CHAPTER 18

BINOMIAL THEOREM

18.1 INTRODUCTION

An algebraic expression containing two terms is called a binomial expression.

For example, $(a+b)$, $(2x-3y)$, $\left(x+\frac{1}{y}\right)$, $\left(x+\frac{3}{x}\right)$, $\left(\frac{2}{x}-\frac{1}{x^2}\right)$ etc. are binomial expressions.

Similarly, an algebraic expression containing three terms is called a *trinomial*. In general, expressions containing more than two terms are known as multinomial expression.

The general form of the binomial expression is $(x+a)$ and the expansion of $(x+a)^n$, $n \in N$ is called the *binomial theorem*. This theorem was first given by **Sir Issac Newton.** It gives a formula for the expansion of the powers of a binomial expression.

In earlier classes, we have learnt that:

$$(x+a)^0 = 1$$

$$(x+a)^1 = x+a$$

$$(x+a)^2 = x^2+2ax+a^2$$

$$(x+a)^3 = x^3+3x^2a+3xa^2+a^3$$

$$(x+a)^4 = x^4+4x^3a+6x^2a^2+4xa^3+a^4$$

We observe that the coefficients in the above expansions follow a particular pattern as given below:

Index of the binomial	*Coefficients of various terms*
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

We also observe that each row is bounded by 1 on both sides. Any entry, except the first and last, in a row is the sum of two entries in the preceding row, one on the immediate left and the other on the immediate right. The above pattern is known as ***Pascal's triangle.*** It has been checked that the above pattern also holds good for the coefficients in the expansions of the binomial expressions having index (exponent) greater than 4 as given below.

Index of the binomial	*Coefficients of various terms*
0	1
1	1 ▽ 1
2	1 ▽ 2 ▽ 1
3	1 ▽ 3 ▽ 3 ▽ 1
4	1 ▽ 4 ▽ 6 ▽ 4 ▽ 1
5	1 ▽ 5 ▽ 10 ▽ 10 ▽ 5 ▽ 1
6	1 ▽ 6 ▽ 15 ▽ 20 ▽ 15 ▽ 6 ▽ 1

....

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Pascal's Triangle

Using the above Pascal's triangle, we obtain

$$(x+a)^1 = x+a$$

or, $$(x+a)^1 = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1 \qquad \left[\because {}^1C_0 = 1 = {}^1C_1\right]$$

$$(x+a)^2 = x^2 + 2ax + a^2$$

or, $$(x+a)^2 = {}^2C_0\, x^2\, a^0 + {}^2C_1\, x^{2-1}\, a^1 + {}^2C_2\, x^{2-2}\, a^2 \qquad \left[\because {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1\right]$$

$$(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3$$

or, $$(x+a)^3 = {}^3C_0\, x^3\, a^0 + {}^3C_1\, x^{3-1}\, a^1 + {}^3C_2\, x^{3-2}\, a^2 + {}^3C_3\, x^{3-3}\, a^3$$

$$(x+a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$$

or, $$(x+a)^4 = {}^4C_0\, x^4\, a^0 + {}^4C_1\, x^{4-1}\, a^1 + {}^4C_2\, x^{4-2}\, a^2 + {}^4C_3\, x^{4-3}\, a^3 + {}^4C_4\, x^{4-4}\, a^4$$

$$(x+a)^5 = {}^5C_0\, x^5\, a^0 + {}^5C_1\, x^{5-1}\, a^1 + {}^5C_2\, x^{5-2}\, a^2 + {}^5C_3\, x^{5-3}\, a^3 + {}^5C_4\, x^{5-4}\, a^4 + {}^5C_5\, x^{5-5}\, a^5$$

By looking at the above expansions we can easily guess that the general formula would be of the form as given in the following theorem.

18.2 BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX

THEOREM *If x and a are real numbers, then for all n ∈ N,*

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n$$

i.e., $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, a^r$$

PROOF We shall prove the theorem by using the principle of mathematical induction on n.

Let $P(n)$ be the statement:

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_{n-1}\, x^1\, a^{n-1} + {}^nC_n\, x^0\, a^n$$

STEP I We have, $P(1)$: $(x+a)^1 = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1$

We know that: $(x+a)^1 = x + a = {}^1C_0\, x^1\, a^0 + {}^1C_1\, x^0\, a^1$

∴ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$(x+a)^m = {}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + {}^mC_2\, x^{m-2} a^2 + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m \quad \ldots\text{(i)}$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$(x+a)^{m+1} = {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, x^0 a^{m+1}$$

Now, $(x+a)^{m+1}$

$$= (x+a)\,.\,(x+a)^m = (x+a)\left[{}^mC_0\, x^m a^0 + {}^mC_1\, x^{m-1} a^1 + \ldots + {}^mC_r\, x^{m-r} a^r + \ldots + {}^mC_{m-1}\, x^1 a^{m-1} + {}^mC_m\, x^0 a^m\right]$$

$$= {}^mC_0\, x^{m+1} a^0 + ({}^mC_1 + {}^mC_0)\, x^m a^1 + ({}^mC_2 + {}^mC_1)\, x^{m-1} a^2 + \ldots + ({}^mC_r + {}^mC_{r-1})\, x^{m-r+1} a^r + \ldots + ({}^mC_{m-1} + {}^mC_m)\, x^1 a^m + {}^mC_m\, a^{m+1}$$

$$= {}^{m+1}C_0\, x^{m+1} a^0 + {}^{m+1}C_1\, x^m a^1 + {}^{m+1}C_2\, x^{m-1} a^2 + \ldots + {}^{m+1}C_r\, x^{(m+1)-r} a^r + \ldots + {}^{m+1}C_m\, x^1 a^m + {}^{m+1}C_{m+1}\, a^{m+1} \quad \left[\because\ {}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r,\ r = 1, 2, 3, \ldots, m\right]$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the theorem is true for all $n \in N$.

Q.E.D.

18.3 SOME IMPORTANT CONCLUSIONS FROM THE BINOMIAL THEOREM

In this section, we shall draw some useful conclusions from the binomial theorem.

(i) We have,

$$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

or, $$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_r\, x^{n-r} a^r + \ldots + {}^nC_n\, x^0 a^n$$

Since r can have values from 0 to n, the total number of terms in the expansion is $(n+1)$.

(ii) The sum of the indices of x and a in each term is n.

