CHAPTER 28

INTRODUCTION TO THREE DIMENSIONAL COORDINATE GEOMETRY

28.1 INTRODUCTION

Uptill now, we have learnt about two-dimensional coordinate system, which is also denoted by R^2. Because we live in a three-dimensional coordinate system. We call this three-dimensional space and denote it by R^3. We introduce a coordinate system in three-dimensional space by choosing three mutually perpendicular axes as a frame of reference. The orientation of the reference system will be *right-handed* in the sense that if you stand at the origin with your right arm along the positive x-axis and your left arm along the positive y-axis, as shown in Fig. 28.1, your head will then point in the direction of positive z-axis.

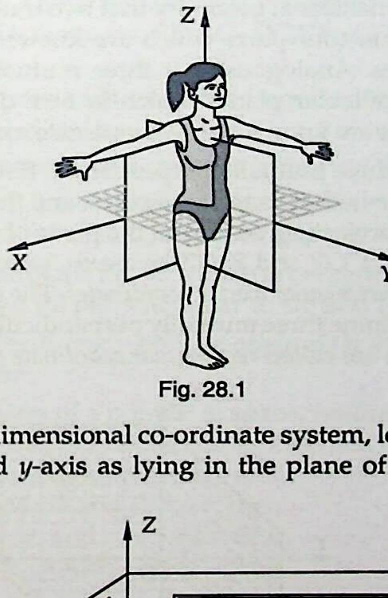

Fig. 28.1

In order to understand a three dimensional co-ordinate system, let us think of a room as shown in Fig. 28.2 and take x-axis and y-axis as lying in the plane of the floor and z-axis as a line

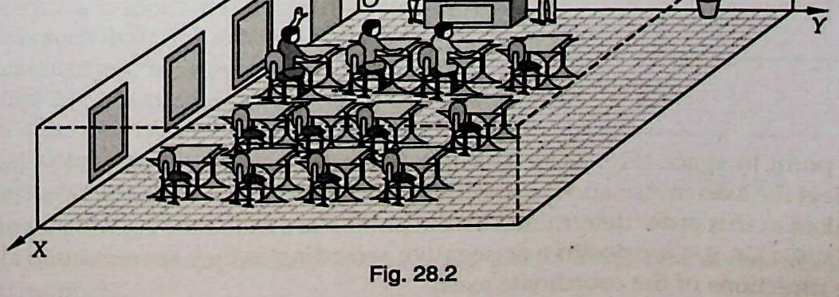

Fig. 28.2

perpendicular to the floor. We observe that the floor has two boundaries as x-axis and y-axis, so we say that it is situated in xy-plane. Similarly, front wall is in the yz-plane and left wall is in xz-plane. The xy, yz and zx-planes are called coordinate planes as shown in Fig. 28.3.

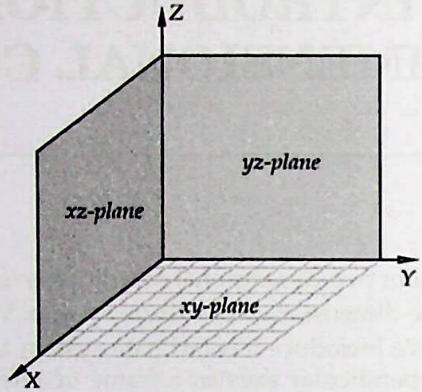

Fig. 28.3

28.2 COORDINATES OF A POINT IN SPACE

As we have studied in two dimensional geometry that two mutually perpendicular lines divide the plane containing them into four parts which are known as quadrants and the lines are known as the coordinate axes. Analogous to it three mutually perpendicular lines in space define three mutually perpendicular planes which in turn divide the space into eight parts known as *octants* and the lines are known as the coordinate axes.

Let $X'OX, Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O such that two of them viz. $Y'OY$ and $Z'OZ$ lie in the plane of the paper and the third $X'OX$ is perpendicular to the plane of the paper and is projecting out from the plane of the paper (see Fig. 28.4). Let O be the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ be x-axis, y-axis and z-axis respectively. These three lines are also called the *rectangular axes of coordinates.* The planes containing the lines $X'OX$, $Y'OY$ and $Z'OZ$ in pairs, determine three mutually perpendicular planes XOY, YOZ and ZOX or simply XY, YZ and ZX which are called *rectangular coordinate planes.*

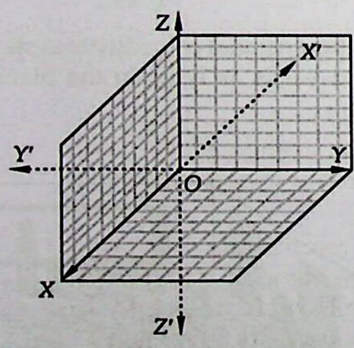

Fig. 28.4

Let P be a point in space (Fig. 28.5). Through P draw three planes parallel to the coordinate planes to meet the axes in A, B and C respectively. Let $OA = x$, $OB = y$ and $OC = z$. These three real numbers taken in this order determined by the point P are called the coordinates of the point P, written as (x, y, z), x, y, z are positive or negative according as they are measured along positive or negative directions of the coordinate axes.

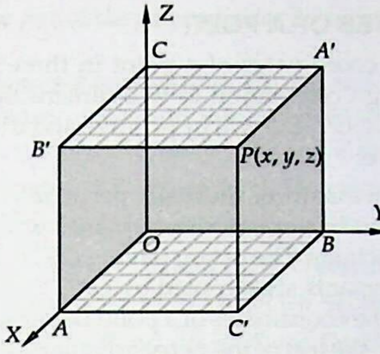

Fig. 28.5

Conversely, given an ordered triad (x, y, z) of real numbers we can always find the point whose coordinates are (x, y, z) in the following manner:

(i) Measure OA, OB, OC along x-axis, y-axis and z-axis respectively.

(ii) Through the points A, B, C draw planes parallel to the coordinate planes YOZ, ZOX and XOY respectively. The point of intersection of these planes is the required point P.

To give another explanation about the coordinates of a point P we draw three planes through P parallel to the coordinate planes. These three planes determine a rectangular parallelopiped which has three pairs of rectangular faces, viz. $PB'AC'$, $OCA'B$; $PA'BC'$, $OAB'C$; $PA'CB'$, $OAC'B$ as shown in Fig. 28.5. Then, we have

$x = OA = CB' = PA'$ = Perpendicular distance from P on the YOZ plane

$y = OB = A'C = PB'$ = Perpendicular distance from P on the ZOX plane

$z = OC = A'B = PC'$ = Perpendicular distance from P on the XOY plane.

Thus, *the coordinates of the point P are the perpendicular distances from P on the three mutually rectangular coordinate planes YOZ, ZOX and XOY respectively.*

Further, since the line PA lies in the plane $PB'AC'$ which is perpendicular to the line OA, we have PA perpendicular to OA. Similarly, PB perpendicular to OB and PC perpendicular to OC.

Thus, *the coordinates of a point are the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.*

Alternatively, to find the coordinates of a point P in space, we first draw perpendicular PM on the xy-plane with M as the foot of this perpendicular as shown in Fig. 28.6. Now, from the point M, we draw perpendicular ML on x-axis with L as the foot of this perpendicular. If $OL = a$, $LM = b$ and $PM = c$, then we say that a, b and c are x, y, and z coordinates, respectively, of the point P in space. In such a case, we say that the point P has coordinates (a, b, c).

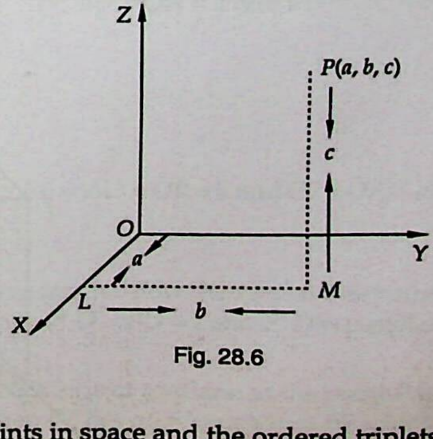

Fig. 28.6

Conversely, if we are given the co-ordinates (a, b, c) of a point P and we have to locate the point, then first fix the point L on x-axis such that $OL = a$. Now, find a point M on the perpendicular to x-axis at point L such that $LM = b$. We can say that M has coordinates (a, b) in xy- plane. Having reached the point M, we draw the perpendicular on xy-plane at point M and locate a point P on this perpendicular such that $PM = a$. The point P so obtained has the coordinates (a, b, c).

Thus, there is one-to-one correspondence between the points in space and the ordered triplets (x, y, z) of real numbers.

28.3 SIGNS OF COORDINATES OF A POINT

To determine the signs of the coordinates of a point in three dimension we follow the sign convention analogous to the sign convention in two dimensional geometry that all distances measured along or parallel to OX, OY, OZ will be positive and distances moved along or parallel to OX', OY', OZ' will be negative.

As discussed in previous section that three mutually perpendicular lines $X'OX, Y'OY$ and $Z'OZ$ determine three mutually perpendicular coordinate planes which in turn divide the space into eight compartments known as octants. The octant having OX, OY and OZ as its edges is denoted by $OXYZ$. Similarly, the other octants are denoted by $OX'YZ$, $OXY'Z$, $OX'Y'Z$, $OXYZ'$, $OXYZ'$, $OXY'Z'$, $OX'Y'Z'$. The signs of the coordinates of a point depend upon the octant in which it lies. Let P be a point and let A, B, C be the feet of the perpendiculars drawn from P on $X'OX, Y'OY$ and $Z'OZ$ respectively. If P lies in octant $OXYZ$, then clearly A, B, C lie on OX, OY and OZ respectively. Therefore, by our sign convention OA, OB and OC are positive. Thus, all the three coordinates of P are positive. If P lies in octant $OX'YZ$, then A, B and C lie on OX', OY and OZ respectively. Therefore, x-coordinate of P is negative and y and z-coordinates are positive.

