log \sin x}{\sqrt{1-x^2}} \right\}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Differentiate the following functions with respect to x:* (i) x^{x^x} (ii) $(x^x)^x$

SOLUTION (i) Let $y = x^{x^x}$. Then, $y = e^{x^x \cdot \log x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x^x \cdot \log x} \frac{d}{dx}(x^x \log x) = x^{x^x} \frac{d}{dx}(e^{x \log x} \log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^x} \left\{ \log x \frac{d}{dx}(e^{x \log x}) + e^{x \log x} \times \frac{d}{dx}(\log x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^x} \left\{ \log x \cdot e^{x \log x} \frac{d}{dx}(x \log x) + e^{x \log x} \times \frac{1}{x} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^x} \left\{ (\log x)\, x^x \left(x \times \frac{1}{x} + \log x \right) + x^x \times \frac{1}{x} \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^x} \left\{ x^x (1 + \log x) \log x + \frac{x^x}{x} \right\} = x^{x^x} x^x \left\{ (1 + \log x) \log x + \frac{1}{x} \right\}$$

(ii) Let $y = (x^x)^x$. Then, $y = x^{x \cdot x} = x^{x^2} \Rightarrow y = e^{x^2 \cdot \log x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x^2 \cdot \log x} \frac{d}{dx}(x^2 \log x) = e^{x^2 \cdot \log x} \left\{ \log x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\log x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^2} \left\{ (\log x)\, 2x + x^2 \times \frac{1}{x} \right\} \qquad \left[\because e^{x^2 \log x} = x^{x^2} \right]$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^2} (2x \log x + x) = x\, x^{x^2} (2 \log x + 1).$$

EXAMPLE 5 *If* $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, *find* $\frac{dy}{dx}$. **[CBSE 2007]**

SOLUTION We have, $y = (\sin x)^{\tan x} + (\cos x)^{\sec x} = e^{\tan x \cdot \log \sin x} + e^{\sec x \cdot \log \cos x}$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(e^{\tan x \log \sin x}) + \frac{d}{dx}(e^{\sec x \log \cos x})$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\tan x \log \sin x} \frac{d}{dx}(\tan x \log \sin x) + e^{\sec x \log \cos x} \frac{d}{dx}(\sec x \log \cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\tan x} \left\{ \frac{d}{dx}(\tan x) \times \log \sin x + \tan x \times \frac{d}{dx}(\log \sin x) \right\}$$
$$+ (\cos x)^{\sec x} \left\{ \frac{d}{dx}(\sec x) \times \log \cos x + \sec x \times \frac{d}{dx}(\log \cos x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\tan x} \left\{ \sec^2 x \log \sin x + \tan x \times \frac{1}{\sin x} \times \cos x \right\}$$
$$+ (\cos x)^{\sec x} \left\{ \sec x \tan x \log \cos x + \sec x \left(\frac{1}{\cos x} \right) (-\sin x) \right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^{\tan x} \left\{ \sec^2 x \log \sin x + 1 \right\} + (\cos x)^{\sec x} \{ \sec x \tan x \cdot \log \cos x - \sec x \tan x \}$$

EXAMPLE 6 *Differentiate:* $(\log x)^x + x^{\log x}$ *with respect to* x. **[CBSE 2020]**

SOLUTION Let $y = (\log x)^x + x^{\log x}$. Then,

$$y = e^{\log(\log x)^x} + e^{\log(x^{\log x})} = e^{x\log(\log x)} + e^{\log x \,.\, \log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x\log(\log x)} \times \frac{d}{dx}\{x\log(\log x)\} + e^{(\log x)^2} \times \frac{d}{dx}(\log x)^2$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^x \left\{\log(\log x) \times \frac{d}{dx}(x) + x \times \frac{d}{dx}\{\log(\log x)\}\right\} + x^{\log x}\left\{2(\log x)\frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^x \left\{\log(\log x) + x \times \frac{1}{\log x} \times \frac{1}{x}\right\} + x^{\log x}\left\{2(\log x)\frac{1}{x}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\log x)^x \left\{\log(\log x) + \frac{1}{\log x}\right\} + x^{\log x}\left\{\frac{2\log x}{x}\right\}$$

EXAMPLE 7 *Differentiate the following functions with respect to* x :

(i) $x^{\cot x} + \dfrac{2x^2 - 3}{x^2 + x + 2}$ **[CBSE 2012]** (ii) $\cos(x^x)$ (iii) $\log(x^x + \operatorname{cosec}^2 x)$ (iv) $x^x e^{2(x+3)}$

SOLUTION (i) Let $y = x^{\cot x} + \dfrac{2x^2 - 3}{x^2 + x + 2}$. Then,

$$y = e^{\cot x \,\log x} + \frac{2x^2 - 3}{x^2 + x + 2} \qquad \left[\because\ x^{\cot x} = e^{\log x^{\cot x}} = e^{\cot x \log x}\right]$$

On differentiating both sides with respect to x, we get

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(e^{\cot x \,\log x}\right) + \frac{d}{dx}\left(\frac{2x^2 - 3}{x^2 + x + 2}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\cot x \,\log x}\frac{d}{dx}(\cot x \,.\, \log x) + \frac{(x^2 + x + 2)\frac{d}{dx}(2x^2 - 3) - (2x^2 - 3)\frac{d}{dx}(x^2 + x + 2)}{(x^2 + x + 2)^2}$$

$$\frac{dy}{dx} = x^{\cot x}\left\{(\log x)\frac{d}{dx}(\cot x) + (\cot x)\frac{d}{dx}(\log x)\right\} + \frac{(x^2 + x + 2)(4x) - (2x^2 - 3)(2x + 1)}{(x^2 + x + 2)^2}$$

$$\frac{dy}{dx} = x^{\cot x}\left\{-\operatorname{cosec}^2 x \,.\, \log x + \frac{\cot x}{x}\right\} + \frac{2x^2 + 14x + 3}{(x^2 + x + 2)^2}$$

(ii) Let $y = \cos(x^x)$. Then, $y = \cos(e^{x\log x})$ $\qquad [\because\ x^x = e^{x\log x}]$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\cos(e^{x\log x})\right\} = -\sin(e^{x\log x})\frac{d}{dx}(e^{x\log x}) = -\sin(x^x)\, e^{x\log x}\frac{d}{dx}(x\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\sin(x^x) \times x^x \left\{\frac{d}{dx}(x) \,.\, \log x + x\frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -x^x \sin(x^x)\left\{\log x + x \cdot \frac{1}{x}\right\} = -x^x \sin(x^x)(\log x + 1)$$

(iii) Let $y = \log(x^x + \operatorname{cosec}^2 x)$. Then,

$$\frac{dy}{dx} = \frac{1}{x^x + \operatorname{cosec}^2 x} \times \frac{d}{dx}(x^x + \operatorname{cosec}^2 x) = \frac{1}{x^x + \operatorname{cosec}^2 x}\left\{\frac{d}{dx}(x^x) + \frac{d}{dx}(\operatorname{cosec}^2 x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^x + \operatorname{cosec}^2 x}\left\{\frac{d}{dx}(e^{x \log x}) + \frac{d}{dx}(\operatorname{cosec}^2 x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^x + \operatorname{cosec}^2 x}\left\{e^{x \log x}\frac{d}{dx}(x \log x) + 2 \operatorname{cosec} x \frac{d}{dx}(\operatorname{cosec} x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x^x + \operatorname{cosec}^2 x}\left\{x^x(1 + \log x) - 2 \operatorname{cosec}^2 x \cot x\right\}$$

(iv) Let $y = x^x e^{2(x+3)}$. Then,

$$y = e^{x \log x} \cdot e^{2(x+3)} \Rightarrow y = e^{x \log x + 2(x+3)} \qquad \left[\because x^x = e^{\log x^x} = e^{x \log x}\right]$$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = e^{x \log x + 2(x+3)} \frac{d}{dx}\{x \log x + 2(x+3)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x \log x} \cdot e^{2(x+3)}\left\{\frac{d}{dx}(x \log x) + 2\frac{d}{dx}(x+3)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^x e^{2(x+3)}(1 + \log x + 2) = x^x e^{2x+3}(3 + \log x)$$

EXAMPLE 8 *If* $x^y = e^{x-y}$, *prove that* $\dfrac{dy}{dx} = \dfrac{\log x}{(1 + \log x)^2}$.

[CBSE 2000 C, 2010 C, 2011, 2013, NCERT EXEMPLAR]

SOLUTION We have,

$$x^y = e^{x-y}$$

$$\Rightarrow \quad e^{y \log x} = e^{x-y} \qquad \left[\because x^y = e^{\log x^y} = e^{y \log x}\right]$$

$$\Rightarrow \quad y \log x = x - y \Rightarrow y \log x + y = x \Rightarrow y(1 + \log x) = x \Rightarrow y = \frac{x}{1 + \log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \times 1 - x\left(0 + \frac{1}{x}\right)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

EXAMPLE 9 *If* $x^y + y^x = 2$, *find* $\dfrac{dy}{dx}$. **[NCERT]**

SOLUTION We have,

$$x^y + y^x = 2 \Rightarrow e^{y \log x} + e^{x \log y} = 2$$

On differentiating both sides with respect to x, we get

$$\frac{d}{dx}\left(e^{y\log x}\right)+\frac{d}{dx}\left(e^{x\log y}\right) = \frac{d}{dx}(2)$$

$$\Rightarrow \quad e^{y\log x}\frac{d}{dx}(y\log x)+e^{x\log y}\frac{d}{dx}(x\log y) = 0$$

$$\Rightarrow \quad x^y\left\{\frac{dy}{dx}\times\log x+y\times\frac{1}{x}\right\}+y^x\left\{1\times\log y+x\times\frac{1}{y}\frac{dy}{dx}\right\} = 0$$

$$\Rightarrow \quad \left\{x^y\log x+y^x\frac{x}{y}\right\}\frac{dy}{dx}+\left\{x^y\times\frac{y}{x}+y^x\times\log y\right\} = 0$$

$$\Rightarrow \quad \left\{x^y\log x+x\,y^{x-1}\right\}\frac{dy}{dx}+\left\{y\,x^{y-1}+y^x\log y\right\} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\left\{\frac{y\,x^{y-1}+y^x\log y}{x^y\log x+x\,y^{x-1}}\right\}$$

EXAMPLE 10 *If* $x^y=y^x$, *find* $\frac{dy}{dx}$. **[NCERT]**

SOLUTION We have, $x^y=y^x$. Taking log on both sides, we get

$$y\log x = x\log y$$

Differentiating both sides with respect to x, we get

$$y\frac{d}{dx}(\log x)+\log x\frac{d}{dx}(y) = x\frac{d}{dx}(\log y)+\log y\frac{d}{dx}(x)$$

$$\Rightarrow \quad y\times\frac{1}{x}+\log x\times\frac{dy}{dx} = x\times\frac{1}{y}\frac{dy}{dx}+(\log y)1$$

$$\Rightarrow \quad \log x\frac{dy}{dx}-\frac{x}{y}\frac{dy}{dx} = \log y-\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\log x-\frac{x}{y}\right) = \log y-\frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{y\log x-x}{y}\right) = \frac{x\log y-y}{x} \Rightarrow \frac{dy}{dx}=\frac{y}{x}\left(\frac{x\log y-y}{y\log x-x}\right).$$

EXAMPLE 11 *If* $(\cos x)^y=(\sin y)^x$, *find* $\frac{dy}{dx}$. **[NCERT, CBSE 2009]**

SOLUTION We have, $(\cos x)^y=(\sin y)^x$. Taking log on both sides, we get

$$y\log\cos x = x\log\sin y$$

Differentiating both sides with respect to x, we get

$$y\frac{d}{dx}(\log\cos x)+\frac{dy}{dx}(\log\cos x) = x\frac{d}{dx}(\log\sin y)+(\log\sin y)1$$

$$\Rightarrow \quad -\frac{y}{\cos x}\sin x+\frac{dy}{dx}\log\cos x = \frac{x}{\sin y}\cos y\frac{dy}{dx}+\log\sin y$$

$$\Rightarrow \quad \frac{dy}{dx}(\log\cos x-x\cot y) = \log\sin y+y\tan x \Rightarrow \frac{dy}{dx} = \frac{\log\sin y+y\tan x}{\log\cos x-x\cot y}.$$

EXAMPLE 12 *If* $y = a^x+e^x+x^x+x^a$, *find* $\frac{dy}{dx}$ at $x = a$. **[NCERT]**

SOLUTION We have,

$$y = a^x + e^x + x^x + x^a$$

$$\Rightarrow \quad y = a^x + e^x + e^{x \log x} + x^a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(a^x) + \frac{d}{dx}(e^x) + \frac{d}{dx}(e^{x \log x}) + \frac{d}{dx}(x^a)$$

$$\Rightarrow \quad \frac{dy}{dx} = a^x \log a + e^x + e^{x \log x} \frac{d}{dx}(x \log x) + a\, x^{a-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = a^x \log a + e^x + x^x (1 + \log x) + a\, x^{a-1}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x = a} = a^a \log a + e^a + a^a (1 + \log a) + a\, a^{a-1} = e^a + 2a^a (1 + \log a)$$

REMARK *In order to find the derivative of a product of a number of functions or a quotient of a number of functions, we first take logarithm of both sides and then differentiate. The procedure is illustrated in the following examples.*

EXAMPLE 13 *If* $y = \dfrac{\sqrt{1-x^2}\,(2x-3)^{1/2}}{(x^2+2)^{2/3}}$, *find* $\dfrac{dy}{dx}$.

