CHAPTER 3

INVERSE TRIGONOMETRIC FUNCTIONS

3.1 INTRODUCTION

In earlier chapters, we have learnt about functions, types of functions, composition of functions and inverse of a function. In this chapter, we shall use these concepts to define the inverses of all trigonometric functions and to study their properties. Let us first recall the definition of inverse of a function.

3.2 INVERSE OF A FUNCTION

In the previous chapter, we have learnt that corresponding to every bijection (one-one onto function) $f : A \to B$ there exists a bijection $g : B \to A$ defined by

$$g(y) = x \text{ if and only } f(x) = y.$$

The function $g : B \to A$ is called the inverse of function $f : A \to B$ and is denoted by f^{-1}.

Thus, we have

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

Also, $(f^{-1} of)(x) = f^{-1}\left(f(x)\right) = f^{-1}(y) = x,$ for all $x \in A$.

and, $(fo f^{-1})(y) = f\left(f^{-1}(y)\right) = f(x) = y,$ for all $y \in B$.

3.3 INVERSES OF TRIGONOMETRIC FUNCTIONS

We know that trigonometric functions are periodic functions, and hence, in general, all trigonometric functions are not bijections. Consequently, their inverses do not exist. However, if we restrict their domains and co-domains, they can be made bijections and we can obtain their inverses. In the following sections, we shall do all these things to obtain the inverses of trigonometric functions.

3.3.1 INVERSE OF SINE FUNCTION

Consider the function $f : R \to R$ given by $f(x) = \sin x$. The graph of this function is shown in Fig. 3.1. Clearly, it is a many-one into function as it attains same value at infinitely many points and its range $[-1, 1]$ is not same as its co-domain. We know that any function can be made an onto function, if we replace its co-domain by its range. Therefore, $f : R \to [-1, 1]$ is a many-one onto function. In order to make f a one-one function, we will have to restrict its domain in such a way that in that domain there is no turn in the graph of the function and the function takes every value between -1 and 1. It is evident from the graph of $f(x) = \sin x$ that if we take the domain as $[-\pi/2, \pi/2]$, then $f(x)$ becomes one-one. Thus, $f : [-\pi/2, \pi/2] \to [-1, 1]$ given by $f(\theta) = \sin \theta$ is a bijection and hence invertible.

The inverse of the sine function is denoted by $\sin^{-1}$. Thus, $\sin^{-1}$ is a function with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ such that $\sin^{-1} x = \theta \Leftrightarrow \sin \theta = x$.

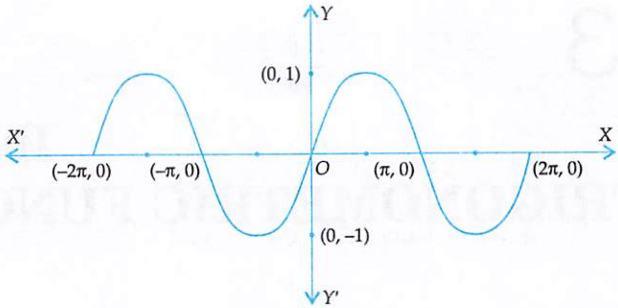


Fig. 3.1 Graph of $y = \sin x, -2\pi \le x \le 2\pi$

Also, $\sin^{-1}(\sin\theta) = \theta$ for all $\theta \in [-\pi/2, \pi/2]$ $[\because f^{-1}of(x) = f^{-1}(f(x)) = x \text{ for all } x \in D(f)]$

and, $\sin(\sin^{-1} x) = x$ for all $x \in [-1, 1]$ $[\because fof^{-1}(y) = f(f^{-1}(y)) = y \text{ for all } y \in D(f^{-1})]$

The graph of the function $f : [-\pi/2, \pi/2] \to [-1, 1]$ given by $f(x) = \sin x$ is shown in Fig. 3.2 and the graph of $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ is shown in Fig. 3.3.

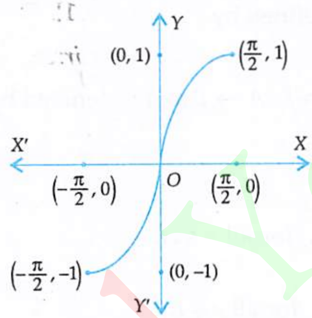


Fig. 3.2 Graph of $y = \sin x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

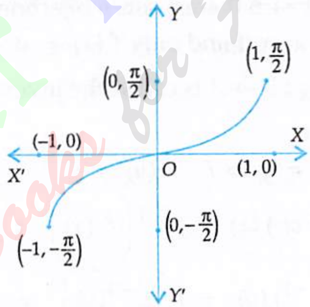


Fig. 3.3 Graph of $y = \sin^{-1} x, -1 \le x \le 1$

REMARK 1 *In the above discussion, we have restricted the domain of sine function to the interval* $[-\pi/2, \pi/2]$ *to make it a bijection. In fact, if we restrict its domain to any one of the intervals* $[-\pi/2, \pi/2]$, $[\pi/2, 3\pi/2]$, $[3\pi/2, 5\pi/2]$, $[-3\pi/2, -\pi/2]$, $[-5\pi/2, -3\pi/2]$ *or, in general* $[n\pi - \pi/2, n\pi + \pi/2]$, $n \in Z$, *then also it becomes a bijection. We can, therefore, define the inverse of the sine function in each of these intervals. Thus,* $\sin^{-1} x$ *is a function with domain* $[-1, 1]$ *and range* $[-\pi/2, \pi/2]$ *or* $[-3\pi/2, -\pi/2]$ *or* $[\pi/2, 3\pi/2]$ *and so on. Corresponding to each such interval, we get a branch of the function* $\sin^{-1} x$. *The branch of the function* $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ *called the principal value branch as shown in Fig. 3.3 and the value* $\sin^{-1} x$ *for given value of* $x \in [-1, 1]$ *is called the principal value.*

REMARK 2 *By considering* $\sin^{-1} x$ *as a function with domain* $[-1, 1]$ *and range* $[-\pi/2, \pi/2]$ *or* $[\pi/2, 3\pi/2]$ or $[3\pi/2, 5\pi/2]$ *and so on, we get different branches. If all these branches are put together and drawn on the same scale, we obtain the graph as shown in Fig. 3.4. Clearly, this graph can be obtained from the graph of sine function by interchanging the coordinate axes. The branch of* $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$ *is the principal value branch and the value of* $\sin^{-1} x$ *lying in* $[-\pi/2, \pi/2]$ *for a given value of* $x \in [-1, 1]$ *is called the principal value.*

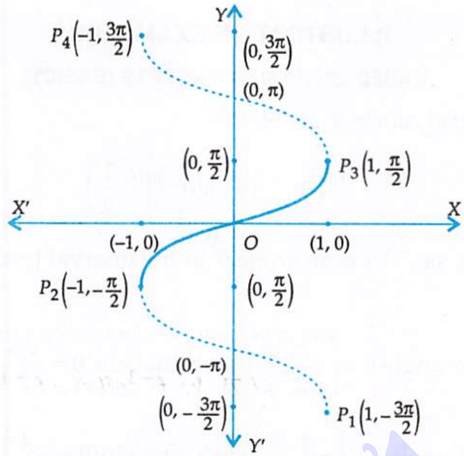


Fig. 3.4 Different branches of $y = \sin^{-1} x$ on the same scale

REMARK 3 *In chapter 2, we have learnt that the graphs of a function and its inverse (if it exists) are mirror images of each other in the line mirror $y = x$. In the above discussion, we have learnt that $\sin^{-1} : [-1, 1] \to [-[\pi/2, \pi/2]$ is the inverse of function $\sin : [-\pi/2, \pi/2] \to [-1, 1]$. Their graphs that is the curves $y = \sin x$ and $y = \sin^{-1} x$ are mirror images of each other in the line mirror $y = x$ as shown in Fig. 3.5.*

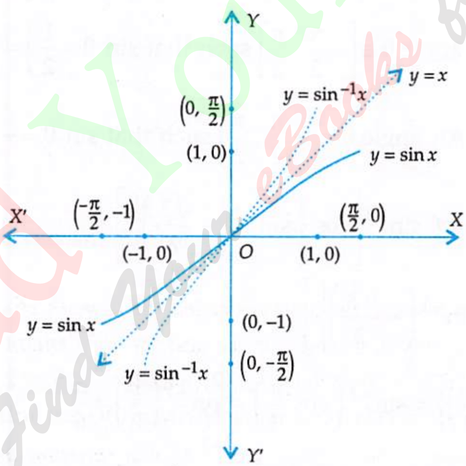


Fig. 3.5 Graphs of $y = \sin x$ and $y = \sin^{-1} x$ as mirror images of each other in line mirror $y = x$

NOTE 1 $\sin^{-1} x$ *is not equal to* $(\sin x)^{-1}$, *or* $\frac{1}{\sin x}$.

SOME OBSERVATIONS From Figures 3.2 and 3.3, we make the following observations :

(i) sin and $\sin^{-1}$ are increasing functions on $[-\pi/2, \pi/2]$ and $[-1, 1]$ respectively.

$\therefore \quad \theta_1 < \theta_2 \Rightarrow \sin \theta_1 < \sin \theta_2$ for all $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$

and, $x_1 < x_2 \Rightarrow \sin^{-1} x_1 < \sin^{-1} x_2$ for all $x_1, x_2 \in [-1, 1]$

(ii) The minimum and the maximum values of $\sin^{-1} x$ are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ respectively.

(iii) $\sin^{-1} x$ attains the minimum value $-\frac{\pi}{2}$ at $x = -1$ and the maximum value $\frac{\pi}{2}$ at $x = 1$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the principal values of*

(i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(-\frac{1}{2}\right)$ [NCERT, CBSE 2011]

SOLUTION For $x \in [-1, 1]$, $\sin^{-1} x$ is an angle θ in the interval $[-\pi/2, \pi/2]$ whose sine is x i.e. $\sin\theta = x$. Therefore,

(i) $$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ such that } \sin\theta = \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

(ii) $$\sin^{-1}\left(-\frac{1}{2}\right) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ such that } \sin\theta = -\frac{1}{2}\right) = -\frac{\pi}{6}$$

EXAMPLE 2 *Find the principal values of*

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ (ii) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

SOLUTION We know that $\sin^{-1} x$ denotes an angle in the interval $[-\pi/2, \pi/2]$ whose sine is x for $x \in [-1, 1]$. Therefore,

(i) $$\sin^{-1}\left(\frac{1}{2}\right) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ such that } \sin\theta = \frac{1}{2}\right) = \frac{\pi}{6}$$

(ii) $$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ such that } \sin\theta = -\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$$

EXAMPLE 3 *Find the value of* $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}\right]$.

SOLUTION $\sin^{-1}\left[\cos\left\{\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right\}\right]$

$$= \sin^{-1}\left\{\cos\left(-\frac{\pi}{3}\right)\right\} = \sin^{-1}\left(\cos\frac{\pi}{3}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \left[\because \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}\right]$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the domain of the function* $f(x) = \sin^{-1}(2x-3)$.

SOLUTION The domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore, $f(x) = \sin^{-1}(2x-3)$ is defined for all x satisfying

$$-1 \le 2x-3 \le 1 \Rightarrow 3-1 \le 2x \le 3+1 \Rightarrow 1 \le x \le 2 \Rightarrow x \in [1, 2]$$

Hence, domain of $f(x) = \sin^{-1}(2x-3)$ is $[1, 2]$.

EXAMPLE 5 *Find the domain of* $f(x) = \sin^{-1}(-x^2)$. [NCERT EXEMPLAR]

SOLUTION The domain of $\sin^{-1} x$ is $[-1, 1]$. Therefore, $f(x) = \sin^{-1}(-x^2)$ is defined for all x satisfying

$$-1 \le -x^2 \le 1$$

$\Rightarrow \quad 1 \geq x^2 \geq -1 \Rightarrow 0 \leq x^2 \leq 1 \Rightarrow x^2 \leq 1 \Rightarrow x^2 - 1 \leq 0 \Rightarrow (x-1)(x+1) \leq 0 \Rightarrow -1 \leq x \leq 1$

Hence, the domain of $f(x) = \sin^{-1}(-x^2)$ is $[-1, 1]$.

EXAMPLE 6 *Find the domain of* $f(x) = \sin^{-1} x + \cos x$. **[NCERT EXEMPLAR]**

SOLUTION The domain of $\sin^{-1} x$ is $[-1, 1]$ and that of $\cos x$ is R. Therefore, domain of $f(x) = \sin^{-1} x + \cos x$ is $[-1, 1] \cap R = [-1, 1]$.

EXAMPLE 7 *Find the domain of the function* $f(x) = \sin^{-1}\sqrt{x-1}$. **[NCERT EXEMPLAR]**

SOLUTION The domain of $\sin^{-1} x$ is $[-1, 1]$. So, the domain of $f(x) = \sin^{-1}\sqrt{x-1}$ is the set of values of x satisfying

$$-1 \leq \sqrt{x-1} \leq 1 \Rightarrow 0 \leq \sqrt{x-1} \leq 1 \Rightarrow 0 \leq x-1 \leq 1 \Rightarrow 1 \leq x \leq 2 \Rightarrow x \in [1, 2] \quad [\because \sqrt{x-1} \geq 0]$$

Hence, the domain of $f(x) = \sin^{-1}\sqrt{x-1}$ is $[1, 2]$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *If* $x, y, z \in [-1, 1]$ *such that* $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = -\frac{3\pi}{2}$, *find the value of* $x^2 + y^2 + z^2$.

SOLUTION We know that the minimum value of $\sin^{-1} x$ for $x \in [-1, 1]$ is $-\frac{\pi}{2}$.

$\therefore \quad \sin^{-1} x \geq -\frac{\pi}{2}, \sin^{-1} y \geq -\frac{\pi}{2}$ and $\sin^{-1} z \geq -\frac{\pi}{2}$ for all $x, y, z \in [-1, 1]$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z \geq \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right) + \left(-\frac{\pi}{2}\right)$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z \geq -\frac{3\pi}{2}$

$\therefore \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = -\frac{3\pi}{2} \Rightarrow \sin^{-1} x = -\frac{\pi}{2}, \sin^{-1} y = -\frac{\pi}{2}, \sin^{-1} z = -\frac{\pi}{2} \Rightarrow x = y = z = -1$

Hence, $x^2 + y^2 + z^2 = (-1)^2 + (-1)^2 + (-1)^2 = 3$.

EXAMPLE 9 *Let* $x, y, z \in [-1, 1]$ *be such that* $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$. *Find the values of*

(i) $x^{2022} + y^{2023} + z^{2024}$ (ii) $x^{2020} + y^{2022} + z^{2024} - \frac{9}{x^{2020} + y^{2022} + z^{2024}}$

SOLUTION For any $x \in [-1, 1]$, the maximum value of $\sin^{-1} x$ is $\frac{\pi}{2}$ and it attains this value at $x = 1$.

$\therefore \quad \sin^{-1} x \leq \frac{\pi}{2}, \sin^{-1} y \leq \frac{\pi}{2}, \sin^{-1} z \leq \frac{\pi}{2}$ for all $x, y, z \in [-1, 1]$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z \leq \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2}$ for all $x, y, z \in [-1, 1]$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z \leq \frac{3\pi}{2}$ for all $x, y, z \in [-1, 1]$

$\therefore \quad \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2} \Rightarrow \sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}, \sin^{-1} z = \frac{\pi}{2} \Rightarrow x = 1, y = 1, z = 1$

(i) $x^{2022} + y^{2023} + z^{2024} = (1)^{2022} + (1)^{2023} + (1)^{2024} = 3$

(ii) $x^{2020} + y^{2022} + z^{2024} - \frac{9}{x^{2020} + y^{2022} + z^{2024}} = 1 + 1 + 1 - \frac{9}{1+1+1} = 3 - 3 = 0$

EXERCISE 3.1

BASIC

1. Find the principal value of each of the following:

(i) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (ii) $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$ (iii) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

(iv) $\sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ (v) $\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$ (vi) $\sin^{-1}\left(\tan\frac{5\pi}{4}\right)$

BASED ON LOTS

2. (i) $\sin^{-1}\frac{1}{2}-2\sin^{-1}\frac{1}{\sqrt{2}}$ (ii) $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$ [NCERT EXEMPLAR]

3. Find the domain of each of the following functions:

(i) $f(x)=\sin^{-1}x^2$ (ii) $f(x)=\sin^{-1}x+\sin x$

(iii) $f(x)=\sin^{-1}\sqrt{x^2-1}$ (iv) $f(x)=\sin^{-1}x+\sin^{-1}2x$

BASED ON HOTS

4. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z+\sin^{-1}t=2\pi$, then find the value of $x^2+y^2+z^2+t^2$.

5. If $(\sin^{-1}x)^2+(\sin^{-1}y)^2+(\sin^{-1}z)^2=\frac{3}{4}\pi^2$, find the value of $x^2+y^2+z^2$.

ANSWERS

1. (i) $-\frac{\pi}{3}$ (ii) $-\frac{\pi}{6}$ (iii) $\frac{\pi}{12}$ (iv) $\frac{5\pi}{12}$ (v) $-\frac{\pi}{4}$

(vi) $\frac{\pi}{2}$ 2. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{6}$ 3. (i) $[-1,1]$ (ii) $[-1,1]$

(iii) $[-\sqrt{2},-1]\cup[1,\sqrt{2}]$ (iv) $[-1/2,1/2]$ 4. 4 5. 3

3.3.2 INVERSE OF COSINE FUNCTION

The graph of cosine function is shown in Fig. 3.6. It is evident from the graph of $y=\cos x$ (see Fig. 3.6) that the function $f:R\to R$ given by $f(\theta)=\cos\theta$ is a many-one into function. However, $f:[0,\pi]\to[-1,1]$ is one-one onto i.e. a bijection and hence it is invertible. The inverse of cosine function is denoted by $\cos^{-1}$.

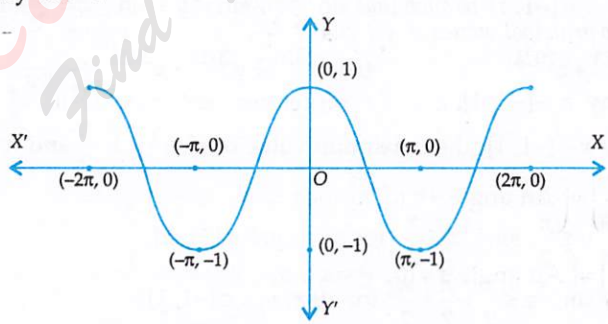


Fig. 3.6 Graph of $y=\cos x$ when $-2\pi\le x\le 2\pi$

Thus, if $\cos:[0,\pi]\to[-1,1]$ is such that $\cos\theta=x$. Then, $\cos^{-1}:[-1,1]\to[0,\pi]$ is defined as $\cos^{-1}x=\theta$. In other words: $\cos\theta=x \Leftrightarrow \cos^{-1}x=\theta$ for all $\theta\in[0,\pi]$ and $x\in[-1,1]$.

The graphs of $\cos:[0,\pi]\to[-1,1]$ and its inverse $\cos^{-1}[-1,1]\to[0,\pi]$ are shown in Figures 3.7 and 3.8 respectively. The branch of $\cos^{-1}:[-1,1]\to[0,\pi]$ is called the principal value branch and the value of $\cos^{-1}x$ lying in $[0,\pi]$ for a given value of $x\in[-1,1]$ is called the principal value.

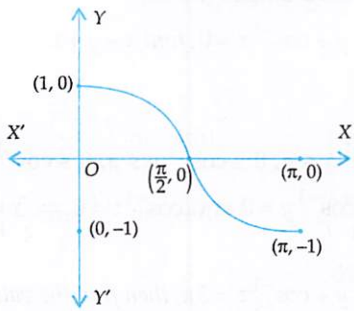


Fig. 3.7 Graph of $y = \cos x$, $0 \le x \le \pi$

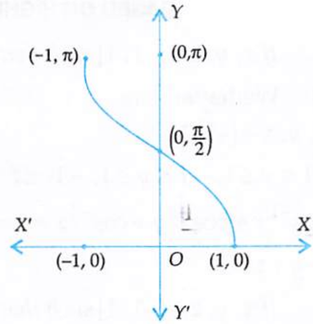


Fig. 3.8 Graph of $y = \cos^{-1} x$

SOME OBSERVATIONS It is evident from the graphs of $\cos x$ and $\cos^{-1} x$ that

(i) the domain and range of $\cos^{-1} x$ are $[-1, 1]$ and $[0, \pi]$ respectively.

(ii) both cos and $\cos^{-1}$ are decreasing functions in their respective domains.

$\therefore \quad \theta_1 < \theta_2 \Rightarrow \cos\theta_1 > \cos\theta_2$ for all $\theta_1, \theta_2 \in [0, \pi]$

and, $\quad x_1 < x_2 \Rightarrow \cos^{-1} x_1 > \cos^{-1} x_2$ for all $x_1, x_2 \in [-1, 1]$

(iii) The minimum and maximum values of $\cos^{-1} x$ are 0 and π respectively which are attained at 1 and –1 respectively i.e. $\cos^{-1}(1) = 0$ and $\cos^{-1}(-1) = \pi$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the domain of* $\cos^{-1}(2x-1)$.

SOLUTION The domain of $\cos^{-1} x$ is $[-1, 1]$. So, the domain of $\cos^{-1}(2x-1)$ is the set of all values of x satisfying: $-1 \le 2x - 1 \le 1 \Rightarrow 0 \le 2x \le 2 \Rightarrow 0 \le x \le 1$.

Hence, the domain of $\cos^{-1}(2x-1)$ is $[0, 1]$.

EXAMPLE 2 *Find the principal values of* (i) $\cos^{-1}\frac{\sqrt{3}}{2}$ (ii) $\cos^{-1}\left(-\frac{1}{2}\right)$. [NCERT]

SOLUTION For any $x \in [-1, 1]$, $\cos^{-1} x$ represents an angle in $[0, \pi]$ whose cosine is x. Therefore,

(i) $$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \left(\text{An angle } \theta \in [0, \pi] \text{ such that } \cos\theta = \frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

(ii) $$\cos^{-1}\left(-\frac{1}{2}\right) = \left(\text{An angle } \theta \in [0, \pi] \text{ such that } \cos\theta = -\frac{1}{2}\right) = \frac{2\pi}{3}$$

EXAMPLE 3 *Find the principal value of* $\cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}$.

SOLUTION We know that $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$.

$$\therefore \quad \cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\} = \cos^{-1}\left(\sin\frac{\pi}{3}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \qquad \left[\because \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}\right]$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *If $x, y, z \in [-1, 1]$ such that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 0$, find $x+y+z$.*

SOLUTION We have,

$$x, y, z \in [-1, 1]$$

$$\Rightarrow \quad -1 \le x \le 1, -1 \le y \le 1, -1, \le z \le 1 \Rightarrow 0 \le \cos^{-1} x \le \pi, 0 \le \cos^{-1} y \le \pi, 0 \le \cos^{-1} z \le \pi$$

$$\therefore \quad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 0 \Rightarrow \cos^{-1}x = 0, \cos^{-1}y = 0 \text{ and } \cos^{-1}z = 0 \Rightarrow x = y = z = 1.$$

Hence, $x+y+z=3$.

EXAMPLE 5 *If $x, y, z \in [-1, 1]$ such that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$, then find the values of*

(i) $xy+yz+zx$ (ii) $x(y+z)+y(z+x)+z(x+y)$ [NCERT EXEMPLAR]

SOLUTION We have,

$$x, y, z \in [-1, 1]$$

$$\Rightarrow \quad -1 \le x \le 1, -1 \le y \le 1, -1, \le z \le 1 \Rightarrow 0 \le \cos^{-1} x \le \pi, 0 \le \cos^{-1} y \le \pi, 0 \le \cos^{-1} z \le \pi$$

$$\therefore \quad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$$

$$\Rightarrow \quad \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi + \pi + \pi \Rightarrow \cos^{-1}x = \pi, \cos^{-1}y = \pi, \cos^{-1}z = \pi$$

$$\Rightarrow \quad x = -1, y = -1, z = -1.$$

Therefore,

(i) $xy+yz+zx = (-1)\times(-1) + (-1)\times(-1) + (-1)\times(-1) = 1+1+1 = 3.$

(ii) $x(y+z)+y(z+x)+z(x+y) = 2(xy+yz+zx) = 2\times 3 = 6$

EXERCISE 3.2

BASIC

1. Find the principal value of each of the following:

(i) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ [NCERT] (ii) $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ [NCERT]

(iii) $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$ (iv) $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$

2. For the principal values, evaluate each of the following :

(i) $\cos^{-1}\frac{1}{2} + 2\sin^{-1}\frac{1}{2}$ [NCERT, CBSE 2012] (ii) $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$ [CBSE 2012]

(iii) $\sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ (iv) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

BASED ON HOTS

3. Find the domain of definition of $f(x) = \cos^{-1}(x^2-4)$.
4. Find the domain of $f(x) = 2\cos^{-1}2x + \sin^{-1}x$.
5. Find the domain of $f(x) = \cos^{-1}x + \cos x$.

ANSWERS

1. (i) $\frac{5\pi}{6}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{5\pi}{6}$ (iv) π
2. (i) $\frac{2\pi}{3}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{2}$ (iv) $-\frac{\pi}{6}$
3. $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$
4. $[-1/2, 1/2]$
5. $[-1, 1]$

3.3.3 INVERSE OF TANGENT FUNCTION

Consider the function $f : R - \left\{(2n+1)\,\dfrac{\pi}{2} : n \in Z\right\} \to R$ given by $f(x) = \tan x$. The graph of this function is shown in Fig. 3.9. It is evident from the graph that $f(x) = \tan x$ is a many-one onto function and hence it is not invertible. However, the function $\tan : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R$ associating each $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ to $\tan x \in R$ is bijection and so it is invertible. The inverse of this function is denoted by $\tan^{-1}$.

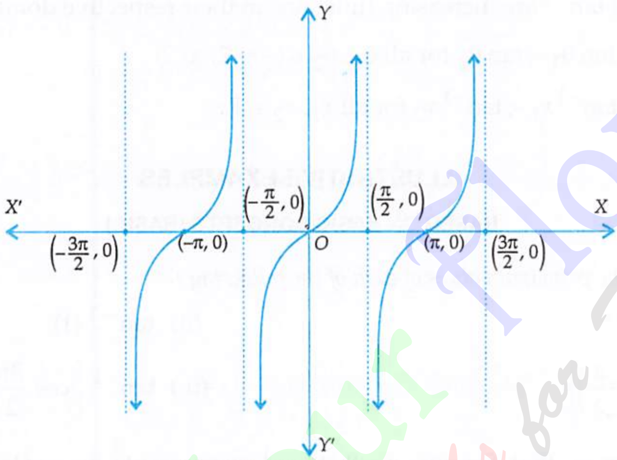


Fig. 3.9 Graph of $y = \tan x$

Clearly, $\tan^{-1} : R \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is such that $\tan^{-1} x = \theta \Leftrightarrow \tan\theta = x$.

Also, $\quad \tan^{-1}(\tan\theta) = \theta$ for all $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and, $\tan(\tan^{-1} x) = x$ for all $x \in R$.

The graphs of the functions $\tan : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to R$ and $\tan^{-1} : R \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ are shown in Figures 3.10 and 3.11 respectively. The branch of $\tan^{-1} : R \to (-\pi/2, \pi/2)$ is called the principal value branch and the value of $\tan^{-1} x$ lying in $(-\pi/2, \pi/2)$ for a given value of $x \in R$ is called the principal value.

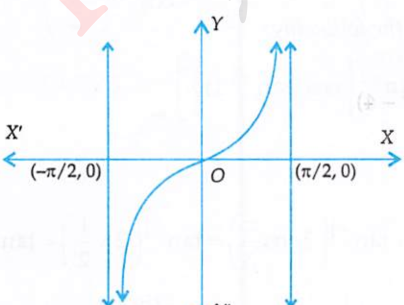


Fig. 3.10 Graph of $y = \tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

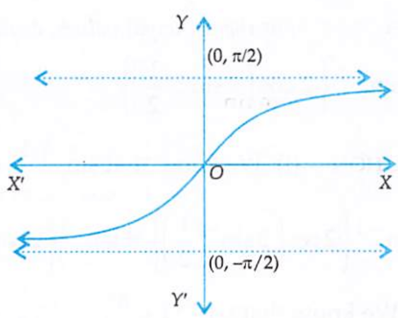


Fig. 3.11 Graph of $y = \tan^{-1} x$

SOME USEFUL OBSERVATIONS *It is evident from the graphs of* $\tan:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ *and* $\tan^{-1}:R\to\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *i.e. the curves* $y=\tan x$ *and* $y=\tan^{-1}x$ *that*

(i) $-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$ for all $x\in R$ i.e. $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are minimum and maximum values of $\tan^{-1}x$ but it not does not attain these values.

(ii) both tan and $\tan^{-1}$ are increasing functions in their respective domains.

$\therefore\quad \theta_1<\theta_2\Rightarrow\tan\theta_1<\tan\theta_2$ for all $\theta_1,\theta_2\in(-\pi/2,\pi/2)$

and, $x_1<x_2\Rightarrow\tan^{-1}x_1<\tan^{-1}x_2$ for all $x_1,x_2\in R$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the principal values of each of the following :*

(i) $\tan^{-1}(-\sqrt{3})$ (ii) $\tan^{-1}(1)$ [NCERT]

(iii) $\tan^{-1}\left\{\sin\left(-\frac{\pi}{2}\right)\right\}$ [NCERT EXEMPLAR] (iv) $\tan^{-1}\left\{\cos\frac{3\pi}{2}\right\}$

SOLUTION We know that for any $x\in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x. Therefore,

(i) $\tan^{-1}(-\sqrt{3})=\left(\text{An angle }\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\text{ such that }\tan\theta=-\sqrt{3}\right)=-\frac{\pi}{3}$

(ii) $\tan^{-1}(1)=\left(\text{An angle }\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\text{ such that }\tan\theta=1\right)=\frac{\pi}{4}$

(iii) We know that $\sin\left(-\frac{\pi}{2}\right)=-1$. Therefore, $\tan^{-1}\left\{\sin\left(-\frac{\pi}{2}\right)\right\}=\tan^{-1}(-1)=-\frac{\pi}{4}$

(iv) $\tan^{-1}\left\{\cos\frac{3\pi}{2}\right\}=\tan^{-1}(0)=0$

EXAMPLE 2 *For the principal values, evaluate each of the following:*

(i) $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$ [NCERT] (ii) $\cot\left[\sin^{-1}\left\{\cos(\tan^{-1}1)\right\}\right]$ [NCERT EXEMPLAR]

SOLUTION (i) We know that $\sin^{-1}\frac{1}{2}=\frac{\pi}{6}$.

$\therefore \tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}=\tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\}=\tan^{-1}\left(2\cos\frac{\pi}{3}\right)=\tan^{-1}\left(2\times\frac{1}{2}\right)=\tan^{-1}1=\frac{\pi}{4}$

(ii) We know that $\tan^{-1}1=\frac{\pi}{4}$.

$\therefore\quad \cot\left[\sin^{-1}\left\{\cos(\tan^{-1}1)\right\}\right]=\cot\left\{\sin^{-1}\left(\cos\frac{\pi}{4}\right)\right\}=\cot\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=\cot\frac{\pi}{4}=1$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Which is greater, $\tan 1$ or $\tan^{-1} 1$?* **[NCERT EXEMPLAR]**

SOLUTION We know that $\tan^{-1} 1 = \frac{\pi}{4}$ and $1 > \frac{\pi}{4}$. Since $\tan\theta$ is an increasing function

$$\therefore \quad 1 > \frac{\pi}{4} \Rightarrow \tan 1 > \tan\frac{\pi}{4} \Rightarrow \tan 1 > 1 \Rightarrow \tan 1 > 1 > \frac{\pi}{4} \qquad \left[\because 1 > \frac{\pi}{4}\right]$$

$$\Rightarrow \quad \tan 1 > \frac{\pi}{4} \Rightarrow \tan 1 > \tan^{-1} 1 \qquad \left[\because \tan^{-1} 1 = \frac{\pi}{4}\right]$$

EXAMPLE 4 *Find the minimum value of n for which $\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$, $n \in N$.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\tan^{-1}\frac{n}{\pi} > \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\frac{n}{\pi} > \tan^{-1} 1 \qquad \left[\because \frac{\pi}{4} = \tan^{-1} 1\right]$$

$$\Rightarrow \quad \tan\left(\tan^{-1}\frac{n}{\pi}\right) > \tan\left(\tan^{-1} 1\right) \qquad [\because \tan\theta \text{ is an increasing function}]$$

$$\Rightarrow \quad \frac{n}{\pi} > 1 \qquad [\because \tan(\tan^{-1} x) = x]$$

$$\Rightarrow \quad n > \pi \cong 3.14 \Rightarrow n = 4, 5, 6, \dots$$

Hence, the minimum value of n is 4.

EXERCISE 3.3

BASIC

1. Find the principal value of each of the following :
   (i) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iii) $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$ (iv) $\tan^{-1}\left(2\cos\frac{2\pi}{3}\right)$
2. For the principal values, evaluate each of the following:
   (i) $\tan^{-1}(-1) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (ii) $\tan^{-1}\left\{2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$
3. Evaluate each of the following :
   (i) $\tan^{-1} 1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ **[NCERT]**
   (ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \tan^{-1}(-\sqrt{3}) + \tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$
   (iii) $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$ **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{6}$ (iii) 0 (iv) $-\frac{\pi}{4}$ 2. (i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{3}$ 3. (i) $\frac{3\pi}{4}$ (ii) $-\frac{3\pi}{4}$ (iii) 0

3.3.4 INVERSE OF SECANT FUNCTION

In Class XI, we have learnt that $\sec\theta$ is not defined at odd multiples of $\pi/2$. Therefore, a rule associating $x \in R - \left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\}$ to $\sec x$ is a function whose graph is shown in Fig. 3.12.

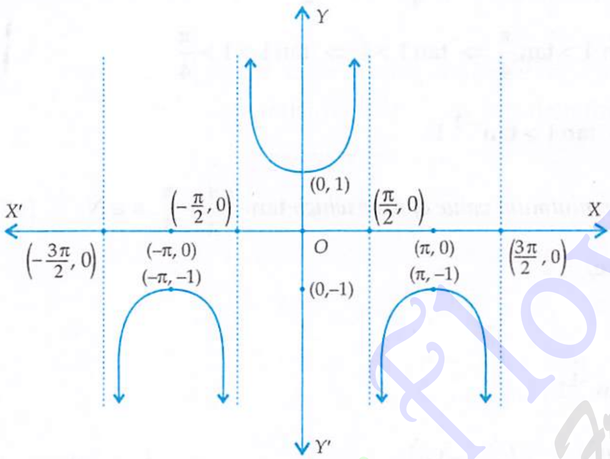


Fig. 3.12 Graph of $y = \sec x$

We observe that the function $\sec : R - \left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\} \to R$ is neither one-one nor onto but, $\sec : R - \left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\} \to (-\infty, -1] \cup [1, \infty)$ is many-one onto. If we restrict the domain to $\left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$, then the function associating each $x \in \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right]$ to $\sec x \in (-\infty, -1] \cup [1, \infty)$ is a bijection as is evident from the graph of $y = \sec x$ shown in Fig. 3.13. the inverse of $\sec : \left[0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{\pi}{2}, \pi\right] \to (-\infty, -1] \cup [1, \infty)$ is denoted by $\sec^{-1}$ such that $\sec^{-1}\theta = x \Leftrightarrow x = \sec\theta$.