(iii) Since ${}^nC_r = {}^nC_{n-r}$, for $r = 0, 1, 2, \ldots, n$

$\therefore\ {}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2} = \ldots$

So, the coefficients of terms equidistant from the beginning and end are equal. These coefficients are known as the binomial coefficients.

(iv) Replacing a by $-a$, we get

$$(x-a)^n = {}^nC_0\, x^n a^0 - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + (-1)^r\, {}^nC_r\, x^{n-r} a^r + \ldots + (-1)^n\, {}^nC_n\, x^0 a^n.$$

i.e. $$(x-a)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

Thus, the terms in the expansion of $(x-a)^n$ are alternatively positive and negative, the last term is positive or negative according as n is even or odd.

(v) Putting $x = 1$ and $a = x$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_n\, x^n$$

i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

This is the expansion of $(1+x)^n$ in ascending powers of x.

(vi) Putting $a = 1$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \ldots + {}^nC_r\, x^{n-r} + \ldots + {}^nC_{n-1}\, x + {}^nC_n$$

i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}$$

This is the expansion of $(1+x)^n$ in descending powers of x.

(vii) Putting $x = 1$ and $a = -x$ in the expansion of $(x+a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1\, x + {}^nC_2\, x^2 - {}^nC_3\, x^3 + \ldots + (-1)^r\, {}^nC_r\, x^r + \ldots + (-1)^n\, {}^nC_n\, x^n.$$

i.e. $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

(viii) The coefficient of $(r+1)$th term in the expansion of $(1+x)^n$ is ${}^nC_r$.

(ix) The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

(x) $$(x+a)^n + (x-a)^n = 2\left\{ {}^nC_0\, x^n\, a^0 + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_4\, x^{n-4}\, a^4 + \ldots \right\}$$

and, $$(x+a)^n - (x-a)^n = 2\left\{ {}^nC_1\, x^{n-1}\, a^1 + {}^nC_3\, x^{n-3}\, a^3 + \ldots \right\}$$

<u>NOTE:</u> *If n is odd than $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas if n is even, then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms and $\left\{(x+a)^n - (x-a)^n\right\}$ has $\left(\frac{n}{2}\right)$ terms.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **DETERMINING THE NUMBER OF TERMS IN THE EXPANSIONS OF BINOMIAL AND TRINOMIAL EXPRESSIONS**

EXAMPLE 1 *Find the number of terms in the expansions of the following:*

(i) $(2x-3y)^9$ (ii) $(1+5\sqrt{2}\,x)^9 + (1-5\sqrt{2}\,x)^9$

(iii) $(\sqrt{x}+\sqrt{y})^{10} + (\sqrt{x}-\sqrt{y})^{10}$ (iv) $(2x+3y-4z)^n$

(v) $[(3x+y)^8 - (3x-y)^8]$ (vi) $(1+2x+x^2)^{20}$

SOLUTION (i) The expansion of $(x+a)^n$ has $(n+1)$ terms. So, the expansion of $(2x-3y)^9$ has 10 terms.

(i) If n is odd, then the expansion of $(x+a)^n + (x-a)^n$ contains $\left(\frac{n+1}{2}\right)$ terms. So, the expansion of $(1+5\sqrt{2}x)^9 + (1-5\sqrt{2}\,x)^9$ has $\left(\frac{9+1}{2}\right) = 5$ terms.

(iii) If n is even, then the expansion of $\left\{(x+a)^n+(x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms.

So, $(\sqrt{x}+\sqrt{y})^{10}+(\sqrt{x}-\sqrt{y})^{10}$ has 6 terms.

(iv) We have,

$$(2x+3y-4z)^n=\left\{2x+(3y-4z)\right\}^n$$
$$={}^nC_0\,(2x)^n\,(3y-4z)^0+{}^nC_1\,(2x)^{n-1}\,(3y-4z)^1+{}^nC_2\,(2x)^{n-2}\,(3y-4z)^2+\ldots$$
$$+{}^nC_{n-1}\,(2x)^1\,(3y-4z)^{n-1}+{}^nC_n\,(3y-4z)^n.$$

Clearly, the first term in the above expansion gives one term, second term gives two terms, third term gives three terms and so on.

So, total number of terms $=1+2+3+\ldots+n+(n+1)=\dfrac{(n+1)(n+2)}{2}$

(v) If n is even, then $\left\{(x+a)^n-(x-a)^n\right\}$ has $\frac{n}{2}$ terms. So, $(3x+y)^8-(3x-y)^8$ has 4 terms.

(vi) We have,

$$(1+2x+x^2)^{20}=\left\{(1+x)^2\right\}^{20}=(1+x)^{40}$$

So, there are 41 terms in the expansion of $(1+2x+x^2)^{20}$

Type II **EXPANDING A GIVEN EXPRESSION USING THE BINOMIAL THEOREM**

EXAMPLE 2 *Expand $(x^2+2a)^5$ by binomial theorem.*

SOLUTION Using binomial theorem,

$$(x^2+2a)^5={}^5C_0\,(x^2)^5\,(2a)^0+{}^5C_1\,(x^2)^4\,(2a)^1+{}^5C_2\,(x^2)^3\,(2a)^2$$
$$+{}^5C_3\,(x^2)^2\,(2a)^3+{}^5C_4\,(x^2)\,(2a)^4+{}^5C_5\,(x^2)^0\,(2a)^5$$
$$=x^{10}+5\,(x^8)\,(2a)+10\,(x^6)\,(4a^2)+10\,(x^4)\,(8a^3)+5\,(x^2)\,(16\,a^4)+32\,a^5$$
$$=x^{10}+10\,x^8\,a+40\,x^6\,a^2+80\,x^4\,a^3+80\,x^2\,a^4+32\,a^5$$

EXAMPLE 3 *Expand $(2x-3y)^4$ by binomial theorem.*

SOLUTION Using binomial theorem, we obtain

$$(2x-3y)^4=\{2x+(-3y)\}^4$$
$$={}^4C_0\,(2x)^4\,(-3y)^0+{}^4C_1\,(2x)^3\,(-3y)+{}^4C_2\,(2x)^2\,(-3y)^2+{}^4C_3\,(2x)^1\,(-3y)^3+{}^4C_4\,(-3y)^4$$
$$=16\,x^4+4\,(8x^3)\,(-3y)+6\,(4x^2)\,(9y^2)+4\,(2x)\,(-27\,y^3)+81\,y^4$$
$$=16x^4-96x^3y+216x^2y^2-216xy^3+81y^4$$