SOLUTION Taking log of both sides, we get

$$\log y = \frac{1}{2}\log(1-x^2) + \frac{1}{2}\log(2x-3) - \frac{2}{3}\log(x^2+2).$$

On differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(1-x^2)}(-2x) + \frac{1}{2(2x-3)} \times 2 - \frac{2}{3} \times \frac{1}{x^2+2} \times 2x$$

$$\therefore \quad \frac{dy}{dx} = y\left\{-\frac{x}{1-x^2} + \frac{1}{2x-3} - \frac{4x}{3(x^2+2)}\right\} = \frac{\sqrt{1-x^2}\,(2x-3)^{1/2}}{(x^2+2)^{2/3}}\left\{-\frac{x}{1-x^2} + \frac{1}{2x-3} - \frac{4x}{3(x^2+2)}\right\}$$

EXAMPLE 14 *Find the derivative of* $\dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$ *with respect to* x.

SOLUTION Let $y = \dfrac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}$. Taking log of both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

On differentiating both sides with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2}\,\frac{1}{x+4}\,\frac{d}{dx}(x+4) - \frac{4}{3} \times \frac{1}{4x-3}\,\frac{d}{dx}(4x-3)$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{4}{3(4x-3)} \times 4\right\} = \frac{\sqrt{x}\,(x+4)^{3/2}}{(4x-3)^{4/3}}\left\{\frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}\right\}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *If* $x^m y^n = (x+y)^{m+n}$, *prove that* $\dfrac{dy}{dx} = \dfrac{y}{x}$.

[CBSE 2000, 2014, 2017, NCERT EXAMPLAR]

SOLUTION We have, $x^m \cdot y^n = (x+y)^{m+n}$

Taking log on both sides, we get

$$m \log x + n \log y = (m+n) \log(x+y)$$

Differentiating both sides with respect to x, we get

$$m \times \frac{1}{x} + n \times \frac{1}{y}\frac{dy}{dx} = \frac{m+n}{x+y}\frac{d}{dx}(x+y)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y} \times \frac{dy}{dx} = \frac{m+n}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left\{\frac{n}{y} - \frac{m+n}{x+y}\right\}\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \left\{\frac{nx+ny-my-ny}{y(x+y)}\right\}\frac{dy}{dx} = \left\{\frac{mx+nx-mx-my}{(x+y)\,x}\right\}$$

$$\Rightarrow \quad \frac{nx-my}{y(x+y)} \cdot \frac{dy}{dx} = \frac{nx-my}{(x+y)\,x} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

EXAMPLE 16 *If* $y = \dfrac{ax^2}{(x-a)(x-b)(x-c)} + \dfrac{bx}{(x-b)(x-c)} + \dfrac{c}{x-c} + 1$, *prove that*

$$\frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{c-x}\right\}.$$

SOLUTION We have,

$$y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \left(\frac{c+x-c}{x-c}\right)$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{x}{x-c}$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx + x(x-b)}{(x-b)(x-c)}$$

$$\Rightarrow \quad y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{x^2}{(x-b)(x-c)}$$

$$\Rightarrow \quad y = \frac{ax^2 + x^2(x-a)}{(x-a)(x-b)(x-c)} \Rightarrow y = \frac{x^3}{(x-a)(x-b)(x-c)} \Rightarrow \log y = \log\left\{\frac{x^3}{(x-a)(x-b)(x-c)}\right\}$$

$$\Rightarrow \quad \log y = 3 \log x - \left\{\log(x-a) + \log(x-b) + \log(x-c)\right\}$$

On differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \left\{\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{\left(\frac{1}{x} - \frac{1}{x-a}\right) + \left(\frac{1}{x} - \frac{1}{x-b}\right) + \left(\frac{1}{x} - \frac{1}{x-c}\right)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left\{-\frac{a}{x(x-a)} - \frac{b}{x(x-b)} - \frac{c}{x(x-c)}\right\} \Rightarrow \frac{dy}{dx} = \frac{y}{x}\left\{\frac{a}{a-x} + \frac{b}{b-x} + \frac{c}{x-c}\right\}$$

EXAMPLE 17 *Prove that the derivative of an even function is an odd function and that of an odd function is an even function.*

SOLUTION Let $f(x)$ be an even function. Then,

$$f(-x) = f(x) \quad \Rightarrow \frac{d}{dx}\{f(-x)\} = \frac{d}{dx}\{f(x)\} \Rightarrow f'(-x)\cdot\frac{d}{dx}(-x) = f'(x)$$

$$\Rightarrow \quad -f'(-x) = f'(x) \Rightarrow f'(-x) = -f'(x) \quad \Rightarrow f'(x) \text{ is an odd function.}$$

Let $f(x)$ be an odd function. Then,

$$f(-x) = -f(x) \quad \Rightarrow \frac{d}{dx}\{f(-x)\} = -\frac{d}{dx}\{f(x)\} \Rightarrow f'(-x)\frac{d}{dx}(-x) = -f'(x)$$

$$\Rightarrow \quad -f'(-x) = -f'(x) \Rightarrow f'(-x) = f'(x) \Rightarrow f'(x) \text{ is an even function.}$$

EXAMPLE 18 *If* $y = f\left(\frac{2x-1}{x^2+1}\right)$ *and* $f'(x) = \sin x^2$, *find* $\frac{dy}{dx}$.

SOLUTION Let $z = \frac{2x-1}{x^2+1}$. Then,

$$y = f(z)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\{f(z)\} = \frac{d}{dz}\{f(z)\}\cdot\frac{dz}{dx} = f'(z)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right) = f'(z)\left\{\frac{2(x^2+1)-(2x-1)\,2x}{(x^2+1)^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin z^2)\,\frac{2(x^2+1)-(4x^2-2x)}{(x^2+1)^2} \qquad [\because f'(x) = \sin x^2 \therefore f'(z) = \sin z^2]$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{1+x-x^2}{(x^2+1)^2}\right\}$$

EXAMPLE 19 *Given that* $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ldots = \frac{\sin x}{x}$, *prove that*

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{4} + \ldots = \text{cosec}^2 x - \frac{1}{x^2}$$

SOLUTION We have, $\cos\frac{x}{2}\cdot\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ldots = \frac{\sin x}{x}$

Taking log on both sides, we get

$$\log\cos\frac{x}{2} + \log\cos\frac{x}{4} + \log\cos\frac{x}{8}\ldots = \log\sin x - \log x$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} - \frac{1}{4}\frac{\sin\frac{x}{4}}{\cos\frac{x}{4}} - \frac{1}{8}\frac{\sin\frac{x}{8}}{\cos\frac{x}{8}}\ldots = \frac{\cos x}{\sin x} - \frac{1}{x}$$

$$\Rightarrow \quad -\frac{1}{2}\tan\frac{x}{2} - \frac{1}{4}\tan\frac{x}{4} - \frac{1}{8}\tan\frac{x}{8}\ldots = \cot x - \frac{1}{x}$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2^2}\sec^2\frac{x}{2}-\frac{1}{4^2}\sec^2\frac{x}{4}-\frac{1}{8^2}\sec^2\frac{x}{8}\ldots=-\operatorname{cosec}^2 x+\frac{1}{x^2}$$

$$\Rightarrow\quad \frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{4^2}\sec^2\frac{x}{4}+\frac{1}{8^2}\sec^2\frac{x}{8}\ldots=\operatorname{cosec}^2 x-\frac{1}{x^2}$$

EXERCISE 10.5

BASIC

Differentiate the following functions with respect to x: (1-18)

1. $x^{1/x}$
2. $x^{\sin x}$
3. $(1+\cos x)^x$
4. $x^{\cos^{-1} x}$
5. $(\log x)^x$
6. $(\log x)^{\cos x}$
7. $(\sin x)^{\cos x}$
8. $e^{x \log x}$
9. $(\sin x)^{\log x}$
10. $10^{\log \sin x}$
11. $(\log x)^{\log x}$ [NCERT]
12. $10^{(10^x)}$
13. $\sin (x^x)$
14. $(\sin^{-1} x)^x$
15. $x^{\sin^{-1} x}$
16. $(\tan x)^{1/x}$
17. $x^{\tan^{-1} x}$
18. (i) $(x^x)\sqrt{x}$
    (ii) $x^{(\sin x-\cos x)}+\dfrac{x^2-1}{x^2+1}$
    (iii) $x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$ [NCERT, CBSE 2011]
    (iv) $(x\cos x)^x+(x\sin x)^{1/x}$ [NCERT]
    (v) $\left(x+\dfrac{1}{x}\right)^x+x^{\left(1+\frac{1}{x}\right)}$ [NCERT]
    (vi) $e^{\sin x}+(\tan x)^x$ [CBSE 2003]
    (vii) $(\cos x)^x+(\sin x)^{1/x}$ [CBSE 2010]
    (viii) $x^{x^2-3}+(x-3)^{x^2}$ [NCERT]

Find $\dfrac{dy}{dx}$, *(19-32) when:*

19. $y=e^x+10^x+x^x$
20. $y=x^n+n^x+x^x+n^n$
21. $y=\dfrac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}$
22. $y=\dfrac{e^{ax}\sec x\,\log x}{\sqrt{1-2x}}$
23. $y=e^{3x}\sin 4x\cdot 2^x$
24. $y=\sin x\sin 2x\sin 3x\sin 4x$
25. $y=x^{\sin x}+(\sin x)^x$ [NCERT]
26. $y=(\sin x)^{\cos x}+(\cos x)^{\sin x}$
27. $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$
28. (i) $y=(\sin x)^x+\sin^{-1}\sqrt{x}$ [NCERT, CBSE 2009, 13, 17]
    (ii) $y=x^{\sin x}+\sin^{-1}\sqrt{x}$ [CBSE 20 20]
29. (i) $y=x^{\cos x}+(\sin x)^{\tan x}$ [CBSE 2009]
    (ii) $y=x^x+(\sin x)^x$ [CBSE 2008]
30. $y=(\tan x)^{\log x}+\cos^2\left(\dfrac{\pi}{4}\right)$
31. $y=x^x+x^{1/x}$
32. $y=x^{\log x}+(\log x)^x$ [NCERT, CBSE 2013, 2019]

BASED ON LOTS

33. If $x^y-y^x=a^b$, find $\dfrac{dy}{dx}$. [CBSE 2019]
34. If $x^{16}y^9=(x^2+y)^{17}$, prove that $x\dfrac{dy}{dx}=2y$

35. If $y = \sin(x^x)$, prove that $\dfrac{dy}{dx} = \cos(x^x) \cdot x^x (1 + \log x)$

36. If $x^x + y^x = 1$, prove that $\dfrac{dy}{dx} = -\left\{ \dfrac{x^x (1 + \log x) + y^x \cdot \log y}{x \cdot y^{(x-1)}} \right\}$