Also, $\sec^{-1}(\sec\theta) = \theta$ for all $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$ and, $\sec(\sec^{-1} x) = x$ for all $x \in (-\infty, -1] \cup [1, \infty)$ The graphs of functions $\sec : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$ and $\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$ are shown in Figures 3.13 and 3.14 respectively. The branch of $\sec^{-1}$ shown in Fig. 3.14 is called the principal value branch and the value of $\sec^{-1} x$ in $[0, \pi/2) \cup (\pi/2, \pi]$ for given value of $x \in (-\infty, -1] \cup [1, \infty)$ is called the principal value.

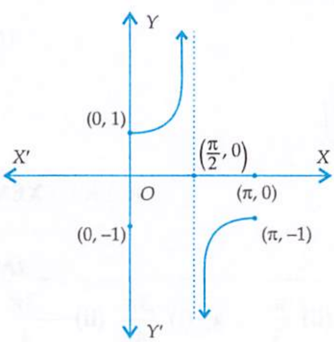


Fig. 3.13 Graph of $y = \sec x, 0 \le x \le \pi, x \ne \pi/2$

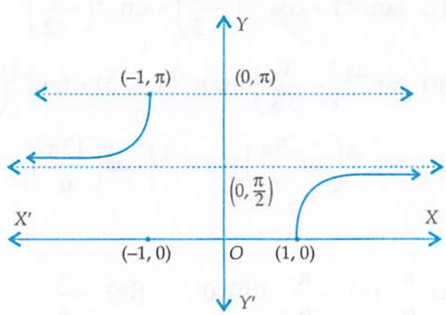


Fig. 3.14 Graph of $y = \sec^{-1} x$

SOME USEFUL OBSERVATION We make the following observations from the graphs of $\sec x$ and $\sec^{-1} x$:

(i) $\sec x$ is an increasing function on the intervals $[0, \pi/2)$ and $(\pi/2, \pi]$ but, it is neither increasing nor decreasing on $[0, \pi/2) \cup (\pi/2, \pi]$.

(ii) $\sec^{-1} x$ is an increasing function the intervals $(-\infty, -1]$ and $[1, \infty)$ but, it is neither increasing nor decreasing on $(-\infty, -1] \cup [1, \infty)$.

(iii) The maximum value of $\sec^{-1} x$ is π which it attains at $x = -1$.

(iv) The minimum value of $\sec^{-1} x$ is 0 which it attains at $x = 1$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the principal values of each of the following:*

(i) $\sec^{-1}(2)$ (ii) $\sec^{-1}\left(\frac{-2}{\sqrt{3}}\right)$.

SOLUTION For any $x \in (-\infty, -1] \cup [1, \infty)$, $\sec^{-1} x$ is an angle $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$ whose secant is x i.e. $\sec\theta = x$. Therefore,

(i) $$\sec^{-1}(2) = \left(\text{An angle } \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, 1\right] \text{ such that } \sec\theta = 2\right) = \frac{\pi}{3}$$

(ii) $$\sec^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \left(\text{An angle } \theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \text{ such that } \sec\theta = -\frac{2}{\sqrt{3}}\right) = \frac{5\pi}{6}$$

EXAMPLE 2 *Find the set of values of* $\sec^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

SOLUTION We know that $\sec^{-1} x$ is defined for all $x \le -1$ or $x \ge 1$ and $\frac{\sqrt{3}}{2} < 1$. Therefore, $\sec^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is not meaningful. Hence, the set of values of $\sec^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is the null set ϕ.

EXAMPLE 3 *Find the principal values of* $\sec^{-1} \frac{2}{\sqrt{3}}$ *and* $\sec^{-1}(-2)$.

SOLUTION Since $\sec^{-1} : R - (-1, 1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$ is a bijection. Therefore, $\sec^{-1} x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is x. Thus,

(i) $$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \left(\text{An angle } \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ such that } \sec\theta = \frac{2}{\sqrt{3}}\right) = \frac{\pi}{6}$$

(ii) $$\sec^{-1}(-2) = \left(\text{An angle } \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\} \text{ such that } \sec\theta = -2\right) = \frac{2\pi}{3}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find the domain of* $\sec^{-1}(2x+1)$.

SOLUTION The domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$. Therefore, $\sec^{-1}(2x+1)$ is meaningful, if

$$2x+1 \ge 1 \text{ or, } 2x+1 \le -1$$

$$\Rightarrow \quad 2x \ge 0 \text{ or, } 2x \le -2 \Rightarrow x \ge 0 \text{ or, } x \le -1 \Rightarrow x \in (-\infty, -1] \cup [0, \infty)$$

Hence, the domain of $\sec^{-1}(2x+1)$ is $(-\infty, -1] \cup [0, \infty)$.

EXERCISE 3.4

BASIC

1. Find the principal values of each of the following:

   (i) $\sec^{-1}(-\sqrt{2})$ (ii) $\sec^{-1}(2)$ (iii) $\sec^{-1}\left(2\sin\frac{3\pi}{4}\right)$ (iv) $\sec^{-1}\left(2\tan\frac{3\pi}{4}\right)$

2. For the principal values, evaluate the following:

   (i) $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$ [CBSE 2012] (ii) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\tan\frac{\pi}{6}\right)$

BASED ON HOTS

3. Find the domain of (i) $\sec^{-1}(3x-1)$ (ii) $\sec^{-1}x-\tan^{-1}x$

ANSWERS

1. (i) $\frac{3\pi}{4}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{2\pi}{3}$ 2. (i) $-\frac{\pi}{3}$ (ii) $-\frac{2\pi}{3}$
3. (i) $(-\infty, 0]\cup[2/3, \infty)$ (ii) $(-\infty, -1]\cup[1, \infty)$

3.3.5 INVERSE OF COSECANT FUNCTION

In Class XI, we have learnt that the function $f(x)=\text{cosec}\, x$ has domain $R-\{n\pi : n\in Z\}$ and range $R-(-1, 1)$. The graph of this function is shown in Fig. 3.15. It is evident from the graph that $f: R-\{n\pi : n\in Z\}\to R$ defined as $f(x)=\text{cosec}\, x$ is a many-one into function and $f: R-\{n\pi : n\in Z\}\to R-(-1, 1)$ is many-one onto.

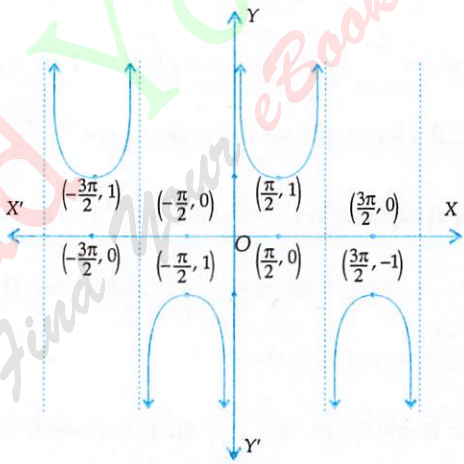


Fig. 3.15 Graph of $y=\text{cosec}\, x$

If we consider $f:[-\pi/2, 0)\cup(0, \pi/2]\to(-\infty, -1]\cup[1, \infty)$, then its is a bijection and hence invertible. The inverse of cosec is denoted by cosec^{-1} and is defined as

$$\text{cosec}^{-1}x=\theta \Leftrightarrow \text{cosec}\,\theta=x \text{ for all } \theta\in[-\pi/2, 0)\cup(0, \pi/2] \text{ and } x\in(-\infty, -1]\cup[1, \infty)$$

Also, $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$ for all $\theta\in\left[-\frac{\pi}{2}, 0\right)\cup\left(0, \frac{\pi}{2}\right]$ and, $\text{cosec}\,(\text{cosec}^{-1}x)=x$ for all $x\in(-\infty, -1]\cup[1, \infty)$. The graphs of $\text{cosec}:[-\pi/2, 0)\cup(0, \pi/2]\to(-\infty, -1]\cup[1, \infty)$ and $\text{cosec}^{-1}:(-\infty -1]\cup[1,\infty)\to[-\pi/2, 0)\cup(0, \pi/2]$ are shown in Fig. 3.16 and 3.17 respectively.

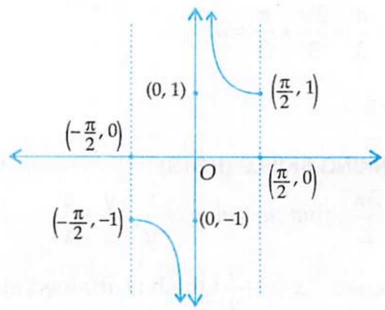


Fig. 3.16 Graph of $y = \operatorname{cosec} x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}, x \ne 0$

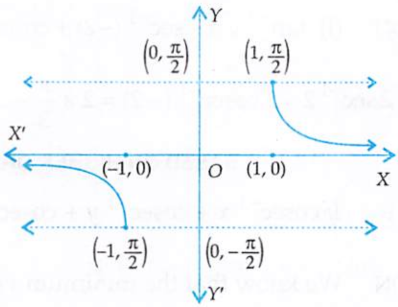


Fig. 3.17 Graph of $y = \operatorname{cosec}^{-1} x$

The branch of $\operatorname{cosec}^{-1} x$ shown in Fig. 3.17 is called the principal value branch and the value of $\operatorname{cosec}^{-1} x$ lying in $[-\pi/2, 0) \cup (0, \pi/2]$ are the principal values.

SOME OBSERVATIONS It is evident from the graphs of cosec x and $\operatorname{cosec}^{-1} x$ that

(i) $\operatorname{cosec} \theta$ is a decreasing function on $[-\pi/2, 0)$ and $(0, \pi/2]$. But, it is neither decreasing nor increasing on $[-\pi/2, 0) \cup (0, \pi/2]$.

(ii) $\operatorname{cosec}^{-1} x$ is decreasing on $(-\infty, -1]$ and $[1, \infty)$. But, it is neither increasing nor decreasing on $(-\infty, -1] \cup [1, \infty)$.

(iii) The maximum value of $\operatorname{cosec}^{-1} x$ is $\pi/2$ which it attains at $x = 1$.

(iv) The minimum value is $\operatorname{cosec}^{-1} x$ is $-\pi/2$ which it attains at $x = -1$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the principal values of* $\operatorname{cosec}^{-1}(2)$ *and* $\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$.

SOLUTION For $x \in (-\infty, -1] \cup [1, \infty)$, $\operatorname{cosec}^{-1} x$ is an angle $\theta \in [-\pi/2, 0) \cup (0, \pi/2]$ such that $\operatorname{cosec} \theta = x$.

$$\therefore \quad \operatorname{cosec}^{-1}(2) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \text{ such that } \operatorname{cosec} \theta = 2\right) = \frac{\pi}{6}$$

$$\text{and,} \quad \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) = \left(\text{An angle } \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \text{ such that } \operatorname{cosec} \theta = -\frac{2}{\sqrt{3}}\right) = -\frac{\pi}{3}$$

EXAMPLE 2 *Find the set of values of* $\operatorname{cosec}^{-1}\left(-\frac{1}{2}\right)$.

SOLUTION We know that $\operatorname{cosec}^{-1} x$ is defined for all $x \le -1$ or $x \ge 1$. So, $\operatorname{cosec}^{-1}\left(-\frac{1}{2}\right)$ is not meaningful. Hence, the set of values of $\operatorname{cosec}^{-1}\left(-\frac{1}{2}\right)$ is the null set ϕ.

EXAMPLE 3 *For the principal values, evaluate each of the following:*

(i) $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) + \operatorname{cosec}^{-1}\frac{2}{\sqrt{3}}$ (ii) $2\sec^{-1}(2) - 2\operatorname{cosec}^{-1}(-2)$

SOLUTION (i) $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)+\operatorname{cosec}^{-1}\dfrac{2}{\sqrt{3}}=\dfrac{\pi}{3}-\dfrac{2\pi}{3}+\dfrac{\pi}{3}=0$

(ii) $2\sec^{-1}2-2\operatorname{cosec}^{-1}(-2)=2\times\dfrac{\pi}{3}-2\times-\dfrac{\pi}{6}=\pi$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *If* $\operatorname{cosec}^{-1}x+\operatorname{cosec}^{-1}y+\operatorname{cosec}^{-1}z=-\dfrac{3\pi}{2}$, *find the value of* $\dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}$.

SOLUTION We know that the minimum value of $\operatorname{cosec}^{-1}x$ is $-\dfrac{\pi}{2}$ which is attained at $x=-1$.

$$\therefore \quad \operatorname{cosec}^{-1}x+\operatorname{cosec}^{-1}y+\operatorname{cosec}^{-1}z=-\frac{3\pi}{2}$$

$$\Rightarrow \quad \cos^{-1}x+\operatorname{cosec}^{-1}+\operatorname{cosec}^{-1}z=\left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}\right)+\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \operatorname{cosec}^{-1}x=-\frac{\pi}{2},\ \operatorname{cosec}^{-1}y=-\frac{\pi}{2},\ \operatorname{cosec}^{-1}z=-\frac{\pi}{2}\Rightarrow x=-1,\ y=-1,\ z=-1$$

$$\therefore \quad \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{(-1)}{(-1)}+\frac{(-1)}{(-1)}+\frac{(-1)}{(-1)}=3$$

EXERCISE 3.5

BASIC

1. Find the principal values of each of the following:
   (i) $\operatorname{cosec}^{-1}(-\sqrt{2})$ [NCERT] (ii) $\operatorname{cosec}^{-1}(-2)$
   (iii) $\operatorname{cosec}^{-1}\left(\dfrac{2}{\sqrt{3}}\right)$ (iv) $\operatorname{cosec}^{-1}\left(2\cos\dfrac{2\pi}{3}\right)$
2. Find the set of values of $\operatorname{cosec}^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$.
3. For the principal values, evaluate the following:
   (i) $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)+\operatorname{cosec}^{-1}\left(-\dfrac{2}{\sqrt{3}}\right)$ (ii) $\sec^{-1}(\sqrt{2})+2\operatorname{cosec}^{-1}(-\sqrt{2})$
   (iii) $\sin^{-1}\left[\cos\left\{2\operatorname{cosec}^{-1}(-2)\right\}\right]$ (iv) $\operatorname{cosec}^{-1}\left(2\tan\dfrac{11\pi}{6}\right)$

ANSWERS

1. (i) $-\dfrac{\pi}{4}$ (ii) $-\dfrac{\pi}{6}$ (iii) $\dfrac{\pi}{3}$ (iv) $-\dfrac{\pi}{2}$ 2. ϕ 3. (i) $-\dfrac{2\pi}{3}$ (ii) $-\dfrac{\pi}{4}$ (iii) $\dfrac{\pi}{6}$ (iv) $-\dfrac{\pi}{3}$

3.3.6 INVERSE OF COTANGENT FUNCTION

We have learnt in earlier classes that the function $f(x)=\cot x$ has domain $=R-\{n\pi : n\in Z\}$ and range R. Therefore, $f:R-\{n\pi : n\in Z\}\to R$ is a many-one onto function as is evident from the curve $y=\cot x$ shown in Fig. 3.18.

If we consider $\cot:(0,\pi)\to R$, then it is a bijection and hence invertible. The inverse of this function is denoted by $\cot^{-1}$ and is defined as $\cot^{-1}x=\theta\Leftrightarrow\cot\theta=x$ for all $\theta\in(0,\pi)$ and all $x\in R$. Also, $\cot^{-1}(\cot\theta)=\theta$ for all $\theta\in(0,\pi)$ and, $\cot(\cot^{-1}x)=x$ for all $x\in R$.

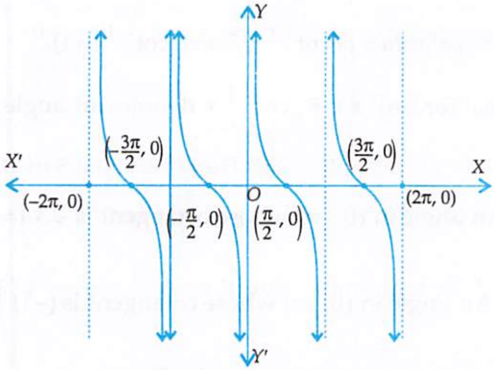


Fig. 3.18 Graph of $y = \cot x$

Graphs of $y = \cot x$ and $y = \cot^{-1} x$ are shown in Figures 3.19 and 3.20 respectively.

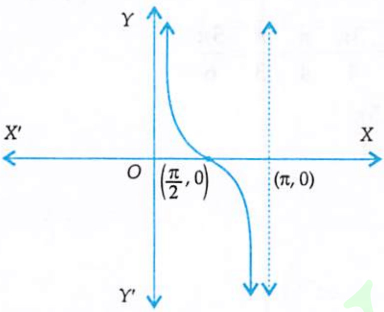


Fig. 3.19 Graph of $y = \cot x, 0 < x < \pi$

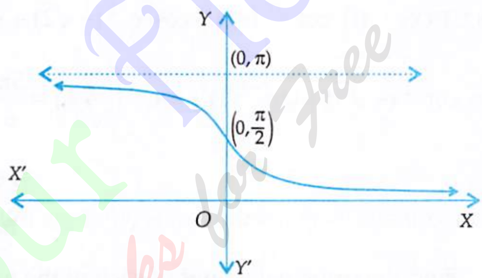


Fig. 3.20 Graph of $y = \cot^{-1} x$

The branch of $\cot^{-1} : R \to (0, \pi)$ is called the principal value branch and the value of $\cot^{-1} x$ for given x is called the principal value.

SOME USEFUL OBSERVATION It is evident from the graphs of $\cot x$ and $\cot^{-1} x$ that

(i) $\cot x$ is a decreasing function on $(0, \pi)$. i.e. $\theta_1 < \theta_2 \Rightarrow \cot \theta_1 > \cot \theta_2$ for all $\theta_1, \theta_2 \in (0, \pi)$

(ii) $\cot^{-1} x$ is a decreasing function on R. i.e. $x_1 < x_2 \Rightarrow \cot^{-1} x_1 > \cot^{-1} x_2$ for all $x_1, x_2 \in R$.

(iii) For all $x \in R$, the values of $\cot^{-1} x$ lie between 0 and π.

(iv) $\cot^{-1} x$ does not attain its minimum value zero and maximum value π at points in R.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the set of values of* $\cot^{-1}(1)$ *and* $\cot^{-1}(-1)$

SOLUTION For any $x \in R$, $\cot^{-1} x$ is an angle $\theta \in (0, \pi)$ such that $\cot \theta = x$.

$\therefore \quad \cot^{-1}(1) = \left(\text{An angle } \theta \in (0, \pi) \text{ such that } \cot\ \theta = 1\right) = \frac{\pi}{4}$

and, $\quad \cot^{-1}(-1) = \left(\text{An angle } \theta \in (0, \pi) \text{ whose cot angent is equal to } -1\right) = \frac{3\pi}{4}$

Hence, required set is $\left\{\frac{\pi}{4}, \frac{3\pi}{3}\right\}$.

EXAMPLE 2 *Find the principal values of* $\cot^{-1}\sqrt{3}$ *and* $\cot^{-1}(-1)$.

SOLUTION We know that for any $x \in R$, $\cot^{-1} x$ denotes an angle in $(0, \pi)$ whose cotangent is x.

$$\therefore \quad \cot^{-1}\sqrt{3} = \left(\text{An angle in } (0,\ \pi) \text{ whose cotangent is } \sqrt{3}\right) = \frac{\pi}{6}$$

$$\text{and,} \quad \cot^{-1}(-1) = \left(\text{An angle in } (0,\ \pi) \text{ whose cotangent is } (-1)\right) = \frac{3\pi}{4}.$$

EXAMPLE 3 *For the principal values, evaluate the following:*

(i) $\cot^{-1}(-1) + \operatorname{cosec}^{-1}(-\sqrt{2}) + \sec^{-1}(2)$ (ii) $\cot^{-1}(-\sqrt{3}) + \tan^{-1}(1) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

SOLUTION (i) $\cot^{-1}(-1) + \operatorname{cosec}^{-1}(-\sqrt{2}) + \operatorname{secs}^{-1}(2) = \frac{3\pi}{4} - \frac{\pi}{4} + \frac{\pi}{3} = \frac{5\pi}{6}$

(ii) $\cot^{-1}(-\sqrt{3}) + \tan^{-1}(1) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{5\pi}{6} + \frac{\pi}{4} + \frac{\pi}{6} = \frac{5\pi}{4}$

EXERCISE 3.6

BASIC

1. Find the principal values of each of the following:

   (i) $\cot^{-1}(-\sqrt{3})$ (ii) $\cot^{-1}(\sqrt{3})$ (iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ (iv) $\cot^{-1}\left(\tan\frac{3\pi}{4}\right)$

2. Find the domain of $f(x) = \cot x + \cot^{-1} x$.

3. Evaluate each of the following:

   (i) $\cot^{-1}\frac{1}{\sqrt{3}} - \operatorname{cosec}^{-1}(-2) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (ii) $\cot^{-1}\left\{2\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$

   (iii) $\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right) + 2\cot^{-1}(-1)$

   (iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$ [NCERT EXEMPLAR]

ANSWERS

1. (i) $\frac{5\pi}{6}$ (ii) $\frac{\pi}{6}$ (iii) $\frac{2\pi}{3}$ (iv) $\frac{3\pi}{4}$

2. $R - \{n\pi : n \in Z\}$

3. (i) $\frac{2\pi}{3}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{7\pi}{6}$ (iv) $-\frac{\pi}{12}$

As a ready reference domains, ranges and principal value branches of all inverse trigonometric functions are tabulated below.

Function	*Domain*	*Range*	*Principal value branch*
$\sin^{-1}$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$y=\sin^{-1}x$ from $\left(-1, -\frac{\pi}{2}\right)$ to $\left(1, \frac{\pi}{2}\right)$
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$y=\cos^{-1}x$ from $(-1, \pi)$ to $(1, 0)$
$\tan^{-1}$	R	$(-\pi/2, \pi/2)$	$y=\tan^{-1}x$ from $\left(-\infty, -\frac{\pi}{2}\right)$ to $\left(\infty, \frac{\pi}{2}\right)$
cosec^{-1}	$(-\infty, -1]\cup[1, \infty)$	$[-\pi/2, \pi/2]-\{0\}$	$y=\text{cosec}^{-1}x$, from $(-\infty, 0)$ to $\left(-1, -\frac{\pi}{2}\right)$ and, from $\left(1, \frac{\pi}{2}\right)$ to $(\infty, 0)$
$\sec^{-1}$	$(-\infty, -1]\cup[1, \infty)$	$[0, \pi]-\left\{\frac{\pi}{2}\right\}$	$y=\sec^{-1}x$, from $\left(-\infty, \frac{\pi}{2}\right)$ to $(-1, \pi)$, and, from $(1, 0)$ to $\left(\infty, \frac{\pi}{2}\right)$
$\cot^{-1}$	R	$(0, \pi)$	$y=\cot^{-1}x$ from $(-\infty, \pi)$ to $(\infty, 0)$

NOTE 1 *If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of that function.*

3.4 PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

In this section, we will learn about various properties of six inverse trigonometric functions defined in the previous section. These properties are very useful in simplifying expressions and solving equations involving inverse trigonometric functions.

3.4.1 PROPERTY-I

In chpater 2, we have learnt that if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ exists such that $f^{-1}of(x)=x$ or, $f^{-1}(f(x))=x$ for all $x\in A$. In the previous section, we have learnt that $\sin:[-\pi/2, \pi/2]\to[-1, 1]$, $\cos:[0, \pi]\to[-1, 1]$, $\tan:(-\pi/2, \pi/2)\to R$, $\cot\,(0, \pi)\to R$ $\sec:[0, \pi/2)\cup(\pi/2, \pi]\to(-\infty, -1]\cup[1, \infty)$ and $\text{cosec}:[-\pi/2, 0)\cup(0, \pi/2]\to(-\infty, -1]\cup[1, \infty)$ are bijections. So, these functions and their inverses satisfy the following property.

PROPERTY-I *(i)* $\sin^{-1}(\sin\theta)=\theta$ *for all* $\theta\in[-\pi/2, \pi/2]$

(ii) $\cos^{-1}(\cos\theta)=\theta$ *for all* $\theta\in[0, \pi]$

(iii) $\tan^{-1}(\tan\theta)=\theta$ *for all* $\theta\in(-\pi/2, \pi/2)$

(iv) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta$ *for all* $\theta\in[-\pi/2, 0)\cup(0, \pi/2]$

(v) $\sec^{-1}(\sec\theta)=\theta$ *for all* $\theta\in[0, \pi/2)\cup(\pi/2, \pi]$

(vi) $\cot^{-1}(\cot\theta)=\theta$ *for all* $\theta\in(0, \pi)$.

In the above property we observe that the relations between trigonometric functions and their inverses hold true for specific values of θ. If θ does not lie in the domain of a trigonometric function in which it is not a bijection, then the above relations do not hold good. For example, $\sin^{-1}(\sin\theta)=\theta$ holds true for $\theta\in[-\pi/2, \pi/2]$. If $\theta\notin[-\pi/2, \pi/2]$, what is the value of $\sin^{-1}(\sin\theta)$? To answer this, we partition real line into sub-intervals so that the sine function with domain any sub interval and co-domain $[-1, 1]$ is a bijection. Clearly, such sub-intervals are

$$\ldots\ldots\left[-\frac{5\pi}{2},-\frac{3\pi}{2}\right],\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right],\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right],\left[\frac{3\pi}{2},\frac{5\pi}{2}\right],\ldots\ldots$$

If $\theta \in \left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ i.e. $\frac{\pi}{2} \le \theta\, \frac{3\pi}{2}$, then $-\frac{3\pi}{2} \le -\theta \le -\frac{\pi}{2} \Rightarrow \pi-\frac{3\pi}{2} \le \pi-\theta \le \pi-\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi-\theta \le \frac{\pi}{2}$

$\therefore \quad \sin^{-1}(\sin\theta) = \sin^{-1}\left(\sin\,(\pi-\theta)\right) = \pi-\theta$

If $\theta \in \left[\frac{3\pi}{2},\frac{5\pi}{2}\right]$ i.e. $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$, then

$$-\frac{5\pi}{2} \le -\theta \le -\frac{3\pi}{2} \Rightarrow 2\pi-\frac{5\pi}{2} \le 2\pi-\theta \le 2\pi-\frac{3\pi}{2} \Rightarrow -\frac{\pi}{2} \le 2\pi-\theta \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \theta-2\pi \le \frac{\pi}{2}$$

$\therefore \quad \sin^{-1}(\sin\theta) = \sin^{-1}\left(-\sin\,(2\pi-\theta)\right) = \sin^{-1}\left(\sin\,(\theta-2\pi)\right) = \theta-2\pi$

If $\theta \in \left[-\frac{3\pi}{2},-\frac{\pi}{2}\right]$ i.e. $-\frac{3\pi}{2} \le \theta \le -\frac{\pi}{2}$, then

$$-\frac{3\pi}{2}+\pi < \pi+\theta < \pi-\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi+\theta \le \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le -\pi-\theta \le \frac{\pi}{2}$$

$\therefore \quad \sin^{-1}(\sin\theta) = \sin^{-1}\left(-\sin\,(\pi+\theta)\right) = \sin^{-1}\left(\sin\,(-\pi-\theta)\right) = -\pi-\theta$

If $\theta \in \left[-\frac{5\pi}{2},-\frac{3\pi}{2}\right]$ i.e. $-\frac{5\pi}{2} \le \theta \le -\frac{3\pi}{2}$, then $-\frac{5\pi}{2}+2\pi \le 2\pi+\theta \le -\frac{3\pi}{2}+2\pi \Rightarrow -\frac{\pi}{2} \le 2\pi+\theta \le \frac{\pi}{2}$.

$\therefore \quad \sin^{-1}(\sin\theta) = \sin^{-1}\left(\sin\,(2\pi+\theta)\right) = 2\pi+\theta$

Thus, we obtain

$$\sin^{-1}(\sin\theta) = \begin{cases} 2\pi+\theta, & \text{if } -5\pi/2 \le \theta \le -3\pi/2 \\ -\pi-\theta, & \text{if } -3\pi/2 \le \theta \le -\pi/2 \\ \theta, & \text{if } -\pi/2 \le \theta \le \pi/2 \\ \pi-\theta, & \text{if } \pi/2 \le \theta \le 3\pi/2 \\ \theta-2\pi, & \text{if } 3\pi/2 \le \theta \le 5\pi/2 \\ 3\pi-\theta, & \text{if } 5\pi/2 \le \theta \le 7\pi/2 \end{cases} \quad \text{and so on.}$$

The graph of $y = \sin^{-1}(\sin x)$ is shown in Fig. 3.21. It is evident from the graph that the function $\sin^{-1}(\sin x)$ is a periodic function with period 2π.

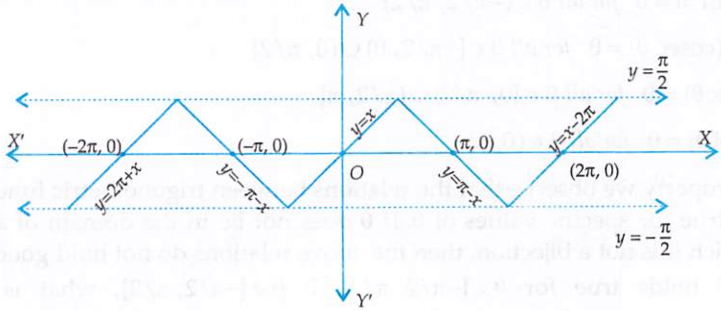


Fig. 3.21 Graph of $f(x) = \sin^{-1}(\sin x)$

Similarly, we find that

$$\cos^{-1}(\cos\theta)=\begin{cases}-2\pi-\theta, & \text{if } -3\pi\le\theta\le-2\pi\\ 2\pi+\theta, & \text{if } -2\pi\le\theta\le-\pi\\ -\theta, & \text{if } -\pi\le\theta\le 0\\ \theta, & \text{if } 0\le\theta\le\pi\\ 2\pi-\theta, & \text{if } \pi\le\theta\le 2\pi\\ \theta-2\pi, & \text{if } 2\pi\le\theta\le 3\pi \quad \text{and so on.}\end{cases}$$

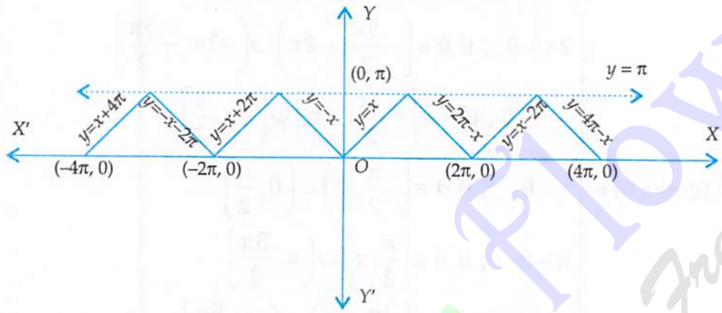


Fig. 3.22 Graph of $y=\cos^{-1}(\cos x)$

$$\tan^{-1}(\tan\theta)=\begin{cases}2\pi+\theta, & \text{if } -5\pi/2<\theta<-3\pi/2\\ \pi+\theta, & \text{if } -3\pi/2<\theta<-\pi/2\\ \theta, & \text{if } -\pi/2<\theta<\pi/2\\ \theta-\pi, & \text{if } \pi/2<\theta<3\pi/2\\ \theta-2\pi, & \text{if } 3\pi/2<\theta<5\pi/2 \quad \text{and so on}\end{cases}$$

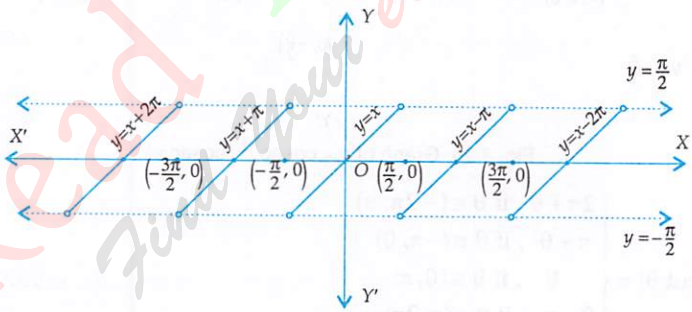


Fig. 3.23 Graph of $y=\tan^{-1}(\tan x)$

$$\sec^{-1}(\sec\theta)=\begin{cases}2\pi+\theta, & \text{if } \theta\in\left[-2\pi,-\frac{3\pi}{2}\right)\cup\left(-\frac{3\pi}{2},-\pi\right]\\ -\theta, & \text{if } \theta\in\left[-\pi,-\frac{\pi}{2}\right)\cup\left(-\frac{\pi}{2},0\right]\\ \theta, & \text{if } \theta\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]\\ 2\pi-\theta, & \text{if } \theta\in\left[\pi,\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right]\\ \theta-2\pi, & \text{if } \theta\in\left[2\pi,\frac{5\pi}{2}\right)\cup\left(\frac{5\pi}{2},3\pi\right] \quad \text{and so on}\end{cases}$$

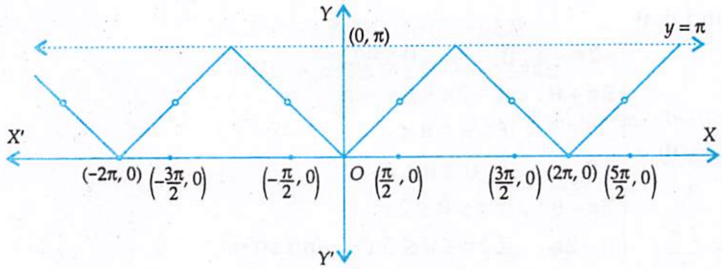


Fig. 3.24 Graph of $y = \sec^{-1}(\sec x)$

$$\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \begin{cases} 2\pi+\theta, & \text{if } \theta \in \left[-\frac{5\pi}{2}, -2\pi\right) \cup \left(-2\pi, -\frac{3\pi}{2}\right] \\ -\pi-\theta, & \text{if } \theta \in \left[-\frac{3\pi}{2}, -\pi\right) \cup \left(-\pi, \frac{\pi}{2}\right] \\ \theta, & \text{if } \theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \\ \pi-\theta, & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right) \cup \left(\pi, \frac{3\pi}{2}\right] \\ \theta-2\pi, & \text{if } \theta \in \left[\frac{3\pi}{2}, 2\pi\right) \cup \left(2\pi, \frac{5\pi}{2}\right] \quad \text{and so on} \end{cases}$$

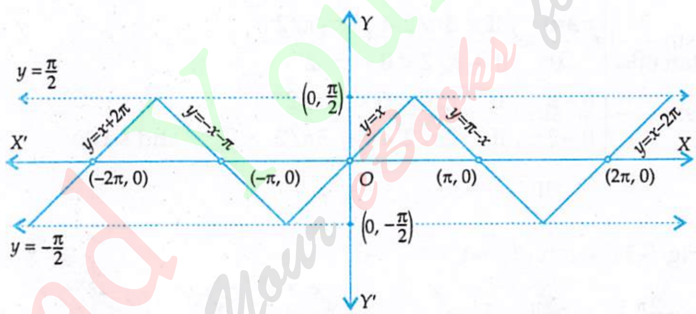


Fig. 3.25 Graph of $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$

$$\cot^{-1}(\cot\theta) = \begin{cases} 2\pi+\theta, & \text{if } \theta \in (-2\pi, \pi) \\ \pi+\theta, & \text{if } \theta \in (-\pi, 0) \\ \theta, & \text{if } \theta \in (0, \pi) \\ \theta-\pi, & \text{if } \theta \in (\pi, 2\pi) \\ \theta-2\pi, & \text{if } \theta \in (2\pi, 3\pi) \quad \text{and so on} \end{cases}$$

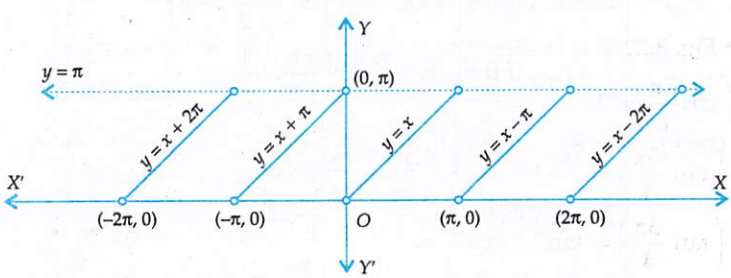


Fig. 3.26 Graph of $y = \cot^{-1}(\cot x)$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate each of the following:*

(i) $\sin^{-1}\left(\sin\frac{\pi}{3}\right)$ (ii) $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$ (iii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ (v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ [CBSE 2009] (vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

(vii) $\sin^{-1}\left(\sin(-600°)\right)$ [NCERT EXEMPLAR]

(viii) $\cos^{-1}\left(\cos(-680°)\right)$ [NCERT EXEMPLAR]

SOLUTION (i) We know that $\sin^{-1}(\sin\theta)=\theta$, if $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. Therefore, $\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$.

