CHAPTER 2

RELATIONS

2.1 INTRODUCTION

In previous chapter, we have discussed various operations on sets to create more sets out of given sets. In this chapter, we shall study one more operation which is known as the cartesian product of sets. This will finally enable us to introduce the concept of relation.

2.2 ORDERED PAIRS

ORDERED PAIR *An ordered pair consists of two objects or elements in a given fixed order.*

For example, if A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) in that order, where $a \in A, b \in B$.

NOTE *An ordered pair is not a set consisting of two elements. The ordering of the two elements in an ordered pair is important and the two elements need not be distinct.*

ILLUSTRATION 1 *The position of a point in a two dimensional plane in cartesian coordinates is represented by an ordered pair. Accordingly, the ordered pairs* (1, 3), (2, 4), (2, 3) *and* (3, 2) *represents different points in a plane.*

EQUALITY OF ORDERED PAIRS *Two ordered pairs* (a_1, b_1) *and* (a_2, b_2) *are equal iff* $a_1 = a_2$ and $b_1 = b_2$.

i.e. $\quad (a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

It is evident from this definition that $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

ILLUSTRATION 2 *Find the values of a and b, if* $(3a - 2, b + 3) = (2a - 1, 3)$.

SOLUTION By the definition of equality of ordered pairs, we obtain

$$(3a - 2, b + 3) = (2a - 1, 3) \Leftrightarrow 3a - 2 = 2a - 1 \text{ and } b + 3 = 3 \Leftrightarrow a = 1 \text{ and } b = 0$$

2.3 CARTESIAN PRODUCT OF SETS

CARTESIAN PRODUCT OF SETS *Let A and B be any two non-empty sets. The set of all ordered pairs* (a, b) *such that* $a \in A$ *and* $b \in B$ *is called the cartesian product of the sets A and B and is denoted by* $A \times B$.

Thus, $\quad A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

If $\quad A = \phi$ or $B = \phi$, then we define $A \times B = \phi$.

ILLUSTRATION 1 *If* $A = \{2, 4, 6\}$ *and* $B = \{1, 2\}$, *then*

$$A \times B = \{2, 4, 6\} \times \{1, 2\} = \{(2, 1), (2, 2), (4, 1), (4, 2), (6, 1), (6, 2)\}$$

and, $\quad B \times A = \{1, 2\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6)\}$

It is evident from the above illustration that to write $A \times B$, we take an element from set A and form all ordered pairs with this element as first element and elements of B as second elements. Next we choose another element from A and corresponding to each element in B we form ordered pairs with this element as first element and elements of B as second elements. This process is continued till all elements of A are exhausted.

ILLUSTRATION 2 *If* $A = \{a, b\}$ *and* $B = \{1, 2, 3\}$, *find* $A \times B$, $B \times A$, $A \times A$, $B \times B$, *and* $(A \times B) \cap (B \times A)$.

SOLUTION We have, $A = \{a, b\}$ and $B = \{1, 2, 3\}$

$$\therefore \quad A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$$
$$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$$
$$B \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

Clearly, $(A \times B) \cap (B \times A) = \phi$.

CARTESIAN PRODUCT OF THREE SETS *Let A, B and C be three sets. Then, $A \times B \times C$ is the set of all ordered triplets having first element from A, second element from B and third element from C.*

i.e. $\quad A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

ILLUSTRATION 3 *If $A = \{1, 2\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$. Then,*

$$A \times B = \{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

and,
$$\begin{aligned} A \times B \times C &= (A \times B) \times C \\ &= \{(1, 3), (1, 4), (2, 3), (2, 4)\} \times \{4, 5, 6\} \\ &= \{(1, 3, 4), (1, 3, 5), (1, 3, 6), (1, 4, 4), (1, 4, 5), (1, 4, 6), \\ &\qquad (2, 3, 4), (2, 3, 5), (2, 3, 6), (2, 4, 4), (2, 4, 5), (2, 4, 6)\} \end{aligned}$$

NOTE *It should be noted that $A \times B \times C = (A \times B) \times C = A \times (B \times C)$.*

If $A_1, A_2, A_3, \ldots, A_n$ are n sets, then the cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of these n sets is the set of all n-tuples of the form $(a_1, a_2, a_3, \ldots, a_n)$, where $a_1 \in A, a_2 \in A_2, \ldots, a_n \in A_n$.

i.e. $\quad A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, a_3, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, a_3 \in A_3, \ldots, a_n \in A_n\}$

2.3.1 NUMBER OF ELEMENTS IN THE CARTESIAN PRODUCT OF TWO SETS

THEOREM *If A and B are two finite sets, then $n(A \times B) = n(A) \times n(B)$.*

PROOF Let $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$ be two sets having m and n elements respectively. Then,

$$\begin{aligned} A \times B = \{&(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots, (a_1, b_n) \\ &(a_2, b_1), (a_2, b_2), (a_2, b_3) \ldots, (a_2, b_n) \\ &\quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ &(a_m, b_1), (a_m, b_2), (a_m, b_3) \ldots, (a_m, b_n)\} \end{aligned}$$

Clearly, in the tabular representation of $A \times B$ there are m rows of ordered pairs and each row has n distinct ordered pairs. So, $A \times B$ has mn elements.

Hence, $n(A \times B) = mn = n(A) \times n(B)$ **Q.E.D.**

REMARK (i) *If either A or B is an infinite set, then $A \times B$ is an infinite set.*

(ii) *If A, B, C are finite sets, then $n(A \times B \times C) = n(A) \times n(B) \times n(C)$*

2.3.2 GRAPHICAL REPRESENTATION OF CARTESIAN PRODUCT OF SETS

Let A and B be any two non-empty sets. To represent $A \times B$ graphically, we draw two mutually perpendicular lines, one horizontal and other vertical. On the horizontal line, we represent the elements of set A and on the vertical line, the elements of B. If $a \in A$, $b \in B$, we draw a vertical line through a and a horizontal line through b. These two lines will meet in a point which will denote the ordered pair (a, b). In this manner we mark points corresponding to each ordered pair in $A \times B$. The set of points so obtained represents $A \times B$ graphically as illustrated below.

ILLUSTRATION *If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, find $A \times B$ and show it graphically.*

SOLUTION Clearly, $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

In order to represent $A \times B$ graphically, we draw two perpendicular lines OX and OY as shown in Fig. 2.1. Now, we represent the set A by three points on OX and the set B by two points on OY. The set $A \times B$ is represented by the six points as shown in Fig. 2.1.

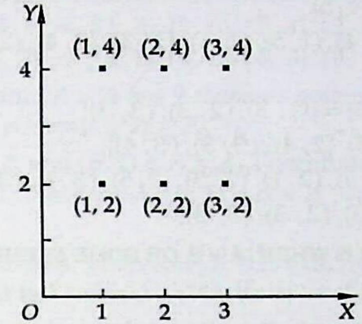

Fig. 2.1

2.3.3 DIAGRAMATIC REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in Fig. 2.2. Now, we draw line segments starting from each element of A and terminating to each element of set B.

If $A = \{1, 3, 5\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.

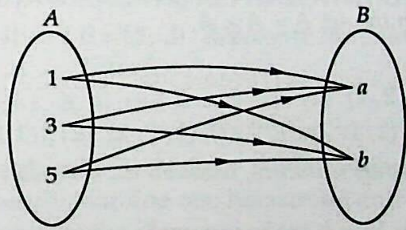

Fig. 2.2

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON EQUALITY OF ORDERED PAIRS**

EXAMPLE 1 *Find x and y, if* $(x + 3, 5) = (6, 2x + y)$.

SOLUTION By the definition of equality of ordered pairs

$$(x+3, 5) = (6, 2x+y)$$

$$\Rightarrow \quad x + 3 = 6 \text{ and } 5 = 2x + y$$

$$\Rightarrow \quad x = 3 \text{ and } 5 = 2x + y$$

$$\Rightarrow \quad x = 3, 5 = 6 + y$$

$$\Rightarrow \quad x = 3 \text{ and } y = -1$$

Type II **ON FINDING THE CARTESIAN PRODUCT OF TWO SETS**

EXAMPLE 2 *If* $A = \{1, 3, 5, 6\}$ *and* $B = \{2, 4\}$, *find* $A \times B$ *and* $B \times A$.

SOLUTION We have, $A = \{1, 3, 5, 6\}$ and $B = \{2, 4\}$. Therefore,

$$A \times B = \{(1, 2), (1, 4), (3, 2), (3, 4), (5, 2), (5, 4), (6, 2), (6, 4)\}$$

and, $$B \times A = \{(2, 1), (2, 3), (2, 5), (2, 6), (4, 1), (4, 3), (4, 5), (4, 6)\}$$

EXAMPLE 3 *If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{1, 3, 5\}$, find*
(i) $A \times (B \cup C)$ (ii) $A \times (B \cap C)$ (iii) $(A \times B) \cap (A \times C)$

SOLUTION (i) Clearly, $B \cup C = \{1, 3, 4, 5\}$

$\therefore$ $A \times (B \cup C) = \{1, 2, 3\} \times \{1, 3, 4, 5\}$
$= \{(1, 1), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 3), (3, 4), (3, 5)\}$

(ii) Clearly, $B \cap C = \{3\}$.

$\therefore$ $A \times (B \cap C) = \{1, 2, 3\} \times \{3\} = \{(1, 3), (2, 3), (3, 3)\}$

(iii) $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$,

and, $A \times C = \{(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3 5)\}$

$\therefore$ $(A \times B) \cap (A \times C) = \{(1, 3), (2, 3), (3, 3)\}$.

Type III **ON FINDING SETS A AND B WHEN A × B OR SOME ELEMENTS OF A × B ARE GIVEN**

EXAMPLE 4 *Let $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x$ is prime less than $5\}$. Find $A \times B$ and $B \times A$.*

SOLUTION We have, $A = \{1, 2, 3\}$ and, $B = \{x : x \in N, x \text{ is prime less than } 5\} = \{2, 3\}$

$\therefore$ $A \times B = \{1, 2, 3\} \times \{2, 3\} = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$

and, $B \times A = \{2, 3\} \times \{1, 2, 3\} = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

EXAMPLE 5 *If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, find $B \times A$.*

SOLUTION Clearly, $B \times A$ can be obtained from $A \times B$ by interchanging the entries (or components) or ordered pair in $A \times B$.

