CHAPTER 24

THE CIRCLE

24.1 DEFINITION

A circle is defined as the locus of a point which moves in a plane such that its distance from a fixed point in that plane is always constant.

The fixed point is called the centre of the *circle* and the constant distance is called the *radius* of the circle.

In Fig. 24.1, P is the moving point, C is the fixed point and CP is equal to the radius.

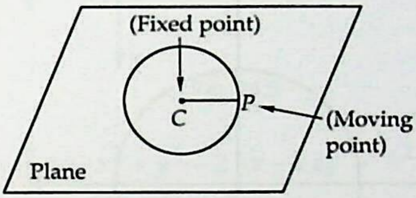

Fig. 24.1

EQUATION OF A CIRCLE By the equation of a circle is meant the equation of the circumference; it is a relation between the coordinates x, y of the moving point P, involving some constants depending upon the position of the centre and the length of the radius. In set theoretical notations it is the set of all points lying on the circumference of the circle.

24.2 STANDARD EQUATION OF A CIRCLE

In this section, we will find the equation of any circle whose centre and radius are given.

Let C be the centre of the circle and its coordinates be (h, k). Let the radius of the circle be a and let $P(x, y)$ be any point on the circumference. Then,

$$CP = a$$

$$\Rightarrow \quad CP^2 = a^2$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = a^2$$

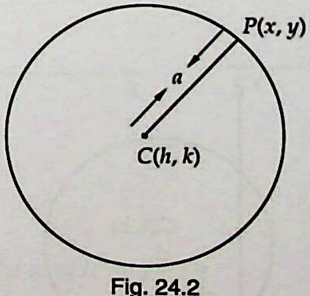

Fig. 24.2

This is the relation between the coordinates of any point on the circumference and hence it is the required equation of the circle having centre at (h, k) and radius equal to a.

NOTE 1 *The above equation is known as the central form of the equation of a circle.*

NOTE 2 *If the centre of the circle is at the origin and radius is a, then from the above form the equation of the circle is* $x^2 + y^2 = a^2$.

ILLUSTRATION 1 *Find the equation of a circle whose centre is (2, –3) and radius 5.*

SOLUTION The equation of the required circle is

$$(x-2)^2+(y+3)^2=5^2 \text{ or, } x^2+y^2-4x+6y-12=0.$$

ILLUSTRATION 2 *Find the equation of a circle whose radius is 6 and the centre is at the origin.*

SOLUTION The equation of the required circle is

$$x^2+y^2=6^2 \text{ or, } x^2+y^2=36.$$

24.3 SOME PARTICULAR CASES

The equation of a circle with centre at (h, k) and radius equal to a, is

$$(x-h)^2+(y-k)^2=a^2 \quad \text{...(i)}$$

(i) *When the centre of the circle coincides with the origin (Fig. 24.3).*

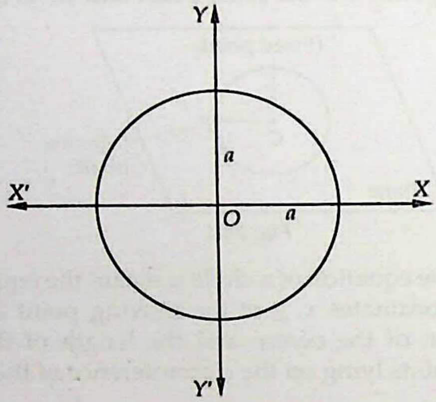

Fig. 24.3

In this case, $h=k=0$. Putting $h=0$, $k=0$ in equation (i), we obtain $x^2 + y^2 = a^2$ as the equation of the circle having centre at the origin and radius equal to 'a'.

(ii) *When the circle passes through the origin (Fig. 24.4):*

Let O be the origin and C (h, k) be the centre of the circle. Draw $CM \perp OX$.

Using Pythagoras Theorem in ΔOCM, we obtain

$$OC^2 = OM^2 + CM^2$$

$$\Rightarrow \quad a^2 = h^2 + k^2$$

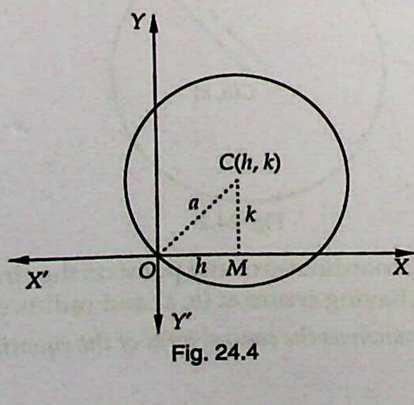

Fig. 24.4

The equation of the circle (i) then becomes

$$(x-h)^2 + (y-k)^2 = h^2 + k^2 \text{ or, } x^2 + y^2 - 2hx - 2ky = 0.$$

(iii) *When the circle touches x-axis (Fig. 24.5):*

Let C (h, k) be the centre of the circle. Since the circle touches the x-axis. Therefore, $a = k$

Hence, the equation of the circle is

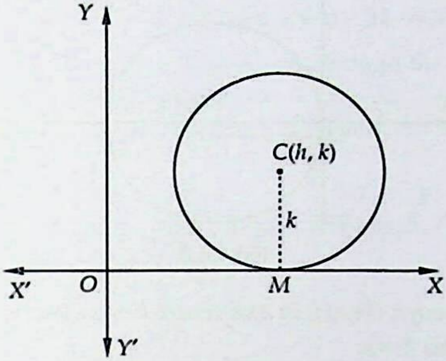

Fig. 24.5

$$(x-h)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2hx - 2ay + h^2 = 0$$

(iv) *When the circle touches y-axis (Fig. 24.6):*

Let C (h, k) be the centre of the circle. Since the circle touches the y-axis. Therefore, $h = a$

Hence, the equation of the circle is

$$(x-a)^2 + (y-k)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax - 2ky + k^2 = 0.$$

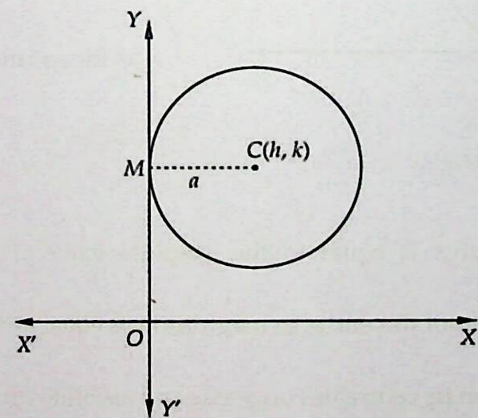

Fig. 24.6

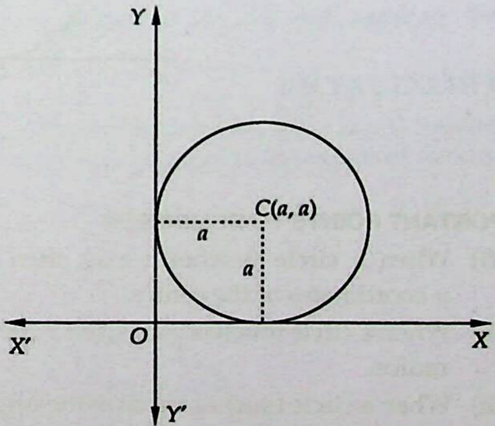

Fig. 24.7

(v) *When the circle touches both the axes (Fig. 24.7):*

In this case we have, $h = k = a$

Hence, the equation of the circle is

$$(x-a)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax - 2ay + a^2 = 0.$$

(vi) *When the circle passes through the origin and centre lies on x-axis (Fig. 24.8):*
In this case, we have $k = 0$ and $h = a$.
Hence, the equation of the circle is

$$(x-a)^2 + (y-0)^2 = a^2 \text{ or, } x^2 + y^2 - 2ax = 0.$$

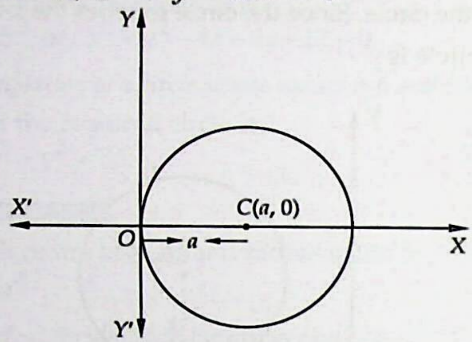

Fig. 24.8

(vii) *When the circle passes through the origin and centre lies on y-axis (Fig. 24.9):*
In this case, we have $h = 0$ and $k = a$.
Hence, the equation of the circle is

$$(x-0)^2 + (y-a)^2 = a^2 \text{ or, } x^2 + y^2 - 2ay = 0.$$

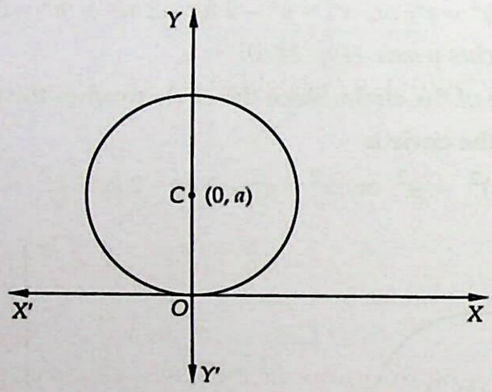

Fig. 24.9

IMPORTANT POINTS TO REMEMBER

(i) When a circle touches x-axis, then its radius is equal to the absolute value of the y-coordinates of the centre.

(ii) When a circle touches y-axis, the x-coordinates of its centre, in magnitude, is equal to the radius.

(iii) When a circle touches x-axis at the origin, then its centre lies on y-axis and absolute value of y-coordinates of the centre is equal to the radius.

(iv) When a circle touches y-axis at the origin, then its centre lies on x-axis at a distance equal to the radius of the circle.

(v) When a circle touches both the axis, then the coordinates of its centre are $(\pm a, \pm a)$, where a is the radius of the circle.

(vi) When a circle touches a line, then length of the perpendicular from its centre on the given line is equal to the radius of the circle.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE EQUATION OF A CIRCLE WHEN ITS CENTRE AND RADIUS ARE KNOWN**

EXAMPLE 1 *Find the equation of the circle whose centre is* (2, – 3) *and radius is 8.*

SOLUTION The equation of the circle is

$$(x-2)^2 + (y-(-3))^2 = 8^2 \qquad [\text{Using: } (x-h)^2+(y-k)^2=a^2]$$

$$\Rightarrow \quad (x-2)^2+(y+3)^2 = 8^2 \text{ or, } x^2+y^2-4x+6y-51=0.$$

EXAMPLE 2 *Find the equation of the circle which passes through the point of intersection of the lines* $3x-2y-1=0$ *and* $4x+y-27=0$ *and whose centre is* $(2,-3)$.

SOLUTION Let P be the point of intersection of the lines AB and LM whose equations are respectively

$$3x-2y-1=0 \quad \ldots(\text{i}) \quad \text{and} \quad 4x+y-27=0 \quad \ldots(\text{ii})$$

Solving (i) and (ii), we get $x=5, y=7$. So, coordinates of P are (5, 7). Let $C(2,-3)$ be the centre of the circle. Since the circle passes through P.

$$\therefore \quad CP = \text{Radius}$$

$$\Rightarrow \quad \sqrt{(5-2)^2+(7+3)^2} = \text{Radius}$$

$$\Rightarrow \quad \text{Radius} = \sqrt{109}.$$

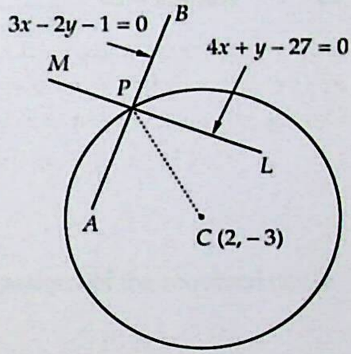

Fig. 24.10

Thus, the required circle has its centre at $C(2,-3)$

and, radius $=\sqrt{109}$. So, its equation is

$$(x-2)^2+(y+3)^2 = (\sqrt{109})^2 \text{ or, } x^2+y^2-4x+6y-96=0$$

EXAMPLE 3 *Find the equation of the circle having centre at* $(3,-4)$ *and touching the line* $5x+12y-12=0$.

[NCERT EXEMPLAR]

SOLUTION Let $C(3,-4)$ be the centre of the circle. If the line $5x+12y-12=0$ touches the required circle at P. Then, CP is perpendicular to the line and is equal to the radius of the circle.

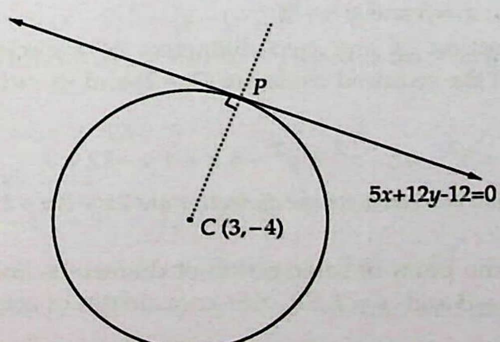

Fig. 24.11

$\therefore$ Radius $= CP =$ Length of perpendicular from $C(3,-4)$ on the line $5x+12y-12=0$

$$\Rightarrow \quad \text{Radius} = \left|\frac{5\times 3+12\times -4-12}{\sqrt{5^2+12^2}}\right| = \frac{45}{13}$$

Thus, the required circle has its centre at C $(3, -4)$ and radius $= \frac{45}{13}$.

Hence, its equation is $(x-3)^2+(y+4)^2 = \left(\frac{45}{13}\right)^2$

EXAMPLE 4 *Find the equation of a circle with origin as centre and which circumscribes an equilateral triangle whose median is of length 3a.*

SOLUTION Let the circle circumscribes an equilateral triangle ABC and let $AD = 3a$ be a median of ΔABC. It is given that the centre of the circle is at the origin O. Clearly, O lies on the median AD and coincides with the centroid of ΔABC.

$$\therefore \quad OA = \frac{2}{3} AD = \frac{2}{3}\times 3a = 2a$$

$$\Rightarrow \quad \text{Radius} = 2a$$

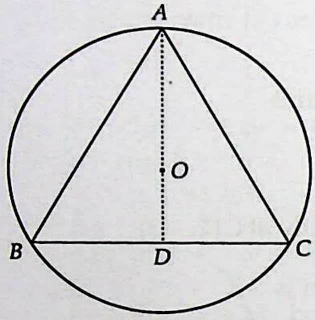

Fig. 24.12

Thus, the given circle has its centre at the origin $O\ (0, 0)$ and radius $= 2a$.

Hence, the equation of the circle is $(x-0)^2+(y-0)^2 = (2a)^2$ or, $x^2+y^2 = 4a^2$.

EXAMPLE 5 *If the equations of the two diameters of a circle are $x - y = 5$ and $2x + y = 4$ and the radius of the circle is 5, find the equation of the circle.*

SOLUTION Let the diameters of the circle be AB and LM whose equations are respectively

$$x - y = 5 \quad \text{...(i)} \qquad 2x + y = 4 \quad \text{...(ii)}$$

Solving (i) and (ii), we get : $x = 3$ and $y = -2$.

Since the point of intersection of any two diameters of a circle is its centre. Therefore, coordinates of the centre of the required circle are $(3, -2)$ and its radius is 5 (given).

Hence, its equation is

$$(x-3)^2+(y+2)^2 = 5^2 \quad \text{or,} \quad x^2+y^2-6x+4y-12=0$$

EXAMPLE 6 *Find the equation of a circle whose diameters are $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$ and area is 154 square units.*

SOLUTION The centre is the point of intersection of diameters. Solving $2x - 3y + 12 = 0$ and $x + 4y - 5 = 0$, we get $x = -3$ and $y = 2$. So, the coordinates of centre are $(-3, 2)$. Let r be the radius of the circle. Then,

$$\text{Area} = 154 \Rightarrow \pi r^2 = 154 \Rightarrow \frac{22}{7}\times r^2 = 154 \Rightarrow r = 7$$

Hence, the equation of the required circle is $(x+3)^2+(y-2)^2 = 49$.