(ii) We know that $\cos^{-1}(\cos\theta)=\theta$ for $0\le\theta\le\pi$. Therefore, $\cos^{-1}\left(\cos\frac{2\pi}{3}\right)=\frac{2\pi}{3}$.

(iii) We know that $\tan^{-1}(\tan\theta)=\theta$ for $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$. Therefore, $\tan^{-1}\left(\tan\frac{\pi}{4}\right)=\frac{\pi}{4}$.

(iv) $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)\ne\frac{2\pi}{3}$ as $\frac{2\pi}{3}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

Now, $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\sin^{-1}\left\{\sin\left(\pi-\frac{\pi}{3}\right)\right\}$ $\left[\because \sin\frac{2\pi}{3}=\sin\left(\pi-\frac{\pi}{3}\right)\right]$

$=\sin^{-1}\left(\sin\frac{\pi}{3}\right)=\frac{\pi}{3}$ $[\because \sin(\pi-\theta)=\sin\theta]$

ALITER From Fig. 3.21, we find that $\sin^{-1}(\sin\theta)=\pi-\theta$ for $\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$.

$\therefore$ $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$

(v) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)\ne\frac{7\pi}{6}$, because $\frac{7\pi}{6}$ does not lie between 0 and π.

$\therefore$ $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left\{\cos\left(2\pi-\frac{5\pi}{6}\right)\right\}$ $\left[\because \frac{7\pi}{6}=2\pi-\frac{5\pi}{6}\right]$

$=\cos^{-1}\left(\cos\frac{5\pi}{6}\right)=\frac{5\pi}{6}$ $[\because \cos(2\pi-\theta)=\cos\theta]$

ALITER From Fig. 3.22, it is evident that $\cos^{-1}(\cos\theta)=2\pi-\theta$ for $\pi\le\theta\le2\pi$.

$\therefore$ $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=2\pi-\frac{7\pi}{6}=\frac{5\pi}{6}$

(vi) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)\ne\frac{3\pi}{4}$, because $\frac{3\pi}{4}$ does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

$\therefore$ $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left\{\tan\left(\pi-\frac{\pi}{4}\right)\right\}$ $\left[\because \frac{3\pi}{4}=\pi-\frac{\pi}{4}\right]$

$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\tan^{-1}\left\{\tan\left(-\frac{\pi}{4}\right)\right\}=-\frac{\pi}{4}$ $[\because \tan(\pi-\theta)=-\tan\theta]$

ALITER It is evident from Fig. 3.23 that $\tan^{-1}(\tan\theta) = \theta - \pi$ for $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

$$\therefore \quad \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \frac{3\pi}{4} - \pi = -\frac{\pi}{4}.$$

(vii) $\sin^{-1}\left(\sin(-600°)\right) = \sin^{-1}\left\{\sin\left(-600 \times \frac{\pi}{180}\right)\right\} = \sin^{-1}\left\{\sin\left(-\frac{10\pi}{3}\right)\right\} = \sin^{-1}\left(-\sin\frac{10\pi}{3}\right)$

$$= \sin^{-1}\left\{-\sin\left(3\pi + \frac{\pi}{3}\right)\right\} = \sin^{-1}\left\{-\left(-\sin\frac{\pi}{3}\right)\right\} = \sin^{-1}\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$$

ALITER From Fig. 3.21, we observe that $\sin^{-1}(\sin\theta) = -3\pi - \theta$ for $-\frac{7\pi}{2} \le \theta \le -\frac{5\pi}{2}$

$$\therefore \quad \sin^{-1}\left\{\sin(-600°)\right\} = \sin^{-1}\left\{\sin\left(-\frac{10\pi}{3}\right)\right\} = -3\pi + \frac{10\pi}{3} = \frac{\pi}{3}$$

(viii) $\cos^{-1}\left\{\cos(-680°)\right\} = \cos^{-1}(\cos 680°) = \cos^{-1}\left\{\cos\left(680 \times \frac{\pi}{180}\right)\right\} = \cos^{-1}\left(\cos\frac{34\pi}{9}\right)$

$$= \cos^{-1}\left\{\cos\left(4\pi - \frac{2\pi}{9}\right)\right\} = \cos^{-1}\left(\cos\frac{2\pi}{9}\right) = \frac{2\pi}{9}$$

ALITER From Fig. 3.22, we find that $\cos^{-1}(\cos\theta) = \theta + 4\pi$, if $-4\pi \le \theta \le -3\pi$ and $-680° = -\frac{34\pi}{9}$

$$\therefore \quad \cos^{-1}\left\{\cos(-680°)\right\} = \cos^{-1}\left\{\cos\left(-\frac{34\pi}{9}\right)\right\} = -\frac{34\pi}{9} + 4\pi = \frac{2\pi}{9}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Express each of the following in the simplest form:*

(i) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, -\pi < x < \pi$ [NCERT] (ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ [CBSE 2012]

(iii) $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ [NCERT] (iv) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

SOLUTION (i) We have, [NCERT]

$$\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\} = \tan^{-1}\left\{\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right\} = \tan^{-1}\left\{\sqrt{\tan^2\frac{x}{2}}\right\} = \tan^{-1}\left\{\left|\tan\frac{x}{2}\right|\right\}$$

$$= \begin{cases} \tan^{-1}\left(-\tan\frac{x}{2}\right), & \text{if } -\pi < x < 0 \\ \tan^{-1}\left(\tan\frac{x}{2}\right), & \text{if } 0 \le x < \pi \end{cases} = \begin{cases} \tan^{-1}\left\{\tan\left(-\frac{x}{2}\right)\right\} = -\frac{x}{2}, & \text{if } -\pi < x < 0 \\ \tan^{-1}\left\{\tan\frac{x}{2}\right\} = \frac{x}{2}, & \text{if } 0 < x < \pi \end{cases}$$

(ii) We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\} = \tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2} \qquad \left[\because -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow -\frac{\pi}{4}<-\frac{x}{2}<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}-\frac{x}{2}<\frac{\pi}{2}\right]$$

ALITER We have,

$$\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}+x\right)}{1-\cos\left(\frac{\pi}{2}+x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}+\frac{x}{2}\right)\cos\left(\frac{\pi}{4}+\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}\right\} = \tan^{-1}\left\{\cot\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left\{\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \frac{\pi}{4}-\frac{x}{2}$$

(iii) We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left\{\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\} = \tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}$$

$$= \frac{\pi}{4}+\frac{x}{2} \qquad \left[\because -\frac{\pi}{2}<x<\frac{\pi}{2} \Rightarrow -\frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{4} \Rightarrow 0<\frac{\pi}{4}+\frac{x}{2}<\frac{\pi}{2}\right]$$

ALITER We have,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left\{\frac{\sin\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right\} = \tan^{-1}\left\{\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}$$

$$= \tan^{-1}\left[\tan\left\{\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}\right] = \tan^{-1}\left\{\tan^{-1}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2}$$

(iv) We have,

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-x\right)\right\}$$

$$= \frac{\pi}{4} - x \qquad \left[\because -\frac{\pi}{4} < x < \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} < -x < \frac{\pi}{4} \Rightarrow 0 < \frac{\pi}{4} - x < \frac{\pi}{2}\right]$$

EXAMPLE 3 *Prove that:*

(i) $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\} = \frac{\pi}{4}+\frac{x}{2},\ 0 < x < \frac{\pi}{2}$

(ii) $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{x}{2},\ 0 < x < \frac{\pi}{2}$ [NCERT, CBSE 2009, 2014, 2016]

SOLUTION (i) We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\} = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right\} \qquad \left[\because 0 < \frac{x}{2} < \frac{\pi}{4} \therefore \cos\frac{x}{2} > 0,\ \sin\frac{x}{2} > 0\right]$$

$$= \tan^{-1}\left\{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\} = \frac{\pi}{4}+\frac{x}{2} \qquad \left[\because 0 < x < \frac{\pi}{2} \therefore \frac{\pi}{4} < \frac{\pi}{4}+\frac{x}{2} < \frac{\pi}{2}\right]$$

(ii) We know that: $1 \pm \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} \pm 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} \pm \sin\frac{x}{2}\right)^2$

$$\therefore \quad \cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$= \cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\} \qquad \left[\because \sqrt{x^2} = |x|\right]$$

$$= \cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \qquad \left[\because\ 0<\frac{x}{2}<\frac{\pi}{4}\ \therefore \cos\frac{x}{2}>\sin\frac{x}{2}\right]$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2} \qquad \left[\because\ 0<\frac{x}{2}<\frac{\pi}{4}\right]$$

EXAMPLE 4 *Prove that:*

(i) $\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}=\frac{\pi}{4}-\frac{x}{2}$, if $\pi<x<\frac{3\pi}{2}$

(ii) $\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\frac{\pi}{2}-\frac{x}{2}$, if $\frac{\pi}{2}<x<\pi$ **[CBSE 2011]**

SOLUTION (i) We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\}$$

$$= \tan^{-1}\left\{\frac{\sqrt{2\cos^2\frac{x}{2}}+\sqrt{2\sin^2\frac{x}{2}}}{\sqrt{2\cos^2\frac{x}{2}}-\sqrt{2\sin^2\frac{x}{2}}}\right\}=\tan^{-1}\left\{\frac{\sqrt{2}\left|\cos\frac{x}{2}\right|+\sqrt{2}\left|\sin\frac{x}{2}\right|}{\sqrt{2}\left|\cos\frac{x}{2}\right|-\sqrt{2}\left|\sin\frac{x}{2}\right|}\right\}$$

$$= \tan^{-1}\left\{\frac{-\cos\frac{x}{2}+\sin\frac{x}{2}}{-\cos\frac{x}{2}-\sin\frac{x}{2}}\right\} \qquad \left[\begin{array}{l}\because\ \pi<x<\frac{3\pi}{2}\Rightarrow\frac{\pi}{2}<\frac{x}{2}<\frac{3\pi}{4}\\ \therefore\ \left|\cos\frac{x}{2}\right|=-\cos\frac{x}{2},\ \left|\sin\frac{x}{2}\right|=\sin\frac{x}{2}\end{array}\right]$$

$$= \tan^{-1}\left\{\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\right\}=\tan^{-1}\left\{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\}=\frac{\pi}{4}-\frac{x}{2} \qquad \left[\because\ \pi<x<\frac{3\pi}{2}\ \therefore\ -\frac{\pi}{2}<\frac{\pi}{4}-\frac{x}{2}<-\frac{\pi}{4}\right]$$

(ii) We have,

$$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\}=\cot^{-1}\left\{\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right\}$$

$$= \cot^{-1}\left\{\frac{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|+\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}{\left|\cos\frac{x}{2}+\sin\frac{x}{2}\right|-\left|\cos\frac{x}{2}-\sin\frac{x}{2}\right|}\right\}$$

$$= \cot^{-1}\left\{\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right\} \quad \left[\because \frac{\pi}{2}<x<\pi \Rightarrow \frac{\pi}{4}<\frac{x}{2}<\frac{\pi}{2} \Rightarrow \cos\frac{x}{2}<\sin\frac{x}{2}\right]$$

$$= \cot^{-1}\left(\tan\frac{x}{2}\right) = \cot^{-1}\left\{\cot\left(\frac{\pi}{2}-\frac{x}{2}\right)\right\} = \frac{\pi}{2}-\frac{x}{2} \qquad \left[\because \frac{\pi}{2}<x<\pi \Rightarrow 0<\frac{\pi}{2}-\frac{x}{2}<\frac{\pi}{4}\right]$$

REMARK *In order to simplify trigonometrical expressions involving inverse trigonometrical functions, following substitutions are very useful:*

Expression	*Substitution*
a^2+x^2	$x=a\tan\theta$ or, $x=a\cot\theta$
a^2-x^2	$x=a\sin\theta$ or, $x=a\cos\theta$
x^2-a^2	$x=a\sec\theta$ or, $x=a\,\mathrm{cosec}\,\theta$
$\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x=a\cos 2\theta$
$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$ or, $\sqrt{\frac{a^2+x^2}{a^2-x^2}}$	$x^2=a^2\cos 2\theta$

EXAMPLE 5 *Write the following functions in the simplest form:*

(i) $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a<x<a$ **[NCERT]** (ii) $\tan^{-1}\left\{\sqrt{\frac{a-x}{a+x}}\right\}, -a<x<a$

(iii) $\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$ (iv) $\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\}$

SOLUTION (i) Putting $x=a\sin\theta$, we obtain

$$\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\} = \tan^{-1}\left\{\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right\} = \tan^{-1}\left\{\frac{a\sin\theta}{a\cos\theta}\right\}$$

$$= \tan^{-1}(\tan\theta) = \theta = \sin^{-1}\frac{x}{a} \qquad \left[\because x=a\sin\theta \Rightarrow \sin\theta=\frac{x}{a} \Rightarrow \theta=\sin^{-1}\frac{x}{a}\right]$$

(ii) Putting $x=a\cos\theta$, we obtain

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}} = \tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}} = \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \tan^{-1}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}$$

$$= \tan^{-1}\left(\left|\tan\frac{\theta}{2}\right|\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \qquad \left[\because -a<x<a \Rightarrow 0<\theta<\pi \Rightarrow 0<\frac{\theta}{2}<\frac{\pi}{2} \therefore \left|\tan\frac{\theta}{2}\right|=\tan\frac{\theta}{2}\right]$$

$$= \frac{\theta}{2} = \frac{1}{2}\cos^{-1}\frac{x}{a} \qquad \left[\because x=a\cos\theta \Rightarrow \cos\theta=\frac{x}{a} \Rightarrow \theta=\cos^{-1}\frac{x}{a}\right]$$

(iii) Putting $x = a \tan \theta$, we obtain

$$\sin^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\} = \sin^{-1}\left\{\frac{a\tan\theta}{\sqrt{a^2\tan^2\theta+a^2}}\right\} = \sin^{-1}\left\{\frac{a\tan\theta}{a\sec\theta}\right\} = \sin^{-1}(\sin\theta)$$

$$= \theta = \tan^{-1}\frac{x}{a} \qquad \left[\because\ x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a}\right]$$

(iv) Putting $x = a \cot \theta$, we obtain

$$\cos^{-1}\left\{\frac{x}{\sqrt{x^2+a^2}}\right\} = \cos^{-1}\left\{\frac{a\cot\theta}{\sqrt{a^2\cot^2\theta+a^2}}\right\} = \cos^{-1}\left\{\frac{a\cot\theta}{a\,\mathrm{cosec}\,\theta}\right\}$$

$$= \cos^{-1}(\cos\theta) = \theta = \cot^{-1}\frac{x}{a} \left[\because x = a\cot\theta \Rightarrow \cot\theta = \frac{x}{a} \Rightarrow \cot^{-1}\frac{x}{a} = \theta\right]$$

EXAMPLE 6 *Prove that:*

(i) $\tan^{-1}\left\{\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x,\ 0 < x < 1$

[NCERT, CBSE 2010, 2011, 2014, 2017]

(ii) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right\} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2,\ -1 < x < 1$ **[NCERT EXEMPLAR]**

SOLUTION (i) Putting $x = \cos 2\theta$, we obtain

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right\} = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right\} \qquad \left[\because 0 < x < 1 \Rightarrow 0 < \cos 2\theta < 1 \Rightarrow 0 < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \theta < \frac{\pi}{4}\right]$$

$$= \tan^{-1}\left\{\frac{1-\tan\theta}{1+\tan\theta}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\theta\right)\right\} = \frac{\pi}{4} - \theta \quad \left[\because 0 < \theta < \frac{\pi}{4} \Rightarrow 0 < \frac{\pi}{4} - \theta < \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x \qquad \left[\because\ \cos 2\theta = x\ \therefore\ 2\theta = \cos^{-1}x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x\right]$$

(ii) Putting $x^2 = \cos 2\theta$, we obtain

$$\tan^{-1}\left\{\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right\} = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right\} = \tan^{-1}\left\{\frac{1+\tan\theta}{1-\tan\theta}\right\} = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}+\theta\right)\right\}$$

$$= \frac{\pi}{4} + \theta \qquad \left[\because\ -1 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow 0 < 2\theta < \frac{\pi}{2} \Rightarrow 0 < \theta < \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2 \qquad \left[\because\ x^2 = \cos 2\theta \Rightarrow 2\theta = \cos^{-1}x^2 \Rightarrow \theta = \frac{1}{2}\cos^{-1}x^2\right]$$

EXAMPLE 7 *Simplify each of the following:*

(i) $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, *where* $\frac{\pi}{2}\le x\le\frac{3\pi}{4}$ **[NCERT EXEMPLAR]**

(ii) $\sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)$

SOLUTION (i) In order to simplify $\cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)$, we will have to express $\frac{3}{5}\cos x+\frac{4}{5}\sin x$ in the form of cosine of some expression. For this, let $\frac{3}{5}=r\cos\theta$ and $\frac{4}{5}=r\sin\theta$. Then,

$$r=\sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}=1 \text{ and, } \tan\theta=\frac{r\sin\theta}{r\cos\theta}=\frac{4}{3}$$

$$\Rightarrow\quad r=1 \text{ and, } \theta=\tan^{-1}\frac{4}{3}\Rightarrow\frac{3}{5}=\cos\theta \text{ and } \frac{4}{5}=\sin\theta, \text{ where } \theta=\tan^{-1}\frac{4}{3}$$

$$\therefore\quad \cos^{-1}\left(\frac{3}{5}\cos x+\frac{4}{5}\sin x\right)=\cos^{-1}(\cos\theta\cos x+\sin\theta\sin x)=\cos^{-1}\{\cos(x-\theta)\}$$

$$=x-\theta=x-\tan^{-1}\frac{4}{3}.$$

(ii) In order to simplify $\sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)$, we will have to express $\frac{5}{13}\cos x+\frac{12}{13}\sin x$ in the form of sine of some expression. For this, let $\frac{5}{13}=r\sin\theta$ and $\frac{12}{13}=r\cos\theta$. Then,

$$r=\sqrt{\left(\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2}=1 \text{ and, } \tan\theta=\frac{r\sin\theta}{r\cos\theta}=\frac{5}{12}$$

$$\Rightarrow\quad r=1 \text{ and, } \theta=\tan^{-1}\frac{5}{12}\Rightarrow\frac{5}{13}=\sin\theta \text{ and } \frac{12}{13}=\cos\theta, \text{ where } \theta=\tan^{-1}\frac{5}{12}$$

$$\therefore\quad \sin^{-1}\left(\frac{5}{13}\cos x+\frac{12}{13}\sin x\right)=\sin^{-1}(\sin\theta\cos x+\cos\theta\sin x)=\sin^{-1}\{\sin(x+\theta)\}$$

$$=x+\theta=x+\tan^{-1}\frac{5}{12}$$

EXAMPLE 8 *Simplify each of the following:*

(i) $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), -\frac{\pi}{4}<x<\frac{\pi}{4}$ (ii) $\cos^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right), \frac{\pi}{4}<x<\frac{5\pi}{4}$

SOLUTION (i) $\sin^{-1}\left(\frac{\sin x+\cos x}{\sqrt{2}}\right)$

$$=\sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)=\sin^{-1}\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)$$

$$=\sin^{-1}\left\{\sin\left(x+\frac{\pi}{4}\right)\right\}=x+\frac{\pi}{4}\qquad\left[\because -\frac{\pi}{4}<x<\frac{\pi}{4}\Rightarrow 0<x+\frac{\pi}{4}<\frac{\pi}{2}\right]$$

(ii) $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right) = \cos^{-1}\left(\sin x\sin\frac{\pi}{4}+\cos x\cos\frac{\pi}{4}\right)$$

$$= \cos^{-1}\left\{\cos\left(x-\frac{\pi}{4}\right)\right\} = x-\frac{\pi}{4} \qquad \left[\because \frac{\pi}{4}<x<\frac{5\pi}{4} \Rightarrow 0<x-\frac{\pi}{4}<\pi\right]$$

REMARK *This example can also be solved by using the procedure used in the earlier example.*

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Simplify each of the the following:*

(i) $\sin^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), \dfrac{\pi}{4}<x<\dfrac{5\pi}{4}$ (ii) $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), \dfrac{5\pi}{4}<x<\dfrac{9\pi}{4}$

SOLUTION (i) $\sin^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right)$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right) = \sin^{-1}\left(\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x\right) = \sin^{-1}\left\{\sin\left(x+\frac{\pi}{4}\right)\right\}$$

$$= \pi-\left(x+\frac{\pi}{4}\right) = \frac{3\pi}{4}-x \qquad \left[\because \frac{\pi}{4}<x<\frac{5\pi}{4} \Rightarrow \frac{\pi}{2}<x+\frac{\pi}{4}<\frac{3\pi}{2} \Rightarrow -\frac{\pi}{2}<\pi-\left(x+\frac{\pi}{4}\right)<\frac{\pi}{2}\right]$$

(ii) $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right)$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right) = \cos^{-1}\left(\cos x\cos\frac{\pi}{4}+\sin x\sin\frac{\pi}{4}\right) = \cos^{-1}\left\{\cos\left(x-\frac{\pi}{4}\right)\right\}$$

$$= 2\pi-\left(x-\frac{\pi}{4}\right) = \frac{9\pi}{4}-x \qquad \left[\because \frac{5\pi}{4}<x<\frac{9\pi}{4} \Rightarrow \pi<x-\frac{\pi}{4}<2\pi \Rightarrow 0<2\pi-\left(x-\frac{\pi}{4}\right)<\pi\right]$$

EXAMPLE 10 *Evaluate the following:*

(i) $\sin^{-1}(\sin 10)$ (ii) $\sin^{-1}(\sin 5)$ (iii) $\cos^{-1}(\cos 10)$ (iv) $\tan^{-1}\{\tan(-6)\}$

SOLUTION (i) We know that $\sin^{-1}(\sin\theta) = \theta$, if $-\dfrac{\pi}{2}\leq\theta\leq\dfrac{\pi}{2}$. Here, $\theta=10$ radians which does not lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. But, $3\pi-\theta$ i.e. $3\pi-10$ lies between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. Also, $\sin(3\pi-10)=\sin 10$.

$$\therefore \quad \sin^{-1}(\sin 10) = \sin^{-1}(\sin(3\pi-10)) = 3\pi-10.$$

ALITER We know that $3\pi<10^c<\dfrac{7\pi}{2}$ and $\sin^{-1}(\sin\theta)=3\pi-\theta$ for $\dfrac{5\pi}{2}\leq\theta\leq\dfrac{7\pi}{2}$.

$$\therefore \quad \sin^{-1}(\sin 10) = 3\pi-10.$$

(ii) Here, $\theta=5$ radians. Clearly, it does not lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$. But, $2\pi-5$ and $5-2\pi$ both lie between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ such that $\sin(5-2\pi) = \sin(-(2\pi-5)) = -\sin(2\pi-5) = -(-\sin 5) = \sin 5$.

$$\therefore \quad \sin^{-1}(\sin 5) = \sin^{-1}(\sin(5-2\pi)) = 5-2\pi.$$

ALITER We know that $\frac{3\pi}{2} < 5^c < \frac{5\pi}{2}$ and $\sin^{-1}(\sin\theta) = \theta - 2\pi$ for $\frac{3\pi}{2} \le \theta \le \frac{5\pi}{2}$ (see Fig. 3.21)

$\therefore \quad \sin^{-1}(\sin 5) = 5 - 2\pi.$

(iii) We know that $\cos^{-1}(\cos\theta) = \theta$, if $0 \le \theta \le \pi$. Here, $\theta = 10$ radians. Clearly, it does not lie between 0 and π. However, $(4\pi - 10)$ lies between 0 and π such that $\cos(4\pi - 10) = \cos 10$.

$\therefore \quad \cos^{-1}(\cos 10) = \cos^{-1}(\cos(4\pi - 10)) = 4\pi - 10$

ALITER We know that $3\pi < 10^c < 4\pi$ and $\cos^{-1}(\cos\theta) = 4\pi - \theta$ for $3\pi < \theta < 4\pi$ (see Fig. 3.22)

$\therefore \quad \cos^{-1}(\cos 10) = 4\pi - 10.$

(iv) We know that $\tan^{-1}(\tan\theta) = \theta$, if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Here, $\theta = -6$ radians which does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. However, $2\pi - 6$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $\tan(2\pi - 6) = -\tan 6 = \tan(-6)$.

$\therefore \quad \tan^{-1}\{\tan(-6)\} = \tan^{-1}\{\tan(2\pi - 6)\} = 2\pi - 6$

ALITER We know that $-2\pi < -6^c < -\frac{3\pi}{2}$ and $\tan^{-1}(\tan\theta) = \theta + 2\pi$ for $-2\pi < \theta < -\frac{3\pi}{2}$ (see Fig. 3.23). Therefore, $\tan^{-1}(\tan(-6)) = 2\pi - 6$.

EXERCISE 3.7

BASIC

1. Evaluate each of the following:

(i) $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$ (ii) $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$ (iii) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

(iv) $\sin^{-1}\left(\sin\frac{13\pi}{7}\right)$ (v) $\sin^{-1}\left(\sin\frac{17\pi}{8}\right)$ (vi) $\sin^{-1}\left\{\left(\sin -\frac{17\pi}{8}\right)\right\}$

(vii) $\sin^{-1}(\sin 3)$ (viii) $\sin^{-1}(\sin 4)$ (ix) $\sin^{-1}(\sin 12)$

(x) $\sin^{-1}(\sin 2)$

2. Evaluate each of the following:

(i) $\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\}$ (ii) $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$ (iii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

(iv) $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$ (v) $\cos^{-1}(\cos 3)$ (vi) $\cos^{-1}(\cos 4)$

(vii) $\cos^{-1}(\cos 5)$ (viii) $\cos^{-1}(\cos 12)$

3. Evaluate each of the following:

(i) $\tan^{-1}\left(\tan\frac{\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\frac{6\pi}{7}\right)$ (iii) $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ [NCERT]

(iv) $\tan^{-1}\left(\tan\frac{9\pi}{4}\right)$ (v) $\tan^{-1}(\tan 1)$ (vi) $\tan^{-1}(\tan 2)$

(vii) $\tan^{-1}(\tan 4)$ (viii) $\tan^{-1}(\tan 12)$

4. Evaluate each of the following:

(i) $\sec^{-1}\left(\sec\frac{\pi}{3}\right)$ (ii) $\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$ (iii) $\sec^{-1}\left(\sec\frac{5\pi}{4}\right)$

(iv) $\sec^{-1}\left(\sec\frac{7\pi}{3}\right)$ (v) $\sec^{-1}\left(\sec\frac{9\pi}{5}\right)$ (vi) $\sec^{-1}\left\{\sec\left(-\frac{7\pi}{3}\right)\right\}$

(vii) $\sec^{-1}\left(\sec\frac{13\pi}{4}\right)$ (viii) $\sec^{-1}\left(\sec\frac{25\pi}{6}\right)$

5. Evaluate each of the following:

(i) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$ (ii) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)$ (iii) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{6\pi}{5}\right)$

(iv) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{11\pi}{6}\right)$ (v) $\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{13\pi}{6}\right)$ (vi) $\operatorname{cosec}^{-1}\left\{\operatorname{cosec}\left(-\frac{9\pi}{4}\right)\right\}$

6. Evaluate each of the following:

(i) $\cot^{-1}\left(\cot\frac{\pi}{3}\right)$ (ii) $\cot^{-1}\left(\cot\frac{4\pi}{3}\right)$ (iii) $\cot^{-1}\left(\cot\frac{9\pi}{4}\right)$

(iv) $\cot^{-1}\left(\cot\frac{19\pi}{6}\right)$ (v) $\cot^{-1}\left\{\cot\left(-\frac{8\pi}{3}\right)\right\}$ (vi) $\cot^{-1}\left\{\cot\left(\frac{21\pi}{4}\right)\right\}$

BASED ON LOTS

7. Write each of the following in the simplest form:

(i) $\cot^{-1}\left\{\frac{a}{\sqrt{x^2-a^2}}\right\}, |x|>a$ (ii) $\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}, x \in R$

(iii) $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\}, x \in R$ (iv) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}, x \neq 0$

(v) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\}, x \neq 0$ (vi) $\tan^{-1}\sqrt{\frac{a-x}{a+x}}, -a<x<a$

(vii) $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}, -a<x<a$ (viii) $\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}, -\frac{1}{2}<x<\frac{1}{\sqrt{2}}$

(ix) $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}, 0<x<1$ (x) $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$

ANSWERS

1. (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{6}$ (iii) $\frac{\pi}{6}$ (iv) $-\frac{\pi}{7}$ (v) $\frac{\pi}{8}$ (vi) $-\frac{\pi}{8}$
(vii) $\pi-3$ (viii) $\pi-4$ (ix) $12-4\pi$ (x) $\pi-2$

2. (i) $\frac{\pi}{4}$ (ii) $\frac{3\pi}{4}$ (iii) $\frac{2\pi}{3}$ (iv) $\frac{\pi}{6}$ (v) 3 (vi) $2\pi-4$
(vii) $2\pi-5$ (viii) $4\pi-12$

3. (i) $\frac{\pi}{3}$ (ii) $-\frac{\pi}{7}$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{4}$ (v) 1 (vi) $2-\pi$
(vii) $4-\pi$ (viii) $12-4\pi$

4. (i) $\frac{\pi}{3}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $\frac{\pi}{3}$ (v) $\frac{\pi}{5}$ (vi) $\frac{\pi}{3}$
(vii) $\frac{3\pi}{4}$ (viii) $\frac{\pi}{6}$

5. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $-\frac{\pi}{5}$ (iv) $-\frac{\pi}{6}$ (v) $\frac{\pi}{6}$ (vi) $-\frac{\pi}{4}$

6. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{6}$ (v) $\frac{\pi}{3}$ (vi) $\frac{\pi}{4}$

7. (i) $\sec^{-1}\frac{x}{a}$ (ii) $\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$ (iii) $\frac{1}{2}\cot^{-1}x$ (iv) $\frac{1}{2}\tan^{-1}x$

(v) $\frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$ (vi) $\frac{1}{2}\cos^{-1}\frac{x}{a}$ (vii) $\frac{1}{2}\sin^{-1}\frac{x}{a}$

(viii) $\frac{\pi}{4}+\sin^{-1}x$ (ix) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$ (x) $\sqrt{1-x^2}$

HINTS TO SELECTED PROBLEMS

7. (ii) Putting $x=\cot\theta$, we obtain

$$\tan^{-1}\left\{x+\sqrt{1+x^2}\right\} = \tan^{-1}(\cot\theta+\operatorname{cosec}\theta) = \tan^{-1}\left(\frac{1+\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} = \tan^{-1}\left(\cot\frac{\theta}{2}\right) = \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} = \frac{\pi}{2}-\frac{\theta}{2}$$

$$=\frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$$

(ix) Putting $x=\sin\theta$, we obtain

$$\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{2}\right\} = \sin^{-1}\left\{\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right\} = \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\theta\right)\right\} = \frac{\pi}{4}+\theta$$

$$=\frac{\pi}{4}+\sin^{-1}x$$

REMARK *Let b, p and h denote respectively the base, perpendicular and hypotenuse of a right triangle PQR and let $\angle QPR=\theta$. Then,*

$$\sin\theta=\frac{p}{h},\ \cos\theta=\frac{b}{h},\ \tan\theta=\frac{p}{b},\ \operatorname{cosec}\theta=\frac{h}{p},\ \sec\theta=\frac{h}{b} \text{ and } \cot\theta=\frac{b}{p}$$

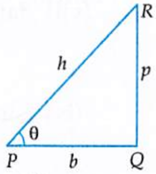


Fig. 3.27

$$\therefore\quad \theta=\sin^{-1}\left(\frac{p}{h}\right),\ \theta=\cos^{-1}\left(\frac{b}{h}\right),\ \theta=\tan^{-1}\left(\frac{p}{b}\right),\ \theta=\operatorname{cosec}^{-1}\left(\frac{h}{p}\right),$$

$$\theta=\sec^{-1}\left(\frac{h}{b}\right) \text{ and, } \theta=\cot^{-1}\left(\frac{b}{p}\right)$$

$$\Rightarrow\quad \sin^{-1}\left(\frac{p}{h}\right)=\cos^{-1}\left(\frac{b}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)=\operatorname{cosec}^{-1}\left(\frac{h}{p}\right)=\sec^{-1}\left(\frac{h}{b}\right)=\cot^{-1}\left(\frac{b}{p}\right)$$

It follows from the above result that any inverse trigonometric function can be expressed interms of the remaining five inverse trigonometric functions. For example, if $\sin^{-1}\left(\frac{5}{13}\right)$ is to be expressed interms of other five inverse trigonometric functions, then we construct a right triangle with perpendicular $p=5$ and hypotenuse $h=13$. The base b of this triangle is $b=12$.

$$\therefore\quad \sin^{-1}\left(\frac{5}{13}\right)=\cos^{-1}\left(\frac{12}{13}\right)=\tan^{-1}\left(\frac{5}{12}\right)=\operatorname{cosec}^{-1}\left(\frac{13}{5}\right)=\sec^{-1}\left(\frac{13}{12}\right)=\cot^{-1}\left(\frac{12}{5}\right)$$

3.4.2 PROPERTY-II

In Chpater 2, we have learnt that if $f: A \to B$ is a bijection, then $f^{-1}: B \to A$ exists such that $fof^{-1}(x) = x$ or, $f(f^{-1}(x)) = x$ for all $x \in B$. Applying this property on various trigonometric functions and their inverses, we obtain the following property.

PROPERTY

(i) $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$ (ii) $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$

(iii) $\tan(\tan^{-1} x) = x$, for all $x \in R$ (iv) $\cot(\cot^{-1} x) = x$, for all $x \in R$.

(v) $\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x$, for all $x \in (\infty, -1] \cup [1, \infty)$

(vi) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

This property and the above Remark help us in finding the values of expression of the form $f(g^{-1}(x))$, where f and g are trigonometric functions. We may use the following algorithm for the same.

ALGORITHM

Step I *Obtain the expression and express it in the form $f(g^{-1}(x))$, where f and g are trigonometric functions.*

Step II *Express $g^{-1}(x)$ in terms of f^{-1} by using the following results:*

$$\sin^{-1}\left(\frac{p}{h}\right) = \cos^{-1}\left(\frac{b}{h}\right) = \tan^{-1}\left(\frac{p}{b}\right) = \operatorname{cosec}^{-1}\left(\frac{h}{p}\right) = \sec^{-1}\left(\frac{h}{b}\right) = \cot^{-1}\left(\frac{b}{p}\right),$$

where p, b and h denote respectively the perpendicular, base and hypotenuse of a right triangle.

