7. $\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C$ 8. $\tan^{-1}(\tan^2 x) + C$ 9. $\log|\tan x + 2| + C$

10. $\frac{1}{2} \log\left|\frac{\tan x}{\tan x + 2}\right| + C$ 11. $\frac{1}{5} \log\left|\frac{\tan x - 2}{2 \tan x + 1}\right| + C$

18.10.9 INTEGRALS OF THE FORM $\int \frac{1}{a \sin x + b \cos x} dx$, $\int \frac{1}{a + b \sin x} dx$, $\int \frac{1}{a + b \cos x} dx$, $\int \frac{1}{a \sin x + b \cos x + c} dx$

To evaluate this type of integrals we use the following algorithm.

ALGORITHM

Step I *Put* $\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}$ *and*, $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$ *and simplify.*

Step II *Replace* $1 + \tan^2 \frac{x}{2}$ *in the numerator by* $\sec^2 \frac{x}{2}$.

Step III *Put* $\tan \frac{x}{2} = t$ *so that* $\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$. *This substitution reduces the integral in the form* $\int \frac{1}{at^2 + bt + c} dt$.

Step IV *Evaluate the integral obtained in step III by using methods discussed earlier.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{1}{1 + \sin x + \cos x} dx$ (ii) $\int \frac{1}{2 + \cos x} dx$ (iii) $\int \frac{1 + \sin x}{\sin x (1 + \cos x)} dx$ (iv) $\int \frac{1}{1 - 2 \sin x} dx$

SOLUTION (i) Putting $\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}$ and $\cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$$I = \int \frac{1}{1 + \sin x + \cos x} dx = \int \frac{1}{1 + \frac{2 \tan x/2}{1 + \tan^2 x/2} + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} dx$$

$$\Rightarrow \quad I = \int \frac{1 + \tan^2 x/2}{1 + \tan^2 x/2 + 2 \tan x/2 + 1 - \tan^2 x/2} dx = \int \frac{\sec^2 x/2}{2 + 2 \tan x/2} dx$$

Putting $\tan \frac{x}{2} = t$ and $\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$ or, $\sec^2 \frac{x}{2} dx = 2dt$, we get

$$I = \int \frac{2dt}{2 + 2t} = \int \frac{1}{t + 1} dt = \log|t + 1| + C = \log\left|\tan \frac{x}{2} + 1\right| + C$$

(ii) Let $I = \int \frac{1}{2 + \cos x} dx = \int \frac{1}{2 + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} dx$ $\left[\because \cos x = \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}\right]$

$$\Rightarrow \quad I = \int \frac{1+\tan^2 x/2}{2(1+\tan^2 x/2)+1-\tan^2 x/2}\,dx = \int \frac{\sec^2 x/2}{\tan^2 x/2+3}\,dx$$

Putting $\tan x/2 = t$ and $(1/2 \sec^2 (x/2) = dt$ or, $\sec^2 (x/2)\,dx = 2dt$, we get

$$I = \int \frac{2dt}{t^2+3} = 2\int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$$

(iii) Let $I = \int \frac{1+\sin x}{\sin x\,(1+\cos x)}\,dx$. Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$ and $\cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2}$, we get

$$I = \int \frac{1+\frac{2\tan x/2}{1+\tan^2 x/2}}{\frac{2\tan x/2}{1+\tan^2 x/2}\left(1+\frac{1-\tan^2 x}{1+\tan^2 x/2}\right)}\,dx$$

$$\Rightarrow \quad I = \int \frac{(1+\tan^2 x/2+2\tan x/2)(1+\tan^2 x/2)}{2\tan x/2\,(1+\tan^2 x/2+1-\tan^2 x/2)}\,dx = \int \frac{(1+\tan x/2)^2 \sec^2 x/2}{4\tan x/2}\,dx$$

Putting $\tan x/2 = t$ and $(1/2)\sec^2 (x/2)\,dx = dt$ *or*, $\sec^2 (x/2)\,dx = 2\,dt$, we get

$$I = \int \frac{(1+t)^2}{4t}\,2\,dt = \frac{1}{2}\int \frac{1+t^2+2t}{t}\,dt = \frac{1}{2}\int \left(\frac{1}{t}+t+2\right)dt$$

$$\Rightarrow \quad I = \frac{1}{2}\left\{\log|t| + \frac{t^2}{2} + 2t\right\} + C = \frac{1}{2}\left\{\log|\tan x/2| + \frac{\tan^2 x/2}{2} + 2\tan x/2\right\} + C$$

(iv) Let $I = \int \frac{1}{1-2\sin x}\,dx$. Putting $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$, we get

$$I = \int \frac{1}{1-\frac{4\tan x/2}{1+\tan^2 x/2}}\,dx = \int \frac{1+\tan^2 x/2}{1+\tan^2 x/2-4\tan x/2}\,dx = \int \frac{\sec^2 x/2}{1+\tan^2 x/2-4\tan x/2}\,dx$$

Putting $\tan x/2 = t$ and $\frac{1}{2}\sec^2 (x/2)\,dx = dt$ or, $\sec^2 (x/2)\,dx = 2dt$, we get

$$I = \int \frac{2}{1+t^2-4t}\,dt = 2\int \frac{1}{t^2-4t+4-4+1}\,dt = 2\int \frac{1}{(t-2)^2-(\sqrt{3})^2}\,dt$$

$$\Rightarrow \quad I = 2\times\frac{1}{2\sqrt{3}}\log\left|\frac{t-2-\sqrt{3}}{t-2+\sqrt{3}}\right| + C = \frac{1}{\sqrt{3}}\log\left|\frac{\tan x/2-2-\sqrt{3}}{\tan x/2-2+\sqrt{3}}\right| + C$$

EXERCISE 18.23

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{5+4\cos x}\,dx$ [CBSE 2003] **2.** $\int \frac{1}{5-4\sin x}\,dx$ **3.** $\int \frac{1}{1-2\sin x}\,dx$

4. $\int \frac{1}{4 \cos x - 1}\, dx$

5. $\int \frac{1}{1 - \sin x + \cos x}\, dx$

6. $\int \frac{1}{3 + 2 \sin x + \cos x}\, dx$ [CBSE 2004]

7. $\int \frac{1}{13 + 3 \cos x + 4 \sin x}\, dx$

8. $\int \frac{1}{\cos x - \sin x}\, dx$

9. $\int \frac{1}{\sin x + \cos x}\, dx$

10. $\int \frac{1}{5 - 4 \cos x}\, dx$

11. $\int \frac{1}{2 + \sin x + \cos x}\, dx$

12. $\int \frac{1}{\sin x + \sqrt{3} \cos x}\, dx$

13. $\int \frac{1}{\sqrt{3} \sin x + \cos x}\, dx$

14. $\int \frac{1}{\sin x - \sqrt{3} \cos x}\, dx$

15. $\int \frac{1}{5 + 7 \cos x + \sin x}\, dx$

ANSWERS

1. $\frac{2}{3} \tan^{-1}\left(\frac{\tan x/2}{3}\right) + C$

2. $\frac{2}{3} \tan^{-1}\left(\frac{5 \tan x/2 - 4}{3}\right) + C$

3. $\frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}} \right| + C$

4. $\frac{1}{\sqrt{15}} \log \left| \frac{\sqrt{3} + \sqrt{5} \tan x/2}{\sqrt{3} - \sqrt{5} \tan x/2} \right| + C$

5. $-\log \left| 1 - \tan \frac{x}{2} \right| + C$

6. $\tan^{-1}\left(1 + \tan \frac{x}{2}\right) + C$

7. $\frac{1}{6} \tan^{-1}\left(\frac{5 \tan x/2 + 2}{6}\right) + C$

8. $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan x/2 + 1}{\sqrt{2} - \tan x/2 - 1} \right| + C$

9. $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \tan x/2 - 1}{\sqrt{2} - \tan x/2 + 1} \right| + C$

10. $\frac{2}{3} \tan^{-1}\left(3 \tan \frac{x}{2}\right) + C$

11. $\sqrt{2} \tan^{-1}\left(\frac{\tan (x/2) + 1}{\sqrt{2}}\right) + C$

12. $\frac{1}{2} \log \left| \frac{1 + \sqrt{3} \tan x/2}{3 - \sqrt{3} \tan x/2} \right| + C$

13. $\frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{12}\right) \right| + C$

14. $\frac{1}{2} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{6}\right) \right| + C$

15. $\frac{1}{5} \log_e \left| \frac{\tan x/2 + 2}{\tan x/2 - 3} \right| + C$

18.10.10 ALTERNATIVE METHOD TO EVALUATE INTEGRALS OF THE FORM $\int \frac{1}{a \sin x + b \cos x}\, dx$

To evaluate this type of integrals, we substitute

$$a = r \cos \theta, b = r \sin \theta \text{ so that } r = \sqrt{a^2 + b^2} \text{ and, } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \quad a \sin x + b \cos x = r \cos \theta \sin x + r \sin \theta \cos x = r \sin (x + \theta)$$

So, $$I = \int \frac{1}{a \sin x + b \cos x}\, dx = \frac{1}{r} \int \frac{1}{\sin (x + \theta)}\, dx = \frac{1}{r} \int \operatorname{cosec} (x + \theta)\, dx$$

$$\Rightarrow \quad I = \frac{1}{r} \log \left| \tan \left(\frac{x}{2} + \frac{\theta}{2}\right) \right| + C = \frac{1}{\sqrt{a^2 + b^2}} \log \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{b}{a}\right) \right| + C$$

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* (i) $\int \frac{1}{\sqrt{3}\sin x+\cos x}\,dx$ (ii) $\int \frac{1}{\sin x+\sqrt{3}\cos x}\,dx$

SOLUTION (i) Let $\sqrt{3}=r\sin\theta$ and $1=r\cos\theta$. Then

$$r=\sqrt{(\sqrt{3})^2+1^2} \text{ and } \tan\theta=\frac{\sqrt{3}}{1}\Rightarrow r=2 \text{ and } \theta=\frac{\pi}{3}$$

$$\therefore \quad I=\int \frac{1}{\sqrt{3}\sin x+\cos x}\,dx=\int \frac{1}{r\sin\theta\sin x+r\cos\theta\cos x}\,dx$$

$$\Rightarrow \quad I=\frac{1}{r}\int \frac{1}{\cos(x-\theta)}\,dx=\frac{1}{r}\int \sec(x-\theta)\,dx=\frac{1}{r}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\theta}{2}\right)\right|+C$$

$$\Rightarrow \quad I=\frac{1}{2}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}-\frac{\pi}{6}\right)\right|+C=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)\right|+C$$

(ii) Let $1=r\cos\theta$ and $\sqrt{3}=r\sin\theta$. Then, $r=\sqrt{1^2+(\sqrt{3})^2}$ and $\tan\theta=\frac{\sqrt{3}}{1}\Rightarrow r=2$ and $\theta=\frac{\pi}{3}$

$$\therefore \quad I=\int \frac{1}{\sin x+\sqrt{3}\cos x}\,dx=\frac{1}{r}\int \frac{1}{\sin x\cos\theta+\cos x\sin\theta}\,dx=\frac{1}{r}\int \frac{1}{\sin(x+\theta)}\,dx$$

$$\Rightarrow \quad I=\frac{1}{r}\int \operatorname{cosec}(x+\theta)\,dx=\frac{1}{r}\log\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right|+C=\frac{1}{2}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right|+C$$

18.10.11 INTEGRALS OF THE FORM $\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx$

To evaluate this type of integrals, we use the following algorithm.

ALGORITHM

Step I *Write, Numerator* $=\lambda$ *(Diff. of denominator)* $+\,\mu$ *(Denominator)*

i.e. $a\sin x+b\cos x=\lambda(c\cos x-d\sin x)+\mu\,(c\sin x+d\cos x)$

Step II *Obtain the values of* λ *and* μ *by equating the coefficients of* $\sin x$ *and* $\cos x$ *on both the sides.*

Step III *Replace numerator in the integrand by* $\lambda\,(c\cos x-d\sin x)+\mu\,(c\sin x+d\cos x)$ *to obtain*

$$\int \frac{a\sin x+b\cos x}{c\sin x+d\cos x}\,dx=\lambda\int \frac{c\cos x-d\sin x}{c\sin x+d\cos x}\,dx+\mu\int \frac{c\sin x+d\cos x}{c\sin x+d\cos x\,dx}$$

$$=\lambda\log|c\sin x+d\cos x|+\mu\,x+C$$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* (i) $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$ (ii) $\int \frac{1}{1+\tan x}\,dx$

SOLUTION (i) Let $I=\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

Let $3\sin x+2\cos x=\lambda\frac{d}{dx}(3\cos x+2\sin x)+\mu\,(3\cos x+2\sin x)$

i.e. $3\sin x + 2\cos x = \lambda\,(-3\sin x + 2\cos x) + \mu\,(3\cos x + 2\sin x)$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$-3\lambda + 2\mu = 3 \text{ and } 2\lambda + 3\mu = 2 \Rightarrow \mu = \frac{12}{13} \text{ and } \lambda = -\frac{5}{13}$$

$$\therefore \quad I = \int \frac{\lambda(-3\sin x + 2\cos x) + \mu\,(3\cos x + 2\sin x)}{3\cos x + 2\sin x}\,dx$$

$$\Rightarrow \quad I = \mu\int 1\cdot dx + \lambda\int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx = \mu x + \lambda\int \frac{dt}{t}, \text{ where } t = 3\cos x + 2\sin x$$

$$\Rightarrow \quad I = \mu\,x + \lambda\log|t| + C = \frac{12}{13}x - \frac{5}{13}\log|3\cos x + 2\sin x| + C$$

(ii) $$I = \int \frac{1}{1+\tan x}\,dx = \int \frac{\cos x}{\cos x + \sin x}\,dx$$

Let $\cos x = \lambda \frac{d}{dx}(\cos x + \sin x) + \mu\,(\cos x + \sin x)$. Then,

$$\cos x = \lambda(-\sin x + \cos x) + \mu\,(\sin x + \cos x)$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get :

$$-\lambda + \mu = 0 \text{ and } \lambda + \mu = 1 \Rightarrow \lambda = \mu = \frac{1}{2}$$

$$\therefore \quad I = \int \frac{\cos x}{\cos x + \sin x}\,dx = \int \frac{1/2\,(-\sin x + \cos x) + 1/2\,(\cos x + \sin x)}{\cos x + \sin x}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{-\sin x + \cos x}{\cos x + \sin x}\,dx + \frac{1}{2}\int \frac{\cos x + \sin x}{\cos x + \sin x}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{dt}{t} + \frac{1}{2}\int 1\cdot dx, \text{ where } t = \cos x + \sin x$$

$$\Rightarrow \quad I = \frac{1}{2}\log|t| + \frac{1}{2}x + C = \frac{1}{2}x + \frac{1}{2}\log|\sin x + \cos x| + C$$

EXAMPLE 2 *Evaluate:* $\displaystyle\int \frac{1}{1+\cot x}\,dx$

SOLUTION Let $$I = \int \frac{1}{1+\cot x}\,dx = \int \frac{1}{1 + \frac{\cos x}{\sin x}}\,dx = \int \frac{\sin x}{\sin x + \cos x}\,dx$$

Let $\sin x = \lambda \frac{d}{dx}(\sin x + \cos x) + \mu\,(\sin x + \cos x)$ i.e. $\sin x = \lambda(\cos x - \sin x) + \mu\,(\sin x + \cos x)$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$0 = \lambda + \mu \text{ and } 1 = -\lambda + \mu \Rightarrow \lambda = -1/2, \mu = 1/2$$

$$\therefore \quad I = \int \frac{\lambda(\cos x - \sin x) + \mu\,(\sin x + \cos x)}{\sin x + \cos x}\,dx = \lambda\int \frac{\cos x - \sin x}{\sin x + \cos x}\,dx + \mu\int \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$

$$\Rightarrow I = \lambda\int \frac{dt}{t} + \mu\int 1\cdot dx, \quad \text{where } t = \sin x + \cos x$$

$$\Rightarrow I = \lambda\log|t| + \mu\,x + C = -(1/2)\log|\sin x + \cos x| + (1/2)\,x + C$$

18.10.12 INTEGRALS OF THE FORM $\displaystyle\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$

To evaluate this type of integrals, we use the following algorithm.

ALGORITHM

Step I *Write: Numerator* $= \lambda$ *(Diff. of denominator)* $+ \mu$ *(Denominator)* $+ \nu$

i.e. $a\sin x + b\cos x + c = \lambda\,(p\cos x - q\sin x) + \mu\,(p\sin x + q\cos x + r) + \nu$

Step II *Obtain the values of* λ and μ *by equating the coefficients of* $\sin x$ and $\cos x$ *and the constant terms on both the sides.*

Step III *Replace the numerator in the integrand by* $\lambda\,(p\cos x - q\sin x) + \mu\,(p\sin x + q\cos x + r) + \nu$ *to obtain*

$$I = \int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx$$

$$\Rightarrow \quad I = \lambda \int \frac{p\cos x - q\sin x}{p\sin x + q\cos x + r}\,dx + \mu \int \frac{p\sin x + q\cos x + r}{p\sin x + q\cos x + r} + \nu \int \frac{1}{p\sin x + q\cos x + r}\,dx$$

$$\Rightarrow \quad I = \lambda \log|p\sin x + q\cos x + r| + \mu\, x + \nu \int \frac{1}{p\sin x + q\cos x + r}\,dx$$

Step IV *Evaluate the integral on RHS in step III by using the method discussed earlier.*

The following example will illustrates the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE *Evaluate:* $\displaystyle\int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

SOLUTION Let $\displaystyle I = \int \frac{3\cos x + 2}{\sin x + 2\cos x + 3}\,dx$

Let $3\cos x + 2 = \lambda\,(\sin x + 2\cos x + 3) + \mu\,(\cos x - 2\sin x) + \nu$

Comparing the coefficients of $\sin x$, $\cos x$ and constant term on both sides, we get

$$\lambda - 2\mu = 0,\ 2\lambda + \mu = 3,\ 3\lambda + \nu = 2 \Rightarrow \lambda = \frac{6}{5},\ \mu = \frac{3}{5} \text{ and } \nu = -\frac{8}{5}$$

$$\therefore \quad I = \int \frac{\lambda\,(\sin x + 2\cos x + 3) + \mu\,(\cos x - 2\sin x) + \nu}{\sin x + 2\cos x + 3}\,dx$$

$$\Rightarrow \quad I = \lambda \int dx + \mu \int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx + \nu \int \frac{1}{\sin x + 2\cos x + 3}\,dx$$

$$\Rightarrow \quad I = \lambda\, x + \mu \log|\sin x + 2\cos x + 3| + \nu\, I_1, \text{ where } I_1 = \int \frac{1}{\sin x + 2\cos x + 3}\,dx$$

Putting $\sin x = \dfrac{2\tan x/2}{1 + \tan^2 x/2}$ and $\cos x = \dfrac{1 - \tan^2 x/2}{1 + \tan^2 x/2}$, we get

$$I_1 = \int \frac{1}{\dfrac{2\tan x/2}{1 + \tan^2 x/2} + \dfrac{2\,(1 - \tan^2 x/2)}{1 + \tan^2 x/2} + 3}\,dx$$

$$\Rightarrow I_1 = \int \frac{1 + \tan^2 x/2}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3\,(1 + \tan^2 x/2)}\,dx = \int \frac{\sec^2 x/2}{\tan^2 x/2 + 2\tan x/2 + 5}\,dx$$

Putting $\tan\dfrac{x}{2} = t$ and $\dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx = dt$ or, $\sec^2\dfrac{x}{2}\,dx = 2\,dt$, we get

$$I_1 = \int \frac{2dt}{t^2 + 2t + 5} = 2\int \frac{dt}{(t+1)^2 + 2^2} = \frac{2}{2}\tan^{-1}\left(\frac{t+1}{2}\right) = \tan^{-1}\left(\frac{\tan\frac{x}{2} + 1}{2}\right)$$

$$\therefore \quad I = \lambda x + \mu \log|\sin x + 2\cos x + 3| + \nu \tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$$

$$\Rightarrow \quad I = \frac{6}{5}x + \frac{3}{5}\log|\sin x + 2\cos x + 3| - \frac{8}{5}\tan^{-1}\left(\frac{\tan\frac{x}{2}+1}{2}\right) + C$$

EXERCISE 18.24

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{1-\cot x}\,dx$
2. $\int \frac{1}{1-\tan x}\,dx$
3. $\int \frac{3+2\cos x+4\sin x}{2\sin x+\cos x+3}\,dx$
4. $\int \frac{1}{p+q\tan x}\,dx$
5. $\int \frac{5\cos x+6}{2\cos x+\sin x+3}\,dx$
6. $\int \frac{2\sin x+3\cos x}{3\sin x+4\cos x}\,dx$
7. $\int \frac{1}{3+4\cot x}\,dx$
8. $\int \frac{2\tan x+3}{3\tan x+4}\,dx$
9. $\int \frac{1}{4+3\tan x}\,dx$
10. $\int \frac{8\cot x+1}{3\cot x+2}\,dx$
11. $\int \frac{4\sin x+5\cos x}{5\sin x+4\cos x}\,dx$

ANSWERS

1. $\frac{1}{2}x + \frac{1}{2}\log|\sin x - \cos x| + C$
2. $\frac{1}{2}x - \frac{1}{2}\log|\sin x - \cos x| + C$
3. $2x - 3\tan^{-1}\left(\tan\frac{x}{2}+1\right) + C$
4. $\frac{p}{p^2+q^2}x + \frac{q}{p^2+q^2}\log|p\cos x + q\sin x| + C$
5. $2x + \log|2\cos x + \sin x + 3| + C$
6. $\frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x| + C$
7. $\frac{3}{25}x - \frac{4}{25}\log|3\sin x + 4\cos x| + C$
8. $\frac{18}{25}x + \frac{1}{25}\log|3\sin x + 4\cos x| + C$
9. $\frac{4}{25}x + \frac{3}{25}\log|4\cos x + 3\sin x| + C$
10. $2x + \log|2\sin x + 3\cos x| + C$
11. $\frac{40}{41}x + \frac{9}{41}\log|5\sin x + 4\cos x| + C$

18.11 INTEGRATION BY PARTS

THEOREM *If u and v are two functions of x, then*

$$\int uv\,dx = u\left(\int v\,dx\right) - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

i.e. *The integral of the product of two functions = (First function) × (Integral of second function) – Integral of {(Diff. of first function) × (integral of second function)}*

PROOF For any two functions $f(x)$ and $g(x)$, we know that

$$\frac{d}{dx}\{f(x)\cdot g(x)\} = f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}$$

$$\therefore \quad \int\left\{f(x)\cdot\frac{d}{dx}\{g(x)\} + g(x)\cdot\frac{d}{dx}\{f(x)\}\right\}dx = f(x)\,g(x)$$

$$\Rightarrow \quad \int \left\{ f(x) \cdot \frac{d}{dx}\{g(x)\} \right\} dx + \int \left\{ g(x) \cdot \frac{d}{dx}\{f(x)\} \right\} dx = f(x)\, g(x)$$

$$\Rightarrow \quad \int \left\{ f(x) \cdot \frac{d}{dx}\{g(x)\} \right\} dx = f(x) \cdot g(x) - \int \left\{ g(x) \cdot \frac{d}{dx}\{f(x)\} \right\} dx \quad \ldots\text{(i)}$$

Let $f(x) = u$ and $\frac{d}{dx}\{g(x)\} = v$ so that $g(x) = \int v\, dx$. Substituting these in (i), we get

$$\therefore \quad \int uv\, dx = u\left\{\int v\, dx\right\} - \int \left\{\frac{du}{dx} \cdot \int v\, dx\right\} dx$$

NOTE 1 *Proper choice of first and second function —*

Integration with the help of the above rule is called the integration by parts. In the above rule there are two terms on RHS and in both the terms the integral of the second function is involved. Therefore in the product of two functions if one of the two functions is not directly integrable (e.g., $\log x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc.), we take it as the first function and the remaining function is taken as the second function. If there is no other function, then unity is taken as the second function. If in the integral both the functions are easily integrable, then the first function is chosen in such a way that the derivative of the function is a simple function and the function thus obtained under the integral sign is easily integrable than the original function.

NOTE 2 *We can also choose the first function as the function which comes first in the word* **ILATE**, *where*

I – stands for the inverse trigonometric function ($\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ etc.)

L – stands for the logarithmic functions

A – stands for the algebraic functions

T – stands for the trigonometric functions

E – stands for the exponential functions

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int x \sin 3x\, dx$ [NCERT] (ii) $\int x \sec^2 x\, dx$ [NCERT]

(iii) $\int x \log x\, dx$ [NCERT] (iv) $\int x \sin^2 x\, dx$

SOLUTION (i) Here both the functions viz. x and $\sin 3x$ are easily integrable and the derivative of x is one, a less complicated function. Therefore, we take x as the first function and $\sin 3x$ as the second function.

$$\therefore \quad I = \int \underset{\text{I}}{x} \underset{\text{II}}{\sin 3x}\, dx = x\left\{\int \sin 3x\, dx\right\} - \int \left\{\frac{d}{dx}(x) \times \int \sin 3x\, dx\right\} dx$$

$$\Rightarrow \quad I = x \times -\frac{1}{3}\cos 3x - \int \left\{-\frac{1}{3}\cos 3x\right\} dx$$

$$\Rightarrow \quad I = -\frac{1}{3} x \cos 3x + \frac{1}{3}\int \cos 3x\, dx = -\frac{1}{3} x \cos 3x + \frac{1}{9} \sin 3x + C.$$

(ii) Let $I = \int \underset{\text{I}}{x} \underset{\text{II}}{\sec^2 x}\, dx$. Then,

$$I = x\left\{\int \sec^2 x\, dx\right\} - \int \left\{\frac{d}{dx}(x) \times \int \sec^2 x\, dx\right\} dx$$

$\Rightarrow \quad I = x \tan x - \int 1 \times \tan x \, dx = x \tan x + \log|\cos x| + C$

(iii) Let $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\log x} \, dx$. Then,

$$I = \log x \left\{\int x \, dx\right\} - \int \left\{\frac{d}{dx}(\log x) \times \int x \, dx\right\} dx = (\log x) \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2} \, dx$$

$$\Rightarrow \quad I = \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \log x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C = \frac{x^2}{2} \log x - \frac{1}{4} x^2 + C$$

(iv) Let $I = \int x \sin^2 x \, dx$. Then,

$$I = \int x \left\{\frac{1 - \cos 2x}{2}\right\} dx = \frac{1}{2} \int x \, dx - \frac{1}{2} \int \underset{\text{I}}{x} \underset{\text{II}}{\cos 2x} \, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left(\frac{x^2}{2}\right) - \frac{1}{2}\left[x\left\{\int \cos 2x \, dx\right\} - \int \left\{\frac{d}{dx}(x) \times \int \cos 2x \, dx\right\} dx\right]$$

$$\Rightarrow \quad I = \frac{1}{4} x^2 - \frac{1}{2}\left\{\frac{x}{2} \sin 2x - \int \frac{\sin 2x}{2} \, dx\right\} = \frac{x^2}{4} - \frac{1}{2}\left\{\frac{x}{2} \sin 2x - \frac{1}{2} \int \sin 2x \, dx\right\}$$

$$\Rightarrow \quad I = \frac{x^2}{4} - \frac{1}{2}\left\{\frac{x}{2} \sin 2x - \frac{1}{2}\left(-\frac{1}{2} \cos 2x\right)\right\} + C = \frac{1}{4} x^2 - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C$$

EXAMPLE 2 *Evaluate:* (i) $\int x^2 \sin x \, dx$ (ii) $\int x^2 e^x \, dx$ **[NCERT]**

SOLUTION (i) Let $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sin x} \, dx$. Then,

$$I = x^2 \left\{\int \sin x \, dx\right\} - \int \left\{\frac{d}{dx}(x^2) \int \sin x \, dx\right\} dx$$

$$\Rightarrow \quad I = -x^2 \cos x - \int 2x(-\cos x) \, dx = -x^2 \cos x + 2 \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x} \, dx$$

$$\Rightarrow \quad I = -x^2 \cos x + 2\left[x\left\{\int \cos x \, dx\right\} - \int \left\{\frac{d}{dx}(x) \times \int \cos x \, dx\right\} dx\right]$$

$$\Rightarrow \quad I = -x^2 \cos x + 2\left\{x \sin x - \int \sin x \, dx\right\} = -x^2 \cos x + 2(x \sin x + \cos x) + C$$

(ii) Let $I = \int \underset{\text{I}}{x^2} \underset{\text{II}}{e^x} \, dx$. Then,

$$I = x^2 \left\{\int e^x \, dx\right\} - \int \left\{\frac{d}{dx}(x^2) \times \int e^x \, dx\right\} dx$$

$$\Rightarrow \quad I = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int \underset{\text{I}}{x} \underset{\text{II}}{e^x} \, dx$$

$$\Rightarrow \quad I = x^2 e^x - 2\left[x\left\{\int e^x \, dx\right\} - \int \left\{\frac{d}{dx}(x) \times \int e^x \, dx\right\} dx\right]$$

$$\Rightarrow \quad I = x^2 e^x - 2\left\{x e^x - \int e^x \, dx\right\} = x^2 e^x - 2(x e^x - e^x) + C$$

EXAMPLE 3 *Evaluate:* (i) $\int \log x \, dx$ **[NCERT]** (ii) $\int (\log x)^2 \, dx$

SOLUTION (i) Let $I = \int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1} \, dx = \log x \cdot \left\{\int 1 \cdot dx\right\} - \int \left\{\frac{d}{dx}(\log x) \cdot \int 1 \, dx\right\} dx$

$\Rightarrow \quad I = (\log x)\, x - \int \frac{1}{x} \cdot x \, dx = x(\log x) - \int 1 \cdot dx = x(\log x) - x + C$

(ii) Let $I = \int \underset{\text{I}}{(\log x)^2} \cdot \underset{\text{II}}{1} \, dx$. Then,

$$I = (\log x)^2 \left\{\int 1 \cdot dx\right\} - \int \left\{\frac{d}{dx}(\log x)^2 \cdot \int 1 \cdot dx\right\} dx$$

$\Rightarrow \quad I = (\log x)^2\, x - \int 2 \log x \cdot \frac{1}{x} \cdot x \, dx = x(\log x)^2 - 2\int \underset{\text{I}}{\log x} \cdot \underset{\text{II}}{1} \, dx$

$\Rightarrow \quad I = x(\log x)^2 - 2\left[(\log x)\left\{\int 1 \cdot dx\right\} - \int \left\{\frac{d}{dx}(\log x) \int 1 \cdot dx\right\} dx\right]$

$\Rightarrow \quad I = x(\log x)^2 - 2\left\{(\log x)\, x - \int \frac{1}{x} \cdot x \, dx\right\} = x(\log x)^2 - 2(x \log x - x) + C$

EXAMPLE 4 *Evaluate:*

(i) $\int \sin^{-1} x \, dx$ **[NCERT, CBSE 2022]** (ii) $\int \tan^{-1} x \, dx$ **[NCERT, CBSE 2022]**

(iii) $\int \sec^{-1} x \, dx$ **[NCERT]**

SOLUTION (i) Let $\sin^{-1} x = t$. Then, $x = \sin t \Rightarrow dx = d(\sin t) = \cos t \, dt$

$\therefore \quad I = \int \sin^{-1} x \, dx$

$\Rightarrow \quad I = \int \underset{\text{I}}{t} \underset{\text{II}}{\cos t} \, dt = t \sin t - \int 1 \cdot (\sin t) \, dt = t \sin t - \int \sin t \, dt = t \sin t + \cos t + C$

