CHAPTER 26

DIRECTIONS COSINES AND DIRECTION RATIOS

26.1 INTRODUCTION

In class XI, we have had a brief introduction of three dimensional geometry in which we used Cartesian methods only. In previous chapters, we have studied some basic concepts of vectors. In this chapter and two more chapters to follow, we will use vector algebra to three dimensional geometry.

26.2 RECAPITULATION

26.2.1 COORDINATES OF A POINT IN SPACE

In this section, we will recapitulate various concepts learnt in class XI. We have leant in class XI that three mutually perpendicular lines in space define three mutually perpendicular planes which in turn divide the space into eight parts known as *octants* and the lines are known as the coordinate axes.

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O such that two of them viz. $Y'OY$ and $Z'OZ$ lie in the plane of the paper and the third $X'OX$ is perpendicular to the plane of the paper and is projecting out from the plane of the paper (see Fig. 26.1).

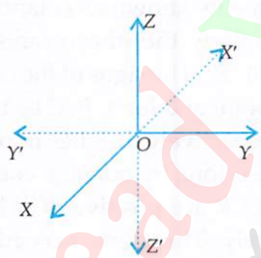

Fig. 26.1

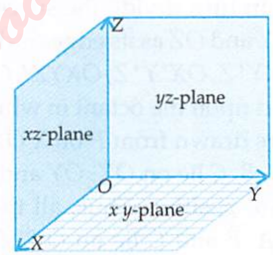

Fig. 26.2

Let O be the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ be x-axis, y-axis and z-axis respectively. These three lines are also called the rectangular axes of coordinates. The planes containing the lines $X'OX$, $Y'OY$ and $Z'OZ$ in pairs, determine three mutually perpendicular planes XOY, YOZ and ZOX or simply XY, YZ and ZX which are called rectangular coordinate planes as shown in Fig. 26.2.

Let P be a point in space (Fig. 26.3). Through P draw three planes parallel to the coordinate planes to meet the axes in A, B and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken in this order determined by the point P are called the coordinates of the point P, written as (x, y, z), x, y, z are positive or negative according as they are measured along positive or negative directions of the coordinate axes.

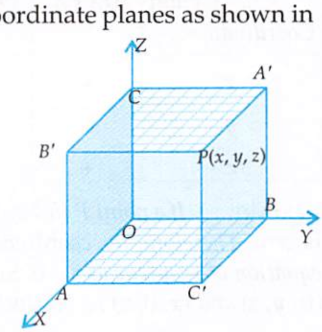

Fig. 26.3

Conversely, given an ordered triad (x, y, z) of real numbers we can always find the point whose coordinates are (x, y, z) in the following manner:

(i) Measure OA, OB, OC along x-axis, y-axis and z-axis respectively.

(ii) Through the points A, B, C draw planes parallel to the coordinate planes YOZ, ZOX and XOY respectively. The point of intersection of these planes is the required point P.

To give another explanation about the coordinates of a point P, we draw three planes through P parallel to the coordinate planes. These three planes determine a rectangular parallelopiped which has three pairs of rectangular faces viz. $PB'AC'$, $OCA'B$; $PA'BC'$, $OAB'C$; $PA'CB'$; $OAC'B$ as shown in Fig. 26.3.

Clearly, $x = OA = CB' = PA'$ = Perpendicular distance from P on the YOZ plane.

$y = OB = A'C = PB'$ = Perpendicular distance from P on the ZOX plane.

$z = OC = A'B = PC'$ = Perpendicular distance from P on the XOY plane.

Thus, the coordinates of the point P are the perpendicular distances from P on the three mutually rectangular coordinate planes YOZ, ZOX and XOY respectively.

Further, since the line PA lies in the plane $PB'AC'$ which is perpendicular to the line OA, we have PA perpendicular to OA. Similarly, PB perpendicular to OB and PC perpendicular to OC.

Thus, *the coordinates of a point are the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.*

26.2.2 SIGNS OF COORDINATES OF A POINT

To determine the signs of the coordinates of a point in three dimension, we follow the sign convention analogous to the sign convention in two dimensional geometry that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative. As discussed in previous section that three mutually perpendicular lines $X'OX$, $Y'OY$ and $Z'OZ$ determine three mutually perpendicular coordinate planes which in turn divide the space into eight compartments known as octants. The octant having OX, OY and OZ as its edges is denoted by $OXYZ$. Similarly, the other octants are denoted by $OX'YZ, OXY'Z, OX'Y'Z, OXYZ', OX'YZ', OXY'Z', OX'Y'Z'$. The signs of the coordinates of a point depend upon the octant in which it lies. Let P be a point and let A, B, C be the feet of the perpendiculars drawn from P on $X'OX$, $Y'OY$ and $Z'OZ$ respectively. If P lies in octant $OXYZ$, then clearly A, B, C lie on OX, OY and OZ respectively. Therefore, by our sign convention OA, OB and OC are positive. Thus, all the three coordinates of P are positive. If P lies in octant $OX'YZ$, then A, B and C lie on OX', OY and OZ respectively. Therefore, x-coordinate of P is negative and y and z coordinates are positive.

Similarly, we can find the signs of coordinates of points in other quadrants. The following table shows the signs of coordinates of points in various octants:

Octant / Coordinate	$OXYZ$	$OX'YZ$	$OXY'Z$	$OX'Y'Z$	$OXYZ'$	$OX'YZ'$	$OXY'Z'$	$OX'Y'Z'$
x	+	−	+	−	+	−	+	−
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

REMARK 1 *If a point P lies in xy-plane, then by the definition of coordinates of a point, z-coordinate of P is zero. Therefore, the coordinates of a point on xy-plane are of the form* $(x, y, 0)$ *and we may take the equation of xy-plane as* $z = 0$. *Similarly, the coordinates of any point in yz and zx-planes are of the forms* $(0, y, z)$ *and* $(x, 0, z)$ *respectively and their equations may be taken as* $x = 0$ *and* $y = 0$ *respectively.*

REMARK 2 *If a point lies on the x-axis, then its y and z-coordinates are both zero. Therefore, the coordinates of a point on x-axis are of the form* $(x, 0, 0)$ *and we may take the equations of x-axis as* $y = 0$,

$z = 0$. Similarly, the coordinates of a point on y and z-axes are of the form (0, y, 0) and (0, 0, z) *respectively and their equations may be taken as* $x = 0, z = 0$ *and* $x = 0, y = 0$ *respectively.*

26.2.3 DISTANCE FORMULA

THEOREM *Prove that the distance between the points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

PROOF Let O be the origin and let $P(x_1, y_1 z_1)$ and $Q(x_2, y_2, z_2)$ be two given points. Then,

$$\vec{OP} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k},\ \vec{OQ} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

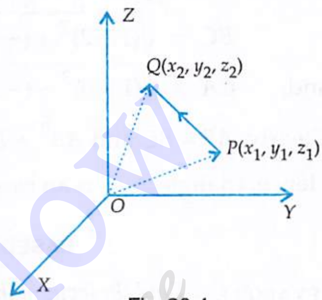

Fig. 26.4

Now,

$$\vec{PQ} = \text{Position vector of } Q - \text{Position vector of } P$$

$$\Rightarrow \quad \vec{PQ} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1)\hat{k}$$

$$\Rightarrow \quad \vec{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\therefore \quad PQ = |\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Hence, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the distance between the points* $P(-2, 4, 1)$ *and* $Q(1, 2, -5)$.

SOLUTION Using distance formula, we obtain

$$PQ = \sqrt{(1 - (-2))^2 + (2 - 4)^2 + (-5 - 1)^2} = \sqrt{9 + 4 + 36} = 7 \text{ units}$$

EXAMPLE 2 *Prove by using distance formula that the points* $P(1, 2, 3)$, $Q(-1, -1, -1)$ *and* $R(3, 5, 7)$ *are collinear.*

SOLUTION Using distance formula, we obtain

$$PQ = \sqrt{(-1 - 1)^2 + (-1 - 2)^2 + (-1 - 3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29},$$

$$QR = \sqrt{(3 + 1)^2 + (5 + 1)^2 + (7 + 1)^2} = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}$$

and, $$PR = \sqrt{(3 - 1)^2 + (5 - 2)^2 + (7 - 3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

Clearly, $QR = PQ + PR$. Therefore, points P, Q and R are collinear.

EXAMPLE 3 *Determine the point in XY-plane which is equidistant from three points* $A(2, 0, 3)$, $B(0, 3, 2)$ *and* $C(0, 0, 1)$.

SOLUTION We know that z-coordinate of every point on xy-plane is zero. So, let $P(x, y, 0)$ be a point on xy-plane such that $PA = PB = PC$.

Now, $PA = PB$

$\Rightarrow \quad PA^2 = PB^2$

$\Rightarrow \quad (x - 2)^2 + (y - 0)^2 + (0 - 3)^2 = (x - 0)^2 + (y - 3)^2 + (0 - 2)^2$

$\Rightarrow \quad 4x - 6y = 0 \Rightarrow 2x - 3y = 0$ ...(i)

and, $PB = PC$

$\Rightarrow \quad PB^2 = PC^2$

$\Rightarrow \quad (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$

$\Rightarrow \quad -6y + 12 = 0 \Rightarrow y = 2$ ...(ii)

Putting $y = 2$ in (i), we obtain $x = 3$. Hence, the required point is (3, 2, 0).