The following table shows the signs of coordinates of points in various octants:

Octant / coordinate	OXYZ	OX'YZ	OXY'Z	OX'Y'Z	OXYZ'	OX'YZ'	OXY'Z'	OX'Y'Z'
x	+	−	+	−	+	−	+	−
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−

<u>REMARK 1</u> *If a point P lies in x y-plane, then by the definition of coordinates of a point, z-coordinate of P is zero. Therefore, the coordinates of a point on xy-plane are of the form $(x, y, 0)$ and we may take the equation of xy-plane as $z = 0$. Similarly, the coordinates of any point in yz and zx-planes are of the forms $(0, y, z)$ and $(x, 0, z)$ respectively and their equations may be taken as $x = 0$ and $y = 0$ respectively.*

<u>REMARK 2</u> *If a point lies on the x-axis, then its y and z-coordinates are both zero. Therefore, the coordinates of a point on x-axis are of the form $(x, 0, 0)$ and we may take the equation of x-axis as $y = 0$, $z = 0$. Similarly, the coordinates of a point on y and z-axes are of the form $(0, y, 0)$ and $(0, 0, z)$ respectively and their equations may be taken as $x = 0, z = 0$ and $x = 0, y = 0$ respectively.*

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *In Fig. 28.7, if the coordinates of point P are (a, b, c), then*

(i) write the coordinates of points A, B, C, D, E and F.

(ii) write the coordinates of the feet of the perpendiculars from the point P to the coordinate axes.

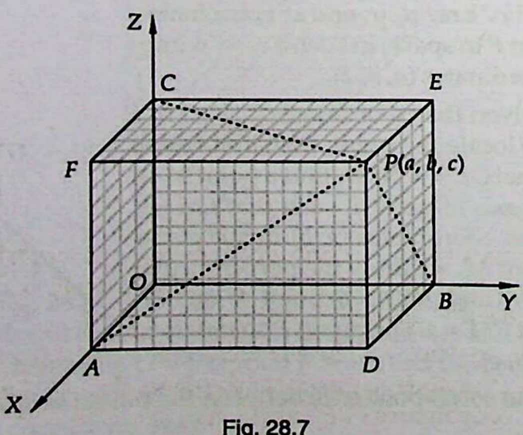

Fig. 28.7

(iii) *write the coordinates of the feet of the perpendicular from the point P on the coordinate planes XY, YZ and ZX.*

(iv) *find the perpendicular distances of point P from XY, YZ and ZX-planes.*

(v) *find the perpendicular distances of the point P from the coordinate axes.*

(vi) *find the coordinates of the reflection of P in XY, YZ* and *ZX-planes.*

SOLUTION (i) Since the coordinates of P are (a, b, c). Therefore, $OA = a$, $OB = b$ and $OC = c$.

Now, A lies on OX such that $OA = a$. So, the coordinates of A are $(a, 0, 0)$.

Similarly, coordinates of B and C are $(0, b, 0)$ and $(0, 0, c)$ respectively.

Since D lies in XY-plane such that $OA = a$ and $AD = OB = b$. So, the coordinates of D are $(a, b, 0)$.

Point E lies in YZ-plane such that $OB = b$ and $BE = OC = c$. So, the coordinates of E are $(0, b, c)$.

Similarly, coordinates of F are $(a, 0, c)$ as it lies in XZ-plane.

(ii) PA, PB and PC are perpendiculars from P on OX, OY and OZ respectively. So, A, B and C are the feet of perpendiculars from P on OX, OY and OZ respectively. Their coordinates are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ as discussed in (i).

(iii) Clearly, PD, PE and PF are the perpendiculars from P on XY, YZ and ZX-planes respectively. So, D, E and F are the feet of the perpendiculars from P on XY, YZ and ZX-planes. The coordinates of D, E and F are $D(a, b, 0)$, $E(0, b, c)$ and $F(a, 0, c)$ respectvely as discussed in (i).

(iv) PD, PE and PF are the perpendicular distances of P from XY, YZ and ZX-planes respectively.

$\therefore \quad PD = OC = c,\ PE = OA = a$ and $PF = OB = b$.

Hence, the perpendicular distances of $P(a, b, c)$ from XY, YZ and ZX planes are c, a and b respectively.

(v) PA, PB and PC are the perpendicular distances of point P from OX, OY and OZ respectively.

In right-angled triangle ADP, we have

$$AP^2 = AD^2 + DP^2$$

$$\Rightarrow \quad PA = \sqrt{AD^2 + DP^2} = \sqrt{b^2 + c^2} \qquad [\because AD = OB = b \text{ and } PD = OC = c]$$

In right-angled triangle PDB right angled at D, we have

$$PB^2 = BD^2 + PD^2$$

$$\Rightarrow \quad PB = \sqrt{BD^2 + PD^2} = \sqrt{a^2 + c^2} \qquad [\because BD = OA = a \text{ and } PD = OC = c]$$

In right-angled triangle PCF right angled at F, we have

$$PC^2 = PF^2 + CF^2$$

$$\Rightarrow \quad PC = \sqrt{PF^2 + CF^2}$$

$$\Rightarrow \quad PC = \sqrt{b^2 + a^2} \qquad [\because PF = AD = OB = b \text{ and } CF = OA = a]$$

$$\Rightarrow \quad PC = \sqrt{a^2 + b^2}$$

(vi) The reflection or image of $P(a, b, c)$ in xy-plane will be as much below the xy-plane as point P is above it, that is, if P' is the reflection of P in xy-plane, then $P'D = PD = c$ and $P'D$ is parallel to OZ'. So, the coordinates of P' are $(a, b, -c)$.

The image of $P(a, b, c)$ in yz-plane will be as much on the back side of yz-plane as the point P is on its front side. Thus, if P'' is the image of P in yz-plane, then P'' lies on PE such that $PE = EP''$. But, $PE = OA = a$. So, the coordinates of P'' are $(-a, b, c)$.

The image of $P(a, b, c)$ in zx-plane will be as much as on the left side of xz-plane as the point P is on its right side. Thus, if P''' is the image of P in zx-plane, then P''' lies on PF produced such that $PF = FP'''$. But, $PF = OB = b$. So, the coordinates of P''' are $(a, -b, c)$.

EXAMPLE 2 *Planes are drawn parallel to the coordinate planes through the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. Find the length of the edges of the parallelopiped so formed.*

SOLUTION Clearly, PA, PB and PC are the lengths of the edges of the parallelopiped shown in Fig. 28.8.

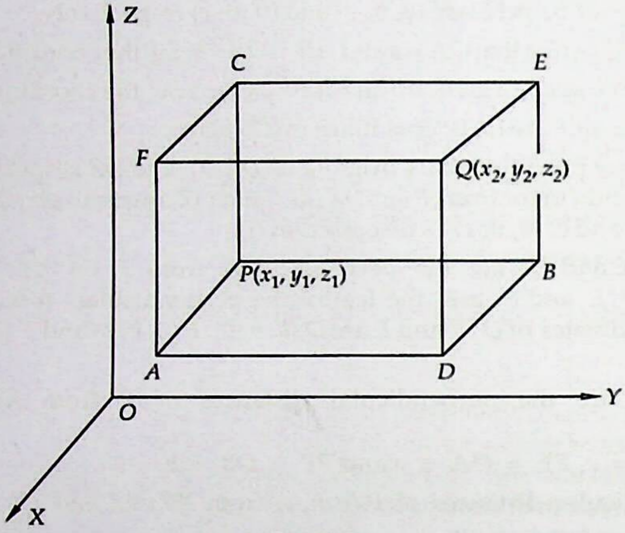

Fig. 28.8

Clearly, $PBEC$, $QDAF$ are planes parallel to yz-plane such that their distances from yz-plane are x_1 and x_2 respectively. So,

$$PA = (\text{Distance between the planes } PBEC \text{ and } QDAF) = x_2 - x_1.$$

PB is the distance between the planes $PAFC$ and $BDQE$ which are parallel to zx-plane and are at distances y_1 and y_2, respectively, from zx-plane.

$$\therefore \quad PB = y_2 - y_1$$

Similarly, PC is the distance between the parallel planes $PBDA$ and $CEQF$ which are at distances z_1 and z_2, respectively, from xy-plane.

$$\therefore \quad PC = z_2 - z_1$$

EXERCISE 28.1

1. Name the octants in which the following points lie:
   (i) $(5, 2, 3)$ (ii) $(-5, 4, 3)$ (iii) $(4, -3, 5)$ (iv) $(7, 4, -3)$
   (v) $(-5, -4, 7)$ (vi) $(-5, -3, -2)$ (vii) $(2, -5, -7)$ (viii) $(-7, 2 -5)$ [NCERT]
2. Find the image of:
   (i) $(-2, 3, 4)$ in the yz-plane. (ii) $(-5, 4, -3)$ in the xz-plane.
   (iii) $(5, 2, -7)$ in the xy-plane. (iv) $(-5, 0, 3)$ in the xz-plane.
   (v) $(-4, 0, 0)$ in the xy-plane.
3. A cube of side 5 has one vertex at the point $(1, 0, -1)$, and the three edges from this vertex are, respectively, parallel to the negative x and y axes and positive z-axis. Find the coordinates of the other vertices of the cube.
4. Planes are drawn parallel to the coordinate planes through the points $(3, 0, -1)$ and $(-2, 5, 4)$. Find the lengths of the edges of the parallelopiped so formed.

