CHAPTER 22

# ALGEBRA OF VECTORS

## 22.1 INTRODUCTION

Physical quantities are divided into two categories — scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space are called *scalar quantities*, or briefly scalars. Examples of scalars are mass, volume, density work, temperature etc. To represent a scalar quantity we assign a real number to it which gives its magnitude in terms of a certain basic unit of the quantity. Throughout this chapter by scalars we shall mean real numbers. Second kind of quantities are those which have both magnitude and direction. Such quantities are called *vectors*. Displacement, velocity, acceleration, momentum, weight, force etc. are examples of vector quantities.

NOTE *It is to note here that in addition to magnitude and direction, two vector quantities of the same kind should be capable of being compounded according to the parallelogram law of addition. Quantities having magnitude and direction but not obeying the parallelogram law of addition will not be treated as vectors. For example, the rotations of a rigid body through finite angles have both magnitudes and directions but do not satisfy the parallelogram law of addition.*

## 22.2 REPRESENTATION OF VECTORS

Vectors are represented by directed line segments such that the length of the line segment is the magnitude of the vector and the direction of arrow marked at one end emphasizes the direction of the vector. A vector, denoted by $\vec{PQ}$, is determined by two points $P$, $Q$ such that the magnitude of the vector is the length of the straight line $PQ$ and its direction is that from $P$ to $Q$. The point $P$ is called the initial point of vector $\vec{PQ}$ and $Q$ is called the terminal point or tip. Vectors are generally denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$ etc.

The modulus, or module, or magnitude of a vector $\vec{a}$ is the positive number which is the measure of its length and is denoted by $|\vec{a}|$. The modulus $|\vec{a}|$ of a vector $\vec{a}$ is sometimes written as **a**.

Every vector $\vec{PQ}$ has the following three characteristics:

**LENGTH** *The length of $\vec{PQ}$ will be denoted by $|\vec{PQ}|$ or $PQ$.*

**SUPPORT** *The line of unlimited length of which $PQ$ is segment is called the support of the vector $\vec{PQ}$.*


Fig. 22.1

**SENSE** *The sense of $\vec{PQ}$ is from $P$ to $Q$ and that of $\vec{QP}$ is from $Q$ to $P$. Thus, the sense of a directed line segment is from its initial point to the terminal point.*

**EQUALITY VECTORS** *Two vectors $\vec{a}$ and $\vec{b}$ are said to be equal, written as $\vec{a}=\vec{b}$, if they have (i) the same length (ii) the same or parallel support, and (iii) the same sense.*

## 22.3 TYPES OF VECTORS

**ZERO OR NULL VECTOR** *A vector whose initial and terminal points are coincident is called the zero or the null vector.*

Thus, the modulus of the null vector is zero but it can be thought of as having any line as its line of support. The null vector is denoted by $\vec{0}$.

Vectors other than the null vector are called proper vectors.

**UNIT VECTOR** *A vector whose modulus is unity, is called a unit vector. The unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$, read as 'a cap'.*

*Thus,* $|\hat{a}|=1$.

**LIKE AND UNLIKE VECTORS** *Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions.*

**COLLINEAR OR PARALLEL VECTORS** *Vectors having the same or parallel supports are called collinear vectors.*

**CO-INITIAL VECTORS** *Vectors having the same initial point are called co-initial vectors.*

**CO-PLANAR VECTORS** *A system of vectors is said to be coplanar, if their supports are parallel to the same plane.*

Note that two vectors are always coplanar.

**COTERMINOUS VECTORS** *Vectors having the same terminal point are called coterminous vectors.*

**NEGATIVE OF A VECTOR** *The vector which has the same magnitude as the vector $\vec{a}$ but opposite direction, is called the negative of $\vec{a}$ and is denoted by $-\vec{a}$.*

Thus, if $\overrightarrow{PQ}=\vec{a}$, then $\overrightarrow{QP}=-\vec{a}$.

**RECIPROCAL OF A VECTOR** *A vector having the same direction as that of a given vector $\vec{a}$ but magnitude equal to the reciprocal of the given vector is known as the reciprocal of $\vec{a}$ and is denoted by $\vec{a}^{-1}$.*

Thus, if $|\vec{a}|=a$, $|\vec{a}^{-1}|=\frac{1}{a}$.

**LOCALIZED AND FREE VECTORS** *A vector which is drawn parallel to a given vector through a specified point in space is called a localized vector.*

For example, a force acting on a rigid body is a localized vector as its effect depends on the line of action of the force.

*If the value of a vector depends only on its length and direction and is independent of its position in the space, it is called a free vector.*

In this chapter, we will be dealing with free vectors, unless otherwise stated. Thus, a free vector can be taken anywhere in space by choosing an arbitrary initial point.

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *Represent graphically (i) a displacement of* 40 *km,* 30° *west of south. (ii)* 60 *km,* 40° *east of north (iii)* 50 *km south-east.* **[NCERT]**

SOLUTION In Fig. 22.2

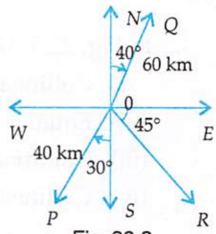

Fig. 22.2

(i) The vector $\vec{OP}$ represents the required displacement vector.

(ii) The vector $\vec{OQ}$ represents the required vector.

(iii) $\vec{OR}$ represents the required vector.

EXAMPLE 2 *Classify the following measures as scalars and vectors*

(i) 10 kg (ii) 10 meters north-west (iii) 10 Newton

(iv) 30 km/hr (v) 50 m/sec towards north (vi) $10^{-19}$ coloumb. [NCERT]

SOLUTION (i) Mass-scalar (ii) Directed distance-vector (iii) Force-vector

(iv) Speed-scalar (v) Velocity-vector (vi) Electric charge-scalar.

EXAMPLE 3 *In Fig. 22.3, which of the vectors are:*

(i) *collinear* (ii) *Equal* (iii) *Co-initial* [NCERT]

SOLUTION Clearly,

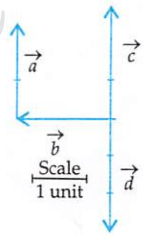

Fig. 22.3

(i) $\vec{a}$, $\vec{c}$ and $\vec{d}$ are collinear vectors.

(ii) $\vec{a}$ and $\vec{c}$ are equal vectors each of magnitude 2 units.

(iii) $\vec{b}$, $\vec{c}$ and $\vec{d}$ are co-initial vectors.

EXAMPLE 4 *In Fig. 22.4 (a square), identify the following vectors:*

(i) *Coinitial* (ii) *Equal* (iii) *Collinear but not equal.* [NCERT]

SOLUTION Clearly,

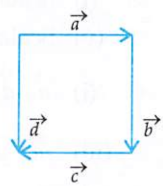

Fig. 22.4

(i) $\vec{a}$, $\vec{d}$ are co-initial vectors

(ii) $\vec{d}$ and $\vec{b}$ are equal vectors.

(iii) $\vec{a}$ and $\vec{c}$ are collinear but not equal vectors.

## EXERCISE 22.1

### BASIC

1. Represent the following graphically: [NCERT]
   (i) a displacement of 40 km, 30° east of north
   (ii) a displacement of 50 km south-east
   (iii) a displacement of 70 km, 40° north of west.
2. Classify the following measures as scalars and vectors: [NCERT]
   (i) 15 kg (ii) 20 kg weight (iii) 45°
   (iv) 10 meters south-east (v) 50 m/sec$^2$
3. Classify the following as scalars and vector quantities: [NCERT]
   (i) Time period (ii) Distance (iii) Displacement
   (iv) Force (v) Work (vi) Velocity
   (vii) Acceleration

4. In Fig. 22.5 *ABCD* is a regular hexagon, which vectors are:

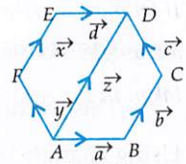

Fig. 22.5

(i) Collinear
(ii) Equal
(iii) Coinitial
(iv) Collinear but not equal.

5. Answer the following as true or false:

(i) $\vec{a}$ and $\vec{a}$ are collinear.
(ii) Two collinear vectors are always equal in magnitude. **[NCERT]**
(iii) Zero vector is unique.
(iv) Two vectors having same magnitude are collinear.
(v) Two collinear vectors having the same magnitude are equal.

### ANSWERS

1.

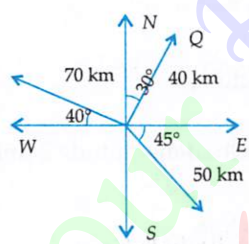

Fig. 22.6

2. (i) Mass — scalar (ii) Weight (Force) — vector (iii) Angle — scalar (iv) Directed Distance — vector (v) Magnitude of acceleration — scalar.

3. (i) Scalar (ii) Scalar (iii) Vector (iv) Vector (v) Scalar (vi) Vector (vii) Vector.

4. (i) $\vec{a}, \vec{d};\ \vec{b}, \vec{x}, \vec{z};\ \vec{c}, \vec{y}$ (ii) $\vec{b}, \vec{x};\ \vec{a}, \vec{d};\ \vec{c}, \vec{y}$

(iii) $\vec{a}, \vec{y}, \vec{z};\ \vec{x}, \vec{d}$ (iv) $\vec{b}, \vec{z};\ \vec{x}, \vec{z}$

5. (i) T (ii) F (iii) F (iv) F (v) F

## 22.4 PARALLELOGRAM LAW OF ADDITION OF VECTORS

*If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their sum $\vec{c}$ is represented by the diagonal of the parallelogram which is coinitial with the given vectors.*

Symbolically, we have

$$\vec{OP} + \vec{OQ} = \vec{OR} \text{ or, } \vec{a} + \vec{b} = \vec{c}$$

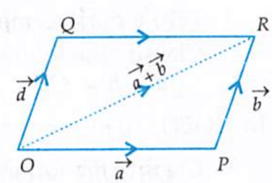

Fig. 22.7

From Fig. 22.7, we obtain

$$\vec{PR} = \vec{OQ} = \vec{b}$$

Therefore, in triangle *OPR*, we obtain

$$\vec{OP} + \vec{PR} = \vec{OR}.$$

*It follows from this result that if two vectors are represented in magnitude and direction by the two sides of a triangle taken in the same order, then their sum is represented by the third side taken in the reverse order. This is called the triangle law of addition of vectors.*

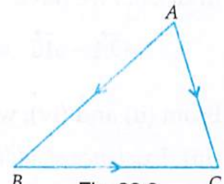

Fig. 22.8

Using triangle law of addition of vectors in $\Delta\, ABC$, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC},\ \vec{BC} + \vec{CA} = \vec{BA} \text{ and, } \vec{BA} + \vec{AC} = \vec{BC}$$

NOTE *It should be noted that the magnitude of $\vec{a} + \vec{b}$ is not equal to the sum of the magnitudes of $\vec{a}$ and $\vec{b}$.*

## 22.5 PROPERTIES OF ADDITION OF VECTORS

In this section, we shall learn some properties of addition of vectors.

*Commutativity: For any two vectors $\vec{a}$ and $\vec{b}$, prove that $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.*

PROOF Let the vectors $\vec{a}$ and $\vec{b}$ be represented by $\vec{OA}$ and $\vec{AB}$ respectively. Complete the parallelogram $OABC$.

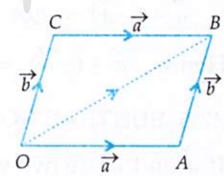

Fig. 22.9

We have, $\vec{CB} = \vec{OA} = \vec{a}$ and $\vec{OC} = \vec{AB} = \vec{b}$

In $\Delta\, OAB$, we have

$$\vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB} \qquad \text{...(i)}$$

In $\Delta\, OCB$, we have

$$\vec{OC} + \vec{CB} = \vec{OB} \qquad \text{[By triangle law]}$$

$$\Rightarrow \quad \vec{b} + \vec{a} = \vec{OB} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

(ii) *Associativity: For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, we prove that $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.*

PROOF Let the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ be represented by $\vec{OA}$, $\vec{AB}$ and $\vec{BC}$ respectively. Join $O, B$; $O, C$ and $A, C$.

In triangle $OAB$, we have

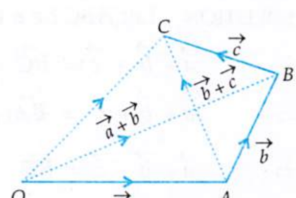

Fig. 22.10

$$\vec{OA} + \vec{AB} = \vec{OB} \qquad \text{[By triangle law]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} = \vec{OB} \qquad \text{...(i)}$$

In $\Delta OBC$,

$$\vec{OB} + \vec{BC} = \vec{OC} \qquad \text{[By triangle law ]}$$

$$\Rightarrow \quad (\vec{a} + \vec{b}) + \vec{c} = \vec{OC} \qquad \text{[Using (i)]} \quad \text{...(ii)}$$

In $\Delta ABC$, we have

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{b} + \vec{c} = \vec{AC} \qquad \text{...(iii)}$$

In $\Delta OAC$, we have

$$\vec{OA} + \vec{AC} = \vec{OC} \Rightarrow \vec{a} + (\vec{b} + \vec{c}) = \vec{OC} \qquad \text{[Using (iii)]} \quad \text{...(iv)}$$

From (ii) and (iv), we obtain: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

(iii) *Existence of additive identity: For every vector $\vec{a}$, prove that:*

$$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}, \text{ where } \vec{0} \text{ is the null vector.}$$

PROOF Let $\vec{OA} = \vec{a}$. Then, $\vec{a} + \vec{0} = \vec{OA} + \vec{AA} = \vec{OA}$ and $\vec{0} + \vec{a} = \vec{OO} + \vec{OA} = \vec{OA} = \vec{a}$.

Hence, $\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$.

(iv) *Existence of additive inverse: Prove that for every vector $\vec{a}$, there corresponds a vector $-\vec{a}$ such that*

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$$

PROOF Let $\vec{OA} = \vec{a}$. Then, $\vec{AO} = -\vec{a}$

$\therefore \quad \vec{a} + (-\vec{a}) = \vec{OA} + \vec{AO} = \vec{OO} = \vec{a}$ and, $(-\vec{a}) + \vec{a} = \vec{AO} + \vec{OA} = \vec{AA} = \vec{0}$ [By triangle law]

Hence, $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = \vec{0}$

## 22.6 SUBTRACTION OF VECTORS

If $\vec{a}$ and $\vec{b}$ are two vectors, then the subtraction of $\vec{b}$ from $\vec{a}$ is defined as the vector sum of $\vec{a}$ and $-\vec{b}$ and it is denoted by $\vec{a} - \vec{b}$

i.e., $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$.

Thus, to subtract vector $\vec{b}$ from vector $\vec{a}$, we reverse the direction of vector $\vec{b}$ and add it to vector $\vec{a}$ as shown in Fig. 22.11.

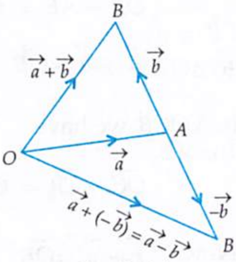


Fig. 22.11

### ILLUSTRATIVE EXAMPLES

#### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the vectors represented by the sides of a triangle, taken in order, then prove that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.*

SOLUTION Let $ABC$ be a triangle such that $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then,

$$\vec{a} + \vec{b} + \vec{c} = \vec{BC} + \vec{CA} + \vec{AB}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{BA} + \vec{AB} \qquad [\because \vec{BC} + \vec{CA} = \vec{BA}]$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{BB} \qquad \text{[By triangle law]}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{0} \qquad \text{[By definition of null vector]}$$

Hence, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

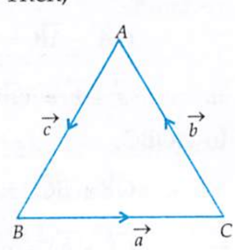


Fig. 22.12

**EXAMPLE 2** *If $P_1, P_2, P_3, P_4$ are points in a plane or space and O is the origin of vectors, show that $P_4$ coincides with O iff $\vec{OP}_1 + \vec{P_1P_2} + \vec{P_2P_3} + \vec{P_3P_4} = \vec{0}$.*

SOLUTION We have,

$$\vec{OP}_1 + \vec{P_1P_2} + \vec{P_2P_3} + \vec{P_3P_4} = \vec{0}$$

$\Rightarrow \quad (\vec{OP}_1 + \vec{P_1P_2}) + \vec{P_2P_3} + \vec{P_3P_4} = \vec{0}$ [By associativity of vector addition]

$\Rightarrow \quad (\vec{OP}_2 + \vec{P_2P_3}) + \vec{P_3P_4} = \vec{0}$ [By triangle law: $\vec{OP}_1 + \vec{P_1P_2} = \vec{OP}_2$]

$\Rightarrow \quad \vec{OP}_3 + \vec{P_3P_4} = \vec{0}$ [By triangle law: $\vec{OP}_2 + \vec{P_2P_3} = \vec{OP}_3$]

$\Rightarrow \quad \vec{OP}_4 = \vec{0}$ [By triangle law: $\vec{OP}_3 + \vec{P_3P_4} = \vec{OP}_4$]

$\Rightarrow \quad P_4$ coincides with $O$ [By definition of null vector]

**EXAMPLE 3** *If $\vec{PO} + \vec{OQ} = \vec{QO} + \vec{OR}$, show that the points P, Q, R are collinear.*

SOLUTION We have,

$\vec{PO} + \vec{OQ} = \vec{QO} + \vec{OR} \Rightarrow \vec{PQ} = \vec{QR}$ [By triangle law of addition of vectors]

Thus, $\vec{PQ}$ and $\vec{QR}$ are either parallel or collinear. But, $Q$ is a point common to them. So, $\vec{PQ}$ and $\vec{QR}$ are collinear. Hence, points $P, Q, R$ are collinear.

**EXAMPLE 4** *If $\vec{a}, \vec{b}$ are any two vectors, then give the geometrical interpretation of the relation*

$$|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$$

SOLUTION Let $\vec{OA} = \vec{a}$ and $\vec{AB} = \vec{b}$. Complete the parallelogram $OACB$. Then, $\vec{OC} = \vec{b}$ and $\vec{CB} = \vec{a}$.

In $\Delta OAB$, we have

$$\vec{OA} + \vec{AB} = \vec{OB} \Rightarrow \vec{a} + \vec{b} = \vec{OB} \quad ...(i)$$

In $\Delta OCA$, we have

$$\vec{OC} + \vec{CA} = \vec{OA} \Rightarrow \vec{b} + \vec{CA} = \vec{a} \Rightarrow \vec{CA} = \vec{a} - \vec{b} \quad ...(ii)$$

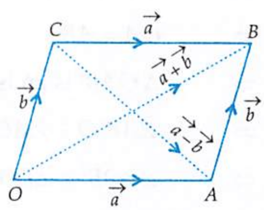

Fig. 22.13

Now, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$\Rightarrow \quad |\vec{OB}| = |\vec{CA}|$

$\Rightarrow \quad OB = CA$

$\Rightarrow \quad$ Diagonals of parallelogram $OACB$ are equal $\Rightarrow OABC$ is a rectangle $\Rightarrow OA \perp OC$

$\Rightarrow \quad \vec{a} \perp \vec{b}$.

Thus, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \Rightarrow \vec{a} \perp \vec{b}$ i.e. if the diagonals of a parallelogram are equal, then it is a rectangle.

**EXAMPLE 5** *If $\vec{a}$ and $\vec{b}$ are the vectors determined by two adjacent sides of a regular hexagon, what are the vectors determined by the other sides taken in order?*

SOLUTION Let $ABCDEF$ be a regular hexagon such that $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.

We know that $AD$ is parallel to $BC$ such that $AD = 2\,BC$.

$\therefore \quad \vec{AD} = 2\,\vec{BC} = 2\,\vec{b}$

In $\Delta ABC$, we have

$$\vec{AB} + \vec{BC} = \vec{AC} \qquad \text{[By triangle law of addition]}$$

$$\Rightarrow \qquad \vec{a} + \vec{b} = \vec{AC}$$

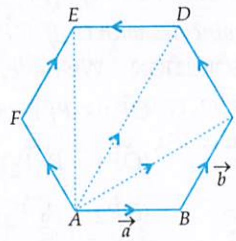

Fig. 22.14

In $\Delta\, ACD$, we have

$$\vec{AC} + \vec{CD} = \vec{AD}$$

$$\Rightarrow \qquad \vec{CD} = \vec{AD} - \vec{AC} = 2\vec{b} - (\vec{a} + \vec{b}) = \vec{b} - \vec{a}$$

Clearly,

$$\vec{DE} = -\vec{AB} = -\vec{a}, \; \vec{EF} = -\vec{BC} = -\vec{b}, \text{ and, } \vec{FA} = -\vec{CD} = -(\vec{b} - \vec{a}) = \vec{a} - \vec{b}$$

EXAMPLE 6 *If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.* **[CBSE 2019]**

SOLUTION Let $\hat{a}$ and $\hat{b}$ be two unit vectors represented by sides $OA$ and $AB$ of a triangle $OAB$. Then, $\vec{OA} = \hat{a}$, $\vec{AB} = \hat{b}$ and $\vec{OB} = \vec{OA} + \vec{AB} = \hat{a} + \hat{b}$. [By triangle law of addition]

It is given that $\hat{a}, \hat{b}$ and $\hat{a} + \hat{b}$ are unit vectors.

$\therefore \qquad |\hat{a}| = |\hat{b}| = |\hat{a} + \hat{b}| = 1 \Rightarrow OA = AB = OB = 1 \Rightarrow \Delta OAB$ is an equilateral triangle

Since $\hat{a}$ and $\hat{b}$ are unit vectors. Therefore, $\hat{a}$ and $-\hat{b}$ are also unit vectors.

$$\therefore \qquad |\hat{a}| = 1 \text{ and } |-\hat{b}| = |\hat{b}| = 1$$

$$\Rightarrow \qquad |\hat{a}| = |-\hat{b}|$$

$$\Rightarrow \qquad OA = AB'$$

$\Rightarrow \qquad \Delta\, OAB'$ is an isosceles triangle

$$\Rightarrow \qquad \angle AB'O = \angle AOB' = 30^\circ$$

$$\Rightarrow \qquad \angle BOB' = \angle BOA + \angle AOB' = 60^\circ + 30^\circ = 90^\circ$$

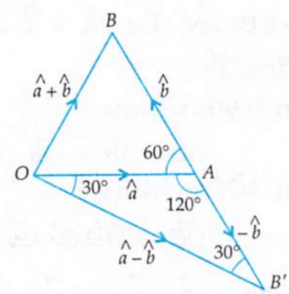

Fig. 22.15

Applying Pythagoras theorem in $\Delta BOB'$, we obtain

$$BB'^2 = OB^2 + OB'^2 \Rightarrow 2^2 = |\hat{a} + \hat{b}|^2 + |\hat{a} - \hat{b}|^2 \Rightarrow 2^2 = 1^2 + |\hat{a} - \hat{b}|^2 \Rightarrow |\hat{a} - \hat{b}| = \sqrt{3}.$$

EXAMPLE 7 *If $\vec{a}$ and $\vec{b}$ represent two adjacent sides $\vec{AB}$ and $\vec{BC}$ respectively of a parallelogram $ABCD$, then show that its diagonals $\vec{AC}$ and $\vec{DB}$ are equal to $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ respectively.*

SOLUTION It is given that $ABCD$ is a parallelogram.

$$\therefore \qquad AB = DC \text{ and } AD = BC \Rightarrow \vec{DC} = \vec{AB} = \vec{a} \text{ and } \vec{AD} = \vec{BC} = \vec{b}$$

In $\Delta\, ABC$, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{a} + \vec{b} = \vec{AC}$$

In $\Delta\, ABD$, we obtain

$$\vec{AD} + \vec{DB} = \vec{AB} \Rightarrow \vec{b} + \vec{DB} = \vec{a} \Rightarrow \vec{DB} = \vec{a} - \vec{b}$$

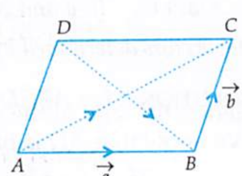

Fig. 22.16

Hence, $\vec{AC} = \vec{a} + \vec{b}$ and $\vec{DB} = \vec{a} - \vec{b}$.

**EXAMPLE 8** *Vectors drawn from the origin O to the points A, B and C are respectively $\vec{a}$, $\vec{b}$ and $4\vec{a} - 3\vec{b}$. Find $\vec{AC}$ and $\vec{BC}$.*

SOLUTION It is given that $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = 4\vec{a} - 3\vec{b}$.

In $\Delta OAC$, we obtain

$$\vec{OA} + \vec{AC} = \vec{OC}$$

$$\Rightarrow \quad \vec{AC} = \vec{OC} - \vec{OA} = 4\vec{a} - 3\vec{b} - \vec{a} = 3\vec{a} - 3\vec{b} = 3(\vec{a} - \vec{b})$$

In $\Delta OBC$, we obtain

$$\vec{OB} + \vec{BC} = \vec{OC}$$

$$\Rightarrow \quad \vec{BC} = \vec{OC} - \vec{OB} = 4\vec{a} - 3\vec{b} - \vec{b} = 4(\vec{a} - \vec{b})$$

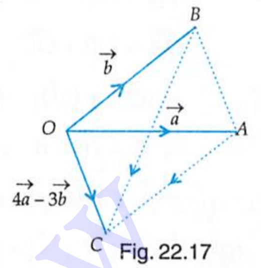

Fig. 22.17

**BASED ON LOWER ORDER THINKING SKILLS (LOTS)**

**EXAMPLE 9** *If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are distinct non-zero vectors represented by directed line segments from the origin to the points A, B, C and D respectively, and if $\vec{b} - \vec{a} = \vec{c} - \vec{d}$, then prove that ABCD is a parallelogram.*

SOLUTION Let $O$ be the origin. It is given that $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$ and $\vec{OD} = \vec{d}$ such that

$$\vec{b} - \vec{a} = \vec{c} - \vec{d} \Rightarrow \vec{OB} - \vec{OA} = \vec{OC} - \vec{OD} \qquad \text{...(i)}$$

In $\Delta OAB$, we obtain

$$\vec{OA} + \vec{AB} = \vec{OB} \Rightarrow \vec{OB} - \vec{OA} = \vec{AB} \qquad \text{...(ii)}$$

In $\Delta OCD$, we obtain

$$\vec{OC} + \vec{CD} = \vec{OD} \Rightarrow \vec{OC} - \vec{OD} = -\vec{CD} \Rightarrow \vec{OC} - \vec{OD} = \vec{DC} \qquad \text{...(iii)}$$

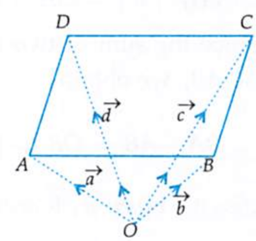

Fig. 22.18

From (i), (ii) and (iii), we get: $\vec{AB} = \vec{DC}$. Hence, $ABCD$ is a parallelogram.

**EXAMPLE 10** *A B, P, Q and R are five points in a plane. Show that the sum of the vectors $\vec{AP}$, $\vec{AQ}$, $\vec{AR}$, $\vec{PB}$, $\vec{QB}$ and $\vec{RB}$ is $3\vec{AB}$.*

SOLUTION Applying triangle law of addition of vectors in $\Delta$s, $APB$, $AQB$ and $ARB$, we obtain

$$\vec{AP} + \vec{PB} = \vec{AB}, \vec{AQ} + \vec{QB} = \vec{AB} \text{ and, } \vec{AR} + \vec{RB} = \vec{AB}$$

Adding all these, we get

$$\vec{AP} + \vec{PB} + \vec{AQ} + \vec{QB} + \vec{AR} + \vec{RB} = 3\vec{AB}$$

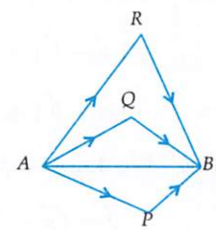

Fig. 22.19

Hence, the sum of the vectors $\vec{AP}$, $\vec{AQ}$, $\vec{AR}$, $\vec{PB}$, $\vec{QB}$ and $\vec{RB}$ is $3\vec{AB}$.

**EXAMPLE 11** *Let O be the centre of a regular hexagon ABCDEF. Find the sum of the vectors $\vec{OA}$, $\vec{OB}$, $\vec{OC}$, $\vec{OD}$, $\vec{OE}$ and $\vec{OF}$.*

SOLUTION We know that the centre of a regular hexaong bisects all the diagonals passing through it.

$\therefore \quad \vec{OA} = -\vec{OD}, \vec{OB} = -\vec{OE}$ and $\vec{OC} = -\vec{OF}$

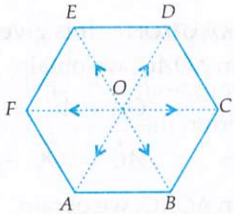

Fig. 22.20

$\Rightarrow \quad \vec{OA} + \vec{OD} = \vec{0}, \vec{OB} + \vec{OE} = \vec{0}$ and $\vec{OC} + \vec{OF} = \vec{0}$ ...(i)

$\therefore \quad \vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} + \vec{OE} + \vec{OF}$

$= (\vec{OA} + \vec{OD}) + (\vec{OB} + \vec{OE}) + (\vec{OC} + \vec{OF})$

$= \vec{0} + \vec{0} + \vec{0} = \vec{0}$ [Using (i)]

**EXAMPLE 12** *For any two vectors $\vec{a}$ and $\vec{b}$, prove that*

(i) $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ (ii) $|\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$ (iii) $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

SOLUTION (i) We have following cases:

Case I *When $\vec{a}, \vec{b}$ are non-collinear vectors:* Let the vectors $\vec{a}$ and $\vec{b}$ be represented by sides $\vec{OA}$ and $\vec{AB}$ of a triangle $OAB$. Then,

$\vec{OA} + \vec{AB} = \vec{OB} \Rightarrow \vec{a} + \vec{b} = \vec{OB}$. [By triangle law of addition of vectors]

Clearly, $|\vec{a}| = OA$, $|\vec{b}| = AB$ and $|\vec{a} + \vec{b}| = OB$.