$\Rightarrow \quad I = x \sin^{-1} x + \sqrt{1 - \sin^2 t} + C = x \sin^{-1} x + \sqrt{1 - x^2} + C$

(ii) Let $\tan^{-1} x = t$. Then, $x = \tan t$ and $dx = \sec^2 t \, dt$

$\therefore \quad I = \int \tan^{-1} x \, dx = \int \underset{\text{I}}{t} \underset{\text{II}}{\sec^2 t} \, dt = t(\tan t) - \int 1 \cdot \tan t \, dt = t \tan t + \log|\cos t| + C$

$\Rightarrow \quad I = x \tan^{-1} x + \log\left|\frac{1}{\sqrt{1+x^2}}\right| + C \qquad \left[\because \tan t = x \Rightarrow \cos t = \frac{1}{\sqrt{1+\tan^2 t}} = \frac{1}{\sqrt{1+x^2}}\right]$

$\Rightarrow \quad I = x \tan^{-1} x - \frac{1}{2}\log|1 + x^2| + C$

(iii) Let $\sec^{-1} x = t$ then, $x = \sec t \Rightarrow dx = \sec t \tan t \, dt$

$\therefore \quad I = \int \sec^{-1} x \, dx = \int \underset{\text{I}}{t} \underset{\text{II}}{(\sec t \tan t)} \, dt = t(\sec t) - \int 1 \cdot \sec t \, dt = t \sec t - \log|(\sec t + \tan t)| + C$

$\Rightarrow I = t \sec t - \log|\sec t + \sqrt{\sec^2 t - 1}| + C = x(\sec^{-1} x) - \log|x + \sqrt{x^2 - 1}| + C$

EXAMPLE 5 *Evaluate:*

(i) $\int x \tan^{-1} x \, dx$ **[NCERT, CBSE 2000C, 2019]** (ii) $\int \frac{\log x}{x^2} \, dx$

(iii) $\int \frac{x - \sin x}{1 - \cos x} \, dx$ (iv) $\int \log(1 + x^2) \, dx$

SOLUTION (i) Let $I = \int \underset{\text{II}}{x} \underset{\text{I}}{\tan^{-1} x}\, dx$. Then,

$$I = (\tan^{-1} x)\frac{x^2}{2} - \int \frac{1}{1+x^2} \times \frac{x^2}{2}\, dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\, dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\, dx = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

(ii) Let $I = \int \frac{\log x}{x^2}\, dx = \int \underset{\text{II}}{\frac{1}{x^2}} \underset{\text{I}}{\log x}\, dx = (\log x)\left(-\frac{1}{x}\right) - \int \frac{1}{x}\left(-\frac{1}{x}\right) dx$

$$\Rightarrow \quad I = -\frac{1}{x}\log x + \int x^{-2}\, dx = -\frac{1}{x}\log x - \frac{1}{x} + C = -\frac{1}{x}(1 + \log x) + C$$

(iii) Let $I = \int \frac{x - \sin x}{1 - \cos x}\, dx$. Then,

$$I = \int \frac{x}{1-\cos x}\, dx - \int \frac{\sin x}{1 - \cos x}\, dx = \int \frac{x}{2}\operatorname{cosec}^2 \frac{x}{2}\, dx - \int \frac{2 \sin x/2 \cos x/2}{2\sin^2 x/2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \underset{\text{I}}{x} \underset{\text{II}}{\operatorname{cosec}^2 \frac{x}{2}} - \int \cot\frac{x}{2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left\{ x\left(-2\cot\frac{x}{2}\right) - \int 1\cdot\left(-2\cot\frac{x}{2}\right) dx\right\} - \int \cot\frac{x}{2}\, dx + C$$

$$\Rightarrow \quad I = -x\cot\frac{x}{2} + \int \cot\frac{x}{2}\, dx - \int \cot\frac{x}{2}\, dx + C = -x\cot\frac{x}{2} + C$$

(iv) Let $I = \int \log(1+x^2)\, dx = \int \underset{\text{I}}{\log(1+x^2)}\cdot \underset{\text{II}}{1}\, dx = x\log(1+x^2) - \int\left(\frac{1}{1+x^2}\cdot 2x\right) x\, dx$

$$\Rightarrow \quad I = x\log(1+x^2) - 2\int \frac{x^2}{x^2+1}\, dx = x\log(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\, dx$$

$$\Rightarrow \quad I = x\log(x^2+1) - 2\int 1 - \frac{1}{x^2+1}\, dx = x\log(x^2+1) - 2\left(x - \tan^{-1} x\right) + C$$

$$\Rightarrow \quad I = x\log(x^2+1) - 2x + 2\tan^{-1} x + C$$

EXAMPLE 6 *Evaluate:*

(i) $\int \sec^3 x\, dx$ (ii) $\int \sin\sqrt{x}\, dx$

(iii) $\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$ (iv) $\int (\sin^{-1} x)^2\, dx$ [NCERT, CBSE 2004]

SOLUTION (i) Let $I = \int \sec^3 x\, dx$

$$\Rightarrow \quad I = \int \underset{\text{I}}{\sec x}\cdot \underset{\text{II}}{\sec^2 x}\, dx = \sec x\left\{\int \sec^2 x\, dx\right\} - \int \left\{\left(\frac{d}{dx}(\sec x)\right)\left(\int \sec^2 x\, dx\right)\right\} dx$$

$$\Rightarrow \quad I = \sec x\tan x - \int \sec x\tan x\tan x\, dx = \sec x\tan x - \int \sec x\tan^2 x\, dx$$

$$\Rightarrow \quad I = \sec x\tan x - \int \sec x(\sec^2 x - 1)\, dx = \sec x\tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$\Rightarrow \quad I = \sec x \tan x - I + \log|\sec x + \tan x| + C$$

$$\Rightarrow \quad 2I = \sec x \tan x + \log|\sec x + \tan x| + C = \frac{1}{2}\sec\tan x + \frac{1}{2}\log|\sec x + \tan x| + C$$

(ii) Let $I = \int \sin\sqrt{x}\, dx$. Let $\sqrt{x} = t$. Then, $d(\sqrt{x}) = dt \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2\sqrt{x}\, dt$

$$\therefore \quad I = \int \sin\sqrt{x}\, dx = \int (\sin t)\, 2t\, dt = 2\int \underset{\mathrm{I}}{t} \underset{\mathrm{II}}{\sin t}\, dt$$

$$\Rightarrow \quad I = 2\left\{t(-\cos t) - \int 1\ (-\cos t)\, dt\right\} = 2\left\{-t\cos t + \int \cos t\, dt\right\}$$

$$\Rightarrow \quad I = 2(-t\cos t + \sin t) + C = 2(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x}) + C$$

(iii) Let $x = \sin t$ Then, $dx = \cos t\, dt$

$$\therefore \quad I = \int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx = \int \frac{t}{(1-\sin^2 t)^{3/2}} \cos t\, dt = \int \underset{\mathrm{I}}{t}\ \underset{\mathrm{II}}{\sec^2 t}\, dt$$

$$\Rightarrow \quad I = t\tan t - \int 1 \times \tan t\, dt = t\tan t + \log|\cos t| + C$$

$$\Rightarrow \quad I = \frac{t\sin t}{\sqrt{1-\sin^2 t}} + \log\left|\sqrt{1-\sin^2 t}\right| + C$$

$$\Rightarrow \quad I = (\sin^{-1} x)\frac{x}{\sqrt{1-x^2}} + \log\left|\sqrt{1-x^2}\right| + C = \frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2}\log|1-x^2| + C$$

(iv) Let $\sin^{-1} x = t$ Then, $x = \sin t \Rightarrow dx = \cos t\, dt$

$$\therefore \quad I = \int (\sin^{-1} x)^2\, dx$$

$$\Rightarrow \quad I = \int \underset{\mathrm{I}}{t^2} \underset{\mathrm{II}}{\cos t}\, dt = t^2(\sin t) - \int 2t\sin t\, dt = t^2\sin t - 2\int \underset{\mathrm{I}}{t}\underset{\mathrm{II}}{\sin t}\, dt$$

$$\Rightarrow \quad I = t^2\sin t - 2\left\{t(-\cos t) - \int 1 \times (-\cos t)\, dt\right\}$$

$$\Rightarrow \quad I = t^2\sin t - 2\left\{-t\cos t + \int \cos t\, dt\right\}$$

$$\Rightarrow \quad I = t^2\sin t - 2\left(-t\cos t + \sin t\right) + C = t^2\sin t - 2\left\{-t\sqrt{1-\sin^2 t} + \sin t\right\} + C$$

$$\Rightarrow \quad I = x(\sin^{-1} x)^2 - 2\left\{-\sqrt{1-x^2}\sin^{-1} x + x\right\} + C$$

EXAMPLE 7 *Evaluate:*

(i) $\int x\log(1+x)\, dx$ (ii) $\int x\cot^{-1} x\, dx$

(iii) $\int x\sin^{-1} x\, dx$ [NCERT, CBSE 2000, 2009] (iv) $\int x^2\tan^{-1} x\, dx$

(v) $\int x^3\log 2x\, dx$ (vi) $\int x^3 e^x\, dx$

SOLUTION (i) Let $I = \int \underset{\mathrm{II}}{x} \underset{\mathrm{I}}{\log(x+1)}\, dx$. Then,

$$I = \log(x+1)\frac{x^2}{2} - \int \frac{1}{x+1} \times \frac{x^2}{2}\, dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{x+1}\,dx = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2-1+1}{x+1}\,dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2-1}{x+1} + \frac{1}{x+1}\,dx = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left\{\int \left((x-1) + \frac{1}{x+1}\right)dx\right\}$$

$$\Rightarrow \quad I = \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left\{\frac{x^2}{2} - x + \log|x+1|\right\} + C$$

(ii) Let $I = \int \underset{\text{II}}{x}\ \underset{\text{I}}{\cot^{-1} x}\,dx$. Then,

$$I = (\cot^{-1} x)\left(\frac{x^2}{2}\right) - \int \frac{-1}{1+x^2} \times \frac{x^2}{2}\,dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int \frac{x^2+1-1}{x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right)dx = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\left(x - \tan^{-1} x\right) + C$$

(iii) Let $I = \int \underset{\text{II}}{x}\ \underset{\text{I}}{\sin^{-1} x}\,dx$. Then,

$$I = (\sin^{-1} x)\frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \times \frac{x^2}{2}\,dx = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad I = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left\{\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right\}$$

$$\Rightarrow \quad I = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left\{\int \sqrt{1-x^2}\,dx - \int \frac{1}{\sqrt{1-x^2}}\,dx\right\}$$

$$\Rightarrow \quad I = \frac{x^2}{2}\sin^{-1} x + \frac{1}{2}\left[\left\{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right\} - \sin^{-1} x\right] + C$$

$$\Rightarrow \quad I = \frac{1}{2}x^2 \sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} - \frac{1}{4}\sin^{-1} x + C$$

(iv) Let $I = \int \underset{\text{II}}{x^2}\ \underset{\text{I}}{\tan^{-1} x}\,dx$. Then,

$$I = (\tan^{-1} x)\frac{x^3}{3} - \int \frac{1}{1+x^2} \times \frac{x^3}{3}\,dx = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \left(x - \frac{x}{x^2+1}\right)dx = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\left\{\int x\,dx - \int \frac{x}{x^2+1}\,dx\right\}$$

$$\Rightarrow \quad I = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int \frac{x}{x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left(\frac{x^2}{2}\right) + \frac{1}{3}\int \frac{1}{2t}\,dt, \text{ where } x^2+1=t$$

$$\Rightarrow \quad I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\log|t| + C = \frac{x^3}{3}\tan^{-1}x - \frac{1}{6}x^2 + \frac{1}{6}\log|x^2+1| + C$$

(v) Let $I = \int \underset{\text{II}}{x^3}\,\underset{\text{I}}{\log 2x}\,dx$. Then,

$$I = (\log 2x)\,\frac{x^4}{4} - \int \frac{1}{2x}\times 2\times\frac{x^4}{4}\,dx = \frac{x^4}{4}\log 2x - \frac{1}{4}\int x^3\,dx = \frac{x^4}{4}\log 2x - \frac{x^4}{16} + C$$

(vi) Let $I = \int \underset{\text{I}}{x^3}\,\underset{\text{II}}{e^x}\,dx$. Then,

$$I = x^3e^x - \int 3x^2e^x\,dx = x^3e^x - 3\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{e^x}\,dx$$

$$\Rightarrow \quad I = x^3e^x - 3\left\{x^2e^x - \int 2x\,e^x\,dx\right\} = x^3e^x - 3\left\{x^2e^x - 2\int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\,dx\right\}$$

$$\Rightarrow \quad I = x^3e^x - 3\left[x^2e^x - 2\left\{x\,e^x - \int 1\cdot e^x\,dx\right\}\right]$$

$$\Rightarrow \quad I = x^3e^x - 3\left\{x^2e^x - 2\left(x\,e^x - e^x\right)\right\} + C$$

$$\Rightarrow \quad I = x^3e^x - 3x^2e^x + 6x\,e^x - 6e^x + C = (x^3 - 3x^2 + 6x - 6)\,e^x + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *Evaluate :* $\int \sin 2x \tan^{-1}(\sin x)\,dx$

SOLUTION Let $I = \int \sin 2x\tan^{-1}(\sin x)\,dx$. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \sin 2x\tan^{-1}(\sin x)\,dx = 2\int \underset{\text{II}}{t}\,\underset{\text{I}}{\tan^{-1}t}\,dt = 2\left\{(\tan^{-1}t)\,\frac{t^2}{2} - \int \frac{1}{1+t^2}\times\frac{t^2}{2}\,dt\right\}$$

$$\Rightarrow \quad I = t^2(\tan^{-1}t) - \int \frac{(t^2+1)-1}{1+t^2}\,dt = t^2\tan^{-1}t - \int\left(1 - \frac{1}{1+t^2}\right)dt$$

$$\Rightarrow \quad I = t^2\tan^{-1}t - t + \tan^{-1}t + C$$

$$\Rightarrow \quad I = \sin^2 x\tan^{-1}(\sin x) - \sin x + \tan^{-1}(\sin x) + C$$

EXAMPLE 9 *Evaluate :* $\int \cot^{-1}(1 - x + x^2)\,dx$

SOLUTION Let $I = \int \cot^{-1}(1 - x + x^2)\,dx$. Then,

$$I = \int \cot^{-1}\left\{1 - x(1-x)\right\}dx = \int \tan^{-1}\left\{\frac{1}{1 - x(1-x)}\right\}dx$$

$$\Rightarrow \quad I = \int \tan^{-1}\left\{\frac{x + (1-x)}{1 - x(1-x)}\right\}dx = \int\left\{\tan^{-1}x + \tan^{-1}(1-x)\right\}dx$$

$$\Rightarrow \quad I = \int \tan^{-1}x\,dx + \int \tan^{-1}(1-x)\,dx = I_1 + I_2 \qquad \text{...(i)}$$

where $I_1 = \int \tan^{-1}x\,dx$ and $I_2 = \int \tan^{-1}(1-x)\,dx$.

Now,

$$I_1 = \int \tan^{-1} x \, dx = \int \underset{\mathrm{I}}{\tan^{-1} x} \, \underset{\mathrm{II}}{1} \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$

$$\Rightarrow \quad I_1 = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{1+x^2} 2x \, dx = x \tan^{-1} x - \frac{1}{2} \log (1+x^2) \qquad \text{...(ii)}$$

and,

$$I_2 = \int \tan^{-1}(1-x) \, dx = -\int \tan^{-1} t \, dt, \text{ where } t = 1-x.$$

$$\Rightarrow \quad I_2 = -\left\{ t \tan^{-1} t - \frac{1}{2} \log (1+t^2) \right\} \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad I_2 = -\left[(1-x) \tan^{-1}(1-x) - \frac{1}{2} \log \left\{ 1 + (1-x)^2 \right\} \right]$$

Substituting the values of I_1 and I_2 in (i), we get

$$\int \cot^{-1}(1-x+x^2) \, dx = x \tan^{-1} x - \frac{1}{2} \log (1+x^2) - (1-x) \tan^{-1}(1-x) + \frac{1}{2} \log \left\{ 1 + (1-x)^2 \right\} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Evaluate* : $\int \left(3x^2 \tan \frac{1}{x} - x \sec^2 \frac{1}{x} \right) dx$

SOLUTION Let $I = \int \left\{ 3x^2 \tan \frac{1}{x} - x \sec^2 \frac{1}{x} \right\} dx$

$$\Rightarrow \quad I = \int \underset{\mathrm{II}}{3x^2} \, \underset{\mathrm{I}}{\tan} \frac{1}{x} \, dx - \int x \sec^2 \frac{1}{x} \, dx$$

$$\Rightarrow \quad I = x^3 \tan \left(\frac{1}{x} \right) - \int \left(\sec^2 \frac{1}{x} \right) \times -\frac{1}{x^2} \times x^3 \, dx - \int x \sec^2 \frac{1}{x} \, dx$$

$$\Rightarrow \quad I = x^3 \tan \frac{1}{x} + \int x \sec^2 \frac{1}{x} \, dx - \int x \sec^2 \frac{1}{x} \, dx + C = x^3 \tan \left(\frac{1}{x} \right) + C$$

EXAMPLE 11 *Evaluate* : $\int \frac{\log (1+x^2)}{x^3} dx$

SOLUTION Let $I = \int \frac{\log (1+x^2)}{x^3} dx$. Then,

$$I = \int \underset{\mathrm{I}}{\log (1+x^2)} \, \underset{\mathrm{II}}{x^{-3}} \, dx = -\frac{1}{2x^2} \log (1+x^2) + \int \frac{x \, dx}{x^2 (1+x^2)}$$

$$\Rightarrow \quad I = -\frac{1}{2x^2} \log (1+x^2) + \frac{1}{2} \int \frac{dt}{t(1+t)}, \quad \text{where } t = x^2$$

$$\Rightarrow \quad I = -\frac{1}{2x^2} \log (1+x^2) + \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt = -\frac{1}{2x^2} \log (1+x^2) + \frac{1}{2} \left\{ \log t - \log (t+1) \right\} + C$$

$$\Rightarrow \quad I = -\frac{1}{2x^2} \log (1+x^2) + \frac{1}{2} \log \left| \frac{x^2}{x^2+1} \right| + C$$

EXAMPLE 12 *Evaluate:* $\displaystyle\int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$ **[NCERT]**

SOLUTION We know that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$

$$\therefore \quad I = \int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx = \int \frac{\sin^{-1}\sqrt{x}-\left(\frac{\pi}{2}-\sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\,dx$$

$$\Rightarrow \quad I = \frac{2}{\pi}\int\left(2\sin^{-1}\sqrt{x}-\frac{\pi}{2}\right)dx = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - \int 1\cdot dx$$

$$\Rightarrow \quad I = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x + C = \frac{4}{\pi}I_1 - x + C, \text{ where } I_1=\int \sin^{-1}\sqrt{x}\,dx \quad ...(i)$$

Putting $x=\sin^2\theta$ and $dx = 2\sin\theta\cos\theta\,d\theta=\sin 2\theta\,d\theta$, we get

$$I_1 = \int \underset{\text{I}}{\theta}\,\underset{\text{II}}{\sin 2\theta}\,d\theta = -\theta\frac{\cos 2\theta}{2}+\int\frac{1}{2}\cos 2\theta\,d\theta = -\frac{\theta}{2}\cos 2\theta+\frac{1}{4}\sin 2\theta$$

$$\Rightarrow \quad I_1 = -\frac{1}{2}\theta(1-2\sin^2\theta)+\frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$\Rightarrow \quad I_1 = -\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x}\sqrt{1-x} \quad ...(ii)$$

From (i) and (ii), we get

$$I = \frac{4}{\pi}\left\{-\frac{1}{2}(1-2x)\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}\right\}-x+C$$

$$\Rightarrow \quad I = \frac{2}{\pi}\left\{\sqrt{x-x^2}-(1-2x)\sin^{-1}\sqrt{x}\right\}-x+C$$

EXAMPLE 13 *Evaluate:* $\displaystyle\int \frac{\sqrt{x^2+1}\left\{\log(x^2+1)-2\log x\right\}}{x^4}\,dx$ **[CBSE 2012, NCERT]**

SOLUTION Let

$$I = \int \frac{\sqrt{x^2+1}\left\{\log(x^2+1)-2\log x\right\}}{x^4}\,dx$$

$$\Rightarrow \quad I = \int \frac{\sqrt{x^2+1}}{x^4}\left\{\log(x^2+1)-\log x^2\right\}dx = \int\sqrt{1+\frac{1}{x^2}}\,\log\left(1+\frac{1}{x^2}\right)\frac{1}{x^3}\,dx$$

Let $1+\frac{1}{x^2}=t$. Then, $d\left(1+\frac{1}{x^2}\right)=dt \Rightarrow -\frac{2}{x^3}dx=dt \Rightarrow \frac{1}{x^3}dx=-\frac{1}{2}dt$

$$\therefore \quad I = -\frac{1}{2}\int \underset{\text{II}}{\sqrt{t}}\;\underset{\text{I}}{\log t}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{2}\left\{\frac{2}{3}(\log t)\,t^{3/2}-\frac{2}{3}\int\frac{1}{t}\times t^{3/2}\,dt\right\} = -\frac{1}{2}\left\{\frac{2}{3}(\log t)\,t^{3/2}-\frac{4}{9}t^{3/2}\right\}+C$$

$$\Rightarrow \quad I = -\frac{1}{3}t^{3/2}\left\{\log t-\frac{2}{3}\right\}+C = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left\{\log\left(1+\frac{1}{x^2}\right)-\frac{2}{3}\right\}+C.$$

EXAMPLE 14 *Evaluate:* $\int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$

SOLUTION Let $I = \int \frac{x^2}{(x \sin x + \cos x)^2}\, dx$. Then, $I = \int \underset{\text{I}}{(x \sec x)} \times \underset{\text{II}}{\frac{x \cos x}{(x \sin x + \cos x)^2}}\, dx$

Let $t = x \sin x + \cos x$. Then,

$$\int \frac{x \cos x}{(x \sin x + \cos x)^2}\, dx = \int \frac{1}{t^2}\, dt = -\frac{1}{t} = -\frac{1}{x \sin x + \cos x}$$

Using this as a result and integrating by parts, we obtain

$$I = (x \sec x) \times \frac{-1}{x \sin x + \cos x} - \int (\sec x + x \sec x \tan x) \times \frac{-1}{x \sin x + \cos x}\, dx$$

$$\Rightarrow \quad I = -\frac{x \sec x}{x \sin x + \cos x} + \int \sec^2 x\, dx$$

$$\Rightarrow \quad I = \frac{-x}{\cos x\,(x \sin x + \cos x)} + \tan x + C = \frac{(\sin x - x \cos x)}{x \sin x + \cos x} + C$$

EXAMPLE 15 *Find an anti-derivative of the function* $f(x)\, g''(x) - f''(x)\, g(x)$

SOLUTION Required anti-derivative of $f(x)\, g''(x) - f''(x)\, g(x)$ is given by

$$\int \{f(x)\, g''(x) - f''(x)\, g(x)\}\, dx$$

$$= \int \underset{\text{I}}{f(x)}\, \underset{\text{II}}{g''(x)}\, dx - \int \underset{\text{II}}{f''(x)}\, \underset{\text{I}}{g(x)}\, dx$$

$$= \left\{ f(x)\, g'(x) - \int f'(x)\, g'(x)\, dx \right\} - \left\{ g(x)\, f'(x) - \int f'(x)\, g'(x)\, dx \right\} + C$$

$$= f(x)\, g'(x) - g(x)\, f'(x) + C$$

EXAMPLE 16 *Evaluate:* $\int \sin^{-1} \left\{ \frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right\} dx$

SOLUTION Let

$$I = \int \sin^{-1} \left\{ \frac{2x+2}{\sqrt{4x^2 + 8x + 13}} \right\} dx = \sin^{-1} \left\{ \frac{2x+2}{\sqrt{(2x+2)^2 + 3^2}} \right\} dx$$

Substituting $2x + 2 = 3 \tan \theta$ and $dx = \frac{3}{2} \sec^2 \theta\, d\theta$, we get

$$I = \int \sin^{-1} \left(\frac{3 \tan \theta}{3 \sec \theta} \right) \times \frac{3}{2} \sec^2 \theta\, d\theta = \frac{3}{2} \int \underset{\text{I}}{\theta}\, \underset{\text{II}}{\sec^2 \theta}\, d\theta$$

$$\Rightarrow \quad I = \frac{3}{2} \left\{ \theta \tan \theta - \int \tan \theta\, d\theta \right\} = \frac{3}{2} \left\{ \theta \tan \theta - \log |\sec \theta| \right\}$$

$$\Rightarrow \quad I = \frac{3}{2} \left\{ \left(\frac{2x+2}{3} \right) \tan^{-1} \left(\frac{2x+2}{3} \right) - \log \sqrt{1 + \left(\frac{2x+2}{3} \right)^2} \right\} + C$$

$$\Rightarrow \quad I = \frac{3}{2} \left\{ \left(\frac{2x+2}{3} \right) \tan^{-1} \left(\frac{2x+2}{3} \right) - \log \sqrt{4x^2 + 8x + 13} \right\} + C$$

$$\Rightarrow \quad I = (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log (4x^2 + 8x + 13) + C$$

EXERCISE 18.25

BASIC

Evaluate the following integrals:

1. $\int x \cos x \, dx$
2. $\int \log (x+1) \, dx$
3. $\int x^3 \log x \, dx$
4. $\int x e^x \, dx$ [NCERT]
5. $\int x e^{2x} \, dx$
6. $\int x^2 e^{-x} \, dx$
7. $\int x^2 \cos x \, dx$
8. $\int x^2 \cos 2x \, dx$
9. $\int x \sin 2x \, dx$
10. $\int \dfrac{\log (\log x)}{x} \, dx$
11. $\int \cos x \tan^{-1}(\sin x) \, dx$ [CBSE 2022]
12. $\int x \operatorname{cosec}^2 x \, dx$
13. $\int x^n \log x \, dx$ [CBSE 2020]
14. $\int \dfrac{\log x}{x^n} \, dx$
15. $\int x \sin x \cos x \, dx$
16. $\int (\log x)^2 \, x \, dx$
17. $\int \dfrac{\log (x+2)}{(x+2)^2} \, dx$
18. $\int \log_{10} x \, dx$
19. $\int \dfrac{\log x}{(x+1)^2} \, dx$ [CBSE 2015]
20. $\int x \tan^2 x \, dx$
21. $\int x \left(\dfrac{\sec 2x - 1}{\sec 2x + 1} \right) dx$
22. $\int (x+1) \log x \, dx$ [CBSE 2002C]
23. $\int (e^{\log x} + \sin x) \cos x \, dx$
24. $\int \sin^{-1} 2x \, dx$ [CBSE2019]
25. $\int x \cos^2 x \, dx$

BASED ON LOTS

26. $\int x^2 \sin^2 x \, dx$
27. $\int 2x^3 e^{x^2} \, dx$
28. $\int x^3 \cos x^2 \, dx$
29. $\int \sin x \log (\cos x) \, dx$ [CBSE 2019]
30. $\int e^{\sqrt{x}} \, dx$
31. $\int \dfrac{x + \sin x}{1 + \cos x} \, dx$
32. $\int \cos \sqrt{x} \, dx$
33. $\int \dfrac{x \cos^{-1} x}{\sqrt{1-x^2}} \, dx$ [CBSE 2014]
34. $\int \operatorname{cosec}^3 x \, dx$
35. $\int \sec^{-1} \sqrt{x} \, dx$
36. $\int \sin^{-1} \sqrt{x} \, dx$
37. $\int (x+1) \, e^x \log (x e^x) \, dx$
38. $\int \sin^{-1}(3x - 4x^3) \, dx$
39. $\int \sin^{-1} \left(\dfrac{2x}{1+x^2} \right) dx$ [NCERT]
40. $\int \tan^{-1} \left(\dfrac{3x - x^3}{1 - 3x^2} \right) dx$
41. $\int x^2 \sin^{-1} x \, dx$
42. $\int \dfrac{\sin^{-1} x}{x^2} \, dx$ [CBSE 2004]
43. $\int \dfrac{x^2 \tan^{-1} x}{1 + x^2} \, dx$
44. $\int \cos^{-1} (4x^3 - 3x) \, dx$
45. $\int \cos^{-1} \left(\dfrac{1 - x^2}{1 + x^2} \right) dx$
46. $\int \tan^{-1} \left(\dfrac{2x}{1 - x^2} \right) dx$
47. $\int x^2 \tan^{-1} x \, dx$ [CBSE 2012]
48. $\int \dfrac{(x \tan^{-1} x)}{(1 + x^2)^{3/2}} \, dx$
49. $\int \tan^{-1} (\sqrt{x}) \, dx$
50. $\int x \sin x \cos 2x \, dx$
51. $\int (\tan^{-1} x^2) \, x \, dx$
52. $\int \dfrac{x \sin^{-1} x}{\sqrt{1 - x^2}} \, dx$ [CBSE 2012, 2016, NCERT]

BASED ON HOTS

53. $\int \sin^3 \sqrt{x}\, dx$

54. $\int x \sin^3 x\, dx$

55. $\int \cos^3 \sqrt{x}\, dx$

56. $\int x \cos^3 x\, dx$

57. $\int \tan^{-1} \sqrt{\frac{1-x}{1+x}}\, dx$ [NCERT]

58. $\int \sin^{-1} \sqrt{\frac{x}{a+x}}\, dx$

59. $\int \frac{x^3 \sin^{-1} x^2}{\sqrt{1-x^4}}\, dx$

60. $\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

ANSWERS

1. $x \sin x + \cos x + C$
2. $x \log (x+1) - x + \log (x+1) + C$
3. $\frac{x^4}{4} \log x - \frac{x^4}{16} + C$
4. $(x-1)\, e^x + C$
5. $\left(\frac{x}{2} - \frac{1}{4}\right) e^{2x} + C$
6. $-e^{-x} (x^2 + 2x + 2) + C$
7. $x^2 \sin x + 2x \cos x - 2 \sin x + C$
8. $\frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{\sin 2x}{4} + C$
9. $-\frac{x}{2} \cos 2x + \frac{1}{4} \sin 2x + C$
10. $\log x \{\log (\log x) - 1\} + C$
11. $\sin x (\tan^{-1} x) - \log (1 + \sin^2 x) + C$
12. $-x \cot x + \log |\sin x| + C$
13. $\frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} + C$
14. $\frac{x^{1-n}}{1-n} \log x - \frac{x^{1-n}}{(1-n)^2} + C$
15. $-\frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$
16. $\frac{x^2}{2} \left\{(\log x)^2 - \log x + \frac{1}{2}\right\} + C$
17. $-\frac{1}{(x+2)} - \frac{\log (x+2)}{(x+2)} + C$
18. $\frac{1}{\log 10} \left\{x (\log x - 1)\right\} + C$
19. $-\frac{\log x}{x+1} + \log \left|\frac{x}{x+1}\right| + C$
20. $x \tan x - \log |\sec x| - \frac{x^2}{2} + C$
21. $x \tan x - \log |\sec x| - \frac{x^2}{2} + C$
22. $\left(x + \frac{x^2}{2}\right) \log x - \left(x + \frac{x^2}{4}\right) + C$
23. $x \sin x + \cos x + \frac{1}{2} \sin^2 x + C$
24. $x \sin^{-1} 2x + \frac{1}{2} \sqrt{1 - 4x^2} + C$
25. $\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + C$
26. $\frac{1}{6} x^3 - \frac{1}{4} x^2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{8} \sin 2x + C$
27. $e^{x^2} (x^2 - 1) + C$
28. $\frac{1}{2} x^2 \sin x^2 + \frac{1}{2} \cos x^2 + C$
29. $\cos x (1 - \log \cos x) + C$
30. $2\, e^{\sqrt{x}} (\sqrt{x} - 1) + C$
31. $x \tan \frac{x}{2} + C$
32. $2 \left\{\sqrt{x} \sin \sqrt{x} + \cos \sqrt{x}\right\} + C$
33. $-\left\{\sqrt{1-x^2} \cos^{-1} x + x\right\} + C$
34. $-\frac{1}{2} \operatorname{cosec} x \cot x + \frac{1}{2} \log \left|\tan \frac{x}{2}\right| + C$
35. $x \sec^{-1} \sqrt{x} - \sqrt{x-1} + C$
36. $\frac{1}{2} (2x-1) \sin^{-1} \sqrt{x} + \frac{1}{2} \sqrt{x - x^2} + C$

37. $x e^x \left\{\log(x e^x) - 1\right\} + C$

38. $3x \sin^{-1} x + 3\sqrt{1-x^2} + C$

39. $2x \tan^{-1} x - \log|1+x^2| + C$

40. $3x \tan^{-1} x - \frac{3}{2}\log|x^2+1| + C$

41. $\frac{x^3}{3}\sin^{-1} x + \frac{1}{3}\sqrt{1-x^2} - \frac{1}{9}(1-x^2)^{3/2} + C$

42. $-\frac{\sin^{-1} x}{x} + \log\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C$

43. $x \tan^{-1} x - \frac{1}{2}\log|1+x^2| - \frac{1}{2}(\tan^{-1} x)^2 + C$

44. $3x \cos^{-1} x - 3\sqrt{1-x^2} + C$

45. $2x \tan^{-1} x - \log|1+x^2| + C$

46. $2x \tan^{-1} x - \log|1+x^2| + C$

47. $\frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\log|x^2+1| + C$

48. $-\frac{\tan^{-1} x}{\sqrt{1+x^2}} + \frac{x}{\sqrt{1+x^2}} + C$

49. $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

50. $\frac{1}{2}\left\{-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + x\cos x - \sin x\right\} + C$

51. $\frac{1}{2}x^2\tan^{-1} x^2 - \frac{1}{4}\log(1+x^4) + C$

52. $-\sqrt{1-x^2}\sin^{-1} x + x + C$

53. $-\frac{3}{2}\sqrt{x}\cos\sqrt{x} + \frac{3}{2}\sin\sqrt{x} + \frac{1}{6}\sqrt{x}\cos 3\sqrt{x} - \frac{1}{18}\sin 3\sqrt{x} + C$

54. $-\frac{3x\cos x}{4} + \frac{3\sin x}{4} + \frac{x\cos 3x}{12} - \frac{\sin 3x}{36} + C$

55. $\frac{\sqrt{x}}{6}\sin 3\sqrt{x} + \frac{\cos 3\sqrt{x}}{18} + \frac{3}{2}\sqrt{x}\sin\sqrt{x} + \frac{3}{2}\cos\sqrt{x} + C$

56. $\frac{x\sin 3x}{12} + \frac{\cos 3x}{36} + \frac{3}{4}x\sin x + \frac{3}{4}\cos x + C$

57. $\frac{1}{2}x(\cos^{-1} x) - \frac{1}{2}\sqrt{1-x^2} + C$

58. $x\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + a\tan^{-1}\sqrt{\frac{x}{a}} + C$

59. $\frac{1}{2}\left\{x^2 - \sqrt{1-x^4}\sin^{-1} x^2\right\} + C$

60. $\frac{x\sin^{-1} x}{\sqrt{1-x^2}} - \frac{1}{2}(\sin^{-1} x)^2 + \frac{1}{2}\log(1-x^2) + C$

18.11.1 INTEGRALS OF THE FORM $\int e^x\left\{f(x) + f'(x)\right\}dx$

THEOREM *Prove that:* $\int e^x\left\{f(x) + f'(x)\right\}dx = e^x f(x) + C.$

PROOF We have,

$$\int e^x\{f(x) + f'(x)\}\,dx = \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{f(x)}\,dx + \int e^x f'(x)\,dx$$

$$\Rightarrow \int e^x\{f(x) + f'(x)\}\,dx = f(x)\cdot e^x - \int f'(x)\,e^x\,dx + \int e^x f'(x)\,dx + C$$

$$\Rightarrow \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + C$$

Q.E.D.