Step III *Let $g^{-1}(x) = f^{-1}(y)$. Replace $g^{-1}(x)$ by $f^{-1}(y)$ in $f(g^{-1}(x))$ and use property-II to get $f(g^{-1}(x)) = f(f^{-1}(y)) = y$.*

Following example will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate each of the following:*

(i) $\sin\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\sin\left(\cos^{-1}\frac{4}{5}\right)$ (iii) $\sin\left(\tan^{-1}\frac{15}{8}\right)$

(iv) $\sin\left(\cot^{-1}\frac{4}{3}\right)$ (v) $\sin\left(\sec^{-1}\frac{17}{15}\right)$ (vi) $\sin\left(\operatorname{cosec}^{-1}\frac{17}{8}\right)$

SOLUTION (i) Using $\sin(\sin^{-1} x) = x$, $x \in [-1, 1]$, we obtain: $\sin\left(\sin^{-1}\frac{5}{13}\right) = \frac{5}{13}$.

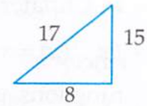
(ii) In order to express $\cos^{-1}\frac{4}{5}$ in terms of $\sin^{-1}$, let us construct a right triangle with base $b = 4$ and hypotenuse $h = 5$. The perpendicular of such triangle is $p = 3$.

$$\therefore \quad \cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5} \qquad \left[\because \cos^{-1}\frac{b}{h} = \sin^{-1}\frac{p}{h}\right]$$

Hence, $\sin\left(\cos^{-1}\frac{4}{5}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

(iii) The right triangle with base $b = 15$ and perpendicular $p = 8$ has hypotenuse $h = 17$.

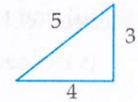
$\therefore \quad \tan^{-1}\frac{15}{8} = \sin^{-1}\frac{15}{17}$



Hence, $\sin\left(\tan^{-1}\frac{15}{8}\right) = \sin\left(\sin^{-1}\frac{15}{17}\right) = \frac{15}{17}$

(iv) The hypotenuse of the right triangle with base $b = 4$, perpendicular $p = 3$ is $h = 5$.

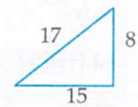
$\therefore \quad \cot^{-1}\frac{4}{3} = \sin^{-1}\frac{3}{5}$



Hence, $\sin\left(\cot^{-1}\frac{4}{3}\right) = \sin\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{5}$

(v) The right triangle with base $b = 15$ and hypotenuse $h = 17$ has perpendicular $p = 8$ has hypotenuse $h = 17$.

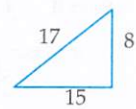
$\therefore \quad \sec^{-1}\frac{17}{15} = \sin^{-1}\frac{8}{17}$



Hence, $\sin\left(\sec^{-1}\frac{17}{15}\right) = \sin\left(\sin^{-1}\frac{8}{17}\right) = \frac{8}{17}$.

(vi) The right triangle with base $b = 15$ and perpendicular $p = 8$ has hypotenuse $h = 17$.

$\therefore \quad \operatorname{cosec}^{-1}\left(\frac{17}{8}\right) = \sin^{-1}\left(\frac{8}{17}\right)$



Hence, $\sin\left(\operatorname{cosec}^{-1}\frac{17}{8}\right) = \sin\left(\sin^{-1}\frac{8}{17}\right) = \frac{8}{17}$

EXAMPLE 2 *Evaluate each of the following:*

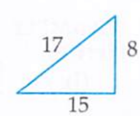
(i) $\cos\left(\cos^{-1}\frac{5}{13}\right)$ (ii) $\cos\left(\sin^{-1}\frac{8}{17}\right)$ (iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$

(iv) $\cos\left(\cot^{-1}\frac{15}{8}\right)$ (v) $\cos\left(\sec^{-1}\frac{5}{3}\right)$ (vi) $\cos\left(\operatorname{cosec}^{-1}\frac{13}{12}\right)$

SOLUTION (i) $\cos\left(\cos^{-1}\frac{5}{13}\right) = \frac{5}{13}$

(ii) The right triangle with perpendicular $p = 8$ and hypotenuse $h = 17$ has base $b = 15$.

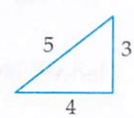
$\therefore \quad \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{15}{17}\right)$



Hence, $\cos\left(\sin^{-1}\frac{8}{17}\right) = \cos\left(\cos^{-1}\frac{15}{17}\right) = \frac{15}{17}$

(iii) The right triangle with perpendicular $p = 3$ and base $b = 4$ has hypotenuse $h = 5$.

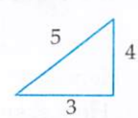
$\therefore \quad \tan^{-1}\left(\frac{3}{4}\right) = \cos^{-1}\left(\frac{4}{5}\right)$



Hence, $\cos\left(\tan^{-1}\frac{3}{4}\right) = \cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}$

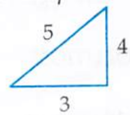
(iv) The right triangle with base $b = 15$ and perpendicular $p = 8$ has hypotenuse $h = 17$.

$\therefore \quad \cot^{-1}\left(\frac{15}{8}\right) = \cos^{-1}\left(\frac{15}{17}\right)$



Hence, $\cos\left(\cot^{-1}\frac{15}{8}\right) = \cos\left(\cos^{-1}\frac{15}{17}\right) = \frac{15}{17}$

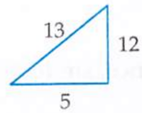
(v) We find that the right triangle with hypotenuse $h = 5$ and base $b = 3$ has perpendicular $p = 4$.



$\therefore \quad \sec^{-1}\frac{5}{3} = \cos^{-1}\frac{3}{5}$

Hence, $\cos\left(\sec^{-1}\frac{5}{3}\right) = \cos\left(\cos^{-1}\frac{3}{5}\right) = \frac{3}{5}$

(vi) Clearly, the right triangle with hypotenuse $h = 13$ and perpendicular $p = 12$ has base $b = 5$.



$\therefore \quad \text{cosec}^{-1}\frac{13}{12} = \cos^{-1}\frac{5}{13}$

Hence, $\cos\left(\text{cosec}^{-1}\frac{13}{12}\right) = \cos\left(\cos^{-1}\frac{5}{13}\right) = \frac{5}{13}$

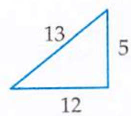
EXAMPLE 3 *Evaluate each of the following:*

(i) $\tan\left(\tan^{-1}\frac{3}{4}\right)$ (ii) $\tan\left(\sin^{-1}\frac{5}{13}\right)$ (iii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$

(iv) $\tan\left(\text{cosec}^{-1}\frac{13}{5}\right)$ (v) $\tan\left(\sec^{-1}\frac{13}{12}\right)$ (vi) $\tan\left(\cot^{-1}\frac{8}{15}\right)$

SOLUTION (i) $\tan\left(\tan^{-1}\frac{3}{4}\right) = \frac{3}{4}$

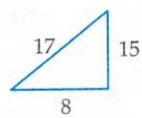
(ii) The right triangle with perpendicular $p = 5$ and hypotenuse $h = 13$ has base $b = 12$.



$\therefore \quad \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{5}{12}$

Hence, $\tan\left(\sin^{-1}\frac{5}{13}\right) = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

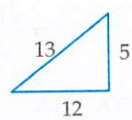
(iii) The right triangle with base $b = 8$, hypotenuse $h = 17$ has perpendicular $p = 15$.



$\therefore \quad \cos^{-1}\frac{8}{17} = \tan^{-1}\frac{15}{8}$

Hence, $\tan\left(\cos^{-1}\frac{8}{17}\right) = \tan\left(\tan^{-1}\frac{15}{8}\right) = \frac{15}{8}$

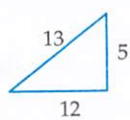
(iv) We find that the right triangle with perpendicular $p = 5$ and hypotenuse $h = 13$ has its base $b = 12$.



$\therefore \quad \text{cosec}^{-1}\left(\frac{13}{5}\right) = \tan^{-1}\left(\frac{5}{12}\right)$

Hence, $\tan\left\{\text{cosec}^{-1}\frac{13}{5}\right\} = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

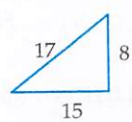
(v) The right triangle with base $b = 12$ and hypotenuse $h = 13$ has perpendicular $p = 5$.



$\therefore \quad \sec^{-1}\frac{13}{12} = \tan^{-1}\frac{5}{12}$

Hence, $\tan\left(\sec^{-1}\frac{13}{12}\right) = \tan\left(\tan^{-1}\frac{5}{12}\right) = \frac{5}{12}$

(vi) Clearly, the right triangle with base $b = 8$ and perpendicular $p = 15$ has hypotenuse $h = 17$.

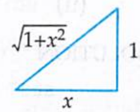


$\therefore \quad \cot^{-1}\left(\frac{8}{15}\right) = \tan^{-1}\frac{15}{8}$

Hence, $\tan\left(\cot^{-1}\frac{8}{15}\right) = \tan\left(\tan^{-1}\frac{15}{8}\right) = \frac{15}{8}$

EXAMPLE 4 *Evaluate:* (i) $\sin\ (\cot^{-1} x)$ (ii) $\cos\ (\tan^{-1} x)$

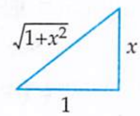
SOLUTION (i) We have to find the value of $\sin\ (\cot^{-1} x) = \sin\left(\cot^{-1}\frac{x}{1}\right)$. The right triangle with base $b = x$, perpendicular $p = 1$ has hypotenuse $h = \sqrt{1+x^2}$.



$$\therefore \qquad \cot^{-1} x = \sin^{-1}\frac{1}{\sqrt{1+x^2}}$$

Hence, $\sin\ (\cot^{-1} x) = \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$

(ii) $\cos\ (\tan^{-1} x) = \cos\left(\tan^{-1}\frac{x}{1}\right)$. The right triangle with perpendicular $p = x$ and base $b = 1$ has its hypotenuse $h = \sqrt{1+x^2}$.



$$\therefore \qquad \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

Hence, $\cos\ (\tan^{-1} x) = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Evaluate:* $\cos\left(\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right)$. **[NCERT EXEMPLAR]**

SOLUTION $\cos\left(\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right) = \cos\left(\sin^{-1}\frac{1}{4} + \cos^{-1}\frac{3}{4}\right)$ $\left[\because \sec^{-1}\frac{4}{3} = \cos^{-1}\frac{3}{4}\right]$

$$= \cos\left(\sin^{-1}\frac{1}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\cos^{-1}\frac{3}{4}\right)$$

$$= \cos\left(\cos^{-1}\frac{\sqrt{15}}{4}\right)\cos\left(\cos^{-1}\frac{3}{4}\right) - \sin\left(\sin^{-1}\frac{1}{4}\right)\sin\left(\sin^{-1}\frac{\sqrt{7}}{4}\right) \quad \left[\begin{array}{l}\because \sin^{-1}\frac{1}{4} = \cos^{-1}\frac{\sqrt{15}}{4}\\ \& \cos^{-1}\frac{3}{4} = \sin^{-1}\frac{\sqrt{7}}{4}\end{array}\right]$$

$$= \frac{\sqrt{15}}{4}\times\frac{3}{4} - \frac{1}{4}\times\frac{\sqrt{7}}{4} = \frac{3\sqrt{15}-\sqrt{7}}{16}$$

EXAMPLE 6 *Evaluate :* $\sin\left(\cos^{-1}\frac{3}{5} + \operatorname{cosec}^{-1}\frac{13}{5}\right)$.

SOLUTION $\sin\left(\cos^{-1}\frac{3}{5} + \operatorname{cosec}^{-1}\frac{13}{5}\right)$

$$= \sin\left(\cos^{-1}\frac{3}{5}\right)\cos\left(\operatorname{cosec}^{-1}\frac{13}{5}\right) + \cos\left(\cos^{-1}\frac{3}{5}\right)\sin\left(\operatorname{cosec}^{-1}\frac{13}{5}\right)$$

$$= \sin\left(\sin^{-1}\frac{4}{5}\right)\cos\left(\cos^{-1}\frac{12}{13}\right) + \cos\left(\cos^{-1}\frac{3}{5}\right)\sin\left(\sin^{-1}\frac{5}{13}\right) = \frac{4}{5}\times\frac{12}{13} + \frac{3}{5}\times\frac{5}{13} = \frac{63}{65}$$

EXAMPLE 7 *Find the value of the expression* $\sin\left[\cot^{-1}\left\{\cos\ (\tan^{-1} 1)\right\}\right]$ **[NCERT EXEMPLAR]**

SOLUTION $\sin\left[\cot^{-1}\left\{\cos\ (\tan^{-1} 1)\right\}\right] = \sin\left\{\cot^{-1}\left(\cos\frac{\pi}{4}\right)\right\}$ $\left[\because \tan^{-1} 1 = \frac{\pi}{4}\right]$

$$= \sin\left(\cot^{-1}\frac{1}{\sqrt{2}}\right) = \sin\left(\sin^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right) \qquad \left[\because \cot^{-1}\frac{1}{\sqrt{2}} = \sin^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right]$$

EXAMPLE 8 *Prove that:*

(i) $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 15$

(ii) $\tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3) = 11$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$$

$$= \{\sec(\tan^{-1} 2)\}^2 + \{\operatorname{cosec}(\cot^{-1} 3)\}^2 = \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\operatorname{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$$

$$= \{\sec(\sec^{-1}\sqrt{5})\}^2 + \{\operatorname{cosec}(\operatorname{cosec}^{-1}\sqrt{10})\}^2 = (\sqrt{5})^2 + (\sqrt{10})^2 = 15$$

ALITER $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) = 1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3)$

$$= 1 + \{\tan(\tan^{-1} 2)\}^2 + 1 + \{\cot(\cot^{-1} 3)\}^2 = 1 + 2^2 + 1 + 3^2 = 15$$

(ii) $\tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3) = \sec^2(\sec^{-1} 2) - 1 + \operatorname{cosec}^2(\operatorname{cosec}^{-1} 3) - 1$

$$= \{\sec(\sec^{-1} 2)\}^2 - 1 + \{\operatorname{cosec}(\operatorname{cosec}^{-1} 3)\}^2 - 1 = 2^2 - 1 + 3^2 - 1 = 11$$

ALITER $\tan^2(\sec^{-1} 2) + \cot^2(\operatorname{cosec}^{-1} 3)$

$$= \left\{\tan\left(\sec^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\cot\left(\operatorname{cosec}^{-1}\frac{3}{1}\right)\right\}^2 = \left\{\tan\left(\tan^{-1}\frac{\sqrt{3}}{1}\right)\right\}^2 + \left\{\cot\left(\cot^{-1}\frac{2\sqrt{2}}{1}\right)\right\}^2$$

$$= (\sqrt{3})^2 + (2\sqrt{2})^2 = 3 + 8 = 11$$

EXAMPLE 9 *Prove that:* (i) $\sin\left[\cot^{-1}\left\{\cos(\tan^{-1} x)\right\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$

(ii) $\cos\left[\tan^{-1}\left\{\sin(\cot^{-1} x)\right\}\right] = \sqrt{\frac{x^2+1}{x^2+2}}$ **[CBSE 2010]**

SOLUTION (i) We find that

$$\cos(\tan^{-1} x) = \cos\left\{\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}} \qquad \left[\because \tan^{-1} x = \cos^{-1}\frac{1}{\sqrt{1+x^2}}\right]$$

$$\therefore \quad \sin\left[\cot^{-1}\left\{\cos(\tan^{-1} x)\right\}\right]$$

$$= \sin\left\{\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \sin\left\{\sin^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

(ii) We find that $\sin(\cot^{-1} x) = \sin\left\{\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \frac{1}{\sqrt{1+x^2}}$

$$\therefore \quad \cos\left[\tan^{-1}\left\{\sin(\cot^{-1} x)\right\}\right]$$

$$= \cos\left\{\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right\} = \cos\left\{\cos^{-1}\frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}\right\} = \sqrt{\frac{1+x^2}{2+x^2}} = \sqrt{\frac{x^2+1}{x^2+2}}$$

EXAMPLE 10 *If* $\sin\left\{\cot^{-1}(x+1)\right\} = \cos(\tan^{-1}x)$, *then find* x. **[CBSE 2015]**

SOLUTION We have,

$$\sin\left\{\cot^{-1}(x+1)\right\} = \cos(\tan^{-1}x)$$

$$\Rightarrow \sin\left\{\sin^{-1}\frac{1}{\sqrt{x^2+2x+2}}\right\} = \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{\sqrt{1+x^2}} \Rightarrow \sqrt{x^2+2x+2} = \sqrt{1+x^2} \Rightarrow x^2+2x+2 = x^2+1 \Rightarrow 2x = -1 \Rightarrow x = -\frac{1}{2}$$

Hence, $x = -\frac{1}{2}$ is a root of the given equation.

EXAMPLE 11 *Solve the following equation for* x *:*

(i) $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

(ii) $\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right)$ **[CBSE 2013, 2014, 2017, NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right) \Rightarrow \cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \sin\left(\sin^{-1}\frac{4}{5}\right)$$

$$\Rightarrow \frac{1}{\sqrt{1+x^2}} = \frac{4}{5} \Rightarrow 4\sqrt{1+x^2} = 5 \Rightarrow 16(1+x^2) = 25 \Rightarrow 16x^2 = 9 \Rightarrow x = \pm\frac{3}{4}$$

(ii) We have,

$$\tan(\cos^{-1}x) = \sin\left(\cot^{-1}\frac{1}{2}\right) \Rightarrow \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$$

$$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}} \Rightarrow \frac{1-x^2}{x^2} = \frac{4}{5} \Rightarrow 4x^2 = 5-5x^2 \Rightarrow 9x^2 = 5 \Rightarrow x = \pm\frac{\sqrt{5}}{3}$$

But, for $x = -\frac{\sqrt{5}}{3}$ we find that LHS is negative where as RHS is positive. Hence, $x = \frac{\sqrt{5}}{3}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *If* $x = \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right\}\right]$

and, $y = \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right\}\right]$

where $a \in [0, 1]$. *Find the relationship between* x *and* y *in terms of* a.

SOLUTION We have,

$$x = \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right\}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sec^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \quad \left[\because \sin^{-1}a = \sec^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cos^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \qquad \left[\because \cot^{-1}\frac{1}{\sqrt{1-a^2}} = \cos^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \operatorname{cosec}\left(\tan^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \operatorname{cosec}\left(\operatorname{cosec}^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

and, $$y = \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right\}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right)\right)\right\}\right] \qquad \left[\because \cos^{-1}a = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-a^2}}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\frac{1}{\sqrt{1-a^2}}\right)\right\}\right]$$

$$= \sec\left[\cot^{-1}\left\{\sin\left(\sin^{-1}\frac{1}{\sqrt{2-a^2}}\right)\right\}\right] \qquad \left[\because \tan^{-1}\frac{1}{\sqrt{1-a^2}} = \sin^{-1}\frac{1}{\sqrt{2-a^2}}\right]$$

$$= \sec\left(\cot^{-1}\frac{1}{\sqrt{2-a^2}}\right) = \sec\left(\sec^{-1}\sqrt{3-a^2}\right) = \sqrt{3-a^2}$$

Thus, we obtain: $x = y = \sqrt{3-a^2} \Rightarrow x^2 = y^2 = 3-a^2$.

EXAMPLE 13 *If* $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$, *prove that*

(i) $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$ (ii) $x^4 + y^4 + z^4 + 4x^2y^2z^2 = 2(x^2y^2 + y^2z^2 + z^2x^2)$

SOLUTION (i) Let $\sin^{-1}x = A$, $\sin^{-1}y = B$ and $\sin^{-1}z = C$. Then, $x = \sin A$, $y = \sin B$ and $z = \sin C$

Now,

$$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi \Rightarrow A + B + C = \pi$$

$$\Rightarrow \quad \sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C$$

$$\Rightarrow \quad 2\sin A\cos A + 2\sin B\cos B + 2\sin C\cos C = 4\sin A\sin B\sin C$$

$$\Rightarrow \quad \sin A\sqrt{1-\sin^2 A} + \sin B\sqrt{1-\sin^2 B} + \sin C\sqrt{1-\sin^2 C} = 2\sin A\sin B\sin C$$

$$\Rightarrow \quad x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

(ii) We have,

$$\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$$

$$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}z$$

$$\Rightarrow \quad \cos\left(\sin^{-1}x + \sin^{-1}y\right) = \cos\left(\pi - \sin^{-1}z\right)$$

$$\Rightarrow \cos(\sin^{-1}x)\cos(\sin^{-1}y)-\sin(\sin^{-1}x)\sin(\sin^{-1}y) = -\cos(\sin^{-1}z)$$

$$\Rightarrow \cos\left(\cos^{-1}\sqrt{1-x^2}\right)\cos\left(\cos^{-1}\sqrt{1-y^2}\right)-\sin\left(\sin^{-1}x\right)\sin\left(\sin^{-1}y\right)=-\cos\left(\cos^{-1}\sqrt{1-z^2}\right)$$

$$\Rightarrow \sqrt{1-x^2}\sqrt{1-y^2}-xy = -\sqrt{1-z^2} \Rightarrow \sqrt{(1-x^2)(1-y^2)} = xy-\sqrt{1-z^2}$$

$$\Rightarrow (1-x^2)(1-y^2)=\left(xy-\sqrt{1-z^2}\right)^2 \qquad \text{[On squaring both sides]}$$

$$\Rightarrow 1-x^2-y^2+x^2y^2 = x^2y^2+1-z^2-2xy\sqrt{1-z^2}$$

$$\Rightarrow x^2+y^2-z^2 = 2xy\sqrt{1-z^2}$$

$$\Rightarrow (x^2+y^2-z^2)^2 = 4x^2y^2(1-z^2)$$

$$\Rightarrow x^4+y^4+z^4-2x^2z^2-2y^2z^2+2x^2y^2 = 4x^2y^2-4x^2y^2z^2$$

$$\Rightarrow x^4+y^4+z^4+4x^2y^2z^2=2(x^2y^2+y^2z^2+z^2x^2)$$

EXAMPLE 14 *If* $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\alpha$, *then prove that* $x^2=\sin 2\alpha$.

SOLUTION We have,

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha \Rightarrow \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \tan\alpha$$

$$\Rightarrow \frac{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)+\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2}-\sqrt{1-x^2}\right)-\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)}=\frac{\tan\alpha+1}{\tan\alpha-1}$$

$$\Rightarrow \frac{2\sqrt{1+x^2}}{-2\sqrt{1-x^2}} = \frac{\tan\alpha+1}{\tan\alpha-1} \Rightarrow \frac{\sqrt{1-x^2}}{\sqrt{1+x^2}}=\frac{1-\tan\alpha}{1+\tan\alpha} \Rightarrow \sqrt{\frac{1-x^2}{1+x^2}} = \frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}$$

$$\Rightarrow \frac{1-x^2}{1+x^2}=\left(\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}\right)^2 \Rightarrow \frac{1-x^2}{1+x^2} = \frac{1-\sin 2\alpha}{1+\sin 2\alpha} \Rightarrow x^2 = \sin 2\alpha$$

EXAMPLE 15 *Prove that:* $\tan\left\{\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right\}+\tan\left\{\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}\dfrac{a}{b}\right\} = \dfrac{2b}{a}$ **[CBSE 2017]**

SOLUTION Let $\cos^{-1}\left(\dfrac{a}{b}\right) = \theta$. Then, $\cos\theta = \dfrac{a}{b}$

$$\therefore \quad \text{LHS} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \frac{1+\tan\theta/2}{1-\tan\theta/2}+\frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$= \frac{(1+\tan\theta/2)^2+(1-\tan\theta/2)^2}{1-\tan^2\theta/2} = 2\left(\frac{1+\tan^2\theta/2}{1-\tan^2\theta/2}\right) = \frac{2}{\cos\theta} = \frac{2b}{a} = \text{RHS}.$$

EXERCISE 3.8

BASIC

1. Evaluate each of the following:

(i) $\sin\left(\sin^{-1}\dfrac{7}{25}\right)$ (ii) $\sin\left(\cos^{-1}\dfrac{5}{13}\right)$ (iii) $\sin\left(\tan^{-1}\dfrac{24}{7}\right)$

(iv) $\sin\left(\sec^{-1}\frac{17}{8}\right)$ (v) $\operatorname{cosec}\left(\cos^{-1}\frac{3}{5}\right)$ (vi) $\sec\left(\sin^{-1}\frac{12}{13}\right)$

(vii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$ (viii) $\cot\left(\cos^{-1}\frac{3}{5}\right)$ (ix) $\cos\left(\tan^{-1}\frac{24}{7}\right)$

2. Prove the following results:

(i) $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)=\frac{17}{6}$ (ii) $\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$ **[CBSE 2012]**

(iii) $\tan\left(\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\right)=\frac{63}{16}$ (iv) $\sin\left(\cos^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)=\frac{63}{65}$

BASED ON LOTS

3. Solve : $\cos\left(\sin^{-1}x\right)=\frac{1}{6}$ 4. Solve : $\cos\left\{2\sin^{-1}(-x)\right\}=0$

ANSWERS

1. (i) $\frac{7}{25}$ (ii) $\frac{12}{13}$ (iii) $\frac{24}{25}$ (iv) $\frac{15}{17}$
(v) $\frac{5}{4}$ (vi) $\frac{13}{5}$ (vii) $\frac{15}{8}$ (viii) $\frac{3}{4}$
(ix) $\frac{7}{25}$ 3. $\pm\frac{\sqrt{35}}{6}$ 4. $\pm\frac{1}{\sqrt{2}}$

HINTS TO SELECTED PROBLEMS

2. (i) $\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$

$$=\frac{\tan\left(\cos^{-1}\frac{4}{5}\right)+\tan\left(\tan^{-1}\frac{2}{3}\right)}{1-\tan\left(\cos^{-1}\frac{4}{5}\right)\tan\left(\tan^{-1}\frac{2}{3}\right)}=\frac{\tan\left(\tan^{-1}\frac{3}{4}\right)+\tan\left(\tan^{-1}\frac{2}{3}\right)}{1-\tan\left(\tan^{-1}\frac{3}{4}\right)\tan\left(\tan^{-1}\frac{2}{3}\right)}=\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}=\frac{17}{6}$$

(ii) $$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\cos\left(\sin^{-1}\frac{3}{5}\right)\cos\left(\cot^{-1}\frac{3}{2}\right)-\sin\left(\sin^{-1}\frac{3}{5}\right)\sin\left(\cot^{-1}\frac{3}{2}\right)$$

$$=\cos\left(\cos^{-1}\frac{4}{5}\right)\cos\left(\cos^{-1}\frac{3}{\sqrt{13}}\right)-\sin\left(\sin^{-1}\frac{3}{5}\right)\sin\left(\sin^{-1}\frac{2}{\sqrt{13}}\right)$$

$$=\frac{4}{5}\times\frac{3}{\sqrt{13}}-\frac{3}{5}\times\frac{2}{\sqrt{13}}=\frac{6}{5\sqrt{13}}$$

3.3.3 PROPERTIES III & IV

PROPERTY-III *Prove that:*

(i) $\sin^{-1}(-x)=-\sin^{-1}x$, for all $x\in[-1,1]$ (ii) $\cos^{-1}(-x)=\pi-\cos^{-1}x$, for all $x\in[-1,1]$

(iii) $\tan^{-1}(-x)=-\tan^{-1}x$, for all $x\in R$ (iv) $\cot^{-1}(-x)=\pi-\cot^{-1}x$, for all $x\in R$

(v) $\operatorname{cosec}^{-1}(-x)=-\operatorname{cosec}^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$

(vi) $\sec^{-1}(-x)=\pi-\sec^{-1}x$, for all $x\in(-\infty,-1]\cup[1,\infty)$

PROOF (i) Clearly, $-x\in[-1,1]$ for all $x\in[-1,1]$. Let $\sin^{-1}(-x)=\theta$ ...(i)

Then, $-x=\sin\theta \Rightarrow x=-\sin\theta \Rightarrow x=\sin(-\theta)$

$\Rightarrow$ $-\theta=\sin^{-1}x$ [$\because$ x $[-1,1]$ and $-\theta\in[-\pi/2,\pi/2]$ for all $\theta\in[-\pi/2,\pi/2]$]

$\Rightarrow \quad \theta = -\sin^{-1} x$ ...(ii)

From (i) and (ii), we get: $\sin^{-1}(-x) = -\sin^{-1} x$

(ii) Clearly, $-x \in [-1, 1]$ for all $x \in [-1, 1]$. Let $\cos^{-1}(-x) = \theta$. ...(i)

Then, $-x = \cos\theta \Rightarrow x = -\cos\theta \Rightarrow x = \cos(\pi - \theta)$

$\Rightarrow \quad \cos^{-1} x = \pi - \theta$ $[\because x \in [-1, 1]$ and $\pi - \theta \in [0, \pi]$ for all $\theta \in [0, \pi]]$

$\Rightarrow \quad \theta = \pi - \cos^{-1} x$ ...(ii)

From (i) and (ii), we get: $\cos^{-1}(-x) = \pi - \cos^{-1} x$

Similarly, other results can be proved.

PROPERTY IV *Prove that:*

(i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

PROOF (i) Let $\operatorname{cosec}^{-1} x = \theta$ ...(i)

Then,

$x = \operatorname{cosec}\theta \Rightarrow \frac{1}{x} = \sin\theta \Rightarrow \theta = \sin^{-1}\frac{1}{x}$ ...(ii) $\left[\begin{array}{l}\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow 1/x \ [-1,1]-\{0\} \\ \text{and } \operatorname{cosec}^{-1} x = \theta \Rightarrow \theta \in [-\pi/2, \pi/2]-\{0\}\end{array}\right]$

From (i) and (ii), we get: $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x$

(ii) Let $\sec^{-1} x = \theta$ ...(i)

Then, $x \in (-\infty, -1] \cup [1, \infty)$ and $\theta \in [0, \pi] - \{\pi/2\}$.

Now, $\sec^{-1} x = \theta \Rightarrow x = \sec\theta \Rightarrow \frac{1}{x} = \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1}\frac{1}{x}$ $\left[\because x \in (-\infty, -1] \cup [1, \infty) \Rightarrow \frac{1}{x} \in [-1, 1] - \{0\} \text{ and } \theta \in [0, \pi]\right]$ ...(ii)

From (i) and (ii), we get: $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x$

(iii) Let $\cot^{-1} x = \theta$. ...(i)

Then, $x \in R$, $x \neq 0$ and $\theta \in (0, \pi)$. Now, two cases arise:

Case I *When* $x > 0$: In this case, $\theta \in (0, \pi/2)$

$\therefore \quad \cot^{-1} x = \theta \Rightarrow x = \cot\theta \Rightarrow \frac{1}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right)$ $[\because \theta \in (0, \pi/2)]$ ...(ii)

From (i) and (ii), we get: $\tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1} x$, for all $x > 0$.

Case II *When* $x < 0$: In this case, $\theta \in (\pi/2, \pi)$ $[\because x = \cot\theta < 0]$

Now, $\frac{\pi}{2} < \theta < \pi$

$\Rightarrow \quad -\frac{\pi}{2} < \theta - \pi < 0 \Rightarrow \theta - \pi \in (-\pi/2, 0)$

$\therefore \quad \cot^{-1} x = \theta \Rightarrow x = \cot\theta \Rightarrow \dfrac{1}{x} = \tan\theta \Rightarrow \dfrac{1}{x} = -\tan(\pi - \theta) \quad [\because \tan(\pi-\theta) = -\tan\theta]$

$\Rightarrow \quad \dfrac{1}{x} = \tan(\theta - \pi) \Rightarrow \theta - \pi = \tan^{-1}\left(\dfrac{1}{x}\right) \quad [\because \theta - \pi \in (-\pi/2, 0)]$

$\Rightarrow \quad \tan^{-1}\left(\dfrac{1}{x}\right) = -\pi + \theta \quad ...(iii)$

From (i) and (iii), we get: $\tan^{-1}\left(\dfrac{1}{x}\right) = -\pi + \cot^{-1} x$, if $x < 0$.

