CHAPTER 29

LIMITS

29.1 INFORMAL APPROACH TO LIMIT

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Clearly, this function is defined for all x except at $x = 2$ as it assumes the form $\frac{0}{0}$ (known as an indeterminate form) at $x = 2$. However, if $x \neq 2$, then

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x + 2$$

The following table exhibits the values of $f(x)$ at points which are close to 2 on its two sides *viz.* left and right on the real line.

x	1.4	1.5	1.6	1.7	1.8	1.9	1.99	2	2.01	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	3.4	3.5	3.6	3.7	3.8	3.9	3.99	$\frac{0}{0}$	4.01	4.1	4.2	4.3	4.4	4.5	4.6

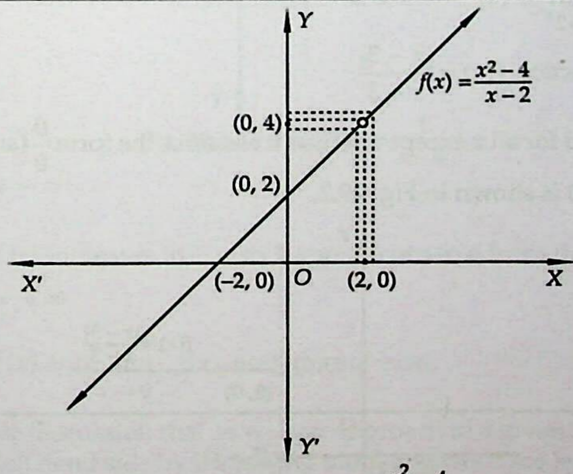

Fig. 29.1 Graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$

The graph of this function is shown in Fig. 29.1.

It is evident from the above table and the graph of $f(x)$ that as x increases and comes closer to 2 from left hand side of 2, the values of $f(x)$ increase and come closer to 4. This is interpreted as:

When x approaches to 2 from its left hand side, the function $f(x)$ tends to the limit 4.

It we use the notation '$x \to 2^-$' to denote 'x tends to 2 from left hand side', the above statement can be restated as:

as $x \to 2^-$, $f(x) \to 4$

or, $\lim_{x \to 2^-} f(x) = 4$

or, Left hand limit of $f(x)$ *at* $x = 2$ is 4.

Thus, $\lim_{x \to 2^-} f(x) = 4$ means that as x tends to 2 from left hand side, the values of $f(x)$ are tending to 4.

From the above table as well as the graph of $f(x)$, shown in Fig. 29.1, we observe that as x decreases and comes closer to 2 from right hand side, the values of $f(x)$ decrease and come closer to 4. This is interpreted as:

When x approaches to 2 from its right hand side, the function f(x) tends to the limit 4.

Using the notation '$x \to 2^+$' to denote 'x tends to 2 from right hand side', the above statement can be re-stated as:

as $x \to 2^+$, $f(x) \to 4$

or, $\lim_{x \to 2^+} f(x) = 4$

or, Right hand limit of $f(x)$ at $x = 2$ is 4.

Thus, $\lim_{x \to 2^+} f(x) = 4$ means that as x tends to 2 from right hand side, the values of $f(x)$ are tending to 4.

It follows from the above discussion that for the function $f(x)$ given by $f(x) = \dfrac{x^2 - 4}{x - 2}$:

(i) $\lim_{x \to 2^-} f(x) = 4$ (ii) $\lim_{x \to 2^+} f(x) = 4$

(iii) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ (iv) $f(2)$ does not exist i.e. $f(x)$ is not defined at $x = 2$.

Now, consider the function $f(x) = \dfrac{|x-3|}{x-3}$.

This function is defined for all x except $x = 3$, as it assumes the form $\dfrac{0}{0}$ (an indeterminate form) at $x = 3$. The graph of $f(x)$ is shown in Fig. 29.2.

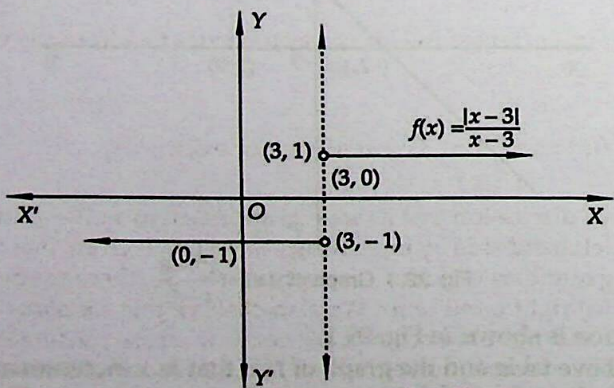

Fig. 29.2 Graph of $f(x) = \dfrac{|x-3|}{x-3}$

The following table shows the values of $f(x)$ at points which are close to 3 and are on its two sides.

x	2.3	2.4	2.5	2.6	2.7	2.8	2.9	2.99	3	3.01	3.1	3.2	3.3	3.4	3.5	3.6
$f(x)$	-1	-1	-1	-1	-1	-1	-1	-1	$\frac{0}{0}$	1	1	1	1	1	1	1

It is evident from the table and the graph of $f(x)$ that as $x \to 3$ from its left hand side the values of $f(x)$ are everywhere -1.

i.e. $\lim\limits_{x \to 3^-} f(x) = -1$ or, Left hand limit (LHL) of $f(x)$ at $x = 3$ is -1.

We also observe that at every point on the right hand side of 3, the function assumes value 1.

$\therefore \quad \lim\limits_{x \to 3^+} f(x) = 1$

Let us now consider the function $f(x) = \dfrac{1}{x-4}$, $x \neq 4$. Here also the function is undefined at $x = 4$ as $f(4)$ assumes the form $\dfrac{1}{0}$. In this case it is evident from the graph shown in Fig. 29.3 that as x approaches to 4 from the left hand side, $f(x)$ decreases to $-\infty$.

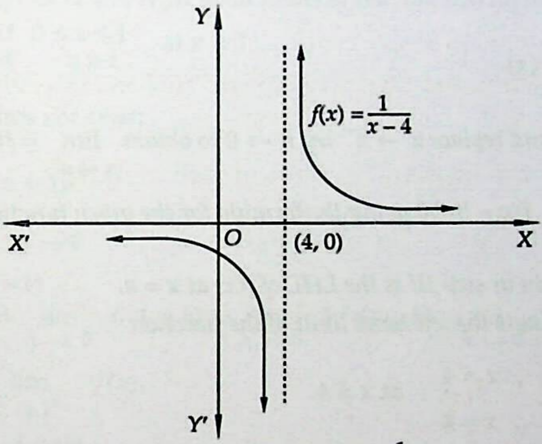

Fig. 29.3 Graph of $f(x) = \dfrac{1}{x-4}$

i.e. $\lim\limits_{x \to 4^-} f(x) = -\infty$

Also, we observe that $f(x)$ increases to $+\infty$ as x approaches to 4 from the right

i.e. $\lim\limits_{x \to 4^+} f(x) = +\infty$

So, we say that $\lim\limits_{x \to 4^-} f(x)$ and $\lim\limits_{x \to 4^+} f(x)$ both do not exist.

It follows from the above discussion that as we can approach to a given number 'a' (say) on the real line either from its left hand side by increasing numbers which are less than 'a' or from right hand side by decreasing numbers which are greater than 'a'. So, there are two types of limits viz. (i) left hand limit and, (ii) right hand limit. We also observe that for some functions at a given point 'a' (say) left hand and right hand limits are equal whereas for some functions these two limits are not equal and even sometimes either left hand limit or right hand limit or both do not exist.

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ i.e. (LHL at $x = a$) = (RHL at $x = a$), then we say that $\lim\limits_{x \to a} f(x)$ exists.

Otherwise, $\lim\limits_{x \to a} f(x)$ does not exist.

29.2 EVALUATION OF LEFT HAND AND RIGHT HAND LIMITS

In the previous sections, we have learnt that a real number l_1 is the left hand limit of function $f(x)$ at $x = a$ if the values of $f(x)$ can be made as close as desired to the number l_1 at points closed to a and on the left of a. In such a case, we write $\lim_{x \to a^-} f(x) = l_1$. Also, a real number l_2 is the right hand limit of $f(x)$ at $x = a$ i.e. $\lim_{x \to a^+} f(x) = l_2$, if the values of $f(x)$ can be made as close as desired to the number l_2 at points close to 'a' on the right of 'a'.

In this section, we shall discuss methods of evaluation of left hand and right hand limits of a function at a given point.

As discussed earlier that statement $x \to a^-$ means that x is tending to a from the left hand side i.e. x is a number less than a but very very close to a. Therefore, $x \to a^-$ is equivalent to $x = a - h$ where $h > 0$ such that $h \to 0$.

Similarly, $x \to a^+$ is equivalent to $x = a + h$ where $h \to 0$. Thus, we have the following algorithms for finding left hand and right hand limits at $x = a$.

ALGORITHM

STEP I *Write* $\lim_{x \to a^-} f(x)$

STEP II *Put* $x = a - h$ *and replace* $x \to a^-$ by $h \to 0$ *to obtain* $\lim_{x \to a^-} = f(x) = \lim_{h \to 0} f(a - h)$.

STEP III *Simplify* $\lim_{h \to 0} f(a - h)$ *by using the formula for the given function.*

STEP IV *The value obtain in step III is the LHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 1 *Evaluate the left hand limit of the function*

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4} & , x \neq 4 \\ 0 & , x = 4 \end{cases} \quad \text{at } x = 4.$$

SOLUTION We have,

(LHL of $f(x)$ at $x = 4$)

$$= \lim_{x \to 4^-} f(x) \qquad \text{(Step I)}$$

$$= \lim_{h \to 0} f(4 - h) \qquad \text{(Step II)}$$

$$= \lim_{h \to 0} \frac{|4-h-4|}{4-h-4} = \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1. \qquad \text{(Step III)}$$

To evaluate RHL of $f(x)$ at $x = a$ i.e. $\lim_{x \to a^+} f(x)$ we use the following algorithm.

ALGORITHM

STEP I *Write* $\lim_{x \to a^+} f(x)$

STEP II *Put* $x = a + h$ *and replace* $x \to a^+$ by $h \to 0$ *to obtain* $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h)$.

STEP III *Simplify* $\lim_{h \to 0} f(a + h)$ *by using the formula for the given function.*

STEP IV *The value obtained in step III is the RHL of* $f(x)$ *at* $x = a$.

ILLUSTRATION 2 *Evaluate the right hand limit of the function*

$$f(x)=\begin{cases}\dfrac{|x-4|}{x-4} & , x\neq 4\\ 0 & , x=4\end{cases} \quad \text{at } x=4.$$

SOLUTION We have,

(RHL of $f(x)$ at $x=4$)

$= \lim\limits_{x\to 4^+} f(x)$ (Step I)

$= \lim\limits_{h\to 0} f(4+h)$ (Step II)

$= \lim\limits_{h\to 0}\dfrac{|4+h-4|}{4+h-4} = \lim\limits_{h\to 0}\dfrac{|h|}{h} = \lim\limits_{h\to 0}\dfrac{h}{h} = \lim\limits_{h\to 0} 1 = 1$ (Step III)

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the left hand and right hand limits of the function defined by*

$$f(x) = \begin{cases}1+x^2, & \text{if } 0\le x\le 1\\ 2-x, & \text{if } x>1\end{cases} \quad \text{at } x=1.$$

Also, show that $\lim\limits_{x\to 1} f(x)$ *does not exist.*

SOLUTION (LHL of $f(x)$ at $x=1$)

$= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} 1+(1-h)^2 = \lim\limits_{h\to 0} 2-2h+h^2 = 2.$

and, (RHL of $f(x)$ at $x=1$)

$= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} 2-(1+h) = \lim\limits_{h\to 0} 1-h = 1.$

Clearly, $\lim\limits_{x\to 1^-} f(x) \neq \lim\limits_{x\to 1^+} f(x).$

Hence, $\lim\limits_{x\to 1} f(x)$ does not exist.