This theorem suggests the following algorithm to evaluate integrals of the form

$$\int e^x\left\{f(x) + f'(x)\right\}dx.$$

ALGORITHM

Step I *Express the integral as the sum of two integrals, one consisting of $f(x)$ and other containing $f'(x)$.*

i.e. $\int e^x \left\{f(x) + f'(x)\right\} dx = \int e^x f(x)\, dx + \int e^x f'(x)\, dx$

Step II *Evaluate the first integral on RHS using integration by parts by taking e^x as the second function. The second integral on RHS will cancel out from the second term obtained by evaluating the first integral.*

NOTE *The above theorem is also true if we have e^{kx} in place of e^x.*

i.e. $\int e^{kx} \left\{k\, f(x) + f'(x)\right\} dx = e^{kx} f(x) + C$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ [NCERT] (ii) $\int e^x (\sin x + \cos x)\, dx$ [NCERT]

(iii) $\int \{\sin(\log x) + \cos(\log x)\}\, dx$

SOLUTION (i) Let $I = \int e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$. Then, $I = \int e^x \left(\underset{f}{\frac{1}{x}} + \underset{f'}{\frac{(-1)}{x^2}}\right) dx$

$$\Rightarrow \quad I = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\frac{1}{x}}\, dx - \int e^x \frac{1}{x^2}\, dx = \frac{1}{x} \cdot e^x - \int -\frac{1}{x^2} e^x\, dx - \int e^x \cdot \frac{1}{x^2}\, dx + C$$

$$\Rightarrow \quad I = \frac{1}{x} \cdot e^x + \int \frac{1}{x^2} \cdot e^x\, dx - \int \frac{1}{x^2} \cdot e^x\, dx + C = \frac{1}{x} e^x + C$$

(ii) Let $I = \int e^x (\underset{f}{\sin x} + \underset{f'}{\cos x})\, dx = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\sin x}\, dx + \int e^x \cos x\, dx$

$$\Rightarrow \quad I = (\sin x) e^x - \int (\cos x) e^x\, dx + \int e^x \cos x\, dx + C = e^x \sin x + C$$

(iii) Let $I = \int \{\sin(\log x) + \cos(\log x)\}\, dx$. Let $\log x = t$. Then, $x = e^t \Rightarrow dx = d(e^t) = e^t\, dt$

$$\therefore \quad I = \int e^t (\underset{f}{\sin t} + \underset{f'}{\cos t})\, dt = \int \underset{\text{II}}{e^t} \underset{\text{I}}{\sin t}\, dt + \int e^t \cos t\, dt$$

$$\Rightarrow \quad I = \sin t \cdot e^t - \int \cos t \cdot e^t\, dt + \int \cos t \cdot e^t\, dt + C$$

$$\Rightarrow \quad I = e^t \cdot \sin t + C = e^{\log x} \sin(\log x) + C = x \sin(\log x) + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int e^x (\tan x + \log \sec x)\, dx$ (ii) $\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) dx$ [NCERT]

(iii) $\int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x}\right) dx$ (iv) $\int e^x \left(\frac{1 - \sin x}{1 - \cos x}\right) dx$ [NCERT]

SOLUTION (i) Let $I = \int e^x (\tan x + \log \sec x)\, dx = \int e^x \left(\underset{f}{\log \sec x} + \underset{f'}{\tan x} \right) dx$

$\Rightarrow \quad I = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\log \sec x}\, dx + \int e^x \tan x\, dx$

$\Rightarrow \quad I = (\log \sec x)\, e^x - \int \frac{1}{\sec x} \times \sec x \tan x\, e^x\, dx + \int e^x \tan x\, dx + C$

$\Rightarrow \quad I = e^x \log \sec x - \int e^x \tan x\, dx + \int e^x \tan x\, dx + C = e^x \cdot \log \sec x + C$

(ii) Let $I = \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$. Then,

$I = \int e^x \left(\frac{2 + 2 \sin x \cos x}{2 \cos^2 x} \right) dx$

$\Rightarrow \quad I = \int e^x (\underset{f'}{\sec^2 x} + \underset{f}{\tan x})\, dx = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\tan x}\, dx + \int e^x \sec^2 x\, dx$

$\Rightarrow \quad I = (\tan x)\, e^x - \int \sec^2 x\, e^x\, dx + \int e^x \sec^2 x\, dx + C = e^x \tan x + C$

(iii) Let $I = \int e^x \left(\frac{1 + \sin x \cos x}{\cos^2 x} \right) dx = \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} \right) dx$

$\Rightarrow \quad I = \int e^x (\underset{f'}{\sec^2 x} + \underset{f}{\tan x})\, dx = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\tan x}\, dx + \int e^x \sec^2 x\, dx$

$\Rightarrow \quad I = (\tan x)\, e^x - \int \sec^2 x\, e^x\, dx + \int e^x \sec^2 x\, dx + C = e^x \tan x + C$

(iv) Let $I = \int e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$. Then,

$I = \int e^x \left(\frac{1 - 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}} \right) dx = \int e^x \left(\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} - \cot \frac{x}{2} \right) dx$

$\Rightarrow \quad I = \int e^x \left\{ \underset{f}{\left(- \cot \frac{x}{2} \right)} + \underset{f'}{\frac{1}{2} \operatorname{cosec}^2 \frac{x}{2}} \right\} dx = - \int \underset{\text{II}}{e^x} \underset{\text{I}}{\cot} \frac{x}{2}\, dx + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx$

$\Rightarrow \quad I = - \left\{ \cot \frac{x}{2} \cdot e^x - \int - \operatorname{cosec}^2 \frac{x}{2} \cdot \frac{1}{2} e^x\, dx \right\} + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx$

$\Rightarrow \quad I = - e^x \cot \frac{x}{2} - \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx + \frac{1}{2} \int e^x \operatorname{cosec}^2 \frac{x}{2}\, dx + C = - e^x \cot \frac{x}{2} + C$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:*

(i) $\int e^x \frac{x}{(x+1)^2}\, dx$ [NCERT] (ii) $\int e^x \frac{x-3}{(x-1)^3}\, dx$ [NCERT] (iii) $\int \frac{\log x}{(1 + \log x)^2}\, dx$

SOLUTION (i) Let $I = \int e^x \cdot \frac{x}{(x+1)^2}\, dx$. Then,

$$I = \int e^x \frac{x+1-1}{(x+1)^2}\, dx = \int e^x \left\{ \underbrace{\frac{1}{x+1}}_{f} + \underbrace{\frac{-1}{(x+1)^2}}_{f'} \right\} dx = \int \underset{\text{II}}{e^x} \underset{\text{I}}{\frac{1}{x+1}}\, dx + \int e^x \frac{(-1)}{(x+1)^2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{x+1} e^x - \int \frac{-1}{(x+1)^2} e^x\, dx + \int e^x \frac{(-1)}{(x+1)^2}\, dx + C = \frac{1}{x+1} e^x + C$$

(ii) Let $I = \int e^x \frac{x-3}{(x-1)^3}\, dx$. Then,

$$I = \int e^x \frac{(x-1)-2}{(x-1)^3}\, dx = \int e^x \left\{ \underbrace{\frac{1}{(x-1)^2}}_{f} + \underbrace{\frac{(-2)}{(x-1)^3}}_{f'} \right\} dx$$

$$\Rightarrow \quad I = \int \underset{\text{II}}{e^x} \times \underset{\text{I}}{\frac{1}{(x-1)^2}}\, dx + \int e^x \times \frac{-2}{(x-1)^3}\, dx$$

$$\Rightarrow \quad I = \frac{1}{(x-1)^2} e^x - \int \frac{-2}{(x-1)^3} \times e^x dx + \int e^x \times \frac{-2}{(x-1)^3}\, dx + C$$

$$\Rightarrow \quad I = \frac{e^x}{(x-1)^2} + 2\int \frac{e^x}{(x-1)^3}\, dx - 2\int \frac{e^x}{(x-1)^3}\, dx + C = \frac{e^x}{(x-1)^2} + C$$

(iii) $I = \int \frac{\log x}{(1+\log x)^2}\, dx$. Let $\log x = t$. Then, $x = e^t \Rightarrow dx = d(e^t) = e^t\, dt$

$$\therefore \quad I = \int \frac{t\, e^t}{(t+1)^2}\, dt = \int \frac{(t+1)-1}{(t+1)^2} e^t\, dt = \int \left\{ \underbrace{\frac{1}{t+1}}_{f} + \underbrace{\frac{-1}{(t+1)^2}}_{f'} \right\} e^t dt$$

$$\Rightarrow \quad I = \int \underset{\text{I}}{\frac{1}{t+1}} \underset{\text{II}}{e^t}\, dt + \int \frac{-1}{(t+1)^2} e^t dt = \frac{1}{t+1} e^t - \int \frac{-1}{(t+1)^2} e^t dt + \int \frac{-1}{(t+1)^2} e^t dt + C$$

$$\Rightarrow \quad I = \frac{e^t}{t+1} + C = \frac{x}{(\log x + 1)} + C$$

EXAMPLE 4 *Evaluate:*

(i) $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$ (ii) $\int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$ **[CBSE 2010]**

SOLUTION (i) Let $I = \int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$. Putting $\log_e x = t$ or $x = e^t$ and $dx = e^t\, dt$, we obtain

$$I = \int \left(\underset{f}{\frac{1}{t}} + \underset{f'}{\frac{(-1)}{t^2}}\right) e^t dt = \int \underset{\text{II}}{e^t} \underset{\text{I}}{\frac{1}{t}} dt - \int e^t \frac{1}{t^2} dt$$

$$\Rightarrow \quad I = \frac{1}{t} e^t - \int -\frac{1}{t^2} \times e^t \, dt - \int e^t \times \frac{1}{t^2} dt + C = \frac{1}{t} e^t + C = \frac{x}{\log x} + C$$

(ii) Let $I = \int \left\{ \log(\log x) + \frac{1}{(\log x)^2} \right\} dx$. Let $\log x = t$. Then, $x = e^t \Rightarrow dx = d(e^t) = e^t \, dt$.

$$\therefore \quad I = \int \left\{ \log t + \frac{1}{t^2} \right\} e^t \, dt$$

$$\Rightarrow \quad I = \int \left\{ \log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right\} e^t \, dt = \int \left(\underset{f}{\log t} + \underset{f'}{\frac{1}{t}} \right) e^t \, dt + \int \left(\underset{f}{-\frac{1}{t}} + \underset{f'}{\frac{1}{t^2}} \right) e^t \, dt$$

$$\Rightarrow \quad I = \int \underset{\text{II}}{e^t} \underset{\text{I}}{\log t} \, dt + \int e^t \cdot \frac{1}{t} dt + \int \underset{\text{II}}{e^t} \underset{\text{I}}{(-1/t)} \, dt + \int e^t \frac{1}{t^2} dt$$

$$\Rightarrow \quad I = (\log t) \, e^t - \int \frac{1}{t} \cdot e^t \, dt + \int e^t \cdot \frac{1}{t} dt + \left(\frac{-1}{t}\right) e^t - \int \frac{1}{t^2} . e^t \, dt + \int e^t \, \frac{1}{t^2} dt + C$$

$$\Rightarrow \quad I = e^t . \log t - \frac{1}{t} e^t + C = x \log(\log x) - \frac{x}{\log x} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Evaluate:* (i) $\int e^x \frac{x^2+1}{(x+1)^2} dx$ (ii) $\int e^x \frac{(1-x)^2}{(1+x^2)^2} dx$

SOLUTION (i) Let $I = \int e^x \frac{x^2+1}{(x+1)^2} dx = \int e^x \left\{ 1 - \frac{2x}{(x+1)^2} \right\} dx = \int e^x dx - 2 \int e^x \frac{x}{(x+1)^2} dx$

$$\Rightarrow \quad I = e^x - 2 \int e^x \cdot \frac{x+1-1}{(x+1)^2} dx = e^x - 2 \int e^x \left\{ \underset{f}{\frac{1}{x+1}} + \underset{f'}{\frac{-1}{(x+1)^2}} \right\} dx$$

$$\Rightarrow \quad I = e^x - 2 \left\{ \int \underset{\text{II}}{e^x} \cdot \underset{\text{I}}{\frac{1}{x+1}} dx - \int e^x \frac{1}{(x+1)^2} dx \right\}$$

$$\Rightarrow \quad I = e^x - 2 \left\{ \frac{1}{x+1} e^x - \int -\frac{1}{(x+1)^2} e^x \, dx - \int e^x \frac{1}{(x+1)^2} dx \right\}$$

$$\Rightarrow \quad I = e^x - 2 \left\{ \frac{1}{x+1} \cdot e^x + \int e^x \cdot \frac{1}{(x+1)^2} dx - \int e^x \cdot \frac{1}{(x+1)^2} dx \right\} + C$$

$$\Rightarrow \quad I = e^x - \frac{2 e^x}{x+1} + C$$

(ii) Let $I = \int e^x \frac{(1-x)^2}{(1+x^2)^2}\, dx$. Then,

$$I = \int e^x \frac{1-2x+x^2}{(1+x^2)^2}\, dx = \int e^x \frac{(1+x^2)+(-2x)}{(1+x^2)^2}\, dx = \int e^x \left\{ \underbrace{\frac{1}{1+x^2}}_{f} + \underbrace{\frac{(-2x)}{(1+x^2)^2}}_{f'} \right\} dx$$

$$\Rightarrow \quad I = \int \underset{\text{II}}{e^x}\, \underset{\text{I}}{\frac{1}{(1+x^2)}}\, dx + \int e^x \frac{(-2x)}{(1+x^2)^2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{1+x^2}\, e^x - \int \frac{(-2x)}{(1+x^2)^2}\, e^x dx + \int e^x \frac{(-2x)}{(1+x^2)^2}\, dx + C = \frac{e^x}{1+x^2} + C$$

EXAMPLE 6 *Evaluate:* $\int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x} \right) dx$ [CBSE 2010]

SOLUTION Let $I = \int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x} \right) dx = \int e^{2x} \left\{ \frac{1+2\sin x \cos x}{2\cos^2 x} \right\} dx$

$$\Rightarrow \quad I = \int e^{2x} \left\{ \frac{1}{2}\sec^2 x + \tan x \right\} dx = \int e^{2x} \left\{ 2\left(\underbrace{\frac{1}{2}\tan x}_{f} \right) + \underbrace{\frac{1}{2}\sec^2 x}_{f'} \right\} dx$$

$$\Rightarrow \quad I = \int \underset{\text{II}}{e^{2x}} \cdot \underset{\text{I}}{\tan x}\, dx + \frac{1}{2}\int e^{2x} \cdot \sec^2 x\, dx$$

$$\Rightarrow \quad I = (\tan x)\frac{e^{2x}}{2} - \int \sec^2 x \cdot \frac{e^{2x}}{2}\, dx + \frac{1}{2}\int e^{2x} \sec^2 x\, dx + C$$

$$\Rightarrow \quad I = \frac{1}{2} e^{2x} \tan x - \frac{1}{2}\int e^{2x} \sec^2 x\, dx + \frac{1}{2}\int e^{2x} \sec^2 x\, dx + C = \frac{1}{2} e^{2x} \tan x + C$$

EXERCISE 18.26

BASIC

Evaluate the following integrals:

1. $\int e^x (\cos x - \sin x)\, dx$
2. $\int e^x \left(\frac{1}{x^2} - \frac{2}{x^3} \right) dx$
3. $\int e^x \left(\frac{1+\sin x}{1+\cos x} \right) dx$ [NCERT]
4. $\int e^x (\cot x - \operatorname{cosec}^2 x)\, dx$
5. $\int e^x \left(\frac{x-1}{2x^2} \right) dx$
6. $\int e^x \sec x (1 + \tan x)\, dx$
7. $\int e^x (\tan x - \log \cos x)\, dx$
8. $\int e^x [\sec x + \log (\sec x + \tan x)]\, dx$
9. $\int e^x (\cot x + \log \sin x)\, dx$
10. $\int e^x \left(\frac{\sin 4x - 4}{1 - \cos 4x} \right) dx$ [CBSE 2010]
11. $\int e^x \left(\log x + \frac{1}{x} \right) dx$
12. $\int e^x \cdot \frac{\sqrt{1-x^2}\, \sin^{-1} x + 1}{\sqrt{1-x^2}}\, dx$

13. $\int e^x \left(\tan^{-1} x + \dfrac{1}{1+x^2} \right) dx$

14. $\int e^x \left(\dfrac{\sin x \cos x - 1}{\sin^2 x} \right) dx$

BASED ON LOTS

15. $\int e^x \dfrac{x-1}{(x+1)^3} dx$

16. $\int \dfrac{2-x}{(1-x)^2} e^x \, dx$

17. $\int e^x \dfrac{1+x}{(2+x)^2} dx$

18. $\int \dfrac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} dx$ [CBSE 2013]

19. $\int e^x \left(\log x + \dfrac{1}{x^2} \right) dx$

20. $\int \dfrac{e^x}{x} \left\{ x (\log x)^2 + 2 \log x \right\} dx$

21. $\int e^{2x} (-\sin x + 2 \cos x) \, dx$

22. $\int \{\tan (\log x) + \sec^2 (\log x)\} \, dx$

23. $\int e^{2x} \left(\dfrac{1-\sin 2x}{1-\cos 2x} \right) dx$ [CBSE 2013]

24. $\int \dfrac{\log x - 3}{(\log x)^4} dx$ [CBSE 2022]

BASED ON HOTS

25. $\int e^x \dfrac{(x-4)}{(x-2)^3} dx$ [CBSE 2009]

ANSWERS

1. $e^x \cos x + C$
2. $\dfrac{e^x}{x^2} + C$
3. $e^x \tan \dfrac{x}{2} + C$
4. $e^x \cot x + C$
5. $\dfrac{e^x}{2x} + C$
6. $e^x \sec x + C$
7. $e^x \log \sec x + C$
8. $e^x \log (\sec x + \tan x) + C$
9. $e^x \log \sin x + C$
10. $e^x \cot 2x + C$
11. $e^x \log x + C$
12. $e^x \sin^{-1} x + C$
13. $e^x \tan^{-1} x + C$
14. $e^x \cot x + C$
15. $\dfrac{e^x}{(x+1)^2} + C$
16. $\dfrac{e^x}{1-x} + C$
17. $\dfrac{e^x}{x+2} + C$
18. $-e^{-x/2} \sec (x/2) + C$
19. $e^x \left(\log x - \dfrac{1}{x} \right) + C$
20. $e^x (\log x)^2 + C$
21. $e^{2x} \cos x + C$
22. $x \tan (\log x) + C$
23. $-\dfrac{1}{2} e^{2x} \cot x + C$
24. $\dfrac{x}{(\log x)^3} + C$
25. $\dfrac{e^x}{(x-2)^2} + C$

18.11.2 INTEGRALS OF THE FORM $\int e^{ax} \sin bx \, dx$ AND $\int e^{ax} \cos bx \, dx$

In this section, we will discuss problems based upon integrals of the form $\int e^{ax} \sin bx \, dx$ and $\int e^{ax} \cos bx \, dx$. In order to evaluate this type of integrals, we may use the following formulae:

THEOREM *Prove that:*

(i) $\int e^{ax} \sin bx \, dx = \dfrac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$ [CBSE 2002]

(ii) $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$

PROOF (i) Let $I = \int e^{ax} \sin bx \, dx$. Then,

$$I = \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\sin bx} \, dx = -e^{ax} \frac{\cos bx}{b} - \int a e^{ax} \left(\frac{-\cos bx}{b} \right) dx$$

$$\Rightarrow \quad I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\cos bx} \, dx = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b} \left\{ e^{ax} \frac{\sin bx}{b} - \int a e^{ax} \frac{\sin bx}{b} dx \right\}$$

$$\Rightarrow \quad I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} \int e^{ax} \sin bx \, dx$$

$$\Rightarrow \quad I = -\frac{1}{b} e^{ax} \cos bx + \frac{a}{b^2} e^{ax} \sin bx - \frac{a^2}{b^2} I$$

$$\Rightarrow \quad I + I \cdot \frac{a^2}{b^2} = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx) \Rightarrow I \left(\frac{a^2+b^2}{b^2} \right) = \frac{e^{ax}}{b^2}(a \sin bx - b \cos bx)$$

$$\Rightarrow \quad I = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + C$$

(ii) Similarly, we can prove that $\int e^{ax} \cos bx \, dx = \dfrac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + C$ **Q.E.D.**

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int e^{2x} \sin 3x \, dx$ (ii) $\int e^{-x} \cos x \, dx$

SOLUTION (i) Let $I = \int e^{2x} \sin 3x \, dx$. Then,

$$I = \int \underset{\text{I}}{e^{2x}} \underset{\text{II}}{\sin 3x} \, dx$$

$$\Rightarrow \quad I = e^{2x}\left(-\frac{\cos 3x}{3}\right) - \int 2e^{2x}\left(-\frac{\cos 3x}{3}\right) dx = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int \underset{\text{I}}{e^{2x}} \underset{\text{II}}{\cos 3x} \, dx$$

$$\Rightarrow \quad I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left\{ e^{2x} \frac{\sin 3x}{3} - \int 2e^{2x} \frac{\sin 3x}{3} dx \right\}$$

$$\Rightarrow \quad I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} \int e^{2x} \sin 3x \, dx$$

$$\Rightarrow \quad I = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x - \frac{4}{9} I$$

$$\Rightarrow \quad I + \frac{4}{9} I = \frac{e^{2x}}{9}(2 \sin 3x - 3 \cos 3x)$$

$$\Rightarrow \quad \frac{13}{9} I = \frac{e^{2x}}{9}(2 \sin 3x - 3 \cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(2 \sin 3x - 3 \cos 3x) + C$$

(ii) Let $I = \int e^{-x} \cos x \, dx$. Then,

$$I = \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\cos x}\, dx$$

$$\Rightarrow \quad I = e^{-x} \sin x - \int -e^{-x} \sin x\, dx$$

$$\Rightarrow \quad I = e^{-x} \sin x + \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\sin x}\, dx$$

$$\Rightarrow \quad I = e^{-x} \sin x + e^{-x}(-\cos x) - \int (-e^{-x})(-\cos x)\, dx$$

$$\Rightarrow \quad I = e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x\, dx$$

$$\Rightarrow \quad I = e^{-x} \sin x - e^{-x} \cos x - I$$

$$\Rightarrow \quad 2I = e^{-x}(\sin x - \cos x) \Rightarrow I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int e^{ax} \cos(bx + c)\, dx$ (ii) $\int \sin(\log x)\, dx$ (iii) $\int e^{x} \cos^2 x\, dx$

SOLUTION (i) Let $I = \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\cos(bx + c)}\, dx$. Integrating by parts, we get

$$I = \frac{e^{ax}}{b} \sin(bx + c) - \int a e^{ax} \frac{\sin(bx + c)}{b}\, dx$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b} \sin(bx + c) - \frac{a}{b} \int \underset{\text{I}}{e^{ax}} \underset{\text{II}}{\sin(bx + c)}\, dx$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b} \sin(bx + c) - \frac{a}{b} \left\{ -e^{ax} \frac{\cos(bx + c)}{b} - \int a e^{ax} \frac{\cos(bx + c)}{b}\, dx \right\}$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b} \sin(bx + c) - \frac{a}{b} \left\{ -\frac{e^{ax}}{b} \cos(bx + c) + \frac{a}{b} \int e^{ax} \cos(bx + c)\, dx \right\}$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b} \sin(bx + c) - \frac{a}{b} \left\{ -\frac{e^{ax}}{b} \cos(bx + c) + \frac{a}{b} I \right\}$$

$$\Rightarrow \quad I = \frac{e^{ax}}{b} \sin(bx + c) + \frac{a}{b^2} e^{ax} \cos(bx + c) - \frac{a^2}{b^2} I$$

$$\Rightarrow \quad I + \frac{a^2}{b^2} I = \frac{b^{ax}}{b^2} \left\{ b \sin(bx + c) + a \cos(bx + c) \right\}$$

$$\Rightarrow \quad I \left(\frac{a^2 + b^2}{b^2} \right) = \frac{e^{ax}}{b^2} \left\{ a \cos(bx + c) + b \sin(bx + c) \right\}$$

$$\Rightarrow \quad I = \frac{e^{ax}}{a^2 + b^2} \left\{ a \cos(bx + c) + b \sin(bx + c) \right\}$$

Hence, $\int e^{ax} \cos(bx + c)\, dx = \dfrac{e^{ax}}{a^2 + b^2} \left\{ a \cos(bx + c) + b \sin(bx + c) \right\} + C_1$

(ii) Let $I = \int \sin(\log x)\, dx$. Let $\log x = t$. Then, $x = e^t \Rightarrow dx = d(e^t) = e^t\, dt$

$$\therefore \quad I = \int \underset{II}{\sin t}\ \underset{I}{e^t}\, dt = -e^t \cos t - \int e^t (-\cos t)\, dt \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad I = -e^t \cos t + \int \underset{I}{e^t}\ \underset{II}{\cos t}\, dt = -e^t \cos t + \left\{ e^t \sin t - \int e^t \sin t\, dt \right\}$$

$$\Rightarrow \quad I = -e^t \cos t + e^t \sin t - I \Rightarrow 2I = e^t (\sin t - \cos t) \Rightarrow I = \frac{e^t}{2} (\sin t - \cos t) + C$$

Hence, $\int \sin (\log x)\, dx = \frac{x}{2} \{\sin (\log x) - \cos (\log x)\} + C.$

(iii) Let $I = \int e^x \cos^2 x\, dx$. Then,

$$\Rightarrow \quad I = \int e^x \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2} \int e^x (1 + \cos 2x)\, dx = \frac{1}{2} \int e^x\, dx + \frac{1}{2} \int e^x \cos 2x\, dx$$

$$\Rightarrow \quad I = \frac{1}{2} e^x + \frac{1}{2} I_1 + C \qquad \text{...(i)}$$

where $I_1 = \int e^x \cos 2x\, dx.$

Now,

$$I_1 = \int \underset{I}{e^x}\ \underset{II}{\cos 2x}\, dx = e^x \frac{\sin 2x}{2} - \int e^x \frac{\sin 2x}{2}\, dx \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad I_1 = \frac{1}{2} e^x \sin 2x - \frac{1}{2} \int \underset{I}{e^x}\ \underset{II}{\sin 2x}\, dx$$

$$\Rightarrow \quad I_1 = \frac{1}{2} e^x \sin 2x - \frac{1}{2} \left\{ -e^x \frac{\cos 2x}{2} - \int e^x \left(\frac{-\cos 2x}{2} \right) dx \right\}$$

$$\Rightarrow \quad I_1 = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} \int e^x \cos 2x\, dx$$

$$\Rightarrow \quad I_1 = \frac{1}{2} e^x \sin 2x + \frac{1}{4} e^x \cos 2x - \frac{1}{4} I_1$$

$$\Rightarrow \quad I_1 + \frac{1}{4} I_1 = \frac{1}{4} e^x (\cos 2x + 2 \sin 2x) \Rightarrow \frac{5}{4} I_1 = \frac{1}{4} e^x (\cos 2x + 2 \sin 2x)$$

$$\Rightarrow \quad I_1 = \frac{e^x}{5} (\cos 2x + 2 \sin 2x) \qquad \text{...(ii)}$$

$$\therefore \quad I = \frac{1}{2} e^x + \frac{e^x}{10} (\cos 2x + 2 \sin 2x) + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 If $I_1 = \int e^{ax} \cos b\, x\, dx$ and $I_2 = \int e^{ax} \sin bx\, dx$, prove that

(i) $(a^2 + b^2)(I_1{}^2 + I_2{}^2) = e^{2ax}$ (ii) $\tan^{-1} \left(\frac{I_2}{I_1} \right) + \tan^{-1} \frac{b}{a} = bx.$

SOLUTION (i) We have,

$$I_1 = \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$$

and, $$I_2 = \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$$

$$\therefore \quad I_1^2 + I_2^2 = \frac{e^{2ax}}{(a^2+b^2)^2}\left\{(a\cos bx + b\sin bx)^2 + (a\sin bx - b\cos bx)^2\right\}$$

$$\Rightarrow \quad I_1^2 + I_2^2 = \frac{e^{2ax}}{(a^2+b^2)^2}(a^2+b^2) = \frac{e^{2ax}}{a^2+b^2}$$

$$\Rightarrow \quad (a^2+b^2)(I_1^2+I_2^2) = e^{2ax}$$

(ii) $$\frac{I_2}{I_1} = \frac{a\sin bx - b\cos bx}{a\cos bx + b\sin bx} = \frac{\tan bx - \frac{b}{a}}{1 + \frac{b}{a}\tan bx}$$ [Diving N^r and D^r by cos bx]

$$\Rightarrow \quad \frac{I_2}{I_1} = \frac{\tan bx - \tan\{\tan^{-1}(b/a)\}}{1 + \tan bx \,.\, \tan^{-1}(b/a)} = \tan\left\{bx - \tan^{-1}\left(\frac{b}{a}\right)\right\}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{I_2}{I_1}\right) = bx - \tan^{-1}\frac{b}{a} \Rightarrow \tan^{-1}\left(\frac{I_2}{I_1}\right) + \tan^{-1}\left(\frac{b}{a}\right) = bx.$$

EXERCISE 18.27

BASIC

Evaluate the following integrals:

1. $\int e^{ax}\cos bx\, dx$ [CBSE 2002]
2. $\int e^{ax}\sin(bx+c)\, dx$
3. $\int \cos(\log x)\, dx$
4. $\int e^{2x}\cos(3x+4)\, dx$
5. $\int e^{2x}\sin x\cos x\, dx$
6. $\int e^{2x}\sin x\, dx$ [NCERT]
7. $\int e^{2x}\sin(3x+1)\, dx$ [CBSE 2015]
8. $\int e^{-2x}\sin x\, dx$

BASED ON LOTS

9. $\int \frac{1}{x^3}\sin(\log x)\, dx$
10. $\int e^{2x}\cos^2 x\, dx$
11. $\int e^x \sin^2 x\, dx$
12. $\int x^2 e^{x^3}\cos x^3\, dx$

ANSWERS

1. $\frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$
2. $\frac{e^{ax}}{a^2+b^2}\left\{a\sin(bx+c) - b\cos(bx+c)\right\} + C_1$
3. $\frac{x}{2}\left\{\cos(\log x) + \sin(\log x)\right\} + C$
4. $\frac{e^{2x}}{13}\left\{2\cos(3x+4) + 3\sin(3x+4)\right\} + C$
5. $\frac{e^{2x}}{8}(\sin 2x - \cos 2x) + C$
6. $\frac{e^{2x}}{5}(2\sin x - \cos x) + C$
7. $\frac{e^{2x}}{13}\left\{2\sin(3x+1) - 3\cos(3x+1)\right\} + C$
8. $\frac{e^{-2x}}{5}(-2\sin x - \cos x) + C$
9. $-\frac{1}{5x^2}\left\{\cos(\log x) + 2\sin(\log x)\right\} + C$
10. $\frac{e^{2x}}{4} + \frac{e^{2x}}{8}(\cos 2x + \sin 2x) + C$
11. $\frac{1}{2}e^x - \frac{e^x}{10}(\cos 2x + 2\sin 2x) + C$
12. $\frac{e^{x^3}}{6}(\sin x^3 + \cos x^3) + C$

18.12 SOME IMPORTANT INTEGRALS

In this section, we will prove three formulae which will be used in evaluating integrals of the form $\int \sqrt{ax^2+bx+c}\,dx$ and $\int (px+q)\sqrt{ax^2+bx+c}\,dx$.

