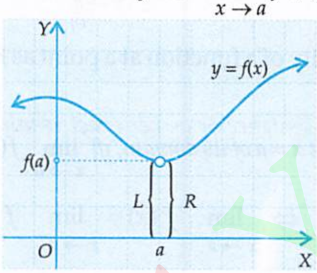
CHAPTER 8

CONTINUITY

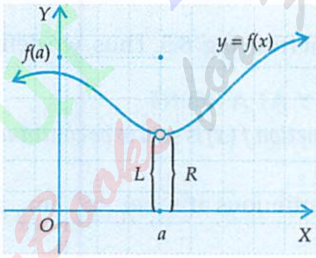
8.1 INTUTIVE NOTION OF CONTINUITY

Intutively a function is continuous in its domain if its graph is a curve without breaks or jumps throughout its domain and a function is continuous at a point in its domain if its graph does not have breaks or jumps in the immediate neighbourhood of the point. Consider the graph of a function $f(x)$ shown in Fig. 8.1. It is evident from the graph that $f(x)$ is not defined at $x = a$. Consequently, there is hole in the curve $y = f(x)$ and so $f(x)$ is not continuous at $x = a$. We also observe that $L = R$ i.e. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$ and so $\lim\limits_{x \to a} f(x)$ exists. Thus, the continuity of $f(x)$ at $x = a$ is destroyed, if $\lim\limits_{x \to a} f(x)$ exists but $f(x)$ is not defined at $x = a$.



$f(a)$ is not defined, $\lim\limits_{x \to a} f(x)$ exists

Fig. 8.1



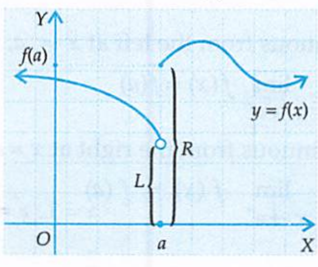
$f(a)$ is defined, $\lim\limits_{x \to a} f(x)$ exists and not equal to $f(a)$

Fig. 8.2

Let us now consider the function f whose graph is shown in Fig. 8.2. Clearly, $L = R$ i.e. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$. Consequently $\lim\limits_{x \to a} f(x)$ exists. But, there is hole in the curve because $\lim\limits_{x \to a} f(x)$ is not equal to $f(a)$. So, $f(x)$ becomes discontinuous at $x = a$ if, $\lim\limits_{x \to a} f(x)$ exists but it is not equal to the value of f at $x = a$.

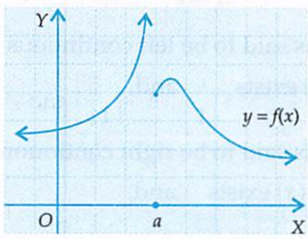
In Fig. 8.3, we observe that $L \neq R$ i.e. $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$. So, $\lim\limits_{x \to a} f(x)$ does not exist.

Also, $f(x)$ is not continuous at $x = a$. Thus, the continuity of f at $x = a$ is also destroyed, if $\lim\limits_{x \to a} f(x)$ does not exist. This happens due to the jump in the values of $f(x)$ as x crosses 'a'.



$\lim\limits_{x \to a^-} f(x)$ is not same as $\lim\limits_{x \to a^+} f(x)$

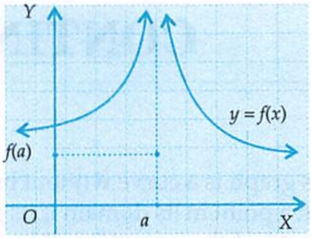
Fig. 8.3



$f(x)$ is defined at $x = a$, $\lim\limits_{x \to a^-} f(x) = +\infty$

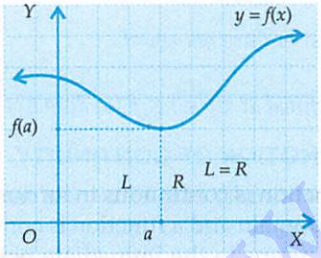
Fig. 8.4 (i)

The continuity of a function f is also destroyed if either of the two limits $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ or both tend to $+\infty$ or $-\infty$ and $f(a)$ is finite as is evident from Fig. 8.4 (i), (ii).



$f(x)$ is defined at $x = a$, $\lim\limits_{x \to a^-} f(x) = +\infty$ and $\lim\limits_{x \to a^+} f(x) = +\infty$

Fig. 8.4 (ii)



$\lim\limits_{x \to a} f(a)$ exists and is equal to $f(a)$.
Function $f(x)$ is continuous at point $x = a$

Fig. 8.5

It follows from the above discussion that a function $f(x)$ can be continuous at a point $x = a$ iff

(i) $f(a)$ is defined, (ii) $\lim\limits_{x \to a} f(x)$ exists and, (iii) $\lim\limits_{x \to a} f(x) = f(a)$.

This is also evident from Fig. 8.5. Thus, we define continuity of a function at a point as follows.

8.2 CONTINUITY AT A POINT

DEFINITION *A function $f(x)$ is said to be continuous at a point $x = a$ of its domain, iff* $\lim\limits_{x \to a} f(x) = f(a)$.

Thus,

$$(f(x) \text{ is continuous at } x = a) \Leftrightarrow \lim_{x \to a} f(x) = f(a) \Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

If $f(x)$ is not continuous at a point $x = a$, then it is said to be discontinuous at $x = a$.

If $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \neq f(a)$, then the discontinuity is known as the *removable discontinuity*, because $f(x)$ can be made continuous by redefining it at point $x = a$ in such a way that $f(a) = \lim\limits_{x \to a} f(x)$.

If $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$, then $f(x)$ is said to have a *discontinuity of first kind*.

A function $f(x)$ is said to have a discontinuity of *the second kind* at $x = a$ iff

$\lim\limits_{x \to a^-} f(x)$ or, $\lim\limits_{x \to a^+} f(x)$ or, both do not exist.

A function $f(x)$ is said to be left continuous or continuous from the left at $x = a$, iff

(i) $\lim\limits_{x \to a^-} f(x)$ exists and, (ii) $\lim\limits_{x \to a^-} f(x) = f(a)$

A function $f(x)$ is said to be right continuous or continuous from the right at $x = a$, iff

(i) $\lim\limits_{x \to a^+} f(x)$ exists and, (ii) $\lim\limits_{x \to a^+} f(x) = f(a)$

It follows from the above definitions that

$f(x)$ is continuous at $x = a$ iff it is both left as well as right continuous at $x = a$.

REMARK *A function $f(x)$ fails to be continuous at $x = a$ for any of the following reasons.*

(i) $\lim_{x \to a} f(x)$ *exists but it is not equal to $f(a)$.* (ii) $\lim_{x \to a} f(x)$ *does not exist.*

This happens if either $\lim_{x \to a^-} f(x)$ *does not exist or,* $\lim_{x \to a^+} f(x)$ *does not exist or both* $\lim_{x \to a^-} f(x)$ *and* $\lim_{x \to a^+} f(x)$ *exist but are not equal.*

(iii) *f is not defined at $x = a$ i.e. $f(a)$ does not exist.*

8.3 ALGEBRA OF CONTINUOUS FUNCTIONS

Regarding the continuity of the sum, difference, product and quotient of functions, we have the following theorems.

THEOREM 1 *Let f and g be two real functions, continuous at $x = a$. Let α be a real number. Then,*

(i) *$f + g$ is continuous at $x = a$.* (ii) *$f - g$ is continuous at $x = a$.*

(iii) *αf is continuous at $x = a$.* (iv) *fg is continuous at $x = a$.*

(v) *$\frac{1}{f}$ is continuous at $x = a$, provided that $f(a) \neq 0$.*

(vi) *$\frac{f}{g}$ is continuous at $x = a$, provided that $g(a) \neq 0$.*

PROOF Since f and g are continuous at $x = a$. Therefore, $\lim_{x \to a} f(x) = f(a)$ and $\lim_{x \to a} g(x) = g(a)$.

(i) We find that

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a) \qquad \left[\because \lim_{x \to a} f(x) = f(a) \text{ and, } \lim_{x \to a} g(x) = g(a)\right]$$

$$= (f+g)(a)$$

$\therefore$ $f + g$ is continuous at $x = a$.

(ii) We find that

$$\lim_{x \to a} (f-g)(x) = \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$= f(a) - g(a) \qquad \left[\because \lim_{x \to a} f(x) = f(a) \text{ and, } \lim_{x \to a} g(x) = g(a)\right]$$

$$= (f-g)(a)$$

$\therefore$ $f - g$ is continuous at $x = a$.

(iii) We find that

$$\lim_{x \to a} (\alpha f)(x) = \lim_{x \to a} \alpha f(x) = \alpha \lim_{x \to a} f(x) = \alpha f(a) \qquad \left[\because \lim_{x \to a} f(x) = f(a)\right]$$

$\therefore$ αf is continuous at $x = a$.

(iv) We find that

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} \{f(x) g(x)\} = \lim_{x \to a} f(x) \lim_{x \to a} g(x) = f(a) g(a) = (fg)(a)$$

So, fg is continuous at $x = a$.

(v) We find that

$$\lim_{x \to a} \left(\frac{1}{f}\right)(x) = \lim_{x \to a} \left(\frac{1}{f(x)}\right) = \frac{1}{\lim\limits_{x \to a} f(x)} = \frac{1}{f(a)} = \left(\frac{1}{f}\right)(a)$$

So, $\frac{1}{f}$ is continuous at $x = a$

(vi) We find that

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

So, $\frac{f}{g}$ is continuous at $x = a$.

THEOREM 2 *Let f and g be real functions such that fog is defined. If g is continuous at $x = a$ and f is continuous at $g(a)$, show that fog is continuous at $x = a$.*

PROOF Since fog is defined. Therefore, Range $(g) \subset$ Domain $(f) \Rightarrow g(x) \in$ Domain (f) for all $x \in$ Domain (g)

Now, $g(x)$ is continuous at $x = a \Rightarrow \lim\limits_{x \to a} g(x) = g(a)$ ...(i)

f is continuous at $g(a)$

$$\Rightarrow \lim_{g(x) \to g(a)} f(g(x)) = f(g(a))$$

$$\Rightarrow \lim_{x \to a} f(g(x)) = f(g(a)) \qquad [\text{From (i)}, x \to a \Rightarrow g(x) \to g(a)]$$

$$\Rightarrow \lim_{x \to a} (fog)(x) = (fog)(a) \Rightarrow fog \text{ is continuous at } x = a.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON TESTING CONTINUITY OF A FUNCTION AT A POINT WHEN THE FUNCTION HAS SAME DEFINITION ON BOTH SIDES OF THE GIVEN POINT

EXAMPLE 1 *Test the continuity of the following function at the origin:* $f(x) = \begin{cases} \frac{|x|}{x} & ; \quad x \neq 0 \\ 1 & ; \quad x = 0 \end{cases}$

SOLUTION We observe that:

$$(\text{LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{|-h|}{-h} = \lim_{h \to 0} \frac{h}{-h} = \lim_{h \to 0} -1 = -1$$

and, $$(\text{RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{|h|}{h} = \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

Thus, we obtain : $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$. Hence, $f(x)$ is not continuous at the origin.

ALITER We have,

$$f(x) = \begin{cases} \dfrac{|x|}{x} & ; \ x \neq 0 \\ 1 & ; \ x = 0 \end{cases} \text{ or, } f(x) = \begin{cases} \dfrac{x}{x} = 1 & ; x > 0 \\ \dfrac{-x}{x} = -1 & ; x < 0 \\ 1 & ; x = 0 \end{cases} \qquad \left[\because |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}\right]$$

$$\therefore \ (\text{LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1 \qquad \left[\begin{array}{l}\because f(x) = -1 \text{ for } x < 0 \text{ and } x \to 0^- \\ \text{means that } x < 0 \text{ such that } x \to 0\end{array}\right]$$

$$(\text{RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} 1 = 1 \qquad \left[\begin{array}{l}\because f(x) = 1 \text{ for } x > 0 \text{ and } x \to 0^+ \\ \text{means that } x > 0 \text{ such that } x \to 0\end{array}\right]$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x).$$

Hence, $f(x)$ is not continuous at the origin.

EXAMPLE 2 *Show that the function $f(x)$ given by* $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2 & , \ x = 0 \end{cases}$ *is continuous at $x = 0$.*

SOLUTION We observe that

$$(\text{LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{\sin(-h)}{-h} + \cos(-h) = \lim_{h \to 0} \frac{\sin h}{h} + \lim_{h \to 0} \cos h = 1 + 1 = 2$$

$$(\text{RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{\sin h}{h} + \cos h = \lim_{h \to 0} \frac{\sin h}{h} + \lim_{h \to 0} \cos h = 1 + 1 = 2$$

and, $f(0) = 2$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE 3 *Examine the function $f(t)$ given by* $f(t) = \begin{cases} \dfrac{\cos t}{\pi/2 - t} & ; \ t \neq \pi/2 \\ 1 & ; \ t = \pi/2 \end{cases}$ *for continuity at $t = \pi/2$.*

SOLUTION We observe that:

$$(\text{LHL at } t = \pi/2) = \lim_{t \to \pi/2^-} f(t)$$

$$= \lim_{h \to 0} f\left(\frac{\pi}{2} - h\right) = \lim_{h \to 0} \frac{\cos(\pi/2 - h)}{\pi/2 - (\pi/2 - h)} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

and, $(\text{RHL at } t = \pi/2) = \lim_{t \to \pi/2^+} f(t)$

$$= \lim_{h \to 0} f\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0} \frac{\cos(\pi/2 + h)}{\pi/2 - (\pi/2 + h)} = \lim_{h \to 0} \frac{-\sin h}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

and, $f(\pi/2) = 1$.