$\therefore$ $B \times A = \{(1, a), (5, a), (2, a), (2, b), (5, b), (1, b)\}$

EXAMPLE 6 *If $A = \{1, 2\}$, form the set $A \times A \times A$.*

SOLUTION We have, $A = \{1, 2\}$.

$\therefore$ $A \times A = \{(1,1), (1, 2), (2, 1), (2, 2)\}$

and, $A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

EXAMPLE 7 *If R is the set of all real numbers, what do the cartesian products $R \times R$ and $R \times R \times R$ represent?*

SOLUTION The cartesian product of the set R of all real numbers with itself i.e. $R \times R$ is the set of all ordered pairs (x, y) where $x, y \in R$. In other words, $R \times R = \{(x, y) : x, y \in R\}$.

Clearly, $R \times R$ is the set of all points in XY-plane. The set $R \times R$ is also denoted by R^2.

Similarly, we have

$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$

Clearly, it represents the set of all points in space. The set $R \times R \times R$ is also denoted by R^3.

EXAMPLE 8 *Express $A = \{(a, b) : 2a + b = 5, a, b \in W\}$ as the set of ordered pairs.*

SOLUTION Here, W denotes the set of whole numbers (non-negative integers).

We have,

$2a + b = 5$, where $a, b \in W$.

$\therefore$ $a = 0 \Rightarrow b = 5, a = 1 \Rightarrow b = 3$ and, $a = 2 \Rightarrow b = 1$

For $a > 3$, the values of b given by the above relation are not whole numbers.

$\therefore$ $A = \{(0, 5), (1, 3), (2, 1)\}$

EXAMPLE 9 *If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$, find A and B.*

SOLUTION Clearly, A is the set of all first entries in ordered pairs in $A \times B$ and B is the set of all second entries in ordered pairs in $A \times B$.

$\therefore$ $A = \{a, b\}$ and $B = \{1, 2, 3\}$

EXAMPLE 10 *Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are : (1, 4), (2, 6), (3, 6). Find $A \times B$ and $B \times A$.*

SOLUTION Since (1, 4), (2, 6) and (3, 6) are elements of $A \times B$. It follows that 1, 2, 3 are elements of A and 4, 6 are elements of B. It is given that $A \times B$ has 6 elements. So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$.

Hence, $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

and, $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$

EXAMPLE 11 *The cartesian product $A \times A$ has 9 elements among which are found $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.*

SOLUTION Since $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$. Therefore,

$$(-1, 0) \in A \times A \Rightarrow -1, 0 \in A \text{ and, } (0, 1) \in A \times A \Rightarrow 0, 1 \in A$$

$$\therefore \quad -1, 0, 1 \in A$$

It is given that $A \times A$ has 9 elements. Therefore, A has exactly three elements.

Hence, $A = \{-1, 0, 1\}$.

EXAMPLE 12 *Let A and B be two sets such that $n(A) = 5$ and $n(B) = 2$. If a, b, c, d, e are distinct and $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are in $A \times B$, find A and B.*

SOLUTION Since $(a, 2), (b, 3), (c, 2), (d, 3), (e, 2)$ are elements of $A \times B$. Therefore, $a, b, c, d, e \in A$ and $2, 3 \in B$.

It is given that $n(A) = 5$ and $n(B) = 2$

$\therefore \quad a, b, c, d, e \in A$ and $n(A) = 5 \Rightarrow A = \{a, b, c, d, e\}$

$\quad 2, 3 \in B$ and $n(B) = 2 \Rightarrow B = \{2, 3\}$

Type IV ON GRAPHICAL AND DIAGRAMATIC REPRESENTATION OF A × B

EXAMPLE 13 *Let $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$. Represent the following products graphically i.e. by lattices:* (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$

SOLUTION (i) We have, $A = \{-1, 3, 4\}$ and $B = \{2, 3\}$.

$\therefore \quad A \times B = \{(-1, 2), (-1, 3), (3, 2), (3, 3), (4, 2), (4, 3)\}$

In order to represent $A \times B$ graphically, we follow the following steps:

(a) Draw two mutually perpendicular line one horizontal and other vertical.

(b) On the horizontal line represent the elements of set A and on the vertical line represent the elements of set B.

(c) Draw vertical dotted lines through points representing elements of A on horizontal line and horizontal lines through points representing elements of B on the vertical line. Points of intersection of these lines will represent $A \times B$ graphically as shown in Fig. 2.3.

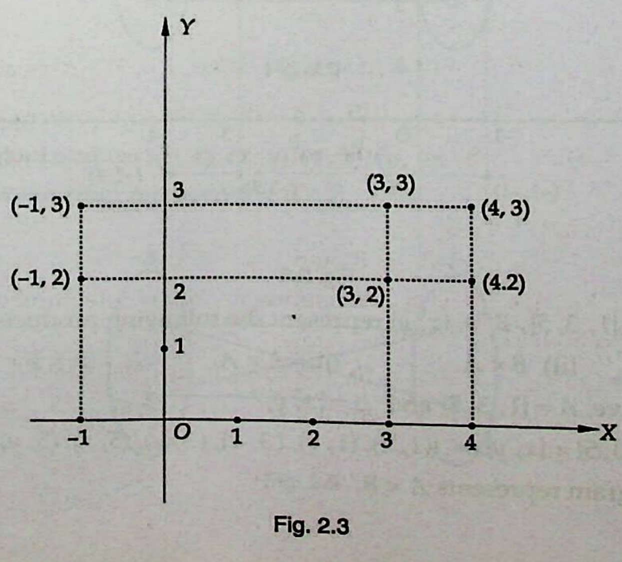

Fig. 2.3

(ii) Clearly, $B \times A = \{2, 3\} \times \{-1, 3, 4\} = \{(2, -1), (2, 3), (2, 4), (3, -1), (3, 3), (3, 4)\}$

Here, we represent B on the horizontal line and A on vertical line. Graphical representation of $B \times A$ is as shown in Fig. 2.4.

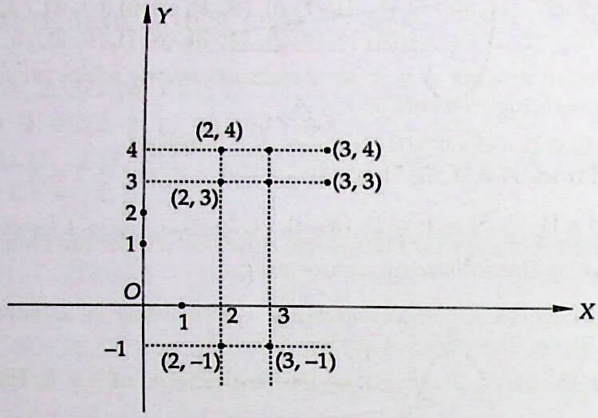

Fig. 2.4

(iii) We have, $A = \{1, 3, 4\}$

$\therefore \quad A \times A = \{-1, 3, 4\} \times \{-1, 3, 4\}$

$= \{(-1, -1), (-1, 3), (-1, 4), (3, -1), (3, 3), (3, 4), (4, -1), (4, 3), (4, 4)\}$

Graphical representation of $A \times A$ is shown in Fig. 2.5.

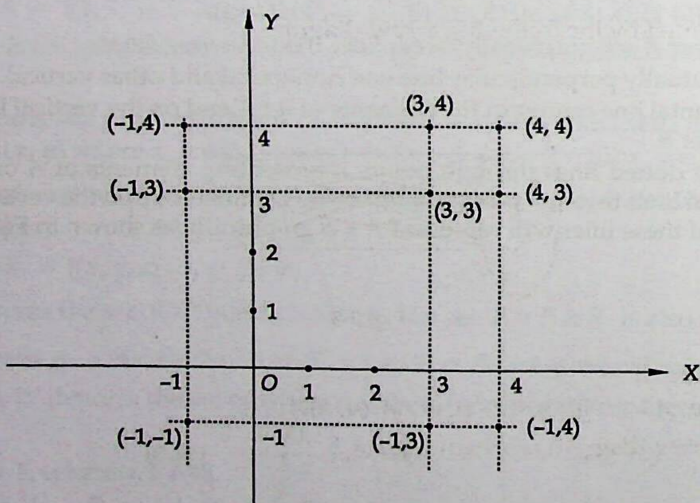

Fig. 2.5

EXAMPLE 14 If $A = \{1, 3, 5\}$, $B = \{x, y\}$ represent the following products by arrow diagrams:

(i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

SOLUTION (i) We have, $A = \{1, 3, 5\}$ and $B = \{x, y\}$

$\therefore \quad A \times B = \{1, 3, 5\} \times \{x, y\} = \{(1, x), (1, y), (3, x), (3, y), (5, x), (5, y)\}$

Following arrow diagram represents $A \times B$.

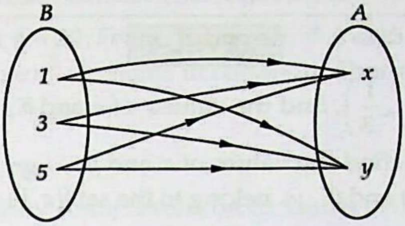

Fig. 2.6

(ii) We have, $B = \{x, y\}$ and $A = \{1, 3, 5\}$.

$\therefore \quad B \times A = \{x, y\} \times \{1, 3, 5\} = \{(x, 1), (x, 3), (x, 5), (y, 1), (y, 3), (y, 5)\}$

It has been represented by the following arrow diagram.

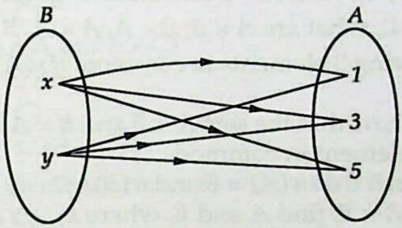

Fig. 2.7

(iii) We have, $A = [1, 3, 5\}$

$\therefore \quad A \times A = \{1, 3, 5\} \times \{1, 3, 5\} = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

It has been represented by the following arrow diagram.

