CHAPTER 15

TANGENTS AND NORMALS

15.1 PRELIMINARIES

SLOPE (GRADIENT) OF A LINE *The trigonometrical tangent of the angle that a line makes with the positive direction of x-axis in anticlockwise sense is called the slope or gradient of the line.*

The slope of a line is generally denoted by m.

Thus, $m = \tan\theta$, where θ is the angle which a line makes with the positive direction of x-axis in anticlockwise sense.

Since a line parallel to x-axis makes an angle of $0°$ with x-axis. Therefore, its slope is $\tan 0° = 0$.

A line perpendicular to x-axis or parallel to y-axis makes an angle of $90°$ with x-axis, so its slope is $\tan \pi/2 = \infty$.

Also, the slope of a line equally inclined with axes is $+1$ or, -1 as it makes either $45°$ or $135°$ angle with x-axis.

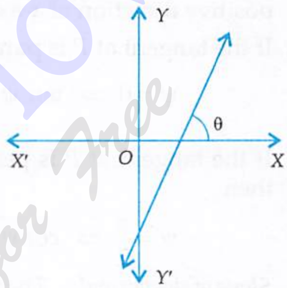

Fig.15.1

Slope of a line in terms of coordinates of any two points on it: Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line. Then its slope m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

For example, the slope of a line passing through $(2, -1)$ and $(3, 4)$ is $m = \frac{4-(-1)}{3-2} = 5$.

Slope of a line when its equation is given: The slope of a line whose equation is $ax + by + c = 0$ is given by

$$m = -\frac{a}{b} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

For example, the slope of the line $3x - 2y + 5 = 0$ is $m = \frac{-3}{-2} = \frac{3}{2}$.

Angle between two lines: The angle θ between two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

Condition of parallelism: If the lines are parallel, then $\theta = 0°$.

$$\therefore \quad \tan\theta = \tan 0 = 0 \Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \Rightarrow m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

Condition of perpendicularity: If two lines of slopes m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$.

Thus, when two lines are perpendicular, the product of their slopes is – 1. If m is the slope of a line, then the slope of a line perpendicular to it is $-\frac{1}{m}$.

Equation of a straight line: The equation of a straight line passing through a point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.

15.2 SLOPES OF TANGENT AND NORMAL

Slope of the tangent: Let $y = f(x)$ be a continuous curve, and let $P(x_1, y_1)$ be a point on it. Then, as discussed in section 10.1, $\left(\frac{dy}{dx}\right)_P$ is the slope of the tangent to the curve $y = f(x)$ at point P

i.e., $\left(\frac{dy}{dx}\right)_P = \tan\psi =$ Slope of the tangent at P,

where ψ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of x-axis.

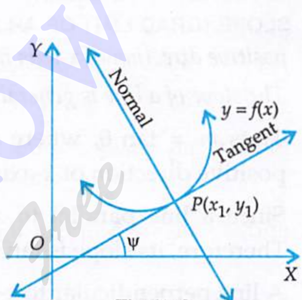

Fig.15.2

If the tangent at P is parallel to x-axis, then

$$\psi = 0 \Rightarrow \tan\psi = 0 \Rightarrow Slope = 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 0$$

If the tangent at P is perpendicular to x-axis, or parallel to y-axis then

$$\psi = \frac{\pi}{2} \Rightarrow \cot\psi = 0 \Rightarrow \frac{1}{\tan\psi} = 0 \Rightarrow \left(\frac{dx}{dy}\right)_P = 0$$

Slope of the normal: The normal to a curve at $P(x_1, y_1)$ is a line perpendicular to the tangent at P and passing through P.

$$\therefore \quad \text{Slope of the normal at } P = -\frac{1}{\text{Slope of the tangent at } P} = -\frac{1}{\left(\frac{dy}{dx}\right)_P} = -\left(\frac{dx}{dy}\right)_P$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE SLOPES OF TANGENT AND NORMAL AT A GIVEN POINT

EXAMPLE 1 *Find the slopes of the tangent and the normal to the curve $x^2 + 3y + y^2 = 5$ at $(1, 1)$.*

SOLUTION The equation of the curve is $x^2 + 3y + y^2 = 5$. Differentiating with respect to x, we get

$$2x + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{2y+3} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{2+3} = -\frac{2}{5}$$

$$\therefore \quad \text{Slope of the tangent at } (1, 1) = \left(\frac{dy}{dx}\right)_{(1,1)} = -\frac{2}{5}$$

and, $$\text{Slope of the normal at } (1, 1) = \frac{-1}{\left(\frac{dy}{dx}\right)_{(1,1)}} = \frac{-1}{\frac{-2}{5}} = \frac{5}{2}.$$

EXAMPLE 2 *Show that the tangents to the curve $y = 2x^3 - 3$ at the points where $x = 2$ and $x = -2$ are parallel.*

SOLUTION The equation of the curve is $y = 2x^3 - 3$. Differentiating with respect to x, we get:

$$\frac{dy}{dx} = 6x^2.$$

Now, $m_1 = (\text{Slope of the tangent at } x = 2) = \left(\dfrac{dy}{dx}\right)_{x=2} = 6 \times (2)^2 = 24$

and, $m_2 = (\text{Slope of the tangent at } x = -2) = \left(\dfrac{dy}{dx}\right)_{x=-2} = 6\,(-2)^2 = 24.$

Clearly, $m_1 = m_2$. Thus, the tangents to the given curve at the points where $x = 2$ and $x = -2$ are parallel.

EXAMPLE 3 *Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points $(2, 0)$ and $(3, 0)$ are at right angles.*

SOLUTION The equation of the curve is $y = x^2 - 5x + 6$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2x - 5$$

Now, $m_1 = \text{Slope of the tangent at } (2, 0) = \left(\dfrac{dy}{dx}\right)_{(2,0)} = 2 \times 2 - 5 = -1$

and, $m_2 = \text{Slope of the tangent at } (3, 0) = \left(\dfrac{dy}{dx}\right)_{(3,0)} = 2 \times 3 - 5 = 1$

Clearly, $m_1 m_2 = -1 \times 1 = -1$. Thus, the tangents to the given curve at $(2, 0)$ and $(3, 0)$ are at right angles.

EXAMPLE 4 *Find the slope of the normal to the curve $x = 1 - a \sin\theta$, $y = b\cos^2\theta$ at $\theta = \dfrac{\pi}{2}$.* **[NCERT]**

SOLUTION The parametric equations of the curve are

$$x = 1 - a\sin\theta \text{ and } y = b\cos^2\theta \Rightarrow \frac{dx}{d\theta} = -a\cos\theta \text{ and } \frac{dy}{d\theta} = -2b\cos\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2b\sin\theta\cos\theta}{-a\cos\theta} = \frac{2b}{a}\sin\theta \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\pi/2} = \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$$

Hence, $\left(\text{Slope of the normal at } \theta = \dfrac{\pi}{2}\right) = \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{\theta=\pi/2}} = -\dfrac{a}{2b}$

EXAMPLE 5 *Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$.* **[NCERT]**

SOLUTION The parametric equation of the curve are

$$x = a\cos^3\theta,\ y = a\sin^3\theta \Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta,\ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \Rightarrow \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\therefore$ Slope of the normal at any point on the curve $= -\dfrac{1}{\frac{dy}{dx}} = \dfrac{-1}{-\tan\theta} = \cot\theta$

Hence, $\left(\text{Slope of the normal at } \theta = \dfrac{\pi}{4}\right) = \cot\dfrac{\pi}{4} = 1.$

Type II ON FINDING THE POINT (s) ON A GIVEN CURVE AT WHICH TANGENT(s) IS ARE PARALLEL OR PERPENDICULAR TO A GIVEN LINE

EXAMPLE 6 *Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.*

SOLUTION Let the required point be $P(x_1, y_1)$. The given curve is

$$y = 2x^2 - 6x - 4 \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 4x - 6 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1 - 6$$

Since the tangent at (x_1, y_1) is parallel to x-axis. Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow 4x_1 - 6 = 0 \Rightarrow x_1 = \frac{3}{2}$$

Since (x_1, y_1) lies on curve (i). Therefore, $y_1 = 2x_1^2 - 6x_1 - 4 = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) - 4 = -\frac{17}{2}$

So, the required point is $(3/2, -17/2)$.

EXAMPLE 7 *At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangents are parallel to the y-axis?* **[NCERT EXEMPLAR]**

SOLUTION Let $P(x_1, y_1)$ be the required point. As $P(x_1, y_1)$ lies on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$.

$$\therefore \quad x_1^2 + y_1^2 - 2x_1 - 4y_1 + 1 = 0 \quad \text{...(i)}$$

If the tangent to the given curve at P is parallel to y-axis, then $\left(\frac{dx}{dy}\right)_P = 0$.

The equation of the curve is $x^2 + y^2 - 2x - 4y + 1 = 0$.

Differentiating both sides with respect to y, we obtain

$$2x\frac{dx}{dy} + 2y - 2\frac{dx}{dy} - 4 = 0 \Rightarrow 2\frac{dx}{dy}(x-1) = 2(2-y) \Rightarrow \frac{dx}{dy} = \frac{2-y}{x-1} \Rightarrow \left(\frac{dx}{dy}\right)_P = \frac{2-y_1}{x_1-1}$$

But, $\left(\frac{dx}{dy}\right)_P = 0$. Therefore, $\frac{2-y_1}{x_1-1} = 0 \Rightarrow 2 - y_1 = 0 \Rightarrow y_1 = 2.$

Putting $y_1 = 2$ in (i), we obtain

$$x_1^2 + 4 - 2x_1 - 8 + 1 = 0 \Rightarrow x_1^2 - 2x_1 - 3 = 0 \Rightarrow (x_1 - 3)(x_1 + 1) \Rightarrow x_1 = -1, 3$$

Hence, the coordinates of required points $(-1, 2)$ and $(3, 2)$.

EXAMPLE 8 *Find points on the curve $\frac{x^2}{9} - \frac{y^2}{16} = 1$ at which the tangents are parallel to the (i) x-axis (ii) y-axis.*

SOLUTION Let $P(x_1, y_1)$ be a point on the curve $\frac{x^2}{9} - \frac{y^2}{16} = 1$. Then,

$$\frac{x_1^2}{9} - \frac{y_1^2}{16} = 1 \quad \text{...(i)}$$

The equation of the curve is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

Differentiating both sides with respect to x, we get

$$\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{16x}{9y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{16x_1}{9y_1}$$

(i) *If the tangent at* $P(x_1, y_1)$ *is parallel to x-axis, then*

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow \frac{16x_1}{9y_1} = 0 \Rightarrow 16x_1 = 0 \Rightarrow x_1 = 0$$

Putting $x_1 = 0$ in (i), we get $y_1^2 = -16$, which is impossible as y_1 is real. Hence, there is no point on the curve where tangent is parallel to x-axis.

(ii) *If the tangent at* $P(x_1, y_1)$ *is parallel to y-axis, then*

$$\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0 \Rightarrow \frac{9y_1}{16x_1} = 0 \Rightarrow y_1 = 0.$$

Putting $y_1 = 0$ in (i), we get $x_1 = \pm 3$. Hence, required points are $(3, 0)$ and $(-3, 0)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON FINDING THE SLOPES OF TANGENT AND NORMAL AT A GIVEN POINT

EXAMPLE 9 *The slope of the curve* $2y^2 = ax^2 + b$ *at* $(1, -1)$ *is* -1. *Find* a, b.

SOLUTION The equation of the curve is $2y^2 = ax^2 + b$ ...(i)

Differentiating with respect to x, we get

$$4y\frac{dy}{dx} = 2ax \Rightarrow \frac{dy}{dx} = \frac{ax}{2y} \Rightarrow \left(\frac{dy}{dx}\right)_{(1, -1)} = \frac{-a}{2}$$

It is given that the slope of the tangent at $(1, -1)$ is -1.

$\therefore \quad -\frac{a}{2} = -1 \Rightarrow a = 2$

Since the point $(1, -1)$ lies on (i). Therefore,

$$2(-1)^2 = a(1)^2 + b \Rightarrow a + b = 2$$

Putting $a = 2$ in $a + b = 2$, we obtain $b = 0$. Hence, $a = 2$ and $b = 0$.

Type II ON FINDING POINT(s) ON A GIVEN CURVE AT WHICH TANGENT(s) IS (ARE) PARALLEL OR PERPENDICULAR TO A GIVEN LINE

EXAMPLE 10 *Find the points on the curve* $y = x^3 - 2x^2 - x$ *at which the tangent lines are parallel to the line* $y = 3x - 2$.

SOLUTION Let $P(x_1, y_1)$ be the required point. The given curve is

$$y = x^3 - 2x^2 - x \quad \text{(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 4x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 4x_1 - 1$$

Since the tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at (x_1, y_1) = Slope of the line $y = 3x - 2$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3$$

$$\Rightarrow \quad 3x_1^2 - 4x_1 - 1 = 3 \Rightarrow 3x_1^2 - 4x_1 - 4 = 0 \Rightarrow (x_1 - 2)(3x_1 + 2) = 0 \Rightarrow x_1 = 2, -\frac{2}{3}$$

Since (x_1, y_1) lies on curve (i). Therefore, $y_1 = x_1^3 - 2x_1^2 - x_1$.

Now, $x_1 = 2 \Rightarrow y_1 = 2^3 - 2(2)^2 - 2 = -2$.

and, $\quad x_1 = -\frac{2}{3} \Rightarrow y_1 = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 + \frac{2}{3} = \frac{-14}{27}$

Thus, required points are $(2, -2)$ and $\left(\frac{-2}{3}, \frac{-14}{27}\right)$.