5. Planes are drawn through the points $(5, 0, 2)$ and $(3, -2, 5)$ parallel to the coordinate planes. Find the lengths of the edges of the rectangular parallelopiped so formed.
6. Find the distances of the point $P(-4, 3, 5)$ from the coordinate axes.
7. The coordinates of a point are $(3, -2, 5)$. Write down the coordinates of seven points such that the absolute values of their coordinates are the same as those of the coordinates of the given point.

ANSWERS

1. (i) $XOYZ$ (ii) $X'OYZ$ (iii) $XOY'Z$ (iv) $XOYZ'$
   (v) $X'OY'Z$ (vi) $X'OY'Z'$ (vii) $XOY'Z'$ (viii) $X'OYZ'$
2. (i) $(2, 3, 4)$ (ii) $(-5, -4, -3)$ (iii) $(5, 2, 7)$ (iv) $(-5, 0, 3)$
   (v) $(-4, 0, 0)$
3. (i) $(1, 0, 4), (1, -5, -1), (1, -5, 4), (-4, 0, -1), (-4, -5, 4), (-4, -5, -1), (4, 0, 4)$
4. 5,5, 5 5. 2, 2, 3 6. x-axis : $\sqrt{34}$; y-axis : $\sqrt{41}$; z-axis: 5
7. $(-3, -2, -5), (-3, -2, 5), (3, -2, -5), (-3, 2 -5), (3, 2, 5), (3, 2, -5), (-3, 2, 5)$

28.4 DISTANCE FORMULA

THEOREM *The distance between the points* $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ *is given by*

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

PROOF Let O be the origin and let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two given points in space. Let L and M be the feet of the perpendiculars from P and Q on the XOY plane. Then in the XOY plane the coordinates of L and M are (x_1, y_1) and (x_2, y_2) respectively. Therefore, by using distance formula in two dimensional geometry, we get

$$LM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad \text{...(i)}$$

Draw PN perpendicular to QM. Then

$$PN = LM \text{ and } NQ = z_2 - z_1$$

Clearly, ΔPNQ is a right triangle right angled at N.

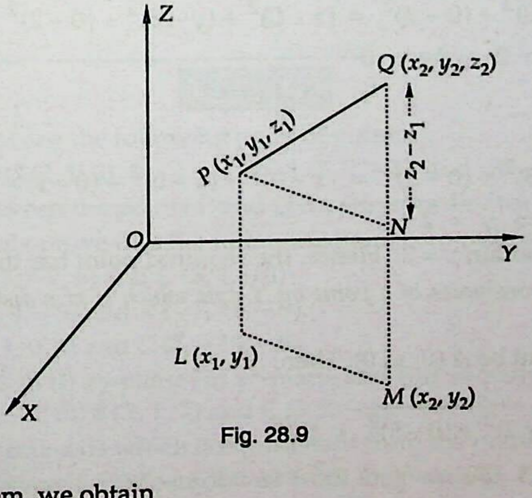

Fig. 28.9

So, by Pythagoras theorem, we obtain

$$PQ^2 = PN^2 + NQ^2$$

$$\Rightarrow \quad PQ^2 = LM^2 + NQ^2 \quad [\because PN = LM]$$

$$\Rightarrow \quad PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \quad \text{[Using (i)]}$$

Thus, the distance between points $P(x_1, y_1, z_1)$ and, $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_1 - y_1)^2 + (z_2 - z_1)^2}$$

Q.E.D.

REMARK *If O is the origin and $P(x, y, z)$ is a point in space, then*

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points $P(-2, 4, 1)$ and $Q(1, 2, -5)$.*

SOLUTION Here, $x_1 = -2,\ y_1 = 4,\ z_1 = 1,\ x_2 = 1,\ y_2 = 2$ and $z_2 = -5$

$$\therefore \quad PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(1-(-2))^2 + (2-4)^2 + (-5-1)^2}$$

$$\Rightarrow \quad PQ = \sqrt{9 + 4 + 36} = 7 \text{ units}$$

EXAMPLE 2 *Prove by using distance formula that the points $P(1, 2, 3)$, $Q(-1, -1, -1)$ and $R(3, 5, 7)$ are collinear.*

SOLUTION Using distance formula, we obtain

$$PQ = \sqrt{(-1-1)^2 + (-1-2)^2 + (-1-3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$QR = \sqrt{(3+1)^2 + (5+1)^2 + (7+1)^2} = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}$$

and, $$PR = \sqrt{(3-1)^2 + (5-2)^2 + (7-3)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

Clearly, $QR = PQ + PR$. Therefore, points Q, P, R are collinear and P lies between Q and R.

EXAMPLE 3 *Determine the point in XY-plane which is equidistant from three points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.*

SOLUTION We know that z-coordinate of every point on xy-plane is zero. So, let $P(x, y, 0)$ be a point on xy-plane such that $PA = PB = PC$.

Now, $PA = PB$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (x-2)^2 + (y-0)^2 + (0-3)^2 = (x-0)^2 + (y-3)^2 + (0-2)^2$$

$$\Rightarrow \quad 4x - 6y = 0 \Rightarrow 2x - 3y = 0 \quad \text{...(i)}$$

$$PB = PC$$

$$\Rightarrow \quad PB^2 = PC^2$$

$$\Rightarrow \quad (x-0)^2 + (y-3)^2 + (0-2)^2 = (x-0)^2 + (y-0)^2 + (0-1)^2$$

$$\Rightarrow \quad -6y + 12 = 0 \Rightarrow y = 2 \quad \text{...(ii)}$$

Putting $y = 2$ in (i), we obtain $x = 3$. Hence, the required point has the coordinates $(3, 2, 0)$.

EXAMPLE 4 *Find the coordinates of a point on Y-axis which is at a distance of $5\sqrt{2}$ from the point $P(3, -2, 5)$.* [NCERT]

SOLUTION Let the point be $A(0, y, 0)$. Then,

$$AP = 5\sqrt{2}$$

$$\Rightarrow \quad \sqrt{(0-3)^2 + (y+2)^2 + (0-5)^2} = 5\sqrt{2}$$

$$\Rightarrow \quad 9 + (y+2)^2 + 25 = 50 \Rightarrow (y+2)^2 = 16 \Rightarrow y + 2 = \pm 4 \Rightarrow y = 2, -6$$

Hence, the coordinates of the required point are $(0, 2, 0)$ and $(0, -6, 0)$.

EXAMPLE 5 *Show that the points $A(0, 1, 2)$, $B(2, -1, 3)$ and $C(1, -3, 1)$ are vertices of an isosceles right-angled triangle.*

SOLUTION Using distance formula, we obtain

$$AB = \sqrt{(2-0)^2 + (-1-1)^2 + (3-2)^2} = \sqrt{4+4+1} = 3$$

$$BC = \sqrt{(1-2)^2 + (-3+1)^2 + (1-3)^2} = \sqrt{1+4+4} = 3$$

and, $$CA = \sqrt{(1-0)^2 + (-3-1)^2 + (1-2)^2} = \sqrt{1+16+1} = 3\sqrt{2}$$

Clearly, $AB = BC$ and $AB^2 + BC^2 = AC^2$. Hence, triangle ABC is an isosceles right-angled triangle which is right angled at B.

EXAMPLE 6 *Find the locus of the point which is equidistant from the points A (0, 2, 3) and (2, −2, 1).*

SOLUTION Let $P(x, y, z)$ be any point which is equidistant from $A(0, 2, 3)$ and $B(2, -2, 1)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad \sqrt{(x-0)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y+2)^2 + (z-1)^2}$$

$$\Rightarrow \quad 4x - 8y - 4z + 4 = 0 \Rightarrow x - 2y - z + 1 = 0$$

Hence, the required locus is $x - 2y - z + 1 = 0$.

EXAMPLE 7 *Find the coordinates of a point equidistant from the four points O (0, 0, 0), A (a, 0, 0), B (0, b, 0) and C (0, 0, c).*

SOLUTION Let $P(x, y, z)$ be the required point. Then ,$OP = PA = PB = PC$.

Now, $$OP = PA$$

$$\Rightarrow \quad OP^2 = PA^2$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = (x-a)^2 + (y-0)^2 + (z-0)^2$$

$$\Rightarrow \quad 0 = -2ax + a^2 \Rightarrow x = a/2$$

Similarly, $OP = PB \Rightarrow y = b/2$ and $OP = PC \Rightarrow z = c/2$.

Hence, the coordinates of the required point are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

EXERCISE 28.2

LEVEL-1

1. Find the distance between the following pairs of points:
   (i) $P(1, -1, 0)$ and $Q(2, 1, 2)$ (ii) $A(3, 2, -1)$ and $B(-1, -1, -1)$.
2. Find the distance between the points P and Q having coordinates $(-2, 3, 1)$ and $(2, 1, 2)$.
3. Using distance formula prove that the following points are collinear:
   (i) $A(4, -3, -1)$, $B(5, -7, 6)$ and $C(3, 1, -8)$
   (ii) $P(0, 7, -7)$, $Q(1, 4, -5)$ and $R(-1, 10, -9)$.
   (iii) $A(3, -5, 1)$, $B(-1, 0, 8)$ and $C(7, -10, -6)$
4. Determine the points in (i) xy-plane (ii) yz-plane and (iii) zx-plane which are equidistant from the points $A(1, -1, 0)$, $B(2, 1, 2)$ and $C(3, 2, -1)$.
5. Determine the point on z-axis which is equidistant from the points $(1, 5, 7)$ and $(5, 1, -4)$.
6. Find the point on y-axis which is equidistant from the points $(3, 1, 2)$ and $(5, 5, 2)$.
7. Find the points on z-axis which are at a distance $\sqrt{21}$ from the point $(1, 2, 3)$.
8. Prove that the triangle formed by joining the three points whose coordinates are $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$ is an equilateral triangle.