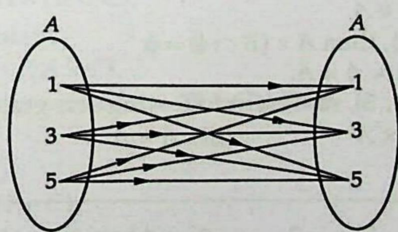

Fig. 2.8

(iv) We have, $B = \{x, y\}$

$\therefore \quad B \times B = \{x, y\} \times \{x, y\} = \{(x, x), (x, y), (y, x), (y, y)\}$

Following is the arrow diagram representing $B \times B$.

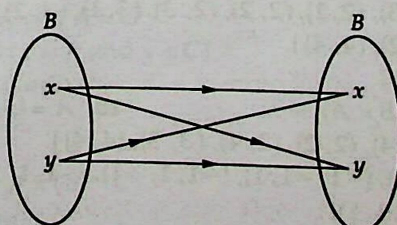

Fig. 2.9

EXERCISE 2.1

LEVEL-1

1. (i) If $\left(\frac{a}{3}+1, b-\frac{2}{3}\right)=\left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of a and b.
   (ii) If $(x+1, 1)=(3, y-2)$, find the values of x and y.
2. If the ordered pairs $(x, -1)$ and $(5, y)$ belong to the set $\{(a, b): b=2a-3\}$, find the values of x and y.
3. If $a \in \{-1, 2, 3, 4, 5\}$ and $b \in \{0, 3, 6\}$, write the set of all ordered pairs (a, b) such that $a+b=5$.
4. If $a \in \{2, 4, 6, 9\}$ and $b \in \{4, 6, 18, 27\}$, then form the set of all ordered pairs (a, b) such that a divides b and $a<b$.
5. If $A=\{1, 2\}$ and $B=\{1, 3\}$, find $A \times B$ and $B \times A$.
6. Let $A=\{1, 2, 3\}$ and $B=\{3, 4\}$. Find $A \times B$ and show it graphically.
7. If $A=\{1, 2, 3\}$ and $B=\{2, 4\}$, what are $A \times B$, $B \times A$, $A \times A$, $B \times B$, and $(A \times B) \cap (B \times A)$?
8. If A and B are two sets having 3 elements in common. *If* $n(A)=5, n(B)=4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.
9. Let A and B be two sets. Show that the sets $A \times B$ and $B \times A$ have an element in common *iff* the sets A and B have an element in common.
10. Let A and B be two sets such that $n(A)=3$ and $n(B)=2$.
    If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.
11. Let $A=\{1, 2, 3, 4\}$ and $R=\{(a, b): a \in A, b \in A, a \text{ divides } b\}$. Write R explicitly.
12. If $A=\{-1, 1\}$, find $A \times A \times A$.
13. State whether each of the following statements are true or false. If the statement is false, re-write the given statement correctly:
    (i) If $P=\{m, n\}$ and $Q=\{n, m\}$, then $P \times Q=\{(m, n), (n, m)\}$
    (ii) If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$.
    (iii) If $A=\{1, 2\}, B=\{3, 4\}$, then $A \times (B \cap \phi)=\phi$.
14. If $A=\{1, 2\}$, form the set $A \times A \times A$.
15. If $A=\{1, 2, 4\}$ and $B=\{1, 2, 3\}$, represent following sets graphically:
    (i) $A \times B$ (ii) $B \times A$ (iii) $A \times A$ (iv) $B \times B$

ANSWERS

1. (i) $a=2, b=1$ (ii) $x=2, y=3$ 2. $x=1, y=7$ 3. $\{(-1, 6), (2, 3), (5, 0)\}$
4. $\{(2, 4), (2, 6), (2, 18), (6, 18), (9, 18), (9, 27)\}$
5. $A \times B=\{(1, 1), (1, 3), (2, 1), (2, 3)\}$ and $B \times A=\{(1, 1), (1, 2), (3, 1), (3, 2)\}$
6. $A \times B=\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$.
7. $A \times B=\{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
   $B \times A=\{(2, 1), (2, 2), (2, 3), (4, 1), (4, 2), (4, 3)\}$
   $A \times A=\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$
   $B \times B=\{(2, 2), (2, 4), (4, 2), (4, 4)\}$.
   $(A \times B) \cap (B \times A)=\{(2, 2)\}$
8. $n(A \times B)=20, n[(A \times B) \cap (B \times A)]=9$. 10. $A=\{x, y, z\}, B=\{1, 2\}$
11. $R=\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
12. $A \times A \times A=\{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$
13. (i) F (ii) F (iii) T
14. $A \times A \times A=\{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$

HINTS TO SELECTED PROBLEM

8. $n(A \times B) = n(A) \times n(B) = 5 \times 4 = 20$. From theorem 9, *if* A and B have n elements in common, then $(A \times B)$ and $B \times A$ have n^2 elements in common. Therefore,

$$n[(A \times B) \cap (B \times A)] = 3^2 = 9.$$

2.4 SOME USEFUL RESULTS

In this section, we intend to study some results on cartesian product of sets which are given as theorems.

THEOREM 1 *For any three sets A, B, C, prove that:*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

PROOF (i) Let (a, b) be an arbitrary element of $A \times (B \cup C)$. Then,

$(a, b) \in A \times (B \cup C)$

$\Rightarrow$ $a \in A$ and $b \in B \cup C$ [By definition]

$\Rightarrow$ $a \in A$ and $(b \in B$ or $b \in C)$ [By definition of union]

$\Rightarrow$ $(a \in A$ and $b \in B)$ or $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ or $(a, b) \in A \times C$

$\Rightarrow$ $(a, b) \in (A \times B) \cup (A \times C)$

$\therefore$ $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cup (A \times C)$. Then,

$(x, y) \in (A \times B) \cup (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ or, $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ or, $(x \in A$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ or $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cup C)$

$\Rightarrow$ $(x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(ii) Let (a, b) be an arbitrary element of $A \times (B \cap C)$. Then,

$(a, b) \in A \times (B \cap C)$

$\Rightarrow$ $a \in A$ and $b \in (B \cap C)$ [By definition]

$\Rightarrow$ $a \in A$ and $(b \in B$ and $b \in C)$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in A$ and $b \in C)$

$\Rightarrow$ $(a, b) \in A \times B$ and $(a, b) \in A \times C$ [By definition]

$\Rightarrow$ $(a, b) \in (A \times B) \cap (A \times C)$

$\therefore$ $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) \cap (A \times C)$. Then,

$(x, y) \in (A \times B) \cap (A \times C)$

$\Rightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \in A \times C$

$\Rightarrow$ $(x \in A$ and $y \in B)$ and $(x \in A$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $(y \in B$ and $y \in C)$

$\Rightarrow$ $x \in A$ and $y \in (B \cap C)$

$\Rightarrow$ $(x, y) \in A \times (B \cap C)$

$\therefore$ $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$ ...(ii)

Hence, from (i) and (ii), we obtain

$A \times (B \cap C) = (A \times B) \cap (A \times C)$ Q.E.D.

THEOREM 2 *For any three sets A, B, C, prove that:* $A \times (B - C) = (A \times B) - (A \times C)$.

PROOF Let (a, b) be an arbitrary element of $A \times (B - C)$. Then,

$(a, b) \in A \times (B - C)$

$\Rightarrow$ $a \in A$ and $b \in (B - C)$

$\Rightarrow$ $a \in A$ and ($b \in B$ and $b \notin C$)

$\Rightarrow$ ($a \in A$ and $b \in B$) and ($a \in A$ and $b \notin C$)

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \notin (A \times C)$

$\Rightarrow$ $(a, b) \in (A \times B) - (A \times C)$

$\therefore$ $A \times (B - C) \subseteq (A \times B) - (A \times C)$ ...(i)

Now, let (x, y) be an arbitrary element of $(A \times B) - (A \times C)$. Then,

$(x, y) \in (A \times B) - (A \times C)$

$\Rightarrow$ $(x, y) \in A \times B$ and $(x, y) \notin A \times C$

$\Rightarrow$ ($x \in A$ and $y \in B$) and ($x \in A$ and $y \notin C$)

$\Rightarrow$ $x \in A$ and ($y \in B$ and $y \notin C$)

$\Rightarrow$ $x \in A$ and $y \in (B - C)$

$\Rightarrow$ $(x, y) \in A \times (B - C)$

$\therefore$ $(A \times B) - (A \times C) \subseteq A \times (B - C)$ ...(ii)

Hence, from (i) and (ii), we get

$A \times (B - C) = (A \times B) - (A \times C)$ **Q.E.D.**

THEOREM 3 *If A and B are any two non-empty sets, then prove that:* $A \times B = B \times A \Leftrightarrow A = B$.

PROOF *First, let $A = B$. Then we have to prove that $A \times B = B \times A$.*

Now, $A = B$

$\Rightarrow$ $A \times B = A \times A$ and $B \times A = A \times A$ [$\because B = A$]

$\Rightarrow$ $A \times B = B \times A$

Conversely, let $A \times B = B \times A$. Then we have to prove that $A = B$.

Let x be an arbitrary element of A. Then,

$x \in A$

$\Rightarrow$ $(x, b) \in A \times B$ for all $b \in B$

$\Rightarrow$ $(x, b) \in B \times A$ [$\because A \times B = B \times A$]

$\Rightarrow$ $x \in B$ [By definition]

$\therefore$ $A \subseteq B$

Now, let y be an arbitrary element of B. Then,

$y \in B$

$\Rightarrow$ $(a, y) \in A \times B$ for all $a \in A$

$\Rightarrow$ $(a, y) \in B \times A$ [$\because A \times B = B \times A$]

$\Rightarrow$ $y \in A$ [By definition]

$\therefore$ $B \subseteq A$

Hence, $A = B$. **Q.E.D.**

THEOREM 4 *If $A \subseteq B$, show that $A \times A \subseteq (A \times B) \cap (B \times A)$.*

PROOF *Let (a, b) be an arbitrary element of $A \times A$. Then,*

$(a, b) \in A \times A$

$\Rightarrow$ $a \in A$ and $b \in A$

$\Rightarrow$ $(a \in A, b \in A)$ and $(a \in A, b \in A)$

$\Rightarrow$ $(a \in A, b \in B)$ and $(a \in B, b \in A)$ [$\because A \subseteq B \therefore a, b \in A \Rightarrow a, b \in B$]

$\Rightarrow$ $(a, b) \in (A \times B)$ and $(a, b) \in (B \times A)$

$\Rightarrow$ $(a, b) \in (A \times B) \cap (B \times A)$

$\therefore$ $A \times A \subseteq (A \times B) \cap (B \times A)$

Hence, $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$. **Q.E.D.**

THEOREM 5 *If $A \subseteq B$, prove that $A \times C \subseteq B \times C$ for any set C.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C$

$\Rightarrow$ $a \in A$ and $b \in C$

$\Rightarrow$ $a \in B$ and $b \in C$ $[\because A \subseteq B \therefore a \in A \Rightarrow a \in B]$

$\Rightarrow$ $(a, b) \in B \times C$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times C$ for all $(a, b) \in (A \times C)$.