EXAMPLE 11 *Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.* **[NCERT, CBSE 2010]**

SOLUTION Let the required point on the curve $y = x^3$ be $P(x_1, y_1)$.

We have,

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^{\ 2}$$

It is given that:

Slope of the tangent at $P(x_1, y_1)$ = Ordinate of $P(x_1, y_1)$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = y_1 \Rightarrow 3x_1^{\ 2} = y_1 \Rightarrow 3x_1^{\ 2} = x_1^{\ 3} \quad \left[\because (x_1, y_1) \text{ lies on } y = x^3 \therefore y_1 = x_1^{\ 3}\right]$$

$$\Rightarrow \quad x_1^{\ 2}(x_1 - 3) = 0 \Rightarrow x_1 = 0, x_1 = 3$$

Since (x_1, y_1) lies on $y = x^3$. Therefore, $y_1 = x_1^{\ 3}$.

$$\therefore \quad x_1 = 0 \Rightarrow y_1 = 0 \text{ and, } x_1 = 3 \Rightarrow y_1 = 3^3 = 27.$$

Hence, required points are $(0, 0)$ and $(3, 27)$.

EXAMPLE 12 *Find a point on the curve $y = (x-3)^2$, where the tangent is parallel to the line joining $(4, 1)$ and $(3, 0)$.*

SOLUTION Let the required point be $P(x_1, y_1)$. The equation of the given curve is

$$y = (x-3)^2 \qquad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2(x-3) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x_1 - 3)$$

Since the tangent at P is parallel to the line joining $(4, 1)$ and $(3, 0)$. Therefore,

Slope of the tangent at P = Slope of the line joining $(4, 1)$ and $(3, 0)$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{0-1}{3-4} \Rightarrow 2(x_1 - 3) = 1 \Rightarrow x_1 = 7/2$$

Since the point $P(x_1, y_1)$ lies on (i). Therefore, $y_1 = (x_1 - 3)^2$.

$$\therefore \quad x_1 = \frac{7}{2} \Rightarrow y_1 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}$$

Hence, the required point is $(7/2, 1/4)$.

EXAMPLE 13 *Find the coordinates of the point on the curve $\sqrt{x} + \sqrt{y} = 4$ at which tangent is equally inclined to the axes.* **[NCERT EXEMPLAR]**

SOLUTION Let the required point be $P(x_1, y_1)$. The tangent to the curve $\sqrt{x} + \sqrt{y} = 4$ at P is equally inclined with the coordinate axes. Therefore, slope of the tangent to the curve at P is ± 1.

i.e. $\quad \left(\frac{dy}{dx}\right)_P = \pm 1$

The equation of the curve is $\sqrt{x} + \sqrt{y} = 4$. Differentiating with respect to x, we obtain

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}} \Rightarrow \left(\frac{dy}{dx}\right)_P=-\frac{\sqrt{y_1}}{\sqrt{x_1}}$$

But, $\left(\frac{dy}{dx}\right)_P=\pm 1$. Therefore, $-\frac{\sqrt{y_1}}{\sqrt{x_1}}=\pm 1 \Rightarrow \sqrt{x_1}=\pm\sqrt{y_1}$ ...(i)

Clearly, $P(x_1, y_1)$ lies on the curve $\sqrt{x}+\sqrt{y}=4$.

$\therefore \quad \sqrt{x_1}+\sqrt{y_1}=4$ ...(ii)

Now two cases arise:

Case I *When* $\sqrt{x_1}=\sqrt{y_1}$: Putting $\sqrt{x_1}=\sqrt{y_1}$ in (ii), we obtain

$2\sqrt{x_1}=4 \Rightarrow \sqrt{x_1}=2 \Rightarrow x_1=4$. So, the coordinates of P are (4, 4).

Case II *When* $\sqrt{x_1}=-\sqrt{y_1}$: Putting $\sqrt{x_1}=-\sqrt{y_1}$ in (ii), we obtain

$-\sqrt{y_1}+\sqrt{y_1}=4$ or $0=4$, which is absurd. So, $\sqrt{x_1}=-\sqrt{y_1}$ is not possible.

Hence, the coordinates of the required point are (4, 4).

EXAMPLE 14 *Find the points on the curve* $4x^2+9y^2=1$, *where the tangents are perpendicular to the line* $2y+x=0$.

SOLUTION Let the required point be $P(x_1, y_1)$. The equation of the given curve is

$$4x^2+9y^2 = 1 \quad \text{...(i)}$$

Differentiating with respect to x, we get

$$8x+18y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{4x}{9y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4x_1}{9y_1}$$

It is given that the tangent at (x_1, y_1) is perpendicular to the line $2y+x=0$.

$\therefore$ Slope of the tangent at $(x_1, y_1)\times$ Slope of the line $=-1$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \times -\frac{1}{2} = -1 \qquad \left[\because \text{ Slope of a line} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}\right]$$

$$\Rightarrow \frac{-4x_1}{9y_1}\times-\frac{1}{2} = -1 \Rightarrow y_1 = \frac{-2x_1}{9} \quad \text{..(ii)}$$

Since $P(x_1, y_1)$ lies on the curve (i). Therefore,

$$4x_1^2+9y_1^2 = 1$$

$$\Rightarrow 4x_1^2+9\left(\frac{-2x_1}{9}\right)^2 = 1 \quad \text{[Using (ii)]}$$

$$\Rightarrow 4x_1^2+\frac{4x_1^2}{9} = 1 \Rightarrow x_1^2 = \frac{9}{40} \Rightarrow x_1 = \pm\frac{3}{2\sqrt{10}}$$

Now, $x_1 = \frac{3}{2\sqrt{10}} \Rightarrow y_1 = \frac{-2}{9}\left(\frac{3}{2\sqrt{10}}\right) = -\frac{1}{3\sqrt{10}}$ [Using (ii)]

and, $x_1 = -\frac{3}{2\sqrt{10}} \Rightarrow y_1 = \frac{-2}{9}\left(-\frac{3}{2\sqrt{10}}\right) = \frac{1}{3\sqrt{10}}$ [Using (ii)]

Hence, the required points are $\left(\frac{3}{2\sqrt{10}}, \frac{-1}{3\sqrt{10}}\right)$ and $\left(\frac{-3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type II ON FINDING POINT(s) ON A GIVEN CURVE AT WHICH TANGENT(s) IS (ARE) PARALLEL OR PERPENDICULAR TO A GIVEN LINE

EXAMPLE 15 *Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent has the equation $y = x - 11$.* **[CBSE 2012, NCERT]**

SOLUTION Let the required point be $P(x_1, y_1)$. Since (x_1, y_1) lies on $y = x^3 - 11x + 5$.

$$\therefore \quad y_1 = x_1^3 - 11x_1 + 5 \quad \text{...(i)}$$

Now, $y = x^3 - 11x + 5 \Rightarrow \frac{dy}{dx} = 3x^2 - 11 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3x_1^2 - 11$

Since the line $y = x - 11$ is tangent at the point (x_1, y_1). Therefore,

Slope of the tangent at (x_1, y_1) = (Slope of the line $y = x - 11$).

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (\text{Slope of the line } x - y - 11 = 0)$$

$$\Rightarrow \quad 3x_1^2 - 11 = \frac{-1}{-1} \qquad \left[\because \text{Slope} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}\right]$$

$$\Rightarrow \quad 3x_1^2 = 12 \Rightarrow x_1 = \pm 2$$

Now, $x_1 = 2 \Rightarrow y_1 = 2^3 - 22 + 5 = -9$ [Using (i)]

$x_1 = -2 \Rightarrow y_1 = (-2)^3 - 11(-2) + 5 = 19$ [Using (i)]

So, two points are $(2, -9)$ and $(-2, 19)$. Of these two points, $(-2, 19)$ does not lie on $y = x - 11$. Therefore, the required point is $(2, -9)$.

EXAMPLE 16 *Find the points on the curve $9y^2 = x^3$ where normal to the curve makes equal intercepts with the axes.*

SOLUTION Let the required point be (x_1, y_1). The equation of the curve is $9y^2 = x^3$.

Since (x_1, y_1) lies on the curve. Therefore,

$$9y_1^2 = x_1^3 \quad \text{...(i)}$$

Now, $9y^2 = x^3 \Rightarrow \frac{dy}{dx} = \frac{x^2}{6y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{x_1^2}{6y_1}$

Since the normal to the curve at (x_1, y_1) make equal intercepts with the coordinate axes.

$\therefore$ Slope of the normal $= \pm 1$

$$\Rightarrow \quad -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = \pm 1$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm 1 \Rightarrow \frac{x_1^2}{6y_1} = \pm 1 \Rightarrow x_1^2 = \pm 6y_1 \Rightarrow x_1^4 = 36y_1^2 \Rightarrow x_1^4 = 36\left(\frac{x_1^3}{9}\right)$$

[Using (i)]

$$\Rightarrow \quad x_1^4 = 4x_1^3 \Rightarrow x_1^3(x_1 - 4) = 0 \Rightarrow x_1 = 0, 4$$

Putting $x_1 = 0$ in (i), we get: $9\,y_1^{\,2} = 0 \Rightarrow y_1 = 0.$

Putting $x_1 = 4$ in (i), we get: $9\,y_1^{\,2} = 4^3 \Rightarrow y_1 = \pm\frac{8}{3}.$

But, the line making equal intercepts with the coordinate axes cannot pass through the origin. Hence, the required points are $(4, 8/3)$ and $(4, -8/3)$.

EXERCISE 15.1

BASIC

1. Find the slopes of the tangent and the normal to the following curves at the indicated points:
   (i) $y = \sqrt{x^3}$ at $x = 4$
   (ii) $y = \sqrt{x}$ at $x = 9$
   (iii) $y = x^3 - x$ at $x = 2$ [NCERT]
   (iv) $y = 2x^2 + 3\sin x$ at $x = 0$
   (v) $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$ at $\theta = -\pi/2$
   (vi) $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \pi/4$
   (vii) $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ at $\theta = \pi/2$
   (viii) $y = (\sin 2x + \cot x + 2)^2$ at $x = \pi/2$
   (ix) $x^2 + 3y + y^2 = 5$ at $(1, 1)$
   (x) $xy = 6$ at $(1, 6)$
2. At what points on the circle $x^2 + y^2 - 2x - 4y + 1 = 0$, the tangent is parallel to x-axis. [CBSE 2002C]
3. At what point of the curve $y = x^2$ does the tangent make an angle of 45° with the x-axis?
4. Find the points on the curve $2a^2 y = x^3 - 3ax^2$ where the tangent is parallel to x-axis.
5. Find the points on the curve $\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are parallel to the
   (i) x-axis (ii) y-axis. [NCERT]
6. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the
   (i) x-axis. (ii) y-axis [NCERT, CBSE 2011]
7. Find the points on the curve $\frac{x^2}{9} + \frac{y^2}{16} = 1$ at which the tangents are (i) parallel to x-axis (ii) parallel to y-axis. [NCERT]
8. Show that the tangents to the curve $y = 7x^3 + 11$ at the points $x = 2$ and $x = -2$ are parallel. [NCERT]

BASED ON LOTS

9. Find the values of a and b if the slope of the tangent to the curve $xy + ax + by = 2$ at $(1, 1)$ is 2.
10. If the tangent to the curve $y = x^3 + ax + b$ at $(1, -6)$ is parallel to the line $x - y + 5 = 0$, find a and b. [CBSE 2005]
11. Find a point on the curve $y = x^3 - 3x$ where the tangent is parallel to the chord joining $(1, -2)$ and $(2, 2)$.
12. Find the points on the curve $y = x^3 - 2x^2 - 2x$ at which the tangent lines are parallel to the line $y = 2x - 3$.
13. Find the points on the curve $y^2 = 2x^3$ at which the slope of the tangent is 3.
14. Find the points on the curve $xy + 4 = 0$ at which the tangents are inclined at an angle of 45° with the x-axis.
15. Find the point on the curve $y = x^2$ where the slope of the tangent is equal to the x-coordinate of the point.
16. Find the points on the curve $y = 3x^2 - 9x + 8$ at which the tangents are equally inclined with the axes.

17. At what points on the curve $y = 2x^2 - x + 1$ is the tangent parallel to the line $y = 3x + 4$?

18. Find the point on the curve $y = 3x^2 + 4$ at which the tangent is perpendicular to the line whose slope is $-\frac{1}{6}$.

19. Find the points on the curve $x^2 + y^2 = 13$, the tangent at each one of which is parallel to the line $2x + 3y = 7$.

20. At what points on the curve $y = x^2 - 4x + 5$ is the tangent perpendicular to the line $2y + x = 7$? **[NCERT]**

21. Find the points on the curve $y = x^3$ where the slope of the tangent is equal to x-coordinate of the point. **[CBSE 2008]**

ANSWERS

1.