9. Show that the points (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) are the vertices of an isosceles right-angled triangle.
10. Show that the points $A(3,3,3)$, $B(0,6,3)$, $C(1,7,7)$ and $D(4,4,7)$ are the vertices of a square.
11. Prove that the point $A(1,3,0)$, $B(-5,5,2)$, $C(-9,-1,2)$ and $D(-3,-3,0)$ taken in order are the vertices of a parallelogram. Also, show that $ABCD$ is not a rectangle.
12. Show that the points $A(1,3,4)$, $B(-1,6,10)$, $C(-7,4,7)$ and $D(-5,1,1)$ are the vertices of a rhombus.
13. Prove that the tetrahedron with vertices at the points $O(0,0,0)$, $A(0,1,1)$, $B(1,0,1)$ and C (1, 1, 0) is a regular one.
14. Show that the points (3, 2, 2), (–1, 4, 2), (0, 5, 6), (2, 1, 2) lie on a sphere whose centre is (1, 3, 4). Find also its radius.
15. Find the coordinates of the point which is equidistant from the four points $O(0,0,0)$, $A(2,0,0)$, $B(0,3,0)$ and C (0, 0, 8).
16. If $A(-2,2,3)$ and $B(13,-3,13)$ are two points. Find the locus of a point P which moves in such a way that $3PA = 2PB$.
17. Find the locus of P if $PA^2 + PB^2 = 2k^2$, where A and B are the points (3, 4, 5) and (–1,3,–7). [NCERT]
18. Show that the points (a, b, c), (b, c, a) and (c, a, b) are the vertices of an equilateral triangle.
19. Are the points $A(3, 6, 9)$, $B(10, 20, 30)$ and $C(25, -41, 5)$, the vertices of a right-angled triangle? [NCERT]
20. Verify the following:
    (i) (0,7,–10), (1,6,–6) and (4,9,–6) are vertices of an isosceles triangle. [NCERT]
    (ii) (0,7,10), (–1,6, 6) and (–4,9, 6) are vertices of a right-angled triangle [NCERT]
    (iii) (–1,2,1), (1,–2,5), (4,–7,8) and (2,–3,4) are vertices of a parallelogram. [NCERT]
    (iv) (5, –1, 1), (7, –4, 7), (1, –6, 10) and (–1, – 3, 4) are the vertices of a rhombus.
21. Find the locus of the points which are equidistant from the points (1,2,3) and (3,2,–1). [NCERT]
22. Find the locus of the point, the sum of whose distances from the points $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10. [NCERT]
23. Show that the points $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$ are the vertices of a parallelogram $ABCD$, but not a rectangle. [NCERT]
24. Find the equation of the set of the points P such that its distances from the points $A(3, 4, -5)$ and $B(-2, 1, 4)$ are equal. [NCERT]

ANSWERS

1. (i) 3 (ii) 5
2. $\sqrt{21}$
4. (i) $\left(\frac{3}{2}, 1, 0\right)$ (ii) $\left(0, \frac{31}{16}, -\frac{3}{16}\right)$ (iii) $\left(\frac{31}{10}, 0, \frac{1}{5}\right)$
5. $\left(0, 0, \frac{3}{2}\right)$
6. (0, 5, 0)
7. (0, 0, 7), (0, 0, – 1)
15. $\left(1, \frac{3}{2}, 4\right)$
16. $5(x^2 + y^2 + z^2) + 140x - 60y + 50z - 1235 = 0$
17. $2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - 2k^2 = 0$
19. No
21. $x - 2z = 0$
22. $9x^2 + 25y^2 + 25z^2 - 225 = 0$
23. $10x + 6y - 18z - 29 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

10. Show that $AB = BC = CD = DA$, $AC^2 = AB^2 + BC^2$ and, $BD^2 = AB^2 + AD^2$.

11. Show that $AB = CD$, $BC = DA$ and $AC \neq BD$.

12. Show that $AB = BC = CD = DA$.

13. Show that $OA = OB = OC = AB = BC = CA$

17. Let the coordinates of P be (α, β, γ). Then,

$$PA^2 + PB^2 = 2k^2$$

$$\Rightarrow \quad (\alpha - 3)^2 + (\beta - 4)^2 + (\gamma - 5)^2 + (\alpha + 1)^2 + (\beta - 3)^2 + (\gamma + 7)^2 = 2k^2$$

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 - 2\alpha - 7\beta + 2\gamma + \frac{109}{2} = k^2$$

Hence, the locus of (α, β, γ) is

$$x^2 + y^2 + z^2 - 2x - 7y + 2z + \frac{109}{2} = k^2$$

or, $\quad 2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - 2k^2 = 0.$

19. $\quad AB = \sqrt{(10-3)^2 + (20-6)^2 + (30-9)^2} = \sqrt{686}$

$$BC = \sqrt{(25-10)^2 + (-41-20)^2 + (5-30)^2} = \sqrt{4571}$$

and, $\quad CA = \sqrt{(25-3)^2 + (-41-6)^2 + (5-9)^2} = \sqrt{2709}$

Clearly, $BC^2 \neq AB^2 + CA^2$, $AB^2 \neq BC^2 + CA^2$ and $AC^2 \neq AB^2 + BC^2$.

Hence, ΔABC is not a right triangle.

20. (i) Let $A(0, 7, -10)$, $B(1, 6, -6)$ and $C(4, 9, -6)$ be the given points. Then,

$AB = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$, $BC = \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2}$

and, $AC = \sqrt{16+4+16} = 6$

We observe that $AB + BC > AC$, $AB + AC > BC$ and $BC + CA > AB$. So, points, A, B, C form a triangle.

Also, $AB = BC$. So, triangle ABC is isosceles.

(ii) Let $P(0, 7, 10)$, $Q(-1, 6, 6)$ and $R(-4, 9, 6)$ be the given points. Then,

$PQ = 3\sqrt{2}$, $QR = 3\sqrt{2}$ and $PR = 6$.

Clearly, $PQ + QR = 6\sqrt{2} > PR$, $QR + PR > PQ$ and $PQ + PR > QR$.

So, given points form a triangle.

Also, $PR^2 = PQ^2 + QR^2$.

So, ΔPQR is a right triangle right angled at Q.

(iii) Let $A(-1, 2, 1)$ $B(1, -2, 5)$, $C(4, -7, 8)$ and $D(2, -3, 4)$ be the given points. Then,

$AB = 6$, $BC = \sqrt{43}$, $CD = 6$ and $AD = \sqrt{43}$

$\therefore \quad AB = CD = 6$ and $BC = AD = \sqrt{43}$

Hence, $ABCD$ is a parallelogram.

21. Let $P(\alpha, \beta, \gamma)$ be a point equidistant from the points $A(1, 2, 3)$ and $B(3, 2, -1)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (\alpha - 1)^2 + (\beta - 2)^2 + (\gamma - 3)^2 = (\alpha - 3)^2 + (\beta - 2)^2 + (\gamma + 1)^2$$

$$\Rightarrow \quad 4\alpha - 8\gamma = 0$$

$$\Rightarrow \quad \alpha - 2\gamma = 0$$

Hence, the locus of $P(\alpha, \beta, \gamma)$ is $x - 2z = 0$.

22. Let $P(\alpha, \beta, \gamma)$ be the point such that the sum of its distances from the points $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10.

i.e. $PA + PB = 10$

$$\Rightarrow \sqrt{(\alpha-4)^2+\beta^2+\gamma^2}+\sqrt{(\alpha+4)^2+\beta^2+\gamma^2} = 10$$

$$\Rightarrow \sqrt{(\alpha+4)^2+\beta^2+\gamma^2} = 10-\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$$

$$\Rightarrow (\alpha+4)^2+\beta^2+\gamma^2 = 100-20\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}+(\alpha-4)^2+\beta^2+\gamma^2$$

$$\Rightarrow 16\alpha = 100-20\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$$

$$\Rightarrow (4\alpha-25) = 5\sqrt{(\alpha-4)^2+\beta^2+\gamma^2}$$

$$\Rightarrow (4\alpha-25)^2 = 25\left\{(\alpha-4)^2+\beta^2+\gamma^2\right\}$$

$$\Rightarrow 9\alpha^2+25\beta^2+25\gamma^2-225 = 0$$

Hence, the locus of (α, β, γ) is $9x^2+25y^2+25z^2-225=0$.

23. Given points are $A(1, 2, 3)$, $B(-1, -2, -1)$, $C(2, 3, 2)$ and $D(4, 7, 6)$.

$\therefore$ $AB = 6$, $BC = \sqrt{43}$, $CD = 6$ and $AD = \sqrt{43}$

$\Rightarrow$ $AB = CE = 6$ and $BC = AD = \sqrt{43}$

So, $ABCD$ is a parallelogram.

