CHAPTER 11

HIGHER ORDER DERIVATIVES

11.1 DEFINITION AND NOTATIONS

If $y = f(x)$, then $\frac{dy}{dx}$, the derivative of y with respect to x, is itself, in general, a function of x and can be differentiated again. To fix up the idea, we shall call $\frac{dy}{dx}$ as the first order derivative of y with respect to x and the derivative of $\frac{dy}{dx}$ with respect to x as the second order derivative of y with respect to x and will be denoted by $\frac{d^2y}{dx^2}$. Similarly the derivative of $\frac{d^2y}{dx^2}$ with respect to x will be termed as the third order derivative of y with respect to x and will be denoted by $\frac{d^3y}{dx^3}$ and so on. The n^{th} order derivative of y with respect to x will be denoted by $\frac{d^n y}{dx^n}$.

If $y = f(x)$, then the other alternative notations for

$$\begin{array}{lllll} \frac{dy}{dx}, & \frac{d^2y}{dx^2}, & \frac{d^3y}{dx^3}, & \ldots, & \frac{d^ny}{dx^n} \text{ are} \\ y_1, & y_2, & y_3, & \ldots, & y_n \\ y', & y'', & y''', & \ldots, & y^{(n)} \\ Dy, & D^2 y, & D^3 y, & \ldots, & D^n y \\ f'(x), & f''(x), & f'''(x), & \ldots, & f^n(x) \end{array}$$

The values of these derivatives at $x = a$ are denoted by $y_n(a)$, $y^n(a)$, $D^n y(a)$, $f^n(a)$ or, $\left(\frac{d^ny}{dx^n}\right)_{x=a}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON PROVING RELATIONS INVOLVING VARIOUS ORDER DERIVATIVES OF CARTESIAN FUNCTIONS**

EXAMPLE 1 *If $y = \sin^{-1} x$, show that* $\frac{d^2 y}{dx^2} = \frac{x}{(1-x^2)^{3/2}}$.

SOLUTION We have, $y = \sin^{-1} x$. On differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

On differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{d}{dx}\{(1-x^2)^{-1/2}\} = -\frac{1}{2}(1-x^2)^{-3/2} \times \frac{d}{dx}(1-x^2)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{2(1-x^2)^{3/2}}(-2x) = \frac{x}{(1-x^2)^{3/2}}.$$

EXAMPLE 2 *If* $y = \tan x + \sec x$, *prove that* $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$. **[NCERT EXEMPLAR]**

SOLUTION We have, $y = \tan x + \sec x$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x + \sec x \tan x = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x} = \frac{1+\sin x}{\cos^2 x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\sin x}{1-\sin^2 x} = \frac{1}{1-\sin x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1}{1-\sin x}\right\} = \frac{d}{dx}\{(1-\sin x)^{-1}\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (-1)(1-\sin x)^{-2}\frac{d}{dx}(1-\sin x) = \frac{-1}{(1-\sin x)^2}(-\cos x) = \frac{\cos x}{(1-\sin x)^2}.$$

EXAMPLE 3 *If* $y = \tan x$, *prove that* $y_2 = 2yy_1$.

SOLUTION We have, $y = \tan x$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x \text{ or, } y_1 = \sec^2 x \qquad \left[\because y_1 = \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(y_1) = \frac{d}{dx}(\sec^2 x)$$

$$\Rightarrow \quad y_2 = 2\sec x \frac{d}{dx}(\sec x) = 2\sec x \sec x \tan x = 2\tan x \sec^2 x$$

$$\Rightarrow \quad y_2 = 2yy_1 \qquad [\because y = \tan x \text{ and } y_1 = \sec^2 x]$$

EXAMPLE 4 *If* $y = x^x$, *find* $\dfrac{d^2y}{dx^2}$.

SOLUTION We have, $y = x^x$

$$\therefore \quad \log y = x \log x$$

Differentiating with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 \times \log x + x \times \frac{1}{x} \Rightarrow \frac{dy}{dx} = y(1+\log x) \qquad ...(i)$$

Differentiating both sides of (i) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}(1+\log x) + y\frac{d}{dx}(1+\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}(1+\log x) + y \times \frac{1}{x} = y(1+\log x)^2 + \frac{y}{x} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^x\left\{(1+\log x)^2 + \frac{1}{x}\right\}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *If $y = A\cos nx + B\sin nx$, show that $\frac{d^2y}{dx^2} + n^2 y = 0$.* **[CBSE 2001C]**

SOLUTION We have, $y = A\cos nx + B\sin nx$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = -An\sin nx + Bn\cos nx$$

On differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = -An^2\cos nx - Bn^2\sin nx = -n^2(A\cos nx + B\sin nx) = -n^2 y$$

$$\therefore \quad \frac{d^2y}{dx^2} + n^2 y = 0.$$

EXAMPLE 6 *If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$.*

[NCERT, CBSE 2007, 2014]

SOLUTION We have, $y = Ae^{mx} + Be^{nx}$

$$\therefore \quad \frac{dy}{dx} = Ame^{mx} + Bne^{nx} \qquad \text{[Differentiating with respect to } x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = Am^2e^{mx} + Bn^2e^{nx}$$

$$\therefore \quad \frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = (Am^2e^{mx} + Bn^2e^{nx}) - (m+n)(Ame^{mx} + Bne^{nx}) + mn(Ae^{mx} + Be^{nx}) = 0.$$

EXAMPLE 7 *If $y = A\cos(\log x) + B\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$.*

[CBSE 2007, 2009]

SOLUTION We have, $y = A\cos(\log x) + B\sin(\log x)$.

On differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{1}{x}A\sin(\log x) + \frac{B}{x}\cos(\log x) \Rightarrow x\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x)$$

On differentiating again with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -A\frac{\cos(\log x)}{x} - B\frac{\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\{A\cos(\log x) + B\sin(\log x)\}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

EXAMPLE 8 *If $y = \log\left\{x + \sqrt{x^2 + a^2}\right\}$, prove that: $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.* **[CBSE 2013]**

SOLUTION We have, $y = \log\left\{x + \sqrt{x^2 + a^2}\right\}$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 + a^2}} \times \frac{d}{dx}\left\{x + \sqrt{x^2 + a^2}\right\} = \frac{1}{x + \sqrt{x^2 + a^2}}\left\{1 + \frac{2x}{2\sqrt{x^2 + a^2}}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \times \left\{ \frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}} \right\}$$

$$\Rightarrow \quad y_1 = \frac{1}{\sqrt{x^2+a^2}}, \text{ where } y_1 = \frac{dy}{dx} \Rightarrow y_1^2 \ (x^2+a^2) = 1$$

Differentiating with respect to x, we get

$$y_1^2 \ \frac{d}{dx}(x^2+a^2) + (x^2+a^2)\frac{d}{dx}(y_1^2) = 0$$

$$\Rightarrow y_1^2 (2x) + (x^2+a^2) \times 2\, y_1\, y_2 = 0 \quad \left[\because \frac{d}{dx}(y_1^2) = 2\,(y_1)^{2-1}\frac{d}{dx}(y_1) = 2\, y_1 \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\, y_1\, y_2\right]$$

$$\Rightarrow 2\, y_1 \left\{ y_2\,(x^2+a^2) + xy_1 \right\} = 0 \Rightarrow y_2\,(x^2+a^2) + xy_1 = 0 \quad [\because y_1 \neq 0]$$

EXAMPLE 9 *If $y = \sin^{-1} x$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$* **[CBSE 2012, NCERT]**

SOLUTION We have, $y = \sin^{-1} x$

Differentiating with respect to x, we obtain

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,\frac{dy}{dx} = 1$$

Differentiating both sides with respect to x, we get

$$\sqrt{1-x^2}\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0 \quad \left[\text{Multiplying both sides by } \sqrt{1-x^2}\right]$$

ALITER We have, $y = \sin^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = 1$

Differentiating both sides with respect to x, we get

$$(1-x^2)\left\{2\frac{dy}{dx} \times \frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} - 2x\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow \quad 2\,(1-x^2)\frac{dy}{dx}\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0.$$

EXAMPLE 10 *If $y = e^{m \sin^{-1} x}$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2 y = 0.$* **[CBSE 2015]**

SOLUTION We have, $y = e^{m\sin^{-1} x}$.

Differentiating with respect to x, we obtain

$$\frac{dy}{dx} = e^{m\sin^{-1}x} \times \frac{m}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{my}{\sqrt{1-x^2}} \quad [\because e^{m\sin^{-1}x} = y]$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{m^2y^2}{1-x^2} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2 \Rightarrow (1-x^2)\ y_1^2 = m^2y^2, \text{ where } y_1 = \frac{dy}{dx}$$

Differentiating with respect to x, we obtain

$$(1-x^2)\frac{d}{dx}(y_1^2)+(y_1^2)\frac{d}{dx}(1-x^2)=m^2\frac{d}{dx}(y^2)$$

$$\Rightarrow \quad (1-x^2)\,2y_1y_2+y_1^2(-2x)=m^2\,(2yy_1) \qquad \left[\because \frac{d}{dx}(y_1^2)=2y_1y_2 \text{ and } \frac{d}{dx}(y^2)=2yy_1\right]$$

$$\Rightarrow \quad 2y_1\left\{(1-x^2)\,y_2-xy_1-m^2y\right\}=0 \Rightarrow (1-x^2)\,y_2-xy_1-m^2y=0 \qquad [\because y_1\neq 0]$$

EXAMPLE 11 *If* $y=\left\{x+\sqrt{x^2+1}\right\}^m$, *show that* $(x^2+1)\,y_2+xy_1-m^2y=0.$ **[CBSE 2013, 2015]**

SOLUTION We have, $y=\left\{x+\sqrt{x^2+1}\right\}^m$. Differentiating with respect to x, we get

$$\frac{dy}{dx}=m\left\{x+\sqrt{x^2+1}\right\}^{m-1}\times\frac{d}{dx}\left\{x+\sqrt{x^2+1}\right\}$$

$$\Rightarrow \quad \frac{dy}{dx}=m\left\{[x+\sqrt{x^2+1}\right\}^{m-1}\times\left\{1+\frac{2x}{2\sqrt{x^2+1}}\right\}=\frac{m\left\{\sqrt{x^2+1}+x\right\}^m}{\sqrt{x^2+1}}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{my}{\sqrt{x^2+1}} \Rightarrow y_1=\frac{my}{\sqrt{x^2+1}} \Rightarrow y_1\sqrt{x^2+1}=my$$

$$\Rightarrow \quad y_1^2\,(x^2+1)=m^2y^2 \qquad \text{[Squaring both sides]}$$

Differentiating with respect to x, we get

$$2y_1\,y_2\,(1+x^2)+y_1^2\,(2x)=2m^2\,yy_1 \Rightarrow y_2\,(1+x^2)+xy_1-m^2\,y=0$$

EXAMPLE 12 *If* $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, *show that* $(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0.$ **[CBSE 2013]**

SOLUTION We have, $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}} \Rightarrow y\sqrt{1-x^2}=\sin^{-1}x$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx}\sqrt{1-x^2}-\frac{x}{\sqrt{1-x^2}}\,y=\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx}(1-x^2)-xy=1$$

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2}(1-x^2)-2x\frac{dy}{dx}-x\frac{dy}{dx}-y=0 \Rightarrow (1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$$

EXAMPLE 13 *If* $x=\tan\left(\frac{1}{a}\log y\right)$, *show that* $(1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx}=0.$ **[CBSE 2011, 2013]**

SOLUTION We have,

$$x=\tan\left(\frac{1}{a}\log y\right) \Rightarrow \tan^{-1}x=\frac{1}{a}\log y \Rightarrow a\tan^{-1}x=\log y$$

Differentiating with respect to x, we get:

$$\frac{a}{1+x^2}=\frac{1}{y}\frac{dy}{dx} \Rightarrow (1+x^2)\frac{dy}{dx}=ay$$

Differentiating with respect to x

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=a\frac{dy}{dx} \Rightarrow (1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx}=0$$

Type II **ON FINDING SECOND ORDER DERIVATIVES OF PARAMETRIC FUNCTIONS**

EXAMPLE 14 *Find* $\frac{d^2y}{dx^2}$, *if* $x=at^2$, $y=2at$.