EXAMPLE 4 *Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are vertices of an isosceles right-angled triangle.*

SOLUTION Using distance formula, we obtain

$$AB = \sqrt{(2-0)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{4+4+1} = 3$$

$$BC = \sqrt{(1-2)^2 + (-3+1)^2 + (1-3)^2} = \sqrt{1+4+4} = 3$$

and, $$CA = \sqrt{(1-0)^2 + (-3-1)^2 + (1-2)^2} = \sqrt{1+16+1} = 3\sqrt{2}$$

Clearly, $AB = BC$ and $AB^2 + BC^2 = AC^2$.

Hence, triangle ABC is an isosceles right-angled triangle.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Find the locus of the point which is equidistant from the points $A(0, 2, 3)$ and $B(2, -2, 1)$.*

SOLUTION Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2}$$

$$\Rightarrow \quad 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$$

Hence, the required locus is $x - 2y - z + 1 = 0$.

EXAMPLE 6 *Find the coordinates of a point equidistant from the four points $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.*

SOLUTION Let $P(x, y, z)$ be the required point. Then, $OP = PA = PB = PC$.

Now, $OP = PA$

$$\Rightarrow \quad OP^2 = PA^2 \Rightarrow x^2 + y^2 + z^2 = (x-a)^2 + (y-0)^2 + (z-0)^2 \Rightarrow 0 = -2ax + a^2 \Rightarrow x = a/2$$

Similarly, $OP = PB \Rightarrow y = b/2$ and $OP = PC \Rightarrow z = c/2$.

Hence, the coordinates of the required point are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

EXAMPLE 7 *Using vector method: prove that the points $A(3, -2, 4)$, $B(1, 1, 1)$ and $C(-1, 4, -2)$ are collinear.*

SOLUTION We find that

$$\vec{AB} = (\hat{i} + \hat{j} + \hat{k}) - (3\hat{i} - 2\hat{j} + 4\hat{k}) = -2\hat{i} + 3\hat{j} - 3\hat{k}$$

and, $$\vec{BC} = (-\hat{i} + 4\hat{j} - 2\hat{k}) - (\hat{i} + \hat{j} + \hat{k}) = -2\hat{i} + 3\hat{j} - 3\hat{k}$$

Clearly, $\vec{AB} = \vec{BC}$. This shows that $\vec{AB}$ is parallel to $\vec{BC}$. But, B is common to $\vec{AB}$ and $\vec{BC}$.

Hence, A, B, C are collinear.

EXAMPLE 8 *Find the distance between the points A and B with position vectors $\hat{i} - \hat{j}$ and $2\hat{i} + \hat{j} + 2\hat{k}$.*

SOLUTION We find that

$$\vec{AB} = \text{Position vector of } B - \text{Position vector of } A = (2\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} - \hat{j} + 0\hat{k}) = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore \quad AB = |\vec{AB}| = \sqrt{1+4+4} = 3$$

26.2.4 SECTION FORMULAE

THEOREM 1 *(Internal Division) Let* $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ *be two points. Let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio* $m_1 : m_2$. *Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$$

PROOF Let the coordinates of R be (x, y, z). Let $\vec{r_1}$, $\vec{r_2}$ and $\vec{r}$ be the position vectors of P, Q and R respectively. Then,

$$\vec{r_1} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k},\ \vec{r_2} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \text{ and, } \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

Since R divides PQ internally in the ratio $m_1 : m_2$. Therefore, position vector $\vec{r}$ of point R is given by

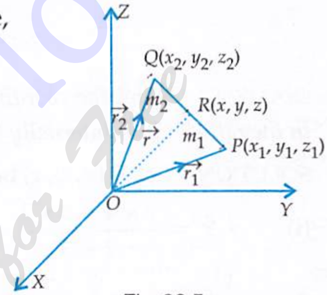

Fig. 26.5

$$\vec{r} = \frac{m_1 \vec{r_2} + m_2 \vec{r_1}}{m_1 + m_2}$$

$$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = \frac{m_1 (x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}) + m_2 (x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k})}{m_1 + m_2}$$

$$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = \left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}\right)\hat{i} + \left(\frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}\right)\hat{j} + \left(\frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)\hat{k}$$

$$\Rightarrow x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2},\ y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2},\ z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

Hence, the coordinates of R are $\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$ **Q.E.D**

COROLLARY *If R is the mid-point of the segment joining* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$, *then* $m_1 = m_2 = 1$ *and the coordinates of R are given by*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

THEOREM 2 *(External Division) Let* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *be two points, and let R be a point on PQ produced dividing it externally in the ratio* $m_1 : m_2$ $(m_1 \neq m_2)$. *Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)$$

PROOF Let the coordinates of R be (x, y, z). Let $\vec{r_1}$, $\vec{r_2}$ and $\vec{r}$ be the position vectors of P, Q and R respectively. Then,

$$\vec{r_1} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k},\ \vec{r_2} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k} \text{ and } \vec{r} = x \hat{i} + y \hat{j} + z \hat{k}.$$

Since R divides PQ externally in the ratio $m_1 : m_2$. Therefore, position vector $\vec{r}$ of point R is given by

$$\vec{r} = \frac{m_1 \vec{r_2} - m_2 \vec{r_1}}{m_1 - m_2}$$

$$\Rightarrow \quad x\hat{i} + y\hat{i} + z\hat{k} = \frac{m_1(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - m_2(x_1\hat{i} + y_1\hat{j} + z_1\hat{k})}{m_1 - m_2}$$

$$\Rightarrow \quad x\hat{i} + y\hat{j} + z\hat{k} = \left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}\right)\hat{i} + \left(\frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}\right)\hat{j} + \left(\frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)\hat{k}$$

$$\Rightarrow \quad x = \frac{m_1 x_2 - m_2 x_1}{m_1 - m_2},\ y = \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2},\ z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

Hence, the coordinates of R are $\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}\right)$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the coordinates of the point which divides the join of $P(2, -1, 4)$ and $Q(4, 3, 2)$ in the ratio $2:3$ (i) internally (ii) externally.*

SOLUTION Let $R(x, y, z)$ be the required point. Then,

(i) $$x = \frac{2 \times 4 + 3 \times 2}{2 + 3},\ y = \frac{2 \times 3 + 3 \times -1}{2 + 3},\ z = \frac{2 \times 2 + 3 \times 4}{2 + 3} \Rightarrow x = \frac{14}{5},\ y = \frac{3}{5},\ z = \frac{16}{5}$$

So, coordinates of the required point R are $\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$

(ii) $$x = \frac{2 \times 4 - 3 \times 2}{2 - 3},\ y = \frac{2 \times 3 - 3 \times -1}{2 - 3},\ z = \frac{2 \times 2 - 3 \times 4}{2 - 3} \Rightarrow x = -2, y = -9, z = 8$$

So, coordinates of the required point R are $(-2, -9, 8)$.

EXAMPLE 2 *Find the ratio in which the line joining the points $(1, 2, 3)$ and $(-3, 4, -5)$ is divided by the xy-plane. Also, find the coordinates of the point of division.*

SOLUTION Suppose the line segment joining the points $P(1, 2, 3)$ and $Q(-3, 4 -5)$ is divided by the xy-plane at a point R in the ratio $\lambda : 1$. Then, the coordinates of R are

i.e. $$\left(\frac{-3\lambda + 1}{\lambda + 1}, \frac{4\lambda + 2}{\lambda + 1}, \frac{-5\lambda + 3}{\lambda + 1}\right) \quad ...(i)$$

Since R lies on xy-plane i.e. $z = 0$. Therefore, z-coordinate of R must be zero.

i.e. $$\frac{-5\lambda + 3}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{3}{5}$$

So, the required ratio is $\frac{3}{5} : 1$ or, $3 : 5$. Putting $\lambda = \frac{3}{5}$ in (i), we obtain the coordinates of R as $(-1/2, 11/4, 0)$.

EXAMPLE 3 *Find the ratio in which join of $A(2, 1, 5)$ and $B(3, 4, 3)$ is divided by the plane $2x + 2y - 2z = 1$. Also, find the coordinates of the point of division.*

SOLUTION Suppose the plane $2x + 2y - 2z = 1$ divides the line joining the points $A(2, 1, 5)$ and $B(3, 4, 3)$ at a point C in the ratio $\lambda : 1$. Then, the coordinates of C are

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$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right) \quad ...(i)$$

Since point C lies on the plane $2x+2y-2z=1$. Therefore, coordinates of C must satisfy the equation of the plane

i.e. $$2\left(\frac{3\lambda+2}{\lambda+1}\right)+2\left(\frac{4\lambda+1}{\lambda+1}\right)-2\left(\frac{3\lambda+5}{\lambda+1}\right)=1 \Rightarrow 8\lambda-4=\lambda+1 \Rightarrow \lambda=\frac{5}{7}$$

So, the required ratio is $\frac{5}{7}:1$ or, $5:7$. Putting $\lambda=\frac{5}{7}$ in (i), the coordinates of the point of division C are $(29/12, 9/4, 25/6)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Using section formula, prove that the three points* $A(-2, 3, 5)$, $B(1, 2, 3)$ *and* $C(7, 0, -1)$ *are collinear.*

SOLUTION Suppose the given points are collinear and C divides AB in the ratio $\lambda:1$. Then, coordinates of C are $\left(\frac{\lambda-2}{\lambda+1}, \frac{2\lambda+3}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right)$. But, coordinates of C are $(7, 0, -1)$.

$$\therefore \quad \frac{\lambda-2}{\lambda+1}=7, \frac{2\lambda+3}{\lambda+1}=0 \text{ and } \frac{3\lambda+5}{\lambda+1}=-1$$

From each of these equations, we get: $\lambda=-\frac{3}{2}$. Since each of these equations give the same value of λ. Therefore, the given points are collinear and C divides AB externally (because λ is negative) in the ratio $3:2$.