EXAMPLE 4 *By using binomial theorem, expand:*

(i) $(1+x+x^2)^3$ (ii) $(1-x+x^2)^4$

SOLUTION (i) Let $y=x+x^2$. Then, **[NCERT EXEMPLAR]**

$$(1+x+x^2)^3=(1+y)^3={}^3C_0+{}^3C_1\,y+{}^3C_2\,y^2+{}^3C_3\,y^3=1+3y+3y^2+y^3$$
$$=1+3\,(x+x^2)+3\,(x+x^2)^2+(x+x^2)^3$$
$$=1+3\,(x+x^2)+3\,(x^2+2x^3+x^4)+\left\{{}^3C_0\,x^3\,(x^2)^0+{}^3C_1\,x^{3-1}\,(x^2)^1\right.$$
$$\left.+{}^3C_2\,x^{3-2}\,(x^2)^2+{}^3C_3\,x^0\,(x^2)^3\right\}$$

$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$

$= x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$

(ii) Let $y = -x + x^2$. Then,

$(1 - x + x^2)^4 = (1 + y)^4 = {}^4C_0 + {}^4C_1 y + {}^4C_2 y^2 + {}^4C_3 y^3 + {}^4C_4 y^4$

$= 1 + 4y + 6y^2 + 4y^3 + y^4 = 1 + 4(-x + x^2) + 6(-x + x^2)^2 + 4(-x + x^2)^3 + (-x + x^2)^4$

$= 1 - 4x(1 - x) + 6x^2(1 - x)^2 - 4x^3(1 - x)^3 + x^4(1 - x)^4$

$= 1 - 4x(1 - x) + 6x^2(1 - 2x + x^2) - 4x^3({}^3C_0 - {}^3C_1 x + {}^3C_2 x^2 - {}^3C_3 x^3)$

$+ x^4\left({}^4C_0 - {}^4C_1 x + {}^4C_2 x^2 - {}^4C_3 x^3 + {}^4C_4 x^4\right)$

$= 1 - 4x + 4x^2 + 6x^2(1 - 2x + x^2) - 4x^3(1 - 3x + 3x^2 - x^3) + x^4(1 - 4x + 6x^2 - 4x^3 + x^4)$

$= 1 - 4x + 4x^2 + 6x^2 - 12x^3 + 6x^4 - 4x^3 + 12x^4 - 12x^5 + 4x^6 + x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$

$= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$

EXAMPLE 5 *Using binomial theorem, expand* $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$, $x \neq 0$.

SOLUTION We have, **[NCERT]**

$$\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \left\{1 + \left(\frac{x}{2} - \frac{2}{x}\right)\right\}^4$$

$$= {}^4C_0 + {}^4C_1\left(\frac{x}{2} - \frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2} - \frac{2}{x}\right)^2 + {}^4C_3\left(\frac{x}{2} - \frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2} - \frac{2}{x}\right)^4$$

$$= 1 + 4\left(\frac{x}{2} - \frac{2}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left\{\frac{x^3}{8} - \frac{8}{x^3} - 3\left(\frac{x}{2} - \frac{2}{x}\right)\right\}$$

$$+ \left\{{}^4C_0\left(\frac{x}{2}\right)^4\left(-\frac{2}{x}\right)^0 + {}^4C_1\left(\frac{x}{2}\right)^3\left(-\frac{2}{x}\right) + {}^4C_2\left(\frac{x}{2}\right)^2\left(-\frac{2}{x}\right)^2\right.$$

$$\left. + {}^4C_3\left(\frac{x}{2}\right)\left(-\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^0\left(-\frac{2}{x}\right)^4\right\}$$

$$= 1 + \left(2x - \frac{8}{x}\right) + 6\left(\frac{x^2}{4} - 2 + \frac{4}{x^2}\right) + 4\left(\frac{x^3}{8} - \frac{8}{x^3} - \frac{3x}{2} - \frac{6}{x}\right)$$

$$+ \left(\frac{x^4}{16} + 4 \times \frac{x^3}{8} \times -\frac{2}{x} + 6 \times \frac{x^2}{4} \times \frac{4}{x^2} + 4 \times \frac{x}{2} \times -\frac{8}{x^3} + \frac{16}{x^4}\right)$$

$$= 1 + \left(2x - \frac{8}{x}\right) + \left(\frac{3}{2}x^2 - 12 + \frac{24}{x^2}\right) + \left(\frac{x^3}{2} - \frac{32}{x^3} - 6x + \frac{24}{x}\right) + \left(\frac{x^4}{16} - x^2 + 6 - \frac{16}{x^2} + \frac{16}{x^4}\right)$$

$$= (1-12+6)+(2x-6x)+\left(\frac{3}{2}x^2-x^2\right)+\frac{x^3}{2}+\frac{x^4}{16}+\left(\frac{-8}{x}+\frac{24}{x}\right)+\left(\frac{24}{x^2}-\frac{16}{x^2}\right)-\frac{32}{x^3}+\frac{16}{x^4}$$

$$= -5-4x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{16}+\frac{16}{x}+\frac{8}{x^2}-\frac{32}{x^3}+\frac{16}{x^4}$$

EXAMPLE 6 *Find the expansion of* $(3x^2-2ax+3a^2)^3$ *using binomial theorem.* **[NCERT]**

SOLUTION We have,

$$(3x^2-2ax+3a^2)^3$$

$$= \left\{(3x^2-2ax)+3a^2\right\}^3$$

$$= {}^3C_0\,(3x^2-2ax)^3\,(3a^2)^0 + {}^3C_1\,(3x^2-2ax)^2\,(3a^2) + {}^3C_2\,(3x^2-2ax)^1\,(3a^2)^2 + {}^3C_3\,(3x^2-2ax)^0\,(3a^2)^3$$

$$= (3x^2-2ax)^3 + 9a^2\,(3x^2-2ax)^2 + 27a^4\,(3x^2-2ax) + 27a^6$$

$$= \left\{{}^3C_0\,(3x^2)^3(-2ax)^0 + {}^3C_1(3x^2)^2\,(-2ax)^1 + {}^3C_2\,(3x^2)\,(-2ax)^2 + {}^3C_3\,(3x^2)^0\,(-2ax)^3\right\} + 9a^2\,(9x^4-12ax^3+4a^2x^2) + 27a^4\,(3x^2-2ax) + 27a^6$$

$$= (27x^6-54x^5a+36x^4a^2-8x^3a^3)+(81x^4a^2-108x^3a^3+36x^2a^4) + (81x^2a^4-54xa^5)+27a^6$$

$$= 27x^6-54x^5a+117x^5a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$$

EXAMPLE 7 *Using binomial theorem, expand* $\left(x+\frac{1}{y}\right)^{11}$.