$\therefore \quad \lim_{t \to \pi/2^-} f(t) = \lim_{t \to \pi/2^+} f(t) = f(\pi/2)$. So, $f(t)$ is continuous at $t = \pi/2$.

Type II ON TESTING CONTINUITY OF A FUNCTION AT A POINT WHEN THE FUNCTION HAS DIFFERENT DEFINITIONS ON BOTH SIDES OF THE GIVEN POINT

Let a function $f(x)$ be defined as

$$f(x)=\begin{cases}\phi(x) & ; \text{ if } x<a \\ \psi(x) & ; \text{ if } x\geq a\end{cases} \text{ or, } f(x)=\begin{cases}\phi(x) & ; \text{ if } x\leq a \\ \psi(x) & ; \text{ if } x>a\end{cases} \text{ or, } f(x)=\begin{cases}\phi(x) & ; \text{ if } x<a \\ k & ; \text{ if } x=a \\ \psi(x) & ; \text{ if } x>a\end{cases}$$

To test the continuity of such functions at $x=a$, we have to find left hand and right hand limits of $f(x)$ at $x=a$. For finding these two limits one can use the method which we have used in previous examples or we can use the following method:

$$(\text{LHL at } x=a)=\lim_{x\to a^-} f(x)=\lim_{x\to a}\phi(x) \quad \left[\begin{array}{l}\because x\to a^- \Leftrightarrow x< \text{ and } x\to a \\ \therefore \lim\limits_{x\to a^-} f(x)=\lim\limits_{x\to a}\phi(x)\ [\because f(x)=\phi(x) \text{ for } x<a]\end{array}\right]$$

Now, $\lim\limits_{x\to a}\phi(x)$ can be calculated by various methods of evaluating limits as discussed in the chapter on limits.

Similarly, we have

$$(\text{RHL at } x=a)=\lim_{x\to a^+} f(x)=\lim_{x\to a}\psi(x) \quad \left[\begin{array}{l}\because x\to a^+ \Leftrightarrow x>a\ \&\ x\to a \\ \therefore \lim\limits_{x\to a^+} f(x)=\lim\limits_{x\to a}\psi(x)\ [\because f(x)=\psi(x) \text{ for } x>a]\end{array}\right]$$

Now, $\lim\limits_{x\to a}\psi(x)$ can be calculated by various methods of evaluating limits.

EXAMPLE 4 *Discuss the continuity of the function $f(x)$ at $x=1/2$, where*

$$f(x)=\begin{cases}1/2-x & ; \ 0\leq x<1/2 \\ 1 & ; \ x=1/2 \\ 3/2-x & ; \ 1/2<x\leq 1\end{cases}$$

[CBSE 2011]

SOLUTION We observe that:

$$(\text{LHL at } x=1/2)=\lim_{x\to 1/2^-} f(x)=\lim_{x\to 1/2}(1/2-x) \quad \left[\because f(x)=\frac{1}{2}-x \text{ for } 0\leq x<\frac{1}{2}\right]$$

$$=1/2-1/2=0 \quad \text{[Using direct substitution method]}$$

and, $$(\text{RHL at } x=1/2)=\lim_{x\to 1/2^+} f(x)=\lim_{x\to 1/2}(3/2-x) \quad \left[\because f(x)=\frac{3}{2}-x \text{ for } \frac{1}{2}<x\leq 1\right]$$

$$=3/2-1/2=1 \quad \text{[Using direct substitution method]}$$

Clearly, $\lim\limits_{x\to 1/2^-} f(x)\neq \lim\limits_{x\to 1/2^+} f(x)$.

Hence, $f(x)$ is not continuous at $x=1/2$. Clearly, $f(x)$ has discontinuity of first kind at $x=1/2$.

EXAMPLE 5 *Discuss the continuity of the function f(x) given by* $f(x)=\begin{cases}2-x, & x<2 \\ 2+x, & x\geq 2\end{cases}$ *at $x=2$.*

SOLUTION We observe that:

$$(\text{LHL at } x=2)=\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(2-x)=2-2=0 \quad [\because f(x)=2-x \text{ for } x<2]$$

and, $$(\text{RHL at } x=2)=\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(2+x)=2+2=4 \quad [\because f(x)=2+x= \text{ for } x\geq 2]$$

$\because \quad \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$. Hence, $f(x)$ is not continuous at $x=2$.

EXAMPLE 6 *Show that* $f(x) = \begin{cases} 5x-4 & , \text{ when } 0<x\le 1 \\ 4x^3-3x & , \text{ when } 1<x<2 \end{cases}$ *is continuous at* $x=1$.

SOLUTION We have,

$$(\text{LHL at } x=1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1} 5x-4 = 5\times 1-4=1 \quad [\because f(x)=5x-4, \text{ when } x\le 1]$$

$$(\text{RHL at } x=1) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} 4x^3-3x = 4(1)^3-3(1)=1 \quad [\because f(x)=4x^3-3x,\ x>1]$$

and, $f(1) = 5\times 1-4 = 1 \quad [\because f(x)=5x-4, \text{ where } x\le 1]$

$\therefore \quad \lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$. So, $f(x)$ is continuous at $x=1$.

EXAMPLE 7 *Show that the function* $f(x) = 2x-|x|$ *is continuous at* $x=0$. **[CBSE 2002]**

SOLUTION We have,

$$f(x) = 2x-|x| = \begin{cases} 2x-x & , \text{ if } x\ge 0 \\ 2x-(-x) & , \text{ if } x<0 \end{cases} = \begin{cases} x & , \text{ if } x\ge 0 \\ 3x & , \text{ if } x<0 \end{cases}$$

Now,

$$(\text{LHL at } x=0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 3x = 3\times 0 = 0$$

$$(\text{RHL at } x=0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 \text{ and, } f(0) = 0$$

$\therefore \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$. So, $f(x)$ is continuous at $x=0$.

Type III ON FINDING THE VALUE(S) OF A CONSTANT GIVEN IN THE DEFINITION OF A FUNCTION WHEN IT IS CONTINUOUS AT AN INDICATED POINT

A function $f(x)$ is continuous at a point $x=a$ iff $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$.

But, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \Rightarrow \lim_{x\to a} f(x)$ exists.

Thus, $f(x)$ is continuous at $x=a$ iff $\lim_{x\to a} f(x) = f(a)$.

We will use this result in finding unknown quantity in the definition of a function when it is given to be continuous at a given point.

EXAMPLE 8 *Determine the value of k for which the function* $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & , \ x\ne 3 \\ k & , \ x=3 \end{cases}$ *is continuous at* $x=3$.

SOLUTION It is given that $f(x)$ is continuous at $x=3$.

$$\therefore \quad \lim_{x\to 3} f(x) = f(3)$$

$$\Rightarrow \quad \lim_{x\to 3} f(x) = k \quad [\because f(3)=k]$$

$$\Rightarrow \quad \lim_{x\to 3} \frac{x^2-9}{x-3} = k \Rightarrow \lim_{x\to 3} \frac{(x-3)(x+3)}{x-3} = k \Rightarrow \lim_{x\to 3} (x+3) = k \Rightarrow 6 = k$$

Thus, $f(x)$ is continuous at $x=3$, if $k=6$.

EXAMPLE 9 *Find the value of the constant λ so that the function given below is continuous at $x = -1$.*

$$f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x+1} & , \quad x \neq -1 \\ \lambda & , \quad x = -1 \end{cases}$$

SOLUTION Given that $f(x)$ is continuous at $x = -1$.

$$\therefore \quad \lim_{x \to -1} f(x) = f(-1)$$

$$\Rightarrow \quad \lim_{x \to -1} \frac{x^2 - 2x - 3}{x+1} = \lambda \qquad [\because f(-1) = \lambda]$$

$$\Rightarrow \quad \lim_{x \to -1} \frac{(x-3)(x+1)}{x+1} = \lambda \Rightarrow \lim_{x \to -1} (x-3) = \lambda \Rightarrow -4 = \lambda$$

So, $f(x)$ is continuous at $x = -1$, if $\lambda = -4$.

EXAMPLE 10 *Find the value of 'a' if the function f(x) defined by* $f(x) = \begin{cases} 2x-1 & , \quad x < 2 \\ a & , \quad x = 2 \\ x+1 & , \quad x > 2 \end{cases}$ *is continuous at $x = 2$.*

SOLUTION We find that:

$$(\text{LHL at } x = 2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} (2x - 1) = 2 \times 2 - 1 = 3,$$

$$(\text{RHL at } x = 2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x+1) = 2 + 1 = 3, \text{ and, } f(2) = a$$

Since $f(x)$ is continuous at $x = 2$. Therefore,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \Rightarrow 3 = 3 = a \Rightarrow a = 3$$

Thus, $f(x)$is continuous at $x = 2$, if $a = 3$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON TESTING CONTINUITY OF A FUNCTION AT A POINT WHEN THE FUNCTION HAS SAME DEFINITION ON BOTH SIDES OF THE GIVEN POINT

EXAMPLE 11 *Show that the function $f(x)$ given by* $f(x) = \begin{cases} x \sin \dfrac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$ *is continuous at $x = 0$.*

[NCERT EXEMPLAR]

SOLUTION We observe that:

$$(\text{LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} f(-h) = \lim_{h \to 0} -h \sin\left(\frac{1}{-h}\right)$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \times (\text{an oscillating number between} -1 \text{ and } 1) = 0$$

$$(\text{RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} f(h) = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right)$$

$$= 0 \times (\text{an oscillating number between} -1 \text{ and } 1) = 0$$

and, $f(0) = 0$.

Thus, we find that : $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. Hence, $f(x)$ is continuous at $x = 0$.

EXAMPLE 12 *Show that the function f(x) given by* $f(x) = \begin{cases} \dfrac{e^{1/x}-1}{e^{1/x}+1} & , \text{ when } x \neq 0 \\ 0 & , \text{ when } x = 0 \end{cases}$ *is discontinuous at* $x = 0$. [NCERT EXEMPLAR]

SOLUTION We observe that:

$$(\text{LHL at } x=0) = \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{e^{-1/h}-1}{e^{-1/h+1}} = \lim_{h \to 0} \frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1} = \frac{0-1}{0+1} = -1 \quad \left[\because \lim_{h \to 0} \frac{1}{e^{1/h}} = 0\right]$$

and, $$(\text{RHL at } x=0) = \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h \to 0} \frac{1-1/e^{1/h}}{1+1/e^{1/h}} = \frac{1-0}{1+0} = 1$$

$\therefore$ (LHL at $x=0$) $\neq$ (RHL at $x=0$)

So, $f(x)$ is not continuous at $x=0$ and has a discontinuity of first kind at $x=0$.

Type II ON TESTING CONTINUITY OF A FUNCTION AT A POINT WHEN THE FUNCTION HAS DIFFERENT DEFINITIONS ON BOTH SIDES OF THE GIVEN POINT

EXAMPLE 13 *Discuss the continuity of the function of given by* $f(x) = |x-1| + |x-2|$ *at* $x=1$ *and* $x=2$.

SOLUTION We have,

$$f(x) = |x-1| + |x-2|$$

$$\Rightarrow \quad f(x) = \begin{cases} -(x-1)-(x-2) & , \text{ if } x < 1 \\ (x-1)-(x-2) & , \text{ if } 1 \le x < 2 \\ (x-1)+(x-2) & , \text{ if } x \ge 2 \end{cases} \Rightarrow f(x) = \begin{cases} -2x+3 & , \text{ if } x < 1 \\ 1 & , \text{ if } 1 \le x < 2 \\ 2x-3 & , x \ge 2 \end{cases}$$

Continuity at $x = 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-2x+3) = -2 \times 1 + 3 = 1, \ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 1 = 1 \text{ and, } f(1) = 1.$$

$\therefore \quad \lim\limits_{x \to 1^-} f(x) = f(1) = \lim\limits_{x \to 1^+} f(x)$. So, $f(x)$ is continuous at $x = 1$.

Continuity at $x = 2$:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 1 = 1, \ \lim_{x \to 2^+} f(x) = \lim_{x \to 2} (2x-3) = 2 \times 2 - 3 = 1$$

and, $f(2) = 2 \times 2 - 3 = 1$.

$\therefore \quad \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$. So, $f(x)$ is continuous at $x = 2$.

Type III ON FINDING THE VALUE(S) OF A CONSTANT GIVEN IN THE DEFINITION OF A FUNCTION WHEN IT IS CONTINUOUS AT AN INDICATED POINT

EXAMPLE 14 *Find the value of the constant k so that the function given below is continuous at* $x = 0$.

$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{2x^2} & , x \neq 0 \\ k & , x = 0 \end{cases}$$

SOLUTION It is given that the function $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{1-\cos 2x}{2x^2} = k \qquad [\because f(0) = k]$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{2\sin^2 x}{2x^2} = k \Rightarrow \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 = k \Rightarrow 1^2 = k \Rightarrow k = 1$$

Thus, $f(x)$ is continuous at $x = 0$, if $k = 1$.