Also,

$$AB^2+BC^2 = 36+43 = 79 \text{ and } AC = \sqrt{3}$$

$\therefore$ $AB^2+BC^2 \neq AC^2$ i.e. $\angle B$ is not a right angle.

Hence, $ABCD$ is not a rectangle.

24. Let $P(\alpha, \beta, \gamma)$ be one of the points equidistant from $A(3, 4, -5)$ and $B(-2, 1, 4)$. Then,

$$PA = PB$$

$$\Rightarrow PA^2 = PB^2$$

$$\Rightarrow (\alpha-3)^2+(\beta-4)^2+(\gamma+5)^2 = (\alpha+2)^2+(\beta-1)^2+(\gamma-4)^2$$

$$\Rightarrow 10\alpha+6\beta-18\gamma-29 = 0$$

Hence, the locus of P is $10x+6y-18z-29=0$.

28.5 SECTION FORMULAE

THEOREM (FOR INTERNAL DIVISION) *Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in space and let R be a point on the line segment joining P and Q such that it divides the join of P and Q internally in the ratio $m_1 : m_2$. Then, the coordinates of R are:*

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$$

PROOF Let the coordinates of R be (x, y, z). Let PL, QM and RN be perpendiculars from P, Q and R respectively on XOY plane. Clearly, PL, QM and RN lie in a plane which contains the line PQ and is perpendicular to XOY plane. Therefore, points L, M, N are in a straight line which is the intersection of this plane with XOY-plane. Through R draw a line parallel to LM and meeting LP produced in L' and MQ in M'. Clearly, triangles RPL' and $RM'Q$ are equiangular and hence similar;

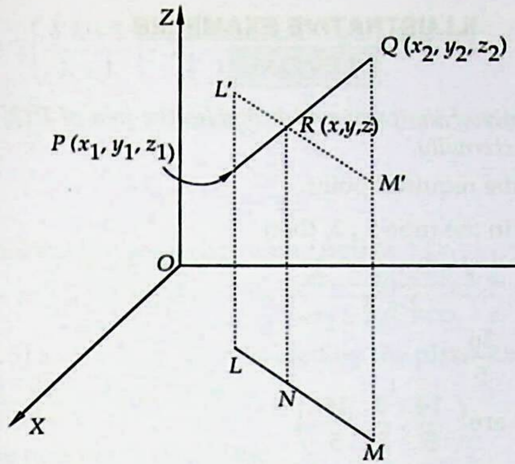

Fig. 28.10

$$\therefore \quad \frac{PL'}{M'Q} = \frac{PR}{PQ}$$

$$\Rightarrow \quad \frac{LL' - LP}{MQ - MM'} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad \frac{NR - LP}{MQ - NR} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z} = \frac{m_1}{m_2}$$

$$\Rightarrow \quad z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

Similarly, by drawing planes containing PQ and perpendicular to YOZ and ZOX planes respectively, we get

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \quad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}$$

Hence, the coordinates of R are $\left(\dfrac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \dfrac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \dfrac{m_1 z_2 + m_2 z_1}{m_1 + m_2}\right)$

REMARK 1 *If R is the mid-point of the segment joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, then $m_1 = m_2 = 1$ and so the coordinates of R are given by*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

REMARK 2. *If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points, and let R be a point on PQ produced dividing it externally in the ratio $m_1 : m_2$ $(m_1 \neq m_2)$. Then, the coordinates of R are*

$$\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2}, \frac{m_1 z_2 - m_2 z_1}{m_1 + m_2}\right).$$

REMARK 3 *The xy, yz and zx planes divide the segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in the ratio $-z_1 : z_2$, $-x_1 : x_2$ and $-y_1 : y_2$ respectively.*

REMARK 4 *The line segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is divided by the plane $ax + by + cz + d = 0$ in the ratio $-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the point which divides the join of $P(2,-1,4)$ and $Q(4,3,2)$ in the ratio 2 : 3 (i) internally (ii) externally.*

SOLUTION Let $R(x, y, z)$ be the required point.

(i) If R divides PQ internally in the ratio 2 : 3, then

$$x = \frac{2\times 4+3\times 2}{2+3}, y = \frac{2\times 3+3\times -1}{2+3}, z = \frac{2\times 2+3\times 4}{2+3}$$

$$\Rightarrow \quad x = \frac{14}{5}, y = \frac{3}{5}, z = \frac{16}{5}$$

So, the coordinates of point R are $\left(\frac{14}{5}, \frac{3}{5}, \frac{16}{5}\right)$.

(ii) If R divides PQ externally in the ratio 2 : 3, then

$$x = \frac{2\times 4-3\times 2}{2-3}, y = \frac{2\times 3-3\times -1}{2-3}, z = \frac{2\times 2-3\times 4}{2-3}$$

$$\Rightarrow \quad x = -2, y = -9, z = 8.$$

So, the coordinates of R are $(-2,-9,8)$.

EXAMPLE 2 *Find the ratio in which the line joining the points $(1,2,3)$ and $(-3,4,-5)$ is divided by the xy-plane. Also, find the coordinates of the point of division.*

SOLUTION Suppose the line joining the points $P(1, 2, 3)$ and $Q(-3, 4-5)$ is divided be the xy-plane at a point R in the ratio $\lambda:1$. Then, the coordinates of R are

$$\left(\frac{-3\lambda+1}{\lambda+1}, \frac{4\lambda+2}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}\right) \qquad \text{...(i)}$$

Since R lies on xy-plane i.e. $z=0$. Therefore,

$$\frac{-5\lambda+3}{\lambda+1} = 0 \Rightarrow \lambda = \frac{3}{5}$$

So, the required ratio is $\frac{3}{5}:1$ or, 3 : 5. Putting $\lambda=\frac{3}{5}$ in (i), we obtain the coordinates of R as $(-1/2, 11/4, 0)$.

ALITER We know that the xy-plane divides the line segment joining points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $-z_1 : z_2$. Hence, the required ratio is $-3:-5$ i.e. 3 : 5 internally and the coordinates of the point of division are

$$\left(\frac{3\times -3+5\times 1}{3+5}, \frac{3\times 4+5\times 2}{3+5}, \frac{3\times -5+5\times 3}{3+5}\right) = \left(-\frac{1}{2}, \frac{11}{4}, 0\right)$$

EXAMPLE 3 *Find the ratio in which the join the $A(2,1,5)$ and $B(3, 4, 3)$ is divided by the plane $2x+2y-2z=1$. Also, find the coordinates of the point of division.*

SOLUTION Suppose the plane $2x+2y-2z=1$ divides the line joining the points $A(2,1,5)$ and $B(3, 4, 3)$ at a point C in the ratio $\lambda:1$. Then, the coordinates of C are

$$\left(\frac{3\lambda+2}{\lambda+1}, \frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+5}{\lambda+1}\right) \qquad \text{...(i)}$$

Since point C lies on the plane $2x+2y-2z=1$. Therefore, coordinates of C must satisfy the equation of the plane i.e. $2\left(\frac{3\lambda+2}{\lambda+1}\right)+2\left(\frac{4\lambda+1}{\lambda+1}\right)-2\left(\frac{3\lambda+5}{\lambda+1}\right)=1 \Rightarrow 8\lambda-4=\lambda+1 \Rightarrow \lambda=\frac{5}{7}$

So, the required ratio is $\frac{5}{7}:1$ or $5:7$. Putting $\lambda=\frac{5}{7}$ in (i), the coordinates of the point C of division are $\left(\frac{29}{12},\frac{9}{4},\frac{25}{6}\right)$.

ALITER We know that the line segment joining points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is divided by the plane $ax+by+cz+d=0$ in the ratio $-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}$. Therefore, the line segment joining points $A(2,1,5)$ and $B(3,4,3)$ is divided by the plane $2x+2y-2z-1=0$ in the ratio

$$-\frac{2\times2+2\times1-2\times5-1}{2\times3+2\times4-2\times3-1}=\frac{5}{7} \text{ i.e. } 5:7.$$

The coordinates of the point of division are

$$\left(\frac{5\times3+7\times2}{5+7},\frac{5\times4+7\times1}{5+7},\frac{5\times3+7\times5}{5+7}\right)=\left(\frac{29}{12},\frac{9}{4},\frac{25}{6}\right)$$

EXAMPLE 4 *Using section formula, prove that the three points $A(-2,3,5)$, $B(1,2,3)$ and $C(7,0,-1)$ are collinear.*

SOLUTION Suppose the given points are collinear and C divides AB in the ratio $\lambda:1$. Then, coordinates of C are

$$\left(\frac{\lambda-2}{\lambda+1},\frac{2\lambda+3}{\lambda+1},\frac{3\lambda+5}{\lambda+1}\right)$$

But, coordinates of C are given as $(7,0,-1)$. Therefore,

$$\frac{\lambda-2}{\lambda+1}=7,\ \frac{2\lambda+3}{\lambda+1}=0 \text{ and, } \frac{3\lambda+5}{\lambda+1}=-1$$

From each of these equations, we obtain $\lambda=-\frac{3}{2}$.

Since each of these equations give the same value of λ. Therefore, the given points are collinear and C divides AB externally in the ratio $3:2$.

EXAMPLE 5 *The mid-points of the sides of a triangle are $(1,5,-1)$, $(0,4,-2)$ and $(2,3,4)$. Find its vertices.*

SOLUTION Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ be the vertices of the given triangle, and let $D(1,5,-1)$, $E(0,4,-2)$ and $F(2,3,4)$ be the mid-points of the sides BC, CA and AB respectively.