THEOREM *Prove that:*

(i) $\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \log\left|x+\sqrt{a^2+x^2}\right| + C$

(iii) $\int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2 \log\left|x+\sqrt{x^2-a^2}\right| + C$

PROOF (i) Let $I = \int \sqrt{a^2-x^2}\,dx$. Then, $I = \int \underset{\text{I}}{\sqrt{a^2-x^2}}\ \underset{\text{II}}{1}\cdot dx$. Integrating by parts, we get

$$I = \sqrt{a^2-x^2}\,x - \int \frac{1}{2}(a^2-x^2)^{-1/2}(0-2x)\,x\,dx = x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2-x^2} - \int \frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2-x^2} - I + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\Rightarrow \quad 2I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + C = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + C$$

(ii) Let $I = \int \sqrt{a^2+x^2}\,dx$. Then, $I = \int \underset{\text{I}}{\sqrt{a^2+x^2}}\ \underset{\text{II}}{1}\cdot dx$. Integrating by parts, we obtain

$$I = \sqrt{a^2+x^2}\,x - \int \frac{1}{2}(a^2+x^2)^{-1/2}(0+2x)\,x\,dx = x\sqrt{a^2+x^2} - \int \frac{x^2}{\sqrt{a^2+x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2+x^2} - \int \frac{(a^2+x^2)-a^2}{\sqrt{a^2+x^2}}\,dx = x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{a^2+x^2} - I + a^2 \log\left|x+\sqrt{a^2+x^2}\right|$$

$$\Rightarrow \quad 2I = x\sqrt{a^2+x^2} + a^2 \log\left|x+\sqrt{a^2+x^2}\right|$$

$$\Rightarrow \quad I = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \log\left|x+\sqrt{a^2+x^2}\right| + C$$

(iii) Let $I = \sqrt{x^2-a^2}\,dx$. Then, $I = \int \underset{\text{I}}{\sqrt{x^2-a^2}}\ \underset{\text{II}}{1}\cdot dx$. Integrating by parts, we obtain

$$I = \sqrt{x^2-a^2}\cdot x - \int \frac{1}{2}(x^2-a^2)^{-1/2}(2x)\,x\,dx$$

$$\Rightarrow \quad I = x\sqrt{x^2-a^2} - \int \frac{x^2}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int \frac{(x^2-a^2)+a^2}{\sqrt{x^2-a^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,dx - a^2 \int \frac{1}{\sqrt{x^2-a^2}}\,dx$$

$$\Rightarrow \quad I = x\sqrt{x^2-a^2} - I - a^2 \log|x+\sqrt{x^2-a^2}|$$

$$\Rightarrow \quad 2I = x\sqrt{x^2-a^2} - a^2\log|x+\sqrt{x^2-a^2}| \Rightarrow I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\log|x+\sqrt{x^2-a^2}| + C$$

18.12.1 INTEGRALS OF THE FORM $\int \sqrt{ax^2+bx+c}\,dx$

In order to evaluate the above type of integrals, we use the following algorithm.

ALGORITHM

Step I *Make coefficient of x^2 as one by taking 'a' common to obtain $x^2 + \frac{b}{a}x + \frac{c}{a}$.*

Step II *Add and subtract $\left(\frac{b}{2a}\right)^2$ in $x^2+\frac{b}{a}x+\frac{c}{a}$ to obtain $\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$.*

After applying these two steps the integral reduces to one of the following three forms:
$\int \sqrt{a^2+x^2}\,dx$, $\int \sqrt{a^2-x^2}\,dx$, $\int \sqrt{x^2-a^2}\,dx$.

Step III *Use the appropriate formula.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*
(i) $\int \sqrt{4x^2+9}\,dx$ (ii) $\int \sqrt{x^2+2x+5}\,dx$ [NCERT]

SOLUTION (i) Let $I = \int \sqrt{4x^2+9}\,dx$. Then,

$$I = 2\int \sqrt{x^2+\frac{9}{4}}\,dx = 2\int \sqrt{x^2+\left(\frac{3}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I = 2\left\{\frac{1}{2}x\sqrt{x^2+\frac{9}{4}} + \frac{1}{2}\left(\frac{3}{2}\right)^2 \log\left|x+\sqrt{x^2+\frac{9}{4}}\right|\right\} + C$$

$$\Rightarrow \quad I = \frac{x}{2}\sqrt{4x^2+9} + \frac{9}{4}\log|2x+\sqrt{4x^2+9}| + C$$

(ii)
$$I = \int \sqrt{x^2+2x+5}\,dx = \int \sqrt{x^2+2x+1+4}\,dx = \int \sqrt{(x+1)^2+2^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}(x+1)\sqrt{(x+1)^2+2^2} + \frac{1}{2}(2)^2\log\left|(x+1)+\sqrt{(x+1)^2+2^2}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log\left|(x+1)+\sqrt{x^2+2x+5}\right| + C$$

EXAMPLE 2 *Evaluate:* (i) $\int \sqrt{7x-10-x^2}\,dx$ (ii) $\int \sqrt{(x-3)(5-x)}\,dx$

SOLUTION (i) Let $I = \int \sqrt{-(x^2-7x+10)}\,dx$. Then,

$$I = \int \sqrt{-\left(x^2-7x+\frac{49}{4}-\frac{49}{4}+10\right)}\,dx$$

$$\Rightarrow \quad I = \int \sqrt{-\left\{\left(x-\frac{7}{2}\right)^2 - \left(\frac{3}{2}\right)^2\right\}}\, dx = \int \sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{7}{2}\right)^2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left(x-\frac{7}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2 - \left(x-\frac{7}{2}\right)^2} + \frac{1}{2}\left(\frac{3}{2}\right)^2 \sin^{-1}\left(\frac{x-7/2}{3/2}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{4}(2x-7)\sqrt{7x-10-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{2x-7}{3}\right) + C.$$

(ii) Let $I = \int \sqrt{(x-3)(5-x)}\, dx$. Then,

$$I = \int \sqrt{-x^2+8x-15}\, dx = \int \sqrt{-\left\{x^2-8x+16-16+15\right\}}\, dx$$

$$\Rightarrow \quad I = \int \sqrt{-\left\{(x-4)^2-1^2\right\}}\, dx = \int \sqrt{1^2-(x-4)^2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}(x-4)\sqrt{1^2-(x-4)^2} + \frac{1}{2}(1)^2 \sin^{-1}\left(\frac{x-4}{1}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{2}(x-4)\sqrt{(x-3)(5-x)} + \frac{1}{2}\sin^{-1}(x-4) + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate* : $\int x\sqrt{\frac{1+x}{1-x}}\, dx$

SOLUTION Let $I = \int x\sqrt{\frac{1+x}{1-x}}\, dx$. Then,

$$I = \int \frac{x(1+x)}{\sqrt{1-x^2}}\, dx = \int \frac{x}{\sqrt{1-x^2}}\, dx + \int \frac{x^2}{\sqrt{1-x^2}}\, dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\, dx - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\, dx$$

$$\Rightarrow I = -\sqrt{1-x^2} - \int \sqrt{1-x^2}\, dx + \int \frac{1}{\sqrt{1-x^2}}\, dx = -\sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + \sin^{-1}x + C$$

$$\Rightarrow I = -\sqrt{1-x^2} - \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x + C$$

EXAMPLE 4 *Evaluate:* $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ **[NCERT]**

SOLUTION Let $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$. Putting $x = t^2$ and $dx = 2t\, dt$, we get

$$I = 2\int \sqrt{\frac{1-t}{1+t}}\,.\,t\, dt = 2\int \frac{t(1-t)}{\sqrt{1-t^2}}\, dt = \int \frac{2t}{\sqrt{1-t^2}}\, dt + 2\int \frac{-t^2}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow I = \int \frac{2t}{\sqrt{1-t^2}}\, dt + 2\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\, dt = -\int \frac{-2t}{\sqrt{1-t^2}}\, dt + 2\int \sqrt{1-t^2}\, dt - 2\int \frac{1}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow I = -2\sqrt{1-t^2} + 2\times\frac{1}{2}\left\{t\sqrt{1-t^2} + \sin^{-1}t\right\} - 2\sin^{-1}t + C$$

$$\Rightarrow I = -2\sqrt{1-t^2} + t\sqrt{1-t^2} - \sin^{-1} t + C = (1-x)(\sqrt{x}-2) - \sin^{-1}\sqrt{x} + C$$

EXAMPLE 5 *Evaluate :* $\int \frac{x^2}{\sqrt{1-2x-x^2}}\, dx$

SOLUTION Let $I = \int \frac{x^2}{\sqrt{1-2x-x^2}}\, dx$. Then,

$$I = -\int \frac{-x^2}{\sqrt{1-2x-x^2}}\, dx = -\int \frac{(1-2x-x^2)+(2x-1)}{\sqrt{1-2x-x^2}}\, dx$$

$$\Rightarrow I = -\int \sqrt{1-2x-x^2}\, dx - \int \frac{2x-1}{\sqrt{1-2x-x^2}}\, dx = -\int \sqrt{1-2x-x^2}\, dx + \int \frac{-2x-2+3}{\sqrt{1-2x-x^2}}\, dx$$

$$\Rightarrow I = -\int \sqrt{1-2x-x^2}\, dx + \int \frac{-2x-2}{\sqrt{1-2x-x^2}}\, dx + 3\int \frac{1}{\sqrt{1-2x-x^2}}\, dx$$

$$\Rightarrow I = -\int \sqrt{(\sqrt{2})^2-(x+1)^2}\, dx + \int \frac{1}{\sqrt{1-2x-x^2}}\, d(1-2x-x^2) + 3\int \frac{1}{\sqrt{(\sqrt{2})^2-(x+1)^2}}\, dx$$

$$\Rightarrow I = -\frac{1}{2}\left\{(x+1)\sqrt{1-2x-x^2} + 2\sin^{-1}\frac{x+1}{\sqrt{2}}\right\} + 2\sqrt{1-2x-x^2} + 3\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$$

EXERCISE 18.28

BASIC

Evaluate the following integrals:

1. $\int \sqrt{3+2x-x^2}\, dx$ [NCERT]
2. $\int \sqrt{x^2+x+1}\, dx$
3. $\int \sqrt{x-x^2}\, dx$
4. $\int \sqrt{1+x-2x^2}\, dx$
5. $\int \cos x\sqrt{4-\sin^2 x}\, dx$
6. $\int e^x\sqrt{e^{2x}+1}\, dx$
7. $\int \sqrt{9-x^2}\, dx$
8. $\int \sqrt{16x^2+25}\, dx$
9. $\int \sqrt{4x^2-5}\, dx$
10. $\int \sqrt{2x^2+3x+4}\, dx$
11. $\int \sqrt{3-2x-2x^2}\, dx$
12. $\int \sqrt{3-x^2}\, dx$
13. $\int \sqrt{x^2-2x}\, dx$ [CBSE 2017]
14. $\int \sqrt{2x-x^2}\, dx$ [CBSE 2017]

BASED ON LOTS

15. $\int x\sqrt{x^4+1}\, dx$
16. $\int x^2\sqrt{a^6-x^6}\, dx$
17. $\int \frac{\sqrt{16+(\log x)^2}}{x}\, dx$ [CBSE 2005]
18. $\int \sqrt{2ax-x^2}\, dx$

ANSWERS

1. $\frac{(x-1)}{2}\sqrt{3+2x-x^2} + 2\sin^{-1}\left(\frac{x-1}{2}\right) + C$
2. $\left(\frac{2x+1}{4}\right)\sqrt{x^2+x+1} + \frac{3}{8}\log\left|(2x+1)+\sqrt{x^2+x+1}\right| + C$

3. $\frac{1}{4}(2x-1)\sqrt{x-x^2}+\frac{1}{8}\sin^{-1}(2x-1)+C$

4. $\frac{1}{8}(4x-1)\sqrt{1+x-2x^2}+\frac{9\sqrt{2}}{32}\sin^{-1}\left(\frac{4x-1}{3}\right)+C$

5. $\frac{1}{2}\sin x\sqrt{4-\sin^2 x}+2\sin^{-1}\left(\frac{\sin x}{2}\right)+C$

6. $\frac{1}{2}e^x\sqrt{e^{2x}+1}+\frac{1}{2}\log\left|e^x+\sqrt{e^{2x}+1}\right|+C$

7. $\frac{1}{2}x\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+C$

8. $2x\sqrt{x^2+\frac{25}{16}}+\frac{25}{8}\log\left|x+\sqrt{x^2+\frac{25}{16}}\right|+C$

9. $x\sqrt{x^2-\frac{5}{4}}-\frac{5}{4}\log\left|x+\sqrt{x^2-\frac{5}{4}}\right|+C$

10. $\left(\frac{4x+3}{8}\right)\sqrt{2x^2+3x+4}+\frac{23\sqrt{2}}{32}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right|+C$

11. $\frac{1}{4}(2x+1)\sqrt{3-2x-2x^2}+\frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x+1}{\sqrt{7}}\right)+C$

12. $\frac{1}{2}x\sqrt{3-x^2}+\frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$

13. $\frac{1}{2}(x-1)\sqrt{x^2-2x}-\frac{1}{2}\log\left|(x-1)+\sqrt{x^2-2x}\right|+C$

14. $\frac{1}{2}(x-1)\sqrt{2x-x^2}+\frac{1}{2}\sin^{-1}(x-1)+C$

15. $\frac{1}{4}x^2\sqrt{x^4+1}+\frac{1}{4}\log\left|x^2+\sqrt{x^4+1}\right|+C$

16. $\frac{1}{6}x^3\sqrt{a^6-x^6}+\frac{a^6}{6}\sin^{-1}\left(\frac{x^3}{a^3}\right)+C$

17. $\frac{1}{2}\log x\sqrt{(\log x)^2+16}+8\log|\log x+\sqrt{(\log x)^2+16}|+C$

18. $\frac{1}{2}(x-a)\sqrt{2ax-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)+C$

18.12.2 INTEGRALS OF THE FORM $\int (px+q)\sqrt{ax^2+bx+c}\,dx$

In order to evaluate this type of integrals, we use the following algorithm.

ALGORITHM

Step I *Express $px+q$ as*

$$px+q = \lambda\frac{d}{dx}(ax^2+bx+c)+\mu \quad i.e.\ px+q = \lambda(2ax+b)+\mu$$

Step II *Obtain the values of λ and μ by equating the coefficients of x and constant terms on both sides.*

Step III *Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the integral to obtain*

$$\int(px+q)\sqrt{ax^2+bx+c}\,dx=\lambda\int(2ax+b)\sqrt{ax^2+bx+c}\,dx+\mu\int\sqrt{ax^2+bx+c}\,dx$$

Step IV *To evaluate first integral on RHS, use the formula* $\int(f(x))^n f'(x)\,dx=\frac{\{f(x)\}^{n+1}}{n+1}$.

Evaluate second integral on RHS by the method discussed in the previous section.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int (x-5)\sqrt{x^2+x}\,dx$ (ii) $\int (3x-2)\sqrt{x^2+x+1}\,dx$ [CBSE 2014]

SOLUTION Let $(x-5) = \lambda \frac{d}{dx}(x^2+x) + \mu$ i.e. $x-5 = \lambda(2x+1)+\mu$

Comparing coefficients of like powers of x, we get

$$1 = 2\lambda \text{ and } \lambda+\mu = -5 \Rightarrow \lambda = \frac{1}{2} \text{ and } \mu = -\frac{11}{2}$$

$$\therefore \quad I = \int (x-5)\sqrt{x^2+x}\,dx$$

$$\Rightarrow \quad I = \int \left(\frac{1}{2}(2x+1) - \frac{11}{2}\right)\sqrt{x^2+x}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{2}(2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int \sqrt{x^2+x}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (2x+1)\sqrt{x^2+x}\,dx - \frac{11}{2}\int \sqrt{x^2+x}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \sqrt{t}\,dt - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\,dx, \quad \text{where } t = x^2+x$$

$$\Rightarrow \quad I = \frac{1}{2}\times\frac{t^{3/2}}{3/2} - \frac{11}{2}\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right.$$
$$\left. - \frac{1}{2}\left(\frac{1}{2}\right)^2 \log\left|\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|\right\} + C$$

$$\Rightarrow \quad I = \frac{1}{3}t^{3/2} - \frac{11}{2}\left\{\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|\right\} + C$$

$$\Rightarrow \quad I = \frac{1}{3}(x^2+x)^{3/2} - \frac{11}{2}\left\{\frac{2x+1}{4}\sqrt{x^2+x} - \frac{1}{8}\log\left|\left(x+\frac{1}{2}\right)\sqrt{x^2+x}\right|\right\}$$

(ii) Let $3x-2 = \lambda \frac{d}{dx}(x^2+x+1)+\mu$ i.e. $3x-2 = \lambda(2x+1)+\mu$

Comparing the coefficients of like powers of x, we get

$$2\lambda = 3 \text{ and } \lambda+\mu = -2 \Rightarrow \lambda = \frac{3}{2} \text{ and } \mu = -\frac{7}{2}$$

$$\therefore \quad I = \int (3x-2)\sqrt{x^2+x+1}\,dx$$

$$\Rightarrow \quad I = \int \left\{\frac{3}{2}(2x+1) - \frac{7}{2}\right\}\sqrt{x^2+x+1}\,dx$$

$$\Rightarrow \quad I = \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx - \frac{7}{2}\int \sqrt{x^2+x+1}\,dx$$

$$\Rightarrow \quad I = \frac{3}{2}\int \sqrt{t}\,dt - \frac{7}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx, \quad \text{where } t = x^2+x+1$$

$$\Rightarrow \quad I = t^{3/2} - \frac{7}{4}\left\{\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \left(\frac{\sqrt{3}}{2}\right)^2 \log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right\} + C$$

$$\Rightarrow \quad I = (x^2+x+1)^{3/2} - \frac{7}{2}\left\{\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{8}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+1}\right|\right\} + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int x\sqrt{1+x-x^2}\,dx$ (ii) $\int (x+1)\sqrt{1-x-x^2}\,dx$

SOLUTION Let $x = \lambda \frac{d}{dx}(1+x-x^2) + \mu$ i.e. $x = \lambda(1-2x)+\mu$

Comparing the coefficients of like powers of x, we get

$$1 = -2\lambda \text{ and } \lambda+\mu = 0 \Rightarrow \lambda = -\frac{1}{2} \text{ and } \mu = \frac{1}{2}$$

$$\therefore \quad I = \int x\sqrt{1+x-x^2}\,dx$$

$$\Rightarrow \quad I = \int \left\{-\frac{1}{2}(1-2x)+\frac{1}{2}\right\}\sqrt{1+x-x^2}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\int (1-2x)\sqrt{1+x-x^2}\,dx + \frac{1}{2}\int \sqrt{1+x-x^2}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\int \sqrt{t}\,dt + \frac{1}{2}\int \sqrt{-\left(x^2-x+\frac{1}{4}-\frac{1}{4}-1\right)}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x-\frac{1}{2}\right)^2}\,dx, \text{ where } t = 1+x-x^2$$

$$\Rightarrow \quad I = -\frac{1}{3}t^{3/2} + \frac{1}{2}\left\{\left(x-\frac{1}{2}\right)\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2} + \frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2\sin^{-1}\left(\frac{x-1/2}{\sqrt{5}/2}\right)\right\} + C$$

$$\Rightarrow \quad I = -\frac{1}{3}(1+x-x^2)^{3/2} + \frac{1}{2}\left\{\frac{1}{2}\left(x-\frac{1}{2}\right)\sqrt{1+x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)\right\} + C$$

(ii) Let $x+1 = \lambda\frac{d}{dx}(1-x-x^2)+\mu$ i.e. $x+1 = \lambda(-1-2x)+\mu$

Comparing the coefficients of like powers of x, we get

$$-2\lambda = 1 \text{ and } \mu-\lambda = 1 \Rightarrow \lambda = -\frac{1}{2} \text{ and } \mu = \frac{1}{2}$$

$$\therefore \quad I = \int (x+1)\sqrt{1-x-x^2}\,dx = \int \left\{-\frac{1}{2}(-1-2x)+\frac{1}{2}\right\}\sqrt{1-x-x^2}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\int (-1-2x)\sqrt{1-x-x^2}\,dx + \frac{1}{2}\int \sqrt{1-x-x^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (-1-2x)\sqrt{1-x-x^2}\,dx + \frac{1}{2}\int \sqrt{\left\{1-\left(x^2+x+\frac{1}{4}-\frac{1}{4}\right)\right\}}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\int \sqrt{t}\,dt + \frac{1}{2}\int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}\,dx, \text{ where } t = 1-x-x^2$$

$$\Rightarrow \quad I = -\frac{1}{2}\left(\frac{t^{3/2}}{3/2}\right) + \frac{1}{2}\left\{\frac{1}{2}\left(x+\frac{1}{2}\right)\sqrt{1-x-x^2} + \frac{1}{2}\times\frac{5}{4}\sin^{-1}\left(\frac{x+1/2}{\sqrt{5}/2}\right)\right\} + C$$

$$\Rightarrow \quad I = -\frac{1}{3}(1-x-x^2)^{3/2} + \frac{1}{8}(2x+1)\sqrt{1-x-x^2} + \frac{5}{16}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:* $\int \frac{x}{x-\sqrt{x^2-1}}\,dx$

SOLUTION Let

$$I = \int \frac{x}{x-\sqrt{x^2-1}}\,dx = \int x \frac{\left(x+\sqrt{x^2-1}\right)}{\left(x-\sqrt{x^2-1}\right)\left(x+\sqrt{x^2-1}\right)}\,dx$$

$$\Rightarrow \quad I = \int \frac{x\left(x+\sqrt{x^2-1}\right)}{x^2-(x^2-1)}\,dx = \int \left(x^2 + x\sqrt{x^2-1}\right)dx = \int x^2\,dx + \int x\sqrt{x^2-1}\,dx$$

$$\Rightarrow \quad I = \int x^2\,dx + \frac{1}{2}\int \sqrt{t}\,dt, \text{ where } t = x^2-1$$

$$\Rightarrow \quad I = \frac{x^3}{3} + \frac{1}{3}t^{3/2} + C = \frac{x^3}{3} + \frac{1}{3}(x^2-1)^{3/2} + C$$

EXERCISE 18.29

BASIC

Evaluate the following integrals:

1. $\int (x+1)\sqrt{x^2-x+1}\,dx$
2. $\int (x+1)\sqrt{2x^2+3}\,dx$
3. $\int (2x-5)\sqrt{2+3x-x^2}\,dx$
4. $\int (x+2)\sqrt{x^2+x+1}\,dx$
5. $\int (4x+1)\sqrt{x^2-x-2}\,dx$
6. $\int (x-2)\sqrt{2x^2-6x+5}\,dx$
7. $\int (x+1)\sqrt{x^2+x+1}\,dx$
8. $\int (2x+3)\sqrt{x^2+4x+3}\,dx$
9. $\int (2x-5)\sqrt{x^2-4x+3}\,dx$
10. $\int x\sqrt{x^2+x}\,dx$
11. $\int (x-3)\sqrt{x^2+3x-18}\,dx$ [CBSE 2014]
12. $\int (x+3)\sqrt{3-4x-x^2}\,dx$ [CBSE 14, 2015]
13. $\int (3x+1)\sqrt{4-3x-2x^2}\,dx$ [CBSE 2016]
14. $\int (2x+5)\sqrt{10-4x-3x^2}\,dx$ [CBSE 2016]

ANSWERS

1. $\frac{(x^2-x+1)^{3/2}}{3} + \frac{3}{8}(2x-1)\sqrt{x^2-x+1} + \frac{9}{16}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + C$

2. $\frac{1}{6}(2x^2+3)^{3/2} + \frac{x}{2}\sqrt{2x^2+3} + \frac{3\sqrt{2}}{4}\log\left|\frac{\sqrt{2}\,x+\sqrt{2x^2+3}}{\sqrt{3}}\right| + C$

3. $-\frac{2}{3}(2+3x-x^2)^{3/2} - \left(\frac{2x-3}{2}\right)\sqrt{2+3x-x^2} - \frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)+C$

4. $\frac{1}{3}(x^2+x+1)^{3/2} + \frac{3(2x+1)}{8}\sqrt{x^2+x+1} + \frac{9}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|+C$

5. $\frac{4}{3}(x^2-x-2)^{3/2}+\frac{3}{4}(2x-1)\sqrt{x^2-x-2}-\frac{27}{8}\log\left|\left(x-\frac{1}{2}\right)+\sqrt{x^2-x-2}\right|+C$

6. $\frac{1}{6}(2x^2-6x+5)^{3/2} - \frac{1}{\sqrt{2}}\left\{\frac{2x-3}{4}\sqrt{x^2-3x+\frac{5}{2}}+\frac{1}{8}\log\left|\frac{2x-3}{2}+\sqrt{x^2-3x+\frac{5}{2}}\right|\right\}+C$

7. $\frac{1}{3}(x^2+x+1)^{3/2}+\frac{1}{2}\left\{\frac{2x+1}{4}\sqrt{x^2+x+1}+\frac{3}{8}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right\}+C$

8. $\frac{2}{3}(x^2+4x+3)^{3/2}-\left\{\frac{1}{2}(x+2)\sqrt{x^2+4x+3}-\frac{1}{2}\log\left|(x+2)+\sqrt{x^2+4x+3}\right|\right\}+C$

9. $\frac{2}{3}(x^2-4x+3)^{3/2}-\left\{\frac{1}{2}(x-2)\sqrt{x^2-4x+3}-\frac{1}{2}\log\left|x-2+\sqrt{x^2-4x+3}\right|\right\}+C$

10. $\frac{1}{3}(x+x^2)^{3/2}-\frac{1}{8}(2x+1)\sqrt{x^2+x}+\frac{1}{16}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right|+C$

11. $\frac{1}{3}(x^2+3x-18)^{3/2}-\frac{3}{4}(2x+3)\sqrt{x^2+3x-18}+\frac{243}{8}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{x^2+3x-18}\right|+C$

12. $-\frac{1}{3}(3-4x-x^2)^{3/2}+\frac{1}{2}(x+2)\sqrt{3-4x-x^2}+\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+C$

13. $-\frac{1}{2}(4-3x-2x^2)^{3/2}-\frac{5}{32}(4x+3)\sqrt{4-3x-2x^2}-\frac{205}{64\sqrt{2}}\sin^{-1}\frac{4x+3}{\sqrt{41}}+C$

14. $-\frac{2}{9}(10-4x-3x^2)^{3/2}+\frac{11}{18}(3x+2)\sqrt{10-4x-3x^2}+\frac{187}{9\sqrt{3}}\sin^{-1}\left(\frac{3x+2}{\sqrt{34}}\right)+C$

18.13 INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS

PARTIAL FRACTIONS *If $f(x)$ and $g(x)$ are two polynomials, then $\frac{f(x)}{g(x)}$ defines a rational algebraic function or a rational function of x.*

If degree of $f(x) <$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called a proper rational function.

If degree of $f(x) \geq$ degree of $g(x)$, then $\frac{f(x)}{g(x)}$ is called an improper rational function.