EXAMPLE 15 *If the function $f(x)$ defined by* $f(x) = \begin{cases} \dfrac{\log(1+ax) - \log(1-bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ *is continuous at $x = 0$, find k.*

SOLUTION Since $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{\log(1+ax) - \log(1-bx)}{x} = k \qquad [\because f(0) = k]$$

$$\Rightarrow \quad \lim_{x\to 0} \left\{\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right\} = k$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{\log(1+ax)}{x} - \lim_{x\to 0} \frac{\log(1-bx)}{x} = k$$

$$\Rightarrow \quad a \lim_{x\to 0} \frac{\log(1+ax)}{ax} - (-b) \lim_{x\to 0} \frac{\log(1-bx)}{(-b)\,x} = k$$

$$\Rightarrow \quad a(1) - (-b)(1) = k \Rightarrow a + b = k \qquad \left[\text{Using: } \lim_{x\to 0} \frac{\log(1+x)}{x} = 1\right]$$

Thus, $f(x)$ is continuous at $x = 0$, if $k = a + b$.

EXAMPLE 16 *Find the values of 'a' so that the function f(x) defined by* $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ *may be continuous at $x = 0$.*

SOLUTION The function $f(x)$ will be continuous at $x = 0$, iff

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{\sin^2 ax}{x^2} = 1 \qquad [\because f(0) = 1]$$

$$\Rightarrow \quad a^2 \lim_{x\to 0} \left(\frac{\sin ax}{ax}\right)^2 = 1 \Rightarrow a^2 (1)^2 = 1 \Rightarrow a = \pm 1$$

Thus, $f(x)$ will be continuous at $x = 0$, if $a = \pm 1$.

EXAMPLE 17 *If the function $f(x)$ given by* $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ *is continuous at $x = 1$, find the values of a and b.* **[CBSE 2002, 2010, 2011, 2012]**

SOLUTION We find that:

$$(\text{LHL at } x=1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5ax - 2b) = 5a - 2b$$

$$(\text{RHL at } x=1) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3ax + b) = 3a + b \text{ and, } f(1) = 11.$$

Since $f(x)$ is continuous at $x = 1$.

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\lim_{x \to 1^-} f(x) = f(1) \text{ and, } \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \quad 5a - 2b = 3a + b = 11 \Rightarrow 5a - 2b = 11 \text{ and } 3a + b = 11 \Rightarrow a = 3 \text{ and } b = 2$$

EXAMPLE 18 *Let* $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} &, \text{ if } x < 0 \\ a &, \text{ if } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} &, \text{ if } x > 0 \end{cases}$. *Determine the value of* a *so that* $f(x)$ *is continuous at* $x = 0$.

[CBSE 2010, 2012, 2013, NCERT EXEMPLAR]

SOLUTION For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = a \qquad \text{...(i)}$$

$$\text{Now,} \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{1-\cos 4x}{x^2} \qquad \left[\because f(x) = \frac{1-\cos 4x}{x^2} \text{ for } x < 0\right]$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{2\sin^2 2x}{x^2}$$

$$\Rightarrow \quad \lim_{x \to 0^-} f(x) = 2 \lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^2 = 2 \times 4 \times \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8 \qquad \text{...(ii)}$$

$$\text{and,} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \qquad \left[\because f(x) = \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \text{ for } x > 0\right]$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \frac{\sqrt{x}}{16+\sqrt{x}-16}\left(\sqrt{16+\sqrt{x}}+4\right)$$

$$\Rightarrow \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \left(\sqrt{16+\sqrt{x}}+4\right) = 4 + 4 = 8 \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we get $a = 8$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 19 *Determine* $f(0)$ *so that the function* $f(x)$ *defined by* $f(x) = \dfrac{(4^x - 1)^3}{\sin \dfrac{x}{4} \log\left(1 + \dfrac{x^2}{3}\right)}$ *becomes continuous at* $x = 0$.

SOLUTION For $f(x)$ to be continuous at $x = 0$, we must have

$$\lim_{x \to 0} f(x) = f(0)$$

$$\Rightarrow \quad f(0) = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{(4^x - 1)^3}{\sin \frac{x}{4} \log\left(1 + \frac{x^2}{3}\right)} = \lim_{x \to 0} \frac{\left(\frac{4^x - 1}{x}\right)^3}{\left(\frac{\sin \frac{x}{4}}{4 \times \frac{x}{4}}\right)\left(\frac{\log\left(1 + \frac{x^2}{3}\right)}{\frac{x^2}{3} \times 3}\right)}$$

$$\Rightarrow \quad f(0) = \frac{(\log_e 4)^3}{\frac{1}{4} \times \frac{1}{3}} = 12(\log_e 4)^3.$$

EXAMPLE 20 *If* $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$, $x \neq \frac{\pi}{4}$. *Find the value of* $f\left(\frac{\pi}{4}\right)$ *so that* $f(x)$ *becomes continuous at* $x = \pi/4$. **[NCERT EXEMPLAR]**

SOLUTION For $f(x)$ to be continuous at $x = \frac{\pi}{4}$, we must have

$$\lim_{x \to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \lim_{x \to \pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \lim_{x \to \pi/4} \frac{\sqrt{2}\left(\cos x - \cos\frac{\pi}{4}\right)}{\cot x - \cot\frac{\pi}{4}} = -\sqrt{2} \lim_{x \to \pi/4} \frac{2\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{\left(\cos x \sin\frac{\pi}{4} - \sin x \cos\frac{\pi}{4}\right)} \times \sin x \sin\frac{\pi}{4}$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = -2\sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{-\sin\left(x - \frac{\pi}{4}\right)} \times \sin x \sin\frac{\pi}{4}$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = -2\sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{-2\sin\left(\frac{x}{2} - \frac{\pi}{8}\right)\cos\left(\frac{x}{2} - \frac{\pi}{8}\right)} \times \sin x \sin\frac{\pi}{4}$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(\frac{x}{2} + \frac{\pi}{8}\right)}{\cos\left(\frac{x}{2} - \frac{\pi}{8}\right)} \times \sin x \sin\frac{\pi}{4} = \sqrt{2}\, \frac{\sin\left(\frac{\pi}{8} + \frac{\pi}{8}\right)}{\cos\left(\frac{\pi}{8} - \frac{\pi}{8}\right)} \times \sin\frac{\pi}{4} \sin\frac{\pi}{4} = \left(\sin\frac{\pi}{4}\right)^2 = \frac{1}{2}$$

EXAMPLE 21 *Prove that the greatest integer function [x] is continuous at all points except at integer points.* **[NCERT]**

SOLUTION Let $f(x) = [x]$ be the greatest integer function and let k be any integer. Then,

$$f(x) = [x] = \begin{cases} k - 1, & \text{if } k - 1 \leq x < k \\ k, & \text{if } k \leq x < k + 1 \end{cases} \qquad \text{[By definition of } [x]\text{]}$$

Now,

$$(\text{LHL at } x=k) = \lim_{x \to k^-} f(x) = \lim_{h \to 0} f(k-h) = \lim_{h \to 0} [k-h]$$

$$= \lim_{h \to 0} (k-1) = k-1 \qquad [\because\ k-1 \le k-h < k\ \therefore\ [k-h] = k-1]$$

and,

$$(\text{RHL at } x=k) = \lim_{x \to k^+} f(x) = \lim_{h \to 0} f(k+h) = \lim_{h \to 0} [k+h]$$

$$= \lim_{h \to 0} k = k \qquad [\because k \le k+h < k+1\ \therefore\ [k+h] = k]$$

$$\therefore \quad \lim_{x \to k^-} f(x) \neq \lim_{x \to k^+} f(x).$$

So, $f(x)$ is not continuous at $x=k$.

Since k is an arbitrary integer. Therefore, $f(x)$ is not continuous at integer points.

Let a be any real number other than an integer. Then, there exists an integer k such that $k-1 < a < k$.

Now,

$$(\text{LHL at } x=a) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} [a-h]$$

$$= \lim_{h \to 0} k-1 = k-1 \qquad [\because\ k-1 < a-h < k\ \therefore\ [a-h] = k-1]$$

$$(\text{RHL at } x=a) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} [a+h] = \lim_{h \to 0} (k-1) = k-1 \qquad \left[\begin{array}{l} \because\ k-1 < a+h < k \\ \therefore\ [a+h] = k-1 \end{array}\right]$$

and, $\quad f(a) = [a] = k-1 \qquad [\because\ k-1 < a < k\ \therefore\ [a] = k-1]$

Thus, $\displaystyle \lim_{x - a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

So, $f(x)$ is continuous at $x=a$. Since a is an arbitrary real number, other than an integer. Therefore, $f(x)$ is continuous at all real points except integer points.

EXAMPLE 22 *Let $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. If $f(x)$ is continuous at $x=0$, show that $f(x)$ is continuous at all x.*

SOLUTION Since $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \quad \lim_{h \to 0} f(0-h) = \lim_{h \to 0} f(0+h) = f(0)$$

$$\Rightarrow \quad \lim_{h \to 0} f(0+(-h)) = \lim_{h \to 0} f(0+h) = f(0)$$

$$\Rightarrow \quad \lim_{h \to 0} [f(0) + f(-h)] = \lim_{h \to 0} [f(0) + f(h)] = f(0) \qquad [\text{Using}: f(x+y) = f(x) + f(y)]$$

$$\Rightarrow \quad f(0) + \lim_{h \to 0} f(-h) = f(0) + \lim_{h \to 0} f(h) = f(0)$$

$$\Rightarrow \quad \lim_{h \to 0} f(-h) = \lim_{h \to 0} f(h) = 0 \qquad \text{...(i)}$$

Let a be any real number. Then,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+(-h))$$

$\Rightarrow \quad \lim_{x \to a^-} f(x) = \lim_{h \to 0} [f(a) + f(-h)]$ $\quad [\because f(x+y) = f(x) + f(y)]$

$\Rightarrow \quad \lim_{x \to a^-} f(x) = f(a) + \lim_{h \to 0} f(-h)$

$\Rightarrow \quad \lim_{x \to a^-} f(x) = f(a) + 0$ $\quad$ [Using (i)]

$\Rightarrow \quad \lim_{x \to a^-} f(x) = f(a).$

and, $\quad \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$

$\Rightarrow \quad \lim_{x \to a^+} f(x) = \lim_{h \to 0} [f(a) + f(h)]$ $\quad [\because f(x+y) = f(x) + f(y)]$

$\Rightarrow \quad \lim_{x \to a^+} f(x) = f(a) + \lim_{h \to 0} f(h)$

$\Rightarrow \quad \lim_{x \to a^+} f(x) = f(a) + 0 = f(a)$ $\quad$ [Using (i)]

Thus, we have

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

$\therefore \quad f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. So, $f(x)$ is continuous at all $x \in R$.

Type IV ON CONTINUITY OF COMPOSITE FUNCTION

EXAMPLE 23 *Show that the function $f(x) = |\sin x + \cos x|$ is continuous at $x = \pi$.*

SOLUTION Let $g(x) = \sin x + \cos x$ and $h(x) = |x|$. Then,

$$(hog)(x) = h(g(x)) = h(\sin x + \cos x) = |\sin x + \cos x| = f(x)$$

In order to prove that $f(x)$ is continuous at $x = \pi$. It is sufficient to prove that $g(x)$ is continuous at $x = \pi$ and $h(x)$ is continuous at $g(\pi) = \sin \pi + \cos \pi = -1$.

Now, $\lim_{x \to \pi} g(x) = \lim_{x \to \pi} (\sin x + \cos x) = \sin \pi + \cos \pi = -1$ and, $g(\pi) = -1$

$\therefore \quad \lim_{x \to \pi} g(x) = g(\pi)$. So, $g(x)$ is continuous at $x = \pi$.

Let $y = g(\pi) = -1$. Then,

$$\lim_{y \to -1} h(y) = \lim_{y \to -1} |y| = \lim_{y \to -1} -y = -(-1) = 1 \text{ and, } h(g(\pi)) = h(-1) = |-1| = 1.$$

$\therefore \quad \lim_{y \to -1} h(y) = h(g(\pi))$

$\Rightarrow \quad \lim_{g(x) \to -1} h(g(x)) = h(g(\pi)) \Rightarrow \lim_{g(x) \to g(\pi)} h(g(x)) = h(g(\pi)) \Rightarrow h(x)$ is continuous at $g(\pi)$

Hence, $f(x) = hog(x)$ is continuous at $x = \pi$.

EXERCISE 8.1

BASIC

1. A function $f(x)$ is defined as $f(x)=\begin{cases}\dfrac{x^2-x-6}{x-3}; & \text{if } x\neq 3\\ 5 ; & \text{if } x=3\end{cases}$. Show that $f(x)$ is continuous at $x=3$.

2. A function $f(x)$ is defined as $f(x)=\begin{cases}\dfrac{x^2-9}{x-3}; & \text{if } x\neq 3\\ 6 ; & \text{if } x=3\end{cases}$. Show that $f(x)$ is continuous at $x=3$.

3. If $f(x)=\begin{cases}\dfrac{x^2-1}{x-1} ; & \text{for } x\neq 1\\ 2 ; & \text{for } x=1\end{cases}$. Find whether $f(x)$ is continuous at $x=1$.

4. If $f(x)=\begin{cases}\dfrac{\sin 3x}{x} , & \text{when } x\neq 0\\ 1 , & \text{when } x=0\end{cases}$. Find whether $f(x)$ is continuous at $x=0$.

5. If $f(x)=\begin{cases}e^{1/x} , & \text{if } x\neq 0\\ 1 , & \text{if } x=0\end{cases}$. Find whether f is continuous at $x=0$.