Since the sum of two sides of a triangle is always greater than the third side. Therefore, in $\Delta OAB$, we obtain

$OA + AB > OB \Rightarrow |\vec{OB}| < |\vec{OA}| + |\vec{AB}| \Rightarrow |\vec{a} + \vec{b}| < |\vec{a}| + |\vec{b}|$

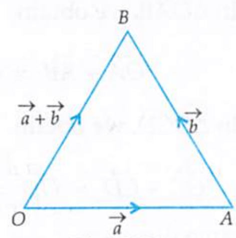

Fig. 22.21

Case II *When $\vec{a}, \vec{b}$ are collinear vectors:* Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$. Then,

$\vec{a} + \vec{b} = \vec{OA} + \vec{AB} = \vec{OB}$ [See Fig. 22.22]

Clearly, $OA = |\vec{a}|$, $AB = |\vec{b}|$ and, $OB = |\vec{a} + \vec{b}|$.

Now, $OB = OA + AB \Rightarrow |\vec{OB}| = |\vec{OA}| + |\vec{AB}| \Rightarrow |\vec{a} + \vec{b}| = |\vec{a}| + |\vec{b}|$

Hence, in general, $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

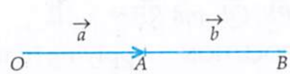

Fig. 22.22

(ii) We have,

$|\vec{a} - \vec{b}| = |\vec{a} + (-\vec{b})| \le |\vec{a}| + |-\vec{b}|$ [Using (i)]

$\Rightarrow \quad |\vec{a} - \vec{b}| \le |\vec{a}| + |\vec{b}|$ $[\because |-\vec{b}| = |\vec{b}|]$

(iii) We know that

$\vec{a} = (\vec{a} - \vec{b}) + \vec{b}$

$\Rightarrow \quad |\vec{a}| = |\vec{a} - \vec{b} + \vec{b}| \le |\vec{a} - \vec{b}| + |\vec{b}|$ [Using (i)]

$\Rightarrow \quad |\vec{a}| - |\vec{b}| \le |\vec{a} - \vec{b}|$ or, $|\vec{a} - \vec{b}| \ge |\vec{a}| - |\vec{b}|$

## 22.7 MULTIPLICATION OF A VECTOR BY A SCALAR

**DEFINITION** *Let $m$ be a scalar and $\vec{a}$ be a vector, then $m\vec{a}$ is defined as a vector having the same support as that of $\vec{a}$ such that its magnitude is $|m|$ times the magnitude of $\vec{a}$ and its direction is same as or opposite to the direction of $\vec{a}$ according as $m$ is positive or negative.*

From the above definition it is evident that

$$\vec{a} = |\vec{a}|\,\hat{a} \Rightarrow \hat{a} = \frac{1}{|\vec{a}|}\vec{a}$$

It is also evident that two vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel iff $\vec{a} = m\vec{b}$ for some non-zero scalar $m$.

For any vector $\vec{a}$, we also define

$$1\,\vec{a} = \vec{a},\ (-1)\,\vec{a} = -\vec{a} \text{ and } 0\,\vec{a} = \vec{0}$$

REMARK *If $\vec{a}$ is a vector, then $3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is same as that of $\vec{a}$. Also, $-3\vec{a}$ is a vector whose magnitude is 3 times the magnitude of $\vec{a}$ and whose direction is opposite to that of $\vec{a}$.*

*A geometric visualization of multiplication of a vector by a scalar is given in Fig. 22.23.*

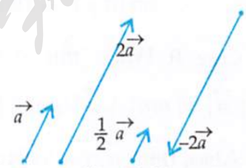


Fig. 22.23

### 22.7.1 PROPERTIES OF MULTIPLICATION OF A VECTOR BY A SCALAR

**THEOREM** *For vectors $\vec{a}, \vec{b}$ and scalars $m, n$, we have*

(i) $m(-\vec{a}) = (-m)\,\vec{a} = -(m\vec{a})$ (ii) $(-m)(-\vec{a}) = m\vec{a}$ (iii) $m(n\vec{a}) = (mn)\,\vec{a} = n(m\vec{a})$

(iv) $(m+n)\,\vec{a} = m\vec{a} + n\vec{a}$ (v) $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$

PROOF (i) Recall that two vectors are equal if their magnitudes are equal and they have the same direction.

Case I *When $m > 0$:* In this case,

$$|m(-\vec{a})| = |m|\,|-\vec{a}| = |m|\,|\vec{a}| = m|\vec{a}| \qquad [\because m > 0 \therefore |m| = m]$$

$$|(-m)\,\vec{a}| = |-m|\,|\vec{a}| = |m|\,|\vec{a}| = m|\vec{a}| \qquad [\because m > 0 \therefore |m| = m]$$

and, $$|(-m\vec{a})| = |m\vec{a}| = |m|\,|\vec{a}| = m|\vec{a}| \qquad [\because m > 0 \therefore |m| = m]$$

Thus, $m(-\vec{a}), (-m)\,\vec{a}$ and $-(m\vec{a})$ are vectors of equal magnitude. Also, they have the same direction which is opposite to that of $\vec{a}$. Hence, $m(-\vec{a}) = (-m)\,\vec{a} = -(m\vec{a})$.

Case II *When $m < 0$:* In this case,

$$|m(-\vec{a})| = |m|\,|-\vec{a}| = |m|\,|\vec{a}| = -m|\vec{a}| \qquad [\because m < 0 \therefore |m| = -m]$$

$$|(-m)\,\vec{a}| = |-m|\,|\vec{a}| = |m|\,|\vec{a}| = -m|\vec{a}| \qquad [\because m < 0 \therefore |m| = -m]$$

and, $$|-(m\vec{a})| = |m\vec{a}| = |m|\,|\vec{a}| = -m|\vec{a}| \qquad [\because m < 0 \therefore |m| = -m]$$

Thus, $m(-\vec{a}), (-m)\,\vec{a}$ and $-(m\vec{a})$ are vectors of equal magnitude. The direction of $m(-\vec{a})$ is opposite to that of $-\vec{a}$ and therefore, it is same as that of $\vec{a}$.

Similarly, $(-m)\vec{a}$ and $-(m\vec{a})$ have the same direction as that of $\vec{a}$.

Hence, $m(-\vec{a}) = (-m)\vec{a} = -(m\vec{a})$.

(ii) Clearly, $(-m)(-\vec{a}) = (-m)\vec{b}$, where $\vec{b} = -\vec{a}$

$\Rightarrow \quad (-m)(-\vec{a}) = -(m\vec{b}) \qquad [\because (-m)\vec{b} = -(m\vec{b})]$

$\Rightarrow \quad (-m)(-\vec{a}) = -\{m(-\vec{a})\} = -\{-(m\vec{a})\} \qquad [\because m(-\vec{a}) = -(m\vec{a})]$

$\Rightarrow \quad (-m)(-\vec{a}) = m\vec{a} \qquad [\because -(-\vec{a}) = \vec{a}]$

Hence, $(-m)(-\vec{a}) = m\vec{a}$

(iii) Following cases arise:

Case I *When* $mn < 0$: In this case, $m(n\vec{a})$ is a vector of magnitude $m|n\vec{a}| = mn|\vec{a}|$ and its direction is same as that of $\vec{a}$. Also, $(mn)\vec{a}$ is a vector of magnitude $mn|\vec{a}|$ and its direction is same as that of $\vec{a}$.

$\therefore \quad m(n\vec{a}) = (mn)\vec{a}$

Case II *When* $mn < 0$: In this case, $m(n\vec{a})$ is a vector of magnitude $|m|\,|n\vec{a}| = |m|\,|n|\,|\vec{a}| = |mn|\,|\vec{a}|$ such that its direction is opposite to that of $\vec{a}$.

Also, $(mn)\vec{a}$ is a vector whose direction is opposite to that of $\vec{a}$ and its magnitude is $|mn|\,|\vec{a}|$.

$\therefore \quad m(n\vec{a}) = (mn)\vec{a}$

Thus, $m(n\vec{a}) = (mn)\vec{a}$ in both the cases. Similarly, we obtain $n(m\vec{a}) = (mn)\vec{a}$

Hence, $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$

(iv) Following cases arise:

Case I *When* $(m+n) > 0$: In this case, $(m+n)\vec{a}$ is a vector of magnitude $(m+n)|\vec{a}|$ and its direction is same as that of $\vec{a}$.

Since $m\vec{a}$ and $n\vec{a}$ are collinear vectors, therefore the magnitude of $m\vec{a} + n\vec{a}$ is $(m+n)$ times that of $\vec{a}$. The direction of $m\vec{a} + n\vec{a}$ is clearly same as that of $\vec{a}$.

$\therefore \quad (m+n)\vec{a} = m\vec{a} + n\vec{a}$

Case II *When* $(m+n) < 0$: In this case, $(m+n)\vec{a}$ is a vector of magnitude $|m+n|\,|\vec{a}|$ and the direction is opposite to that of $\vec{a}$. Also, $m\vec{a}$ and $n\vec{a}$ being collinear vectors, the magnitude of $m\vec{a} + n\vec{a}$ is $|m+n|\,|\vec{a}|$ and the direction is opposite to that of $\vec{a}$.

$\therefore \quad (m+n)\vec{a} = m\vec{a} + n\vec{a}$

Hence, in both the cases, we obtain: $(m+n)\vec{a} = m\vec{a} + n\vec{a}$

(v) Let $\vec{OA} = \vec{a}$ and $\vec{AB} = \vec{b}$. Then, $\vec{OB} = \vec{OA} + \vec{AB} = \vec{a} + \vec{b}$. Following cases arise:

Case I *When* $m > 1$.

Produce $OA$ to $C$ such that $OC = m.\,OA$ and draw $CD$ parallel to $AB$, meeting $OB$ produced at $D$.

Clearly, triangles $OAB$ and $OCD$ are similar.

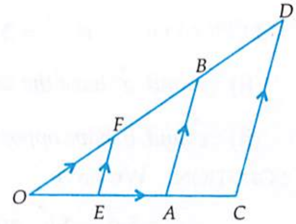

Fig. 22.24

$$\therefore \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA}$$

$$\Rightarrow \quad \frac{OD}{OB} = \frac{CD}{AB} = m \qquad [\because\ OC = m\,OA]$$

$$\Rightarrow \quad OD = m\,OB,\ CD = m\,AB \text{ and } OC = m\,OA$$

$$\Rightarrow \quad \vec{OD} = m(\vec{OB}),\ \vec{CD} = m(\vec{AB}) \text{ and } \vec{OC} = m(\vec{OA})$$

In $\Delta OCD$, using triangle law of addition of vectors, we obtain

$$\vec{OD} = \vec{OC} + \vec{CD} \Rightarrow m(\vec{OB}) = m(\vec{OA}) + m(\vec{AB}) \Rightarrow m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}.$$

Case II *When* $0 < m < 1$: Take a point $E$ on $OA$ such that $OE = m(OA)$ and draw $EF$ parallel to $AB$, meeting $OB$ at $F$. Clearly, triangles $OAB$ and $OEF$ are similar.

$$\therefore \quad \frac{OE}{OA} = \frac{EF}{AB} = \frac{OF}{OB}$$

$$\Rightarrow \quad \frac{OF}{OB} = \frac{EF}{AB} = m \qquad [\because\ OE = m \cdot OA]$$

$$\Rightarrow \quad OF = m\,OB,\ EF = m\,AB \text{ and } OE = m\,OA$$

$$\Rightarrow \quad \vec{OF} = m(\vec{OB}),\ \vec{EF} = m(\vec{AB}) \text{ and } \vec{OE} = m(\vec{OA})$$

Now, $\quad \vec{OF} = \vec{OE} + \vec{EF}$ [By triangle law of addition of vectors]

$$\Rightarrow \quad m(\vec{OB}) = m(\vec{OA}) + m(\vec{AB}) \Rightarrow m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$$

Case III *When* $m < 0$: Take a point $C$ on $AO$ produced such that $OC = |\,m\,|\,OA$. From $C$ draw $CD$ parallel to $AB$ but in direction opposite to that of $\vec{AB}$. Now, produce $BO$ to meet $CD$ at $D$. Clearly, triangles $OAB$ and $OCD$ are similar.

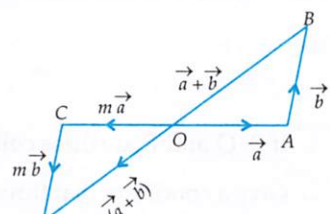

Fig. 22.25

$$\therefore \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA}$$

$$\Rightarrow \quad \frac{OD}{OB} = \frac{CD}{AB} = \frac{OC}{OA} = |m| \qquad [\because\ OC = m\,OA]$$

$$\Rightarrow \quad \frac{|\vec{OD}|}{|\vec{OB}|} = \frac{|\vec{CD}|}{|\vec{AB}|} = \frac{|\vec{OC}|}{|\vec{OA}|} = |m|$$

$$\Rightarrow \quad |\vec{OD}| = |m|\,|\vec{OB}|,\ |\vec{CD}| = |m|\,|\vec{AB}| \text{ and, } |\vec{OC}| = |m|\,|\vec{OA}|$$

$$\Rightarrow \quad \vec{OD} = m\,\vec{OB},\ \vec{CD} = m\,\vec{AB} \text{ and } \vec{OC} = m\,\vec{OA}$$

Now, $\quad \vec{OD} = \vec{OC} + \vec{CD}$ [By triangle law of addition]

$$\Rightarrow \quad m(\vec{OB}) = m(\vec{OA}) + m(\vec{AB}) \Rightarrow m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$$

Case IV *When* $m = 0$: In this case,

$$m(\vec{a} + \vec{b}) = 0(\vec{a} + \vec{b}) = \vec{0} \text{ and } m\vec{a} + m\vec{b} = \vec{0a} + 0\vec{b} = \vec{0}. \text{ Therefore, } m(\vec{a} + \vec{b} = m\vec{a} + m\vec{b}$$

Hence, in all the cases, we obtain $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$.

**ILLUSTRATION** *If $\vec{c} = 3\vec{a} + 4\vec{b}$ and $2\vec{c} = \vec{a} - 3\vec{b}$, show that*

(i) *$\vec{c}$ and $\vec{a}$ have the same direction and* $|\vec{c}| > |\vec{a}|$

(ii) *$\vec{c}$ and $\vec{b}$ have opposite direction and* $|\vec{c}| > |\vec{b}|$

SOLUTION We have,

$$\vec{c} = 3\vec{a} + 4\vec{b} \text{ and } 2\vec{c} = \vec{a} - 3\vec{b}$$

$$\Rightarrow \quad 2(3\vec{a} + 4\vec{b}) = \vec{a} - 3\vec{b} \qquad [\text{On eliminating } \vec{c}]$$

$$\Rightarrow \quad 6\vec{a} + 8\vec{b} = \vec{a} - 3\vec{b} \Rightarrow 5\vec{a} = -11\vec{b} \Rightarrow \vec{a} = -\frac{11}{5}\vec{b} \text{ and } \vec{b} = -\frac{5}{11}\vec{a}$$

Substituting $\vec{b} = -\frac{5}{11}\vec{a}$ in $\vec{c} = 3\vec{a} + 4\vec{b}$, we get

$$\vec{c} = 3\vec{a} + 4\left(-\frac{5}{11}\vec{a}\right) = 3\vec{a} - \frac{20}{11}\vec{a} = \frac{13}{11}\vec{a}$$

This shows that $\vec{c}$ and $\vec{a}$ have the same direction and $|\vec{c}| = \frac{13}{11}|\vec{a}|$.

Now, $|\vec{c}| = \frac{13}{11}|\vec{a}| \Rightarrow |\vec{c}| > |\vec{a}| \qquad \left[\because \frac{13}{11}|\vec{a}| > |\vec{a}|\right]$

Hence, $\vec{c}$ and $\vec{a}$ have the same direction and $|\vec{c}| > |\vec{a}|$.

(ii) We have,

$$\vec{c} = 3\vec{a} + 4\vec{b} \text{ and } \vec{a} = -\frac{11}{5}\vec{b} \Rightarrow \vec{c} = -\frac{33}{5}\vec{b} + 4\vec{b} = -\frac{13}{5}\vec{b}$$

This shows that $\vec{c}$ and $\vec{b}$ have opposite directions and, $|\vec{c}| = \left|-\frac{13}{5}\vec{b}\right| = \frac{13}{5}|\vec{b}| > |\vec{b}|$

## EXERCISE 22.2

### BASIC

1. If $P$, $Q$ and $R$ are three collinear points such that $\vec{PQ} = \vec{a}$ and $\vec{QR} = \vec{b}$. Find the vector $\vec{PR}$.
2. Give a condition that three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ form the three sides of a triangle. What are the other possibilities ?
3. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors having the same initial point. What are the vectors represented by $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.
4. If $\vec{a}$ is a vector and $m$ is a scalar such that $m\vec{a} = \vec{0}$, then what are the alternatives for $m$ and $\vec{a}$?
5. If $\vec{a}, \vec{b}$ are two vectors, then write the truth value of the following statements:

   (i) $\vec{a} = -\vec{b} \Rightarrow |\vec{a}| = |\vec{b}|$ (ii) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm\vec{b}$ (ii) $|\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \vec{b}$

### BASED ON LOTS

6. $ABCD$ is a quadrilateral. Find the sum of the vectors $\vec{BA}, \vec{BC}, \vec{CD}$ and $\vec{DA}$.
7. $ABCDE$ is a pentagon, prove that

   (i) $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = \vec{0}$ (ii) $\vec{AB} + \vec{AE} + \vec{BC} + \vec{DC} + \vec{ED} + \vec{AC} = 3\vec{AC}$
8. Prove that the sum of all vectors drawn from the centre of a regular octagon to its vertices is the zero vector.

9. If $P$ is a point and $ABCD$ is a quadrilateral and $\overrightarrow{AP}+\overrightarrow{PB}+\overrightarrow{PD}=\overrightarrow{PC}$, show that $ABCD$ is a parallelogram.
10. Five forces $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{AE}$ and $\overrightarrow{AF}$ act at the vertex of a regular hexagon $ABCDEF$. Prove that the resultant is $6\,\overrightarrow{AO}$, where $O$ is the centre of hexagon.

### ANSWERS

1. $\vec{a}+\vec{b}$   2. $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, Other possibilities are $\vec{a}+\vec{b}=\vec{c}$, $\vec{b}+\vec{c}=\vec{a}$ and $\vec{c}+\vec{a}=\vec{b}$
3. Diagonals of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$.
4. Either $m=0$ or, $\vec{a}=\vec{0}$   5. (i) $T$ (ii) F (iii) F   6. $2\,\overrightarrow{BA}$

## 22.8 POSITION VECTOR OF A POINT

**POSITION VECTOR** *If a point O is fixed as the origin in space (or plane) and P is any point, then $\overrightarrow{OP}$ is called the position vector of P with respect to O.*

If we say that $P$ is the point $\vec{r}$, then we mean that the position vector of $P$ is $\vec{r}$ with respect to some origin $O$.

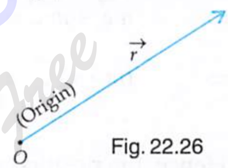

Fig. 22.26

### 22.8.1 A VECTOR IN TERMS OF POSITION VECTORS OF ITS END POINTS

Let $\vec{a}$ and $\vec{b}$ be the position vectors of points $A$ and $B$ respectively. Then, $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$.

In $\Delta OAB$, we obtain

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$

$$\Rightarrow \overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\vec{b}-\vec{a}=(\text{Position vector of } B)-(\text{Position vector of } A)$$

or, $\overrightarrow{AB}=$ (Position vector of head) – (Position vector of tail)

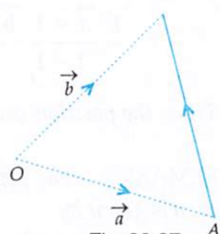

Fig. 22.27

**ILLUSTRATION 1** *The position vectors of points A, B, C, D are $\vec{a}$, $\vec{b}$, $2\vec{a}+3\vec{b}$ and $\vec{a}-2\vec{b}$ respectively. Show that $\overrightarrow{DB}=3\vec{b}-\vec{a}$ and $\overrightarrow{AC}=\vec{a}+3\vec{b}$.*

SOLUTION We find that

$$\overrightarrow{DB}=\text{Position vector of } B-\text{Position vector of } D=\vec{b}-(\vec{a}-2\vec{b})=3\vec{b}-\vec{a}$$

and, $$\overrightarrow{AC}=\text{Position vector of } C-\text{Position vector of } A=(2\vec{a}+3\vec{b})-\vec{a}=\vec{a}+3\vec{b}$$

**ILLUSTRATION 2** *Let ABCD be a parallelogram. If $\vec{a}$, $\vec{b}$, $\vec{c}$ be the position vectors of A, B, C respectively with reference to the origin O, find the position vector of D with reference to O.*

SOLUTION We have, $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=\vec{b}$ and $\overrightarrow{OC}=\vec{c}$

Let $\vec{d}$ be the position vector of point $D$.
Since opposite sides of a parallelogram are parallel and equal.

$$\therefore \quad \overrightarrow{AB}=\overrightarrow{DC} \Rightarrow \vec{b}-\vec{a}=\vec{c}-\vec{d} \Rightarrow \vec{d}=\vec{c}+\vec{a}-\vec{b}$$

Hence, the position vector of $D$ is $\vec{c}+\vec{a}-\vec{b}$.

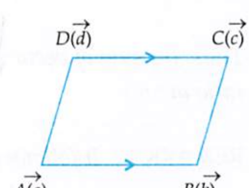

Fig. 22.28

## 22.9 SECTION FORMULAE

**THEOREM 1** *(Internal Division) Let A and B be two points with position vectors $\vec{a}$ and $\vec{b}$ respectively, and let C be a point dividing AB internally in the ration m : n. Then the position vector of C is given by* $\overrightarrow{OC} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$.

PROOF Let $O$ be the Origin. Then, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of C which divides $AB$ internally in the ratio $m : n$. Then,

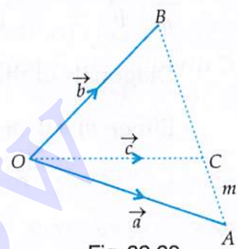

Fig. 22.29

$$\frac{AC}{CB} = \frac{m}{n}$$

$$\Rightarrow \quad n \cdot AC = m \cdot CB$$

$$\Rightarrow \quad n\overrightarrow{AC} = m\overrightarrow{CB}$$

$$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{b} - \vec{c})$$

$$\Rightarrow \quad n\vec{c} + m\vec{c} = m\vec{b} + n\vec{a}$$

$$\Rightarrow \quad (m+n)\vec{c} = m\vec{b} + n\vec{a} \Rightarrow \vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n} \Rightarrow \overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

Hence, the position vector of point C is $\dfrac{m\vec{b} + n\vec{a}}{m+n}$.

REMARK 1 *If C is the mid point of AB, then it divides AB in the ratio 1 : 1. Therefore, position vector of C is given by*

$$\frac{1 \cdot \vec{a} + 1 \cdot \vec{b}}{1+1} = \frac{\vec{a} + \vec{b}}{2}$$

*Thus, the position vector of the mid point of AB is* $\dfrac{1}{2}(\vec{a} + \vec{b})$.

REMARK 2 *The position vector $\vec{c}$ of the point dividing segment joining A $(\vec{a})$ and B $(\vec{b})$ in the ratio m : n is given by*

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n} = \left(\frac{m}{m+n}\right)\vec{b} + \left(\frac{n}{m+n}\right)\vec{a} = \lambda\vec{a} + \mu\vec{b}, \text{ where } \lambda = \frac{n}{m+n}, \ \mu = \frac{m}{m+n}$$

*Thus, position vector of any point C on AB can always be taken as* $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, *where* $\lambda + \mu = 1$.

REMARK 3 *In Fig. 22.30, the position vector of C is given by*

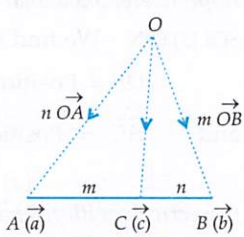

Fig. 22.30

$$\vec{c} = \frac{m\vec{b} + n\vec{a}}{m+n}$$

$$\Rightarrow \quad (m+n)\vec{c} = m\vec{b} + n\vec{a} \Rightarrow n\overrightarrow{OA} + m\overrightarrow{OB} = (n+m)\overrightarrow{OC}$$

*Thus, the sum of vectors $n\overrightarrow{OA}$ and $m\overrightarrow{OB}$ is $(n+m)\overrightarrow{OC}$, where C is a point on AB dividing it in the ratio m : n.*

REMARK 4 *It follows from the above remark that, if C is the mid point of AB, then m = n*

$$\therefore \quad n\overrightarrow{OA} + m\overrightarrow{OB} = (m+n)\overrightarrow{OC} \Rightarrow m\overrightarrow{OA} + m\overrightarrow{OB} = 2m\overrightarrow{OC} \Rightarrow \overrightarrow{OA} + \overrightarrow{OB} = 2\overrightarrow{OC}$$

*Thus, if A and B are two points and O is the origin, then* $\vec{OA} + \vec{OB} = 2\,\vec{OC}$, *where C is the mid-point of AB.*

**THEOREM 2** *(External Division) Let A and B be two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *respectively and let C be a point dividing AB externally in the ratio m : n. Then, the position vector of C is given by*

$$\vec{OC} = \frac{m\vec{b} - n\vec{a}}{m-n}.$$

PROOF Let $O$ be the origin. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$. Let $\vec{c}$ be the position vector of a point C dividing $AB$ externally in the ratio $m : n$. Then,

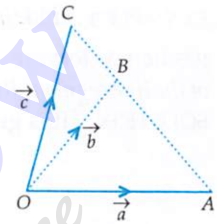


Fig. 22.31

$$\frac{AC}{BC} = \frac{m}{n}$$

$$\Rightarrow \quad n \cdot AC = m \cdot BC$$

$$\Rightarrow \quad n\,\vec{AC} = m\,\vec{BC}$$

$$\Rightarrow \quad n(\vec{c} - \vec{a}) = m(\vec{c} - \vec{b})$$

$$\Rightarrow \quad m\vec{c} - n\vec{c} = m\vec{b} - n\vec{a}$$

$$\Rightarrow \quad (m-n)\,\vec{c} = m\vec{b} - n\vec{a} \Rightarrow \vec{c} = \frac{m\vec{b} - n\vec{a}}{m-n} \Rightarrow \vec{OC} = \frac{m\vec{b} - n\vec{a}}{m-n}$$

Hence, the position vector of point C is $\dfrac{m\vec{b} - n\vec{a}}{m-n}$.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *Find the position vectors of the points which divide the join of the points* $2\vec{a} - 3\vec{b}$ *and* $3\vec{a} - 2\vec{b}$ *internally and externally in the ratio 2 : 3.*

SOLUTION Let $A$ and $B$ be the given points with position vectors $2\vec{a} - 3\vec{b}$ and $3\vec{a} - 2\vec{b}$ respectively. Let $P$ and $Q$ be the points dividing $AB$ in the ratio 2 : 3 internally and externally respectively. Then,

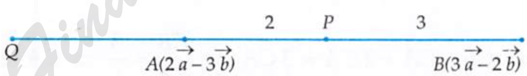


Fig. 22.32

$$\text{Position vector of } P = \frac{3(2\vec{a} - 3\vec{b}) + 2(3\vec{a} - 2\vec{b})}{3+2} = \frac{12\,\vec{a}}{5} - \frac{13\,\vec{b}}{5}$$

$$\text{Position vector of } Q = \frac{3(2\vec{a} - 3\vec{b}) - 2(3\vec{a} - 2\vec{b})}{3-2} = -5\,\vec{b}$$

**EXAMPLE 2** *If* $\vec{a}$ *and* $\vec{b}$ *are position vectors of points A and B respectively, then find the position vector of points of trisection of AB.*

SOLUTION Let $P$ and $Q$ be points of trisection of $AB$. Then, $AP = PQ = QB = \lambda$ (say).


Fig. 22.33

$$\therefore \quad PB = PQ + QB = \lambda + \lambda = 2\lambda$$

Thus, we have

$$AP = \lambda \text{ and } PB = 2\lambda \Rightarrow AP : PB = \lambda : 2\lambda = 1:2 \text{ i.e. } P \text{ divides } AB \text{ in the ratio } 1:2.$$

$\therefore$ Position vector of $P = \dfrac{1\cdot \vec{b} + 2\cdot \vec{a}}{1+2} = \dfrac{\vec{b} + 2\vec{a}}{3}$

Clearly, $PQ = QB = \lambda$. So, point $Q$ is the mid-point of $PB$.