D is the mid-point of BC

$\therefore \quad \frac{x_2+x_3}{2}=1,\ \frac{y_2+y_3}{2}=5,\ \frac{z_2+z_3}{2}=-1$

$\Rightarrow \quad x_2+x_3=2,\ y_2+y_3=10,\ z_2+z_3=-2$ ...(i)

E is the mid-point of CA

$\therefore \quad \frac{x_1+x_3}{2}=0,\ \frac{y_1+y_3}{2}=4,\ \frac{z_1+z_3}{2}=-2$

$\Rightarrow \quad x_1+x_3=0,\ y_1+y_3=8,\ z_1+z_3=-4$ ...(ii)

F is the mid-point of AB

$\therefore \quad \frac{x_1+x_2}{2}=2,\ \frac{y_1+y_2}{2}=3,\ \frac{z_1+z_2}{2}=4$

$\Rightarrow \quad x_1+x_2=4,\ y_1+y_2=6,\ z_1+z_2=8$ ...(iii)

Adding first three equations in (i), (ii) and (iii), we obtain

$$2(x_1 + x_2 + x_3) = 2 + 0 + 4 \Rightarrow x_1 + x_2 + x_3 = 3.$$

Solving first equations in (i), (ii) and (iii) with $x_1 + x_2 + x_3 = 3$, we obtain:

$$x_1 = 1, x_2 = 3, x_3 = -1.$$

Adding second equations in (i), (ii) and (iii), we obtain

$$2(y_1 + y_2 + y_3) = 10 + 8 + 6 \Rightarrow y_1 + y_2 + y_3 = 12$$

Solving second equations in (i), (ii) and (iii) with $y_1 + y_2 + y_3 = 12$, we obtain

$$y_1 = 2,\ y_2 = 4,\ y_3 = 6.$$

Adding last equations in (i), (ii) and (iii), we obtain

$$2(z_1 + z_2 + z_3) = -2 - 4 + 8 \Rightarrow z_1 + z_2 + z_3 = 1.$$

Solving last equations in (i), (ii) and (iii) with $z_1 + z_2 + z_3 = 1$, we obtain

$$z_1 = 3,\ z_2 = 5,\ z_3 = -7.$$

Thus, the vertices of the triangle are $A(1, 2, 3)$, $B(3, 4, 5)$ and $C(-1, 6, -7)$.

EXAMPLE 6 *Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR.* [NCERT]

SOLUTION Suppose Q divides PR in the ratio $\lambda : 1$. Then, coordinates of Q are

$$\left(\frac{9\lambda + 3}{\lambda + 1}, \frac{8\lambda + 2}{\lambda + 1}, \frac{-10\lambda - 4}{\lambda + 1}\right)$$

But, coordinates of Q are $(5, 4, -6)$. Therefore,

$$\frac{9\lambda + 3}{\lambda + 1} = 5,\ \frac{8\lambda + 2}{\lambda + 1} = 4,\ \frac{-10\lambda - 4}{\lambda + 1} = 6.$$

These three equations give $\lambda = \frac{1}{2}$. So, Q divides PR in the ratio $\frac{1}{2} : 1$ or $1 : 2$.

EXAMPLE 7 *Find the coordinates of the points which trisect the line segment AB, given that $A(2, 1, -3)$ and $B(5, -8, 3)$.*

SOLUTION Let P and Q be the points which trisect AB. Then, $AP = PQ = QB$. Therefore, P divides AB in the ratio $1 : 2$ and Q divides it in the ratio $2 : 1$.

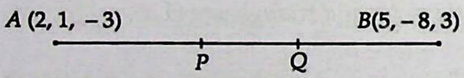

Fig. 28.11

As P divides AB in the ratio $1 : 2$, so coordinates of P are

$$\left(\frac{1 \times 5 + 2 \times 2}{1 + 2}, \frac{1 \times -8 + 2 \times 1}{1 + 2}, \frac{1 \times 3 + 2 \times -3}{1 + 2}\right) = (3, -2, -1)$$

Since Q divides AB in the ratio $2 : 1$, so coordinates of Q are

$$\left(\frac{2 \times 5 + 1 \times 2}{2 + 1}, \frac{2 \times -8 + 1 \times 1}{2 + 1}, \frac{2 \times 3 + 1 \times -3}{1 + 2}\right) = (4, -5, 1)$$

EXAMPLE 8 *Three vertices of a parallelogram ABCD are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.* [NCERT]

SOLUTION Let the coordinates of the fourth vertex D be (x, y, z). Since diagonals of a parallelogram bisect each other. Therefore, mid-point of AC and BD coincide.

$$\therefore \quad \left(\frac{3 - 1}{2}, \frac{-1 + 1}{2}, \frac{2 + 2}{2}\right) = \left(\frac{1 + x}{2}, \frac{2 + y}{2}, \frac{-4 + z}{2}\right)$$

$\Rightarrow \quad (1, 0, 2) = \left(\frac{x+1}{2}, \frac{y+2}{2}, \frac{z-4}{2}\right)$

$\Rightarrow \quad \frac{x+1}{2} = 1, \frac{y+2}{2} = 0, \frac{z-4}{2} = 2$

$\Rightarrow \quad x = 1, y = -2,$ and $z = 8$

Hence, the coordinates of the fourth vertex are $(1, -2, 8)$.

EXAMPLE 9 *Find the lengths of the medians of the triangle with vertices* $A(0, 0, 6)$, $B(0, 4, 0)$ *and* $C(6, 0, 0)$. [NCERT]

SOLUTION Let D, E and F be the mid-points of sides BC, CA and AB respectively.

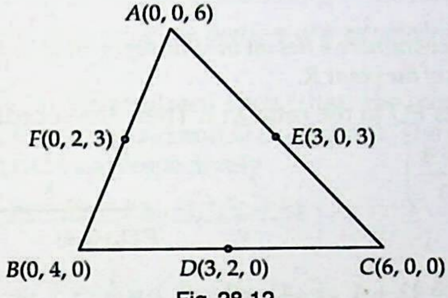

Fig. 28.12

The coordiantes of D, E and F are

$$\left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right) = (3, 2, 0),\ E\left(\frac{6+0}{2}, \frac{0+0}{2}, \frac{0+6}{2}\right) = (0, 2, 3)$$

and $F\left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = (0, 2, 3)$ respectively.

$\therefore \quad AD = \sqrt{(0-3)^2 + (0-2)^2 + (6-0)^2} = \sqrt{9+4+36} = 7$

$BE = \sqrt{(0-3)^2 + (4-0)^2 + (0-3)^2} = \sqrt{9+16+9} = \sqrt{34}$

and, $\quad CF = \sqrt{(6-0)^2 + (0-2)^2 + (0-3)^2} = \sqrt{36+4+9} = 7$

EXAMPLE 10 *Let* $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ *be three points forming a triangle. The bisector* AD *of* $\angle BAC$ *meets side* BC *in* D. *Find the coordinates of* D. [NCERT]

SOLUTION The bisector AD of $\angle BAC$ divides BC in the ratio $AB : AC$.

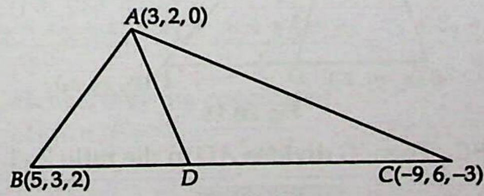

Fig. 28.13

Now, $\quad AB = \sqrt{(3-5)^2 + (2-3)^2 + (0-2)^2} = 3$ and, $AC = \sqrt{(3+9)^2 + (2-6)^2 + (0+3)^2} = 13$

Thus, D divides BC in the ratio $AB : AC$ i.e. $3 : 13$. Hence, the coordinates of D are

$$\left(\frac{3 \times -9 + 13 \times 5}{3+13}, \frac{3 \times 6 + 3 \times 13}{3+13}, \frac{3 \times -3 + 13 \times 2}{3+13}\right) = \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

EXAMPLE 11 *If the origin is the centroid of the triangle with vertices P (2a, 2, 6), Q (− 4, 3b, − 10) and R (8, 14, 2c), find the values of a, b and c.* **[NCERT]**

SOLUTION The coordinates of the centroid of ΔPQR are

$$\left(\frac{2a-4+8}{3}, \frac{2+3b+14}{3}, \frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3}, \frac{3b+16}{3}, \frac{2c-4}{3}\right)$$

It is given that the origin is the centroid of ΔPQR.

$$\therefore \quad \frac{2a+4}{3}=0, \frac{16+3b}{3}=0, \frac{2c-4}{3}=0$$

$$\Rightarrow \quad 2a+4=0, 16+3b=0, 2c-4=0$$

$$\Rightarrow \quad a=-2, b=-\frac{16}{3} \text{ and } c=2$$

EXAMPLE 12 *A point R with x-coordinate 4 lies on the line segment joining the points P (2, − 3, 4) and Q (8, 0, 10). Find the coordinates of the point R.* **[NCERT]**

SOLUTION Suppose R divides PQ in the ratio $\lambda:1$. Then, the coordinates of R are

$$\left(\frac{8\lambda+2}{\lambda+1}, \frac{-3}{\lambda+1}, \frac{10\lambda+4}{\lambda+1}\right)$$

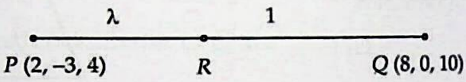

Fig. 28.14

Since x-coordinate of R is 4.

$$\therefore \quad \frac{8\lambda+2}{\lambda+1}=4 \Rightarrow 8\lambda+2=4\lambda+4 \Rightarrow 4\lambda=2 \Rightarrow \lambda=\frac{1}{2}$$

Hence, the coordinates of R are $(4, -2, 6)$.