6. Let $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2} , & \text{when } x\neq 0\\ 1 , & \text{when } x=0\end{cases}$. Show that $f(x)$ is discontinuous at $x=0$.

7. Show that $f(x)=\begin{cases}\dfrac{x-|x|}{2} , & \text{when } x\neq 0\\ 2 , & \text{when } x=0\end{cases}$ is discontinuous at $x=0$.

8. Show that $f(x)=\begin{cases}\dfrac{|x-a|}{x-a} , & \text{when } x\neq a\\ 1 , & \text{when } x=a\end{cases}$ is discontinuous at $x=a$.

9. Show that $f(x)=\begin{cases}1+x^2 , & \text{if } 0\le x\le 1\\ 2-x , & \text{if } x>1\end{cases}$ is discontinuous at $x=1$.

10. Examine the continuity of the function $f(x)=\begin{cases}3x-2 , & x\le 0\\ x+1 , & x>0\end{cases}$ at $x=0$.

    Also sketch the graph of this function.

11. Discuss the continuity of the function $f(x)=\begin{cases}x , & x>0\\ 1 , & x=0\\ -x , & x<0\end{cases}$ at the point $x=0$.

12. Discuss the continuity of the function $f(x)=\begin{cases}x , & 0\le x<1/2\\ 1/2 , & x=1/2\\ 1-x , & 1/2<x\le 1\end{cases}$ at the point $x=1/2$.

13. Discuss the continuity of $f(x)=\begin{cases}2x-1 , x<0\\ 2x+1 , x\ge 0\end{cases}$ at $x=0$. [CBSE 2002]

14. For what value of k is the function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$ continuous at $x = 1$?

15. Determine the value of the constant k so that the function $f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$

16. If $f(x) = \begin{cases} \dfrac{x^2}{2}, & \text{if } 0 \le x \le 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & \text{if } 1 < x \le 2 \end{cases}$. Show that f is continuous at $x = 1$.

17. Prove that $f(x) = \begin{cases} \dfrac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ is discontinuous at $x = 0$.

18. If $f(x) = \begin{cases} 2x^2 + k, & \text{if } x \ge 0 \\ -2x^2 + k, & \text{if } x < 0 \end{cases}$, then what should be the value of k so that $f(x)$ is continuous at $x = 0$.

19. For what value of λ is the function $f(x) = \begin{cases} \lambda(x^2 - 2x), & \text{if } x \le 0 \\ 4x + 1, & \text{if } x > 0 \end{cases}$ continuous at $x = 0$? What about continuity at $x = \pm 1$? [NCERT]

20. For what value of k is the function $f(x) = \begin{cases} 2x+1; & \text{if } x < 2 \\ k; & x = 2 \\ 3x - 1; & x > 2 \end{cases}$ continuous at $x = 2$? [CBSE 2008]

21. For what value of k is the function $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$?

22. Determine the value of the constant k so that the function $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. [NCERT]

23. Determine the value of the constant k so that the function $f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$. [CBSE 2007]

24. Find the values of a so that the function $f(x) = \begin{cases} ax + 5, & \text{if } x \le 2 \\ x - 1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$. [CBSE 2002]

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25. Discuss the continuity of the following functions at the indicated point(s):

(i) $f(x) = \begin{cases} |x| \cos\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$ [NCERT EXEMPLAR]

(ii) $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(iii) $f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right), & x \neq a \\ 0, & x = a \end{cases}$ at $x = a$

(iv) $f(x) = \begin{cases} \frac{e^x - 1}{\log(1+2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$ at $x = 0$

(v) $f(x) = \begin{cases} \frac{1-x^n}{1-x}, & x \neq 1 \\ n-1, & x = 1 \end{cases}$ $n \in N$ at $x = 1$

(vi) $f(x) = \begin{cases} \frac{|x^2-1|}{x-1}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$ at $x = 1$

(vii) $f(x) = \begin{cases} \frac{2|x| + x^2}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$

(viii) $f(x) = \begin{cases} |x-a| \sin\left(\frac{1}{x-a}\right), & \text{for } x \neq a \\ 0, & \text{for } x = a \end{cases}$ at $x = a$ [NCERT EXEMPLAR]

26. Prove that the function $f(x) = \begin{cases} \frac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ remains discontinuous at $x = 0$, regardless the choice of k. [NCERT EXEMPLAR]

27. Find the value of k if $f(x)$ is continuous at $x = \pi/2$, where $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x}, & x \neq \pi/2 \\ 3, & x = \pi/2 \end{cases}$ [NCERT]

28. For what value of k is the function $f(x) = \begin{cases} \frac{\sin 2x}{x}, & x \neq 0 \\ k, & x = 0 \end{cases}$ continuous at $x = 0$?

29. If $f(x) = \frac{2x + 3 \sin x}{3x + 2 \sin x}$, $x \neq 0$ is continuous at $x = 0$, then find $f(0)$.

30. Find the value of k for which $f(x) = \begin{cases} \frac{1 - \cos 4x}{8x^2}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at $x = 0$. [CBSE 2000 C, 2017, NCERT EXEMPLAR]

31. In each of the following, find the value of the constant k so that the given function is continuous at the indicated point:

(i) $f(x) = \begin{cases} \frac{1 - \cos 2kx}{x^2}, & \text{if } x \neq 0 \\ 8, & \text{if } x = 0 \end{cases}$ at $x = 0$

(ii) $f(x) = \begin{cases} (x-1) \tan \frac{\pi x}{2}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ at $x = 1$

(iii) $f(x) = \begin{cases} k(x^2 - 2x), & \text{if } x < 0 \\ \cos x, & \text{if } x \geq 0 \end{cases}$ at $x = 0$ [NCERT]

(iv) $f(x) = \begin{cases} kx + 1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ at $x = \pi$ [NCERT]

(v) $f(x) = \begin{cases} kx+1 \ , & \text{if } x \le 5 \\ 3x-5 \ , & \text{if } x > 5 \end{cases}$ at $x=5$

(vi) $f(x) = \begin{cases} \dfrac{x^2-25}{x-5}, & x \ne 5 \\ k \ , & x=5 \end{cases}$ at $x=5$ [CBSE 2007]

(vii) $f(x) = \begin{cases} kx^2 \ , & x \ge 1 \\ 4 \ , & x < 1 \end{cases}$ at $x=1$ [CBSE 2007]

(viii) $f(x) = \begin{cases} k(x^2+2) \ , & \text{if } x \le 0 \\ 3x+1 \ , & \text{if } x > 0 \end{cases}$ at $x=0$. [CBSE 2010]

(ix) $f(x) = \begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2}, & x \ne 2 \\ k, & x=2 \end{cases}$ at $x=2$. [NCERT EXEMPLAR]

32. Find the values of a and b so that the function f given by

$f(x) = \begin{cases} 1 \ , & \text{if } x \le 3 \\ ax+b \ , & \text{if } 3 < x < 5 \\ 7 \ , & \text{if } x \ge 5 \end{cases}$ is continuous at $x = 3$ and $x = 5$. [CBSE 2013]

33. Find the relationship between 'a' and 'b' so that the function 'f' defined by

$f(x) = \begin{cases} ax+1 \ , & \text{if } x \le 3 \\ bx+3 \ , & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. [CBSE 2011]

34. If the functions $f(x)$, defined below is continuous at $x = 0$, find the value of k:

$$f(x) = \begin{cases} \dfrac{1-\cos 2x}{2x^2}, & x < 0 \\ k, & x = 0 \\ \dfrac{x}{|x|}, & x > 0 \end{cases}$$ [CBSE 2010]

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35. Show that $f(x) = \begin{cases} \dfrac{\sin 3x}{\tan 2x}, & \text{if } x < 0 \\ \dfrac{3}{2}, & \text{if } x = 0 \\ \dfrac{\log(1+3x)}{e^{2x}-1}, & \text{if } x > 0 \end{cases}$ is continuous at $x = 0$

36. Find the value of 'a' for which the function f defined by

$f(x) = \begin{cases} a \sin \dfrac{\pi}{2}(x+1), & x \le 0 \\ \dfrac{\tan x - \sin x}{x^3}, & x > 0 \end{cases}$ is continuous at $x = 0$. [CBSE 2011]

37. Determine the values of a, b, c for which the function

$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{for } x < 0 \\ c, & \text{for } x = 0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & \text{for } x > 0 \end{cases}$ is continuous at $x = 0$.

38. If $f(x) = \begin{cases} \dfrac{1-\cos kx}{x \sin x} & , \quad x \neq 0 \\ \dfrac{1}{2} & , \quad x = 0 \end{cases}$ is continuous at $x = 0$, find k. [NCERT EXEMPLAR]

39. If $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a & , \quad \text{if } x < 4 \\ a + b & , \quad \text{if } x = 4 \\ \dfrac{x-4}{|x-4|} + b & , \quad \text{if } x > 4 \end{cases}$ is continuous at $x = 4$, find a, b. [NCERT EXEMPLAR]

40. Let $f(x) = \dfrac{\log\left(1+\dfrac{x}{a}\right) - \log\left(1-\dfrac{x}{b}\right)}{x}$, $x \neq 0$. Find the value of f at $x = 0$ so that f becomes continuous at $x = 0$.

41. If $f(x) = \begin{cases} \dfrac{2^{x+2} - 16}{4^x - 16} & , \text{ if } x \neq 2 \\ k & , \text{ if } x = 2 \end{cases}$ is continuous at $x = 2$, find k. [NCERT EXEMPLAR]

42. If $f(x) = \begin{cases} \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1} & , x \neq 0 \\ k & , x = 0 \end{cases}$ is continuous at $x = 0$, find k.

43. Extend the definition of the following by continuity $f(x) = \dfrac{1 - \cos 7(x-\pi)}{5(x-\pi)^2}$ at the point $x = \pi$.

44. Discuss the continuity of the $f(x)$ at the indicated points:

(i) $f(x) = |x| + |x-1|$ at $x = 0, 1$.

(ii) $f(x) = |x-1| + |x+1|$ at $x = -1, 1$. [NCERT EXEMPLAR]

45. Let $f(x) = \begin{cases} \dfrac{1 - \sin^3 x}{3\cos^2 x} & , \text{ if } x < \dfrac{\pi}{2} \\ a & , \text{ if } x = \dfrac{\pi}{2} \\ \dfrac{b(1-\sin x)}{(\pi - 2x)^2} & , \text{ if } x > \dfrac{\pi}{2} \end{cases}$. If $f(x)$ is continuous at $x = \dfrac{\pi}{2}$, find a and b.

[CBSE 2008, 2016]

ANSWERS

3. Continuous	4. Discontinuous	5. Discontinuous	10. Discontinuous
11. Discontinuous	12. Continuous	13. Discontinuous	14. 2
15. -1	18. k is any real number.		

19. There is no value of λ for which it is continuous at $x = 0$. At $x = \pm 1$, $f(x)$ is continuous

20. $k = 5$	21. 5/3	22. 3/4	23. 2/5
24. -2	25. (i) Continuous	(ii) Continuous	(iii) Continuous
(iv) Discontinuous	(v) Discontinuous	(vi) Discontinuous	(vii) Discontinuous
(viii) Continuous	27. 6	28. 2	29. 1 30. 1

31. (i) $k = \pm 2$ (ii) $k = \frac{-2}{\pi}$ (iii) No value of k (iv) $k = \frac{-2}{\pi}$

(v) $k = \frac{9}{5}$ (vi) $k = 10$ (vii) $k = 4$ (viii) $k = 1/2$ (ix) $k = 7$

32. $a = 3, b = -8$ **33.** $3a - 3b = 2$ **34.** $k = 1$ **36.** $a = \frac{1}{2}$

37. $a = -\frac{3}{2}, b \in R - \{0\}, c = \frac{1}{2}$ **38.** ± 1 **39.** $a = 1, b = -1$

40. $\frac{a+b}{ab}$ **41.** $1/2$ **42.** -4 **43.** $49/10$

44. (i) Continuous, (ii) Continuous **45.** $a = \frac{1}{2}, b = 4$

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22. If $f(x) = \begin{cases} kx^2 & , x \le 2 \\ 3 & , x > 2 \end{cases}$ is continuous at $x = 2$, then

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \Rightarrow \lim_{x \to 2^-} kx^2 = \lim_{x \to 2^+} 3 = k(2)^2 \Rightarrow 4k = 3 \Rightarrow k = \frac{3}{4}$$

27. It is given that $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & , x \ne \dfrac{\pi}{2} \\ 3 & , x = \dfrac{\pi}{2} \end{cases}$ is continuous at $x = \frac{\pi}{2}$.

$$\therefore \lim_{x \to \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \to \pi/2} \frac{k \cos x}{\pi - 2x} = 3 \Rightarrow k \lim_{x \to \pi 2} \frac{\sin(\pi/2 - x)}{2(\pi/2 - x)} = 3 \Rightarrow \frac{k}{2} \times 1 = 3 \Rightarrow k = 6$$

31. (iii) It is given that $f(x) = \begin{cases} k(x^2 - 2x), & x < 0 \\ \cos x & , x \ge 0 \end{cases}$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} k(x^2 - 2x) = 0 \text{ for all } k$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \cos x = 1 \text{ and, } f(0) = \cos 0 = 1$$

Clearly, there is no value of k for which $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$ may hold good.

Hence, there is no value of k for which $f(x)$ is continuous at $x = 0$.