$\therefore$ $A \times C \subseteq B \times C$. **Q.E.D.**

THEOREM 6 *If $A \subseteq B$ and $C \subseteq D$, prove that $A \times C \subseteq B \times D$.*

PROOF Let (a, b) be an arbitrary element of $A \times C$. Then,

$(a, b) \in A \times C$

$\Rightarrow$ $a \in A$ and $b \in C$

$\Rightarrow$ $a \in B$ and $b \in D$ $[\because A \subseteq B$ and $C \subseteq D]$

$\Rightarrow$ $(a, b) \in B \times D$

Thus, $(a, b) \in A \times C \Rightarrow (a, b) \in B \times D$ for all $(a, b) \in (A \times C)$.

$\therefore$ $A \times C \subseteq B \times D$ **Q.E.D.**

THEOREM 7 *For any sets A, B, C, D prove that:* $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

PROOF Let (a, b) be an arbitrary element of $(A \times B) \cap (C \times D)$. Then,

$(a, b) \in (A \times B) \cap (C \times D)$

$\Rightarrow$ $(a, b) \in A \times B$ and $(a, b) \in C \times D$

$\Rightarrow$ $(a \in A$ and $b \in B)$ and $(a \in C$ and $b \in D)$

$\Rightarrow$ $(a \in A$ and $a \in C)$ and $(b \in B$ and $b \in D)$

$\Rightarrow$ $a \in (A \cap C)$ and $b \in B \cap D$

$\Rightarrow$ $(a, b) \in (A \cap C) \times (B \cap D)$

$\therefore$ $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$

Similarly, $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$

Hence, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ **Q.E.D.**

COROLLARY *For any sets A and B, prove that* $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$.

THEOREM 8 *For any three sets A, B, C prove that:*

(i) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$ (ii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$.

PROOF (i) We have,

$A \times (B' \cup C')' = A \times ((B')' \cap (C')')$ [By De-Morgan's law]

$= A \times (B \cap C) = (A \times B) \cap (A \times C)$ [See Theorem 1]

(ii) $A \times (B' \cap C')' = A \times ((B')' \cup (C')')$ [By De-Morgan's Law]

$= A \times (B \cup C) = (A \times B) \cup (A \times C)$ [See Theorem 1]

Q.E.D.

THEOREM 9 *Let A and B be two non-empty sets having n elements in common, then prove that $A \times B$ and $B \times A$ have n^2 elements in common.*

PROOF We have,

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ [See Theorem 7]

$\Rightarrow$ $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ [On replacing C by B and D by A]

$\Rightarrow$ $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$

It is given that $A \cap B$ has n elements, so $(A \cap B) \times (B \cap A)$ has n^2 elements.

But, $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$ [Proved above]

$\therefore$ $(A \times B) \cap (B \times A)$ has n^2 elements.

Hence, $A \times B$ and $B \times A$ have n^2 elements in common. Q.E.D.

THEOREM 10 *Let A be a non-empty set such that $A \times B = A \times C$. Show that $B = C$.*

PROOF Let b be an arbitrary element of B. Then,

$(a, b) \in A \times B$ for all $a \in A$

$\Rightarrow$ $(a, b) \in A \times C$ for all $a \in A$ $[\because A \times B = A \times C]$

$\Rightarrow$ $b \in C$

Thus, $b \in B \Rightarrow b \in C$

$\therefore$ $B \subset C$ ...(i)

Now, let c be an arbitrary element of C. Then,

$(a, c) \in A \times C$ for all $a \in A$

$\Rightarrow$ $(a, c) \in A \times B$ for all $a \in A$ $[\because A \times B = A \times C]$

$\Rightarrow$ $c \in B$

Thus, $c \in C \Rightarrow c \in B$

$\therefore$ $C \subset B$ ...(ii)

From (i) and (ii), we get $B = C$.

EXERCISE 2.2

LEVEL-1

1. Given $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, find $(A \times B) \cap (B \times C)$.
2. If $A = \{2, 3\}$, $B = \{4, 5\}$, $C = \{5, 6\}$, find $A \times (B \cup C)$, $A \times (B \cap C)$, $(A \times B) \cup (A \times C)$.
3. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that:
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = (A \times B) - (A \times C)$.
4. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that:
   (i) $A \times C \subset B \times D$ (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
5. If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, find
   (i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$ (iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

LEVEL-2

6. Prove that: (i) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ (ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$
7. If $A \times B \subseteq C \times D$ and $A \times B \neq \phi$, prove that $A \subseteq C$ and $B \subseteq D$.

ANSWERS

1. $\{3, 4\}$.
2. $A \times (B \cup C) = \{(2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$
   $A \times (B \cap C) = \{(2, 5), (3, 5)\}$,
   $(A \times B) \cup (A \times C) = \{(2, 4), (2, 5), (3, 4), (3, 5), (2, 6), (3, 6)\}$.
4. $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
5. (i) $\{(1, 4), (2, 4), (3, 4)\}$ (ii) $\{(1, 4), (2, 4), (3, 4)\}$
   (iii) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
   (iv) $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$

2.5 RELATIONS

Let A and B denote the sets of all male and female members in the royal family of Dasrath's kingdom. Clearly, A = { Dasrath, Ram, Laxman, Shatrughan, Bharat } and B = { Kaushalya, Kaikai, Sumitra, Sita, Urmila, Shrutkirti, Mandvi }.

If we write R for the relation "was husband of" then the fact that Dasrath was husband of Kaushalya, Kaikai and Sumitra, Ram was husband of Sita, Laxman was husband of Urmila, Bharat was husband of Mandvi and Shatrughan was husband of Shrutkirti can be represented as:

Dasrath R Kaushlya, Dasrath R Kaikai, Dasrath R Sumitra, Ram R Sita, Laxman R Urmila, Bharat R Mandvi and Shatrughan R Shrutkirti.

Now, if we omit the letter R between the pairs of names and write them as ordered pairs, then the above fact can also be written as a set R of ordered pairs as given below:

R = { (Dasrath, Kaushalya), (Dasrath, Kaikai), (Dasrath, Sumitra), (Ram, Sita), (Laxman, Urmila), (Bharat, Mandvi), (Shatrughan, Shrutkirti) }.

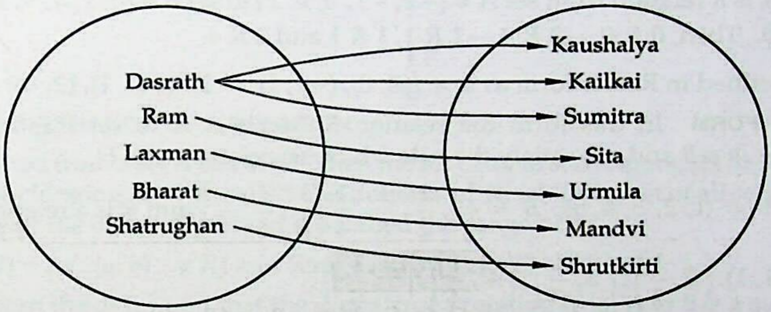

Fig. 2.10

Clearly, $R \subseteq A \times B$.

A visual representation of this relation R in the form of an arrow diagram is as follows:

Thus, we see that the relation "was husband of" from set A to set B gives rise to a subset R of $A \times B$ such that $(x, y) \in R$ *iff* xRy.

Keeping this example in mind, we may define a relation as follows.

RELATION *Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.*

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If R is a relation from a non-void set A to a non-void set B and *if* $(a, b) \in R$, then we write aRb which is read as `a is related to b by the relation R. *If* $(a, b) \notin R$, then we write aRb and we say that a is not related to b by the relation R.

ILLUSTRATION 1 *If $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$, then $R = \{(1, b), (2, c), (1, a), (3, a)\}$, being a subset of $A \times B$, is a relation from A to B. Here, $(1, b)$, $(2, c)$, $(1, a)$ and $(3, a) \in R$, so we write 1Rb, 2Rc, 1Ra and 3Ra. But, $(2, b) \notin R$, so we write 2Rb*

ILLUSTRATION 2 *If $A = \{a, b, c, d\}$, $B = \{p, q, r, s\}$, then which of the following are relations from A to B? Give reasons for your answer.*

(i) $R_1 = \{(a, p), (b, r), (c, s)\}$ *(ii)* $R_2 = \{(q, b), (c, s), (d, r)\}$

(iii) $R_3 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$ *(iv)* $R_4 = \{(a, p), (q, a), (b, s), (s, b)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, R_1 is a relation from A to B.

(ii) Since $(q, b) \in R_2$ but $(q, b) \notin A \times B$. So, $R_2 \not\subseteq A \times B$. Thus, R_2 is not a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So it is a relation from A to B.

(iv) R_4 is not a relation from A to B, because (q, a) and (s, b) are elements of R_4 but (q, a) and (s, b) are not in $A \times B$. As such $R_4 \not\subseteq A \times B$.

TOTAL NUMBER OF RELATIONS Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then, $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}. Since each subset of $A \times B$ defines a relation from A to B, so total number of relations from A to B is 2^{mn}. Among these 2^{mn} relations the void relation ϕ and the universal relation $A \times B$ are trivial relations from A to B.

2.5.1 REPRESENTATION OF A RELATION

A relation from a set A to a set B can be represented in any one of the following forms:

(I) ROSTER FORM In this form a relation is represented by the set of all ordered pairs belonging to R.