$\therefore$ Position vector of $Q = \dfrac{\dfrac{\vec{b}+2\vec{a}}{3} + \vec{b}}{2} = \dfrac{4\vec{b}+2\vec{a}}{6} = \dfrac{\vec{a}+2\vec{b}}{3}$

**EXAMPLE 3** *Find the position vector of a point R which divides the line segment joining P and Q whose position vectors are $2\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$, externally in the ratio 1 : 2. Also, show that P is the mid-point of the line segment RQ.* **[CBSE 2010]**

SOLUTION It is given that $R$ divides $PQ$ externally in the ratio 1 : 2.

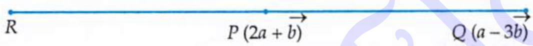


Fig. 22.34

$\therefore$ The position vector of $R = \dfrac{1\times(\vec{a}-3\vec{b}) - 2(2\vec{a}+\vec{b})}{1-2} = 3\vec{a}+5\vec{b}$

Now,

$$\frac{\text{Position vector of } R + \text{Position vector of } Q}{2} = \frac{3\vec{a}+5\vec{b}+\vec{a}-3\vec{b}}{2} = 2\vec{a}+\vec{b}$$
$$= \text{Position vector of } P$$

Hence, $P$ is the mid-point of $RQ$.

**EXAMPLE 4** *If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B respectively, find the position vector of a point C on BA produced such that BC = 1.5 BA.* **[NCERT EXEMPLAR]**

SOLUTION Let the position vector of point $C$ be $\vec{c}$. It is given that

$BC = 1.5\,BA$

$\Rightarrow$ $BA + CA = 1.5\,BA$


Fig. 22.35

$\Rightarrow$ $CA = \dfrac{1}{2}BA$

$\therefore$ $CB = CA + AB \Rightarrow CB = CA + 2\,CA = 3\,CA \Rightarrow \dfrac{CB}{CA} = \dfrac{3}{1} \Rightarrow CA : CB = 1:3$

$\Rightarrow$ $C$ divides $AB$ externally in the ratio $CA : CB = 1:3$.

So, the position vector $\vec{c}$ of point C is given by $\vec{c} = \dfrac{1\times\vec{b}-3\vec{a}}{1-3} = -\dfrac{1}{2}(\vec{b}-3\vec{a}) = \dfrac{1}{2}(3\vec{a}-\vec{b})$

ALITER In Fig. 22.35, $CA = \dfrac{1}{2}BA$ or, $BA = 2\,CA$. Therefore, $AB : AC = 2\,CA : AC = 2:1$

So, $A$ divides $BC$ internally in the ratio 2 : 1. Therefore, position vector of $A$ is $\dfrac{2\vec{c}+\vec{b}}{2+1}$. But, the position vector of $A$ is $\vec{a}$.

$\therefore$ $\vec{a} = \dfrac{2\vec{c}+\vec{b}}{3} \Rightarrow \vec{c} = \dfrac{1}{2}(3\vec{a}-\vec{b})$

## BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 5** *Four points A, B, C, D with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively are such that $3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = \vec{0}$. Show that the four points are coplanar. Also, find the position vector of the point of intersection of line segments AC and BD.*

SOLUTION We have, $3\vec{a} - \vec{b} + 2\vec{c} - 4\vec{d} = \vec{0} \Rightarrow 3\vec{a} + 2\vec{c} = \vec{b} + 4\vec{d}$

We note that the sum of the coefficients on both sides of the above result is 5. We therefore, divide both the sides by 5 to get

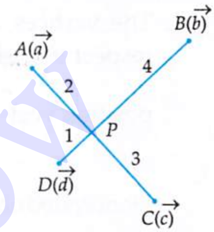


Fig. 22.36

$$\frac{3\vec{a} + 2\vec{c}}{5} = \frac{\vec{b} + 4\vec{d}}{5}$$

$$\Rightarrow \quad \frac{3\vec{a} + 2\vec{c}}{3+2} = \frac{\vec{b} + 4\vec{d}}{1+4}$$

$\Rightarrow$ Position vector of a point $P$ dividing $AC$ in the ratio $2:3$

= Position vector of a point $P$ dividing $BD$ in the ratio $4:1$

Consequently, point $P$ is common to $AC$ and $BD$. Therefore, $AC$ and $BD$ intersect. Hence, points $A$, $B$ ,$C$ and $D$ are coplanar. Since $P$ is the point of intersection of $AC$ and $BD$. Therefore, the position vector of the point of intersection of $AC$ and $BD$ is

$$\frac{3\vec{a} + 2\vec{c}}{5} \text{ or, } \frac{\vec{b} + 4\vec{d}}{5}.$$

**EXAMPLE 6** *Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of three distinct points A,B,C. If there exist scalars x, y, z (not all zero) such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ and $x + y + z = 0$, then show that A, B and C lie on a line.*

SOLUTION It is given that $x$, $y$, $z$ are not all zero. So, let $z$ be non-zero. Then,

$$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$$

$$\Rightarrow \quad z\vec{c} = -(x\vec{a} + y\vec{b}) \Rightarrow \vec{c} = -\frac{(x\vec{a} + y\vec{b})}{z} \Rightarrow \vec{c} = \frac{x\vec{a} + y\vec{b}}{x+y} \left[\because\ x + y + z = 0 \ \therefore z = -(x+y)\right]$$

This shows that the point $C$ divides the line joining the points $A$ and $B$ in the ratio $y : x$. Hence, $A$, $B$ and $C$ lie on the same line.

## EXERCISE 22.3

### BASIC

1. Find the position vector of a point $R$ which divides the line joining the two points $P$ and $Q$ with position vectors $\vec{OP} = 2\vec{a} + \vec{b}$ and $\vec{OQ} = \vec{a} - 2\vec{b}$, respectively in the ratio $1:2$ internally and externally. **[NCERT EXEMPLAR, CBSE 2016]**
2. $X$ and $Y$ are two points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ respectively. Write down the position vector of a point $Z$ which divides the line segment $XY$ in the ration $2:1$ externally. **[CBSE 2019]**
3. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of the four distinct points $A, B, C, D$. If $\vec{b} - \vec{a} = \vec{c} - \vec{d}$, then show that $ABCD$ is a parallelogram.
4. If $\vec{a}, \vec{b}$ are the position vectors of $A$, $B$ respectively, find the position vector of a point $C$ in $AB$ produced such that $AC = 3AB$ and that a point $D$ in $BA$ produced such that $BD = 2BA$.

### BASED ON HOTS

5. Show that the four points $A, B, C, D$ with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively such that $3\vec{a}-2\vec{b}+5\vec{c}-6\vec{d}=\vec{0}$, are coplanar. Also, find the position vector of the point of intersection of the line segments $AC$ and $BD$.
6. Show that the four points $P, Q, R, S$ with position vectors $\vec{p}, \vec{q}, \vec{r}, \vec{s}$ respectively such that $5\vec{p}-2\vec{q}+6\vec{r}-9\vec{s}=\vec{0}$, are coplanar. Also, find the position vector of the point of intersection of the line segments $PR$ and $QS$.
7. The vertices $A, B, C$ of triangle $ABC$ have respectively position vectors $\vec{a}, \vec{b}, \vec{c}$ with respect to a given origin $O$. Show that the point $D$ where the bisector of $\angle A$ meets $BC$ has position vector $\vec{d}=\dfrac{\beta\vec{b}+\gamma\vec{c}}{\beta+\gamma}$, where $\beta=|\vec{c}-\vec{a}|$ and, $\gamma=|\vec{a}-\vec{b}|$.

   Hence, deduce that the incentre I has position vector $\dfrac{\alpha\vec{a}+\beta\vec{b}+\gamma\vec{c}}{\alpha+\beta+\gamma}$, where $\alpha=|\vec{b}-\vec{c}|$.

### ANSWERS

1. $\dfrac{5}{3}\vec{a},\ 3\vec{a}+4\vec{b}$ 2. $-\vec{a}-7\vec{b}$ 4. $3\vec{b}-2\vec{a},\ 2\vec{a}-\vec{b}$ 5. $\dfrac{3\vec{a}+5\vec{c}}{8}$ or, $\dfrac{2\vec{b}+6\vec{c}}{8}$

6. $\dfrac{5\vec{p}+6\vec{q}}{11}$ or, $\dfrac{2\vec{q}+9\vec{s}}{11}$

## 22.10 LINEAR COMBINATION OF VECTORS

**DEFINITION** *A vector $\vec{r}$ is said to be a linear combination of vectors $\vec{a}, \vec{b}, \vec{c}$ ... etc. if there exist scalars $x, y, z$ etc., such that $\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}+\dots$.*

Note that a linear combination of vectors involves two linear compositions of the addition of vectors and the multiplication of vectors with scalars.

In the following sections, the linear combinations of the form $x\vec{a}$, $x\vec{a}+y\vec{b}$ and $x\vec{a}+y\vec{b}+z\vec{c}$ will be of special interest to us.

### ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *If D is the mid-point of the side BC of a triangle ABC, prove that $\vec{AB}+\vec{AC}=2\vec{AD}$.*

SOLUTION Let $A$ be the origin and let the position vectors of $B$ and $C$ be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vector of the mid-point of $BC$ is $\dfrac{\vec{b}+\vec{c}}{2}$.

$\therefore$ Position vector of $D=\dfrac{\vec{b}+\vec{c}}{2}$

Now, $\vec{AB}+\vec{AC}=\vec{b}+\vec{c}$

$$\Rightarrow \quad \vec{AB}+\vec{AC}=2\left(\frac{\vec{b}+\vec{c}}{2}\right)=2\vec{AD}$$

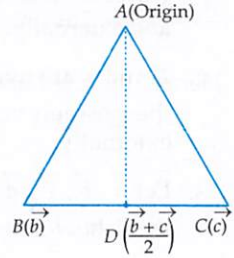

Fig. 22.37

**EXAMPLE 2** *Points $L, M, N$ divide the sides $BC, CA, AB$ of $\Delta ABC$ in the ratios $1:4$, $3:2$, $3:7$ respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, where $K$ divides $AB$ in the ratio $1:3$.*

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices $A$, $B$ and $C$ of $\Delta ABC$. Then, the position vectors of $L, M$ and $N$ are $\dfrac{4\vec{b}+\vec{c}}{5}$, $\dfrac{3\vec{a}+2\vec{c}}{5}$ and $\dfrac{7\vec{a}+3\vec{b}}{10}$ respectively. The position vector of $K$ is $\dfrac{\vec{b}+3\vec{a}}{4}$.

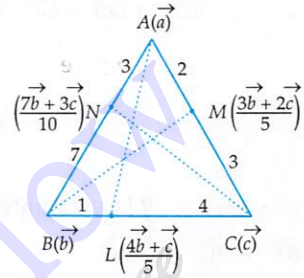

Fig. 22.38

Now,

$$\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$$

$$= \frac{4\vec{b}+\vec{c}}{5} - \vec{a} + \frac{3\vec{a}+2\vec{c}}{5} - \vec{b} + \frac{7\vec{a}+3\vec{b}}{10} - \vec{c}$$

$$= \frac{3\vec{a}+\vec{b}-4\vec{c}}{10} = \frac{4}{10}\left(\frac{3\vec{a}+\vec{b}-4\vec{c}}{4}\right) = \frac{4}{10}\overrightarrow{CK}$$

Hence, $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is parallel to $\overrightarrow{CK}$.

**EXAMPLE 3** *Prove using vectors:* Medians *of a triangle are concurrent.*

SOLUTION Let $ABC$ be a triangle and let $D, E, F$ be the mid-points of its sides $BC$, $CA$ and $AB$ respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of $A$, $B$ and $C$ respectively. Then, the position vectors of $D$, $E$ and $F$ are $\dfrac{\vec{b}+\vec{c}}{2}$, $\dfrac{\vec{c}+\vec{a}}{2}$ and $\dfrac{\vec{a}+\vec{b}}{2}$ respectively.

The position vector of a point dividing $AD$ in the ratio 2 : 1 is

$$\frac{1\cdot\vec{a} + 2\left(\dfrac{\vec{b}+\vec{c}}{2}\right)}{1+2} = \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

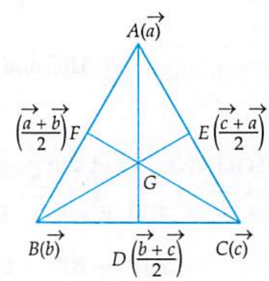

Fig. 22.39

Similarly, position vectors of points dividing $BE$ and $CF$ in the ratio 2 :1 are each equal to $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$. Thus, the point dividing $AD$ in the ratio $2:1$ also divides $BE$ and $CF$ in the same ratio.

Hence, the medians of a triangle are concurrent and the position vector of the centroid is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.

NOTE *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle, then the position vector of its centroid is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.*

## BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 4** *If G is the centroid of a triangle ABC, prove that* $\vec{GA} + \vec{GB} + \vec{GC} = \vec{0}$.

SOLUTION Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices $A$, $B$ and $C$ respectively.

Then, the position vector of the centroid G is $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$ (see Example 3).

$$\therefore \quad \vec{GA} + \vec{GB} + \vec{GC} = \vec{a} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) + \vec{b} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) + \vec{c} - \left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)$$

$$= (\vec{a}+\vec{b}+\vec{c}) - 3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right) = (\vec{a}+\vec{b}+\vec{c}) - (\vec{a}+\vec{b}+\vec{c}) = \vec{0}$$

**EXAMPLE 5** *If D and E are the mid-points of sides AB and AC of a triangle ABC respectively, show that* $\vec{BE} + \vec{DC} = \dfrac{3}{2}\vec{BC}$.

SOLUTION Taking $A$ as the origin, let the position vectors of $B$ and $C$ be $\vec{b}$ and $\vec{c}$ respectively. Since $D$ and $E$ are the mid-points of $AB$ and $AC$. Therefore, the position vectors of $D$ and $E$ are $\dfrac{\vec{b}}{2}$ and $\dfrac{\vec{c}}{2}$ respectively.

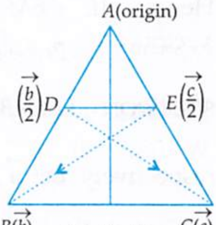

Fig. 22.40

$$\therefore \quad \vec{BE} + \vec{DC} = \left(\frac{\vec{c}}{2} - \vec{b}\right) + \left(\vec{c} - \frac{\vec{b}}{2}\right) = \frac{3}{2}(\vec{c} - \vec{b}) = \frac{3}{2}\vec{BC}$$

**EXAMPLE 6** *If ABC and A′B′C′ are two triangles and G, G′ be their centroids, prove that*

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\,\vec{GG'}$$

SOLUTION Let the position vectors of $A$, $B$, $C$ and $A'$, $B'$, $C'$ with reference to some origin be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ respectively. Then,

$$\vec{AA'} + \vec{BB'} + \vec{CC'} = (\vec{a'} - \vec{a}) + (\vec{b'} - \vec{b}) + (\vec{c'} - \vec{c}) = (\vec{a'} + \vec{b'} + \vec{c'}) - (\vec{a} + \vec{b} + \vec{c}) \quad \text{...(i)}$$

The position vectors of the centroids $G$ and $G'$ of triangles $ABC$ and $A'B'C'$ are $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$ and $\dfrac{\vec{a'}+\vec{b'}+\vec{c'}}{3}$ respectively.

$$\therefore \quad \vec{GG'} = \frac{\vec{a'}+\vec{b'}+\vec{c'}}{3} - \frac{\vec{a}+\vec{b}+\vec{c}}{3}$$

$$\Rightarrow \quad \vec{GG'} = \frac{1}{3}\left\{(\vec{a'}+\vec{b'}+\vec{c'}) - (\vec{a}+\vec{b}+\vec{c})\right\} \Rightarrow 3\,\vec{GG'} = (\vec{a'}+\vec{b'}+\vec{c'}) - (\vec{a}+\vec{b}+\vec{c}) \quad \text{...(ii)}$$

From (i) and (ii), we obtain: $\vec{AA'} + \vec{BB'} + \vec{CC'} = 3\,\vec{GG'}$

**EXAMPLE 7** *Prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and equal to half of it.*

SOLUTION Let $ABC$ be a triangle and let $D$ and $E$ be the mid points of its sides $AB$ and $AC$ respectively. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of vertices $A$, $B$ and $C$ respectively. Then, position vectors of $D$ and $E$ are $\frac{\vec{a}+\vec{b}}{2}$ and $\frac{\vec{a}+\vec{c}}{2}$ respectively.

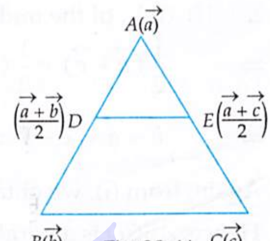

Fig. 22.41

Now,

$$\overrightarrow{DE} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{b}) = \frac{1}{2}\overrightarrow{BC}$$

$\therefore \quad \overrightarrow{DE} \parallel \overrightarrow{BC}$

Now, $DE = |\overrightarrow{DE}| = \left|\frac{1}{2}\overrightarrow{BC}\right| = \frac{1}{2}|\overrightarrow{BC}| = \frac{1}{2}BC$. Hence, $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

**EXAMPLE 8** *Prove that the sum of the vectors directed from the vertices to the mid-points of opposite sides of a triangle is zero.*

SOLUTION Let $ABC$ be a triangle and let the position vectors of vertices $A$, $B$, $C$ be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Let $D$, $E$, $F$ be mid points of sides $BC$, $CA$ and $AB$ respectively. Then, position vectors of $D$, $E$ and $F$ are $\frac{\vec{b}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{a}}{2}$ and $\frac{\vec{a}+\vec{b}}{2}$ respectively.

We have to prove that $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \vec{0}$.

We find that

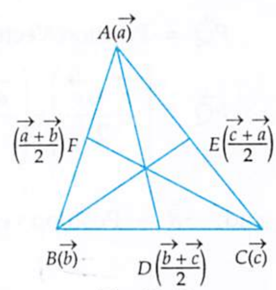

Fig. 22.42

$$\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF} = \left(\frac{\vec{b}+\vec{c}}{2} - \vec{a}\right) + \left(\frac{\vec{c}+\vec{a}}{2} - \vec{b}\right) + \left(\frac{\vec{a}+\vec{b}}{2} - \vec{c}\right)$$

$$= \frac{1}{2}(\vec{b}+\vec{c}-2\vec{a}+\vec{c}+\vec{a}-2\vec{b}+\vec{a}+\vec{b}-2\vec{c}) = \vec{0}$$

**EXAMPLE 9** *Prove using vectors: The diagonals of a quadrilateral bisect each other iff it is a parallelogram.*

SOLUTION First, let us assume that $ABCD$ is a parallelogram. Then, we have to prove that its diagonals bisect each other. Let the position vectors of $A$, $B$, $C$ and $D$ be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Since $ABCD$ is a parallelogram.

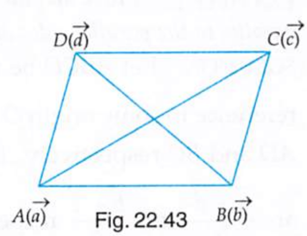

Fig. 22.43

$\therefore \quad \overrightarrow{AB} = \overrightarrow{DC}$

$$\Rightarrow \vec{b}-\vec{a} = \vec{c}-\vec{d} \Rightarrow \vec{b}+\vec{d} = \vec{a}+\vec{c} \Rightarrow \frac{1}{2}(\vec{b}+\vec{d}) = \frac{1}{2}(\vec{a}+\vec{c})$$

$\Rightarrow$ P.V. of the mid-point of $BD$ = P.V. of the mid-point of $AC$.

Thus, the point which bisects $AC$ also bisects $BD$. Hence, diagonals of parallelogram $ABCD$ bisect each other.

Conversely, let $ABCD$ be a quadrilateral such that its diagonals bisect each other. Then, we have to prove that it is a parallelogram.

Let $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ be the position vectors of its vertices $A, B, C$ and $D$ respectively. Since diagonals $AC$ and $BD$ bisect each other.

$\therefore$ P.V. of the mid-point of $AC$ = P.V. of the mid-point of $BD$

$$\Rightarrow \quad \frac{1}{2}(\vec{a}+\vec{c}) = \frac{1}{2}(\vec{b}+\vec{d}) \Rightarrow \vec{a}+\vec{c} = \vec{b}+\vec{d} \qquad \text{...(i)}$$

$$\Rightarrow \quad \vec{b}-\vec{a} = \vec{c}-\vec{d} \Rightarrow \vec{AB} = \vec{DC}$$

Again, from (i), we obtain: $\vec{d}-\vec{a} = \vec{c}-\vec{b} \Rightarrow \vec{AD} = \vec{BC}$

Hence, $ABCD$ is a parallelogram.

**EXAMPLE 10** *Using vector method, prove that the line segments joining the mid-points of the adjacent sides of a quadrilateral taken in order form a parallelogram.*

SOLUTION Let $ABCD$ be a quadrilateral and let $P, Q, R, S$ be the mid-points of the sides $AB$, $BC$, $CD$ and $DA$ respectively. Then, the position vectors of $P, Q, R, S$ are

$$\frac{\vec{a}+\vec{b}}{2}, \frac{\vec{b}+\vec{c}}{2}, \frac{\vec{c}+\vec{d}}{2} \text{ and } \frac{\vec{d}+\vec{a}}{2} \text{ respectively.}$$

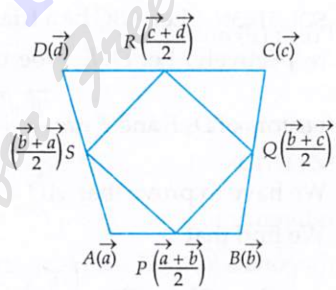

Fig. 22.44

In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that $\vec{PQ} = \vec{SR}$.

Now,

$$\vec{PQ} = \text{Position Vector of } Q - \text{Position Vector of } P$$

$$\Rightarrow \vec{PQ} = \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{b}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$$

and, $\vec{SR}$ = Position vector of $R$ – Position vector of $S = \left(\frac{\vec{c}+\vec{d}}{2}\right) - \left(\frac{\vec{d}+\vec{a}}{2}\right) = \frac{1}{2}(\vec{c}-\vec{a})$

Clearly, $\vec{PQ} = \vec{SR}$ Consequently, $PQ \parallel SR$ and $PQ = SR$. Hence, $PQRS$ is a parallelogram.

## BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 11** *Prove that the segment joining the middle points of two non-parallel sides of a trapezium is parallel to the parallel sides and half of their sum.*

SOLUTION Let $ABCD$ be the given trapezium. Let the position vectors of $A$, $B$, $C$ and $D$ with reference to some origin $O$ be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively. Let $P$ and $Q$ be the mid-points of $AD$ $AD$ and $BC$ respectively. Then, the position vectors of $P$ and $Q$ are $\frac{\vec{a}+\vec{d}}{2}$ and $\frac{\vec{b}+\vec{c}}{2}$ respectively.

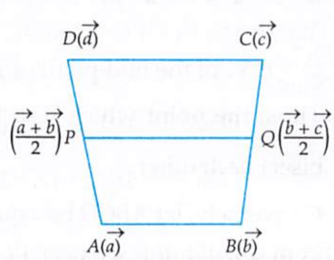

Fig. 22.45

Clearly, $\vec{AB} = \vec{b} - \vec{a}$ and $\vec{DC} = \vec{c} - \vec{d}$

Since $\vec{DC}$ is parallel to $\vec{AB}$. Therefore, there exists a scalar $\lambda$ such that

$$\vec{DC} = \lambda \vec{AB} \Rightarrow (\vec{c}-\vec{d}) = \lambda(\vec{b}-\vec{a}) \qquad \text{...(i)}$$

Now,

$\vec{PQ}$ = Position vector of $Q$ – Position vector of $P$

$$\Rightarrow \vec{PQ} = \left(\frac{\vec{b}+\vec{c}}{2}\right) - \left(\frac{\vec{a}+\vec{d}}{2}\right) = \frac{1}{2}\left\{(\vec{b}-\vec{a})+(\vec{c}-\vec{d})\right\} = \frac{1}{2}\left\{(\vec{b}-\vec{a})+\lambda(\vec{b}-\vec{a})\right\} \quad \text{[Using (i)]}$$

$$\Rightarrow \vec{PQ} = \frac{1}{2}(\lambda+1)(\vec{b}-\vec{a}) = \frac{1}{2}(\lambda+1)\vec{AB} \quad \text{...(ii)}$$

This shows that $PQ$ is parallel to $AB$. But, $AB$ is parallel to $CD$. Consequently, $PQ$ is parallel to $CD$.

Now,

$$|\vec{AB}| + |\vec{DC}| = |\vec{AB}| + |\lambda \vec{AB}| \quad [\because \vec{DC} = \lambda \vec{AB} \text{ from (i)}]$$

$$\Rightarrow |\vec{AB}| + |\vec{DC}| = |\vec{AB}| + \lambda|\vec{AB}| \quad [\because \lambda > 0]$$

$$\Rightarrow |\vec{AB}| + |\vec{DC}| = (1+\lambda)|\vec{AB}| \Rightarrow \frac{1}{2}\left[|\vec{AB}| + |\vec{DC}|\right] = \frac{1}{2}(\lambda+1)|\vec{AB}| \quad \text{...(iii)}$$

From (ii) and (iii), we get: $\frac{1}{2}\left[|\vec{AB}| + |\vec{DC}|\right] = |\vec{PQ}|$ or, $PQ = \frac{1}{2}(AB + DC)$.

**EXAMPLE 12** *Prove by vector method that the line segment joining the mid-points of the diagonals of a trapezium is parallel to the parallel sides and equal to half of their difference.*

SOLUTION Let $ABCD$ be a trapezium and let the position vectors of $A$, $B$, $C$ and $D$ with reference to some origin $O$ be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Let $P$ and $Q$ be the mid-points of diagonals $BD$ and $AC$ respectively. Then, the position vectors of $P$ and $Q$ are $\frac{\vec{b}+\vec{d}}{2}$ and $\frac{\vec{a}+\vec{c}}{2}$ respectively.

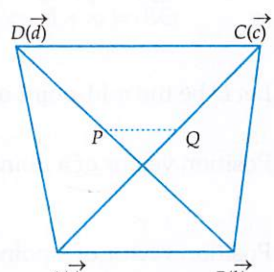

Fig. 26.46

$\therefore$ $\vec{PQ}$ = Position vector of $Q$ – Position vector of $P$

$$\Rightarrow \vec{PQ} = \left(\frac{\vec{a}+\vec{c}}{2}\right) - \left(\frac{\vec{b}+\vec{d}}{2}\right)$$

$$\Rightarrow \vec{PQ} = \frac{1}{2}\left\{(\vec{c}-\vec{d})+(\vec{a}-\vec{b})\right\} = \frac{1}{2}\left\{(\vec{c}-\vec{d})-(\vec{b}-\vec{a})\right\} = \frac{1}{2}\left(\vec{DC} - \vec{AB}\right)$$

Now, $\vec{DC} \parallel \vec{AB} \Rightarrow \vec{DC} = \lambda \vec{AB}$ for some scalar $\lambda$ ...(i)

$$\therefore \quad \vec{PQ} = \frac{1}{2}\left(\lambda\vec{AB} - \vec{AB}\right) = \frac{1}{2}(\lambda-1)\vec{AB} \quad \text{...(ii)}$$

This shows that $PQ$ is parallel to $AB$. But, $AB$ is parallel to $DC$. Consequently, $PQ$ is parallel to $DC$.

Now,

$$|\vec{DC}| - |\vec{AB}| = \lambda|\vec{AB}| - |\vec{AB}| \quad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{DC}| - |\vec{AB}| = (\lambda-1)|\vec{AB}| \Rightarrow \frac{1}{2}\left\{|\vec{DC}| - |\vec{AB}|\right\} = \frac{1}{2}(\lambda-1)|\vec{AB}| \quad \text{...(iii)}$$

From (ii) and (iii), we get: $|\vec{PQ}| = \frac{1}{2}\left\{|\vec{DC}| - |\vec{AB}|\right\}$ or, $PQ = \frac{1}{2}(DC - AB)$

Hence, $PQ$ is half of the difference of parallel sides.

**EXAMPLE 13** *If ABCD is quadrilateral and E and F are the mid-points of AC and BD respectively, prove that* $\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 4\,\vec{EF}$.