SOLUTION We have, $x=at^2$ and $y=2at$

$$\Rightarrow \quad \frac{dx}{dt}=2at \text{ and } \frac{dy}{dt}=2a \qquad \text{...(i)}$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{2a}{2at}=\frac{1}{t}$$

Differentiating both sides with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{t}\right)=-\frac{1}{t^2}\frac{dt}{dx}=-\frac{1}{t^2}\times\frac{1}{2at}=-\frac{1}{2at^3} \qquad \left[\text{From (i)}, \frac{dx}{dt}=2at \therefore \frac{dt}{dx}=\frac{1}{2at}\right]$$

EXAMPLE 15 *If* $x=a\cos^3\theta$, $y=a\sin^3\theta$, *find* $\frac{d^2y}{dx^2}$. *Also, find its value at* $\theta=\frac{\pi}{6}$. **[CBSE 2013]**

SOLUTION We have, $x=a\cos^3\theta$ and $y=a\sin^3\theta$

$$\therefore \quad \frac{dx}{d\theta}=-3a\cos^2\theta\sin\theta \text{ and } \frac{dy}{d\theta}=3a\sin^2\theta\cos\theta$$

$$\text{So,} \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$$

Differentiating both sides with respect to x, we obtain,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-\tan\theta)=-\sec^2\theta\frac{d\theta}{dx}=-\sec^2\theta\times\frac{1}{-3a\cos^2\theta\sin\theta}=\frac{1}{3a}\sec^4\theta\operatorname{cosec}\theta$$

$$\therefore \quad \left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{6}}=\frac{1}{3a}\sec^4\frac{\pi}{6}\operatorname{cosec}\frac{\pi}{6}=\frac{1}{3a}\times\left(\frac{2}{\sqrt{3}}\right)^4\times 2=\frac{32}{27a}$$

EXAMPLE 16 *If* $x=a\sin t$ *and* $y=a\left(\cos t+\log\tan\frac{t}{2}\right)$, *find* $\frac{d^2y}{dx^2}$. **[CBSE 2013]**

SOLUTION We have,

$$x=a\sin t \text{ and } y=a\left(\cos t+\log\tan\frac{t}{2}\right)$$

$$\Rightarrow \quad \frac{dx}{dt}=a\cos t \text{ and } \frac{dy}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{t}{2}}\times\sec^2\frac{t}{2}\times\frac{1}{2}\right)$$

$$\Rightarrow \quad \frac{dx}{dt}=a\cos t \text{ and } \frac{dy}{dt}=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$\Rightarrow \quad \frac{dx}{dt}=a\cos t \text{ and } \frac{dy}{dt}=\frac{a(1-\sin^2 t)}{\sin t} \Rightarrow \frac{dx}{dt}=a\cos t \text{ and } \frac{dy}{dt}=\frac{a\cos^2 t}{\sin t}$$

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\frac{a\cos^2 t}{\sin t}}{a\cos t}=\frac{\cos t}{\sin t}=\cot t$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\cot t) = -\operatorname{cosec}^2 t \frac{dt}{dx} = -\operatorname{cosec}^2 t \times \frac{1}{a\cos t} = -\frac{1}{a\sin^2 t \cos t}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 17 *If $y = \tan^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$y = \tan^{-1} x \Rightarrow x = \tan y$$

Differentiating with respect to y, we obtain

$$\frac{dx}{dy} = \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y \qquad \left[\because \frac{dy}{dx} = \frac{1}{dx/dy}\right]$$

Differentiating both sides with respect to x, we obtain

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\cos^2 y) = -2\cos y \sin y \frac{dy}{dx} = -2\cos y \sin y \times \cos^2 y \qquad \left[\because \frac{dy}{dx} = \cos^2 y\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -2\sin y \cos^3 y$$

EXAMPLE 18 *If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$x^m y^n = (x+y)^{m+n} \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$ [See Example 15 on page 10.75]

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\frac{dy}{dx} - y \times 1}{x^2} = \frac{x\left(\frac{y}{x}\right) - y}{x^2} = 0 \qquad \left[\text{Using: } \frac{dy}{dx} = \frac{y}{x}\right]$$

EXAMPLE 19 *If $y^3 - y = 2x$, prove that $\frac{d^2y}{dx^2} = -\frac{24y}{(3y^2-1)^3}$.*

SOLUTION We have, $y^3 - y = 2x$

Differentiating both sides with respect to y, we obtain

$$(3y^2 - 1) = 2\frac{dx}{dy} \Rightarrow \frac{dy}{dx} = \frac{2}{(3y^2-1)}$$

Differentiating both sides with respect to x, we obtain

$$\frac{d^2y}{dx^2} = -\frac{2}{(3y^2-1)^2}\frac{d}{dx}(3y^2-1) = -\frac{2}{(3y^2-1)^2} \times 6y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{12y}{(3y^2-1)^2} \times \frac{2}{(3y^2-1)} = -\frac{24y}{(3y^2-1)^3} \qquad \left[\because \frac{dy}{dx} = \frac{2}{3y^2-1}\right]$$

EXAMPLE 20 *If $e^y(x+1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.* **[NCERT]**

SOLUTION We have,

$$e^y(x+1) = 1 \Rightarrow e^y = \frac{1}{x+1} \Rightarrow \log e^y = \log\left(\frac{1}{x+1}\right) \Rightarrow y = -\log(x+1)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x+1} \text{ and } \frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} \Rightarrow \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$

EXAMPLE 21 *If* $(ax+b)\, e^{y/x} = x$ *or,* $y = x \log\left(\frac{x}{a+bx}\right)$, *prove that* $x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

[CBSE 2005, 2013, 2015]

SOLUTION We have, $(ax+b)\, e^{y/x} = x$

$$\Rightarrow \quad e^{y/x} = \frac{x}{ax+b} \Rightarrow \frac{y}{x} = \log\left(\frac{x}{ax+b}\right) \Rightarrow y = x \log\left(\frac{x}{a+bx}\right)$$

$$\Rightarrow \quad y = x\{\log x - \log(a+bx)\} \Rightarrow \frac{y}{x} = \log x - \log(a+bx)$$

On differentiating with respect to x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\frac{d}{dx}(a+bx) = \frac{1}{x} - \frac{b}{a+bx}$$

$$\Rightarrow \quad x\frac{dy}{dx} - y = x^2\left\{\frac{1}{x} - \frac{b}{a+bx}\right\} \Rightarrow x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad \text{...(i)}$$

Differentiating both sides of (i) with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)\,a - ax\,(0+b)}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} \qquad \text{[Multiplying both sides by } x^2\text{]}$$

$$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2 \qquad \text{...(ii)}$$

From (i) and (ii), we obtain : $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$.