For example, if R is a relation from set $A = \{-2, -1, 0, 1, 2\}$ to set $B = \{0, 1, 4, 9, 10\}$ by the rule $a\,R\,b \Leftrightarrow a^2 = b$. Then, $0\,R\,0, -2\,R\,4, -1\,R\,1, 1\,R\,1$ and $2\,R\,4$.

So, R can be described in Roster form as $R = \{(0, 0), (-1, 1), (-2, 4), (1, 1), (2, 4)\}$

(II) SET-BUILDER FORM In this form the relation R from set A to set B is represented as $R = \{(a, b) : a \in A, b \in B$ and a, b satisfy the rule which associates a and $b\}$.

For example, if $A = \{1, 2, 3, 4, 5\}$, $B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$ and R is a relation from A to B given by $R = \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$.

Then, R in set-builder form can be described as: $R = \left\{(a, b) : a \in A, b \in B \text{ and } b = \frac{1}{a}\right\}$.

It should be noted that it is not possible to express every relation from set A to set B in set-builder form. For example, the relation $R = \{(1, a), (1, c), (3, b)\}$ from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$ cannot be described in set-builder form.

(III) BY ARROW DIAGRAM In order to represent a relation from set A to a set B by an arrow diagram, we draw arrows from first components to the second components of all ordered pairs belonging to R.

For example, relation $R = \{(1, 2), (2, 4), (3, 2), (1, 3), (3, 4)\}$ from set $A = \{1, 2, 3, 4, 5\}$ to set $B = \{2, 3, 4, 5, 6, 7\}$ can be represented by the following arrow diagram:

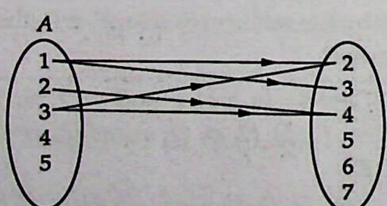

Fig. 2.11

(IV) BY LATTICE In this form, the relation R from set A to set B is represented by darking the dots in the lattice for $A \times B$ which represent the ordered pairs in R.

For example, if $R = \{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$ is a relation from set $A = \{-3, -2, -1, 0, 1, 2, 3\}$ to set $B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then R can be represented by the following lattice.

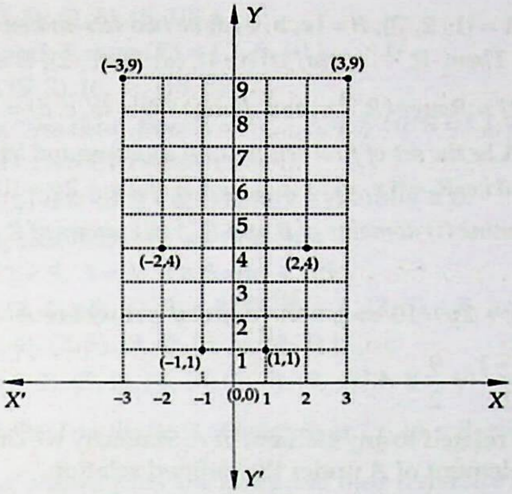

Fig. 2.12

2.5.2 DOMAIN AND RANGE OF A RELATION

Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, Dom $(R) = \{ a : (a, b) \therefore \in R\}$ and Range $(R) = \{b : (a, b) \in R\}$.

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B. The set B is called the co-domain of relation R.

ILLUSTRATION 1 *If* $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ *and let* $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ *be a relation from A to B. Then,*

$$\text{Domain } (R) = \{1, 3, 5\} \text{ and Range} (R) = \{8, 6, 2, 4\}$$

ILLUSTRATION 2 *Let* $A = \{ 1, 3, 5, 7 \}$ *and* $B = \{ 2, 4, 6, 8 \}$ *be two sets and let R be a relation from A to B defined by the phrase* "$(x, y) \in R \Leftrightarrow x > y$". *Under this relation R, we obtain* $3R2, 5R2, 5R4, 7R2, 7R4$ *and* $7R6$.

i.e. $R = \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)\}$.

$\therefore$ Domain $(R) = \{3, 5, 7\}$ and Range $(R) = \{2, 4, 6\}$

ILLUSTRATION 3 *If R is a relation from set* $A = \{2, 4, 5\}$ *to set* $B = \{1, 2, 3, 4, 6, 8\}$ *defined by* $xRy \Leftrightarrow x$ *divides* y.

(i) Write R as a set of ordered pairs, *(ii) Find the domain and the range of R.*

SOLUTION (i) Clearly, $2R2, 2R4, 2R6, 2R8, 4R4$, and $4R8$.

$\therefore$ $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$

(ii) Clearly, Domain $(R) = \{ 2, 4 \}$ and Range $(R) = \{ 2, 4, 6, 8 \}$

RELATION ON A SET *Let A be a non-void set. Then, a relation from A to itself i.e. a subset of* $A \times A$, *is called a relation on set A.*

2.5.3 INVERSE OF A RELATION

INVERSE RELATION *Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by* R^{-1}, *is a relation from B to A and is defined by* $R^{-1} = \{(b, a) : (a, b) \in R\}$.

Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, Dom (R) = Range (R^{-1}) and, Range (R) = Dom (R^{-1}).

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ be two sets and let $R = \{(1, a), (1, c), (2, d), (2, c)\}$ be a relation from A to B. Then, $R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$ is a relation from B to A.*

Also, $\text{Dom}(R) = \{1, 2\} = \text{Range}(R^{-1})$, *and* $\text{Range}(R) = \{a, c, d\} = \text{Dom}(R^{-1})$.

ILLUSTRATION 2 *Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$ i.e. $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Also, determine (i) domains of R and R^{-1} (ii) ranges of R and R^{-1}.*

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \frac{10 - x}{2}, x, y \in A \text{ where } A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Now, $x = 1 \Rightarrow y = \frac{10-1}{2} = \frac{9}{2} \notin A.$

This shows that 1 is not related to any element in A. Similarly we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation.

Further we find that:

For $x = 2$, $y = \frac{10-2}{2} = 4 \in A$. Therefore, $(2, 4) \in R$

For $x = 4$, $y = \frac{10-4}{2} = 3 \in A$. Therefore, $(4, 3) \in R$

For $x = 6$, $y = \frac{10-6}{2} = 2 \in A$. Therefore, $(6, 2) \in R$

For $x = 8$, $y = \frac{10-8}{2} = 1 \in A$. Therefore, $(8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\} \Rightarrow R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $\text{Dom}(R) = \{2, 4, 6, 8\} = \text{Range}(R^{-1})$ and, $\text{Range}(R) = \{4, 3, 2, 1\} = \text{Dom}(R^{-1})$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON EXAMINING WHETHER A SET OF ORDERED PAIRS REPRESENTS A RELATION OR NOT**

EXAMPLE 1 *If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer:*

(i) $R_1 = \{(1, 4), (1, 5), (1, 6)\}$ (ii) $R_2 = \{(1, 5), (2, 4), (3, 6)\}$

(iii) $R_3 = \{(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)\}$ (iv) $R_4 = \{(4, 2), (2, 6), (5, 1), (2, 4)\}$.

SOLUTION (i) Clearly, $R_1 \subseteq A \times B$. So, it is a relation from A to B.

(ii) Clearly, $R_2 \subseteq A \times B$. So, it is a relation from A to B.

(iii) Clearly, $R_3 \subseteq A \times B$. So, it is a relation from A to B.

(iv) Since $(4, 2) \in R_4$ but $(4, 2) \notin A \times B$. So, R_4 is not a relation from A to B.

Type II **ON DESCRIBING A RELATION AND ITS INVERSE AS A SET OF ORDERED PAIRS AND FINDING THEIR DOMAINS AND RANGES**

EXAMPLE 2 *A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows: $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R. Also, find R^{-1}.*

SOLUTION Recall that $a \mid b$ stands for 'a divides b'. For the elements of the given sets A and B, we find that $2 \mid 6$, $2 \mid 10$, $3 \mid 3$, $3 \mid 6$, and $5 \mid 10$.

$\therefore$ $(2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$.

Thus, $R = \{ (2, 6), (2, 10), (3, 3), (3, 6), (5, 10) \}$.
Clearly, Dom $(R) = \{ 2, 3, 5 \}$ and, Range $(R) = \{ 3, 6, 10 \}$.
Also, $R^{-1} = \{ (6, 2), (10, 2), (3, 3), (6, 3), (10, 5) \}$.

EXAMPLE 3 *If R is the relation "less than" from $A = \{ 1, 2, 3, 4, 5 \}$ to $B = \{ 1, 4, 5 \}$, write down the set of ordered pairs corresponding to R. Find the inverse of R.*

SOLUTION It is given that $(x, y) \in R \Leftrightarrow x < y$, where $x \in A$ and $y \in B$.
For the elements of the given sets A and B, we find that

$$1 < 4,\ 1 < 5,\ 2 < 4,\ 2 < 5,\ 3 < 4,\ 3 < 5 \text{ and } 4 < 5$$

$\therefore$ $(1, 4) \in R, (1, 5) \in R, (2, 4) \in R, (2, 5) \in R, (3, 4) \in R, (3, 5) \in R$ and $(4, 5) \in R$.
Thus, $R = \{ (1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5) \}$.
$\therefore$ $R^{-1} = \{ (4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4) \} = \{(x, y) : x \in B, y \in A \text{ and } x > y\}$.

EXAMPLE 4 *A relation R is defined on the set Z of integers as: $(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$.*

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION We have,

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \Leftrightarrow y = \pm \sqrt{25 - x^2}$$

We observe that from the above relation $x = 0$ gives $y = \pm 5$.
$\therefore$ $(0, 5) \in R$ and $(0, -5) \in R$
Similarly, $x = \pm 3 \Rightarrow y = \sqrt{25 - 9} = \pm 4$
$\therefore$ $(3, 4) \in R, (-3, 4) \in R, (3, -4) \in R$ and $(-3, -4) \in R$
$x = \pm 4 \Rightarrow y = \sqrt{25 - 16} = \pm 3$
$\therefore$ $(4, 3) \in R, (-4, 3) \in R, (4, -3) \in R$ and $(-4, -3) \in R$
$x = \pm 5 \Rightarrow y = \sqrt{25 - 25} = 0$
$\therefore$ $(5, 0) \in R$ and $(-5, 0) \in R$

We also notice that for any other integral value of x, the value of y given by $y = \pm \sqrt{25 - x^2}$ is not an integer.