EXAMPLE 2 *If* $f(x) = \begin{cases}\dfrac{x-|x|}{x}, & x\neq 0\\ 2, & x=0\end{cases}$ *show that* $\lim\limits_{x\to 0} f(x)$ *does not exist.*

SOLUTION We have,

(LHL of $f(x)$ at $x=0$)

$= \lim\limits_{x\to 0^-} f(x)$

$= \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0}\dfrac{-h-|-h|}{(-h)} = \lim\limits_{h\to 0}\dfrac{-h-h}{-h} = \lim\limits_{h\to 0}\dfrac{-2h}{-h} = \lim\limits_{h\to 0} 2 = 2.$

(RHL of $f(x)$ at $x=0$)

$= \lim\limits_{x\to 0^+} f(x)$

$= \lim\limits_{x\to 0} f(0+h) = \lim\limits_{h\to 0}\dfrac{h-|h|}{h} = \lim\limits_{h\to 0}\dfrac{h-h}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = \lim\limits_{h\to 0} 0 = 0.$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x).$

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

EXAMPLE 3 *If* $f(x) = \begin{cases} 5x - 4, & 0 < x \le 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$, *show that* $\lim_{x \to 1} f(x)$ *exists.*

SOLUTION We have,

(LHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 5(1-h) - 4 = \lim_{h \to 0} 1 - 5h = 1.$$

(RHL of $f(x)$ at $x = 1$)

$$= \lim_{x \to 1^+} f(x)$$

$$= \lim_{h \to 0} f(1+h) = \lim_{h \to 0} 4(1+h)^3 - 3(1+h) = 4(1)^3 - 3(1) = 1$$

Clearly, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$.

Hence, $\lim_{x \to 1} f(x)$ exists and is equal to 1.

EXAMPLE 4 *Discuss the existence of each of the following limits:*

(i) $\lim_{x \to 0} \frac{1}{x}$ (ii) $\lim_{x \to 0} \frac{1}{|x|}$

SOLUTION (i) The graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 29.4. We observe that as x approaches to 0 from the LHS i.e. x is negative and very close to zero, then the values of $1/x$ are negative and very large in magnitude.

$$\therefore \quad \lim_{x \to 0^-} \frac{1}{x} \to -\infty.$$

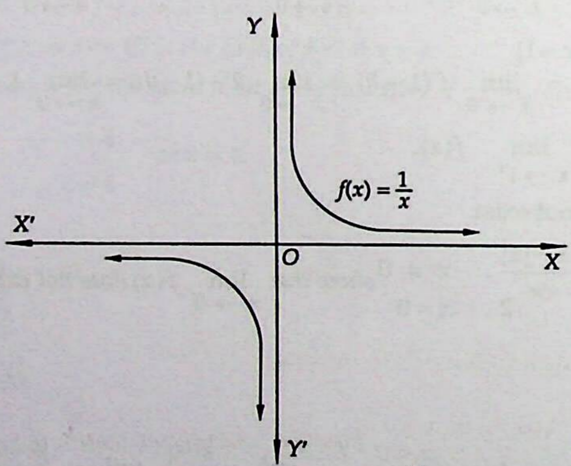

Fig. 29.4 Graph of $f(x) = \frac{1}{x}$

Similarly, when x approaches to 0 from the right i.e. x is positive and very close to 0, then the values of $\frac{1}{x}$ are very large and positive.

$$\therefore \quad \lim_{x \to 0^+} \frac{1}{x} \to \infty.$$

Thus we have, $\lim_{x \to 0^-} \frac{1}{x} \ne \lim_{x \to 0^+} \frac{1}{x}$. Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist.

(ii) The graph of $f(x) = \frac{1}{|x|}$ is shown in Fig. 29.5. We observe that as x approaches to 0 from LHS i.e. x is negative and close to 0, then $|x|$ is close to zero and is positive. Consequently, $\frac{1}{|x|}$ is large and positive.

$$\therefore \quad \lim_{x \to 0^-} \frac{1}{|x|} \to \infty$$

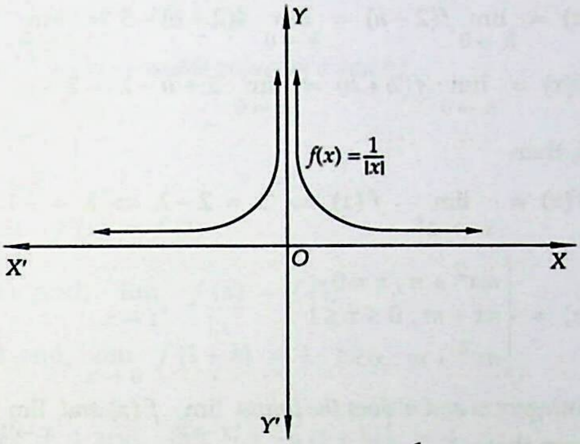

Fig. 29.5 Graph of $f(x) = \frac{1}{|x|}$

Also, if x approaches to 0 from RHS i.e. x is positive and close to 0, then $|x|$ is close to zero and is positive.

Consequently, $\frac{1}{|x|}$ is large and positive.

$$\therefore \quad \lim_{x \to 0^+} \frac{1}{|x|} \to \infty$$

Thus, we have

$$\lim_{x \to 0^-} \frac{1}{|x|} = \lim_{x \to 0^+} \frac{1}{|x|}$$

Hence, $\lim\limits_{x \to 0} \frac{1}{|x|}$ exists and it tends to infinity.

EXAMPLE 5 *Let* $f(x) = \begin{cases} \cos x, & \text{if } x > 0 \\ x + k, & \text{if } x < 0 \end{cases}$ *Find the value of constant k, given that* $\lim\limits_{x \to 0} f(x)$ *exists.*

SOLUTION It is given that

$$\lim_{x \to 0} f(x) \text{ exists}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \quad \lim_{x \to 0} x + k = \lim_{x \to 0} \cos x \qquad \text{[Using definition of } f(x)\text{]}$$

$$\Rightarrow \quad 0 + k = \cos 0 \Rightarrow k = 1$$

EXAMPLE 6 *Let $f(x)$ be a function defined by* $f(x) = \begin{cases} 4x-5, & \text{if } x \le 2 \\ x-\lambda, & \text{if } x > 2 \end{cases}$

Find λ, if $\lim\limits_{x \to 2} f(x)$ exists.

SOLUTION We have,

$$f(x) = \begin{cases} 4x-5, & \text{if } x \le 2 \\ x-\lambda, & \text{if } x > 2 \end{cases}$$

$\therefore$ $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{h \to 0} f(2-h) = \lim\limits_{h \to 0} 4(2-h)-5 = \lim\limits_{h \to 0} 3-4h = 3$

and, $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{h \to 0} f(2+h) = \lim\limits_{h \to 0} 2+h-\lambda = 2-\lambda$

If $\lim\limits_{x \to 2} f(x)$ exists, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \Rightarrow 3 = 2-\lambda \Rightarrow \lambda = -1.$$

EXAMPLE 7 If $f(x) = \begin{cases} mx^2+n, & x<0 \\ nx+m, & 0 \le x \le 1 \\ nx^3+m, & x>1 \end{cases}$ **[NCERT]**

For what values of integers m and n does the limits $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ exist.

SOLUTION It is given that

$\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ both exist

$\Leftrightarrow$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x)$ and, $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^+} f(x)$

$\Leftrightarrow$ $\lim\limits_{h \to 0} f(0-h) = \lim\limits_{h \to 0} f(0+h)$ and, $\lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0} f(1+h)$

$\Leftrightarrow$ $\lim\limits_{h \to 0} m(-h)^2+n = \lim\limits_{h \to 0} n(h)+m$ and, $\lim\limits_{h \to 0} n(1-h)+m = \lim\limits_{h \to 0} n(1+h)^3+m$

$\Leftrightarrow$ $n = m$, and $n+m = n+m$

$\Leftrightarrow$ $m = n$

Hence, $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$ both sides for $n = m$.

EXAMPLE 8 *If* $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$. *For what value (s) of 'a' does $\lim\limits_{x \to a} f(x)$ exist?*

SOLUTION We have, **[NCERT]**

$$f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -x+1, & x<0 \\ 0, & x=0 \\ x-1, & x>0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}\right]$$

Clearly, $\lim\limits_{x \to a} f(x)$ exists for all $a \ne 0$. So, let us see whether $\lim\limits_{x \to 0} f(x)$ exist or not.

We observe that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} f(0-h) = \lim_{h \to 0} -(-h)+1 = 1$$

and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h - 1 = -1$

$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

So, $\lim_{x \to 0} f(x)$ does not exist. Hence, $\lim_{x \to a} f(x)$ exists for all $a \neq 0$.

EXAMPLE 9 *Suppose* $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$

and, if $\lim_{x \to 1} f(x) = f(1)$. *What are possible values of a and b?* **[NCERT]**

SOLUTION We have,

$$\lim_{x \to 1} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{x \to 1^-} f(x) = f(1) \text{ and, } \lim_{x \to 1^+} f(x) = f(1)$$

$$\Leftrightarrow \quad \lim_{h \to 0} f(1-h) = 4 \text{ and, } \lim_{h \to 0} f(1+h) = 4$$

$$\Leftrightarrow \quad \lim_{h \to 0} \left\{ a + b(1-h) \right\} = 4 \text{ and, } \lim_{h \to 0} \left\{ b - a(1+h) \right\} = 4$$

$$\Leftrightarrow \quad a + b = 4 \text{ and, } b - a = 4$$

$$\Leftrightarrow \quad a = 0, b = 4$$

LEVEL-2

EXAMPLE 10 *Find the left hand and right hand limits of the greatest integer function* $f(x) = [x] =$ *greatest integer less than or equal to x, at x = k, where k is an integer. Also, show that* $\lim_{x \to k} f(x)$ *does not exist.*

SOLUTION We have,

$$(\text{LHL at } x = k) = \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h]$$

$$= \lim_{h \to 0} k - 1 = k - 1 \qquad [\because k-1 < k-h < k \therefore [k-h] = k-1]$$

$$(\text{RHL at } x = k) = \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$$

$$= \lim_{h \to 0} k = k \qquad [\because k < k+h < k+1 \therefore [k+h] = k]$$

Clearly, $\lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x)$.

Hence, $\lim_{x \to k} f(x)$ does not exist.

EXAMPLE 11 *Prove that* $\lim_{x \to a^+} [x] = [a]$ *for all* $a \in R$, *where* [.] *denotes the greatest integer function.*

SOLUTION Since $a \in R$. Therefore, there exists an integer k such that $k \le a < k+1$.

Now, $\lim_{x \to a^+} [x] = \lim_{h \to 0} [a+h] = k \qquad [\because k \le a < k+1 \therefore k \le a+h < k+1 \Rightarrow [a+h] = k]$

$= [a] \qquad [\because k \le a < k+1 \Rightarrow [a] = k]$

EXAMPLE 12 *Show that* $\lim_{x \to 0} \frac{e^{1/x}-1}{e^{1/x}+1}$ *does not exist.*

SOLUTION Let $f(x) = \frac{e^{1/x}-1}{e^{1/x}+1}$. Then,

(LHL of $f(x)$ at $x=0$)

$$= \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h \to 0} \left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1} \right) = \frac{0-1}{0+1} = -1 \qquad \left[\because h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty \Rightarrow \frac{1}{e^{1/h}} \to 0\right]$$

and, (RHL of $f(x)$ at $x=0$)

$$= \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} \frac{e^{1/h}-1}{e^{1/h}+1}$$

$$= \lim_{h \to 0} \left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}} \right) = \frac{1-0}{1+0} = 1$$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Hence, $\lim_{x \to 0} f(x)$ does not exist.

EXAMPLE 13 *If f is an odd function and if* $\lim_{x \to 0} f(x)$ *exists. Prove that this limit must be zero.*

SOLUTION It is given that

$\lim_{x \to 0} f(x)$ exists

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$\Rightarrow \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h)$$

$$\Rightarrow \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h)$$

$$\Rightarrow -\lim_{h \to 0} f(h) = \lim_{h \to 0} f(h) \qquad [\because f(x) \text{ is odd } \therefore f(-h) = -f(h)]$$

$$\Rightarrow 2\lim_{h \to 0} f(h) = 0 \Rightarrow \lim_{h \to 0} f(h) = 0 \Rightarrow \lim_{x \to 0} f(x) = 0$$

EXAMPLE 14 *If f is an even function, then prove that* $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$.

SOLUTION Clearly,

$$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} f(h) \qquad [\because f \text{ is even } \therefore f(-h) = f(h)]$$

$$= \lim_{h \to 0} f(0+h) = \lim_{x \to 0^+} f(x).$$

EXERCISE 29.1

LEVEL-1

1. Show that $\lim\limits_{x \to 0} \dfrac{x}{|x|}$ does not exist. **[NCERT]**

2. Find k so that $\lim\limits_{x \to 2} f(x)$ may exist, where $f(x) = \begin{cases} 2x+3, & x \le 2 \\ x+k, & x > 2 \end{cases}$.

3. Show that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist.

4. Let $f(x)$ be a function defined by $f(x) = \begin{cases} \dfrac{3x}{|x|+2x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

   Show that $\lim\limits_{x \to 0} f(x)$ does not exist.

5. Let $f(x) = \begin{cases} x+1, & \text{if } x > 0 \\ x-1, & \text{if } x < 0 \end{cases}$. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

6. Let $f(x) = \begin{cases} x+5, & \text{if } x > 0 \\ x-4, & \text{if } x < 0 \end{cases}$. Prove that $\lim\limits_{x \to 0} f(x)$ does not exist.