LEVEL-2

EXAMPLE 13 *Show that the coordinates of the centroid of the triangle with vertices A (x_1, y_1, z_1), B (x_2, y_2, z_2) and C (x_3, y_3, z_3) are* $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$ **[NCERT]**

SOLUTION *Let D be the mid-point of AC. Then, coordinates of D are*

$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right).$$

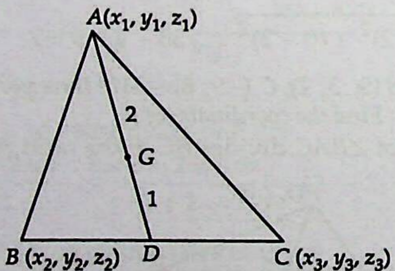

Fig. 28.15

Let G be the centroid of $\Delta\, ABC$. Then, G divides AD in the ratio $2:1$.
So, coordinates of D are

$$\left(\frac{1\cdot x_1+2\left(\frac{x_2+x_3}{2}\right)}{1+2}, \frac{1\cdot y_1+2\left(\frac{y_2+y_3}{2}\right)}{1+2}, \frac{1\cdot z_1+2\left(\frac{z_2+z_3}{2}\right)}{1+2}\right)$$

i.e., $$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

EXAMPLE 14 *Let P and Q be any two points. Find the coordinates of the point R which divides PQ externally in the ratio 2 : 1 and verify that Q is the mid point of PR.*

SOLUTION Let the coordinates of points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then, the coordinates of the point R which divides PQ externally in the ratio $2:1$ are

$$\left(\frac{2x_2 - x_1}{2-1}, \frac{2y_2 - y_1}{2-1}, \frac{2z_2 - z_1}{2-1}\right) = (2x_2 - x_1, 2y_2 - y_1, 2z_2 - z_1)$$

The coordinates of the mid-point of PR are

$$\left(\frac{x_1 + 2x_2 - x_1}{2}, \frac{y + 2y_2 - y_1}{2}, \frac{z_1 + 2z_2 - z_1}{2}\right) = (x_2, y_2, z_2)$$

Clearly, these are the coordinates of point Q. Hence, Q is the mid-point of PR.

EXAMPLE 15 *Prove that the lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent.*

SOLUTION Let $ABCD$ be a tetrahedron such that the coordinates of its vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$. The coordinates of the centroids of faces ABC, DAB, DBC and DCA are respectively

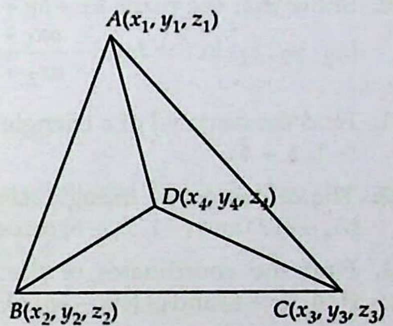

Fig. 28.16

$$G_1\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

$$G_2\left(\frac{x_1 + x_2 + x_4}{3}, \frac{y_1 + y_2 + y_4}{3}, \frac{z_1 + z_2 + z_4}{3}\right)$$

$$G_3\left(\frac{x_2 + x_3 + x_4}{3}, \frac{y_2 + y_3 + y_4}{3}, \frac{z_2 + z_3 + z_4}{3}\right)$$

and, $$G_4\left(\frac{x_4 + x_3 + x_1}{3}, \frac{y_4 + y_3 + y_1}{3}, \frac{z_4 + z_3 + z_1}{3}\right)$$

Now, coordinates of point G dividing DG_1 in the ratio $3:1$ are

$$\left(\frac{1.x_4 + 3\left(\frac{x_1 + x_2 + x_3}{3}\right)}{1+3}, \frac{1.y_4 + 3\left(\frac{y_1 + y_2 + y_3}{3}\right)}{1+3}, \frac{1.z_4 + 3\left(\frac{z_1 + z_2 + z_3}{3}\right)}{1+3}\right)$$

$$= \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

Similarly, the points dividing CG_2, AG_3 and BG_4 in the ratio $3:1$ have the same coordinates.

Thus, the point $G\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$ is common to DG_1, CG_2, AG_3 and BG_4. Hence, they are concurrent.

EXERCISE 28.3

LEVEL-1

1. The vertices of the triangle are $A(5, 4, 6)$, $B(1, -1, 3)$ and $C(4, 3, 2)$. The internal bisector of angle A meets BC at D. Find the coordinates of D and the length AD.
2. A point C with z-coordinate 8 lies on the line segment joining the points $A(2, -3, 4)$ and $B(8, 0, 10)$. Find its coordinates.

3. Show that the three points $A(2, 3, 4)$, $B(-1, 2-3)$ and $C(-4, 1, -10)$ are collinear and find the ratio in which C divides AB.
4. Find the ratio in which the line joining $(2, 4, 5)$ and $(3, 5, 4)$ is divided by the yz-plane.
5. Find the ratio in which the line segment joining the points $(2, -1, 3)$ and $(-1, 2, 1)$ is divided by the plane $x + y + z = 5$.
6. If the points $A(3, 2, -4)$, $B(9, 8, -10)$ and $C(5, 4, -6)$ are collinear, find the ratio in which C divides AB.
7. The mid-points of the sides of a triangle ABC are given by $(-2, 3, 5)$, $(4, -1, 7)$ and $(6, 5, 3)$. Find the coordinates of A, B and C.
8. $A(1, 2, 3)$, $B(0, 4, 1)$, $C(-1, -1, -3)$ are the vertices of a triangle ABC. Find the point in which the bisector of the angle $\angle BAC$ meets BC.
9. Find the ratio in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points $(12, -4, 8)$ and $(27, -9, 18)$.
10. Show that the plane $ax + by + cz + d = 0$ divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $-\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.
11. Find the centroid of a triangle, mid-points of whose sides are $(1, 2, -3)$, $(3, 0, 1)$ and $(-1, 1, -4)$.
12. The centroid of a triangle ABC is at the point $(1, 1, 1)$. If the coordinates of A and B are $(3, -5, 7)$ and $(-1, 7, -6)$ respectively, find the coordinates of the point C. [NCERT]
13. Find the coordinates of the points which tisect the line segment joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$. [NCERT]
14. Using section formula, show that the points $A(2, -3, 4)$, $B(-1, 2, 1)$ and $C(0, 1/3, 2)$ are collinear. [NCERT]
15. Given that $P(3, 2, -4)$, $Q(5, 4, -6)$ and $R(9, 8, -10)$ are collinear. Find the ratio in which Q divides PR. [NCERT]
16. Find the ratio in which the line segment joining the points $(4, 8, 10)$ and $(6, 10, -8)$ is divided by the yz-plane. [NCERT]

ANSWERS

1. $\left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right), \frac{\sqrt{1530}}{8}$
2. $(6, -1, 8)$
3. $2:1$ externally
4. $2:3$ externally
5. $1:3$ externally
6. $1:2$
7. $A(12, 1, 5)$, $B(0, 9, 1)$, $C(-4, -3, 9)$
8. $\left(-\frac{3}{10}, \frac{5}{2}, -\frac{1}{5}\right)$
9. $2:3, -2:3$
11. $(1, 1, -2)$
12. $(1, 1, 2)$
13. $(6, -4, -2)$, $(8, -10, 2)$
15. $1:2$
16. $2:3$ externally

HINTS TO NCERT & SELECTED PROBLEMS

1. Use the fact that D divides BC in the ratio $AB : AC$.
2. Suppose C divides AB in the ratio $\lambda : 1$. Then, the coordinates of C are

$$\left(\frac{8\lambda + 2}{\lambda + 1}, \frac{-3}{\lambda + 1}, \frac{10\lambda + 4}{\lambda + 1}\right).$$

It is given that the z-coordinate of C is 8.

$$\therefore \quad \frac{10\lambda + 4}{\lambda + 1} = 8 \Rightarrow \lambda = 2.$$

Hence, the coordinates of C are $(6, -1, 8)$.

12. Let the coordinates of C be (α, β, γ). Then centroid of triangle ABC has the coordinates $\left(\frac{\alpha+2}{3}, \frac{\beta+2}{3}, \frac{\gamma+1}{3}\right)$. But, coordinates of the centroid are given as $(1, 1, 1)$.

$$\therefore \quad \frac{\alpha+2}{3} = 1, \frac{\beta+2}{3} = 1, \frac{\gamma+1}{3} = 1 \Rightarrow \alpha = 1, \beta = 1, \gamma = 2$$

Hence, the coordinates of C are $(1, 1, 2)$.

13. Let A and B be the points of trisection of PQ. Then, A divides PQ internally in the ratio 1 : 2. So, coordinates of A are

$$\left(\frac{1\times 10 + 2\times 4}{1+2}, \frac{1\times -16 + 2\times 2}{1+2}, \frac{1\times 6 + 2\times -6}{1+2}\right) = (6, -4, -2).$$

Clearly, B is the mid-point of AQ. So, its coordinates are

$$\left(\frac{6+10}{2}, \frac{-4-16}{2}, \frac{-2+6}{2}\right) = (8, -10, 2).$$

14. Suppose $C\left(0, \frac{1}{3}, 2\right)$ divides the segment joining $A\,(2, -3, 4)$ and $B\,(-1, 2, 1)$ in the ratio $\lambda : 1$. Then, the coordinates of C are $\left(\frac{-\lambda+2}{\lambda+1}, \frac{2\lambda-3}{\lambda+1}, \frac{\lambda+4}{\lambda+1}\right)$.