If $\frac{f(x)}{g(x)}$ is an improper rational function, we divide $f(x)$ by $g(x)$ so that the rational function $\frac{f(x)}{g(x)}$ is expressed in the form $\phi(x)+\frac{\psi(x)}{g(x)}$ where $\phi(x)$ and $\psi(x)$ are polynomials such that the degree of $\psi(x)$ is less than that of $g(x)$. Thus, $\frac{f(x)}{g(x)}$ is expressible as the sum of a polynomial and a proper rational function.

Any proper rational function $\frac{f(x)}{g(x)}$ can be expressed as the sum of rational functions, each having a simple factor of $g(x)$. Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $\frac{f(x)}{g(x)}$ into partial fractions.

The resolution of $\frac{f(x)}{g(x)}$ into partial fractions depends mainly upon the nature of the factors of $g(x)$ as discussed below.

Case I When denominator is expressible as the product of non-repeating linear factors.

Let $g(x) = (x - a_1)(x - a_2)\dots(x - a_n)$. Then, we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \dots + \frac{A_n}{x - a_n}$$

where $A_1, A_2, .. A_n$ are constants and can be determined by equating the numerator on RHS to the numerator on LHS after taking LCM on RHS and then substituting $x = a_1, a_2, \dots, a_n$.

ILLUSTRATION 1 *Resolve* $\frac{3x+2}{x^3 - 6x^2 + 11x - 6}$ *into partial fractions.*

SOLUTION We have, $\frac{3x+2}{x^3 - 6x^2 + 11x - 6} = \frac{3x+2}{(x-1)(x-2)(x-3)}$

Let $\frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$. Then,

$$\Rightarrow \quad \frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow \quad 3x + 2 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2) \quad \text{...(i)}$$

Putting $x - 1 = 0$ or, $x = 1$ in (i), we get $5 = A(1-2)(1-3) \Rightarrow A = \frac{5}{2}$

Putting $x - 2 = 0$ or, $x = 2$ in (i), we obtain $8 = B(2-1)(2-3) \Rightarrow B = -8$

Putting $x - 3 = 0$ or, $x = 3$ in (i), we obtain $11 = C(3-1)(3-2) \Rightarrow C = \frac{11}{2}$

$$\therefore \quad \frac{3x+2}{x^3 - 6x^2 + 11x - 6} = \frac{3x+2}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{8}{x-2} + \frac{11}{2(x-3)}$$

REMARK *In order to determine the values of constants in the numerator of the partial fraction corresponding to the non-repeated linear factor $px + q$ in the denominator of a rational expression, we may proceed as follows:*

Replace $x = -q/p$ (obtained by putting $px + q = 0$) everywhere in the given rational expression except in the factor $px + q$ itself. For example, in the above illustration the value of A is obtained by replacing x by 1 in all factors of $\frac{3x+2}{(x-1)(x-2)(x-3)}$ *except* $(x-1)$.

i.e. $$A = \frac{3 \times 1 + 2}{(1-2)(1-3)} = \frac{5}{2}$$

Similarly, B is obtained by putting $x = 2$ in all factors of $\frac{3x+2}{(x-1)(x-2)(x-2)}$ except $(x-2)$ in denominator.

$$B = \frac{3 \times 2 + 1}{(2-2)(2-3)} = -8$$

To find C, we put $x=3$ in all factors of $\dfrac{3x+2}{(x-1)(x-2)(x-3)}$ except $(x-3)$ in denominator.

$$\therefore \quad C = \frac{3\times 3+2}{(3-1)(3-2)} = \frac{11}{2}$$

ILLUSTRATION 2 *Resolve* $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6}$ *into partial fractions.*

SOLUTION Here, the given function is an improper rational function. On dividing, we get

$$\frac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1+\frac{(-x+4)}{x^2-5x+6} \quad \text{...(i)}$$

Now, $\dfrac{-x+4}{x^2-5x+6} = \dfrac{-x+4}{(x-2)(x-3)}$

So, let $\dfrac{-x+4}{(x-2)(x-3)} = \dfrac{A}{x-2}+\dfrac{B}{x-3}$

$$\therefore \quad -x+4 = A(x-3)+B(x-2) \quad \text{...(ii)}$$

Putting $x-3=0$ or, $x=3$ in (ii), we get: $1=B(1) \Rightarrow B=1$.

Putting $x-2=0$ or, $x=2$ in (ii), we get: $2=A(2-3) \Rightarrow A=-2$

$$\therefore \quad \frac{-x+4}{(x-2)(x-3)} = \frac{-2}{x-2}+\frac{1}{x-3}$$

Hence, $\dfrac{x^3-6x^2+10x-2}{x^2-5x+6} = x-1-\dfrac{2}{x-2}+\dfrac{1}{x-3}$

Case II *When the denominator $g(x)$ is expressible as the product of the linear factors such that some of them are repeating.*

Let $g(x)=(x-a)^k(x-a_1)(x-a_2)\dots(x-a_r)$. Then we assume that

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\dots+\frac{A_k}{(x-a)^k}+\frac{B_1}{x-a}+\frac{B_2}{x-a_2}+\dots+\frac{B_r}{(x-a_r)}$$

i.e., corresponding to non-repeating factors we assume as in Case I and for each repeating factor $(x-a)^k$, we assume partial fractions

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\dots+\frac{A_k}{(x-a)^k}, \text{ where } A_1, A_2, \dots, A_k \text{ are constants.}$$

Now, to determine constants we equate numerators on both sides. Some of the constants are determined by substitution as in case I and remaining are obtained by comparing coefficients of equal powers of x on both sides.

Following illustration illustrates the procedure.

ILLUSTRATION 3 *Resolve* $\dfrac{3x-2}{(x-1)^2(x+1)(x+2)}$ *into partial fractions.*

SOLUTION Let $\dfrac{3x-2}{(x-1)^2(x+1)(x+2)} = \dfrac{A_1}{x-1}+\dfrac{A_2}{(x-1)^2}+\dfrac{A_3}{x+1}+\dfrac{A_4}{x+2}$

$$\Rightarrow 3x-2 = A_1(x-1)(x+1)(x+2)+A_2(x+1)(x+2)+A_3(x-1)^2(x+2)+A_4(x-1)^2(x+1) \quad \text{...(i)}$$

Putting $x-1=0$ or, $x=1$ in (i), we get: $1=A_2(1+1)(1+2) \Rightarrow A_2=\dfrac{1}{6}$

Putting $x+1=0$ or, $x=-1$ in (i), we get: $-5=A_3(-2)^2(-1+2) \Rightarrow A_3=-\dfrac{5}{4}$

Putting $x+2=0$ or, $x=-2$ in (i), we get: $-8=A_4(-3)^2(-1) \Rightarrow A_4=\dfrac{8}{9}$

Now, equating coefficient of x^3 on both sides, we get

$$0 = A_1 + A_3 + A_4 \Rightarrow A_1 = -A_3 - A_4 = \frac{5}{4} - \frac{8}{9} = \frac{13}{36}$$

$$\therefore \quad \frac{3x-2}{(x-1)^2 (x+1)(x+2)} = \frac{13}{36(x-1)} + \frac{1}{6(x-1)^2} - \frac{5}{4(x+1)} + \frac{8}{9(x+2)}$$

Case III *When some of the factors of denominator $g(x)$ are quadratic but non-repeating.*

Corresponding to each quadratic factor $ax^2 + bx + c$, we assume partial fraction of the type $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be determined by comparing coefficients of similar powers of x in the numerator of both sides. In practice it is advisable to assume partial fractions of the type $\frac{A(2ax+b)}{ax^2+bx+c} + \frac{B}{ax^2+bx+c}$.

Following illustration illustrates the procedure.

ILLUSTRATION 4 *Resolve* $\frac{2x-1}{(x+1)(x^2+2)}$ *into partial fractions.*

SOLUTION Let $\frac{2x-1}{(x+1)(x^2+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2}$. Then,

$$\frac{2x-1}{(x+1)(x^2+2)} = \frac{A(x^2+2) + (Bx+C)(x+1)}{(x+1)(x^2+2)} \Rightarrow 2x-1 = A(x^2+2) + (Bx+C)(x+1) \quad \text{...(i)}$$

Putting $x+1=0$ or, $x=-1$ in (i), we get: $-3 = A(3) \Rightarrow A = -1$.

Comparing coefficients of like powers of x on both sides of (i), we get

$$A + B = 0,\ C + 2A = -1 \text{ and } C + B = 2$$

$$\therefore \quad -1 + B = 0,\ C - 2 = -1 \Rightarrow B = 1,\ C = 1 \qquad \text{[Putting } A = -1\text{]}$$

$$\therefore \quad \frac{2x-1}{(x+1)(x^2+2)} = -\frac{1}{x+1} + \frac{x+1}{x^2+2}$$

Case IV *When some of the factors of the denominator $g(x)$ are quadratic and repeating.*

For every quadratic repeating factor of the type $(ax^2+bx+c)^k$, we assume $2k$ partial fractions of the form

$$\left\{\frac{A_0(2ax+b)}{ax^2+bx+c} + \frac{A_1}{ax^2+bx+c}\right\} + \left\{\frac{A_1(2ax+b)}{(ax^2+bx+c)^2} + \frac{A_2}{(ax^2+bx+c)^2}\right\}$$
$$+ \dots + \left\{\frac{A_{2k-1}(2ax+b)}{(ax^2+bx+c)^k} + \frac{A_{2k}}{(ax^2+bx+c)^k}\right\}$$

Following illustrations will illustrate the procedure.

ILLUSTRATION 5 *Resolve* $\frac{2x-3}{(x-1)(x^2+1)^2}$ *into partial fractions.*

SOLUTION Let $\frac{2x-3}{(x-1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$. Then,

$$2x - 3 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1) \quad \text{...(i)}$$

Putting $x = 1$ in (i), we get: $-1 = A(1+1)^2 \Rightarrow A = -\frac{1}{4}$

Equating coefficients of like powers of x, we get

$$A + B = 0,\ C - B = 0,\ 2A + B - C + D = 0,\ C + E - B - D = 2 \text{ and } A - C - E = -3$$

Putting $A = -\frac{1}{4}$ and solving these equations, we get: $B = \frac{1}{4} = C$, $D = \frac{1}{2}$ and $E = \frac{5}{2}$.

$$\therefore \quad \frac{2x-3}{(x-1)(x^2+1)^2} = \frac{-1}{4(x-1)} + \frac{x+1}{4(x^2+1)} + \frac{x+5}{2(x^2+1)^2}$$

ILLUSTRATION 6 *Resolve* $\frac{2x}{x^3-1}$ *into partial fractions.*

SOLUTION We have, $\frac{2x}{x^3-1} = \frac{2x}{(x-1)(x^2+x+1)}$

So, let $\frac{2x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$. Then,

$$2x = A(x^2+x+1) + (Bx+C)(x-1) \qquad \text{...(i)}$$

Putting $x-1=0$ or, $x=1$ in (i), we get : $2 = 3A \Rightarrow A = \frac{2}{3}$

Putting $x=0$ in (i), we get : $A - C = 0 \Rightarrow C = A = \frac{2}{3}$

Putting $x=-1$ in (i), we get : $-2 = A + 2B - 2C \Rightarrow -2 = \frac{2}{3} - 2B - \frac{4}{3} \Rightarrow B = \frac{2}{3}$

$$\therefore \quad \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2/3\,x + 2/3}{x^2+x+1} \text{ or, } \frac{2x}{x^3-1} = \frac{2}{3}\cdot\frac{1}{x-1} + \frac{2}{3}\cdot\frac{x+1}{x^2+x+1}$$

Let us now use partial fractions in evaluating integrals containing rational algebraic functions.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **WHEN THE DENOMINATOR IS EXPRESSIBLE AS A PRODUCT OF DISTINCT LINEAR FACTORS**

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{x-1}{(x+1)(x-2)}\,dx$ (ii) $\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$ [CBSE 2005] (iii) $\int \frac{x^3}{(x-1)(x-2)}\,dx$

SOLUTION (i) Let $\frac{x-1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$ ...(i)

$$\Rightarrow \quad x-1 = A(x-2) + B(x+1) \qquad \text{...(ii)}$$

Putting $x-2=0$ or, $x=2$ in (ii), we get: $1 = 3B \Rightarrow B = 1/3$

Putting $x+1=0$ or, $x=-1$ in (ii), we get: $-2 = -3A \Rightarrow A = 2/3$

Substituting the values of A and B in (i), we get

$$\frac{x-1}{(x+1)(x-2)} = \frac{2}{3}\cdot\frac{1}{x+1} + \frac{1}{3}\cdot\frac{1}{x-2}$$

$$\therefore \quad I = \int \frac{x-1}{(x+1)(x-2)}\,dx = \frac{2}{3}\int\frac{1}{x+1}\,dx + \frac{1}{3}\int\frac{1}{x-2}\,dx = \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C$$

(ii) Let $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$ ...(i)

$$\Rightarrow \quad 2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2) \qquad \text{...(ii)}$$

Putting $x+2=0$ or, $x=-2$ in (ii), we get: $-5 = B(-3)(-5) \Rightarrow B = -1/3$

Putting $x-3=0$ or, $x=3$ in (ii), we get: $5=C\,(2)\,(5) \Rightarrow C=1/2$

Putting $x-1=0$ or, $x=1$ in (ii), we get: $1=A\,(3)\,(-2) \Rightarrow A=-1/6$

Substituting the values of A, B and C in (i), we obtain

$$\frac{2x-1}{(x-1)(x+2)(x-3)} = -\frac{1}{6}\cdot\frac{1}{x-1}-\frac{1}{3}\cdot\frac{1}{x+2}+\frac{1}{2}\cdot\frac{1}{x-3}$$

$$\therefore \quad I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx = -\frac{1}{6}\int\frac{1}{x-1}\,dx-\frac{1}{3}\int\frac{1}{x+2}\,dx+\frac{1}{2}\int\frac{1}{x-3}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{6}\log|x-1|-\frac{1}{3}\log|x+2|+\frac{1}{2}\log|x-3|+C$$

(iii) Here, the degree of numerator is greater than that of denominator. So, we divide the numerator by denominator to obtain

$$\frac{x^3}{(x-1)(x-2)} = x+3+\frac{7x-6}{(x-1)(x-2)} \quad \text{...(i)}$$

Now, let $\dfrac{7x-6}{(x-1)(x-2)} = \dfrac{A}{x-1}+\dfrac{B}{x-2}$ ...(ii)

$$\Rightarrow \quad 7x-6 = A\,(x-2)+B\,(x-1) \quad \text{...(iii)}$$

Putting $x-2=0$ or, $x=2$ in (iii), we get: $B=8$. Putting $x-1=0$ or, $x=1$ in (iii), we get: $A=-1$

Substituting the values of A and B in (ii), we obtain

$$\frac{7x-6}{(x-1)(x-2)} = -\frac{1}{x-1}+\frac{8}{x-2}$$

$$\therefore \quad \frac{x^3}{(x-1)(x-2)} = x+3-\frac{1}{x-1}+\frac{8}{x-2} \quad \text{[From (i)]}$$

$$\therefore \quad I = \int \frac{x^3}{(x-1)(x-2)}\,dx$$

$$\Rightarrow \quad I = \int\left(x+3-\frac{1}{x-1}+\frac{8}{x-2}\right)dx = \frac{x^2}{2}+3x-\log|x-1|+8\log|x-2|+C$$

EXAMPLE 2 *Evaluate:*

(i) $\displaystyle\int \frac{2x}{(x^2+1)(x^2+2)}\,dx$ [NCERT] (ii) $\displaystyle\int \frac{\cos\theta}{(2+\sin\theta)(3+4\sin\theta)}\,d\theta$ [CBSE 2019]

(iii) $\displaystyle\int \frac{1}{\sin x-\sin 2x}\,dx$ [CBSE 2010] (iv) $\displaystyle\int \frac{1-\cos x}{\cos x\,(1+\cos x)}\,dx$

SOLUTION (i) Let $I=\displaystyle\int \frac{2x}{(x^2+1)(x^2+2)}\,dx$. Putting $x^2=t$ and $2x\,dx=dt$, we get

$$I=\int \frac{dt}{(t+1)(t+2)}$$

Let $\dfrac{1}{(t+1)(t+2)} = \dfrac{A}{t+1}+\dfrac{B}{t+2}$ ...(i)

$$\Rightarrow \quad 1 = A\,(t+2)+B\,(t+1) \quad \text{...(ii)}$$

Putting $t=-2$ in (ii), we obtain: $B=-1$. Putting $t=-1$ in (ii), we obtain: $A=1$.

Substituting the values of A and B in (i), we get

$$\frac{1}{(t+1)(t+2)} = \frac{1}{t+1}-\frac{1}{t+2}$$

$$\therefore \quad I = \int \frac{1}{(t+1)(t+2)}\,dt = \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right)dt$$

$$\Rightarrow \quad I = \log|t+1| - \log|t+2| + C = \log|x^2+1| - \log|x^2+2| + C$$

(ii) Let $I = \int \frac{\cos\theta}{(2+\sin\theta)(3+4\sin\theta)}\,d\theta$. Putting $\sin\theta = t$, and $\cos\theta\,d\theta = dt$, we get

$$I = \int \frac{dt}{(2+t)(3+4t)}$$

Let $\frac{1}{(2+t)(3+4t)} = \frac{A}{2+t} + \frac{B}{3+4t}$ ...(i)

$\Rightarrow \quad 1 = A(3+4t) + B(2+t)$ ...(ii)

Putting $3+4t=0$ i.e. $t = -\frac{3}{4}$ in (ii), we get: $1 = B\left(2-\frac{3}{4}\right) \Rightarrow B = \frac{4}{5}$

Putting $2+t=0$ i.e. $t=-2$ in (ii), we get: $1 = A(3-8) \Rightarrow A = -\frac{1}{5}$

Substituting the values of A and B in (i), we obtain

$$\frac{1}{(2+t)(3+4t)} = -\frac{1}{5}\cdot\frac{1}{2+t} + \frac{4}{5}\cdot\frac{1}{3+4t}$$

$$\therefore \quad I = \int \frac{1}{(2+t)(3+4t)}\,dt = -\frac{1}{5}\int \frac{1}{2+t}\,dt + \frac{4}{5}\int \frac{1}{3+4t}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{5}\log|2+t| + \frac{4}{5}\cdot\frac{1}{4}\log|3+4t| + C = -\frac{1}{5}\log|2+\sin\theta| + \frac{1}{5}\log|3+4\sin\theta| + C$$

(iii) We have,

$$I = \int \frac{1}{\sin x - \sin 2x}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{(\sin x - 2\sin x\cos x)}\,dx = \int \frac{1}{\sin x(1-2\cos x)}\,dx = \int \frac{\sin x}{\sin^2 x(1-2\cos x)}\,dx$$

$$\Rightarrow \quad I = \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)}\,dx$$

Putting $\cos x = t$, and $-\sin x\,dx = dt$ or, $\sin x\,dx = -dt$, we get

$$I = \int \frac{-dt}{(1-t^2)(1-2t)} = \int \frac{-1}{(1-t)(1+t)(1-2t)}\,dt$$

Let $\frac{-1}{(1-t)(1+t)(1-2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1-2t}$. Then,

$-1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$ ...(i)

Putting $t+1=0$ or, $t=-1$ in (i), we get: $-1 = 6B \Rightarrow B = -\frac{1}{6}$.

Putting $1-t=0$ or, $t=1$ in (i), we get: $-1 = -2A \Rightarrow A = \frac{1}{2}$.

Putting $1-2t=0$ or, $t=\frac{1}{2}$ in (i), we get: $-1 = C\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \Rightarrow C = -\frac{4}{3}$.

$$\therefore \quad \frac{-1}{(1-t)(1+t)(1-2t)} = \frac{1}{2}\cdot\frac{1}{1-t} - \frac{1}{6}\cdot\frac{1}{1+t} - \frac{4}{3}\cdot\frac{1}{1-2t}$$

$$\Rightarrow \quad I = \int \frac{-\,dt}{(1-t)(1+t)(1-2t)} = \frac{1}{2}\int \frac{1}{1-t}\,dt - \frac{1}{6}\int \frac{1}{1+t}\,dt - \frac{4}{3}\int \frac{1}{1-2t}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{2}\log|1-t| - \frac{1}{6}\log|1+t| - \frac{4}{3}\times -\frac{1}{2}\log|1-2t| + C$$

$$\Rightarrow \quad I = -\frac{1}{2}\log|1-\cos x| - \frac{1}{6}\log|1+\cos x| + \frac{2}{3}\log|1-2\cos x| + C$$

(iv) Let $I = \int \frac{1-\cos x}{\cos x(1+\cos x)}\,dx$ and let $\cos x = y$. Then, $\frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1-y}{y(1+y)}$

Let $\quad \frac{1-y}{y(1+y)} = \frac{A}{y} + \frac{B}{1+y}$ ...(i)

$\Rightarrow \quad 1-y = A(1+y) + By$ ...(ii)

Putting $y = 0$ in (ii), we get $A = 1$. Putting $y = -1$ in (ii), we get $B = -2$.

Substituting the values of A and B in (i), we obtain

$$\frac{1-y}{y(1+y)} = \frac{1}{y} - \frac{2}{1+y} \Rightarrow \frac{1-\cos x}{\cos x(1+\cos x)} = \frac{1}{\cos x} - \frac{2}{1+\cos x} \quad [\because y = \cos x]$$

$$\therefore \quad I = \int \frac{1-\cos x}{\cos x(1+\cos x)}\,dx = \int \frac{1}{\cos x}\,dx - \int \frac{2}{1+\cos x}\,dx$$

$$\Rightarrow \quad I = \int \sec x\,dx - \int \frac{2}{2\cos^2 x/2}\,dx = \int \sec x - \int \sec^2 x/2\,dx$$

$$\Rightarrow \quad I = \log|\sec x + \tan x| - 2\tan x/2 + C$$

EXAMPLE 3 *Evaluate:* $\int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx.$

SOLUTION Let $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{A}{(x-4)} + \frac{B}{x-5} + \frac{C}{x-6}$ ...(i)

Then, $(x-1)(x-2)(x-3) = (x-4)(x-5)(x-6) + A(x-5)(x-6) + B(x-4)(x-6) + C(x-4)(x-5)$ ...(ii)

Putting $x = 4, 5$ and 6 successively in (ii), we obtain: $A = 3$, $B = -24$ and $C = 30$

Substituting values of A, B and C in (i), we obtain

$$\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + \frac{3}{x-4} - \frac{24}{x-5} + \frac{30}{x-6}$$

$$\therefore \quad I = \int \frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}\,dx = \int 1\cdot dx + 3\int \frac{1}{x-4}\,dx - 24\int \frac{1}{x-5}\,dx + 30\int \frac{1}{x-6}\,dx$$

$$\Rightarrow \quad I = x + 3\log|x-4| - 24\log|x-5| + 30\log|x-6| + C$$

Type II WHEN DENOMINATOR CONTAINS SOME REPEATING LINEAR FACTORS

EXAMPLE 4 *Evaluate:*

(i) $\int \frac{3x+1}{(x-2)^2(x+2)}\,dx$ [CBSE 2007] (ii) $\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$ [CBSE 2012]

SOLUTION (i) Let $\frac{3x+1}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$ ...(i)

$\Rightarrow \quad 3x+1 = A(x-2)(x+2) + B(x+2) + C(x-2)^2$ ...(ii)

Putting $x-2=0$ i.e. $x=2$ in (ii), we get: $7=4B \Rightarrow B=\frac{7}{4}$

Putting $x+2=0$ i.e. $x=-2$ in (ii), we get: $-5=16C \Rightarrow C=-\frac{5}{16}$

Comparing coefficients of x^2 on both sides of the identity (ii), we get

$$A+C=0 \Rightarrow A=-C \Rightarrow A=\frac{5}{16}$$

Substituting the values of A, B and C in (i), we get

$$\frac{3x+1}{(x-2)^2(x+2)}=\frac{5}{16}\cdot\frac{1}{x-2}+\frac{7}{4}\frac{1}{(x-2)^2}-\frac{5}{16(x+2)}$$

$$\therefore \quad I=\int \frac{3x+1}{(x-2)^2(x+2)}\,dx=\frac{5}{16}\int \frac{1}{x-2}\,dx+\frac{7}{4}\int \frac{1}{(x-2)^2}\,dx-\frac{5}{16}\int \frac{1}{x+2}\,dx$$

$$\Rightarrow \quad I=\frac{5}{16}\log|x-2|-\frac{7}{4(x-2)}-\frac{5}{16}\log|x+2|+C$$

(ii) We have, $I=\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx$

Let $$\frac{x^2+1}{(x-1)^2(x+3)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3} \quad \text{...(i)}$$

$$\Rightarrow \quad x^2+1=A(x-1)(x+3)+B(x+3)+C(x-1)^2 \quad \text{...(ii)}$$

Putting $x-1=0$ i.e. $x=1$ in (ii), we get: $2=4B \Rightarrow B=\frac{1}{2}$

Putting $x+3=0$ i.e. $x=-3$ in (ii), we get: $10=16C \Rightarrow C=\frac{5}{8}$

Equating the coefficients of x^2 on both sides of the identity (ii), we get

$$1=A+C \Rightarrow A=1-C=1-\frac{5}{8}=\frac{3}{8}$$

Substituting the values of A, B and C in (i), we get

$$\frac{x^2+1}{(x-1)^2(x+3)}=\frac{3}{8}\cdot\frac{1}{x-1}+\frac{1}{2}\cdot\frac{1}{(x-1)^2}+\frac{5}{8(x+3)}$$

$$\Rightarrow \quad I=\int \frac{x^2+1}{(x-1)^2(x+3)}\,dx=\frac{3}{8}\int \frac{1}{x-1}\,dx+\frac{1}{2}\int \frac{1}{(x-1)^2}\,dx+\frac{5}{8}\int \frac{1}{x+3}\,dx$$

$$\Rightarrow \quad I=\frac{3}{8}\log|x-1|-\frac{1}{2(x-1)}+\frac{5}{8}\log|x+3|+C$$

EXAMPLE 5 *Evaluate:* $\int \frac{x^2+x+1}{(x-1)^3}\,dx.$

SOLUTION We have, $I=\int \frac{x^2+x+1}{(x-1)^3}\,dx$. Putting $x-1=t$ and $dx=dt$, we get

$$I=\int \frac{(t+1)^2+(t+1)+1}{t^3}\,dt=\int \frac{t^2+3t+3}{t^3}\,dt=\int \left(\frac{1}{t}+\frac{3}{t^2}+\frac{3}{t^3}\right)dt$$

$$\Rightarrow \quad I = \log|t| - \frac{3}{t} - \frac{3}{2t^2} + C = \log|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C$$

NOTE *This sum can also be done by using partial fractions. We write*

$$\frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

EXAMPLE 6 *Evaluate:* $\int \frac{x^2}{(x-1)^3(x+1)}\,dx$

SOLUTION We have, $I = \int \frac{x^2}{(x-1)^3(x+1)}\,dx$

Let $$\frac{x^2}{(x-1)^3(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1} \quad \text{...(i)}$$

$$\Rightarrow \quad x^2 = A(x-1)^2(x+1) + B(x-1)(x+1) + C(x+1) + D(x-1)^3 \quad \text{...(ii)}$$

Putting $x-1=0$ i.e. $x=1$ in (ii), we get: $1 = 2C \Rightarrow C = \frac{1}{2}$

Putting $x+1=0$ i.e. $x=-1$ in (ii), we get: $1 = -8D \Rightarrow D = -\frac{1}{8}$

Putting $x=0$ in (iii), we get: $0 = A - B + C - D \Rightarrow A - B = -\frac{5}{8}$

Putting $x=2$ in (ii), we get: $4 = 3A + 3B + 3C + D \Rightarrow 4 = 3(A+B+C) + D \Rightarrow A + B = \frac{7}{8}$

Now, $A - B = -\frac{5}{8}$ and $A + B = \frac{7}{8} \Rightarrow A = \frac{1}{8}$ and $B = \frac{3}{4}$

Thus, we obtain $A = \frac{1}{8}, B = \frac{3}{4}, C = \frac{1}{2}, D = -\frac{1}{8}$

Substituting the values of A, B, C and D in (i), we get

$$\frac{x^2}{(x-1)^3(x+1)} = \frac{1}{8(x-1)} + \frac{3}{4(x-1)^2} + \frac{1}{2(x-1)^3} - \frac{1}{8(x+1)}$$

$$\therefore \quad I = \int \frac{x^2}{(x-1)^3(x+1)}\,dx$$

$$\Rightarrow \quad I = \frac{1}{8}\int \frac{1}{x-1}\,dx + \frac{3}{4}\int \frac{1}{(x-1)^2}\,dx + \frac{1}{2}\int \frac{1}{(x-1)^3}\,dx - \frac{1}{8}\int \frac{1}{x+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{8}\log|x-1| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} - \frac{1}{8}\log|x+1| + C$$

$$\Rightarrow \quad I = \frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} + C$$

Type III THE DENOMINATOR CONTAINS IRREDUCIBLE QUADRATIC FACTORS

EXAMPLE 7 *Evaluate:*

(i) $\int \frac{8}{(x+2)(x^2+4)}\,dx$ [CBSE 2013] (ii) $\int \frac{x}{(x-1)(x^2+4)}\,dx$

SOLUTION (i) Let $\frac{8}{(x+2)(x^2+4)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$ ...(i)

Then, $8 = A(x^2+4) + (Bx+C)(x+2)$ (ii)

Putting $x+2=0$ i.e. $x=-2$ in (ii), we get: $8 = 8A \Rightarrow A = 1$

Putting $x=0$ and 1 respectively in (ii), we get: $8 = 4A + 2C$ and $8 = 5A + 3B + 3C$

Solving these equations, we obtain: $A=1, C=2$ and $B=-1$.

Substituting the values of A, B and C in (i), we obtain

$$\frac{8}{(x+2)(x^2+4)} = \frac{1}{x+2} + \frac{-x+2}{x^2+4}$$

$$\therefore \quad I = \int \frac{8}{(x+2)(x^2+4)}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{x+2}\,dx + \int \frac{-x+2}{x^2+4}\,dx = \int \frac{1}{x+2}\,dx - \int \frac{x}{x^2+4}\,dx + 2\int \frac{1}{x^2+4}\,dx$$

$$\Rightarrow \quad I = \log|x+2| - \frac{1}{2}\int \frac{1}{t}\,dt + 2 \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C, \text{ where } t = x^2+4$$

$$\Rightarrow \quad I = \log|x+2| - \frac{1}{2}\log t + \tan^{-1}\frac{x}{2} + C = \log|x+2| - \frac{1}{2}\log(x^2+4) + \tan^{-1}\frac{x}{2} + C$$

(ii) Let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$ ...(i)

$$\Rightarrow \quad x = A(x^2+4) + (Bx+C)(x-1) \qquad \text{...(ii)}$$

Putting $x=1$ in (ii), we get: $1 = 5A$. Putting $x=0$ in (ii), we get: $0 = 4A - C$. Putting $x=-1$ in (ii), we get: $-1 = 5A + 2B - 2C$

Solving these equations, we obtain: $A = \frac{1}{5}$, $B = -\frac{1}{5}$ and $C = \frac{4}{5}$.