$\therefore$ $R = \{ (0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0) \}$
$\Rightarrow R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5) \}$
Clearly, Domain $(R) = \{ 0, 3, -3, 4, -4, 5, -5 \} =$ Domain (R^{-1}).

EXAMPLE 5 *Let R be the relation on the set N of natural numbers defined by*

$$R = \{(a, b) : a + 3b = 12, a \in N, b \in N \}.$$

Find: (i) R *(ii) Domain of R* *(iii) Range of R*

SOLUTION (i) We have,

$$a + 3b = 12 \Rightarrow a = 12 - 3b$$

Putting $b = 1, 2, 3$ respectively in the above relation, we get $a = 9, 6, 3$ respectively.
For $b = 4$, $a = 12 - 3b$ gives $a = 0$ which does not belong to N. Also, values of a given by $a = 12 - 3b$ do not belong to N for all $b > 4$.
$\therefore$ $R = \{(9, 1), (6, 2), (3, 3)\}$
(ii) Clearly, Domain of $R = \{9, 6, 3\}$
(iii) Clearly, Range of $R = \{1, 2, 3\}$

Type III ON REPRESENTING A RELATION BY USING AN ARROW DIAGRAM

EXAMPLE 6 *Let $A = \{1, 2, 3, 4, 5, 6\}$. Define a relation R on set A by $R = \{(x, y) : y = x + 1\}$.*
(i) Depict this relation using an arrow diagram *(ii) Write down the domain, co-domain and range of R.*

SOLUTION (i) Putting $x = 1, 2, 3, 4, 5, 6$ respectively in $y = x + 1$, we get $y = 2, 3, 4, 5, 6, 7$ respectively.

$\therefore$ $(1, 2) \in R, (2, 3) \in R, (3, 4) \in R, (4, 5) \in R, (5, 6) \in R$ and $(6, 7) \notin R$.

For $x = 6$, we get $y = 7$ which does not belong to set A.

Hence, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$

The arrow diagram representing R is as follows.

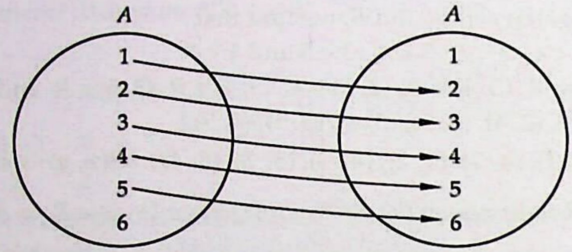

Fig. 2.13

(ii) Clearly, Domain $(R) = \{1, 2, 3, 4, 5\}$, Range $(R) = \{2, 3, 4, 5, 6\}$.

EXAMPLE 7 *Figure 2.14 shows a relation R between the sets P and Q. Write this relation R in (i) Roster form (ii) Set builder form. What is its domain and range?*

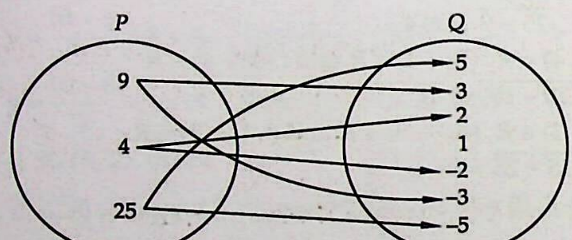

Fig. 2.14

SOLUTION (i) It is evident from the figure that

$$R = \{(9, 3), (9, -3), (4, 2), (4 - 2), (25, 5), (25 - 5)\}$$

(ii) It is evident from R, that it consists of elements (x, y), where x is the square of y i.e. $x = y^2$.

Therefore, relation R in set builder form is $R = \{(x, y) : x = y^2, x \in P, y \in Q\}$.

The domain and range of R are $\{9, 4, 25\}$ and $\{-5, -3, -2, 2, 3, 5\}$ respectively.

REMARK *In the above example, the range of relation R is not same as the set Q. The set Q is known as the co-domain.*

Type IV ON PROVING RESULTS BASED ON THE DEFINITION OF A RELATION

EXAMPLE 8 *Let R be a relation on Q defined by* $R = \{(a, b) : a, b \in Q \text{ and } a - b \in Z\}$.

Show that:

(i) $(a, a) \in R$ *for all* $a \in Q$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. [NCERT]

SOLUTION (i) For any $a \in Q$, we have

$a - a = 0 \in Z$

$\Rightarrow$ $(a, a) \in R$

Hence, $(a, a) \in R$ for all $a \in Q$.

(ii) Let $(a, b) \in R$. Then,

$(a, b) \in R$

$a - b \in Z$, where $a, b \in Q$

$\Rightarrow$ $b - a \in Z$ $[\because\ b - a = -(a - b)]$

$\Rightarrow$ $(b, a) \in R$

(iii) Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b \in Z$ and $b - c \in Z$

$\Rightarrow$ $(a - b) + (b - c) \in Z$

$\Rightarrow$ $a - c \in Z$

$\Rightarrow$ $(a, c) \in R$

EXAMPLE 9 *Let R be a relation on N defined by* $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$.
Are the following true:

(i) $(a, a) \in R$ *for all* $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$

(iii) $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$ **[NCERT]**

Justify your answer in each case.

SOLUTION (i) We observe that $a = a^2$ is true for $a = 1 \in N$ only. Therefore, $(1, 1) \in R$. But, $(2, 2), (3, 3), (4, 4)$ etc do not belong to R. So, $(a, a) \in R$ for all $a \in N$ is not true.

(ii) We observe that $(4, 2) \in R$, because $4 = 2^2$. But, $(2, 4) \notin R$ as $2 \neq 4^2$.

So, $(a, b) \in R \Rightarrow (b, a) \in R$ is not true for all $a, b \in N$.

(iii) We observe that $(16, 4) \in R$ and $(4, 2) \in R$. However, $(16, 2) \notin R$.

So, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ is not true for all $a, b, c \in N$.

EXAMPLE 10 *Let a relation* R_1 *on the set R of all real numbers be defined as* $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$ *for all* $a, b \in R$.

Show that: (i) $(a, a) \in R_1$ *for all* $a \in R$ (ii) $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ *for all* $a, b \in R$

SOLUTION (i) For any $a \in R$, we have

$1 + a^2 > 0 \Rightarrow (a, a) \in R_1$

Thus, $(a, a) \in R_1$ for all $a \in R$.

(ii) Let $(a, b) \in R_1$. Then,

$(a, b) \in R_1 \Rightarrow 1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R_1$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$.

EXAMPLE 11 *Let R be the relation on the set Z of all integers defined by* $(x, y) \in R \Rightarrow x - y$ *is divisible by n. Prove that:*

(i) $(x, x) \in R$ *for all* $x \in Z$

(ii) $(x, y) \in R \Rightarrow (y, x) \in R$ *for all* $x, y \in Z$

(iii) $(x, y) \in R$ *and* $(y, z) \in R \Rightarrow (x, z) \in R$ *for all* $x, y, z \in R$.

SOLUTION (i) For any $x \in Z$, we have

$x - x = 0 = 0 \times n$

$\Rightarrow$ $x - x$ is divisible by n

$\Rightarrow$ $(x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then,

$(x, y) \in R$

$\Rightarrow$ $x - y$ is divisible by n

$\Rightarrow$ $x - y = \lambda n$ for same $\lambda \in Z$

$\Rightarrow$ $y - x = (-\lambda) n$

$\Rightarrow$ $y - x$ is divisible by n $[\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\Rightarrow$ $(y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R \Rightarrow x - y$ is divisible by $n \Rightarrow x - y = \lambda n$ for some $\lambda \in Z$

$(y, z) \in R \Rightarrow y - z$ is divisible by $n \Rightarrow y - z = \mu n$ for some $\mu \in Z$

$\therefore$ $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $x - y = \lambda n$ and $y - z = \mu n$

$\Rightarrow$ $(x - y) + (y - z) = \lambda n + \mu n$

$\Rightarrow$ $x - z = (\lambda + \mu) n$

$\Rightarrow$ $x - z$ is divisible by n $[\because \lambda + \mu \in Z]$

$\Rightarrow$ $(x, z) \in R$

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

EXERCISE 2.3

LEVEL-1

1. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following are relations from A to B? Give reasons in support of your answer.
   (i) $\{(1, 6), (3, 4), (5, 2)\}$ (ii) $\{(1, 5), (2, 6), (3, 4), (3, 6)\}$
   (iii) $\{4, 2), (4, 3), (5, 1)\}$ (iv) $A \times B$.
2. A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows:
   $(x, y) \in R \Leftrightarrow x$ is relatively prime to y
   Express R as a set of ordered pairs and determine its domain and range.
3. Let A be the set of first five natural numbers and let R be a relation on A defined as follows:
   $(x, y) \in R \Leftrightarrow x \leq y$
   Express R and R^{-1} as sets of ordered pairs. Determine also
   (i) the domain of R^{-1} (ii) the range of R.
4. Find the inverse relation R^{-1} in each of the following cases:
   (i) $R = \{(1, 2), (1, 3), (2, 3), (3, 2), (5, 6)\}$
   (ii) $R = \{(x, y) : x, y \in N, x + 2y = 8\}$
   (iii) R is a relation from $\{11, 12, 13\}$ to $(8, 10, 12\}$ defined by $y = x - 3$.
5. Write the following relations as the sets of ordered pairs:
   (i) A relation R from the set $\{2, 3, 4, 5, 6\}$ to the set $\{1, 2, 3\}$ defined by $x = 2y$.
   (ii) A relation R on the set $\{1, 2, 3, 4, 5, 6, 7\}$ defined by
   $(x, y) \in R \Leftrightarrow x$ is relatively prime to y.
   (iii) A relation R on the set $\{0, 1, 2, ..., 10\}$ defined by $2x + 3y = 12$.
   (iv) A relation R from a set $A = \{5, 6, 7, 8\}$ to the set $B = \{10, 12, 15, 16, 18\}$ defined by
   $(x, y) \in R \Leftrightarrow x$ divides y.
6. Let R be a relation in N defined by $(x, y) \in R \Leftrightarrow x + 2y = 8$. Express R and R^{-1} as sets of ordered pairs.