But, the coordinates of C are $\left(0, \frac{1}{3}, 2\right)$.

$$\therefore \quad \frac{-\lambda+2}{\lambda+1} = 0, \frac{2\lambda-3}{\lambda+1} = \frac{1}{3} \text{ and } \frac{\lambda+4}{\lambda+1} = 2$$

All these equations give the same value of λ. Hence, A, B, C are collinear points.

15. Suppose Q divides PR in the ratio $\lambda : 1$. Then, the coordinates of Q are

$$\left(\frac{9\lambda+3}{\lambda+1}, \frac{8\lambda+2}{\lambda+1}, \frac{-10\lambda-4}{\lambda+1}\right)$$

But, the coordinates of Q are $(5, 4, -6)$.

$$\therefore \quad \frac{9\lambda+3}{\lambda+1} = 5, \frac{8\lambda+2}{\lambda+1} = 4, \frac{-10\lambda-4}{\lambda+1} = -6$$

All these equations give $\lambda = \frac{1}{2}$. Hence, Q divides PR in the ratio $\frac{1}{2} : 1$ i.e. 1 : 2.

16. Let the required ratio be $\lambda : 1$. Then, the coordinates of point of division are $\left(\frac{6\lambda+4}{\lambda+1}, \frac{10\lambda+8}{\lambda+1}, \frac{-8\lambda+10}{\lambda+1}\right)$. This point lies of yz-plane. So, its x-coordinate must be zero.

$$\text{i.e.} \quad \frac{6\lambda+4}{\lambda+1} = 0 \Rightarrow \lambda = -\frac{2}{3}$$

Hence, yz-plane divides the segment joining given points externally in the ratio 2 : 3.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the distance of the point $P(2, 3, 5)$ from the xy-plane.
2. Write the distance of the point $P(3, 4, 5)$ from z-axis.
3. If the distance between the points $P(a, 2, 1)$ and $Q(1, -1, 1)$ is 5 units, find the value of a.
4. The coordinates of the mid-points of sides AB, BC and CA of ΔABC are $D(1, 2, -3)$, $E(3, 0, 1)$ and $F(-1, 1, -4)$ respectively. Write the coordinates of its centroid.
5. Write the coordinates of the foot of the perpendicular from the point $(1, 2, 3)$ on y-axis.
6. Write the length of the perpendicular drawn from the point $P(3, 5, 12)$ on x-axis.
7. Write the coordinates of third vertex of a triangle having centroid at the origin and two vertices at $(3, -5, 7)$ and $(3, 0, 1)$.
8. What is the locus of a point (x, y, z) for which $y = 0, z = 0$?
9. Find the ratio in which the line segment joining the points $(2, 4, 5)$ and $(3, -5, 4)$ is divided by the yz-plane.
10. Find the point on y-axis which is at a distance of $\sqrt{10}$ units from the point $(1, 2, 3)$.
11. Find the point on x-axis which is equidistant from the points $A(3, 2, 2)$ and $B(5, 5, 4)$.
12. Find the coordinates of a point equidistant from the origin and points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.
13. Write the coordinates of the point P which is five-sixth of the way from $A(-2, 0, 6)$ to $B(10, -6, -12)$.
14. If a parallelopiped is formed by the planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$ parallel to the coordinate planes, then write the lengths of edges of the parallelopiped and length of the diagonal.
15. Determine the point on yz-plane which is equidistant from points $A(2, 0, 3)$, $B(0, 3, 2)$ and $C(0, 0, 1)$.
16. If the origin is the centroid of a triangle ABC having vertices $A(a, 1, 3)$, $B(-2, b -5)$ and $C(4, 7, c)$, find the values of a, b, c.

ANSWERS

1. 5	2. 5	3. $5, -3$	4. $(1, 1, -2)$	5. $(0, 2, 0)$
6. 13	7. $(-6, 5, -8)$	8. x-axis	9. $4:5$ internally	
10. $(0, 2, 0)$	11. $(49/4, 0, 0)$	12. $(a/2, b/2, c/2)$	13. $(8, -5, -9)$	
14. $3, 6, 2, 7$	15. $(0, 1, 3)$	16. $a = -2, b = -8, c = 2$		

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternatives in each of the following:

1. The ratio in which the line joining $(2, 4, 5)$ and $(3, 5, -9)$ is divided by the yz-plane is
   (a) $2:3$ (b) $3:2$ (c) $-2:3$ (d) $4:-3$
2. The ratio in which the line joining the points (a, b, c) and $(-a, -c, -b)$ is divided by the xy-plane is
   (a) $a:b$ (b) $b:c$ (c) $c:a$ (d) $c:b$
3. If $P(0, 1, 2)$, $Q(4, -2, 1)$ and $O(0, 0, 0)$ are three points, then $-POQ =$
   (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
4. If the extremities of the diagonal of a square are $(1, -2, 3)$ and $(2, -3, 5)$, then the length of the side is

(a) $\sqrt{6}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{7}$

5. The points $(5, -4, 2)$, $(4, -3, 1)$, $(7, 6, 4)$ and $(8, -7, 5)$ are the vertices of
   (a) a rectangle (b) a square (c) a parallelogram (d) none of these

6. In a three dimensional space the equation $x^2 - 5x + 6 = 0$ represents
   (a) points (b) planes (c) curves (d) pair of straight lines

7. Let $(3, 4, -1)$ and $(-1, 2, 3)$ be the end points of a diameter of a sphere. Then, the radius of the sphere is equal to
   (a) 2 (b) 3 (c) 6 (d) 7

8. XOZ-plane divides the join of $(2, 3, 1)$ and $(6, 7, 1)$ in the ratio
   (a) $3:7$ (b) $2:7$ (c) $-3:7$ (d) $-2:7$

9. What is the locus of a point for which $y = 0, z = 0$?
   (a) x-axis (b) y-axis (c) z-axis (d) yz-plane

10. The coordinates of the foot of the perpendicular drawn from the point $P(3, 4, 5)$ on the yz-plane are
    (a) $(3, 4, 0)$ (b) $(0, 4, 5)$ (c) $(3, 0, 5)$ (d) $(3, 0, 0)$

11. The coordinates of the foot of the perpendicular from a point $P(6, 7, 8)$ on x-axis are
    (a) $(6, 0, 0)$ (b) $(0, 7, 0)$ (c) $(0, 0, 8)$ (d) $(0, 7, 8)$

12. The perpendicular distance of the point $P(6, 7, 8)$ from xy-plane is
    (a) 8 (b) 7 (c) 6 (d) 10

13. The length of the perpendicular drawn from the point $P(3, 4, 5)$ on y-axis is
    (a) 10 (b) $\sqrt{34}$ (c) $\sqrt{113}$ (d) $5\sqrt{2}$

14. The perpendicular distance of the point $P(3, 3, 4)$ from the x-axis is
    (a) $3\sqrt{2}$ (b) 5 (c) 3 (d) 4

15. The length of the perpendicular drawn from the point $P(a, b, c)$ from z-axis is
    (a) $\sqrt{a^2 + b^2}$ (b) $\sqrt{b^2 + c^2}$ (c) $\sqrt{a^2 + c^2}$ (d) $\sqrt{a^2 + b^2 + c^2}$

ANSWERS

1. (c) 2. (d) 3. (d) 4. (b) 5. (a) 6. (b) 7. (c) 8. (c)
9. (a) 10. (b) 11. (a) 12. (a) 13. (b) 14. (b) 15. (a)

SUMMARY

1. In three dimensions, the coordinate axes of a rectangular Cartesian coordinate system are three mutually perpendicular lines. The axes are called the x, y and z axes.

2. The three planes determined by the pair of axes are the coordinate planes. These planes are called xy, yz and zx planes and they divide the space into eight regions known as octants.

3. The coordinates of a point P in the space are the perpendicular distances from P on three mutaually perpendicular coordinates planes YZ, ZX and XY respectively. The coordinates of a point P are written in the form of triplet like (x, y, z).

4. The coordinates of a point are also the distances from the origin of the feet of the perpendiculars from the point on the respective coordinate axes.

5. The coordinates of any point on:
   (i) x-axis are of the form $(x, 0, 0)$ (ii) y-axis are of the form $(0, y, 0)$
   (iii) z-axis are of the form $(0, 0, z)$ (iv) xy-plane are of the form $(x, y, 0)$
   (v) yz-plane are of the form $(0, y, z)$ (vi) zx – plane are of the form $(x, 0, z)$

6. The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

7. The distance of a point $P(x, y, z)$ from the origin $O(0, 0, 0)$ is given by 3 $OP = \sqrt{x^2 + y^2 + z^2}$.

8. The coordinates of the point R which divides the line segment joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally and externally in the ratio $m:n$ are given by

$$\left(\frac{m x_2 + n x_1}{m+n}, \frac{m y_2 + n y_1}{m+n}, \frac{m z_2 + n z_1}{m+n}\right) \text{and,} \left(\frac{m x_2 - n x_1}{m-n}, \frac{m y_2 - n y_1}{m-n}, \frac{m z_2 - n z_1}{m-n}\right)$$

respectively.

9. The coordinates of the mid-point of the line segment joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

10. The coordinates of the centroid of the triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and $(x_3 + y_3 + z_3)$ are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$.