	Slope of tangent	Slope of normal		Slope of tangent	Slope of normal
(i)	3	$-1/3$	(vi)	-1	1
(ii)	$1/6$	-6	(vii)	1	-1
(iii)	11	$-1/11$	(viii)	-12	$1/12$
(iv)	3	$-1/3$	(ix)	$-2/5$	$5/2$
(v)	1	-1	(x)	-6	$1/6$

2. (1, 0), (1, 4) **3.** $(1/2, 1/4)$ **4.** $(0, 0)$, $(2a, -2a)$

5. (i) $(0, 5), (0, -5)$ (ii) $(2, 0), (-2, 0)$ **6.** (i) $(1, \pm 2)$ (ii) $(-1, 0), (3, 0)$

7. (i) $(0, 4), (0, -4)$ (ii) $(3, 0), (-3, 0)$ **9.** $a = 5, b = -4$ **10.** $a = -2, b = -5$

11. $\left(\pm\sqrt{\frac{7}{3}}, \mp\frac{2}{3}\sqrt{\frac{7}{3}}\right)$ **12.** $(2, -4); (-2/3, 4/27)$ **13.** $(2, 4)$ **14.** $(2, -2)$ and $(-2, 2)$

15. $(0, 0)$ **16.** $(5/3, 4/3)$ and $(4/3, 4/3)$ **17.** $(1, 2)$ **18.** $(1, 7)$

19. $(2, 3); (-2, -3)$ **20.** $(3, 2)$ **21.** $(0, 0), (1/3, 1/27)$

15.3 EQUATIONS OF TANGENT AND NORMAL

We know that the equation of a line passing through a point (x_1, y_1) and having slope m is

$$y - y_1 = m(x - x_1)$$

As discussed in article 15.2 that the slopes of the tangent and the normal to the curve $y = f(x)$ at a point $P(x_1, y_1)$ are $\left(\frac{dy}{dx}\right)_P$ and $-\frac{1}{\left(\frac{dy}{dx}\right)_P}$ respectively. Therefore, the equation of the tangent at $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1) \quad \text{...(i)}$$

Since the normal at $P(x_1, y_1)$ passes through P and has slope $-\frac{1}{\left(\frac{dy}{dx}\right)_P}$. Therefore, the equation of the normal at $P(x_1, y_1)$ to the curve $y = f(x)$ is

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_1) \quad \text{...(ii)}$$

REMARK 1 *If* $\left(\frac{dy}{dx}\right)_P = \infty$, *then the tangent at* (x_1, y_1) *is parallel to y-axis and its equation is* $x = x_1$.

REMARK 2 *If* $\left(\frac{dy}{dx}\right)_P = 0$, *then the normal at* (x_1, y_1) *is parallel to y-axis and its equation is* $x = x_1$.

In order to find the equations of tangent and normal to a given curve at a given point, we may use the following algorithm.

ALGORITHM

Step I *Find* $\frac{dy}{dx}$ *from the given equation* $y = f(x)$.

Step II *Find the value of* $\frac{dy}{dx}$ *at the given point* $P(x_1, y_1)$.

Step III *If* $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ *is a non-zero finite number, then obtain the equations of tangent and normal at* (x_1, y_1) *by using the formulae* $y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$ and

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$$ *respectively. Otherwise go to step IV.*

Step IV If $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$, then the equations of the tangent and normal at (x_1, y_1) are $y - y_1 = 0$ and $x - x_1 = 0$ respectively. *If* $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \pm\infty$, *then the equations of the tangent and normal at* (x_1, y_1) *are* $x - x_1 = 0$ *and* $y - y_1 = 0$ *respectively.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE EQUATIONS OF TANGENT AND NORMAL TO A CURVE AT A POINT

EXAMPLE 1 *Find the equation of the tangent to the curve* $y = -5x^2 + 6x + 7$ *at the point* $(1/2, 35/4)$.

SOLUTION The equation of the given curve is

$$y = -5x^2 + 6x + 7 \Rightarrow \frac{dy}{dx} = -10x + 6 \Rightarrow \left(\frac{dy}{dx}\right)_{(1/2,\, 35/4)} = -\frac{10}{2} + 6 = 1$$

The required equation of the tangent at $(1/2, 35/4)$ is

$$y - \frac{35}{4} = \left(\frac{dy}{dx}\right)_{(1/2,\, 35/4)}\left(x - \frac{1}{2}\right) \Rightarrow y - \frac{35}{4} = 1\left(x - \frac{1}{2}\right) \Rightarrow y = x + \frac{33}{4}$$

EXAMPLE 2 *Find the equations of the tangent and normal to the parabola* $y^2 = 4ax$ *at the point* $(at^2, 2at)$. **[NCERT, CBSE 2013]**

SOLUTION The equation of the given curve is $y^2 = 4ax$ ...(i)

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} = \frac{2a}{2at} = \frac{1}{t}$$

So, the equation of the tangent at $(at^2, 2at)$ is

$$y - 2at = \left(\frac{dy}{dx}\right)_{(at^2,\, 2at)} (x - at^2) \Rightarrow y - 2at = \frac{1}{t}(x - at^2) \Rightarrow ty = x + at^2$$

The equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -\frac{1}{\left(\frac{dy}{dx}\right)_{(at^2,\, 2at)}}(x - at^2)$$

$$\Rightarrow \quad y - 2at = -\frac{1}{1/t}(x - at^2) \Rightarrow y - 2at = -t(x - at^2) \Rightarrow y + tx = 2at + at^3$$

EXAMPLE 3 *Find the equation of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$.*

SOLUTION The equation of the given curve is $y = 2x^2 + 3\sin x$ ...(i)

Putting $x = 0$ in (i), we get $y = 0$. So, the point of contact is (0, 0).

Now, $y = 2x^2 + 3\sin x \Rightarrow \frac{dy}{dx} = 4x + 3\cos x \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 4 \times 0 + 3\cos 0 = 3$

So, the equation of the normal at (0, 0) is $y - 0 = -\frac{1}{3}(x - 0)$ or, $x + 3y = 0$.

EXAMPLE 4 *Find the equations of the tangent and the normal to $16x^2 + 9y^2 = 144$ at (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$.* **[CBSE 2018]**

SOLUTION The equation of the given curve is

$$16x^2 + 9y^2 = 144 \qquad ...(i)$$

Since (x_1, y_1) lies on (i). Therefore,

$$16x_1^2 + 9y_1^2 = 144 \Rightarrow 16(2)^2 + 9y_1^2 = 144 \Rightarrow y_1^2 = \frac{80}{9} \Rightarrow y_1 = \frac{4\sqrt{5}}{3} \qquad [\because y_1 > 0]$$

So, coordinates of the given point are $\left(2, \frac{4\sqrt{5}}{3}\right)$.

Now, $16x^2 + 9y^2 = 144$

$$\Rightarrow \quad 32x + 18y\frac{dy}{dx} = 0 \qquad \text{[Differentiating with respect to } x]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-16x}{9y} \Rightarrow \left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)} = -\frac{16 \times 2}{9 \times \frac{4\sqrt{5}}{3}} = -\frac{8}{3\sqrt{5}}$$

The equation of the tangent at $\left(2, \frac{4\sqrt{5}}{3}\right)$ is

$$y - \frac{4\sqrt{5}}{3} = \left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}(x - 2) \text{ or, } y - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x - 2) \text{ or, } 8x + 3\sqrt{5}y - 36 = 0$$

The equation of the normal at $\left(2, \frac{4\sqrt{5}}{3}\right)$ is $y - \frac{4\sqrt{5}}{3} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}}(x - 2)$.

or, $\quad y-\frac{4\sqrt{5}}{3}=\frac{-1}{\frac{-8}{3\sqrt{5}}}(x-2)$ or, $y-\frac{4\sqrt{5}}{3}=\frac{3\sqrt{5}}{8}(x-2)$ or, $9\sqrt{5}\,x-24y+14\sqrt{5}=0.$

EXAMPLE 5 *Find the equations of tangent and normal to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at* (x_1, y_1).

SOLUTION We have,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(i)}$$

Since $P(x_1, y_1)$ lies on the curve (i). Therefore,

$$\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}=1 \quad \text{...(ii)}$$

Differentiating (i) with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{b^2 x}{a^2 y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)}=-\frac{b^2 x_1}{a^2 y_1}$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

$$\Rightarrow \quad y-y_1=-\frac{b^2 x_1}{a^2 y_1}(x-x_1)$$

$$\Rightarrow \quad \frac{y y_1-y_1^{\ 2}}{b^2}=-\left(\frac{xx_1-x_1^{\ 2}}{a^2}\right) \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2} \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{[Using (ii)]}$$

The equation of the normal at $P(x_1, y_1)$ is

$$y-y_1=-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x-x_1)$$

$$\Rightarrow \quad y-y_1=\frac{a^2 y_1}{b^2 x_1}(x-x_1)$$

$$\Rightarrow \quad \frac{b^2(y-y_1)}{y_1}=\frac{a^2(x-x_1)}{x_1} \Rightarrow \frac{b^2 y}{y_1}-b^2=\frac{a^2 x}{x_1}-a^2 \Rightarrow \frac{a^2 x}{x_1}-\frac{b^2 y}{y_1}=a^2-b^2$$

EXAMPLE 6 *Find the equation of the tangent line to the curve* $x=1-\cos\theta$, $y=\theta-\sin\theta$ *at* $\theta=\pi/4$. **[CBSE 2004]**

SOLUTION Putting $\theta=\frac{\pi}{4}$ in $x=1-\cos\theta$ and $y=\theta-\sin\theta$, we get

$$x=1-\cos\frac{\pi}{4}=1-\frac{1}{\sqrt{2}} \text{ and } y=\frac{\pi}{4}-\sin\frac{\pi}{4}=\frac{\pi}{4}-\frac{1}{\sqrt{2}}.$$

So, coordinates of the point of contact are $\left(1-\frac{1}{\sqrt{2}}, \frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)$.

Now, $\quad x=1-\cos\theta$ and $y=\theta-\sin\theta \Rightarrow \frac{dx}{d\theta}=\sin\theta$ and $\frac{dy}{d\theta}=1-\cos\theta$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{1-\cos\theta}{\sin\theta} \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}}=\frac{1-\cos\pi/4}{\sin\pi/4}=\frac{1-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}=\sqrt{2}-1$$

So, the equation of the tangent line at $\theta = \frac{\pi}{4}$ is

$$y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2}-1)\left\{x - \left(1 - \frac{1}{\sqrt{2}}\right)\right\} \text{ or, } (\sqrt{2}-1)\,x - y = 2(\sqrt{2}-1) - \pi/4$$

EXAMPLE 7 *Find the equations of the tangent and the normal at the point 't' on the curve* $x = a\sin^3 t,\ y = b\cos^3 t$. **[NCERT, CBSE 2010, 2014]**

SOLUTION The parametric equations of the curve are

$$x = a\sin^3 t \text{ and, } y = b\cos^3 t \Rightarrow \frac{dx}{dt} = 3a\sin^2 t\cos t \text{ and, } \frac{dy}{dt} = -3b\cos^2 t\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b}{a}\cdot\frac{\cos t}{\sin t}$$

So, the equation of the tangent at the point 't' is

$$y - b\cos^3 t = \left(\frac{dy}{dx}\right)(x - a\sin^3 t)$$

or, $$y - b\cos^3 t = -\frac{b}{a}\frac{\cos t}{\sin t}(x - a\sin^3 t) \text{ or, } bx\cos t + ay\sin t = ab\sin t\cos t$$

The equation of the normal at the point 't' is

$$y - b\cos^3 t = \frac{-1}{\left(\frac{dy}{dx}\right)}(x - a\sin^3 t)$$

or, $$y - b\cos^3 t = -\frac{1}{\frac{-b\cos t}{a\sin t}}(x - a\sin^3 t) \text{ or, } ax\sin t - by\cos t = a^2\sin^4 t - b^2\cos^4 t$$

EXAMPLE 8 *Find the equations of the tangent and the normal to the curve* $y = \frac{x-7}{(x-2)(x-3)}$ *at the point, where it cuts x-axis.* **[NCERT, CBSE 2010]**

SOLUTION The equation of the given curve is

$$y(x-2)(x-3) - x + 7 = 0 \quad \text{...(i)}$$

This cuts the x-axis at the point, where $y = 0$. Putting $y = 0$ in (i), we get: $-x + 7 = 0 \Rightarrow x = 7$

So, the point of contact is (7, 0).

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx}(x-2)(x-3) + y(2x-5) - 1 = 0 \quad \text{...(ii)}$$

Putting $x = 7$ and $y = 0$ in (ii), we get

$$\left(\frac{dy}{dx}\right)_{(7,0)}(7-2)(7-3) - 1 = 0 \Rightarrow \left(\frac{dy}{dx}\right)_{(7,0)} = \frac{1}{20}$$

So, the equation of the tangent at (7, 0) is

$$y - 0 = \left(\frac{dy}{dx}\right)_{(7,0)}(x-7) \Rightarrow y - 0 = \frac{1}{20}(x-7) \Rightarrow x - 20y - 7 = 0$$

The equation of the normal at (7, 0) is

$$y-0=-\frac{1}{\left(\frac{dy}{dx}\right)_{(7,0)}}(x-7) \Rightarrow y-0=-20(x-7) \Rightarrow 20x+y-140=0$$

EXAMPLE 9 *Find the equation of the tangent to the curve $y=(x^3-1)(x-2)$ at the points where the curve cuts the x-axis.*

SOLUTION The equation of the curve is $y=(x^3-1)(x-2)$ ...(i)

It cuts x-axis at $y=0$. So, putting $y=0$ in (i), we get

$$(x^3-1)(x-2)=0$$

$$\Rightarrow \quad (x-1)(x-2)(x^2+x+1)=0$$

$$\Rightarrow \quad x-1=0, x-2=0 \Rightarrow x=1, 2. \qquad [\because x^2+x+1 \neq 0]$$

Thus, the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0).