Hence, $\tan^{-1}\left(\dfrac{1}{x}\right) = \begin{cases} \cot^{-1} x & \text{, for } x > 0 \\ -\pi + \cot^{-1} x & \text{, for } x < 0 \end{cases}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\cos\left\{\sin^{-1}\left(-\dfrac{5}{13}\right)\right\}$ (ii) $\cot\left\{\sin^{-1}\left(-\dfrac{7}{25}\right)\right\}$ (iii) $\sec\left\{\sin^{-1}\left(-\dfrac{8}{17}\right)\right\}$

SOLUTION We know that $\sin^{-1}(-x) = -\sin^{-1} x$ for all $x \in [-1, 1]$. Therefore,

(i) $\cos\left\{\sin^{-1}\left(-\dfrac{5}{13}\right)\right\} = \cos\left(-\sin^{-1}\dfrac{5}{13}\right) = \cos\left(\sin^{-1}\dfrac{5}{13}\right) = \cos\left(\cos^{-1}\dfrac{12}{13}\right) = \dfrac{12}{13}$

(ii) $\cot\left\{\sin^{-1}\left(-\dfrac{7}{25}\right)\right\} = \cot\left(-\sin^{-1}\dfrac{7}{25}\right) = -\cot\left(\sin^{-1}\dfrac{7}{25}\right) = -\cot\left(\cot^{-1}\dfrac{24}{7}\right) = -\dfrac{24}{7}$

(iii) $\sec\left\{\sin^{-1}\left(-\dfrac{8}{17}\right)\right\} = \sec\left(-\sin^{-1}\dfrac{8}{17}\right) = \sec\left(\sin^{-1}\dfrac{8}{17}\right) = \sec\left(\sec^{-1}\dfrac{17}{15}\right) = \dfrac{17}{15}$

EXAMPLE 2 *Evaluate:*

(i) $\sin\left\{\cos^{-1}\left(-\dfrac{3}{5}\right)\right\}$ (ii) $\tan\left\{\cos^{-1}\left(-\dfrac{12}{13}\right)\right\}$ (iii) $\operatorname{cosec}\left\{\cos^{-1}\left(-\dfrac{12}{13}\right)\right\}$

SOLUTION We know that $\cos^{-1}(-x) = \pi - \cos^{-1} x$ for all $x \in [-1, 1]$. Therefore,

(i) $\sin\left\{\cos^{-1}\left(-\dfrac{3}{5}\right)\right\} = \sin\left\{\pi - \cos^{-1}\dfrac{3}{5}\right\} = \sin\left(\cos^{-1}\dfrac{3}{5}\right) = \sin\left(\sin^{-1}\dfrac{4}{5}\right) = \dfrac{4}{5}$

(ii) $\tan\left\{\cos^{-1}\left(-\dfrac{12}{13}\right)\right\} = \tan\left\{\pi - \cos^{-1}\dfrac{12}{13}\right\} = -\tan\left(\cos^{-1}\dfrac{12}{13}\right) = -\tan\left(\tan^{-1}\dfrac{5}{12}\right) = -\dfrac{5}{12}$

(iii) $\operatorname{cosec}\left\{\cos^{-1}\left(-\dfrac{12}{13}\right)\right\} = \operatorname{cosec}\left\{\pi - \cos^{-1}\dfrac{12}{13}\right\} = \operatorname{cosec}\left(\cot^{-1}\dfrac{12}{13}\right) = \operatorname{cosec}\left(\operatorname{cosec}^{-1}\dfrac{13}{5}\right) = \dfrac{13}{5}$

EXAMPLE 3 *Evaluate:*

(i) $\sin\left\{\tan^{-1}\left(-\dfrac{7}{24}\right)\right\}$ (ii) $\cos\left\{\cot^{-1}\left(-\dfrac{5}{12}\right)\right\}$ (iii) $\operatorname{cosec}\left\{\cot^{-1}\left(-\dfrac{4}{3}\right)\right\}$

SOLUTION We know that $\tan^{-1}(-x) = -\tan^{-1} x$ and $\cot^{-1}(-x) = \pi - \cot^{-1} x$ for all $x \in R$. Therefore,

(i) $\sin\left\{\tan^{-1}\left(-\dfrac{7}{24}\right)\right\} = \sin\left(-\tan^{-1}\dfrac{7}{24}\right) = -\sin\left(\tan^{-1}\dfrac{7}{24}\right) = -\sin\left(\sin^{-1}\dfrac{7}{25}\right) = -\dfrac{7}{25}$

(ii) $\cos\left\{\cot^{-1}\left(-\frac{5}{12}\right)\right\} = \cos\left(\pi - \cot^{-1}\frac{5}{12}\right) = -\cos\left(\cot^{-1}\frac{5}{12}\right) = -\cos\left(\cos^{-1}\frac{5}{13}\right) = -\frac{5}{13}$

(iii) $\operatorname{cosec}\left\{\cot^{-1}\left(-\frac{4}{3}\right)\right\} = \operatorname{cosec}\left(\pi - \cot^{-1}\frac{4}{3}\right) = \operatorname{cosec}\left(\cot^{-1}\frac{4}{3}\right) = \operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{5}{3}\right) = \frac{5}{3}$

EXAMPLE 4 *Prove that :* $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

SOLUTION We find that :

$$\sin^{-1}\left(-\frac{4}{5}\right) = -\sin^{-1}\left(\frac{4}{5}\right) = -\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(-\frac{4}{3}\right)$$

and, $\cos^{-1}\left(-\frac{3}{5}\right) - \pi = \left(\pi - \cos^{-1}\frac{3}{5}\right) - \pi = -\cos^{-1}\frac{3}{5} = -\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(-\frac{4}{3}\right)$

Hence, $\sin^{-1}\left(-\frac{4}{5}\right) = \tan^{-1}\left(-\frac{4}{3}\right) = \cos^{-1}\left(-\frac{3}{5}\right) - \pi$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Prove that* $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$

SOLUTION We have,

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$$

$$\therefore \quad \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \tan^{-1} x + \cot^{-1} x = \pi/2, & \text{if } x > 0 \\ \tan^{-1} x + \cot^{-1} x - \pi = \pi/2 - \pi = -\pi/2, & \text{if } x < 0 \end{cases}$$

EXERCISE 3.9

BASIC

1. Evaluate :

   (i) $\cos\left\{\sin^{-1}\left(-\frac{7}{25}\right)\right\}$ (ii) $\sec\left\{\cot^{-1}\left(-\frac{5}{12}\right)\right\}$ (iii) $\cot\left\{\sec^{-1}\left(-\frac{13}{5}\right)\right\}$

2. Evaluate :

   (i) $\tan\left\{\cos^{-1}\left(-\frac{7}{25}\right)\right\}$ (ii) $\operatorname{cosec}\left\{\cot^{-1}\left(-\frac{12}{5}\right)\right\}$ (iii) $\cos\left\{\tan^{-1}\left(-\frac{3}{4}\right)\right\}$

3. Evaluate : $\sin\left\{\cos^{-1}\left(-\frac{3}{5}\right) + \cot^{-1}\left(-\frac{5}{12}\right)\right\}$.

ANSWERS

1. (i) $\frac{24}{25}$ (ii) $-\frac{13}{5}$ (iii) $-\frac{5}{12}$ **2.** (i) $-\frac{24}{7}$ (ii) $\frac{13}{5}$ (iii) $\frac{4}{5}$ **3.** $-\frac{56}{65}$

3.3.4 PROPERTY V

PROPERTY V *Prove that:*

(i) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, *for all* $x \in [-1, 1]$ (ii) $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$, *for all* $x \in R$

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$, *for all* $x \in (-\infty, -1] \cup [1, \infty)$.

PROOF (i) Let $\sin^{-1} x = \theta$ ...(i)

Then,

$\theta \in [-\pi/2, \pi/2] \Rightarrow -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leq -\theta \leq \frac{\pi}{2} \Rightarrow 0 \leq \frac{\pi}{2} - \theta \leq \pi \Rightarrow \frac{\pi}{2} - \theta \in [0, \pi]$ $[\because x \in [-1, 1]]$

Now, $\sin^{-1} x = \theta$

$\Rightarrow \quad x = \sin\theta \Rightarrow x = \cos\left(\frac{\pi}{2} - \theta\right) \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \theta$ $[\because x\ [-1, 1] \text{ and } (\pi/2 - \theta) \in [0, \pi]]$

$\Rightarrow \quad \theta + \cos^{-1} x = \frac{\pi}{2}$ ...(ii)

From (i) and (ii), we get: $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(ii) Let $\tan^{-1} x = \theta$ ...(i)

Then,

$\theta \in (-\pi/2, \pi/2) \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \pi \Rightarrow \left(\frac{\pi}{2} - \theta\right) \in (0, \pi)$ $[\because x \in R]$

Now, $\tan^{-1} x = \theta \Rightarrow x = \tan\theta \Rightarrow x = \cot\left(\frac{\pi}{2} - \theta\right) \Rightarrow \cot^{-1} x = \frac{\pi}{2} - \theta$ $\left[\because \frac{\pi}{2} - \theta \in (0, \pi)\right]$

$\Rightarrow \quad \theta + \cot^{-1} x = \frac{\pi}{2}$ ...(ii)

From (i) and (ii), we get: $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

(iii) Let $\sec^{-1} x = \theta$ ...(i)

Then, $\theta \in [0, \pi] - \{\pi/2\}$ $[\because x \in (-\infty, -1] \cup [1, \infty)]$

$\Rightarrow \quad 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2} \Rightarrow -\pi \leq -\theta \leq 0, \theta \neq \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \leq \frac{\pi}{2} - \theta \leq \frac{\pi}{2}, \frac{\pi}{2} - \theta \neq 0$

$\Rightarrow \quad \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\frac{\pi}{2} - \theta \neq 0$.

Now, $\sec^{-1} x = \theta$

$\Rightarrow \quad x = \sec\theta$

$\Rightarrow \quad x = \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) \Rightarrow \operatorname{cosec}^{-1} x = \frac{\pi}{2} - \theta$ $\left[\because \left(\frac{\pi}{2} - \theta\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and, } \frac{\pi}{2} - \theta \neq 0\right]$

$\Rightarrow \quad \theta + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$ ...(ii)

From (i) and (ii), we get: $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the value of* $\cot(\tan^{-1} a + \cot^{-1} a)$. **[CBSE 2012, NCERT]**

SOLUTION We know that $\tan^{-1} a + \cot^{-1} a = \frac{\pi}{2}$.

$\therefore \quad \cot(\tan^{-1} a + \cot^{-1} a) = \cot\frac{\pi}{2} = 0$

EXAMPLE 2 *If* $-1 \leq x, y \leq 1$ *such that* $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, *find the value of* $\cos^{-1} x + \cos^{-1} y$.

[NCERT EXEMPLAR]

SOLUTION We have,

$$\sin^{-1}x+\sin^{-1}y=\frac{\pi}{2}$$

$\Rightarrow \quad \left(\frac{\pi}{2}-\cos^{-1}x\right)+\left(\frac{\pi}{2}-\cos^{-1}y\right)=\frac{\pi}{2} \qquad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2} \text{ and } \sin^{-1}y+\cos^{-1}y=\frac{\pi}{2}\right]$

$\Rightarrow \quad \pi-(\cos^{-1}x+\cos^{-1}y)=\frac{\pi}{2} \Rightarrow \cos^{-1}x+\cos^{-1}y=\frac{\pi}{2}$

EXAMPLE 3 *If* $\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$, *find* $\cot^{-1}x+\cot^{-1}y$. [NCERT EXEMPLAR]

SOLUTION We have,

$$\tan^{-1}x+\tan^{-1}y=\frac{4\pi}{5}$$

$\Rightarrow \quad \left(\frac{\pi}{2}-\cot^{-1}x\right)+\left(\frac{\pi}{2}-\cot^{-1}y\right)=\frac{4\pi}{5} \qquad \left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}, \tan^{-1}y+\cot^{-1}y=\frac{\pi}{2}\right]$

$\Rightarrow \quad \pi-(\cot^{-1}x+\cot^{-1}y)=\frac{4\pi}{5} \Rightarrow \cot^{-1}x+\cot^{-1}y=\pi-\frac{4\pi}{5} \Rightarrow \cot^{-1}x+\cot^{-1}y=\frac{\pi}{5}$

EXAMPLE 4 *If* $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\frac{1}{\sqrt{3}}$, *find the value of x.* [NCERT EXEMPLAR]

SOLUTION We have,

$$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow \tan^{-1}x-\cot^{-1}x=\frac{\pi}{6} \qquad \ldots(i)$$

We know that: $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ ...(ii)

Adding (i) and (ii), we obtain

$$2\tan^{-1}x=\frac{2\pi}{3} \Rightarrow \tan^{-1}x=\frac{\pi}{3} \Rightarrow x=\tan\frac{\pi}{3}=\sqrt{3}$$

EXAMPLE 5 *If* $\sin\left(\cos^{-1}\frac{5}{13}+\sin^{-1}x\right)=1$, *find the value of x.*

SOLUTION We have,

$$\sin\left(\cos^{-1}\frac{5}{13}+\sin^{-1}x\right)=1$$

$\Rightarrow \quad \cos^{-1}\frac{5}{13}+\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow \quad \sin^{-1}x=\frac{\pi}{2}-\cos^{-1}\frac{5}{13} \Rightarrow \sin^{-1}x=\sin^{-1}\frac{5}{13} \Rightarrow x=\frac{5}{13} \qquad \left[\because \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{5}{13}=\frac{\pi}{2}\right]$

EXAMPLE 6 *If* $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, *then find the value of x.* [NCERT, CBSE 2014]

SOLUTION We have, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$

$\Rightarrow \quad \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$

$\Rightarrow \quad \cos^{-1}x=\cos^{-1}\frac{1}{5} \Rightarrow x=\frac{1}{5} \qquad \left[\because \sin^{-1}\frac{1}{5}+\cos^{-1}\frac{1}{5}=\frac{\pi}{2}\right]$

EXAMPLE 7 *If* $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$, *find the value of x.* [NCERT EXEMPLAR]

SOLUTION We have, $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$

$\Rightarrow \quad \sin^{-1}\frac{2}{5}+\cos^{-1}x=\frac{\pi}{2} \Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{2}{5}$

$\Rightarrow \quad \cos^{-1}x=\cos^{-1}\frac{2}{5} \Rightarrow x=\frac{2}{5}$ $\quad\left[\because \sin^{-1}\frac{2}{5}+\cos^{-1}\frac{2}{5}=\frac{\pi}{2}\right]$

EXAMPLE 8 *Evaluate:* $\cos(2\cos^{-1}x+\sin^{-1}x)$ *at* $x=\frac{1}{5}$.

SOLUTION We have,

$$\cos(2\cos^{-1}x+\sin^{-1}x)$$

$$=\cos(\cos^{-1}x+\cos^{-1}x+\sin^{-1}x)=\cos\left(\cos^{-1}x+\frac{\pi}{2}\right) \quad \left[\because \cos^{-1}x+\sin^{-1}x=\frac{\pi}{2}\right]$$

$$=-\sin(\cos^{-1}x)=-\sin\left(\sin^{-1}\sqrt{1-x^2}\right)=-\sqrt{1-x^2}$$

Putting $x=\frac{1}{5}$, we obtain: $\cos\left(2\cos^{-1}x+\sin^{-1}x\right)=-\sqrt{1-\frac{1}{25}}=-\sqrt{\frac{24}{25}}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *If* $(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$, *then find* x. **[CBSE 2015]**

SOLUTION We have,

$$(\tan^{-1}x)^2+(\cot^{-1}x)^2=\frac{5\pi^2}{8}$$

$$\Rightarrow (\tan^{-1}x)^2+(\cot^{-1}x)^2+2\tan^{-1}x\cot^{-1}x-2\tan^{-1}x\cot^{-1}x=\frac{5\pi^2}{8}$$

$$\Rightarrow (\tan^{-1}x+\cot^{-1}x)^2-2\tan^{-1}x\cot^{-1}x=\frac{5\pi^2}{8}$$

$$\Rightarrow \frac{\pi^2}{4}-2\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)=\frac{5\pi^2}{8} \quad \left[\because \tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}\right]$$

$$\Rightarrow \frac{\pi^2}{4}-\pi\tan^{-1}x+2(\tan^{-1}x)^2=\frac{5\pi^2}{8} \Rightarrow 2(\tan^{-1}x)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0$$

$$\Rightarrow 16(\tan^{-1}x)^2-8\pi(\tan^{-1}x)-3\pi^2=0 \Rightarrow 16(\tan^{-1}x)^2-12\pi\tan^{-1}x+4\pi\tan^{-1}x-3\pi^2=0$$

$$\Rightarrow 4\tan^{-1}x(4\tan^{-1}x-3\pi)+\pi(4\tan^{-1}x-3\pi)=0 \Rightarrow (4\tan^{-1}x-3\pi)(4\tan^{-1}x+\pi)=0$$

$$\Rightarrow 16\left(\tan^{-1}x-\frac{3\pi}{4}\right)\left(\tan^{-1}x+\frac{\pi}{4}\right)=0 \Rightarrow \tan^{-1}x+\frac{\pi}{4}=0 \left[\because -\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2} \therefore \tan^{-1}x-\frac{3\pi}{4}\neq 0\right]$$

$$\Rightarrow \tan^{-1}x=-\frac{\pi}{4} \Rightarrow x=\tan\left(-\frac{\pi}{4}\right) \Rightarrow x=-1.$$

EXAMPLE 10 *Prove that* $\tan(\cot^{-1}x)=\cot(\tan^{-1}x)$. *State with the reason weather the equality is valid for all values of* x. **[NCERT EXEMPLAR]**

SOLUTION We know that $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ or, $\cot^{-1}x=\frac{\pi}{2}-\tan^{-1}x$ for all $x\in R$.

$$\therefore \quad \tan(\cot^{-1}x)=\tan\left(\frac{\pi}{2}-\tan^{-1}x\right)=\cot(\tan^{-1}x) \text{ for all } x\in R$$

Clearly, the equality holds for all $x\in R$ as $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$ hold for all $x\in R$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *Find the greatest and least values of* $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$. **[NCERT EXEMPLAR]**

SOLUTION $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$

$$= \left\{\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2 + 2\sin^{-1} x \cos^{-1} x\right\} - 2\sin^{-1} x \cos^{-1} x$$

$$= \left(\sin^{-1} x + \cos^{-1} x\right)^2 - 2\sin^{-1} x \cos^{-1} x$$

$$= \frac{\pi^2}{4} - 2\sin^{-1} x\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$= \frac{\pi^2}{4} - \pi \sin^{-1} x + 2\left(\sin^{-1} x\right)^2 = 2\left\{\left(\sin^{-1} x\right)^2 - \frac{\pi}{2}\sin^{-1} x + \frac{\pi^2}{8}\right\}$$

$$= 2\left\{\left(\sin^{-1} x\right)^2 - 2\left(\frac{\pi}{4}\right)\sin^{-1} x + \frac{\pi^2}{16} - \frac{\pi^2}{16} + \frac{\pi^2}{8}\right\} = 2\left\{\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right\}$$

Now,

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \text{ for all } x \in [-1, 1] \Rightarrow -\frac{\pi}{2} - \frac{\pi}{4} \le \sin^{-1} x - \frac{\pi}{4} \le \frac{\pi}{2} - \frac{\pi}{4} \text{ for all } x \in [-1, 1]$$

$$\Rightarrow \quad -\frac{3\pi}{4} \le \left(\sin^{-1} x - \frac{\pi}{4}\right) \le \frac{\pi}{4} \text{ for all } x \in [-1, 1] \Rightarrow 0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16}$$

$$\Rightarrow \quad \frac{\pi^2}{16} \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16} \le \frac{9\pi^2}{16} + \frac{\pi^2}{16} \Rightarrow \frac{\pi^2}{16} \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16} \le \frac{5\pi^2}{8}$$

$$\Rightarrow \quad \frac{\pi^2}{8} \le 2\left\{\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{16}\right\} \le \frac{5\pi^2}{4} \Rightarrow \frac{\pi^2}{8} \le (\sin^{-1} x)^2 + (\cos^{-1} x)^2 \le \frac{5\pi^2}{4}$$

Hence, the greatest and the least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ are $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$ respectively.

EXAMPLE 12 *Find the maximum and minimum values of* $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$, *where* $-1 \le x \le 1$.

SOLUTION Let $y = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$. Then,

$$y = (\sin^{-1} x + \cos^{-1} x)^3 - 3\sin^{-1} x \cos^{-1} x (\sin^{-1} x + \cos^{-1} x)$$

$$\Rightarrow \quad y = \left(\frac{\pi}{2}\right)^3 - \frac{3\pi}{2}\sin^{-1} x\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad y = \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1} x + \frac{3\pi}{2}(\sin^{-1} x)^2$$

$$\Rightarrow \quad \frac{3\pi}{2}(\sin^{-1} x)^2 - \frac{3\pi^2}{4}(\sin^{-1} x) + \left(\frac{\pi^3}{8} - y\right) = 0 \Rightarrow (\sin^{-1} x)^2 - \frac{\pi}{2}(\sin^{-1} x) + \frac{2}{3\pi}\left(\frac{\pi^3}{8} - y\right) = 0$$

$$\Rightarrow \quad \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16} + \frac{\pi^2}{12} - \frac{2y}{3\pi} = 0$$

$$\Rightarrow \quad \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{48} - \frac{2y}{3\pi} = 0 \Rightarrow \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{2y}{3\pi} - \frac{\pi^2}{48} \qquad \text{...(i)}$$

We know that

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2} \text{ for all } x \in [-1, 1]$$

$$\Rightarrow \quad -\frac{3\pi}{4} < \sin^{-1} x - \frac{\pi}{4} \le \frac{\pi}{4} \Rightarrow 0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9\pi^2}{16} \qquad \text{...(ii)}$$

From (i) and (ii), we find that

$$0 \le \frac{2y}{3\pi} - \frac{\pi^2}{48} \le \frac{9\pi^2}{16} \Rightarrow \frac{\pi^2}{48} \le \frac{2y}{3\pi} \le \frac{9\pi^2}{16} + \frac{\pi^2}{48} \Rightarrow \frac{\pi^3}{32} \le y \le \frac{7\pi^3}{8}$$

Hence, the maximum and minimum values of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ are $\frac{\pi^3}{32}$ and $\frac{7\pi^3}{8}$.

ALITER $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$

$$= (\sin^{-1} x + \cos^{-1} x)\left\{(\sin^{-1} x)^2 + (\cos^{-1} x)^2 - \sin^{-1} x \cos^{-1} x\right\}$$

$$= \frac{\pi}{2}\left\{\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2 - \sin^{-1} x \cos^{-1} x\right\} \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}\left\{2\left(\sin^{-1} x\right)^2 + 2\left(\cos^{-1} x\right)^2 - 2\sin^{-1} x \cos^{-1} x\right\}$$

$$= \frac{\pi}{4}\left[2\left(\sin^{-1} x\right)^2 + 2\left(\cos^{-1} x\right)^2 - \left\{\left(\sin^{-1} x + \cos^{-1} x\right)^2 - \left(\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2\right)\right\}\right]$$

$$= \frac{\pi}{4}\left[2\left(\sin^{-1} x\right)^2 + 2\left(\cos^{-1} x\right)^2 - \left\{\frac{\pi^2}{4} - \left(\sin^{-1} x\right)^2 - \left(\cos^{-1} x\right)^2\right\}\right]$$

$$= \frac{\pi}{4}\left\{3\left(\sin^{-1} x\right)^2 + 3\left(\cos^{-1} x\right)^2 - \frac{\pi^2}{4}\right\} = \frac{3\pi}{4}\left\{\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2\right\} - \frac{\pi^3}{16} \qquad \text{...(i)}$$

From Example 11, we obtain

$$\frac{\pi^2}{8} \le \left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2 \le \frac{5\pi^2}{4} \Rightarrow \frac{3\pi^3}{32} \le \frac{3\pi}{4}\left\{\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2\right\} \le \frac{15\pi^3}{16}$$

$$\Rightarrow \quad \frac{3\pi^3}{32} - \frac{\pi^3}{16} \le \frac{3\pi}{4}\left\{\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2\right\} - \frac{\pi^3}{16} \le \frac{15\pi^3}{16} - \frac{\pi^3}{16}$$

$$\Rightarrow \quad \frac{\pi^3}{32} \le \left(\sin^{-1} x\right)^3 + \left(\cos^{-1} x\right)^3 \le \frac{7\pi^3}{8}$$

Hence, the maximum and minimum values of $\left(\sin^{-1} x\right)^3 + \left(\cos^{-1} x\right)^3$ are $\frac{7\pi^3}{8}$ and $\frac{\pi^3}{32}$ respectively.

EXERCISE 3.10

BASIC

1. Evaluate:

(i) $\cot\left(\sin^{-1}\frac{3}{4}+\sec^{-1}\frac{4}{3}\right)$ (ii) $\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)$ for $x<0$

(iii) $\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)$ for $x>0$ (iv) $\cot\left(\tan^{-1}a+\cot^{-1}a\right)$ [CBSE 2012]

(v) $\cos\left(\sec^{-1}x+\operatorname{cosec}^{-1}x\right), |x|\geq 1$

2. If $\cos^{-1}x+\cos^{-1}y=\frac{\pi}{4}$, find the value of $\sin^{-1}x+\sin^{-1}y$.

BASED ON LOTS

3. If $\sin^{-1}x+\sin^{-1}y=\frac{\pi}{3}$ and $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{6}$, find the values of x and y.

4. If $\cot\left(\cos^{-1}\frac{3}{5}+\sin^{-1}x\right)=0$, find the values of x.

5. If $(\sin^{-1}x)^2+(\cos^{-1}x)^2=\frac{17\pi^2}{36}$, find x.

6. Solve : $\sin\left\{\sin^{-1}\frac{1}{5}+\cos^{-1}x\right\}=1$ [CBSE 2014, NCERT]

7. Solve the following equations:

(i) $\sin^{-1}x=\frac{\pi}{6}+\cos^{-1}x$ (ii) $4\sin^{-1}x=\pi-\cos^{-1}x$

(iii) $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$ (iv) $5\tan^{-1}x+3\cot^{-1}x=2\pi$

ANSWERS

1. (i) 0 (ii) −1 (iii) 1 (iv) 0 (v) 0

2. $\frac{3\pi}{4}$ **3.** $x=\frac{\sqrt{3}-1}{2\sqrt{2}}, y=\frac{1}{\sqrt{2}}$ **4.** $x=\frac{3}{5}$ **5.** $x=-\frac{1}{2}$

6. $x=\frac{1}{5}$ **7.** (i) $x=\frac{\sqrt{3}}{2}$ (ii) $x=\frac{1}{2}$ (iii) $x=\sqrt{3}$ (iv) 1

HINTS TO SELECTED PROBLEMS

6. We have, $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$

$$\Rightarrow \sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2} \Rightarrow \cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5} \Rightarrow \cos^{-1}x=\cos^{-1}\frac{1}{5} \Rightarrow x=\frac{1}{5}$$

7. (i) We have, $\sin^{-1}x=\frac{\pi}{6}+\cos^{-1}x$

$$\Rightarrow \sin^{-1}x-\cos^{-1}x=\frac{\pi}{6} \Rightarrow \sin^{-1}x-\left(\frac{\pi}{2}-\sin^{-1}x\right)=\frac{\pi}{6} \qquad \left[\because \sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]$$

$$\Rightarrow 2\sin^{-1}x=\frac{2\pi}{3} \Rightarrow \sin^{-1}x=\frac{\pi}{3} \Rightarrow x=\frac{\sqrt{3}}{2}$$

3.3.6 PROPERTY-VI

PROPERTY VI *Prove that:*

(i) $$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

(ii) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

PROOF (i) Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$. Then, $x = \tan A$ and $y = \tan B$ and $A, B \in (-\pi/2, \pi/2)$.

$$\therefore \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{x+y}{1-xy} \quad \text{...(i)}$$

Now, the following cases arise.

Case I *When $x > 0, y > 0$ and $xy < 1$:* In this case, we have

$$x > 0, y > 0 \text{ and } xy < 1 \Rightarrow \frac{x+y}{1-xy} > 0 \Rightarrow \tan(A+B) > 0 \quad \text{[Using (i)]}$$

$\Rightarrow$ $A + B$ lies either in I quadrant or in III quadrant

$$\Rightarrow \quad 0 < A + B < \frac{\pi}{2} \qquad \left[\because \left.\begin{matrix} x > 0 \Rightarrow 0 < A < \dfrac{\pi}{2} \\ y > 0 \Rightarrow 0 < B < \pi/2 \end{matrix}\right\} \Rightarrow 0 < A + B < \pi\right]$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \quad \text{[From (i)]}$$

$$\Rightarrow \quad A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \qquad \left[\because 0 < A + B < \frac{\pi}{2}\right]$$

$$\Rightarrow \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case II *When $x < 0, y < 0$ and $xy < 1$:* In this case, we have

$$x < 0, y < 0 \text{ and } xy < 1 \Rightarrow \frac{x+y}{1-xy} < 0 \Rightarrow \tan(A+B) < 0 \quad \text{[From (i)]}$$

$\Rightarrow$ $A + B$ lies in II quadrant or in IV quadrant.

$$\Rightarrow \quad A + B \text{ lies in IV quadrant} \qquad \left[\because \left.\begin{matrix} x < 0 \Rightarrow -\pi/2 < A < 0 \\ y < 0 \Rightarrow -\pi/2 < B < 0 \end{matrix}\right\} \Rightarrow -\pi < A + B < 0\right]$$

$$\Rightarrow \quad -\frac{\pi}{2} < A + B < 0$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \quad \text{[From (i)]}$$

$$\Rightarrow \quad A+B = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Case III *When $x>0$ and $y<0$ or $x<0$ and $y>0$:* In this case, we have

$$x>0 \text{ and } y<0 \Rightarrow A \in (0, \pi/2) \text{ and } B \in (-\pi/2, 0) \Rightarrow A+B \in (-\pi/2, \pi/2)$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad A+B = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Similarly, if $x<0$ and $y>0$, we obtain: $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

It follows from the above three cases that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$, if $xy<1$.

Case IV *When $x>0$, $y>0$ and $xy>1$:* In this case, we have

$$x>0, y>0 \text{ and } xy>1 \Rightarrow \frac{x+y}{1-xy} < 0 \Rightarrow \tan(A+B) < 0 \qquad \left[\text{From (i)}, \tan(A+B) = \frac{x+y}{1-xy}\right]$$

$\Rightarrow$ $A+B$ lies either in II quadrant or in IV quadrant

$\Rightarrow$ $A+B$ lies in II quadrant $\qquad [\because x>0, y>0 \Rightarrow A, B \in (-\pi/2, 0) \Rightarrow A+B \in (-0, \pi)]$

$$\Rightarrow \quad \frac{\pi}{2} < A+B < \pi \Rightarrow \frac{\pi}{2} - \pi < (A+B) - \pi < 0 \Rightarrow -\frac{\pi}{2} < (A+B) - \pi < 0$$

$$\therefore \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad -\tan\{\pi - (A+B)\} = \frac{x+y}{1-xy} \qquad [\because \tan\{\pi-(A+B)\} = -\tan(A+B)]$$

$$\Rightarrow \quad \tan\{(A+B) - \pi\} = \frac{x+y}{1-xy} \Rightarrow A+B-\pi = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad A+B = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right).$$

Case V *When $x<0$, $y<0$ and $xy>1$:* In this case, we have

$$x<0, y<0 \text{ and } xy>1 \Rightarrow \frac{x+y}{1-xy} > 0 \Rightarrow \tan(A+B) > 0 \qquad \left[\text{From (i)}, \tan(A+B) = \frac{x+y}{1-xy}\right]$$

$\Rightarrow$ $A+B$ lies either in I quadrant or III quadrant

$\Rightarrow$ $A+B$ lies in III quadrant $\qquad [\because x<0, y<0 \Rightarrow A, B \in (-\pi/2, 0) \Rightarrow A+B \in (-\pi, 0)]$

$$\Rightarrow \quad -\pi < A+B < -\frac{\pi}{2} \Rightarrow \pi - \pi < \pi + (A+B) < \pi - \frac{\pi}{2} \Rightarrow 0 < \pi + (A+B) < \frac{\pi}{2}$$

$$\text{Now,} \quad \tan(A+B) = \frac{x+y}{1-xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad \tan(\pi + A + B) = \frac{x+y}{1-xy} \qquad [\because \tan(\pi+\theta) = \tan\theta]$$

$$\Rightarrow \quad \pi + A + B = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$\Rightarrow \quad A+B = -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right) \Rightarrow \tan^{-1}x+\tan^{-1}y = -\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

(ii) Let $\tan^{-1}x = A$ and $\tan^{-1}y = B$. Then, $x=\tan A$, $y=\tan B$ and $A, B \in (-\pi/2, \pi/2)$

$$\therefore \quad \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B} \Rightarrow \tan(A-B) = \frac{x-y}{1+xy} \qquad \text{...(i)}$$

Case I *When $xy>-1$: If $x>0$ and $y>0$, then*

$$A\in(0, \pi/2), B\in(0, \pi/2) \Rightarrow A-B\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore \quad \tan(A-B) = \frac{x-y}{1+xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad A-B = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \Rightarrow \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$\Rightarrow \quad \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ for all x, y with $xy>-1$.

Case II *When $x>0, y<0$ and $xy<-1$*: In this case, we have

$$x>0, y<0 \Rightarrow A\in(0, \pi/2), B\in(-\pi/2, 0) \Rightarrow A\in(0, \pi/2), -B\in(0, \pi/2) \Rightarrow A-B\in(0, \pi)$$

Again, $x>0, y<0$ and $xy<-1 \Rightarrow x>0, -y>0$ and $1+xy<0 \Rightarrow x-y>0$ and $1+xy<0$

$$\Rightarrow \quad \frac{x-y}{1+xy} < 0 \Rightarrow \tan(A-B) < 0 \Rightarrow A-B\in(\pi/2, \pi) \qquad [\because A-B\in(0, \pi)]$$

$$\Rightarrow \quad \frac{\pi}{2}<A-B<\pi \Rightarrow -\frac{\pi}{2} < (A-B)-\pi < 0$$

$$\therefore \quad \tan(A-B) = \frac{x-y}{1+xy} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad -\tan\{\pi-(A-B)\} = \frac{x-y}{1+xy} \Rightarrow \tan\{(A-B)-\pi\} = \frac{x-y}{1+xy} \Rightarrow (A-B)-\pi = \tan^{-1}\frac{x-y}{1+xy}$$

$$\Rightarrow \quad A-B = \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right) \Rightarrow \tan^{-1}x-\tan^{-1}y = \pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Case III *When $x<0, y>0$ and $xy<-1$*: In this case, we have

$$x<0, y>0 \text{ and } xy<-1 \Rightarrow x-y < 0 \text{ and } 1+xy < 0 \Rightarrow \frac{x-y}{1+xy} > 0$$

$$\Rightarrow \quad \tan(A-B) > 0 \qquad \text{[From (i)]}$$

$\Rightarrow \quad (A-B)$ lies either in I quadrant or in III quadrant

$$\Rightarrow \quad -\pi<A-B<-\frac{\pi}{2} \qquad [\because x<0, y>0 \Rightarrow A\in(-\pi/2, 0), B\in(0, \pi/2) \Rightarrow -\pi<A-B<0]$$

$$\Rightarrow \quad 0<\pi+(A-B) < \frac{\pi}{2}$$

$$\therefore \quad \tan(A-B) = \frac{x-y}{1+xy} \Rightarrow \tan\{\pi+(A-B)\} = \frac{x-y}{1+xy} \Rightarrow \pi+A-B = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

$$\Rightarrow \quad A-B = -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right) \Rightarrow \tan^{-1}x-\tan^{-1}y = -\pi+\tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

REMARK *If $x_1, x_2, x_3, \ldots, x_n \in R$, then*

$$\tan^{-1} x_1 + \tan^{-1} x_2 + \ldots + \tan^{-1} x_n = \tan^{-1}\left(\frac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right)$$

where S_k denotes the sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that :* $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

SOLUTION $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24}$

$$= \tan^{-1}\left\{\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right\} \qquad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{If } xy<1\right]$$

$$= \tan^{-1}\left\{\frac{48+77}{264-14}\right\} = \tan^{-1}\left(\frac{125}{250}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

EXAMPLE 2 *Prove that:* $\tan^{-1} 2 + \tan^{-1} 3 = \frac{3\pi}{4}$

SOLUTION $\tan^{-1} 2 + \tan^{-1} 3$

$$= \pi + \tan^{-1}\left\{\frac{2+3}{1-2\times 3}\right\} \qquad \left[\because \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy>1\right]$$

$$= \pi + \tan^{-1}(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

EXAMPLE 3 *Prove that:* $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$ **[CBSE 2010]**

SOLUTION $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \tan^{-1} 1 + (\tan^{-1} 2 + \tan^{-1} 3)$

$$= \frac{\pi}{4} + \frac{3\pi}{4} = \pi \qquad \text{[See Example 2]}$$

EXAMPLE 4 *Prove that:* $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

SOLUTION We have,

$$\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16}$$

$$= \tan^{-1}\frac{12}{5} + \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{63}{16} \qquad \left[\because \sin^{-1}\frac{12}{13} = \tan^{-1}\frac{12}{5} \ \& \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4}\right]$$

$$= \pi + \tan^{-1}\left\{\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right\} + \tan^{-1}\frac{63}{16} \qquad \left[\because \tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy>1\right]$$

$$= \pi + \tan^{-1}\left(\frac{63}{-16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$$

$$= \pi - \tan^{-1}\frac{63}{16} + \tan^{-1}\frac{63}{16} = \pi \qquad [\because \tan^{-1}(-x) = -\tan^{-1} x]$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *If* $\tan^{-1}2+\tan^{-1}3+\theta=\pi$, *find the value of* θ.