EXAMPLE 5 *The mid-points of the sides of a triangle are* $(1, 5, -1)$, $(0, 4, -2)$ *and* $(2, 3, 4)$. *Find its vertices.*

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1, 5, -1)$, $E(0, 4, -2)$ and $F(2, 3, 4)$ be the mid-points of the sides BC, CA and AB respectively.

Now, D is the mid-point of BC.

$$\therefore \quad \frac{x_2+x_3}{2}=1, \frac{y_2+y_3}{2}=5, \frac{z_2+z_3}{2}=-1 \Rightarrow x_2+x_3=2, y_2+y_3=10, z_2+z_3=-2 \quad ...(i)$$

E is the mid-point of CA

$$\therefore \quad \frac{x_1+x_3}{2}=0, \frac{y_1+y_3}{2}=4, \frac{z_1+z_3}{2}=-2 \Rightarrow x_1+x_3=0, y_1+y_3=8, z_1+z_3=-4 \quad ...(ii)$$

F is the mid-point of AB

$$\therefore \quad \frac{x_1+x_2}{2}=2, \frac{y_1+y_2}{2}=3, \frac{z_1+z_2}{2}=4 \Rightarrow x_1+x_2=4, y_1+y_2=6, z_1+z_2=8 \quad ...(iii)$$

Adding first three equations in (i), (ii) and (iii), we obtain

$$2(x_1+x_2+x_3)=2+0+4 \Rightarrow x_1+x_2+x_3=3$$

Solving first three equations in (i), (ii) and (iii) with $x_1+x_2+x_3=3$, we obtain $x_1=1, x_2=3, x_3=-1$.

Adding next three equations in (i), (ii) and (iii), we obtain

$$2(y_1+y_2+y_3)=10+8+6 \Rightarrow y_1+y_2+y_3=12$$

Solving next three equations in (i), (ii) and (iii) with $y_1+y_2+y_3=12$, we obtain

$$y_1=2, y_2=4, y_3=6.$$

Adding last three equations in (i), (ii) and (iii), we obtain

$$2(z_1+z_2+z_3) = -2-4+8 \Rightarrow z_1+z_2+z_3 = 1.$$

Solving last three equations in (i), (ii) and (iii) with $z_1+z_2+z_3=1$, we obtain

$$z_1 = 3,\ z_2 = 5,\ z_3 = -7.$$

Thus, the vertices of the triangle are $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXAMPLE 6 *Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.*

SOLUTION Suppose Q divides PR in the ratio $\lambda:1$. Then, coordinates of Q are

$$\left(\frac{9\lambda+3}{\lambda+1}, \frac{8\lambda+2}{\lambda+1}, \frac{-10\lambda-4}{\lambda+1}\right)$$

But, coordinates of Q are given as $(5, 4, -6)$.

$$\therefore \quad \frac{9\lambda+3}{\lambda+1} = 5,\ \frac{8\lambda+2}{\lambda+1} = 4,\ \frac{-10\lambda-4}{\lambda+1} = 6.$$

All these equations give the same value of λ equal to $\frac{1}{2}$. So, Q divides PR in the ratio $\frac{1}{2}:1$ or, $1:2$.

EXAMPLE 7 *Find the coordinates of the points which trisect the line segment AB, given that $A(2, 1, -3)$ and $B(5, -8, 3)$.*

SOLUTION Let P and Q be the points which trisect AB. Then, $AP = PQ = QB$. Therefore, P divides AB in the ratio 1 : 2 and Q divides it in the ratio 2 : 1.

As P divides AB in the ratio 1 : 2. So coordinates of P are

$$\left(\frac{1\times 5+2\times 2}{1+2}, \frac{1\times -8+2\times 1}{1+2}, \frac{1\times 3+2\times -3}{1+2}\right) \text{ or, } (3, -2, -1)$$

Fig. 26.6

Since Q divides AB in the ratio 2 : 1. So coordinates of Q are

$$\left(\frac{2\times 5+1\times 2}{2+1}, \frac{2\times -8+1\times 1}{2+1}, \frac{2\times 3+1\times -3}{2+1}\right) \text{ or, } (4, -5, 1)$$

EXAMPLE 8 *The x-coordinates of a point on the line joining the points $Q(2, 2, 1)$ and $R(5, 1, -2)$ is 4. Find its z-coordinate.* **[NCERT EXEMPLAR]**

SOLUTION Let the point P divide QR in the ratio $\lambda:1$. Then, the coordinates of P are $\left(\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$.

Fig. 26.7

It is given that the x-coordinates of P is 4.

$$\therefore \quad \frac{5\lambda+2}{\lambda+1} = 4 \Rightarrow 5\lambda+2 = 4\lambda+4 \Rightarrow \lambda = 2.$$

Hence, the z-coordinate of P is $\frac{-2\lambda+1}{\lambda+1} = \frac{-4+1}{2+1} = -1$.

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BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Show that the centroid of the triangle with vertices* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ *and* $C(x_3, y_3, z_3)$ *has the coordinates*

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

SOLUTION Let D be the mid-point of AC. Then, coordinates of D are

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right).$$

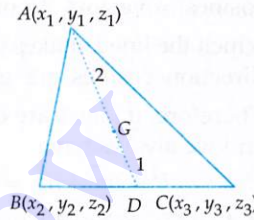

Fig. 26.8

Let G be the centroid of ΔABC. Then, G divides AD in the ratio $2:1$.
So, coordinates of G are

$$\left(\frac{1\times x_1 + 2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1\times y_1 + 2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1\times z_1 + 2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

i.e. $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$

EXAMPLE 10 *Find the coordinates of the foot of the perpendicular drawn from the point* $A(1, 2, 1)$ *to the line joining* $B(1, 4, 6)$ *and* $C(5, 4, 4)$.

SOLUTION Let D be the foot of the perpendicular drawn from A on BC, and let D divide BC in the ratio $\lambda:1$. Then, coordinates of D are

$$\left(\frac{5\lambda+1}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{4\lambda+6}{\lambda+1}\right) \qquad ...(i)$$

Now, $\vec{AD}$ = Position vector of D – Position vector of A

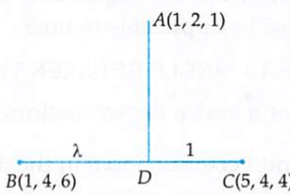

Fig. 26.9

$$\Rightarrow \quad \vec{AD} = \left(\frac{5\lambda+1}{\lambda+1} - 1\right)\hat{i} + \left(\frac{4\lambda+4}{\lambda+1} - 2\right)\hat{j} + \left(\frac{4\lambda+6}{\lambda+1} - 1\right)\hat{k}$$

$$\Rightarrow \quad \vec{AD} = \left(\frac{4\lambda}{\lambda+1}\right)\hat{i} + \left(\frac{2\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{3\lambda+5}{\lambda+1}\right)\hat{k}$$

and, $\vec{BC}$ = Position vector of C – Position vector of B

$$\Rightarrow \quad \vec{BC} = (5\hat{i} + 4\hat{j} + 4\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k}) = 4\hat{i} + 0\hat{j} - 2\hat{k}.$$

It is given that AD is perpendicular to BC.

$$\therefore \quad \vec{AD} \perp \vec{BC}$$

$$\Rightarrow \quad \vec{AD}\cdot\vec{BC} = 0$$

$$\Rightarrow \quad \left\{\left(\frac{4\lambda}{\lambda+1}\right)\hat{i} + \left(\frac{2\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{3\lambda+5}{\lambda+1}\right)\hat{k}\right\}\cdot(4\hat{i} + 0\hat{j} - 2\hat{k}) = 0$$

$$\Rightarrow \quad 4\left(\frac{4\lambda}{\lambda+1}\right) + 0\left(\frac{2\lambda+2}{\lambda+1}\right) - 2\left(\frac{3\lambda+5}{\lambda+1}\right) = 0$$

$$\Rightarrow \quad \frac{16\lambda}{\lambda+1} + 0 - 2\,\frac{(3\lambda+5)}{\lambda+1} = 0 \Rightarrow 16\lambda - 6\lambda - 10 = 0 \Rightarrow \lambda = 1$$

Putting $\lambda = 1$ in (i), we obtain that the coordinates of D are $(3, 4, 5)$.

26.3 DIRECTION COSINES AND DIRECTION RATIOS OF A LINE

In chapter 23, we have learnt about the direction cosines and direction ratios of a vector. In this section, we will introduce the notion of direction cosines and direction ratios of a line.