Now, $y=(x^3-1)(x-2) \Rightarrow \frac{dy}{dx}=3x^2(x-2)+(x^3-1) \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-3$ and, $\left(\frac{dy}{dx}\right)_{(2,0)}=7.$

The equations of the tangents at (1, 0) and (2 , 0) are respectively

$$y-0=\left(\frac{dy}{dx}\right)_{(1,0)}(x-1) \text{ and } y-0=\left(\frac{dy}{dx}\right)_{(2,0)}(x-2)$$

$$\Rightarrow \quad y-0=-3(x-1) \text{ and } y-0=7(x-2) \Rightarrow y+3x-3=0 \text{ and } 7x-y-14=0.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON FINDING THE EQUATIONS OF TANGENT AND NORMAL TO A CURVE AT A POINT

EXAMPLE 10 *Show that the line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=be^{-x/a}$ at the point where it crosses the y-axis.* [CBSE 2005, 2007, NCERT EXEMPLAR]

SOLUTION The equation of the given curve is $y=be^{-x/a}$ ...(i)

It crosses y-axis at the point, where $x=0$. Putting $x=0$ in (i), we get: $y=be^0=b$. So, the point of contact is $(0, b)$.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx}=be^{-x/a}\frac{d}{dx}\left(-\frac{x}{a}\right) \Rightarrow \frac{dy}{dx}=\frac{-b}{a}e^{-x/a} \Rightarrow \left(\frac{dy}{dx}\right)_{(0,b)}=-\frac{b}{a}e^0=-\frac{b}{a}$$

The equation of the tangent at $(0, b)$ is

$$y-b=\left(\frac{dy}{dx}\right)_{(0,b)}(x-a)$$

$$\Rightarrow \quad y-b=-\frac{b}{a}(x-0) \Rightarrow ay-ab=-bx \Rightarrow bx+ay=ab \Rightarrow \frac{x}{a}+\frac{y}{b}=1.$$

Hence, $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=b\,e^{-x/a}$ at the point where it crosses the axis of y.

Type II ON FINDING TANGENT AND NORMAL PARALLEL OR PERPENDICULAR TO A GIVEN LINE

EXAMPLE 11 *Find the equation of the tangent line to the curve $y=\sqrt{5x-3}-2$ which is parallel to the line $4x-2y+3=0$.*

SOLUTION Let the point of contact of the tangent line parallel to the given line be $P(x_1, y_1)$.

The equation of the curve is $y=\sqrt{5x-3}-2$. Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{5}{2\sqrt{5x-3}} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{5}{2\sqrt{5x_1-3}}$$

Since the tangent at (x_1, y_1) is parallel to the line $4x - 2y + 3 = 0$. Therefore,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (\text{Slope of the line } 4x - 2y + 3 = 0)$$

$$\Rightarrow \quad \frac{5}{2\sqrt{5x_1-3}} = \frac{-4}{-2} \Rightarrow 4\sqrt{5x_1-3} = 5 \Rightarrow 16(5x_1-3) = 25 \Rightarrow x_1 = \frac{73}{80}$$

Since (x_1, y_1) lies on $y = \sqrt{5x-3} - 2$. Therefore,

$$y_1 = \sqrt{5x_1-3} - 2 \Rightarrow y_1 = \sqrt{5 \times \frac{73}{80} - 3} - 2 = -\frac{3}{4} \qquad \left[\because x_1 = \frac{73}{80}\right]$$

So, the coordinates of the point of contact are $\left(\frac{73}{80}, -\frac{3}{4}\right)$. Hence, the required equation of the tangent is

$$y - \left(-\frac{3}{4}\right) = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$$\Rightarrow \quad y - \left(-\frac{3}{4}\right) = 2\left(x - \frac{73}{80}\right) \Rightarrow 80x - 40y - 103 = 0 \qquad \left[\because \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2\right]$$

EXAMPLE 12 *Find the equation of tangent line to $y = 2x^2 + 7$ which is parallel to the line $4x - y + 3 = 0$.*

SOLUTION Let the point of contact of the required tangent line be (x_1, y_1). The equation of the given curve is

$$y = 2x^2 + 7$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 4x \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1$$

Since the line $4x - y + 3 = 0$ is parallel to the tangent at (x_1, y_1).

$\therefore$ Slope of the tangent at (x_1, y_1) = (Slope of the line $4x - y + 3 = 0$)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4}{-1} \Rightarrow 4x_1 = 4 \Rightarrow x_1 = 1 \qquad \left[\because \text{Slope} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}\right]$$

Now, (x_1, y_1) lies on $y = 2x^2 + 7$. Therefore,

$$\therefore \quad y_1 = 2{x_1}^2 + 7 \Rightarrow y_1 = 2 + 7 = 9 \qquad [\because x_1 = 1]$$

So, the coordinates of the point of contact are $(1, 9)$. Hence, the required equation of the tangent line is

$$y - 9 = 4(x - 1) \Rightarrow 4x - y + 5 = 0$$

EXAMPLE 13 *Find the equation(s) of normal(s) to the curve $3x^2 - y^2 = 8$ which is (are) parallel to the line $x + 3y = 4$.* **[NCERT EXEMPLAR]**

SOLUTION Let the required normal be drawn at the point (x_1, y_1). The equation of the given curve is

$$3x^2 - y^2 = 8 \qquad \text{...(i)}$$

Differentiating both sides with respect to x, we get

$$6x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}$$

Since the normal at (x_1, y_1) is parallel to the line $x + 3y = 4$. Therefore,

Slope of the normal at (x_1, y_1) = (Slope of the line $x + 3y = 4$)

$$\Rightarrow \quad \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}} = \frac{-1}{3} \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{-y_1}{3x_1} = -\frac{1}{3} \Rightarrow y_1 = x_1 \qquad \text{...(iii)}$$

Since (x_1, y_1) lies on (i). Therefore, $3x_1^2 - y_1^2 = 8$ ...(iv)

Eliminating y_1 between (iii) and (iv), we get

$$3x_1^2 - x_1^2 = 8 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$$

Now, $x_1 = 2 \Rightarrow y_1 = 2$ and, $x_1 = -2 \Rightarrow y_1 = -2$ [Using (iii)]

Thus, the coordinates of the point are $(2, 2)$ and $(-2, -2)$. The equation of the normal at $(2, 2)$ is

$$y - 2 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1,y_1)}}(x-2) \Rightarrow y - 2 = -\frac{1}{3}(x-2) \Rightarrow x + 3y - 8 = 0 \qquad \text{[Using (ii)]}$$

The equation of the normal at $(-2, -2)$ is $y - (-2) = -\frac{1}{3}(x - (-2))$ or, $x + 3y + 8 = 0$ [Using (ii)]

EXAMPLE 14 *Find the equation(s) of tangent(s) to the curve $y = x^3 + 2x + 6$ which is perpendicular to the line $x + 14y + 4 = 0$.* **[NCERT, CBSE 2010]**

SOLUTION Let the coordinates of the point of contact be (x_1, y_1). As it lies on $y = x^3 + 2x + 6$

$$\therefore \quad y_1 = x_1^3 + 2x_1 + 6 \qquad \text{...(i)}$$

The equation of the curve is $y = x^3 + 2x + 6$. Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 3x^2 + 2 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = 3x_1^2 + 2 \qquad \text{...(ii)}$$

Since the tangent at (x_1, y_1) is perpendicular to the line $x + 14y + 4 = 0$.

$\therefore$ Slope of the tangent at (x_1, y_1) × Slope of the line $= -1$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} \times -\frac{1}{14} = -1 \Rightarrow (3x_1^2 + 2)\left(-\frac{1}{14}\right) = -1 \Rightarrow 3x_1^2 + 2 = 14 \Rightarrow x_1 = \pm 2$$

Now, $x_1 = 2 \Rightarrow y_1 = 2^3 + 2 \times 2 + 6 = 18$ and, $x_1 = -2 \Rightarrow y_1 = (-2)^3 + 2(-2) + 6 = -6$ [Using (i)]

So, the coordinates of the points of contact are $(2, 18)$ and $(-2, -6)$.

From (ii), we obtain

$$\left(\frac{dy}{dx}\right)_{(2,18)} = 3(2)^2 + 2 = 14 \text{ and, } \left(\frac{dy}{dx}\right)_{(-2,-6)} = 3(-2)^2 + 2 = 14$$

The equation of the tangent at $(2, 18)$ is

$$y - 18 = \left(\frac{dy}{dx}\right)_{(2,18)}(x-2) \Rightarrow y - 18 = 14(x-2) \Rightarrow 14x - y - 10 = 0$$

The equation of the tangent at $(-2, -6)$ is

$$y-(-6)=\left(\frac{dy}{dx}\right)_{(-2,-6)}(x-(-2)) \Rightarrow y-(-6) = 14(x-(-2)) \Rightarrow 14x-y+22 = 0$$

Type III **ON FINDING TANGENT OR NORMAL PASSING THROUGH A GIVEN POINT**

EXAMPLE 15 *Find the equations of the tangents drawn to the curve $y^2-2x^3-4y+8=0$ from the point $(1, 2)$.*

SOLUTION Suppose the tangent drawn from $(1, 2)$ to the curve $y^2-2x^3-4y+8=0$ touches the curve at (h, k). Then, (h, k) lies on the given curve.

$$\therefore \quad k^2-2h^3-4k+8 = 0 \qquad \text{...(i)}$$

The equation of the curve is $y^2-2x^3-4y+8=0$. Differentiating with respect to x, we get

$$2y\frac{dy}{dx}-6x^2-4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{y-2} \Rightarrow \left(\frac{dy}{dx}\right)_{(h,k)} = \frac{3h^2}{k-2}$$

So, the equation of the tangent at (h, k) is

$$y-k=\left(\frac{dy}{dx}\right)_{(h,k)}(x-h) \text{ or, } y-k = \frac{3h^2}{k-2}(x-h) \qquad \text{...(ii)}$$

It passes through $(1, 2)$. Therefore,

$$2-k = \frac{3h^2}{k-2}(1-h) \Rightarrow -(k-2)^2 = 3h^2(1-h) \Rightarrow 3h^3-3h^2-k^2+4k-4=0 \qquad \text{...(iii)}$$

Adding (i) and (iii), we get

$$h^3-3h^2+4 = 0 \Rightarrow (h-2)^2(h+1) = 0 \Rightarrow h = -1, 2.$$

Putting $h = 2$ in (iii), we get

$$24-12-k^2+4k-4 = 0 \Rightarrow k^2-4k-8=0 \Rightarrow k=2\pm 2\sqrt{3}$$

Putting $h=-1$ in (iii) we obtain imaginary values of k. Thus, the points contact are $(2, 2\pm 2\sqrt{3})$.

Putting the values of h and k in (ii), we obtain the following equations of the tangent

$$y-(2+2\sqrt{3}) = 2\sqrt{3}(x-2) \text{ and } y-(2-\sqrt{3}) = -2\sqrt{3}(x-2).$$

EXAMPLE 16 *Find the equation of the normal to the curve $x^2=4y$ which passes through the point $(1, 2)$.* **[NCERT, CBSE 2013]**

SOLUTION Suppose the normal at $P(x_1, y_1)$ on the parabola $x^2=4y$ passes through the point $(1, 2)$. Since $P(x_1, y_1)$ lies on $x^2=4y$.

$$\therefore \quad {x_1}^2 = 4y_1 \qquad \text{...(i)}$$

The equation of the curve is $x^2=4y$. Differentiating with respect to, x we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{x_1}{2}$$

The equation of the normal at $P(x_1, y_1)$ is

$$y-y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x-x_1) \Rightarrow y-y_1 = -\frac{2}{x_1}(x-x_1) \qquad \text{...(ii)}$$

It passes through $(1, 2)$.

$\therefore \quad 2 - y_1 = -\dfrac{2}{x_1}(1 - x_1) \Rightarrow 2 - y_1 = -\dfrac{2}{x_1} + 2 \Rightarrow x_1 y_1 = 2$ ...(iii)

Eliminating y_1 between (i) and (iii), we obtain

$$\frac{x_1^{\ 3}}{4} = 2 \Rightarrow x_1^{\ 3} = 8 \Rightarrow x_1 = 2$$

Putting $x_1 = 2$ in (ii), we get $y_1 = 1$. Putting the values of x_1 and y_1 in (ii), we get

$y - 1 = -1(x - 2) \Rightarrow x + y - 3 = 0$, which is the required equation of the normal.

EXAMPLE 17 *Find the coordinates of the points on the curve $y = x^2 + 3x + 4$, the tangents at which pass through the origin.*

SOLUTION Let $P(x_1, y_1)$ be a point on the given curve such that the tangent at P passes through the origin. Since $P(x_1, y_1)$ lies on $y = x^2 + 3x + 4$.

$\therefore \quad y_1 = x_1^{\ 2} + 3x_1 + 4$ ...(i)

The equation of the curve is $y = x^2 + 3x + 4$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2x + 3 \Rightarrow \left(\frac{dy}{dx}\right)_P = 2x_1 + 3.$$

The equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1) \text{ or, } y - y_1 = (2x_1 + 3)(x - x_1)$$

It passes through the origin i.e. $(0, 0)$.

$\therefore \quad 0 - y_1 = (2x_1 + 3)(0 - x_1) \Rightarrow y_1 = 2x_1^{\ 2} + 3x_1$ ...(ii)

Subtracting (ii) from (i), we get: $-x_1^{\ 2} + 4 = 0 \Rightarrow x_1 = \pm 2$.

From (ii), we find that

$$x_1 = 2 \Rightarrow y_1 = 4 + 6 + 4 = 14 \text{ and, } x_1 = -2 \Rightarrow y_1 = 4 - 6 + 4 = 2$$

Hence, the required points are $(2, 14)$ and $(-2, 2.)$.