SOLUTION We have,

$$\tan^{-1}2+\tan^{-1}3+\theta=\pi \Rightarrow \tan^{-1}2+\tan^{-1}3=\pi-\theta$$

$$\Rightarrow \quad \pi+\tan^{-1}\left(\frac{2+3}{1-2\times 3}\right)=\pi-\theta \Rightarrow \pi+\tan^{-1}(-1)=\pi-\theta \Rightarrow \pi-\frac{\pi}{4}=\pi-\theta \Rightarrow \theta=\frac{\pi}{4}$$

EXAMPLE 6 *Prove that:*

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$ (ii) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

[CBSE 2011, 2013]

(iii) $\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\frac{\pi}{4}$ (iv) $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

[NCERT, CBSE 2008, 2010, 2016]

(v) $\cot^{-1}7+\cot^{-1}8+\cot^{-1}18=\cot^{-1}3$ [NCERT EXEMPLAR, CBSE 2014]

SOLUTION (i) LHS $=\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}$

$$=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right\} \quad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}, \text{ if } xy<1\right]$$

$$=\tan^{-1}\frac{20}{90}=\tan^{-1}\frac{2}{9}=\text{R.H.S.}$$

(ii) $\text{LHS}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\left\{\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}\right\}+\tan^{-1}\frac{1}{8}$

$$=\tan^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\times\frac{1}{5}}\right\}+\tan^{-1}\frac{1}{8} \quad \left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy<1\right]$$

$$=\tan^{-1}\frac{7}{9}+\tan^{-1}\frac{1}{8}=\tan^{-1}\left\{\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}\times\frac{1}{8}}\right\}=\tan^{-1}\left(\frac{65}{65}\right)=\tan^{-1}1=\frac{\pi}{4}=\text{R.H.S.}$$

(iii) $\text{LHS}=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}-\tan^{-1}\frac{8}{19}=\left\{\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{3}{5}\right\}-\tan^{-1}\frac{8}{19}$

$$=\tan^{-1}\left\{\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}\times\frac{3}{5}}\right\}-\tan^{-1}\frac{8}{19}=\tan^{-1}\frac{27}{11}-\tan^{-1}\frac{8}{19}=\tan^{-1}\left\{\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}\times\frac{8}{19}}\right\}$$

$$=\tan^{-1}\frac{425}{425}=\tan^{-1}1=\frac{\pi}{4}=\text{R.H.S.}$$

(iv) $\text{LHS}=\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$

$$= \left(\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7}\right) + \left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) = \tan^{-1}\frac{6}{17} + \tan^{-1}\frac{11}{23}$$

$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right) = \tan^{-1}\left(\frac{325}{325}\right) = \tan^{-1} 1 = \frac{\pi}{4} = \text{R.H.S.}$$

(v) $\text{LHS} = \cos^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$$= \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} \qquad \left[\because \cot^{-1}(x) = \tan^{-1}\frac{1}{x}, \text{ if } x > 0\right]$$

$$= \left\{\tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{8}\right\} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\times\frac{1}{8}}\right\} + \tan^{-1}\frac{1}{18}\left[\because xy = \frac{1}{7}\times\frac{1}{8} < 1\right]$$

$$= \tan^{-1}\frac{3}{11} + \tan^{-1}\frac{1}{18} = \tan^{-1}\left\{\frac{\frac{3}{11}+\frac{1}{18}}{1-\frac{3}{11}\times\frac{1}{18}}\right\} \qquad \left[\because xy = \frac{3}{11}\times\frac{1}{18} < 1\right]$$

$$= \tan^{-1}\frac{65}{195} = \tan^{-1}\frac{1}{3} = \cot^{-1}3.$$

EXAMPLE 7 *Simplify each of the following:*

(i) $\tan^{-1}\left(\dfrac{a+bx}{b-ax}\right), x < \dfrac{b}{a}$

(ii) $\tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}, \dfrac{a}{b}\tan x > -1$ [NCERT]

(iii) $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\dfrac{1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$ [NCERT]

SOLUTION (i) $\tan^{-1}\left(\dfrac{a+bx}{b-ax}\right) = \tan^{-1}\left(\dfrac{\frac{a}{b}+x}{1-\frac{a}{b}x}\right) = \tan^{-1}\dfrac{a}{b} + \tan^{-1}x$

(ii) $\tan^{-1}\left(\dfrac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\dfrac{\frac{a}{b} - \tan x}{1+\frac{a}{b}\tan x}\right) = \tan^{-1}\dfrac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\dfrac{a}{b} - x$

(iii) $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left\{\dfrac{3\left(\frac{x}{a}\right) - \left(\frac{x}{a}\right)^3}{1 - 3\left(\frac{x}{a}\right)^2}\right\} = 3\tan^{-1}\dfrac{x}{a} \quad \left[\because \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right) = 3\tan^{-1}x\right]$

EXAMPLE 8 *Prove that:* $\tan^{-1}\dfrac{1-x}{1+x} - \tan^{-1}\dfrac{1-y}{1+y} = \sin^{-1}\dfrac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$.

SOLUTION We have,

$$\text{LHS} = \tan^{-1}\frac{1-x}{1+x} - \tan^{-1}\frac{1-y}{1+y} = (\tan^{-1}1 - \tan^{-1}x) - (\tan^{-1}1 - \tan^{-1}y)$$

$$= \tan^{-1}y - \tan^{-1}x = \tan^{-1}\left(\frac{y-x}{1+yx}\right)$$

$$= \sin^{-1}\frac{y-x}{\sqrt{(1+yx)^2+(y-x)^2}} = \sin^{-1}\left\{\frac{y-x}{\sqrt{(1+x^2)(1+y^2)}}\right\} = \text{RHS}.$$

EXAMPLE 9 *If $a > b > c > 0$, prove that:* $\cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right) = \pi$

SOLUTION We know that

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x & \text{, for } x > 0 \\ -\pi + \cot^{-1}x & \text{, for } x < 0 \end{cases} \Rightarrow \cot^{-1}x = \begin{cases} \tan^{-1}\frac{1}{x} & \text{, for } x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & \text{, for } x < 0 \end{cases}$$

It is given that $a > b > c > 0$. Therefore, $a - b > 0$, $b - c > 0$ and $c - a < 0$.

$$\Rightarrow \quad \cot^{-1}\left(\frac{ab+1}{a-b}\right) = \tan^{-1}\left(\frac{a-b}{1+ab}\right), \cot^{-1}\left(\frac{bc+1}{b-c}\right) = \tan^{-1}\left(\frac{b-c}{1+bc}\right)$$

and, $$\cot^{-1}\left(\frac{ca+1}{c-a}\right) = \pi + \tan^{-1}\left(\frac{c-a}{1+ac}\right)$$

$$\therefore \quad \cot^{-1}\left(\frac{ab+1}{a-b}\right) + \cot^{-1}\left(\frac{bc+1}{b-c}\right) + \cot^{-1}\left(\frac{ca+1}{c-a}\right)$$

$$= \tan^{-1}\left(\frac{a-b}{1+ab}\right) + \tan^{-1}\left(\frac{b-c}{1+bc}\right) + \pi + \tan^{-1}\left(\frac{c-a}{1+ca}\right)$$

$$= \tan^{-1}a - \tan^{-1}b + \tan^{-1}b - \tan^{-1}c + \pi + \tan^{-1}c - \tan^{-1}a = \pi.$$

EXAMPLE 10 *Solve the following equations:*

(i) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$ **[NCERT, CBSE 2010]**

(ii) $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$ **[NCERT, CBSE 2009, 2012]**

(iii) $\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$

SOLUTION (i) We have,

$$\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4} \Rightarrow \tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \tan^{-1}1$$

$$\Rightarrow \quad \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}1 - \tan^{-1}\frac{x+1}{x+2} \Rightarrow \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$$

$$\Rightarrow \quad \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\frac{x+2-x-1}{x+2+x+1} \Rightarrow \tan^{-1}\frac{x-1}{x-2} = \tan^{-1}\frac{1}{2x+3}$$

$$\Rightarrow \quad \frac{x-1}{x-2} = \frac{1}{2x+3}$$

$$\Rightarrow \quad (2x+3)(x-1) = x-2 \Rightarrow 2x^2 + x - 3 = x - 2 \Rightarrow 2x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

(ii) We have,

$$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4} \Rightarrow \tan^{-1}\left\{\frac{2x+3x}{1-2x\times 3x}\right\} = \tan^{-1} 1, \text{ if } 6x^2 < 1$$

$$\Rightarrow \quad \frac{5x}{1-6x^2} = 1, \text{ if } 6x^2 < 1 \Rightarrow 6x^2 + 5x - 1 = 0 \text{ and } x^2 < \frac{1}{6}$$

$$\Rightarrow \quad (6x-1)(x+1) = 0 \text{ and } -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}} \Rightarrow x = -1, \frac{1}{6} \text{ and } -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}} \Rightarrow x = \frac{1}{6}$$

(iii) We have,

$$\tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{\frac{x-1}{x+1} + \frac{2x-1}{2x+1}}{1 - \frac{x-1}{x+1}\cdot\frac{2x-1}{2x+1}}\right\} = \tan^{-1}\frac{23}{36}, \text{ if } \frac{(x-1)(2x-1)}{(x+1)(2x+1)} < 1$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x^2-1}{3x}\right) = \tan^{-1}\frac{23}{36} \text{ and } \frac{(x-1)(2x-1)}{(x+1)(2x+1)} - 1 < 0$$

$$\Rightarrow \quad \frac{2x^2-1}{3x} = \frac{23}{36} \text{ and } \frac{-6x}{(x+1)(2x+1)} < 0$$

$$\Rightarrow \quad 24x^2 - 23x - 12 = 0 \text{ and } \frac{x}{(x+1)(2x+1)} > 0$$

$$\Rightarrow \quad (3x-4)(8x+3) = 0 \text{ and } x \in (-1, -1/2) \cup (0, \infty) \Rightarrow x = \frac{4}{3}$$

EXAMPLE 11 *If $a, b, c > 0$ such that $a+b+c = abc$, find the value of $\tan^{-1} a + \tan^{-1} b + \tan^{-1} c$.*

SOLUTION It is given that

$$a+b+c = abc \Rightarrow \frac{abc}{c} = \frac{a}{c} + \frac{b}{c} + 1 \Rightarrow ab = 1 + \left(\frac{a}{c} + \frac{b}{c}\right) \Rightarrow ab - 1 = \frac{a+b}{c}$$

$$\Rightarrow \quad ab - 1 > 0 \quad \Rightarrow ab > 1 \qquad \left[\because a, b, c > 0 \ \therefore \frac{a+b}{c} > 0\right]$$

Now,

$$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c = \pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right) + \tan^{-1} c \qquad [\because ab > 1]$$

$$= \pi + \tan^{-1}\left(\frac{abc-c}{1-ab}\right) + \tan^{-1} c \qquad [\because a+b+c = abc \Rightarrow a+b = abc - c]$$

$$= \pi + \tan^{-1}\left\{\frac{-c(1-ab)}{1-ab}\right\} + \tan^{-1} c = \pi + \tan^{-1}(-c) + \tan^{-1} c = \pi - \tan^{-1} c + \tan^{-1} c = \pi$$

EXAMPLE 12 *Solve the equation:* $\tan^{-1}\sqrt{x^2+x} + \sin^{-1}\sqrt{x^2+x+1} = \frac{\pi}{2}$

[NCERT EXEMPLAR]

SOLUTION This equation holds, if $x^2+x \geq 0$ and $0 \leq x^2+x+1 \leq 1$

Now, $x^2+x \geq 0$ and $0 \leq x^2+x+1 \leq 1$

$\Rightarrow$ $x^2+x \geq 0$ and $x^2+x+1 \leq 1$ [$\because$ $x^2+x+1>0$ for all x]

$\Rightarrow$ $x^2+x \geq 0$ and $x^2+x \leq 0 \Rightarrow x^2+x=0 \Rightarrow x=0,-1$

Clearly, these two values satisfy the given equation.

Hence, $x=0,-1$ are the solutions of the given equation.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 13 *If $a_1, a_2, a_3, \ldots, a_n$ are in arithmetic progression with common difference d, then evaluate the following expression :*

$$\tan\left\{\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right\}$$

[NCERT EXEMPLAR]

SOLUTION It is given that $a_1, a_2, a_3, \ldots, a_n$ are in arithmetic progression with common difference d. Therefore, $d=a_2-a_1=a_3-a_2=a_4-a_3=\ldots=a_n-a_{n-1}$.

$$\therefore \quad \tan\left\{\tan^{-1}\left(\frac{d}{1+a_1a_2}\right)+\tan^{-1}\left(\frac{d}{1+a_2a_3}\right)+\tan^{-1}\left(\frac{d}{1+a_3a_4}\right)+\ldots+\tan^{-1}\left(\frac{d}{1+a_{n-1}a_n}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\tan^{-1}\left(\frac{a_3-a_2}{1+a_3a_2}\right)+\tan^{-1}\left(\frac{a_4-a_3}{1+a_4a_3}\right)+\ldots+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_n a_{n-1}}\right)\right\}$$

$$=\tan\left\{\left(\tan^{-1}a_2-\tan^{-1}a_1\right)+\left(\tan^{-1}a_3-\tan^{-1}a_2\right)\ldots\ldots+\ldots+\left(\tan^{-1}a_n-\tan^{-1}a_{n-1}\right)\right\}$$

$$=\tan\left(\tan^{-1}a_n-\tan^{-1}a_1\right)=\tan\left\{\tan^{-1}\left(\frac{a_n-a_1}{1+a_na_1}\right)\right\}=\frac{a_n-a_1}{1+a_na_1}=\frac{(n-1)\,d}{1+a_1a_n} \quad [\because a_n=a_1+(n-1)\,d]$$

EXAMPLE 14 *Prove that:* $\sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)=\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)$

SOLUTION L.H.S. $=\sum_{m=1}^{n}\tan^{-1}\left(\frac{2m}{m^4+m^2+2}\right)=\sum_{m=1}^{n}\tan^{-1}\left\{\frac{2m}{1+(m^4+m^2+1)}\right\}$

$$=\sum_{m=1}^{n}\tan^{-1}\left\{\frac{2m}{1+(m^2+1)^2-m^2}\right\}=\sum_{m=1}^{n}\tan^{-1}\left\{\frac{2m}{1+(m^2+m+1)(m^2-m+1)}\right\}$$

$$=\sum_{m=1}^{n}\tan^{-1}\left\{\frac{(m^2+m+1)-(m^2-m+1)}{1+(m^2+m+1)(m^2-m+1)}\right\}$$

$$=\sum_{m=1}^{n}\left\{\tan^{-1}(m^2+m+1)-\tan^{-1}(m^2-m+1)\right\}$$

$$=(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}7-\tan^{-1}3)+(\tan^{-1}13-\tan^{-1}7)$$
$$+\ldots\ldots+\left\{\tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1)\right\}$$

$$=\tan^{-1}(n^2+n+1)-\tan^{-1}1=\tan^{-1}\left\{\frac{n^2+n+1-1}{1+(n^2+n+1)}\right\}=\tan^{-1}\left(\frac{n^2+n}{n^2+n+2}\right)=\text{RHS}$$

EXAMPLE 15 *Sum the following series to infinity :*

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\dots\dots$$

SOLUTION Let T_n be the nth term of the given series and S be the sum to infinity. Then,

$$T_n = \tan^{-1}\left\{\frac{1}{1+n+n^2}\right\} = \tan^{-1}\left\{\frac{(n+1)-n}{1+(n+1)n}\right\} = \tan^{-1}(n+1)-\tan^{-1}n$$

and, $$S=\sum_{n=1}^{\infty} T_n = \lim_{n\to\infty}\sum_{r=1}^{n} T_r = \lim_{n\to\infty}(T_1+T_2+\dots\dots+T_n)$$

$$=\lim_{n\to\infty}\left\{(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+\dots+(\tan^{-1}(n+1)-\tan^{-1}n)\right\}$$

$$=\lim_{n\to\infty}\left\{\tan^{-1}(n+1)-\tan^{-1}1\right\} = \tan^{-1}\infty-\tan^{-1}1 = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

EXAMPLE 16 *If $c_i>0$ for $i=1,2,\dots,n$, prove that*

$$\tan^{-1}\left(\frac{c_1x-y}{c_1y+x}\right)+\tan^{-1}\left(\frac{c_2-c_1}{1+c_2c_1}\right)+\tan^{-1}\left(\frac{c_3-c_2}{1+c_3c_2}\right)+\dots\dots+\tan^{-1}\frac{1}{c_n}=\tan^{-1}\frac{x}{y}$$

SOLUTION We have,

$$\text{LHS}=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}.\frac{1}{c_1}}\right)+\tan^{-1}\left(\frac{\frac{1}{c_1}-\frac{1}{c_2}}{1+\frac{1}{c_1c_2}}\right)+\dots\dots+\tan^{-1}\left(\frac{\frac{1}{c_{n-1}}-\frac{1}{c_n}}{1+\frac{1}{c_{n-1}c_n}}\right)+\tan^{-1}\frac{1}{c_n}$$

$$=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)+\left(\tan^{-1}\frac{1}{c_2}-\tan^{-1}\frac{1}{c_3}\right)+\dots\dots$$

$$+\left(\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}\right)+\tan^{-1}\frac{1}{c_n}$$

$$=\tan^{-1}\frac{x}{y}=\text{R.H.S.}$$

EXAMPLE 17 *Prove that* $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$, *where $x,y,z>0$ such that $x^2+y^2+z^2=r^2$*

SOLUTION We have, $x^2+y^2+z^2=r^2$. Therefore, $\frac{yz}{xr}\times\frac{zx}{yr}=\frac{z^2}{r^2}=\frac{z^2}{x^2+y^2+z^2}<1$.

$$\therefore\quad \tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{xz}{yr}+\tan^{-1}\frac{xy}{zr}=\tan^{-1}\left\{\frac{\frac{yz}{xr}+\frac{xz}{yr}}{1-\frac{yz}{xr}\times\frac{zx}{yr}}\right\}+\tan^{-1}\frac{xy}{zr}$$

$$=\tan^{-1}\left\{\frac{\frac{z(x^2+y^2)}{xyr}}{1-\frac{z^2}{r^2}}\right\}+\tan^{-1}\left(\frac{xy}{zr}\right)=\tan^{-1}\left\{\frac{z(x^2+y^2)}{xyr}\times\frac{r^2}{x^2+y^2}\right\}+\tan^{-1}\left(\frac{xy}{zr}\right)$$

$$=\tan^{-1}\left(\frac{zr}{xy}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)=\cot^{-1}\left(\frac{xy}{zr}\right)+\tan^{-1}\left(\frac{xy}{zr}\right)=\frac{\pi}{2}$$

EXAMPLE 18 *If $a, b, c > 0$ and $s = \frac{a+b+c}{2}$, prove that* $\tan^{-1}\sqrt{\frac{2as}{bc}} + \tan^{-1}\sqrt{\frac{2bs}{ca}} + \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi$

SOLUTION We find that

$$\sqrt{\frac{2as}{bc}} \times \sqrt{\frac{2bs}{ca}} = \sqrt{\frac{4abs^2}{abc^2}} = \frac{2s}{c} = \frac{a+b+c}{c} \Rightarrow \sqrt{\frac{2as}{bc}} \times \sqrt{\frac{2bs}{ca}} = 1 + \frac{a+b}{c} > 1 \qquad \left[\because \frac{a+b}{c} > 0\right]$$

$$\therefore \quad \tan^{-1}\sqrt{\frac{2as}{bc}} + \tan^{-1}\sqrt{\frac{2bs}{ca}} + \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi + \tan^{-1}\left\{\frac{\sqrt{\frac{2as}{bc}} + \sqrt{\frac{2bs}{ca}}}{1 - \sqrt{\frac{2as}{bc}} \times \sqrt{\frac{2bs}{ca}}}\right\} + \tan^{-1}\sqrt{\frac{2cs}{ab}}$$

$$= \pi + \tan^{-1}\left\{\frac{\sqrt{\frac{2s}{abc}}}{1 - \frac{2s}{c}}\right\} + \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi + \tan^{-1}\left\{\frac{\sqrt{\frac{2s}{abc}}\,(a+b)}{1 - \frac{a+b+c}{c}}\right\} + \tan^{-1}\sqrt{\frac{2cs}{ab}}$$

$$= \pi + \tan^{-1}\left(-\sqrt{\frac{2cs}{ab}}\right) + \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi - \tan^{-1}\sqrt{\frac{2cs}{ab}} + \tan^{-1}\sqrt{\frac{2cs}{ab}} = \pi$$

EXAMPLE 19 *Evaluate:* $\tan^{-1}\left(\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

SOLUTION Using $\cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha}$ and, $\sin 2\alpha = \frac{2\tan\alpha}{1 + \tan^2\alpha}$, we obtain

$$\tan^{-1}\left(\frac{3\sin 2\alpha}{5 + 3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right) = \tan^{-1}\left(\frac{6\tan\alpha}{8 + 2\tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$$

$$= \tan^{-1}\left(\frac{3\tan\alpha}{4 + \tan^2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right) = \tan^{-1}\left\{\frac{\frac{3\tan\alpha}{4 + \tan^2\alpha} + \frac{1}{4}\tan\alpha}{1 - \frac{3\tan^2\alpha}{16 + 4\tan^2\alpha}}\right\}$$

$$= \tan^{-1}\left\{\frac{(16 + \tan^2\alpha)\tan\alpha}{16 + \tan^2\alpha}\right\} = \tan^{-1}(\tan\alpha) = \alpha \qquad [\because -\pi/2 < \alpha < \pi/2]$$

EXAMPLE 20 *Prove that:* $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \begin{cases} 0, \text{ if } \frac{\pi}{4} < A < \frac{\pi}{2} \\ \pi, \text{ if } 0 < A < \pi/4 \end{cases}$

SOLUTION We know that $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy > 1 \end{cases}$

Also, $\cot A > 1$, if $0 < A < \frac{\pi}{4}$ and, $0 < \cot A < 1$, if $\frac{\pi}{4} < A < \frac{\pi}{2}$

$$\therefore \quad \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \begin{cases} \tan^{-1}\left(\dfrac{\cot A + \cot^3 A}{1 - \cot^4 A}\right), & \text{if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi + \tan^{-1}\left(\dfrac{\cot A + \cot^3 A}{1 - \cot^4 A}\right), & \text{if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} \tan^{-1}\left(\dfrac{\cot A}{1 - \cot^2 A}\right), & \text{if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi + \tan^{-1}\left(\dfrac{\cot A}{1 - \cot^2 A}\right), & \text{if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} \tan^{-1}\left(-\dfrac{1}{2}\tan 2A\right), & \text{if } \dfrac{\pi}{4} < A < \dfrac{\pi}{2} \\ \pi + \tan^{-1}\left(-\dfrac{1}{2}\tan 2A\right), & \text{if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

$$= \begin{cases} -\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right), & \text{if } \dfrac{\pi}{4} < A < \dfrac{\pi}{4} \\ \pi - \tan^{-1}\left(\dfrac{1}{2}\tan 2A\right), & \text{if } 0 < A < \dfrac{\pi}{4} \end{cases}$$

Adding $\tan^{-1}\left(\dfrac{1}{2}\tan 2A\right)$ on both sides, we get

$$\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \begin{cases} 0, \text{ if } \pi/4 < A < \pi/2 \\ \pi, \text{ if } 0 < A < \pi/4 \end{cases}$$

EXERCISE 3.11

BASIC

1. Prove the following results:

(i) $\tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{13} = \tan^{-1}\dfrac{2}{9}$ (ii) $\sin^{-1}\dfrac{12}{13} + \cos^{-1}\dfrac{4}{5} + \tan^{-1}\dfrac{63}{16} = \pi$

(iii) $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \sin^{-1}\dfrac{1}{\sqrt{5}}$ [NCERT EXEMPLAR, CBSE 2020]

2. Find the value of $\tan^{-1}\left(\dfrac{x}{y}\right) - \tan^{-1}\left(\dfrac{x-y}{x+y}\right)$ [CBSE 2011]

BASED ON LOTS

3. Solve the following equations for x :

(i) $\tan^{-1} 2x + \tan^{-1} 3x = n\pi + \dfrac{3\pi}{4}$

(ii) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\dfrac{8}{31}$

(iii) $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$

(iv) $\tan^{-1}\left(\dfrac{1-x}{1+x}\right) - \dfrac{1}{2}\tan^{-1} x = 0$, where $x > 0$ [NCERT, CBSE 2008, 2010, 2011]

(v) $\cot^{-1} x - \cot^{-1}(x+2) = \frac{\pi}{12}$, where $x > 0$

(vi) $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), x > 0$ [CBSE 2010]

(vii) $\tan^{-1}\frac{x}{2} + \tan^{-1}\frac{x}{3} = \frac{\pi}{4}, 0 < x < \sqrt{6}$ [CBSE 2010C]

(viii) $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$ [CBSE 2014]

(ix) $\tan^{-1}(2+x) + \tan^{-1}(2-x) = \tan^{-1}\frac{2}{3}$, where $x < -\sqrt{3}$ or, $x > \sqrt{3}$

(x) $\tan^{-1}\frac{x-2}{x-1} + \tan^{-1}\frac{x+2}{x+1} = \frac{\pi}{4}$ [CBSE 2016]

(xi) $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$ [CBSE 2019]

BASED ON HOTS

4. Sum the series : $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \tan^{-1}\frac{4}{33} + \ldots + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$

ANSWERS

3. (i) $-\frac{1}{6}$ (ii) $\frac{1}{4}$ (iii) $0, \pm\frac{1}{2}$ (iv) $\frac{1}{\sqrt{3}}$ (v) $\sqrt{3}$ (vi) $\frac{1}{4}$

(vii) 1 (viii) $\pm\sqrt{2}$ (ix) ± 3 (x) $\pm\sqrt{\frac{7}{2}}$ (xi) $\frac{1}{12}$ 4. $\tan^{-1} 2^n - \frac{\pi}{4}$

HINTS TO SELECTED PROBLEMS

2. $\tan^{-1}\frac{x}{y} - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\frac{x}{y} + \tan^{-1}\left(\frac{y-x}{y+x}\right) = \tan^{-1}\frac{x}{y} + \tan^{-1}\left(\frac{1-x/y}{1+x/y}\right)$

$$= \tan^{-1}\frac{x}{y} + \tan^{-1} 1 - \tan^{-1}\frac{x}{y} = \tan^{-1} 1 = \frac{\pi}{4}$$

3. (iv) We have, $\tan^{-1}\left(\frac{1-x}{1+x}\right) - \frac{1}{2}\tan^{-1} x = 0,\ x > 0$

$$\Rightarrow \tan^{-1} 1 - \tan^{-1} x - \frac{1}{2}\tan^{-1} x = 0 \Rightarrow \frac{\pi}{4} - \frac{3}{2}\tan^{-1} x = 0 \Rightarrow \tan^{-1} x = \frac{\pi}{6} \Rightarrow x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

(v) $\tan^{-1}\frac{1}{x} - \tan^{-1}\frac{1}{x+2} = \frac{\pi}{12} \Rightarrow \tan^{-1}\left\{\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x(x+2)}}\right\} = \frac{\pi}{12}$

$$\Rightarrow \tan^{-1}\left(\frac{2}{x^2+2x+1}\right) = \frac{\pi}{12} \Rightarrow \frac{2}{x^2+2x+1} = \tan\frac{\pi}{12}$$

$$\Rightarrow \frac{2}{x^2+2x+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \Rightarrow \frac{2}{x^2+2x+1} = \frac{2}{(\sqrt{3}+1)^2} \Rightarrow (x+1)^2 = (\sqrt{3}+1)^2 \Rightarrow x = \sqrt{3}$$

3.3.7 PROPERTY VII

PROPERTY VII *Prove that:*

(i) $$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy<0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,\ y<0 \text{ and } x^2+y^2>1\end{cases}$$

(ii) $$\sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \\ & \text{or} \\ & \text{if } xy>0 \text{ and } x^2+y^2>1 \\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1 \\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}, & \text{if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$$

PROOF Let $\sin^{-1}x = A$ and $\sin^{-1}y = B$. Then, $x=\sin A$, $y=\sin B$ and $A, B\in[-\pi/2, \pi/2]$

$\Rightarrow \quad \cos A=\sqrt{1-x^2}$, $\cos B=\sqrt{1-y^2}$ $\quad [\because A, B\in[-\pi/2,\pi/2] \therefore \cos A, \cos B\ge 0]$

$\therefore \quad \sin(A+B) = \sin A\cos B+\cos A\sin B = x\sqrt{1-y^2}+y\sqrt{1-x^2}$ ...(i)

$\sin(A-B) = x\sqrt{1-y^2}-y\sqrt{1-x^2}$ ...(ii)

$\cos(A+B) = \sqrt{1-x^2}\sqrt{1-y^2}-xy$ ...(iii)

and, $\cos(A-B) = \sqrt{1-x^2}\sqrt{1-y^2}+xy$ ...(iv)

Case I *When $-1\le x, y\le 1$ and $x^2+y^2\le 1$:* In this case, we have

$x^2+y^2\le 1 \Rightarrow 1-x^2\ge y^2$ and $1-y^2\ge x^2 \Rightarrow (1-x^2)(1-y^2)\ge x^2y^2$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2}\ge xy \Rightarrow \sqrt{1-x^2}\sqrt{1-y^2}-xy\ge 0 \Rightarrow \cos(A+B)\ge 0$ [Using (iii)]

$\Rightarrow \quad A+B$ lies either in I quadrant or in IV quadrant

$\Rightarrow \quad A+B\in[-\pi/2, \pi/2]$ $\quad [\because A, B\in[-\pi/2,\pi/2] \Rightarrow -\pi\le A+B\le\pi]$

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2}+y\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad A+B = \sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$ $\quad \left[\because -\frac{\pi}{2}\le A+B\le\frac{\pi}{2}\right]$

$\Rightarrow \quad \sin^{-1}x+\sin^{-1}y = \sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\}$

Case II *When $xy<0$ and $x^2+y^2>1$:* In this case, we have

$xy<0 \Rightarrow x>0$ and $y<0$ or $x<0$ and $y>0$

$\Rightarrow \quad \{A\in(0,\pi/2]$ and $B\in[-\pi/2, 0)\}$ or $\left\{A\in\left[-\frac{\pi}{2}, 0\right)\text{ and } B\in\left(0, \frac{\pi}{2}\right]\right\}$

$\Rightarrow \quad -\frac{\pi}{2} \leq A + B \leq \frac{\pi}{2}$ ...(v)

and, $\quad x^2 + y^2 > 1 \Rightarrow 1 - x^2 < y^2$ and $1 - y^2 < x^2 \Rightarrow (1 - x^2)(1 - y^2) < x^2 y^2$

$\Rightarrow \quad (\sqrt{1 - x^2}\sqrt{1 - y^2})^2 < (|xy|)^2$ $[\because xy < 0]$

$\Rightarrow \quad -|xy| < \sqrt{1 - x^2}\sqrt{1 - y^2} < |xy| \Rightarrow xy < \sqrt{1 - x^2}\sqrt{1 - y^2} < -xy$ $[\because xy < 0 \therefore |xy| = -xy]$

$\Rightarrow \quad \sqrt{1 - x^2}\sqrt{1 - y^2} - xy > 0$

$\Rightarrow \quad \cos(A + B) > 0$

$\Rightarrow \quad A + B$ lies either in I quadrant or in IV quadrant

$\Rightarrow \quad A + B \in [-\pi/2, \pi/2]$ [Using (v)]

$\therefore \quad \sin(A + B) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$

$\Rightarrow \quad A + B = \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}$ $[\because A + B \in [-\pi/2, \pi/2]]$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}$

Case III *When* $0 < x, y \leq 1$ *and* $x^2 + y^2 > 1$: In this case, we have

$0 < x, y \leq 1 \Rightarrow A \in (0, \pi/2]$ and $B \in (0, \pi/2] \Rightarrow A + B \in (0, \pi]$ ...(vi)

and, $\quad x^2 + y^2 > 1$

$\Rightarrow \quad 1 - x^2 < y^2$ and $1 - y^2 < x^2 \Rightarrow (1 - x^2)(1 - y^2) < x^2 y^2$

$\Rightarrow \quad \sqrt{1 - x^2}\sqrt{1 - y^2} < xy$ $[\because xy > 0]$

$\Rightarrow \quad \sqrt{1 - x^2}\sqrt{1 - y^2} - xy < 0$

$\Rightarrow \quad \cos(A + B) < 0$ [Using (iii)]

$\Rightarrow \quad A + B$ lies either in II quadrant or in III quadrant

$\Rightarrow \quad \frac{\pi}{2} \leq A + B \leq \pi$ $[\because A + B \; (0, \pi$, from (vi)]

$\Rightarrow \quad -\pi \leq -(A + B) \leq -\frac{\pi}{2} \Rightarrow 0 \leq \pi - (A + B) \leq \frac{\pi}{2}$

$\therefore \quad \sin(A + B) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ [From (i)]

$\Rightarrow \quad \sin\{\pi - (A + B)\} = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}$ $[\because \sin(\pi - \theta) = \sin\theta]$

$\Rightarrow \quad \pi - (A + B) = \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}$

$\Rightarrow \quad A + B = \pi - \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}$

$\Rightarrow \quad \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left\{x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right\}$

Case IV *When* $-1 \leq x, y < 0$ *and* $x^2 + y^2 > 1$: In this case, we have

$-1 \leq x, y < 0 \Rightarrow A \in [-\pi/2, 0)$ and $B \in [-\pi/2, 0) \Rightarrow A + B \in [-\pi, 0)$ ...(vii)

and, $\quad x^2 + y^2 > 1 \Rightarrow 1 - x^2 < y^2$ and $1 - y^2 < x^2 \Rightarrow (1 - x^2)(1 - y^2) < x^2 y^2$

$\Rightarrow \quad \sqrt{1 - x^2}\sqrt{1 - y^2} < xy$ $[\because xy > 0]$

$\Rightarrow \quad \sqrt{1-x^2}\sqrt{1-y^2} - xy < 0$

$\Rightarrow \quad \cos(A+B) < 0$ [Using (iii)]

$\Rightarrow \quad A+B$ lies either in II quadrant or in III quadrant

$\Rightarrow \quad -\pi \le A+B \le \frac{\pi}{2}$ [Using (vii)]

$\Rightarrow \quad \frac{\pi}{2} \le -(A+B) \le \pi \Rightarrow -\frac{\pi}{2} \le -\pi-(A+B) \le 0$

$\therefore \quad \sin(A+B) = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad -\sin\{\pi+(A+B)\} = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad \sin\{-\pi-(A+B)\} = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$\Rightarrow \quad -\pi-(A+B) = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

$\Rightarrow \quad A+B = -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

$\Rightarrow \quad \sin^{-1}x + \sin^{-1}y = -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

(ii) Do yourself.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:* $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85} = \tan^{-1}\left(\frac{77}{36}\right)$ **[CBSE 2012, NCERT]**

SOLUTION $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\left\{\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}\right\}$

$= \sin^{-1}\left\{\frac{8}{17}\times\frac{4}{5} + \frac{3}{5}\times\frac{15}{17}\right\} = \sin^{-1}\left(\frac{77}{85}\right) = \tan^{-1}\left(\frac{77}{36}\right)$

EXAMPLE 2 *Prove that:* $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$ **[NCERT, CBSE 2010, 2012]**

SOLUTION We have,

$\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{3}{5}$ $\left[\because \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{5}{13}\right]$

$= \sin^{-1}\left\{\frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2}\right\} = \sin^{-1}\left\{\frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13}\right\} = \sin^{-1}\frac{56}{65}$

EXAMPLE 3 *Prove that:*

(i) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \sin^{-1}\frac{77}{85}$ **[NCERT EXEMPLAR]**

(ii) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65} = \frac{\pi}{2}$ **[CBSE 2009]**

(iii) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{36}{85}$ **[CBSE 2010]**

(iv) $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65} = \sin^{-1}\frac{56}{65}$ **[CBSE 2010]**

SOLUTION Using $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$, we obtain

(i) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$

$$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\} = \sin^{-1}\left\{\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right\} = \sin^{-1}\frac{77}{85}$$

(ii) $\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}$

$$= \left\{\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13}\right\} + \sin^{-1}\frac{16}{65} = \sin^{-1}\left\{\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right\} + \sin^{-1}\frac{16}{65}$$

$$= \sin^{-1}\left\{\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right\} + \sin^{-1}\frac{16}{65} = \sin^{-1}\frac{63}{65} + \sin^{-1}\frac{16}{65}$$

$$= \cos^{-1}\frac{16}{65} + \sin^{-1}\frac{16}{65} \qquad \left[\because \sin^{-1}\frac{63}{65} = \cos^{-1}\sqrt{1-\left(\frac{63}{65}\right)^2} = \cos^{-1}\frac{16}{65}\right]$$

$$= \frac{\pi}{2} \qquad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}\right]$$

(iii) $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{8}{17}$

$$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\} = \sin^{-1}\left\{\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right\}$$

$$= \sin^{-1}\frac{77}{85} = \cos^{-1}\sqrt{1-\left(\frac{77}{85}\right)^2} = \cos^{-1}\frac{36}{85} \qquad \left[\because \sin^{-1} x = \cos^{-1}\sqrt{1-x^2}\right]$$

(iv) $\sin^{-1}\frac{3}{5} + \cos^{-1}\frac{12}{13}$

$$= \sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} \qquad \left[\because \cos^{-1} x = \sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\left\{\frac{3}{5}\times\sqrt{1-\left(\frac{5}{13}\right)^2} + \frac{5}{13}\times\sqrt{1-\left(\frac{3}{5}\right)^2}\right\}$$

$$= \sin^{-1}\left\{\frac{3}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{4}{5}\right\} = \sin^{-1}\frac{56}{65} = \cos^{-1}\sqrt{1-\left(\frac{56}{65}\right)^2} = \cos^{-1}\frac{33}{65}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Solve the following equations:*

(i) $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x$

(ii) $\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1}\left\{\frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}\right\}=\sin^{-1}x$$

$$\Rightarrow \quad \frac{3x}{5}\sqrt{1-\frac{16x^2}{25}}+\frac{4x}{5}\sqrt{1-\frac{9x^2}{25}}=x$$

$$\Rightarrow \quad 3x\sqrt{25-16x^2}+4x\sqrt{25-9x^2}=25x \Rightarrow x=0 \text{ or, } 3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$$

Now, $3\sqrt{25-16x^2}+4\sqrt{25-9x^2}=25$

$$\Rightarrow \quad 4\sqrt{25-9x^2}=25-3\sqrt{25-16x^2}$$

$$\Rightarrow \quad 16(25-9x^2)=625+9(25-16x^2)-150\sqrt{25-16x^2}$$

$$\Rightarrow \quad 150\sqrt{25-16x^2}=450 \Rightarrow 25-16x^2=9 \Rightarrow x=\pm 1$$

Hence, $x=0, 1, -1$ are roots of the given equation.