7. Find $\lim\limits_{x \to 3} f(x)$, where $f(x) = \begin{cases} 4, & \text{if } x > 3 \\ x+1, & \text{if } x < 3 \end{cases}$

8. If $f(x) = \begin{cases} 2x+3, & x \le 0 \\ 3(x+1), & x > 0 \end{cases}$. Find $\lim\limits_{x \to 0} f(x)$ and $\lim\limits_{x \to 1} f(x)$. **[NCERT]**

9. Find $\lim\limits_{x \to 1} f(x)$, if $f(x) = \begin{cases} x^2 - 1, & x \le 1 \\ -x^2 - 1, & x > 1 \end{cases}$. **[NCERT]**

10. Evaluate $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ **[NCERT]**

11. Let $a_1, a_2, \ldots, a_n$ be fixed real numbers such that $f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$

    What is $\lim\limits_{x \to a_1} f(x)$? For $a \ne a_1, a_2, \ldots, a_n$ compute $\lim\limits_{x \to a} f(x)$. **[NCERT]**

12. Find $\lim\limits_{x \to 1^+} \dfrac{1}{x-1}$.

13. Evaluate the following one sided limits:

    (i) $\lim\limits_{x \to 2^+} \dfrac{x-3}{x^2-4}$ (ii) $\lim\limits_{x \to 2^-} \dfrac{x-3}{x^2-4}$ (iii) $\lim\limits_{x \to 0^+} \dfrac{1}{3x}$

    (iv) $\lim\limits_{x \to -8^+} \dfrac{2x}{x+8}$ (v) $\lim\limits_{x \to 0^+} \dfrac{2}{x^{1/5}}$ (vi) $\lim\limits_{x \to \frac{\pi}{2}^-} \tan x$

    (vii) $\lim\limits_{x \to -\pi/2^+} \sec x$ (viii) $\lim\limits_{x \to 0^-} \dfrac{x^2-3x+2}{x^3-2x^2}$ (ix) $\lim\limits_{x \to -2^+} \dfrac{x^2-1}{2x+4}$

    (x) $\lim\limits_{x \to 0^-} (2 - \cot x)$ (xi) $\lim\limits_{x \to 0^-} 1 + \operatorname{cosec} x$

LEVEL-2

14. Show that $\lim_{x \to 0} e^{-1/x}$ does not exist.

15. Find:

(i) $\lim_{x \to 2} [x]$ (ii) $\lim_{x \to \frac{5}{2}} [x]$ (iii) $\lim_{x \to 1} [x]$

16. Prove that $\lim_{x \to a^+} [x] = [a]$ for all $a \in R$. Also, prove that $\lim_{x \to 1^-} [x] = 0$.

17. Show that $\lim_{x \to 2^-} \frac{x}{[x]} \neq \lim_{x \to 2^+} \frac{x}{[x]}$.

18. Find $\lim_{x \to 3^+} \frac{x}{[x]}$. Is it equal to $\lim_{x \to 3^-} \frac{x}{[x]}$.

19. Find $\lim_{x \to -5/2} [x]$.

20. Evaluate $\lim_{x \to 2} f(x)$ (if it exists), where $f(x) = \begin{cases} x - [x] & , \quad x < 2 \\ 4 & , \quad x = 2. \\ 3x - 5 & , \quad x > 2 \end{cases}$

21. Show that $\lim_{x \to 0} \sin \frac{1}{x}$ does not exist.

22. Let $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & , \text{ where } x \neq \dfrac{\pi}{2} \\ 3 & , \text{ where } x \neq \dfrac{\pi}{2} \end{cases}$ and if $\lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$, find the value of k.

ANSWERS

2. $k = 5$ 7. 4 8. 3, Does not exist 9. Does not exist 10. Does not exist

11. $0, (a - a_1)(a - a_2) \dots (a - a_n)$ 12. ∞ 13. (i) $-\infty$ (ii) ∞ (iii) ∞ (iv) $-\infty$

(v) ∞ (vi) ∞ (vii) $-\infty$ (viii) $-\infty$ (ix) ∞ (x) ∞ (xi) $-\infty$

15. (i) Does not exist (ii) 2 (iii) Does not exist

18. 1, No 19. -3 20. 1 22. $k = 6$

HINTS TO NCERT & SELECTED PROBLEMS

1. $\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} \frac{x}{-x} = -1$ and, $\lim_{x \to 0^+} = \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1$

$\therefore \quad \lim_{x \to 0^-} \frac{x}{|x|} \neq \lim_{x \to 0^+} \frac{x}{|x|}$

Hence, $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

3. We have,

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{1}{-h} = -\infty$ and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{1}{h} = \infty$

Clearly, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$. Hence, $\lim_{x \to 0} f(x)$ does not exist.

8. We have,

$$f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1) & x \geq 0 \end{cases}$$

$\therefore \quad \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (2x+3) = 2 \times 0 + 3 = 3$

and, $\quad \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 3(x+1) = 3(0+1) = 3$

So, $\quad \lim\limits_{x \to 0} f(x)$ exists and is equal to 3.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} 2x+3 = 2 \times 1 + 3 = 5$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} 3(x+1) = 3(1+1) = 6.$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$

Hence, $\lim\limits_{x \to 1} f(x)$ does not exist.

9. We have,

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x \geq 1 \end{cases}$$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^2 - 1 = 1^2 - 1 = 0$ and, $\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} -x^2 - 1 = -1 - 1 = -2$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x).$

Hence, $\lim\limits_{x \to 1} f(x)$ does not exist.

10. We have,

$$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\therefore \quad \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = -1$ and, $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0} \dfrac{x}{x} = 1.$

$\therefore \quad \lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x).$

Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

11. We have,

$$f(x) = (x - a_1)(x - a_2) \ldots (x - a_n)$$

$\therefore \quad \lim_{x \to a_1^-} f(x) = \lim_{x \to 0} f(a_1 - h) = \lim_{x \to 0} -h\,(a_1 - h - a_2)(a_1 - h - a_3) \ldots (a_1 - h - a_n)$

$= 0\,(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) = 0$

$\lim_{x \to a_1^+} f(x) = \lim_{x \to 0} (a_1 + h - a_1)(a_1 + h - a_2)(a_1 + h - a_3) \ldots (a_1 + h - a_n)$

$= \lim_{x \to 0} h(a_1 + h - a_2)(a_1 + h - a_3) \ldots (a_1 + h - a_n)$

$= 0\,(a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n) = 0$

$\therefore \quad \lim_{x \to a_1^-} f(x) = \lim_{x \to a_1^+} f(x) = 0.$

Hence, $\lim_{x \to a_1} f(x) = 0.$

For any $a \neq a_1, a_2, \ldots a_n$.

$\lim_{x \to a} f(x) = \lim_{x \to a} (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n) = (a - a_1)(a - a_2)(a - a_3) \ldots (a - a_n)$

12. $\lim_{x \to 1^+} \frac{1}{x - 1} = \lim_{h \to 0} \frac{1}{1 + h - 1} = \lim_{h \to 0} \frac{1}{h} = \infty.$

14. We have,

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} e^{1/h} = \infty$

and, $\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} e^{-1/h} = \lim_{h \to 0} \frac{1}{e^{1/h}} = 0$

21. We have,

$\lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h - 0} \sin\left(\frac{1}{-h}\right) = - \lim_{h \to 0} \sin \frac{1}{h}$

$\Rightarrow \quad \lim_{x \to 0^-} f(x) = -$ (An oscillating number which oscillates between -1 and 1).

So, $\lim_{x \to 0^-} f(x)$ does not exist. Similarly, $\lim_{x \to 0^+} f(x)$ does not exist.

29.3 DIFFERENCE BETWEEN THE VALUE OF A FUNCTION AT A POINT AND THE LIMIT AT THAT POINT

Let $f(x)$ be a function and let a be a point. Then, we have the following possibilities:

(i) $\lim_{x \to a} f(x)$ *exists but* $f(a)$ *(the value of* $f(x)$ *at* $x = a$*) does not exist*:

Consider the function $f(x)$ defined by $f(x) = \frac{x^2 - 9}{x - 3}$.

Clearly, this function is not defined at $x = 3$ i.e. $f(3)$ does not exist, because it attains the form $\frac{0}{0}$. But, it can be easily seen that $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = 6$. So, $\lim_{x \to 3} f(x)$ exists.

Thus, the $\lim_{x \to 3} f(x)$ exists but $f(3)$ does not exist.

(ii) *The value $f(a)$ exists but $\lim_{x \to a} f(x)$ does not exist:*

In example 4 on page 29.6, we have seen that $\lim_{x \to k} f(x)$ does not exist but $f(k) = k$ exists.

(iii) $\lim_{x \to a} f(x)$ *and $f(a)$ both exist but are unequal:*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$$

It can be easily seen that $\lim_{x \to 2^-} f(x) = 4 = \lim_{x \to 2^+} f(x)$.

So, $\lim_{x \to 2} f(x)$ exists and is equal to 4. Also, the value $f(2)$ exists and is equal to 3.

Thus, $\lim_{x \to 2} f(x)$ and $f(2)$ both exist but are unequal.

(iv) $\lim_{x \to a} f(x)$ *and $f(a)$ both exist and are equal*

Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$$

For this function, it can be easily seen that $\lim_{x \to 2} f(x)$ and $f(2)$ both exist and are equal to 4.

29.4 THE ALGEBRA OF LIMITS

Let f and g be two real functions with common domain D. In the chapter on functions, we have defined four new function $f \pm g, fg, f/g$ on domain D by setting

$$(f \pm g)(x) = f(x) \pm g(x),$$
$$(fg)(x) = f(x)\, g(x)$$
$$(f/g)(x) = f(x)/g(x), \text{ if } g(x) \neq 0 \text{ for any } x \in D.$$

Following are some results concerning the limits of these functions.

Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$.

If l and m exist, then

(i) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m$

(ii) $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = lm$

(iii) $\lim_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$.

(iv) $\lim_{x \to a} k\, f(x) = k \lim_{x \to a} f(x)$, where k is constant

(v) $\lim_{x \to a} |f(x)| = |\lim_{x \to a} f(x)| = |l|$

(vi) $\lim\limits_{x \to a} \left\{f(x)\right\}^{g(x)} = l^m$

(vii) If $f(x) \le g(x)$ for every x in the deleted neighbourhood *of a*, then $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$

(viii) If $f(x) \le g(x) \le h(x)$ for every x in the deleted neighbourhood of a and

$\lim\limits_{x \to a} f(x) = l = \lim\limits_{x \to a} h(x)$, then $\lim\limits_{x \to a} g(x) = l$.

This result is often stated as *Sandwich Theorem*.

(ix) If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$.

29.5 INDETERMINATE FORMS AND EVALUATION OF LIMITS

Uptill now we have been discussing left hand and right hand limits and the existence of limits. In what follows, we will be assuming that the limit of a function at a given point exists. In the previous section, we have stated that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}, \text{ provided that } \lim_{x \to a} g(x) \neq 0.$$

An interesting situation now arises. If $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ takes the form $\dfrac{0}{0}$, which is undefined or meaningless. But, this does not imply that $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ is meaningless or it does not exist. In fact, in many cases this limit exists and has a finite value. The determination of limit in such a case is traditionally referred to as the evaluation of the indeterminate form $\dfrac{0}{0}$, though literally speaking nothing is indeterminate involved here. Sometimes $\dfrac{0}{0}$ is referred to as undetermined form or illusory form. In addition to $\dfrac{0}{0}$ there are six other indeterminate forms, namely, $\dfrac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0 and 1^∞. Among all these seven indeterminate forms $\dfrac{0}{0}$ is the fundamental one because all the remaining six forms can easily be reduced to this form. In this chapter, we shall study how to evaluate a limit which belongs to one of following in determinate forms:

$\dfrac{0}{0}$, $0 \times \infty$ and $\infty - \infty$.

To facilitate the job of evaluation of limits we categorize problems on limits in the following categories:

(i) Algebraic Limits. (ii) Non-algebraic Limits.

If a problem on limits does not involve trigonometric, inverse trigonometric, exponential and logarithmic function, then it is a problem on algebraic limits, otherwise, it is a problem on non-algebraic limits.

For example,

(i) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (iii) $\lim\limits_{x \to 2} \dfrac{x^5 - 32}{x - 2}$

(iv) $\lim\limits_{x \to \infty} \dfrac{x^2 + x + 1}{2x^2 + 5}$ etc. are problems on algebraic limits.