Type IV MISCELLANEOUS EXAMPLES

EXAMPLE 18 *For the curve $y = 4x^3 - 2x^5$ find all points at which the tangent passes through the origin.* **[NCERT, CBSE 2013]**

SOLUTION Let (x_1, y_1) be the required point on $y = 4x^3 - 2x^5$. Then,

$y_1 = 4x_1^{\ 3} - 2x_1^{\ 5}$ ...(i)

The equation of the given curve is $y = 4x^3 - 2x^5$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 12x_1^{\ 2} - 10x_1^{\ 4}$$

So, the equation of the tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1) \Rightarrow y - y_1 = (12x_1^{\ 2} - 10x_1^{\ 4})(x - x_1)$$

This passes through the origin. Therefore,

$0 - y_1 = (12x_1^{\ 2} - 10x_1^{\ 4})(0 - x_1) \Rightarrow y_1 = 12x_1^{\ 3} - 10x_1^{\ 5}$ ...(ii)

Subtracting (ii) from (i), we get

$$0 = -8x_1^3 + 8x_1^5 \Rightarrow 8x_1^3(x_1^2 - 1) = 0 \Rightarrow x_1 = 0 \text{ or, } x_1 = \pm 1$$

From (*ii*), we find that

$$x_1 = 0 \Rightarrow y_1 = 0; \; x_1 = 1 \Rightarrow y_1 = 12 - 10 = 2; \; x_1 = -1 \Rightarrow y_1 = -12 + 10 = -2$$

Hence, the required points are (0, 0), (1, 2) and (−1, −2).

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I MISCELLANEOUS EXAMPLES

EXAMPLE 19 *Find the equations of all lines of slope −1 that are tangents to the curve* $y = \frac{1}{x-1}, x \neq 1$. **[NCERT]**

SOLUTION Let (x_1, y_1) be the point of contact of a line of slope −1 which touches the curve $y = \frac{1}{x-1}$. Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -1$.

Now, $$y = \frac{1}{x-1} \Rightarrow \frac{dy}{dx} = -\frac{1}{(x-1)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{1}{(x_1-1)^2}$$

It is given that

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -1 \Rightarrow -\frac{1}{(x_1-1)^2} = -1 \Rightarrow (x_1-1)^2 = 1 \Rightarrow x_1 - 1 = \pm 1 \Rightarrow x_1 = 0, x_1 = 2.$$

Since (x_1, y_1) lies on the curve $y = \frac{1}{x-1}$. Therefore, $y_1 = \frac{1}{x_1 - 1}$. ...(i)

Using (i), we find that

$$x_1 = 0 \Rightarrow y_1 = \frac{1}{-1} = -1 \text{ and, } x_1 = 2 \Rightarrow y_1 = \frac{1}{2-1} = 1$$

Thus, the coordinates of the points of contact are (0, −1) and (2, 1). The equations of the tangents at (0, −1) and (2, 1) are respectively.

$$y + 1 = \left(\frac{dy}{dx}\right)_{(0,-1)} (x - 0) \text{ and, } y - 1 = \left(\frac{dy}{dx}\right)_{(2,1)} (x - 2)$$

$$\Rightarrow \quad y + 1 = -1(x - 0) \text{ and } (y - 1) = -1(x - 2) \Rightarrow x + y + 1 = 0 \text{ and } x + y - 3 = 0$$

EXAMPLE 20 *Prove that all normals to the curve* $x = a\cos t + at\sin t$, $y = a\sin t - at\cos t$ *are at a distance a from the origin.* **[NCERT, CBSE 2013]**

SOLUTION The parametric equations of the curve are:

$$x = a\cos t + at\sin t \text{ and } y = a\sin t - at\cos t \Rightarrow \frac{dx}{dt} = at\cos t \text{ and } \frac{dy}{dt} = at\sin t$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at\sin t}{at\cos t} = \tan t$$

The equation of the normal at any point t is given by

$$y - (a\sin t - at\cos t) = -\frac{1}{\frac{dy}{dx}}\{x - (a\cos t + at\sin t)\}$$

$$\Rightarrow \quad y - (a\sin t - at\cos t) = -\frac{1}{\tan t}\{x - (a\cos t + at\sin t)\}$$

$$\Rightarrow \quad y-(a \sin t - at \cos t) = -\frac{\cos t}{\sin t}\{x-(a \cos t + at \sin t)\}$$

$$\Rightarrow \quad y \sin t - (a \sin^2 t - at \sin t \cos t) = -x \cos t + a \cos^2 t + at \sin t \cos t$$

$$\Rightarrow \quad x \cos t + y \sin t = a \qquad \text{...(i)}$$

$\therefore$ Length of the perpendicular from the origin to (i) $= \dfrac{|0 \cos t + 0 \sin t - a|}{\sqrt{\cos^2 t + \sin^2 t}} = a$

Hence, all normals to the given curve are at a distance 'a' from the origin.

Type II ON FINDING THE EQUATIONS OF TANGENT AND NORMAL

EXAMPLE 21 *Find the equation of the normal to the curve* $y=(1+x)^y + \sin^{-1}(\sin^2 x)$ *at* $x=0$.

SOLUTION The equation of the curve is

$$y = (1+x)^y + \sin^{-1}(\sin^2 x) \qquad \text{...(i)}$$

Putting $x = 0$, we get

$$y = (1+0)^y + \sin^{-1}(\sin^2 0) \Rightarrow y = 1.$$

Thus, we have to write the equation of the normal to (i) at $P(0, 1)$.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = e^{y \log(1+x)} \frac{d}{dx}\{y \log(1+x)\} + \frac{1}{\sqrt{1-\sin^4 x}} \frac{d}{dx}(\sin^2 x)$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x)^y \left\{\frac{dy}{dx} \log(1+x) + \frac{y}{1+x}\right\} + \frac{2 \sin x \cos x}{|\cos x| \sqrt{1+\sin^2 x}}$$

Putting $x=0$ and $y=1$, we obtain

$$\left(\frac{dy}{dx}\right)_P = \left\{\left(\frac{dy}{dx}\right)_P \times 0 + 1\right\} + 0 \Rightarrow \left(\frac{dy}{dx}\right)_P = 1$$

Hence, the equation of the normal at $P(0, 1)$ is

$$y-1 = \left(\frac{dy}{dx}\right)_P (x-0) \Rightarrow y-1 = -1(x-0) \Rightarrow x+y = 1$$

EXAMPLE 22 *Find all the tangents to the curve* $y = \cos(x+y)$, $-2\pi \le x \le 2\pi$ *that are parallel to the line* $x+2y=0$. **[NCERT EXEMPLAR, CBSE 2016, 2017]**

SOLUTION Let the point of contact of one of the tangents be (x_1, y_1). Then, (x_1, y_1) lies on $y=\cos(x+y)$

$$\therefore \quad y_1 = \cos(x_1+y_1) \qquad \text{...(i)}$$

Since the tangents are parallel to the line $x+2y=0$. Therefore,

Slope of the tangent at (x_1, y_1) = (Slope of the line $x+2y=0$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-1}{2}$$

The equation of the curve is $y = \cos(x+y)$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\sin(x+y)\left(1+\frac{dy}{dx}\right) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\sin(x_1+y_1)\left\{1+\left(\frac{dy}{dx}\right)_{(x_1, y_1)}\right\}$$

$$\Rightarrow \quad -\frac{1}{2} = -\sin(x_1+y_1)\left(1-\frac{1}{2}\right) \Rightarrow \sin(x_1+y_1) = 1 \qquad \text{...(ii)}$$

Squaring (i) and (ii) and then adding, we get

$$\cos^2(x_1+y_1)+\sin^2(x_1+y_1) = {y_1}^2+1 \Rightarrow 1 = {y_1}^2+1 \Rightarrow y_1 = 0$$

Putting $y_1 = 0$ in (i) and (ii), we get

$$\cos x_1 = 0 \text{ and } \sin x_1 = 1 \Rightarrow x_1 = \frac{\pi}{2}, \frac{-3\pi}{2} \qquad [\because -2\pi \le x_1 \le 2\pi]$$

Hence, the points of contact are $(\pi/2, 0)$ and $(-3\pi/2, 0)$.

The slope of the tangent is $(-1/2)$. Therefore, equations of tangents at $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{3\pi}{2}, 0\right)$ are

$$y-0 = -\frac{1}{2}\left(x-\frac{\pi}{2}\right) \text{ and } y-0 = -\frac{1}{2}\left(x+\frac{3\pi}{2}\right) \text{respectively}$$

or, $\quad 2x+4y-\pi = 0$ and $2x+4y+3\pi = 0$ respectively.

Type III ON FINDING THE EQUATION OF THE CURVE

EXAMPLE 23 *The curve $y = ax^3+bx^2+cx+5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at the point Q where its gradient is 3. Find the equation of the curve completely.*

SOLUTION The equation of the given curve is

$$y = ax^3+bx^2+cx+5 \Rightarrow \frac{dy}{dx} = 3ax^2+2bx+c$$

Since the curve $y = ax^3+bx^2+cx+5$ touches the x-axis at $P(-2, 0)$. This means that the curve passes through $P(-2, 0)$ and x-axis is the tangent at $P(-2, 0)$.

$$\therefore \quad 0 = -8a+4b-2c+5 \text{ and, } \left(\frac{dy}{dx}\right)_P = 0$$

$$\Rightarrow \quad 8a-4b+2c = 5 \text{ and, } 3a(-2)^2+2b\times(-2)+c = 0$$

$$\Rightarrow \quad 8a-4b+2c = 5 \quad \text{...(i)} \quad \text{and,} \quad 12a-4b+c = 0 \qquad \text{...(ii)}$$

The curve $y = ax^3+bx^2+cx+5$ meets y-axis at Q. Putting $x = 0$ in $y = ax^3+bx^2+cx+5$, we get: $y = 5$. Thus, the coordinates of Q are $(0, 5)$. It is given that the gradient of the curve at Q is 3.

$$\therefore \quad \left(\frac{dy}{dx}\right)_Q = 3 \Rightarrow 3a\times 0+2b\times 0+c = 3 \Rightarrow c = 3$$

Putting $c = 3$ in (i) and (ii), we get: $8a-4b = -1$ and $12a-4b = -3$.

Solving these two equations, we get: $a = -\frac{1}{2}$ and $b = -\frac{3}{4}$.

Substituting the values of a, b and c in the equation of the curve, we obtain

$$y = -\frac{1}{2}x^3-\frac{3}{4}x^2+3x+5 \text{ as the equation of the curve.}$$

EXAMPLE 24 *Determine the quadratic curve $y = f(x)$ if it touches the line $y = x$ at the point $x = 1$ and passes through the point $(-1, 0)$.*

SOLUTION Let the required quadratic curve be $y = ax^2+bx+c$ ...(i)

It passes through $(-1, 0)$. Therefore, $0 = a-b+c$ ...(ii)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = 2ax+b \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 2a+b$$

Since the line $y = x$ touches curve (i) at $x = 1$. Therefore, (Slope of the tangent at $x=1$) = (Slope of the line $y=x$)

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = 1 \Rightarrow 2a+b = 1 \qquad \text{...(iii)}$$

Putting $x = 1$ in $y = x$, we get $y=1$. Thus, the curve (i) passes through (1, 1).

$$\therefore \quad 1 = a+b+c \qquad \text{...(iv)}$$

Solving (ii), (iii) and (iv), we get: $a=\frac{1}{4}$, $b=\frac{1}{2}$ and $c=\frac{1}{4}$.

Substituting these values in (i), we get $y = \frac{x^2}{4}+\frac{x}{2}+\frac{1}{4}$ as the required quadratic curve.

EXERCISE 15.2

BASIC

1. Find the equation of the tangent to the curve $\sqrt{x}+\sqrt{y}=a$, at the point $(a^2/4, a^2/4)$.
2. Find the equation of the normal to $y=2x^3-x^2+3$ at (1, 4).
3. Find the equations of the tangent and the normal to the following curves at the indicated points:
   (i) $y=x^4-bx^3+13x^2-10x+5$ at (0, 5) [NCERT]
   (ii) $y=x^4-6x^3+13x^2-10x+5$ at $x=1$ [NCERT, CBSE 2011]
   (iii) $y=x^2$ at (0, 0) [NCERT]
   (iv) $y=2x^2-3x-1$ at (1, − 2)
   (v) $y^2=\frac{x^3}{4-x}$ at (2, − 2)
   (vi) $y=x^2+4x+1$ at $x=3$ [CBSE 2004]
   (vii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(a\cos\theta, b\sin\theta)$
   (viii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(a\sec\theta, b\tan\theta)$
   (ix) $y^2=4ax$ at $(a/m^2, 2a/m)$
   (x) $c^2(x^2+y^2)=x^2y^2$ at $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$
   (xi) $xy=c^2$ at $(ct, c/t)$
   (xii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at (x_1, y_1)
   (xiii) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at (x_0, y_0) [NCERT]
   (xiv) $x^{2/3}+y^{2/3}=2$ at (1, 1) [NCERT]
   (xv) $x^2=4y$ at (2, 1)
   (xvi) $y^2=4x$ at (1, 2) [NCERT]
   (xvii) $4x^2+9y^2=36$ at $(3\cos\theta, 2\sin\theta)$ [CBSE 2011]
   (xviii) $y^2=4ax$ at (x_1, y_1) [CBSE 2012]
   (xix) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at $(\sqrt{2}a, b)$ [CBSE 2014]
4. Find the equation of the tangent to the curve $x=\theta+\sin\theta$, $y=1+\cos\theta$ at $\theta=\pi/4$.
5. Find the equations of the tangent and the normal to the following curves at the indicated points:
   (i) $x=\theta+\sin\theta$, $y=1+\cos\theta$ at $\theta=\pi/2$.
   (ii) $x=\frac{2at^2}{1+t^2}$, $y=\frac{2at^3}{1+t^2}$ at $t=1/2$.
   (iii) $x=at^2$, $y=2at$ at $t=1$.
   (iv) $x=a\sec t$, $y=b\tan t$ at t.
   (v) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ at θ.
   (vi) $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$ [NCERT EXEMPLAR, CBSE 2016]

6. Find the equation of the normal to the curve $x^2 + 2y^2 - 4x - 6y + 8 = 0$ at the point whose abscissa is 2.

7. Find the equation of the normal to the curve $ay^2 = x^3$ at the point (am^2, am^3). [CBSE 2012, NCERT]

8. Find the equation of the tangent to the curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$. [CBSE 2008]

9. At what points will be tangents to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to x-axis? Also, find the equations of the tangents to the curve at these points. [CBSE 2011]

BASED ON LOTS

10. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b. [CBSE 2016]

11. Find the equation of the tangent line to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$.

12. Find an equation of normal line to the curve $y = x^3 + 2x + 6$ which is parallel to the line $x + 14y + 4 = 0$. [CBSE 2013]

13. Determine the equation(s) of tangent(s) line to the curve $y = 4x^3 - 3x + 5$ which are perpendicular to the line $9y + x + 3 = 0$.