(ii) We have,

$$\sin^{-1}6x+\sin^{-1}6\sqrt{3}x=-\frac{\pi}{2} \Rightarrow \sin^{-1}6x=-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}x$$

$$\Rightarrow \quad \sin(\sin^{-1}6x)=\sin\left(-\frac{\pi}{2}-\sin^{-1}6\sqrt{3}x\right) \Rightarrow 6x=-\sin\left(\frac{\pi}{2}+\sin^{-1}6\sqrt{3}\,x\right)$$

$$\Rightarrow \quad 6x=-\cos\left(\sin^{-1}6\sqrt{3}\,x\right) \Rightarrow 6x=-\cos\left\{\cos^{-1}\sqrt{1-(6\sqrt{3}x)^2}\right\} \quad \left[\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \quad 6x=-\sqrt{1-108x^2} \Rightarrow 36x^2=1-108x^2 \Rightarrow 144x^2=1 \Rightarrow x=\pm\frac{1}{12}$$

We observe that $x=\frac{1}{12}$ does not satisfy the given equation. Hence, $x=-\frac{1}{12}$ is the only root of the given equation.

EXERCISE 3.12

BASIC

1. Evaluate : $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$

2. Prove the following results :

(i) $\sin^{-1}\left(\frac{63}{65}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$ [CBSE 2012]

(ii) $\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\frac{3}{5}=\tan^{-1}\frac{63}{16}$ [NCERT EXEMPLAR]

(iii) $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$ [NCERT]

BASED ON LOTS

3. Solve the following :

(i) $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

(ii) $\cos^{-1}x+\sin^{-1}\frac{x}{2}-\frac{\pi}{6}=0$ [CBSE 2010]

(iii) $\sin^{-1}\frac{5}{x}+\sin^{-1}\frac{12}{x}=\frac{\pi}{2}, x>0$ [CBSE 2020]

ANSWERS

1. $\frac{33}{65}$

3. (i) $\frac{1}{2}\sqrt{\frac{3}{7}}$ (ii) 1 (iii) 13

3.3.8 PROPERTY-VIII

PROPERTY VIII *Prove that:*

(i) $$\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x+y\ge 0\\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x+y\le 0\end{cases}$$

(ii) $$\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy+\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}, & \text{if } -1\le x, y\le 1 \text{ and } x\le y\\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}, & \text{if } -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y\end{cases}$$

PROOF Let $\cos^{-1}x = A$ and $\cos^{-1}y = B$. Then, $x = \cos A$, $y=\cos B$ and $A, B\in[0,\pi]$.

$\therefore\quad \sin A = \sqrt{1-x^2}$ and $\sin B = \sqrt{1-y^2}$ $\quad[\because \sin A, \sin B\ge 0 \text{ for } A, B\in[0,\pi]]$

and so, $\cos(A+B) = xy-\sqrt{1-x^2}\,\sqrt{1-y^2}$ ...(i)

$\cos(A-B) = xy+\sqrt{1-x^2}\,\sqrt{1-y^2}$ ...(ii)

Case I *When* $-1\le x, y\le 1$ *and* $x+y\ge 0$: In this case, we have

$-1\le x, y\le 1 \Rightarrow A, B\in[0,\pi]\Rightarrow 0\le A+B\le 2\pi$ ...(iii)

and, $x+y\ge 0$

$\Rightarrow\quad \cos A+\cos B\ge 0\Rightarrow \cos A\ge-\cos B\Rightarrow \cos A\ge\cos(\pi-B)$

$\Rightarrow\quad A\le\pi-B$ $\quad[\because \cos\theta$ is decreasing on $[0,\pi]]$

$\Rightarrow\quad A+B\le\pi$ ...(iv)

From (iii) and (iv), we get

$0\le A+B\le\pi$

$\therefore\quad \cos(A+B) = xy-\sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow\quad A+B=\cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

$\Rightarrow\quad \cos^{-1}x+\cos^{-1}y = \cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

Case II *When* $-1\le x, y<0$ *and* $x+y\le 0$: In this case, we have

$-1\le x, y<1\Rightarrow A, B\in[0,\pi]\Rightarrow 0\le A+B\le 2\pi$ ...(v)

and, $x+y\le 0\Rightarrow \cos A+\cos B\le 0\Rightarrow \cos A\le-\cos B\Rightarrow\cos A\le\cos(\pi-B)$

$\Rightarrow\quad A\ge\pi-B$ $\quad[\because \cos\theta$ is decreasing on $[0,\pi]]$

$\Rightarrow\quad A+B\ge\pi$ ...(vi)

From (v) and (vi), we get

$\pi\le A+B\le 2\pi\Rightarrow -\pi\ge-(A+B)\ge-2\pi\Rightarrow \pi\ge 2\pi-(A+B)\ge 0\Rightarrow 0\le 2\pi-(A+B)\le\pi$

$\therefore\quad \cos(A+B) = xy-\sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow\quad \cos\{2\pi-(A+B)\} = xy-\sqrt{1-x^2}\,\sqrt{1-y^2}$

$\Rightarrow\quad 2\pi-(A+B) = \cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

$\Rightarrow\quad A+B = 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

$\Rightarrow\quad \cos^{-1}x+\cos^{-1}y = 2\pi-\cos^{-1}(xy-\sqrt{1-x^2}\,\sqrt{1-y^2})$.

(ii) Do yourself.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:* $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$ [NCERT, CBSE 2010, 2012]

SOLUTION $\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right\}$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{12}{13}-\frac{3}{5}\times\frac{5}{13}\right\}=\cos^{-1}\left\{\frac{48}{65}-\frac{15}{65}\right\}=\cos^{-1}\frac{33}{65}$$

EXAMPLE 2 *Prove that:* $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{84}{85}$ [NCERT]

SOLUTION $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{8}{17}$

$$=\cos^{-1}\frac{4}{5}-\cos^{-1}\frac{15}{17} \qquad \left[\because \sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5},\ \sin^{-1}\frac{8}{17}=\cos^{-1}\frac{15}{17}\right]$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\sqrt{1-\left(\frac{4}{5}\right)^2}\times\sqrt{1-\left(\frac{15}{17}\right)^2}\right\}$$

$$=\cos^{-1}\left\{\frac{4}{5}\times\frac{15}{17}+\frac{3}{5}\times\frac{8}{17}\right\}=\cos^{-1}\left\{\frac{60}{85}+\frac{24}{85}\right\}=\cos^{-1}\frac{84}{85}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *If* $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, *prove that* $\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$

[CBSE 2016]

SOLUTION We have,

$$\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$$

$$\Rightarrow \cos^{-1}\left\{\frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right\}=\alpha \Rightarrow \frac{xy}{ab}-\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}=\cos\alpha$$

$$\Rightarrow \frac{xy}{ab}-\cos\alpha=\sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} \quad \Rightarrow \left(\frac{xy}{ab}-\cos\alpha\right)^2=\left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$$

$$\Rightarrow \frac{x^2y^2}{a^2b^2}-\frac{2xy}{ab}\cos\alpha+\cos^2\alpha=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{x^2y^2}{a^2b^2}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=1-\cos^2\alpha \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos\alpha=\sin^2\alpha$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *If* $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, *prove that* $x^2+y^2+z^2+2xyz=1.$

SOLUTION We have,

$$\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$$

$\Rightarrow \quad \cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1} z$

$\Rightarrow \quad \cos^{-1} x + \cos^{-1} y = \cos^{-1}(-z) \qquad [\because \cos^{-1}(-z) = \pi - \cos^{-1} z]$

$\Rightarrow \quad \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\} = \cos^{-1}(-z)$

$\Rightarrow \quad xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow \quad (xy+z)^2 = (1-x^2)(1-y^2)$

$\Rightarrow \quad x^2y^2 + z^2 + 2xyz = 1 - x^2 - y^2 + x^2y^2 \Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

EXERCISE 3.13

BASED ON LOTS

1. If $\cos^{-1}\frac{x}{2} + \cos^{-1}\frac{y}{3} = \alpha$, then prove that $9x^2 - 12xy\cos\alpha + 4y^2 = 36\sin^2\alpha$.
2. Solve the equation: $\cos^{-1}\frac{a}{x} - \cos^{-1}\frac{b}{x} = \cos^{-1}\frac{1}{b} - \cos^{-1}\frac{1}{a}$
3. Solve : $\cos^{-1}\sqrt{3}x + \cos^{-1}x = \frac{\pi}{2}$
4. Prove that:
   (i) $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$ [CBSE 2012]
   (ii) $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$ [CBSE 2019]

ANSWERS

2. $x = ab$ 3. $x = \frac{1}{2}$

3.3.9 PROPERTY-IX

PROPERTY IX *Prove that:*

(i) $2\sin^{-1}x = \begin{cases} \sin^{-1}(2x\sqrt{1-x^2}) & , \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}(2x\sqrt{1-x^2}) & , \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}(2x\sqrt{1-x^2}) & , \text{if } -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$ [NCERT]

(ii) $3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3) & , \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3) & , \text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3) & , \text{if } -1 \le x < -\frac{1}{2} \end{cases}$ [CBSE 2018]

PROOF (i) Let $\sin^{-1}x = \theta$. Then,

$x = \sin\theta \Rightarrow \cos\theta = \sqrt{1-x^2} \qquad [\because \cos\theta > 0 \text{ for } \theta \in [-\pi/2, \pi/2]]$

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta \Rightarrow \sin 2\theta = 2x\sqrt{1-x^2} \qquad \text{...(i)}$

Case I *When* $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$: In this case, $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

Also, $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 1$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad 2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \Rightarrow 2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Case II *When* $\frac{1}{\sqrt{2}} \le x \le 1$: In this case,

$\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow \frac{1}{\sqrt{2}} \le \sin\theta \le 1 \Rightarrow \frac{\pi}{4} \le \theta \le \frac{\pi}{2} \Rightarrow \frac{\pi}{2} \le 2\theta \le \pi \Rightarrow -\pi \le -2\theta \le -\frac{\pi}{2} \Rightarrow 0 \le \pi - 2\theta \le \frac{\pi}{2}$

Also, $\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow 0 \le 2x\sqrt{1-x^2} < 1$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ [From (i)]

$\Rightarrow \quad \sin(\pi - 2\theta) = 2x\sqrt{1-x^2} \Rightarrow \pi - 2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow \pi - 2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \Rightarrow 2\sin^{-1}x = \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Case III *When* $-1 \le x < -\frac{1}{\sqrt{2}}$: In this case,

$-1 \le x < -\frac{1}{\sqrt{2}} \Rightarrow -1 \le \sin\theta \le -\frac{1}{\sqrt{2}} \Rightarrow -\frac{\pi}{2} \le \theta \le -\frac{\pi}{4} \Rightarrow -\pi \le 2\theta \le -\frac{\pi}{2} \Rightarrow 0 \le \pi + 2\theta \le \frac{\pi}{2}$

Also, $-1 \le x \le -\frac{1}{\sqrt{2}} \Rightarrow -1 \le 2x\sqrt{1-x^2} \le 0$

$\therefore \quad \sin 2\theta = 2x\sqrt{1-x^2}$ [From (i)]

$\Rightarrow -\sin(\pi + 2\theta) = 2x\sqrt{1-x^2} \quad \Rightarrow \sin(-\pi - 2\theta) = 2x\sqrt{1-x^2}$

$\Rightarrow -\pi - 2\theta = \sin^{-1}\left(2x\sqrt{1-x^2}\right) \Rightarrow 2\theta = -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$\Rightarrow 2\sin^{-1}x = -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(ii) Let $\sin^{-1} x = \theta$. Then, $x = \sin\theta$ and, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 3x - 4x^3$

Case I *When* $-\frac{1}{2} \le x \le \frac{1}{2}$: In this case

$-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \sin\theta \le \frac{1}{2} \Rightarrow -\frac{\pi}{6} \le \theta \le \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} \le 3\theta \le \frac{\pi}{2}$

Also, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \sin 3\theta = 3x - 4x^3 \Rightarrow 3\theta = \sin^{-1}(3x - 4x^3) \Rightarrow 3\sin^{-1}x = \sin^{-1}(3x - 4x^3)$

Case II *When* $1/2 < x \le 1$: In this case,

$\frac{1}{2} < x \le 1 \Rightarrow \frac{1}{2} < \sin\theta \le 1 \Rightarrow \frac{\pi}{6} < \theta \le \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 3\theta \le \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} \le -3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi - 3\theta < \frac{\pi}{2}$

Also, $\frac{1}{2} < x \le 1 \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \sin 3\theta = 3x - 4x^3 \Rightarrow \sin(\pi - 3\theta) = (3x - 4x^3) \Rightarrow \pi - 3\theta = \sin^{-1}(3x - 4x^3)$

$\Rightarrow \pi - 3\sin^{-1}x = \sin^{-1}(3x-4x^3) \Rightarrow 3\sin^{-1}x = \pi - \sin^{-1}(3x-4x^3)$.

Case III *When* $-1 \le x < -\frac{1}{2}$: In this case, $-1 \le x < -\frac{1}{2} \Rightarrow -1 \le \sin\theta < -\frac{1}{2}$

$$\Rightarrow \quad -\frac{\pi}{2} \le \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} \le 3\theta \le -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} \le \pi + 3\theta \le \theta \Rightarrow 0 \le -\pi - \theta \le \frac{\pi}{2}$$

Also, $-\frac{1}{2} \le x < -\frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$$\therefore \quad \sin 3\theta = 3x - 4x^3$$

$$\Rightarrow \quad -\sin(\pi + 3\theta) = 3x - 4x^3 \qquad [\sin(\pi+3\theta) = -\sin 3\theta]$$

$$\Rightarrow \quad \sin(-\pi - 3\theta) = 3x - 4x^3 \qquad \Rightarrow -\pi - 3\theta = \sin^{-1}(3x-4x^3)$$

$$\Rightarrow \quad -\pi - 3\sin^{-1}x = \sin^{-1}(3x-4x^3) \Rightarrow 3\sin^{-1}x = -\pi - \sin^{-1}(3x-4x^3)$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* (i) $\sin(2\sin^{-1}0.6)$ (ii) $\sin(2\sin^{-1}0.8)$

SOLUTION (i) $\sin(2\sin^{-1}0.6)$

$$= \sin\left[\sin^{-1}\left\{2\times 0.6\times\sqrt{1-(0.6)^2}\right\}\right] \quad \left[\because 2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{ if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}\right]$$

$$= \sin(\sin^{-1}0.96) = 0.96$$

(ii) $\sin(2\sin^{-1}0.8)$

$$= \sin\left[\pi - \sin^{-1}\left\{2\times 0.8\times\sqrt{1-(0.8)^2}\right\}\right] \left[\because 2\sin^{-1}x = \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{ if, } \frac{1}{\sqrt{2}} \le x < 1\right]$$

$$= \sin(\pi - \sin^{-1}0.96) = \sin\left\{\sin^{-1}(0.96)\right\} = 0.96 \qquad [\because \sin(\pi-\theta) = \sin\theta]$$

EXAMPLE 2 *Evaluate:* $\sin(3\sin^{-1}0.4)$

SOLUTION Using $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$, we obtain

$$\sin(3\sin^{-1}0.4) = \sin\left[\sin^{-1}\left\{3\times 0.4 - 4\times(0.4)^3\right\}\right]$$

$$= \sin\left\{\sin^{-1}(1.2-0.256)\right\} = \sin\left\{\sin^{-1}(0.944)\right\} = 0.944$$

3.3.10 PROPERTIES X-XII

PROPERTY X *Prove that*

(i) $$2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1) & , \text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1) & , \text{if } -1 \le x \le 0 \end{cases}$$

(ii) $$3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x) & , \text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x) & , \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x) & , \text{if } -1 \le x \le -\frac{1}{2} \end{cases}$$

PROOF (i) Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

$\therefore \quad \cos 2\theta = 2\cos^2\theta - 1 \Rightarrow \cos 2\theta = 2x^2 - 1$

Case I *When* $0 \le x \le 1$: In this case,

$$0 \le x \le 1 \Rightarrow 0 \le \cos\theta \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{2} \Rightarrow 0 \le 2\theta \le \pi \text{ Also, } 0 \le x \le 1 \Rightarrow -1 \le 2x^2 - 1 \le 1$$

$$\therefore \quad \cos 2\theta = 2x^2 - 1 \Rightarrow 2\theta = \cos^{-1}(2x^2 - 1) \Rightarrow 2\cos^{-1} x = \cos^{-1}(2x^2 - 1).$$

Case II *When* $-1 \le x \le 0$: In this case,

$$-1 \le x \le 0 \Rightarrow -1 \le \cos\theta \le 0 \Rightarrow \frac{\pi}{2} \le \theta \le \pi \Rightarrow \pi \le 2\theta \le 2\pi \Rightarrow -2\pi \le -2\theta \le -\pi \Rightarrow 0 \le 2\pi - 2\theta \le \pi$$

Also, $\quad -1 \le x \le 0 \Rightarrow -1 \le 2x^2 - 1 \le 1$

$\therefore \quad \cos 2\theta = (2x^2 - 1)$

$\Rightarrow \quad \cos(2\pi - 2\theta) = (2x^2 - 1)$

$$\Rightarrow 2\pi - 2\theta = \cos^{-1}(2x^2 - 1) \Rightarrow 2\theta = 2\pi - \cos^{-1}(2x^2 - 1) \Rightarrow 2\cos^{-1} x = 2\pi - \cos^{-1}(2x^2 - 1).$$

(ii) Let $\cos^{-1} x = \theta$. Then, $x = \cos\theta$

$\therefore \quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos 3\theta = 4x^3 - 3x$

Case I *When* $\frac{1}{2} \le x \le 1$: In this case,

$$\frac{1}{2} \le x \le 1 \Rightarrow \frac{1}{2} \le \cos\theta \le 1 \Rightarrow 0 \le \theta \le \frac{\pi}{3} \Rightarrow 0 \le 3\theta \le \pi. \text{ Also, } \frac{1}{2} \le x \le 1 \Rightarrow -1 \le 4x^3 - 3x \le 1$$

$$\therefore \quad \cos 3\theta = 4x^3 - 3x \Rightarrow 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$$

Case II *When* $-\frac{1}{2} \le x \le \frac{1}{2}$: In this case,

$$-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -\frac{1}{2} \le \cos\theta \le \frac{1}{2} \Rightarrow \frac{\pi}{3} \le \theta \le \frac{2\pi}{3} \Rightarrow \pi \le 3\theta \le 2\pi \Rightarrow -2\pi \le -3\theta \le -\pi \Rightarrow 0 \le 2\pi - 3\theta \le \pi$$

$\therefore \quad \cos 3\theta = 4x^3 - 3x$

$\Rightarrow \cos(2\pi - 3\theta) = 4x^3 - 3x$

$$\Rightarrow 2\pi - 3\theta = \cos^{-1}(4x^3 - 3x) \Rightarrow 3\theta = 2\pi - \cos^{-1}(4x^3 - 3x) \Rightarrow 3\cos^{-1} x = 2\pi - \cos^{-1}(4x^3 - 3x)$$

Case III *When* $-1 \le x \le -\frac{1}{2}$: In this case,

$$-1 \le x \le -\frac{1}{2} \Rightarrow -1 \le \cos\theta \le -\frac{1}{2}$$

$$\Rightarrow \frac{2\pi}{3} \le \theta \le \pi \Rightarrow 2\pi \le 3\theta \le 3\pi \Rightarrow -3\pi \le -3\theta \le -2\pi \Rightarrow -\pi \le 2\pi - 3\theta \le 0 \Rightarrow 0 \le 3\theta - 2\pi \le \pi$$

$\therefore \quad \cos 3\theta = 4x^3 - 3x$

$\Rightarrow \cos(2\pi - 3\theta) = 4x^3 - 3x$

$\Rightarrow \cos(3\theta - 2\pi) = 4x^3 - 3x \Rightarrow 3\theta - 2\pi = \cos^{-1}(4x^3 - 3x)$

$\Rightarrow 3\theta = 2\pi + \cos^{-1}(4x^3 - 3x) \Rightarrow 3\cos^{-1} x = 2\pi + \cos^{-1}(4x^3 - 3x).$

PROPERTY XI *Prove that:*

(i) $$2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } x > 1 \\ -\pi + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) & \text{, if } x < -1 \end{cases}$$

(ii) $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x > \dfrac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) & \text{, if } x < -\dfrac{1}{\sqrt{3}} \end{cases}$$

PROOF (i) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$.

$$\therefore \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \Rightarrow \tan 2\theta = \frac{2x}{1-x^2}$$

Case I *When* $-1 < x < 1$: In this case,

$$-1 < x -1 \Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{2} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 2\theta = \frac{2x}{1-x^2} \Rightarrow 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \Rightarrow 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Case II *When* $x > 1$: In this case,

$$x > 1 \Rightarrow \tan\theta > 1$$

$$\Rightarrow \quad \frac{\pi}{2} > \theta > \frac{\pi}{4} \Rightarrow \pi > 2\theta > \frac{\pi}{2} \Rightarrow -\pi < -2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi + 2\theta < 0$$

$$\therefore \quad \tan 2\theta = \frac{2x}{1-x^2} \Rightarrow -\tan(\pi - 2\theta) = \frac{2x}{1-x^2} \Rightarrow \tan(-\pi + 2\theta) = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad -\pi + 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \Rightarrow 2\theta = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \Rightarrow 2\tan^{-1}x = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Case III *When* $x < -1$: In this case,

$$x < -1 \Rightarrow \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi + 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \tan 2\theta = \frac{2x}{1-x^2}$$

$$\Rightarrow \quad \tan(\pi + 2\theta) = \frac{2x}{1-x^2} \qquad [\because \tan(\pi + \alpha) = \tan\alpha]$$

$$\Rightarrow \quad \pi + 2\theta = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$\Rightarrow \quad \pi + 2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \Rightarrow 2\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(ii) Let $\tan^{-1} x = \theta$. Then, $x = \tan\theta$.

$\therefore \quad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \Rightarrow \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$

Case I *When* $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$: In this case

$-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2}$

$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2} \Rightarrow 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) \Rightarrow 3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Case II *When* $x > \frac{1}{\sqrt{3}}$: In this case,

$x > \frac{1}{\sqrt{3}} \Rightarrow \tan\theta > \frac{1}{\sqrt{3}} \Rightarrow \frac{\pi}{2} > \theta > \frac{\pi}{6}$

$\Rightarrow \quad \frac{\pi}{2} < 3\theta < \frac{3\pi}{2} \Rightarrow -\frac{3\pi}{2} < -3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi - 3\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < 3\theta - \pi < \frac{\pi}{2}$

$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$

$\Rightarrow \quad -\tan(\pi - 3\theta) = \frac{3x - x^3}{1 - 3x^2} \qquad [\because \tan(\pi - 3\theta) = -\tan 3\theta]$

$\Rightarrow \quad \tan(3\theta - \pi) = \frac{3x - x^3}{1 - 3x^2}$

$\Rightarrow \quad 3\theta - \pi = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow \quad 3\tan^{-1} x - \pi = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) \Rightarrow 3\tan^{-1} x = \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

Case III *When* $x < -\frac{1}{\sqrt{3}}$: In this case

$x < -\frac{1}{\sqrt{3}} \Rightarrow \tan\theta < -\frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{6} \Rightarrow -\frac{3\pi}{2} < 3\theta < -\frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < \pi + 3\theta < \frac{\pi}{2}$

$\therefore \quad \tan 3\theta = \frac{3x - x^3}{1 - 3x^2}$

$\Rightarrow \quad \tan(\pi + 3\theta) = \frac{3x - x^3}{1 - 3x^2} \qquad [\because \tan(\pi + x) = \tan x]$

$\Rightarrow \quad \pi + 3\theta = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

$\Rightarrow \quad \pi + 3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) \Rightarrow 3\tan^{-1} x = -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

PROPERTY XII *Prove that:*

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), & \text{if } -\infty < x \le 0 \end{cases}$$

PROOF (i) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$

$$\therefore \quad \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \Rightarrow \sin 2\theta = \frac{2x}{1+x^2}$$

Case I *When* $-1 \le x \le 1$: In this case,

$$-1 \le x \le 1 \Rightarrow -1 \le \tan\theta \le 1 \Rightarrow -\frac{\pi}{4} \le \theta \le \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2} \Rightarrow 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Case II *When* $x > 1$: In this case,

$$x > 1 \Rightarrow \tan\theta > 1 \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi \Rightarrow -\pi < -2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi - 2\theta < \frac{\pi}{2}$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2} \Rightarrow \sin(\pi - 2\theta) = \frac{2x}{1+x^2} \Rightarrow \pi - 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad \pi - 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow 2\tan^{-1}x = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

Case III When $x < -1$: In this case,

$$x < -1 \Rightarrow \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2} \Rightarrow 0 < \pi + 2\theta < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -\pi - 2\theta < 0$$

$$\therefore \quad \sin 2\theta = \frac{2x}{1+x^2}$$

$$\Rightarrow \quad -\sin(\pi + 2\theta) = \frac{2x}{1+x^2} \Rightarrow \sin(-\pi - 2\theta) = \frac{2x}{1+x^2} \Rightarrow -\pi - 2\theta = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\Rightarrow \quad -\pi - 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow 2\tan^{-1}x = -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

(ii) Let $\tan^{-1}x = \theta$. Then, $x = \tan\theta$.

$$\therefore \quad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta} \Rightarrow \cos 2\theta = \frac{1-x^2}{1+x^2}$$

Case I *When* $0 \le x < \infty$: In this case,

$$0 \le x < \infty \Rightarrow 0 \le \tan\theta < \infty \Rightarrow 0 \le \theta < \frac{\pi}{2} \Rightarrow 0 \le 2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2} \Rightarrow 2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \Rightarrow 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right).$$

Case II *When* $-\infty < x \le 0$: In this case,

$$-\infty < x \le 0 \Rightarrow -\infty < \tan\theta \le 0 \Rightarrow -\frac{\pi}{2} < \theta \le 0 \Rightarrow -\pi < 2\theta \le 0 \Rightarrow 0 \le -2\theta < \pi$$

$$\therefore \quad \cos 2\theta = \frac{1-x^2}{1+x^2} \Rightarrow \cos(-2\theta) = \frac{1-x^2}{1+x^2} \Rightarrow -2\theta = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad -2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) \quad \Rightarrow 2\tan^{-1}x = -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:*

$$\tan^{-1}x + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), |x| < \frac{1}{\sqrt{3}}$$ [NCERT, CBSE 2010]

SOLUTION $\tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

$$= \tan^{-1}\left\{\frac{x + \dfrac{2x}{1-x^2}}{1 - \dfrac{2x^2}{1-x^2}}\right\} \quad \left[\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{ if } xy < 1\right]$$

$$= \tan^{-1}\left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2}\right) = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

ALITER LHS $= \tan^{-1}x + \tan^{-1}\left(\dfrac{2x}{1-x^2}\right) = \tan^{-1}x + 2\tan^{-1}x \quad \left[\because -\dfrac{1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}\right]$

$$= 3\tan^{-1}x = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \text{RHS}$$

EXAMPLE 2 *Simplify:* $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right), -\dfrac{1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$ [NCERT]

SOLUTION $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$

$$= \tan^{-1}\left\{\frac{3\left(\dfrac{x}{a}\right) - \left(\dfrac{x}{a}\right)^3}{1 - 3\left(\dfrac{x}{a}\right)^2}\right\} = 3\tan^{-1}\frac{x}{a} \quad \left[\because \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = 3\tan^{-1}x\right]$$

EXAMPLE 3 *Prove that:* $2\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{31}{17}$

SOLUTION $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

$$= \tan^{-1}\left\{\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right\}+\tan^{-1}\frac{1}{7} \qquad \left[\because\ 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right), \text{ if } -1<x<1\right]$$

$$= \tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}\right\} = \tan^{-1}\frac{31}{17}$$

EXAMPLE 4 *Evaluate:* $\tan\left(2\tan^{-1}\frac{1}{5}\right)$ **[CBSE 2013]**

SOLUTION $\tan\left(2\tan^{-1}\frac{1}{5}\right) = \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right)\right\} = \tan\left(\tan^{-1}\frac{5}{12}\right)=\frac{5}{12}$

EXAMPLE 5 *Prove that:* $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$ **[NCERT EXEMPLAR]**

SOLUTION $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}$

$$=2\tan^{-1}\frac{3}{4}-\tan^{-1}\frac{17}{31} \qquad \left[\because\ \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}\right]$$

$$=\tan^{-1}\left\{\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right\}-\tan^{-1}\frac{17}{31} \qquad \left[\because\ 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2} \text{ for } |x|<1\right]$$

$$=\tan^{-1}\frac{24}{7}-\tan^{-1}\frac{17}{13}=\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)=\tan^{-1}1=\frac{\pi}{4}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 6 *Prove that:* $\tan\frac{1}{2}\left\{\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right\}=\frac{x+y}{1-xy}$, *if* $|x|<1, y>0$ *and* $xy<1$. **[NCERT, CBSE 2013]**

SOLUTION We know that

$$\cos^{-1}\frac{1-y^2}{1+y^2}=2\tan^{-1}y \text{ for all } y\ge 0 \text{ and, } \sin^{-1}\frac{2x}{1+x^2}=2\tan^{-1}x \text{ for all } x\in[-1,1]$$

$$\therefore\quad \text{LHS}=\tan\frac{1}{2}\left\{\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right\} = \tan\frac{1}{2}\left\{2\tan^{-1}x+2\tan^{-1}y\right\}$$

$$=\tan\left(\tan^{-1}x+\tan^{-1}y\right)=\tan\left\{\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}=\frac{x+y}{1-xy}=\text{RHS} \qquad [\because\ xy<1]$$

EXAMPLE 7 *Find the value of:* $\sin\left\{2\cot^{-1}\left(-\frac{5}{12}\right)\right\}$. **[NCERT EXEMPLAR]**

SOLUTION $\sin\left\{2\cot^{-1}\left(-\frac{5}{12}\right)\right\}=\sin\left\{2\left(\pi-\cot^{-1}\frac{5}{12}\right)\right\}$ $\quad[\because \cot^{-1}(-x)=\pi-\cot^{-1}x]$

$$=\sin\left(2\pi-2\cot^{-1}\frac{5}{12}\right)=-\sin\left(2\cot^{-1}\frac{5}{12}\right) \qquad [\because \sin(2\pi-\theta)=-\sin\theta]$$

$$=-\sin\left(2\tan^{-1}\frac{12}{5}\right) \qquad \left[\because \cot^{-1}x=\tan^{-1}\frac{1}{x}\text{ for } x>0\right]$$

$$=-\sin\left\{\pi-\sin^{-1}\left(\frac{2\times\frac{12}{5}}{1+\left(\frac{12}{5}\right)^2}\right)\right\} \qquad \left[\because 2\tan^{-1}x=\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right)\text{ for } x>1\right]$$

$$=-\sin\left(\pi-\sin^{-1}\frac{120}{169}\right)=-\sin\left(\sin^{-1}\frac{120}{169}\right)=-\frac{120}{169}$$

EXAMPLE 8 *Show that:* $2\tan^{-1}(-3)=-\frac{\pi}{2}+\tan^{-1}\left(-\frac{4}{3}\right)$. **[NCERT EXEMPLAR]**

SOLUTION $\text{LHS}=2\tan^{-1}(-3)=-2\tan^{-1}3$ $\quad[\because \tan^{-1}(-x)=-\tan^{-1}x]$

$$=-\left\{\pi+\tan^{-1}\left(\frac{2\times 3}{1-3^2}\right)\right\} \qquad \left[\because 2\tan^{-1}x=\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right),\text{ if } x>1\right]$$

$$=-\pi-\tan^{-1}\left(-\frac{3}{4}\right)=-\pi+\tan^{-1}\frac{3}{4} \qquad \left[\because \tan^{-1}(-x)=-\tan^{-1}x\right]$$

$$=-\pi+\left(\frac{\pi}{2}-\cot^{-1}\frac{3}{4}\right) \qquad \left[\because \tan^{-1}x=\frac{\pi}{2}-\cot^{-1}x\right]$$

$$=-\frac{\pi}{2}-\cot^{-1}\frac{3}{4}=-\frac{\pi}{2}-\tan^{-1}\frac{4}{3} \qquad \left[\because \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\text{ for } x>0\right]$$

$$=-\frac{\pi}{2}+\tan^{-1}\left(-\frac{4}{3}\right)=\text{RHS}$$

EXAMPLE 9 *Show that:* $\cos\left(2\tan^{-1}\frac{1}{7}\right)=\sin\left(4\tan^{-1}\frac{1}{3}\right)$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\text{LHS}=\cos\left(2\tan^{-1}\frac{1}{7}\right)=\cos\left\{\cos^{-1}\left(\frac{1-\frac{1}{49}}{1+\frac{1}{49}}\right)\right\} \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\text{ for } 0\le x<\infty\right]$$

$$=\cos\left\{\cos^{-1}\left(\frac{24}{25}\right)\right\}=\frac{24}{25} \qquad \ldots(i)$$

and,

$$\text{RHS}=\sin\left(4\tan^{-1}\frac{1}{3}\right)=\sin\left\{2\left(2\tan^{-1}\frac{1}{3}\right)\right\}$$

$$=\sin\left\{2\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\frac{1}{9}}\right)\right\} \qquad \left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\text{ for } -1<x<1\right]$$

$$= \sin\left(2\tan^{-1}\frac{3}{4}\right) = \sin\left\{\sin^{-1}\left(\frac{2\times\frac{3}{4}}{1+\frac{9}{16}}\right)\right\} \quad \left[\because 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ if } -1 < x < 1\right]$$

$$= \sin\left(\sin^{-1}\frac{24}{25}\right) = \frac{24}{25} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$$

EXAMPLE 10 *Prove that:* $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x \in [0, 1]$ **[NCERT, CBSE 2010]**

SOLUTION We have,

$$\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}\left\{\frac{1-(\sqrt{x})^2}{1+(\sqrt{x})^2}\right\} = \frac{1}{2}\times 2\tan^{-1}\sqrt{x} = \tan^{-1}\sqrt{x}.$$

<u>ALITER</u> Putting $x = \tan^2\theta$, we obtain

$$\text{RHS} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \frac{1}{2}\cos^{-1}(\cos 2\theta) = \theta = \tan^{-1}\sqrt{x} = \text{LHS}$$

EXAMPLE 11 *Find the value of the expression:* $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1} 2\sqrt{2}\right)$.