Following are some examples of non-algebraic limits:

(i) $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$ (ii) $\lim\limits_{x \to 0} \dfrac{3 \sin^{-1} 2x}{\sin x}$ (iii) $\lim\limits_{x \to \pi/4} \dfrac{\sin x - \cos x}{4x - \pi}$

(iv) $\lim\limits_{x \to 0} \dfrac{2^x - 3^x}{x}$

29.6 EVALUATION OF ALGEBRAIC LIMITS

In order to evaluate algebraic limits we have the following methods.

(i) Direct substitution method.
(ii) Factorisation method.
(iii) Rationalisation method.
(iv) By using some standard limits.
(v) Method of evaluation of algebraic limits at infinity.

We shall now discuss these methods with suitable illustrations in the following sub- sections.

29.6.1 DIRECT SUBSTITUTION METHOD

Consider the following limits:

(i) $\lim\limits_{x \to a} f(x)$ (ii) $\lim\limits_{x \to a} \dfrac{\Phi(x)}{\psi(x)}$

If $f(a)$ and $\dfrac{\varphi(a)}{\psi(a)}$ exist and are fixed real numbers, then we say that

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} \frac{\varphi(x)}{\psi(a)} = \frac{\phi(a)}{\psi(a)}$$

In other words, if the direct substitution of the point, to which the variable tends to, we obtain a fixed real number, then the number obtained is the limit of the function. In fact, if the point to which the variable tends to is a point in the domain of the function, then the value of the function at that point is its limit.

Following examples will illustrate the above method.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate :* $\lim\limits_{x \to 1} 3x^2 + 4x + 5.$

SOLUTION $\lim\limits_{x \to 1} 3x^2 + 4x + 5 = 3(1)^2 + 4(1) + 5 = 12.$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3}$

SOLUTION Using direct substitution method, we obtain

$$\lim_{x \to 2} \frac{x^2 - 4}{x + 3} = \frac{4 - 4}{2 + 3} = \frac{0}{5} = 0.$$

EXAMPLE 3 *Evaluate* $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x}+\sqrt{1-x}}{1+x}$

SOLUTION Using direct substitution method, we obtain

$$\lim_{x \to 0} \frac{\sqrt{1+x}+\sqrt{1-x}}{1+x} = \frac{\sqrt{1+0}+\sqrt{1-0}}{1+0} = \frac{1+1}{1} = 2.$$

EXERCISE 29.2

LEVEL-1

Evaluate the following limits:

1. $\lim\limits_{x \to 1} \dfrac{x^2+1}{x+1}$
2. $\lim\limits_{x \to 0} \dfrac{2x^2+3x+4}{x^2+3x+2}$
3. $\lim\limits_{x \to 3} \dfrac{\sqrt{2x+3}}{x+3}$
4. $\lim\limits_{x \to 1} \dfrac{\sqrt{x+8}}{\sqrt{x}}$
5. $\lim\limits_{x \to a} \dfrac{\sqrt{x}+\sqrt{a}}{x+a}$
6. $\lim\limits_{x \to 1} \dfrac{1+(x-1)^2}{1+x^2}$
7. $\lim\limits_{x \to 0} \dfrac{x^{2/3}-9}{x-27}$
8. $\lim\limits_{x \to 0} 9$
9. $\lim\limits_{x \to 2} (3-x)$
10. $\lim\limits_{x \to -1} (4x^2+2)$
11. $\lim\limits_{x \to 1} \dfrac{x^3-3x+1}{x-1}$
12. $\lim\limits_{x \to 0} \dfrac{3x+1}{x+3}$
13. $\lim\limits_{x \to 3} \dfrac{x^2-9}{x+2}$
14. $\lim\limits_{x \to 0} \dfrac{ax+b}{cx+d}, d \neq 0$

ANSWERS

1. 1 2. 2 3. $\frac{1}{2}$ 4. 3 5. $\frac{1}{\sqrt{a}}$ 6. $\frac{1}{2}$ 7. $\frac{1}{3}$ 8. 9
9. 1 10. 6 11. $\frac{-3}{2}$ 12. $\frac{1}{3}$ 13. 0 14 $\frac{b}{d}$

29.6.2 FACTORIZATION METHOD

Consider the limit: $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$.

If by substituting $x = a$, $\dfrac{f(x)}{g(x)}$, reduces to the form $\dfrac{0}{0}$, then $(x-a)$ is a factor of $f(x)$ and $g(x)$ both. So, we first factorize $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

Following algorithm may be used to evaluate the limit by factorization method.

ALGORITHM

STEP I Obtain the problem, say, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, where $\lim\limits_{x \to a} f(x) = 0$ *and* $\lim\limits_{x \to a} g(x) = 0$.

STEP II Factorize $f(x)$ and $g(x)$.

STEP III Cancel out the common factor(s) of $f(x)$ and $g(x)$.

STEP IV Use direct substitution method to obtain the limit.

Some useful results to remember:

(i) $a^2 - b^2 = (a-b)(a+b)$ (ii) $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

(iii) $a^3 + b^3 = (a+b)(a^2-ab+b^2)$ (iv) $a^4 - b^4 = (a^2-b^2)(a^2+b^2) = (a+b)(a-b)(a^2+b^2)$

(v) If $f(\alpha) = 0$, then $x - \alpha$ is a factor of $f(x)$.

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate :* $\lim\limits_{x \to 2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

SOLUTION When $x = 2$ the expression $\dfrac{x^2-5x+6}{x^2-4}$ assumes the indeterminate form $\dfrac{0}{0}$.

Therefore, $(x-2)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^2-5x+6}{x^2-4} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{(x-2)(x-3)}{(x+2)(x-2)} = \lim_{x \to 2} \frac{x-3}{x+2} = \frac{2-3}{2+2} = -\frac{1}{4}.$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x - 1}$.

SOLUTION When $x = 1$ the expression $\dfrac{x^3-1}{x-1}$ assumes the indeterminate form $\dfrac{0}{0}$. Therefore, $(x-1)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1} \frac{x^3-1}{x-1} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \to 1} x^2+x+1 = 1^2+1+1 = 3.$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^3 - 3x^2 + 4}{x^4 - 8x^2 + 16}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^3-3x^2+4}{x^4-8x^2+16}$ assumes the indeterminate form $\dfrac{0}{0}$.

Therefore, $(x-2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 2} \frac{x^3-3x^2+4}{x^4-8x^2+16}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2-x-2)}{(x^2-4)^2}$$

$$= \lim_{x \to 2} \frac{(x-2)(x-2)(x+1)}{(x-2)^2(x+2)^2} = \lim_{x \to 2} \frac{x+1}{(x+2)^2} = \frac{2+1}{(2+2)^2} = \frac{3}{16}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to 2} \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$.

SOLUTION When $x = 2$, the expression $\dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$ assumes the form $\dfrac{0}{0}$. Therefore, $(x - 2)$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 2} \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 6x + 8}$$

$$= \lim_{x \to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)} = \lim_{x \to 2} \frac{(x-1)(x-3)}{(x-4)} = \frac{(2-1)(2-3)}{(2-4)} = \frac{1}{2}.$$

EXAMPLE 5 *Evaluate:* $\lim\limits_{x \to 1/2} \dfrac{8x^3 - 1}{16x^4 - 1}$.

SOLUTION When $x = 1/2$, the expression $\dfrac{8x^3 - 1}{16x^4 - 1}$ assumes the form $\dfrac{0}{0}$. Therefore, $\left(x - \dfrac{1}{2}\right)$ or, $2x - 1$ is a factor common to numerator and denominator. Factorising the numerator and denominator, we obtain

$$\lim_{x \to 1/2} \frac{8x^3 - 1}{16x^4 - 1} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x)^3 - 1^3}{(4x^2)^2 - 1^2}$$

$$= \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2 + 1)(4x^2 - 1)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1/2} \frac{(2x-1)(4x^2 + 2x + 1)}{(4x^2 + 1)(2x-1)(2x+1)} = \lim_{x \to 1/2} \frac{4x^2 + 2x + 1}{(4x^2 + 1)(2x + 1)} = \frac{3}{4}.$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{x \to 1} \left(\dfrac{2}{1 - x^2} + \dfrac{1}{x - 1}\right)$.

SOLUTION We have,

$$\lim_{x \to 1} \left(\frac{2}{1 - x^2} + \frac{1}{x - 1}\right) = \lim_{x \to 1} \left(\frac{2}{1 - x^2} - \frac{1}{1 - x}\right).$$

When $x = 1$, the expression $\dfrac{2}{1 - x^2} - \dfrac{1}{1 - x}$ assumes the form $\infty - \infty$. So, we need some simplification to express it in the form $\dfrac{0}{0}$. Taking LCM, we get

$$\lim_{x \to 1} \left(\frac{2}{1 - x^2} - \frac{1}{1 - x}\right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \frac{2 - (1 + x)}{1 - x^2} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{1 - x}{1 - x^2} = \lim_{x \to 1} \frac{1}{1 + x} = \frac{1}{2}.$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 1} \left(\dfrac{1}{x^2 + x - 2} - \dfrac{x}{x^3 - 1} \right)$.

SOLUTION When $x = 1$, the expression $\dfrac{1}{x^2 + x - 2} - \dfrac{x}{x^3 - 1}$ assumes the indeterminate form $\infty - \infty$. So, we need simplification to reduce the expression in the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x \to 1} \left(\frac{1}{x^2 + x - 2} - \frac{x}{x^3 - 1} \right) \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{ \frac{1}{(x+2)(x-1)} - \frac{x}{(x-1)(x^2 + x + 1)} \right\} \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 1} \left\{ \frac{(x^2 + x + 1) - x(x+2)}{(x+2)(x-1)(x^2 + x + 1)} \right\} \qquad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \to 1} \frac{-(x-1)}{(x+2)(x-1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{-1}{(x+2)(x^2 + x + 1)} = -\frac{1}{9}.$$

EXAMPLE 8 *Evaluate:* $\lim_{x \to \sqrt{2}} \dfrac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = \sqrt{2}$, the expression $\dfrac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8}$ assumes the indeterminate form $\dfrac{0}{0}$. So, $(x - \sqrt{2})$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to \sqrt{2}} \frac{x^4 - 4}{x^2 + 3x\sqrt{2} - 8} \qquad \left(\text{form } \frac{0}{0} \right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^2 + 2)}{(x + 4\sqrt{2})(x - \sqrt{2})} \qquad \left(\text{form } \frac{0}{0} \right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})(x - \sqrt{2})} \qquad \left(\text{form } \frac{0}{0} \right)$$

$$= \lim_{x \to \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})} = \frac{(2\sqrt{2})(2 + 2)}{5\sqrt{2}} = \frac{8}{5}.$$

EXAMPLE 9 *Evaluate:* $\lim_{x \to 4} \dfrac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$.

SOLUTION When $x = 4$, the expression $\dfrac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$ assumes the form $\dfrac{0}{0}$. So, $(x - 4)$ is a common factor in numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{(x^2 - x - 12)^{18}}{(x^3 - 8x^2 + 16x)^9}$$

$$= \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{[x(x^2 - 8x + 16)]^9} = \lim_{x \to 4} \frac{[(x-4)(x+3)]^{18}}{x^9 (x-4)^{18}}$$

$$= \lim_{x \to 4} \frac{(x-4)^{18}(x+3)^{18}}{x^9 (x-4)^{18}} = \lim_{x \to 4} \frac{(x+3)^{18}}{x^9} = \frac{7^{18}}{4^9}$$

LEVEL-2

EXAMPLE 10 *Evaluate:* $\lim\limits_{x \to 3} \dfrac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$.