(iv) It is given that $f(x) = \begin{cases} kx + 1 & , x \le \pi \\ \cos x & , x > \pi \end{cases}$ is continuous at $x = \pi$.

$$\therefore \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$$

$$\Rightarrow \lim_{x \to \pi^-} kx + 1 = \lim_{x \to \pi^+} \cos x = k\pi + 1 \Rightarrow k\pi + 1 = \cos \pi \Rightarrow k\pi + 1 = -1 \Rightarrow k = -\frac{2}{\pi}$$

8.4 CONTINUITY ON AN INTERVAL

CONTINUITY ON AN OPEN INTERVAL *A function $f(x)$ is said to be continuous on an open interval (a, b) iff it is continuous at every point on the interval (a, b).*

CONTINUITY ON A CLOSED INTERVAL *A function $f(x)$ is said to be continuous on a closed interval $[a, b]$ iff* *(i) f is continuous on the open interval (a, b) (ii)* $\lim\limits_{x \to a^+} f(x) = f(a)$ *and, (iii)* $\lim\limits_{x \to b^-} f(x) = f(b)$.

In other words, f(x) is continuous on [a, b] iff it is continuous on (a, b) and it is continuous at a from the right and at b from the left.

CONTINUOUS FUNCTION *A function f(x) is said to be continuous, if it is continuous at each point of its domain.*

EVERYWHERE CONTINUOUS FUNCTION *A function f(x) is said to be everywhere continuous if it is continuous on the entire real line* $(-\infty, \infty)$.

8.5 PROPERTIES OF CONTINUOUS FUNCTIONS

In this section, we shall learn some properties of continuous functions and prove the continuity of some standard real functions in their domains.

THEOREM 1 *If f and g are two continuous functions on their common domain D, then*

(i) $f+g$ *is continuous on D* (ii) $f-g$ *is continuous on D*

(iii) fg *is continuous on D* (iv) αf *is continuous on D, where α is any real number.*

(v) $\dfrac{f}{g}$ *is continuous on* $D - \{x : g(x) \neq 0\}$ (vi) $\dfrac{1}{f}$ *is continuous on* $D - \{x : f(x) \neq 0\}$

PROOF Let a be an arbitrary point in common domain D. Since f and g are continuous on D. So, they are also continuous at 'a'.

$\therefore$ $\lim\limits_{x \to a} f(x) = f(a)$ and $\lim\limits_{x \to a} g(x) = g(a)$ ...(i)

(i) We find that

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a) = (f+g)(a) \quad \text{[Using (i)]}$$

$\therefore$ $f+g$ is continuous at $x = a$.

Since a is an arbitrary point in D. Hence, $f+g$ is continuous on D.

(ii) We find that

$$\lim_{x \to a} (f-g)(x) = \lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$= f(a) - g(a) = (f-g)(a) \quad \text{[Using (i)]}$$

$\therefore$ $f-g$ is continuous at $x = a$.

Since a is an arbitrary point in D. Hence, $f-g$ is continuous in D.

(iii) We find that

$$\lim_{x \to a} (fg)(x) = \lim_{x \to a} (f(x)\ g(x)) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$$

$$= f(a)\, g(a) = (fg)(a) \quad \text{[Using (i)]}$$

$\therefore$ fg is continuous at $x = a$.

Since a is an arbitrary point in D. Hence, fg is continuous in D.

(iv) We find that

$$\lim_{x \to a} (\alpha f)(x) = \lim_{x \to a} (\alpha f(x)) = \alpha \lim_{x \to a} f(x) = \alpha f(a) = (\alpha f)(a) \quad \text{[Using (i)]}$$

$\therefore$ αf is continuous at $x = a$.

Since a is an arbitrary point in D. Hence, αf is continuous in D.

(v) Let $a \in D$ such that $g(a) \neq 0$. Then,

$$\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

$\therefore$ $\frac{f}{g}$ is continuous at $x = a$.

Since a is an arbitrary point in D such that $g(a) \neq 0$. Hence, $\frac{f}{g}$ is continuous on $D - \{x : g(x) \neq 0\}$.

(vi) Let $a \in D$ such that $f(a) \neq 0$. Then,

$$\lim_{x \to a} \left(\frac{1}{f}\right)(x) = \lim_{x \to a} \frac{1}{f(x)} = \frac{1}{\lim_{x \to a} f(x)} = \frac{1}{f(a)} = \left(\frac{1}{f}\right)(a) \qquad \text{[Using (i)]}$$

$\therefore$ $\frac{1}{f}$ is continuous at $x = a$.

Since a is arbitrary point in D such that $f(a) \neq 0$. Hence, $1/f$ is continuous on $D - \{x : f(x) \neq 0\}$.

Q.E.D.

THEOREM 2 *The composition of two continuous functions is a continuous function.*

PROOF Let f and g be two real functions such that gof exists. Then, Range $(f) \subseteq$ Domain (g).

Let a be an arbitrary point in the domain of f. Then,

$$a \in \text{Domain}(f) \Rightarrow f(a) \in \text{Range}(f) \Rightarrow f(a) \in \text{Domain}(g) \qquad [\because \text{Range}(f) \subseteq \text{Domain}(g)]$$

Since f and g are continuous on their domains. Therefore,

$a \in$ Domain (f) and $f(a) \in$ Domain (g)

$\Rightarrow$ f is continuous at $x = a$ and g is continuous at $f(a)$

$$\Rightarrow \lim_{x \to a} f(x) = f(a) \text{ and } \lim_{y \to f(a)} g(y) = g(f(a))$$

$$\Rightarrow \lim_{x \to a} f(x) = f(a) \text{ and } \lim_{f(x) \to f(a)} g(f(x)) = g(f(a)), \text{ where } y = f(x)$$

$$\Rightarrow \lim_{x \to a} g(f(x)) = g(f(a)) \qquad [\because x \to a \Rightarrow f(x) \to f(a)]$$

$$\Rightarrow \lim_{x \to a} gof(x) = gof(a) \Rightarrow gof \text{ is continuous at } x = a.$$

Since a is an arbitrary point in its domain. Hence, gof is continuous. Q.E.D.

THEOREM 3 *If f is continuous on its domain D, then $|f|$ is also continuous on D.*

PROOF Recall that $|f|$ (known as absolute function) is defined as $|f|(x) = |f(x)|$.

Let a be an arbitrary real number in D. Then, f is continuous at a.

$$\therefore \lim_{x \to a} f(x) = f(a)$$

Now,

$$\lim_{x \to a} |f|(x) = \lim_{x \to a} |f(x)| \qquad \text{[By definition of } |f|\text{]}$$

$$\Rightarrow \lim_{x \to a} |f|(x) = \left|\lim_{x \to a} f(x)\right| = |f(a)| = |f|(a)$$

$\therefore$ $|f|$ is continuous at $x = a$.

Since a is an arbitrary point in D. Therefore, $|f|$ is continuous in D. Q.E.D.

REMARK *The converse of the above theorem may not be true. For example, consider the function*

$$f(x) = \begin{cases} 1, & \text{if } x \in Z \\ -1, & \text{if } x \in R - Z \end{cases}$$

Let a be an arbitrary integer. Then,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} -1 = -1 \quad [\because h > 0,\ a - h \notin Z \text{ as } h \text{ is very small}]$$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} -1 = -1 \text{ and, } f(a) = 1.$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$$

So, f is discontinuous at $x = a$.

Now, $|f|(x) = |f(x)| = 1$ *for all* $x \in R$. *So,* $|f|$ *is a constant function and hence, it is everywhere continuous.*

THEOREM 4 *A constant function is everywhere continuous.*

PROOF Let $f(x) = c$, where c is a constant. Clearly, the domain of a constant function is R.

Let a be any real number. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c \text{ and, } f(a) = c.$$

$$\therefore \quad \lim_{x \to a} f(x) = f(a) \Rightarrow f(x) \text{ is continuous at } x = a.$$

But, a is an arbitrary real number. Hence, $f(x)$ is continuous on R. **Q.E.D.**

REMARK 1 *It is evident from the graph of a constant function that is everywhere continuous.*

THEOREM 5 *The identity function is everywhere continuous.*

PROOF Let $f(x) = x$ for all $x \in R$ be the identity function. Let a be any real number. Then,

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a \text{ and, } f(a) = a.$$

$$\therefore \quad \lim_{x \to a} f(x) = f(a). \text{ So, } f(x) \text{ is continuous at } x = a.$$

Since a is an arbitrary real number. Hence, $f(x)$ is continuous on R i.e. it is everywhere continuous.

REMARK 2 *The above fact can be easily observed from the graph of the identity function.* **Q.E.D.**

THEOREM 6 *A polynomial function is everywhere continuous.*

PROOF Let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$, $n \in Z$, $n \geq 0$, $x \in R$ be a polynomial function.

We shall prove the theorem by induction on n.

Step I *When* $n = 0$, *we have*

$$f(x) = a_0$$

Clearly, $f(x)$ is a constant function which is everywhere continuous

When $n = 1$, we have

$$f(x) = a_0 + a_1 x.$$

Clearly, $f(x)$ is the sum of a constant function and a multiple of the identity function. So, being the sum of two everywhere continuous functions, $f(x)$ is everywhere continuous.

Step II *Let every polynomial function of degree at most n be everywhere continuous.*

Consider a general polynomial function of degree $(n + 1)$.

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + a_{n+1} x^{n+1}, \text{ where } a_{n+1} \neq 0.$$

$$\Rightarrow \quad g(x) = a_0 + x(a_1 + a_2 x + \ldots + a_n x^{n-1} + a_{n+1} x^n)$$

Clearly, it is the sum of a constant function a_0 (which is everywhere continuous) and the product of the identity function x (which is everywhere continuous) and the polynomial function $a_1 + a_2 x + \ldots + a_{n+1} x^n$ of degree at most n (which is everywhere continuous by induction assumption). Therefore, $g(x)$ is everywhere continuous.

Hence, by the principle of mathematical induction, a polynomial function is everywhere continuous.

A simple consequence of the above theorem is the following: Q.E.D.

COROLLARY *Every rational function is continuous at every point in its domain.*

PROOF Let $f(x) = \dfrac{g(x)}{h(x)}$, $h(x) \neq 0$ be a rational function. Then, $g(x)$ and $h(x)$ are polynomial functions.

The domain of $f(x)$ is the set $D = R - \{x : g(x) = 0\}$.

Since polynomial functions are everywhere continuous. Therefore, $g(x)$ and $h(x)$ are continuous on D.

Hence, by theorem 1, $f(x) = \dfrac{g(x)}{h(x)}$ is continuous on D.

THEOREM 7 *The modulus function is everywhere continuous.*

PROOF We know that the identify function is everywhere continuous.

Also, if f is continuous, then $|f|$ is also continuous. Therefore, $|x|$ is everywhere continuous.

Q.E.D.

THEOREM 8 *The exponential function a^x, $a > 0$ is everywhere continuous.*

PROOF *Let* $f(x) = a^x$. Then,

$$\lim_{x \to 0} a^x = \lim_{x \to 0} \left\{ \left(\frac{a^x - 1}{x} \right) x + 1 \right\} = \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) \times \lim_{x \to 0} x + \lim_{x \to 0} 1 = (\log_e a \times 0) + 1 = 1.$$

Let c be an arbitrary real number. Then,

$$\lim_{x \to c^-} f(x) = \lim_{h \to 0} f(c-h) = \lim_{h \to 0} a^{c-h} = a^c \lim_{h \to 0} a^{-h} = a^c \lim_{h \to 0} \frac{1}{a^h} = a^c \times \frac{1}{1} = a^c = f(c)$$

and, $$\lim_{x \to c^+} f(x) = \lim_{h \to 0} f(c+h) = \lim_{h \to 0} a^{c+h} = a^c \lim_{h \to 0} a^h = a^c \times 1 = a^c = f(c)$$

$$\therefore \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$$

So, $f(x)$ is continuous at $x = c$.

Since c is an arbitrary real number. Hence, $f(x) = a^x$ is everywhere continuous.

Q.E.D.

COROLLARY *e^x is everywhere continuous.*

THEOREM 9 *The logarithmic function is continuous in its domain.*

PROOF Let $f(x) = \log_c x$, where $c > 0$ be the logarithmic function. Clearly, domain $(f) = (0, \infty)$.

Let a be an arbitrary point in $(0, \infty)$. Then,

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \lim_{h \to 0} \log_c (a+h)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \lim_{h \to 0} \log_c a\left(1+\frac{h}{a}\right)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \lim_{h \to 0} \left\{\log_c a + \log_c \left(1+\frac{h}{a}\right)\right\}$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \log_c a + \lim_{h \to 0} \log_c \left(1+\frac{h}{a}\right)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \log_c a + \lim_{h \to 0} \left\{\frac{\log_c \left(1+\frac{h}{a}\right)}{\frac{h}{a}}\right\} \times \frac{h}{a}$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \log_c a + \lim_{h \to 0} \frac{\log_c \left(1+\frac{h}{a}\right)}{\frac{h}{a}} \times \lim_{h \to 0} \frac{h}{a}$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \log_c a + \log_e c \times 0 = \log_c a = f(a)$$

Similarly, we obtain : $\lim_{x \to a^-} f(x) = f(a)$

$\therefore \lim_{x \to a^-} f(x) = f(a) = \lim_{h \to a^+} f(x)$. So, $f(x)$ is continuous at $x = a$.