EXAMPLE 7 *Find the equation of a circle of radius 5 whose centre lies on x-axis and passes through the point (2, 3).*

SOLUTION Let the coordinates of the centre of the required circle be $C(a, 0)$. Since it passes through $P(2, 3)$.

$\therefore \quad CP = \text{radius}$

$\Rightarrow \quad CP = 5$

$\Rightarrow \quad \sqrt{(a-2)^2 + (0-3)^2} = 5$

$\Rightarrow \quad (a-2)^2 + 9 = 25 \Rightarrow a - 2 = \pm 4 \Rightarrow a = 6 \text{ or, } a = -2$

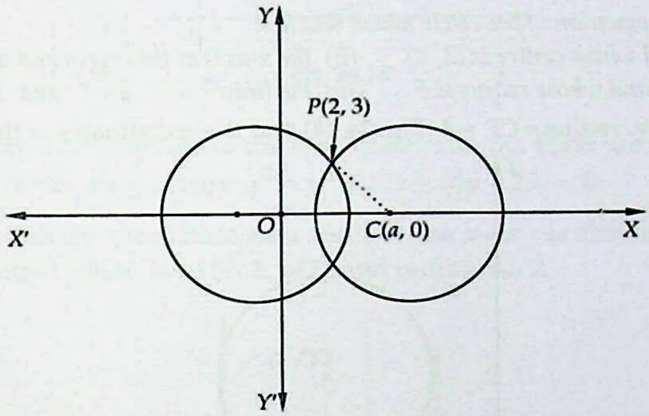

Fig. 24.13

Thus, the coordinates of the centre are (6, 0) or (–2, 0). Hence, the equations of the required circle are

$$(x-6)^2 + (y-0)^2 = 5^2 \quad \text{and} \quad (x+2)^2 + (y-0)^2 = 5^2$$

$$\Rightarrow \quad x^2 + y^2 - 12x + 11 = 0 \quad \text{and} \quad x^2 + y^2 + 4x - 21 = 0$$

Type II ON FINDING THE CENTRE AND RADIUS OF A GIVEN CIRCLE

EXAMPLE 8 *Find the centre and radius of each of the following circles:*

(i) $x^2 + (y+2)^2 = 9$ (ii) $x^2 + y^2 - 4x + 6y = 12$

(iii) $(x+1)^2 + (y-1)^2 = 4$ (iv) $x^2 + y^2 + 6x - 4y + 4 = 0.$

SOLUTION (i) We have,

$$x^2 + (y+2)^2 = 9 \Rightarrow (x-0)^2 + \{y-(-2)\}^2 = 3^2$$

Comparing this equation with $(x-a)^2 (y-b)^2 = r^2$, we find that the given circle has its centre at (0, –2) and radius 3.

(ii) We have, $x^2 + y^2 - 4x + 6y = 12$

$\Rightarrow \quad (x^2 - 4x) + (y^2 + 6y) = 12$

$\Rightarrow \quad (x^2 - 4x + 4) + (y^2 + 6y + 9) = 12 + 4 + 9$

$\Rightarrow \quad (x-2)^2 + (y+3)^2 = 5^2 \Rightarrow (x-2)^2 + \{y-(-3)\}^2 = 5^2$

Comparing this equation with $(x-a)^2 + (y-b)^2 = r^2$, we find that the given circle has its centre at (2, –3) and radius 5.

(iii) We have, $(x+1)^2 + (y-1)^2 = 4$

$\Rightarrow \quad \{x-(-1)\}^2 + (y-1)^2 = 2^2$

Clearly, the given circle has its centre at (–1, 1) and radius 2.

(iv) We have, $x^2 + y^2 + 6x - 4y + 4 = 0$

$$\Rightarrow \quad (x^2 + 6x) + (y^2 - 4y) = -4$$

$$\Rightarrow \quad (x^2 + 6x + 9) + (y^2 - 4y + 4) = -4 + 9 + 4$$

$$\Rightarrow \quad (x + 3^2) + (y - 2)^2 = 3^2 \Rightarrow \{x - (-3)\}^2 + (y - 2)^2 = 3^2.$$

Clearly, this circle has its centre at (–3, 2) and radius 3.

Type III ON FINDING THE EQUATION OF A CIRCLE SATISFYING SOME GIVEN GEOMETRICAL CONDITIONS

EXAMPLE 9 *Find the equation of the circle which touches:*

(i) the x-axis and whose centre is (3, 4) *(ii) the x-axis at the origin and whose radius is 5*

(iii) both the axes and whose radius is 5 *(iv) the lines $x = 0$, $y = 0$ and $x = a$.*

SOLUTION (i) Clearly, radius $= CP = 4$ (Fig. 24.14) and the coordinates of the centre are (3, 4).

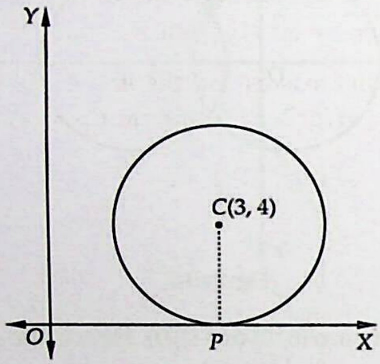

Fig. 24.14

Hence, the equation of the required circle is

$$(x - 3)^2 + (y - 4)^2 = 4^2 \text{ or, } x^2 + y^2 - 6x - 8y + 9 = 0$$

(ii) Since the circle touches the x-axis at the origin and has radius 5. So, the coordinates of the centre are (0, 5) as shown in Fig. 24.15. Hence, the equation of the circle is

$$(x - 0)^2 + (y - 5)^2 = 5^2 \text{ or, } x^2 + y^2 - 10y = 0.$$

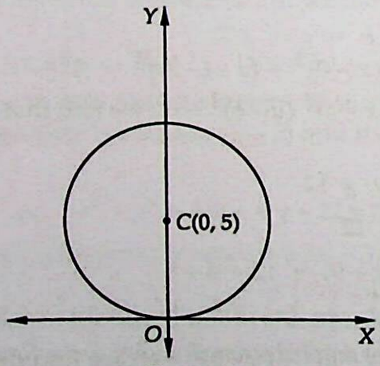

Fig. 24.15

(iii) The circle touches both the axes and has radius 5. So, the coordinates of the centre are (5, 5) and radius = 5 as shown in Fig. 24.16. So, the equation of the required circle is

$$(x - 5)^2 + (y - 5)^2 = 5^2 \text{ or, } x^2 + y^2 - 10x - 10y + 25 = 0.$$

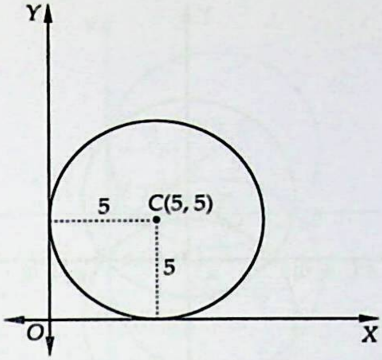

Fig. 24.16

Since, the circle may lie in any one of the four quadrants. So, there are four such circles. The equations of these circles are given by $x^2 + y^2 \pm 10x \pm 10y + 25 = 0.$

(iv) The circle touches the coordinate axes and the line $x = a$ as shown in Fig. 24.17. So, the centre of the required circle is at $(a/2, a/2)$ and radius $= a/2$.

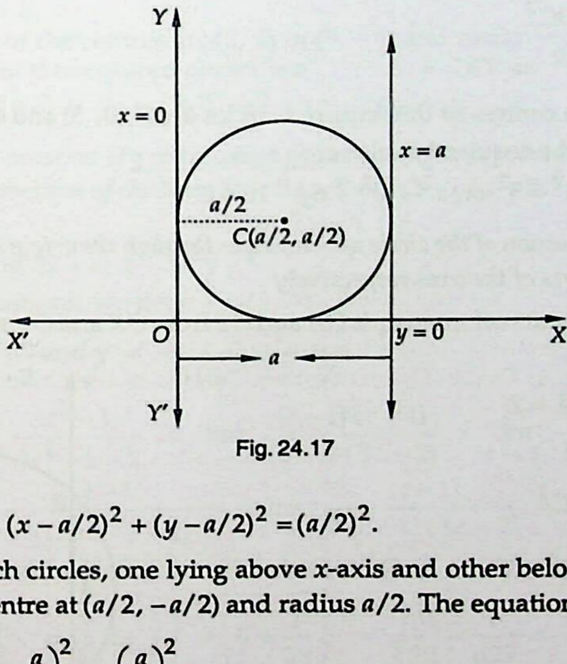

Fig. 24.17

Hence, its equation is $(x - a/2)^2 + (y - a/2)^2 = (a/2)^2$.

There may be two such circles, one lying above x-axis and other below x-axis. The circle lying below x-axis has its centre at $(a/2, -a/2)$ and radius $a/2$. The equation of this circles is

$$\left(x-\frac{a}{2}\right)^2+\left(y+\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$$

Hence, the equations of the circles are $\left(x-\frac{a}{2}\right)^2+\left(y\mp\frac{a}{2}\right)^2=\left(\frac{a}{2}\right)^2$.

EXAMPLE 10 *Find the equations of the circles which passes through two points on the x-axis which are at distances 4 from the origin and whose radius is 5.*

SOLUTION As is evident from Fig. 24.18 there are two circles which pass through two points A and A' on x-axis which are at a distance 4 from the origin. The centres of these circles lie on y-axis.

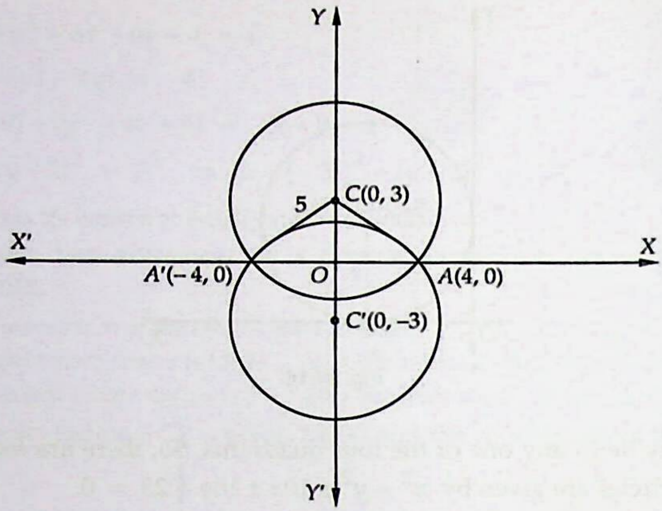

Fig. 24.18

Applying Pythagoras Theorem in $\Delta\ OAC$, we get

$$AC^2 = OA^2 + OC^2$$

$$\Rightarrow \quad 5^2 = 4^2 + OC^2 \Rightarrow OC = 3.$$

So, the coordinates of the centres of the required circles are $C\ (0, 3)$ and $C'\ (0, -3)$.

Hence, the equations of the required circles are

$$(x-0)^2 + (y \mp 3)^2 = 5^2 \text{ or, } x^2 + y^2 \mp 6y - 16 = 0.$$

EXAMPLE 11 *Find the equation of the circle which passes through the origin and cuts off intercepts 3 and 4 from the positive parts of the axes respectively.*

SOLUTION Let the circle cuts off intercepts OA and OB from OX and OY respectively. It is given that $OA = 3$ and $OB = 4$.

$$\therefore \quad OL = \frac{3}{2} \text{ and, } CL = 2$$

In $\Delta\ OLC$, we have

$$OC^2 = OL^2 + LC^2$$

$$\Rightarrow \quad OC^2 = \left(\frac{3}{2}\right)^2 + 2^2$$

$$\Rightarrow \quad OC = \frac{5}{2}.$$

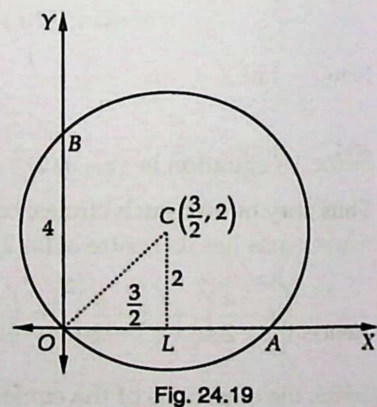

Fig. 24.19

Thus, the required circle has its centre at $(3/2, 2)$ and radius $5/2$.

Hence, its equation is $\left(x - \frac{3}{2}\right)^2 + (y-2)^2 = \left(\frac{5}{2}\right)^2$.

EXAMPLE 12 *Find the equation of a circle which touches y-axis at a distance of 4 units from the origin and cuts an intercept of 6 units along the positive direction of x-axis.*

SOLUTION The given circle touches y-axis at $L(0, 4)$ and cuts an intercept $AB = 6$ along the positive direction of x-axis. As shown in Fig. 24.20, there are two such circles.

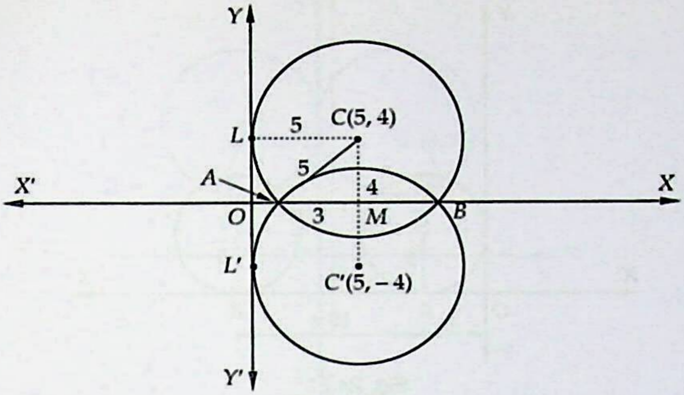

Fig. 24.20

In ΔCAM, we have

$$CA^2 = CM^2 + AM^2$$

$$\Rightarrow \quad CA^2 = 4^2 + 3^2$$

$$\Rightarrow \quad CA = 5$$

Also, $CL = CA = 5$.

Thus, the coordinates of the centres are (5, 4) or (5, – 4) and radius = 5.

Hence, the equations of the required circles are

$$(x-5)^2 + (y \mp 4)^2 = 5 \Rightarrow x^2 + y^2 - 10x \mp 8y + 16 = 0.$$

EXAMPLE 13 *Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the lines $3x + 5y = 1$ and $(2 + c)\,x + 5c^2 y = 1$ as $c \to 1$.*

SOLUTION We have,

$$3x + 5y = 1 \text{ and } (2 + c)\,x + 5c^2 y = 1.$$

Solving these two equations, we get

$$x = \frac{c^2 - 1}{3c^2 - c - 2} \text{ and } y = -\frac{c-1}{5\,(3c^2 - c - 2)}$$

Now, $\displaystyle \lim_{c\to 1} x = \lim_{c\to 1} \frac{c^2-1}{3c^2-c-2} = \lim_{c\to 1} \frac{(c-1)(c+1)}{(c-1)(3c+2)} = \lim_{c\to 1} \frac{c+1}{3c+2} = \frac{2}{5}$

and, $\displaystyle \lim_{c\to 1} y = \lim_{c\to 1} -\frac{c-1}{5\,(3c^2-c-2)} = -\lim_{c\to 1} \frac{(c-1)}{5\,(c-1)(3c+2)} = \lim_{c\to 1} \frac{1}{5\,(3c+2)} = -\frac{1}{25}$

Thus, the coordinates of the centre of the circle are C (2/5, – 1/25). It passes through P (2, 0).

$$\therefore \quad \text{Radius} = CP = \sqrt{\left(2 - \frac{2}{5}\right)^2 + \left(0 + \frac{1}{25}\right)^2} = \sqrt{\frac{64}{25} + \frac{1}{625}} = \frac{\sqrt{1601}}{25}$$

Hence, the equation of the required circle is

$$\left(x - \frac{2}{5}\right)^2 + \left(y + \frac{1}{25}\right)^2 = \frac{1601}{625} \text{ or, } 25x^2 + 25y^2 - 20x + 2y - 60 = 0$$

LEVEL-2

EXAMPLE 14 *A circle of radius 5 units touches the coordinate axes in the first quadrant. If the circle makes one complete roll on x-axis along the positive direction of x-axis, find its equation in new position.*

SOLUTION Let C and C_1 be the centres of the circle in its initial and final positions. The coordinates of C are (5, 5). In making one complete roll on x-axis, the centre C moves through the distance $CC_1 = AB$ = Circumference of the circle = $10\,\pi$.