DEFINITION *The direction cosines of a line are defined as the direction cosines of any vector whose support is the given line.*

It follows from the above definition if A and B are two points on a given line L, then direction cosines of vectors $\vec{AB}$ or, $\vec{BA}$ are the direction cosines of line L. Thus, if α, β, γ are the angles which the line L makes with positive directions of x-axis, y-axis and z-axis respectively, then its direction cosines are either, $\cos\alpha, \cos\beta \cos\gamma$ or, $-\cos\alpha, -\cos\beta, -\cos\gamma$.

Therefore, if l, m, n are direction cosines of a line, then $-l, -m, -n$ are also its direction cosines and we always have

$$l^2 + m^2 + n^2 = 1$$

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line L, then its direction cosines are

$$\frac{x_2 - x_1}{AB}, \frac{y_2 - y_1}{AB}, \frac{z_2 - z_1}{AB} \text{ or } \frac{x_1 - x_2}{AB}, \frac{y_1 - y_2}{AB}, \frac{z_1 - z_2}{AB}.$$

DEFINITION *The direction ratios of a line are proportional to the direction ratios of any vector whose support is the given line*

If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points on a line, then its direction ratios are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

26.4 ANGLE BETWEEN TWO VECTORS

In this section, we will find the formula for the angle between two vectors in terms of their direction cosines and also in terms of their direction ratios. The angle between two lines is defined as the angle between two vectors parallel to them. So, the results derived for vectors will also be applicable to lines.

26.4.1 ANGLE BETWEEN TWO VECTORS IN TERMS OF THEIR DIRECTION COSINES

Let $\vec{a}$ and $\vec{b}$ be two vectors with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively. Then, the unit vectors $\hat{a}$ and $\hat{b}$ in the directions of $\vec{a}$ and $\vec{b}$ respectively are given by

$$\hat{a} = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k} \text{ and, } \hat{b} = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, θ is also the angle between $\hat{a}$ and $\hat{b}$.

$$\therefore \quad \cos\theta = \frac{\hat{a} \cdot \hat{b}}{|\hat{a}|\,|\hat{b}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) \cdot (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k})}{(1)(1)} \quad [\because |\hat{a}| = |\hat{b}| = 1]$$

$$\Rightarrow \quad \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Condition for perpendicularity: If $\vec{a}$ and $\vec{b}$ are perpendicular, then

$$\vec{a} \perp \vec{b} \Leftrightarrow \hat{a} \perp \hat{b} \Leftrightarrow \hat{a} \cdot \hat{b} = 0 \Leftrightarrow (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) \cdot (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}) = 0 \Leftrightarrow l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Condition for parallelism: If $\vec{a}$ and $\vec{b}$ are parallel, then

$\hat{a}$ and $\hat{b}$ are parallel

$\Leftrightarrow \quad \hat{a} = \lambda \hat{b}$ for some scalar λ

$\Leftrightarrow \quad (l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k}) = \lambda (l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}) \Leftrightarrow l_1 = \lambda l_2, m_1 = \lambda m_2, n_1 = \lambda n_2 \Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

26.4.2 ANGLE BETWEEN TWO VECTORS IN TERMS OF THEIR DIRECTION RATIOS

Let $\vec{a}$ and $\vec{b}$ be two vectors with direction ratios proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Then,

$$\vec{A} = \text{A vector along } \vec{a} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k} \text{ and, } \vec{B} = \text{A vector along } \vec{b} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}.$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then, θ is also the angle between $\vec{A}$ and $\vec{B}$.

$$\therefore \quad \cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|\,|\vec{B}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) \cdot (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k})}{|a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}|\,|a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}|} \Rightarrow \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition for perpendicularity: If $\vec{a}$ and $\vec{b}$ are perpendicular, then

$$\vec{A} \perp \vec{B} \Leftrightarrow \vec{A} \cdot \vec{B} = 0 \Leftrightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

Condition for parallelism: If $\vec{a}$ and $\vec{b}$ are parallel, then

$\vec{A}$ and $\vec{B}$ are parallel

$\Leftrightarrow \quad \vec{A} = \lambda \vec{B}$ for some scalar λ

$\Leftrightarrow \quad (a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) = \lambda (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}) \Leftrightarrow a_1 = \lambda a_2, b_1 = \lambda b_2, c_1 = \lambda c_2 \Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

In order to find the angle between two vectors when their direction ratios or cosines are given, we may use the following algorithm.

ALGORITHM

Step I *Obtain direction ratios or direction cosines of two vectors. Let the direction ratios of two vectors be proportional to a_1, b_1, c_1 and a_2, b_2, c_2 respectively.*

Step II *Write vectors parallel to the given vectors.*

Let $\vec{a}$ = A vector parallel to the vector having direction ratios a_1, b_1, c_1. Then,

$\vec{a} = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$

and, $\vec{b}$ = A vector parallel to the vector having direction ratios a_2, b_2, c_2. Then,

$\vec{b} = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$

Step III *Use the formula:* $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the angle between the vectors with direction ratios proportional to* $4, -3, 5$ *and* $3, 4, 5$.

SOLUTION Let $\vec{a}$ be a vector parallel to the vector having direction ratios $4, -3, 5$. Then, $\vec{a} = 4\hat{i} - 3\hat{j} + 5\hat{k}$. Further, let $\vec{b}$ be a vector parallel to the vector having direction ratios $3, 4, 5$. Then, $\vec{b} = 3\hat{i} + 4\hat{j} + 5\hat{k}$. Let θ be the angle between the given vectors. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{12 - 12 + 25}{\sqrt{16+9+25}\sqrt{9+16+25}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

Thus, the measure of the angle between the given vectors is $60°$.

EXAMPLE 2 *Find the angle between the lines whose direction ratios are proportional to* $4, -3, 5$ *and* $3, 4, 5$.

SOLUTION Let θ be the angle between the given lines. We have,

$$a_1 = 4, b_1 = -3, c_1 = 5 \text{ and } a_2 = 3, b_2 = 4, c_2 = 5$$

$$\therefore \quad \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{12 - 12 + 25}{\sqrt{16+9+25}\sqrt{9+16+25}} = \frac{25}{50} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

Thus, the measure of the angle between the given lines is $60°$.

EXAMPLE 3 $P(6, 3, 2), Q(5, 1, 4)$ *and* $R(3, 3, 5)$ *are the vertices of a triangle PQR. Find* $\angle PQR$.

SOLUTION We know that the direction ratios of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Therefore,

Direction ratios of QP are proportional to $6-5, 3-1, 2-4$ i.e. $1, 2, -2$

Direction ratios of QR are proportional to $3-5, 3-1, 5-4$ i.e. $-2, 2, 1$

Let $\angle PQR = \theta$. Then,

$$\cos\theta = \frac{1\times -2 + 2\times 2 + -2\times 1}{\sqrt{1^2 + 2^2 + (-2)^2}\sqrt{(-2)^2 + 2^2 + (-1)^2}} = \frac{-2+4-2}{\sqrt{9}\sqrt{9}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the coordinates of the foot of the perpendicular drawn from the point* $A(1, 2, 1)$ *to the line joining* $B(1, 4, 6)$ *and* $C(5, 4, 4)$.

SOLUTION Let D be the foot of the perpendicular drawn from A on BC, and let D divide BC in the ratio $\lambda : 1$. Then, the coordinates of D are

$$\left(\frac{5\lambda+1}{\lambda+1}, \frac{4\lambda+4}{\lambda+1}, \frac{4\lambda+6}{\lambda+1}\right) \quad \text{...(i)}$$

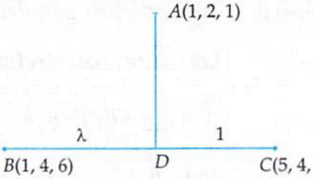

Fig. 26.10

The direction ratios of BC are proportional to
$5-1, 4-4, 4-6$ i.e $4, 0, -2$.

The direction ratios of AD are proportional to

$$\frac{5\lambda+1}{\lambda+1} - 1, \frac{4\lambda+4}{\lambda+1} - 2, \frac{4\lambda+6}{\lambda+1} - 1, \text{ i.e. } \frac{4\lambda}{\lambda+1}, \frac{2\lambda+2}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}$$

Since $AD \perp BC$. Therefore,

$$4\times\left(\frac{4\lambda}{\lambda+1}\right) + 0\times\left(\frac{2\lambda+2}{\lambda+1}\right) + (-2)\left(\frac{3\lambda+5}{\lambda+1}\right) = 0 \quad [\text{Using}: a_1 a_2 + b_1 b_2 + c_1 c_2 = 0]$$

$\Rightarrow \quad 16\lambda - 6\lambda - 10 = 0 \Rightarrow \lambda = 1$

Putting $\lambda = 1$ in (i) we obtain that the coordinates of D are (3, 4, 5).

EXAMPLE 5 *Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to* $1, -2, -2$ *and* $0, 2, 1$.

SOLUTION Let l, m, n be the direction cosines of the required line. Since it is perpendicular to the lines whose direction cosines are proportional to $1, -2, -2$ and $0, 2, 1$ respectively.