14. Find the equation of a normal to the curve $y = x \log_e x$ which is parallel to the line $2x - 2y + 3 = 0$.

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$ (ii) perpendicular to the line $5y - 15x = 13$.

[NCERT, CBSE 2014]

16. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$. [NCERT, CBSE 2005, 2009, 2019]

17. Find the equation of the tangent to the curve $x^2 + 3y - 3 = 0$, which is parallel to the line $y = 4x - 5$. [CBSE 2005]

18. Find the equation of the tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $(4/3, 0)$. [CBSE 2013]

BASED ON HOTS

19. Find the equations of all lines having slope 2 and that are tangent to the curve $y = \dfrac{1}{x-3}$, $x \neq 3$. [NCERT]

20. Find the equations of all lines of slope zero and that are tangent to the curve $y = \dfrac{1}{x^2 - 2x + 3}$. [NCERT]

21. Prove that $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$ touches the straight line $\dfrac{x}{a} + \dfrac{y}{b} = 2$ for all $n \in N$, at the point (a, b). [CBSE 2007 C]

ANSWERS

1. $x + y = a^2/2$

2. $x + 4y = 17$

3.

	Tangent	Normal
(i)	$y + 10x - 5 = 0$	$x - 10y + 50 = 0$
(ii)	$2x - y + 1 = 0$	$x + 2y - 7 = 0$
(iii)	$y = 0$	$x = 0$

	Tangent	Normal
(iv)	$x-y-3=0$	$x+y+1=0$
(v)	$2x+y-2=0$	$x-2y-6=0$
(vi)	$10x-y-8=0$	$x+10y-223=0$
(vii)	$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$	$ax\sec\theta-by\,\text{cosec}\,\theta=a^2-b^2$
(viii)	$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$	$ax\cos\theta+by\cot\theta=a^2+b^2$
(ix)	$m^2x-my+a=0$	$m^2x+m^3y-2am^2-a=0$
(x)	$x\cos^3\theta+y\sin^3\theta=c$	$x\sin^3\theta-y\cos^3\theta+2c\cot 2\theta=0$
(xi)	$x+yt^2=2ct$	$xt^3-ty=ct^4-c$
(xii)	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$	$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$
(xiii)	$\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=1$	$\frac{a^2x}{x_0}+\frac{b^2y}{y_0}=a^2+b^2$
(xiv)	$x+y-2=0$	$y-x=0$
(xv)	$x-y-1=0$	$x+y-3=0$
(xvi)	$x-y+1=0$	$x+y-3=0$
(xvii)	$2x\cos\theta+3y\sin\theta=6$	$3x\sin\theta-2y\cos\theta-5\sin\theta\cos\theta=0$
(xviii)	$yy_1=2a(x+x_1)$	$y-y_1=\frac{-y_1}{2a}(x-x_1)$
(xix)	$\frac{\sqrt{2}x}{a}-\frac{y}{b}=1$	$\frac{ax}{\sqrt{2}}+by=a^2+b^2$

4. $\left(y-1-\frac{1}{\sqrt{2}}\right)=(1-\sqrt{2})\left(x-\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)$

5.

	Tangent	*Normal*
(i)	$2x+2y-\pi-4=0$	$2x-2y=\pi$
(ii)	$13x-16y-2a=0$	$16x+13y-9a=0$
(iii)	$x-y+a=0$	$x+y=3a$
(iv)	$bx\sec t-ay\tan t=ab$	$ax\cos t+by\cot t=a^2+b^2$
(v)	$y=(x-a\theta)\tan(\theta/2)$	$(y-2a)\tan(\theta/2)+x-a\theta=0$
(vi)	$4(y\cos^3\theta-x\sin^3\theta)=3\sin 4\theta$	

6. $x=2$

7. $2x+3my-am^2(2+3m^2)=0$

8. $2\sqrt{2}x-3y-2=0$

9. $(2,7),(3,6)$

10. $a=2, b=-7$

11. $12x-4y-65=0$

12. $x+14y+86=0,\ x+14y-254=0$

13. $9x-y-3=0,\ 9x-y+13=0$

14. $x-y=3e^{-2}$

15. (i) $2x-y+3=0$ (ii) $12x+36y-227=0$

16. $48x-24y=23$

17. $4x-y+13=0$

18. $y=3x-4$

19. There is no tangent to the curve that has slope 2.

20. $y=1/2$

3. (i) The equation of the curve is $y = x^4 - bx^3 + 13x^2 - 10x + 5$.

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 3bx^2 + 26x - 10 \Rightarrow \left(\frac{dy}{dx}\right)_{(0,5)} = -10$$

The equation of tangent at $(0, 5)$ is

$$y - 5 = \left(\frac{dy}{dx}\right)_{(0,5)} (x - 0) \Rightarrow y - 5 = -10\,(x - 0) \Rightarrow 10x + y - 5 = 0$$

The equation of the normal at $(0, 5)$ is

$$y - 5 = -\frac{1}{\left(\frac{dy}{dx}\right)_{0,5)}} (x - 0) \Rightarrow y - 5 = \frac{1}{10}(x - 0) \Rightarrow x - 10y + 50 = 0$$

(ii) The equation of the curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ ...(i)

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10 \Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = 4 - 18 + 26 - 10 = 2$$

Putting $x = 1$ in (i), we get $y = 3$. The equation of the tangent at $(1, 3)$ is

$$y - 3 = \left(\frac{dy}{dx}\right)_{x=1} (x - 1) \Rightarrow y - 3 = 2\,(x - 1) \Rightarrow 2x - y + 1 = 0$$

The equation of the normal at $(1, 3)$ is

$$y - 3 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x=1}} (x - 1) \Rightarrow y - 3 = -\frac{1}{2}(x - 1) \Rightarrow x + 2y - 7 = 0$$

(iii) The equation of the curve is $y = x^2$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2x \Rightarrow \left(\frac{dy}{dx}\right)_{(0,0)} = 0$$

So, the tangent at $(0, 0)$ is parallel to x-axis and hence the normal there at is parallel to y-axis. So, their equations are $y = 0$ and $x = 0$ respectively.

(xiii) The equation of the curve is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ...(i)

Point $P(x_0, y_0)$ lies on (i). Therefore,

$$\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1 \qquad \text{...(ii)}$$

Differentiating (i) with respect to x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_P = \frac{b^2 x_0}{a^2 y_0}$$

The equation of the tangent at $P\,(x_0, y_0)$ is

$$y - y_0 = \left(\frac{dy}{dx}\right)_P (x - x_0) \Rightarrow y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\Rightarrow \frac{yy_0 - y_0^2}{b^2} = \frac{xx_0 - x_0^2}{a^2} \Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1 \qquad \text{[Using (ii)]}$$

The equation of the normal at $P(x_0, y_0)$ is

$$y - y_0 = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x_0)$$

$$\Rightarrow \quad y - y_0 = -\frac{a^2 y_0}{b^2 x_0}(x - x_0) \Rightarrow \frac{b^2}{y_0}(y - y_0) = -\frac{a^2}{x_0}(x - x_0) \Rightarrow \frac{a^2 x}{x_0} + \frac{b^2 y}{y_0} = a^2 + b^2$$

(xiv) We have,

$$x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,1)} = -1$$

The equation of the tangent at (1,1) is

$$y - 1 = \left(\frac{dy}{dx}\right)_{(1,1)}(x - 1) \Rightarrow y - 1 = -1(x - 1) \Rightarrow x + y - 2 = 0$$

The equation of the normal at (1, 1) is

$$y - 1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,1)}}(x - 1) \Rightarrow y - 1 = \frac{-1}{-1}(x - 1) \Rightarrow x = y$$

(xvi) The equation of the curve is $y^2 = 4x$. Differentiating with respect to x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{2} = 1$$

The equation of the tangent at (1, 2) is

$$y - 2 = \left(\frac{dy}{dx}\right)_{(1,2)}(x - 1) \Rightarrow y - 2 = (x - 1) \Rightarrow x - y + 1 = 0$$

The equation of the normal at (1, 2) is

$$y - 2 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(1,2)}}(x - 1) \Rightarrow y - 2 = -\frac{1}{1}(x - 1) \Rightarrow x + y - 3 = 0$$

7. We have, $ay^2 = x^3$. Differentiating with respect to x, we get

$$2ay\frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{3x^2}{2ay} \Rightarrow \left(\frac{dy}{dx}\right)_{(am^2, am^3)} = \frac{3a^2 m^4}{2a^2 m^3} = \frac{3m}{2}$$

The equation of the normal at (am^2, am^3) is

$$y - am^3 = -\frac{2}{3m}(x - am^2) \text{ or, } 2x + 3my - am^2(2 + 3m^2) = 0$$

15. The equation of the curve is $y = x^2 - 2x + 7$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = 2x - 2$$

(i) Let $P(x_1, y_1)$ be a point on $y = x^2 - 2x + 7$ such that tangent at P is parallel to the line $2x - y + 9 = 0$. Then,

$$\left(\frac{dy}{dx}\right)_P = 2 \Rightarrow 2x_1 - 2 = 2 \Rightarrow x_1 = 2$$

Since $P(x_1, y_1)$ lies on $y = x^2 - 2x + 7$. Therefore,

$$y_1 = x_1^2 - 2x_1 + 7 \Rightarrow y_1 = 4 - 4 + 7 = 7$$

Hence, required point is (2, 7). The equation of the tangent at (2, 7) is

$$y - 7 = \left(\frac{dy}{dx}\right)_P (x-2) \Rightarrow y - 7 = 2(x-2) \Rightarrow 2x - y + 3 = 0$$

(ii) If the tangent at $P(x_1, y_1)$ is perpendicular to the line $5y - 15x = 13$. Then,

$$\left(\frac{dy}{dx}\right)_P \times 3 = -1 \Rightarrow (2x_1 - 2) \times 3 = -1 \Rightarrow x_1 = \frac{5}{6}$$

Since (x_1, y_1) lies on $y = x^2 - 2x + 7$.

$$\therefore \quad y_1 = x_1^2 - 2x_1 + 7 \Rightarrow y_1 = \frac{25}{36} - \frac{5}{3} + 7 = \frac{217}{36}$$

The equation of the tangent at $P\left(\frac{5}{6}, \frac{217}{36}\right)$ is

$$y - \frac{217}{36} = -\frac{1}{3}\left(x - \frac{5}{6}\right) \text{ or, } 12x + 36y - 227 = 0 \qquad \left[\because \left(\frac{dy}{dx}\right)_P = -\frac{1}{3}\right]$$

16. Let (x_1, y_1) be the point of contact of tangent to the curve $y = \sqrt{3x-2}$ which is parallel to the line $4x - 2y + 5 = 0$. Then,

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (\text{Slope of the line } 4x - 2y + 5 = 0) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2 \qquad \text{...(i)}$$

Now, $y = \sqrt{3x-2} \Rightarrow \frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3}{2\sqrt{3x_1 - 2}}$ ...(ii)

From (i) and (ii), we get

$$\frac{3}{2\sqrt{3x_1-2}} = 2 \Rightarrow 9 = 16(3x_1 - 2) \Rightarrow x_1 = \frac{41}{48}$$

Since (x_1, y_1) lies on $y = \sqrt{3x-2}$. Therefore, $y_1 = \sqrt{3x_1 - 2} \Rightarrow y_1 = \sqrt{\frac{41}{16} - 2} = \frac{3}{4}$

So, the point of contact is $\left(\frac{41}{48}, \frac{3}{4}\right)$. The equation of tangent at $\left(\frac{41}{48}, \frac{3}{4}\right)$ is

$$y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right) \text{ or, } 48x - 24y = 23$$

19. Let (x_1, y_1) be the point of contact of a line of slope 2 which touches the curve $y = \frac{1}{x-3}$, $x \neq 3$. Now,

$$y = \frac{1}{x-3} \Rightarrow \frac{dy}{dx} = -\frac{1}{(x-3)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{1}{(x_1-3)^2}$$

But, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2$. Therefore, $-\frac{1}{(x_1-3)^2} = 2 \Rightarrow 2(x_1-3)^2 = -1$, which is not possible as LHS is positive and RHS is negative. Hence, there is no tangent line of slope 2 to the given curve.