37. If $x^y \cdot y^x = 1$, prove that $\dfrac{dy}{dx} = -\dfrac{y(y + x\log y)}{x(y \log x + x)}$

38. If $x^y + y^x = (x + y)^{x+y}$, find $\dfrac{dy}{dx}$

39. If $x^m y^n = 1$, prove that $\dfrac{dy}{dx} = -\dfrac{my}{nx}$

40. If $y^x = e^{y-x}$, prove that $\dfrac{dy}{dx} = \dfrac{(1 + \log y)^2}{\log y}$ [NCERT EXEMPLAR]

41. If $(\sin x)^y = (\cos y)^x$, prove that $\dfrac{dy}{dx} = \dfrac{\log \cos y - y \cot x}{\log \sin x + x \tan y}$

42. If $(\cos x)^y = (\tan y)^x$, prove that $\dfrac{dy}{dx} = \dfrac{\log \tan y + y \tan x}{\log \cos x - x \sec y \operatorname{cosec} y}$

43. If $e^x + e^y = e^{x+y}$, prove that $\dfrac{dy}{dx} + e^{y-x} = 0$ [CBSE 2014]

44. If $e^y = y^x$, prove that $\dfrac{dy}{dx} = \dfrac{(\log y)^2}{\log y - 1}$

45. If $e^{x+y} - x = 0$, prove that $\dfrac{dy}{dx} = \dfrac{1-x}{x}$

46. If $y = x\sin(a + y)$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin(a + y) - y\cos(a + y)}$

47. If $x\sin(a + y) + \sin a \cos(a + y) = 0$, prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a + y)}{\sin a}$ [CBSE 2013]

48. If $(\sin x)^y = x + y$, prove that $\dfrac{dy}{dx} = \dfrac{1 - (x + y)\, y \cot x}{(x + y)\log \sin x - 1}$

49. If $xy \log(x + y) = 1$, prove that $\dfrac{dy}{dx} = -\dfrac{y(x^2 y + x + y)}{x(xy^2 + x + y)}$

50. If $y = x\sin y$, prove that $\dfrac{dy}{dx} = \dfrac{y}{x(1 - x\cos y)}$

51. Find the derivative of the function $f(x)$ given by $f(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$ and hence find $f'(1)$

52. If $y = \log \dfrac{x^2 + x + 1}{x^2 - x + 1} + \dfrac{2}{\sqrt{3}} \tan^{-1}\left(\dfrac{\sqrt{3}\,x}{1 - x^2}\right)$, find $\dfrac{dy}{dx}$.

53. If $y = (\sin x - \cos x)^{\sin x - \cos x}$, $\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$, find $\dfrac{dy}{dx}$. [CBSE 2010]

54. If $xy = e^{x-y}$, find $\dfrac{dy}{dx}$. [NCERT]

55. If $y^x + x^y + x^x = a^b$, find $\dfrac{dy}{dx}$. [NCERT]

56. If $(\cos x)^y = (\cos y)^x$ find $\dfrac{dy}{dx}$. [CBSE 2012]

57. If $\cos y = x\cos(a+y)$, where $\cos a \neq \pm 1$, prove that $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$. [CBSE 2014]

BASED ON HOTS

58. If $(x-y)\, e^{\frac{x}{x-y}} = a$, prove that $y\dfrac{dy}{dx} + x = 2y$. [CBSE 2014]

59. If $x = e^{x/y}$, prove that $\dfrac{dy}{dx} = \dfrac{x-y}{x\log x}$ [NCERT EXEMPLAR]

60. If $y = x^{\tan x} + \sqrt{\dfrac{x^2+1}{2}}$, find $\dfrac{dy}{dx}$ [NCERT EXEMPLAR]

61. If $y = 1 + \dfrac{\alpha}{\left(\frac{1}{x}-\alpha\right)} + \dfrac{\beta/x}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)} + \dfrac{\gamma/x^2}{\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)}$, find $\dfrac{dy}{dx}$

ANSWERS

1. $x^{1/x}\left(\dfrac{1-\log x}{x^2}\right)$

2. $x^{\sin x}\left\{\dfrac{\sin x}{x} + (\cos x)\log x\right\}$

3. $(1+\cos x)^x\left\{\log(1+\cos x) - \dfrac{x\sin x}{1+\cos x}\right\}$

4. $x^{\cos^{-1}x}\left\{\dfrac{\cos^{-1}x}{x} - \dfrac{\log x}{\sqrt{1-x^2}}\right\}$

5. $(\log x)^x\left\{\log(\log x) + \dfrac{1}{\log x}\right\}$

6. $(\log x)^{\cos x}\left\{-\sin x\cdot\log(\log x) + \dfrac{\cos x}{x\log x}\right\}$

7. $(\sin x)^{\cos x}\{-\sin x\log\sin x + \cos x\cot x\}$

8. $x^x(1+\log x)$

9. $(\sin x)^{\log x}\left\{\dfrac{1}{x}\log\sin x + \log x\cdot\cot x\right\}$

10. $10^{\log\sin x}\cdot\log 10\cdot\cot x$

11. $(\log x)^{\log x}\left\{\dfrac{1+\log(\log x)}{x}\right\}$

12. $10^{10^x}10^x(\log_e 10)^2$

13. $x^x(1+\log x)\cos(x^x)$

14. $(\sin^{-1}x)^x\left\{\log\sin^{-1}x + \dfrac{x}{\sin^{-1}x}\cdot\dfrac{1}{\sqrt{1-x^2}}\right\}$

15. $x^{\sin^{-1}x}\left\{\dfrac{\sin^{-1}x}{x} + \dfrac{\log x}{\sqrt{1-x^2}}\right\}$

16. $(\tan x)^{1/x}\left\{-\dfrac{1}{x^2}\log\tan x + \dfrac{1}{x}\dfrac{\sec^2 x}{\tan x}\right\}$

17. $x^{\tan^{-1}x}\left\{\dfrac{\tan^{-1}x}{x} + \dfrac{\log x}{1+x^2}\right\}$

18. (i) $x^{x+1/2}\left\{\left(\dfrac{2x+1}{2x}\right) + \log x\right\}$

(ii) $x^{(\sin x-\cos x)}\left\{\frac{\sin x-\cos x}{x}+(\cos x+\sin x)\log x\right\}+\frac{4x}{(x^2+1)^2}$

(iii) $x^{x\cos x}\{(1+\log x)\cos x-x\log x\sin x\}-\frac{4x}{(x^2-1)^2}$

(iv) $(x\cos x)^x\{1-x\tan x+\log(x\cos x)\}+(x\sin x)^{1x}\frac{\{1+x\cot x-\log(x\sin x)\}}{x^2}$

(v) $\left(x+\frac{1}{x}\right)^x\left\{\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right\}+x^{1+\frac{1}{x}}\left\{\frac{x+1}{x^2}-\frac{\log x}{x^2}\right\}$

(vi) $e^{\sin x}\cos x+(\tan x)^x\{\log\tan x+x\sec x\operatorname{cosec} x\}$

(vii) $(\cos x)^x(\log\cos x-x\tan x)+(\sin x)^{1/x}\left(-\frac{1}{x^2}\log\sin x+\frac{\cot x}{x}\right)$

(viii) $x^{x^2-3}\left\{\frac{x^2-3}{x}+2x\log x\right\}+(x-3)^{x^2}\left\{\frac{x^2}{x-3}+2x\log(x-3)\right\}$

19. $e^x+10^x\log 10+x^x\log(ex)$

20. $nx^{n-1}+n^x\log n+x^x\log(ex)$

21. $\frac{(x^2-1)^3(2x-1)}{\sqrt{(x-3)(4x-1)}}\left\{\frac{6x}{x^2-1}+\frac{2}{2x-1}-\frac{1}{2(x-3)}-\frac{2}{4x-1}\right\}$

22. $\frac{e^{ax}\sec x\log x}{\sqrt{1-2x}}\left\{a+\tan x+\frac{1}{x\log x}+\frac{1}{1-2x}\right\}$

23. $e^{3x}\sin 4x\, 2^x(3+4\cot 4x+\log 2)$

24. $\sin x\sin 2x\sin 3x\sin 4x(\cot x+2\cot 2x+3\cot 3x+4\cot 4x)$

25. $x^{\sin x}\left\{\cos x\log x+\frac{\sin x}{x}\right\}+(\sin x)^x\{\log\sin x+x\cot x\}$

26. $(\sin x)^{\cos x}\{-\sin x\log\sin x+\cos x\cot x\}+(\cos x)^{\sin x}\{\cos x\log\cos x-\sin x\tan x\}$

27. $(\tan x)^{\cot x}\operatorname{cosec}^2 x(1-\log\tan x)+(\cot x)^{\tan x}\sec^2 x\{\log\cot x-1\}$

28. (i) $(\sin x)^x(x\cot x+\log\sin x)+\frac{1}{2\sqrt{x-x^2}}$ (ii) $x^{\sin x}\left(\frac{\sin x}{x}+\cos x.\log x\right)+\frac{1}{2\sqrt{x-x^2}}$

29. (i) $x^{\cos x}\left\{\frac{\cos x}{x}-\sin x\log x\right\}+(\sin x)^{\tan x}\left\{1+\sec^2 x\log\sin x\right\}$

(ii) $x^x(1+\log x)+(\sin x)^x(x\cot x+\log\sin x)$

30. $(\tan x)^{\log x}\left\{\log x\frac{\sec^2 x}{\tan x}+\frac{\log\tan x}{x}\right\}$

31. $x^x(1+\log x)+x^{1/x}\left\{\frac{1-\log x}{x^2}\right\}$

32. $x^{\log x}\left\{\frac{2\log x}{x}\right\}+(\log x)^x\left\{\log(\log x)+\frac{1}{\log x}\right\}$

33. $\frac{y^x\log y-yx^{y-1}}{x^y\log x-xy^{x-1}}$

38. $\dfrac{(x+y)^{(x+y)}\{1+\log(x+y)\}-yx^{y-1}-y^{x}\log y}{x^{y}\log x+xy^{x-1}-(x+y)^{x+y}\{1+\log(x+y)\}}$

51. $1+2x+3x^2+\ldots+15x^{14}, f'(1)=120$ 52. $\dfrac{4}{x^4+x^2+1}$

53. $(\sin x-\cos x)^{\sin x-\cos x}\{(\sin x+\cos x)\log(\sin x-\cos x)+(\cos x+\sin x)\}$

54. $\dfrac{y(x-1)}{x(y+1)}$ 55. $-\dfrac{y^x\log y+yx^{y-1}+x^x(1+\log x)}{xy^{x-1}+x^y\log x}$ 56. $\dfrac{\log\cos y+y\tan x}{\log\cos x+x\tan y}$

60. $x^{\tan x}\left\{\sec^2 x\log x+\dfrac{\tan x}{x}\right\}+\dfrac{x}{\sqrt{2x^2+2}}$ 61. $\dfrac{y}{x}\left\{\dfrac{\alpha}{\frac{1}{x}-\alpha}+\dfrac{\beta}{\frac{1}{x}-\beta}+\dfrac{\gamma}{\frac{1}{x}-\gamma}\right\}$

HINTS TO SELECTED PROBLEMS

11. Let $y=(\log x)^{\log x}$. Then, $y=e^{\log\left\{(\log x)^{\log x}\right\}}\Rightarrow y=e^{(\log x)\log(\log x)}$

$$\Rightarrow \frac{dy}{dx}=e^{(\log x)\log(\log x)}\frac{d}{dx}\left\{(\log x)\ \log(\log x)\right\}$$

$$=(\log x)^{\log x}\left\{\frac{d}{dx}(\log x)\cdot\log(\log x)+\log x\frac{d}{dx}\log(\log x)\right\}$$

$$=(\log x)^{\log x}\left\{\frac{\log(\log x)}{x}+\log x\times\frac{1}{\log x}\times\frac{1}{x}\right\}$$

$$=(\log x)^{\log x}\left\{\frac{\log(\log x)}{x}+\frac{1}{x}\right\}=(\log x)^{\log x}\left\{\frac{1+\log(\log x)}{x}\right\}$$