EXAMPLE 22 $y = x^x$, *prove that* $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. **[CBSE 2014, 2016]**

SOLUTION We have, $y = x^x$ or, $y = e^{\log x^x} = e^{x\log x}$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = e^{x\log x}\frac{d}{dx}(x\log x) \Rightarrow \frac{dy}{dx} = x^x(1+\log x) \Rightarrow \frac{dy}{dx} = y(1+\log x) \qquad \text{...(i)}$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = y \times \frac{d}{dx}(1+\log x) + \frac{dy}{dx} \times (1+\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = y \times \frac{1}{x} + \frac{dy}{dx} \times (1+\log x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{dy}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right) \qquad \left[\text{From (i)}, 1+\log x = \frac{1}{y}\frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \Rightarrow \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$

Type III ON PROVING RELATIONS INVOLVING VARIOUS ORDER DERIVATIVES OF PARAMETRIC FUNCTIONS

EXAMPLE 23 *If* $x = a\cos\theta + b\sin\theta$ *and* $y = a\sin\theta - b\cos\theta$, *prove that* $y^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 0.$

[CBSE 2013, 2014, 2015]

SOLUTION We have, $x = a\cos\theta + b\sin\theta$ and $y = a\sin\theta - b\cos\theta$

$$\therefore \quad x^2+y^2 = (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2$$

$$\Rightarrow \quad x^2+y^2 = a^2(\cos^2\theta+\sin^2\theta) + b^2(\sin^2\theta+\cos^2\theta) \Rightarrow x^2+y^2 = a^2+b^2$$

Differentiating with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \qquad \text{...(i)}$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = -\left\{\frac{y\times 1 - x\dfrac{dy}{dx}}{y^2}\right\} = -\left\{\frac{y - x\left(-\dfrac{x}{y}\right)}{y^2}\right\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{(x^2+y^2)}{y^3} \qquad \text{...(ii)}$$

$$\therefore \quad y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = -y^2\left(\frac{x^2+y^2}{y^3}\right) - x\left(-\frac{x}{y}\right) + y = 0 \qquad \text{[Using (i) and (ii)]}$$

EXAMPLE 24 *If* $x = \sin t$ *and* $y = \sin pt$, *prove that* $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0.$

[NCERT EXEMPLAR, CBSE 2016, 2019]

SOLUTION We have,

$$x = \sin t,\ y = \sin pt \Rightarrow \frac{dx}{dt} = \cos t \text{ and}, \frac{dy}{dt} = p\cos pt$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{p\cos pt}{\cos t} = \frac{p\sqrt{1-\sin^2 pt}}{\sqrt{1-\sin^2 t}} = \frac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{p^2(1-y^2)}{1-x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2(1-x^2) = p^2(1-y^2)$$

Differentiating with respect to x, we obtain

$$(1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2) = p^2\left(0 - 2y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(-2x) = -2p^2y\frac{dy}{dx}$$

$$\Rightarrow \quad 2\frac{dy}{dx}\left\{(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y\right\}=0 \Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0 \qquad \left[\because\ 2\frac{dy}{dx}\neq 0\right]$$

EXAMPLE 25 *If* $x=\sin\theta$, $y=\cos p\theta$, *prove that* $(1-x^2)\,y_2-xy_1+p^2\,y = 0$, *where* $y_2=\frac{d^2y}{dx^2}$ *and* $y_1=\frac{dy}{dx}$.

SOLUTION We have, $x = \sin\theta$ and $y=\cos p\,\theta$.

$$\therefore \quad \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-p\sin p\theta}{\cos\theta}=\frac{-p\sqrt{1-\cos^2 p\,\theta}}{\sqrt{1-\sin^2\theta}}=\frac{-p\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2=\frac{p^2\,(1-y^2)}{(1-x^2)} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2=p^2\,(1-y^2)$$

Differentiating both sides with respect to x, we get

$$(1-x^2)\,2\frac{dy}{dx}\times\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=p^2\left(0-2y\frac{dy}{dx}\right)\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0$$

Type IV ON PROVING RELATIONS INVOLVING VARIOUS ORDER DERIVATIVES

EXAMPLE 26 *If* $(x-a)^2+(y-b)^2 = c^2$, *prove that* $\dfrac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$ *is a constant independent of a and b.* **[NCERT]**

SOLUTION We have,

$$(x-a)^2+(y-b)^2=c^2 \qquad \text{...(i)}$$

Differentiating with respect to x, we get

$$2\,(x-a)+2\,(y-b)\frac{dy}{dx}=0\Rightarrow (x-a)+(y-b)\frac{dy}{dx}=0 \qquad \text{...(ii)}$$

Differentiating with respect to x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

$$\Rightarrow \quad (y-b)\frac{d^2y}{dx^2}=-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\Rightarrow \frac{d^2y}{dx^2}=-\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{y-b} \qquad \text{...(iii)}$$

From (ii), we obtain: $\frac{dy}{dx}=-\left(\frac{x-a}{y-b}\right)$

$$\therefore \quad 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(x-a)^2}{(y-b)^2}=\frac{(x-a)^2+(y-b)^2}{(y-b)^2}=\frac{c^2}{(y-b)^2} \qquad \text{[Using (i)]} \qquad \text{...(iv)}$$

$$\Rightarrow \quad \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}=\left\{\frac{c^2}{(y-b)^2}\right\}^{3/2}=\frac{c^3}{(y-b)^3} \qquad \text{...(v)}$$

From (iii) and (iv), we obtain

$$\frac{d^2y}{dx^2} = -\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{y-b} = -\frac{c^2/(y-b)^2}{(y-b)} = \frac{-c^2}{(y-b)^3} \qquad \text{...(vi)}$$

From (v) and (vi), we obtain

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\frac{c^3}{(y-b)^3}}{\frac{-c^2}{(y-b)^3}} = -c, \text{ which is independent of } a \text{ and } b.$$