20. Let $P(x_1, y_1)$ be the point of contact of a line of slope zero which touches the curve $y = \frac{1}{x^2 - 2x + 3}$ at point P. The equation of the curve is $y = \frac{1}{x^2 - 2x + 3}$. Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{-2(x-1)}{(x^2-2x+3)^2} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2(x_1-1)}{(x_1^{\ 2}-2x_1+3)^2}$$

It is given that $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \quad \Rightarrow \quad \frac{-2(x_1-1)}{(x_1^{\ 2}-2x_1+3)^2} = 0 \Rightarrow x_1 = 1$

Since (x_1, y_1) lies on $y = \frac{1}{x^2-2x+3} \Rightarrow y_1 = \frac{1}{x_1^{\ 2}-2x_1+3} \Rightarrow y_1 = \frac{1}{1-2+3} = \frac{1}{2}$

Hence, the equation of the tangent is $y - \frac{1}{2} = 0(x-1)$ or, $y = \frac{1}{2}$.

21. We have, $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

Differentiating both sides with respect to x, we get

$$n\left(\frac{x}{a}\right)^{n-1}\frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1}\frac{1}{b}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b}{a}\left(\frac{x}{a}\right)^{n-1}\left(\frac{b}{y}\right)^{n-1} \Rightarrow \left(\frac{dy}{dx}\right)_{(a,b)} = \frac{-b}{a}$$

The equation of the tangent at (a, b) is

$$y - b = \frac{-b}{a}(x-a) \Rightarrow ay - ab = -bx + ab \Rightarrow bx + ay = 2ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

Hence, $\frac{x}{a} + \frac{y}{b} = 2$ touches the given curve at (a, b) for all $n \in N$.

15.4 ANGLE OF INTERSECTION OF TWO CURVES

DEFINITION *The angle of intersection of two curves is defined to be the angle between the tangents to the two curves at their point of intersection.*

Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$ respectively. Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 respectively at their common point of intersection. Then, the angle ϕ between PT_1 and PT_2 is the angle of intersection of C_1 and C_2. Let ψ_1 and ψ_2 be the angles made by PT_1 and PT_2 with the positive direction of x-axis in anticlockwise sense. Then,

$$m_1 = \tan\psi_1 \Rightarrow m_1 = (\text{Slope of the tangent to } y = f(x) \text{ at } P) = \left(\frac{dy}{dx}\right)_{C_1}$$

and, $$m_2 = \tan\psi_2 \Rightarrow m_2 = (\text{Slope of the tangent to } y = g(x) \text{ at } P) = \left(\frac{dy}{dx}\right)_{C_2}$$

From Fig. 15.3, it is evident that

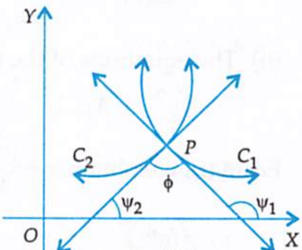

Fig.15.3

$$\phi = \psi_1 - \psi_2$$

$$\Rightarrow \quad \tan\phi = \tan(\psi_1 - \psi_2)$$

$$\Rightarrow \quad \tan\phi = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}$$

$$\Rightarrow \quad \tan\phi = \frac{\left(\frac{dy}{dx}\right)_{C_1} - \left(\frac{dy}{dx}\right)_{C_2}}{1 + \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2}}$$

The other angle between the tangents is $180° - \phi$. Generally, the smaller of these two angles is taken to be the angle of intersection.

ORTHOGONAL CURVES *If the angle of intersection of two curves is a right angle, the two curves are said to intersect orthogonally and the curves are called orthogonal curves.*

If the curves are orthogonal, then $\phi = \frac{\pi}{2}$.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow \left(\frac{dy}{dx}\right)_{C_1} \times \left(\frac{dy}{dx}\right)_{C_2} = -1$$

REMARK *If the angle of intersection of two curves is zero, then* $\left(\frac{dy}{dx}\right)_{C_1} = \left(\frac{dy}{dx}\right)_{C_2}$ *at the point of intersection and the two curve touch each other at the point of intersection.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the angle of intersection of the following curves:*

(i) $xy = 6$ *and* $x^2 y = 12$ (ii) $y^2 = 4x$ *and* $x^2 = 4y$

SOLUTION (i) The equations of the two curves are

$$xy = 6 \quad \text{...(i)} \qquad \text{and,} \quad x^2 y = 12 \quad \text{...(ii)}$$

From (i), we obtain $y = \frac{6}{x}$. Putting this value of y in (ii), we obtain

$$x^2\left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2$$

Putting $x = 2$ in (i) or (ii), we get $y = 3$. Thus, the two curves intersect at P (2, 3).

Differentiating (i) with respect to x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(2,\,3)} = -\frac{3}{2}$$

Differentiating (ii) with respect to x, we get

$$x^2\frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -\frac{2y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(2,\,3)} = -3$$

Let θ be the angle of intersection of curves (i) and (ii) at point P, then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-(3/2) + 3}{1 + (-3/2)(-3)} = \frac{3}{11} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{11}\right)$$

(ii) The equations of the two curves are

$$y^2 = 4x \quad \text{...(i)} \qquad \text{and,} \qquad x^2 = 4y \quad \text{...(ii)}$$

From (i), we obtain $x = \frac{y^2}{4}$. Putting $x = \frac{y^2}{4}$ in (ii), we get

$$\left(\frac{y^2}{4}\right)^2 = 4y \Rightarrow y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0 \Rightarrow y = 0,\ y = 4$$

From (i), when $y = 0$, we get $x = 0$ and when $y = 4$, we get $x = 4$. Thus the two curves intersect at (0, 0) and (4, 4).

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y} \qquad ...(iii)$$

Differentiating (ii) with respect to x, we get

$$2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \qquad ...(iv)$$

Angle of Intersection at (0, 0): From (iii), we get: $m_1 = \left(\frac{dy}{dx}\right)_{(0,0)} = \infty$. Therefore, the tangent to curve (i) at (0, 0) is parallel to y-axis. From (iv), we get: $m_2 = \left(\frac{dy}{dx}\right)_{(0,0)} = 0$. Therefore, the tangent to curve (ii) at (0, 0) is parallel to x-axis. Hence, the angle between the tangents to two curves at (0, 0) is a right angle. Consequently, the two curves intersect at right angle at (0, 0).

Angle of Intersection at (4, 4): From (iii), we obtain: $m_1 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{2}{4} = \frac{1}{2}$.

From (iv), we obtain: $m_2 = \left(\frac{dy}{dx}\right)_{(4,4)} = \frac{4}{2} = 2$.

Let θ be the angle of intersection of the two curves. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{2-(1/2)}{1+2\times(1/2)}\right| = \frac{3}{4}.$$

EXAMPLE 2 *Find the angle between the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at their point of intersection other than the origin.* **[CBSE 2016]**

SOLUTION The equations of two parabolas are $y^2 = 4ax$ and $x^2 = 4by$.

Now, $x^2 = 4by \Rightarrow y = \frac{x^2}{4b}$. Substituting this value of y in $y^2 = 4ax$, we obtain

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \Rightarrow x^4 - 64ab^2\, x = 0 \Rightarrow x(x^3 - 64ab^2) = 0 \Rightarrow x = 0,\, x^3 = 64ab^2$$

$$\Rightarrow \quad x = 0,\, x = 4\,a^{1/3}\,b^{2/3}$$

Putting $x = 0$ and $x = 4a^{1/3}\,b^{2/3}$ successively in $y = \frac{x^2}{4b}$, we get:

$y = 0$ and $y = 4a^{2/3}\,b^{1/3}$ respectively.

Thus, the two curves intersect at $P(4a^{1/3}\,b^{2/3}, 4a^{2/3}\,b^{1/3})$ other than the origin O (0, 0).

Now, $y^2 = 4ax$ and $x^2 = 4by$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 4a \text{ and } 2x = 4b\frac{dy}{dx} \qquad \text{[Differentiating both with respect to } x]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y} \text{ and } \frac{dy}{dx} = \frac{x}{2b}$$

$$\Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_P = \frac{2a}{4a^{2/3}\,b^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3} \text{ and } m_2 = \left(\frac{dy}{dx}\right)_P = \frac{4a^{1/3}\,b^{2/3}}{2b} = 2\left(\frac{a}{b}\right)^{1/3}$$

Let θ be the angle between the tangents to the parabolas $y^2 = 4ax$ and $x^2 = 4by$ at P. Then,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad \tan\theta = \left|\frac{\frac{1}{2}\left(\frac{a}{b}\right)^{1/3} - 2\left(\frac{a}{b}\right)^{1/3}}{1 + \frac{1}{2}\left(\frac{a}{b}\right)^{1/3} \times 2\left(\frac{a}{b}\right)^{1/3}}\right| = \left|\frac{-\frac{3}{2}\left(\frac{a}{b}\right)^{1/3}}{1 + \left(\frac{a}{b}\right)^{2/3}}\right| = \left|\frac{3a^{1/3}b^{1/3}}{2\,(a^{2/3} + b^{2/3})}\right|$$

$$\Rightarrow \quad \theta = \tan^{-1}\left\{\frac{3\,(ab)^{1/3}}{2\,(a^{2/3} + b^{2/3})}\right\}$$

EXAMPLE 3 *Show that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.*

[CBSE 2004, 2005, 2013]

SOLUTION The given curves are

$$x = y^2 \quad ...(i) \qquad \text{and,} \qquad xy = k \quad ...(ii)$$

From (i), we obtain $x = y^2$. Putting this value of x in (ii), we obtain: $y^3 = k \Rightarrow y = k^{1/3}$.

Putting $y = k^{1/3}$ in (i), we get $x = k^{2/3}$. So, the two curves intersect at the point $P\,(k^{2/3}, k^{1/3})$.

Differentiating (i) with respect to x, we get

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_P = \frac{1}{2k^{1/3}}$$

Differentiating (ii) with respect to x, we get

$$1 \cdot y + x\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_P = -\frac{k^{1/3}}{k^{2/3}} = -\frac{1}{k^{1/3}}$$

For the curves (i) and (ii) to cut at right angles at P, we must have

$$m_1 m_2 = -1 \Rightarrow \frac{1}{2k^{1/3}} \times -\frac{1}{k^{1/3}} = -1 \Rightarrow 2k^{2/3} = 1 \Rightarrow (2k^{2/3})^3 = 1^3 \Rightarrow 8k^2 = 1.$$

EXAMPLE 4 *Find the values of p for which the curves $x^2 = 9p\,(9 - y)$ and $x^2 = p\,(y + 1)$ cut each other at right angles.*

[CBSE 2015]

SOLUTION The equations of the given curves are

$$x^2 = 9p\,(9 - y) \quad ...(i) \qquad \text{and,} \qquad x^2 = p\,(y + 1) \quad ...(ii)$$

To find the coordinates of the point(s) of intersection of (i) and (ii), we solve the two equations simultaneously. On eliminating x^2, we obtain

$$9p\,(9 - y) = p\,(y + 1) \Rightarrow 81 - 9y = y + 1 \Rightarrow 10y = 80 \Rightarrow y = 8.$$

Putting $y = 8$ in (i) or (ii), we obtain: $x^2 = 9p \Rightarrow x = \pm\, 3\sqrt{p}$.

Thus, curves (i) and (ii) intersect at $P\,(3\sqrt{p}, 8)$ and $Q\,(-3\sqrt{p}, 8)$.

Differentiating (i) and respect to x, we obtain

$$2x = -9p\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{2x}{9p} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{C_1} = -\frac{2 \times 3\sqrt{p}}{9p} = -\frac{2}{3\sqrt{p}}$$

Differentiating (ii) with respect to x, we obtain

$$2x = p\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{p} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{C_2} = \frac{2(3\sqrt{p})}{p} = \frac{6}{\sqrt{p}}.$$

If curves (i) and (ii) cut each other at P at right angles, then

$$m_1 m_2 = -1 \Rightarrow \frac{-2}{3\sqrt{p}} \times \frac{6}{\sqrt{p}} = -1 \Rightarrow p = 4$$

Similarly, by using the condition of orthogonality of the curves at Q, we obtain $p = 4$.

Hence, the two curves cut each other at right angles, if $p = 4$.

EXAMPLE 5 *Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.* **[CBSE 2002]**

SOLUTION The given curves are

$$xy = a^2 \quad ...(i) \qquad x^2 + y^2 = 2a^2 \quad ..(ii)$$

From (i), we get $y = \frac{a^2}{x}$. Substituting this value of y in equation (ii), we get

$$x^2 + \frac{a^4}{x^2} = 2a^2 \Rightarrow x^4 - 2a^2x^2 + a^4 = 0 \Rightarrow (x^2 - a^2)^2 = 0 \Rightarrow x = \pm a$$

From (i), we get: $y = a$ for $x = a$ and, $y = -a$ for $x = -a$. Thus, the two curves intersect at $P(a, a)$ and $Q(-a, -a)$.

Differentiating both sides of curve (i) with respect to x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x} \quad ...(iii)$$

Differentiating both sides of curve (ii) with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \quad ...(iv)$$

Angle of intersection at $P(a, a)$: Substituting $x = a$, $y = a$ in (iii) and (iv), we get

$$\left(\frac{dy}{dx}\right)_{C_1} = -\frac{a}{a} = -1 \text{ and, } \left(\frac{dy}{dx}\right)_{C_2} = -\frac{a}{a} = -1$$

Clearly, $\left(\frac{dy}{dx}\right)_{C_1} = \left(\frac{dy}{dx}\right)_{C_2}$ at P. So, the two curves touch each other at P.