18. (iii) Let $y=x^{x\cos x}+\dfrac{x^2+1}{x^2-1}$. Then, $y=e^{x\cos x\log x}+\dfrac{x^2+1}{x^2-1}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left\{e^{x\cos x\log x}\right\}+\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)=e^{x\cos x\log x}\frac{d}{dx}(x\cos x\log x)+\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=x^{x\cos x}\{\cos x\log x-x\sin x\log x+\cos x\}+\frac{(x^2-1)\,2x-(x^2+1)\,2x}{(x^2-1)^2}$$

$$=x^{x\cos x}\{\cos x\log x-x\sin x\log x+\cos x\}-\frac{4x}{(x^2-1)^2}$$

18. (iv) Let $y=(x\cos x)^x+(x\sin x)^{1/x}$. Then,

$$y=e^{\log(x\cos x)^x}+e^{\log(x\sin x)^{1/x}}$$

$$\Rightarrow y=e^{x\log(x\cos x)}+e^{1/x\log(x\sin x)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left\{e^{x\log(x\cos x)}\right\}+\frac{d}{dx}\left\{e^{1/x\ \log(x\sin x)}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x \log (x \cos x)} \frac{d}{dx}\{x \log (x \cos x)\} + e^{1/x \log (x \sin x)} \frac{d}{dx}\left\{\frac{1}{x} \log (x \sin x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x \cos x)^x \frac{d}{dx}\{x (\log x + \log \cos x)\} + (x \sin x)^{1/x} \frac{d}{dx}\left\{\frac{1}{x}(\log x + \log \sin x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x \cos x)^x \left\{(\log x + \log \cos x) + x\left(\frac{1}{x} - \tan x\right)\right\}$$

$$+ (x \sin x)^{1/x}\left\{\frac{1}{x}\left(\frac{1}{x} + \cot x\right) - \frac{1}{x^2}(\log x + \log \sin x)\right\}$$

$$= (x \cos x)^x \{\log (x \cos x) + (1 - x \tan x)\} + (x \sin x)^{1/x}\left\{\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log x}{x^2} - \frac{\log \sin x}{x^2}\right\}$$

$$= (x \cos x)^x \{\log (x \cos x) + 1 - x \tan x\} + (x \sin x)^{1/x}\left\{\frac{1 + x \cot x - \log (x \sin x)}{x^2}\right\}$$

18. (v) Let $y = \left(x + \frac{1}{x}\right)^x + x^{1 + 1/x}$. Then,

$$y = e^{x \log (x + 1/x)} + e^{(1 + 1/x) \log x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x \log (x + 1/x)} \frac{d}{dx}\left\{x \log\left(x + \frac{1}{x}\right)\right\} + e^{(1 + 1/x) \log x} \frac{d}{dx}\left\{\left(1 + \frac{1}{x}\right) \log x\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left\{\log\left(x + \frac{1}{x}\right) + \frac{x}{x + \frac{1}{x}} \frac{d}{dx}\left(x + \frac{1}{x}\right)\right\} + x^{1 + 1/x}\left\{-\frac{1}{x^2} \log x + \left(1 + \frac{1}{x}\right)\frac{1}{x}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left\{\log\left(x + \frac{1}{x}\right) + \frac{x^2}{x^2 + 1}\left(1 - \frac{1}{x^2}\right)\right\} + x^{1 + 1/x}\left\{\frac{1 + x - \log x}{x^2}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \left(x + \frac{1}{x}\right)^x \left\{\log\left(x + \frac{1}{x}\right) + \frac{x^2 - 1}{x^2 + 1}\right\} + x^{1/x - 1}\{1 + x - \log x\}$$

18. (viii) Let $y = x^{x^2 - 3} + (x - 3)^{x^2}$. Then,

$$y = e^{\log x^{(x^2 - 3)}} + e^{\log (x - 3)^{x^2}}$$

$$\Rightarrow \quad y = e^{(x^2 - 3) \log x} + e^{x^2 \log (x - 3)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left\{e^{(x^2 - 3) \log x}\right\} + \frac{d}{dx}\left\{e^{x^2 \log (x - 3)}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{(x^2 - 3) \log x} \frac{d}{dx}\left\{(x^2 - 3) \log x\right\} + e^{x^2 \log (x - 3)} \frac{d}{dx}\left\{x^2 \log (x - 3)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^2 - 3}\left\{2x \log x + \frac{x^2 - 3}{x}\right\} + (x - 3)^{x^2}\left\{2x \log (x - 3) + \frac{x^2}{x - 3}\right\}$$

25. We have, $y = x^{\sin x} + (\sin x)^x$

$$\Rightarrow \quad y = e^{\log x^{\sin x}} + e^{\log (\sin x)^x} \Rightarrow y = e^{\sin x \log x} + e^{x \log \sin x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{\sin x \log x}) + \frac{d}{dx}(e^{x \log \sin x})$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{\sin x \log x}\frac{d}{dx}(\sin x \log x) + e^{x \log \sin x}\frac{d}{dx}(x \log \sin x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\sin x}\left\{\cos x \log x + \frac{\sin x}{x}\right\} + (\sin x)^x \{\log \sin x + x \cot x\}$$

28. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$. Then, $y = e^{x \log \sin x} + \sin^{-1}\sqrt{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{x \log \sin x}) + \frac{d}{dx}(\sin^{-1}\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{x \log \sin x}\frac{d}{dx}(x \log \sin x) + \frac{d}{dx}(\sin^{-1}\sqrt{x})$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^x \{\log \sin x + x \cot x\} + \frac{1}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{dy}{dx} = (\sin x)^x \{\log \sin x + x \cot x\} + \frac{1}{2\sqrt{x - x^2}}$$

32. We have, $y = x^{\log x} + (\log x)^x$

$$y = e^{\log (x^{\log x})} + e^{\log (\log x)^x} = e^{\log x \log x} + e^{x \log (\log x)} = e^{(\log x)^2} + e^{x \log (\log x)}$$

$$\therefore \quad \frac{dy}{dx} = e^{(\log x)^2}\frac{d}{dx}(\log x)^2 + e^{x \log (\log x)}\frac{d}{dx}\{x \log (\log x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{\log x}\left(2 \log x \times \frac{1}{x}\right) + (\log x)^x \left\{\log (\log x) + \frac{x}{\log x} \times \frac{1}{x}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x^{\log x}}{x}\log x + (\log x)^x \left\{\log (\log x) + \frac{1}{\log x}\right\}$$

54. We have,

$$xy = e^{x-y} \Rightarrow \log (xy) = \log (e^{x-y}) \Rightarrow \log x + \log y = x - y$$

Differentiating with respect to x, we get

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx} \Rightarrow \frac{dy}{dx}\left(\frac{1}{y} + 1\right) = 1 - \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

55. We have,

$$y^x + x^y + x^x = a^b$$

$$\Rightarrow \quad e^{\log y^x} + e^{\log x^y} + e^{\log x^x} = a^b \Rightarrow e^{x \log y} + e^{y \log x} + e^{x \log x} = a^b$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(e^{x \log y}) + \frac{d}{dx}(e^{y \log x}) + \frac{d}{dx}(e^{x \log x}) = \frac{d}{dx}(a^b)$$

$$\Rightarrow \quad e^{x \log y}\frac{d}{dx}(x \log y) + e^{y \log x}\frac{d}{dx}(y \log x) + e^{x \log x}\frac{d}{dx}(x \log x) = 0$$

$$\Rightarrow \quad y^x\left(\log y + \frac{x}{y}\frac{dy}{dx}\right) + x^y\left(\frac{dy}{dx}\log x + \frac{y}{x}\right) + x^x(1 + \log x) = 0$$

$$\Rightarrow \quad \frac{dy}{dx}(xy^{x-1} + x^y \log x) = -\{y^x \log y + y\, x^{y-1} + x^x (1 + \log x)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{\{y^x \log y + y\, x^{y-1} + x^x (1 + \log x)\}}{xy^{x-1} + x^y \log x}$$

10.9 DIFFERENTIATION OF INFINITE SERIES

Sometimes the value of y is given as an infinite series and we are asked to find $\frac{dy}{dx}$. In such cases we use the fact that if a term is deleted from an infinite series, it remains unaffected. The method of finding $\frac{dy}{dx}$ is explained in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If* $y = x^{x^{x^{\cdots\infty}}}$, *find* $\frac{dy}{dx}$.

SOLUTION Since by deleting a single term from an infinite series, it remains same. Therefore, the given function may be written as

$$y = x^y$$

$$\Rightarrow \quad \log y = y \log x \qquad \text{[On taking log of both sides]}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + y\frac{d}{dx}(\log x) \qquad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + \frac{y}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{y} - \log x\right\} = \frac{y}{x} \Rightarrow \frac{dy}{dx}\frac{(1 - y\log x)}{y} = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1 - y\log x)}$$

EXAMPLE 2 *If* $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots \text{to } \infty}}}$, *prove that* $\frac{dy}{dx} = \frac{\cos x}{2y - 1}$.

SOLUTION The given series may be written as

$$y = \sqrt{\sin x + y}$$

$$\Rightarrow \quad y^2 = \sin x + y \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = \cos x + \frac{dy}{dx} \qquad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{dy}{dx}(2y - 1) = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y - 1}.$$

EXAMPLE 3 *If* $y = a^{x^{a^{x^{\cdots\infty}}}}$, *prove that* $\frac{dy}{dx} = \frac{y^2 \log y}{x(1 - y\log x \cdot \log y)}$.

SOLUTION The given series may be written as

$$y = a^{(x^y)}$$

$$\Rightarrow \quad \log y = x^y \log a \qquad \text{[Taking log of both sides]}$$

$$\Rightarrow \quad \log(\log y) = y\log x + \log(\log a) \qquad \text{[Taking log of both sides]}$$

$$\Rightarrow \quad \frac{1}{\log y}\frac{d}{dx}(\log y) = \frac{dy}{dx}\log x + y\frac{d}{dx}(\log x) + 0 \qquad \text{[Differentiating both sides w.r.t. } x]$$

$$\Rightarrow \quad \frac{1}{\log y}\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx}\log x + y \times \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{y\log y} - \log x\right\} = \frac{y}{x} \Rightarrow \frac{dy}{dx}\left\{\frac{1 - y\log y \log x}{y \log y}\right\} = \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y^2 \log y}{x\{1 - y\log y \log x\}}.$$

EXAMPLE 4 *If* $y = e^{x + e^{x + e^{x + \dots \text{ to } \infty}}}$, *show that* $\frac{dy}{dx} = \frac{y}{1-y}$.

SOLUTION The given function may be written as

$$y = e^{x+y}$$

$$\Rightarrow \quad \log y = (x+y)\log e \qquad \text{[Taking log of both sides]}$$

$$\Rightarrow \quad \log y = x + y \qquad [\because \log e = 1]$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1 + \frac{dy}{dx} \qquad \text{[Differentiating with respect to } x]$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - 1\right) = 1 \Rightarrow \frac{dy}{dx} = \frac{y}{1-y}$$

EXAMPLE 5 *If* $y = (\sqrt{x})^{(\sqrt{x})^{(\sqrt{x})^{\dots\infty}}}$, *show that* $\frac{dy}{dx} = \frac{y^2}{x(2 - y\log x)}$.

SOLUTION The given function can be written as

$$y = (\sqrt{x})^y$$

$$\Rightarrow \quad y = x^{y/2}$$

$$\Rightarrow \quad \log y = \frac{y}{2}\log x \qquad \text{[On taking log of both sides]}$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{y}{2} \times \frac{1}{x} + \frac{1}{2}\log x\frac{dy}{dx} \qquad \text{[Differentiating both sides with respect to } x]$$

$$\Rightarrow \quad \frac{dy}{dx}\left\{\frac{1}{y} - \frac{1}{2}\log x\right\} = \frac{y}{2x} \Rightarrow \frac{dy}{dx}\left\{\frac{2 - y\log x}{2y}\right\} = \frac{y}{2x} \Rightarrow \frac{dy}{dx} = \frac{y^2}{x(2 - y\log x)}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *If* $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \dots}}}$, *prove that* $\frac{dy}{dx} = \frac{y}{2y - x}$.