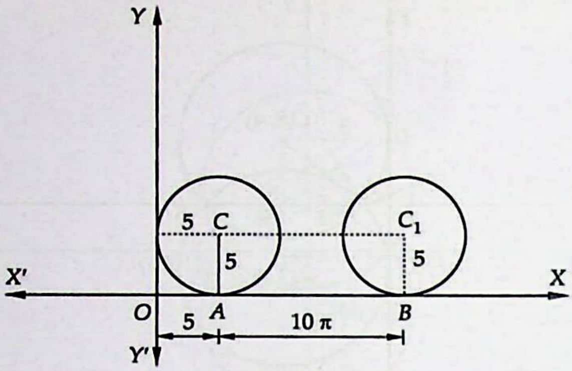

Fig. 24.21

So, the coordinates of the centre of the circle in the new position are $(5 + 10\pi, 5)$.
Radius of the circle in its new position is 5 units.
Hence, its equation is $\{x - (5 + 10\pi)\}^2 + (y - 5)^2 = 5^2$.

EXAMPLE 15 *A circle of radius 6 units touches the coordinate axes in the first quadrant. Find the equation of its image in the line mirror $y = 0$.*

SOLUTION The given circle has radius 6 and the co- ordinates of its centre C are (6, 6). The coordinates of its image C_1 in the line mirror $y = 0$ i.e. x-axis are $(6, -6)$.

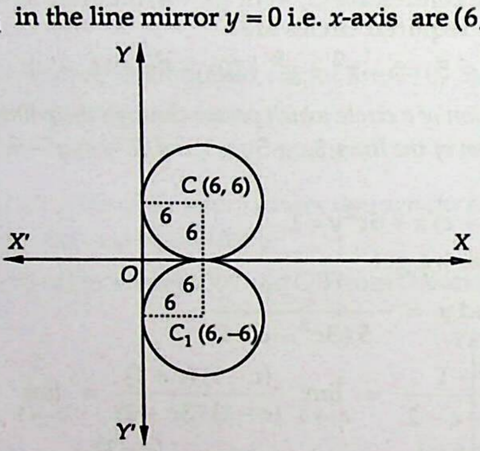

Fig. 24.22

So, the centre of the required circle is at C_1 $(6, -6)$ and its radius is 6.
Hence, its equation is

$$(x - 6)^2 + (y + 6)^2 = 6^2 \text{ or, } x^2 + y^2 - 12x + 12y + 36 = 0.$$

EXAMPLE 16 *Find the equation of the image of the circle $x^2 + y^2 + 8x - 16y + 64 = 0$ in the line mirror $x = 0$.*

SOLUTION The equation of the given circle is

$$x^2 + y^2 + 8x - 16y + 64 = 0$$

$$\Rightarrow \quad (x^2 + 8x + 16) + (y^2 - 16y + 64) = 16$$

$$\Rightarrow \quad (x + 4)^2 + (y - 8)^2 = 4^2$$

$$\Rightarrow \quad (x - (-4))^2 + (y - 8)^2 = 4^2$$

Clearly, its centre is at $(-4, 8)$ and radius = 4.

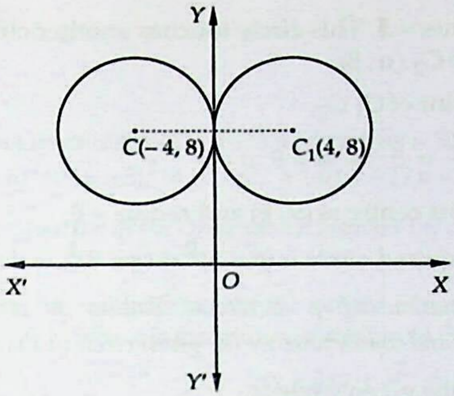

Fig. 24.23

The image of this circle in the line mirror has its centre C_1 (4, 8) and radius 4. So, its equation is

$$(x-4)^2 + (y-8)^2 = 4^2 \text{ or, } x^2 + y^2 - 8x - 16y + 64 = 0$$

EXAMPLE 17 *The circle $(x-a)^2 + (y-a)^2 = a^2$ is rolled on the y-axis in the positive direction through one complete revolution. Find the equation of the circle in its new-position.*

SOLUTION The given circle has its centre C (a, a) and radius $= a$. Clearly, it touches both the axes. When this circle rolls on y-axis and completes one revolution, its centre moves vertically through the distance equal to its circumference i.e. $2\pi a$. So, the coordinates of the centre of the new-circle are C_1 $(a, a + 2\pi a)$. Clearly, radius of the new circle is same as that of the given circle i.e. a.

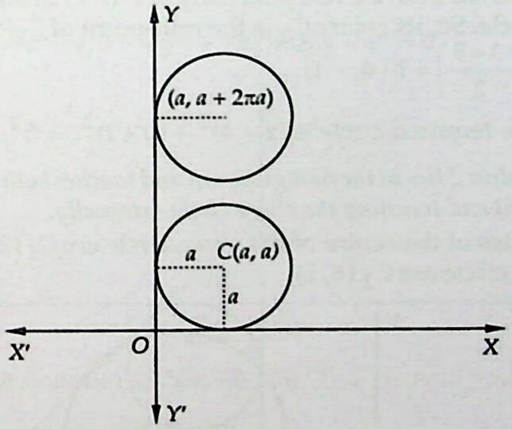

Fig. 24.24

Hence, the equation of the new circle is $(x-a)^2 + \{y-(a+2\pi a)\}^2 = a^2$.

EXAMPLE 18 *Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at the point (5, 5).*

SOLUTION The equation of the given circle is

$$x^2 + y^2 - 2x - 4y - 20 = 0 \text{ or, } (x-1)^2 + (y-2)^2 = 5^2.$$

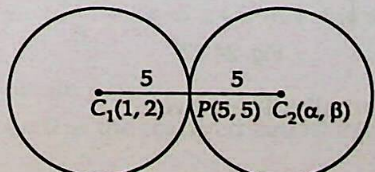

Fig. 24.25

Its centre is C_1 (1, 2) and radius = 5. This circle touches another circle of radius 5 externally at point P (5, 5). Let its centre be C_2 (α, β).

Clearly, P (5, 5) is the mid-point of $C_1 C_2$.

$$\therefore \quad \frac{\alpha+1}{2} = 5 \text{ and } \frac{\beta+2}{2} = 5 \Rightarrow \alpha = 9, \beta = 8$$

Thus, the required circle has its centre at (9, 8) and radius = 5.

Hence, the equation of the required circle is $(x-9)^2 + (y-8)^2 = 5^2$.

EXAMPLE 19 *Find the equation of a circle of radius 5 which lies within the circle* $x^2 + y^2 + 14x + 10y - 26 = 0$ *and which touches the given circle at the point* $(-1, 3)$.

SOLUTION The equation of the given circle is

$$x^2 + y^2 + 14x + 10y - 26 = 0 \text{ or, } (x-(-7))^2 + (y-(-5))^2 = 10^2$$

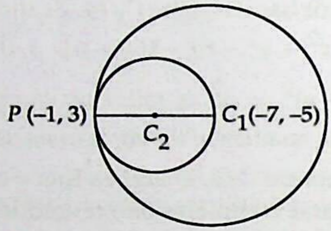

Fig. 24.26

Its centre is at $C_1(-7, -5)$ and, radius =10.

The required circle touches the above circle internally at $P(-1, 3)$ and has radius = 5 i.e. half of the radius of the given circle. So, its centre C_2 is the mid-point of C_1P. Therefore, coordinates of its centre C_2 are $\left(\frac{-1-7}{2}, \frac{3-5}{2}\right) = (-4, -1)$.

Hence, the equation of the required circle is $(x+4)^2 + (y+1)^2 = 5^2$.

EXAMPLE 20 *A circle of radius 2 lies in the first quadrant and touches both the axes. Find the equation of the circle with centre at (6, 5) and touching the above circle externally.*

SOLUTION The coordinates of the centre of the given circle are C_1 (2, 2) and the coordinates of the centre of the required circle are C_2 (6, 5).

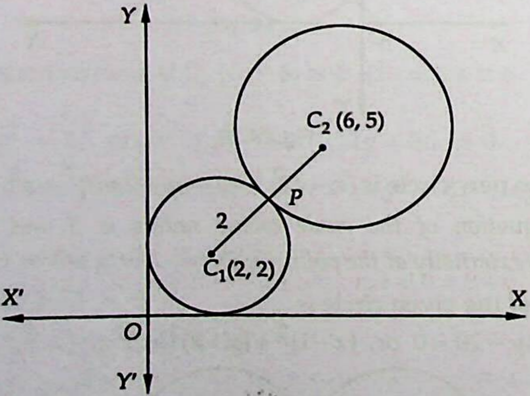

Fig. 24.27

Since it touches the given circle externally. Therefore,

$$C_1C_2 = C_1P + C_2P$$

$\Rightarrow \quad \sqrt{(6-2)^2+(5-2)^2} = 2+C_2P$

$\Rightarrow \quad 5 = 2+C_2P$

$\Rightarrow \quad C_2P = 3$

Thus, the required circle has its centre at $C_2(6, 5)$ and radius = 3.

Hence, its equation is $(x-6)^2+(y-5)^2=3^2$ or, $x^2+y^2-12x-10y+52=0$

EXAMPLE 21 *Show that the equation of the circle which touches the coordinate axes and whose centre lies on the line $lx+my+n=0$ is $(l+m)^2(x^2+y^2)+2n(l+m)(x+y)+n^2 = 0$.*

SOLUTION We know that the coordinates of the centre of a circle touching the coordinates axes in first quadrant are (a, a), where a is the radius of the circle. So, the equation of the circle is

$$(x-a)^2+(y-a)^2 = a^2 \text{ or, } x^2+y^2-2ax-2ay+a^2 = 0 \quad \text{...(i)}$$

Since the centre (a, a) lies on $lx+my+n=0$. Therefore,

$$la+ma+n = 0 \Rightarrow a = -\frac{n}{l+m}$$

Putting the value of a in (i), we obtain the equation of the circle as

$$x^2+y^2+\frac{2nx}{l+m}+\frac{2ny}{l+m}+\frac{n^2}{(1+m)^2}=0 \text{ or, } (l+m)^2(x^2+y^2)+2n(x+y)(l+m)+n^2=0.$$

EXAMPLE 22 *Find the equation of the circle which touches both the axes and the line $3x-4y+8=0$ and lies in the third quadrant.* **[NCERT EXEMPLAR]**

SOLUTION Let a be the radius of the circle. It is given that the circle touches both the axes and lies in the third quadrant. So, the coordinates of its centre are $(-a, -a)$ and the equation of the circle is

$$(x+a)^2+(y+a)^2=a^2 \text{ or, } x^2+y^2+2ax+2ay+a^2=0 \quad \text{...(i)}$$

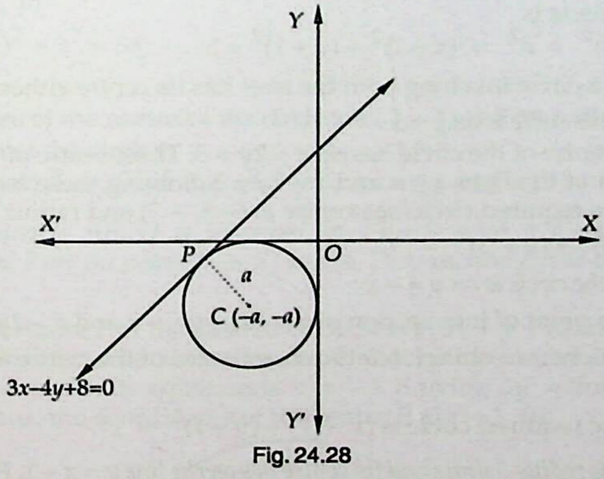

Fig. 24.28

The circle touches the line $3x-4y+8=0$. Therefore, length of the perpendicular from the centre $(-a, -a)$ to the line $3x-4y+8=0$ is equal to the radius of the circle.

i.e. $\quad CP = a$

$$\Rightarrow \quad \left|\frac{-3a+4a+8}{\sqrt{3^2+(-4)^2}}\right| = a \Rightarrow \frac{|a+8|}{5} = a \Rightarrow a+8=5a \Rightarrow a=2 \quad [\because a>0 \therefore a+8>0]$$

Substituting $a=2$ in (i), we obtain

$x^2+y^2+4x+4y+4=0$ as the required equation of the circle.

EXAMPLE 23 *Find the equation of the circle which touches the coordinate axes and whose centre lies on the line $x - 2y = 3$.*

SOLUTION Since the circle touches the coordinate axes and the line $x - 2y + 3 = 0$. So, its centre lies in third or in fourth quadrant. Let a be the radius of the circle.

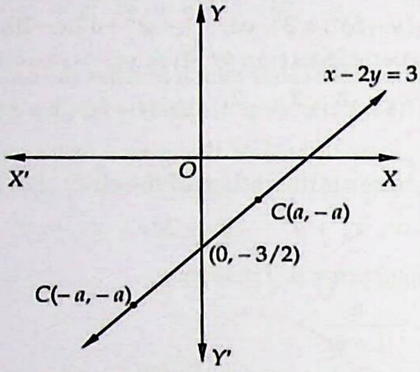

Fig. 24.29

CASE I *When centre is in third quadrant:*
In this case, the coordinates of the centre are $(-a, -a)$. As it lies on $x - 2y = 3$.

$\therefore \quad -a + 2a = 3 \Rightarrow a = 3.$

So, the equation of the circle is

$$(x + a)^2 + (y + a)^2 = a^2 \Rightarrow (x + 3)^2 + (y + 3)^2 = 3^2.$$

CASE II *When centre is in fourth quadrant:*
In this case, the coordinates of the centre are $(a, -a)$. As it lies on $x - 2y = 3$.

$\therefore \quad a + 2a = 3 \Rightarrow a = 1$

So, the equation of the circle is

$$(x - a)^2 + (y + a)^2 = a^2 \Rightarrow (x - 1)^2 + (y + 1)^2 = 1$$

ALITER We know that a circle touching both the axes has its centre either on $y = x$ or, $y = -x$.

CASE I *When centre of the circle is on $y = x$:*
It is also given that the centre of the circle lies on $x - 2y = 3$. Thus, centre of the required circle is the point of intersection of the lines $y = x$ and $x - 2y = 3$. Solving these two equations, we get $x = -3$, $y = -3$. Thus, the required circle has centre at $(-3, -3)$ and radius 3.
So, its equation is $(x + 3)^2 + (y + 3)^2 = 3^2$.

CASE II *When centre of the circle is on $y = -x$:*
In this case, centre is the point of intersection of the lines $y = -x$ and $x - 2y = 3$.
Solving these two equations, we obtain that the coordinates of the centre are $(1, -1)$. Radius of the required circle is 1 unit.
Thus, the equation of the required circle is $(x - 1)^2 + (y + 1)^2 = 1^2$.