EXAMPLE 27 *If $y^2 = a^2\cos^2 x + b^2 \sin^2 x$, then prove that $\frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3}$.*

SOLUTION We have,

$$y^2 = a^2\cos^2 x + b^2\sin^2 x$$

$$\Rightarrow \quad 2y^2 = a^2(2\cos^2 x) + b^2(2\sin^2 x) = a^2(1+\cos 2x) + b^2(1-\cos 2x)$$

$$\Rightarrow \quad 2y^2 = (a^2+b^2) + (a^2-b^2)\cos 2x \qquad \text{...(i)}$$

Differentiating with respect to x, we get

$$4y\frac{dy}{dx} = -2(a^2-b^2)\sin 2x \Rightarrow 2y\frac{dy}{dx} = -(a^2-b^2)\sin 2x \qquad \text{...(ii)}$$

From (i), we obtain

$$2y^2 - (a^2+b^2) = (a^2-b^2)\cos 2x \qquad \text{...(iii)}$$

Squaring (ii) and (iii) and adding, we get

$$4y^2\left(\frac{dy}{dx}\right)^2 + \left\{2y^2-(a^2+b^2)\right\}^2 = (a^2-b^2)^2\{\sin^2 2x + \cos^2 2x\}$$

$$\Rightarrow \quad 4y^2\left(\frac{dy}{dx}\right)^2 + 4y^4 - 4y^2(a^2+b^2) + (a^2+b^2)^2 = (a^2-b^2)^2$$

$$\Rightarrow \quad 4y^2\left\{\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2)\right\} = (a^2-b^2)^2 - (a^2+b^2)^2$$

$$\Rightarrow \quad 4y^2\left\{\left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2)\right\} = -4a^2b^2 \Rightarrow \left(\frac{dy}{dx}\right)^2 + y^2 - (a^2+b^2) = -\frac{a^2b^2}{y^2}$$

Differentiating both sides with respect to x, we get

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = \frac{2a^2b^2}{y^3}\frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} + y = \frac{a^2b^2}{y^3} \quad \left[\text{Dividing both sides by } 2\frac{dy}{dx}\right]$$

EXAMPLE 28 *If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.*

SOLUTION We have, [NCERT]

$$f(x) = |x|^3 = \begin{cases} x^3, & \text{if } x \geq 0 \\ (-x)^3 = -x^3, & \text{if } x < 0 \end{cases}$$

Now,

$$(\text{LHD of } f(x) \text{ at } x=0) = \lim_{x \to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \frac{-x^3-0}{x} = \lim_{x \to 0^-} -x^2 = 0$$

$$(\text{RHD of } f(x) \text{ at } x=0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{x^3-0}{x} = \lim_{x \to 0} x^2 = 0$$

$\therefore$ (LHD of $f(x)$ at $x=0$) = (RHD of $f(x)$ at $x=0$)

So, $f(x)$ is differentiable at $x=0$ and the derivative of $f(x)$ is given by

$$f'(x) = \begin{cases} 3x^2 \ , & \text{if } x \geq 0 \\ -3x^2 \ , & \text{if } x < 0 \end{cases}$$

Now,

$$(\text{LHD of } f'(x) \text{ at } x=0) = \lim_{x \to 0^-} \frac{f'(x)-f'(0)}{x-0} = \lim_{x \to 0^-} \frac{-3x^2-0}{x} = \lim_{x \to 0^-} -3x = 0$$

$$(\text{RHD of } f'(x) \text{ at } x=0) = \lim_{x \to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x \to 0^+} \frac{3x^2-0}{x-0} = \lim_{x \to 0^+} 3x = 0$$

$\therefore$ (LHD of $f'(x)$ at $x=0$) = (RHD of $f'(x)$ at $x=0$)

So, $f'(x)$ is differentiable at $x=0$.

Hence, $f''(x) = \begin{cases} 6x \ , & \text{if } x \geq 0 \\ -6x \ , & \text{if } x < 0 \end{cases}$.

Type V MISCELLANEOUS PROBLEMS

EXAMPLE 29 *In $\frac{dy}{dx}$, x is independent variable and y is the dependent variable. If independent and dependent variables are interchanged $\frac{dy}{dx}$ becomes $\frac{dx}{dy}$ and these two are connected by the relation $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$. Find a relation between $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$.*

SOLUTION We know that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{dx/dy}\right) = \frac{d}{dx}\left\{\left(\frac{dx}{dy}\right)^{-1}\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dy}\left\{\left(\frac{dx}{dy}\right)^{-1}\right\}\frac{dy}{dx} = \left\{-\left(\frac{dx}{dy}\right)^{-2}\frac{d}{dy}\left(\frac{dx}{dy}\right)\right\} \times \frac{1}{dx/dy}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left\{-\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\right\} \times \left(\frac{dx}{dy}\right)^{-1} = -\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}$$

Hence, $\frac{d^2y}{dx^2} = -\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}$ or, $\left(\frac{dy}{dx}\right)^3 \frac{d^2y}{dx^2} = -\frac{d^2x}{dy^2}$

EXAMPLE 30 *Find the equation to which the equation* $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$ *is transformed by interchanging the independent and dependent variables.*

SOLUTION We know that

$$\frac{dy}{dx}=\frac{1}{dx/dy} \text{ and } \frac{d^2y}{dx^2}=-\frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2} \qquad \text{[See Example 29]}$$

Substituting these values in the equation $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-\left(\frac{dy}{dx}\right)=0$, we get

$$-\frac{x}{\left(\frac{dx}{dy}\right)^3}\frac{d^2x}{dy^2}+\left(\frac{1}{\frac{dx}{dy}}\right)^2-\frac{y}{\left(\frac{dx}{dy}\right)}=0$$

$$\Rightarrow \quad -x\frac{d^2x}{dy^2}+\frac{dx}{dy}-y\left(\frac{dx}{dy}\right)^2=0 \qquad \left[\text{Multiplying both sides by } \left(\frac{dx}{dy}\right)^3\right]$$

$$\Rightarrow \quad x\frac{d^2x}{dy^2}+y\left(\frac{dx}{dy}\right)^2-\frac{dx}{dy}=0$$