SOLUTION We have,

$$y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \dots}}} \Rightarrow y = x + \frac{1}{y} \Rightarrow y^2 = xy + 1$$

Differentiating both sides with respect to x, we obtain

$$2y\frac{dy}{dx} = y + x\frac{dy}{dx} + 0 \Rightarrow \frac{dy}{dx}(2y - x) = y \Rightarrow \frac{dy}{dx} = \frac{y}{2y - x}.$$

EXAMPLE 7 *If* $y = \dfrac{\sin x}{1 + \dfrac{\cos x}{1 + \dfrac{\sin x}{1 + \dfrac{\cos x}{1 + \ldots \text{ to } \infty}}}}$, *prove that* $\dfrac{dy}{dx} = \dfrac{(1+y)\cos x + y \sin x}{1 + 2y + \cos x - \sin x}$.

SOLUTION We have,

$$y = \frac{\sin x}{1 + \dfrac{\cos x}{1+y}} \Rightarrow y = \frac{(1+y)\sin x}{1 + y + \cos x} \Rightarrow y + y^2 + y\cos x = (1+y)\sin x$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y \sin x = \frac{dy}{dx}\sin x + (1+y)\cos x$$

$$\Rightarrow \quad \frac{dy}{dx}\{1 + 2y + \cos x - \sin x\} = (1+y)\cos x + y \sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1 + 2y + \cos x - \sin x}.$$

EXERCISE 10.6

BASIC

1. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{2y - 1}$.

2. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \text{ to } \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{\sin x}{1 - 2y}$.

3. If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \text{ to } \infty}}}$, prove that $(2y-1)\dfrac{dy}{dx} = \dfrac{1}{x}$.

4. If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \text{ to } \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{\sec^2 x}{2y - 1}$.

5. If $y = (\sin x)^{(\sin x)^{(\sin x)^{\cdots \infty}}}$, prove that $\dfrac{dy}{dx} = \dfrac{y^2 \cot x}{(1 - y \log \sin x)}$.

6. If $y = (\tan x)^{(\tan x)^{(\tan x)^{\cdots \infty}}}$, prove that $\dfrac{dy}{dx} = 2$ at $x = \dfrac{\pi}{4}$.

BASED ON LOTS

7. If $y = (\cos x)^{(\cos x)^{(\cos x)\ldots \infty}}$, prove that $\dfrac{dy}{dx} = -\dfrac{y^2 \tan x}{(1 - y \log \cos x)}$. **[NCERT EXEMPLAR]**

BASED ON HOTS

8. If $y = e^{x^{e^x}} + x^{e^{e^x}} + e^{x^{x^e}}$, prove that

$$\frac{dy}{dx} = e^{x^{e^x}} \cdot x^{e^x}\left\{\frac{e^x}{x} + e^x \cdot \log x\right\} + x^{e^{e^x}} \cdot e^{e^x}\left\{\frac{1}{x} + e^x \cdot \log x\right\} + e^{x^{x^e}} x^{x^e} \cdot x^{e-1}\{1 + e \log x\}$$

10.10 DIFFERENTIATION OF PARAMETRIC FUNCTIONS

Sometimes x and y are given as functions of a single variable e.g. $x = \phi(t)$, $y = \psi(t)$ are two functions of a single variable. In such a case x and y are called parametric functions or parametric equations and t is called the parameter. To find $\frac{dy}{dx}$ in case of parametric functions, we first obtain the relationship between x and y by eliminating the parameter t and then we differentiate it with respect to x. But, it is not always convenient to eliminate the parameter. Therefore, $\frac{dy}{dx}$ can also be obtained by the following formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

In order to prove this, let Δx and Δy be the changes in x and y respectively corresponding to a small change Δt in t. Then,

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y/\Delta t}{\Delta x/\Delta t} \Rightarrow \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim\limits_{\Delta t \to 0} \frac{\Delta y}{\Delta t}}{\lim\limits_{\Delta t \to 0} \frac{\Delta x}{\Delta t}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find $\frac{dy}{dx}$ in each of the following:*

(i) $x = a\left\{\cos t + \frac{1}{2}\log \tan^2 \frac{t}{2}\right\}$ and $y = a \sin t$. [CBSE 2011, 2019 NCERT]

(ii) $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$. [NCERT]

SOLUTION (i) $x = a\left\{\cos t + \frac{1}{2}\log \tan^2 \frac{t}{2}\right\}$ and $y = a \sin t$

$$\Rightarrow \quad x = a\left\{\cos t + \frac{1}{2} \times 2 \log \tan \frac{t}{2}\right\} \text{ and } y = a \sin t$$

$$\Rightarrow \quad x = a\left\{\cos t + \log \tan \frac{t}{2}\right\} \text{ and } y = a \sin t.$$

Differentiating with respect to t, we get

$$\frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\tan t/2}\left(\sec^2 \frac{t}{2}\right) \times \frac{1}{2}\right\} \text{ and } \frac{dy}{dt} = a \cos t$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{2 \sin (t/2) \cos (t/2)}\right\} \text{ and } \frac{dy}{dt} = a \cos t$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{-\sin t + \frac{1}{\sin t}\right\} \text{ and } \frac{dy}{dt} = a \cos t$$

$$\Rightarrow \quad \frac{dx}{dt} = a\left\{\frac{-\sin^2 t + 1}{\sin t}\right\} \text{ and } \frac{dy}{dt} = a \cos t$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{a \cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a \cos t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \cos t}{\frac{a \cos^2 t}{\sin t}} = \tan t$$

(ii) We have, $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\frac{\theta}{2}.$$

EXAMPLE 2 *If $x = a\sec^3\theta$ and $y = a\tan^3\theta$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.* [NCERT EXEMPLAR]

SOLUTION We have, $x = a\sec^3\theta$ and $y = a\tan^3\theta$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = 3a\sec^2\theta\frac{d}{d\theta}(\sec\theta) \text{ and } \frac{dy}{d\theta} = 3a\tan^2\theta\frac{d}{d\theta}(\tan\theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a\sec^3\theta\tan\theta \text{ and } \frac{dy}{d\theta} = 3a\tan^2\theta\sec^2\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

EXAMPLE 3 *Find $\frac{dy}{dx}$, when $x = a\cos^3 t$ and $y = a\sin^3 t$.*

SOLUTION We have, $x = a\cos^3 t$ and $y = a\sin^3 t$

$$\therefore \quad \frac{dx}{dt} = 3a\cos^2 t\frac{d}{dt}\cos(t) = -3a\cos^2 t\sin t \text{ and, } \frac{dy}{dt} = 3a\sin^2 t\frac{d}{dt}(\sin t) = 3a\sin^2 t\cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\tan t$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *If $x = \sqrt{a^{\sin^{-1}t}}$, $y = \sqrt{a^{\cos^{-1}t}}$, $a > 0$ and $-1 < t < 1$, show that $\frac{dy}{dx} = -\frac{y}{x}$.* [CBSE 2012, NCERT]

SOLUTION We have,

$$x = \sqrt{a^{\sin^{-1}t}} \text{ and } y = \sqrt{a^{\cos^{-1}t}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\sin^{-1}t}\right) \text{ and } \frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\frac{d}{dt}\left(a^{\cos^{-1}t}\right)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1}t}\right)^{-1/2}\left(a^{\sin^{-1}t}\log_e a\right)\frac{d}{dt}(\sin^{-1}t)$$

and,

$$\frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1}t}\right)^{-1/2}\left(a^{\cos^{-1}t}\log_e a\right)\frac{d}{dt}(\cos^{-1}t)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1} t}\right)^{1/2} (\log_e a) \times \frac{1}{\sqrt{1-t^2}} = \frac{x \log_e a}{2\sqrt{1-t^2}}$$

and, $$\frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1} t}\right)^{1/2} (\log_e a) \times \frac{-1}{\sqrt{1-t^2}} = \frac{-y \log_e a}{2\sqrt{1-t^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-y \log_e a}{2\sqrt{1-t^2}} \times \frac{2\sqrt{1-t^2}}{x \log_e a} = \frac{-y}{x}.$$

ALITER Clearly, $x^2 y^2 = a^{\sin^{-1} t + \cos^{-1} t} \Rightarrow x^2 y^2 = a^{\pi/2}$ $\left[\because \sin^{-1} t + \cos^{-1} t = \frac{\pi}{2}\right]$

Differentiating with respect to x, we get

$$2xy^2 + 2x^2 y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *If* $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ *and* $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, $t > 1$. *Prove that* $\frac{dy}{dx} = -1$.

SOLUTION Let $t = \tan\theta$. Then,

$$t > 1 \Rightarrow \tan\theta > 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$$\therefore \quad x = \sin^{-1}\left\{\frac{2t}{1+t^2}\right\} = \sin^{-1}\left\{\frac{2\tan\theta}{1+\tan^2\theta}\right\}$$

$$\Rightarrow \quad x = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{\sin(\pi - 2\theta)\} = \pi - 2\theta = \pi - 2\tan^{-1} t$$

$$\Rightarrow \quad \frac{dx}{dt} = 0 - \frac{2}{1+t^2} = \frac{-2}{1+t^2}$$

and, $$y = \tan^{-1}\left\{\frac{2t}{1-t^2}\right\}$$

$$\Rightarrow \quad y = \tan^{-1}\left\{\frac{2\tan\theta}{1-\tan^2\theta}\right\} = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{-\tan(\pi - 2\theta)\}$$

$$\Rightarrow \quad y = -\tan^{-1}\{\tan(\pi - 2\theta)\} = -(\pi - 2\theta) = -\pi + 2\tan^{-1} t$$

$$\Rightarrow \quad \frac{dy}{dt} = 0 + \frac{2}{1+t^2} = \frac{2}{1+t^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{-\frac{2}{1+t^2}} = -1.$$

EXAMPLE 6 *If* $u = \sin(m\cos^{-1} x)$, $v = \cos(m \sin^{-1} x)$, *prove that* $\frac{du}{dv} = \frac{\sqrt{1-u^2}}{1-v^2}$.

SOLUTION We have,

$$u = \sin(m\cos^{-1}x) \text{ and } v = \cos(m\sin^{-1}x)$$

$$\Rightarrow \quad \sin^{-1}u = m\cos^{-1}x \text{ and } \cos^{-1}v = m\sin^{-1}x$$

$$\Rightarrow \quad \sin^{-1}u + \cos^{-1}v = m(\cos^{-1}x + \sin^{-1}x) \Rightarrow \sin^{-1}u + \cos^{-1}v = \frac{m\pi}{2} \quad \left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Differentiating both sides with respect to v, we obtain

$$\frac{1}{\sqrt{1-u^2}}\frac{du}{dv} - \frac{1}{\sqrt{1-v^2}} = 0 \Rightarrow \frac{du}{dv} = \sqrt{\frac{1-u^2}{1-v^2}}$$

EXAMPLE 7 *If* $x = \sec\theta - \cos\theta$ *and* $y = \sec^n\theta - \cos^n\theta$, *prove that* $(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$

SOLUTION We have,

$$x = \sec\theta - \cos\theta \text{ and } y = \sec^n\theta - \cos^n\theta$$

$$\therefore \quad \frac{dx}{d\theta} = \sec\theta\tan\theta + \sin\theta \text{ and, } \frac{dy}{d\theta} = n\sec^n\theta\tan\theta + n\cos^{n-1}\theta\sin\theta$$

$$\Rightarrow \quad \frac{dx}{d\theta} = \tan\theta(\sec\theta + \cos\theta) \text{ and, } \frac{dy}{d\theta} = n\tan\theta(\sec^n\theta + \cos^n\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)} = n\,\frac{\sec^n\theta + \cos^n\theta}{\sec\theta + \cos\theta}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = n^2\,\frac{(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = n^2\left\{\frac{(\sec^n\theta - \cos^n\theta)^2 + 4\sec^n\theta\cos^n\theta}{(\sec\theta - \cos\theta)^2 + 4\sec\theta\cos\theta}\right\} \quad [\because (a+b)^2 = (a-b)^2 + 4ab]$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = n^2\left(\frac{y^2+4}{x^2+4}\right) \Rightarrow (x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$

EXERCISE 10.7

BASIC

Find $\frac{dy}{dx}$, *when*

1. $x = at^2$ and $y = 2at$ [NCERT]
2. $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$
3. $x = a\cos\theta$ and $y = b\sin\theta$ [NCERT]
4. $x = ae^{\theta}(\sin\theta - \cos\theta)$, $y = ae^{\theta}(\sin\theta + \cos\theta)$ [CBSE 2014]
5. $x = b\sin^2\theta$ and $y = a\cos^2\theta$
6. $x = a(1 - \cos\theta)$ and $y = a(\theta + \sin\theta)$ at $\theta = \frac{\pi}{2}$ [NCERT]
7. $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$
8. $x = \frac{3at}{1+t^2}$ and $y = \frac{3at^2}{1+t^2}$

9. $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ [NCERT]

10. $x = e^{\theta}\left(\theta + \frac{1}{\theta}\right)$ and $y = e^{-\theta}\left(\theta - \frac{1}{\theta}\right)$. [NCERT EXEMPLAR]

11. $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$.