SOLUTION When $x = 3$, the expression $\dfrac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$ assumes the form $\dfrac{0}{0}$. So, $(x-3)$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 3} \frac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x^2 - 4x + 3)}{(x-3)(x^3 - 2x^2 - 6x + 9)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^3 - 2x^2 - 6x + 9} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x-1)}{(x-3)(x^2 + x - 3)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x-1}{x^2 + x - 3} = \frac{3-1}{9+3-3} = \frac{2}{9}.$$

EXAMPLE 11 *Evaluate:*

$$\lim_{x \to \sqrt{2}} \frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$$

SOLUTION When $x = \sqrt{2}$, the expression $\dfrac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$ assume the form $\dfrac{0}{0}$. Therefore, $(x - \sqrt{2})$ is a factor of numerator and denominator. But, irrational roots occur in pairs. So, $(x + \sqrt{2})$ will also be a factor of both numerator and denominator, consequently, $(x^2 - 2)$ will be a common factor of numerator and denominator, Dividing numerator and denominator by $(x^2 - 2)$, we get

$$\lim_{x \to \sqrt{2}} \frac{x^9 - 3x^8 + x^6 - 9x^4 - 4x^2 - 16x + 84}{x^5 - 3x^4 - 4x + 12}$$

$$= \lim_{x \to \sqrt{2}} \frac{(x^2 - 2)(x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42)}{(x^2 - 2)(x^3 - 3x^2 + 2x - 6)}$$

$$= \lim_{x \to \sqrt{2}} \frac{x^7 - 3x^6 + 2x^5 - 5x^4 + 4x^3 - 19x^2 + 8x - 42}{x^3 - 3x^2 + 2x - 6}$$

$$= \frac{8\sqrt{2} - 24 + 8\sqrt{2} - 20 + 8\sqrt{2} - 38 + 8\sqrt{2} - 42}{2\sqrt{2} - 6 + 2\sqrt{2} - 6} = \frac{8\sqrt{2} - 31}{\sqrt{2} - 3}.$$

EXERCISE 29.3

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 5} \dfrac{2x^2+9x-5}{x+5}$

2. $\lim_{x \to 3} \dfrac{x^2-4x+3}{x^2-2x-3}$

3. $\lim_{x \to 3} \dfrac{x^4-81}{x^2-9}$

4. $\lim_{x \to 2} \dfrac{x^3-8}{x^2-4}$

5. $\lim_{x \to -1/2} \dfrac{8x^3+1}{2x+1}$

6. $\lim_{x \to 4} \dfrac{x^2-7x+12}{x^2-3x-4}$

7. $\lim_{x \to 2} \dfrac{x^4-16}{x-2}$

8. $\lim_{x \to 5} \dfrac{x^2-9x+20}{x^2-6x+5}$

9. $\lim_{x \to -1} \dfrac{x^3+1}{x+1}$

10. $\lim_{x \to 5} \dfrac{x^3-125}{x^2-7x+10}$

11. $\lim_{x \to \sqrt{2}} \dfrac{x^2-2}{x^2+\sqrt{2}x-4}$

12. $\lim_{x \to \sqrt{3}} \dfrac{x^2-3}{x^2+3\sqrt{3}\,x-12}$

13. $\lim_{x \to \sqrt{3}} \dfrac{x^4-9}{x^2+4\sqrt{3}x-15}$

14. $\lim_{x \to 2} \left(\dfrac{x}{x-2}-\dfrac{4}{x^2-2x}\right)$

15. $\lim_{x \to 1} \left(\dfrac{1}{x^2+x-2}-\dfrac{x}{x^3-1}\right)$

16. $\lim_{x \to 3} \left(\dfrac{1}{x-3}-\dfrac{2}{x^2-4x+3}\right)$

17. $\lim_{x \to 2} \left(\dfrac{1}{x-2}-\dfrac{2}{x^2-2x}\right)$

18. $\lim_{x \to 1/4} \dfrac{4x-1}{2\sqrt{x}-1}$

19. $\lim_{x \to 4} \dfrac{x^2-16}{\sqrt{x}-2}$

20. $\lim_{x \to 0} \dfrac{(a+x)^2-a^2}{x}$

21. $\lim_{x \to 2} \left(\dfrac{1}{x-2}-\dfrac{4}{x^3-2x^2}\right)$

22. $\lim_{x \to 3} \left(\dfrac{1}{x-3}-\dfrac{3}{x^2-3x}\right)$

23. $\lim_{x \to 1} \left(\dfrac{1}{x-1}-\dfrac{2}{x^2-1}\right)$

24. $\lim_{x \to 3} (x^2-9)\left(\dfrac{1}{x+3}+\dfrac{1}{x-3}\right)$

LEVEL-2

25. $\lim_{x \to 1} \dfrac{x^4-3x^3+2}{x^3-5x^2+3x+1}$

26. $\lim_{x \to 2} \dfrac{x^3+3x^2-9x-2}{x^3-x-6}$

27. $\lim_{x \to 1} \dfrac{1-x^{-1/3}}{1-x^{-2/3}}$

28. $\lim_{x \to 3} \dfrac{x^2-x-6}{x^3-3x^2+x-3}$

29. $\lim_{x \to -2} \dfrac{x^3+x^2+4x+12}{x^3-3x+2}$

30. $\lim_{x \to 1} \dfrac{x^3+3x^2-6x+2}{x^3+3x^2-3x-1}$

31. $\lim\limits_{x \to 2} \left\{ \frac{1}{x-2} - \frac{2(2x-3)}{x^3 - 3x^2 + 2x} \right\}$ **[NCERT EXEMPLAR]**

32. $\lim\limits_{x \to 1} \frac{\sqrt{x^2-1} + \sqrt{x-1}}{\sqrt{x^2-1}}, x > 1$

33. $\lim\limits_{x \to 1} \left\{ \frac{x-2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x} \right\}$ **[NCERT]**

34. $\lim\limits_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ **[NCERT EXEMPLAR]**

ANSWERS

1. -11 2. $\frac{1}{2}$ 3. 18 4. 3 5. 3 6. $\frac{1}{5}$ 7. 32 8. $\frac{1}{4}$
9. 3 10. 25 11. $\frac{2}{3}$ 12. $\frac{2}{5}$ 13. 2 14. 2 15. $-\frac{1}{9}$ 16. $\frac{1}{2}$
17. $\frac{1}{2}$ 18. 2 19. 32 20. $2a$ 21. 1 22. $\frac{1}{3}$ 23. $\frac{1}{2}$ 24. 6
25. $\frac{5}{4}$ 26. $\frac{15}{11}$ 27. $\frac{1}{2}$ 28. $\frac{1}{2}$ 29. $\frac{4}{3}$ 30. $\frac{1}{2}$ 31. $-\frac{1}{2}$ 32. $\frac{\sqrt{2}+1}{\sqrt{2}}$
33. 2 34. 1

HINTS TO NCERT & SELECTED PROBLEM

33. $\lim\limits_{x \to 1} \left(\frac{x-2}{x^2 - x} - \frac{1}{x^3 - 3x^2 + 2x} \right)$

$$= \lim_{x \to 1} \left\{ \frac{x-2}{x(x-1)} - \frac{1}{x(x-1)(x-2)} \right\}$$

$$= \lim_{x \to 1} \frac{(x-2)^2 - 1^2}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{(x-3)(x-1)}{x(x-1)(x-2)} = \lim_{x \to 1} \frac{x-3}{x(x-2)} = \frac{-2}{-1} = 2$$

34. $\lim\limits_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim\limits_{x \to 1} \frac{(x-1)(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{(x-1)(x^2 - 2x - 2)}$

$$= \lim_{x \to 1} \frac{(x^6 + x^5 - x^4 - x^3 - x^2 - x - 1)}{x^2 - 2x - 2} = \frac{-3}{-3} = 1$$

29.6.3 RATIONALISATION METHOD

This is particularly used when either the numerator or denominator or both involve expression consisting of square roots and substituting the value of x the rational expression takes the form $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc.

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{\sqrt{2+x} - \sqrt{2}}{x}$.

SOLUTION When $x = 0$, the expression $\dfrac{\sqrt{2+x} - \sqrt{2}}{x}$ takes the form $\dfrac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x \to 0} \frac{(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})}{x(\sqrt{2+x} + \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 + x - 2}{x(\sqrt{2+x} + \sqrt{2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

EXAMPLE 2 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{x}{\sqrt{a+x} - \sqrt{a-x}}$.

SOLUTION When $x = 0$, the expression $\dfrac{x}{\sqrt{a+x} - \sqrt{a-x}}$ takes the form $\dfrac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x \to 0} \frac{x}{\sqrt{a+x} - \sqrt{a-x}} = \lim_{x \to 0} \frac{x}{(\sqrt{a+x} - \sqrt{a-x})} \times \frac{(\sqrt{a+x} + \sqrt{a-x})}{(\sqrt{a+x} + \sqrt{a-x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{x(\sqrt{a+x} + \sqrt{a-x})}{(a + x - a + x)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{(\sqrt{a+x} + \sqrt{a-x})}{2} = \frac{2\sqrt{a}}{2} = \sqrt{a}$$

EXAMPLE 3 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}{x^2}$.

SOLUTION When $x = 0$, the expression $\dfrac{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}{x^2}$ takes the form $\dfrac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to 0} \frac{\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2}}{x^2} = \lim_{x \to 0} \frac{(\sqrt{a^2 + x^2} - \sqrt{a^2 - x^2})}{x^2} \frac{(\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2})}{(\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2})} \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{a^2 + x^2 - a^2 + x^2}{x^2(\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{(\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2})} = \frac{2}{\sqrt{a^2} + \sqrt{a^2}} = \frac{1}{a}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{\sqrt{x^2 + 9} - 5}$.

SOLUTION When $x = 4$, the expression $\dfrac{x^2 - 16}{\sqrt{x^2 + 9} - 5}$ assumes the form $\dfrac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{x^2+9} - 5} = \lim_{x \to 4} \frac{(x^2-16)}{(\sqrt{x^2+9}-5)} \frac{(\sqrt{x^2+9}+5)}{(\sqrt{x^2+9}+5)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(x^2-16)(\sqrt{x^2+9}+5)}{(x^2+9-25)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \left(\sqrt{x^2+9}+5\right) = \left(\sqrt{16+9}+5\right) = 5+5 = 10$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = a$, the expression $\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$ assumes the form $\frac{0}{0}$.

Rationalising the numerator, we get

$$\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} = \lim_{x \to a} \frac{(\sqrt{a+2x} - \sqrt{3x})(\sqrt{a+2x} + \sqrt{3x})}{(\sqrt{3a+x} - 2\sqrt{x})(\sqrt{3a+x} + 2\sqrt{x})} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})}$$

$$= \lim_{x \to a} \frac{\sqrt{3a+x} + 2\sqrt{x}}{3\left\{\sqrt{a+2x} + \sqrt{3x}\right\}} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \frac{\sqrt{3a+a} + 2\sqrt{a}}{3(\sqrt{a+2a} + \sqrt{3a})} = \frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3a}} = \frac{2}{3\sqrt{3}}$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to 4} \frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$.

SOLUTION When x = 4, the expression $\frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}}$ assumes the form $\frac{0}{0}$.

Rationalising the numerator and denominator, we get

$$\lim_{x \to 4} \frac{3 - \sqrt{5+x}}{1 - \sqrt{5-x}} = \lim_{x \to 4} \frac{(3 - \sqrt{5+x})(3 + \sqrt{5+x})}{(1 - \sqrt{5-x})(1 + \sqrt{5-x})} \frac{(1 + \sqrt{5-x})}{(3 + \sqrt{5+x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{(9-5-x)}{(1-5+x)} \left(\frac{1 + \sqrt{5-x}}{3 + \sqrt{5+x}}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(x-4)(1 + \sqrt{5-x})}{(x-4)(3 + \sqrt{5+x})} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 4} \frac{-(1 + \sqrt{5-x})}{(3 + \sqrt{5+x})} = \frac{-(1+1)}{(3+3)} = -\frac{1}{3}$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 2} \frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$. **[NCERT EXEMPLAR]**

SOLUTION When $x = 2$, the expression $\frac{x^2 - 4}{\sqrt{3x-2} - \sqrt{x+2}}$ assumes the form $\frac{0}{0}$.

Rationalising the denominator, we get

$$\lim_{x\to 2}\frac{x^2-4}{\sqrt{3x-2}-\sqrt{x+2}}=\lim_{x\to 2}\frac{(x-2)(x+2)\,(\sqrt{3x-2}+\sqrt{x+2})}{(\sqrt{3x-2}-\sqrt{x+2})(\sqrt{3x-2}+\sqrt{x+2})}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{(3x-2-x-2)}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2(x-2)}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to 2}\frac{(x+2)(\sqrt{3x-2}+\sqrt{x+2})}{2}=\frac{(2+2)(2+2)}{2}=8.$$

EXAMPLE 8 *Evaluate:* $\lim\limits_{x\to 1}\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$.

SOLUTION When $x=1$, the expression $\dfrac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ takes the form $\dfrac{0}{0}$.