[NCERT EXEMPLAR]

SOLUTION $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1} 2\sqrt{2}\right)$

$$= \sin\left\{\sin^{-1}\left(\frac{2\times\frac{1}{3}}{1+\frac{1}{9}}\right)\right\} + \cos\left(\cos^{-1}\frac{1}{3}\right) \quad \left[\because 2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{ if } -1 < x < 1\right]$$

$$= \sin\left(\sin^{-1}\frac{3}{5}\right) + \cos\left(\cos^{-1}\frac{1}{3}\right) = \frac{3}{5} + \frac{1}{3} = \frac{14}{15}$$

EXAMPLE 12 *Prove the following:*

(i) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = \frac{\pi}{4}$ (ii) $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

[CBSE 2014]

SOLUTION (i) We have,

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\left\{2\tan^{-1}\frac{1}{5}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}$$

$$= 2\tan^{-1}\left\{\frac{2\times 1/5}{1-(1/5)^2}\right\} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} \quad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}, \text{ if } |x| < 1\right]$$

$$= 2\tan^{-1}\frac{5}{12} - \left\{\tan^{-1}\frac{1}{70} - \tan^{-1}\frac{1}{99}\right\}$$

$$= \tan^{-1}\left\{\frac{2\times 5/12}{1-(5/12)^2}\right\} - \tan^{-1}\left\{\frac{\frac{1}{70}-\frac{1}{99}}{1+\frac{1}{70}\times\frac{1}{99}}\right\} = \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{29}{6931}$$

$$= \tan^{-1}\frac{120}{119} - \tan^{-1}\frac{1}{239} = \tan^{-1}\left\{\frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\times\frac{1}{239}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

(ii) $2\tan^{-1}\frac{1}{5} + \sec^{-1}\frac{5\sqrt{2}}{7} + 2\tan^{-1}\frac{1}{8}$

$$= 2\left\{\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}\right\} + \sec^{-1}\frac{5\sqrt{2}}{7}$$

$$= 2\tan^{-1}\left\{\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right\} + \tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2 - 1} \qquad \left[\because \sec^{-1}x = \tan^{-1}\sqrt{x^2-1}\right]$$

$$= 2\tan^{-1}\frac{13}{39} + \tan^{-1}\frac{1}{7} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left\{\frac{2\times 1/3}{1-(1/3)^2}\right\} + \tan^{-1}\frac{1}{7} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2},\ \text{if } |x|<1\right]$$

$$= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7} = \tan^{-1}\left\{\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right\} = \tan^{-1}1 = \frac{\pi}{4}$$

EXAMPLE 13 *Evaluate:*

(i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ (ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

SOLUTION (i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$

$$= \tan\left\{\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\frac{1}{25}}\right) - \tan^{-1}1\right\} \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right),\ \text{if } |x|<1\right]$$

$$= \tan\left\{\tan^{-1}\frac{5}{12} - \tan^{-1}1\right\} = \tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}}\right)\right\} = \tan\left\{\tan^{-1}\left(\frac{-7}{17}\right)\right\} = \frac{-7}{17}$$

(ii) Let $\cos^{-1}\frac{\sqrt{5}}{3} = \theta$. Then, $\cos\theta = \frac{\sqrt{5}}{3}$.

$$\therefore \quad \tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right) = \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}$$

$$= \sqrt{\frac{(3-\sqrt{5})^2}{(3+\sqrt{5})(3-\sqrt{5})}} = \sqrt{\frac{(3-\sqrt{5})^2}{9-5}} = \frac{3-\sqrt{5}}{2}$$

EXAMPLE 14 *Solve for* x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\,x)$.

[NCERT EXEMPLAR, CBSE 2009, 2010 C, 2014, 2016]

SOLUTION We have,

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\,x)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\,\mathrm{cosec}\,x)$$

$$\Rightarrow \quad \frac{2\cos x}{\sin^2 x} = 2\,\mathrm{cosec}\,x \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$$

EXAMPLE 15 *Solve the following equations:*

(i) $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$ [CBSE 2020] (ii) $\sin\left[2\cos^{-1}\left\{\cot(2\tan^{-1}x)\right\}\right] = 0$

(iii) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$ [NCERT EXEMPLAR, CBSE 2016]

SOLUTION (i) We have,

$$\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$$

$$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$$

$$\Rightarrow \quad \sin\left\{\sin^{-1}(1-x)\right\} = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$$

$$\Rightarrow \quad 1-x = \cos(2\sin^{-1}x)$$

$$\Rightarrow \quad 1-x = \cos\left\{\cos^{-1}(1-2x^2)\right\} \qquad [\because 2\sin^{-1}x = \cos^{-1}(1-2x^2)]$$

$$\Rightarrow \quad 1-x = (1-2x^2) \Rightarrow x = 2x^2 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0, \frac{1}{2}$$

For, $x = \frac{1}{2}$, we obtain

$$\text{LHS} = \sin^{-1}(1-x) - 2\sin^{-1}x = \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6} \neq \text{R.H.S.}$$

So, $x = 1/2$ is not a root of the given equation.

Clearly, $x = 0$ satisfies the given equation. Hence, $x = 0$ is a root of the given equation.

(ii) We have,

$$\sin\left[2\cos^{-1}\left\{\cot(2\tan^{-1}x)\right\}\right] = 0$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)\right\}\right] = 0 \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left\{\cot\left(\cot^{-1}\left(\frac{1-x^2}{2x}\right)\right)\right\}\right] = 0 \qquad \left[\because \cot^{-1}x = \tan^{-1}\frac{1}{x}\right]$$

$$\Rightarrow \quad \sin\left[2\cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$$

$$\Rightarrow \quad \sin\left[\sin^{-1}\left\{2\left(\frac{1-x^2}{2x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right\}\right] = 0 \qquad \left[\because 2\cos^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})\right]$$

$$\Rightarrow \quad \left(\frac{1-x^2}{x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2} = 0 \Rightarrow \frac{1-x^2}{x} = 0 \text{ or, } \sqrt{1-\left(\frac{1-x^2}{2x}\right)^2} = 0$$

$\Rightarrow \quad 1-x^2 = 0$ or, $\left(\frac{1-x^2}{2x}\right)^2 = 1 \Rightarrow x = \pm 1$ or, $(1-x^2)^2 = 4x^2$

Now, $(1-x^2)^2 = 4x^2 \Rightarrow (1-x^2)^2 - (2x)^2 = 0 \Rightarrow (1-x^2-2x)(1-x^2+2x) = 0$

$\Rightarrow \quad 1-x^2-2x = 0$ or, $1-x^2+2x = 0 \Rightarrow x^2+2x-1 = 0$ or, $x^2-2x-1 = 0$

$\Rightarrow \quad x = -1 \pm \sqrt{2}$ or, $x = 1 \pm \sqrt{2}$

Hence, $x = \pm 1, -1 \pm \sqrt{2}, 1 \pm \sqrt{2}$ are the roots of the given equation.

(iii) We have,

$$\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$$

$$\Rightarrow \quad \sin^{-1} x + \sin^{-1}(1-x) = \frac{\pi}{2} - \sin^{-1} x \qquad \left[\because \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x\right]$$

$$\Rightarrow \quad \sin^{-1}(1-x) = \frac{\pi}{2} - 2\sin^{-1} x$$

$$\Rightarrow \quad \sin\left\{\sin^{-1}(1-x)\right\} = \sin\left(\frac{\pi}{2} - 2\sin^{-1} x\right)$$

$$\Rightarrow \quad 1-x = \cos(2\sin^{-1} x)$$

$$\Rightarrow \quad 1-x = \cos\left\{\cos^{-1}(1-2x^2)\right\} \qquad \left[\because 2\sin^{-1} x = \cos^{-1}(1-2x^2)\right]$$

$$\Rightarrow \quad 1-x = 1-2x^2 \Rightarrow 2x^2 - x = 0 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0 \text{ or, } x = \frac{1}{2}$$

Clearly, these values satisfy the given equation. Hence, $x = 0, \frac{1}{2}$ are the roots of the given equation.

EXAMPLE 16 *Solve for* x: $\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}, -1 < x < 1$ **[CBSE 2011]**

SOLUTION We know that

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x & , \text{if } x > 0 \\ -\pi + \cot^{-1} x, & \text{if } x < 0 \end{cases} \quad \text{i.e. } \cot^{-1} x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right), & \text{if } x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right), & \text{if } x < 0 \end{cases}$$

So, following cases arise:

Case I *When* $0 < x < 1$: In this case, we have

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \qquad \ldots\text{(i)}$$

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3}$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 4\tan^{-1} x = \frac{\pi}{3} \Rightarrow \tan^{-1} x = \frac{\pi}{12} \Rightarrow x = \tan\frac{\pi}{12} = \tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

Case II *When* $-1 < x < 0$: In this case, we have

$$\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \text{...(ii)}$$

Given equation is

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{3} \Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = -\frac{2\pi}{3} \Rightarrow 4\tan^{-1}x = -\frac{2\pi}{3} \Rightarrow \tan^{-1}x = -\frac{\pi}{6} \Rightarrow x = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

Case III *When* $x = 0$: In this case, we have

$$\text{LHS} = \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}(0) + \cot^{-1}(\infty) = \frac{\pi}{2} \text{ and, RHS } = \frac{\pi}{3}$$

So, $x = 0$ is not a solution of the given equation.

Hence, $x = \frac{\sqrt{3}-1}{\sqrt{3}+1}$ and $x = \frac{1}{\sqrt{3}}$ are solutions of the given equation.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 17 *Solve :* $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$

SOLUTION The given equation is

$$\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \quad \pi - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) - \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3} \Rightarrow \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3} \quad \text{...(i)}$$

We know that

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & x \geq 0 \\ -2\tan^{-1}x, & x < 0 \end{cases} \text{ and, } \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x, & x < -1 \\ 2\tan^{-1}x, & -1 < x < 1 \\ -\pi + 2\tan^{-1}x, & x > 1 \end{cases}$$

So, we have the following cases:

Case I *When* $x < -1$: In this case, we have

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = -2\tan^{-1}x \quad \text{and} \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \pi + 2\tan^{-1}x$$

$$\therefore \quad \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$

$$\Rightarrow \quad -2\tan^{-1}x + \pi + 2\tan^{-1}x = \frac{\pi}{3} \Rightarrow \pi = \frac{\pi}{3}, \text{which is absurd.}$$

So, the equation (i) has no solution for $x < -1$.

Case II *When* $-1 < x < 0$: In this case, we have

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = -2\tan^{-1}x \quad \text{and,} \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \pi + 2\tan^{-1}x$$

$\therefore \quad \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

$\Rightarrow \quad -2\tan^{-1}x+2\tan^{-1}x=\frac{\pi}{3} \Rightarrow 0=\frac{\pi}{3}$, which is also an absurd result.

So, the equation (i) has no solution for $1<x<0$.

Case III *When* $0<x<1$: In this case, we have

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x \quad \text{and} \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$$

$\therefore \quad \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}x+2\tan^{-1}x=\frac{\pi}{3} \Rightarrow 4\tan^{-1}x=\frac{\pi}{3} \Rightarrow \tan^{-1}x=\frac{\pi}{12} \Rightarrow x=\tan\frac{\pi}{12}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=2-\sqrt{3}$

Clearly, $x=2-\sqrt{3}$ satisfies the condition $0<x<1$. Hence, $x=2-\sqrt{3}$ is a solution of equation (i).

Case IV *When* $x>1$. In this case, we have

$$\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x \quad \text{and} \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)=-\pi+2\tan^{-1}x$$

$\therefore \quad \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$

$\Rightarrow \quad 2\tan^{-1}x-\pi+2\tan^{-1}x=\frac{\pi}{3} \Rightarrow 4\tan^{-1}x=\frac{4\pi}{3} \Rightarrow \tan^{-1}x=\frac{\pi}{3} \Rightarrow x=\tan\frac{\pi}{3}=\sqrt{3}.$

Hence, $x=2-\sqrt{3}, \sqrt{3}$ are solutions of the given equation.

EXAMPLE 18 *If* $y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$, *prove that* $\sin y=\tan^2\frac{x}{2}$.

SOLUTION We have,

$$y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)=\frac{\pi}{2}-\tan^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$$

$\Rightarrow \quad y=\frac{\pi}{2}-2\tan^{-1}\left(\sqrt{\cos x}\right)$

$\Rightarrow \quad y=\frac{\pi}{2}-\cos^{-1}\left\{\frac{1-\left(\sqrt{\cos x}\right)^2}{1+\left(\sqrt{\cos x}\right)^2}\right\} \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

$\Rightarrow \quad y=\frac{\pi}{2}-\cos^{-1}\left(\frac{1-\cos x}{1+\cos x}\right) \Rightarrow y=\frac{\pi}{2}-\cos^{-1}\left(\tan^2\frac{x}{2}\right)$

$\Rightarrow \quad \cos^{-1}\left(\tan^2\frac{x}{2}\right)=\frac{\pi}{2}-y \Rightarrow \tan^2\frac{x}{2}=\cos\left(\frac{\pi}{2}-y\right) \Rightarrow \tan^2\frac{x}{2}=\sin y.$

EXAMPLE 19 *Prove that:* $\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)=2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$.

SOLUTION We have,

$$\text{RHS}=2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$$

$$= \cos^{-1}\left\{\frac{1-\tan^2\alpha/2\tan^2\beta/2}{1+\tan^2\alpha/2\tan^2\beta/2}\right\} \qquad \left[\because\ 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$= \cos^{-1}\left\{\frac{\cos^2\alpha/2\cos^2\beta/2-\sin^2\alpha/2\sin^2\beta/2}{\cos^2\alpha/2\cos^2\beta/2+\sin^2\alpha/2\sin^2\beta/2}\right\}$$

$$= \cos^{-1}\left\{\frac{(2\cos^2\alpha/2)\ (2\cos^2\beta/2)-(2\sin^2\alpha/2)\,(2\sin^2\beta/2)}{(2\cos^2\alpha/2)\,(2\cos^2\beta/2)+(2\sin^2\alpha/2)\,(2\sin^2\beta/2)}\right\}$$

$$= \cos^{-1}\left\{\frac{(1+\cos\alpha)\,(1+\cos\beta)-(1-\cos\alpha)\,(1-\cos\beta)}{(1+\cos\alpha)\,(1+\cos\beta)+(1-\cos\alpha)\,(1-\cos\beta)}\right\}=\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)=\text{LHS}.$$

EXAMPLE 20 *Show that:* $2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}=\tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right)$.

[NCERT EXEMPLAR]

SOLUTION $\text{LHS}=2\tan^{-1}\left\{\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\right\}$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}{1-\tan^2\frac{\alpha}{2}\tan^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)}\right\}=\tan^{-1}\left[\frac{2\tan\frac{\alpha}{2}\left(\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)}{1-\tan^2\frac{\alpha}{2}\left(\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\right)^2}\right]$$

$$=\tan^{-1}\left[\frac{2\tan\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\beta}{2}\right)^2-\tan^2\frac{\alpha}{2}\left(1-\tan\frac{\beta}{2}\right)^2}\right]$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)-\tan^2\frac{\alpha}{2}\left(1-2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1+2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)-\tan^2\frac{\alpha}{2}\left(1-2\tan\frac{\beta}{2}+\tan^2\frac{\beta}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)+\tan^2\frac{\beta}{2}\left(1-\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$=\tan^{-1}\left\{\frac{2\tan\frac{\alpha}{2}\left(1-\tan^2\frac{\beta}{2}\right)}{\left(1-\tan^2\frac{\alpha}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)+2\tan\frac{\beta}{2}\left(1+\tan^2\frac{\alpha}{2}\right)}\right\}$$

$$\Rightarrow \quad \text{LHS} = \tan^{-1}\left\{\frac{\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \times \dfrac{1-\tan^2\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}{\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} + \dfrac{2\tan\frac{\beta}{2}}{1+\tan^2\frac{\beta}{2}}}\right\} = \tan^{-1}\left(\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}\right) = \text{RHS}$$

EXAMPLE 21 *Prove that:* $\dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right) = (\alpha+\beta)(\alpha^2+\beta^2)$.

SOLUTION $\quad \text{LHS} = \dfrac{\alpha^3}{2}\operatorname{cosec}^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\alpha}{\beta}\right) + \dfrac{\beta^3}{2}\sec^2\left(\dfrac{1}{2}\tan^{-1}\dfrac{\beta}{\alpha}\right)$

$$= \frac{\alpha^3}{2\sin^2\theta} + \frac{\beta^3}{2\cos^2\phi}, \text{ where } \theta = \frac{1}{2}\tan^{-1}\frac{\alpha}{\beta} \text{ and } \phi = \frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}$$

$$= \frac{\alpha^3}{1-\cos 2\theta} + \frac{\beta^3}{1+\cos 2\phi} = \frac{\alpha^3}{1-\cos(\tan^{-1}\alpha/\beta)} + \frac{\beta^3}{1+\cos(\tan^{-1}\beta/\alpha)}$$

$$= \frac{\alpha^3}{1-\cos\left(\cos^{-1}\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)} + \frac{\beta^3}{1+\cos\left(\cos^{-1}\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\right)}$$

$$= \frac{\alpha^3}{1-\dfrac{\beta}{\sqrt{\alpha^2+\beta^2}}} + \frac{\beta^3}{1+\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}} = \left\{\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}-\beta} + \frac{\beta^3}{\sqrt{\alpha^2+\beta^2}+\alpha}\right\}\sqrt{\alpha^2+\beta^2}$$

$$= \left[\frac{\alpha^3\left\{\sqrt{\alpha^2+\beta^2}+\beta\right\}}{\alpha^2+\beta^2-\beta^2} + \frac{\beta^3\left\{\sqrt{\alpha^2+\beta^2}-\alpha\right\}}{\alpha^2+\beta^2-\alpha^2}\right]\sqrt{\alpha^2+\beta^2}$$

$$= \left\{\alpha\left(\sqrt{\alpha^2+\beta^2}+\beta\right) + \beta\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\right\}\sqrt{\alpha^2+\beta^2}$$

$$= \alpha(\alpha^2+\beta^2) + \beta(\alpha^2+\beta^2) = (\alpha+\beta)(\alpha^2+\beta^2) = \text{RHS}$$

EXAMPLE 22 *Prove that:*

$$\tan^{-1}\left\{\frac{\cos 2\alpha\sec 2\beta + \cos 2\beta\sec 2\alpha}{2}\right\} = \tan^{-1}\left\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\right\} + \tan^{-1}1$$

SOLUTION We have,

$$\text{RHS} = \tan^{-1}\left\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\right\} + \tan^{-1}1 = \tan^{-1}\left\{\frac{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)+1}{1-\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)+\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)}{\cos^2(\alpha+\beta)\cos^2(\alpha-\beta)-\sin^2(\alpha+\beta)\sin^2(\alpha-\beta)}\right\}$$

$$= \tan^{-1}\left\{\frac{\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2+\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2}{\{2\cos(\alpha+\beta)\cos(\alpha-\beta)\}^2-\{2\sin(\alpha+\beta)\sin(\alpha-\beta)\}^2}\right\}$$

$$= \tan^{-1}\left\{\frac{(\cos 2\beta - \cos 2\alpha)^2 + (\cos 2\alpha + \cos 2\beta)^2}{(\cos 2\alpha + \cos 2\beta)^2 - (\cos 2\beta - \cos 2\alpha)^2}\right\}$$

$$= \tan^{-1}\left\{\frac{\cos^2 2\alpha + \cos^2 2\beta}{2\cos 2\alpha \cos 2\beta}\right\} = \tan^{-1}\left\{\frac{\cos 2\alpha \sec 2\beta + \cos 2\beta \sec 2\alpha}{2}\right\} = \text{LHS}.$$

EXERCISE 3.14

BASIC

1. Evaluate the following:

(i) $\tan\left\{2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right\}$ (ii) $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$ **[CBSE 2013, NCERT EXEMPLAR]**

(iii) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$ (iv) $\sin\left(2\tan^{-1}\frac{2}{3}\right) + \cos\left(\tan^{-1}\sqrt{3}\right)$ **[NCERT EXEMPLAR]**

2. Prove the following results:

(i) $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$ (ii) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$ **[NCERT EXEMPLAR]**

(iii) $\tan^{-1}\frac{2}{3} = \frac{1}{2}\tan^{-1}\frac{12}{5}$ (iv) $\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{4}$ **[CBSE 2010]**

(v) $\sin^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{2}$ (vi) $2\sin^{-1}\frac{3}{5} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

(vii) $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{4}{7}$ **[NCERT]**

(viii) $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$ **[CBSE 2011]**

(ix) $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$ **[CBSE 2011, 2020]**

(x) $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\cos^{-1}\frac{3}{5} = \frac{1}{2}\sin^{-1}\frac{4}{5}$ **[CBSE 2010 C]**

3. Find the values of each of the following:

(i) $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$ (ii) $\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right), |x| \ge 1$

4. If $\sin^{-1}\frac{2a}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2} = \tan^{-1}\frac{2x}{1-x^2}$, then prove that $x = \frac{a-b}{1+ab}$.

BASED ON LOTS

5. Prove that:

(i) $\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{\pi}{2}$ (ii) $\sin\left\{\tan^{-1}\frac{1-x^2}{2x} + \cos^{-1}\frac{1-x^2}{1+x^2}\right\} = 1$

(iii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1$ **[CBSE 2020]**

6. If $\sin^{-1}\frac{2a}{1+a^2} + \sin^{-1}\frac{2b}{1+b^2} = 2\tan^{-1}x$, prove that $x = \frac{a+b}{1-ab}$.

7. Show that $2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$ is constant for $x \ge 1$, find that constant.

8. Solve the following equations for x:

(i) $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$

(ii) $3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$

(iii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{2\pi}{3}, x>0$ [CBSE 2010]

(iv) $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x), x\neq\frac{\pi}{2}$ [CBSE 2012]

(v) $\cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)+\frac{1}{2}\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3}$

(vi) $\tan^{-1}\left(\frac{x-2}{x-1}\right)+\tan^{-1}\left(\frac{x+2}{x+1}\right)=\frac{\pi}{4}$ [CBSE 2016]

BASED ON HOTS

9. Prove that: $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$.

10. Prove that:

$$\tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2xy}{x^2-y^2}=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2}, \text{ where } \alpha=ax-by \text{ and } \beta=ay+bx.$$

11. For any $a, b, x, y>0$, prove that:

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2},$$

where $\alpha=-ax+by, \beta=bx+ay$.

ANSWERS

1. (i) $-\frac{7}{17}$ (ii) $\frac{4-\sqrt{7}}{3}$ (iii) $\frac{1}{\sqrt{10}}$ (iv) $\frac{37}{26}$ 3. (i) $\frac{\pi}{4}$ (ii) 0 7. π

8. (i) $-\frac{461}{9}$ (ii) $\frac{1}{\sqrt{3}}$ (iii) $\frac{1}{\sqrt{3}}$ (iv) $x=n\pi+\frac{\pi}{4}, n\in Z$ (v) $x=\sqrt{3}$ (vi) $\pm\sqrt{\frac{7}{2}}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of $\sec^2(\tan^{-1}2)+\operatorname{cosec}^2(\cot^{-1}3)$ is ....

2. If $\sin^{-1}x-\cos^{-1}x=\frac{\pi}{6}$, then $x=$ ....

3. The range of $\sin^{-1}x+\cos^{-1}x+\tan^{-1}x$ is ....

4. If $\sin^{-1}x=\frac{\pi}{5}$ for some $x\in(-1,1)$, then the value of $\cos^{-1}x$ is ....

5. If $x<0$, then $\tan^{-1}x+\tan^{-1}\frac{1}{x}$ is equal to ....

6. The value of $\tan^{-1}2+\tan^{-1}3$ is ....

7. If $\tan^{-1}\frac{1}{\sqrt{3}}+\cot^{-1}x=\frac{\pi}{2}$, then $x=$ ....

8. If $\tan^{-1}x - \tan^{-1}y = \frac{\pi}{4}$, then $x - y - xy =$ ....

9. The value of $\cot(\tan^{-1}x + \cot^{-1}x)$ for all $x \in R$, is.............................. .

10. If $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{3}$, then $\sin^{-1}x + \sin^{-1}y =$ ....

11. If $x > 0$, $y > 0$, $xy > 1$, then $\tan^{-1}x + \tan^{-1}y =$ ....

12. If $3\sin^{-1}x = \pi - \cos^{-1}x$, then $x =$ ....

13. If $\tan^{-1}x + \tan^{-1}y = \frac{5\pi}{6}$, then $\cot^{-1}x + \cot^{-1}y =$ ....

14. If $\tan^{-1}x - \cot^{-1}x = \tan^{-1}\sqrt{3}$, then $x =$ ....

15. If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = -\frac{3\pi}{2}$, then $xyz =$ ....

16. The value of $\cos^{-1}\left\{\sin\left(\cos^{-1}\frac{1}{2}\right)\right\}$ is ....

17. The value of $\tan\left[\cos^{-1}\left\{\sin\left(\cot^{-1}1\right)\right\}\right]$ is ....

18. The value of $\tan^2(\sec^{-1}3) + \cot^2(\operatorname{cosec}^{-1}4)$ is ....

19. If $\tan^{-1}(\cot\theta) = 2\theta$, then $\theta =$ ....

20. The value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is ....

21. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y =$ ....

22. If $3\tan^{-1}x + \cot^{-1}x = \pi$, then $x =$..............................

23. If $\tan^{-1}2$, $\tan^{-1}3$ are measures of two angles of a triangle, then the measure of its third angle is ....

24. If $\tan^{-1}\frac{a}{x} + \tan^{-1}\frac{b}{x} = \frac{\pi}{2}$, then $x =$.................................. .

25. If $\cos(2\sin^{-1}x) = \frac{1}{9}$, then the value of x is ....

26. If $0 < x < \frac{\pi}{2}$, then $\sin^{-1}(\cos x) + \cos^{-1}(\sin x) =$ ....

27. If $\tan^{-1}x = \frac{\pi}{4} - \tan^{-1}\frac{1}{3}$, then $x =$ ....

28. If $\tan^{-1}x + \tan^{-1}\frac{1}{2} = \frac{\pi}{4}$, then $x =$.................................. .

29. $\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right)$ is equal to ....

30. $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$ is equal to ....

31. If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ for all x, then y lies in the interval....... **[NCERT EXEMPLAR]**

32. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ is true when value of xy is...........

[NCERT EXENMPLAR]

33. The value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$ is.......... **[NCERT EXEMPLAR]**

34. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is .... [CBSE 2020]

35. The range of the principal value branch of $y = \sec^{-1} x$ is .... [CBSE 2020]

ANSWERS

1. 15	2. $\frac{\sqrt{3}}{2}$	3. $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$	4. $\frac{3\pi}{10}$	5. $-\frac{\pi}{2}$
6. $\frac{3\pi}{4}$	7. $\frac{1}{\sqrt{3}}$	8. 1	9. 0	10. $\frac{2\pi}{3}$
11. $\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$		12. $\frac{1}{\sqrt{2}}$	13. $\frac{\pi}{6}$	14. $\tan\frac{5\pi}{12}$
15. -1	16. $\frac{\pi}{6}$	17. 1	18. 23	19. $\frac{\pi}{6}$
20. $-\frac{\pi}{10}$	21. $\frac{\pi}{5}$	22. 1	23. $\frac{\pi}{4}$	24. $\sqrt{ab}$
25. $\pm\frac{2}{3}$	26. $\pi - 2x$	27. $\frac{1}{2}$	28. $\frac{1}{3}$	29. 7
30. $-\frac{\pi}{3}$	31. $(-2\pi, 2\pi)$	32. $xy > -1$	33. $\pi - \cot^{-1} x$	34. $\frac{2\pi}{3}$
35. $(-\infty, -1] \cup [1, \infty)$				

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$.
2. Write the difference between maximum and minimum values of $\sin^{-1} x$ for $x \in [-1, 1]$.
3. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then write the value of $x + y + z$.
4. If $x > 1$, then write the value of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in terms of $\tan^{-1} x$.
5. If $x < 0$, then write the value of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ in terms of $\tan^{-1} x$.
6. Write the value of $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x > 0$.
7. Write the value of $\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$ for $x < 0$.
8. What is the value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$?
9. If $-1 < x < 0$, then write the value of $\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.
10. Write the value of $\sin(\cot^{-1} x)$.
11. Write the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$.
12. Write the range of $\tan^{-1} x$.
13. Write the value of $\cos^{-1}(\cos 1540°)$.

14. Write the value of $\sin^{-1}(\sin(-600°))$.

15. Write the value of $\cos\left(2\sin^{-1}\frac{1}{3}\right)$.

16. Write the value of $\sin^{-1}(\sin 1550°)$.

17. Evaluate: $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$.

18. Evaluate: $\sin\left(\tan^{-1}\frac{3}{4}\right)$.

19. Write the value of $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$.

20. Write the value of $\cos\left(2\sin^{-1}\frac{1}{2}\right)$.

21. Write the value of $\cos^{-1}(\cos 350°) - \sin^{-1}(\sin 350°)$

22. Write the value of $\cos^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$.

23. If $\tan^{-1}x + \tan^{-1}y = \frac{\pi}{4}$, then write the value of $x + y + xy$. [CBSE 2014]

24. Write the value of $\cos^{-1}(\cos 6)$.

25. Write the value of $\sin^{-1}\left(\cos\frac{\pi}{9}\right)$.

26. Write the value of $\sin\left\{\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right\}$. [CBSE 2011]

27. Write the value of $\tan^{-1}\left\{\tan\left(\frac{15\pi}{4}\right)\right\}$.

28. Write the value of $2\sin^{-1}\frac{1}{2} + \cos^{-1}\left(-\frac{1}{2}\right)$. [CBSE 2014]

29. Write the value of $\tan^{-1}\frac{a}{b} - \tan^{-1}\left(\frac{a-b}{a+b}\right)$.

30. Write the value of $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$.

31. Show that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\sin^{-1}x$

32. Evaluate: $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. [CBSE 2009]

33. If $\tan^{-1}(\sqrt{3}) + \cot^{-1}x = \frac{\pi}{2}$, find x. [CBSE 2010]

34. If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$, then find x. [CBSE 2010]

35. Write the value of $\sin^{-1}\left(\frac{1}{3}\right) - \cos^{-1}\left(-\frac{1}{3}\right)$.

36. If $4\sin^{-1}x + \cos^{-1}x = \pi$, then what is the value of x?

37. If $x < 0$, $y < 0$ such that $xy = 1$, then write the value of $\tan^{-1}x + \tan^{-1}y$.

38. What is the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$? [CBSE 2010]

39. Write the principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$. [CBSE 2011]

40. Write the principal value of $\cos^{-1}\left(\cos\frac{2\pi}{3}\right) + \sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

41. Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$. [CBSE 2013]

42. Write the principal value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$. [CBSE 2013]

43. Write the value of $\tan^{-1}\left\{2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right\}$. [CBSE 2013]

44. Write the principal value of $\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}$. [CBSE 21013]

45. Write the principal value of $\cos^{-1}(\cos 680°)$. [CBSE 2014]

46. Write the value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$. [NCERT EXEMPLAR]

47. Write the value of $\sec^{-1}\left(\frac{1}{2}\right)$. [NCERT EXEMPLAR]

48. Write the value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$. [NCERT EXEMPLAR]

49. Write the value of $\cos\left(\sin^{-1}x + \cos^{-1}x\right), |x| \le 1$. [NCERT EXEMPLAR]

50. Write the value of the expression $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right)$, when $x = \frac{\sqrt{3}}{2}$.

[NCERT EXEMPLAR]

51. Write the principal value of $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{1}{2}\right)\right\}$. [NCERT EXEMPLAR]

52. The set of values of $\operatorname{cosec}^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

53. Write the value of $\tan^{-1}\left(\frac{1}{x}\right)$ for $x < 0$ in terms of $\cot^{-1}(x)$.

54. Write the value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$.

55. Write the value of $\cos\left(\frac{\tan^{-1}x + \cot^{-1}x}{3}\right)$, when $x = -\frac{1}{\sqrt{3}}$.

56. If $\cos(\tan^{-1}x + \cot^{-1}\sqrt{3}) = 0$, find the value of x. [NCERT EXEMPLAR]

57. Find the value of $2\sec^{-1}2 + \sin^{-1}\left(\frac{1}{2}\right)$.

58. If $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right)=0$, find the value of x.

59. Find the value of $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

60. Find the value of $\tan^{-1}\left(\tan\frac{9\pi}{8}\right)$.

61. Find the value of $\tan^{-1}\sqrt{3}-\cot^{-1}(-\sqrt{3})$. [CBSE 2018]

62. Find the value of $\sin^{-1}\left\{\sin\left(\frac{-17\pi}{8}\right)\right\}$. [CBSE 2020]

63. Two angles of a triangle are $\cot^{-1}2$ and $\cot^{-1}3$. The third angle of the triangle is...... [CBSE 2020]

ANSWERS

1. $\frac{\pi}{3}$	2. π	3. 3	4. $\pi-2\tan^{-1}x$		5. $-2\tan^{-1}x$
6. $\frac{\pi}{2}$	7. $-\frac{\pi}{2}$	8. π	9. 0	10. $\frac{1}{\sqrt{1+x^2}}$	11. $\frac{2\pi}{3}$
12. $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	13. $100°$	14. $60°$	15. $\frac{7}{9}$	16. 70	17. $\frac{1}{\sqrt{10}}$
18. $\frac{3}{5}$	19. π	20. $\frac{1}{2}$	21. $20°$	22. $\frac{4}{5}$	23. 1
24. $2\pi-6$	25. $\frac{7\pi}{18}$	26. 1	27. $-\frac{\pi}{4}$	28. π	29. $\frac{\pi}{4}$
30. $\frac{3\pi}{4}$	32. $\frac{2\pi}{5}$	33. $\sqrt{3}$	34. $\frac{1}{3}$	35. $-\frac{\pi}{2}$	36. $\frac{1}{2}$
37. $-\frac{\pi}{2}$	38. $-\frac{\pi}{3}$	39. $\frac{-\pi}{6}$	40. π	41. $\frac{5}{12}$	42. $\frac{11\pi}{12}$
43. $\frac{\pi}{3}$	44. $\frac{\pi}{2}$	45. $40°$	46. $\frac{2\pi}{5}$	47. ϕ	48. $\frac{2\pi}{3}$
49. 0	50. 1	51. $\frac{\pi}{3}$	52. ϕ	53. $-\pi+\cot^{-1}x$	54. $\pi-\cot^{-1}x$
55. $\frac{\sqrt{3}}{2}$	56. $\sqrt{3}$	57. $\frac{5\pi}{6}$	58. $\frac{2}{5}$	59. $\frac{\pi}{6}$	60. $\frac{\pi}{8}$
61. $-\frac{\pi}{2}$	62. $-\frac{\pi}{8}$	63. $\frac{3\pi}{4}$			