SOLUTION Since $F$ is the mid-point of $BD$. Therefore, in triangle $ABD$, we obtain

$$1 \cdot \vec{AB} + 1 \cdot \vec{AD} = (1+1)\,\vec{AF} \Rightarrow \vec{AB} + \vec{AD} = 2\,\vec{AF} \quad \text{...(i)}$$

Similarly, in triangle BCD, we obtain

$$1 \cdot \vec{CB} + 1 \cdot \vec{CD} = (1+1)\,\vec{CF} \Rightarrow \vec{CB} + \vec{CD} = 2\,\vec{CF} \quad \text{...(ii)}$$

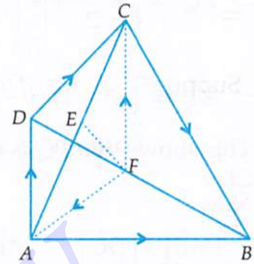

Fig. 26.47

Adding (i) and (ii), we obtain

$$\vec{AB} + \vec{AD} + \vec{CB} + \vec{CD} = 2\,\vec{AF} + 2\,\vec{CF}$$

$$= -2\,\vec{FA} - 2\,\vec{FC}$$

$$= -2\,(\vec{FA} + \vec{FC})$$

$$= -2\,(2\,\vec{FE}) = 4\,\vec{EF} \quad [\because E \text{ is the mid-point of } AC]$$

**EXAMPLE 14** *Show that the line joining one vertex of a parallelogram to the mid-point of an opposite side trisects the diagonal and is trisected thereat.*

SOLUTION Let $OABC$ be a parallelogram. Taking $O$ as the origin let the position vectors of $A$ and C be $\vec{a}$ and $\vec{b}$ respectively. In $\Delta OAB$, we have

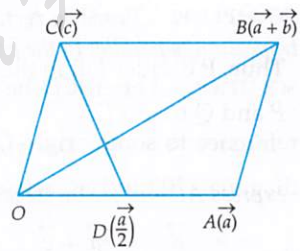

Fig. 26.48

$$\vec{OA} + \vec{AB} = \vec{OB}$$

$$\Rightarrow \quad \vec{OA} + \vec{OC} = \vec{OB} \quad [\because \vec{AB} = \vec{OC}]$$

$$\Rightarrow \quad \vec{OB} = \vec{a} + \vec{b} \Rightarrow \text{Position vector of } B \text{ is } \vec{a} + \vec{b}.$$

Let $D$ be the mid-point of $OA$. Then, the position vector of $D$ is $\dfrac{\vec{a}}{2}$.

Position vector of a point dividing $CD$ in the ratio $2:1$ is $\dfrac{2 \cdot (\vec{a}/2) + 1 \cdot \vec{b}}{2+1} = \dfrac{\vec{a} + \vec{b}}{3}$

Position vector of a point dividing $OB$ in the ratio $1:2$ is $\dfrac{1 \cdot (\vec{a} + \vec{b}) + 2 \cdot \vec{0}}{1+2} = \dfrac{\vec{a} + \vec{b}}{3}$

Thus, the position vectors of points trisecting $OB$ and $DC$ are same. Hence, $DC$ trisects $OB$ and $DC$ is trisected thereat.

**EXAMPLE 15** *ABCD is a parallelogram. E, F are mid-points of BC, CD respectively. AE, AF meet the diagonal BD at points Q and P respectively. Show that points P and Q trisect DB.*

SOLUTION Let $A$ be the origin and let the position vectors of $B$ and $D$ be $\vec{b}$ and $\vec{d}$ respectively.

Then, $\vec{AB} = \vec{b}$, $\vec{AD} = \vec{d}$. In $\Delta ABC$, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AB} + \vec{AD} = \vec{AC} \quad [\because \vec{BC} = \vec{AD}]$$

$$\Rightarrow \quad \vec{b} + \vec{d} = \vec{AC} \Rightarrow \text{Position vector of C is } \vec{b} + \vec{d}$$

Since $E$ and $F$ are the mid-points of $BC$ and $CD$ respectively. Therefore, position vectors of $E$ and $F$ are $\dfrac{\vec{b} + (\vec{b} + \vec{d})}{2} = \vec{b} + \dfrac{\vec{d}}{2}$ and $\dfrac{(\vec{b} + \vec{d}) + \vec{d}}{2} = \dfrac{\vec{b}}{2} + \vec{d}$ respectively.

Since $A$ is the origin and $P$ lies on $AF$.

$\therefore \quad \vec{AP} = \lambda\, \vec{AF}$ for some scalar $\lambda$

$$\Rightarrow \quad \vec{AP} = \lambda\left(\frac{\vec{b}}{2} + \vec{d}\right) \qquad \text{...(i)}$$

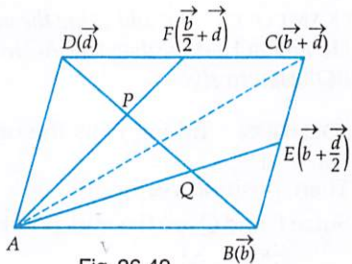

Fig. 26.49

Suppose $P$ divides $DB$ in the ratio $\mu : 1$. Then,

$$\text{Position vector of } P \text{ is } = \frac{\mu\,\vec{b} + 1\cdot\vec{d}}{\mu + 1}$$

$$\Rightarrow \quad \vec{AP} = \frac{\mu\,\vec{b} + \vec{d}}{\mu + 1} \qquad [\because\ A \text{ is the origin}] \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\lambda\left(\frac{\vec{b}}{2} + \vec{d}\right) = \frac{\mu\,\vec{b} + \vec{d}}{\mu + 1} \Rightarrow \left(\frac{\lambda}{2} - \frac{\mu}{\mu+1}\right)\vec{b} + \left(\lambda - \frac{1}{\mu+1}\right)\vec{d} = \vec{0}$$

$$\Rightarrow \quad \frac{\lambda}{2} - \frac{\mu}{\mu+1} = 0 \text{ and } \lambda - \frac{1}{\mu+1} = 0 \qquad [\because\ \vec{b} \text{ and } \vec{d} \text{ are non-collinear vectors}]$$

$$\Rightarrow \quad \mu = \frac{1}{2} \text{ and } \lambda = \frac{2}{3}$$

Thus, P divides DB in the ratio $\frac{1}{2} : 1$ i.e. $1 : 2$. Similarly, $Q$ divides $DB$ in the ratio $2 : 1$. Hence, $P$ and $Q$ trisect $DB$.

**EXAMPLE 16** *ABCD is a parallelogram. If L and M are the mid-points of BC and DC respectively, then express $\vec{AL}$ and $\vec{AM}$ in terms of $\vec{AB}$ and $\vec{AD}$. Also, prove that $\vec{AL} + \vec{AM} = \frac{3}{2}\vec{AC}$.*

SOLUTION Taking $A$ as the origin, let the position vectors of $B$ and $D$ be $\vec{b}$ and $\vec{d}$ respectively. Then, $\vec{AB} = \vec{b}$ and $\vec{AD} = \vec{d}$. In triangle $ABC$, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AC} = \vec{AB} + \vec{AD} = \vec{b} + \vec{d} \qquad [\because\ \vec{BC} = \vec{AD}]$$

$\therefore$ Position vector of $C$ is $\vec{b} + \vec{d}$

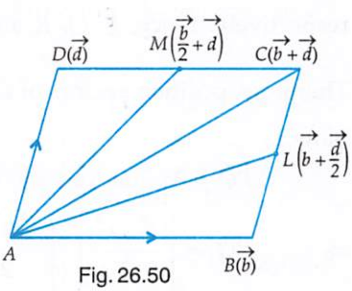

Fig. 26.50

Since $L$ and $M$ are mid-points of $BC$ and $CD$ respectively.

$$\therefore \text{ Position vector of } L = \frac{\vec{b} + (\vec{b} + \vec{d})}{2} = \vec{b} + \frac{1}{2}\vec{d}, \text{ Position vector of } M = \frac{(\vec{b} + \vec{d}) + \vec{d}}{2} = \frac{\vec{b}}{2} + \vec{d}$$

$$\therefore \vec{AL} = \text{Position vector of } L - \text{Position vector of } A = \vec{b} + \frac{1}{2}\vec{d} - \vec{0} = \vec{b} + \frac{1}{2}\vec{d} = \vec{AB} + \frac{1}{2}\vec{AD}$$

$$\text{and, } \vec{AM} = \text{Position vector of } M - \text{Position vector of } A = \frac{\vec{b}}{2} + \vec{d} - \vec{0} = \frac{\vec{b}}{2} + \vec{d} = \frac{1}{2}\vec{AB} + \vec{AD}$$

$$\therefore \quad \vec{AL} + \vec{AM} = \left(\vec{b} + \frac{1}{2}\vec{d}\right) + \left(\frac{1}{2}\vec{b} + \vec{d}\right) = \frac{3}{2}\vec{b} + \frac{3}{2}\vec{d} = \frac{3}{2}(\vec{b} + \vec{d}) = \frac{3}{2}\vec{AC}$$

**EXAMPLE 17** *If P and Q are the mid-points of the sides AB and CD of a parallelogram ABCD, prove that DP and BQ cut the diagonal AC in its points of trisection which are also the points of trisection of DP and BQ respectively.*

SOLUTION Taking $O$ as the origin, let the position vector $B$ and $D$ be $\vec{b}$ and $\vec{d}$ respectively. Then, position vector of $C$ is $\vec{b}+\vec{d}$.

Since $P$ and $Q$ are the mid-points of $AB$ and $CD$ respectively. Therefore, position vectors of $P$ and $Q$ are $\frac{\vec{b}}{2}$ and $\frac{\vec{b}}{2}+\vec{d}$ respectively.

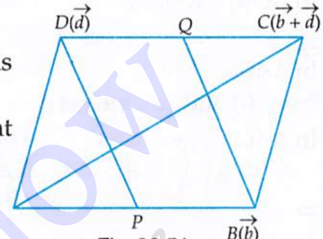

Fig. 26.51

The position vector of a point dividing $AC$ in the ratio $1:2$ is $\frac{1\cdot(\vec{b}+\vec{d})+2\cdot\vec{0}}{1+2}=\frac{\vec{b}+\vec{d}}{3}$. Also, the position vector of the point dividing $DP$ in the ratio $2:1$ is $\frac{2\left(\frac{\vec{b}}{2}\right)+1\cdot\vec{d}}{2+1}=\frac{\vec{b}+\vec{d}}{3}$

Thus, the point of trisection of $AC$ coincides with the point of trisection of $DP$.

Hence, $DP$ cuts the diagonal $AC$ in its point of trisection, which is also the point of trisection of $DP$. Similarly, $BQ$ cuts the diagonal $AC$ in its point of trisection, which is also the point of trisection of $BQ$.

**EXAMPLE 18** *"The mid-points of two opposite sides of a quadrilateral and the mid-points of the diagonals are the vertices of a parallelogram". Prove using vectors.*

SOLUTION Let $ABCD$ be a quadrilateral and let $P$, $R$ be the mid-points of the sides $AB$ and $CD$ respectively. Let $Q$ and $S$ be the mid-points of diagonals $AC$ and $BD$ respectively.

With reference to some origin $O$, let the position vectors of $A$, $B$, $C$ and $D$ be $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Since, $P$, $Q$, $R$ and $S$ are the mid-points of $AB$, $AC$, $CD$ and $BD$ respectively. Therefore, position vectors of $P$, $Q$, $R$ and $S$ are $\frac{\vec{a}+\vec{b}}{2}$, $\frac{\vec{a}+\vec{c}}{2}$, $\frac{\vec{c}+\vec{d}}{2}$ and $\frac{\vec{b}+\vec{d}}{2}$ respectively.

Now,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = \left(\frac{\vec{a}+\vec{c}}{2}\right)-\left(\frac{\vec{a}+\vec{b}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

and,

$$\vec{SR} = \text{Position vector of } R - \text{Position vector of } S$$

$$\Rightarrow \quad \vec{SR} = \left(\frac{\vec{c}+\vec{d}}{2}\right)-\left(\frac{\vec{b}+\vec{d}}{2}\right)=\frac{1}{2}(\vec{c}-\vec{b})$$

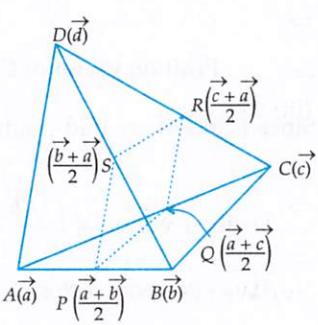

Fig. 26.52

Clearly, $\vec{PQ}=\vec{SR}$. Hence, $PQRS$ is a parallelogram.

**EXAMPLE 19** *If O is the circumcentre and O′ the orthocentre of a triangle ABC, prove that*

(i) $\vec{SA}+\vec{SB}+\vec{SC}=3\,\vec{SG}$, *where S is any point in the plane of triangle ABC whose centroid is at G.*

(ii) $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OO'}$

(iii) $\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\,\overrightarrow{O'O}$

(iv) $\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = \overrightarrow{AP}$, *where $\overrightarrow{AP}$ is the diameter of the circumcircle.*

SOLUTION Let $G$ be the centroid of triangle $ABC$. First we will show that the circumcentre $O$, orthocentre $O'$ and centroid $G$ are collinear and $O'G = 2OG$.

Let $AL$ and $BM$ be perpendiculars on the sides $BC$ and $CA$ respectively. Let $AD$ be the median and $OD$ be the perpendicular from $O$ on side $BC$. If $R$ is the circum radius of circumcircle of $\Delta ABC$, then $OB = OC = R$.

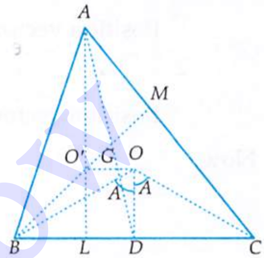

Fig. 26.53

In $\Delta OBD$, we have

$$OD = R\cos A \quad \text{...(i)}$$

In $\Delta ABM$, we have

$$AM = AB\cos A = c\cos A \quad \text{...(ii)}$$

In $\Delta AO'M$, we have

$$AO' = AM\sec\angle O'AM$$

$$\Rightarrow \quad AO' = c\cos A\ \sec(90° - C) \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad AO' = c\cos A\,\text{cosec}\, C$$

$$\Rightarrow \quad AO' = \frac{c}{\sin C}\cos A = 2R\cos A \qquad \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{C}{\sin C} = 2R\right]$$

$$\therefore \quad AO' = 2\,OD \quad \text{...(iii)} \qquad \text{[Using (i)]}$$

Triangles $AGO'$ and $OGD$ are similar

$$\therefore \quad \frac{OG}{O'G} = \frac{GD}{GA} = \frac{OD}{AO'} = \frac{1}{2} \qquad \text{[Using (iii)]}$$

$$\Rightarrow \quad 2\cdot OG = O'G \quad \text{...(iv)}$$

(i) Clearly,

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \overrightarrow{SA} + (\overrightarrow{SB} + \overrightarrow{SC})$$

$$\Rightarrow \quad \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \overrightarrow{SA} + 2\overrightarrow{SD} \qquad [\because D \text{ is the mid-point of } BC]$$

$$\Rightarrow \quad \overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = (1+2)\,\overrightarrow{SG} = 3\,\overrightarrow{SG} \qquad [\because G \text{ divides } AD \text{ in the ratio } 2:1]$$

(ii) Replacing $S$ by $O$ in (i), we have

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 3\overrightarrow{OG}$$

$$\Rightarrow \quad \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 2\overrightarrow{OG} + \overrightarrow{OG} = \overrightarrow{GO'} + \overrightarrow{OG} \qquad [\because 2OG = GO']$$

$$\Rightarrow \quad \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OG} + \overrightarrow{GO'} = \overrightarrow{OO'}$$

(iii) We have,

$$\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 3\overrightarrow{O'G} = 2\overrightarrow{O'G} + \overrightarrow{O'G} \qquad \text{[From (i)]}$$

$$\Rightarrow \quad \overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'G} + 2\overrightarrow{GO} \qquad [\because 2OG = GO']$$

$$\Rightarrow \quad \overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{O'O}$$

(iv) We have,

$$\overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\,\overrightarrow{AO'} + (\overrightarrow{O'A} + \overrightarrow{O'B} + \overrightarrow{O'C})$$

$$\Rightarrow \quad \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO'} + 2\,\overrightarrow{O'O} \qquad \text{[From (iii)]}$$

$$\Rightarrow \quad \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\,(\overrightarrow{AO'} + \overrightarrow{O'O})$$

$$\Rightarrow \quad \overrightarrow{AO'} + \overrightarrow{O'B} + \overrightarrow{O'C} = 2\overrightarrow{AO} = \overrightarrow{AP} \qquad [\because AO \text{ is the circum-radius of } \Delta ABC]$$

EXAMPLE 20 *The lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent.*

SOLUTION Let $OABC$ be a tetrahedron. Taking $O$ as the origin, let the position vectors of the vertices $A, B, C$ be $\vec{a}, \vec{b}$, and $\vec{c}$ respectively. Let $G, G_1, G_2, G_3$ be the centroids of the faces $ABC$, $OAB$, $OBC$ and $OCA$ respectively. Then,

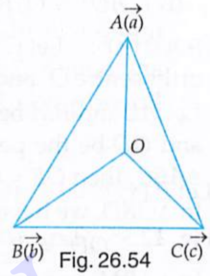

Fig. 26.54

$$\text{Position vector of } G = \frac{\vec{a}+\vec{b}+\vec{c}}{3}, \text{ Position vector of } G_1 = \frac{\vec{a}+\vec{b}}{3}$$

$$\text{Position vector of } G_2 = \frac{\vec{b}+\vec{c}}{3}, \text{ Position vector of } G_3 = \frac{\vec{c}+\vec{a}}{3}$$

Now,

$$\text{P.V. of a point dividing } OG \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{a}+\vec{b}+\vec{c}}{3}\right)+1\cdot\vec{0}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } AG_2 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{b}+\vec{c}}{3}\right)1\,.\,\vec{a}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } BG_3 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{c}+\vec{a}}{3}\right)+1\cdot\vec{b}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

$$\text{P.V. of a point dividing } CG_1 \text{ in the ratio } 3:1 = \frac{3\left(\frac{\vec{a}+\vec{b}}{3}\right)+1\cdot\vec{c}}{3+1} = \frac{\vec{a}+\vec{b}+\vec{c}}{4}$$

Thus, the point having position vector $\frac{\vec{a}+\vec{b}+\vec{c}}{4}$ is common to $OG$, $AG_2$, $BG_3$ and $CG_1$. Hence, the line joining the vertices of a tetrahedrun of the centroids of opposite faces are concurrent.

## EXERCISE 22.4

### BASIC

1. If $O$ is a point in space, $ABC$ is a triangle and $D, E, F$ are the mid-points of the sides $BC, CA$ and $AB$ respectively of the triangle, prove that $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OD}+\vec{OE}+\vec{OF}$.
2. Show that the sum of three vectors determined by the medians of a triangle directed from the vertices is zero.

### BASED ON LOTS

3. $ABCD$ is a parallelogram and $P$ is the point of intersection of its diagonals. If $O$ is the origin of reference, show that $\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=4\,\vec{OP}$.
4. Show that the line segments joining the mid-points of opposite sides of a quadrilateral bisects each other.

**BASIC ON HOTS**

5. $ABCD$ are four points in a plane and $Q$ is the point of intersection of the lines joining the mid-points of $AB$ and $CD$; $BC$ and $AD$. Show that $\vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} = 4\,\vec{PQ}$, where $P$ is any point.
6. Prove by vector method that the internal bisectors of the angles of a triangle are concurrent.

## 22.11 COMPONENTS OF A VECTOR IN TWO DIMENSION

Let $P(x, y)$ be a point in a plane with reference to $OX$ and $OY$ as the coordinate axes as show in Fig. 22.55. Then, $OM = x$ and $PM = y$. Let $\hat{i}, \hat{j}$ be unit vectors along $OX$ and $OY$ respectively. Then, $\vec{OM} = x\hat{i}$ and $\vec{MP} = y\hat{j}$. Vectors $\vec{OM}$ and $\vec{MP}$ are known as the components of $\vec{OP}$ along $x$-axis and $y$-axis respectively.

Now,

$$\vec{OP} = \vec{OM} + \vec{MP} \Rightarrow \vec{OP} = x\hat{i} + y\hat{j}$$

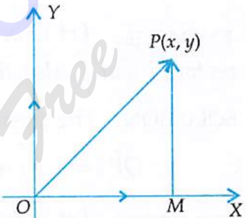

Fig. 22.55

Let $\vec{OP} = \vec{r}$. Then, $\vec{r} = x\hat{i} + y\hat{j}$.

Applying Pythagoras Theorem in $\Delta OMP$, we obtain

$$OP^2 = OM^2 + MP^2 = x^2 + y^2$$

$$\Rightarrow \quad OP = \sqrt{x^2 + y^2} \Rightarrow |\vec{r}| = \sqrt{x^2 + y^2}$$

*Thus, if a point P in a plane has coordinates (x, y), then*

*(i)* $\vec{OP} = x\hat{i} + y\hat{j}$ *(ii)* $|\vec{OP}| = \sqrt{x^2 + y^2}$

*(iii) The components of $\vec{OP}$ along x-axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and whose direction is along $OX$ or $OX'$ according as x is positive or negative.*

*(iv) The component of $\vec{OP}$ along y-axis is a vector $y\hat{j}$, whose magnitude is $|y|$ and whose direction is along $OY$ or $OY'$ according as y is positive or negative.*

### 22.11.1 COMPONENTS OF A VECTOR IN TERMS OF COORDINATES OF ITS END POINTS

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in $XOY$ plane. Let $\hat{i}$ and $\hat{j}$ be unit vectors along $OX$ and $OY$ respectively. From Fig. 22.56, we get

$$AF = x_2 - x_1,\ BF = y_2 - y_1 \Rightarrow \vec{AF} = (x_2 - x_1)\hat{i} \text{ and } \vec{FB} = (y_2 - y_1)\hat{j}$$

Now, $\vec{AB} = \vec{AF} + \vec{FB}$

$\Rightarrow \quad \vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$

$\Rightarrow$ Component of $\vec{AB}$ along $x$-axis $= (x_2 - x_1)\hat{i}$

and, Component of $\vec{AB}$ along $y$-axis $= (y_2 - y_1)\hat{j}$

Also, $|\vec{AB}| = AB = \sqrt{AF^2 + FB^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

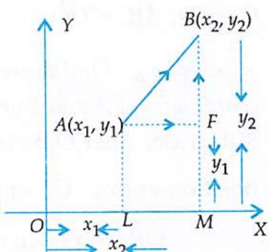

Fig. 22.56

### 22.11.2 ADDITION, SUBTRACTION, MULTIPLICATION OF A VECTOR BY A SCALAR AND EQUALITY IN TERMS OF COMPONENTS

For any two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j}$, we define

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$ (ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j}$

(iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$, where $m$ is a scalar (iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1$ and $a_2 = b_2$

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.*

SOLUTION We know that: $a_1\hat{i} + b_1\hat{j} = a_2\hat{i} + b_2\hat{j} \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

$\therefore \quad 2\hat{i} + 3\hat{j} = x\hat{i} + y\hat{j} \Rightarrow x = 2$ and $y = 3$.

**EXAMPLE 2** *Let O be the origin and let P(– 4, 3) be a point in the xy-plane. Express $\vec{OP}$ in terms of vectors $\hat{i}$ and $\hat{j}$. Also, find $|\vec{OP}|$.*

SOLUTION The position vector of point $P$ is $-4\hat{i} + 3\hat{j}$.

$\therefore \quad \vec{OP} = -4\hat{i} + 3\hat{j} \Rightarrow |\vec{OP}| = \sqrt{(-4)^2 + 3^2} = 5$

**EXAMPLE 3** *Let $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$. Is $|\vec{a}| = |\vec{b}|$? Are the vectors $\vec{a}$ and $\vec{b}$ equal?*

SOLUTION We have $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$.

$\therefore \quad |\vec{a}| = \sqrt{1 + 4} = \sqrt{5}$ and $|\vec{b}| = \sqrt{4 + 1} = \sqrt{5} \Rightarrow |\vec{a}| = |\vec{b}|$.

But, given vectors are not equal as their corresponding components are not equal.

**EXAMPLE 4** *If the position vector $\vec{a}$ of a point (12, n) is such that $|\vec{a}| = 13$, find the value of n.*

SOLUTION The position vector of the point (12, $n$) is $12\hat{i} + n\hat{j}$.

$\therefore \quad \vec{a} = 12\hat{i} + n\hat{j} \Rightarrow |\vec{a}| = \sqrt{12^2 + n^2}$

But, it is given that $|\vec{a}| = 13 \Rightarrow 13 = \sqrt{12^2 + n^2} \Rightarrow 169 = 144 + n^2 \Rightarrow n^2 = 25 \Rightarrow n = \pm 5$.

**EXAMPLE 5** *If A = (0, 1) B = (1, 0), C = (1, 2), D = (2, 1), prove that $\vec{AB} = \vec{CD}$.*

SOLUTION Clearly,

$$\vec{AB} = (\text{Position vector of } B - \text{Positive vector of } A) = (\hat{i} + 0\hat{j}) - (0\hat{i} + \hat{j}) = \hat{i} - \hat{j}$$

and, $$\vec{CD} = (\text{Position vector of } D - \text{Position vector of } C) = (2\hat{i} + \hat{j}) - (\hat{i} + 2\hat{j}) = \hat{i} - \hat{j}.$$

Clearly, $\vec{AB} = \vec{CD}$.

**EXAMPLE 6** *Find the coordinates of the tip of the position vector which is equivalent to $\vec{AB}$, where the coordinates of A and B are (3, 1) and (5, 0) respectively.*

SOLUTION Let $O$ be the origin and let $P(x, y)$ be the required point. Then, $P$ is the tip of the position vector $\vec{OP}$ of point P. We find that

$$\vec{OP} = x\hat{i} + y\hat{j}$$

and, $$\vec{AB} = (\text{Position vector of } B - \text{Position vector of } A) = (5\hat{i} + 0\hat{j}) - (3\hat{i} + \hat{j}) = 2\hat{i} - \hat{j}$$

It is given that $\vec{OP} = \vec{AB} \Rightarrow x\hat{i} + y\hat{j} = 2\hat{i} - \hat{j} \Leftrightarrow x = 2$ and $y = -1$.

Hence, the coordinates of the required point are (2, –1).

**EXAMPLE 7** *If $\vec{a}$ is a position vector whose tip is (1, –3). Find the coordinates of the point B such that $\vec{AB} = \vec{a}$. If A has coordinates (– 1, 5).*

SOLUTION Let $O$ be the origin and let $P(1, -3)$ be the tip of the position vector $\vec{a}$. Then, $\vec{a} = \vec{OP} = \hat{i} - 3\hat{j}$. Let the coordinates of $B$ be $(x, y)$ and $A$ has coordinates (–1, 5).

$\therefore \quad \vec{AB} =$ Position vector of $B -$ Position vector of $A = (x\hat{i} + y\hat{j}) - (-\hat{i} + 5\hat{j}) = (x+1)\hat{i} + (y-5)\hat{j}$

It is given that: $\vec{AB} = \vec{a}$

$\Rightarrow \quad (x+1)\hat{i} + (y-5)\hat{j} = \hat{i} - 3\hat{j} \Rightarrow x + 1 = 1$ and $y - 5 = -3 \Rightarrow x = 0$ and $y = 2$

Hence, the coordinates of B are (0, 2).

**EXAMPLE 8** *ABCD is a parallelogram. If the coordinates of A, B, C are (2, 3), (1, 4) and (0, –2) respectively, find the coordinates of D.*

SOLUTION Let the coordinates of $D$ be $(x, y)$. Since $ABCD$ is a parallelogram

$\therefore \quad \vec{AB} = \vec{DC}$

$\Rightarrow \quad (\hat{i} + 4\hat{j}) - (2\hat{i} + 3\hat{j}) = (0\hat{i} - 2\hat{j}) - (x\hat{i} + y\hat{j})$

$\Rightarrow \quad -\hat{i} + \hat{j} = -x\hat{i} - (y+2)\hat{j} \Rightarrow -1 = -x$ and $1 = -(y+2) \Rightarrow x = 1$ and $y = -3$

Hence, the coordinates of $D$ are ( 1, – 3).

**EXAMPLE 9** *Find a unit vector parallel to the vector $-3\hat{i} + 4\hat{j}$.*

SOLUTION Let $\vec{a} = -3\hat{i} + 4\hat{j}$. Then, $|\vec{a}| = \sqrt{(-3)^2 + (4)^2} = 5$

$$\therefore \quad \text{Unit vector parallel to } \vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{5}(-3\hat{i} + 4\hat{j}) = -\frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

**EXAMPLE 10** *Find a vector of magnitude 5 units which is parallel to the vector $2\hat{i} - \hat{j}$.* **[NCERT]**

SOLUTION Let $\vec{a} = 2\hat{i} - \hat{j}$. Then, $|\vec{a}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$. Let $\hat{a}$ be the unit vector parallel to $\vec{a}$.

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{5}}(2\hat{i} - \hat{j}) = \frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}$$

$$\text{Hence, required vector} = 5\hat{a} = 5\left(\frac{2}{\sqrt{5}}\hat{i} - \frac{1}{\sqrt{5}}\hat{j}\right) = 2\sqrt{5}\,\hat{i} - \sqrt{5}\,\hat{j}.$$

**BASED ON LOWER ORDER THINKING SKILLS (LOTS)**

**EXAMPLE 11** *Find the components along the coordinate axes of the position vector of each of the following points:*

(i) $P$ (5, 4) (ii) $Q$ (– 4, 3) (iii) $R$ (5, –7) (iv) $S$ (– 4, –5).

SOLUTION (i) Let $O$ be the origin. Then, the position vector of point $P$ (5, 4) is $\vec{OP} = 5\hat{i} + 4\hat{j}$.

So, component of $\vec{OP}$ along $x$-axis is a vector of magnitude 5 having its direction along the positive direction of $x$-axis. Also, the component of $\vec{OP}$ along $y$-axis is a vector of magnitude 4 along the positive direction of $y$-axis.