EXERCISE 11.1

BASIC

1. Find the second order derivatives of each of the following functions:
 (i) $x^3+\tan x$ (ii) $\sin(\log x)$ [NCERT] (iii) $\log(\sin x)$ [NCERT]
 (iv) $e^x \sin 5x$ [NCERT] (v) $e^{6x}\cos 3x$ [NCERT] (vi) $x^3\log x$ [NCERT]
 (vii) $\tan^{-1}x$ [NCERT] (viii) $x\cos x$ [NCERT] (ix) $\log(\log x)$ [NCERT]
2. If $y=e^{-x}\cos x$, show that $\frac{d^2y}{dx^2}=2e^{-x}\sin x$.
3. If $y=x+\tan x$, show that $\cos^2 x\frac{d^2y}{dx^2}-2y+2x=0$. [CBSE 2007]
4. If $y=x^3\log x$, prove that $\frac{d^4y}{dx^4}=\frac{6}{x}$.
5. If $y=\log(\sin x)$, prove that $\frac{d^3y}{dx^3}=2\cos x\,\mathrm{cosec}^3\,x$.
6. If $y=2\sin x+3\cos x$, show that $\frac{d^2y}{dx^2}+y=0$.
7. If $y=\frac{\log x}{x}$, show that $\frac{d^2y}{dx^2}=\frac{2\log x-3}{x^3}$.
8. If $x=a\sec\theta,\ y=b\tan\theta$, prove that $\frac{d^2y}{dx^2}=-\frac{b^4}{a^2y^3}$.
9. If $x=a(\cos\theta+\theta\sin\theta),\ y=a(\sin\theta-\theta\cos\theta)$, prove that
 $\frac{d^2x}{d\theta^2}=a(\cos\theta-\theta\sin\theta),\ \frac{d^2y}{d\theta^2}=a(\sin\theta+\theta\cos\theta)$ and $\frac{d^2y}{dx^2}=\frac{\sec^3\theta}{a\theta}$.
 [NCERT, CBSE 2012, 2017]

10. If $y = e^x \cos x$, prove that $\frac{d^2y}{dx^2} = 2\, e^x \cos\left(x + \frac{\pi}{2}\right)$. [CBSE 2012]

11. If $x = a \cos\theta, y = b \sin\theta$, show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2\, y^3}$. [CBSE 2020]

12. If $x = a\,(1 - \cos^3\theta), y = a \sin^3\theta$, prove that $\frac{d^2y}{dx^2} = \frac{32}{27\, a}$ at $\theta = \frac{\pi}{6}$.

13. If $x = a\,(\theta + \sin\theta), y = a\,(1 + \cos\theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$.

14. If $x = a\,(\theta - \sin\theta), y = a\,(1 + \cos\theta)$ find $\frac{d^2y}{dx^2}$. [CBSE 2011]

15. If $x = a\,(1 - \cos\theta),\; y = a\,(\theta + \sin\theta)$, prove that $\frac{d^2y}{dx^2} = -\frac{1}{a}$ at $\theta = \frac{\pi}{2}$.

16. If $x = a\,(1 + \cos\theta), y = a\,(\theta + \sin\theta)$, prove that $\frac{d^2y}{dx^2} = \frac{-1}{a}$ at $\theta = \frac{\pi}{2}$.

BASED ON LOTS

17. If $x = \cos\theta, y = \sin^3\theta$, prove that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta\,(5\cos^2\theta - 1)$. [CBSE 2013]

18. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x = 0$. [CBSE 2018]

19. If $x = \sin t, y = \sin pt$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0$.

20. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2)\, y_2 - x y_1 - 2 = 0$. [CBSE 2019]

21. If $y = e^{\tan^{-1} x}$, prove that $(1 + x^2)\, y_2 + (2x - 1)\, y_1 = 0$.

22. If $y = 3\cos(\log x) + 4\sin(\log x)$, prove that $x^2 y_2 + x\, y_1 + y = 0$. [NCERT, CBSE 2009, 2012, 2016]

23. If $y = e^{2x}(ax + b)$, show that $y_2 - 4\,y_1 + 4\,y = 0$.

24. If $x = \sin\left(\frac{1}{a}\log y\right)$, show that $(1 - x^2)\, y_2 - x\, y_1 - a^2\, y = 0$. [CBSE 2010]

25. If $\log y = \tan^{-1} x$, show that $(1 + x^2)\, y_2 + (2x - 1)\, y_1 = 0$.

26. If $y = \tan^{-1} x$, show that $(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$.

27. If $y = \left\{\log(x + \sqrt{x^2 + 1})\right\}^2$, show that $(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 2$. [CBSE 2008]

28. If $y = (\tan^{-1} x)^2$, then prove that $(1 + x^2)^2\, y_2 + 2x\,(1 + x^2)\, y_1 = 2$. [CBSE 2012, [NCERT]

29. If $y = \cot x$ show that $\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} = 0$.

30. Find $\frac{d^2y}{dx^2}$, where $y = \log\left(\frac{x^2}{e^2}\right)$. [CBSE 2000]

31. If $y = ae^{2x} + be^{-x}$, show that, $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$. [CBSE 2000C]

32. If $y = e^x (\sin x + \cos x)$ prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$ [CBSE 2002, 2009]

33. If $y = \cos^{-1} x$, find $\frac{d^2y}{dx^2}$ in terms of y alone. [NCERT]

34. If $y = e^{a\cos^{-1} x}$, prove that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2 y = 0.$ [NCERT, CBSE 2012, 2020]

35. If $y = 500\, e^{7x} + 600\, e^{-7x}$, show that $\frac{d^2y}{dx^2} = 49\, y.$ [NCERT]

36. If $x = 2\cos t - \cos 2t,\ y = 2\sin t - \sin 2t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

37. If $x = 4z^2 + 5,\ y = 6z^2 + 7z + 3$, find $\frac{d^2y}{dx^2}$.

38. If $y = \log(1 + \cos x)$, prove that $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}\cdot\frac{dy}{dx} = 0.$ [CBSE 2005]

39. If $y = \sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$ [CBSE 2007]

40. If $y = 3\, e^{2x} + 2\, e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$ [NCERT, CBSE 2007, 2009]

41. If $y = (\cot^{-1} x)^2$, prove that $y_2 (x^2 + 1)^2 + 2x(x^2 + 1)\, y_1 = 2.$

42. If $y = \text{cosec}^{-1} x,\ x > 1$, then show that $x(x^2 - 1)\frac{d^2y}{dx^2} + (2x^2 - 1)\frac{dy}{dx} = 0.$ [CBSE 2010]