Substituting the values of A, B and C in (i), we obtain

$$\frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} + \frac{-\frac{1}{5}x + \frac{4}{5}}{x^2+4}$$

$$\Rightarrow \quad \frac{x}{(x-1)(x^2+4)} = \frac{1}{5(x-1)} - \frac{1}{5}\frac{(x-4)}{(x^2+4)}$$

$$\therefore \quad I = \int \frac{x}{(x-1)(x^2+4)}\,dx = \int \left\{ \frac{1}{5(x-1)} - \frac{1}{5}\cdot\frac{x-4}{x^2+4} \right\} dx = \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{5}\int \frac{x-4}{x^2+4}\,dx$$

$$\Rightarrow \quad I = \frac{1}{5}\int \frac{1}{x-1}\,dx - \frac{1}{10}\int \frac{2x}{x^2+4}\,dx + \frac{4}{5}\int \frac{1}{x^2+4}\,dx$$

$$\Rightarrow \quad I = \frac{1}{5}\log|x-1| - \frac{1}{10}\log(x^2+4) + \frac{4}{5} \times \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$\Rightarrow \quad I = \frac{1}{5}\log|x-1| - \frac{1}{10}\log(x^2+4) + \frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right) + C$$

IMPORTANT NOTE *If a rational function contains only even powers of x in both the numerator and denominator, then to resolve it into partial fractions, we proceed as follows:*

Step I *Put $x^2 = y$ in the given rational function.*

Step II *Resolve the rational function obtained in step I into partial fractions.*

Step III *Replace y by x^2.*

EXAMPLE 8 *Evaluate:*

(i) $\int \frac{x^2}{(x^2+1)(x^2+4)}\,dx$ **[CBSE 2013, 2014]** (ii) $\int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx$

SOLUTION Let $x^2 = y$. Then,

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$$

Let $$\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4} \quad \text{...(i)}$$

$$\Rightarrow \quad y = A(y+4) + B(y+1) \quad \text{...(ii)}$$

Putting $y = -1$ and $y = -4$ successively in (ii), we get: $A = -\frac{1}{3}$ and $B = \frac{4}{3}$.

Substituting the values of A and B in (i), we obtain

$$\frac{y}{(y+1)(y+4)} = -\frac{1}{3(y+1)} + \frac{4}{3(y+4)}$$

Replacing y by x^2, we obtain

$$\frac{x^2}{(x^2+1)(x^2+4)} = -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)}$$

$$\therefore \quad I = \int \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \int \left\{ -\frac{1}{3(x^2+1)} + \frac{4}{3(x^2+4)} \right\} dx = -\frac{1}{3}\int \frac{1}{x^2+1}\,dx + \frac{4}{3}\int \frac{1}{x^2+4}\,dx$$

$$\Rightarrow I = -\frac{1}{3}\tan^{-1} x + \frac{4}{3} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C = -\frac{1}{3}\tan^{-1} x + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + C$$

(ii) Let $x^2 = y$. Then,

$$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{y+1}{(y+2)(2y+1)}$$

Let $$\frac{y+1}{(y+2)(2y+1)} = \frac{A}{y+2} + \frac{B}{2y+1} \quad \text{...(i)}$$

$$\Rightarrow y+1 = A(2y+1) + B(y+2) \quad \text{...(ii)}$$

Putting $y + 2 = 0$ i.e. $y = -2$ in (ii), we get: $-1 = -3A \Rightarrow A = \frac{1}{3}$.

Putting $2y + 1 = 0$ i.e. $y = -\frac{1}{2}$ in (ii), we get: $\frac{1}{2} = B\left(\frac{3}{2}\right) \Rightarrow B = \frac{1}{3}$.

Substituting the values of A and B in (i), we obtain

$$\frac{y+1}{(y+2)(2y+1)} = \frac{1}{3} \cdot \frac{1}{y+2} + \frac{1}{3}\,\frac{1}{(2y+1)}$$

Replacing y by x^2, we get

$$\frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{1}{3}\cdot\frac{1}{x^2+2}+\frac{1}{3(2x^2+1)}$$

$$\therefore \quad I = \int \frac{x^2+1}{(x^2+2)(2x^2+1)}\,dx = \frac{1}{3}\int \frac{1}{x^2+2}\,dx + \frac{1}{3}\int \frac{1}{(\sqrt{2}x)^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+\frac{1}{3\sqrt{2}}\tan^{-1}(\sqrt{2}x)+C = \frac{1}{3\sqrt{2}}\left\{\tan^{-1}\frac{x}{\sqrt{2}}+\tan^{-1}\sqrt{2}\,x\right\}+C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Evaluate* : $\displaystyle\int \frac{x+1}{x(1+xe^x)^2}\,dx$

SOLUTION Let $\displaystyle I = \int \frac{x+1}{x(1+xe^x)^2}\,dx = \int \frac{(x+1)e^x}{xe^x(1+xe^x)^2}\,dx$

Let $xe^x = t$. Then $d(xe^x) = dt$ or, $(x+1)e^x dx = dt$.

$$\therefore \quad I = \int \frac{1}{t(1+t)^2}\,dt, \quad \text{where } t = xe^x.$$

Let $$\frac{1}{t(1+t)^2} = \frac{A}{t}+\frac{B}{1+t}+\frac{C}{(1+t)^2} \qquad \ldots\text{(i)}$$

$$\Rightarrow \quad 1 = A(1+t)^2 + Bt(1+t) + Ct \qquad \ldots\text{(ii)}$$

Putting $t = 0$ and $t = -1$ respectively in (ii) we get: $A = 1, C = -1$

Now, putting $t = 1$ in (ii) we get: $1 = 4A + 2B + C \Rightarrow 1 = 4 + 2B - 1 \Rightarrow B = -1$

Substituting the values of A, B and C in (i), we get

$$\frac{1}{t(1+t)^2} = \frac{1}{t}-\frac{1}{1+t}-\frac{1}{(1+t)^2}$$

$$\therefore \quad I = \int \frac{1}{t(1+t)^2}\,dt = \int\left(\frac{1}{t}-\frac{1}{1+t}-\frac{1}{(1+t)^2}\right)dt$$

$$\Rightarrow \quad I = \log|t| - \log|1+t| + \frac{1}{1+t} + C = \log(xe^x) - \log(1+xe^x) + \frac{1}{1+xe^x} + C$$

EXAMPLE 10 *Evaluate* : $\displaystyle\int \frac{1}{x+\sqrt{x^2-x+1}}\,dx$

SOLUTION Let $\displaystyle I = \int \frac{1}{x+\sqrt{x^2-x+1}}\,dx$. Let $x+\sqrt{x^2-x+1} = t$. Then,

$$\sqrt{x^2-x+1} = t - x \Rightarrow x^2-x+1 = (t-x)^2 \Rightarrow -2x+1 = t^2-2tx \Rightarrow x = \frac{t^2-1}{2t-1}$$

$$\therefore \quad dx = \frac{(2t-1)\,2t - 2(t^2-1)}{(2t-1)^2}\,dt = \frac{2t^2-2t+2}{(2t-1)^2}\,dt$$

Substituting these values, we get

$$I = \int \frac{1}{t}\times\frac{2t^2-2t+2}{(2t-1)^2}\,dt = 2\int \frac{t^2-t+1}{t(2t-1)^2}\,dt$$

Let $\dfrac{t^2-t+1}{t(2t-1)^2} = \dfrac{A}{t}+\dfrac{B}{2t-1}+\dfrac{C}{(2t-1)^2}$ ...(i)

Using cover-up method, we obtain $A=1$ and $C=\dfrac{3}{2}$.

From (i), we obtain: $t^2-t+1 = A(2t-1)^2+B(2t-1)t+Ct$

On equating the coefficient of t^2 on both sides, we get: $1 = 4A+2B \Rightarrow B = -\dfrac{3}{2}$

Substituting the values of A, B, C in (i), we get

$$\frac{t^2-t+1}{t(2t-1)^2} = \frac{1}{t}-\frac{3}{2(2t-1)}+\frac{3}{2}\,\frac{1}{(2t-1)^2}$$

$$\therefore\ I = 2\int \frac{1}{t}\,dt - 3\int \frac{1}{2t-1}\,dt+\frac{3}{2}\int \frac{1}{(2t-1)^2}\,dt = 2\log t-\frac{3}{2}\log(2t-1)-\frac{3}{4}\,\frac{1}{(2t-1)}+C$$

$$\Rightarrow I = 2\log\left(x+\sqrt{x^2-x+1}\right)-\frac{3}{2}\log\left\{(2x-1)+2\sqrt{x^2-x+1}\right\}-\frac{3}{4\left\{(2x-1)+2\sqrt{x^2-x+1}\right\}}+C$$

EXAMPLE 11 *Evaluate:* $\displaystyle\int \frac{\sin x}{\sin 4x}\,dx$

SOLUTION We have,

$$I = \int \frac{\sin x}{\sin 4x}\,dx = \int \frac{\sin x}{2\sin 2x\cos 2x}\,dx = \int \frac{\sin x}{4\sin x\cos x\cos 2x}\,dx$$

$$\Rightarrow\quad I = \frac{1}{4}\int \frac{1}{\cos x\cos 2x}\,dx = \frac{1}{4}\int \frac{\cos x}{\cos^2 x\cos 2x}\,dx = \frac{1}{4}\int \frac{\cos x}{(1-\sin^2 x)(1-2\sin^2 x)}\,dx$$

Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)}$$

Let $t^2 = y$. Then,

$$\frac{1}{(1-t^2)(1-2t^2)} = \frac{1}{(1-y)(1-2y)}$$

Let $\dfrac{1}{(1-y)(1-2y)} = \dfrac{A}{1-y}+\dfrac{B}{1-2y}$. Then, $1 = A(1-2y)+B(1-y)$ ...(i)

Putting $y=1$ and $y=\dfrac{1}{2}$ respectively in (i), we get: $A=-1$ and $B=2$.

$$\therefore\quad \frac{1}{(1-y)(1-2y)} = \frac{-1}{1-y}+\frac{2}{1-2y}$$

$$\Rightarrow\quad \frac{1}{(1-t^2)(1-2t^2)} = -\frac{1}{1-t^2}+\frac{2}{1-2t^2}$$

$$\Rightarrow\quad I = \frac{1}{4}\int \frac{dt}{(1-t^2)(1-2t^2)} = \frac{1}{4}\int -\frac{1}{1-t^2}+\frac{2}{1-2t^2}\,dt$$

$$\Rightarrow I = -\frac{1}{4}\int \frac{1}{1-t^2}\,dt + \frac{2}{4}\int \frac{1}{1-(\sqrt{2}\,t)^2}\,dt = -\frac{1}{4}\times\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}\,t}{1-\sqrt{2}\,t}\right| + C$$

$$\Rightarrow I = -\frac{1}{8}\log\left|\frac{1+t}{1-t}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}t}{1-\sqrt{2}t}\right| + C = -\frac{1}{8}\log\left|\frac{1+\sin x}{1-\sin x}\right| + \frac{1}{4\sqrt{2}}\log\left|\frac{1+\sqrt{2}\sin x}{1-\sqrt{2}\sin x}\right| + C$$

EXAMPLE 12 *Evaluate:* $\displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta$

SOLUTION We have,

$$I = \int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\,d\theta = \int \frac{\tan\theta\,(1+\tan^2\theta)}{1+\tan^3\theta}\,d\theta = \int \frac{\tan\theta\sec^2\theta}{(1+\tan^3\theta)}\,d\theta$$

Putting $\tan\theta = t$ and $\sec^2\theta\,d\theta = dt$, we get

$$I = \int \frac{t\,dt}{(1+t^3)} = \int \frac{t}{(1+t)\,(t^2-t+1)}\,dt$$

Let $\displaystyle\frac{t}{(1+t)\,(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$. Then, $t = A\,(1-t+t^2) + (Bt+C)\,(t+1)$ ...(i)

Putting $1+t=0$ or, $t=-1$ in (i), we get: $A = -\dfrac{1}{3}$.

Comparing the coefficients of t^2 on both sides of (i), we get: $A + B = 0 \Rightarrow B = -A = \dfrac{1}{3}$

Comparing, constant terms on both sides of (i), we get: $A + C = 0 \Rightarrow C = -A = \dfrac{1}{3}$

$$\therefore \quad \frac{t}{1+t^3} = -\frac{1}{3\,(1+t)} + \frac{\frac{1}{3}t + \frac{1}{3}}{1-t+t^2} = -\frac{1}{3\,(t+1)} + \frac{1}{3}\cdot\frac{t+1}{(1-t+t^2)}$$

$$\Rightarrow \quad I = -\frac{1}{3}\int \frac{1}{1+t}\,dt + \frac{1}{3}\int \frac{t+1}{t^2-t+1}\,dt = -\frac{1}{3}\int \frac{1}{1+t}\,dt + \frac{1}{6}\int \frac{2t-1+3}{t^2-t+1}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{3}\int \frac{1}{1+t}\,dt + \frac{1}{6}\int \frac{2t-1}{t^2-t+1}\,dt + \frac{3}{6}\int \frac{1}{t^2-t+1}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{2}\times\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$\Rightarrow \quad I = -\frac{1}{3}\log|1+t| + \frac{1}{6}\log|t^2-t+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \quad I = -\frac{1}{3}\log|1+\tan\theta| + \frac{1}{6}\log|\tan^2\theta - \tan\theta + 1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta - 1}{\sqrt{3}}\right) + C$$

EXAMPLE 13 *Evaluate:* $\displaystyle\int \frac{1}{\sin x\,(2\cos^2 x - 1)}\,dx$

SOLUTION Putting $\cos x = t$ and $d(\cos x) = dt$ or, $-\sin x\, dx = dt$, we get

$$I = \int \frac{1}{\sin x\,(2\cos^2 x - 1)}\,dx = \int \frac{1}{\sin x\,(2t^2-1)} \times -\frac{dt}{\sin x}$$

$$\Rightarrow \quad I = -\int \frac{1}{(1-t^2)(2t^2-1)}\,dt$$

$$\therefore \quad I = -\int \left(\frac{1}{1-t^2} + \frac{2}{2t^2-1}\right) dt = -\int \frac{1}{1-t^2}\,dt - 2\int \frac{1}{2t^2-1}\,dt$$

$$\Rightarrow \quad I = -\int \frac{1}{1^2-t^2}\,dt - \int \frac{1}{t^2-(1/\sqrt{2})^2}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{2}\log\left|\frac{1+t}{1-t}\right| - \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + C = -\frac{1}{2}\log\left|\frac{1+\cos x}{1-\cos x}\right| - \frac{1}{\sqrt{2}}\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C$$

EXERCISE 18.30

BASIC

Evaluate the following integrals:

1. $\int \frac{2x+1}{(x+1)(x-2)}\,dx$
2. $\int \frac{1}{x(x-2)(x-4)}\,dx$
3. $\int \frac{x^2}{(x-1)(x-2)(x-3)}\,dx$
4. $\int \frac{5x}{(x+1)(x^2-4)}\,dx$ [NCERT]
5. $\int \frac{x^2+1}{x(x^2-1)}\,dx$
6. $\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$
7. $\int \frac{x^2+x+1}{(x^2+1)(x+2)}\,dx$ [CBSE 2015]
8. $\int \frac{ax^2+bx+c}{(x-a)(x-b)(x-c)}\,dx$, where a, b, c are distinct.
9. $\int \frac{x}{(x^2+1)(x-1)}\,dx$ [CBSE 2013]
10. $\int \frac{1}{(x-1)(x+1)(x+2)}\,dx$
11. $\int \frac{5x^2-1}{x(x-1)(x+1)}\,dx$
12. $\int \frac{x^2+6x-8}{x^3-4x}\,dx$
13. $\int \frac{x^2+1}{(2x+1)(x^2-1)}\,dx$
14. $\int \frac{3x-2}{(x+1)^2(x+3)}\,dx$ [CBSE 2013]
15. $\int \frac{2x+1}{(x+2)(x-3)^2}\,dx$
16. $\int \frac{x^2+1}{(x-2)^2(x+3)}\,dx$
17. $\int \frac{x}{(x-1)^2(x+2)}\,dx$ [NCERT, CBSE 2020]
18. $\int \frac{x^2}{(x-1)(x+1)^2}\,dx$
19. $\int \frac{x^2+x-1}{(x+1)^2(x+2)}\,dx$
20. $\int \frac{5x^2+20x+6}{x^3+2x^2+x}\,dx$

21. $\int \frac{18}{(x+2)(x^2+4)}\,dx$ [CBSE 2013]

22. $\int \frac{5}{(x^2+1)(x+2)}\,dx$

23. $\int \frac{x}{(x+1)(x^2+1)}\,dx$ [CBSE 2002, 05]

24. $\int \frac{1}{1+x+x^2+x^3}\,dx$

25. $\int \frac{1}{(x+1)^2(x^2+1)}\,dx$

26. $\int \frac{2x}{x^3-1}\,dx$

27. $\int \frac{3x+5}{x^3-x^2-x+1}\,dx$ [NCERT, CBSE 2013]

28. $\int \frac{x^3-1}{x^3+x}\,dx$

29. $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$ [NCERT, CBSE 2014]

30. $\int \frac{3}{(1-x)(1+x^2)}\,dx$ [CBSE 2012]

31. $\int \frac{\cos x}{(1-\sin x)^3(2+\sin x)}\,dx$

32. $\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$ [CBSE 2007]

33. $\int \frac{2x+1}{(x-2)(x-3)}\,dx$ [CBSE 2007]

34. $\int \frac{2x}{(x^2+1)(x^2+2)^2}\,dx$ [CBSE 2017]

BASED ON LOTS

35. $\int \frac{x^2+x-1}{x^2+x-6}\,dx$

36. $\int \frac{3+4x-x^2}{(x+2)(x-1)}\,dx$

37. $\int \frac{x^2+1}{x^2-1}\,dx$

38. $\int \frac{x^3}{(x-1)(x-2)(x-3)}\,dx$

39. $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)}\,dx$ [CBSE 2004]

40. $\int \frac{2x}{(x^2+1)(x^2+3)}\,dx$ [CBSE 2004, 11]

41. $\int \frac{1}{x\log x(2+\log x)}\,dx$

42. $\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$ [CBSE 2013]

43. $\int \frac{1}{x\left\{6(\log x)^2+7\log x+2\right\}}\,dx$

44. $\int \frac{1}{x(x^n+1)}\,dx$ [NCERT]

45. $\int \frac{x}{(x^2-a^2)(x^2-b^2)}\,dx$

46. $\int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$ [CBSE 2013]

47. $\int \frac{x^3+x+1}{x^2-1}\,dx$

48. $\int \frac{2x^2+7x-3}{x^2(2x+1)}\,dx$

49. $\int \frac{1}{(x^2+1)(x^2+4)}\,dx$

50. $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$

51. $\int \frac{1}{x(x^4+1)}\,dx$

52. $\int \frac{1}{x(x^3+8)}\,dx$ [CBSE 2013]

53. $\int \frac{2x^2+1}{x^2(x^2+4)}\,dx$ [CBSE 2013]

54. $\int \frac{1}{(x^2+1)(x^2+2)}\,dx$ [CBSE 2010]

55. $\int \frac{1}{x(x^4-1)}\,dx$ [NCERT]

56. $\int \frac{1}{x^4-1}\,dx$ [NCERT]

57. $\int \frac{x^2}{(x-1)(x^2+1)}\,dx$ [CBSE 2017]

BASED ON HOTS

58. $\int \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx$

59. $\int \frac{1}{\cos x\,(5-4\sin x)}\,dx$

60. $\int \frac{1}{\sin x\,(3+2\cos x)}\,dx$

61. $\int \frac{1}{\sin x+\sin 2x}\,dx$ [CBSE 2015]

62. $\int \frac{x+1}{x(1+x e^x)}\,dx$

63. $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}\,dx$

64. $\int \frac{4x^4+3}{(x^2+2)(x^2+3)(x^2+4)}\,dx$

65. $\int \frac{x^4}{(x-1)(x^2+1)}\,dx$

66. $\int \frac{x^2}{x^4-x^2-12}\,dx$

67. $\int \frac{x^2}{1-x^4}\,dx$

68. $\int \frac{x^2}{x^4+x^2-2}\,dx$ [NCERT, CBSE 2016]

69. $\int \frac{(x^2+1)(x^2+4)}{(x^2+3)(x^2-5)}\,dx$ [CBSE 2016]

70. $\int \frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}\,dx$ [CBSE 2018]

ANSWERS

1. $\frac{1}{3}\log|x+1|+\frac{5}{3}\log|x-2|+C$

2. $\frac{1}{8}\log\left|\frac{x(x-4)}{(x-2)^2}\right|+C$

3. $\frac{1}{2}\log|x-1|-4\log|x-2|+\frac{9}{2}\log|x-3|+C$

4. $\frac{5}{6}\log\left|\frac{(x+1)^2(x-2)}{(x+2)^3}\right|+C$

5. $\log\left|\frac{x^2-1}{x}\right|+C$

6. $\frac{5}{2}\log|x+1|-\frac{1}{10}\log|x-1|-\frac{12}{5}\log|2x+3|+C$

7. $\frac{3}{5}\log|x+2|+\frac{1}{5}\log(x^2+1)+\frac{1}{5}\tan^{-1}x+C$

8. $\frac{a^3+ab+c}{(a-b)(a-c)}\log|x-a|+\frac{ab^2+b^2+c}{(b-a)(b-c)}\log|x-b|+\frac{ac^2+bc+c}{(c-a)(c-b)}\log|x-c|+K$

9. $\frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+C$

10. $\frac{1}{6}\log\left|\frac{(x-1)(x+2)^2}{(x+1)^3}\right|+C$

11. $\log|x(x^2-1)^2|+C$

12. $\log\left|\frac{x^2(x-2)}{(x+2)^2}\right|+C$

13. $-\frac{5}{6}\log|2x+1|+\frac{1}{3}\log|x-1|+\log|x+1|+C$

14. $\frac{11}{4}\log|x+1|+\frac{5}{2(x+1)}-\frac{11}{4}\log|x+3|+C$

15. $-\frac{3}{25}\log|x+2|+\frac{3}{25}\log|x-3|-\frac{7}{5(x-3)}+C$

16. $\frac{2}{5}\log|x+3|+\frac{3}{5}\log|x-2|-\frac{1}{x-2}+C$

17. $\frac{2}{9}\log\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+C$

18. $\frac{1}{4}\log|x-1|+\frac{3}{4}\log|x+1|+\frac{1}{2(x+1)}+C$

19. $\frac{1}{x+1}+\log|x+2|+C$

20. $6\log|x|-\log|x+1|-\frac{9}{(x+1)}+C$

21. $\frac{9}{4}\left\{\log|x+2|-\frac{1}{2}\log|x^2+4|+\tan^{-1}\left(\frac{x}{2}\right)\right\}+C$

22. $2\tan^{-1}x-\frac{1}{2}\log|x^2+1|+\log|x+2|+C$

23. $-\frac{1}{2}\log|x+1|+\frac{1}{4}\log|x^2+1|+\frac{1}{2}\tan^{-1}x+C$

24. $\frac{1}{2}\log|x+1|-\frac{1}{4}\log|1+x^2|+\frac{1}{2}\tan^{-1}x+C$

25. $\frac{1}{2}\log|x+1|-\frac{1}{2(x+1)}-\frac{1}{4}\log|x^2+1|+C$

26. $\frac{2}{3}\log|x-1|-\frac{1}{3}\log|x^2+x+1|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$

27. $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+C$

28. $x-\log|x|-\tan^{-1}x+\frac{1}{2}\log(x^2+1)+C$

29. $-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C$

30. $\frac{3}{4}\left\{\log\frac{x^2+1}{(1-x)^2}+2\tan^{-1}x\right\}+C$

31. $-\frac{1}{27}\log|1-\sin x|+\frac{1}{9(1-\sin x)}+\frac{1}{6(1-\sin x)^2}+\frac{1}{27}\log|2+\sin x|+C$

32. $\log\left|\frac{2-\sin x}{1-\sin x}\right|+C$

33. $\log\left|\frac{(x-3)^7}{(x-2)^5}\right|+C$

34. $\log\left(\frac{x^2+1}{x^2+2}\right)+\frac{1}{x^2+2}+C$

35. $x-\log|x+3|+\log|x-2|+C$

36. $-x+3\log|x+2|+2\log|x-1|+C$

37. $x+\log\left|\frac{x-1}{x+1}\right|+C$

38. $x+\frac{1}{2}\log|x-1|-8\log|x-2|+\frac{27}{2}\log|x-3|+C$

39. $\log\left|\frac{(2+\sin x)^4}{(1+\sin x)^2}\right|+C$

40. $\frac{1}{2}\log\left|\frac{x^2+1}{x^2+3}\right|+C$

41. $\frac{1}{2}\log\left|\frac{\log x}{2+\log x}\right|+C$

42. $-\frac{2}{5}\tan^{-1}\frac{x}{2}+\frac{3}{5}\tan^{-1}\frac{x}{3}+C$

43. $\log|2\log x+1|-\log|3\log x+2|+C$

44. $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right|+C$

45. $\frac{1}{2(a^2-b^2)}\log\left|\frac{x^2-a^2}{x^2-b^2}\right|+C$

46. $-\frac{1}{14}\tan^{-1}\frac{x}{2}+\frac{8}{35}\tan^{-1}\frac{x}{5}+C$

47. $\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$

48. $\frac{3}{x}+13\log|x|-12\log|2x+1|+C$

49. $\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right)+C$

50. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right)-\tan^{-1}x+C$

51. $\frac{1}{4}\log\left(\frac{x^4}{x^4+1}\right)+C$

52. $\frac{1}{8}\log|x|-\frac{1}{24}\log\left|x^3+8\right|+C$

53. $-\frac{1}{4x}+\frac{7}{8}\tan^{-1}\left(\frac{x}{2}\right)+C$

54. $-\frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}}+\tan^{-1}x+C$

55. $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right|+C$

56. $\frac{1}{4}\log\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\tan^{-1}x+C$

57. $\frac{1}{2}\log|x-1|+\frac{1}{4}\log(x^2+1)+\frac{1}{2}\tan^{-1}x+C$

58. $\frac{a}{a^2-b^2}\tan^{-1}\frac{x}{a}-\frac{b}{a^2-b^2}\tan^{-1}\frac{x}{b}+C$

59. $\frac{1}{18}\log|1+\sin x|-\frac{1}{2}\log|1-\sin x|+\frac{4}{9}\log|5-4\sin x|+C$

60. $-\frac{1}{2}\log|1+\cos x|+\frac{1}{10}\log|1-\cos x|+\frac{2}{5}\log|3+2\cos x|+C$

61. $\frac{1}{6}\log|1-\cos x|+\frac{1}{2}\log|1+\cos x|-\frac{2}{3}\log|1+2\cos x|+C$

62. $\log\left|\frac{xe^x}{1+xe^x}\right|+C$

63. $x+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)-3\tan^{-1}\left(\frac{x}{2}\right)+C$

64. $\frac{19}{2\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{39}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+\frac{67}{4}\tan^{-1}\left(\frac{x}{2}\right)+C$

65. $\frac{x^2}{2}+x+\frac{1}{2}\log|x-1|-\frac{1}{4}\log(x^2+1)-\frac{1}{2}\tan^{-1}x+C$

66. $\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C$

67. $\frac{1}{4}\log\left|\frac{1-x}{1+x}\right|-\frac{1}{2}\tan^{-1}x+C$

68. $\frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}+\frac{1}{6}\log\left|\frac{x-1}{x+1}\right|+C$

69. $x+\frac{1}{4\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}+\frac{27}{8\sqrt{5}}\log\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right|+C$

70. $-\log(1-\sin x)+\frac{1}{2}\log(1+\sin^2 x)+\tan^{-1}(\sin x)+C$

18.14 INTEGRALS OF THE FORM

$$\int\frac{x^2+1}{x^4+\lambda x^2+1}\,dx,\ \int\frac{x^2-1}{x^4+\lambda x^2+1}\,dx,\ \int\frac{1}{x^4+\lambda x^2+1}\,dx, \text{where } \lambda\in R$$

To evaluate this type of integrals, we use the following algorithm.

ALGORITHM

Step I *Divide numerator and denominator by* x^2.

Step II *Express the denominator of integrand in the form* $\left(x+\frac{1}{x}\right)^2 \pm k^2$.

Step III *Introduce* $d\left(x+\frac{1}{x}\right)$ *or,* $d\left(x-\frac{1}{x}\right)$ *or both in the numerator.*

Step IV *Substitute* $x+\frac{1}{x}=t$ *or,* $x-\frac{1}{x}=t$ *as the case may be.*

This substitution reduces the integral in one of the following forms $\int \frac{1}{x^2+a^2}\,dx$, $\int \frac{1}{x^2-a^2}\,dx$.