(ii) If $O$ is the origin, then the position vector of point $Q(-4, 3)$ is given by $\vec{OQ} = -4\hat{i} + 3\hat{j}$. So, its component along $x$-axis is a vector of magnitude 4, having its direction along the negative direction of $x$-axis. Also, the component of $\vec{OQ}$ along $y$-axis is a vector of magnitude 3, having its direction along the positive direction of $y$-axis.

(iii) Let $O$ be the origin. Then, the position vector of point $R(5, -7)$ is given by $\vec{OR} = 5\hat{i} - 7\hat{j}$. So, its component along $x$-axis is a vector of magnitude 5, having its direction along the positive direction of $x$-axis. Also, the component of $\vec{OR}$ along $y$-axis is a vector of magnitude 7, having its direction along the (–) negative direction of $y$-axis.

(iv) If $O$ is the origin, then the position vector of point $S(-4, -5)$ is given by $\vec{OS} = -4\hat{i} - 5\hat{j}$. Clearly, its component along $x$-axis is a vector of magnitude 4, having its direction along the negative direction of $x$-axis. Also, the component of $\vec{OS}$ along $y$-axis is a vector of magnitude 5, having its direction along the negative direction of $y$-axis.

**EXAMPLE 12** *Find the scalar and vector components of the vector with initial point $A(2, 1)$ and terminal point $B(-5, 7)$.*

SOLUTION Clearly,

$$\vec{AB} = \text{Position Vector of } B - \text{Position vector of } A = (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j}) = -7\hat{i} + 6\hat{j}$$

So, scalar components of $\vec{AB}$ along $OX$ and $OY$ are $-7$ and $6$ respectively. The vector components of $\vec{AB}$ are $-7\hat{i}$ and $6\hat{j}$ along $OX'$ and $OY$ respectively.

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 13** *Write all the unit vectors in XY-plane.*

SOLUTION Let $\vec{r} = x\hat{i} + y\hat{j}$ be a unit vector in $XY$-plane. Then,

$$|\vec{r}| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

Clearly, $P(x, y)$ lies on the circle $x^2 + y^2 = 1$ whose centre is at the origin $O$. Suppose $OP$ makes an angle $\theta$ with $OX$.

In $\Delta OLP$, we obtain

$$\cos\theta = \frac{OL}{OP} \text{ and } \sin\theta = \frac{LP}{OP}$$

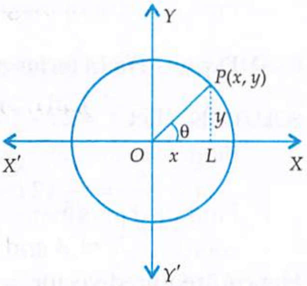


Fig. 22.57

$$\Rightarrow \quad x = OL = \cos\theta \text{ and } y = LP = \sin\theta$$

$$\therefore \quad \vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$

Clearly, as $\theta$ varies from 0 to $2\pi$, the point $P$ traces the circle $x^2 + y^2 = 1$ in counter clockwise sense and this covers all possible directions of $\vec{r}$. Hence, $\vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$ gives every unit vector in $XY$-plane.

**EXAMPLE 14** *Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.* **[NCERT]**

SOLUTION Let $P(x, y)$ be a point in $XY$-plane such that $OP = 1$ and $\angle XOP = 30°$. Then,

$$x = OP \cos 30° \text{ and } y = OP \sin 30° \Rightarrow x = \frac{\sqrt{3}}{2} \text{ and } y = \frac{1}{2}$$

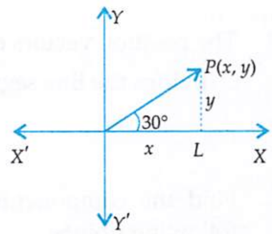

Fig. 22.58

$\therefore \ \vec{OP} = x\hat{i} + y\hat{j} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$ is the required unit vector.

**EXAMPLE 15** *A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.* **[NCERT]**

SOLUTION Let $B(x, y)$ be the final position of the girl and $O$ be the initial point of departure. Then,

$$AL = AB \cos 60° = \frac{3}{2} \text{ and, } BL = AB \sin 60° = \frac{3\sqrt{3}}{2}$$

$$\therefore \quad OL = OA - AL = \left(4 - \frac{3}{2}\right) = \frac{5}{2} \text{ and, } BL = \frac{3\sqrt{3}}{2}$$

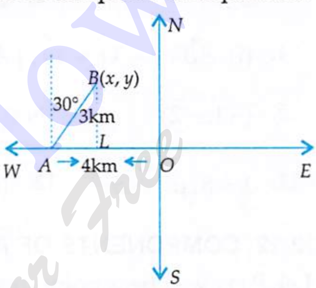

Fig. 22.59

Clearly, $B(x, y)$ lies in second quadrant. So, coordinates of $B$ are $\left(-\frac{5}{2}, \frac{3\sqrt{3}}{2}\right)$. Hence, position vector of $B$ is $-\frac{5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

## EXERCISE 22.5

### BASIC

1. If the position vector of a point $(-4, -3)$ be $\vec{a}$, find $|\vec{a}|$.
2. If the position vector $\vec{a}$ of a point $(12, n)$ is such that $|\vec{a}| = 13$, find the value(s) of $n$.
3. Find a vector of magnitude 4 units which is parallel to the vector $\sqrt{3}\hat{i} + \hat{j}$.
4. Express $\vec{AB}$ in terms of unit vectors $\hat{i}$ and $\hat{j}$, when the points are:
   (i) $A(4, -1)$, $B(1, 3)$ (ii) $A(-6, 3)$, $B(-2, -5)$
   Find $|\vec{AB}|$ in each case.
5. Find the coordinates of the tip of the position vector which is equivalent to $\vec{AB}$, where the coordinates of $A$ and $B$ are $(-1, 3)$ and $(-2, 1)$ respectively.
6. $ABCD$ is a parallelogram. If the coordinates of $A$, $B$, $C$ are $(-2, -1)$, $(3, 0)$ and $(1, -2)$ respectively, find the coordinates of $D$.
7. If the position vectors of the points $A(3, 4)$, $B(5, -6)$ and $C(4, -1)$ are $\vec{a}, \vec{b}, \vec{c}$ respectively, compute $\vec{a} + 2\vec{b} - 3\vec{c}$.
8. If $\vec{a}$ be the position vector whose tip is $(5, -3)$, find the coordinates of a point $B$ such that $\vec{AB} = \vec{a}$, the coordinates of $A$ being $(4, -1)$.
9. Show that the points $2\hat{i}$, $-\hat{i} - 4\hat{j}$ and $-\hat{i} + 4\hat{j}$ form an isosceles triangle.
10. Find a unit vector parallel to the vector $\hat{i} + \sqrt{3}\hat{j}$.

**11.** The position vectors of points $A$, $B$ and $C$ are $\lambda\hat{i}+3\hat{j}$, $12\hat{i}+\mu\hat{j}$ and $11\hat{i}-3\hat{j}$ respectively. If $C$ divides the line segment joining $A$ and $B$ in the ratio 3 : 1, find the values of $\lambda$ and $\mu$. **[CBSE 2017]**

**BASED ON LOTS**

**12.** Find the components along the coordinate axes of the position vector of each of the following points:

(i) $P(3, 2)$ (ii) $Q(-5, 1)$ (iii) $R(-11, -9)$ (iv) $S(4, -3)$

***ANSWERS***

**1.** 5 **2.** $\pm 5$ **3.** $2\sqrt{3}\,\hat{i}+2\hat{j}$

**4.** (i) $\vec{AB}=-3\hat{i}+4\hat{j}$, $|\vec{AB}|=5$ (ii) $\vec{AB}=4\hat{i}-8\hat{j}$, $|\vec{AB}|=4\sqrt{5}$

**5.** $(-1,-2)$ **6.** $(-4,-3)$ **7.** $\hat{i}-5\hat{j}$ **8.** $(9,-4)$ **10.** $\frac{1}{2}\hat{i}+\frac{\sqrt{3}}{2}\hat{j}$

**11.** $\lambda=8, \mu=-5$ **12.** (i) $3i, 2j$ (ii) $-5\hat{i}, \hat{j}$ (iii) $-11\hat{i}, -9\hat{j}$ (iv) $4\hat{i}, -3\hat{j}$

## 22.12 COMPONENTS OF A VECTOR IN THREE DIMENSIONS

Let $P(x, y, z)$ be a point in space with reference to $OX$, $OY$ and $OZ$ as the coordinate axes as shown in Fig. 22.60. Then, $OA = x$, $OB = y$ and $OC = z$. Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along $OX$, $OY$ and $OZ$ respectively. Then, $\vec{OA}=x\hat{i}$, $\vec{OB}=y\hat{j}$ and $\vec{OC}=z\hat{k}$.

From Fig. 22.60, we have

$$\vec{BC'}=\vec{OA}=x\hat{i},\ \vec{C'P}=\vec{OC}=z\hat{k}$$

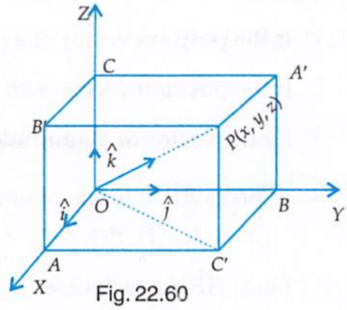

Fig. 22.60

Now, $\vec{OP}=\vec{OC'}+\vec{C'P}$

$\Rightarrow \vec{OP}=\vec{OB}+\vec{BC'}+\vec{C'P}$

$\Rightarrow \vec{OP}=\vec{OB}+\vec{OA}+\vec{OC}$

$\Rightarrow \vec{OP}=\vec{OA}+\vec{OB}+\vec{OC}$

$\Rightarrow \vec{OP}=x\hat{i}+y\hat{j}+z\hat{k}$

If $\vec{OP}=\vec{r}$. Then, $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Thus, the position vector of a point $P(x, y, z)$ in space is given by $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Now,

$$OP^2=OC'^2+C'P^2=(OB^2+BC'^2)+C'P^2=(OB^2+OA^2)+OC^2=OA^2+OB^2+OC^2$$

$$\Rightarrow OP^2=x^2+y^2+z^2 \Rightarrow OP=\sqrt{x^2+y^2+z^2} \Rightarrow |\vec{r}|=\sqrt{x^2+y^2+z^2}$$

*Thus, if a point $P$ in space has coordinates $(x, y, z)$, then its position vector $\vec{r}$ is $x\hat{i}+y\hat{j}+z\hat{k}$ and $|\vec{r}|=\sqrt{x^2+y^2+z^2}$.*

*The vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are known as the component vectors of $\vec{r}$ along x, y and z axes respectively.*

### 22.12.1 ADDITION, SUBTRACTION AND MULTIPLICATION OF A VECTOR BY A SCALAR AND EQUALITY IN TERMS OF COMPONENTS

*For any two vectors* $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ *and* $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$, *we define*

(i) $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

(ii) $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

(iii) $m\vec{a} = (ma_1)\hat{i} + (ma_2)\hat{j} + (ma_3)\hat{k}$, *where m is a scalar*

(iv) $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1, a_2 = b_2$ and $a_3 = b_3$.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Then,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\therefore \quad PQ = |\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the values of x, y and z so that the vectors* $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ *and* $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ *are equal.* [NCERT]

SOLUTION Two vectors $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are equal iff $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$.

$\therefore$ $\vec{a} = x\hat{i} + 2\hat{j} + z\hat{k}$ and $\vec{b} = 2\hat{i} + y\hat{j} + \hat{k}$ are equal iff. $x = 2$, $y = 2$ and $z = 1$

EXAMPLE 2 *Find the sum of vectors* $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ *and* $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

SOLUTION We have,

$$\vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = \{(\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k})\} + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$= \{(1-2)\hat{i} + (-2+4)\hat{j} + (1+5)\hat{k}\} + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$= (-\hat{i} + 2\hat{j} + 6\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k}) = (-1+1)\hat{i} + (2-6)\hat{j} + (6-7)\hat{k} = 0\hat{i} - 4\hat{j} - \hat{k}$$

EXAMPLE 3 *Find the magnitude of the vector* $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. [NCERT]

SOLUTION We have, $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. Therefore, $|\vec{a}| = \sqrt{3^2 + (-2)^2 + 6^2} = \sqrt{49} = 7$

EXAMPLE 4 *Find the distance between the points A (2, 3, 1) and B(–1, 2, –3), using vector method.*

SOLUTION We have,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A = (-\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k}) = -3\hat{i} - \hat{j} - 4\hat{k}$$

$$\therefore \quad AB = |\vec{AB}| = \sqrt{(-3)^2 + (-1)^2 + (-4)^2} = \sqrt{26}$$

Hence, the distance between $A$ and $B$ is $\sqrt{26}$.

EXAMPLE 5 *If* $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ *and* $\vec{b} = 2\hat{i} - 4\hat{j} - 3\hat{k}$, *find* $|\vec{a} - 2\vec{b}|$.

SOLUTION We have $\vec{a} - 2\vec{b} = (3\hat{i} - 2\hat{j} + \hat{k}) - 2(2\hat{i} - 4\hat{j} - 3\hat{k}) = -\hat{i} + 6\hat{j} + 7\hat{k}$

$\therefore \ |\vec{a}-2\vec{b}| = |-\hat{i}+6\hat{j}+7\hat{k}| = \sqrt{(-1)^2+6^2+7^2} = \sqrt{86}$.

**EXAMPLE 6** *If A, B, C have position vectors (2, 0, 0) (0, 1, 0), (0, 0, 2), show that $\Delta ABC$ is isosceles.*

SOLUTION We have,

$\vec{AB}$ = Position vector of $B$ – Position vector of $A = (0\hat{i}+\hat{j}+0\hat{k}) - (2\hat{i}+0\hat{j}+0\hat{k}) = -2\hat{i}+\hat{j}+0\hat{k}$

$\therefore \ AB = |\vec{AB}| = \sqrt{(-2)^2+1^2+0^2} = \sqrt{5}$

$\vec{BC}$ = Position vector of C – Position vector of $B = (0\hat{i}+0\hat{j}+2\hat{k}) - (0\hat{i}+\hat{j}+0\hat{k}) = 0\hat{i}-\hat{j}+2\hat{k}$

$\therefore \ BC = |\vec{BC}| = \sqrt{0^2+(-1)^2+2^2} = \sqrt{5}$

Clearly, $AB = BC$. Hence, $\Delta ABC$ is isosceles.

**EXAMPLE 7** *Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$ respectively, form the vertices of a right angled triangle.* **[NCERT]**

SOLUTION We find that

$$\vec{AB} = \vec{b}-\vec{a} = (2\hat{i}-\hat{j}+\hat{k}) - (3\hat{i}-4\hat{j}-4\hat{k}) = -\hat{i}+3\hat{j}+5\hat{k}$$

$$\vec{BC} = \vec{c}-\vec{b} = (\hat{i}-3\hat{j}-5\hat{k}) - (2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$$

and, $\vec{CA} = \vec{a}-\vec{c} = (3\hat{i}-4\hat{j}-4\hat{k}) - (\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

Clearly, $\vec{AB}+\vec{BC}+\vec{CA} = \vec{0}$. So, points $A$, $B$ and $C$ form a triangle.

Now, $|\vec{AB}| = \sqrt{(-1)^2+3^2+5^2} = \sqrt{35}$, $|\vec{BC}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{41}$

and, $|\vec{CA}| = \sqrt{2^2+(-1)^2+(1)^2} = \sqrt{6}$

Clearly, $|\vec{BC}|^2 = |\vec{AB}|^2 + |\vec{CA}|^2$. Hence, $\Delta ABC$ is a right-angled triangle.

**EXAMPLE 8** *Find the unit vector in the direction of $3\hat{i}-6\hat{j}+2\hat{k}$.*

SOLUTION Let $\vec{a} = 3\hat{i}-6\hat{j}+2\hat{k}$. Then, $|\vec{a}| = \sqrt{3^2+(-6)^2+2^2} = 7$

So, a unit vector in the direction of $\vec{a}$ is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}) = \dfrac{3}{7}\hat{i}-\dfrac{6}{7}\hat{j}+\dfrac{2}{7}\hat{k}$.

**EXAMPLE 9** *Find the unit vector in the direction of $\vec{a}+\vec{b}$, if $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}-\hat{k}$.* **[NCERT]**

SOLUTION We have, $\vec{a} = 2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}+\hat{j}-\hat{k}$.

$\therefore \ \vec{a}+\vec{b} = (2\hat{i}-\hat{j}+2\hat{k}) + (-\hat{i}+\hat{j}-\hat{k}) = (2-1)\hat{i} + (-1+1)\hat{j} + (2-1)\hat{k} = \hat{i}+0\hat{j}+\hat{k}$

$\Rightarrow \ |\vec{a}+\vec{b}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$\therefore$ Required unit vector $= \dfrac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \dfrac{1}{\sqrt{2}}(\hat{i}+0\hat{j}+\hat{k}) = \dfrac{1}{\sqrt{2}}\hat{i} + \dfrac{1}{\sqrt{2}}\hat{k}$.

**EXAMPLE 10** *Find the unit vector in the direction of $\vec{PQ}$, where P and Q are the points (1, 2, 3) and (4, 5, 6) respectively.* **[NCERT]**

SOLUTION Clearly, $\vec{PQ}$ = Position vector of $Q$ – Position vector of $P$

$$\Rightarrow \quad \vec{PQ} = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k} \Rightarrow |\vec{PQ}| = \sqrt{9+9+9} = 3\sqrt{3}$$

$\therefore$ The unit vector in the direction of $\vec{PQ}$ is

$$\frac{1}{|\vec{PQ}|}\vec{PQ} = \frac{1}{3\sqrt{3}}(3\hat{i} + 3\hat{j} + 3\hat{k}) = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

**EXAMPLE 11** *Find a vector of magnitude 11 in the direction opposite to that of $\vec{PQ}$, where P and Q are the points (1, 3, 2) and (−1, 0, 8) respectively.* **[NCERT EXEMPLAR]**

SOLUTION We find that $\vec{PQ} = (-\hat{i} + 0\hat{j} + 8\hat{k}) - (\hat{i} + 3\hat{j} + 2\hat{k}) = -2\hat{i} - 3\hat{j} + 6\hat{k}$

$$\therefore \quad \vec{QP} = -\vec{PQ} = 2\hat{i} + 3\hat{j} - 6\hat{k} \Rightarrow |\vec{QP}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

So, the unit vector in the direction of $\vec{QP}$ is $\hat{QP} = \frac{1}{7} = (2\hat{i} + 3\hat{j} - 6\hat{k})$.

Hence, the required vector of magnitude 11 in the direction of $\vec{QP}$ is $11\hat{QP} = \frac{11}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$.

**EXAMPLE 12** *The two vectors $\hat{j} + \hat{i}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the sides $\vec{AB}$ and $\vec{AC}$ respectively of triangle ABC. Find the length of the median through A.* **[CBSE 2015, 2016]**

SOLUTION Let $D$ be the mid-point of side $BC$ of triangle $ABC$. Then,

$$\vec{AB} + \vec{AC} = 2\vec{AD} \qquad \text{[See Remark 4 on page 22.17]}$$

$$\Rightarrow \quad (\vec{j} + \vec{i}) + (3\hat{i} - \hat{j} + 4\hat{k}) = 2\vec{AD} \Rightarrow \vec{AD} = 2\vec{j} + 0\vec{j} + 2\hat{k} \Rightarrow |\vec{AD}| = \sqrt{4+0+4} = 2\sqrt{2}$$

## BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 13** *If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ represent two adjacent sides of a parallelogram, find unit vectors parallel to the diagonals of the parallelogram.* **[CBSE 2020]**

SOLUTION Let $ABCD$ be a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$. Then,

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{AC} = \vec{a} + \vec{b} = 3\hat{i} + 6\hat{j} - 2\hat{k}$$

and, $$\vec{AB} + \vec{BD} = \vec{AD}$$

$$\Rightarrow \quad \vec{BD} = \vec{AD} - \vec{AB} \Rightarrow \vec{BD} = \vec{b} - \vec{a} = \hat{i} + 2\hat{j} - 8\hat{k}$$

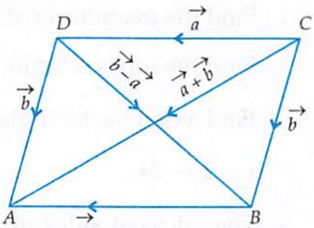

Fig. 22.61

Now, $$\vec{AC} = 3\hat{i} + 6\hat{j} - 2\hat{k} \Rightarrow |\vec{AC}| = \sqrt{9+36+4} = 7$$

and, $$\vec{BD} = \hat{i} + 2\hat{j} - 8\hat{k} \Rightarrow |\vec{BD}| = \sqrt{1+4+64} = \sqrt{69}$$

$\therefore$ $$\text{Unit vector along } \vec{AC} = \frac{\vec{AC}}{|\vec{AC}|} = \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$$

and, $$\text{Unit vector along } \vec{BD} = \frac{\vec{BD}}{|\vec{BD}|} = \frac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} - 8\hat{k}).$$

### BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

**EXAMPLE 14** *Three vectors of magnitude a, 2a, 3a meet in a point and their directions are along the diagonals of the adjacent faces of a cube. Determine their resultant.*

SOLUTION Consider a unit cube whose one vertex is at the origin and three coterminous edges $OA$, $OB$ and $OC$ along the coordinate exes $OX$, $OY$ and $OZ$ respectively. Then, $\vec{OA} = \hat{i}$, $\vec{OB} = \hat{j}$ and $\vec{OC} = \hat{k}$. Let $OD$, $OE$ and $OF$ be the diagonals of three adjacent faces of the cube passing through $O$ along which act the vectors of magnitude $a$, $2a$ and $3a$ respectively.

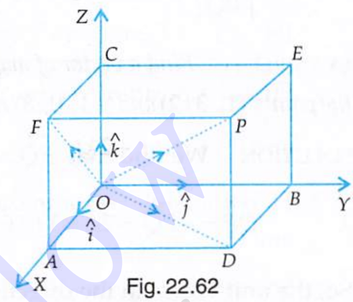

Fig. 22.62

We have,

$$\vec{OD} = \vec{OB} + \vec{BD} = \vec{OB} + \vec{OA} = \hat{j} + \hat{i}$$

$$\therefore \quad |\vec{OD}| = \sqrt{1+1} = \sqrt{2}$$

Thus, the unit vector along $\vec{OD}$ is $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$.

Similarly, unit vectors along $\vec{OE}$ and $\vec{OF}$ are $\frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$ and $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$ respectively.

A vector of magnitude '$a$' along $OD$ is given by $\vec{r_1} = a \times \frac{1}{\sqrt{2}}(\hat{i} + \hat{j}) = \frac{a}{\sqrt{2}}(\hat{i} + \hat{j})$

Similarly, vectors of magnitude $2a$ and $3a$ along $OE$ and $OF$ are given by

$$\vec{r_2} = 2a \times \frac{1}{\sqrt{2}}(\hat{j} + \hat{k}) = \frac{2a}{\sqrt{2}}(\hat{j} + \hat{k}) \text{ and, } \vec{r_3} = \frac{3a}{\sqrt{2}}(\hat{i} + \hat{k}) \text{ respectively.}$$

Let $\vec{r}$ be the resultant of $\vec{r_1}$, $\vec{r_2}$ and $\vec{r_3}$. Then,

$$\vec{r} = \vec{r_1} + \vec{r_2} + \vec{r_3} \Rightarrow \vec{r} = \frac{a}{\sqrt{2}}(\hat{i} + \hat{j}) + \frac{2a}{\sqrt{2}}(\hat{j} + \hat{k}) + \frac{3a}{\sqrt{2}}(\hat{k} + \hat{i}) = \frac{a}{\sqrt{2}}(4\hat{i} + 3\hat{j} + 5\hat{k})$$

$$\therefore \quad |\vec{r}| = \frac{a}{\sqrt{2}}\sqrt{16+9+25} = 5a.$$

## EXERCISE 22.6

### BASIC

1. Find the magnitude of the vector $\vec{a} = 2\hat{i} + 3\hat{j} - 6\hat{k}$.
2. Find the unit vector in the direction of $3\hat{i} + 4\hat{j} - 12\hat{k}$.
3. Find a unit vector in the direction of the resultant of the vectors $\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + 2\hat{j} - 2\hat{k}$.
4. The adjacent sides of a parallelogram are represented by the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} + 2\hat{k}$. Find unit vectors parallel to the diagonals of the parallelogram. [NCERT]
5. If $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, find $|3\vec{a} - 2\vec{b} + 4\vec{c}|$.
6. If $\vec{PQ} = 3\hat{i} + 2\hat{j} - \hat{k}$ and the coordinates of $P$ are $(1, -1, 2)$, find the coordinates of $Q$.
7. Prove that the points $\hat{i} - \hat{j}$, $4\hat{i} - 3\hat{j} + \hat{k}$ and $2\hat{i} - 4\hat{j} + 5\hat{k}$ are the vertices of a right-angled triangle.

8. If the vertices $A, B, C$ of a triangle $ABC$ are the points with position vectors $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ respectively, what are the vectors determined by its sides? Find the length of these vectors.
9. Find the vector from the origin $O$ to the centroid of the triangle whose vertices are $(1, -1, 2)$, $(2, 1, 3)$ and $(-1, 2, -1)$.
10. Find the position vector of a point $R$ which divides the line segment joining points $P(\hat{i} + 2\hat{j} + \hat{k})$ and $Q(-\hat{i} + \hat{j} + \hat{k})$ in the ratio 2:1. (i) internally (ii) externally **[NCERT]**
11. Find the position vector of the mid-point of the vector joining the points $P(2\hat{i} - 3\hat{j} + 4\hat{k})$ and $Q(4\hat{i} + \hat{j} - 2\hat{k})$.
12. Find the unit vector in the direction of vector $\overrightarrow{PQ}$, where $P$ and $Q$ are the points $(1, 2, 3)$ and $(4, 5, 6)$.
13. Show that the points $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$, $C(3\hat{i} - 4\hat{j} - 4\hat{k})$ are the vertices of a right angled triangle. **[NCERT]**
14. Find the position vector of the mid-point of the vector joining the points $P(2, 3, 4)$ and $Q(4, 1, -2)$. **[NCERT]**
15. Find the value of $x$ for which $x(\hat{i} + \hat{j} + \hat{k})$ is a unit vector. **[NCERT]**
16. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to $2\vec{a} - \vec{b} + 3\vec{c}$. **[NCERT]**

**BASED ON LOTS**

17. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude 6 units which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$. **[CBSE 2010]**
18. Find a vector of magnitude of 5 units parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$. **[CBSE 2011]**
19. The two vectors $\hat{j} + \hat{i}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively of triangle $ABC$. Find the length of the median through $A$. **[CBSE 2015, 2016]**
20. In a parallelogram $PQRS$, $\overrightarrow{PQ} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{PS} = -\hat{i} - 2\hat{k}$. Find $|\overrightarrow{PR}|$ and $|\overrightarrow{QS}|$. **[CBSE 2022]**

## ANSWERS

1. 7
2. $\frac{3}{13}\hat{i} + \frac{4}{13}\hat{j} - \frac{12}{13}\hat{k}$
3. $\frac{1}{\sqrt{21}}(4\hat{i} + 2\hat{j} - \hat{k})$
4. $\frac{1}{\sqrt{6}}(-\hat{i} + 2\hat{j} + \hat{k})$, $\frac{1}{\sqrt{2}}(\hat{i} - \hat{k})$
5. $\sqrt{398}$
6. $(4, 1, 1)$
8. $\overrightarrow{AB} = (b_1 - a_1)\hat{i} + (b_2 - a_2)\hat{j} + (b_3 - a_3)\hat{k}$, $\overrightarrow{BC} = (c_1 - b_1)\hat{i} + (c_2 - b_2)\hat{j} + (c_3 - b_3)\hat{k}$

   $\overrightarrow{CA} = (a_1 - c_1)\hat{i} + (a_2 - c_2)\hat{j} + (a_3 - c_3)\hat{k}$

   $|\overrightarrow{AB}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$ etc.

9. $\frac{2}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{4}{3}\hat{k}$ 10. (i) $-\frac{1}{3}\hat{i}+\frac{4}{3}\hat{j}+\hat{k}$ (ii) $-3\hat{i}+\hat{k}$ 11. $3\hat{i}-\hat{j}+\hat{k}$

12. $\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$ 14. $3\hat{i}+2\hat{j}+\hat{k}$ 15. $\pm\frac{1}{\sqrt{3}}$ 16. $\frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}$

17. $2\hat{i}-4\hat{j}+4\hat{k}$ 18. $\sqrt{\frac{5}{2}}(3\hat{i}+\hat{j})$ 19. $2\sqrt{2}$ 20. $2\sqrt{2}, 6$

## 22.13 COLLINEARITY

In this section, we will discuss collinearity of vectors and points in a plane and in space as well.

### 22.13.1 COLLINEARITY OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two collinear or parallel vectors. Then their supports are parallel. Let $\hat{a}$ be the unit vector in the direction of $\vec{a}$. As $\vec{a}$ and $\vec{b}$ may be either like parallel vectors or unlike parallel vectors. So, the unit vector in the direction of $\vec{b}$ is either $\hat{a}$ or $-\hat{a}$.