43. If $x = \cos t + \log\tan\frac{t}{2},\ y = \sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. [CBSE 2012]

44. If $x = a\sin t$ and $y = a\left(\cos t + \log\tan\frac{t}{2}\right)$, find $\frac{d^2y}{dx^2}$. [CBSE 2013]

45. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. [CBSE 2014]

46. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$, evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. [CBSE 2014]

47. If $x = a(\cos 2t + 2t\sin 2t)$ and $y = a(\sin 2t - 2t\cos 2t)$, then find $\frac{d^2y}{dx^2}$. [CBSE 2015]

48. If $x = 3\cos t - 2\cos^3 t,\ y = 3\sin t - 2\sin^3 t$, find $\frac{d^2y}{dx^2}$. [CBSE 2017]

BASED ON HOTS

49. If $x = a\sin t - b\cos t,\ y = a\cos t + b\sin t$, prove that $\frac{d^2y}{dx^2} = -\frac{x^2 + y^2}{y^3}$.

50. Find A and B so that $y = A\sin 3x + B\cos 3x$ satisfies the equation
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 10\cos 3x.$$

51. If $y = A\, e^{-kt}\cos(pt + c)$, prove that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + n^2 y = 0$, where $n^2 = p^2 + k^2$.

52. If $y = x^n\{a\cos(\log x) + b\sin(\log x)\}$, prove that $x^2\frac{d^2y}{dx^2} + (1 - 2n)\,x\frac{dy}{dx} + (1 + n^2)\,y = 0.$

53. If $y = a\left\{x+\sqrt{x^2+1}\right\}^n + b\left\{x-\sqrt{x^2+1}\right\}^{-n}$, prove that $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y = 0.$

ANSWERS

1. (i) $6x + 2\sec^2 x \tan x$ (ii) $\frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$ (iii) $-\operatorname{cosec}^2 x$

(iv) $2e^x(5\cos 5x - 12\sin 5x)$ (v) $9e^{6x}(3\cos 3x - 4\sin 3x)$ (vi) $x(5 + 6\log x)$

(vii) $\frac{-2x}{(1+x^2)^2}$ (viii) $-x\cos x - 2\sin x$ (ix) $-\frac{(1+\log x)}{(x\log x)^2}$

14. (ii) $\frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}$ **30.** $-\frac{2}{x^2}$ **33.** $-\cot y \operatorname{cosec}^2 y$

36. $-\frac{3}{2}$ **37.** $-\frac{7}{64z^3}$ **43.** $\left(\frac{d^2y}{dt^2}\right)_{t=\frac{\pi}{4}} = -\frac{1}{\sqrt{2}}$, $\left(\frac{d^2y}{dx^2}\right)_{t=\frac{\pi}{4}} = 2\sqrt{2}$

44. $-\frac{1}{a\sin^2 t\cos t}$ **45.** $\frac{8\sqrt{2}}{\pi a}$ **46.** $\frac{8\sqrt{3}}{a}$ **47.** $\frac{1}{2a}\sec^3 2t$

48. $-\frac{1}{3\sin^3 t\cos 2t}$ **50.** $A = \frac{2}{3}, B = -\frac{1}{3}$

HINTS TO SELECTED PROBLEMS

1. (ii) Let $y = \sin(\log x)$. Then,

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x} \Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right)\cos(\log x) + \frac{1}{x}\frac{d}{dx}\{\cos(\log x)\}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{\cos(\log x)}{x^2} - \frac{\sin(\log x)}{x^2} = -\frac{1}{x^2}\left\{\cos(\log x) + \sin(\log x)\right\}$$

(iii) Let $y = \log(\sin x)$. Then, $\frac{dy}{dx} = \cot x \Rightarrow \frac{d^2y}{dx^2} = -\operatorname{cosec}^2 x$

(iv) Let $y = e^x \sin 5x$. Then,

$$\frac{dy}{dx} = e^x\sin 5x + e^x(5\cos 5x) = e^x(\sin 5x + 5\cos 5x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}(e^x)\cdot(\sin 5x + 5\cos 5x) + e^x\frac{d}{dx}(\sin 5x + 5\cos 5x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = e^x(\sin 5x + 5\cos 5x) + e^x(5\cos 5x - 25\sin 5x) = e^x(-24\sin 5x + 10\cos 5x)$$

(v) Let $y = e^{6x}\cos 3x$. Then,

$$\frac{dy}{dx} = \frac{d}{dx}(e^{6x})\cos 3x + e^{6x}\frac{d}{dx}(\cos 3x)$$

$$\Rightarrow \frac{dy}{dx} = 6e^{6x}\cos 3x - 3e^{6x}\sin 3x \Rightarrow \frac{dy}{dx} = 3e^{6x}(2\cos 3x - \sin 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\frac{d}{dx}(e^{6x})(2\cos 3x - \sin 3x) + 3e^{6x}\frac{d}{dx}(2\cos 3x - \sin 3x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 18e^{6x}(2\cos 3x - \sin 3x) + 3e^{6x}(-6\sin 3x - 3\cos 3x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 9\,e^{6x}\,\{4\cos 3x - 2\sin 3x - 2\sin 3x - \cos 3x\} = 9\,e^{6x}\,(3\cos 3x - 4\sin 3x)$$

(vi) Let $y = x^3 \log x$. Then,

$$\frac{dy}{dx} = \log x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 \log x + x^3 \times \frac{1}{x} \Rightarrow \frac{dy}{dx} = x^2(3\log x + 1)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = (3\log x + 1)\frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(3\log x + 1)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 2x(3\log x + 1) + x^2 \times \frac{3}{x} = x(6\log x + 5)$$

(vii) Let $y = \tan^{-1} x$. Then,

$$\frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -(1+x^2)^{-2}\frac{d}{dx}(1+x^2) = -\frac{2x}{(1+x^2)^2}$$

(viii) Let $y = x\cos x$. Then,

$$\frac{dy}{dx} = \cos x - x\sin x \Rightarrow \frac{d^2y}{dx^2} = -\sin x - (\sin x + x\cos x) = -2\sin x - x\cos x$$