EXAMPLE 24 *A circle has radius 3 units and its centre lies on the line $y = x - 1$. Find the equation of the circle, if it passes through $(7, 3)$.* **[NCERT EXEMPLAR]**

SOLUTION The coordinates of any point on the line $y = x - 1$ can be taken as $(t, t - 1)$. So, let $C(t, t - 1)$ be the centre of required circle. Its radius is 3. Therefore, equation of the required circle is

$$(x - t)^2 + \{y - (t - 1)\}^2 = 3^2 \quad \text{...(i)}$$

It passes through $(7, 3)$.

$$\therefore \quad (7 - t)^2 + \{3 - (t - 1)\}^2 = 3^2$$

$$\Rightarrow \quad (7 - t)^2 + (4 - t)^2 = 9 \Rightarrow t^2 - 11t + 28 = 0 \Rightarrow (t - 4)(t - 7) = 0 \Rightarrow t = 4, 7$$

Substituting the values of t in (i), we obtain that the equations of the required circles are

$$(x-4)^2+(y-3)^2 = 3^2 \text{ and } (x-7)^2+(y-6)^2 = 3^2$$

EXAMPLE 25 *Find the equation of the circle whose centre is at* $(3, -1)$ *and which cuts off a chord of length 6 units on the line* $2x-5y+18=0$ **[NCERT EXEMPLAR]**

SOLUTION Let PQ be the chord cut off by the circle on the line $2x-5y+18=0$. Let $C(3, -1)$ be the centre of the circle and CL perpendicular drawn from C on the chord PQ. Then, L bisects PQ.

$$\therefore \quad PL = QL = \frac{1}{2}PQ \Rightarrow PL = QL = 3$$

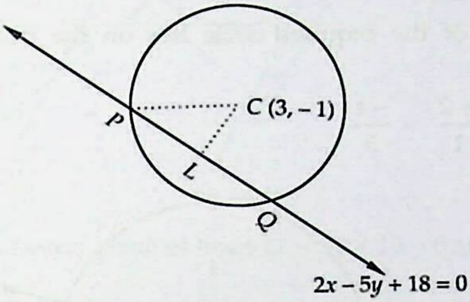

Fig. 24.30

Now,

$CL =$ Length of perpendicular from $C\ (3, -1)$ on $2x-5y+18=0$

$$\Rightarrow \quad CL = \left|\frac{2\times 3-5\times -1+18}{\sqrt{2^2+(-5)^2}}\right| = \sqrt{29}$$

Applying Pythagoras theorem in ΔCLP, we obtain

$$CP^2 = CL^2 + PL^2$$

$$\Rightarrow \quad CP^2 = (\sqrt{29})^2 + 3^2 = 38$$

$$\Rightarrow \quad CP = \sqrt{38}.$$

Thus, the coordinates of the centre of the circle are $(3, -1)$ and its radius is $\sqrt{38}$.

Hence, the equation of the circle is

$$(x-3)^2+(y+1)^2 = (\sqrt{38})^2 \text{ or, } x^2+y^2-6x+2y-28=0$$

EXAMPLE 26 *A rectangle ABCD is inscribed in a circle with a diameter lying along the line* $3y = x+10$. *If A and B are the points* $(-6, 7)$ *and* $(4, 7)$ *respectively, find the area of the rectangle and equation of the circle.*

SOLUTION Clearly, centre P of the desired circle lies on $3y = x+10$ and perpendicular bisector of AB. As AB is parallel to x-axis, therefore perpendicular bisector of AB passes through $(-1, 7)$ and is parallel to y-axis. So, its equation is $x = -1$. Solving $3y = x+10$ and $x = -1$, we get $x = -1$ and $y = 3$. Thus, the coordinates of the centre P are $(-1, 3)$.

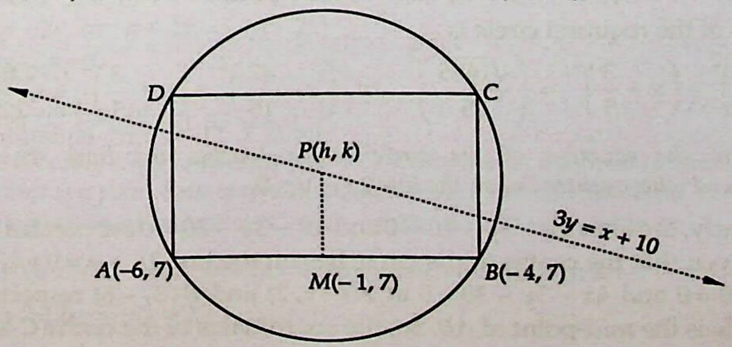

Fig. 24.31

Also, AP = Radius of the circle $= \sqrt{(-6+1)^2 + (7-3)^2} = \sqrt{41}$

Hence, the equation of the circle is

$$(x+1)^2 + (y-3)^2 = (\sqrt{41})^2 \text{ or, } x^2 + y^2 + 2x - 6y - 31 = 0$$

Now, $AD = 2\,PM = 2\sqrt{(-1+1)^2 + (3-7)^2} = 8$ and $AB = \sqrt{(-6-4)^2 + (7-7)^2} = 10$

$\therefore$ Area of rectangle $ABCD = AB \times AD = 10 \times 8 = 80$ square units.

EXAMPLE 27 *Find the equation of the circle passing through the points* $(1, -2)$ *and* $(4, -3)$ *and whose centre lies on the line* $3x + 4y = 7$.

SOLUTION Clearly, centre of the required circle lies on the perpendicular bisector of AB.

Clearly,

$$\text{Slope of } AB = \frac{-3+2}{4-1} = \frac{-1}{3}$$

$\therefore$ Slope of $CP = 3$

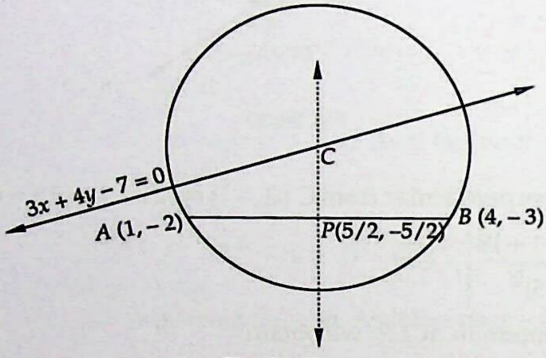

Fig. 24.32

The coordinates of the mid-point P of AB are $(5/2, -5/2)$. The perpendicular bisector of AB passes through $P(5/2, -5/2)$ and is perpendicular to AB. So, the equation of perpendicular bisector of AB is

$$y + \frac{5}{2} = 3\left(x - \frac{5}{2}\right) \text{ or, } 3x - y - 10 = 0$$

Solving $3x + 4y - 7 = 0$ and $3x - y - 10 = 0$, we get:

$$x = 47/15 \text{ and, } y = -3/5$$

So, the coordinates of C are $(47/15, -3/5)$.

Clearly, radius of the circle is AC.

$$\therefore \quad \text{Radius} = AC = \sqrt{\left(\frac{47}{15} - 1\right)^2 + \left(-\frac{3}{5} + 2\right)^2} = \sqrt{\left(\frac{32}{15}\right)^2 + \left(\frac{7}{5}\right)^2} = \frac{\sqrt{1465}}{15}$$

Hence, equation of the required circle is

$$\left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \left(\frac{\sqrt{1465}}{15}\right)^2 \text{ or, } \left(x - \frac{47}{15}\right)^2 + \left(y + \frac{3}{5}\right)^2 = \frac{1465}{225}$$

EXAMPLE 28 *Find the equation of the circle which touches the lines* $4x - 3y + 10 = 0$ *and* $4x - 3y - 30 = 0$ *and whose centre lies on the line* $2x + y = 0$.

SOLUTION Clearly, the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ are parallel and are touching the circle. It is given that the centre of the circle lies on the line $2x + y = 0$ which intersects the lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$ at $A(-1, 2)$ and $B(3, -6)$ respectively. Therefore, centre of the circle is the mid-point of AB. So, the coordinates of the centre C are $(1, -2)$.

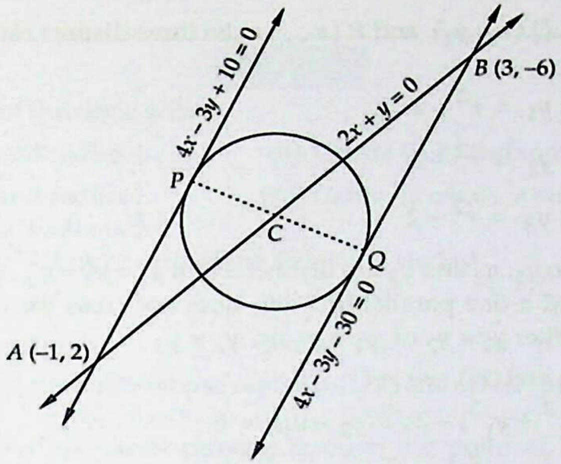

Fig. 24.33

Let d be the distance between parallel lines $4x - 3y + 10 = 0$ and $4x - 3y - 30 = 0$. Then,

$$d = \left| \frac{10 - (-30)}{\sqrt{4^2 + (-3)^2}} \right| = 8$$

$\therefore$ Radius $= \frac{1}{2}(PQ) = \frac{1}{2} \times d = 4$

Thus, the required circle has its centre at $C(1, -2)$ and radius = 4.
Hence, its equation is $(x-1)^2 + (y+2)^2 = 4^2$.

EXAMPLE 29 *Find the locus of the centre of the circle touching the line $x + 2y = 0$ and $x - 2y = 0$.*

SOLUTION Let (h, k) be the centre of the circle touching the lines $x + 2y = 0$ and $x - 2y = 0$. Let r be the radius of the circle. We know that the length of the perpendicular from the centre of a circle on the tangent line is equal to the radius of the circle.

$\therefore$ (Length of the perpendicular from (h, k) on $x + 2y = 0$) $= r$

and, (Length of the perpendicular from (h, k) on $x - 2y = 0$) $= r$.

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{1+2^2}} = r$ and, $\left| \frac{h-2k}{\sqrt{1^2+(-2)^2}} \right| = r$

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{5}} = r$ and, $\frac{|h-2k|}{\sqrt{5}} = r$

$\Rightarrow \quad \frac{|h+2k|}{\sqrt{5}} = \frac{|h-2k|}{\sqrt{5}}$

$\Rightarrow \quad |h+2k| = |h-2k|$

$\Rightarrow \quad h + 2k = \pm(h - 2k)$

$\Rightarrow \quad h + 2k = h - 2k$ or, $h + 2k = -(h - 2k)$

$\Rightarrow \quad 4k = 0$ or, $2h = 0$

$\Rightarrow \quad h = 0$ or, $k = 0$

Hence, the locus of (h, k) is $x = 0$ or $y = 0$.

EXAMPLE 30 *Let C be any circle with centre $(0, \sqrt{2})$. Prove that at most two rational points can be there on C. (A rational point is a point both of whose coordinates are rational numbers)*

SOLUTION The equation of any circle C with centre $(0, \sqrt{2})$ is given by

$(x-0)^2 + (y-\sqrt{2})^2 = r^2$, where r is any positive real number.

or, $\quad x^2 + y^2 - 2\sqrt{2}y = r^2 - 2 \quad$ ...(i)

If possible, let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be three distinct rational points on circle C. Then,

$$x_1^2 + y_1^2 - 2\sqrt{2}\, y_1 = r^2 - 2 \quad \text{...(ii)}$$

$$x_2^2 + y_2^2 - 2\sqrt{2}\, y_2 = r^2 - 2 \quad \text{...(iii)}$$

$$x_3^2 + y_3^2 - 2\sqrt{2}\, y_3 = r^2 - 2 \quad \text{...(iv)}$$

We claim that at least two y_1, y_2 and y_3 are distinct. For if $y_1 = y_2 = y_3$, then P, Q and R lie on a line parallel to x-axis and a line parallel to x-axis does not cross the circle in more than two points. Thus, we have either $y_1 \neq y_2$ or, $y_1 \neq y_3$ or, $y_2 \neq y_3$.

Subtracting (ii) from (iii) and (iv), we get

$$(x_2^2 + y_2^2) - (x_1^2 + y_1^2) - 2\sqrt{2}(y_2 - y_1) = 0$$

and, $$(x_3^2 + y_3^2) - (x_1^2 + y_1^2) - 2\sqrt{2}(y_3 - y_1) = 0$$

$\Rightarrow$ $$a_1 - \sqrt{2} b_1 = 0 \text{ and } a_2 - \sqrt{2}\, b_2 = 0 \quad \text{...(v)}$$

where, $a_1 = (x_2^2 + y_2^2) - (x_1^2 + y_1^2)$, $b_1 = 2(y_2 - y_1)$

$a_2 = (x_3^2 + y_3^2) - (x_1^2 + y_1^2)$, $b_2 = 2(y_3 - y_1)$

Clearly, a_1, a_2, b_1, b_2 are rational numbers as $x_1, x_2, x_3, y_1, y_2, y_3$ are rational numbers.

Since either $y_1 \neq y_2$ or, $y_1 \neq y_3$. Therefore, either $b_1 \neq 0$ or, $b_2 \neq 0$.

If $b_1 \neq 0$, then

$$a_1 - \sqrt{2}\, b_1 = 0 \Rightarrow \frac{a_1}{b_1} = \sqrt{2}$$

This is not possible because $\frac{a_1}{b_1}$ is a rational number and $\sqrt{2}$ is an irrational number.

If $b_2 \neq 0$, then

$$a_2 - \sqrt{2}\, b_2 = 0 \Rightarrow \frac{a_2}{b_2} = \sqrt{2}$$

This is not possible because $\frac{a_2}{b_2}$ is a rational number and $\sqrt{2}$ is an irrational number.

Thus, in both the cases we arrive at a contradiction. This means that our supposition is wrong. Hence, there can be at most two rational points on circle C.

<u>ALITER</u> Let there be three points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ with rational coordinates on circle C having its equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Since P, Q, R lie on circle (i). Therefore,

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

These are three linear equations in g, f and c with rational coefficients. So, we get rational values of g, f, c. But, $f = \sqrt{2}$. Thus, we arrive at a contradiction. Hence, there can be at most two rational points on circle C.

EXERCISE 24.1

LEVEL-1

1. Find the equation of the circle with:
   (i) Centre (–2, 3) and radius 4.
   (ii) Centre (a, b) and radius $\sqrt{a^2+b^2}$.
   (iii) Centre (0, –1) and radius 1.
   (iv) Centre $(a \cos \alpha,\ a \sin \alpha)$ and radius a.
   (v) Centre (a, a) and radius $\sqrt{2}\,a$.
2. Find the centre and radius of each of the following circles:
   (i) $(x-1)^2+y^2 = 4$
   (ii) $(x+5)^2+(y+1)^2 = 9$
   (iii) $x^2+y^2-4x+6y = 5$
   (iv) $x^2+y^2-x+2y-3 = 0$.
3. Find the equation of the circle whose centre is (1, 2) and which passes through the point (4, 6).
4. Find the equation of the circle passing through the point of intersection of the lines $x+3y=0$ and $2x-7y=0$ and whose centre is the point of intersection of the lines $x+y+1=0$ and $x-2y+4=0$.
5. Find the equation of the circle whose centre lies on the positive direction of y-axis at a distance 6 from the origin and whose radius is 4.
6. If the equations of two diameters of a circle are $2x+y=6$ and $3x+2y=4$ and the radius is 10, find the equation of the circle.
7. Find the equation of a circle
   (i) which touches both the axes at a distance of 6 units from the origin.
   (ii) which touches x-axis at a distance 5 from the origin and radius 6 units
   (iii) which touches both the axes and passes through the point (2, 1).
   (iv) passing through the origin, radius 17 and ordinate of the centre is –15.
8. Find the equation of the circle which has its centre at the point (3, 4) and touches the straight line $5x+12y-1=0$.
9. Find the equation of the circle which touches the axes and whose centre lies on $x-2y=3$.
10. A circle whose centre is the point of intersection of the lines $2x-3y+4=0$ and $3x+4y-5=0$ passes through the origin. Find its equation.
11. A circle of radius 4 units touches the coordinate axes in the first quadrant. Find the equations of its images with respect to the line mirrors $x=0$ and $y=0$.
12. Find the equations of the circles touching y-axis at (0, 3) and making an intercept of 8 units on the x-axis.
13. Find the equations of the circles passing through two points on y-axis at distances 3 from the origin and having radius 5.
14. If the lines $2x-3y=5$ and $3x-4y=7$ are the diameters of a circle of area 154 square units, then obtain the equation of the circle.
15. If the line $y=\sqrt{3}x+k$ touches the circle $x^2+y^2=16$, then find the value of k. [NCERT EXEMPLAR]
16. Find the equation of the circle having $(1,-2)$ as its centre and passing through the intersection of the lines $3x+y=14$ and $2x+5y=18$. [NCERT EXEMPLAR]
17. If the lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to a circle, then find the radius of the circle. [NCERT EXEMPLAR]

LEVEL-2

18. Show that the point (x, y) given by $x=\dfrac{2at}{1+t^2}$ and $y=a\left(\dfrac{1-t^2}{1+t^2}\right)$ lies on a circle for all real values of t such that $-1 \le t \le 1$, where a is any given real number. [NCERT EXEMPLAR]

19. The circle $x^2 + y^2 - 2x - 2y + 1 = 0$ is rolled along the positive direction of x-axis and makes one complete roll. Find its equation in new-position.
20. One diameter of the circle circumscribing the rectangle $ABCD$ is $4y = x + 7$. If the coordinates of A and B are $(-3, 4)$ and $(5, 4)$ respectively, find the equation of the circle.
21. If the line $2x - y + 1 = 0$ touches the circle at the point $(2, 5)$ and the centre of the circle lies on the line $x + y - 9 = 0$. Find the equation of the circle.