$\therefore \quad \vec{a}=|\vec{a}|\,\hat{a}$ and $\vec{b}=\pm|\vec{b}|\,\hat{a}$

Now, $\quad \vec{a}=|\vec{a}|\,\hat{a} \Rightarrow \vec{a}=\pm\frac{|\vec{a}|}{|\vec{b}|}\left\{\pm|\vec{b}|\,\hat{a}\right\} \Rightarrow \vec{a}=\lambda\vec{b}$, where $\lambda=\pm\frac{|\vec{a}|}{|\vec{b}|}$

Also, $\quad \vec{b}=\pm|\vec{b}|\,\hat{a} \Rightarrow \vec{b}=\left\{\pm\frac{|\vec{b}|}{|\vec{a}|}\right\}|\vec{a}|\,\hat{a} \Rightarrow \vec{b}=\mu\,\vec{a}$, where $\mu=\pm\frac{|\vec{b}|}{|\vec{a}|}$

Thus, if $\vec{a}$ and $\vec{b}$ are two collinear or parallel vectors, then there exists a scalar $\lambda$ such that $\vec{a}=\lambda\vec{b}$ or, $\vec{b}=\lambda\vec{a}$.

In the following theorem, we prove the general criterion for the coplanarity of two vectors.

**THEOREM 1** *Two non-zero vectors $\vec{a}$ and $\vec{b}$ are collinear iff there exist scalars $x, y$ not both zero such that $x\vec{a}+y\vec{b}=\vec{0}$.*

PROOF First, let $\vec{a}$ and $\vec{b}$ be two collinear vectors. Then, there exists a scalar $\lambda$ such that

$\vec{a}=\lambda\vec{b} \Rightarrow 1\cdot\vec{a}+(-\lambda)\vec{b}=\vec{0} \Rightarrow x\vec{a}+y\vec{b}=\vec{0}$, where $x=1$ and $y=-\lambda$.

Conversely, let $\vec{a}$ and $\vec{b}$ be two non-zero vectors such that $x\vec{a}+y\vec{b}=\vec{0}$ for some scalars $x, y$ not both zero. Then, we have to prove that $\vec{a}$ and $\vec{b}$ are collinear vectors. Let $x\neq 0$. Then,

$$x\vec{a}+y\vec{b}=\vec{0} \Rightarrow x\vec{a}=-y\vec{b} \Rightarrow \vec{a}=\left(-\frac{y}{x}\right)\vec{b} \Rightarrow \vec{a}=\lambda\vec{b}, \text{ where } \lambda=-\frac{y}{x}$$

$\Rightarrow \quad \vec{a}$ and $\vec{b}$ are collinear vectors. Q.E.D.

It follows from the above theorem that if two non-zero vectors are non-collinear, we cannot express one in terms of the other. In other words, their linear combination can never be the zero vector.

Following theorem proves that the linear combination of two non-zero vectors is zero if each scalars in the linear combination is zero.

**THEOREM 2** *If $\vec{a}$, $\vec{b}$ are any two non-zero non-collinear vectors and $x, y$ are scalars, then*

$$x\vec{a}+y\vec{b}=\vec{0} \Rightarrow x=y=0$$

PROOF If possible, let $x \neq 0$. Then,

$$x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x\vec{a} = -y\vec{b} \Rightarrow \vec{a} = \left(-\frac{y}{x}\right)\vec{b} \Rightarrow \vec{a} \text{ and } \vec{b} \text{ are collinear vectors.}$$

This is a contradiction to the hypothesis that $\vec{a}$ and $\vec{b}$ are non-collinear vectors. Therefore, $x = 0$.

Similarly, we obtain $y = 0$. Hence, $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$. **Q.E.D.**

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

**EXAMPLE 1** *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors such that $x_1\vec{a} + y_1\vec{b} = x_2\vec{a} + y_2\vec{b}$, then prove that $x_1 = x_2$ and $y_1 = y_2$.*

SOLUTION We have, $x_1\vec{a} + y_1\vec{b} = x_2\vec{a} + y_2\vec{b}$

$\Rightarrow \quad (x_1 - x_2)\vec{a} + (y_1 - y_2)\vec{b} = \vec{0}$

$\Rightarrow \quad x_1 - x_2 = 0$ and $y_1 - y_2 = 0 \Rightarrow x_1 = x_2$ and $y_1 = y_2$. $\quad [\because \vec{a}$ and $\vec{b}$ are non-collinear$]$

**EXAMPLE 2** *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of $x$ for which vectors $\vec{\alpha} = (x-2)\vec{a} + \vec{b}$ and $\vec{\beta} = (3+2x)\vec{a} - 2\vec{b}$ are collinear.*

SOLUTION Since vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear. Therefore, there exist scalar $\lambda$ such that

$\vec{\alpha} = \lambda\vec{\beta}$

$\Rightarrow (x-2)\vec{a} + \vec{b} = \lambda\{(3+2x)\vec{a} - 2\vec{b}\}$

$\Rightarrow \{x - 2 - \lambda(3+2x)\}\vec{a} + (1+2\lambda)\vec{b} = \vec{0}$

$\Rightarrow x - 2 - \lambda(3+2x) = 0$ and $1 + 2\lambda = 0$ $\quad [\because \vec{a}$ and $\vec{b}$ are non-collinear$]$

$\Rightarrow x - 2 - \lambda(3+2x) = 0$ and $\lambda = -\frac{1}{2} \Rightarrow x - 2 + \frac{1}{2}(3+2x) = 0 \Rightarrow 4x - 1 = 0 \Rightarrow x = \frac{1}{4}$

**EXAMPLE 3** *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors, find the value of $x$ for which the vectors $\vec{\alpha} = (2x+1)\vec{a} - \vec{b}$ and $\vec{\beta} = (x-2)\vec{a} + \vec{b}$ are collinear.*

SOLUTION Given vectors $\vec{\alpha}$ and $\vec{\beta}$ will be collinear, if

$\vec{\alpha} = m\vec{\beta}$ for some scalar $m$

$\Rightarrow \quad (2x+1)\vec{a} - \vec{b} = m\{(x-2)\vec{a} + \vec{b}\}$

$\Rightarrow \quad \{(2x+1) - m(x-2)\}\vec{a} - (m+1)\vec{b} = \vec{0}$

$\Rightarrow \quad (2x-1) - m(x-2) = 0$ and $-(m+1) = 0$ $\quad [\because \vec{a}$ and $\vec{b}$ are non-collinear$]$

$\Rightarrow \quad m = -1$ and $x = \frac{1}{3}$.

**EXAMPLE 4** *If $\vec{a}, \vec{b}$ are the position vectors of the points $(1, -1)$, $(-2, m)$, find the value of $m$ for which $\vec{a}$ and $\vec{b}$ are collinear.*

SOLUTION It is given that the position vectors of points (1, – 1) and (–2, $m$) are $\vec{a}$ and $\vec{b}$ respectively. Therefore, $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -2\hat{i} + m\hat{j}$. It is given that $\vec{a}$ and $\vec{b}$ are collinear vectors.

$\therefore \quad \vec{a} = \lambda \vec{b}$ for some scalar $\lambda$

$\Rightarrow \quad \hat{i} - \hat{j} = \lambda(-2\hat{i} + m\hat{j})$

$\Rightarrow \quad \hat{i} - \hat{j} = (-2\lambda)\hat{i} + (m\lambda)\hat{j} \Rightarrow 1 = -2\lambda$ and $-1 = m\lambda$ [$\because$ $\hat{i}$ and $\hat{j}$ are non-collinear]

$\Rightarrow \quad \lambda = -\frac{1}{2}$ and $\lambda = -\frac{1}{m} \Rightarrow -\frac{1}{2} = -\frac{1}{m} \Rightarrow m = 2$

**EXAMPLE 5** *Let $\vec{u} = \hat{i} + 2\hat{j}$, $\vec{v} = -2\hat{i} + \hat{j}$ and $\vec{w} = 4\hat{i} + 3\hat{j}$. Find scalars $x$ and $y$ such that $\vec{w} = x\vec{u} + y\vec{v}$.*

SOLUTION We have, $\vec{w} = x\vec{u} + y\vec{v}$

$\Rightarrow \quad 4\hat{i} + 3\hat{j} = x(\hat{i} + 2\hat{j}) + y(-2\hat{i} + \hat{j})$

$\Rightarrow \quad (x - 2y - 4)\hat{i} + (2x + y - 3)\hat{j} = \vec{0}$

$\Rightarrow \quad x - 2y - 4 = 0$ and $2x + y - 3 = 0$ [$\because$ $\hat{i}$ and $\hat{j}$ are non-collinear vectors]

$\Rightarrow \quad x = 2$ and $y = -1$.

**EXAMPLE 6** *Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.* **[NCERT]**

SOLUTION Let $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k}$. Then,

$\vec{b} = -4\hat{i} + 6\hat{j} - 8\hat{k} = -2(2\hat{i} - 3\hat{j} + 4\hat{k}) = -2\vec{a}$. Hence, $\vec{a}$ and $\vec{b}$ are collinear.

**BASED ON LOWER ORDER THINKING SKILLS (LOTS)**

**EXAMPLE 7** *If $\vec{a}, \vec{b}, \vec{c}$ are three non-null vectors such that any two of them are non-collinear. If $\vec{a} + \vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + \vec{c}$ is collinear with $\vec{a}$, then find $\vec{a} + \vec{b} + \vec{c}$.*

SOLUTION It is given that:

$\vec{a} + \vec{b}$ is collinear with $\vec{c} \Rightarrow \vec{a} + \vec{b} = \lambda_1 \vec{c}$ for some scalar $\lambda_1$ ...(i)

It is also given that $\vec{b} + \vec{c}$ is collinear with $\vec{a} \Rightarrow \vec{b} + \vec{c} = \lambda_2 \vec{a}$ for some scalar $\lambda_2$ ...(ii)

From (i), we get: $\vec{a} = \lambda_1 \vec{c} - \vec{b}$

Substituting this value of $\vec{a}$ in (ii), we get

$$\vec{b} + \vec{c} = \lambda_2 (\lambda_1 \vec{c} - \vec{b}) \Rightarrow (1 + \lambda_2)\vec{b} + (1 - \lambda_1\lambda_2)\vec{c} = \vec{0}$$

$\Rightarrow \quad 1 + \lambda_2 = 0$ and $1 - \lambda_1\lambda_2 = 0$ [$\because$ $\vec{b}$ and $\vec{c}$ are non-collinear]

$\Rightarrow \quad \lambda_2 = -1, \lambda_1 = -1$

Substituting the values of $\lambda_1$ and $\lambda_2$ in (i) and (ii) respectively, we get

$$\vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

**EXAMPLE 8** *Let $\vec{a}, \vec{b}, \vec{c}$ be three non-zero vectors such that any two of them are non-collinear. If $\vec{a} + 2\vec{b}$ is collinear with $\vec{c}$ and $\vec{b} + 3\vec{c}$ is collinear with $\vec{a}$, then prove that $\vec{a} + 2\vec{b} + 6\vec{c} = \vec{0}$.*

SOLUTION It is given that

$\vec{a}+2\vec{b}$ is collinear with $\vec{c} \Rightarrow \vec{a}+2\vec{b}=\lambda\vec{c}$ for some scalar $\lambda$ ...(i)

and, $\vec{b}+3\vec{c}$ is collinear with $\vec{a} \Rightarrow \vec{b}+3\vec{c}=\mu\vec{a}$ for some scalar $\mu$ ...(ii)

From (i), we get: $\vec{a}=\lambda\vec{c}-2\vec{b}$. Substituting this value of $\vec{a}$ in (ii), we get

$$\vec{b}+3\vec{c}=\mu(\lambda\vec{c}-2\vec{b})$$

$$\Rightarrow \quad (1+2\mu)\vec{b}+(3-\mu\lambda)\vec{c}=\vec{0}$$

$$\Rightarrow \quad 1+2\mu=0 \text{ and } 3-\mu\lambda=0 \Rightarrow \mu=-\frac{1}{2} \text{ and } \lambda=-6 \quad [\because \vec{b} \text{ and } \vec{c} \text{ are non-collinear vectors}]$$

Substituting the values of $\lambda$ and $\mu$ in (i) and (ii), respectively, we get $\vec{a}+2\vec{b}+6\vec{c}=\vec{0}$.

**EXAMPLE 9** *If $\vec{a}$ and $\vec{b}$ are non-collinear vectors and vectors $\vec{\alpha}=(x+4y)\vec{a}+(2x+y+1)\vec{b}$ and $\vec{\beta}=(-2x+y+2)\vec{a}+(2x-3y-1)\vec{b}$ are connected by the relation $3\vec{\alpha}=2\vec{\beta}$, find $x, y$.*

SOLUTION We have, $3\vec{\alpha}=2\vec{\beta}$

$$\Rightarrow \quad 3\left\{(x+4y)\vec{a}+(2x+y+1)\vec{b}\right\}=2\left\{(-2x+y+2)\vec{a}+(2x-3y-1)\vec{b}\right\}$$

$$\Rightarrow \quad (3x+12y+4x-2y-4)\vec{a}+(6x+3y+3-4x+6y+2)\vec{b}=\vec{0}$$

$$\Rightarrow \quad (7x+10y-4)\vec{a}+(2x+9y+5)\vec{b}=\vec{0}$$

$$\Rightarrow \quad 7x+10y-4=0 \text{ and } 2x+9y+5=0 \quad [\because \vec{b} \text{ and } \vec{b} \text{ are non-collinear}]$$

$$\Rightarrow \quad x=2,\ y=-1.$$

### 22.13.2 COLLINEARITY OF POINTS

Let $A, B, C$ be three collinear points. Then, each pair of the vectors $\vec{AB}, \vec{BC}$ ; $\vec{AB}, \vec{AC}$ and $\vec{BC}, \vec{AC}$ is a pair of collinear vectors. Thus, to check the collinearity of three points, we can check the collinearity of any two vectors obtained with the help of three points.

Following theorem provides the general criterion for the collinearity of three points.

**THEOREM** *Three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear if and only if there exist three scalars $x, y, z$ not all zero simultaneously such that $x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}$, together with $x+y+z=0$.*

PROOF First, let three points $A, B, C$ with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively be collinear. Then, vectors $\vec{AB}$ and $\vec{BC}$ are collinear. Therefore, there exist scalar $\lambda$ such that

$$\vec{AB}=\lambda\vec{BC}$$

$$\Rightarrow \quad \vec{b}-\vec{a}=\lambda(\vec{c}-\vec{a})$$

$$\Rightarrow \quad (\lambda-1)\vec{a}+1\cdot\vec{b}+(-\lambda)\vec{c}=\vec{0}$$

$$\Rightarrow \quad x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}, \text{ where } x=\lambda-1,\ y=1 \text{ and } z=-\lambda$$

$$\Rightarrow \quad x\vec{a}+y\vec{b}+z\vec{c}=\vec{0}, \text{ where } x+y+z=\lambda-1+1+(-\lambda)=0$$

Conversely, let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors of points $A$, $B$ and $C$ respectively such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$, where $x + y + z = 0$.

Now, $\quad x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$

$\Rightarrow \quad x\vec{a} + y\vec{b} = -z\vec{c}$

$\Rightarrow \quad x\vec{a} + y\vec{b} = (x + y)\vec{c} \qquad [\because x + y + z = 0 \Rightarrow -z = x + y]$

$\Rightarrow \quad \vec{c} = \dfrac{x\vec{a} + y\vec{b}}{x + y} \Rightarrow$ Point $C$ divides $AB$ in the ratio $y : x$. $\Rightarrow$ $A$, $B$, $C$ are collinear points.

Q.E.D.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that the three points $A(-2, 3, 5)$, $B(1, 2, 3)$ and $C(7, 0, -1)$ are collinear.*

SOLUTION We have, $\vec{AB}$ = Position vector of $B$ – Position vector of $A$

$\Rightarrow \quad \vec{AB} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = 3\hat{i} - \hat{j} - 2\hat{k}$

and, $\quad \vec{BC}$ = Position vector of $C$ – Position vector of $B$

$\Rightarrow \quad \vec{BC} = (7\hat{i} + 0\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 6\hat{i} - 2\hat{j} - 4\hat{k} = 2(3\hat{i} - \hat{j} - 2\hat{k})$

Clearly, $\vec{BC} = 2\vec{AB}$. This shows that the vectors $\vec{AB}$ and $\vec{BC}$ are parallel. But, $B$ is a common point of $\vec{AB}$ and $\vec{BC}$. So, the given points $A$, $B$ and $C$ are collinear.

EXAMPLE 2 *The position vectors of the points P, Q, R are $\hat{i} + 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} - \hat{k}$ respectively. Prove that P, Q and R are collinear points.* [NCERT]

SOLUTION We have, $\vec{PQ}$ = Position vector of $Q$ – Position vector of $P$

$\Rightarrow \quad \vec{PQ} = (-2\hat{i} + 3\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = -3\hat{i} + \hat{j} + 2\hat{k}$

and, $\quad \vec{QR}$ = Position vector of $R$ – Position vector of $Q$

$\Rightarrow \quad \vec{QR} = (7\hat{i} - \hat{k}) - (-2\hat{i} + 3\hat{j} + 5\hat{k}) = 9\hat{i} - 3\hat{j} - 6\hat{k}$

Clearly, $\vec{QR} = -3\vec{PQ}$. This shows that the vectors $\vec{PQ}$ and $\vec{QR}$ are collinear. But, $Q$ is common point between $\vec{PQ}$ and $\vec{QR}$. Therefore, given points $P$, $Q$ and $R$ are collinear.

EXAMPLE 3 *If the position vectors of the points A, B, C, D are $2\hat{i} + 4\hat{k}$, $5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}$, $-2\sqrt{3}\hat{j} + \hat{k}$ and $2\hat{i} + \hat{k}$ respectively, prove that CD is parallel to AB and $CD = \dfrac{2}{3}AB$.*

SOLUTION We have, $\vec{AB}$ = Position vector of $B$ – Position vector of $A$

$\Rightarrow \quad \vec{AB} = (5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}) - (2\hat{i} + 4\hat{k}) = 3\hat{i} + 3\sqrt{3}\hat{j} + 0\hat{k} = 3(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k})$

and, $\vec{CD}$ = Position vector of $D$ − Position vector of $C$

$\Rightarrow \quad \vec{CD} = (2\hat{i} + \hat{k}) - (-2\sqrt{3}\hat{j} + \hat{k}) = 2\hat{i} + 2\sqrt{3}\hat{j} + 0\hat{k} = 2(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k})$

$\therefore \quad \vec{CD} = 2(\hat{i} + \sqrt{3}\hat{j} + 0\hat{k}) = \frac{2}{3}(3\hat{i} + 3\sqrt{3}\hat{j} + 0\hat{k}) = \frac{2}{3}\vec{AB}$

Hence, $CD$ is parallel to $AB$ and $CD = \frac{2}{3}AB$.

**EXAMPLE 4** *If the points $(-1, -1, 2)$, $(2, m, 5)$ and $(3, 11, 6)$ are collinear, find the value of $m$.*

[NCERT EXEMPLAR, CBSE 2022]

SOLUTION Let the given points be $P(-1, -1, 2)$, $Q(2, m, 5)$ and $R(3, 11, 6)$. Then,

$$\vec{PQ} = (2\hat{i} + m\hat{j} + 5\hat{k}) - (-\hat{i} - \hat{j} + 2\hat{k}) = 3\hat{i} + (m+1)\hat{j} + 3\hat{k}$$

and, $\vec{PR} = (3\hat{i} + 11\hat{j} + 6\hat{k}) - (-\hat{i} - \hat{j} + 2\hat{k}) = 4\hat{i} + 12\hat{j} + 4\hat{k}$

If points $P, Q, R$ are collinear, then

$\vec{PQ} = \lambda\vec{PR}$ for some scalar $\lambda$

$\Rightarrow \quad 3\hat{i} + (m+1)\hat{j} + 3\hat{k} = \lambda(4\hat{i} + 12\hat{j} + 4\hat{k})$

$\Rightarrow \quad 3 = 4\lambda$ and $m + 1 = 12\lambda \Rightarrow m + 1 = 9 \Rightarrow m = 8$ [On equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$]

**EXAMPLE 5** *If the points with position vectors $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$ and $a\hat{i} - 52\hat{j}$ are collinear, find the value of $a$.*

SOLUTION Let the points be $A$, $B$ and $C$ respectively. It is given that points $A, B, C$ are collinear.

$\therefore \quad \vec{AB}$ and $\vec{BC}$ are collinear

$\Rightarrow \quad \vec{AB} = \lambda\vec{BC}$ for some scalar $\lambda$

$\Rightarrow \quad (-20\hat{i} - 11\hat{j}) = \lambda\{(a - 40)\hat{i} - 44\hat{j}\}$

$\Rightarrow \quad \{\lambda(a - 40) + 20\}\hat{i} - (44\lambda - 11)\hat{j} = \vec{0}$

$\Rightarrow \quad \lambda(a - 40) + 20 = 0$ and, $44\lambda - 11 = 0$ [$\because \hat{i}, \hat{j}$ are non-collinear]

$\Rightarrow \quad \lambda = \frac{1}{4}$ and $\lambda(a - 40) + 20 = 0 \Rightarrow \frac{1}{4}(a - 40) + 20 = 0 \Rightarrow a = -40.$

Hence, the given points will be collinear, if $a = -40$.

**EXAMPLE 6** *Using, vectors, show that the points $A(-2, 1)$, $B(-5, -1)$ and $C(1, 3)$ are collinear.*

SOLUTION We have,

$\vec{AB}$ = (Position vector of $B$ − Position vector of $A$) $= (-5\hat{i} - \hat{j}) - (-2\hat{i} + \hat{j}) = -3\hat{i} - 2\hat{j}$

and, $\vec{BC}$ = (Position vector of $C$ − Position vector of $B$) $= (\hat{i} + 3\hat{j}) - (-5\hat{i} - \hat{j}) = 6\hat{i} + 4\hat{j}$

Clearly, $\vec{BC} = -2\vec{AB}$. Therefore, $\vec{AB}$ and $\vec{BC}$ are parallel vectors. But, $B$ is a common point of $\vec{a}\,\vec{AB}$ and $\vec{BC}$. Hence, the points $A, B, C$ are collinear.

### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 7** *Show that the points with position vectors* $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-\vec{c}$ *and* $4\vec{a}-7\vec{b}+7\vec{c}$ *are collinear.*

SOLUTION Let $P$, $Q$, $R$ be the points with position vectors $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-\vec{c}$ and $4\vec{a}-7\vec{b}+7\vec{c}$ respectively. Then,

$$\vec{PQ} = \text{P.V. of } Q - \text{P.V. of } P = (-2\vec{a}+3\vec{b}-\vec{c})-(\vec{a}-2\vec{b}+3\vec{c}) = -3\vec{a}+5\vec{b}-4\vec{c}$$

and, $$\vec{QR} = \text{P.V. of R} - \text{P.V. of } Q = (4\vec{a}-7\vec{b}+7\vec{c})-(-2\vec{a}+3\vec{b}-\vec{c}) = 6\vec{a}-10\vec{b}+8\vec{c}$$

Clearly, $\vec{QR} = -2\,\vec{PQ}$. This shows that $\vec{PQ}$ and $\vec{QR}$ are parallel vectors. But, $Q$ is a point common to them. So, $\vec{PQ}$ and $\vec{QR}$ are collinear. Hence, points $P$, $Q$ and $R$ are collinear.

**EXAMPLE 8** *Show that the points with position vectors* $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}+2\vec{c}$ *and* $-8\vec{a}+13\vec{b}$ *are collinear whatever be* $\vec{a}, \vec{b}, \vec{c}$.

SOLUTION Let $P$, $Q$, $R$ be the points with position vectors $\vec{a}-2\vec{b}+3\vec{c}$ $-2\vec{a}+3\vec{b}+2\vec{c}$ and $-8\vec{a}+13\vec{b}$ respectively. Then,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (-2\vec{a}+3\vec{b}+2\vec{c})-(\vec{a}-2\vec{b}+3\vec{c}) = -3\vec{a}+5\vec{b}-\vec{c} \quad \text{...(i)}$$

and, $$\vec{QR} = \text{Position vector of } R - \text{Position vector of } Q$$

$$\Rightarrow \quad \vec{QR} = (-8\vec{a}+13\vec{b})-(-2\vec{a}+3\vec{b}+2\vec{c}) = 6\vec{a}+10\vec{b}-2\vec{c} = 2(-3\vec{a}+5\vec{b}-\vec{c}) \quad \text{...(ii)}$$

From (i) and (ii), we get $\vec{QR} = 2\,\vec{PQ}$. This shows that $\vec{PQ}$ and $\vec{QR}$ are parallel vectors. But, $Q$ is a point common to them.

So, $\vec{PQ}$ and $\vec{QR}$ are collinear. Hence, $P$, $Q$ and $R$ are collinear points.

**EXAMPLE 9** *Show that the points A, B, C with position vectors* $2\vec{a}+3\vec{b}+5\vec{c}$, $\vec{a}+2\vec{b}+3\vec{c}$ *and* $7\vec{a}-\vec{c}$ *respectively, are collinear.*

SOLUTION We have,

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A$$

$$\Rightarrow \quad \vec{AB} = (\vec{a}+2\vec{b}+3\vec{c})-(-2\vec{a}+3\vec{b}+5\vec{c}) = 3\vec{a}-\vec{b}-2\vec{c} \quad \text{...(i)}$$

and, $$\vec{BC} = \text{Position vector of } C - \text{Position vector of } B$$

$$\Rightarrow \quad \vec{BC} = (7\vec{a}-\vec{c})-(\vec{a}+2\vec{b}+3\vec{c}) = 6\vec{a}-2\vec{b}-4\vec{c} = 2(3\vec{a}-\vec{b}-2\vec{c}) \quad \text{...(ii)}$$

From (i) and (ii), we obtain $2\,\vec{AB} = \vec{BC}$. Therefore, $\vec{AB}$ and $\vec{BC}$ are parallel vectors. But, $B$ is a point common to $\vec{AB}$ and $\vec{BC}$. Therefore, $\vec{AB}$ and $\vec{BC}$ are collinear vectors. Hence, points $A$, $B$ and $C$ are collinear

**BASED ON HIGHER ORDER THINKING SKILLS (HOTS)**

**EXAMPLE 10** *Show that the points $A(6, -7, 0)$, $B(16, -19, -4)$, $C(0, 3, -6)$ and $D(2, -5, 10)$ are such that AB and CD intersect at the point $P(1, -1, 2)$.*

SOLUTION We have,

$\vec{AP}$ = Position vector of $P$ – Position vector of $A = (\hat{i} - \hat{j} + 2\hat{k}) - (6\hat{i} - 7\hat{j} + 0\hat{k}) = -5\hat{i} + 6\hat{j} + 2\hat{k}$

and, $\vec{PB}$ = Position vector of $B$ – Position vector of $P$

$\Rightarrow \quad \vec{PB} = (16\hat{i} - 19\hat{j} - 4\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k}) = 15\hat{i} - 18\hat{j} - 6\hat{k} = -3(-5\hat{i} + 6\hat{j} + 2\hat{k})$

Clearly, $\vec{PB} = -3\vec{AP}$. So, vectors $\vec{AP}$ and $\vec{PB}$ are collinear. But, $P$ is a point common to $\vec{AP}$ and $\vec{PB}$. Hence, $P, A, B$ are collinear points.

Now, $\vec{CP} = (\hat{i} - \hat{j} + 2\hat{k}) - (0\hat{i} + 3\hat{j} - 6\hat{k}) = \hat{i} - 4\hat{j} + 8\hat{k}$

and, $\vec{PD} = (2\hat{i} - 5\hat{j} + 10\hat{k}) - (\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 4\hat{j} + 8\hat{k}$

Clearly, $\vec{CP} = \vec{PD}$. So, vectors $\vec{CP}$ and $\vec{PD}$ are collinear. But, $P$ is a common point to $\vec{CP}$ and $\vec{CD}$. Hence, $C, P, D$ are collinear points. Thus, $A, B, C, D$ and $P$ are points such that $A, P, B$ and $C, P, D$ are two sets of collinear points. Hence, $AB$ and $CD$ intersect at the point $P$.