Since a is an arbitrary point in $(0, \infty)$. Hence, $f(x)$ is continuous on $(0, \infty)$. **Q.E.D.**

THEOREM 10 *The sine function is everywhere continuous.*

PROOF Let $f(x) = \sin x$ and let a be an arbitrary real number. Then,

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \lim_{h \to 0} \sin(a+h)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \lim_{h \to 0} \{\sin a \cos h + \cos a \sin h\}$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \sin a \left(\lim_{h \to 0} \cos h\right) + \cos a \left(\lim_{h \to 0} \sin h\right)$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{h \to 0} \sin h = 0 \text{ and } \lim_{h \to 0} \cos h = 1\right]$$

$$\Rightarrow \lim_{x \to a^+} f(x) = \sin a = f(a)$$

Similarly, we obtain

$$\lim_{x \to a^-} f(x) = f(a)$$

$\therefore \lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$. $\Rightarrow f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. Hence, $f(x) = \sin x$ is everywhere continuous. **Q.E.D.**

THEOREM 11 *The cosine function is everywhere continuous.*

PROOF Let $f(x) = \cos x$ and let a be any real number. Then,

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

$$\Rightarrow \quad \lim_{x \to a^+} f(x) = \lim_{h \to 0} \cos(a+h)$$

$$\Rightarrow \quad \lim_{x \to a^+} f(x) = \lim_{h \to 0} \{\cos a \cos h - \sin a \sin h\}$$

$$\Rightarrow \quad \lim_{x \to a^+} f(x) = \cos a \left(\lim_{h \to 0} \cos h\right) - \sin a \left(\lim_{h \to 0} \sin h\right)$$

$$\Rightarrow \quad \lim_{x \to a^+} f(x) = (\cos a) \times 1 - \sin a \times 0 \qquad \left[\because \lim_{h \to 0} \cos h = 1 \text{ and } \lim_{h \to 0} \sin h = 0\right]$$

$$\Rightarrow \quad \lim_{x \to a^+} f(x) = \cos a = f(a)$$

Similarly, we obtain

$$\lim_{x \to a^-} f(x) = f(a)$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a).$$ So, $f(x)$ is continuous at $x = a$

Since a is an arbitrary real number. Hence, $f(x)$ is everywhere continuous. **Q.E.D.**

THEOREM 12 *The tangent function is continuous in its domain.*

PROOF Let $f(x) = \tan x$. Clearly, domain $(f) = R - \left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\}$

We have, $f(x) = \tan x = \dfrac{\sin x}{\cos x}$. Since $\sin x$ and $\cos x$ are everywhere continuous. Therefore, $f(x) = \tan x$ is continuous for all $x \in R$ except when $\cos x \neq 0$. But, $\cos x = 0$ at $x = (2n+1)\pi/2, n \in Z$.

Hence, $f(x) = \tan x$ is continuous for all $x \in R - \{(2n+1)\pi/2 : n \in Z\}$. **Q.E.D.**

THEOREM 13 (i) *The cosecant function is continuous in its domain.*

(ii) *The secant function is continuous in its domain.*

(iii) *The cotangent function is continuous in its domain.*

PROOF It is the direct consequence of the above Theorems and Theorem 1.

THEOREM 14 $f(x) = \sin^{-1} x$ *is continuous on* $[-1, 1]$.

PROOF Let a be an arbitrary point in $[-1, -1]$. Let $y = \sin^{-1} x$. Then, $x = \sin y$.

$$\therefore \quad x \to a \Rightarrow \sin y \to a \Rightarrow y \to \sin^{-1} a.$$

Thus, $$\lim_{x \to a} f(x) = \lim_{x \to a} \sin^{-1} x = \lim_{y \to \sin^{-1} a} y = \sin^{-1} a = f(a)$$

So, $f(x)$ is continuous at $x = a$.

Since a is an arbitrary point of $[-1, 1]$. Hence, $f(x) = \sin^{-1} x$ is continuous on $[-1, 1]$.

Q.E.D.

REMARK *Proceeding as above, it can be shown that all inverse trigonometric functions are continuous in their respective domains.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON TESTING THE CONTINUITY OF A FUNCTION IN ITS DOMAIN

EXAMPLE 1 *If a function f is defined as* $f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0 \quad , & x = 4 \end{cases}$. *Show that f is everywhere continuous except at* $x = 4$.

SOLUTION We have,

$$f(x) = \begin{cases} \dfrac{|x-4|}{x-4}, & x \neq 4 \\ 0 \quad , & x = 4 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \dfrac{-(x-4)}{x-4} = -1 \ ; & x < 4 \\ \dfrac{x-4}{x-4} = 1 \ ; & x > 4 \\ 0 \ ; & x = 4 \end{cases} \qquad \left[\because |x-4| = \begin{cases} -(x-4), & x < 4 \\ x-4, & x \geq 4 \end{cases}\right]$$

When $x < 4$, we have $f(x) = -1$, which, being a constant function, is continuous at each point $x < 4$.

Also, when $x > 4$, we have $f(x) = 1$, which, being a constant function, is continuous at each point $x > 4$.

Let us consider the point $x = 4$. We find that

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4} -1 = -1, \quad \lim_{x \to 4^+} f(x) = \lim_{x \to 4} 1 = 1 \text{ and, } f(4) = 0.$$

$$\therefore \quad \lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x). \text{ So, } f(x) \text{ is not continuous at } x = 4.$$

Hence, $f(x)$ is everywhere continuous, except at $x = 4$.

EXAMPLE 2 *Discuss the continuity of the function* $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & , \text{ if } x < 0 \\ x + 2 & , \text{ if } x \geq 0 \end{cases}$.

SOLUTION When $x < 0$, we have $f(x) = \dfrac{\sin 2x}{x}$. We know that $\sin 2x$ as well as the identity function x both are everywhere continuous. So, the quotient function $\dfrac{\sin 2x}{x} = f(x)$ is continuous at each $x < 0$.

When, $x > 0$, we have $f(x) = x + 2$, which being a polynomial function, is continuous at each $x > 0$. Let us now consider the point $x = 0$. We find that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x} = 2 \lim_{x \to 0} \frac{\sin 2x}{2x} = 2(1) = 2, \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} x + 2 = 2$$

and, $f(0) = 0 + 2 = 2$

$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. So, $f(x)$ is also continuous at $x = 0$.

Hence, $f(x)$ is everywhere continuous.

EXAMPLE 3 *Discuss the continuity of the function* $f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

SOLUTION We have,

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \Rightarrow f(x) = \begin{cases} \frac{-x}{x} = -1, & \text{if } x < 0 \\ \frac{x}{x} = 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases} \quad \left[\because |x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases} \right]$$

We observe that $f(x)$ is a constant function for all $x < 0$ as well as for $x > 0$. So, it is continuous for all $x > 0$ and for all $x < 0$. Consider the point $x = 0$. At $x = 0$, we find that

(LHL at $x = 0$) $= \lim_{x \to 0^-} f(x) = \lim_{x \to 0} -1 = -1$ and, (RHL at $x = 0$) $= \lim_{x \to 0^+} f(x) = \lim_{x \to 0} 1 = 1$

$\therefore \quad$ (LHL at $x = 0$) $\neq$ (RHL at $x = 0$)

So, $f(x)$ is not continuous at $x = 0$. Hence, $f(x)$ is continuous at each point, except at $x = 0$.

EXAMPLE 4 *Discuss the continuity of the function* $f(x)$ *given by* $f(x) = \begin{cases} 2x - 1, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$

[CBSE 2002]

SOLUTION When $x < 0$, we have $f(x) = 2x - 1$. Clearly, $f(x)$ is a polynomial function for $x < 0$. So, $f(x)$ is continuous for all $x < 0$.

When $x > 0$, we have $f(x) = 2x + 1$. Clearly, $f(x)$ is a polynomial function for $x > 0$. So, it is continuous for all $x > 0$.

Let us now consider the point $x = 0$. At $x = 0$, we find that

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (2x - 1) = -1 \quad \text{and,} \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0} (2x + 1) = 1.$$

$$\therefore \quad \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$$

So, $f(x)$ is not continuous at $x = 0$. Hence, $f(x)$ is everywhere continuous except at $x = 0$.

Type II ON FINDING THE VALUE(S) OF A CONSTANT GIVEN IN THE DEFINITION OF A FUNCTION WHEN IT IS CONTINUOUS ON ITS DOMAIN

EXAMPLE 5 *Determine the value of the constant* k *so that the function* $f(x) = \begin{cases} k x^2, & \text{if } x \leq 2 \\ 3, & \text{if } x > 2 \end{cases}$ *is continuous.* [NCERT]

SOLUTION When $x \leq 2$, we have $f(x) = k x^2$, which being a polynomial function is continuous at each $x < 2$.

When $x > 2$, we have $f(x) = 3$, which being a constant function is continuous at each $x > 2$.

Let us now consider the point $x = 2$. At $x = 2$, we have

$$(\text{LHL at } x=2) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2} kx^2 = 4k \qquad [\because f(x) = kx^2 \text{ for } x \le 2]$$

$$(\text{RHL at } x=2) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} 3 = 3 \qquad [\because f(x) = 3 \text{ for } x > 2]$$

and, $\quad f(2) = k(2)^2 = 4k.$

As $f(x)$ is continuous in its domain. Therefore, it is also continuous at $x = 2$. Consequently, we have

$$\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x) \Rightarrow 4k = 3 \Rightarrow k = \frac{3}{4}$$

EXAMPLE 6 *If* $f(x) = \begin{cases} 1 & , \text{ if } x \le 3 \\ ax + b & , \text{ if } 3 < x < 5 \\ 7 & , \text{ if } 5 \le x \end{cases}$. *Determine the values of a and b so that f(x) is continuous.*

SOLUTION The given function is a constant function for all $x < 3$ as well as for all $x > 5$. So, it is continuous for all $x < 3$ as well as for all $x > 5$. We know that a polynomial function is continuous. So, the given function is continuous for all $x \in (3, 5)$. Thus, $f(x)$ is continuous at each $x \in R$ except possibly at $x = 3$ and $x = 5$.

At $x = 3$, we have

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} 1 = 1, \quad \lim_{x \to 3^+} f(x) = \lim_{x \to 3} ax + b = 3a + b \quad \text{and, } f(3) = 1$$

For $f(x)$ be continuous at $x = 3$, we must have

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3) \Rightarrow 1 = 3a + b \qquad \text{...(i)}$$

At $x = 5$, we have

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5} ax + b = 5a + b, \quad \lim_{x \to 5^+} f(x) = \lim_{x \to 5} 7 = 7 \quad \text{and, } f(5) = 7$$

For $f(x)$ to be continuous at $x = 5$, we must have

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = f(5) \Rightarrow 5a + b = 7 \qquad \text{...(ii)}$$

Solving (i) and (ii), we get: $a = 3, b = -8$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Show that the function f defined by* $f(x) = |1 - x + |x||$ *is everywhere continuous.* **[NCERT]**

SOLUTION Let $g(x) = 1 - x + |x|$ and $h(x) = |x|$ be two functions defined on R. Then,

$$(hog)(x) = h(g(x)) = h(1 - x + |x|) = |1 - x + |x|| = f(x) \text{ for all } x \in R.$$

Since $(1 - x)$, being a polynomial function and $|x|$ being a modulus function are continuous on R. Therefore, $g(x) = 1 - x + |x|$ is everywhere continuous. Also, $h(x) = |x|$ is everywhere continuous. Hence, $f = hog$ is everywhere continuous.

ALITER We have, $f(x) = |1 - x + |x|| = \begin{cases} |1 - x - x|, & \text{if } x < 0 \\ |1 - x + x|, & \text{if } x \ge 0 \end{cases} = \begin{cases} (1 - 2x), & \text{if } x < 0 \\ 1, & \text{if } x \ge 0 \end{cases}$

For $x < 0$, we have $f(x) = 1 - 2x$. Clearly, $f(x)$ is a polynomial function. So, $f(x)$ is continuous for all $x < 0$.

For $x > 0$, $f(x) = 1$, being a constant function, is continuous. So, $x = 0$ is the only point of possible discontinuity.

Now, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0} (1 - 2x) = 1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} 1 = 1$

Thus, $\lim\limits_{x \to 0^-} f(x) = f(0) = \lim\limits_{x \to 0^+} f(x)$. So, $f(x)$ is continuous at $x = 0$. Hence, $f(x)$ is everywhere discontinuous.

EXAMPLE 8 *Prove that* $f(x) = \sqrt{|x| - x}$ *is continuous for all* $x \geq 0$.

SOLUTION Let $g(x) = |x| - x$ and $h(x) = \sqrt{x}$. Clearly, domain$(g) = R$ and domain $(h) = [0, \infty)$. Also, $g(x)$) and $h(x)$ are continuous in their domains.

We observe that

$$\begin{aligned} \text{Domain } (hog) &= \{x \in \text{Domain } (g) : g(x) \in \text{Domain } (h)\} \\ &= \{x \in R : |x| - x \in [0, \infty)\} = \{x \in R : x \geq 0\} = [0, \infty) \end{aligned}$$

Since $g(x)$ and $h(x)$ are continuous on their respective domains. Therefore, $hog : [0, \infty) \to R$ is also continuous.

ALITER We have, $f(x) = \sqrt{|x| - x} \Rightarrow f(x) = \begin{cases} \sqrt{x - x} = 0 & , \text{ if } x \geq 0 \\ \sqrt{-x - x} = \sqrt{-2x}, & \text{if } x < 0 \end{cases}$

For $x \geq 0$, we have $f(x) = 0$, which being a constant function, is continuous.