$\therefore \quad l - 2m - 2n = 0 \quad$ ...(i)

and, $\quad 0l + 2m + n = 0 \quad$ ...(ii)

On solving (i) and (ii) by cross-multiplication, we get

$$\frac{l}{-2+4} = \frac{m}{0-1} = \frac{n}{2} \Rightarrow \frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$$

Thus, the direction ratios of the required line are proportional to $2, -1, 2$. Hence, its direction cosines are

$$\frac{2}{\sqrt{2^2+(-1)^2+2^2}}, \frac{-1}{\sqrt{2^2+(-1)^2+2^2}}, \frac{2}{\sqrt{2^2+(-1)^2+2^2}} \text{ or, } \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}.$$

EXAMPLE 6 *Find the direction cosines of the sides of the triangle whose vertices are* $(3, 5, -4)$, $(-1, 1, 2)$ *and* $(-5, -5, -2)$ *and also find the angles of the triangle. What types of triangle it is?*

SOLUTION Let ABC be the triangle the coordinates of whose vertices are $A(3, 5, -4)$, $B(-1, 1, 2)$ and $C(-5, -5, -2)$. The direction ratios of AB are proportional to

$$-1-3, 1-5, 2+4 \text{ or, } -4, -4, 6 \text{ or, } -2, -2, 3$$

$\therefore$ Direction cosines of AB are

$$\frac{-2}{\sqrt{(-2)^2+(-2)^2+3^2}}, \frac{-2}{\sqrt{(-2)^2+(-2)^2+3^2}}, \frac{3}{\sqrt{(-2)^2+(-2)^2+3^2}}$$

or, $$\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}. \text{ Let } l_1 = \frac{-2}{\sqrt{17}}, m_1 = \frac{-2}{\sqrt{17}}, n_1 = \frac{3}{\sqrt{17}}$$

The direction ratios of AC are proportional to

$$-5-3, -5-5, -2-(-4) \text{ or, } -8, -10, 2 \text{ or, } -4, -5, 1$$

$\therefore$ Direction cosines of AC are

$$\frac{-4}{\sqrt{(-4)^2+(-5)^2+1^2}}, \frac{-5}{\sqrt{(-4)^2+(-5)^2+1^2}}, \frac{1}{\sqrt{(-4)^2+(-5)^2+1^2}}$$

or, $$\frac{-4}{\sqrt{42}}, -\frac{5}{\sqrt{42}}, \frac{1}{\sqrt{42}}. \text{ Let } l_2 = \frac{-4}{\sqrt{42}}, m_2 = \frac{-5}{\sqrt{42}}, n_2 = \frac{1}{\sqrt{42}}$$

The direction ratios of BC are proportional to

$$-5-(-1), -5-1, -2-2 \text{ or, } -4, -6, -4 \text{ or, } -2, -3, -2$$

$\therefore$ Direction cosines of BC are

$$\frac{-2}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}, \frac{-3}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}, \frac{-2}{\sqrt{(-2)^2+(-3)^2+(-2)^2}}$$

or, $$\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}. \text{ Let } l_3 = \frac{-2}{\sqrt{17}}, m_3 = \frac{-3}{\sqrt{17}}, n_3 = \frac{-2}{\sqrt{17}}$$

The angle between sides AB and AC is A and is given by

$$\cos A = l_1 l_2 + m_1 m_2 + n_1 n_2 = \frac{8}{\sqrt{17}\sqrt{42}} + \frac{10}{\sqrt{17}\sqrt{42}} + \frac{3}{\sqrt{17}\sqrt{42}} = \frac{21}{\sqrt{17}\sqrt{42}} = \sqrt{\frac{21}{34}}$$

$\Rightarrow \quad A = \cos^{-1}\left(\sqrt{\dfrac{21}{34}}\right)$

The angle between sides BA and BC is B and is given by

$\cos B = (-l_1)\, l_3 + (-m_1)\, m_3 + (-n_1)\, n_3 \quad [\because$ Direction cosines of BA are $-l_1, -m_1, -n_1]$

$$\Rightarrow \quad \cos B = \frac{2}{\sqrt{17}} \times \frac{-2}{\sqrt{17}} + \frac{2}{17} \times \frac{-3}{\sqrt{17}} + \frac{-3}{\sqrt{17}} \times \frac{-2}{\sqrt{17}} = \frac{-4-6+6}{17} = -\frac{4}{17}$$

$\Rightarrow \quad B = \cos^{-1}\left(-\dfrac{4}{17}\right)$

The angle between sides CB and CA is C and direction cosines of CB and CA are $-l_3, -m_3, -n_3$ and $-l_2, -m_2, -n_2$ respectively

$\therefore \quad \cos C = (-l_3)(-l_2) + (-m_3)(-m_2) + (-n_3)(-n_2) = l_2\, l_3 + m_2\, m_3 + n_2\, n_3$

$$\Rightarrow \quad \cos C = \frac{-4}{\sqrt{42}} \times \frac{-2}{\sqrt{17}} + \frac{-5}{\sqrt{42}} \times \frac{-3}{\sqrt{17}} + \frac{1}{\sqrt{42}} \times \frac{-2}{\sqrt{17}} = \frac{8+15-2}{\sqrt{42}\,\sqrt{17}} = \frac{21}{\sqrt{42}\,\sqrt{17}} = \sqrt{\frac{21}{34}}$$

$\Rightarrow \quad C = \cos^{-1}\sqrt{\dfrac{21}{34}}$

We observe that $\angle A = \angle C$ and $\cos B$ is negative. Therefore, ΔABC is isosceles obtuse angled triangle.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two lines, show that the direction cosines of the line perpendicular to both of them are proportional to*

$(m_1 n_2 - m_2 n_1), (n_1 l_2 - n_2 l_1), (l_1\, m_2 - l_2\, m_1)$.

SOLUTION Let l, m, n be the direction cosines of the line perpendicular to each of the given lines. Then,

$l l_1 + m m_1 + n n_1 = 0$ ...(i)

and, $\quad l l_2 + m m_2 + n n_2 = 0$ ...(ii)

On solving (i) and (ii) by cross-multiplication, we get

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$

Hence, the direction cosines of the line perpendicular to the given lines are proportional to

$(m_1 n_2 - m_2 n_1), (n_1 l_2 - n_2 l_1), (l_1 m_2 - l_2 m_1)$.

EXAMPLE 8 *If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of them are*

$(m_1 n_2 - m_2 n_1), (n_1 l_2 - n_2 l_1), (l_1 m_2 - l_2 m_1)$ **[NCERT]**

SOLUTION Let l, m, n be the direction cosines of the line perpendicular to each of the given lines. Then, proceeding as in Example 5, we get

$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$

Thus, the direction cosines of the given line are proportional to

$(m_1 n_2 - m_2 n_1),\ (n_1 l_2 - n_2 l_1),\ (l_1 m_2 - l_2 m_1)$.

So, its direction cosines are

$\dfrac{m_1 n_2 - m_2 n_1}{\lambda}, \dfrac{n_1 l_2 - n_2 l_1}{\lambda}, \dfrac{l_1 m_2 - l_2 m_1}{\lambda}$, where

$$\lambda = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}$$

We know that,

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 = (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2 \quad ...(i)$$

It is given that the given lines are perpendicular to each other. Therefore,

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

Also, we have: $l_1^2 + m_1^2 + n_1^2 = 1$ and, $l_2^2 + m_2^2 + n_2^2 = 1$

Putting these values in (i), we get

$$(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2 = 1 \Rightarrow \lambda = 1$$

Hence, the direction cosines of the given line are: $(m_1 n_2 - m_2 n_1)$, $(n_1 l_2 - n_2 l_1)$, $(l_1 m_2 - l_2 m_1)$.

EXAMPLE 9 *If l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angles with them.* **[NCERT EXEMPLAR]**

SOLUTION It is given that l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the direction cosines of three mutually perpendicular lines.

$$\therefore \quad l_1^2 + m_1^2 + n_1 = 1,\ l_2^2 + m_2^2 + n_2^2 = 1,\ l_3^2 + m_3^2 + n_3^2 = 1,$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,\ l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \text{ and } l_1 l_3 + m_1 m_3 + n_1 n_3 = 0 \quad ...(i)$$

$$\therefore \quad (l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2$$

$$= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2) + 2(l_1 l_2 + m_1 m_2 + n_1 n_2) + 2(l_1 l_3 + m_1 m_3 + n_1 n_3) + 2(l_2 l_3 + m_2 m_3 + n_2 n_3)$$

$$= 3 + 2 \times 0 + 2 \times 0 + 2 \times 0 = 3$$

$$\Rightarrow \quad \sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2} = \sqrt{3}$$

Let θ_1 be the angle between the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ and the line whose direction cosines are l_1, m_1, n_1. Then,

$$\cos\theta_1 = \frac{l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{(l_1^2 + m_1^2 + n_1^2) + (l_1 l_2 + m_1 m_2 + n_1 n_2) + (l_1 l_3 + m_1 m_3 + n_1 n_3)}{\sqrt{l_1^2 + m_1^2 + n_1^2}\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}} = \frac{1 + 0 + 0}{\sqrt{1}\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Similarly, let θ_2 be the angle between the line with direction cosines l_2, m_2, n_2 and the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$. Then,

$$\cos\theta_2 = \frac{l_2(l_1 + l_2 + l_3) + m_2(m_1 + m_2 + m_3) + n_2(n_1 + n_2 + n_3)}{\sqrt{l_2^2 + m_2^2 + n_2^2}\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}}$$

$$\Rightarrow \quad \cos\theta_2 = \frac{(l_1 l_2 + m_1 m_2 + n_1 n_2) + (l_2^2 + m_2^2 + n_2^2) + (l_1 l_3 + m_1 m_3 + n_1 n_3)}{\sqrt{l_2^2 + m_2^2 + n_2^2}\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}} = \frac{0 + 1 + 0}{\sqrt{1} \times \sqrt{3}} = \frac{1}{\sqrt{3}}$$