SOLUTION We have,

$$\left(x+\frac{1}{y}\right)^{11} = {}^{11}C_0\,x^{11}\left(\frac{1}{y}\right)^0 + {}^{11}C_1\,x^{10}\left(\frac{1}{y}\right) + {}^{11}C_2\,x^9\left(\frac{1}{y}\right)^2 + {}^{11}C_3\,x^8\left(\frac{1}{y}\right)^3$$
$$+ {}^{11}C_4\,x^7\left(\frac{1}{y}\right)^4 + {}^{11}C_5\,x^6\left(\frac{1}{y}\right)^5 + {}^{11}C_6\,x^5\left(\frac{1}{y}\right)^6 + {}^{11}C_7\,x^4\left(\frac{1}{y}\right)^7 + {}^{11}C_8\,x^3\left(\frac{1}{y}\right)^8$$
$$+ {}^{11}C_9\,x^2\left(\frac{1}{y}\right)^9 + {}^{11}C_{10}\,x\left(\frac{1}{y}\right)^{10} + {}^{11}C_{11}\left(\frac{1}{y}\right)^{11}$$

$$= x^{11}+11\frac{x^{10}}{y}+55\frac{x^9}{y^2}+165\frac{x^8}{y^3}+330\frac{x^7}{y^4}+462\frac{x^6}{y^5}+462\frac{x^5}{y^6} + \frac{330x^4}{y^7}+\frac{165x^3}{y^8}+\frac{55x^2}{y^9}+\frac{11x}{y^{10}}+\frac{1}{y^{11}}$$

EXAMPLE 8 *Prove that* $\sum\limits_{r=0}^{n} {}^nC_r\,3^r = 4^n$ **[NCERT]**

SOLUTION We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1\,x + {}^nC_2\,x^2 + \ldots + {}^nC_n\,x^n$$

or, $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\,x^r$$

Putting $x = 3$ on both sides, we get

$$(1+3)^n = \sum_{r=0}^{n} {}^nC_r\ 3^r \text{ or, } 4^n = \sum_{r=0}^{n} {}^nC_r\ 3^r$$

Type III **ON APPLICATIONS OF BINOMIAL THEOREM**

EXAMPLE 9 *Find an approximation of $(0.99)^5$ using the first three terms of its expansion.* **[NCERT]**

SOLUTION We have,

$$(0.99)^5 = (1-0.01)^5 = \left(1 - \frac{1}{100}\right)^5$$

$$= {}^5C_0 - {}^5C_1 \times \frac{1}{100} + {}^5C_2 \times \left(\frac{1}{100}\right)^2 - {}^5C_3\left(\frac{1}{100}\right)^3 + {}^5C_4\left(\frac{1}{100}\right)^4 - {}^5C_5\left(\frac{1}{100}\right)^5$$

$$= 1 - \frac{5}{100} + \frac{10}{10000} - \frac{10}{1000000} + \frac{5}{(100)^4} - \frac{1}{(100)^5}$$

$$= 1 - 0.05 + 0.001 \quad \text{[Neglecting fourth and other terms]}$$

$$= 0.951$$

EXAMPLE 10 *Using binomial theorem, compute the following:*

(i) $(99)^5$ (ii) $(102)^6$ (iii) $(10.1)^5$

SOLUTION (i) We have,

$$(99)^5 = (100-1)^5$$

$$= {}^5C_0 \times (100)^5 - {}^5C_1 \times (100)^4 + {}^5C_2 \times (100)^3 - {}^5C_3 \times (100)^2 + {}^5C_4 \times (100)^1 - {}^5C_5 \times (100)^0$$

$$= (100)^5 - 5 \times (100)^4 + 10 \times (100)^3 - 10 \times (100)^2 + 5 \times 100 - 1$$

$$= 10^{10} - 5 \times 10^8 + 10^7 - 10^5 + 5 \times 10^2 - 1$$

$$= (10^{10} + 10^7 + 5 \times 10^2) - (5 \times 10^8 + 10^5 + 1) = 10010000500 - 500100001 = 9509900499.$$

(ii) We have,

$$(102)^6 = (100+2)^6$$

$$= {}^6C_0 \times (100)^6 + {}^6C_1 \times (100)^5 \times 2 + {}^6C_2 \times (100)^4 \times 2^2$$
$$+ {}^6C_3 \times (100)^3 \times 2^3 + {}^6C_4 \times (100)^2 \times 2^4 + {}^6C_5 \times (100)^1 \times 2^5 + {}^6C_6 \times (100)^0 \times 2^6$$

$$= (100)^6 + 6 \times (100)^5 \times 2 + 15 \times (100)^4 \times 2^2 + 20 \times (100)^3 \times 2^3 + 15 \times (100)^2 \times 2^4$$
$$+ 6 \times (100)^1 \times 2^5 + 2^6$$

$$= 10^{12} + 12 \times 10^{10} + 6 \times 10^9 + 16 \times 10^7 + 24 \times 10^5 + 192 \times 10^2 + 64$$

$$= 1126162419264.$$

(iii) We have,

$$(10.1)^5 = (10+0.1)^5$$

$$= {}^5C_0 \times (10)^5 \times (0.1)^0 + {}^5C_1 \times (10)^4 \times (0.1) + {}^5C_2 \times (10)^3 \times (0.1)^2 + {}^5C_3 \times (10)^2 \times (0.1)^3$$
$$+ {}^5C_4 \times (10)^1 \times (0.1)^4 + {}^5C_5 \times (10)^0 \times (0.1)^5$$

$$= (10)^5 + 5 \times 10^4 \times 0.1 + 10 \times 10^3 \times (0.1)^2 + 10 \times (10)^2 \times (0.1)^3 + 5 \times 10 \times (0.1)^4 + (0.1)^5$$

$$= 10^5 + 5 \times 10^3 + 10^2 + 1 + 5 \times 0.001 + 0.00001$$

$$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001 = 105101.00501.$$

EXAMPLE 11 *Write down the binomial expansion of $(1+x)^{n+1}$, when $x=8$. Deduce that $9^{n+1}-8n-9$ is divisible by 64, where n is a positive integer.* **[NCERT]**