**EXAMPLE 11** *If $\vec{a}, \vec{b}$ are two non-collinear vectors, show that the points having position vectors $l_1\vec{a} + m_1\vec{b}$, $l_2\vec{a} + m_2\vec{b}$ and $l_3\vec{a} + m_3\vec{b}$ are collinear, if* $\begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0.$

SOLUTION If given points are collinear, then there exist scalars $x, y, z$ such that

$$x(l_1\vec{a} + m_1\vec{b}) + y(l_2\vec{a} + m_2\vec{b}) + z(l_3\vec{a} + m_3\vec{b}) = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow \quad (l_1x + l_2y + l_3z)\vec{a} + (m_1x + m_2y + m_3z)\vec{b} = \vec{0}, \text{ where } x + y + z = 0$$

$$\Rightarrow \quad l_1x + l_2y + l_3z = 0, m_1x + m_2y + m_3z = 0, \quad [\because \vec{a}, \vec{b} \text{ are non-collinear vectors}]$$

where $x + y + z = 0$

Thus, we have

$$x + y + z = 0 \qquad \text{...(i)}$$

$$l_1x + l_2y + l_3z = 0 \qquad \text{...(ii)}$$

$$m_1x + m_2y + m_3z = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{x}{l_2m_3 - l_3m_2} = \frac{y}{l_3m_1 - l_1m_3} = \frac{z}{l_1m_2 - l_2m_1} = \lambda \text{ (say)}$$

$$\Rightarrow \quad x = \lambda(l_2m_3 - l_3m_2), y = \lambda(l_3m_1 - l_1m_3)\ z = \lambda(l_1m_2 - l_2m_1)$$

Substituting the values of $x, y, z$ in (i), we get

$$(l_2m_3 - l_3m_2) + (l_3m_1 - l_1m_3) + (l_1m_2 - l_2m_1) = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{vmatrix} = 0$$

## EXERCISE 22.7

### BASIC

1. Show that the points (3, 4), (− 5, 16), (5, 1) are collinear.
2. If the vectors $\vec{a} = 2\hat{i} - 3\hat{j}$ and $\vec{b} = -6\hat{i} + m\hat{j}$ are collinear, find the value of $m$.
3. If the points $A\ (m, -1)$, $B\ (2, 1)$ and $C\ (4, 5)$ are collinear, find the value of $m$.
4. Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear.
5. Prove that the points having position vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $3\hat{i} + 4\hat{j} + 7\hat{k}$, $-3\hat{i} - 2\hat{j} - 5\hat{k}$ ar collinear.
6. If the points with position vectors $10\hat{i} + 3\hat{j}$, $12\hat{i} - 5\hat{j}$ and $a\hat{i} + 11\hat{j}$ are collinear, find the value of $a$. **[CBSE 2017]**
7. Using vectors, find the value of $\lambda$ such that the points $(\lambda, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear. **[NCERT EXEMPLAR]**
8. If $\vec{AO} + \vec{OB} = \vec{BO} + \vec{OC}$, prove that $A, B, C$ are collinear points.
9. Show that the points $A, B, C$ with position vectors $\vec{a} - 2\vec{b} + 3\vec{c}$, $2\vec{a} + 3\vec{b} - 4\vec{c}$ and $-7\vec{b} + 10\vec{c}$ are collinear.

### BASED ON LOTS

10. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the points having the following position vectors are collinear:
    (i) $\vec{a}, \vec{b}, 3\vec{a} - 2\vec{b}$  (ii) $\vec{a} + \vec{b} + \vec{c}$, $4\vec{a} + 3\vec{b}$, $10\vec{a} + 7\vec{b} - 2\vec{c}$
11. Show that the points $A\ (1, -2, -8)$, $B\ (5, 0, -2)$ and $C\ (11, 3, 7)$ are collinear, and find the ratio in which $B$ divides $AC$. **[NCERT]**
12. If $\vec{a}, \vec{b}$ are two non-collinear vectors, prove that the points with position vectors $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$ and $\vec{a} + \lambda\vec{b}$ are collinear for all real values of $\lambda$.
13. Using vectors show that the points $A\ (-2, 3, 5)$, $B\ (7, 0, -1)$ $C\ (-3, -2, -5)$ and $D\ (3, 4, 7)$ are such that $AB$ and $CD$ intersect at the point $P\ (1, 2, 3)$. **[CBSE 2012]**

## ANSWER

**2.** 9  **3.** 1  **6.** 8  **7.** −2  **11.** 2 : 3

## HINTS TO SELECTED PROBLEMS

11. Given points are $A\ (1, -2, -8)$, $B\ (5, 0, -2)$ and $C\ (11, 3, 7)$.

$\therefore \quad \vec{AB} = 4\hat{i} + 2\hat{j} + 6\hat{k}$, $\vec{BC} = 6\hat{i} + 3\hat{j} + 9\hat{k}$ and $\vec{CA} = 10\hat{i} + 5\hat{j} + 15\hat{k}$

Clearly, $\vec{AB} + \vec{BC} = \vec{CA}$. Hence, points $A, B$ and $C$ are collinear. Suppose $B$ divides $AC$ in the ratio $\lambda : 1$. Then, the position vectors $B$ is

$$\left(\frac{11\lambda + 1}{\lambda + 1}\right)\hat{i} + \left(\frac{3\lambda - 2}{\lambda + 1}\right)\hat{j} + \left(\frac{7\lambda - 8}{\lambda + 1}\right)\hat{k}$$

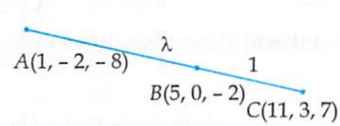

Fig. 22.63

But, the position vector of $B$ is $5\hat{i} + 0\hat{j} - 2\hat{k}$.

$$\therefore \quad \frac{11\lambda+1}{\lambda+1} = 5,\ \frac{3\lambda-2}{\lambda+1} = 0,\ \frac{7\lambda-8}{\lambda+1} = -2$$

$$\Rightarrow \quad 11\lambda+1 = 5\lambda+5,\ 3\lambda-2=0,\ 7\lambda-8=-2\lambda-2 \Rightarrow 6\lambda=4,\ 3\lambda=2,\ 9\lambda=6 \Rightarrow \lambda=\frac{2}{3}.$$

## 22.14 COPLANARITY

In this section, we shall discuss coplanarity of a system of vectors and also that of four or more points as three points are always coplanar.

*A system of vectors is said to be coplanar, if their supports are parallel to the same plane.*

We have already seen that any two vectors are always coplanar. But, three or more vectors may or may not be coplanar. The following theorem gives a test of coplanrity of three vectors.

**THEOREM 1** *(Test of coplanarity of three vectors) Let $\vec{a}$ and $\vec{b}$ be two given non-zero non-collinear vectors. Then, any vector $\vec{r}$ coplanar with $\vec{a}$ and $\vec{b}$ can be uniquely expressed as $\vec{r} = x\vec{a} + y\vec{b}$, for some scalars x and y.*

PROOF Let $\vec{a}$ and $\vec{b}$ be two non-collinear non-zero vectors and let $\vec{r}$ be a vector coplanar with $\vec{a}$ and $\vec{b}$. Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OP} = \vec{r}$. Complete the parallelogram $OLPM$ with $OP$ as diagonal. Since vectors $\vec{OL}$ and $\vec{OM}$ are collinear with $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ respectively.

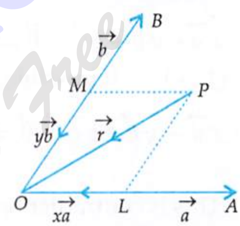

Fig. 22.64

$$\therefore \quad \vec{OL} = x\vec{a} \text{ and } \vec{OM} = y\vec{b} \text{ for some scalars } x, y.$$

Now, $\vec{OP} = \vec{OL} + \vec{LP}$ [By triangle law of addition of vectors]

$\Rightarrow \quad \vec{OP} = \vec{OL} + \vec{OM}$ $[\because \vec{LP} = \vec{OM}]$

$\Rightarrow \quad \vec{r} = x\vec{a} + y\vec{b}$

Thus, $\vec{r} = x\vec{a} + y\vec{b}$ for some scalars $x$ and $y$.

To prove the uniqueness of this representation, let $\vec{r} = x_1\vec{a} + y_1\vec{b}$ for some scalars $x_1$ and $y_1$. Then,

$$x\vec{a} + y\vec{b} = x_1\vec{a} + y_1\vec{b} \Rightarrow (x-x_1)\vec{a} + (y-y_1)\vec{b} = \vec{0}$$

$\Rightarrow \quad x - x_1 = 0$ and $y - y_1 = 0 \Rightarrow x = x_1$ and $y = y_1$ $[\because \vec{a}$ and $\vec{b}$ are non-collinear$]$

Hence, the representation is unique.

Q.E.D

The above theorem can also be re-stated as under:

*Three vectors are coplanar if one of them is expressible as a linear combination of the other two.*

Following theorem provides an alternative test for the coplanarity of three vectors.

**THEOREM 2** *Prove that a necessary and sufficient condition for three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ to be coplanar is that there exist scalars $l, m, n$ not all zero simultaneously such that $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$.*

PROOF *Necessary condition:* First let $\vec{a}, \vec{b}, \vec{c}$ be three coplanar vectors. Then, one of them is expressible as a linear combination of the other two. Let $\vec{c} = x\vec{a} + y\vec{b}$ for some scalars $x, y$. Then,

$$\vec{c} = x\vec{a} + y\vec{b} \text{ for some scalars } x, y \Rightarrow l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}, \text{ where } l = x,\ m = y \text{ and } n = -1.$$

Thus, if $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then there exist scalars $l, m, n$ such that $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$ where $l, m, n$ are not all zero simultaneously.

*Sufficient condition:* Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that there exist scalars $l, m, n$ not all zero simultaneously satisfying $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$. We have to prove that $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors.

Now, $\quad l\vec{a} + m\vec{b} + n\vec{c} = \vec{0} \Rightarrow n\vec{c} = -l\vec{a} - m\vec{b} \Rightarrow \vec{c} = \left(-\frac{l}{n}\right)\vec{a} + \left(-\frac{m}{n}\right)\vec{b}$

$\Rightarrow \quad \vec{c}$ is a linear combination of $\vec{a}$ and $\vec{b} \Rightarrow \vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b}$

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. **Q.E.D.**

**THEOREM 3** *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero non-coplanar vectors and $x, y, z$ are three scalars, then* $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0.$

PROOF If possible, let $x \neq 0$. Then,

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x\vec{a} = -y\vec{b} - z\vec{c} \Rightarrow \vec{a} = \left(-\frac{y}{x}\right)\vec{b} + \left(-\frac{z}{x}\right)\vec{c} \Rightarrow \vec{a}$ is coplanar with $\vec{b}$ and $\vec{c}$

This is contradiction to the fact that $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar. Therefore, $x = 0$. Similarly, we can prove that $y = z = 0$. Hence, $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$. **Q.E.D.**

**THEOREM 4** *(Test of coplanarity of four points) Four points with position vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff there exist scalars $x, y, z, u$ not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where* $x + y + z + u = 0.$

PROOF Let $A, B, C$, and $D$ be four points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively.

First, let $A, B, C, D$ be four coplanar points. Then $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar vectors. So, there exist scalars $\lambda$ and $\mu$ such that

$$\vec{AB} = \lambda\vec{AC} + \mu\vec{AD} \Rightarrow \vec{b} - \vec{a} = \lambda(\vec{c} - \vec{a}) + \mu(\vec{d} - \vec{a})$$

$\Rightarrow \quad (\lambda + \mu - 1)\vec{a} + \vec{b} + (-\lambda)\vec{c} + (-\mu)\vec{d} = \vec{0}$

$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where $x = \lambda + \mu - 1,\ y = 1,\ z = -\lambda$ and $u = -\mu$

$\Rightarrow \quad x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$, where $x + y + z + u = \lambda + \mu - 1 + 1 - \lambda - \mu = 0$

Conversely, let there be scalars $x, y, z, u$ not all zero such that $x\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$ ...(i)

where $\quad x + y + z + u = 0$ ...(ii)

Putting $x = -(y + z + u)$ from (ii) in (i), we get

$$-(y + z + u)\vec{a} + y\vec{b} + z\vec{c} + u\vec{d} = \vec{0}$$

$\Rightarrow \quad y(\vec{b} - \vec{a}) + z(\vec{c} - \vec{a}) + u(\vec{d} - \vec{a}) = \vec{0} \Rightarrow y\vec{AB} + z\vec{AC} + u\vec{AD} = \vec{0}$

Let $y \neq 0$. Then, $\vec{AB} = \left(-\frac{z}{y}\right)\vec{AC} + \left(-\frac{u}{y}\right)\vec{AD}$. This shows that $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ are coplanar vectors. Hence, points $A, B, C, D$ are coplanar points. **Q.E.D.**

## ILLUSTRATIVE EXAMPLES

### BASED ON LOWER ORDER THINKING SKILLS (LOTS)

**EXAMPLE 1** *Show that the vectors* $\vec{a}-2\vec{b}+3\vec{c}$, $\vec{a}-3\vec{b}+5\vec{c}$ *and* $-2\vec{a}+3\vec{b}-4\vec{c}$ *are coplanar, where* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar.*

SOLUTION Recall that three vectors are coplanar if one of the given vectors is expressible as a linear combination of the other two. Let

$$\vec{a}-2\vec{b}+3\vec{c}=x(\vec{a}-3\vec{b}+5\vec{c})+y(-2\vec{a}+3\vec{b}-4\vec{c}) \text{ for some scalars } x \text{ and } y.$$

$$\vec{a}-2\vec{b}+3\vec{c}=(x-2y)\vec{a}+(-3x+3y)\vec{b}+(5x-4y)\vec{c}$$

$$\Rightarrow \quad 1=x-2y,\ -2=-3x+3y \text{ and } 3=5x-4y$$

Solving first two of these equations, we get $x=1/3$, $y=-1/3$. Clearly, these values of $x$ and $y$ satisfy the third equation. Hence, the given vectors are coplanar.

**EXAMPLE 2** *Show that the vectors* $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ *and* $\vec{a}+\vec{b}-3\vec{c}$ *are non-coplanar vectors.*

SOLUTION Let, if possible, the given vectors be coplanar. Then one of the given vectors is expressible in terms of the other two.

Let $\quad 2\vec{a}-\vec{b}+3\vec{c}=x(\vec{a}+\vec{b}-2\vec{c})+y(\vec{a}+\vec{b}-3\vec{c})$ for some scalars $x$ and $y$.

$$\Rightarrow \quad 2\vec{a}-\vec{b}+3\vec{c}=(x+y)\vec{a}+(x+y)\vec{b}+(-2x-3y)\vec{c}$$

$$\Rightarrow \quad 2=x+y,\ -1=x+y \text{ and } 3=-2x-3y$$

Solving, first and third of these equations, we get $x=9$ and $y=-7$. Clearly, these values do not satisfy the third equation. Hence, the given vectors are not coplanar.

**EXAMPLE 3** *Prove that four points:* $2\vec{a}+3\vec{b}-\vec{c}$, $\vec{a}-2\vec{b}+3\vec{c}$, $3\vec{a}+4\vec{b}-2\vec{c}$ *and* $\vec{a}-6\vec{b}+6\vec{c}$ *are coplanar.*

SOLUTION Let the given four points be $P, Q, R$ and $S$ respectively. These points are coplanar if the vectors $\vec{PQ}$, $\vec{PR}$ and $\vec{PS}$ are coplanar. These vectors are coplanar iff one of them can be expressed as a linear combination of other two. So, let

$$\vec{PQ}=x\vec{PR}+y\vec{PS}$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=x(\vec{a}+\vec{b}-\vec{c})+y(-\vec{a}-9\vec{b}+7\vec{c})$$

$$\Rightarrow \quad -\vec{a}-5\vec{b}+4\vec{c}=(x-y)\vec{a}+(x-9y)\vec{b}+(-x+7y)\vec{c}$$

$$\Rightarrow \quad x-y=-1,\ x-9y=-5,\ -x+7y=4 \qquad \text{[Equating coeff. of } \vec{a}, \vec{b}, \vec{c} \text{ on both sides]}$$

Solving the first of these three equations, we get $x=-1/2$, $y=1/2$. These values also satisfy the third equation. Hence the given four points are coplanar.

**EXAMPLE 4** *If* $\vec{a}, \vec{b}, \vec{c}$ *are non-coplanar vectors such that* $x_1\vec{a}+y_1\vec{b}+z_1\vec{c}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$, *then prove that* $x_1=x_2$, $y_1=y_2$ *and* $z_1=z_2$.

SOLUTION We have,

$$x_1\vec{a}+y_1\vec{b}+z_1\vec{c}=x_2\vec{a}+y_2\vec{b}+z_2\vec{c}$$

$$\Rightarrow \quad (x_1-x_2)\vec{a}+(y_1-y_2)\vec{b}+(z_1-z_2)\vec{c}=\vec{0}$$

$$\Rightarrow \quad x_1-x_2=0,\ y_1-y_2=0,\ z_1-z_2=0 \Rightarrow x_1=x_2,\ y_1=y_2 \text{ and } z_1=z_2. \quad \text{[Using Theorem 3]}$$

## EXERCISE 22.8

### BASIC

1. Show that the points whose position vectors are as given below are collinear:

(i) $2\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$ and $\hat{i}+4\hat{j}-3\hat{k}$ (ii) $3\hat{i}-2\hat{j}+4\hat{k}$, $\hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+4\hat{j}-2\hat{k}$

2. Using vector method, prove that the following points are collinear:

(i) $A(6,-7,-1), B(2,-3,1)$ and $C(4,-5,0)$ (ii) $A(2,-1,3), B(4,3,1)$ and $C(3,1,2)$

(iii) $A(1,2,7), B(2,6,3)$ and $C(3,10,-1)$ (iv) $A(-3,-2,-5), B(1,2,3)$ and $C(3,4,7)$

### BASED ON LOTS

3. If $\vec{a}, \vec{b}, \vec{c}$ are non-zero, non-coplanar vectors, prove that the following vectors are coplanar:

(i) $5\vec{a}+6\vec{b}+7\vec{c}$, $7\vec{a}-8\vec{b}+9\vec{c}$ and $3\vec{a}+20\vec{b}+5\vec{c}$

(ii) $\vec{a}-2\vec{b}+3\vec{c}$, $\vec{a}-3\vec{b}+5\vec{c}$ and $-2\vec{a}+3\vec{b}-4\vec{c}$

4. Show that the four points having position vectors $6\hat{i}-7\hat{j}$, $16\hat{i}-19\hat{j}-4\hat{k}$, $3\hat{j}-6\hat{k}$, $2\hat{i}-5\hat{j}+10\hat{k}$ are coplanar.

5. Prove that the following vectors are coplanar:

(i) $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ (ii) $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+3\hat{j}-\hat{k}$ and $-\hat{i}-2\hat{j}+2\hat{k}$

6. Prove that the following vectors are non-coplanar:

(i) $3\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+7\hat{k}$ and $7\hat{i}-\hat{j}+23\hat{k}$ (ii) $\hat{i}+2\hat{j}+3\hat{k}$, $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

7. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, prove that the following vectors are non-coplanar:

(i) $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ and $\vec{a}+\vec{b}-3\vec{c}$

(ii) $\vec{a}+2\vec{b}+3\vec{c}$, $2\vec{a}+\vec{b}+3\vec{c}$ and $\vec{a}+\vec{b}+\vec{c}$

8. Show that the vectors $\vec{a}, \vec{b}, \vec{c}$ given by $\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$ are non-coplanar. Express vector $\vec{d}=2\hat{i}-\hat{j}-3\hat{k}$ as a linear combination of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

### BASED ON HOTS

9. Prove that a necessary and sufficient condition for three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar is that there exist scalars $l, m, n$ not all zero simultaneously such that $l\vec{a}+m\vec{b}+n\vec{c}=\vec{0}$.

10. Show that the four points $A, B, C$ and $D$ with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ respectively are coplanar if and only if $3\vec{a}-2\vec{b}+\vec{c}-2\vec{d}=\vec{0}$.

## ANSWERS

8. $\vec{d}=-\frac{8}{3}\vec{a}+\frac{1}{3}\vec{b}+4\vec{c}$

## 22.15 DIRECTION COSINES AND DIRECTION RATIOS

In this section, we shall discuss about the direction cosines and direction ratios of a vector.

### 22.15.1 DIRECTION COSINES

**DIRECTION COSINES** *If $\alpha, \beta, \gamma$ are the angles which a vector $\vec{OP}$ makes with the positive directions of the coordinate axes OX, OY, OZ respectively, then $\cos\alpha, \cos\beta, \cos\gamma$ are known as the direction cosines of $\vec{OP}$ and are generally denoted by the letters $l, m, n$ respectively.*

$\therefore \quad l = \cos\alpha, m = \cos\beta, n = \cos\gamma.$

The angles $\alpha, \beta, \gamma$ are known as the direction angles and satisfy the condition $0 \le \alpha, \beta, \gamma \le \pi$.

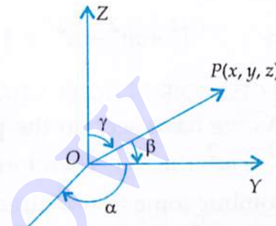

Fig. 22.65

In the above definition the vector $\vec{OP}$ has its initial point at the origin. If a given vector does not have its initial point at the origin, then we can draw a parallel vector of the same magnitude having initial point at the origin.

Clearly, $\vec{PO}$ makes angles $\pi - \alpha, \pi - \beta, \pi - \gamma$ with OX, OY, OZ respectively. Therefore, direction cosines of $\vec{PO}$ are $\cos(\pi - \alpha)$, $\cos(\pi - \beta)$, $\cos(\pi - \gamma)$ or, $-l, -m, -n$.

As the $x$-axis makes angles $0, \frac{\pi}{2}, \frac{\pi}{2}$ with OX, OY and OZ respectively. Therefore, direction cosines of $x$-axis are $\cos 0$, $\cos\frac{\pi}{2}$, $\cos\frac{\pi}{2}$ i.e., 1, 0, 0.

Similarly, the direction cosines of $Y$ and $Z$-axes are 0, 1, 0 and 0, 0, 1 respectively.

**THEOREM** *Let $P(x, y, z)$ be a point in space such that $\vec{r} = \vec{OP}$ has direction cosines $l, m, n$. Then,*

*(i) $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$ are projections of $\vec{r}$ on OX, OY, OZ respectively.*

*(ii) $x = l|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|$*

*(iii) $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$ and $\hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$* (iv) $l^2 + m^2 + n^2 = 1$

PROOF The position vector $\vec{r}$ of point $P(x, y, z)$ is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Suppose $\vec{OP}$ makes angles $\alpha, \beta, \gamma$ with OX, OY and OZ respectively. Then, $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$.

(i) Projection of $\vec{r}$ on $x$-axis $= \vec{r} \cdot \hat{i}$ [$\because$ Projection of $\vec{a}$ on $\vec{b} = \vec{a} \cdot \hat{b}$]

$\Rightarrow$ Projection of $\vec{r}$ on $x$-axis $= |\vec{r}||\hat{i}|\cos\alpha$ [By definition of dot product]

$\Rightarrow$ Projection of $\vec{r}$ on $x$-axis $= l|\vec{r}|$ [$\because l = \cos\alpha$]

Similarly, projections of $\vec{r}$ on OY and OZ axes are $m|\vec{r}|$ and $n|\vec{r}|$ respectively.

(ii) Projection of $\vec{r}$ on $x$ axis $= \vec{r} \cdot \hat{i} = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \hat{i} = x$

Similarly, projections of $\vec{r}$ on OY and OZ axes are $y$ and $z$ respectively.

But, projections of $\vec{r}$ on OX, OY, OZ are $l|\vec{r}|, m|\vec{r}|$ and $n|\vec{r}|$ respectively.

$\therefore \quad x = l|\vec{r}|, \; y = m|\vec{r}|, \; z = n|\vec{r}|$

(iii) Putting $x = l|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|$ in $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we obtain

$$\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k}) \Rightarrow \frac{\vec{r}}{|\vec{r}|} = l\hat{i} + m\hat{j} + n\hat{k} \Rightarrow \hat{r} = l\hat{i} + m\hat{j} + n\hat{k}$$

(iv) We have,

$$OP = |\vec{r}| \Rightarrow \sqrt{x^2 + y^2 + z^2} = |\vec{r}| \Rightarrow x^2 + y^2 + z^2 = |\vec{r}|^2$$

$$\Rightarrow \quad l^2|\vec{r}|^2 + m^2|\vec{r}|^2 + n^2|\vec{r}|^2 = |\vec{r}|^2 \qquad \left[\because x = l|\vec{r}|, y = m|\vec{r}|, z = n|\vec{r}|\right]$$

$$\Rightarrow \quad l^2 + m^2 + n^2 = 1$$

Q.E.D.

### 22.15.2 DIRECTION RATIOS

As we have seen in the previous section that if $l, m, n$ are direction cosines of a vector, then $l^2 + m^2 + n^2 = 1$. Therefore, $l, m, n$ usually involve fractions and radical signs. Also, it is slightly combur some to use direction cosines $l, m, n$ as such. So, we introduce the concept of direction ratios of a vector.

**DIRECTION RATIOS** *Let $l, m, n$ be direction cosines of a vector $\vec{r}$ and $a, b, c$, be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$. Then, we say that the direction ratios or direction numbers of vector $\vec{r}$ are proportional to $a, b, c$.*

For example, if $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ are direction cosines of a vector $\vec{r}$, then its direction ratios are proportional to $2, -2, 1$ or, $-2, 2, -1$ or, $4, -4, 2$, because

$$\frac{2/3}{2} = \frac{-2/3}{-2} = \frac{1/3}{1}, \frac{2/3}{-2} = \frac{-2/3}{2} = \frac{1/3}{-1}, \frac{2/3}{4} = \frac{-2/3}{-4} = \frac{1/3}{2}$$

It is evident from the above definition that to obtain direction ratios of a vector from its direction cosines we just multiply them by a common number. This also shows that there can be infinitely many sets of direction ratios for a given vector. But, the direction cosines are unique.
We shall now obtain the direction cosines from the direction ratios.

Let direction ratios of a vector $\vec{r}$ having direction cosines $l, m, n$ be proportional to $a, b, c$. Then,

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} \qquad \text{[By definition]}$$

Let $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$. Then, $l = a\lambda, m = b\lambda, n = c\lambda$.

$$\therefore \quad l^2 + m^2 + n^2 = 1 \Rightarrow a^2\lambda^2 + b^2\lambda^2 + c^2\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \; m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \; n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the signs should be taken all positive or all negative.
*Thus, if the direction ratios of a vector are proportional to $a, b, c$ then its direction cosines are given by*

$$\pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}, \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

*where the signs should be taken all positive or all negative.*

For example, if direction ratios of a vector $\vec{r}$ are proportional to $3, -4, 12$, then its direction cosines are

$$\frac{3}{\sqrt{(3)^2+(-4)^2+(12)^2}}\ \frac{-4}{\sqrt{(3)^2+(-4)^2+(12)^2}}\ \frac{12}{\sqrt{(3)^2+(-4)^2+(12)^2}} \text{ or } \frac{3}{13}, -\frac{4}{13}, \frac{12}{13}.$$

### 22.15.3 SOME IMPORTANT RESULTS ON DIRECTION RATIOS AND DIRECTION COSINES

In this section, we shall state and prove some important results on direction ratios and cosines of a vector as theorems.

**THEOREM 1** *If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$, then its direction cosines are*

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}} \text{ or, } \frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$$

*and the direction ratios of $\vec{r}$ are proportional to $a, b, c$.*

PROOF Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be a vector having direction cosines $l, m, n$.

If $\vec{r}$ makes angles $\alpha, \beta, \gamma$ with $OX$, $OY$ and $OZ$ respectively. Then,

$$\cos\alpha = \frac{\vec{r}\cdot\hat{i}}{|\vec{r}||\hat{i}|}, \cos\beta = \frac{\vec{r}\cdot\hat{j}}{|\vec{r}||\hat{j}|}, \cos\gamma = \frac{\vec{r}\cdot\hat{k}}{|\vec{r}||\hat{k}|} \qquad \left[\because \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$\Rightarrow \quad \cos\alpha = \frac{(a\hat{i}+b\hat{j}+c\hat{k})\cdot\hat{i}}{|\vec{r}|}, \cos\beta = \frac{(a\hat{i}+b\hat{j}+c\hat{k})\cdot\hat{j}}{|\vec{r}|}, \cos\gamma = \frac{(a\hat{i}+b\hat{j}+c\hat{k})\cdot\hat{k}}{|\vec{r}|}$$

$$\Rightarrow \quad l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|} \qquad [\because \cos\alpha = l, \cos\beta = m, \cos\gamma = n]$$

Thus, direction cosines of $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ are $\frac{a}{|\vec{r}|}, \frac{b}{|\vec{r}|}, \frac{c}{|\vec{r}|}$ and hence its direction ratios are proportional to $a, b, c$. **Q.E.D.**

The vector $\vec{r} = 2\hat{i} + \hat{j} - 2\hat{k}$ has direction ratios proportional to 2, 1, – 2 and its direction cosines are

$$\frac{2}{\sqrt{2^2+1^2+(-2)^2}}, \frac{1}{\sqrt{2^2+(1)^2+(-2)^2}}, \frac{-2}{\sqrt{2^2+1^2+(-2)^2}} \text{ or, } \frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$$

**THEOREM 2** *The direction ratios of the line segment joining points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$*

PROOF Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Then,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

So, direction ratios of $\vec{PQ}$ are proportional to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ and its direction cosines are

$$\frac{x_2 - x_1}{|\vec{PQ}|}, \frac{y_2 - y_1}{|\vec{PQ}|}, \frac{z_2 - z_1}{|\vec{PQ}|}.$$ **Q.E.D.**

**THEOREM 3** *Two parallel vectors have proportional direction ratios.*

PROOF Let $\vec{a}$ and $\vec{b}$ be two parallel vectors. Then, $\vec{b} = \lambda\vec{a}$ for some scalar $\lambda$.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $\vec{b} = \lambda\vec{a} \Rightarrow \vec{b} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$.