12. $x = \cos^{-1}\frac{1}{\sqrt{1+t^2}}$ and $y = \sin^{-1}\frac{t}{\sqrt{1+t^2}}$, $t \in R$

13. $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$

14. If $x = 10(t - \sin t)$, $y = 12(1 - \cos t)$, find $\frac{dy}{dx}$. [NCERT]

15. If $x = a(\theta - \sin\theta)$ and, $y = a(1 + \cos\theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$. [CBSE 2011]

16. If $x = \frac{1 + \log t}{t^2}$, $y = \frac{3 + 2\log t}{t}$, find $\frac{dy}{dx}$. [NCERT EXEMPLAR]

17. If $x = 3\sin t - \sin 3t$, $y = 3\cos t - \cos 3t$, find $\frac{dy}{dx}$ at $t = \frac{\pi}{3}$. [NCERT EXEMPLAR]

18. If $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$. [NCERT EXEMPLAR]

19. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$. [CBSE 2018]

20. If $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$, $-1 < t < 1$, prove that $\frac{dy}{dx} = 1$

BASED ON LOTS

21. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$. [CBSE 2013]

22. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$ [NCERT EXEMPLAR]

23. If $x = a\left(t + \frac{1}{t}\right)$ and $y = a\left(t - \frac{1}{t}\right)$, prove that $\frac{dy}{dx} = \frac{x}{y}$.

24. If $x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$, $y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$, find $\frac{dy}{dx}$ [NCERT]

25. If $x = \left(t + \frac{1}{t}\right)^a$, $y = a^{t + \frac{1}{t}}$, find $\frac{dy}{dx}$

26. If $x = a\left(\frac{1+t^2}{1-t^2}\right)$ and $y = \frac{2t}{1-t^2}$, find $\frac{dy}{dx}$ [CBSE 2005]

27. If $x = a\sin 2t(1 + \cos 2t)$ and $y = b\cos 2t(1 - \cos 2t)$, show that at $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{b}{a}$.

[CBSE 2014, 2016, NCERT EXEMPLAR]

28. If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$ find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$.

[CBSE 2014, NCERT EXEMPLAR]

ANSWERS

1. $\frac{1}{t}$
2. $\tan \frac{\theta}{2}$
3. $-\frac{b}{a}\cot\theta$
4. $\cot\theta$
5. $-\frac{a}{b}$
6. 1
7. $\frac{x}{y}$
8. $\frac{2t}{1-t^2}$
9. $\tan\theta$
10. $e^{-2\theta}\frac{(\theta^2-\theta^3+\theta+1)}{(\theta^3+\theta^2+\theta-1)}$
11. $-\frac{x}{y}$
12. 1
13. $\frac{t^2-1}{2t}$
14. $\frac{6}{5}\cot\left(\frac{t}{2}\right)$
15. $-\sqrt{3}$
16. t
17. $-\frac{1}{\sqrt{3}}$
18. 1
19. $\frac{1}{\sqrt{3}}$
24. $-\cot 3t$
25. $\frac{a^{t+\frac{1}{t}}\log a}{a\left(t+\frac{1}{t}\right)^{a-1}}$
26. $\frac{1+t^2}{2at}$
28. 1

HINTS TO SELECTED PROBLEMS

22. We have, $x = e^{\cos 2t}$, $y = e^{\sin 2t}$

$$\Rightarrow \quad \frac{dx}{dt} = e^{\cos 2t}(-2\sin 2t),\ \frac{dy}{dt} = e^{\sin 2t}(2\cos 2t)$$

$$\Rightarrow \quad \frac{dx}{dt} = -2x\sin 2t,\ \frac{dy}{dt} = 2y\cos 2t$$

$$\Rightarrow \quad \frac{dx}{dt} = -2x\log y,\ \frac{dy}{dt} = 2y\log x \qquad \left[\begin{array}{l}\because\ x = e^{\cos 2t},\ y = e^{\sin 2t}\\ \Rightarrow \log x = \cos 2t,\ \log y = \sin 2t\end{array}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2y\log x}{-2x\log y} = -\left(\frac{y\log x}{x\log y}\right)$$

ALITER We have,

$$x = e^{\cos 2t} \text{ and, } y = e^{\sin 2t}$$

$$\Rightarrow \quad \log x = \cos 2t \text{ and, } \log y = \sin 2t \Rightarrow (\log x)^2 + (\log y)^2 = \cos^2 2t + \sin^2 2t$$

$$\Rightarrow \quad (\log x)^2 + (\log y)^2 = 1$$

Differentiating both sides with respect to x, we get

$$2(\log x)\frac{1}{x} + 2(\log y)\frac{1}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y\log x}{x\log y}$$

24. We have,

$$x = \frac{\sin^3 t}{\sqrt{\cos 2t}},\ y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

$$\Rightarrow \quad x = \sin^3 t\,(\cos 2t)^{-1/2},\ y = \cos^3 t\,(\cos 2t)^{-1/2}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \sin^3 t \times -\frac{1}{2}(\cos 2t)^{-3/2}\frac{d}{dt}(\cos 2t)$$

and, $$\frac{dy}{dt} = \frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}} + \cos^3 t \times -\frac{1}{2}(\cos 2t)^{-3/2}\frac{d}{dt}(\cos 2t)$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t}{\sqrt{\cos 2t}} + \frac{\sin^3 t\sin 2t}{(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{-3\cos^2 t\sin t}{\sqrt{\cos 2t}} + \frac{\cos^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t\cos 2t + \sin^3 t\sin 2t}{(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{-3\cos^2 t\sin t\cos 2t + \cos^3 t\sin 2t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t(1-2\sin^2 t) + 2\sin^4 t\cos t}{(\cos 2t)^{3/2}},$$

$$\frac{dy}{dt} = \frac{-3\cos^2 t\sin t(2\cos^2 t - 1) + 2\cos^4 t\sin t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{3\sin^2 t\cos t - 4\sin^4 t\cos t}{(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{-4\cos^4 t\sin t + 3\cos^2 t\sin t}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\sin t\cos t(3\sin t - 4\sin^3 t)}{(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{-\sin t\cos t(4\cos^3 t - 3\cos t)}{(\cos 2t)^{3/2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{\sin 2t\sin 3t}{2(\cos 2t)^{3/2}}, \frac{dy}{dt} = \frac{-\sin 2t\cos 3t}{(\cos 2t)^{3/2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin 2t\cos 3t}{\sin 2t\sin 3t} = -\cot 3t$$

10.11 DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

So far we have discussed derivative of one variable, say, y with respect to other variable, say, x. In this section, we will discuss derivative of a function with respect to another function.

Let $u = f(x)$ and $v = g(x)$ be two functions of x. Then, to find the derivative of $f(x)$ with respect to $g(x)$ i.e., to find $\frac{du}{dv}$ we use the following formula

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

Thus, to find the derivative of $f(x)$ with respect to $g(x)$, we first differentiate both with respect to x and then divide the derivative of $f(x)$ with respect to x by the derivative of $g(x)$ with respect to x.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Differentiate* $\log\sin x$ *with respect to* $\sqrt{\cos x}$.

SOLUTION Let $u = \log\sin x$ and $v = \sqrt{\cos x}$. Then,

$$\frac{du}{dx} = \cot x \quad \text{and} \quad \frac{dv}{dx} = -\frac{\sin x}{2\sqrt{\cos x}}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = -\frac{\cot x}{\frac{\sin x}{2\sqrt{\cos x}}} = -2\sqrt{\cos x}\cot x \operatorname{cosec} x$$

EXAMPLE 2 *Differentiate* $\tan^{-1}\left(\frac{1+2x}{1-2x}\right)$ *with respect to* $\sqrt{1+4x^2}$.

SOLUTION Let $u = \tan^{-1}\left(\frac{1+2x}{1-2x}\right)$ and $v = \sqrt{1+4x^2}$. Then,

$$u = \tan^{-1} 1 + \tan^{-1} 2x \text{ and } v = \sqrt{1+4x^2} \Rightarrow \frac{du}{dx} = \frac{2}{1+4x^2} \text{ and } \frac{dv}{dx} = \frac{1}{2\sqrt{1+4x^2}} \times 8x = \frac{4x}{\sqrt{1+4x^2}}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+4x^2}}{\frac{4x}{\sqrt{1+4x^2}}} = \frac{1}{2x\sqrt{1+4x^2}}$$

EXAMPLE 3 *Differentiate* $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *with respect to* $\tan^{-1} x$, $x \neq 0$.

SOLUTION Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \tan^{-1} x$. Putting $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad u = \tan^{-1}\left\{\frac{2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right\} = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1} x$$

Thus, we obtain

$$u = \frac{1}{2}\tan^{-1} x \text{ and } v = \tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{1}{2} \times \frac{1}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{1}{1+x^2}.$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{1}{2(1+x^2)} \times (1+x^2) = \frac{1}{2}.$$

EXAMPLE 4 *Differentiate* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ *with respect to* $\tan^{-1} x$, $-1 < x < 1$.

SOLUTION Let $u = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and $v = \tan^{-1} x$. Putting $x = \tan\theta$, we get

$$u = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$\Rightarrow \quad u = 2\theta = 2\tan^{-1} x \qquad \left[\because -1 < x < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}\right]$

Thus, we obtain

$$u = 2\tan^{-1} x \text{ and } v = \tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{2}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{2/1+x^2}{1/1+x^2} = 2$$

EXAMPLE 5 *Differentiate x^x with respect to $x \log x$.*

SOLUTION Let $u = x^x$ and $v = x \log x$. Then,

$$u = x^x = e^{\log x^x} = e^{x \log x} \text{ and } v = x \log x$$

$$\Rightarrow \quad \frac{du}{dx} = e^{x \log x} \times \frac{d}{dx}(x \log x) \text{ and } \frac{dv}{dx} = x \times \frac{1}{x} + 1 \times \log x$$

$$\Rightarrow \quad \frac{du}{dx} = x^x (1 + \log x) \text{ and } \frac{dv}{dx} = 1 + \log x \Rightarrow \frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{x^x (1+\log x)}{(1+\log x)} = x^x$$

ALITER We have, $u = x^x \Rightarrow \log u = x \log x = v \Rightarrow u = e^v$

$$\therefore \quad \frac{du}{dv} = \frac{d}{dv}(e^v) = e^v = u \Rightarrow \frac{du}{dv} = x^x.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *Differentiate* $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ *with respect to* $\cos^{-1} x^2$. [CBSE 2019]

SOLUTION Let $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1} x^2$. Putting $x^2 = \cos\theta$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\} = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}\right\}$$

$$\Rightarrow \quad u = \tan^{-1}\left\{\frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}\right\}$$

$$\Rightarrow \quad u = \tan^{-1}\left\{\frac{1-\tan\theta/2}{1+\tan\theta/2}\right\} \qquad \left[\text{Dividing numerator and denominator by } \cos\frac{\theta}{2}\right]$$

$$\Rightarrow \quad u = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{4} - \frac{1}{2}\theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2 \qquad [\because x^2 = \cos\theta \therefore \theta = \cos^{-1} x^2]$$

$$\Rightarrow \quad \frac{du}{dx} = -\frac{1}{2} \times \frac{-2x}{\sqrt{1-x^4}} = \frac{x}{\sqrt{1-x^4}}$$

Now, $v = \cos^{-1} x^2 \Rightarrow \dfrac{dv}{dx} = \dfrac{-2x}{\sqrt{1-x^4}}$

$$\therefore \quad \frac{du}{dv} = \frac{du/dx}{dv/dx} = -\frac{1}{2}$$

EXAMPLE 7 *Differentiate* $x^{\sin^{-1} x}$ *with respect to* $\sin^{-1} x$.