Similarly, it can be shown that the two curve touch each other at Q.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Show that the condition that the curves*

$$ax^2 + by^2 = 1 \quad ...(i) \qquad \textit{and} \qquad a'x^2 + b'y^2 = 1 \quad ...(ii)$$

Should intersect orthogonally is that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$. **[NCERT EXEMPLAR]**

SOLUTION Let (x_1, y_1) be the point of intersection of the given curves. Then,

$$a{x_1}^2 + b{y_1}^2 = 1 \quad ...(iii)$$

$$a'{x_1}^2 + b'{y_1}^2 = 1 \quad ...(iv)$$

Differentiating (i) with respect to x, we get

$$2ax + 2by\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{ax}{by} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{ax_1}{by_1} \quad \text{...(v)}$$

Differentiating (ii) with respect to x, we obtain

$$2a'x + 2b'y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{a'x}{b'y} \Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{a'x_1}{b'y_1} \quad \text{...(vi)}$$

The two curves will intersect orthogonally, if

$$m_1 m_2 = -1 \Rightarrow -\frac{ax_1}{by_1} \times -\frac{a'x_1}{b'y_1} = -1 \Rightarrow aa'x_1^2 = -bb'y_1^2 \quad \text{...(vii)}$$

Subtracting (iv) from (iii), we obtain

$$(a - a')x_1^2 = -(b - b')y_1^2 \quad \text{...(viii)}$$

Dividing (viii) by (vii), we get: $\frac{a - a'}{aa'} = \frac{b - b'}{bb'} \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

EXAMPLE 7 *If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then prove that $a^2\cos^2\alpha + b^2\sin^2\alpha = p^2$.* **[NCERT EXEMPLAR]**

SOLUTION Suppose the straight line $x\cos\alpha + y\sin\alpha = p$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$. Then it is the equation of tangent to the given curve at $P(x_1, y_1)$. But, the equation of tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$ [See Example 5 on page 15.16]

Thus, equations $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ and $x\cos\alpha + y\sin\alpha = p$ represent the same line.

$$\therefore \quad \frac{x_1/a^2}{\cos\alpha} = \frac{y_1/b^2}{\sin\alpha} = \frac{1}{p} \Rightarrow x_1 = \frac{a^2\cos\alpha}{p}, \; y_1 = \frac{b^2\sin\alpha}{p} \quad \text{...(i)}$$

The point $P(x_1, y_1)$ lies on the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\therefore \quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \Rightarrow \frac{a^4\cos^2\alpha}{p^2a^2} + \frac{b^4\sin^2\alpha}{p^2b^2} = 1 \Rightarrow a^2\cos^2\alpha + b^2\sin^2\alpha = p^2 \quad \text{[Using (i)]}$$

EXAMPLE 8 *Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points on the curve* $\log(x^2 + y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$.

SOLUTION The equation of the curve is $\log x^2 + y^2) = k\tan^{-1}\left(\frac{y}{x}\right)$. Differentiating with respect to x, we get

$$\frac{1}{x^2+y^2}\left\{2x+2y\frac{dy}{dx}\right\} = k\,\frac{1}{1+\frac{y^2}{x^2}}\left\{\frac{x\frac{dy}{dx}-y}{x^2}\right\}$$

$$\Rightarrow \quad 2\left\{x+y\frac{dy}{dx}\right\} = k\left\{x\frac{dy}{dx}-y\right\}$$

$$\Rightarrow \quad 2x+ky = (kx-2y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x+ky}{kx-2y}$$

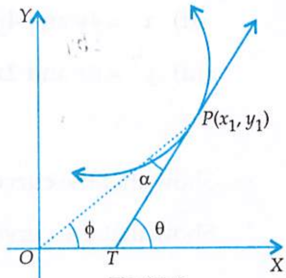
Fig.15.4

Let the coordinates of P be (x_1, y_1). Then, $\left(\frac{dy}{dx}\right)_P = \frac{2x_1+ky_1}{kx_1-2y_1}$.

If the tangent at P makes an angle θ with x-axis, then $\tan\theta = \frac{2x_1+ky_1}{kx_1-2y_1}$.

Suppose OP makes an angle ϕ with x-axis. Then, $\tan\phi$ = Slope of $OP = \frac{y_1}{x_1}$.

Let α be the angle between OP and PT. Then,

$$\theta = \alpha+\phi \Rightarrow \alpha = \theta-\phi \Rightarrow \tan\alpha = \tan(\theta-\phi) \Rightarrow \tan\alpha = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

$$\Rightarrow \quad \tan\alpha = \frac{\frac{2x_1+ky_1}{kx_1-2y_1}-\frac{y_1}{x_1}}{1+\frac{2x_1+ky_1}{kx_1-2y_1}\times\frac{y_1}{x_1}} = \frac{2x_1^2+kx_1y_1-kx_1y_1+2y_1^2}{kx_1^2-2x_1y_1+2x_1y_1+ky_1^2} = \frac{2}{k}$$

$$\Rightarrow \quad \alpha = \tan^{-1}\left(\frac{2}{k}\right) = \text{Constant.}$$

EXERCISE 15.3

BASIC

1. Find the angle of intersection of the following curves:
   (i) $y^2 = x$ and $x^2 = y$ [NCERT EXE-]
   (ii) $y = x^2$ and $x^2+y^2 = 20$
   (iii) $2y^2 = x^3$ and $y^2 = 32x$
   (iv) $x^2+y^2-4x-1 = 0$ and $x^2+y^2-2y-9 = 0$
   (v) $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and $x^2+y^2 = ab$
   (vi) $x^2+4y^2 = 8$ and $x^2-2y^2 = 2$
   (vii) $x^2 = 27y$ and $y^2 = 8x$
   (viii) $x^2+y^2 = 2x$ and $y^2 = x$
   (ix) $y = 4-x^2$ and $y = x^2$ [NCERT EXEMPLAR]
2. Show that the following set of curves intersect orthogonally:
   (i) $y = x^3$ and $6y = 7-x^2$
   (ii) $x^3-3xy^2 = -2$ and $3x^2y-y^3 = 2$
   (iii) $x^2+4y^2 = 8$ and $x^2-2y^2 = 4$.

3. Show that the following curves intersect orthogonally at the indicated points:

(i) $x^2 = 4y$ and $4y + x^2 = 8$ at (2, 1) (ii) $x^2 = y$ and $x^3 + 6y = 7$ at (1, 1)

(iii) $y^2 = 8x$ and $2x^2 + y^2 = 10$ at $(1, 2\sqrt{2})$

BASED ON LOTS

4. Show that the curves $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$.

5. Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles, if $k^2 = 8$.

[NCERT EXEMPLAR]

6. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other. [NCERT EXEMPLAR]

7. Prove that the curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ touch each other at the point (1, 2).

[NCERT EXEMPLAR]

BASED ON HOTS

8. Find the condition for the following set of curves to intersect orthogonally:

(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ [NCERT EXEMPLAR] (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$.

9. Show that the curves $\frac{x^2}{a^2 + \lambda_1} + \frac{y^2}{b^2 + \lambda_1} = 1$ and $\frac{x^2}{a^2 + \lambda_2} + \frac{y^2}{b^2 + \lambda_2} = 1$ intersect at right angles.

10. If the straight line $x \cos\alpha + y \sin\alpha = p$ touches the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then prove that $a^2 \cos^2\alpha - b^2 \sin^2\alpha = p^2$.

ANSWERS

1. (i) $\frac{\pi}{2}$ and $\tan^{-1}\frac{3}{4}$ (ii) $\tan^{-1}\frac{9}{2}$ (iii) $\frac{\pi}{2}$ and $\tan^{-1}\frac{1}{2}$ (iv) $\frac{\pi}{4}$

(v) $\tan^{-1}\left(\frac{a-b}{\sqrt{ab}}\right)$ (vi) $\tan^{-1} 3$ (vii) $\tan^{-1}\frac{9}{13}$ (viii) $\tan^{-1}\frac{1}{2}$ (ix) $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$

8. (i) $b^2 = a^2$ (ii) $a^2 - b^2 = A^2 + B^2$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The equation of the normal to the curve $y = \tan x$ at (0, 0) is ....
2. The value of 'a' for which $y = x^2 + ax + 25$ touches the axis of x are ....
3. The points on the curve $y = 12x - x^3$ at which the gradient is zero are ....
4. The coordinates of a point on the curve $y = x \log_e x$ at which the normal is parallel to the line $2x - 2y = 3$ are ....
5. The coordinates of the point on the curve $y = 2 + \sqrt{4x+1}$ where tangent has slope $\frac{2}{5}$ are ....

6. The slope of the tangent to the curve $x = 3t^2 + 1,\ y = t^3 - 1$ at $x = 1$ is ....

7. The angle of intersection of the curves $y = x^2$ and $x = y^2$ at (0, 0), is ....

8. The slope of the tangent to the curve $y = b\,e^{-x/a}$ where it crosses y-axis is ....

9. The tangent to the curve $y = e^{2x}$ at (0, 1) cuts x-axis at the point ....

10. The slope of the normal to the curve $y^3 - xy - 8 = 0$ at the point (0, 2) is equal to ....

11. If the normal to the curve $y^2 = 5x - 1$, at the point $(1, -2)$ is of the form $ax - 5y + b = 0$, then $a + b =$ ....

12. If the line $ax + by + c = 0$ is normal to the curve $xy = 1$, then the set of values of $\frac{a}{b}$, is ....

13. If the normal to the curve $y = f(x)$ at (3, 4) makes an angle $\frac{3\pi}{4}$ with positive x-axis, then $f'(3)$ is equal to ....

14. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to x-axis, is ....

15. The coordinates of the point on the curve $y = x^2 - 3x + 2$ where the tangent is perpendicular to the line $x - y = 0$ are ....

16. If slope of tangent to curve $y = x^3$ at a point is equal to ordinate of point, then the point is ....

17. The slope of the normal to the curve $x^2 + y^2 - 2x + 4y - 5 = 0$ at (2, 1) is ....

18. The point on the curve $y^2 = x$, the tangent at which makes an angle of 45° with x-axis is ....

19. The curve $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point ....

20. The equation of the normal to the curve $y^2 = 8x$ at the origin is .... [CBSE 2020]

ANSWERS

1. $y = -x$ 2. ± 10 3. $(-2, -16), (2, 16)$ 4. $(e^{-2}, -2e^{-2})$ 5. (6, 7)

6. 0 7. $\frac{\pi}{2}$ 8. $-\frac{b}{a}$ 9. $\left(-\frac{1}{2}, 0\right)$ 10. -6 11. -10

12. $(-\infty, 0)$ 13. 1 14. $y = 3$ 15. (1, 0) 16. (3, 27) 17. 3

18. $\left(\frac{1}{4}, \frac{1}{2}\right)$ 19. (3, 34) 20. $y = 0$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the point on the curve $y = x^2 - 2x + 3$, where the tangent is parallel to x-axis.

2. Find the slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at $t = 2$.

3. If the tangent line at a point (x, y) on the curve $y = f(x)$ is parallel to x-axis, then write the value of $\frac{dy}{dx}$.

4. Write the value of $\frac{dy}{dx}$, if the normal to the curve $y = f(x)$ at (x, y) is parallel to y-axis.

5. If the tangent to a curve at a point (x, y) is equally inclined to the coordinate axes, then write the value of $\frac{dy}{dx}$.

6. If the tangent line at a point (x, y) on the curve $y = f(x)$ is parallel to y-axis, find the value of $\frac{dx}{dy}$.

7. Find the slope of the normal at the point 't' on the curve $x = \frac{1}{t}$, $y = t$.

8. Write the coordinates of the point on the curve $y^2 = x$ where the tangent line makes an angle $\frac{\pi}{4}$ with x-axis.

9. Write the angle made by the tangent to the curve $x = e^t \cos t$, $y = e^t \sin t$ at $t = \frac{\pi}{4}$ with the x-axis.

10. Write the equation of the normal to the curve $y = x + \sin x \cos x$ at $x = \frac{\pi}{2}$.

11. Find the coordinates of the point on the curve $y^2 = 3 - 4x$ where tangent is parallel to the line $2x + y - 2 = 0$.

12. Write the equation of the tangent to the curve $y = x^2 - x + 2$ at the point where it crosses the y-axis.

13. Write the angle between the curves $y^2 = 4x$ and $x^2 = 2y - 3$ at the point $(1, 2)$.

14. Write the angle between the curves $y = e^{-x}$ and $y = e^x$ at their point of intersection.

15. Write the slope of the normal to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

16. Write the coordinates of the point at which the tangent to the curve $y = 2x^2 - x + 1$ is parallel to the line $y = 3x + 9$.

17. Write the equation of the normal to the curve $y = \cos x$ at $(0, 1)$.

18. Write the equation of the tangent drawn to the curve $y = \sin x$ at the point $(0, 0)$. **[CBSE 2017]**

19. Find the slope of the tangent to the curve $y = 2\sin^2 3x$ at $x = \frac{\pi}{6}$. **[CBSE 2020]**

ANSWERS

1. (1, 2)
2. $\frac{6}{7}$
3. 0
4. 0
5. ± 1
6. 0
7. $\frac{1}{t^2}$
8. $\left(\frac{1}{4}, \frac{1}{2}\right)$
9. $\frac{\pi}{2}$
10. $2x = \pi$
11. $\left(\frac{1}{2}, 1\right)$
12. $x + y - 2 = 0$
13. 0
14. 90°
15. 9
16. (1, 2)
17. $x = 0$
18. $y = x$
19. 0