ANSWERS

1. (i) $(x+2)^2 + (y-3)^2 = 16$ (ii) $x^2 + y^2 - 2ax - 2by = 0$ (iii) $x^2 + y^2 + 2y = 0$
   (iv) $x^2 + y^2 - (2a \cos \alpha) . x - (2a \sin \alpha) . y = 0$ (v) $x^2 + y^2 - 2ax - 2ay = 0$
2. (i) $(1, 0); 2$ (ii) $(-5, -1)\,;\,3$ (iii) $(2, -3)\,;\,3\sqrt{2}$ (iv) $\left(\frac{1}{2}, -1\right); \frac{\sqrt{17}}{2}$
3. $x^2 + y^2 - 2x - 4y - 20 = 0$
4. $x^2 + y^2 + 4x - 2y = 0$
5. $x^2 + y^2 - 12y + 20 = 0$
6. $x^2 + y^2 - 16x + 20y + 64 = 0$
7. (i) $x^2 + y^2 - 12x - 12y + 36 = 0$
   (ii) $x^2 + y^2 - 10x - 12y + 25 = 0$ (iii) $x^2 + y^2 - 2x - 2y + 1 = 0,\ x^2 + y^2 - 10x - 10y + 25 = 0$
   (iv) $x^2 + y^2 \pm 16x + 30y = 0$
8. $169(x^2 + y^2 - 6x - 8y) + 381 = 0$
9. $x^2 + y^2 + 6x + 6y + 9 = 0$ or $x^2 + y^2 - 2x + 2y + 1 = 0$
10. $\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \frac{485}{289}$
11. With respect to $x = 0; x^2 + y^2 + 8x - 8y + 16 = 0$
    With respect to $y = 0\ x^2 + y^2 - 8x + 8y + 16 = 0$
12. $x^2 + y^2 \pm 10x - 6y + 9 = 0$
13. $x^2 + y^2 \pm 8x - 9 = 0$
14. $x^2 + y^2 - 2x + 2y - 47 = 0$
15. $k = \pm 8$
16. $x^2 + y^2 - 2x + 4y - 20 = 0$
17. $3/2$
19. $(x - 1 - 2\pi)^2 + (y-1)^2 = 1$
20. $x^2 + y^2 - 2x - 4y - 15 = 0$
21. $(x-6)^2 + (y-3)^2 = 20$

HINTS TO NCERT & SELECTED PROBLEMS

8. Radius = Length of the perpendicular from the centre (3, 4) to the line $5x + 12y - 1 = 0$.
9. Let a be the radius of the circle. Clearly, the required circle lies either in third or in fourth quadrant. So, the coordinates of its centre are $(-a, -a)$ or $(a, -a)$. Since, centre lies on $x - 2y = 3$. Therefore, $a = 3$ or $a = 1$.
17. Clearly, Diameter = Distance between parallel tangents $3x - 4y + 4 = 0$ and $3x - 4y - 7/2 = 0$
18. $x = \frac{2at}{1+t^2}$ and, $y = a\left(\frac{1-t^2}{1+t^2}\right)$ are parametric equations of a curve. In order to obtain the cartesian equation, we will have to eliminate parameter t.

    Clearly, $x^2 + y^2 = \frac{4at^2}{(1+t^2)^2} + \frac{a^2(1-t^2)^2}{(1+t^2)^2} = a^2\frac{(1-t^2)^2}{(1+t^2)^2} = a^2$, which is the cartesian equation of the curve representing a circle having centre at (0, 0) and radius a.

24.4 GENERAL EQUATION OF A CIRCLE

THEOREM *Prove that the equation $x^2+y^2+2gx+2fy+c=0$ always represents a circle whose centre is $(-g,-f)$ and radius $=\sqrt{g^2+f^2-c}$.*

PROOF The given equation is $x^2+y^2+2gx+2fy+c=0$ ...(i)

$\Rightarrow \quad (x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$

$\Rightarrow \quad (x+g)^2+(y+f)^2=\left\{\sqrt{g^2+f^2-c}\right\}^2$

$\Rightarrow \quad \{x-(-g)\}^2+\{y-(-f)\}^2=\left\{\sqrt{g^2+f^2-c}\right\}^2$

This is of the form $(x-h)^2+(y-k)^2=a^2$ which represents a circle having centre at (h, k) and radius equal to a.

Hence, the given equation (i) represents a circle whose centre is at

$$(-g,-f) \text{ i.e. } \left(-\frac{1}{2}\text{ Coefficient of } x, -\frac{1}{2}\text{ Coefficient of } y\right)$$

and, $\quad \text{Radius}=\sqrt{g^2+f^2-c}=\sqrt{\left(\frac{1}{2}\text{ Coeff. of } x\right)^2+\left(\frac{1}{2}\text{ Coeff. of } y\right)^2-\text{Constant term}}$

Q.E.D.

NOTE 1 *The equation $x^2+y^2+2gx+2fy+c=0$ represents a circle of radius $\sqrt{g^2+f^2-c}$.*

If $g^2+f^2-c>0$, then the radius of the circle is real and hence the circle is also real.

If $g^2+f^2-c=0$, then the radius of the circle is zero. Such a circle is known as a point circle.

If $g^2+f^2-c<0$, then the radius $\sqrt{g^2+f^2-c}$ of the circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.

NOTE 2 *Special features of the general equation $x^2+y^2+2gx+2fy+c=0$ of the circle are:*

(i) *it is quadratic in both x and y.*
(ii) *Coefficient of x^2 = Coefficient of y^2.*
(iii) *there is no term containing xy i.e., the coefficient of xy is zero.*
(iv) *it contains three arbitrary constants viz. g, f and c.*

NOTE 3 *The equation $ax^2+ay^2+2gx+2fy+c=0, a \neq 0$ also represents a circle. This equation can also be written as*

$$x^2+y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0.$$

The coordinates of the centre of the circle are $(-g/a, -f/a)$ and, radius $=\sqrt{\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}}$.

NOTE 4 *On comparing the general equation $x^2+y^2+2gx+2fy+c=0$ of a circle with the general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$, we find that it represents a circle if $a=b$ i.e., coefficient of x^2 = coefficient of y^2 and $h=0$ i.e., coefficient of $xy=0$.*

NOTE 5 *While solving problems it is advisable to keep the coefficient of x^2 and y^2 unity.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE CENTRE AND RADIUS OF A CIRCLE WHEN ITS EQUATION IS GIVEN**

RESULT *The coordinates of the centre of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ are*

$$\left(-\frac{1}{2} \text{ Coefficient of } x, \ -\frac{1}{2} \text{ Coefficient of } y\right)$$

and, $\quad$ Radius $= \sqrt{\left(\frac{1}{2} \text{ Coefficient of } x\right)^2 + \left(\frac{1}{2} \text{ Coefficient of } y^2\right)^2 - \text{Constant term}}$

EXAMPLE 1 *Find the centre and radius of the circle* $x^2 + y^2 - 6x + 4y - 12 = 0$.

SOLUTION The coordinates of the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are

$$\left(-\frac{1}{2} \text{ Coeff. of } x, \ -\frac{1}{2} \text{ Coeff. of } y\right) \text{i.e.} \left(-\frac{1}{2} \times -6, -\frac{1}{2} \times 4\right) = (3, -2)$$

and, $\quad$ Radius $= \sqrt{\left(-\frac{6}{2}\right)^2 + \left(\frac{4}{2}\right)^2 - (-12)} = \sqrt{9 + 4 + 12} = 5.$

EXAMPLE 2 *Find the centre and radius of the circle given by the equation*

$$2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0.$$

SOLUTION In the given equation the coefficients of x^2 and y^2 are not unity. So, we re-write the equation to make the coefficients of x^2 and y^2 unity.

We have, $\quad 2x^2 + 2y^2 + 3x + 4y + \frac{9}{8} = 0 \Rightarrow x^2 + y^2 + \frac{3}{2}x + 2y + \frac{9}{16} = 0.$

So, the coordinates of the centre are $(-3/4, -1)$ and, Radius $= \sqrt{\left(\frac{3}{4}\right)^2 + (1)^2 - \frac{9}{16}} = 1.$

Type II **ON FINDING THE EQUATION OF A CIRCLE SATISFYING GIVEN CONDITIONS**

EXAMPLE 3 *Find the equation of the circle whose centre is at the point (4, 5) and which passes through the centre of the circle* $x^2 + y^2 - 6x + 4y - 12 = 0$.

SOLUTION The coordinates of the centre of the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ are $C_1 (3, -2)$. Therefore, the required circle passes through the point $C_1 (3, -2)$ and has its centre at the point $C (4, 5)$. So, its radius is equal to

$$C C_1 = \sqrt{(4-3)^2 + (5+2)^2} = \sqrt{50}$$

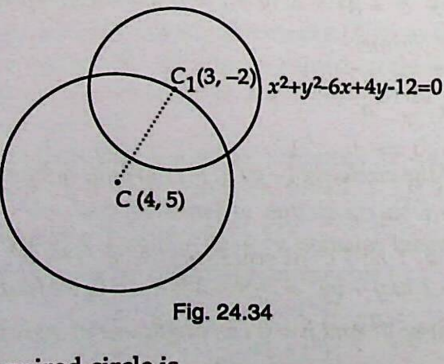

Fig. 24.34

Hence, the equation of the required circle is

$$(x-4)^2 + (y-5)^2 = (\sqrt{50})^2 \quad \text{or, } x^2 + y^2 - 8x - 10y - 9 = 0$$

EXAMPLE 4 *Find the equation of the circle concentric with the circle* $2x^2 + 2y^2 + 8x + 10y - 39 = 0$ *and having its area equal to* 16π *square units.*

SOLUTION The equation of the given circle is

$$2x^2 + 2y^2 + 8x + 10y - 39 = 0 \Rightarrow x^2 + y^2 + 4x + 5y - 39/2 = 0.$$

The coordinates of its centre are $(-2, -5/2)$. The required circle is concentric with the above circle, therefore the coordinates its centre are $(-2, -5/2)$.

Let r be the radius of the required circle. Then, its area is πr^2. But, it is given that its area is 16π sq. units.

$\therefore \quad \pi r^2 = 16\pi \Rightarrow r = 4$

Hence, the equation of the required circle is

$$(x+2)^2 + (y+5/2)^2 = 4^2 \text{ or, } 4x^2 + 4y^2 + 16x + 20y - 23 = 0.$$

Type III ON FINDING THE EQUATION OF A CIRCLE PASSING THROUGH THREE GIVEN POINTS

EXAMPLE 5 *Find the equation of the circle that passes through the points (1, 0), (–1, 0) and (0, 1).*

SOLUTION Let the required circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

It passes through (1, 0), (–1, 0) and (0, 1). Therefore, on substituting the coordinates of three points successively in equation (i), we get

$$1 + 2g + c = 0 \quad \text{...(ii)}, \qquad 1 - 2g + c = 0 \quad \text{...(iii)}, \qquad 1 + 2f + c = 0 \quad \text{...(iv)}$$

Subtracting (iii) from (ii), we get

$$4g = 0 \Rightarrow g = 0$$

Putting $g = 0$ in (ii), we obtain $c = -1$.

Now, putting $c = -1$ in (iv), we get $f = 0$.

Substituting the values of g, f and c in equation (i), we obtain the equation of the required circle as $x^2 + y^2 = 1$.

EXAMPLE 6 *Find the equation of the circle which passes through the points (5, – 8), (2, – 9) and (2, 1). Find also the coordinates of its centre and radius.*

SOLUTION Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It passes through the points (5, – 8), (2, – 9) and (2, 1). Therefore,

$$89 + 10g - 16f + c = 0 \quad \text{...(ii)}$$
$$85 + 4g - 18f + c = 0 \quad \text{...(iii)}$$
$$5 + 4g + 2f + c = 0 \quad \text{...(iv)}$$

Subtracting (iii) from (ii), we obtain

$$4 + 6g + 2f = 0 \Rightarrow 2 + 3g + f = 0 \quad \text{...(v)}$$

Subtracting (iv) from (iii), we get

$$80 + 0g - 20f = 0 \Rightarrow f = 4$$

Putting $f = 4$ in (v), we get $g = -2$. Putting $f = 4, g = -2$ in (iv), we get

$$5 - 8 + 8 + c = 0 \Rightarrow c = -5$$

Substituting the values of g, f and c in equation (i), we obtain the equation of the required circle as

$$x^2 + y^2 - 4x + 8y - 5 = 0.$$

The coordinates of the centre are $(-g, -f)$ i.e $(2, -4)$.

and, $\quad$ Radius $= \sqrt{g^2 + f^2 - c} = \sqrt{4 + 16 + 5} = 5.$

EXAMPLE 7 *The straight line* $\frac{x}{a}+\frac{y}{b}=1$ *cuts the coordinate axes at A and B. Find the equation of the circle passing through O (0, 0), A and B.*

SOLUTION The straight line $\frac{x}{a}+\frac{y}{b}=1$ cuts the coordinate axes at $A(a, 0)$ and $B(0, b)$.

Let $x^2+y^2+2gx+2fy+c=0$ ...(i)

be the circle passing through O, A and B. Then,

$$0 + \ + c = 0 \quad ...(ii)$$

$$a^2+2ga+c=0 \quad ...(iii)$$

$$b^2+2fb+c=0 \quad ...(iv)$$

Solving (ii), (iii) and (iv), we obtain

$$g=-\frac{a}{2}, f=-\frac{b}{2} \text{ and } c=0.$$

Substituting these values in (i), we obtain the equation of the required circle as

$$x^2+y^2-ax-by=0$$

ALITER The line represented by the equation $\frac{x}{a}+\frac{y}{b}=1$ meets the coordinate axes at $A(a, 0)$ and $B(0, b)$. Clearly, $\angle AOB = 90°$. So, AB is a diameter of the circle such that

$$AB=\sqrt{(a-0)^2+(0-b)^2}=\sqrt{a^2+b^2}$$

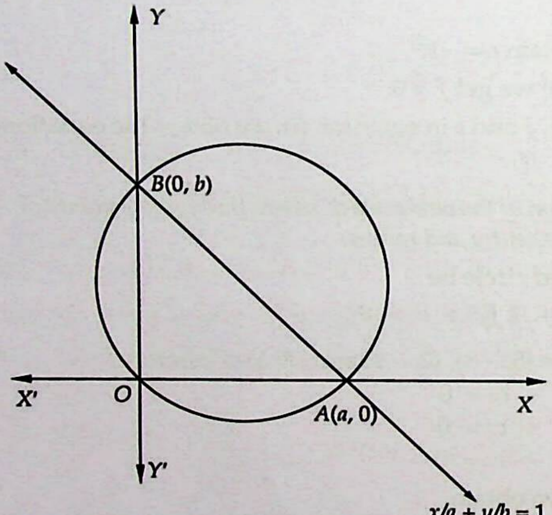

Fig. 24.35

$\therefore$ Radius $=\frac{1}{2}AB=\frac{1}{2}\sqrt{a^2+b^2}$.