SOLUTION Let $u = x^{\sin^{-1} x}$ and $v = \sin^{-1} x$. Then,

$$u = x^{\sin^{-1} x}$$

$$\Rightarrow \quad u = e^{\sin^{-1} x \,.\, \log x}$$

$$\Rightarrow \quad \frac{du}{dx} = e^{\sin^{-1} x \,.\, \log x} \,.\, \frac{d}{dx}\{\sin^{-1} x \,.\, \log x\} = x^{\sin^{-1} x}\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}$$

and, $\quad v = \sin^{-1} x \Rightarrow \dfrac{dv}{dx} = \dfrac{1}{\sqrt{1-x^2}}$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{x^{\sin^{-1} x}\left\{\frac{\sin^{-1} x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right\}}{\frac{1}{\sqrt{1-x^2}}} = x^{\sin^{-1} x}\left\{\log x + \frac{\sqrt{1-x^2}}{x}\sin^{-1} x\right\}$$

EXAMPLE 8 *If* $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$, *differentiate* $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ *with respect to* $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

[CBSE 2014, NCERT EXEMPLAR]

SOLUTION Let $u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ and $v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$. Let $x = \sin\theta$. Then,

$$x \in \left(\frac{1}{\sqrt{2}}, 1\right) \Rightarrow \frac{1}{\sqrt{2}} < \sin\theta < 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2}$$

Now,

$$u = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = \tan^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \tan^{-1}(\cot\theta) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$\Rightarrow \quad u = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sin^{-1} x \qquad \left[\because \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{4}\right]$$

$$\Rightarrow \quad \frac{du}{dx} = 0 - \frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sqrt{1-x^2}}$$

$$v = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$\Rightarrow \quad v = \frac{\pi}{2} - \sin^{-1}\left(2x\sqrt{1-x^2}\right) = \frac{\pi}{2} - \sin^{-1}(\sin 2\theta) \qquad [\because x = \sin\theta]$$

$\Rightarrow \quad v = \frac{\pi}{2} - \sin^{-1}\{\sin(\pi - 2\theta)\} = \frac{\pi}{2} - (\pi - 2\theta) \qquad \left[\because \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2}\right]$

$\Rightarrow \quad v = -\frac{\pi}{2} + 2\theta = -\frac{\pi}{2} + 2\sin^{-1} x \Rightarrow \frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}}$

$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{-1}{\sqrt{1-x^2}}}{\frac{2}{\sqrt{1-x^2}}} = -\frac{1}{2}$

EXAMPLE 9 *Differentiate* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ *with respect to* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, *if*

(i) $x \in (-1, 1)$ (ii) $x \in (1, \infty)$ (iii) $x \in (-\infty, -1)$

SOLUTION Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$. Putting $x = \tan\theta$, we get

$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$ and $v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$

(i) *When* $x \in (-1, 1)$: We have,

$x \in (-1, 1)$ and $x = \tan\theta \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$

$\therefore \quad u = \tan^{-1}(\tan 2\theta) = 2\theta$ and $v = \sin^{-1}(\sin 2\theta) = 2\theta$

$\Rightarrow \quad u = 2\tan^{-1} x$ and $v = 2\tan^{-1} x \qquad [\because x = \tan\theta \Rightarrow \theta = \tan^{-1} x]$

$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+x^2}$ and $\frac{dv}{dx} = \frac{2}{1+x^2}$

$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$

(ii) *When* $x \in (1, \infty)$: We have,

$x \in (1, \infty)$ and $x = \tan\theta \Rightarrow 1 < \tan\theta < \infty \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$

$\therefore \quad u = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{-\tan(\pi - 2\theta)\} = \tan^{-1}\{\tan(2\theta - \pi)\} = 2\theta - \pi$

$\Rightarrow \quad u = 2\tan^{-1} x - \pi \qquad [\because \theta = \tan^{-1} x]$

$\Rightarrow \quad \frac{du}{dx} = \frac{2}{1+x^2} - 0 = \frac{2}{1+x^2}$

and, $v = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{\sin(\pi - 2\theta)\} = \pi - 2\theta = \pi - 2\tan^{-1} x$

$\Rightarrow \quad \frac{dv}{dx} = 0 - \frac{2}{1+x^2} = \frac{-2}{1+x^2}$

$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{-2}{1+x^2}} = -1$

(iii) *When* $x \in (-\infty, -1)$: We have,

$$x = \tan\theta \text{ and } x \in (-\infty, -1) \Rightarrow -\infty < \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2}$$

$$\therefore \quad u = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{\tan(\pi + 2\theta)\} = \pi + 2\theta = \pi + 2\tan^{-1} x$$

$$\Rightarrow \quad \frac{du}{dx} = 0 + \frac{2}{1+x^2} = \frac{2}{1+x^2}$$

and, $v = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{-\sin(\pi + 2\theta)\} = \sin^{-1}(\sin(-\pi - 2\theta)) = -\pi - 2\theta = -\pi - 2\tan^{-1} x$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{2}{1+x^2}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{-\frac{2}{1+x^2}} = -1$$

EXAMPLE 10 *If* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, *differentiate* $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ *with respect to* $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

SOLUTION Let $u = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ and $v = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$. Putting $x = \tan\theta$, we obtain

$$u = \tan^{-1}(\tan 3\theta) \text{ and } v = \tan^{-1}(\tan 2\theta)$$

$$\Rightarrow \quad u = 3\theta \text{ and } v = 2\theta \quad \left[\begin{array}{l} \because -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \\ \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \text{ and } -\frac{\pi}{3} < 2\theta < \frac{\pi}{3} \end{array}\right]$$

$$\Rightarrow \quad u = 3\tan^{-1} x \text{ and } v = 2\tan^{-1} x \Rightarrow \frac{du}{dx} = \frac{3}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\therefore \quad \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{3}{1+x^2}}{\frac{2}{1+x^2}} = \frac{3}{2}.$$

EXERCISE 10.8

BASIC

1. Differentiate $\sec^2(x^2)$ with respect to x^2. [CBSE 2020]
2. Differentiate $\log(1 + x^2)$ with respect to $\tan^{-1} x$.
3. Differentiate $(\log x)^x$ with respect to $\log x$.
4. Differentiate $(\cos x)^{\sin x}$ with respect to $(\sin x)^{\cos x}$.
5. Differentiate $\tan^{-1}\left(\frac{1 + ax}{1 - ax}\right)$ with respect to $\sqrt{1 + a^2 x^2}$.
6. Differentiate $\sin^2 x$ with respect to $e^{\cos x}$. [CBSE 2020]
7. Differentiate $\tan^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$ with respect to $\sec^{-1} x$.

BASED ON LOTS

8. Differentiate $\sin^{-1}\sqrt{1-x^2}$ with respect to $\cos^{-1}x$, if (i) $x\in(0,1)$ (ii) $x\in(-1,0)$.

9. Differentiate $\sin^{-1}(4x\sqrt{1-4x^2})$ with respect to $\sqrt{1-4x^2}$, if

(i) $x\in\left(-\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ (ii) $x\in\left(\frac{1}{2\sqrt{2}},\frac{1}{2}\right)$ (iii) $x\in\left(-\frac{1}{2},-\frac{1}{2\sqrt{2}}\right)$

10. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, if $-1<x<1$, $x\neq 0$. [CBSE 2014, 2016]

11. Differentiate $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$, if:

(i) $x\in(0,1/\sqrt{2})$ (ii) $x\in(1/\sqrt{2},1)$

12. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if $0<x<1$.

13. Differentiate $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$ with respect to $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$.

14. Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, if $0<x<1$.

15. Differentiate $\tan^{-1}\left(\frac{x-1}{x+1}\right)$ with respect to $\sin^{-1}(3x-4x^3)$, if $-\frac{1}{2}<x<\frac{1}{2}$.

16. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, if $-1<x<1$.

17. Differentiate $\cos^{-1}(4x^3-3x)$ with respect to $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$, if $\frac{1}{2}<x<1$.

18. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$, if $-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$. [CBSE 2014]

19. Differentiate $\sin^{-1}\sqrt{1-x^2}$ with respect to $\cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$, if $0<x<1$.

20. Differentiate $\sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$ with respect to $\sqrt{1-a^2x^2}$, if $-\frac{1}{\sqrt{2}}<ax<\frac{1}{\sqrt{2}}$.

21. Differentiate $\tan^{-1}\left(\frac{1-x}{1+x}\right)$ with respect to $\sqrt{1-x^2}$, if $-1<x<1$.

ANSWERS

1. $2\sec^2(x^2)\tan(x^2)$ 2. $2x$ 3. $x(\log x)^{x-1}\{1+\log x\cdot\log(\log x)\}$

4. $\dfrac{(\cos x)^{\sin x}\{\cos x\cdot\log\cos x-\sin x\cdot\tan x\}}{(\sin x)^{\cos x}\{-\sin x\log\sin x+\cos x\cdot\cot x\}}$ 5. $\dfrac{1}{ax\sqrt{1+a^2x^2}}$

6. $-2e^{-\cos x}\cos x$ 7. $\dfrac{-x\sqrt{x^2-1}}{2}$ 8. (i) 1 (ii) -1 9. (i) $-\dfrac{1}{x}$ (ii) $\dfrac{1}{x}$ (iii) $\dfrac{1}{x}$

10. $\dfrac{1}{4}$ 11. (i) 2 (ii) -2 12. 1 13. 2 14. 1 15. $\dfrac{\sqrt{1-x^2}}{3(1+x^2)}$

16. 1 17. 3 18. $\dfrac{1}{2}$ 19. 1 20. $-\dfrac{2}{ax}$ 21. $\dfrac{\sqrt{1-x^2}}{x(1+x^2)}$

10.12 DIFFERENTIATION OF DETERMINANTS

In the previous sections, we have studied differentiation in detail. In this section, we shall discuss the differentiation of determinants.

To differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

For example, if

$$\Delta(x)=\begin{vmatrix} f(x) & g(x)\\ u(x) & v(x)\end{vmatrix}, \text{ then}$$

$$\frac{d}{dx}\{\Delta(x)\}=\begin{vmatrix} f'(x) & g'(x)\\ u(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g(x)\\ u'(x) & v'(x)\end{vmatrix}. \text{ Also, } \frac{d}{dx}\{\Delta(x)\}=\begin{vmatrix} f'(x) & g(x)\\ u'(x) & v(x)\end{vmatrix}+\begin{vmatrix} f(x) & g'(x)\\ u(x) & v'(x)\end{vmatrix}$$

Similar results hold for the differentiation of determinants of higher order. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *If* $f(x)=\begin{vmatrix} x+a^2 & ab & ac\\ ab & x+b^2 & bc\\ ac & bc & x+c^2\end{vmatrix}$, *find* $f'(x)$.