SOLUTION We have,

$$(1+x)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\, x + {}^{n+1}C_2\, x^2 + {}^{n+1}C_3\, x^3 + \ldots + {}^{n+1}C_{n+1}\, x^{n+1}$$

Putting $x=8$, we get

$$(1+8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\,(8)^1 + {}^{n+1}C_2(8)^2 + {}^{n+1}C_3(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1} \quad \ldots\text{(i)}$$

$$\Rightarrow \quad 9^{n+1} = 1 + (n+1)\times 8 + {}^{n+1}C_2\,(8)^2 + {}^{n+1}C_3\,(8)^3 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n+1}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = (8)^2\left\{{}^{n+1}C_2 + {}^{n+1}C_3\,(8) + {}^{n+1}C_4\,(8)^2 + \ldots + {}^{n+1}C_{n+1}\,(8)^{n-1}\right\}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 = 64 \times \text{an integer}$$

$$\Rightarrow \quad 9^{n+1} - 8n - 9 \text{ is divisible by 64.}$$

EXAMPLE 12 *Using binomial theorem, prove that $6^n - 5n$ always leaves the remainder 1 when divided by 25.* **[NCERT]**

SOLUTION We have,

$$6^n - 5n = (1+5)^n - 5n$$

$$\Rightarrow \quad 6^n - 5n = \left\{{}^nC_0 + {}^nC_1\times(5) + {}^nC_2\times(5)^2 + {}^nC_3\times(5)^3 + \ldots + {}^nC_n\times(5)^n\right\} - 5n$$

$$\Rightarrow \quad 6^n - 5n = 1 + 5n + {}^nC_2\times 5^2 + {}^nC_3\times 5^3 + \ldots + {}^nC_n\times 5^n - 5n$$

$$\Rightarrow \quad 6^n - 5n - 1 = {}^nC_2\times 5^2 + {}^nC_3\times 5^3 + \ldots + {}^nC_n\times 5^n$$

$$\Rightarrow \quad 6^n - 5n - 1 = 5^2\left\{{}^nC_2 + {}^nC_3\times 5 + {}^nC_4\times 5^2 + \ldots + {}^nC_n\times 5^{n-2}\right\}$$

$$\Rightarrow \quad 6^n - 5n - 1 = 25\times \text{an integer}$$

$$\Rightarrow \quad 6^n - 5n = 25\times \text{an integer} + 1$$

$$\Rightarrow \quad 6^n - 5n \text{ leaves the remainder 1 when divided by 25.}$$

LEVEL-2

Type IV ON EXPANSION OF A BINOMIAL BY USING BINOMIAL THEOREM

EXAMPLE 13 *Using binomial theorem, expand $\left\{(x+y)^5 + (x-y)^5\right\}$ and hence find the value of $\left\{(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5\right\}$.*

SOLUTION We have,

$$(x+y)^5 + (x-y)^5 = 2\left\{{}^5C_0\, x^5 + {}^5C_2\, x^3 y^2 + {}^5C_4\, x^1 y^4\right\} = 2\left(x^5 + 10x^3y^2 + 5xy^4\right)$$

Putting $x=\sqrt{2}$ and $y=1$, we get

$$(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5 = 2\left\{(\sqrt{2})^5 + 10\,(\sqrt{2})^3 + 5\sqrt{2}\right\} = 2\left(4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}\right) = 58\sqrt{2}$$

EXAMPLE 14 *If O be the sum of odd terms and E that of even terms in the expansion of $(x+a)^n$, prove that:*

(i) $O^2 - E^2 = (x^2 - a^2)^n$ (ii) $4\,OE = (x+a)^{2n} - (x-a)^{2n}$

(iii) $2\,(O^2 + E^2) = (x+a)^{2n} + (x-a)^{2n}$

SOLUTION We have,

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \ldots + {}^nC_{n-1}\, x\, a^{n-1} + {}^nC_n\, a^n$$

$$\Rightarrow \quad (x+a)^n = \left\{{}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \ldots\right\} + \left\{{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\right\}$$

$$\Rightarrow \quad (x+a)^n = O + E \qquad \ldots\text{(i)}$$

and, $$(x-a)^n = {}^nC_0\, x^n - {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \ldots + {}^nC_{n-1}\, x\,(-1)^{n-1} a^{n-1} + {}^nC_n\,(-1)^n a^n$$

$$\Rightarrow \quad (x-a)^n = \left\{{}^nC_0\, x^n + {}^nC_2\, x^{n-2} a^2 + \ldots\right\} - \left\{{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \ldots\right\}$$

$$\Rightarrow \quad (x-a)^n = O - E \qquad \ldots\text{(ii)}$$

(i) Multiplying (i) and (ii), we get

$$(x+a)^n (x-a)^n = (O+E)(O-E)$$

$$\Rightarrow \quad (x^2-a^2)^n = O^2 - E^2$$

(ii) We have,

$$4\,OE = (O+E)^2 - (O-E)^2$$

$$\Rightarrow \quad 4\,OE = \left\{(x+a)^n\right\}^2 - \left\{(x-a)^n\right\}^2 \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad 4\,OE = (x+a)^{2n} - (x-a)^{2n}$$

(iii) Squaring (i) and (ii) and then adding, we get

$$(x+a)^{2n} + (x-a)^{2n} = (O+E)^2 + (O-E)^2 = 2(O^2+E^2).$$

Type V ON APPLICATIONS OF BINOMIAL THEOREM

EXAMPLE 15 *Which is larger $(1.01)^{1000000}$ or, 10,000?* **[NCERT]**

SOLUTION We have,

$$(1.01)^{1000000} - 10000$$

$$= (1+0.01)^{1000000} - 10000$$

$$= {}^{1000000}C_0 + {}^{1000000}C_1\,(0.01) + {}^{1000000}C_2\,(0.01)^2 + \ldots + {}^{1000000}C_{1000000} \times (0.01)^{1000000} - 10000$$

$$= (1 + 1000000 \times 0.01 + \text{other positive terms}) - 10000$$

$$= (1 + 10000 + \text{other positive terms}) - 10000$$

$$= 1 + \text{other positive terms} > 0$$

$$\therefore \quad (1.01)^{1000000} > 10000$$

EXAMPLE 16 *If a and b are distinct integers, prove that $a^n - b^n$ is divisible by $(a-b)$, whenever $n \in N$.* **[NCERT]**

SOLUTION We have,

$$a^n = \{(a-b)+b\}^n$$

$$\Rightarrow a^n = {}^nC_0\,(a-b)^n + {}^nC_1\,(a-b)^{n-1} b^1 + {}^nC_2\,(a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1}\,(a-b)\, b^{n-1} + {}^nC_n\, b^n$$

$$\Rightarrow a^n - b^n = (a-b)^n + {}^nC_1\,(a-b)^{n-1} b^1 + {}^nC_2\,(a-b)^{n-2} b^2 + \ldots + {}^nC_{n-1}\,(a-b)\, b^{n-1}$$

$$\Rightarrow a^n - b^n = (a-b)\left\{(a-b)^{n-1} + {}^nC_1 (a-b)^{n-2} b + {}^nC_2 (a-b)^{n-3} b^2 + \ldots + {}^nC_{n-1} b^{n-1}\right\}$$

Clearly, RHS is divisible by $(a-b)$. Hence, $a^n - b^n$ is divisible by $(a-b)$.