The centre C of the circle is the mid-point of AB and so its coordinates are

$$\left(\frac{a+0}{2}, \frac{0+b}{2}\right)=\left(\frac{a}{2}, \frac{b}{2}\right).$$

Hence, the equation of the circle is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{1}{2}\sqrt{a^2+b^2}\right)^2 \text{ or, } x^2+y^2-ax-by=0$$

EXAMPLE 8 *Find the equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius.*

SOLUTION Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

This passes through the points A (1, 0) and B (0, 1).

$$\therefore \quad 1 + 2g + c = 0 \text{ and, } 1 + 2f + c = 0 \Rightarrow g = -\left(\frac{c+1}{2}\right) \text{ and, } f = -\left(\frac{c+1}{2}\right)$$

Let r be the radius of circle (i). Then,

$$r = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \quad r = \sqrt{\left(\frac{c+1}{2}\right)^2 + \left(\frac{c+1}{2}\right)^2 - c} \Rightarrow r = \sqrt{\frac{c^2+1}{2}} \Rightarrow r^2 = \frac{1}{2}(c^2+1)$$

Now, $\frac{1}{2}c^2 \geq 0 \Rightarrow \frac{1}{2}c^2 + \frac{1}{2} \geq \frac{1}{2} \Rightarrow r^2 \geq \frac{1}{2}$

Thus, the minimum value of r^2 is $\frac{1}{2}$.

Also, $r^2 = \frac{1}{2} \Rightarrow \frac{1}{2}c^2 + \frac{1}{2} = \frac{1}{2} \Rightarrow c = 0$

So, r is minimum when $c = 0$ and in that case, the minimum value of r is $\frac{1}{\sqrt{2}}$.

Putting $c = 0$ in $g = -\frac{c+1}{2}$ and $f = -\frac{c+1}{2}$, we get $g = -\frac{1}{2}$ and $f = -\frac{1}{2}$.

Substituting the values of g, f and c in (i), we get $x^2 + y^2 - x - y = 0$ as the equation of the required circle.

NOTE *To prove that four given points are concyclic; find the equation of the circle passing through any of the three given points and show that the fourth point lies on it.*

Type IV ON CONCYCLIC POINTS

EXAMPLE 9 *Show that the points* (9, 1), (7, 9) (– 2, 12) *and* (6, 10) *are concyclic.*

SOLUTION Let the equation of the circle passing through (9, 1), (7, 9) and (– 2, 12) be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

Then,

$$82 + 18g + 2f + c = 0 \quad \text{...(ii)}$$

$$130 + 14g + 18f + c = 0 \quad \text{...(iii)}$$

$$148 - 4g + 24f + c = 0 \quad \text{...(iv)}$$

Subtracting (ii) from (iii), we get

$$48 - 4g + 16f = 0 \Rightarrow 12 - g + 4f = 0 \quad \text{...(v)}$$

Subtracting (iii) from (iv), we get

$$18 - 18g + 6f = 0 \Rightarrow 3 - 3g + f = 0 \quad \text{...(vi)}$$

Solving (v) and (vi) as simultaneous linear equations in g and f, we get: $f = -3, g = 0$.

Putting $f = -3, g = 0$ in (ii), we get

$$82 + 0 - 6 + c = 0 \Rightarrow c = -76$$

Substituting the values of g, f and c in (i), we get $x^2 + y^2 - 6y - 76 = 0$ as the equation of the circle passing through points (9, 1), (7, 9) and (– 2, 12).

Clearly, point (6, 10) satisfies this equation. Hence, the given points are concyclic.

Type V ON FINDING THE EQUATION OF A CIRCLE SATISFYING THREE GIVEN CONDITIONS

EXAMPLE 10 *Find the equation of the circle which passes through the points $(1, -2)$ and $(4, -3)$ and has its centre on the line $3x + 4y = 7$.*

SOLUTION Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(i)}$$

It passes through $(1, -2)$ and $(4, -3)$.

$$\therefore \quad 5 + 2g - 4f + c = 0 \quad \text{...(ii)}$$

$$\text{and,} \quad 25 + 8g - 6f + c = 0 \quad \text{...(iii)}$$

The centre $(-g, -f)$ of (i) lies on $3x + 4y = 7$.

$$\therefore \quad -3g - 4f = 7 \quad \text{...(iv)}$$

Subtracting (ii) from (iii), we get

$$20 + 6g - 2f = 0 \Rightarrow 10 + 3g - f = 0 \quad \text{...(v)}$$

Solving (iv) and (v) as simultaneous equations, we get

$$g = -\frac{47}{15} \text{ and } f = \frac{3}{5}$$

Substituting the values of g and f in (ii), we get

$$5 - \frac{94}{15} - \frac{12}{5} + c = 0 \Rightarrow c = \frac{55}{15} = \frac{11}{3}$$

Substituting the values of g, f and c in (i) we obtain the required equation of the circle as

$$x^2 + y^2 - \frac{94}{15}x + \frac{6}{5}y + \frac{11}{3} = 0 \text{ or, } 15(x^2 + y^2) - 94x + 18y + 33 = 0$$

EXAMPLE 11 *Find the equation of the circle circumscribing the triangle formed by the lines $x + y = 6$, $2x + y = 4$ and $x + 2y = 5$.*

SOLUTION Let the equations of sides AB, BC and CA of ΔABC are respectively

$$x + y = 6 \quad \text{...(i)} \qquad 2x + y = 4 \quad \text{...(ii)} \qquad \text{and} \qquad x + 2y = 5 \quad \text{...(iii)}$$

Solving (i) and (iii), (i) and (ii); (ii) and (iii) we get the coordinates of A, B and C. The coordinates A, B and C are $(7, -1)$, $(-2, 8)$ and $(1, 2)$ respectively.

Let the equation of the circumcircle of ΔABC be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(iv)}$$

It passes through the points $A(7, -1)$, $B(-2, 8)$ and $C(1, 2)$. Therefore,

$$50 + 14g - 2f + c = 0 \quad \text{...(v)}$$

$$68 - 4g + 16f + c = 0 \quad \text{...(vi)}$$

$$5 + 2g + 4f + c = 0 \quad \text{...(vii)}$$

Subtracting (v) from (vi), we get

$$18 - 18g + 18f = 0 \Rightarrow 1 - g + f = 0 \quad \text{...(viii)}$$

Subtracting (v) from (vii), we get: $-45 - 12g + 6f = 0$ ...(ix)

Solving (viii) and (ix), we get: $g = -17/2, f = -19/2$.

Putting the values of g and f in (v), we get $c = 50$.

Substituting the values of g, f and c in (iv), the equation of the required circumcircle is

$$x^2 + y^2 - 17x - 19y + 50 = 0$$

LEVEL-2

Type VI MISCELLANEOUS EXAMPLES

EXAMPLE 12 *Find the radius of the circle* $(x \cos \alpha + y \sin \alpha - a)^2 + (x \sin \alpha - y \cos \alpha - b)^2 = k^2$, *if* α *varies, the locus of its centre is again a circle. Also, find its centre and radius.*

SOLUTION The given equation is

$$(x \cos \alpha + y \sin \alpha - a)^2 + (x \sin \alpha - y \cos \alpha - b)^2 = k^2$$

$$\Rightarrow \quad x^2 (\cos^2 \alpha + \sin^2 \alpha) + y^2 (\sin^2 \alpha + \cos^2 \alpha) - 2 (a \cos \alpha + b \sin \alpha) x - 2 (a \sin \alpha - b \cos \alpha) y + a^2 + b^2 - k^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 - 2x (a \cos \alpha + b \sin \alpha) - 2y (a \sin \alpha - b \cos \alpha) + a^2 + b^2 - k^2 = 0$$

The coordinates of the centre of this circle are $(a \cos \alpha + b \sin \alpha, a \sin \alpha - b \cos \alpha)$. Let its radius be r. Then,

$$r = \sqrt{(a \cos \alpha + b \sin \alpha)^2 + (a \sin \alpha - b \cos \alpha)^2 - (a^2 + b^2 - k^2)}$$

$$\Rightarrow \quad r = \sqrt{a^2 (\cos^2 \alpha + \sin^2 \alpha) + b^2 (\sin^2 \alpha + \cos^2 \alpha) - (a^2 + b^2 - k^2)}$$

$$\Rightarrow \quad r = \sqrt{a^2 + b^2 - a^2 - b^2 + k^2} = k$$

Let (p, q) be the coordinates of the centre of the given circle. Then,

$$p = a \cos \alpha + b \sin \alpha \text{ and } q = a \sin \alpha - b \cos \alpha$$

To find the locus of (p, q) we have to eliminate α. Squaring and adding these two, we get

$$p^2 + q^2 = (a \cos \alpha + b \sin \alpha)^2 + (a \sin \alpha - b \cos \alpha)^2$$

$$\Rightarrow \quad p^2 + q^2 = a^2 (\cos^2 \alpha + \sin^2 \alpha) + b^2 (\sin^2 \alpha + \cos^2 \alpha)$$

$$\Rightarrow \quad p^2 + q^2 = a^2 + b^2$$

Hence, the locus of (p, q) is $x^2 + y^2 = a^2 + b^2$. This is a circle having centre at (0, 0) and radius equal to $\sqrt{a^2 + b^2}$.

EXAMPLE 13 *Find the area of an equilateral triangle inscribed in the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

SOLUTION Let ABC be an equilateral triangle inscribed in the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{...(i)}$$

Let $O(-g, -f)$ be the centre of the circle. Then,

$$OA = OB = OC = \sqrt{g^2 + f^2 - c}$$

In ΔOBD, we have

$$\sin 60° = \frac{BD}{OB}$$

$$\Rightarrow \quad BD = \frac{\sqrt{3}}{2} OB$$

$$\Rightarrow \quad BD = \frac{\sqrt{3}}{2} \sqrt{g^2 + f^2 - c}$$

$$\therefore \quad BC = 2\,BD = \sqrt{3} \sqrt{g^2 + f^2 - c}$$

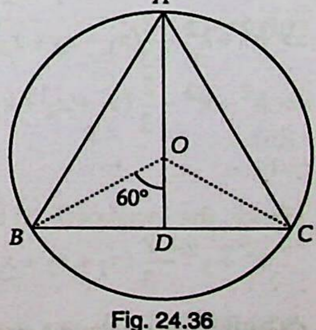

Fig. 24.36

$$\therefore \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4} (\text{Side})^2 = \frac{\sqrt{3}}{4} (BC)^2$$

$$\Rightarrow \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4} \times 3 (g^2 + f^2 - c) \text{ sq. units} = \frac{3\sqrt{3}}{4} (g^2 + f^2 - c) \text{ sq. units.}$$

EXAMPLE 14 *If the line $lx+my=1$ is a tangent to the circle $x^2+y^2=a^2$, then prove that (l, m) lies on a circle.* **[NCERT EXEMPLAR]**

SOLUTION If the line $lx+my-1=0$ touches the circle $x^2+y^2=a^2$, then length of the perpendicular from its centre O (0, 0) is equal to the radius a.

$$\therefore \quad \left|\frac{m+m\times 0-1}{\sqrt{l^2+m^2}}\right|=a$$

$$\Rightarrow \quad \frac{1}{\sqrt{l^2+m^2}}=a$$

$$\Rightarrow \quad l^2+m^2=\frac{1}{a^2}$$

$\Rightarrow$ (l, m) satisfies the equation $x^2+y^2=\dfrac{1}{a^2}$

$\Rightarrow$ (l, m) lies on the circle $x^2+y^2=\dfrac{1}{a^2}$.

Hence, (l, m) lies on a circle.

EXAMPLE 15 *If the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$, then prove that $(l^2+m^2)\,a^2=n^2$.* **[NCERT EXEMPLAR]**

SOLUTION If the line $lx+my+n=0$ touches the circle $x^2+y^2=a^2$, then length of the perpendicular from its centre O (0, 0) is equal to its radius a.

$$\therefore \quad \left|\frac{l\times 0+m\times 0+n}{\sqrt{l^2+m^2}}\right|=a$$

$\Rightarrow$ $(l^2+m^2)\,a^2=n^2$, which is the required condition.

EXAMPLE 16 *Prove that the locus of a point which moves such that the sum of the squares of its distances from the vertices of a triangle is constant is a circle having centre at the centroid of the triangle.* **[NCERT EXEMPLAR]**

SOLUTION Let $A\,(x_1, y_1)$, $B\,(x_2, y_2)$ and $C\,(x_3, y_3)$ be the vertices of ΔABC, and let $P(h, k)$ be a point which moves in such a way that

$$PA^2+PB^2+PC^2=c \text{ (constant)}$$

$$\Rightarrow (h-x_1)^2+(k-y_1)^2+(h-x_2)^2+(k-y_2)^2+(h-x_3)^2+(k-y_3)^2=c$$

$$\Rightarrow 3h^2+3k^2-2h\,(x_1+x_2+x_3)-2k\,(y_1+y_2+y_3)+x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c=0$$

$$\Rightarrow h^2+k^2-\frac{2}{3}(x_1+x_2+x_3)\,h-\frac{2}{3}(y_1+y_2+y_3)\,k+\frac{1}{3}(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c)=0$$

$$\Rightarrow h^2+k^2-\frac{2}{3}(x_1+x_2+x_3)\,h-\frac{2}{3}(y_1+y_2+y_3)\,k+\lambda=0,$$

where $\lambda=\dfrac{1}{3}(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2-c)$

Hence, the locus of (h, k) is

$$x^2+y^2-\frac{2}{3}(x_1+x_2+x_3)\,x-\frac{2}{3}(y_1+y_2+y_3)\,y+\lambda=0$$

Clearly, it represents a circle with centre at $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$, which is the centroid of ΔABC.

EXAMPLE 17 *If a circle of constant radius 3c passes through the origin and meets the axes at A and B, prove that the locus of the centroid of ΔABC is a circle of radius 2c.* **[NCERT EXEMPLAR]**

SOLUTION Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. Clearly, ΔOAB is a right triangle right-angled at O. Therefore, AB is a diameter of the circle.

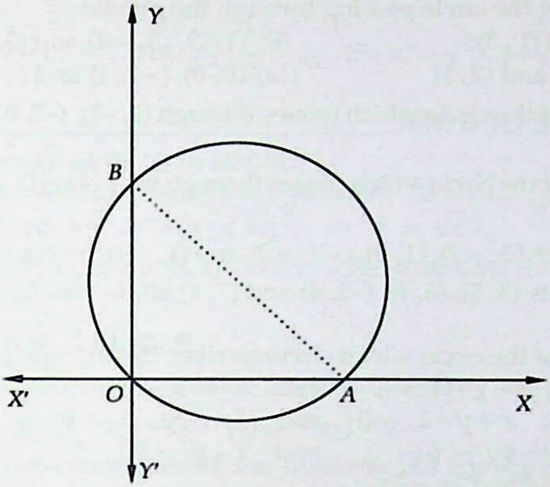

Fig. 24.37

$\therefore \quad AB = 2(3c) = 6c$

In ΔAOB,

$$OA^2 + OB^2 = AB^2 \qquad \text{[Using Pythagoras theorem]}$$

$$\Rightarrow \quad a^2 + b^2 = 36c^2 \qquad \text{...(i)}$$

Let (α, β) be the coordinates of the centroid of ΔOAB. Then,

$$\alpha = \frac{0+a+0}{3} = \frac{a}{3}, \quad \beta = \frac{0+0+b}{3} = \frac{b}{3} \Rightarrow a = 3\alpha \text{ and } b = 3\beta$$

Substituting the values of a and b in (i), we obtain

$$9\alpha^2 + 9\beta^2 = 36c^2 \text{ or, } \alpha^2 + \beta^2 = (2c)^2$$

Hence, the locus of (α, β) is $x^2 + y^2 = (2c)^2$, which is a circle of radius $2c$.

ALITER Let $OA = a$ and $OB = b$. Then, the coordinates of A and B are $(a, 0)$ and $(0, b)$ respectively.