For $x < 0$, we have $f(x) = \sqrt{-2x}$

We know that the square root function is continuous in its domain. So, $f(x) = \sqrt{-2x}$ is continuous for all $x < 0$.

Now,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \sqrt{-2x} = 0 \text{ and, } \lim_{x \to 0^+} f(x) = \lim_{x \to 0} 0 = 0$$

$$\therefore \quad \lim_{x \to 0^-} f(x) = f(0) = \lim_{x \to 0^+} f(x). \text{ So, } f(x) \text{ is continuous at } x = 0.$$

Hence, $f(x)$ is everywhere continuous.

EXAMPLE 9 *Determine the value of the constant m so that the function* $f(x) = \begin{cases} m(x^2 - 2x) & , \text{ if } x < 0 \\ \cos x & , \text{ if } x \geq 0 \end{cases}$ *is continuous.*

SOLUTION When $x < 0$, we have $f(x) = m(x^2 - 2x)$, which being a polynomial is continuous at each $x < 0$.

When $x > 0$, we have $f(x) = \cos x$, which being a cosine function is continuous at each $x > 0$.

Let us now consider the point $x = 0$. At $x = 0$, we have

$$(\text{LHL at } x = 0) = \lim_{x \to 0^-} f(x) = \lim_{x \to 0} m(x^2 - 2x) = 0 \text{ for all values of } m.$$

$$\text{and, } \quad (\text{RHL at } x = 0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0} \cos x = 1$$

Clearly, $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$ for any value of m. So, $f(x)$ cannot be made continuous for any value of m. In other words, the value of m does not exist for which $f(x)$ can be made continuous.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Given* $f(x)=\dfrac{1}{x-1}$. *Find the points of discontinuity of the composite function* $f(f(x))$.

[NCERT EXEMPLAR]

SOLUTION We find that

$$\lim_{x\to 1^-} f(x)=\lim_{h\to 0} f(1-h)=\lim_{h\to 0}\frac{1}{-h}\to -\infty \text{ and, } \lim_{x\to 1^+} f(x)=\lim_{h\to 0} f(1+h)=\lim_{h\to 0}\frac{1}{h}\to\infty$$

$$\therefore \quad \lim_{x\to 1^-} f(x)\neq \lim_{x\to 1^+} f(x)$$

So, $f(x)$ is discontinuous at $x=1$. Also, $f(x)$ is not defined at $x=1$. So, for $x\neq 1$, we obtain

$$f(f(x))=f\left(\frac{1}{x-1}\right)=\frac{1}{\frac{1}{x-1}-1}=\frac{x-1}{2-x}$$

Let $g(x)=f(f(x))=\dfrac{x-1}{2-x}$. Then,

$$\lim_{x\to 2^-} g(x)=\lim_{h\to 0} g(2-h)=\lim_{h\to 0}\frac{2-h-1}{2-(2-h)}=\lim_{h\to 0}\left(\frac{1}{h}-1\right)\to\infty$$

and, $$\lim_{x\to 2^+} g(x)=\lim_{h\to 0} g(2+h)=\lim_{h\to 0}\frac{2+h-1}{2-(2+h)}=\lim_{h\to 0}\left(-1-\frac{1}{h}\right)\to-\infty$$

$$\therefore \quad \lim_{x\to 2^-} g(x)\neq \lim_{x\to 2^+} g(x) \text{ or, } \lim_{x\to 2^-} f\left(f(x)\right)\neq \lim_{x\to 2^+} f\left(f(x)\right)$$

So, $f(f(x))$ is discontinuous at $x=2$. Hence, $f(f(x))$ is discontinuous at $x=1$ and $x=2$.

EXERCISE 8.2

BASIC

1. Prove that the function $f(x)=\begin{cases}\dfrac{\sin x}{x}, & x<0\\ x+1, & x\geq 0\end{cases}$ is everywhere continuous.

2. Discuss the continuity of the function $f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 0, & x=0\end{cases}$.

3. Find the points of discontinuity, if any, of the following functions:

(i) $f(x)=\begin{cases}x^3-x^2+2x-2, & \text{if } x\neq 1\\ 4, & \text{if } x=1\end{cases}$

(ii) $f(x)=\begin{cases}\dfrac{x^4-16}{x-2}, & \text{if } x\neq 2\\ 16, & \text{if } x=2\end{cases}$

(iii) $f(x)=\begin{cases}\dfrac{\sin x}{x}, & \text{if } x<0\\ 2x+3, & x\geq 0\end{cases}$

(iv) $f(x)=\begin{cases}\dfrac{\sin 3x}{x}, & \text{if } x\neq 0\\ 4, & \text{if } x=0\end{cases}$

(v) $f(x)=\begin{cases}\dfrac{\sin x}{x}+\cos x, & \text{if } x\neq 0\\ 5, & \text{if } x=0\end{cases}$

(vi) $f(x)=\begin{cases}\dfrac{x^4+x^3+2x^2}{\tan^{-1}x}, & \text{if } x\neq 0\\ 10, & \text{if } x=0\end{cases}$

(vii) $f(x) = \begin{cases} \dfrac{e^x - 1}{\log_e(1+2x)} & , \text{ if } x \neq 0 \\ 7 & , \text{ if } x = 0 \end{cases}$ (viii) $f(x) = \begin{cases} |x-3| & , \text{ if } x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4} & , \text{ if } x < 1 \end{cases}$

(ix) $f(x) = \begin{cases} |x| + 3 & , \text{ if } x \leq -3 \\ -2x & , \text{ if } -3 < x < 3 \\ 6x + 2 & , \text{ if } x > 3 \end{cases}$ [CBSE 2010]

(x) $f(x) = \begin{cases} x^{10} - 1 & , \text{ if } x \leq 1 \\ x^2 & , \text{ if } x > 1 \end{cases}$ [NCERT]

(xi) $f(x) = \begin{cases} 2x & , \text{ if } x < 0 \\ 0 & , \text{ if } 0 \leq x \leq 1 \\ 4x & , \text{ if } x > 1 \end{cases}$ [NCERT]

(xii) $f(x) = \begin{cases} \sin x - \cos x & , \text{ if } x \neq 0 \\ -1 & , \text{ if } x = 0 \end{cases}$ [NCERT]

(xiii) $f(x) = \begin{cases} -2 & , \text{ if } x \leq -1 \\ 2x & , \text{ if } -1 < x < 1 \\ 2 & , \text{ if } x \geq 1 \end{cases}$ [NCERT]

4. In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous:

(i) $f(x) = \begin{cases} \dfrac{\sin 2x}{5x} & , \text{ if } x \neq 0 \\ 3k & , \text{ if } x = 0 \end{cases}$ (ii) $f(x) = \begin{cases} kx + 5, & \text{if } x \leq 2 \\ x - 1, & \text{if } x > 2 \end{cases}$

(iii) $f(x) = \begin{cases} k(x^2 + 3x) & , \text{ if } x < 0 \\ \cos 2x & , \text{ if } x \geq 0 \end{cases}$ (iv) $f(x) = \begin{cases} 2 & , \text{ if } x \leq 3 \\ ax + b & , \text{ if } 3 < x < 5 \\ 9 & , \text{ if } x \geq 5 \end{cases}$

(v) $f(x) = \begin{cases} 4 & , \text{ if } x \leq -1 \\ ax^2 + b & , \text{ if } -1 < x < 0 \\ \cos x & , \text{ if } x \geq 0 \end{cases}$

(vi) $f(x) = \begin{cases} \dfrac{\sqrt{1+px} - \sqrt{1-px}}{x} & , \text{ if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-2} & , \text{ if } 0 \leq x \leq 1 \end{cases}$ [CBSE 2013, NCERT EXEMPLAR]

(vii) $f(x) = \begin{cases} 5 & , \text{ if } x \leq 2 \\ ax + b & , \text{ if } 2 < x < 10 \\ 21 & , \text{ if } x \geq 10 \end{cases}$ [NCERT]

(viii) $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & , \quad x < \dfrac{\pi}{2} \\ 3 & , \quad x = \dfrac{\pi}{2} \\ \dfrac{3 \tan 2x}{2x - \pi} & , \quad x > \dfrac{\pi}{2} \end{cases}$ [CBSE 2010]

BASED ON LOTS

5. The function $f(x) = \begin{cases} \dfrac{x^2}{a} & , \text{ if } 0 \le x < 1 \\ a & , \text{ if } 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & , \text{ if } \sqrt{2} \le x < \infty \end{cases}$ is continuous on $[0, \infty)$. Find the most suitable values of a and b.

6. Find the values of a and b so that the function $f(x)$ defined by
$f(x) = \begin{cases} x + a\sqrt{2} \sin x & , \text{ if } 0 \le x < \pi/4 \\ 2x \cot x + b & , \text{ if } \pi/4 \le x < \pi/2 \\ a \cos 2x - b \sin x & , \text{ if } \pi/2 \le x \le \pi \end{cases}$ becomes continuous on $[0, \pi]$.

7. The function $f(x)$ is defined by $f(x) = \begin{cases} x^2 + ax + b & , 0 \le x < 2 \\ 3x + 2 & , 2 \le x \le 4 \\ 2ax + 5b & , 4 < x \le 8 \end{cases}$. If f is continuous on $[0, 8]$, find the values of a and b.

8. Discuss the continuity of the function $f(x) = \begin{cases} 2x - 1 & , \text{ if } x < 2 \\ \dfrac{3x}{2} & , \text{ if } x \ge 2 \end{cases}$.

9. Prove that $f(x) = \begin{cases} \dfrac{\sin x}{x} & , x < 0 \\ x + 1 & , x \ge 0 \end{cases}$ is everywhere continuous.

10. Is $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & , x \ne 0 \\ 0 & , x = 0 \end{cases}$ a continuous function? [NCERT]

BASED ON HOTS

11. If $f(x) = \dfrac{\tan\left(\dfrac{\pi}{4} - x\right)}{\cot 2x}$ for $x \ne \dfrac{\pi}{4}$, find the value which can be assigned to $f(x)$ at $x = \pi/4$ so that the function $f(x)$ becomes continuous every where in $[0, \pi/2]$.

12. Discuss the continuity of $f(x) = \sin |x|$. [NCERT]

13. Show that the function $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer function. [NCERT]

14. Discuss the continuity of the following functions:
(i) $f(x) = \sin x + \cos x$ (ii) $f(x) = \sin x - \cos x$ (iii) $f(x) = \sin x \cos x$ [NCERT]

15. Show that $f(x) = \cos x^2$ is a continuous function. [NCERT]

16. Show that $f(x) = |\cos x|$ is a continuous function. [NCERT]

17. Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$. [NCERT]

18. Given the function $f(x) = \dfrac{1}{x + 2}$. Find the points of discontinuity of the function $f(f(x))$.

[NCERT EXEMPLAR]

19. Find all point of discontinuity of the function $f(t) = \frac{1}{t^2 + t - 2}$, where $t = \frac{1}{x-1}$.

[NCERT EXEMPLAR]

ANSWERS

2. Discontinuous at $x = 0$
3. (i) $x = 1$ (ii) $x = 2$ (iii) $x = 0$ (iv) $x = 0$ (v) $x = 0$
(vi) $x = 0$ (vii) $x = 0$ (viii) Nowhere discontinuous
(ix) Discontinuous at $x = 3$ (x) Discontinuous at $x = 1$
(xi) Discontinuous at $x = 1$ (xii) Everywhere continuous
(xiii) Everywhere continuous
4. (i) $k = \frac{2}{15}$ (ii) $k = -2$
(iii) No value of k can make f (iv) $a = 7/2, b = -17/2$ (v) $a = 3, b = 1$
(vi) $p = -1/2$ (vii) $a = 2, b = 1$ (viii) 6
5. $a = -1, b = 1$ or $a = 1, b = 1 \pm \sqrt{2}$
6. $a = \pi/6,\ b = -\pi/12$
7. $a = 3, b = -2$
8. Everywhere continuous.
10. Continuous
11. $f\left(\frac{\pi}{4}\right) = \frac{1}{2}$
12. Everywhere continuous
14. (i) Everywhere continuous (ii) Everywhere continuous
(iii) Everywhere continuous
17. No point of discontinuity
18. Discontinuous at $x = -2$ and $x = -5/2$
19. Discontinuous $x = 1/2, 1, 2$.

HINTS TO SELECTED PROBLEMS

3. (x) We have,

$$f(x) = \begin{cases} x^{10} - 1, & x \le 1 \\ x^2, & x > 1 \end{cases}$$

Clearly, $f(x)$ is a polynomial function for all $x \le 1$ as well as for all $x > 1$. So, $f(x)$ is everywhere continuous except possibly at $x = 1$.

Now, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^{10} - 1 = 1 - 1 = 0$ and, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^2 = 1^2 = 1$

Clearly, $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. So, $f(x)$ is not continuous at $x = 1$.

Hence, $f(x)$ is everywhere continuous except at $x = 1$.

(xi) We have, $f(x) = \begin{cases} 2x, & x < 0 \\ 0, & 0 \le x \le 1 \\ 4x, & x > 1 \end{cases}$

At $x = 0$, we have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x = 2 \times 0 = 0, \quad \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 0 = 0 \text{ and, } f(0) = 0$$

Thus, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$. So, $f(x)$ is continuous at $x = 0$.

At $x = 1$, we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 0 = 0, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4x = 4$$

$\therefore \quad \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$. So, $f(x)$ is not continuous at $x = 1$.