7. Let $A = \{3, 5\}$ and $B = \{7, 11\}$. Let $R = \{(a, b) : a \in A, b \in B, a - b \text{ is odd }\}$. Show that R is an empty relation from A into B.
8. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the total number of relations from A into B.
9. Determine the domain and range of the relation R defined by
   (i) $R = \{(x, x + 5) : x \in \{0, 1, 2, 3, 4, 5\}\}$ **[NCERT]**
   (ii) $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$ **[NCERT]**
10. Determine the domain and range of the following relations:
   (i) $R = \{(a, b) : a \in N, a < 5, b = 4\}$
   (ii) $S = \{(a, b) : b = |a - 1|, a \in Z \text{ and } |a| \leq 3\}$
11. Let $A = \{a, b\}$. List all relations on A and find their number.
12. Let $A = \{x, y, z\}$ and $B = \{a, b\}$. Find the total number of relations from A into B. **[NCERT]**
13. Let R be a relation from N to N defined by $R = \{(a, b) : a, b \in N \text{ and } a = b^2\}$.
   Are the following statements true?
   (i) $(a, a) \in R$ for all $a \in N$ (ii) $(a, b) \in R \Rightarrow (b, a) \in R$
   (iii) $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$
14. Let $A = \{1, 2, 3, \ldots, 14\}$. Define a relation on a set A by
   $R = \{(x, y) : 3x - y = 0, \text{ where } x, y \in A\}$.
   Depict this relationship using an arrow diagram. Write down its domain, co-domain and range.
15. Define a relation R on the set N of natural numbers by
   $R = \{(x, y) : y = x + 5, x \text{ is a natural number less than } 4, x, y \in N\}$.
   Depict this relationship using (i) roster form (ii) an arrow diagram. Write down the domain and range or R. **[NCERT]**
16. $A = \{1, 2, 3, 5\}$ and $B = \{4, 6, 9\}$. Define a relation R from A to B by
   $R = \{(x, y) :$ the difference between x and y is odd, $x \in A, y \in B\}$.
   Write R in Roster form. **[NCERT]**

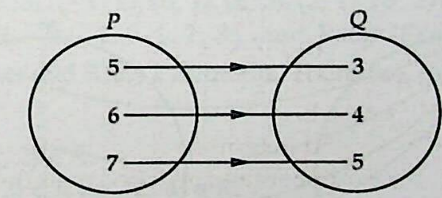

Fig. 2.15

17. Write the relation $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$ in roster form. **[NCERT]**
18. Let $A = \{1, 2, 3, 4, 5, 6\}$. Let R be a relation on A defined by
   $R = \{(a, b) : a, b \in A, b \text{ is exactly divisible by a}\}$
   (i) Write R in roster form (ii) Find the domain of R (iii) Find the range of R. **[NCERT]**
19. Figure 2.15 shows a relationship between the sets P and Q. Write this relation in
   (i) set builder form (ii) roster form. What is its domain and range? **[NCERT]**
20. Let R be the relation on Z defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$. Find the domain and range of R. **[NCERT]**
21. For the relation R_1 defined on R by the rule $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$.
   Prove that: $(a, b) \in R_1$ and $(b, c) \in R_1 \Rightarrow (a, c) \in R_1$ is not true for all $a, b, c \in R$.
22. Let R be a relation on $N \times N$ defined by
   $(a, b)\, R\, (c, d) \Leftrightarrow a + d = b + c$ for all $(a, b), (c, d) \in N \times N$
   Show that:
   (i) $(a, b)\, R\, (a, b)$ for all $(a, b) \in N \times N$
   (ii) $(a, b)\, R\, (c, d) \Rightarrow (c, d)\, R\, (a, b)$ for all $(a, b), (c, d) \in N \times N$
   (iii) $(a, b)\, R\, (c, d)$ and $(c, d)\, R\, (e, f) \Rightarrow (a, b)\, R\, (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$

ANSWERS

1. (i) It is not a relation from A to B.
   (ii) It is a subset of $A \times B$, so it is a relation from A to B.
   (iii) It is not a relation from A to B as it is not a subset of $A \times B$.
   (iv) It is a relation from A to B.
2. $R = \{ (2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 7) \}$
3. $R = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5), (5,5) \}$
   $R^{-1} = \{ (1,1), (2,1), (3,1), (4,1), (5,1), (2,2), (3,2), (4,2), (5,2), (3,3), (4,3), (5,3), (4,4), (5,4), (5,5) \}$
   Domain of $R^{-1} = \{ 1, 2, 3, 4, 5 \}$ = Range of R.
4. (i) $R^{-1} = \{ (2, 1), (3, 1), (3, 2), (2, 3), (6, 5) \}$ (ii) $R^{-1} = \{ (3, 2), (2, 4), (1, 6) \}$
   (iii) $R^{-1} = \{ (8, 11), (10, 13) \}$
5. (i) $\{ (2, 1), (4, 2), (6, 3) \}$
   (ii) $\{ (2,3), (2,5), (2,7), (3,2), (3,4), (3,5), (3,7), (4,3), (4,5), (4,7), (5,2), (5,3), (5,4), (5,6), (5,7), (6,5), (6,7), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7) \}$
   (iii) $\{(0, 4), (3, 2), (6, 0) \}$ (iv) $\{ (5, 10), (5, 15), (6, 12), (6, 18), (8, 16) \}$
6. $R = \{ (2, 3), (4, 2), (6, 1) \}$ $R^{-1} = \{ (3, 2), (2, 4), (1, 6) \}$ **8.** 16
9. (i) Domain $R = \{0, 1, 2, 3, 4, 5\}$, Range $R = \{5, 6, 7, 8, 9, 10\}$
   (ii) Domain $R = \{2, 3, 5, 7\}$, Range $R = \{8, 27, 125, 343\}$
10. (i) Domain $R = \{1, 2, 3, 4\}$, Range $R = \{4\}$
    Domain $S = \{0, -1, -2, -3, 1, 2, 3\}$, Range $S = \{0, 1, 2, 3, 4\}$
    (ii) $S = \{(0, 1)), (-1, 2), (-2, 3), (-3, 4), (1, 0), (2, 1), (3, 2)\}$
11. 16 **12.** 64 **13.** (i) No (ii) No (iii) No
14. Domain $(R) = \{1, 2, 3, 4\}$, Co-domain $(R) = A$, Range$(R) = \{3, 6, 9, 12\}$

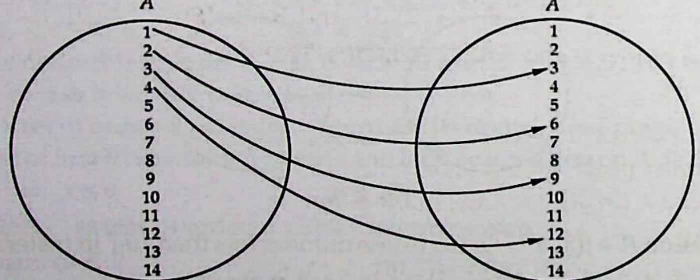

Fig. 2.16

15. (i) $R = \{(1, 6), (2, 7), (3, 8)\}$ (ii) Domain (R) = [1, 2, 3], Range (R) = $\{ 6, 7, 8 \}$

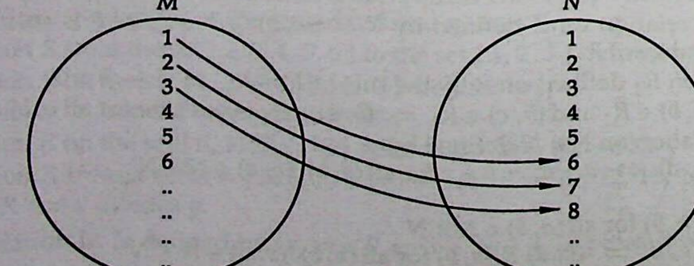

Fig. 2.17

16. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. $R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5.5), (6, 6)\}$

(ii) Domain $(R) = \{1, 2, 3, 4, 5, 6\}$ (iii) Range $(R) = \{1, 2, 3, 4, 5, 6\}$

19. (i) $R = \{(x, y) : y = x - 2, x \in P, y \in Q\}$ (ii) $R = \{(5, 3), (6, 4), (7, 5)\}$

Domain $(R) = \{5, 6, 7\}$, Range $(R) = 3, 4, 5$

20. (i) Domain $(R) = Z$, Range $(R) = Z$

HINTS TO NCERT & SELECTED PROBLEMS

8. We have, $n(A) = 2, n(B) = 2$

$\therefore \quad n(A \times B) = 2 \times 2 = 4$

So, there are $2^4 = 16$ relations from A to B.

9. (i) We have,

$R = \{(x, x+5): x \in \{0, 1, 2, 3, 4, 5\}\} = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$

$\therefore$ Domain $(R) = \{0, 1, 2, 3, 4, 5\}$ and, Range $(R) = \{5, 6, 7, 8, 9, 10\}$

(ii) We have,

$R = \{(x, x^3) : x$ is a prime number less than $10\} = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

$\therefore$ Domain $(R) = \{2, 3, 5, 7\}$, and Range $(R) = \{8, 27, 125, 343\}$

10. (i) We have, $R = \{(1, 4), (2, 4), (3, 4), (4, 4)\}$

$\therefore$ Domain $(R) = \{1, 2, 3, 4\}$, Range $(R) = \{4\}$

(ii) We have,

$S = \{(-3, 4), (-2, 3), (-1, 2), (0, 1), (1, 0), (2, 1), (3, 2)\}$

$\therefore$ Domain $(S) = \{-3, -2, -1, 0, 1, 2, 3\}$, and Range $(S) = \{0, 1, 2, 3, 4\}$

12. Here A has 3 elements and B has 2 elements. Therefore, total number of relations from A to B is $2^{3 \times 2} = 64$.

13. (i) No, because $(2, 2) \notin R$.

(ii) No, because $(4, 2) \in R$ but $(2, 4) \notin R$.