Step V *Use the appropriate formula.*

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{x^2+1}{x^4+1}\,dx$ [CBSE 2007, 11] (ii) $\int \frac{x^2-1}{x^4+x^2+1}\,dx$ (iii) $\int \frac{x^2+4}{x^4+16}\,dx$ [CBSE 2007]

SOLUTION (i) Let $I=\int \frac{x^2+1}{x^4+1}\,dx$. Dividing the numerator and denominator by x^2, we obtain

$$I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx$$

Let $x-\frac{1}{x} = t \Rightarrow d\left(x-\frac{1}{x}\right) = dt \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$.

$$\therefore\ I=\int \frac{dt}{t^2+(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+C$$

(ii) Let $I=\int \frac{x^2-1}{x^4+x^2+1}\,dx$. Dividing the numerator and denominator by x^2, we obtain

$$I = \int \frac{1-\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-1^2}\,dx$$

Let $x+\frac{1}{x} = u$. Then, $d\left(x+\frac{1}{x}\right) = du \Rightarrow \left(1-\frac{1}{x^2}\right)dx = du$

$$\therefore\ I = \int \frac{du}{u^2-1^2} = \frac{1}{2(1)}\log\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\log\left|\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1}\right| + C = \frac{1}{2}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$$

(iii) Let $I = \int \frac{x^2+4}{x^4+16}\,dx$. Dividing the numerator and denominator by x^2, we obtain

$$I = \int \frac{1+\frac{4}{x^2}}{x^2+\frac{16}{x^2}}\,dx = \int \frac{1+\frac{4}{x^2}}{x^2+\left(\frac{4}{x}\right)^2-8+8}\,dx = \int \frac{1+\frac{4}{x^2}}{\left(x-\frac{4}{x}\right)^2+8}\,dx.$$

Let $x-\frac{4}{x}=t$. Then, $d\left(x-\frac{4}{x}\right) = dt \Rightarrow \left(1+\frac{4}{x^2}\right)dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2+(2\sqrt{2})^2}$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{2\sqrt{2}}\right)+C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-\frac{4}{x}}{2\sqrt{2}}\right)+C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-4}{2x\sqrt{2}}\right)+C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{1}{x^4+1}\,dx$ (ii) $\int \frac{x^2}{x^4+1}\,dx$ (iii) $\int \sqrt{\tan\theta}\,d\theta$

SOLUTION (i) Let $I = \int \frac{1}{x^4+1}\,dx$. Dividing numerator and denominator by x^2, we obtain

$$\Rightarrow \quad I = \int \frac{\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{\frac{2}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} - \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2}\,dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-2}\,dx$$

Substituting $x-\frac{1}{x}=u$ in first integral and $x+\frac{1}{x}=v$ in second integral, we obtain

$$I = \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} - \frac{1}{2}\int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right| + C$$

(ii) $I = \int \frac{x^2}{x^4+1}\,dx$. Dividing numerator and denominator by x^2, we obtain

$$\Rightarrow \quad I = \int \frac{1}{x^2+1/x^2}\,dx = \frac{1}{2}\int \frac{2}{x^2+1/x^2}\,dx = \int \frac{(1+1/x^2)+(1-1/x^2)}{x^2+1/x^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int = \frac{1}{2}\int \frac{1+1/x^2}{x^2+1/x^2}\,dx + \frac{1}{2}\int \frac{1-1/x^2}{x^2+1/x^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+1/x^2}{(x-1/x)^2+2}\,dx + \frac{1}{2}\int \frac{1-1/x^2\,dx}{(x+1/x)^2-2}$$

Putting $x-\frac{1}{x}=u$ in first integral and $x+\frac{1}{x}=v$ in second integral, we obtain

$$I = \frac{1}{2}\int \frac{du}{u^2+(\sqrt{2})^2} + \frac{1}{2}\int \frac{dv}{v^2-(\sqrt{2})^2}$$

$$\Rightarrow \quad I = \frac{1}{2}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2}\times\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + C$$

(iii) Let $I=\int\sqrt{\tan\theta}\,d\theta$. Let $\tan\theta = x^2$. Then,

$$d(\tan\theta) = d(x^2) \Rightarrow \sec^2\theta\,d\theta = 2x\,dx \Rightarrow d\theta = \frac{2x\,dx}{\sec^2\theta} = \frac{2x\,dx}{1+\tan^2\theta} = \frac{2x\,dx}{1+x^4}$$

$$I = \int\sqrt{x^2}\cdot\frac{2x\,dx}{1+x^4} = \int\frac{2x^2}{x^4+1}\,dx = \int\frac{2}{x^2+1/x^2}\,dx = \int\frac{1+1/x^2+1-1/x^2}{x^2+1/x^2}\,dx$$

$$\Rightarrow \quad I = \int\frac{1+1/x^2}{x^2+1/x^2}\,dx + \int\frac{1-1/x^2}{x^2+1/x^2}\,dx = \int\frac{1+1/x^2}{(x-1/x)^2+2}\,dx + \int\frac{1-1/x^2\,dx}{(x+1/x)^2-2}$$

Substituting $x-\frac{1}{x}=u$ in first integral and $x+\frac{1}{x}=v$ in second integral, we get

$$I = \int\frac{du}{u^2+(\sqrt{2})^2} + \int\frac{dv}{v^2-(\sqrt{2})^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\,x}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta-1}{\sqrt{2\tan\theta}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{\tan\theta-\sqrt{2\tan\theta}+1}{\tan\theta+\sqrt{2\tan\theta}+1}\right| + C$$

EXAMPLE 3 *Evaluate:* $\int\left\{\sqrt{\tan\theta}+\sqrt{\cot\theta}\right\}d\theta$ **[CBSE 2010, 2013, 2014, 2020]**

SOLUTION Let $I=\int\left\{\sqrt{\tan\theta}+\sqrt{\cot\theta}\right\}d\theta$. Then,

$$I = \int \left\{ \sqrt{\tan\theta} + \frac{1}{\sqrt{\tan\theta}} \right\} d\theta \Rightarrow I = \int \frac{\tan\theta + 1}{\sqrt{\tan\theta}}\, d\theta.$$

Let $\tan\theta = x^2$. Then,

$$d(\tan\theta) = d(x^2) \Rightarrow \sec^2\theta\, d\theta = 2x\, dx \Rightarrow d\theta = \frac{2x\, dx}{\sec^2\theta} = \frac{2x\, dx}{1+\tan^2\theta} = \frac{2x\, dx}{1+x^4}$$

$$\therefore \quad I = \int \frac{x^2+1}{\sqrt{x^2}} \times \frac{2x\, dx}{1+x^4} = 2\int \frac{x^2+1}{x^4+1} = 2\int \frac{1+1/x^2}{x^2+1/x^2}\, dx = 2\int \frac{1+1/x^2}{(x-1/x)^2+2}\, dx$$

$$\Rightarrow \quad I = 2\int \frac{1+1/x^2}{(x-1/x)^2+(\sqrt{2})^2}\, dx = 2\int \frac{du}{u^2+(\sqrt{2})^2} = \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C, \text{ where } x - \frac{1}{x} = u$$

$$\Rightarrow \quad I = \sqrt{2}\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right) + C = \sqrt{2}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}\, x}\right) + C = \sqrt{2}\tan^{-1}\left\{\frac{\tan\theta - 1}{\sqrt{2}\tan\theta}\right\} + C$$

EXAMPLE 4 *Evaluate:* $\int \frac{1}{\sin^4 x + \cos^4 x}\, dx.$ **[CBSE 2014]**

SOLUTION Let $I = \int \frac{1}{\sin^4 x + \cos^4 x}\, dx$. Then,

$$\Rightarrow I = \int \frac{1/\cos^4 x}{\frac{\sin^4 x + \cos^4 x}{\cos^4 x}}\, dx = \int \frac{\sec^4 x}{\tan^4 x + 1}\, dx = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1}\, dx = \int \left\{\frac{1+\tan^2 x}{1+\tan^4 x}\right\} \sec^2 x\, dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I = \int \frac{1+t^2}{1+t^4}\, dt = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right) + C \quad \text{[Proceed as in Example 3]}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Evaluate :* $\int \frac{1}{\cos^6 x + \sin^6 x}\, dx$

SOLUTION Let $I = \int \frac{1}{\cos^6 x + \sin^6 x}\, dx$. Then,

$$I = \int \frac{1}{(\cos^2 x + \sin^2 x)(\cos^4 x + \sin^4 x - \cos^2 x \sin^2 x)}\, dx$$

$$\Rightarrow \quad I = \int \frac{1}{(\cos^2 x + \sin^2 x)^2 - 3\sin^2 x \cos^2 x}\, dx$$

$$\Rightarrow \quad I = \int \frac{1}{1 - 3\sin^2 x \cos^2 x}\, dx = \int \frac{\sec^4 x}{\sec^4 x - 3\tan^2 x}\, dx \quad \left[\begin{array}{l}\text{Dividing N}^r \text{ and} \\ \text{D}^r \text{ by } \cos^4 x\end{array}\right]$$

$$\Rightarrow \quad I = \int \frac{(1+\tan^2 x)\sec^2 x}{(1+\tan^2 x)^2 - 3\tan^2 x}\, dx = \int \frac{t^2+1}{t^4 - t^2 + 1}\, dt, \quad \text{where } t = \tan^2 x$$

$$\Rightarrow \quad I = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-1}\,dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+1} = \int \frac{du}{u^2+1}, \text{ where } u = t-\frac{1}{t}$$

$$\Rightarrow \quad I = \tan^{-1} u + C = \tan^{-1}(\tan x - \cot x) + C$$

EXAMPLE 6 *Evaluate :* $\displaystyle\int \frac{x^4+1}{x^6+1}\,dx$

SOLUTION Let $I = \displaystyle\int \frac{x^4+1}{x^6+1}\,dx$. Then,

$$I = \int \frac{(x^2+1)^2-2x^2}{(x^2+1)(x^4-x^2+1)}\,dx = \int \frac{x^2+1}{x^4-x^2+1}\,dx - 2\int \frac{x^2}{x^6+1}\,dx$$

$$\Rightarrow \quad I = \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx - \frac{2}{3}\int \frac{3x^2\,dx}{(x^3)^2+1}\,dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1^2} - \frac{2}{3}\int \frac{1}{(x^3)^2+1^2}\,3x^2\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{u^2+1^2}\,du - \frac{2}{3}\int \frac{1}{v^2+1^2}\,dv, \text{ where } u = x-\frac{1}{x} \text{ and } v = x^3$$

$$\Rightarrow \quad I = \tan^{-1} u - \frac{2}{3}\tan^{-1} v + C = \tan^{-1}\left(x-\frac{1}{x}\right) - \frac{2}{3}\tan^{-1} x^3 + C$$

EXAMPLE 7 *Evaluate :* $\displaystyle\int \frac{x^3}{x^{16}+4}\,dx$

SOLUTION Let $I = \displaystyle\int \frac{x^3}{x^{16}+4}\,dx = \int \frac{x^3}{(x^4)^4+4}\,dx$. Putting $x^4 = t$ and $4x^3\,dx = dt$, we get

$$I = \frac{1}{4}\int \frac{1}{t^4+4}\,dt = \frac{1}{4}\int \frac{\frac{1}{t^2}}{t^2+\frac{4}{t^2}}\,dt \qquad [\text{Dividing N}^r \text{ and D}^r \text{ by } t^2]$$

$$\Rightarrow I = \frac{1}{16}\int \frac{\frac{4}{t^2}}{t^2+\frac{4}{t^2}}\,dt = \frac{1}{16}\int \frac{\left(1+\frac{2}{t^2}\right)-\left(1-\frac{2}{t^2}\right)}{t^2+\frac{4}{t^2}}\,dt$$

$$\Rightarrow I = \frac{1}{16}\int \frac{1+\frac{2}{t^2}}{t^2+\frac{4}{t^2}}\,dt - \frac{1}{16}\int \frac{1-\frac{2}{t^2}}{t^2+\frac{4}{t^2}}\,dt = \frac{1}{16}\int \frac{1+\frac{2}{t^2}}{\left(t-\frac{2}{t}\right)^2+4}\,dt - \frac{1}{16}\int \frac{1-\frac{2}{t^2}}{\left(t+\frac{2}{t}\right)^2-4}\,dt$$

$$\Rightarrow I = \frac{1}{16}\int \frac{du}{u^2+2^2} - \frac{1}{16}\int \frac{dv}{v^2-2^2}, \text{ where } u = t-\frac{2}{t} \text{ and } v = t+\frac{2}{t}$$

$$\Rightarrow I = \frac{1}{32}\tan^{-1}\left(\frac{u}{2}\right) - \frac{1}{16}\times\frac{1}{2\times 2}\log\left|\frac{v-2}{v+2}\right| + C = \frac{1}{32}\tan^{-1}\left(\frac{t^2-2}{2t}\right) - \frac{1}{64}\log\left|\frac{t^2-2t+2}{t^2+2t+2}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{32}\tan^{-1}\left(\frac{x^8-2}{2x^4}\right) - \frac{1}{64}\log\left|\frac{x^8-2x^4+2}{x^8+2x^4+2}\right| + C$$

EXAMPLE 8 *Evaluate :* $\displaystyle\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\,dx$

SOLUTION Let $I = \displaystyle\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}\,dx$. Then,

$$I = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}}\,dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-2}}\,dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } x^2]$$

Let $x+\frac{1}{x}=t$. Then, $d\left(x+\frac{1}{x}\right)=dt$ or, $\left(1-\frac{1}{x^2}\right)dx = dt$.

$$\therefore\ I = \int \frac{1}{t\sqrt{t^2-2}}\,dt = \int \frac{1}{t\sqrt{t^2-(\sqrt{2})^2}}\,dt = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{t}{\sqrt{2}}\right)+C = \frac{1}{\sqrt{2}}\sec^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+C$$

EXERCISE 18.31

BASED ON LOTS

Evaluate the following integrals:

1. $\displaystyle\int \frac{x^2+1}{x^4+x^2+1}\,dx$
2. $\displaystyle\int \sqrt{\cot\theta}\,d\theta$
3. $\displaystyle\int \frac{x^2+9}{x^4+81}\,dx$
4. $\displaystyle\int \frac{1}{x^4+x^2+1}\,dx$
5. $\displaystyle\int \frac{x^2-3x+1}{x^4+x^2+1}\,dx$
6. $\displaystyle\int \frac{x^2+1}{x^4-x^2+1}\,dx$
7. $\displaystyle\int \frac{x^2-1}{x^4+1}\,dx$ [CBSE 2007]
8. $\displaystyle\int \frac{x^2+1}{x^4+7x^2+1}\,dx$
9. $\displaystyle\int \frac{(x-1)^2}{x^4+x^2+1}\,dx$
10. $\displaystyle\int \frac{1}{x^4+3x^2+1}\,dx$
11. $\displaystyle\int \frac{1}{\sin^4 x+\sin^2 x\cos^2 x+\cos^4 x}\,dx$ [CBSE 2014]

ANSWERS

1. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+C$
2. $-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cot\theta-1}{\sqrt{2\cot\theta}}\right)-\frac{1}{2\sqrt{2}}\log\left|\frac{\cot\theta+1-\sqrt{2\cot\theta}}{\cot\theta+1+\sqrt{2\cot\theta}}\right|+C$
3. $\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x^2-9}{3\sqrt{2}\,x}\right)+C$
4. $\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{3}}\right)-\frac{1}{4}\log\left|\frac{x^2-x+1}{x^2+x+1}\right|+C$
5. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right)+\sqrt{3}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$

6. $\tan^{-1}\left(\frac{x^2-1}{x}\right)+C$

7. $\frac{1}{2\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}\,x+1}{x^2+\sqrt{2}\,x+1}\right|+C$

8. $\frac{1}{3}\tan^{-1}\left(\frac{x^2-1}{3x}\right)+C$

9. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}\,x}\right)-\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$

10. $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{5}x}\right)-\frac{1}{2}\tan^{-1}\left(\frac{x^2+1}{x}\right)+C$

11. $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x-\cot x}{\sqrt{3}}\right)+C$

18.15 INTEGRATION OF SOME SPECIAL IRRATIONAL ALGEBRAIC FUNCTIONS

In this section, we shall discuss four integrals of the form $\int\frac{\phi(x)}{P\sqrt{Q}}\,dx$, where P and Q are polynomial functions of x.

18.15.1 INTEGRALS OF THE FORM $\int\frac{\phi(x)}{P\sqrt{Q}}\,dx$, WHERE P AND Q BOTH ARE LINEAR FUNCTIONS OF x

To evaluate this type of integrals we put $Q=t^2$ i.e., to evaluate integrals of the form $\int\frac{1}{(ax+b)\sqrt{cx+d}}\,dx$, put $cx+d=t^2$.

Following examples illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\int\frac{1}{(x-3)\sqrt{x+1}}\,dx$

SOLUTION Let $I=\int\frac{1}{(x-3)\sqrt{x+1}}\,dx$

Here, P and Q both are linear, so we substitute $Q=t^2$ *i.e.* $x+1=t^2$ and $dx=2t\,dt$.

$$\therefore\quad I=\int\frac{1}{(t^2-1-3)}\frac{2t}{\sqrt{t^2}}\,dt=2\int\frac{dt}{t^2-2^2}=2\times\frac{1}{2\,(2)}\log\left|\frac{t-2}{t+2}\right|+C=\frac{1}{2}\log\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C.$$

EXAMPLE 2 *Evaluate* $\int\frac{\sqrt{x}}{x+1}\,dx.$

SOLUTION Let $I=\int\frac{\sqrt{x}}{x+1}\,dx=\int\frac{x}{\sqrt{x}\,(x+1)}\,dx$. Putting $x=t^2$ and $dx=2t\,dt$, we get

$$I=\int\frac{t^2}{\sqrt{t^2}}\frac{2t\,dt}{(t^2+1)}=2\int\frac{t^2}{t^2+1}\,dt=2\int\frac{t^2+1-1}{t^2+1}\,dt$$

$$\Rightarrow\quad I=2\int 1-\frac{1}{t^2+1}\,dt=2\left(t-\tan^{-1}t\right)+C=2\left(\sqrt{x}-\tan^{-1}\sqrt{x}\right)+C$$

18.15.2 INTEGRALS OF THE FORM $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, WHERE P IS A QUADRATIC EXPRESSION AND Q IS A LINEAR EXPRESSION

To evaluate this type of integrals we put $Q = t^2$ i.e., to evaluate integrals of the form $\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}}\,dx$, we put $px+q=t^2$.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:* $\int \frac{1}{(x^2-4)\sqrt{x+1}}\,dx$

SOLUTION Let $I = \int \frac{1}{(x^2-4)\sqrt{x+1}}\,dx$. Putting $x+1=t^2$ and $dx = 2t\,dt$, we get

$$I = \int \frac{2t\,dt}{\left\{(t^2-1)^2-4\right\}\sqrt{t^2}} = 2\int \frac{dt}{(t^2-1-2)(t^2-1+2)} = 2\int \frac{dt}{(t^2-3)(t^2+1)}$$

Let $t^2 = y$. Then, $\frac{1}{(t^2-3)(t^2+1)} = \frac{1}{(y-3)(y+1)}$

Let $\frac{1}{(y-3)(y+1)} = \frac{A}{y-3} + \frac{B}{y+1}$ ...(i)

$\Rightarrow \quad 1 = A(y+1) + B(y-3)$ ...(ii)

Putting $y=-1, 3$ respectively in (ii), we get: $B = -\frac{1}{4}$ and $A = \frac{1}{4}$.

Substituting the values of A and B in (i), we obtain

$$\frac{1}{(y-3)(y+1)} = \frac{1}{4(y-3)} - \frac{1}{4(y+1)} \Rightarrow \frac{1}{(t^2-3)(t^2+1)} = \frac{1}{4(t^2-3)} - \frac{1}{4(t^2+1)} \qquad [\because y = t^2]$$

$$\therefore \quad I = 2\int \left\{\frac{1}{4(t^2-3)} - \frac{1}{4(t^2+1)}\right\} dt = \frac{1}{2}\int \frac{1}{t^2-(\sqrt{3})^2}\,dt - \frac{1}{2}\int \frac{1}{t^2+1^2}\,dt$$

$$\Rightarrow I = \frac{1}{2}\times\frac{1}{2\sqrt{3}}\log\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}(t) + C = \frac{1}{4\sqrt{3}}\log\left|\frac{\sqrt{x+1}-\sqrt{3}}{\sqrt{x+1}+\sqrt{3}}\right| - \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{x+1}}{1}\right) + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 2 *Evaluate:* $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$

SOLUTION Let $I = \int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}\,dx$. Putting $x+1=t^2$, and $dx = 2t\,dt$, we get

$$I = \int \frac{(t^2+1)\,2t}{\left\{(t^2-1)^2+3(t^2-1)+3\right\}\sqrt{t^2}}\,dt$$

$$\Rightarrow \quad I = 2\int \frac{(t^2+1)}{t^4+t^2+1}\,dt = 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } t^2]$$

$$\Rightarrow \quad I = 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{3})^2}\,dt = 2\int \frac{du}{u^2+(\sqrt{3})^2}, \text{ where } t-\frac{1}{t}=u$$

$$\Rightarrow \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{t-\frac{1}{t}}{\sqrt{3}}\right\}+C$$

$$\Rightarrow \quad I = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{3}\,t}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{x}{\sqrt{3\,(x+1)}}\right\}+C$$

18.15.3 INTEGRALS OF THE FORM $\int \frac{\phi(x)}{P\sqrt{Q}}\,dx$, WHERE P IS A LINEAR EXPRESSION AND Q IS A QUADRATIC EXPRESSION

To evaluate this type of integrals we put $P=1/t$ i.e. to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}}\,dx$, we put $ax+b=\frac{1}{t}$.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate* (i) $\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$ (ii) $\int \frac{1}{(x-1)\sqrt{x^2+4}}\,dx$

SOLUTION (i) Let $I=\int \frac{1}{(x+1)\sqrt{x^2-1}}\,dx$. Putting $x+1=\frac{1}{t}$ and $dx=-\frac{1}{t^2}\,dt$, we get

$$\therefore \quad I = \int \frac{1}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2-1}}\cdot\left(-\frac{1}{t^2}\right)dt = -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2}\,dt = -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)}+C$$

$$\Rightarrow \quad I = \sqrt{1-2t}+C = \sqrt{1-\frac{2}{x+1}}+C = \sqrt{\frac{x-1}{x+1}}+C$$

(ii) Let $I=\int \frac{1}{(x-1)\sqrt{x^2+4}}\,dx$. Putting $x-1=\frac{1}{t}$, and $dx=-\frac{1}{t^2}\,dt$, we get

$$\therefore \quad I = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2+4}} = -\int \frac{dt}{\sqrt{5t^2+2t+1}} = -\frac{1}{\sqrt{5}}\int \frac{dt}{\sqrt{t^2+\frac{2}{5}t+\frac{1}{5}}}$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{\left(t^2 + \frac{2}{5}t + \frac{1}{25}\right) + \frac{1}{5} - \frac{1}{25}}} = -\frac{1}{\sqrt{5}} \int \frac{1}{\sqrt{\left(t + \frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2}}\, dt$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{5}} \log \left| \left(t + \frac{1}{5}\right) + \sqrt{\left(t + \frac{1}{5}\right)^2 + \left(\frac{2}{5}\right)^2} \right| + C = -\frac{1}{\sqrt{5}} \log \left| \left(t + \frac{1}{5}\right) + \sqrt{t^2 + \frac{2}{5}t + \frac{1}{5}} \right| + C$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{5}} \log \left| \left(\frac{1}{x-1} + \frac{1}{5}\right) + \sqrt{\frac{1}{(x-1)^2} + \frac{2}{5(x-1)} + \frac{1}{5}} \right| + C$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{5}} \log \left| \frac{1}{x-1} + \frac{1}{5} + \sqrt{\frac{x^2+4}{5(x-1)^2}} \right| + C$$

18.15.4 INTEGRALS OF THE FORM $\int \frac{\phi(x)}{P\sqrt{Q}}\, dx$, WHERE P AND Q BOTH ARE PURE QUADRATIC EXPRESSION IN x *i.e.* $P = ax^2 + b$ AND $Q = cx^2 + d$

To evaluate this type of integrals we put $x = \frac{1}{t}$ and then $c + dt^2 = u^2$ i.e., to evaluate integrals of the form $\int \frac{1}{(ax^2 + b)\sqrt{cx^2 + d}}\, dx$, we put $x = \frac{1}{t}$ to obtain $\int \frac{-t\,dt}{(a + bt^2)\sqrt{c + dt^2}}$ and then substitute $c + dt^2 = u^2$.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:* $\int \frac{1}{x^2\sqrt{1+x^2}}\, dx$

SOLUTION Let $I = \int \frac{1}{x^2\sqrt{1+x^2}}\, dx$. Putting $x = \frac{1}{t}$ and $-\frac{1}{x^2}\, dx = dt$ or, $dx = -x^2\, dt$, we get

$$I = \int \frac{-dt}{\sqrt{1 + \frac{1}{t^2}}} = -\int \frac{t\,dt}{\sqrt{t^2+1}} = -\int \frac{u\,du}{\sqrt{u^2}}, \quad \text{where } t^2 + 1 = u^2$$

$$\Rightarrow \quad I = \int -1 \cdot du = -u + C = -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2} + 1} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 2 *Evaluate:* $\int \frac{\sqrt{1+x^2}}{1-x^2}\, dx$

SOLUTION Let $I = \int \frac{\sqrt{1+x^2}}{1-x^2}\, dx = \int \frac{\sqrt{1+x^2}}{1-x^2} \times \sqrt{\frac{1+x^2}{1+x^2}}\, dx = \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2}}\, dx$

$$\Rightarrow \quad I = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx + \int \frac{x^2}{(1-x^2)\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{-x^2}{(1-x^2)\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{(1-x^2)-1}{(1-x^2)\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \left\{\frac{1-x^2}{(1-x^2)\sqrt{1+x^2}} - \frac{1}{(1-x^2)\sqrt{1+x^2}}\right\}dx$$

$$\Rightarrow \quad I = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{1}{\sqrt{1+x^2}}\,dx + \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \quad I = 2\int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx - \int \frac{1}{\sqrt{1+x^2}}\,dx$$

$$\Rightarrow \quad I = 2I_1 - \log\left|x+\sqrt{1+x^2}\right| + C, \text{ where } I_1 = \int \frac{1}{(1-x^2)\sqrt{1+x^2}}\,dx$$

Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}\,dt$ in I_1, we get

$$I_1 = \int \frac{\left(-\frac{1}{t^2}\right)dt}{\left(1-\frac{1}{t^2}\right)\sqrt{1+\frac{1}{t^2}}}$$

$$\Rightarrow \quad I_1 = -\int \frac{t\,dt}{(t^2-1)\sqrt{t^2+1}} = -\int \frac{u\,du}{(u^2-2)\sqrt{u^2}}, \text{ where } t^2+1=u^2 \text{ and } t\,dt = u\,du$$

$$\Rightarrow \quad I_1 = -\int \frac{du}{u^2-(\sqrt{2})^2} = -\frac{1}{2\sqrt{2}}\log\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| = -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}}\right|$$

$$\Rightarrow \quad I_1 = -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}}\right| = -\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}\,x}{\sqrt{1+x^2}+\sqrt{2}\,x}\right|$$

$$\therefore \quad I = -\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{1+x^2}-\sqrt{2}\,x}{\sqrt{1+x^2}+\sqrt{2}\,x}\right| - \log\left|x+\sqrt{1+x^2}\right| + C$$

EXAMPLE 3 *Evaluate:* $\int \frac{1}{x\sqrt{ax-x^2}}\,dx$ **[NCERT]**

SOLUTION Let $I = \int \frac{1}{x\sqrt{ax-x^2}}\,dx$. Putting $x = \frac{1}{t}$ and $dx = -\frac{1}{t^2}\,dt$, we get

$$I = \int \frac{1}{\frac{1}{t}\sqrt{\frac{a}{t}-\frac{1}{t^2}}} \times -\frac{1}{t^2}\, dt = -\int \frac{1}{\sqrt{at-1}}\, dt = -\int (at-1)^{-1/2}\, dt = -\frac{(at-1)^{1/2}}{\frac{1}{2}a} + C$$

$$\Rightarrow \quad I = \frac{-2}{a}\sqrt{at-1} + C = \frac{-2}{a}\sqrt{\frac{a}{x}-1} + C = \frac{-2}{a}\sqrt{\frac{a-x}{x}} + C$$

EXERCISE 18.32

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Evaluate the following integrals:

1. $\int \frac{1}{(x-1)\sqrt{x+2}}\, dx$
2. $\int \frac{1}{(x-1)\sqrt{2x+3}}\, dx$
3. $\int \frac{x+1}{(x-1)\sqrt{x+2}}\, dx$
4. $\int \frac{x^2}{(x-1)\sqrt{x+2}}\, dx$
5. $\int \frac{x}{(x-3)\sqrt{x+1}}\, dx$
6. $\int \frac{1}{(x^2+1)\sqrt{x}}\, dx$
7. $\int \frac{x}{(x^2+2x+2)\sqrt{x+1}}\, dx$
8. $\int \frac{1}{(x-1)\sqrt{x^2+1}}\, dx$
9. $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}\, dx$
10. $\int \frac{1}{(x^2-1)\sqrt{x^2+1}}\, dx$
11. $\int \frac{x}{(x^2+4)\sqrt{x^2+1}}\, dx$
12. $\int \frac{1}{(1+x^2)\sqrt{1-x^2}}\, dx$
13. $\int \frac{1}{(2x^2+3)\sqrt{x^2-4}}\, dx$
14. $\int \frac{x}{(x^2+4)\sqrt{x^2+9}}\, dx$

ANSWERS

1. $\frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + C$
2. $\frac{1}{\sqrt{5}} \log \left| \frac{\sqrt{2x+3}-\sqrt{5}}{\sqrt{2x+3}+\sqrt{5}} \right| + C$
3. $2\sqrt{x+2} + \frac{2}{\sqrt{3}} \log \left| \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + C$
4. $\frac{2}{3}(x+2)^{3/2} - 2\sqrt{x+2} + \frac{1}{\sqrt{3}} \log \left| \frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}} \right| + C$
5. $2\sqrt{x+1} + \frac{3}{2} \log \left| \frac{\sqrt{x+1}-2}{\sqrt{x+1}+2} \right| + C$
6. $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-1}{\sqrt{2x}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1} \right| + C$
7. $\frac{1}{\sqrt{2}} \log \left| \frac{(x+2)-\sqrt{2(x+1)}}{(x+2)+\sqrt{2(x+1)}} \right| + C$
8. $-\frac{1}{\sqrt{2}} \log \left| \left(t+\frac{1}{2} \right) + \sqrt{\left(t+\frac{1}{2} \right)^2 + \frac{1}{4}} \right| + C$, where $t = \frac{1}{x-1}$
9. $-\log \left| \frac{1}{x+1} - \frac{1}{2} + \frac{\sqrt{x^2+x+1}}{x+1} \right| + C$
10. $-\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2}\,x + \sqrt{x^2+1}}{\sqrt{2}\,x - \sqrt{x^2+1}} \right| + C$
11. $\frac{1}{\sqrt{3}} \tan^{-1}\left(\sqrt{\frac{x^2+1}{3}} \right) + C$
12. $-\frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{\frac{1-x^2}{2x^2}} \right) + C$
13. $\frac{1}{2\sqrt{33}} \log \left| \frac{\sqrt{11}\,x + \sqrt{3x^2-12}}{\sqrt{11}\,x - \sqrt{3x^2-12}} \right| + C$
14. $\frac{1}{2\sqrt{5}} \log \left| \frac{\sqrt{x^2+9}-\sqrt{5}}{\sqrt{x^2+9}+\sqrt{5}} \right| + C$

REVISION EXERCISE

Evaluate the following integrals :