Rationalising $(\sqrt{x}-1)$ in the numerator, we get

$$\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}=\lim_{x\to 1}\frac{(2x-3)(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}+1)(2x+3)(x-1)}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{(2x-3)(x-1)}{(\sqrt{x}+1)(2x+3)(x-1)}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{2x-3}{(\sqrt{x}+1)(2x+3)}=-\frac{1}{10}.$$

LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim\limits_{x\to\sqrt{10}}\dfrac{\sqrt{7-2x}-(\sqrt{5}-\sqrt{2})}{x^2-10}$

SOLUTION We have,

$$\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-(\sqrt{5}-\sqrt{2})}{x^2-10}$$

$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{(\sqrt{5}-\sqrt{2})^2}}{x^2-10}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{7-2\sqrt{10}}}{x^2-10}\quad\left(\text{form }\frac{0}{0}\right)$$

$$=\lim_{x\to\sqrt{10}}\frac{\sqrt{7-2x}-\sqrt{7-2\sqrt{10}}}{x^2-10}\times\frac{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}}{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}}$$

$$=\lim_{x\to\sqrt{10}}\frac{(7-2x)-(7-2\sqrt{10})}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}\right\}}$$

$$=\lim_{x\to\sqrt{10}}\frac{-2x+2\sqrt{10}}{(x-\sqrt{10})(x+\sqrt{10})\left\{\sqrt{7-2x}+\sqrt{7-2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2(x - \sqrt{10})}{(x - \sqrt{10})(x + \sqrt{10})\left\{\sqrt{7 - 2x} + \sqrt{7 - 2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{(x + \sqrt{10})\left\{\sqrt{7 - 2x} + \sqrt{7 - 2\sqrt{10}}\right\}}$$

$$= \lim_{x \to \sqrt{10}} \frac{-2}{2\sqrt{10}\left\{\sqrt{7 - 2\sqrt{10}} + \sqrt{7 - 2\sqrt{10}}\right\}}$$

$$= \frac{-1}{\sqrt{10} \times 2 \times \sqrt{7 - 2\sqrt{10}}} = \frac{-1}{2\sqrt{10}(\sqrt{5} - \sqrt{2})} \qquad \left[\because (\sqrt{5} - \sqrt{2})^2 = 7 - 2\sqrt{10}\right]$$

$$= \frac{-1}{2\sqrt{10}} \times \frac{(\sqrt{5} + \sqrt{2})}{3} = -\frac{(\sqrt{5} + \sqrt{2})}{6\sqrt{10}}$$

EXERCISE 29.4

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{\sqrt{1 + x + x^2} - 1}{x}$

2. $\lim_{x \to 0} \frac{2x}{\sqrt{a + x} - \sqrt{a - x}}$

3. $\lim_{x \to 0} \frac{\sqrt{a^2 + x^2} - a}{x^2}$

4. $\lim_{x \to 0} \frac{\sqrt{1 + x} - \sqrt{1 - x}}{2x}$

5. $\lim_{x \to 2} \frac{\sqrt{3 - x} - 1}{2 - x}$

6. $\lim_{x \to 3} \frac{x - 3}{\sqrt{x - 2} - \sqrt{4 - x}}$

7. $\lim_{x \to 0} \frac{x}{\sqrt{1 + x} - \sqrt{1 - x}}$

8. $\lim_{x \to 1} \frac{\sqrt{5x - 4} - \sqrt{x}}{x - 1}$

9. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2}$

10. $\lim_{x \to 3} \frac{\sqrt{x + 3} - \sqrt{6}}{x^2 - 9}$

11. $\lim_{x \to 1} \frac{\sqrt{5x - 4} - \sqrt{x}}{x^2 - 1}$

12. $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$ [NCERT]

13. $\lim_{x \to 2} \frac{\sqrt{x^2 + 1} - \sqrt{5}}{x - 2}$

14. $\lim_{x \to 2} \frac{x - 2}{\sqrt{x} - \sqrt{2}}$

15. $\lim_{x \to 7} \frac{4 - \sqrt{9 + x}}{1 - \sqrt{8 - x}}$

16. $\lim_{x \to 0} \frac{\sqrt{a + x} - \sqrt{a}}{x\sqrt{a^2 + ax}}$

17. $\lim_{x \to 5} \frac{x - 5}{\sqrt{6x - 5} - \sqrt{4x + 5}}$

18. $\lim_{x \to 1} \frac{\sqrt{5x - 4} - \sqrt{x}}{x^3 - 1}$

19. $\lim_{x \to 2} \frac{\sqrt{1 + 4x} - \sqrt{5 + 2x}}{x - 2}$

20. $\lim_{x \to 1} \frac{\sqrt{3 + x} - \sqrt{5 - x}}{x^2 - 1}$

21. $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{x}$

22. $\lim_{x\to 0} \frac{\sqrt{1+x+x^2}-\sqrt{x+1}}{2x^2}$

23. $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x}$

24. $\lim_{x\to a} \frac{x-a}{\sqrt{x}-\sqrt{a}}$

25. $\lim_{x\to 0} \frac{\sqrt{1+3x}-\sqrt{1-3x}}{x}$

26. $\lim_{x\to 0} \frac{\sqrt{2-x}-\sqrt{2+x}}{x}$

27. $\lim_{x\to 1} \frac{\sqrt{3+x}-\sqrt{5-x}}{x^2-1}$

28. $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{3x^2+3x-6}$

29. $\lim_{x\to 0} \frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}$

30. $\lim_{x\to 1} \frac{x^2-\sqrt{x}}{\sqrt{x}-1}$ [NCERT EXEMPLAR]

31. $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}, x \neq 0$ [NCERT EXEMPLAR]

LEVEL-2

32. $\lim_{x\to\sqrt{10}} \frac{\sqrt{7+2x}-(\sqrt{5}+\sqrt{2})}{x^2-10}$

33. $\lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6}$

34. $\lim_{x\to\sqrt{2}} \frac{\sqrt{3+2x}-(\sqrt{2}+1)}{x^2-2}$

ANSWERS

1. $\frac{1}{2}$ 2. $2\sqrt{a}$ 3. $\frac{1}{2a}$ 4. $\frac{1}{2}$ 5. $\frac{1}{2}$ 6. 1 7. 1 8. 2

9. 2 10. $\frac{1}{12\sqrt{6}}$ 11. 1 12. $\frac{1}{2}$ 13. $\frac{2}{\sqrt{5}}$ 14. $2\sqrt{2}$ 15. $-\frac{1}{4}$ 16. $\frac{1}{2a\sqrt{a}}$

17. 5 18. $\frac{2}{3}$ 19. $\frac{1}{3}$ 20. $\frac{1}{4}$ 21. 0 22. $\frac{1}{4}$ 23. $\frac{1}{4}$ 24. $2\sqrt{a}$

25. 3 26. $-\frac{1}{\sqrt{2}}$ 27. $\frac{1}{4}$ 28. $-\frac{1}{18}$ 29. 1 30. 3 31. $\frac{1}{2\sqrt{x}}$ 32. $\frac{(\sqrt{5}-\sqrt{2})}{6\sqrt{10}}$

33. $\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{6}}$ 34. $\frac{\sqrt{2}-1}{2\sqrt{2}}$

HINTS TO NCERT & SELECTED PROBLEM

12. $$\lim_{x\to 0} = \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

33. $$\lim_{x\to\sqrt{6}} = \frac{\sqrt{5+2x}-(\sqrt{3}+\sqrt{2})}{x^2-6} = \lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{(\sqrt{3}+\sqrt{2})^2}}{x^2-6} = \lim_{x\to\sqrt{6}} \frac{\sqrt{5+2x}-\sqrt{5+2\sqrt{6}}}{x^2-6}$$

$$= \lim_{x\to\sqrt{6}} \frac{(\sqrt{5+2x})-(\sqrt{5+2\sqrt{6}})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \lim_{x\to\sqrt{6}} \frac{2(x-\sqrt{6})}{(x-\sqrt{6})(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})}$$

$$= \lim_{x\to\sqrt{6}} \frac{2}{(x+\sqrt{6})(\sqrt{5+2x}+\sqrt{5+2\sqrt{6}})} = \frac{2}{(2\sqrt{6})(2\sqrt{5+2\sqrt{6}})} = \frac{1}{2\sqrt{6}(\sqrt{3}+\sqrt{2})}$$

29.6.4 EVALUATION OF ALGEBRAIC LIMITS BY USING SOME STANDARD LIMITS

Following theorem will be used to evaluate some algebraic limits.

THEOREM If $n \in Q$, *then* $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$.

PROOF We have,

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a^+} \frac{x^n - a^n}{x - a} \qquad \left[\because \lim_{x \to a} f(x) \text{ exists } \therefore \lim_{x \to a} f(x) = \lim_{x \to a^+} f(x)\right]$$

$$= \lim_{h \to 0} \frac{(a+h)^n - a^n}{a + h - a}$$

$$= \lim_{h \to 0} \frac{a^n \left\{\left(1 + \frac{h}{a}\right)^n - 1\right\}}{h}$$

$$= a^n \lim_{h \to 0} \frac{\left\{1 + n\frac{h}{a} + \frac{n(n-1)}{2!}\frac{h^2}{a^2} + \dots - 1\right\}}{h} \qquad \left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots\right]$$

$$= a^n \lim_{h \to 0} \left\{\frac{n}{a} + \frac{n(n-1)}{2!}\frac{h}{a^2} + \dots\right\} = a^n \times \frac{n}{a} = n\,a^{n-1}$$

Following examples will illustrate the use of the above result in evaluating algebraic limits.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim_{x \to 2} \frac{x^{10} - 1024}{x - 2}$

SOLUTION When $x = 2$, the expression $\frac{x^{10} - 1024}{x - 2}$ assumes the form $\frac{0}{0}$.

Now, $\lim_{x \to 2} \frac{x^{10} - 1024}{x - 2} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2} = 10(2^{10-1}) = 5120$

EXAMPLE 2 *Evaluate:* $\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32}$.

SOLUTION When $x = 2$, the expression $\frac{x^{10} - 1024}{x^5 - 32}$ assume s the indeterminate form $\frac{0}{0}$.

Now, $\lim_{x \to 2} \frac{x^{10} - 1024}{x^5 - 32}$ $\qquad \left(\text{form } \frac{0}{0}\right)$

$$= \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x^5 - 2^5} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 2} \frac{\dfrac{x^{10} - 2^{10}}{x - 2}}{\dfrac{x^5 - 2^5}{x - 2}} = \lim_{x \to 2} \frac{x^{10} - 2^{10}}{x - 2} \div \lim_{x \to 2} \frac{x^5 - 2^5}{x - 2} = 10 \cdot 2^{10 - 1} \div 5 \cdot 2^{5-1} = 64.$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to 9} \dfrac{x^{3/2} - 27}{x - 9}$.

SOLUTION When $x = 9$, the expression $\dfrac{x^{3/2} - 27}{x - 9}$ assumes the form $\dfrac{0}{0}$

Now, $\lim_{x \to 9} \dfrac{x^{3/2} - 27}{x - 9} = \lim_{x \to 9} \dfrac{x^{3/2} - 9^{3/2}}{x - 9} = \dfrac{3}{2}(9)^{3/2-1} = \dfrac{3}{2}(3) = \dfrac{9}{2}$

EXAMPLE 4 *Evaluate:* $\lim_{x \to a} \dfrac{x\sqrt{x} - a\sqrt{a}}{x - a}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{x\sqrt{x} - a\sqrt{a}}{x - a} = \lim_{x \to a} \frac{x^{3/2} - a^{3/2}}{x - a} = \frac{3}{2} a^{3/2 - 1} = \frac{3}{2}\sqrt{a}$$

EXAMPLE 5 *Evaluate:* $\lim_{x \to a} \dfrac{x^m - a^m}{x^n - a^n}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{x^m - a^m}{x^n - a^n} = \lim_{x \to a} \left\{ \frac{x^m - a^m}{x - a} \cdot \frac{x - a}{x^n - a^n} \right\} = \lim_{x \to a} \left\{ \frac{x^m - a^m}{x - a} \div \frac{x^n - a^n}{x - a} \right\}$$

$$= \lim_{x \to a} \frac{x^m - a^m}{x - a} \div \lim_{x \to a} \frac{x^n - a^n}{x - a} = ma^{m-1} \div na^{n-1} = \frac{m}{n} a^{m-n}.$$

EXAMPLE 6 *Evaluate:* $\lim_{x \to 2} \dfrac{x - 2}{\sqrt[3]{x} - \sqrt[3]{2}}$.

SOLUTION We have,

$$\lim_{x \to 2} \frac{x - 2}{x^{1/3} - 2^{1/3}} = \frac{1}{\lim_{x \to 2} \dfrac{x^{1/3} - 2^{1/3}}{x - 2}} = \frac{1}{\dfrac{1}{3}\left(2^{1/3 - 1}\right)} = \frac{1}{\dfrac{1}{3} \times (2^{-2/3})} = 3\,(2^{2/3})$$

EXAMPLE 7 *Evaluate:* $\lim_{x \to 0} \dfrac{(1 - x)^n - 1}{x}$.

SOLUTION We have,

$$\lim_{x \to 0} \frac{(1 - x)^n - 1}{x} = - \lim_{x \to 0} \frac{(1 - x)^n - 1}{(1 - x) - 1}$$

$$= - \lim_{y \to 1} \frac{y^n - 1^n}{y - 1}, \text{ where } y = 1 - x. \qquad [\because\ x \to 0 \Rightarrow y \to 1]$$

$$= -n\,(1)^{n-1} = -n.$$

EXAMPLE 8 *Evaluate:* $\lim\limits_{x \to a} \dfrac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a}$.