Proceeding as above the angle θ_3 between a line with direction cosines l_3, m_3, n_3 and a line whose direction ratios are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ is given by $\cos\theta_3 = \frac{1}{\sqrt{3}}$.

$$\therefore \quad \cos\theta_1 = \cos\theta_2 = \cos\theta_3 = \frac{1}{\sqrt{3}} \Rightarrow \theta_1 = \theta_2 = \theta_3.$$

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EXAMPLE 10 *Find the angle between the lines whose direction cosines are given by the equations*

$$3l+m+5n = 0,\ 6mn-2nl+5lm = 0$$ **[NCERT EXEMPLAR]**

SOLUTION The given equations are

$$3l+m+5n = 0 \quad ...(i)$$

and, $$6mn-2nl+5\,lm = 0 \quad ...(ii)$$

From (i), we get $m=-3l-5n$. Putting $m=-3l-5n$ in (ii), we get

$$6(-3l-5n)\,n-2nl+5l(-3l-5n) = 0$$

$$\Rightarrow \quad 30n^2+45ln+15l^2 = 0 \Rightarrow 2n^2+3ln+l^2 = 0 \Rightarrow 2n^2+2nl+nl+l^2 = 0$$

$$\Rightarrow \quad 2n(n+l)+l(n+l) = 0 \Rightarrow (n+l)(2n+l) = 0 \Rightarrow \text{either } l=-n \text{ or, } l=-2n.$$

If $l=-n$, then by putting $l=-n$ in (i), we obtain $m=-2n$.

If $l=-2n$, then by putting $l=-2n$ in (i), we obtain $m=n$.

Thus, the direction ratios of two lines are proportional to

$$-n,-2n,n \text{ and } -2n,n,n \text{ or, } 1,2,-1 \text{ and } -2,1,1.$$

So, vectors parallel these lines are $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b}=-2\hat{i}+\hat{j}+\hat{k}$ respectively.

If θ is the angle between the lines, then θ is also the angle between $\vec{a}$ and $\vec{b}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{-2+2-1}{\sqrt{1+4+1}\,\sqrt{4+1+1}} = -\frac{1}{6} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{6}\right)$$

EXAMPLE 11 *Find the direction cosines of the two lines which are connected by the relations.*

$$l-5m+3n = 0 \ \text{and} \ 7l^2+5m^2-3n^2 = 0$$

SOLUTION The given equations are

$$l-5m+3n = 0 \quad ...(i)$$

$$7l^2+5m^2-3n^2 = 0 \quad ...(ii)$$

From (i), we obtain $l=5m-3n$ Putting $l=5m-3n$ in (ii), we get

$$7(5m-3n)^2+5m^2-3n^2 = 0 \Rightarrow 6m^2-7mn+2n^2 = 0$$

$$\Rightarrow \quad 6m^2-3mn-4mn+2n^2 = 0 \Rightarrow (3m-2n)(2m-n) = 0 \Rightarrow m = \frac{2}{3}n \text{ or, } m = \frac{n}{2}$$

If $m=\frac{2}{3}n$, then from (i), we obtain $l=\frac{1}{3}n$. If $m=\frac{n}{2}$, then from (i), we obtain $l=-\frac{n}{2}$.

Thus, direction ratios of two lines are proportional to

$$\frac{n}{3},\frac{2}{3}n,n \text{ and } -\frac{n}{2},\frac{n}{2},n \text{ i.e. } 1,2,3 \text{ and } -1,1,2$$

Hence, the direction cosines of two lines are $\pm\frac{1}{\sqrt{14}},\pm\frac{2}{\sqrt{14}},\pm\frac{3}{\sqrt{14}}$ and $\pm\frac{-1}{\sqrt{6}},\pm\frac{1}{6},\pm\frac{2}{\sqrt{6}}$

EXAMPLE 12 *If a variable line in two adjacent positions has direction cosines l, m, n and $l+\delta l$, $m+\delta m$, $n+\delta n$, show that the small angle $\delta\theta$ between two positions is given by*

$$(\delta\theta)^2 = (\delta l)^2+(\delta m)^2+(\delta n)^2$$ **[NCERT EXEMPLAR]**

SOLUTION Since l,m,n and $l+\delta l, m+\delta m, n+\delta n$ are direction cosines of a variable line in two different positions. Therefore,

$$l^2+m^2+n^2 = 1 \quad ...(i)$$

and, $$(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2 = 1 \quad ...(ii)$$

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Now, $(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2 = 1$

$\Rightarrow \quad (l^2+m^2+n^2)+2(l\,\delta l+n\,\delta m+n\,\delta n)+(\delta l)^2+(\delta m)^2+(\delta n)^2 = 1$

$\Rightarrow \quad 1+2(l\,\delta l+m\,\delta m+n\,\delta n)+(\delta l)^2+(\delta m)^2+(\delta n)^2 = 1$

$\Rightarrow \quad 2(l\,\delta l+m\,\delta m+n\,\delta n) = -\left\{(\delta l)^2+(\delta m)^2+(\delta n)^2\right\}$

$\Rightarrow \quad l\,\delta l+m\,\delta m+n\,\delta n = -\frac{1}{2}\left\{(\delta l)^2+(\delta m)^2+(\delta n)^2\right\}$ ...(iii)

Now, $\hat{a}$ = (Unit vector along a line with direction cosines l, m, n) $= l\hat{i}+m\hat{j}+n\hat{k}$

and, $\hat{b}$ = (Unit vector along a line with direction cosines $l+\delta l, m+\delta m, n+\delta n$)

$= (l+\delta l)\hat{i}+(m+\delta m)\hat{j}+(n+\delta n)\hat{k}$

$\therefore \quad \cos\delta\theta = \dfrac{\hat{a}\cdot\hat{b}}{|\hat{a}|\,|\hat{b}|} = \hat{a}\cdot\hat{b}$ $[\because |\hat{a}|=|\hat{b}| = 1]$

$\Rightarrow \quad \cos\delta\theta = l(l+\delta l)+m(m+\delta m)+n(n+\delta n) = (l^2+m^2+n^2)+(l\,\delta l+m\,\delta m+n\,\delta n)$

$\Rightarrow \quad \cos\delta\theta = 1-\frac{1}{2}\left\{(\delta l)^2+(\delta m)^2+(\delta n)^2\right\}$ [Using (i) and (iii)]

$\Rightarrow \quad 2(1-\cos\delta\theta) = (\delta l)^2+(\delta m)^2+(\delta n)^2$

$\Rightarrow \quad 2\times 2\sin^2\frac{\delta\theta}{2} = (\delta l)^2+(\delta m)^2+(\delta n)^2$ $\left[\because 1-\cos\delta\theta = 2\sin^2\frac{\delta\theta}{2}\right]$

$\Rightarrow \quad 4\left(\frac{\delta\theta}{2}\right)^2 = (\delta l)^2+(\delta m)^2+(\delta n)^2$ $\left[\because \frac{\delta\theta}{2} \text{ is small, } \therefore \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$

$\Rightarrow \quad (\delta\theta)^2 = (\delta l)^2+(\delta m)^2+(\delta n)^2$

EXAMPLE 13 *Prove that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $f\,mn+g\,nl+h\,lm=0$ are perpendicular, if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$ and parallel, if $a^2f^2+b^2g^2+c^2h^2-2abfg-2bcgh-2achf=0$.*

SOLUTION The given relations are

$al+bm+cn = 0$ ...(i)

and, $fmn+fnl+hlm = 0$ ...(ii)

From (i), we obtain $n = -\left(\dfrac{al+bm}{c}\right)$. Putting this value of n in (ii), we get

$$-fm\left(\frac{al+bm}{c}\right)-gl\left(\frac{al+bm}{c}\right)+hlm = 0$$

$\Rightarrow \quad agl^2+(af+bg-ch)\,lm+bfm^2 = 0 \Rightarrow ag\left(\dfrac{l}{m}\right)^2+(af+bg-ch)\dfrac{l}{m}+bf = 0$ ...(iii)

This is a quadratic equation in $\dfrac{l}{m}$. So, it will have two roots, say $\dfrac{l_1}{m_1}$ and $\dfrac{l_2}{m_2}$.