(ix) Let $y = \log(\log x)$. Then,

$$\frac{dy}{dx} = \frac{1}{\log x}\frac{d}{dx}(\log x) = \frac{1}{x\log x} = (x\log x)^{-1}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\{(x\log x)^{-1}\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -(x\log x)^{-2}\frac{d}{dx}(x\log x) = -\frac{1}{(x\log x)^2}(1+\log x) = \frac{-(1+\log x)}{(x\log x)^2}$$

22. We have,

$$y = 3\cos(\log x) + 4\sin(\log x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{3}{x}\sin(\log x) + \frac{4}{x}\cos(\log x) \Rightarrow x\frac{dy}{dx} = -3\sin(\log x) + 4\cos(\log x)$$

Differentiating both sides with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{-3}{x}\cos(\log x) - \frac{4}{x}\sin(\log x)$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\{3\cos(\log x) + 4\sin(\log x)\}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

28. We have,

$$y = (\tan^{-1} x)^2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2(\tan^{-1} x)^{2-1}\frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{1+x^2}\tan^{-1} x \Rightarrow (1+x^2)\frac{dy}{dx} = 2\tan^{-1} x$$

$$\Rightarrow \quad (1+x^2)^2\left(\frac{dy}{dx}\right)^2 = 4(\tan^{-1}x)^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad (1+x^2)^2\left(\frac{dy}{dx}\right)^2 = 4y$$

Differentiating both sides with respect to x, we get

$$2(1+x^2)\times 2x\left(\frac{dy}{dx}\right)^2 + 2(1+x^2)^2\frac{dy}{dx}\frac{d^2y}{dx^2} = 4\frac{dy}{dx}$$

$$\Rightarrow \quad 2x(1+x^2)\frac{dy}{dx} + (1+x^2)^2\frac{d^2y}{dx^2} = 2 \Rightarrow (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$

33. We have,

$$y = \cos^{-1}x$$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} = -(1-x^2)^{-1/2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{2}(1-x^2)^{-3/2}\frac{d}{dx}(1-x^2) \Rightarrow \frac{d^2y}{dx^2} = \frac{-x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-\cos y}{(1-\cos^2 y)^{3/2}} \qquad [\because y = \cos^{-1}x \Rightarrow x = \cos y]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\cot y \operatorname{cosec}^2 y$$

34. We have,

$$y = e^{a\cos^{-1}x}$$

$$\Rightarrow \quad \frac{dy}{dx} = e^{a\cos^{-1}x}\frac{d}{dx}(a\cos^{-1}x) \Rightarrow \frac{dy}{dx} = e^{a\cos^{-1}x}\times -\frac{a}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} = \frac{-ay}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} = -ay \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2 \qquad \text{[On squaring both sides]}$$

Differentiating with respect to x, we get

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2)\,2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2a^2y\frac{dy}{dx}$$

$$\Rightarrow \quad -x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2y \Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$$

35. We have,

$$y = 500e^{7x} + 600e^{-7x}$$

$$\Rightarrow \quad \frac{dy}{dx} = 3500e^{7x} - 4200e^{-7x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 3500\times 7e^{7x} + 4200\times 7e^{-7x} \Rightarrow \frac{d^2y}{dx^2} = 49(500e^{7x}+600e^{-7x}) \Rightarrow \frac{d^2y}{dx^2} = 49y$$

40. We have,

$$y = 3e^{2x} + 2e^{3x} \Rightarrow \frac{dy}{dx} = 6e^{2x} + 6e^{3x} \text{ and } \frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x}$$

$$\therefore \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = (12e^{2x}+18e^{3x}) - 5(6e^{2x}+6e^{3x}) + 6(3e^{2x}+2e^{3x}) = 0$$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $y = t^{10} + 1$ and $x = t^8 + 1$, then $\frac{d^2y}{dx^2} =$ ....

2. If $x = a\sin\theta$ and $y = b\cos\theta$, then $\frac{d^2y}{dx^2} =$ ....

3. If $y = x + e^x$, then $\frac{d^2y}{dx^2} =$ ....

4. If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$ ...., then $\frac{d^2y}{dx^2} =$ ....

5. If $y = x + e^x$, then $\frac{d^2x}{dy^2} =$ ....

6. If $y = \log_e x$, then $\frac{d^2y}{dx^2} =$ .... [CBSE 2020]

ANSWERS

1. $\frac{5}{16t^6}$ 2. $\frac{-b}{a^2}\sec^3\theta$ 3. e^x 4. e^{-x} 5. $-\frac{e^x}{(1+e^x)^3}$ 6. $-\frac{1}{x^2}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $y = a\,x^{n+1} + bx^{-n}$ and $x^2\frac{d^2y}{dx^2} = \lambda y$, then write the value of λ.

2. If $x = a\cos nt - b\sin nt$ and $\frac{d^2x}{dt^2} = \lambda x$, then find the value of λ.

3. If $x = t^2$ and $y = t^3$, find $\frac{d^2y}{dx^2}$.

4. If $x = 2at$, $y = at^2$, where a is a constant, then find $\frac{d^2y}{dx^2}$ at $x = \frac{1}{2}$.

5. If $x = f(t)$ and $y = g(t)$, then write the value of $\frac{d^2y}{dx^2}$.

6. If $y = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!}$ ..... to ∞, then write $\frac{d^2y}{dx^2}$ in terms of y.

7. If $y = x + e^x$, find $\frac{d^2x}{dy^2}$.

8. If $y = |x - x^2|$, then find $\frac{d^2y}{dx^2}$.

9. If $y = |\log_e x|$, find $\frac{d^2y}{dx^2}$.

ANSWERS

1. $n(n+1)$ 2. n^2 3. $\frac{3}{4t}$ 4. $\frac{1}{2a}$ 5. $\frac{f'g'' - g'f''}{f'^3}$

6. y 7. $\frac{-e^x}{(1+e^x)^3}$ 8. $\frac{d^2y}{dx^2} = \begin{cases} -2, & 0 < x < 1 \\ 2, & x > 1, x < 0 \end{cases}$ 9. $\frac{d^2y}{dx^2} = \begin{cases} \frac{1}{x^2}, & 0 < x < 1 \\ -\frac{1}{x^2}, & x > 1 \end{cases}$