SOLUTION We have,

$$\lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{x-a}$$

$$= \lim_{x \to a} \frac{(x+2)^{5/3} - (a+2)^{5/3}}{(x+2)-(a+2)}$$

$$= \lim_{y \to b} \frac{y^{5/3} - b^{5/3}}{y-b}, \text{ where } x+2 = y \text{ and } a+2 = b.$$

$$= \frac{5}{3} b^{5/3-1} = \frac{5}{3} b^{2/3} = \frac{5}{3}(a+2)^{2/3}$$

EXAMPLE 9 *Find the value of k, if* $\lim\limits_{x \to 1} \dfrac{x^4-1}{x-1} = \lim\limits_{x \to k} \dfrac{x^3-k^3}{x^2-k^2}$. **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to 1} \frac{x^4-1^4}{x-1} = 4\,(1)^{4-1} = 4$$

and, $$\lim_{x \to k} \frac{x^3-k^3}{x^2-k^2} = \lim_{x \to k} \frac{x^3-k^3}{x-k} \times \frac{x-k}{x^2-k^2} = \lim_{x \to k} \frac{x^3-k^3}{x-k} \div \frac{x^2-k^2}{x-k}$$

$$= \lim_{x \to k} \frac{x^3-k^3}{x-k} \div \lim_{x \to k} \frac{x^2-k^2}{x-k} = 3k^{3-1} \div 2k^{2-1} = \frac{3}{2}k$$

$$\therefore \quad \lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2}$$

$$\Rightarrow \quad 4 = \frac{3k}{2} \Rightarrow k = \frac{8}{3}$$

EXAMPLE 10 *If* $\lim\limits_{x \to 2} \dfrac{x^n - 2^n}{x-2} = 80$ *and* $n \in N$, *find n.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\lim_{x \to 2} \frac{x^n - 2^n}{x-2} = 80 \Rightarrow n \cdot 2^{n-1} = 80 \Rightarrow n \cdot 2^{n-1} = 5 \cdot 2^{5-1} \Rightarrow n = 5.$$

LEVEL-2

EXAMPLE 11 *If* $\lim\limits_{x \to -a} \dfrac{x^9 + a^9}{x+a} = 9$, *find the real values of a.*

SOLUTION We have,

$$\lim_{x \to -a} \frac{x^9 + a^9}{x+a} = 9$$

$$\Rightarrow \quad \lim_{x \to -a} \frac{x^9 - (-a)^9}{x-(-a)} = 9 \Rightarrow 9(-a)^{9-1} = 9 \Rightarrow 9a^8 = 9 \Rightarrow a^8 = 1 \Rightarrow a = \pm 1$$

EXAMPLE 12 *Evaluate:* $\lim_{x \to 1} \frac{(x + x^2 + x^3 + \ldots + x^n) - n}{x - 1}$.

SOLUTION We have,

$$\lim_{x \to 1} \frac{(x + x^2 + x^3 + \ldots + x^n) - n}{x - 1} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{(x-1) + (x^2 - 1) + (x^3 - 1) + \ldots + (x^n - 1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{x-1}{x-1} + \lim_{x \to 1} \frac{x^2 - 1^2}{x-1} + \lim_{x \to 1} \frac{x^3 - 1}{x-1} + \ldots + \lim_{x \to 1} \frac{x^n - 1^n}{x-1}$$

$$= 1 + 2(1)^{2-1} + 3(1)^{3-1} + \ldots + n(1)^{n-1} = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

EXERCISE 29.5

LEVEL-1

Evaluate the following limits:

1. $\lim_{x \to a} \frac{(x+2)^{5/2} - (a+2)^{5/2}}{x - a}$

2. $\lim_{x \to a} \frac{(x+2)^{3/2} - (a+2)^{3/2}}{x - a}$

3. $\lim_{x \to 0} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

4. $\lim_{x \to a} \frac{x^{2/7} - a^{2/7}}{x - a}$

5. $\lim_{x \to a} \frac{x^{5/7} - a^{5/7}}{x^{2/7} - a^{2/7}}$

6. $\lim_{x \to -1/2} \frac{8x^3 + 1}{2x + 1}$

7. $\lim_{x \to 27} \frac{(x^{1/3} + 3)(x^{1/3} - 3)}{x - 27}$

8. $\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16}$

9. $\lim_{x \to 1} \frac{x^{15} - 1}{x^{10} - 1}$

10. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

11. $\lim_{x \to a} \frac{x^{2/3} - a^{2/3}}{x^{3/4} - a^{3/4}}$

12. If $\lim_{x \to 3} \frac{x^n - 3^n}{x - 3} = 108$, find the value of n.

13. If $\lim_{x \to a} \frac{x^9 - a^9}{x - a} = 9$, find all possible values of a.

14. If $\lim_{x \to a} \frac{x^5 - a^5}{x - a} = 405$, find all possible values of a.

15. If $\lim_{x \to a} \frac{x^9 - a^9}{x - a} = \lim_{x \to 5} (4 + x)$, find all possible values of a.

16. If $\lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to 1} \frac{x^4 - 1}{x - 1}$, find all possible values of a.

ANSWERS

1. $\frac{5}{2}(a+2)^{3/2}$ 2. $\frac{3}{2}(a+2)^{1/2}$ 3. 3 4. $\frac{2}{7}a^{-5/7}$ 5. $\frac{5}{2}a^{3/7}$

6. 3 7. $\frac{2}{9}$ 8. 6 9. $\frac{3}{2}$ 10. 3 11. $\frac{8}{9}a^{-1/12}$ 12. 4

13. 1, –1 14. $a = 3, -3$ 15. 1, –1 16. $\pm\frac{2}{\sqrt{3}}$

HINTS TO SELECTED PROBLEM

9. $\lim\limits_{x \to 1} \dfrac{x^{15}-1}{x^{10}-1} = \lim\limits_{x \to 1} \dfrac{\dfrac{x^{15}-1^{15}}{x-1}}{\dfrac{x^{10}-1^{10}}{x-1}} = \dfrac{15\,(1)^{15-1}}{10\,(1)^{10-1}} = \dfrac{3}{2}$

29.6.5 METHOD OF EVALUATION OF ALGEBRAIC LIMITS AT INFINITY

Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Graphs of these functions are shown in Figures 29.6 and 29.7.

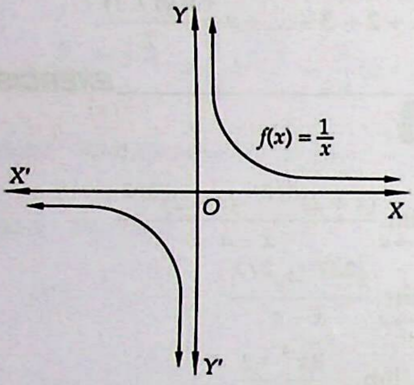

Fig. 29.6 Graph of $f(x) = \dfrac{1}{x}$

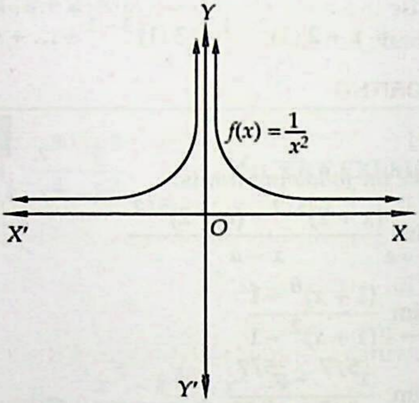

Fig. 29.7 Graph of $f(x) = \dfrac{1}{x^2}$

We observe from the graphs that as x increases, the values of $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ decrease rapidly and when x is indefinitely large $\frac{1}{x}$ and $\frac{1}{x^2}$ are indefinitely small i.e. very close to zero. In such cases, we write

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to +\infty} \frac{1}{x^2} = 0.$$

We also observe from the graphs of these two functions that as x decreases and is very small negative real number, then also the values of $\frac{1}{x}$ and $\frac{1}{x^2}$ approach to zero. So, we write

$$\lim_{x \to -\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^2} = 0.$$

It follows from the above discussion that:

(i) $\lim\limits_{x \to \infty} c = c$ (ii) $\lim\limits_{x \to -\infty} c = c$

(iii) $\lim\limits_{x \to \infty} \dfrac{c}{x^n} = 0, n > 0$ (iv) $\lim\limits_{x \to -\infty} \dfrac{c}{x^n} = 0, n \in N$

From the graphs of real functions, we obtain the following useful results:

(i) $\lim\limits_{x \to +\infty} x \to +\infty$ (ii) $\lim\limits_{x \to -\infty} x \to -\infty$

(iii) $\lim\limits_{x \to +\infty} x^2 \to +\infty$ (iv) $\lim\limits_{x \to -\infty} x^2 \to +\infty$ and so on.

(v) $\lim_{x \to \infty} e^x \to \infty$ or, $\lim_{x \to -\infty} e^{-x} \to \infty$

(vi) $\lim_{x \to \infty} e^{-x} \to 0$ or, $\lim_{x \to -\infty} e^x \to 0$

(vii) $\lim_{x \to \infty} a^x \to 0$, if $|a| < 1$ (viii) $\lim_{x \to \infty} a^x \to \infty$, if $a > 1$

(ix) $\lim_{x \to 0^+} \log_a x \to -\infty$ and $\lim_{x \to \infty} \log_a x \to \infty$, where $a > 1$

(x) $\lim_{x \to 0^+} \log_a x \to \infty$ and $\lim_{x \to \infty} \log_a x \to -\infty$, if $0 < a < 1$.

We use these results to evaluate limits at infinity. Following algorithm may be used to evaluate algebraic limits at infinity.

ALGORITHM

STEP I *Write down the given expression in the form of a rational function. i.e.* $\frac{f(x)}{g(x)}$, *if it is not so.*

STEP II *If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by* x^k.

STEP III *Use the results* $\lim_{x \to \infty} \frac{c}{x^n} = 0$ and $\lim_{x \to \infty} c = c$, where $n > 0$.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate:* $\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f}$.

SOLUTION Here the expression assumes the form $\frac{\infty}{\infty}$. We notice that the highest power of x in both the numerator and denominator is 2. So we divide each term in both the numerator and denominator by x^2.

$$\therefore \quad \lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x \to \infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a + 0 + 0}{d + 0 + 0} = \frac{a}{d}.$$

EXAMPLE 2 *Evaluate:* $\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}}$.

SOLUTION We have,

$$\lim_{x \to \infty} \frac{5x - 6}{\sqrt{4x^2 + 9}} = \lim_{x \to \infty} \frac{5 - 6/x}{\sqrt{4 + 9/x^2}} \quad \text{[Dividing each term in } N^r \text{ and } D^r \text{ by } x]$$

$$= \frac{5 - 0}{\sqrt{4 + 0}} = \frac{5}{2}$$

EXAMPLE 3 *Evaluate:* $\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} + \sqrt{2x^2 - 1}}{4x + 3}$

SOLUTION Dividing each term in the numerator and denominator by x, we get

$$\lim_{x \to \infty} \frac{\sqrt{3x^2 - 1} + \sqrt{2x^2 - 1}}{4x + 3} = \lim_{x \to \infty} \frac{\sqrt{3 - 1/x^2} + \sqrt{2 - 1/x^2}}{4 + 3/x} = \frac{\sqrt{3} + \sqrt{2}}{4}$$

EXAMPLE 4 *Evaluate:* $\lim\limits_{x \to \infty} \sqrt{x}\,(\sqrt{x + c} - \sqrt{x})$.

SOLUTION The given expression is of the form $\infty - \infty$. So we first write it in the rational form $\frac{f(x)}{g(x)}$. So that it reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

$$\therefore \lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x + c} - \sqrt{x}\right\} = \lim_{x \to \infty} \frac{\sqrt{x}\left\{\sqrt{x + c} - \sqrt{x}\right\}\left\{\sqrt{x + c} + \sqrt{x}\right\}}{\left\{\sqrt{x + c} + \sqrt{x}\right\}}$$

$$= \lim_{x \to \infty} \frac{\sqrt{x}\,(x + c - x)}{\sqrt{x + c} + \sqrt{x}}$$

$$= \lim_{x \to \infty} \frac{c\sqrt{x}}{\sqrt{x + c} + \sqrt{x}} \qquad \left(\text{form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{c}{\sqrt{1 + \frac{c}{x}} + 1} \qquad \text{[Diving } N^r \text{ and } D^r \text{ by } \sqrt{x}\text{]}$$

$$= \frac{c}{\sqrt{1 + 0} + 1} = \frac{c}{2}$$

EXAMPLE 5 *Evaluate:* $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}\right)$

SOLUTION Here the expression assumes the form $\infty - \infty$ as $x \to \infty$. So, we first reduce it to the rational form $\frac{f(x)}{g(x)}$.

$$\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 + 1} = \lim_{x \to \infty} \frac{\left\{\sqrt{x^2 + x + 1} - \sqrt{x^2 + 1}\right\}}{\left\{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}\right\}} \left\{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}\right\}$$

$$= \lim_{x \to \infty} \frac{x^2 + x + 1 - x^2 - 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}}$$

$$= \lim_{x \to \infty} \frac{x}{\sqrt{x^2 + x + 1} + \sqrt{x^2 + 1}}$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 + \frac{1}{x^2}}} \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } x\text{]}$$

$$= \frac{1}{1 + 1} = \frac{1}{2}$$

EXAMPLE 6 *Evaluate:* $\lim\limits_{n \to \infty} \frac{1 + 2 + 3 + \ldots + n}{n^2}$.