$\therefore \quad \dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2} = \dfrac{bf}{ag} \Rightarrow \dfrac{l_1l_2}{bf} = \dfrac{m_1m_2}{ag} \Rightarrow \dfrac{l_1l_2}{f/a} = \dfrac{m_1m_2}{g/b}$ ...(iv)

Similarly, by making a quadratic in $\frac{m}{n}$, by using (i) and (ii), we get

$$\frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} \qquad \text{...(v)}$$

From (iv) and (v), we obtain

$$\frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c} = \lambda \text{ (say)} \Rightarrow l_1 l_2 = \lambda\left(\frac{f}{a}\right), m_1 m_2 = \lambda\left(\frac{g}{b}\right), n_1 n_2 = \lambda\left(\frac{h}{c}\right)$$

The given lines will be perpendicular, if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \Rightarrow \lambda\left(\frac{f}{a} + \frac{g}{b} + \frac{h}{c}\right) = 0 \Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$$

The given lines will be parallel, if their direction cosines are same. This is possible only when the roots of equation (iii) are equal. The condition for equal roots is

$$(af + bg - ch)^2 - 4agbf = 0 \qquad \text{[On equating discriminant to zero]}$$

$$\Rightarrow \quad a^2 f^2 + b^2 g^2 + c^2 h^2 - 2abfg - 2bcgh - 2achf = 0$$

EXAMPLE 14 *Show that the straight lines whose direction cosines are given by the equations* $al + bm + cn = 0$ *and* $ul^2 + vm^2 + wn^2 = 0$ *are perpendicular, if*

$$a^2(v + w) + b^2(u + w) + c^2(u + v) = 0 \text{ and, parallel, if } \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0$$

SOLUTION The given equations are

$$al + bm + cn = 0 \qquad \text{...(i)}$$

and, $$ul^2 + vm^2 + wn^2 = 0 \qquad \text{...(ii)}$$

From (i), we obtain $n = -\left(\frac{al + bm}{c}\right)$. Putting $n = -\left(\frac{al + bm}{c}\right)$ in (ii), we get

$$ul^2 + vm^2 + w\frac{(al + bm)^2}{c^2} = 0$$

$$\Rightarrow \quad (c^2 u + a^2 w) l^2 + 2abwlm + (c^2 v + b^2 w) m^2 = 0$$

$$\Rightarrow \quad (a^2 w + c^2 u)\left(\frac{l}{m}\right)^2 + 2abw\left(\frac{l}{m}\right) + (b^2 w + c^2 v) = 0 \qquad \text{...(iii)}$$

This is a quadratic equation in $\frac{l}{m}$. So, it gives two values of $\frac{l}{m}$. Let the two values be $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

$$\therefore \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b^2 w + c^2 v}{a^2 w + c^2 u} \Rightarrow \frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{a^2 w + c^2 u} \qquad \text{...(iv)}$$

Similarly, by making a quadratic equation in $\frac{m}{n}$, we obtain

$$\frac{m_1 m_2}{a^2 w + c^2 u} = \frac{n_1 n_2}{a^2 v + b^2 u} \qquad \text{...(v)}$$

From (iv) and (v), we get

$$\frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{a^2 w + c^2 u} = \frac{n_1 n_2}{a^2 v + b^2 u} = \lambda \qquad \text{(say).}$$

$$\Rightarrow \quad l_1 l_2 = \lambda(b^2 w + c^2 v), m_1 m_2 = \lambda(a^2 w + c^2 u), n_1 n_2 = \lambda(a^2 v + b^2 u)$$

For the given lines to be perpendicular, we must have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \quad \lambda(b^2w + c^2v) + \lambda(a^2w + c^2u) + \lambda(a^2v + b^2u) = 0$$

$$\Rightarrow \quad a^2(v+w) + b^2(u+w) + c^2(u+v) = 0$$

For the given lines to be parallel, the direction cosines must be equal and so the roots of the equation (iii) must be equal.

$$\therefore \quad 4a^2b^2w^2 - 4(a^2w + c^2u)(b^2w + c^2v) = 0 \qquad \text{[On equating discriminant to zero]}$$

$$\Rightarrow \quad a^2c^2vw + b^2c^2uw + c^4uv = 0$$

$$\Rightarrow \quad a^2vw + b^2uw + c^2uv = 0 \Rightarrow \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0 \qquad \text{[Dividing throughout by } uvw\text{]}$$

EXAMPLE 15 *If the edges of a rectangular parallelopiped are a, b, c; prove that the angles between the four diagonals are given by* $\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

SOLUTION Let O be the origin and OX, OY, OZ be the coordinate axes. Let OA, OB, OC be the coterminus edges of the parallelopiped such that $OA = a$, $OB = b$ and $OC = c$. Then, the coordinates of the vertices of the parallelopiped are: $O(0,0,0)$, $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$, $P(a,b,c)$ $Q(a,b,0)$, $R(0,b,c)$, $S(a,0,c)$

Clearly, OP, AR, CQ and BS are four diagonals of the rectangular parallelopiped.

Since the direction ratios of the line segment joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. Therefore, direction ratios of the diagonals OP, AR, BS and CQ are proportional to a, b, c ; $-a, b, c$; $a, -b, c$ and $a, b, -c$ respectively.

Let θ_1 be the angle between OP and AR. Then,

$$\cos\theta_1 = \frac{a \times -a + b \times b + c \times c}{\sqrt{a^2+b^2+c^2}\,\sqrt{(-a)^2+b^2+c^2}}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{-a^2+b^2+c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{-a^2+b^2+c^2}{a^2+b^2+c^2}\right)$$

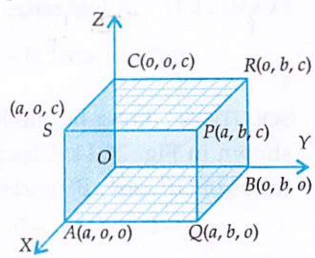

Fig. 26.11

Now, let θ_2 be the angle between OP and BS. Then,

$$\cos\theta_2 = \frac{a \times a + b \times -b + c \times c}{\sqrt{a^2+b^2+c^2}\,\sqrt{a^2+(-b)^2+c^2)^2}}$$

$$\Rightarrow \quad \cos\theta_2 = \frac{a^2-b^2+c^2}{a^2+b^2+c^2} \Rightarrow \theta_2 = \cos^{-1}\left(\frac{a^2-b^2+c^2}{a^2+b^2+c^2}\right)$$

Let θ_3 be the angle between diagonals CQ and BS. Then,

$$\cos\theta_3 = \frac{a \times a + (-b) \times b + c \times -c}{\sqrt{a^2+(-b)^2+c^2}\,\sqrt{a^2+b^2+(-c^2)}}$$

$$\Rightarrow \quad \cos\theta_3 = \frac{a^2-b^2-c^2}{a^2+b^2+c^2} \Rightarrow \theta_3 = \cos^{-1}\left(\frac{a^2-b^2-c^2}{a^2+b^2+c^2}\right)$$

Similarly, the angles between the other pairs of diagonals can be obtained.

Putting all the results together, we obtain that the angles between the diagonals of the parallelopiped are given by $\cos^{-1}\left(\frac{a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$.

EXAMPLE 16 *Show that the angles between the diagonals of a cube is* $\cos^{-1}\left(\frac{1}{3}\right)$.

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. 26.12. Clearly, OP, AR, BS and CQ are the diagonals of the cube.

Consider the diagonals OP and AR.

Direction ratios of OP and AR are proportional to $a-0, a-0, a-0$ and $0-a, a-0, a-0$ i.e. a, a, a and $-a, a, a$ respectively.

Let θ be the angle between OP and AR. Then,

$$\cos\theta = \frac{a\times -a + a\times a + a\times a}{\sqrt{a^2+a^2+a^2}\,\sqrt{(-a)^2+a^2+a^2}}$$

$$\Rightarrow \quad \cos\theta = \frac{-a^2+a^2+a^2}{\sqrt{3a^2}\,\sqrt{3a^2}} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

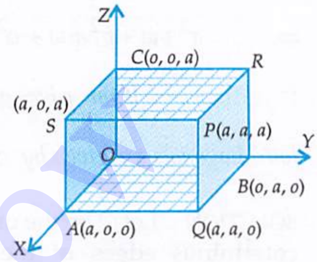

Fig. 26.12

Similarly, the angles between the other pairs of diagonals are each equal to $\cos^{-1}\left(\frac{1}{3}\right)$.

Hence, the angle between any two diagonals of a cube is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 17 *A line makes angles* $\alpha, \beta, \gamma, \delta$ *with the four diagonals of a cube, prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. 26.11. Clearly, OP, AR, BS and CQ are the diagonals of the cube. The direction ratios of OP, AR, BS and CQ are

$a-0, a-0, a-0$ i.e. a, a, a

$0-a, a-0, a-0$ i.e. $-a, a, a$

$a-0, 0-a, a-0$ i.e. $a, -a, a$.

and, $a-0, a-0, 0-a$ i.e. $a, a, -a$ respectively.

Let the direction ratios of a line be proportional to l, m, n. Suppose this line makes angles α, β, γ and δ with OP, AR, BS and CQ respectively.