(iii) No, because $(16, 4) \in R$ and $(4, 2) \in R$ but $(16, 2) \notin R$.

14. $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Domain $(R) = \{1, 2, 3, 4\}$, and Range $(R) = \{3, 6, 9, 12\}$.

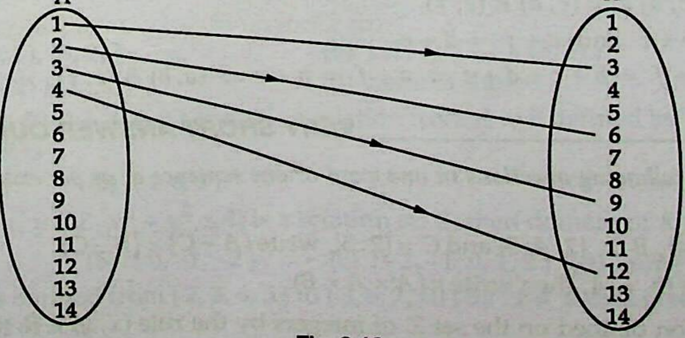

Fig. 2.18

15. (i) We have,

$R = \{(x, y) : y = x + 5, x \text{ is a natural number less than } 4, x, y \in N\} = \{(1, 6), (2, 7), (3, 8)\}$

(ii) Domain $(R) = \{1, 2, 3\}$, and Range $(R) = \{6, 7, 8\}$

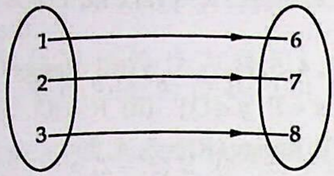

Fig. 2.19

16. We have, $A = \{1, 2, 3, 5\}, B = \{4, 6, 9\}$

$R = \{(x, y) : \text{the difference between } x \text{ and } y \text{ is odd}, x \in A, y \in B\}$

$\therefore \quad R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

17. We have, $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$

$\therefore \quad R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$

18. (i) We have,

$R = \{(a, b) : a, b \in A, b \text{ is exactly divisible by } a\}$, where $A = \{1, 2, 3, 4, 5, 6\}$.

$\therefore R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)\}$

(i) Domain $R = \{1, 2, 3, 4, 5, 6\}$ (ii) Range $R = \{1, 2, 3, 4, 5, 6\}$

19. (i) $\{(x, y) : y = x - 2, x \in \{5, 6, 7\}, y \in \{3, 4, 5\}\}$

(ii) $\{(5, 3), (6, 4), (7, 5)\}$

Domain $R = \{5, 6, 7\}$, and Range $R = \{3, 4, 5\}$

20. The relation R on Z is defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$

Since $a - b$ is an integer for all $a, b \in Z$. So, domain $(R) = Z =$ Range (R).

21. We have,

$\left(1, -\frac{1}{2}\right) \in R_1$ and $\left(-\frac{1}{2}, -4\right) \in R_1$ as $1 + \left(\frac{-1}{2}\right) > 0$ and $1 + \left(\frac{-1}{2}\right)(-4) > 0$.

But, $1 + 1 \times -4 \ngtr 0$. So, $(1, -4) \notin R_1$.

22. (i) We know that

$a + b = b + a$ for all $a, b \in N$

$\therefore \quad (a, b)\ R\ (a, b)$ for all $a, b \in N$

(ii) $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a \Rightarrow (c, d)\ R\ (a, b)$

(iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow a + d = b + c$ and $c + f = d + e$

$\Rightarrow a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $A = \{1, 2, 4\}, B = \{2, 4, 5\}$ and $C = \{2, 5\}$, write $(A - C) \times (B - C)$.
2. If $n(A) = 3, n(B) = 4$, then write $n(A \times A \times B)$.
3. If R is a relation defined on the set Z of integers by the rule $(x, y) \in R \Leftrightarrow x^2 + y^2 = 9$, then write domain of R.

4. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation defined on the set Z of integers, then write domain of R.
5. If R is a relation from set $A = \{11, 12, 13\}$ to set $B = \{8, 10, 12\}$ defined by $y = x - 3$, then write R^{-1}.
6. Let $A = \{1, 2, 3\}$ and $R = \{(a, b) : |a^2 - b^2| \leq 5, a, b \in A\}$. Then write R as set of ordered pairs.
7. Let $R = \{(x, y) : x, y \in Z, y = 2x - 4\}$. If $(a, -2)$ and $(4, b^2) \in R$, then write the values of a and b.
8. If $R = \{(2, 1), (4, 7), (1, -2), \ldots\}$, then write the linear relation between the components of the ordered pairs of the relation R.
9. If $A = \{1, 3, 5\}$ and $B = \{2, 4\}$, list the elements of R, if $R = \{(x, y) : x, y \in A \times B$ and $x > y\}$.
10. If $R = \{(x, y) : x, y \in W, 2x + y = 8\}$, then write the domain and range of R.
11. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, write A and B.
12. Let $A = \{1, 2, 3, 5\}$, $B = \{4, 6, 9\}$ and R be a relation from A to B defined by $R = \{(x, y) : x - y$ is odd $\}$. Write R in roster form.

ANSWERS

1. $\{(1, 4), (4, 4)\}$
2. 36
3. Domain $(R) = \{-3, 0, 3\}$
4. Domain $(R) = \{-2, -1, 0, 1, 2\}$
5. $\{(8, 11), (10, 13)\}$
6. $\{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$
7. $a = 1, b = \pm 2$
8. $y = 3x - 5$
9. $\{(3, 2), (5, 2), (5, 4)\}$
10. Domain $(R) = \{0, 1, 2, 3, 4\}$, Range $(R) = \{0, 2, 4, 6, 8\}$
11. $A = \{x, y, z\}$, $B = \{1, 2\}$
12. $R = \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$, then $(A - B) \times (B - C)$ is
   (a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{(1, 4)\}$
   (c) $(1, 4)$ (d) none of these.
2. If R is a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given by $x\,R\,y \Leftrightarrow y = 3x$, then $R=$
   (a) $\{(3, 1), (6, 2), (8, 2), (9, 3)\}$ (b) $\{(3, 1), (6, 2), (9, 3)\}$
   (c) $\{(3, 1), (2, 6), (3, 9)\}$ (d) none of these.
3. Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$. Then, R^{-1} is
   (a) $\{(3, 3), (3, 1), (5, 2)\}$ (b) $\{(1, 3), (2, 5), (3, 3)\}$
   (c) $\{(1, 3), (5, 2)\}$ (d) none of these.
4. If $A = \{1, 2, 3\}$, $B = \{1, 4, 6, 9\}$ and R is a relation from A to B defined by 'x is greater than y. The range of R is
   (a) $\{1, 4, 6, 9\}$ (b) $\{4, 6, 9\}$ (c) $\{1\}$ (d) none of these.
5. If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation on Z, then domain of R is
   (a) $\{0, 1, 2\}$ (b) $\{0, -1, -2\}$ (c) $\{-2, -1, 0, 1, 2\}$ (d) none of these.
6. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by : $x\,R\,y \Leftrightarrow x$ is relatively prime to y. Then, domain of R is
   (a) $\{2, 3, 5\}$ (b) $\{3, 5\}$ (c) $\{2, 3, 4\}$ (d) $\{2, 3, 4, 5\}$.

7. A relation ϕ from C to R is defined by $x \,\phi\, y \Leftrightarrow |x| = y$. Which one is correct?
   (a) $(2+3i)\,\phi\,13$ (b) $3\,\phi(-3)$ (c) $(1+i)\,\phi\,2$ (d) $i\,\phi\,1$.
8. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is
   (a) $\{2, 4, 8\}$ (b) $\{2, 4, 6, 8\}$ (c) $\{2, 4, 6\}$ (d) $\{1, 2, 3, 4\}$.
9. R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. Then, R^{-1} is
   (a) $\{(8, 11), (10, 13)\}$ (b) $\{(11, 8), (13, 10)\}$
   (c) $\{(10, 13), (8, 11), (12, 10)\}$ (d) none of these.
10. If the set A has p elements, B has q elements, then the number of elements in $A \times B$ is
    (a) $p+q$ (b) $p+q+1$ (c) pq (d) p^2
11. Let R be a relation from a set A to a set B, then
    (a) $R = A \cup B$ (b) $R = A \cap B$ (c) $R \subseteq A \times B$ (d) $R \subseteq B \times A$.
12. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is
    (a) 2^{mn} (b) $2^{mn} - 1$ (c) $2mn$ (d) m^n
13. If R is a relation on a finite set having n elements, then the number of relations on A is
    (a) 2^n (b) 2^{n^2} (c) n^2 (d) n^n.

ANSWERS

1. (b) 2. (d) 3. (a) 4. (c) 5. (c) 6. (d) 7. (d) 8. (c) 9. (a)
10. (c) 11. (c) 12. (a) 13. (b)

SUMMARY

1. An ordered pair consists of two objects or elements in a given fixed order.
2. $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$
3. If A and B are two non-empty sets, then $A \times B = \{(a, b) : a \in A, b \in B\}$ is called the cartesian product of A and B. If A and B are finite sets having m and n elements respectively, then $A \times B$ has mn elements.
4. $R \times R = \{(x, y) : x, y \in R\}$ is the set of all points in xy-plane.
5. $R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ set of all points in three dimensional space.
6. For any three sets A, B, C, we have
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$. (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   (iii) $A \times (B - C) = A \times B - A \times C$ (iv) $A \times B = B \times A \Leftrightarrow A = B$
   (v) $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$ (vi) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
   (vii) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$ (viii) $A \times B = A \times C \Rightarrow B = C$
7. Let A and B be two sets. A relation from A to B is a subset of $A \times B$.
8. If A and B are finite sets having m and n elements respectively. Then, 2^{mn} relations can be defined from A to B.
9. If R is a relation from set A to set B, then
   Domain $(R) = \{x : (x, y) \in R\}$, Range $(R) = \{y : (x, y) \in R\}$
10. A relation from a set A to itself is called a relation on A.
11. Let A, B be two sets and let R be a relation from set A to set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.
    Clearly, $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$
    Domain (R) = Range (R^{-1}), and Range (R) = Domain (R^{-1}).