EXAMPLE 17 *Using binomial theorem, prove that* $(101)^{50} > 100^{50} + 99^{50}$. **[NCERT EXEMPLAR]**

SOLUTION Let $x = 101^{50}$ and $y = 100^{50} + 99^{50}$. Then,

$$x - y = 101^{50} - 100^{50} - 99^{50}$$

$$\Rightarrow \quad x - y = 101^{50} - 99^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = (100+1)^{50} - (100-1)^{50} - 100^{50}$$

$$\Rightarrow \quad x - y = 2\left\{{}^{50}C_1 \times 100^{49} + {}^{50}C_3 \times 100^{47} + \ldots + {}^{50}C_{49} \times 100\right\} - 100^{50}$$

$$\Rightarrow \quad x - y = 100^{50} + 2 \times {}^{50}C_3 \times 100^{47} + \ldots + 2 \times {}^{50}C_{49} \times 100 - 100^{50}$$

$$\Rightarrow \quad x - y = 2 \times {}^{50}C_3 \times 100^{47} + \ldots + 2 \times {}^{50}C_{49} \times 100$$

$$\Rightarrow \quad x - y = a \text{ positive integer}$$

$$\Rightarrow \quad x - y > 0 \Rightarrow x > y \Rightarrow 101^{50} > 100^{50} + 99^{50}$$

EXERCISE 18.1

LEVEL-1

1. Using binomial theorem, write down the expansions of the following:

(i) $(2x+3y)^5$ (ii) $(2x-3y)^4$ (iii) $\left(x - \frac{1}{x}\right)^6$

(iv) $(1-3x)^7$ (v) $\left(ax - \frac{b}{x}\right)^6$ (vi) $\left(\sqrt{\frac{x}{a}} - \sqrt{\frac{a}{x}}\right)^6$

(vii) $(\sqrt[3]{x} - \sqrt[3]{a})^6$ (viii) $(1+2x-3x^2)^5$ (ix) $\left(x + 1 - \frac{1}{x}\right)^3$

(x) $(1-2x+3x^2)^3$

2. Evaluate the following:

(i) $\left(\sqrt{x+1} + \sqrt{x-1}\right)^6 + \left(\sqrt{x+1} - \sqrt{x-1}\right)^6$ (ii) $\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6$

(iii) $(1+2\sqrt{x})^5 + (1-2\sqrt{x})^5$. (iv) $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

(v) $(3+\sqrt{2})^5 - (3-\sqrt{2})^5$ (vi) $(2+\sqrt{3})^7 + (2-\sqrt{3})^7$

(vii) $(\sqrt{3}+1)^5 - (\sqrt{3}-1)^5$ (viii) $(0.99)^5 + (1.01)^5$

(ix) $(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$ **[NCERT]**

(x) $\left\{a^2 + \sqrt{a^2-1}\right\}^4 + \left\{a^2 - \sqrt{a^2-1}\right\}^4$ **[NCERT, NCERT EXEMPLAR]**

3. Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4$. **[NCERT]**

4. Find $(x+1)^6 + (x-1)^6$. Hence, or otherwise evaluate $(\sqrt{2}+1)^6 + \sqrt{2}-1)^6$. **[NCERT]**

5. Using binomial theorem evaluate each of the following:
(i) $(96)^3$ [NCERT] (ii) $(102)^5$ [NCERT] (iii) $(101)^4$ [NCERT] (iv) $(98)^5$ [NCERT]
6. Using binomial theorem, prove that $2^{3n} - 7n - 1$ is divisible by 49, where $n \in N$.
7. Using binomial theorem, prove that $3^{2n+2} - 8n - 9$ is divisible by 64, $n \in N$.
8. If n is a positive integer, prove that $3^{3n} - 26n - 1$ is divisible by 676.

LEVEL-2

9. Using binomial theorem, indicate which is larger $(1.1)^{10000}$ or 1000? [NCERT]
10. Using binomial theorem determine which number is larger $(1.2)^{4000}$ or 800?
11. Find the value of $(1.01)^{10} + (1 - 0.01)^{10}$ correct to 7 places of decimal.
12. Show that $2^{4n+4} - 15n - 16$, where $n \in N$ is divisible by 225. [NCERT EXEMPLAR]

ANSWERS

1. (i) $32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5$
(ii) $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$
(iii) $x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$
(iv) $1 - 21x + 189x^2 - 945x^3 + 2835x^4 - 5103x^5 + 5103x^6 - 2187x^7$
(v) $a^6x^6 - 6a^5x^4b + 15a^4x^2b^2 - 20a^3b^3 + 15\frac{a^2b^4}{x^2} - \frac{6ab^5}{x^4} + \frac{b^6}{x^6}$
(vi) $\frac{x^3}{a^3} - 6\frac{x^2}{a^2} + 15\frac{x}{a} - 20 + 15\frac{a}{x} - 6\frac{a^2}{x^2} + \frac{a^3}{x^3}$
(vii) $x^2 - 6x^{5/3}a^{1/3} + 15x^{4/3}a^{2/3} - 20ax + 15x^{2/3}a^{4/3} - 6x^{1/3}a^{5/3} + a^2$
(viii) $1 + 10x + 25x^2 - 40x^3 - 190x^4 + 92x^5 + 570x^6 - 360x^7 - 675x^8 + 810x^9 - 243x^{10}$
(ix) $x^3 + 3x^2 - 5 + \frac{3}{x^2} - \frac{1}{x^3}$
(x) $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$
2. (i) $16x(4x^2 - 3)$ (ii) $64x^6 - 96x^4 + 36x^2 - 2$ (iii) $2(1 + 40x + 80x^2)$
(iv) 198 (v) $1178\sqrt{2}$ (vi) 10084 (vii) 152
(viii) 2.0020001 (ix) $396\sqrt{6}$ (x) $2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$
3. $8(a^3b + ab^3)$, $40\sqrt{6}$ 4. $2(x^6 + 15x^4 + 15x^2 + 1)$, 198
5. (i) 884736 (ii) 11040808032 (iii) 104060401 (iv) 9039207968
9. $(1.1)^{10000} > 1000$ 10. 800 11. 2.0090042