SOLUTION We have,

$$f(x)=\begin{vmatrix} x+a^2 & ab & ac\\ ab & x+b^2 & bc\\ ac & bc & x+c^2\end{vmatrix}$$

$$\Rightarrow\quad f'(x)=\begin{vmatrix} 1 & 0 & 0\\ ab & x+b^2 & bc\\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac\\ 0 & 1 & 0\\ ac & bc & x+c^2\end{vmatrix}+\begin{vmatrix} x+a^2 & ab & ac\\ ab & x+b^2 & bc\\ 0 & 0 & 1\end{vmatrix}$$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} x+b^2 & bc \\ bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ac \\ ac & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab \\ ab & x+b^2 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = \left\{(x+b^2)(x+c^2) - b^2c^2\right\} + \left\{(x+a^2)(x+c^2) - a^2c^2\right\} + \left\{(x+a)^2(x+b^2) - a^2b^2\right\}$$

$$\Rightarrow \quad f'(x) = x^2 + x(b^2+c^2) + x^2 + x(a^2+c^2) + x^2 + x(a^2+b^2) = 3x^2 + 2x(a^2+b^2+c^2).$$

EXAMPLE 2 *If $f_r(x)$, $g_r(x)$ and $h_r(x)$; $r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a)$; $r = 1, 2, 3$ and*

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}, \text{ find } F'(x) \text{ at } x = a.$$

SOLUTION We have,

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

$$\therefore \quad F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1'(x) & g_2'(x) & g_3'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3'(x) \end{vmatrix}$$

$$\Rightarrow \quad F'(a) = \begin{vmatrix} f_1'(a) & f_2'(a) & f_3'(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \\ h_1(a) & h_2(a) & h_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ h_1'(a) & h_2'(a) & h_3'(a) \end{vmatrix}$$

$$\Rightarrow \quad F'(a) = \begin{vmatrix} f'_1(a) & f'_2(a) & f'_3(a) \\ g_1(a) & g_2(a) & g_3(a) \\ g_1(a) & g_2(a) & g_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(a) \\ g'_1(a) & g'_2(a) & g'_3(a) \\ f_1(a) & f_2(a) & f_3(a) \end{vmatrix} + \begin{vmatrix} f_1(a) & f_2(a) & f_3(x) \\ f_1(a) & f_2(a) & f_3(a) \\ g_1'(a) & g_2'(a) & g_3'(a) \end{vmatrix}$$

[Using: $f_r(a) = g_r(a) = h_r(a)$; $r = 1, 2, 3$]

$\Rightarrow \quad F'(a) = 0 + 0 + 0 = 0$ [$\because$ Two rows are identical in each of the determinants]

EXAMPLE 3 *If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2, then prove that*

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} \text{ is a constant polynomial.}$$

SOLUTION Let $f(x) = a_1x^2 + a_2x + a_3$, $g(x) = b_1x^2 + b_2x + b_3$ and $h(x) = c_1x^2 + c_2x + c_3$.

Then, $f'(x) = 2a_1x + a_2$, $g'(x) = 2b_1x + b_2$ and $h'(x) = 2c_1x + c_2$

$f''(x) = 2a_1$, $g'' = 2b_1$, $h''(x) = 2c_1$ and, $f'''(x) = g'''(x) = h'''(x) = 0$ ...(i)

In order to prove that $\phi(x)$ is a constant polynomial, it is sufficient to show that $\phi'(x) = 0$ for all x.

Now,

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$\Rightarrow \quad \phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$$\Rightarrow \quad \phi'(x) = 0 + 0 + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix} \qquad \text{[Using (i)]}$$

$\Rightarrow \quad \phi'(x) = 0 + 0 + 0 = 0$ for all $x \Rightarrow \phi(x) =$ Constant for all x.

Hence, $\phi(x)$ is a constant polynomial.

EXAMPLE 4 *If f, g, h are differentiable functions of x and* $\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}$,

prove that $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$.

SOLUTION We have,

$$(xf)' = xf' + f, (xg)' = xg' + g, (xh)' = xh' + h$$

$$(x^2 f)' = x^2 f' + 2xf, (x^2 g)' = x^2 g' + 2xg, (x^2 h)' = x^2 h' + 2xh$$

$$(x^2 f)'' = x^2 f'' + 4xf' + 2f, (x^2 g)'' = x^2 g'' + 4xg' + 2g \text{ and}, (x^2 h)'' = x^2 h'' + 4xh' + 2h$$

$$\therefore \quad \Delta = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ x^2 f'' + 4xf' + 2f & x^2 g'' + 4xg' + 2g & x^2 h'' + 4xh' + 2h \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' + 4xf' & x^2 g'' + 4xg' & x^2 h'' + 4xh' \end{vmatrix} \qquad \text{Applying } R_2 \to R_2 - R_1, \text{ and } R_3 \to R_3 - 2R_1$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix} \qquad \text{Applying } R_3 \to R_3 - 4R_2$$

$$\Rightarrow \quad \Delta = x \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix} \qquad [\because \text{ Taking } x \text{ common from } R_2]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} \qquad [\text{Multiplying } R_3 \text{ by } x]$$

$$\therefore \quad \Delta' = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

$$\Rightarrow \quad \Delta' = 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix} = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

EXAMPLE 5 *If* $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, *prove that* $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$. [NCERT]

SOLUTION We have, $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

$$\therefore \quad \frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx}(f(x)) & \frac{d}{dx}(g(x)) & \frac{d}{dx}(h(x)) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

FILL IN THE BLANKS TYPE QUESTIONS (VBQs)

1. If $y = x|x|$, then $\left(\dfrac{dy}{dx}\right)_{x=-1} =$ ....
2. If $y = 2x + |x|$, then $\left(\dfrac{dy}{dx}\right)_{x=-1} =$ .... and $\left(\dfrac{dy}{dx}\right)_{x=1} =$ ....
3. If $f(x) = |x^2 - x|$, then $f'(2) =$ ....
4. If $y = \sin x°$ and $\dfrac{dy}{dx} = k \cos x°$, then $k =$ ....
5. If $f(x) = e^x g(x)$, $g(0) = 2$, $g'(0) = 1$, then $f'(0) =$ ....
6. If $f(x) = 3|x+2|$, then $f'(-3) =$ ....
7. If $f(1) = 3$, $f'(2) = 1$, then $\dfrac{d}{dx}\left\{\ln f(e^x + 2x)\right\} =$ ....
8. If $f(x) = x|x|$, then $f'(x) =$ ....
9. If $f(x) = |x-1| + |x-3|$, then $f'(2) =$ ....
10. If $f(x) = |\cos x - \sin x|$, then $f'\left(\dfrac{\pi}{3}\right) =$ ....

11. If $f(x)=|\cos x|$, then $f'\left(\frac{\pi}{4}\right)=$ ....

12. The derivative of x^2 with respect to x^3 is ....

13. For the curve $\sqrt{x}+\sqrt{y}=1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$ is ....

14. If $f(x)=|\sin x|$, then $f'\left(-\frac{\pi}{4}\right)=$ ....

15. If $f(x)=|\sin x-\cos x|$, then $f'\left(\frac{\pi}{6}\right)=$ ....

16. If $y=\tan x^\circ$, then $\left(\frac{dy}{dx}\right)_{x=45^\circ}=$ ....

17. If $y=\sin^{-1}(e^x)+\cos^{-1}(e^x)$, then $\frac{dy}{dx}=$ ....

18. If $y=\sin^{-1}(3x-4x^3)$, $\frac{1}{2}<x<1$, then $\frac{dy}{dx}=$ ....

19. If $y=\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to ....

20. The derivative of $\cos x$ with respect to $\sin x$ is ....

21. The derivative of $\log_{10} x$ with respect to x is ....

22. If $\frac{d}{dx}(f(x))=\frac{1}{1+x^2}$, then $\frac{d}{dx}\left\{f(x^3)\right\}=$ ....

23. If $y=\cos(\sin x^2)$, then $\frac{dy}{dx}$ at $x=\sqrt{\frac{\pi}{2}}$ is equal to ....

24. If $y=\log|x|, x\neq 0$, then $\frac{dy}{dx}=$ ....

25. If $f(x)=ax^2+bx+c$, then $f'(1)+f'(4)-f'(5)$ is equal to ....

26. If $f'(1)=2$ and $g'(\sqrt{2})=4$, then the derivative of $f(\tan x)$ with respect to $g(\sec x)$ at $x=\frac{\pi}{4}$ is equal to ....

ANSWERS

1. 2
2. 1, 3
3. 3
4. $\frac{\pi}{180}$
5. 3
6. -3
7. 2
8. $2|x|, x\neq 0$
9. 0
10. $\frac{\sqrt{3}+1}{2}$
11. $-\frac{1}{\sqrt{2}}$
12. $\frac{3x}{2}$
13. -1
14. $-\frac{1}{\sqrt{2}}$
15. $-\left(\frac{\sqrt{3}+1}{2}\right)$
16. $\frac{\pi}{90}$
17. 0
18. $\frac{-3}{\sqrt{1-x^2}}$
19. 0
20. $-\tan x$
21. $\frac{1}{x}\log_{10} e$
22. $\frac{3x^2}{1+x^6}$
23. 0
24. $\frac{1}{x}$
25. b
26. $\frac{1}{\sqrt{2}}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $f(x) = \log_e(\log_e x)$, then write the value of $f'(e)$.
2. If $f(x) = x + 1$, then write the value of $\frac{d}{dx}(fof)(x)$. **[CBSE 2019]**
3. If $f'(1) = 2$ and $y = f(\log_e x)$, find $\frac{dy}{dx}$ at $x = e$.
4. If $f(1) = 4$, $f'(1) = 2$, find the value of the derivative of $\log(f(e^x))$ with respect to x at the point $x = 0$.
5. If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then find $\frac{dy}{dx}$ at $x = 1$.
6. Let $g(x)$ be the inverse of an invertible function $f(x)$ which is derivable at $x = 3$. If $f(3) = 9$ and $f'(3) = 9$, write the value of $g'(9)$.
7. If $y = \sin^{-1}(\sin x)$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Then, write the value of $\frac{dy}{dx}$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.
8. If $\frac{\pi}{2} \le x \le \frac{3\pi}{2}$ and $y = \sin^{-1}(\sin x)$, find $\frac{dy}{dx}$.
9. If $\pi \le x \le 2\pi$ and $y = \cos^{-1}(\cos x)$, find $\frac{dy}{dx}$.
10. If $y = \sin^{-1}\left(\frac{2x}{1 + x^2}\right)$, write the value of $\frac{dy}{dx}$ for $x > 1$.
11. If $f(0) = f(1) = 0$, $f'(1) = 2$ and $y = f(e^x)\, e^{f(x)}$, write the value of $\frac{dy}{dx}$ at $x = 0$.
12. If $y = x|x|$, find $\frac{dy}{dx}$ for $x < 0$.
13. If $y = \sin^{-1} x + \cos^{-1} x$, find $\frac{dy}{dx}$.
14. If $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$, find $\frac{dy}{dx}$.
15. If $-\frac{\pi}{2} < x < 0$ and $y = \tan^{-1}\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$, find $\frac{dy}{dx}$.
16. If $y = x^x$, find $\frac{dy}{dx}$ at $x = e$.
17. If $y = \tan^{-1}\left(\frac{1 - x}{1 + x}\right)$, find $\frac{dy}{dx}$.
18. If $y = \log_a x$, find $\frac{dy}{dx}$.
19. If $y = \log\sqrt{\tan x}$, write $\frac{dy}{dx}$.
20. If $y = \sin^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) + \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)$, find $\frac{dy}{dx}$.

21. If $y = \sec^{-1}\left(\dfrac{x+1}{x-1}\right) + \sin^{-1}\left(\dfrac{x-1}{x+1}\right)$, then write the value of $\dfrac{dy}{dx}$.

22. If $|x| < 1$ and $y = 1 + x + x^2 + \ldots$ to ∞, then find the value of $\dfrac{dy}{dx}$.

23. If $u = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $v = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$, where $-1 < x < 1$, then write the value of $\dfrac{du}{dv}$.

24. If $f(x) = \log\left\{\dfrac{u(x)}{v(x)}\right\}$, $u(1) = v(1)$ and $u'(1) = v'(1) = 2$, then find the value of $f'(1)$.

25. If $y = \log|3x|$, $x \neq 0$, find $\dfrac{dy}{dx}$.

26. If $f(x)$ is an even function, then write whether $f'(x)$ is even or odd.

27. If $f(x)$ is an odd function, then write whether $f'(x)$ is even or odd.

28. Write the derivative of $\sin x$ with respect to $\cos x$. [CBSE 2014]

29. If $y = \log(\cos e^x)$, then find $\dfrac{dy}{dx}$. [CBSE 2019]

30. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, then find $\dfrac{d}{dx}(fog)(x)$. [CBSE 2019]

ANSWERS

1. 1	2. 1	3. $\dfrac{2}{e}$	4. $\dfrac{1}{2}$	5. 2
6. $\dfrac{1}{9}$	7. 1	8. -1	9. -1	10. $\dfrac{-2}{1+x^2}$
11. 2	12. $-2x$	13. 0	14. $-\tan\dfrac{\theta}{2}$	15. -1
16. $2e^e$	17. $-\dfrac{1}{1+x^2}$	18. $\dfrac{1}{x\log_e a}$	19. $\operatorname{cosec} 2x$	20. 0
21. 0	22. $\dfrac{1}{(1-x)^2}$	23. 1	24. 0	25. $\dfrac{1}{x}$
26. odd	27. even	28. $-\cot x$	29. $-\tan(e^x)\,.\,e^x$	30. 2