Now, α is the angle between OP and the line whose direction ratios are proportional to l, m, n.

$$\therefore \quad \cos\alpha = \frac{a.l + a.m + a.n}{\sqrt{a^2+a^2+a^2}\,\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\alpha = \frac{l+m+n}{\sqrt{3}\,\sqrt{l^2+m^2+n^2}}$$

Since β is the angle between AR and the line with direction ratios proportional to l, m, n.

$$\therefore \quad \cos\beta = \frac{-a.l + a.m + a.n}{\sqrt{a^2+a^2+a^2}\,\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\beta = \frac{-l+m+n}{\sqrt{3}\,\sqrt{l^2+m^2+n^2}}$$

Similarly,

$$\cos\gamma = \frac{al - am + an}{\sqrt{a^2+a^2+a^2}\,\sqrt{l^2+m^2+n^2}} \Rightarrow \cos\gamma = \frac{l-m+n}{\sqrt{3}\,\sqrt{l^2+m^2+n^2}}$$

and, $\cos\delta = \dfrac{al + am - an}{\sqrt{a^2 + a^2 + a^2}\,\sqrt{l^2 + m^2 + n^2}} \Rightarrow \cos\theta = \dfrac{l + m - n}{\sqrt{3}\,\sqrt{l^2 + m^2 + n^2}}$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{(l+m+n)^2}{3\,(l^2+m^2+n^2)} + \frac{(-l+m+n)^2}{3\,(l^2+m^2+n^2)} + \frac{(l-m+n)^2}{3\,(l^2+m^2+n^2)} + \frac{(l+m-n)^2}{3\,(l^2+m^2+n^2)}$$

$$= \frac{1}{3\,(l^2+m^2+n^2)}\left\{(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\right\}$$

$$= \frac{1}{3\,(l^2+m^2+n^2)}\,4\,(l^2+m^2+n^2) = \frac{4}{3}.$$

EXERCISE 26.1

BASIC

1. If a line makes angles of 90°, 60° and 30° with the positive direction of x, y, and z-axis respectively, find its direction cosines. [NCERT]
2. If a line has direction ratios 2, – 1, – 2, determine its direction cosines. [NCERT]
3. Find the direction cosines of the line passing through two points (– 2, 4, –5) and (1, 2, 3). [NCERT]
4. Using direction ratios show that the points A (2, 3, – 4), B (1, – 2, 3) and C (3, 8, –11) are collinear. [NCERT]
5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, – 4), (–1, 1, 2) and (–5, –5, – 2). [NCERT]
6. Find the angle between the vectors with direction ratios proportional to 1, – 2, 1 and 4, 3, 2.
7. Find the angle between the vectors whose direction cosines are proportional to 2, 3, – 6 and 3, – 4, 5.
8. Find the acute angle between the lines whose direction ratios are proportional to 2 : 3 : 6 and 1 : 2 : 2.
9. Show that the points (2, 3, 4), (–1, – 2, 1), (5, 8, 7) are collinear.
10. Show that the line through points (4, 7, 8) and (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1) and (1, 2, 5).
11. Show that the line through the points (1, – 1, 2) and (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).
12. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, – 1) and (4, 3, – 1).
13. Find the angle between the lines whose direction ratios are proportional to a, b, c and $b - c, c - a, a - b$.
14. If the coordinates of the points A, B, C, D are (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2), then find the angle between AB and CD.

BASED ON LOTS

15. Find the direction cosines of the lines, connected by the relations: $l + m + n = 0$ and $2lm + 2ln - mn = 0$.

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16. Find the angle between the lines whose direction cosines are given by the equations

(i) $l+m+n=0$ and $l^2+m^2-n^2=0$

(ii) $2l-m+2n=0$ and $mn+nl+lm=0$

(iii) $l+2m+3n=0$ and $3lm-4ln+mn=0$

(iv) $2l+2m-n=0, mn+\ln+lm=0$ [NCERT EXEMPLAR]

ANSWERS

1. $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$ 2. $\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$ 3. $\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}$

5. $\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}; \frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}; \frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}$ 6. $\frac{\pi}{2}$

7. $\cos^{-1}\left(-\frac{18\sqrt{2}}{35}\right)$ 8. $\cos^{-1}\left(\frac{20}{21}\right)$ 13. $\frac{\pi}{2}$ 14. 0

15. $\pm\frac{1}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{-2}{56}; \pm\frac{-1}{\sqrt{6}}, \pm\frac{2}{\sqrt{6}}, \pm\frac{1}{\sqrt{6}}$

16. (i) $\frac{\pi}{3}$ (ii) $\frac{\pi}{2}$ (iii) $\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$

FILL IN THE BLANK TYPE QUESTIONS (FBQs)

1. The distance of the point (a, b, c) from y-axis is ....
2. The distance of the point (a, b, c) from z-axis is .... [CBSE 2020]
3. If a line makes angles $\frac{\pi}{2}, \frac{3\pi}{4}$ and $\frac{\pi}{4}$ with x, y, z axes respectively, then its direction cosines are ....
4. If a line makes angles α, β, γ with positive directions of the coordinate axes, then the value of $\cos 2\alpha+\cos 2\beta+\cos 2\gamma$ is ....
5. If a line makes angles α, β, γ with positive directions of the coordinate axes, then the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$ is ....
6. If a line makes an angle $\frac{\pi}{4}$ with each of y and z-axes, then the angle which it makes with x-axis, is ....
7. The direction cosines of the vector $2\hat{i}+2\hat{j}-\hat{k}$ are ....
8. A unit vector making angle $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{3}$ with y-axis and an acute angle with z-axis is.................. .
9. If the projections of a line segment on the coordinate axes are 3, 4 and 5, then its length is equal to ....
10. A vector of magnitude 21 having direction ratios proportional to 2, –3, 6 is ....
11. The direction cosines of the line joining points (4, 3, –5) and (–2, 1, –8) are.................. .

12. If $\frac{1}{c}, \frac{1}{c}, \frac{1}{c}$ are direction cosines of a line, then the values of c are ....

13. If O is the origin and $OP = 6$ with direction ratios proportional to $-1, 2, -2,$ then the coordinates of P are ....

14. The angle between the vectors with direction ratios proportional to 1, 1, 2 and $\sqrt{3}-1, -\sqrt{3}-1, 4$ is ....

15. If $\frac{1}{2}, \frac{1}{3}, n$ are the direction cosines of a line, then the values of n are.................. .

16. If a line a makes angles α, β, γ with x, y and z axes respectively such that $\alpha + \beta = \frac{\pi}{2}$, then $\gamma =$.................. .

17. The total number of straight lines equally inclined with the coordinate axis is ....

18. zx-plane divides the line segment joining (2, 3, 1) and (6, 7, 1) in the ratio ....

ANSWERS

1. $\sqrt{a^2+c^2}$	2. $\sqrt{a^2+b^2}$	3. $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$	4. -1	5. 2
6. $\frac{\pi}{2}$	7. $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$	8. $\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}$	9. $5\sqrt{2}$	10. $6\hat{i}-9\hat{j}+18\hat{k}$
11. $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$	12. $\pm\sqrt{3}$	13. $(-2, 4, -4)$	14. $\frac{\pi}{3}$	15. $\pm\frac{\sqrt{23}}{6}$
16. $\frac{\pi}{2}$	17. 8	18. 3 : 7 externally		

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define direction cosines of a directed line.
2. What are the direction cosines of X-axis?
3. What are the direction cosines of Y-axis?
4. What are the direction cosines of Z-axis?
5. Write the distances of the point $(7, -2, 3)$ from XY, YZ and XZ-planes.
6. Write the distance of the point $(3, -5, 12)$ from X-axis?
7. Write the ratio in which YZ-plane divides the segment joining $P(-2, 5, 9)$ and $Q(3, -2, 4)$.
8. A line makes an angle of 60° with each of X-axis and Y-axis. Find the acute angle made by the line with Z-axis.
9. If a line makes angles α, β and γ with the coordinate axes, find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.
10. Write the ratio in which the line segment joining (a, b, c) and $(-a, -c, -b)$ is divided by the xy-plane.
11. Write the inclination of a line with Z-axis, if its direction ratios are proportional to $0, 1, -1$.
12. Write the angle between the lines whose direction ratios are proportional to $1, -2, 1$ and $4, 3, 2$.

13. Write the distance of the point $P(x, y, z)$ from XOY plane.
14. Write the coordinates of the projection of point $P(x, y, z)$ on XOZ-plane.
15. Write the coordinates of the projection of the point $P(2, -3, 5)$ on Y-axis.
16. Find the distance of the point $(2, 3, 4)$ from the x-axis. **[CBSE 2010]**
17. If a line has direction ratios proportional to $2, -1, -2$, then what are its direction consines? **[CBSE 2012]**
18. Write direction cosines of a line parallel to z-axis. **[CBSE 2012]**
19. If a unit vector $\vec{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the value of θ. **[CBSE 2013]**
20. Write the distance of a point $P(a, b, c)$ from x-axis. **[CBSE 2014]**
21. If a line makes angle 90° and 60° respectively with positive directions of x and y axes, find the angle which it makes with the positive direction of z-axis. **[CBSE 2017]**
22. If a line makes angles 90°, 135°, 45° with the x, y and z axes respectively, find its direction cosines. **[CBSE 2019]**

ANSWERS

2. 1, 0, 0 **3.** 0, 1, 0 **4.** 0, 0, 1 **5.** 3, 7, 2 **6.** 13 units **7.** 2 : 3 internally **8.** 45° **9.** −1
10. $c : b$ **11.** $\frac{3\pi}{4}$ **12.** $\frac{\pi}{2}$ **13.** $|z|$ **14.** $(x, 0, z)$ **15.** $(0, -3, 0)$ **16.** 5
17. $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$ **18.** 0, 0, 1 **19.** $\frac{\pi}{3}$ **20.** $\sqrt{b^2 + c^2}$ **21.** $\frac{\pi}{6}, \frac{5\pi}{6}$ **22.** $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

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