For $x < 0$, $f(x)$ is a polynomial function which is everywhere continuous. For $x \in [0, 1]$, $f(x)$ is a constant function which is also continuous. For $x > 1$, $f(x)$ is a polynomial function which is everywhere continuous. Hence, $f(x)$ is everywhere continuous except at $x = 1$.

(xii) We have, $f(x) = \begin{cases} \sin x - \cos x, & x \neq 0 \\ -1 & , x = 0 \end{cases}$. Clearly, $f(x)$ is continuous for all $x \neq 0$.

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} (\sin x - \cos x) = \sin 0 - \cos 0 = -1 = f(0)$

So, $f(x)$ is continuous at $x = 0$. Hence, $f(x)$ is everywhere continuous.

(xiii) We have, $f(x) = \begin{cases} -2 & , x \leq -1 \\ 2x & , -1 < x < 1 \\ 2 & , x \geq 1 \end{cases}$

As $f(x)$ is a constant function for all $x < -1$ and all $x > 1$. So, $f(x)$ is continuos for all $x < -1$ and all $x > 1$. For $x \in (-1, 1)$, $f(x)$ is a polynomial function which is always continuous. Thus, $f(x)$ is continuous for all x except possible at $x = -1, 1$.

Continuity at $x = -1$: We observe that

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} -2 = -2, \quad \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 2x = -2 \text{ and, } f(-1) = -2$$

$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$. So, $f(x)$ is continuous at $x = -1$.

Continuity at $x = 1$: Clearly,

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 = 2, \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2 \times 1 = 2 \text{ and, } f(1) = 2$$

$$\therefore \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

So, $f(x)$ is continuous at $x = 1$. Hence, $f(x)$ is everywhere continuous.

4. (vii) It is given that $f(x) = \begin{cases} 5 & , x \leq 2 \\ ax + b & , 2 < x < 10 \\ 21 & , x \geq 10 \end{cases}$ is everywhere continuous.

So, it is continuous at $x = 2$ and $x = 10$.

$$\therefore \quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2) \quad \text{and,} \quad \lim_{x \to 10^-} f(x) = \lim_{x \to 10^+} f(x) = f(10)$$

$$\Rightarrow \quad \lim_{x \to 2^-} 5 = \lim_{x \to 2^+} ax + b = 5 \quad \text{and,} \quad \lim_{x \to 10^-} ax + b = \lim_{x \to 10^+} 21 = 21$$

$$\Rightarrow \quad 5 = 2a + b \text{ and } 10a + b = 21 \Rightarrow a = 2, \ b = 1$$

10. We have, $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$. Clearly, $f(x)$ is continuous for all $x \neq 0$.

Now,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin \frac{1}{x} = 0 \times (\text{An oscillating number between } -1 \text{ and } 1) = 0 = f(0)$$

So, $f(x)$ is continuous at $x = 0$. Hence, $f(x)$ is everywhere continuous.

12. Let a be any real number. Then,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \sin|x| = \lim_{h \to 0} \sin|a-h| = \sin|a|$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \sin|x| = \lim_{h \to 0} \sin|a+h| = \sin|a| \text{ and, } f(a) = \sin|a|$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$$

So, $f(x)$ is continuous at $x = a$. Since a is an arbitrary real number. Hence, $f(x)$ is everywhere continuous.

13. Let a be any integer. Then,

$$\lim_{x \to a^+} g(x) = \lim_{h \to 0} = g(a-h) = \lim_{h \to 0} (a-h) - [a-h] = \lim_{h \to 0} (a-h) - (a-1) = a - (a-1) = 1$$

$$\lim_{x \to a^+} g(x) = \lim_{h \to 0} g(a+h) = \lim_{h \to 0} (a+h) - [a+h] = \lim_{h \to 0} (a+h) - a = a - a = 0$$

$$\therefore \quad \lim_{x \to a^-} g(x) \neq \lim_{x \to a^+} g(x). \text{ So, } g(x) \text{ is discontinuous at } x = a.$$

Since a is an arbitrary integer. Hence, $g(x)$ is discontinuous at all integral points.

14. We know that $\sin x$ and $\cos x$ are everywhere continuous. Therefore, $\sin x + \cos x$, $\sin x - \cos x$ and $\sin x \cos x$ are everywhere continuous.

15. Let $f(x) = \cos x^2$ and a be any real number. Then,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} \cos(a-h)^2 = \cos a^2$$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} \cos(a+h)^2 = \cos a^2 \text{ and, } f(a) = \cos a^2$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a). \text{ So, } f(x) \text{ is continuous at } x = a.$$

Since 'a' is an arbitrary real number. Hence, $f(x)$ is everywhere continuous.

16. Let $f(x) = |\cos x|$ and a be any real number. Then,

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h) = \lim_{h \to 0} |\cos(a-h)| = |\cos a|$$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h) = \lim_{h \to 0} |\cos(a+h)| = |\cos a| \text{ and, } f(a) = |\cos a|$$

$$\therefore \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a). \text{ So, } f(x) \text{ is continuous at } x = a.$$

Since 'a' is an arbitrary real number. Therefore, $f(x)$ is everywhere continuous.

17. We have,

$$f(x) = |x| - |x+1| = \begin{cases} -x + x + 1 & , x < -1 \\ -x - (x+1) & , -1 \le x < 0 \\ x - (x+1) & , x \ge 0 \end{cases} \Rightarrow f(x) = \begin{cases} 1 & , x < -1 \\ -2x - 1 & , -1 \le x < 0 \\ -1 & , x \ge 0 \end{cases}$$

Clearly, $f(x)$ is continuous for all x satisfying $x < -1$, $-1 < x < 0$ and $x > 0$. So, possibly points of discontinuity are $x = -1$ and $x = 0$.

Continuity at $x=-1$: Clearly,

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 1 = 1, \quad \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-2x-1) = -2 \times -1 - 1 = 1$$

and, $f(-1) = -2 \times -1 - 1 = 1$

$$\therefore \quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1).$$

So, $f(x)$ is continuous at $x = 1$.

Continuity at $x=0$: Clearly,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2x-1) = -2 \times 0 - 1 = -1,$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} -1 = -1 \text{ and, } f(0) = -2 \times 0 - 1 = -1$$

$$\therefore \quad \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

So, $f(x)$ is continuous at $x=0$. Hence, $f(x)$ is everywhere continuous.

18. Clearly, $f(x) = \dfrac{1}{x+2}$ is discontinuous at $x=-2$. Also, it is not defined at $x=-2$.

For $x \neq -2$, we have $f(x) = f\left(\dfrac{1}{x+2}\right) = \dfrac{1}{\dfrac{1}{x+2}+2} = \dfrac{x+2}{2x+5}$

We observe that $f(f(x))$ is discontinuous and undefined at $x=-5/2$.
Hence, $f(f(x))$ is discontinuous at $x=-2$ and $x=-5/2$.

19. We have, $f(x) = \dfrac{1}{t^2+t-2}$ and $t = \dfrac{1}{x-1}$. Clearly, $t = \dfrac{1}{x-1}$ is discontinuous and undefined at $x=1$.

For $x \neq 1$, we have $f(t) = \dfrac{1}{t^2+t-2} = \dfrac{1}{(t+2)(t-1)}$. This is discontinuous at $t=2$ and $t=1$.

For $t=-2$, $t = \dfrac{1}{x-1} \Rightarrow x = \dfrac{1}{2}$ and, For $t=1$, $t = \dfrac{1}{x-1} \Rightarrow x=2$

Hence, f is discontinuous at $x=1/2$, $x=1$ and $x=2$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $f(x) = \begin{cases} \dfrac{x^3-a^3}{x-a}, & x \neq a \\ b, & x = a \end{cases}$ is continuous at $x=a$, then $b =$ ....

2. If the function $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & x \neq 0 \\ 1, & x=0 \end{cases}$ is continuous at $x=0$, then $a =$ ....

3. If the function $f(x) = \begin{cases} ax^2-b, & 0 \le x < 1 \\ 2, & x=1 \\ x+1, & 1 < x \le 2 \end{cases}$ is continuous at $x=1$, then $a-b =$ ....

4. If $f(x)=\begin{cases} x+k & , x<3 \\ 4 & , x=3 \\ 3x-5, & x>3 \end{cases}$ is continuous at $x=3$, then $k=$ ....

5. Let $f(x)=\begin{cases} \dfrac{x-4}{|x-4|}+a, & x<4 \\ a+b & , x=4 \\ \dfrac{x-4}{|x-4|}+b, & x>4 \end{cases}$. Then $f(x)$ is continuous at $x=4$ when $a+b=$ ....

6. If $f: R \to R$ defined by $f(x)=\begin{cases} \dfrac{\cos 3x-\cos x}{x^2}, & x \neq 0 \\ \lambda & , x=0 \end{cases}$ is continuous at $x=0$, then $\lambda=$ ....

7. If $f(x)=\begin{cases} \dfrac{1-\sin x}{\pi-2x}, & x \neq \dfrac{\pi}{2} \\ k & , x=\dfrac{\pi}{2} \end{cases}$ is continuous at $x=\dfrac{\pi}{2}$, then $k=$ ....

8. If $f(x)=\dfrac{2-\sqrt{x+4}}{\sin 2x}, x \neq 0$, is continuous at $x=0$, then $f(0)=$ ....

9. If $f(x)=\begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ 2x+k, & x=3 \end{cases}$ is continuous at $x=3$, then $k=$ ....

10. If the function $f(x)=\begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ k & , x=1 \end{cases}$ is given to be continuous at $x=1$, then the value of k is ....

11. The set of points where $f(x)=x-[x]$ is discontinuous is ....

12. Let $f(x)=\dfrac{1-\tan x}{4x-\pi}, x \neq \dfrac{\pi}{4}, x \in \left[0, \dfrac{\pi}{2}\right]$. If $f(x)$ is continuous in $\left[0, \dfrac{\pi}{2}\right]$, then $f\left(\dfrac{\pi}{4}\right)=$ ....

13. If $f(x)=x \sin\left(\dfrac{\pi}{x}\right)$ is everywhere continuous, then $f(0)=$ ....

14. The set of points at which the function $f(x)=\dfrac{1}{\log|x|}$ is not continuous, is ....

15. If $f(x)=\begin{cases} ax+1, \text{ if } x \geq 1 \\ x+2, \text{ if } x<1 \end{cases}$ is continuous, then 'a' should be equal to ....

16. If $f(x)$ is continuous at $x=a$ and $\lim\limits_{x \to a^-} f(x)=\lim\limits_{x \to a^+} f(x)=k$, then k is equal to ....

17. If $f(x)=\begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0 \\ \dfrac{k}{2} & , \text{if } x=0 \end{cases}$ is continuous at $x=0$, then k is equal to ....

18. The set of points of discontinuity of $f(x)=\tan x$ is ....

19. The set of points of discontinuity of $f(x)=[x]$ is ....

20. The set of points of discontinuity of $f(x)=\dfrac{1}{x-[x]}$ is ....

ANSWERS

1. $3a^2$	2. ± 1	3. 2	4. 1	5. 0	6. -4	7. 0
8. $-\dfrac{1}{8}$	9. 0	10. 2	11. Z	12. $-\dfrac{1}{2}$	13. 0	14. $\{-1, 0, 1\}$
15. 2	16. $f(a)$	17. 6	18. $\left\{(2n+1)\dfrac{\pi}{2} : n \in Z\right\}$	19. Z	20. Z	

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the questions.

1. Define continuity of a function at a point.
2. What happens to a function $f(x)$ at $x=a$, if $\lim\limits_{x\to a} f(x)=f(a)$?
3. Find $f(0)$, so that $f(x)=\dfrac{x}{1-\sqrt{1-x}}$ becomes continuous at $x=0$.
4. If $f(x)=\begin{cases}\dfrac{x}{\sin 3x}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, then write the value of k.
5. If the function $f(x)=\dfrac{\sin 10x}{x}$, $x\neq 0$ is continuous at $x=0$, find $f(0)$.
6. If $f(x)=\begin{cases}\dfrac{x^2-16}{x-4}, & \text{if } x\neq 4\\ k, & \text{if } x=4\end{cases}$ is continuous at $x=4$, find k.
7. Determine whether $f(x)=\begin{cases}\dfrac{\sin x^2}{x}, & x\neq 0\\ 0, & x=0\end{cases}$ is continuous at $x=0$ or not.
8. If $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, find k.
9. If $f(x)=\begin{cases}\dfrac{\sin^{-1} x}{x}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at $x=0$, write the value of k.
10. Write the value of b for which $f(x)=\begin{cases}5x-4 & 0<x\leq 1\\ 4x^2+3bx & 1<x<2\end{cases}$ is continuous at $x=1$.
11. Determine the value of constant 'k' so that the function $f(x)=\begin{cases}\dfrac{kx}{|x|}, & x<0\\ 3, & x\geq 0\end{cases}$ is continuous at $x=0$. **[CBSE 2017]**

12. Find the value of k for which the function $f(x) = \begin{cases} \dfrac{x^2+3x-10}{x-2} &, x \neq 2 \\ k &, x^2 \end{cases}$ is continuous at $x = 2$.

[CBSE 2017]

ANSWERS

2. $f(x)$ becomes continuous at $x = a$ 3. 2 4. $\frac{1}{3}$ 5. 10 6. 8

7. continuous 8. $\frac{1}{2}$ 9. 1 10. -1 11. -3 12. 7