The equation of the circle passing through O, A and B is

$$x^2 + y^2 - ax - by = 0 \qquad \text{[See Example 7]} \qquad \text{...(i)}$$

Let (α, β) be the coordinates of the centroid of ΔOAB. Then,

$$\alpha = \frac{a}{3} \text{ and } \beta = \frac{b}{3} \Rightarrow a = 3\alpha \text{ and } b = 3\beta \qquad \text{...(ii)}$$

It is given that the radius of circle (i) is $3c$.

$$\therefore \quad \sqrt{\frac{a^2}{4} + \frac{b^2}{4} - 0} = 3c \Rightarrow a^2 + b^2 = 36c^2 \qquad \text{...(iii)}$$

Eliminating a and b between (ii) and (iii), we obtain

$$9\alpha^2 + 9\beta^2 = 36c^2 \quad \text{or,} \quad \alpha^2 + \beta^2 = 4c^2$$

Hence, the locus of (α, β) is $x^2 + y^2 = (2c)^2$, which is a circle of radius $2c$.

EXERCISE 24.2

LEVEL-1

1. Find the coordinates of the centre and radius of each of the following circles :
   (i) $x^2 + y^2 + 6x - 8y - 24 = 0$ (ii) $2x^2 + 2y^2 - 3x + 5y = 7$

(iii) $\frac{1}{2}(x^2 + y^2) + x\cos\theta + y\sin\theta - 4 = 0$ (iv) $x^2 + y^2 - ax - by = 0$

2. Find the equation of the circle passing through the points:
(i) $(5, 7), (8, 1)$ and $(1, 3)$ (ii) $(1, 2), (3, -4)$ and $(5, -6)$
(iii) $(5, -8), (-2, 9)$ and $(2, 1)$ (iv) $(0, 0), (-2, 1)$ and $(-3, 2)$
3. Find the equation of the circle which passes through $(3, -2), (-2, 0)$ and has its centre on the line $2x - y = 3$.
4. Find the equation of the circle which passes through the points $(3, 7), (5, 5)$ and has its centre on the line $x - 4y = 1$.
5. Show that the points $(3, -2), (1, 0), (-1, -2)$ and $(1, -4)$ are concyclic.
6. Show that the points $(5, 5), (6, 4), (-2, 4)$ and $(7, 1)$ all lie on a circle, and find its equation, centre and radius.
7. Find the equation of the circle which circumscribes the triangle formed by the lines
(i) $x + y + 3 = 0$, $x - y + 1 = 0$ and $x = 3$
(ii) $2x + y - 3 = 0$, $x + y - 1 = 0$ and $3x + 2y - 5 = 0$
(iii) $x + y = 2$, $3x - 4y = 6$ and $x - y = 0$.
(iv) $y = x + 2$, $3y = 4x$ and $2y = 3x$. [NCERT EXEMPLAR]
8. Prove that the centres of the three circles $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 2x + 4y - 10 = 0$ and $x^2 + y^2 - 10x - 16y - 1 = 0$ are collinear.
9. Prove that the radii of the circles $x^2 + y^2 = 1$, $x^2 + y^2 - 2x - 6y - 6 = 0$ and $x^2 + y^2 - 4x - 12y - 9 = 0$ are in A.P.
10. Find the equation of the circle which passes through the origin and cuts off chords of lengths 4 and 6 on the positive side of the x-axis and y-axis respectively.
11. Find the equation of the circle concentric with the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and double of its area.
12. Find the equation to the circle which passes through the points $(1, 1)$ $(2, 2)$ and whose radius is 1. Show that there are two such circles.
13. Find the equation of the circle concentric with $x^2 + y^2 - 4x - 6y - 3 = 0$ and which touches the y-axis.
14. If a circle passes through the point $(0, 0), (a, 0), (0, b)$, then find the coordinates of its centre.
15. Find the equation of the circle which passes through the points $(2, 3)$ and $(4, 5)$ and the centre lies on the straight line $y - 4x + 3 = 0$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $(-3, 4); 7$ (ii) $\left(\frac{3}{4}, -\frac{5}{4}\right); \frac{3\sqrt{10}}{4}$
(iii) $(-\cos\theta, -\sin\theta); 3$ (iv) $\left(\frac{a}{2}, \frac{b}{2}\right); \frac{1}{2}\sqrt{a^2 + b^2}$.
2. (i) $3(x^2 + y^2) - 29x - 19y + 56 = 0$ (ii) $x^2 + y^2 - 22x - 4y + 25 = 0$
(iii) $x^2 + y^2 + 116x + 48y - 285 = 0$ (iv) $x^2 + y^2 - 3x - 11y = 0$
3. $x^2 + y^2 + 3x + 12y + 2 = 0$ 4. $x^2 + y^2 + 6x + 2y - 90 = 0$
6. $x^2 + y^2 - 4x - 2y - 20 = 0; (2, 1), 5$
7. (i) $x^2 + y^2 - 6x + 2y - 15 = 0$ (ii) $x^2 + y^2 - 13x - 5y + 16 = 0$
(iii) $x^2 + y^2 + 4x + 6y - 12 = 0$ (iv) $x^2 + y^2 - 46x + 22y = 0$

10. $x^2 + y^2 - 4x - 6y = 0$ 11. $x^2 + y^2 - 6x + 12y - 15 = 0$

12. $x^2 + y^2 - 4x - 2y + 4 = 0,\ x^2 + y^2 - 2x - 4y + 4 = 0$

13. $x^2 + y^2 - 4x - 6y + 9 = 0$ 14. $\left(\frac{a}{2}, \frac{b}{2}\right)$ 15. $x^2 + y^2 - 4x - 10y + 25 = 0$

HINTS TO SELECTED PROBLEMS

10. The circle passes through (0, 0), (4, 0) and (0, 6).
11. Centre of the given circle is (3, – 6), and radius $= \sqrt{9 + 36 - 15} = \sqrt{30}$. Let r be the radius of the required circle. Then, $\pi r^2 = 2(\pi(\sqrt{30})^2) \Rightarrow r = \sqrt{60}$.
13. The centre of the required circle is (2, 3). As it touches y-axis. So, its radius = x-coordinate of centre = 2.

24.5 DIAMETER FORM OF A CIRCLE

THEOREM *The equation of the circle drawn on the straight line joining two given points* (x_1, y_1) *and* (x_2, y_2) *as diameter is* $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

PROOF Let A and B be the extremities of the diameter AB having coordinates (x_1, y_1) and (x_2, y_2) respectively. Let $P(x, y)$ be any point on the circle. Join point P to points A and B. Then,

$$m_1 = \text{Slope of the line } AP = \frac{y - y_1}{x - x_1} \text{ and, } m_2 = \text{Slope of the line } BP = \frac{y - y_2}{x - x_2}$$

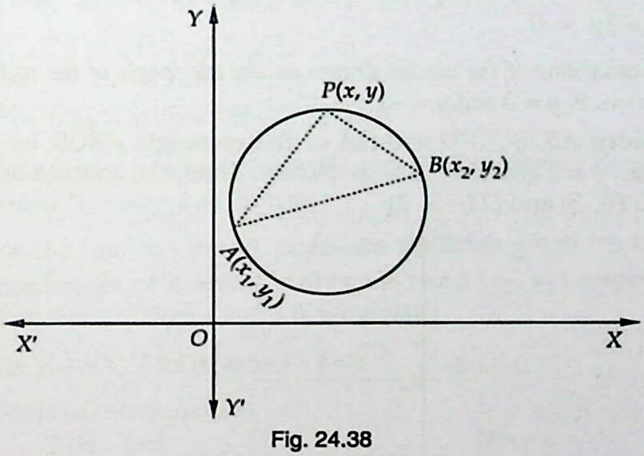

Fig. 24.38

The angle subtended at the point P in the semi-circle APB is a right angle.

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \quad \frac{y - y_1}{x - x_1} \times \frac{y - y_2}{x - x_2} = -1$$

$$\Rightarrow \quad (y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$$\Rightarrow \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0 \quad \text{...(i)}$$

This is the required equation of the circle having (x_1, y_1) and (x_2, y_2) as the coordinates of the end points of a diameter. **Q.E.D.**

REMARK 1 *If the coordinates of the end points of a diameter of a circle are given, we can also find the equation of the circle by finding the coordinates of the centre and radius. The centre is the mid-point of the diameter and radius is half of the length of the diameter.*

REMARK 2 *Equation (i) can also be written as*

$$x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1x_2 + y_1y_2 = 0$$

or, $x^2 + y^2 - x$ *(Sum of the abscissae)* $- y$ *(Sum of the ordinates)* + *Product of the abscissae* + *Product of the ordinates* = 0.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the circle, the coordinates of the end points of whose diameter are* (–1, 2) *and* (4, –3).

SOLUTION We know that the equation of the circle described on the line segment joining (x_1, y_1) and (x_2, y_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.

Here, $x_1 = -1$, $x_2 = 4$, $y_1 = 2$ and $y_2 = -3$.

So, the equation of the required circle is

$$(x + 1)(x - 4) + (y - 2)(y + 3) = 0 \quad \text{or,} \quad x^2 + y^2 - 3x + y - 10 = 0.$$

EXAMPLE 2 *Find the equation of the circle drawn on the intercept made by the line* $2x + 3y = 6$ *between the coordinate axes as diameter.*

SOLUTION The line $2x + 3y = 6$ meets x and y-axes at $A(3, 0)$ and $B(0, 2)$ respectively. Taking AB as a diameter, the equation of the required circle is

$$(x - 3)(x - 0) + (y - 0)(y - 2) = 0 \qquad [\text{Using: } (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0]$$

or, $x^2 + y^2 - 3x - 2y = 0$

EXAMPLE 3 *Find the equations of the circles drawn on the diagonals of the rectangle as its diameter whose sides are* $x = 6$, $x = -3$, $y = 3$ *and* $y = -1$.

SOLUTION Let the sides AB, BC, CD and DA of the rectangle $ABCD$ be represented by the equations $y = -1$, $x = 6$, $y = 3$ and $x = -3$ respectively. Then, the coordinates of the vertices are $A(-3, 1)$, $B(6, -1)$, $C(6, 3)$ and $D(-3, 3)$.

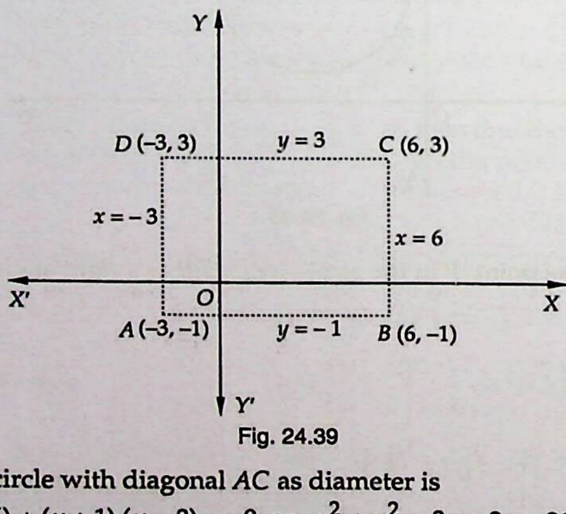

Fig. 24.39

The equation of the circle with diagonal AC as diameter is

$$(x + 3)(x - 6) + (y + 1)(y - 3) = 0 \text{ or, } x^2 + y^2 - 3x - 2y - 21 = 0$$

The equation of the circle with diagonal BD as diameter is

$$(x - 6)(x + 3) + (y + 1)(y - 3) = 0 \text{ or, } x^2 + y^2 - 3x - 2y - 21 = 0.$$

EXAMPLE 4 *If* $y = 2x$ *is a chord of the circle* $x^2 + y^2 - 10x = 0$, *find the equation of a circle with this chord as diameter.*

SOLUTION The points of intersection of the given chord and the given circle are obtained by solving $y = 2x$ and $x^2 + y^2 - 10x = 0$ simultaneously. Putting $y = 2x$ in $x^2 + y^2 - 10x = 0$, we get

$$5x^2 - 10x = 0 \Rightarrow 5x(x-2) = 0 \Rightarrow x = 0, 2.$$

Putting $x = 0$ and $x = 2$ respectively in $y = 2x$, we get $y = 0$ and $y = 4$.

Thus, the coordinates of the points of intersection of the given line and the given circle are $A(0, 0)$ and $B(2, 4)$. The equation of the circle with chord AB as diameter is

$$(x-0)(x-2) + (y-0)(y-4) = 0 \text{ or, } x^2 + y^2 - 2x - 4y = 0.$$

LEVEL-2

EXAMPLE 5 *If the abscissae and the ordinates of two points A and B be the roots of $ax^2 + bx + c = 0$ and $a'y^2 + b'y + c' = 0$ respectively, show that the equation of the circle described on AB as diameter is $aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0$.*

SOLUTION Let (x_1, y_1) and (x_2, y_2) be the coordinates of points A and B respectively.
It is given that x_1, x_2 are roots of $ax^2 + bx + c = 0$ and y_1, y_2 are roots of $a'y^2 + b'y + c' = 0$.

$$\therefore \quad x_1 + x_2 = -\frac{b}{a}, \quad x_1 x_2 = \frac{c}{a}, \quad y_1 + y_2 = -\frac{b'}{a'} \text{ and } y_1 y_2 = \frac{c'}{a'} \quad \text{...(i)}$$

The equation of the circle with AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\Rightarrow \quad x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0.$$

$$\Rightarrow \quad x^2 + y^2 - x\left(-\frac{b}{a}\right) - \left(-\frac{b'}{a'}\right)y + \frac{c}{a} + \frac{c'}{a'} = 0 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad aa'(x^2 + y^2) + a'bx + ab'y + (ca' + c'a) = 0$$

EXAMPLE 6 *Find the equation of the circle on the straight line joining the points of intersection of $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ as diameter.*

SOLUTION Suppose the line $lx + my = 1$ intersects the lines given by $ax^2 + 2hxy + by^2 = 0$ in A and B. Let the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively. Eliminating y between $lx + my = 1$ and $ax^2 + 2hxy + by^2 = 0$, we obtain

$$x^2(am^2 - 2hlm + bl^2) - 2x(bl - hm) + b = 0$$

Clearly, x_1, x_2 are roots of this equation.

$$\therefore \quad x_1 + x_2 = \frac{2(bl - hm)}{am^2 - 2hlm + bl^2} \text{ and, } x_1 x_2 = \frac{b}{am^2 - 2hlm + bl^2}$$

Now, eliminating x between $lx + my = 1$ and $ax^2 + 2hxy + by^2 = 0$, we get

$$y^2(am^2 - 2h\,lm + bl^2) - 2y(am - hl) + a = 0.$$

Since y_1, y_2 are roots of this equation.

$$\therefore \quad y_1 + y_2 = \frac{2(am - hl)}{am^2 - 2hlm + bl^2} \text{ and, } y_1 y_2 = \frac{a}{am^2 - 2hlm + bl^2}$$

The equation of the circle with AB as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$$\text{or, } x^2 - x(x_1 + x_2) + x_1 x_2 + y^2 - y(y_1 + y_2) + y_1 y_2 = 0$$

$$\text{or, } x^2 + y^2 - \frac{2x(bl - hm)}{am^2 - 2hlm + bl^2} - \frac{2y(am - hl)}{am^2 - 2hlm + bl^2} + \frac{b}{am^2 - 2hlm + bl^2} + \frac{a}{am^2 - 2hlm + bl^2} = 0$$

or, $(x^2 + y^2)(am^2 - 2hlm + bl^2) - 2x(bl - hm) - 2y(am - hl) + (a + b) = 0$

This is the required equation of the circle.