SOLUTION We have,

$$\lim_{n\to\infty} \frac{1+2+3+\ldots+n}{n^2} = \lim_{n\to\infty} \frac{1}{n^2}\times\frac{n(n+1)}{2} \qquad \left[\because 1+2+\ldots+n=\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

EXAMPLE 7 *Evaluate:* $\lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!-n!}$

SOLUTION We have,

$$\lim_{n\to\infty} \frac{n!}{(n+1)!-n!} = \lim_{n\to\infty} \frac{n!}{(n+1)\,n!-n!} = \lim_{n\to\infty} \frac{1}{n+1-1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

EXAMPLE 8 *Let* $f(x)=\dfrac{ax+b}{x+1}$, $\lim\limits_{x\to 0} f(x)=2$ *and* $\lim\limits_{x\to\infty} f(x)=1$ *prove that* $f(-2)=0$.

SOLUTION We have,

$$\lim_{x\to 0} f(x) = 2 \Rightarrow \lim_{x\to 0} \frac{ax+b}{x+1}=2 \Rightarrow \frac{b}{1}=2 \Rightarrow b=2$$

It is also given that

$$\lim_{x\to\infty} f(x)=1 \Rightarrow \lim_{x\to\infty} \frac{ax+b}{x+1}=1 \Rightarrow \lim_{x\to\infty} \frac{a+\frac{b}{x}}{1+\frac{1}{x}}=1 \Rightarrow \frac{a+0}{1+0}=1 \Rightarrow a=1.$$

Substituting the values of a and b in $f(x)=\dfrac{ax+b}{x+1}$, we obtain

$$f(x) = \frac{x+2}{x+1} \Rightarrow f(-2) = \frac{-2+2}{-2+1} = 0.$$

LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim\limits_{x\to-\infty} \left(\sqrt{x^2-x+1}+x\right)$.

SOLUTION We have,

$$\lim_{x\to-\infty} \left(\sqrt{x^2-x+1}+x\right) = \lim_{y\to\infty} \left(\sqrt{y^2+y+1}-y\right), \text{ where } y=-x$$

$$= \lim_{y\to\infty} \frac{\left\{\sqrt{y^2+y+1}-y\right\}\left\{\sqrt{y^2+y+1}+y\right\}}{\left\{\sqrt{y^2+y+1}+y\right\}}$$

$$= \lim_{y\to\infty} \frac{y^2+y+1-y^2}{\sqrt{y^2+y+1}+y}$$

$$= \lim_{y\to\infty} \frac{y+1}{\sqrt{y^2+y+1}+y} = \lim_{y\to\infty} \frac{1+\frac{1}{y}}{\sqrt{1+\frac{1}{y}+\frac{1}{y^2}}+1} = \frac{1}{2}$$

<u>ALITER</u> We have,

$$\lim_{x\to-\infty} \left(\sqrt{x^2-x+1}+x\right) = \lim_{x\to-\infty} \frac{\left\{\sqrt{x^2-x+1}+x\right\}\left\{\sqrt{x^2-x+1}-x\right\}}{\left\{\sqrt{x^2-x+1}-x\right\}}$$

$$= \lim_{x \to -\infty} \frac{x^2 - x + 1 - x^2}{\left\{\sqrt{x^2 - x + 1} - x\right\}}$$

$$= \lim_{x \to -\infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} - x}$$

$$= \lim_{x \to -\infty} \frac{-\frac{x}{|x|} + \frac{1}{|x|}}{\frac{\sqrt{x^2 - x + 1}}{|x|} - \frac{x}{|x|}} \quad \left[\text{Dividing } N^r \text{ and } D^r \text{ by } |x|\right]$$

$$= \lim_{x \to -\infty} \frac{\frac{-x}{-x} - \frac{1}{x}}{\sqrt{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}} + \frac{x}{x}} \quad \left[\because \sqrt{x^2} = |x| = -x \text{ for } x < 0\right]$$

$$= \lim_{x \to -\infty} \frac{1 - \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + 1} = \frac{1}{2}$$

EXAMPLE 10 *Evaluate:* $\lim\limits_{x \to \infty} \dfrac{(x+1)^{10} + (x+2)^{10} + ... + (x+100)^{10}}{x^{10} + 10^{10}}$

SOLUTION We have,

$$\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + ... + (x+100)^{10}}{x^{10} + 10^{10}}$$

$$= \lim_{x \to \infty} \frac{\left(1 + \frac{1}{x}\right)^{10} + \left(1 + \frac{2}{x}\right)^{10} + \ldots + \left(1 + \frac{100}{x}\right)^{10}}{1 + \left(\frac{10}{x}\right)^{10}}$$

$$= \frac{1 + 1 + \ldots + 1 \text{ (100 - times)}}{1 + 0} = \frac{100}{1} = 100$$

EXERCISE 29.6

LEVEL-1

Evaluate the following limits:

1. $\lim\limits_{x \to \infty} \dfrac{(3x-1)(4x-2)}{(x+8)(x-1)}$

2. $\lim\limits_{x \to \infty} \dfrac{3x^3 - 4x^2 + 6x - 1}{2x^3 + x^2 - 5x + 7}$

3. $\lim\limits_{x \to \infty} \dfrac{5x^3 - 6}{\sqrt{9 + 4x^6}}$

4. $\lim\limits_{x \to \infty} \sqrt{x^2 + cx} - x$

5. $\lim\limits_{x \to \infty} \sqrt{x+1} - \sqrt{x}$

6. $\lim\limits_{x \to \infty} \sqrt{x^2 + 7x} - x$

7. $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{4x^2 + 1} - 1}$

8. $\lim\limits_{n \to \infty} \dfrac{n^2}{1 + 2 + 3 + \ldots + n}$

9. $\lim_{x \to \infty} \dfrac{3x^{-1} + 4x^{-2}}{5x^{-1} + 6x^{-2}}$

10. $\lim_{x \to \infty} \dfrac{\sqrt{x^2+a^2} - \sqrt{x^2+b^2}}{\sqrt{x^2+c^2} - \sqrt{x^2+d^2}}$

11. $\lim_{n \to \infty} \dfrac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$

12. $\lim_{x \to \infty} x\left\{\sqrt{x^2+1} - \sqrt{x^2-1}\right\}$

13. $\lim_{x \to \infty} \left\{\sqrt{x+1} - \sqrt{x}\right\} \sqrt{x+2}$

14. $\lim_{n \to \infty} \dfrac{1^2 + 2^2 + \ldots + n^2}{n^3}$

15. $\lim_{n \to \infty} \left(\dfrac{1}{n^2} + \dfrac{2}{n^2} + \dfrac{3}{n^2} + \ldots + \dfrac{n-1}{n^2}\right)$

16. $\lim_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{n^4}$

17. $\lim_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{(n-1)^4}$

18. $\lim_{x \to \infty} \sqrt{x}\left\{\sqrt{x+1} - \sqrt{x}\right\}$

19. $\lim_{n \to \infty} \left(\dfrac{1}{3} + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \ldots + \dfrac{1}{3^n}\right)$

20. $\lim_{x \to \infty} \dfrac{x^4 + 7x^3 + 46x + a}{x^4 + 6}$, where a is a non-zero real number.

21. $f(x) = \dfrac{ax^2 + b}{x^2 + 1}$, $\lim_{x \to 0} f(x) = 1$ and $\lim_{x \to \infty} f(x) = 1$, then prove that $f(-2) = f(2) = 1$.

22. Show that $\lim_{x \to \infty} (\sqrt{x^2 + x + 1} - x) \neq \lim_{x \to \infty} (\sqrt{x^2 + 1} - x)$

LEVEL-2

23. $\lim_{x \to -\infty} \left(\sqrt{4x^2 - 7x} + 2x\right)$

24. $\lim_{x \to -\infty} \left(\sqrt{x^2 - 8x} + x\right)$

25. Evaluate : $\lim_{n \to \infty} \dfrac{1^4 + 2^4 + 3^4 + \ldots + n^4}{n^5} - \lim_{n \to \infty} \dfrac{1^3 + 2^3 + \ldots + n^3}{n^5}$

26. Evaluate : $\lim_{n \to \infty} \dfrac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)}{n^3}$

ANSWERS

1. 12 2. $\dfrac{3}{2}$ 3. $\dfrac{5}{2}$ 4. $\dfrac{c}{2}$ 5. 0 6. $\dfrac{7}{2}$ 7. $\dfrac{1}{2}$ 8. 2 9. $\dfrac{3}{5}$

10. $\dfrac{a^2 - b^2}{c^2 - d^2}$ 11. 1 12. 1 13. $\dfrac{1}{2}$ 14. $\dfrac{1}{3}$ 15. $\dfrac{1}{2}$ 16. $\dfrac{1}{4}$ 17. $\dfrac{1}{4}$

18. $\dfrac{1}{2}$ 19. $\dfrac{1}{2}$ 20. 1 23. $\dfrac{7}{4}$ 24. 4 25. $\dfrac{1}{5}$ 26. $\dfrac{1}{3}$

29.7 EVALUATION OF TRIGONOMETRIC LIMITS

In this section, we will be studying various methods of evaluating trigonometric limits. In order to evaluate trigonometric limits we will be using the following results which are stated and proved in the following theorem.

THEOREM *If angle θ is measured in radians, then*

(i) $\lim_{\theta \to 0} \sin \theta = 0$ and $\lim_{\theta \to 0} \cos \theta = 1$

(ii) $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, *where* θ *is measured in radians*

(iii) $\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$

(iv) $\lim_{\theta \to a} \frac{\sin (\theta - a)}{\theta - a} = 1$

(v) $\lim_{\theta \to a} \frac{\tan (\theta - a)}{\theta - a} = 1$

PROOF (i) Let ABC be a right angled triangle such that $\angle C = \frac{\pi}{2}$ and $\angle ABC = \theta$. Then,

$$\sin \theta = \frac{CA}{BA} \text{ and, } \cos \theta = \frac{BC}{BA}.$$

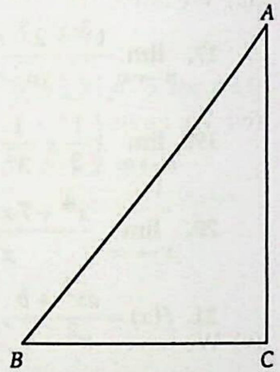

Fig. 29.8

Now, if we keep BC fixed and go on decreasing angle θ, then we find that A goes on coming nearer and nearer to C.

$\therefore \quad A \to C$ as $\theta \to 0$

This means that $CA \to 0$ and $BA \to BC$ as $\theta \to 0$.

$\Rightarrow \quad \frac{CA}{BA} \to 0$ and $\frac{BC}{BA} \to 1$ as $\theta \to 0$

$\Rightarrow \quad \sin \theta \to 0$ and $\cos \theta \to 1$ as $\theta \to 0$

(ii) Consider a circle of radius r. Let O be the centre of the circle such that $\angle AOB = \theta$ where θ is measured in radians and it is very small. Suppose the tangent at A meets OB produced at P.

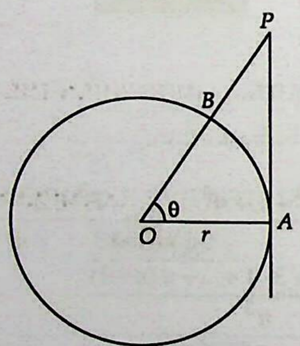

Fig. 29.9

From the figure 29.9, we have

Area of $\Delta\, OAB <$ Area of sector $OAB <$ Area of $\Delta\, OAP$

$\Rightarrow \quad \frac{1}{2} OA \cdot OB \sin \theta < \frac{1}{2} (OA)^2\, \theta < \frac{1}{2} OA \cdot AP$

$\Rightarrow \quad \frac{1}{2} r^2 \sin \theta < \frac{1}{2} r^2\, \theta < \frac{1}{2} r^2 \tan \theta$ [In $\Delta\, OAP$, $AP = OA \tan \theta$]

$\Rightarrow \quad \sin \theta < \theta < \tan \theta$

$\Rightarrow \quad 1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$ [$\because$ θ is small $\sin \theta > 0$]