HINTS TO NCERT & SELECTED PROBLEMS

2. (ix) We know that $(x + a)^n - (x - a)^n = 2\left\{{}^nC_1 x^{n-1}a^1 + {}^nC_3 x^{n-3}a^3 + ...\right\}$

$$\therefore\ (\sqrt{3}+\sqrt{2})^6-(\sqrt{3}-\sqrt{2})^6 = 2\left\{{}^6C_1(\sqrt{3})^5(\sqrt{2})^1+{}^6C_3(\sqrt{3})^3(\sqrt{2})^3+{}^6C_5(\sqrt{3})^1(\sqrt{2})^5\right\}$$

$$= 2(6\times 9\times\sqrt{6}+20\times 6\times\sqrt{6}+6\times 4\times\sqrt{6})$$

$$= 2\left(54\sqrt{6}+120\sqrt{6}+24\sqrt{6}\right) = 2\times 198\sqrt{6} = 396\sqrt{6}$$

(x) Using $(x+a)^n+(x-a)^n = 2\left\{{}^nC_0x^na^0+{}^nC_2x^{n-2}a^2+\ldots\right\}$, we get

$$\left\{a^2+\sqrt{a^2-1}\right\}^4+\left\{a^2-\sqrt{a^2-1}\right\}^4$$

$$= 2\left\{{}^4C_0(a^2)^0\left(\sqrt{a^2-1}\right)^4+{}^4C_2(a^2)^2\left(\sqrt{a^2-1}\right)^2+{}^4C_4(a^2)^4\left(\sqrt{a^2-1}\right)^0\right\}$$

$$= 2\left\{(a^2-1)^2+6a^4(a^2-1)+a^8\right\}$$

$$= 2(a^8+6a^6-5a^4-2a^2+1) = 2a^8+12a^6-10a^4-4a^2+2$$

3. Using $(x+a)^n-(x-a)^n = 2\left\{{}^nC_1x^{n-1}a^1+{}^nC_3x^{x-3}a^3+\ldots\right\}$, we get

$$(a+b)^4-(a-b)^4 = 2\left\{{}^4C_1a^3b^1+{}^4C_3a^1b^2\right\} = 2(4a^3b+4ab^3) = 8ab(a^2+b^2)$$

$$\therefore\quad (\sqrt{3}+\sqrt{2})^4-(\sqrt{3}-\sqrt{2})^4 = 8\sqrt{3}\times\sqrt{2}\left\{(\sqrt{3})^2+(\sqrt{2})^2\right\} = 40\sqrt{6}$$

4. Using $(x+a)^n+(x-a)^n = 2\left({}^nC_0x^n a^0+{}^nC_2x^{x-2}a^2+\ldots\right)$, we get

$$(x+1)^6+(x-1)^6 = 2\left({}^6C_0x^6+{}^6C_2x^4+{}^6C_4x^2+{}^6C_6x^0\right) = 2(x^6+15x^4+15x^2+1)$$

Putting $x=\sqrt{2}$, we get

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6 = 2\left\{(\sqrt{2})^6+15(\sqrt{2})^4+15(\sqrt{2})^2+1\right\} = 2(8+60+30+1) = 198$$

5. (i) $96^3 = (100-4)^3$

$$= {}^3C_0(100)^3(4)^0-{}^3C_1(100)^2(4)^1+{}^3C_2(100)^1(4)^2-{}^3C_3(100)^0(4)^3$$

$$= 10^6-12\times 10^4+4800-64 = 1000000-120000+4800-64 = 884736$$

(ii) $(102)^5 = (100+2)^5$

$$= {}^5C_0(100)^5 2^0+{}^5C_1(100)^4\times 2+{}^5C_2\times(100)^3\times 2^2+{}^5C_3\times(100)^2\times 2^3$$

$$+{}^5C_4\times(100)^1\times 2^4+{}^5C_5\times(100)^0\times 2^5$$

$$= 10^{10}+10^9+40\times 10^6+80\times 10^4+80\times 10^2+32 = 11040808032$$

(iii) $(101)^4 = (10^2+1)^4$

$$= {}^4C_0(10^2)^0+{}^4C_1(10^2)^1+{}^4C_2(10^2)^2+{}^4C_3(10^2)^3+{}^4C_4(10^2)^4$$

$$= 1+400+6\times 10^4+4\times 10^6+10^8 = 104060401$$

(ix) $(98)^5 = (100-2)^5$

$$= {}^5C_0\,(100)^5 - {}^5C_1(100)^4 \times 2 + {}^5C_2 \times (100)^3 \times 2 - {}^5C_3 \times (100)^2 \times 2^3 + {}^5C_4 \times (100)^1 \times 2^4 - {}^5C_5 \times (100)^0 \times 2^5$$

$$= 10^{10} - 10^9 + 40 \times 10^6 + 8000 - 32 = 1039207968$$

9. Using $(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \ldots + {}^nC_n\, x^0\, a^n$, we get

$$(1.1)^{10000} = \left(1 + \frac{1}{10}\right)^{10000}$$

$$= {}^{10000}C_0 + {}^{10000}C_1 \times \frac{1}{10} + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$= 1 + 1000 + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\therefore \quad (1.1)^{10000} - 1000 = 1 + {}^{10000}C_2 \times \left(\frac{1}{10}\right)^2 + \ldots + {}^{10000}C_{10000}\left(\frac{1}{10}\right)^{10000}$$

$$\Rightarrow \quad (1.1)^{10000} - 1000 > 0$$

$$\Rightarrow \quad (1.1)^{10000} > 1000$$

12. $2^{4n+4} - 15n - 16 = 2^{4(n+1)} - 15n - 15 - 1$

$$= (2^4)^{n+1} - 15\,(n+1) - 1$$

$$= 16^{n+1} - 15\,(n+1) - 1$$

$$= (1+15)^{n+1} - 15\,(n+1) - 1$$

$$= \left\{ {}^{n+1}C_0 + {}^{n+1}C_1(15) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3\,(15)^3 + \ldots\ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} \right\} - 15\,(n+1) - 1$$

$$= \left\{ 1 + 15\,(n+1) + {}^{n+1}C_2\,(15)^2 + {}^{n+1}C_3\,(15)^3 + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} \right\} - 15\,(n+1) - 1$$

$$= 225 \left\{ {}^{n+1}C_2 + {}^{n+1}C_3\,(15) + \ldots + {}^{n+1}C_{n+1}\,(15)^{n-1} \right\}$$

$$= 225 \times \text{A natural number.}$$

Hence, $2^{4n+4} - 15n - 16$ is divisible by 225.