EXAMPLE 7 *On the line joining (1, 0) and (3, 0) an equilateral triangle is drawn, having its vertex in the first quadrant. Find the equation to the circles described on its sides as diameter.*

SOLUTION Let (1, 0) and (3, 0) be the coordinates of the points A and B respectively. Then,

$$AB = \sqrt{(1-3)^2 + (0-0)^2} = 2.$$

Let C (x_1, y_1) be the third vertex of the equilateral triangle ABC. Then, $AC = BC = 2$

Now, $AC = \sqrt{(x_1 - 1)^2 + (y_1 - 0)^2}$, $BC = \sqrt{(x_1 - 3)^2 + (y_1 - 0)^2}$

$\therefore \quad AC = BC$

$\Rightarrow \quad AC^2 = BC^2$

$\Rightarrow \quad (x_1 - 1)^2 + {y_1}^2 = (x_1 - 3)^2 + {y_1}^2 \Rightarrow 4x_1 = 8 \Rightarrow x_1 = 2$

Again, $AC = 2$

$\Rightarrow \quad \sqrt{(x_1 - 1)^2 + y_1^2} = 2$

$\Rightarrow \quad (x_1 - 1)^2 + y_1^2 = 4$

$\Rightarrow \quad (2-1)^2 + y_1^2 = 4 \Rightarrow y_1 = \pm\sqrt{3} \Rightarrow y_1 = \sqrt{3}$ $\quad [\because x_1 = 2]$

So, the coordinates of C are $(2, \sqrt{3})$. $\quad [\because$ C (x_1, y_1) lies in first quadrant]

The equation of the circle on AC as diameter is

$$(x-1)(x-2) + (y-0)(y-\sqrt{3}) = 0 \text{ or, } x^2 + y^2 - 3x - \sqrt{3}y + 2 = 0.$$

Similarly, the equations of circles with AB and BC as diameters are

$$(x-1)(x-3) + (y-0)(y-0) = 0 \text{ and, } (x-3)(x-2) + (y-0)(y-\sqrt{3}) = 0$$

or, $\quad x^2 + y^2 - 4x + 3 = 0$ and, $x^2 + y^2 - 5x - \sqrt{3}y + 6 = 0$ respectively.

EXAMPLE 8 *Find the equations to the circles which pass through the origin and cut off equal chords of length 'a' from the straight lines* $y = x$ *and* $y = -x$.

SOLUTION From Fig. 24.40, we see that there will be four such circles which pass through the origin and cut off equal chords of length a from the straight lines $y = \pm x$.

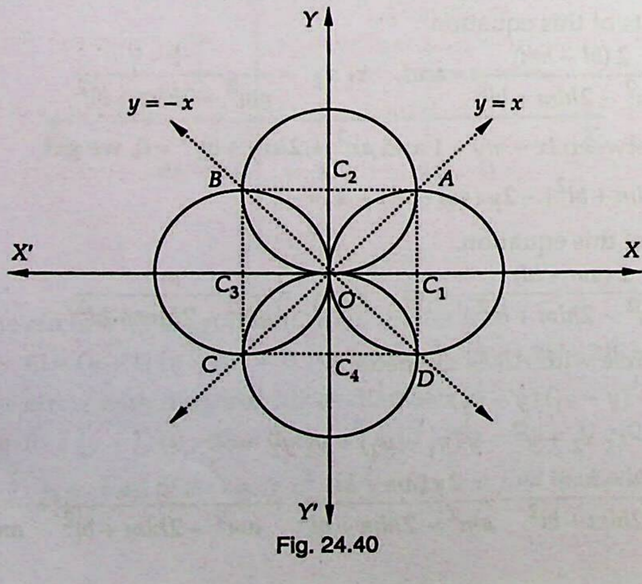

Fig. 24.40

Since $\angle AOB = \angle BOC = \angle COD = \angle DOA = \pi/2$. Therefore, AB, BC, CD and DA are diameters of the four circles.

Now, $\angle XOA = \pi/4$ and, $OA = a$

$\therefore \quad AC_1 = a \sin \frac{\pi}{4} = \frac{a}{\sqrt{2}}$ and, $OC_1 = a \cos \frac{\pi}{4} = \frac{a}{\sqrt{2}}$

So, the coordinates of A are $(a/\sqrt{2}, a/\sqrt{2})$.

Similarly, the coordinates of B, C and D are $(-a/\sqrt{2}, a/\sqrt{2}), (-a/\sqrt{2}, -a/\sqrt{2})$ and $(a/\sqrt{2}, -a/\sqrt{2})$ respectively.

The equation of the circle with AD as diameter is

$$\left(x - \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 - \sqrt{2}\, ax = 0.$$

Similarly, the equations of the required circles with BC, CD and AB as diameters are

$$\left(x + \frac{a}{\sqrt{2}}\right)\left(x + \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 + \sqrt{2}\, ax = 0$$

$$\left(x + \frac{a}{\sqrt{2}}\right)\left(x - \frac{a}{\sqrt{2}}\right) + \left(y + \frac{a}{\sqrt{2}}\right)\left(y + \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 + \sqrt{2}\, ay = 0$$

and, $$\left(x - \frac{a}{\sqrt{2}}\right)\left(x + \frac{a}{\sqrt{2}}\right) + \left(y - \frac{a}{\sqrt{2}}\right)\left(y - \frac{a}{\sqrt{2}}\right) = 0 \quad \text{or,} \quad x^2 + y^2 - \sqrt{2}\, ay = 0$$

respectively.

EXERCISE 24.3

LEVEL-1

1. Find the equation of the circle, the end points of whose diameter are $(2, -3)$ and $(-2, 4)$. Find its centre and radius.
2. Find the equation of the circle the end points of whose diameter are the centres of the circles $x^2 + y^2 + 6x - 14y - 1 = 0$ and $x^2 + y^2 - 4x + 10y - 2 = 0$.
3. The sides of a square are $x = 6,\ x = 9,\ y = 3$ and $y = 6$. Find the equation of a circle drawn on the diagonal of the square as its diameter.
4. Find the equation of the circle circumscribing the rectangle whose sides are $x - 3y = 4$, $3x + y = 22$, $x - 3y = 14$ and $3x + y = 62$.
5. Find the equation of the circle passing through the origin and the points where the line $3x + 4y = 12$ meets the axes of coordinates.
6. Find the equation of the circle which passes through the origin and cuts off intercepts a and b respectively from x and y-axes.
7. Find the equation of the circle whose diameter is the line segment joining $(-4, 3)$ and $(12, -1)$. Find also the intercept made by it on y-axis.

LEVEL-2

8. The abscissae of the two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. Find the equation of the circle with AB as diameter. Also, find its radius.
9. $ABCD$ is a square whose side is a; taking AB and AD as axes, prove that the equation of the circle circumscribing the square is $x^2 + y^2 - a(x + y) = 0$.
10. The line $2x - y + 6 = 0$ meets the circle $x^2 + y^2 - 2y - 9 = 0$ at A and B. Find the equation of the circle on AB as diameter.

11. Find the equation of the circle which circumscribes the triangle formed by the lines $x=0,\ y=0$ and $lx+my=1$.
12. Find the equations of the circles which pass through the origin and cut off equal chords of $\sqrt{2}$ units from the lines $y=x$ and $y=-x$.

ANSWERS

1. $x^2+y^2-y-16=0;\left(0,\frac{1}{2}\right),\ \frac{\sqrt{65}}{2}$
2. $x^2+y^2+x-2y-41=0$
3. $x^2+y^2-15x-9y+72=0$
4. $x^2+y^2-27x-3y+142=0$
5. $x^2+y^2-4x-3y=0$
6. $x^2+y^2\pm ax\pm by=0$
7. $x^2+y^2-8x-2y-51=0,\ 4\sqrt{13}$
8. $x^2+y^2+2ax+2py-b^2-q^2=0,\ \sqrt{a^2+b^2+p^2+q^2}$
10. $x^2+y^2+4x-4y+3=0$
11. $x^2+y^2-\frac{1}{l}x-\frac{1}{m}y=0$
12. $x^2+y^2\pm 2y=0,\ x^2+y^2\pm 2x=0.$

HINTS TO SELECTED PROBLEMS

5. The line $3x+4y=12$ meets the coordinate axes at $A\ (4,\ 0)$ and $B\ (0,\ 3)$. We have to find the equation of the circle with AB as diameter.
6. The coordinates of the end points of a diameter are $(\pm a,\ 0)$ and $(0,\ \pm b)$.
9. The required circle has $(0, 0)$ and (a, a) as the end points of a diameter.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the length of the intercept made by the circle $x^2+y^2+2x-4y-5=0$ on y-axis.
2. Write the coordinates of the centre of the circle passing through (0, 0), (4, 0) and (0, – 6).
3. Write the area of the circle passing through (– 2, 6) and having its centre at (1, 2).
4. If the abscissae and ordinates of two points P and Q are roots of the equations $x^2+2ax-b^2=0$ and $x^2+2px-q^2=0$ respectively, then write the equation of the circle with PQ as diameter.
5. Write the equation of the unit circle concentric with $x^2+y^2-8x+4y-8=0$.
6. If the radius of the circle $x^2+y^2+ax+(1-a)\,y+5=0$ does not exceed 5, write the number of integral values a.
7. Write the equation of the circle passing through (3, 4) and touching y-axis at the origin.
8. If the line $y=mx$ does not intersect the circle $(x+10)^2+(y+10)^2=180$, then write the set of values taken by m.
9. Write the coordinates of the centre of the circle inscribed in the square formed by the lines $x=2,\ x=6,\ y=5$ and $y=9$.

ANSWERS

1. 6 units
2. (2, – 3)
3. 10π sq. units
4. $x^2+y^2+2ax+2py-b^2-q^2=0$
5. $x^2+y^2-8x+4y+19=0$
6. 16
7. $3(x^2+y^2)-25x=0$
8. $m\in\left(-2,-\frac{1}{2}\right)$
9. (4, 7)

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternatives in each of the following:

1. If the equation of a circle is $\lambda x^2 + (2\lambda - 3) y^2 - 4x + 6y - 1 = 0$, then the coordinates of centre are
   (a) $(4/3, -1)$ (b) $(2/3, -1)$ (c) $(-2/3, 1)$ (d) $(2/3, 1)$
2. If $2x^2 + \lambda xy + 2y^2 + (\lambda - 4)x + 6y - 5 = 0$ is the equation of a circle, then its radius is
   (a) $3\sqrt{2}$ (b) $2\sqrt{3}$ (c) $2\sqrt{2}$ (d) none of these
3. The equation $x^2 + y^2 + 2x - 4y + 5 = 0$ represents
   (a) a point (b) a pair of straight lines
   (c) a circle of non-zero radius (d) none of these
4. If the equation $(4a - 3)x^2 + ay^2 + 6x - 2y + 2 = 0$ represents a circle, then its centre is
   (a) $(3, -1)$ (b) $(3, 1)$ (c) $(-3, 1)$ (d) none of these
5. The radius of the circle represented by the equation $3x^2 + 3y^2 + \lambda xy + 9x + (\lambda - 6)y + 3 = 0$ is
   (a) $3/2$ (b) $\sqrt{17}/2$ (c) $2/3$ (d) none of these
6. The number of integral values of λ for which the equation $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius cannot exceed 5, is
   (a) 14 (b) 18 (c) 16 (d) none of these
7. The equation of the circle passing through the point (1, 1) and having two diameters along the pair of lines $x^2 - y^2 - 2x + 4y - 3 = 0$, is
   (a) $x^2 + y^2 - 2x - 4y + 4 = 0$ (b) $x^2 + y^2 + 2x + 4y - 4 = 0$
   (c) $x^2 + y^2 - 2x + 4y + 4 = 0$ (d) none of these
8. If the centroid of an equilateral triangle is (1, 1) and its one vertex is (−1, 2), then the equation of its circumcircle is
   (a) $x^2 + y^2 - 2x - 2y - 3 = 0$ (b) $x^2 + y^2 + 2x - 2y - 3 = 0$
   (c) $x^2 + y^2 + 2x + 2y - 3 = 0$ (d) none of these
9. If the point $(2, k)$ lies outside the circles $x^2 + y^2 + x - 2y - 14 = 0$ and $x^2 + y^2 = 13$ then k lies in the interval
   (a) $(-3, -2) \cup (3, 4)$ (b) $-3, 4$
   (c) $(-\infty, -3) \cup (4, \infty)$ (d) $(-\infty, -2) \cup (3, \infty)$
10. If the point $(\lambda, \lambda + 1)$ lies inside the region bounded by the curve $x = \sqrt{25 - y^2}$ and y-axis, then λ belongs to the interval
    (a) $(-1, 3)$ (b) $(-4, 3)$
    (c) $(-\infty, -4) \cup (3, \infty)$ (d) none of these
11. The equation of the incircle formed by the coordinate axes and the line $4x + 3y = 6$ is
    (a) $x^2 + y^2 - 6x - 6y + 9 = 0$ (b) $4(x^2 + y^2 - x - y) + 1 = 0$
    (c) $4(x^2 + y^2 + x + y) + 1 = 0$ (d) none of these
12. If the circles $x^2 + y^2 = 9$ and $x^2 + y^2 + 8y + c = 0$ touch each other, then c is equal to
    (a) 15 (b) −15 (c) 16 (d) −16
13. If the circle $x^2 + y^2 + 2ax + 8y + 16 = 0$ touches x-axis, then the value of a is
    (a) ± 16 (b) ± 4 (c) ± 8 (d) ± 1

14. The equation of a circle with radius 5 and touching both the coordinate axes is
(a) $x^2 + y^2 \pm 10x \pm 10y + 5 = 0$
(b) $x^2 + y^2 \pm 10x \pm 10y = 0$
(c) $x^2 + y^2 \pm 10x \pm 10y + 25 = 0$
(d) $x^2 + y^2 \pm 10x \pm 10y + 51 = 0$

15. The equation of the circle passing through the origin which cuts off intercept of length 6 and 8 from the axes is
(a) $x^2 + y^2 - 12x - 16y = 0$
(b) $x^2 + y^2 + 12x + 16y = 0$
(c) $x^2 + y^2 + 6x + 8y = 0$
(d) $x^2 + y^2 - 6x - 8y = 0$

16. The equation of the circle concentric with $x^2 + y^2 - 3x + 4y - c = 0$ and passing through $(-1, -2)$ is
(a) $x^2 + y^2 - 3x + 4y - 1 = 0$
(b) $x^2 + y^2 - 3x + 4y = 0$
(c) $x^2 + y^2 - 3x + 4y + 2 = 0$
(d) none of these

17. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not intersect x-axis, if
(a) $g^2 < c$ (b) $g^2 > c$ (c) $g^2 > 2c$ (d) none of these

18. The area of an equilateral triangle inscribed in the circle $x^2 + y^2 - 6x - 8y - 25 = 0$ is
(a) $\frac{225\sqrt{3}}{6}$ (b) 25π (c) $50\pi - 100$ (d) none of these

19. The equation of the circle which touches the axes of coordinates and the line $\frac{x}{3} + \frac{y}{4} = 1$ and whose centres lie in the first quadrant is $x^2 + y^2 - 2cx - 2cy + c^2 = 0$, where c is equal to
(a) 4 (b) 2 (c) 3 (d) 6

20. If the circles $x^2 + y^2 = a$ and $x^2 + y^2 - 6x - 8y + 9 = 0$, touch externally, then $a =$
(a) 1 (b) -1 (c) 21 (d) 16

21. If $(x, 3)$ and $(3, 5)$ are the extremities of a diameter of a circle with centre at $(2, y)$, then the values of x and y are
(a) $(3, 1)$ (b) $x = 4, y = 1$ (c) $x = 8, y = 2$ (d) none of these

22. If $(-3, 2)$ lies on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ which is concentric with the circle $x^2 + y^2 + 6x + 8y - 5 = 0$, then $c =$
(a) 11 (b) -11 (c) 24 (d) none of these

23. Equation of the diameter of the circle $x^2 + y^2 - 2x + 4y = 0$ which passes through the origin is
(a) $x + 2y = 0$ (b) $x - 2y = 0$ (c) $2x + y = 0$ (d) $2x - y = 0$

24. Equation of the circle through origin which cuts intercepts of length a and b on axes is
(a) $x^2 + y^2 + ax + by = 0$
(b) $x^2 + y^2 - ax - by = 0$
(c) $x^2 + y^2 + bx + ay = 0$
(d) none of these

25. If the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other, then
(a) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (c) $a + b = 2c$ (d) $\frac{1}{a} + \frac{1}{b} = \frac{2}{c}$

ANSWERS

1. (b)	2. (a)	3. (a)	4. (c)	5. (a)	6. (c)	7. (a)	8. (a)
9. (c)	10. (a)	11. (b)	12. (a)	13. (b)	14. (c)	15. (d)	16. (b)
17. (a)	18. (a)	19. (d)	20. (a)	21. (a)	22. (b)	23. (c)	24. (b)
25. (a)							