This shows that the direction ratios $\lambda \vec{a}$ are proportional to $\lambda a_1, \lambda a_2, \lambda a_3$ or, $a_1, a_2, a_3$ because

$$\lambda a_1 : \lambda a_2 : \lambda a_3 \Leftrightarrow a_1 : a_2 : a_3.$$

Thus, $\vec{a}$ and $\vec{b}$ have proportional direction ratios and hence equal direction cosines.

Q.E.D.

**THEOREM 4** *If a vector $\vec{r}$ has direction ratios proportional to a, b, c. Then,*

$$\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i} + b\hat{j} + c\hat{k})$$

PROOF Since direction ratios of $\vec{r}$ are proportional to $a, b, c$. Therefore, its direction cosines are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Now, $\quad \vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$

$$\Rightarrow \quad \vec{r} = |\vec{r}|\left\{\frac{a}{\sqrt{a^2+b^2+c^2}}\hat{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\hat{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\hat{k}\right\}$$

$$\Rightarrow \quad \vec{r} = \frac{|\vec{r}|}{\sqrt{a^2+b^2+c^2}}(a\hat{i} + b\hat{j} + c\hat{k})$$

Q.E.D.

**THEOREM 5** *If l, m, n are the direction cosines of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then its projections on the coordinate axes are respectively $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.*

PROOF Let $l, m, n$ be the direction cosines of a vectors $\vec{r}$. If $\vec{r}$ makes angels $\alpha, \beta$ and $\gamma$ with $OX$, $OY$ and $OZ$ respectively. Then, $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.
Now,

$$\text{Projection of } \vec{r} \text{ on } x\text{-axis} = \vec{r}\cdot\hat{i} = |\vec{r}||\hat{i}|\cos\alpha = l|\vec{r}|$$

Similarly, we obtain

$$\text{Projection of } \vec{r} \text{ on } y\text{-axis} = m|\vec{r}| \text{ and, Projection of } \vec{r} \text{ on } z\text{-axis} = n|\vec{r}|$$

Thus, projections of $\vec{r}$ on the coordinate axes are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.

Q.E.D.

## ILLUSTRATIVE EXAMPLES

### BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *A vector $\vec{OP}$ is inclined to OX at 45° and OY at 60°. Find the angle at which $\vec{OP}$ is inclined to OZ.*

SOLUTION Suppose $\vec{OP}$ is inclined at angle $\gamma$ to $OZ$. Let $l, m, n$ be the direction cosines of $\vec{OP}$. Then,

$$l = \cos 45°, \; m = \cos 60°, \; n = \cos\gamma \Rightarrow l = \frac{1}{\sqrt{2}}, \; m = \frac{1}{2}, \; n = \cos\gamma.$$

Now,

$$l^2+m^2+n^2 = 1$$

$$\Rightarrow \quad \frac{1}{2}+\frac{1}{4}+n^2 = 1 \Rightarrow n^2 = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{2} \Rightarrow \gamma = 60^\circ \text{ or, } 120^\circ$$

Hence, $\vec{OP}$ is inclined at angle of 60° or, 120° to $OZ$.

**EXAMPLE 2** *If a vector makes angles $\alpha$, $\beta$, $\gamma$ with OX, OY and OZ respectively, prove that*

$$\sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2.$$

SOLUTION Let $l, m, n$ be the direction cosines of the given vector. Then,

$$l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma.$$

Now, $l^2+m^2+n^2 = 1$

$$\Rightarrow \quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$

$$\Rightarrow \quad (1-\sin^2\alpha)+(1-\sin^2\beta)+(1-\sin^2\gamma) = 1 \Rightarrow \sin^2\alpha+\sin^2\beta+\sin^2\gamma = 2.$$

**EXAMPLE 3** *Find the direction cosines of a vector $\vec{r}$ which is equally inclined with OX, OY and OZ. If $|\vec{r}|$ is given, find the total number of such vectors.* **[NCERT]**

SOLUTION Let $l, m, n$ be the direction cosines of $\vec{r}$. Since $\vec{r}$ is equally inclined with $OX$, $OY$ and $OZ$.

$$\therefore \quad l = m = n \qquad [\because\ \alpha=\beta=\gamma\ \therefore\ \cos\alpha=\cos\beta=\cos\gamma]$$

Now, $l^2+m^2+n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

Hence, direction cosines of $\vec{r}$ are $\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}$.

$$\therefore \quad \vec{r} = |\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \vec{r} = |\vec{r}|\left\{\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}\right\}.$$

Since + and − signs can be arranged at three places in $2\times2\times2=8$ ways. Therefore, there are eight vectors of given magnitude which are equally inclined with the coordinate axes.

**EXAMPLE 4** *A vector $\vec{r}$ is inclined at equal angles to OX, OY and OZ. If the magnitude of $\vec{r}$ is 6 units, find $\vec{r}$.* **[CBSE 2020]**

SOLUTION Suppose $\vec{r}$ makes an angle $\alpha$ with each of the axes $OX, OY$ and $OZ$. Then, its direction cosines are: $l = \cos\alpha, m = \cos\alpha, n = \cos\alpha \Rightarrow l = m = n.$

Now, $l^2+m^2+n^2 = 1 \Rightarrow 3l^2 = 1 \Rightarrow l = \pm\frac{1}{\sqrt{3}}$

$$\therefore \quad \vec{r} = |\vec{r}|\left(\hat{i}\,l+\hat{j}\,m+\hat{k}\,n\right) \Rightarrow \vec{r} = 6\left(\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}\right) = 2\sqrt{3}\,(\pm\hat{i}\pm\hat{j}\pm\hat{k}).$$

**BASED ON LOWER ORDER THINKING SKILLS (LOTS)**

**EXAMPLE 5** *A vector $\vec{r}$ has length 21 and its direction ratios are proportional to 2, −3, 6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.* **[NCERT EXEMPLAR]**

SOLUTION Recall that if the direction ratios of a vector are proportional to $a, b, c$, then its direction cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Therefore, direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2+(-3)^2+6^2}},\ \pm\frac{-3}{\sqrt{2^2+(-3)^2+6^2}},\ \pm\frac{6}{\sqrt{2^2+(-3)^2+6^2}}$$

Since $\vec{r}$ makes an acute angle with $x$-axis. Therefore, $\cos\alpha > 0$ i.e., $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$.

$$\therefore \qquad \vec{r} = |\vec{r}|(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \vec{r} = 21\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) = 6\hat{i}-9\hat{j}+18\hat{k}.$$

So, components of $\vec{r}$ along $OX$, $OY$ and $OZ$ are $6\hat{i}$, $-9\hat{j}$ and $18\hat{k}$ respectively.

**EXAMPLE 6** *Find the angles at which the vector $2\hat{i}-\hat{j}+2\hat{k}$ is inclined to each of the coordinate axes.*

SOLUTION Let $\vec{r}$ be the given vector, and let it make angles $\alpha, \beta, \gamma$ with $OX$, $OY$ and $OZ$ respectively. Then, its direction cosine are $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

So, direction ratios of $\vec{r} = 2\hat{i}-\hat{j}+2\hat{k}$ are proportional to 2, −1, 2. Therefore, direction cosines of $\vec{r}$ are

$$\frac{2}{\sqrt{2^2+(-1)^2+2^2}},\ \frac{-1}{\sqrt{2^2+(-1)^2+2^2}},\ \frac{2}{\sqrt{2^2+(-1)^2+2^2}} \quad \text{i.e. } \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}.$$

$$\therefore \qquad \cos\alpha = \frac{2}{3},\ \cos\beta = -\frac{1}{3},\ \cos\gamma = \frac{2}{3}$$

$$\Rightarrow \qquad \alpha = \cos^{-1}\left(\frac{2}{3}\right),\ \beta = \cos^{-1}\left(-\frac{1}{3}\right),\ \gamma = \cos^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow \qquad \alpha = \cos^{-1}\left(\frac{2}{3}\right),\ \beta = \pi - \cos^{-1}\left(\frac{1}{3}\right),\ \gamma = \cos^{-1}\left(\frac{2}{3}\right)$$

**EXAMPLE 7** *The projection of a vector on the coordinate axes are 6, −3, 2. Find its length and direction cosines.*

SOLUTION Let $l, m, n$ be the direction cosines of the given vector $\vec{r}$ (say). Then, its projections on the coordinate axes are $l|\vec{r}|, m|\vec{r}|, n|\vec{r}|$.

$$\therefore \qquad l|\vec{r}| = 6,\ m|\vec{r}| = -3,\ n|\vec{r}| = 2 \qquad \text{...(i)}$$

$$\Rightarrow \qquad \left\{l|\vec{r}|\right\}^2 + \left\{m|\vec{r}|\right\}^2 + \left\{|(n|\vec{r}|\right\}^2 = 6^2+(-3)^2+(2)^2$$

$$\Rightarrow \qquad |\vec{r}|^2(l^2+m^2+n^2) = 36+9+4 \ \Rightarrow |\vec{r}|^2 = 49 \Rightarrow |\vec{r}| = 7 \qquad [\because\ l^2+m^2+n^2=1]$$

Putting $|\vec{r}| = 7$ in (i), we obtain that the direction cosines of $\vec{r}$ are $l = \frac{6}{7}, m = -\frac{3}{7}, n = \frac{2}{7}$.

**EXAMPLE 8** *Find the direction cosines of the vector joining the points A (1, 2, – 3) and B (–1, –2, 1), directed from A to B.*

SOLUTION Clearly, $\vec{AB}$ = Position vector of $B$ – Position vector of $A$

$$\Rightarrow \quad \vec{AB} = (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 4\hat{j} + 4\hat{k}$$

So, direction ratios of $\vec{AB}$ are proportional to – 2, – 4, 4. Consequently, direction cosines of $\vec{AB}$ are

$$\frac{-2}{\sqrt{(-2)^2 + (-4)^2 + 4^2}}, \frac{-4}{\sqrt{(-2)^2 + (-4)^2 + 4^2}}, \frac{4}{\sqrt{(-2)^2 + (-4)^2 + 4^2}} \text{ or, } -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$$

**EXAMPLE 9** *Find the position vector of a point A in space such that $\vec{OA}$ is inclined at 60° to OX and at 45° to OY and $|\vec{OA}| = 10$ units.* **[NCERT EXEMPLAR]**

SOLUTION Let $l, m, n$ be the direction cosines of $\vec{OA}$. It is given that $\vec{OA}$ is inclined at 60° to $OX$ and at 45° to $OY$.

$$\therefore \quad l = \cos 60° = \frac{1}{2} \text{ and } m = \cos 45° = \frac{1}{\sqrt{2}}$$

Now, $\quad l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{4} + \frac{1}{2} + n^2 = 1 \Rightarrow n^2 = \frac{1}{4} \Rightarrow n = \pm\frac{1}{2}$

Thus, we obtain: $l = \frac{1}{2}, m = \frac{1}{2}, n = \pm\frac{1}{2}$ and $|\vec{OA}| = 10$

$$\therefore \quad \vec{OA} = |\vec{OA}|(l\hat{i} + m\hat{j} + n\hat{k}) \Rightarrow \vec{OA} = 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} \pm \frac{1}{2}\hat{k}\right) = 5\hat{i} + 5\sqrt{2}\,\hat{j} \pm 5\hat{k}$$

## EXERCISE 22.9

### BASIC

1. Can a vector have direction angles 45°, 60°, 120°
2. Prove that 1, 1, 1 cannot be direction cosines of a straight line.
3. A vector makes an angle of $\frac{\pi}{4}$ with each of $x$-axis and $y$-axis. Find the angle made by it with the $z$-axis.
4. A vector $\vec{r}$ is inclined at equal acute angles to $x$-axis, $y$-axis and $z$-axis. If $|\vec{r}| = 6$ units, find $\vec{r}$.
5. A vector $\vec{r}$ is inclined to $x$-axis at 45° and $y$-axis at 60°. If $|\vec{r}| = 8$ units, find $\vec{r}$.
6. Find the direction cosines of the following vectors :

   (i) $2\hat{i} + 2\hat{j} - \hat{k}$ (ii) $6\hat{i} - 2\hat{j} - 3\hat{k}$ (iii) $3\hat{i} - 4\hat{k}$
7. Find the angles at which the following vectors are inclined to each of the coordinate axes :

   (i) $\hat{i} - \hat{j} + \hat{k}$ (ii) $\hat{j} - \hat{k}$ (iii) $4\hat{i} + 8\hat{j} + \hat{k}$
8. Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined with the axes $OX$, $OY$ and $OZ$. **[NCERT]**
9. Show that the direction cosines of a vector equally inclined to the axes $OX$, $OY$ and $OZ$ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$. **[NCERT]**

### BASED ON LOTS

**10.** If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an accute angle $\theta$ with $\hat{k}$, then find $\theta$ and hence, the components of $\vec{a}$. **[NCERT]**

**11.** Find a vector $\vec{r}$ of magnitude $3\sqrt{2}$ units which makes an angle of $\frac{\pi}{4}$ and $\frac{\pi}{2}$ with $y$ and $z$-axes respectively. **[NCERT EXEMPLAR]**

**12.** A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is $2\sqrt{3}$, find $\vec{r}$. **[NCERT EXEMPLAR]**

## ANSWERS

**1.** Yes **2.** No **3.** $\frac{\pi}{2}$ **4.** $\vec{r} = 2\sqrt{3}\,(\hat{i}+\hat{j}+\hat{k})$

**5.** $\vec{r} = 4\,(\sqrt{2}\,\hat{i}+\hat{j}\pm\hat{k})$ **6.** (i) $\frac{2}{3}, \frac{2}{3} - \frac{1}{3}$ (ii) $\frac{6}{7}, \frac{-2}{7}, \frac{-3}{7}$ (iii) $\frac{3}{5}, 0, \frac{-4}{5}$

**7.** (i) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{-1}{\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (ii) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$

(iii) $\cos^{-1}\left(\frac{4}{9}\right)$, $\cos^{-1}\left(\frac{8}{9}\right)$, $\cos^{-1}\left(\frac{1}{9}\right)$ **10.** $\frac{\pi}{3}; \frac{1}{2}\hat{i}, \frac{1}{\sqrt{2}}\hat{j}, \frac{1}{2}\hat{k}$

**11.** $\vec{r} = \pm\, 3\hat{i} + 3\hat{j}$ **12.** $2\,(\pm\,\hat{i} \pm \hat{j} \pm \hat{k})$

## FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

**1.** If $\vec{a}, \vec{b}, \vec{c}$ represent the sides of a triangle taken in order, then $\vec{a}+\vec{b}+\vec{c}$ = ........................ .

**2.** In a parallelogram $ABCD$, if $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$, then $\vec{AC}$ = ................and $\vec{BD}$ =.............. .

**3.** If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of vertices of a triangle having its centroid at the origin, then $\vec{a}+\vec{b}+\vec{c}$ = ........................ .

**4.** If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of an equilateral triangle whose circumcentre is at the origin, then $\vec{a}+\vec{b}+\vec{c}$ = ................ .

**5.** If the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 6\hat{j} + m\hat{k}$ are collinear, then $m$ = ........................ .

**6.** Vectors of magnitude 21 units in the direction of the vector $3\hat{i} - 6\hat{j} + 2\hat{k}$ are ....................... .

**7.** If $D$ is the mid-point of side $BC$ of $\Delta ABC$ and $\vec{AB} + \vec{AC} = k\,\vec{AD}$, then $k$ = ...................... .

**8.** The cosines of the angles made by the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ with the coordinate axes are: ......... .

**9.** A vector of magnitude 6, making angle $\frac{\pi}{4}$ with $x$-axis, $\frac{\pi}{3}$ with $y$-axis and an acute angle with $z$-axis, is .............................. .

**10.** If the vectors $\vec{a} = 2\hat{i} - (y+z)\hat{j} + 5\hat{k}$ and $\vec{b} = (x+y)\,\hat{i} + 3\hat{j} + (z+x)\,\hat{k}$ are equal, then $x+y+z$ = ............... .

**11.** If $\vec{a}$ and $\vec{b}$ are the position vectors of $A$ and $B$ respectively, then the position vector of a point C on $AB$ produced such that $\vec{AC} = 3\,\vec{AB}$ is ............................. .

12. In a regular hexagon $ABCDEF$, $\vec{AC}+\vec{AF}-\vec{AB}=$ ............................ .

13. If $O, A, B, C$ and $D$ are five points, such that $3\vec{OD}+\vec{DA}+\vec{DB}+\vec{DC}=k(\vec{OA}+\vec{OB}+\vec{OC})$, then $k=$ ....................... .

14. $A, B, C, D, E$ are five coplanar points such that $\vec{DA}+\vec{DB}+\vec{DC}+\vec{AE}+\vec{BE}+\vec{CE}=k\,\vec{DE}$, then $k=$ .................... .

15. The vectors $\vec{a}$ and $\vec{b}$ are non-collinear. If vectors $(x-2)\,\vec{a}+\vec{b}$ and $(2x+1)\,\vec{a}-\vec{b}$ are collinear, then $x=$ ............................ .

16. If $A, B, C, D, E$ are five points in a plane such that $\vec{AB}+\vec{AE}+\vec{BC}+\vec{DC}+\vec{ED}=k\,\vec{AC}$, then the value of $k$ is ........................ .

17. Let $P$ be the point of intersection of the diagonal of a parallelogram $ABCD$ and $O$ is any point. If $\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=\lambda\,\vec{OP}$, then $\lambda=$ .................... .

18. $A, B, C, D, E$ are five coplanar points such that $\vec{DA}+\vec{DB}+\vec{DC}+\vec{AE}+\vec{BE}+\vec{CE}=\lambda\,\vec{DE}$, then $\lambda=$ ........................... .

19. If $D, E, F$ are mid-points of the sides $BC, CA$ and $AB$ respectively of $\Delta ABC$, then $\vec{AD}+\vec{BE}+\vec{CF}=$ ............. .

20. The algebraic sum of the vectors directed from the vertices to the mid-points of the opposite side is equal to ........................... .

21. The values of $k$ for which $|k\,\vec{a}|<|\vec{a}|$ and $k\,\vec{a}+\frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true, are ............ .

**[NCERT EXEMPLAR]**

22. The vector $\vec{a}+\vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$, if ............ .

**[NCERT EXEMPLAR]**

23. The position vectors of two points $A$ and $B$ are $\vec{OA}=2\hat{i}-\hat{j}-\hat{k}$ and $\vec{OB}=2\hat{i}-\hat{j}+2\hat{k}$ respectively. The position vector of a point $P$ which divides the line segment joining $A$ and $B$ in the ratio $2:1$ is .................. **[CBSE 2020]**

## ANSWERS

1. $\vec{0}$ 2. $\vec{a}+\vec{b}$, $\vec{b}-\vec{a}$ 3. $\vec{0}$ 4. $\vec{0}$ 5. 9
6. $\pm(9\hat{i}-18\hat{j}+6\hat{k})$ 7. 2 8. $\frac{1}{3}, \frac{-2}{3}, \frac{2}{3}$ 9. $3\sqrt{2}\,\hat{i}+3\hat{j}+3\hat{k}$
10. 2 11. $3\vec{b}-2\vec{a}$ 12. $\vec{AF}$ 13. 1 14. 4
15. 1/3 16. 2 17. 4 18. 3 19. $\vec{0}$
20. $\vec{0}$ 21. $k\in\left(-1,-\frac{1}{2}\right)\cup\left(-\frac{1}{2},1\right)$ 22. $|\vec{a}|=|\vec{b}|$ 23. $2\hat{i}-\hat{j}+\hat{k}$

## VERY SHORT ANSWER QUESTIONS (VSAQs)

*Answer each of the following questions in one word or one sentence or as per exact requirement of the question:*

1. Define "zero vector".
2. Define unit vector.
3. Define position vector of a point.

4. Write $\overrightarrow{PQ} + \overrightarrow{RP} + \overrightarrow{QR}$ in the simplified form.

5. If $\vec{a}$ and $\vec{b}$ are two non-collinear vectors such that $x\,\vec{a} + y\,\vec{b} = \vec{0}$, then write the values of $x$ and $y$.

6. If $\vec{a}$ and $\vec{b}$ represent two adjacent sides of a parallelogram, then write vectors representing its diagonals.

7. If $\vec{a}, \vec{b}, \vec{c}$ represent the sides of a triangle taken in order, then write the value of $\vec{a} + \vec{b} + \vec{c}$. **[CBSE 2011]**

8. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the vertices $A$, $B$ and $C$ respectively, of a triangle $ABC$, write the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

9. If $\vec{a}, \vec{b}, \vec{c}$ are position vectors of the points $A$, $B$ and $C$ respectively, write the value of $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{AC}$.

10. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of a triangle, then write the position vector of its centroid.

11. If $G$ denotes the centroid of $\Delta\, ABC$, then write the value of $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC}$.

12. If $\vec{a}$ and $\vec{b}$ denote the position vectors of points $A$ and $B$ respectively and $C$ is a point on $AB$ such that $3AC = 2AB$, then write the position vector of $C$.

13. If $D$ is the mid-point of side $BC$ of a triangle $ABC$ such that $\overrightarrow{AB} + \overrightarrow{AC} = \lambda\ \overrightarrow{AD}$, write the value of $\lambda$.

14. If $D, E, F$ are the mid-points of the sides $BC$, $CA$ and $AB$ respectively of a triangle $ABC$, write the value of $\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CF}$.

15. If $\vec{a}$ is a non-zero vector of modulus $a$ and $m$ is a non-zero scalar such that $m\,\vec{a}$ is a unit vector, write the value of $m$.

16. If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices of an equilateral triangle whose orthocentre is at the origin, then write the value of $\vec{a} + \vec{b} + \vec{c}$.

17. Write a unit vector making equal acute angles with the coordinates axes.

18. If a vector makes angles $\alpha, \beta, \gamma$ with $OX$, $OY$ and $OZ$ respectively, then write the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

19. Write a vector of magnitude 12 units which makes 45° angle with $X$-axis, 60° angle with $Y$-axis and an obtuse angle with $Z$-axis.

20. Write the length (magnitude) of a vector whose projections on the coordinate axes are 12, 3 and 4 units.

21. Write the position vector of a point dividing the line segment joining points $A$ and $B$ with position vectors $\vec{a}$ and $\vec{b}$ externally in the ratio $1 : 4$, where $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = -\hat{i} + \hat{j} + \hat{k}$.

22. Write the direction cosines of the vector $\vec{r} = 6\hat{i} - 2\hat{j} + 3\hat{k}$.

23. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$ and $\vec{c} = \hat{k} + \hat{i}$, write unit vectors parallel to $\vec{a} + \vec{b} - 2\vec{c}$.

24. If $\vec{a} = \hat{i} + 2\hat{j}$, $\vec{b} = \hat{j} + 2\hat{k}$, write a unit vector along the vector $3\vec{a} - 2\vec{b}$.

25. Write the position vector of a point dividing the line segment joining points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$ and $2\hat{i} - \hat{j} + 3\hat{k}$ externally in the ratio 2:3.

26. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, $\vec{c} = \hat{k} + \hat{i}$, find the unit vector in the direction of $\vec{a} + \vec{b} + \vec{c}$.

27. If $\vec{a} = 3\hat{i} - \hat{j} - 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 3\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, find $|3\vec{a} - 2\vec{b} + 4\vec{c}|$.

28. A unit vector $\vec{r}$ makes angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$ with $\hat{j}$ and $\hat{k}$ respectively and an acute angle $\theta$ with $\hat{i}$. Find $\theta$.

29. Write a unit vector in the direction of $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. **[CBSE 2008]**

30. If $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} + 9\hat{k}$, find a unit vector parallel to $\vec{a} + \vec{b}$. **[CBSE 2008, 2013]**

31. Write a unit vector in the direction of $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$. **[CBSE 2009]**

32. Find the position vector of the mid-point of the line segment $AB$, where $A$ is the point $(3, 4, -2)$ and $B$ is the point $(1, 2, 4)$. **[CBSE 2010]**

33. Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, which has magnitude of 6 units. **[CBSE 2010]**

34. What is the cosine of the angle which the vector $\sqrt{2}\hat{i} + \hat{j} + \hat{k}$ makes with $y$-axis? **[CBSE 2010]**

35. Write two different vectors having same magnitude.

36. Write two different vectors having same direction.

37. Write a vector in the direction of vector $5\hat{i} - \hat{j} + 2\hat{k}$ which has magnitude of 8 unit.

38. Write the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

39. Find a unit vector in the direction of $\vec{a} = 2\hat{i} - 3\hat{j} + 6\hat{k}$. **[CBSE 2011]**

40. For what value of '$a$' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear? **[CBSE 2011]**

41. Write the direction cosines of the vectors $-2\hat{i} + \hat{j} - 5\hat{k}$. **[CBSE 2011]**

42. Find the sum of the following vectors: $\vec{a} = \hat{i} - 2\hat{j}$, $\vec{b} = 2\hat{i} - 3\hat{j}$, $\vec{c} = 2\hat{i} + 3\hat{k}$. **[CBSE 2012]**

43. Find a unit vector in the direction of the vector $\vec{a} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. **[CBSE 2012]**

44. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$. **[CBSE 2013]**

45. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 7\hat{k}$. **[CBSE 2014]**

46. Find the value of '$p$' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. **[CBSE 2014]**

47. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with $x$-axis, $\frac{\pi}{2}$ with $y$-axis and an acute angle $\theta$ with $z$-axis. **[CBSE 2014]**

**48.** Write a unit vector in the direction of $\overrightarrow{PQ}$, where $P$ and $Q$ are the points $(1, 3, 0)$ and $(4, 5, 6)$ respectively. **[CBSE 2014]**

**49.** Find a vector in the direction of vector $2\hat{i} - 3\hat{j} + 6\hat{k}$ which has magnitude 21 units. **[CBSE 2014]**

**50.** If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then write the range of $|\lambda\hat{a}|$.

**51.** In a triangle $OAC$, if $B$ is the mid-point of side $AC$ and $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$, then what is $\overrightarrow{OC}$? **[CBSE 2015]**

**52.** Write the position vector of the point which divides the join of points with position vectors $3\vec{a} - 2\vec{b}$ and $2\vec{a} + 3\vec{b}$ in the ratio $2:1$. **[CBSE 2016]**

## ANSWERS

**5.** $x = 0, y = 0$
**6.** $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$
**7.** $\vec{0}$
**8.** $\vec{0}$
**9.** $2(\vec{c} - \vec{a})$
**10.** $\dfrac{\vec{a} + \vec{b} + \vec{c}}{3}$
**11.** $\vec{0}$
**12.** $\dfrac{\vec{a} + 2\vec{b}}{3}$
**13.** 2
**14.** 0
**15.** $\pm\dfrac{1}{a}$
**16.** $\vec{0}$
**17.** $\dfrac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$
**18.** 2
**19.** $6(\sqrt{2}\,\hat{i} + \hat{j} - \hat{k})$
**20.** 13
**21.** $3\hat{i} + \dfrac{11}{3}\hat{j} + 5\hat{k}$
**22.** $\dfrac{6}{7}, \dfrac{-2}{7}, \dfrac{3}{7}$
**23.** $\pm\dfrac{1}{\sqrt{6}}(\hat{i} - 2\hat{j} + \hat{k})$
**24.** $\dfrac{1}{\sqrt{41}}(3\hat{i} + 4\hat{j} - 4\hat{k})$
**25.** $-\hat{i} + 5\hat{j} - 12\hat{k}$
**26.** $\dfrac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$
**27.** $\sqrt{398}$
**28.** $60°$
**29.** $\hat{a} = \dfrac{3}{7}\hat{i} - \dfrac{2}{7}\hat{j} + \dfrac{6}{7}\hat{k}$
**30.** $\dfrac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$
**31.** $\dfrac{2}{3}\hat{i} + \dfrac{1}{3}\hat{j} + \dfrac{2}{3}\hat{k}$
**32.** $(2, 3, 1)$
**33.** $4\hat{i} - 2\hat{j} + 4\hat{k}$
**34.** $\dfrac{1}{2}$
**35.** $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$
**36.** $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = 4\hat{i} + 2\hat{j} + 4\hat{k}$
**37.** $\pm\dfrac{8}{\sqrt{30}}(5\hat{i} - \hat{j} + 2\hat{k})$
**38.** $\dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}$
**39.** $\dfrac{2}{7}\hat{i} - \dfrac{3}{7}\hat{j} + \dfrac{6}{7}\hat{k}$
**40.** $-4$
**41.** $-\dfrac{2}{\sqrt{30}}, \dfrac{1}{\sqrt{30}}, -\dfrac{5}{\sqrt{30}}$
**42.** $5\hat{i} - 5\hat{j} + 3\hat{k}$
**43.** $\dfrac{3}{7}, -\dfrac{2}{7}, \dfrac{6}{7}$
**44.** 0
**45.** $\dfrac{4}{13}\hat{i} + \dfrac{3}{13}\hat{j} - \dfrac{12}{13}\hat{k}$
**46.** $p = -\dfrac{1}{3}$
**47.** $5(\hat{i} + 0\hat{j} + \hat{k})$
**48.** $\dfrac{1}{7}(3\hat{i} + 2\hat{j} + 6\hat{k})$
**49.** $6\hat{i} - 9\hat{j} + 18\hat{k}$
**50.** $[-12, 8]$
**51.** $2\vec{b} - \vec{a}$
**52.** $\dfrac{7}{3}\vec{a} + \dfrac{4}{3}\vec{b}$