1. $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}\,dx$

2. $\int \frac{1-x^4}{1-x}\,dx$

3. $\int \frac{x+2}{(x+1)^3}\,dx$

4. $\int \frac{8x+13}{\sqrt{4x+7}}\,dx$

5. $\int \frac{1+x+x^2}{x^2(1+x)}\,dx$

6. $\int \frac{(2^x+3^x)^2}{6^x}\,dx$

7. $\int \frac{\sin x}{1+\sin x}\,dx$

8. $\int \frac{x^4+x^2-1}{x^2+1}\,dx$

9. $\int \sec^2 x \cos^2 2x\,dx$

10. $\int \operatorname{cosec}^2 x \cos^2 2x\,dx$

11. $\int \sin^4 2x\,dx$

12. $\int \cos^3 3x\,dx$

13. $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}$

14. $\int \frac{1}{(\sin^{-1} x)\sqrt{1-x^2}}\,dx$

15. $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\,dx$

16. $\int \frac{1}{e^x+1}\,dx$

17. $\int \frac{e^x-1}{e^x+1}\,dx$

18. $\int \frac{1}{e^x+e^{-x}}\,dx$

19. $\int \frac{\cos^7 x}{\sin x}\,dx$

20. $\int \sin x \sin 2x \sin 3x\,dx$

21. $\int \cos x \cos 2x \cos 3x\,dx$

22. $\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$

23. $\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}\,dx$

24. $\int \frac{1}{\sin(x-a)\sin(x-b)}\,dx$

25. $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$

26. $\int \frac{\sin x}{\sqrt{1+\sin x}}\,dx$

27. $\int \frac{\sin x}{\cos 2x}\,dx$

28. $\int \tan^3 x\,dx$

29. $\int \tan^4 x\,dx$

30. $\int \tan^5 x\,dx$

31. $\int \cot^4 x\,dx$

32. $\int \cot^5 x\,dx$

33. $\int \frac{x^2}{(x-1)^3}\,dx$

34. $\int x\sqrt{2x+3}\,dx$

35. $\int \frac{x^3}{(1+x^2)^2}\,dx$

36. $\int x \sin^5 x^2 \cos x^2\,dx$

37. $\int \sin^3 x \cos^4 x\,dx$

38. $\int \sin^5 x\,dx$

39. $\int \cos^5 x\,dx$

40. $\int \sqrt{\sin x}\cos^3 x\,dx$

41. $\int \frac{\sin 2x}{\sin^4 x+\cos^4 x}\,dx$

42. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx$

43. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$

44. $\int \frac{1}{4x^2+4x+5}\,dx$

45. $\int \frac{1}{x^2+4x-5}\,dx$

46. $\int \frac{1}{1-x-4x^2}\,dx$

47. $\int \frac{1}{3x^2+13x-10}\,dx$

48. $\int \frac{\sin x}{\sqrt{\cos^2 x-2\cos x-3}}\,dx$

49. $\int \sqrt{\operatorname{cosec} x-1}\,dx$

50. $\int \frac{1}{\sqrt{3-2x-x^2}}\,dx$

51. $\int \frac{x+1}{x^2+4x+5}\,dx$

52. $\int \frac{5x+7}{\sqrt{(x-5)(x-4)}}\,dx$

53. $\int \sqrt{\frac{1+x}{x}}\,dx$

54. $\int \sqrt{\frac{1-x}{x}}\,dx$

55. $\int \dfrac{\sqrt{a}-\sqrt{x}}{1-\sqrt{ax}}\,dx$

56. $\int \dfrac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)}\,dx$

57. $\int \dfrac{1}{4\sin^2 x + 4\sin x\cos x + 5\cos^2 x}\,dx$

58. $\int \dfrac{1}{a + b\tan x}\,dx$

59. $\int \dfrac{1}{\sin^2 x + \sin 2x}\,dx$

60. $\int \dfrac{\sin x + 2\cos x}{2\sin x + \cos x}\,dx$

61. $\int \dfrac{x^3}{\sqrt{x^8+4}}\,dx$

62. $\int \dfrac{1}{2 - 3\cos 2x}\,dx$

63. $\int \dfrac{\cos x}{\frac{1}{4} - \cos^2 x}\,dx$

64. $\int \dfrac{1}{1 + 2\cos x}\,dx$

65. $\int \dfrac{1}{1 - 2\sin x}\,dx$

66. $\int \dfrac{1}{\sin x\,(2 + 3\cos x)}\,dx$

67. $\int \dfrac{1}{\sin x + \sin 2x}\,dx$

68. $\int \dfrac{1}{\sin^4 x + \cos^4 x}\,dx$

69. $\int \dfrac{1}{5 - 4\sin x}\,dx$

70. $\int \sec^4 x\,dx$

71. $\int \operatorname{cosec}^4 2x\,dx$

72. $\int \dfrac{1 + \sin x}{\sin x\,(1 + \cos x)}\,dx$

73. $\int \dfrac{1}{2 + \cos x}\,dx$

74. $\int \sqrt{\dfrac{a+x}{x}}\,dx$

75. $\int \dfrac{6x+5}{\sqrt{6 + x - 2x^2}}\,dx$

76. $\int \dfrac{\sin^5 x}{\cos^4 x}\,dx$

77. $\int \dfrac{\cos^5 x}{\sin x}\,dx$

78. $\int \dfrac{\sin^6 x}{\cos x}\,dx$

79. $\int \dfrac{\sin^2 x}{\cos^6 x}\,dx$

80. $\int \sec^6 x\,dx$

81. $\int \tan^5 x \sec^3 x\,dx$

82. $\int \tan^3 x \sec^4 x\,dx$

83. $\int \dfrac{1}{\sec x + \operatorname{cosec} x}\,dx$

84. $\int \sqrt{a^2 + x^2}\,dx$

85. $\int \sqrt{x^2 - a^2}\,dx$

86. $\int \sqrt{a^2 - x^2}\,dx$

87. $\int \sqrt{3x^2 + 4x + 1}\,dx$

88. $\int \sqrt{1 + 2x - 3x^2}\,dx$

89. $\int x\sqrt{1 + x - x^2}\,dx$

90. $\int (2x+3)\sqrt{4x^2 + 5x + 6}\,dx$

91. $\int (1 + x^2)\cos 2x\,dx$

92. $\int \log_{10} x\,dx$

93. $\int \dfrac{\log(\log x)}{x}\,dx$

94. $\int x\sec^2 2x\,dx$

95. $\int x\sin^3 x\,dx$

96. $\int (x+1)^2 e^x\,dx$

97. $\int \log\left(x + \sqrt{x^2 + a^2}\right)dx$

98. $\int \dfrac{\log x}{x^3}\,dx$

99. $\int \dfrac{\log(1-x)}{x^2}\,dx$

100. $\int x^3(\log x)^2\,dx$

101. $\int \dfrac{1}{x\sqrt{1+x^n}}\,dx$

102. $\int \dfrac{x^2}{\sqrt{1-x}}\,dx$

103. $\int \dfrac{x^5}{\sqrt{1+x^3}}\,dx$

104. $\int \dfrac{1+x^2}{\sqrt{1-x^2}}\,dx$

05. $\int x\sqrt{\dfrac{1-x}{1+x}}\,dx$

106. $\int \dfrac{1}{x\sqrt{1+x^3}}\,dx$

107. $\int \dfrac{\sin x + \cos x}{\sin^4 x + \cos^4 x}\,dx$

108. $\int x^2 \tan^{-1} x\,dx$

109. $\int \tan^{-1}\sqrt{x}\,dx$

110. $\int \sin^{-1}\sqrt{x}\,dx$

111. $\int \sec^{-1}\sqrt{x}\,dx$

112. $\int \tan^{-1} \sqrt{\dfrac{1-x}{1+x}}\, dx$ 113. $\int \sin^{-1} \sqrt{\dfrac{x}{a+x}}\, dx$ 114. $\int \sin^{-1}(3x-4x^3)\, dx$

115. $\int (\sin^{-1} x)^3\, dx$ 116. $\int \cos^{-1}(1-2x^2)\, dx$ 117. $\int \dfrac{x \sin^{-1} x}{(1-x^2)^{3/2}}\, dx$

118. $\int e^{2x}\left(\dfrac{1+\sin 2x}{1+\cos 2x}\right) dx$ 119. $\int \dfrac{\sqrt{1-\sin x}}{1+\cos x}\, e^{-x/2}\, dx$ 120. $\int e^x \dfrac{(1-x)^2}{(1+x^2)^2}\, dx$

121. $\int \dfrac{e^{m \tan^{-1} x}}{(1+x^2)^{3/2}}\, dx$ 122. $\int \dfrac{x^2}{(x-1)^3(x+1)}\, dx$ 123. $\int \dfrac{x}{x^3-1}\, dx$

124. $\int \dfrac{1}{1+x+x^2+x^3}\, dx$ 125. $\int \dfrac{1}{(x^2+2)(x^2+5)}\, dx$ 126. $\int \dfrac{x^2-2}{x^5-x}\, dx$

127. $\int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$ 128. $\int \dfrac{x^2+x+1}{(x+1)^2(x+2)}\, dx$ 129. $\int \dfrac{\sin 4x - 2}{1-\cos 4x}\, e^{2x}\, dx$

130. $\int \dfrac{\{\cot x + \cot^3\}\, x}{1+\cot^3 x}\, dx$

ANSWERS

1. $\dfrac{2}{3}\left\{(x+1)^{3/2} - x^{3/2}\right\} + C$ 2. $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + C$ 3. $-\dfrac{1}{x+1} - \dfrac{1}{2(x+1)^2} + C$

4. $\dfrac{1}{3}(4x+7)^{3/2} - \dfrac{1}{2}\sqrt{4x+7} + C$ 5. $-\dfrac{1}{x} + \log|x+1| + C$

6. $\left(\dfrac{2}{3}\right)^x \dfrac{1}{\log\left(\dfrac{2}{3}\right)} + \left(\dfrac{3}{2}\right)^x \dfrac{1}{\log\left(\dfrac{3}{2}\right)} + 2x + C$ 7. $x - \tan x + \sec x + C$

8. $\dfrac{x^3}{3} - \tan^{-1} x + C$ 9. $\sin 2x + \tan x - 2x + C$ 10. $-\cot x - \sin 2x - 2x + C$

11. $\dfrac{3}{8}x + \dfrac{\sin 8x}{64} - \dfrac{\sin 4x}{8} + C$ 12. $\dfrac{\sin 3x}{3} - \dfrac{\sin^3 3x}{9} + C$ 13. $\dfrac{1}{b^2}\log(a^2 + b^2 \sin^2 x) + C$

14. $\log|\sin^{-1} x| + C$ 15. $\dfrac{1}{4}(\sin^{-1} x)^4 + C$ 16. $x - \log(e^x + 1) + C$

17. $2\log(e^x + 1) - x + C$ 18. $\tan^{-1}(e^x) + C$

19. $\log|\sin x| - \dfrac{\sin^6 x}{6} - \dfrac{3\sin^2 x}{2} + \dfrac{3\sin^4 x}{4} + C$

20. $\dfrac{\cos 4x}{16} - \dfrac{\cos 2x}{8} + \dfrac{\cos 6x}{24} + C$ 21. $\dfrac{x}{4} + \dfrac{\sin 6x}{24} + \dfrac{\sin 4x}{16} + \dfrac{\sin 2x}{8} + C$

22. $\sin^{-1}(\sin x - \cos x) + C$ 23. $-\log|(\sin x + \cos x) + \sqrt{\sin 2x}| + C$

24. $\dfrac{1}{\sin(a-b)} \log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right| + C$ 25. $\dfrac{1}{\sin(a-b)} \log\left|\dfrac{\cos(x-a)}{\cos(x-b)}\right| + C$

26. $2\left(\sin\dfrac{x}{2} - \cos\dfrac{x}{2}\right) - \sqrt{2}\log\left|\tan\left(\dfrac{x}{4} + \dfrac{\pi}{8}\right)\right| + C$ 27. $\dfrac{1}{2\sqrt{2}} \log\left|\dfrac{1+\sqrt{2}\cos x}{1-\sqrt{2}\cos x}\right| + C$

28. $\frac{1}{2}\tan^2 x - \log|\sec x| + C$

29. $\frac{1}{3}\tan^3 x - \tan x + x + C$

30. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C$

31. $-\frac{1}{3}\cot^3 x + \cot x + x + C$

32. $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log|\sin x| + C$

33. $\log|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C$

34. $\frac{1}{10}(2x+3)^{5/2} - \frac{1}{2}(2x+3)^{3/2} + C$

35. $\frac{1}{2}\left\{\log(1+x^2) + \frac{1}{1+x^2}\right\} + C$

36. $\frac{1}{12}\sin^6 x^2 + C$

37. $-\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

38. $-\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C$

39. $\sin x + \frac{1}{5}\sin^5 x - \frac{2}{3}\sin^3 x + C$

40. $\frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x + C$

41. $\tan^{-1}(\tan^2 x) + C$

42. $\log|x + \sqrt{x^2 - a^2}| + C$

43. $\log|x + \sqrt{x^2 + a^2}| + C$

44. $\frac{1}{4}\tan^{-1}\left(x + \frac{1}{2}\right) + C$

45. $\frac{1}{6}\log\left|\frac{x-1}{x+5}\right| + C$

46. $\frac{1}{2}\log\left|\left(x + \frac{1}{8}\right) + \frac{1}{2}\sqrt{1 - x - 4x^2}\right| + C$

47. $\frac{1}{17}\log\left|\frac{3x-2}{3x+15}\right| + C$

48. $-\log\left|(1-\cos x) + \sqrt{\cos^2 x - 2\cos x - 3}\right| + C$

49. $\log\left|\left(\sin x + \frac{1}{2}\right) + \sqrt{\sin^2 x + \sin x}\right| + C$

50. $\sin^{-1}\left(\frac{x+1}{2}\right) + C$

51. $\frac{1}{2}\log|x^2 + 4x + 5| + \tan^{-1}(x+2) + C$

52. $6\sqrt{x^2 - 9x + 20} + 34\log\left|\left(x - \frac{9}{2}\right) + \sqrt{x^2 - 9x + 20}\right| + C$

53. $\sqrt{x^2 + x} + \frac{1}{2}\log\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right| + C$

54. $\sqrt{x - x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$

55. $-\frac{2}{a\sqrt{a}}\left\{(a-1)\log|1 - \sqrt{ax}| + (2-a)(1-\sqrt{ax}) - \frac{1}{2}(1-\sqrt{ax})^2\right\} + C$

56. $\frac{1}{5}\log\left|\frac{\tan x - 2}{2\tan x + 1}\right| + C$

57. $\frac{1}{4}\tan^{-1}\left(\tan x + \frac{1}{2}\right) + C$

58. $\frac{a}{a^2+b^2}x + \frac{b}{a^2+b^2}\log|a\cos x + b\sin x| + C$

59. $\frac{1}{2}\log\left|\frac{\tan x}{\tan x + 2}\right| + C$

60. $\frac{4}{5}x + \frac{3}{5}\log|2\sin x + \cos x| + C$

61. $\frac{1}{4}\log\left|x^8 + \sqrt{x^8 + 4}\right| + C$

62. $\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x - 1}{\sqrt{5}\tan x + 1}\right| + C$

63. $\frac{1}{\sqrt{3}}\log\left|\frac{2\sin x - \sqrt{3}}{2\sin x + \sqrt{3}}\right| + C$

64. $\frac{1}{\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan\frac{x}{2}}{\sqrt{3}-\tan\frac{x}{2}}\right|+C$

65. $\frac{1}{\sqrt{3}}\log\left|\frac{\tan\frac{x}{2}-2-\sqrt{3}}{\tan\frac{x}{2}-2+\sqrt{3}}\right|+C$

66. $\frac{1}{10}\log|\cos x-1|+\frac{1}{2}\log|\cos x+1|-\frac{3}{5}\log|3\cos x+2|+C$

67. $\frac{1}{6}\log|\cos x-1|+\frac{1}{2}\log|\cos x+1|-\frac{2}{3}\log|2\cos x+1|+C$

68. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan 2x\right)+C$

69. $\frac{2}{3}\tan^{-1}\left\{\frac{5\tan\frac{x}{2}-4}{3}\right\}+C$

70. $\tan x+\frac{1}{3}\tan^3 x+C$

71. $-\frac{1}{2}\cot 2x-\frac{1}{6}\cot^3 2x+C$

72. $\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|+C$

73. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right)+C$

74. $\sqrt{x^2+ax}+\frac{a}{2}\log\left|x+\frac{a}{2}+\sqrt{x^2+ax}\right|+C$

75. $-3\sqrt{6+x-2x^2}+\frac{13}{2\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{7}\right)+C$

76. $-\cos x-\frac{2}{\cos x}+\frac{1}{3\cos^3 x}+C$

77. $\frac{1}{4}\sin^4 x-\sin^2 x+\log|\sin x|+C$

78. $-\frac{1}{5}\sin^5 x-\frac{1}{3}\sin^3 x-\sin x+\frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right|+C$

79. $\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C$

80. $\tan x+\frac{2}{3}\tan^3 x+\frac{1}{5}\tan^5 x+C$

81. $\frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C$

82. $\frac{1}{4}\tan^4 x+\frac{1}{6}\tan^6 x+C$

83. $-\frac{1}{2}\cos x+\frac{1}{2}\sin x-\frac{1}{2\sqrt{2}}\log\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|+C$

84. $\frac{1}{2}x\sqrt{a^2+x^2}+\frac{1}{2}a^2\log|x+\sqrt{x^2+a^2}|+C$

85. $\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\log|x+\sqrt{x^2-a^2}|+C$

86. $\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)+C$

87. $\frac{1}{6}(3x+2)\sqrt{3x^2+4x+1}-\frac{\sqrt{3}}{18}\log\left|\left(x+\frac{2}{3}\right)+\sqrt{x^2+\frac{4x}{3}+\frac{1}{3}}\right|+C$

88. $\left(\frac{3x-1}{6}\right)\sqrt{1+2x-3x^2}+\frac{2\sqrt{3}}{9}\sin^{-1}\left(\frac{3x-1}{2}\right)+C$

89. $\frac{1}{24}(8x^2-2x-11)\sqrt{1+x-x^2}+\frac{5}{16}\sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+C$

90. $\frac{1}{192}(128x^2+328x+297)\sqrt{4x^2+5x+6}+\frac{497}{256}\log\left|\left(x+\frac{5}{8}\right)+\sqrt{x^2+\frac{5}{4}x+\frac{3}{2}}\right|+C$

91. $\frac{1}{2}(1+x^2)\sin 2x + \frac{x}{2}\cos 2x - \frac{\sin 2x}{4} + C$

92. $x(\log x - 1)\cdot \log_{10} e + C$

93. $\log\{\log(\log x)\} - \log x + C$

94. $\frac{1}{2}x\tan 2x - \frac{1}{4}\log|\sec 2x| + C$

95. $\frac{1}{4}\left\{-3x\cos x + 3\sin x + \frac{x}{3}\cos 3x - \frac{\sin 3x}{9}\right\} + C$

96. $e^x(x^2+1) + C$

97. $x\log|x+\sqrt{x^2+a^2}| - \sqrt{x^2+a^2} + C$

98. $-\frac{1}{4x^2}(2\log x + 1) + C$

99. $\left(1-\frac{1}{x}\right)\log(1-x) - \log x + C$

100. $\frac{x^4}{4}(\log x)^2 - \frac{1}{8}x^4\log x + \frac{1}{32}x^4 + C$

101. $\frac{1}{n}\log\left|\frac{\sqrt{1+x^n}-1}{\sqrt{1+x^n}+1}\right| + C$

102. $-\frac{2}{15}\sqrt{1-x}\left(3x^2+4x+8\right) + C$

103. $\frac{2}{9}\sqrt{1+x^3}\,(x^3-2) + C$

104. $\frac{3}{2}\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} + C$

105. $\left(\frac{x}{2}-1\right)\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x + C$

106. $\frac{1}{3}\log\left|\frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}\right| + C$

107. $\frac{1}{\sqrt{2}}\left[\frac{1}{2\sqrt{\sqrt{2}+1}}\log\left|\frac{\sqrt{\sqrt{2}+1}+t}{\sqrt{\sqrt{2}+1}-t}\right| + \frac{1}{\sqrt{\sqrt{2}-1}}\tan^{-1}\left(\frac{t}{\sqrt{\sqrt{2}-1}}\right)\right]$, where $t = \sin x - \cos x$

108. $\frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\log|x^2+1| + C$

109. $(x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + C$

110. $-\frac{1}{2}(1-2x)^2\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x-x^2} + C$

111. $x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C$

112. $\frac{1}{2}\left\{x\cos^{-1}x - \sqrt{1-x^2}\right\} + C$

113. $(x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$

114. $3\left\{x\sin^{-1}x + \sqrt{1-x^2}\right\} + C$

115. $x\sin^{-1}x\left\{(\sin^{-1}x)^2 - 6\right\} + 3\left\{(\sin^{-1}x)^2 - 2\right\}\sqrt{1-x^2} + C$

116. $2\left(x\sin^{-1}x + \sqrt{1-x^2}\right) + C$

117. $\frac{\sin^{-1}x}{\sqrt{1-x^2}} - \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$

118. $\frac{1}{2}e^{2x}\tan x + C$

119. $-e^{-x/2}\sec\left(\frac{x}{2}\right) + C$

120. $\frac{e^x}{1+x^2} + C$

121. $\frac{e^{m\tan^{-1}x}}{\sqrt{m^2+1}}\cos\left\{\tan^{-1}x - \cot^{-1}m\right\} + C$

122. $\frac{1}{8}\log\left|\frac{x-1}{x+1}\right| - \frac{3}{4(x-1)} - \frac{1}{4(x-1)^2} + C$

123. $\frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

124. $\frac{1}{2}\log\left(\frac{|x+1|}{\sqrt{x^2+1}}\right)+\frac{1}{2}\tan^{-1}x+C$ **125.** $\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)-\frac{1}{3\sqrt{5}}\tan^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

126. $2\log|x|-\frac{1}{4}\log|x^2-1|-\frac{3}{4}\log|x^2+1|+C$

127. $-2\sqrt{1-x}+\cos^{-1}\sqrt{x}+\sqrt{x}\sqrt{1-x}+C$ **128.** $-2\log|1+x|-\frac{1}{x+1}+3\log|x+2|+C$

129. $\frac{1}{2}e^{2x}\cot 2x+C$

130. $-\frac{1}{6}\log|\cot^2 x-\cot x+1|+\frac{1}{3}\log|\cot x+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\cot x-1}{\sqrt{3}}\right)+C$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $0<x<\frac{\pi}{4}$, then $\int\sqrt{1-\sin 2x}\,dx$ is equal to ....
2. If $\frac{\pi}{4}<x<\frac{\pi}{2}$, then $\int\sqrt{1-\sin 2x}\,dx$ is equal to ....
3. If $\int\frac{1}{\tan x+\cot x}\,dx=k\cos 2x+C$, then $k=$ ....
4. The value of the integral $\int e^{2\log x}+e^{x\log 2}\,dx$ is ....
5. If the value of integral $\int\frac{\sin^6 x}{\cos^8 x}dx$ is $k\tan^7 x+c$, then $k=$ ....
6. $\int\frac{x+3}{(x+4)^2}e^x dx=$ ....
7. $\int\cos x\,e^{2\sin x}\,dx=$ ....
8. $\int e^x\frac{x}{(x+1)^2}\,dx=$ ....
9. $\int\frac{\sin x}{3+4\cos^2 x}\,dx=$ ....
10. $\int\frac{x^2+1}{x^2-1}\,dx=$ ....
11. $\int\frac{e^{5\log_e x}-e^{4\log_e x}}{e^{3\log_e x}-e^{2\log_e x}}dx=$ ....
12. $\int\frac{x^4+x^2+1}{x^2-x+1}dx=$ ....
13. $\int\frac{\cos x-\sin x}{1+\sin 2x}dx=$ ....
14. $\int e^x\sin e^x\,dx=$ ....
15. $\int\frac{x}{e^{3x^2}}dx=$ ....
16. $\int x^x(1+\log x)\,dx=$ ....
17. $\int\frac{e^x}{e^x+1}dx=$ ....
18. $\int(x+3)(x^2+6x+10)^9\,dx=$ ....
19. $\int\frac{(\tan^{-1}x)^3}{1+x^2}dx=$ ....
20. $\int e^x(1-\cot x+\operatorname{cosec}^2 x)\,dx=$ ....

ANSWERS

1. $\sin x+\cos x+C$ **2.** $-\cos x-\sin x+C$ **3.** $-\frac{1}{4}$

4. $\frac{x^3}{3}+\frac{2^x}{\log 2}+C$ **5.** $\frac{1}{7}$ **6.** $\frac{e^x}{x+4}+C$

7. $\frac{1}{2}e^{2\sin x}+C$

8. $\frac{e^x}{x+1}+C$

9. $-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right)+C$

10. $x+\log\left|\frac{x-1}{x+1}\right|+C$

11. $\frac{x^3}{3}+C$

12. $\frac{1}{3}x^3+\frac{1}{2}x^2+x+C$

13. $-\frac{1}{\sin x+\cos x}+C$

14. $-\cos(e^x)+C$

15. $-\frac{1}{6}e^{-3x^2}+C$

16. x^x+C

17. $\log|e^x+1|+C$

18. $\frac{1}{20}(x^2+6x+10)^{10}+C$

19. $\frac{1}{4}(\tan^{-1}x)^4+C$

20. $e^x(1-\cot x)+C$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or, one sentence or, as per exact requirement of the question:

1. Write a value of $\int \frac{1+\cot x}{x+\log \sin x}\,dx$.
2. Write a value of $\int e^{3\log x}\,x^4\,dx$.
3. Write a value of $\int x^2 \sin x^3\,dx$.
4. Write a value of $\int \tan^3 x \sec^2 x\,dx$.
5. Write a value of $\int e^x(\sin x+\cos x)\,dx$.
6. Write a value of $\int \tan^6 x \sec^2 x\,dx$.
7. Write a value of $\int \frac{\cos x}{3+2\sin x}\,dx$.
8. Write a value of $\int e^x \sec x(1+\tan x)\,dx$.
9. Write a value of $\int \frac{\log x^n}{x}\,dx$.
10. Write a value of $\int \frac{(\log x)^n}{x}\,dx$.
11. Write a value of $\int e^{\log \sin x}\cos x\,dx$.
12. Write a value of $\int \sin^3 x\cos x\,dx$.
13. Write a value of $\int \cos^4 x\sin x\,dx$.
14. Write a value of $\int \tan x\sec^3 x\,dx$.
15. Write a value of $\int \frac{1}{1+e^x}\,dx$.
16. Write a value of $\int \frac{1}{1+2e^x}\,dx$.
17. Write a value of $\int \frac{(\tan^{-1}x)^3}{1+x^2}\,dx$.
18. Write a value of $\int \frac{\sec^2 x}{(5+\tan x)^4}\,dx$.
19. Write a value of $\int \frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}\,dx$.
20. Write a value of $\int \log_e x\,dx$.
21. Write a value of $\int a^x e^x\,dx$.
22. Write a value of $\int e^{2x^2+\ln x}\,dx$.
23. Write a value of $\int (e^{x\log_e a}+e^{a\log_e x})\,dx$.
24. Write a value of $\int \frac{\cos x}{\sin x\log \sin x}\,dx$.
25. Write a value of $\int \frac{\sin 2x}{a^2\sin^2 x+b^2\cos^2 x}\,dx$.
26. Write a value of $\int \frac{a^x}{3+a^x}\,dx$.
27. Write a value of $\int \frac{1+\log x}{3+x\log x}\,dx$.
28. Write a value of $\int \frac{\sin x}{\cos^3 x}\,dx$.
29. Write a value of $\int \frac{\sin x-\cos x}{\sqrt{1+\sin 2x}}\,dx$.
30. Write a value of $\int \frac{1}{x(\log x)^n}\,dx$.
31. Write a value of $\int e^{ax}\sin bx\,dx$.
32. Write a value of $\int e^{ax}\cos bx\,dx$.
33. Write a value of $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$.
34. Write a value of $\int e^{ax}\{a f(x)+f'(x)\}\,dx$.

35. Write a value of $\int \sqrt{4-x^2}\ dx$.

36. Write a value of $\int \sqrt{9+x^2}\ dx$.

37. Write a value of $\int \sqrt{x^2-9}\ dx$.

38. Evaluate: $\int \frac{x^2}{1+x^3}\ dx$ [CBSE 2008]

39. Evaluate: $\int \frac{x^2+4x}{x^3+6x^2+5}\ dx$ [CBSE 2008]

40. Evaluate: $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\ dx$ [CBSE 2009]

41. Evaluate: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\ dx$ [CBSE 2009]

42. Evaluate: $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx$ [CBSE 2009]

43. Evaluate: $\int \frac{(1+\log x)^2}{x}\ dx$ [CBSE 2009]

44. Evaluate: $\int \sec^2(7-4x)\ dx$ [CBSE 2009]

45. Evaluate: $\int \frac{\log x}{x}\ dx$ [CBSE 2010]

46. Evaluate: $\int 2^x\ dx$ [CBSE 2010]

47. Evaluate: $\int \frac{1-\sin x}{\cos^2 x}\ dx$ [CBSE 2010]

48. Evaluate: $\int \frac{x^3-1}{x^2}\ dx$ [CBSE 2010]

49. Evaluate: $\int \frac{x^3-x^2+x-1}{x-1}\ dx$ [CBSE 2011]

50. Evaluate: $\int \frac{e^{\tan^{-1}x}}{1+x^2}\ dx$ [CBSE 2011]

51. Evaluate: $\int \frac{1}{\sqrt{1-x^2}}\ dx$ [CBSE 2011]

52. Evaluate: $\int \sec x\,(\sec x+\tan x)\ dx$ [CBSE 2011]

53. Evaluate: $\int \frac{1}{x^2+16}\ dx$ [CBSE 2011]

54. Evaluate: $\int (1-x)\sqrt{x}\ dx$ [CBSE 2012]

55. Evaluate: $\int \frac{x+\cos 6x}{3x^2+\sin 6x}\ dx$ [CBSE 2012]

56. If $\int \left(\frac{x-1}{x^2}\right) e^x\ dx = f(x)\, e^x + C$, then write the value of $f(x)$. [CBSE 2012]

57. If $\int e^x(\tan x+1)\sec x\ dx = e^x f(x) + C$, then write the value $f(x)$. [CBSE 2012]

58. Evaluate: $\int \frac{2}{1-\cos 2x}\ dx$ [CBSE 2012]

59. Write the anti derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$. [CBSE 2014]

60. Evaluate : $\int \cos^{-1}(\sin x)\ dx$ [CBSE 2014]

61. Evaluate $\int \frac{1}{\sin^2 x\cos^2 x}\ dx$ [CBSE 2014]

62. Evaluate : $\int \frac{1}{x(1+\log x)}\ dx$ [CBSE 2017]

63. $\int \frac{1}{x(x^2+4)}\ dx$ [CBSE 2022]

ANSWERS

1. $\log|x+\log\sin x| + C$
2. $\frac{x^8}{8}+C$
3. $-\frac{1}{3}\cos x^3 + C$
4. $\frac{\tan^4 x}{4}+C$
5. $e^x \sin x + C$
6. $\frac{\tan^7 x}{7}+C$
7. $\frac{1}{2}\log|3+2\sin x| + C$
8. $e^x \sec x + C$
9. $\frac{n}{2}(\log x)^2 + C$

10. $\frac{(\log x)^{n+1}}{n+1}+C$
11. $\frac{\sin^2 x}{2}+C$
12. $\frac{\sin^4 x}{4}+C$
13. $-\frac{\cos^5 x}{5}+C$
14. $\frac{\sec^3 x}{3}+C$
15. $-\log(1+e^{-x})+C$
16. $-\log|2+e^{-x}|+C$
17. $\frac{(\tan^{-1} x)^4}{4}+C$
18. $\frac{1}{-3(5+\tan x)^3}+C$
19. $x+C$
20. $x(\log_e x-1)+C$
21. $\frac{(ae)^x}{\log(ae)}+C$
22. $\frac{1}{4}e^{2x^2}+C$
23. $\frac{a^x}{\log_e a}+\frac{x^{a+1}}{a+1}+C$
24. $\log(\log\sin x)+C$
25. $\frac{1}{a^2-b^2}\log(a^2\sin^2 x+b^2\cos^2 x)+C$
26. $\frac{1}{\log a}\log(3+a^x)+C$
27. $\log(3+x\log x)+C$
28. $\frac{1}{2}\sec^2 x+C$
29. $-\log|\sin x+\cos x|+C$
30. $\frac{(\log x)^{1-n}}{1-n}+C$
31. $\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$
32. $\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$
33. $\frac{e^x}{x}+C$
34. $e^{ax}f(x)+C$
35. $\frac{1}{2}x\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}+C$
36. $\frac{1}{2}x\sqrt{9+x^2}+\frac{9}{2}\log\left|x+\sqrt{9+x^2}\right|+C$
37. $\frac{1}{2}x\sqrt{x^2-9}-\frac{9}{2}\log\left|x+\sqrt{x^2-9}\right|+C$
38. $\frac{1}{3}\log|1+x^3|+C$
39. $\frac{1}{3}\log|x^3+6x^2+5|+C$
40. $2\tan\sqrt{x}+C$
41. $-2\cos\sqrt{x}+C$
42. $2\sin\sqrt{x}+C$
43. $\frac{(1+\log x)^3}{3}+C$
44. $-\frac{1}{4}\tan(7-4x)+C$
45. $\frac{1}{2}(\log x)^2+C$
46. $\frac{2^x}{\log_e 2}+C$
47. $\tan x-\sec x+C$
48. $\frac{x^2}{2}+\frac{1}{x}+C$
49. $\frac{x^3}{3}+x+C$
50. $e^{\tan^{-1}x}+C$
51. $\sin^{-1}x+C$
52. $\sec x+\tan x+C$
53. $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)+C$
54. $\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}+C$
55. $\frac{1}{6}\log|3x^2+\sin 6x|+C$
56. $\frac{1}{x}$
57. $\sec x$
58. $-\cot x+C$
59. $2(x^{3/2}+x^{1/2})+C$
60. $\frac{\pi}{2}x-\frac{x^2}{2}+C$
61. $-2\cot 2x+C$
62. $\log(1+\log x)+C$
63. $\frac{1}{8}\log\left(\frac{x^2}{x^2+4}\right)+C$