18.4 GENERAL TERM AND MIDDLE TERMS IN A BINOMIAL EXPANSION

We have,

$$(x+a)^n = {}^nC_0\, x^n\, a^0 + {}^nC_1\, x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots + {}^nC_n\, x^0\, a^n$$

We find that: The first term $= {}^nC_0\, x^n\, a^0$

The second term $= {}^nC_1\, x^{n-1}\, a^1$

The third term $= {}^nC_2\, x^{n-2}\, a^2$

The fourth term $= {}^nC_3\, x^{n-3}\, a^3$, and so on.

We thus observe that the suffix of C in any term is one less than the number of terms, the index of x is n minus the suffix of C and the index of a is the same as the suffix of C.

Hence, the $(r+1)$th term is given by ${}^nC_r\, x^{n-r}\, a^r$. Thus, if T_{r+1} denotes the $(r+1)^{th}$ term, then

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

This is called the *general term*, because by giving different values to r we can determine all terms of the expansion.

Since, $(x-a)^n - \{x+(-a)\}^n$. So, the general term in the binomial expansion of $(x-a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}(-a)^r = (-1)^r\, {}^nC_r\, x^{n-r} a^r$$

In the binomial expansion of $(1+x)^n$, the general term is given by

$$T_{r+1} = {}^nC_r\, x^r$$

In the binomial expansion of $(1-x)^n$, the general term is given by

$$T_{r+1} = (-1)^r\, {}^nC_r\, x^r$$

NOTE: *In the binomial expansion of $(x+a)^n$, the r^{th} term from the end is $((n+1)-r+1) = (n-r+2)$th term from the beginning.*

18.4.1 MIDDLE TERMS IN A BINOMIAL EXPANSION

The binomial expansion of $(x+a)^n$ contains $(n+1)$ terms. Therefore,

(i) If n is even, then $\left(\frac{n}{2}+1\right)^{th}$ term is the middle term.

(ii) If n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are the two middle terms.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I ON FINDING THE GENERAL TERM OR AN INDICATED TERM IN THE BINOMIAL EXPANSION OF SOME GIVEN EXPRESSION

EXAMPLE 1 *Write the general term in the expansion of $(x^2-y)^6$.* [NCERT]

SOLUTION We have, $(x^2-y)^6 = \left\{x^2+(-y)\right\}^6$

The general term in the expansion of the above binomial is given by

$$T_{r+1} = {}^6C_r\, (x^2)^{6-r}\, (-y)^r \qquad [\because T_{r+1} = {}^nC_r\, x^{n-r}\, a^r]$$

$$\Rightarrow \quad T_{r+1} = (-1)^r\, {}^6C_r\, x^{12-2r}\, y^r$$

EXAMPLE 2 *Find the 10th term in the binomial expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$.*

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$, the tenth term T_{10} is given by

$$T_{10} = T_{9+1} = {}^{12}C_9\,(2x^2)^{12-9}\left(\frac{1}{x}\right)^9 \qquad \left[\text{Here: } n=12, r=9, x=2x^2 \text{ and } a=\frac{1}{x}\right]$$

$$\Rightarrow T_{10} = {}^{12}C_9\,(2x^2)^3\times\frac{1}{x^9} = {}^{12}C_9\times 2^3\left(\frac{1}{x^3}\right)$$

$$\Rightarrow T_{10} = {}^{12}C_3\,\frac{8}{x^3} = \frac{12\times 11\times 10}{3\times 2\times 1}\times\frac{8}{x^3} = \frac{1760}{x^3} \qquad \left[\because {}^{12}C_9 = {}^{12}C_3\right]$$

EXAMPLE 3 *Find the 9th term in the expansion of* $\left(\frac{x}{a}-\frac{3a}{x^2}\right)^{12}$.

SOLUTION We know that the $(r+1)$th term in the expansion of $(x+a)^n$ is given by

$$T_{r+1} = {}^nC_r\, x^{n-r}\, a^r$$

Therefore, in the expansion of $\left(\frac{x}{a}-\frac{3a}{x^2}\right)^{12}$, the 9th term T_9 is given by

$$\Rightarrow T_9 = T_{8+1} = {}^{12}C_8\left(\frac{x}{a}\right)^{12-8}\left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_8\left(\frac{x}{a}\right)^4\left(-\frac{3a}{x^2}\right)^8 = {}^{12}C_4\times 3^8\times\frac{a^4}{x^{12}}$$

$$\Rightarrow T_9 = ({}^{12}C_4\, x^{-12}\, a^4)\, 3^8$$

EXAMPLE 4 *Find the 6th term in the expansion of* $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.

SOLUTION Clearly, $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9 = \left\{\frac{4x}{5}+\left(-\frac{5}{2x}\right)\right\}^9$

$$\therefore T_6 = T_{5+1} = {}^9C_5\left(\frac{4x}{5}\right)^{9-5}\left(-\frac{5}{2x}\right)^5 \qquad \left[\because T_{r+1} = {}^nC_r\, x^{n-r}\, a^r\right]$$

$$\Rightarrow T_6 = {}^9C_5\left(\frac{4x}{5}\right)^4(-1)^5\left(\frac{5}{2x}\right)^5 = -\,{}^9C_4\left(\frac{4x}{5}\right)^4\left(\frac{5}{2x}\right)^5 \qquad \left[\because {}^9C_5 = {}^9C_4\right]$$

$$\Rightarrow T_6 = -\frac{9\times 8\times 7\times 6}{4\times 3\times 2\times 1}\left(\frac{2^8 x^4}{5^4}\right)\left(\frac{5^5}{2^5 x^5}\right) = -\frac{5040}{x}$$

EXAMPLE 5 *Find 13th term in the expansion of* $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$, $x\neq 0$. **[NCERT]**

SOLUTION Clearly,

$$\left(9x-\frac{1}{3\sqrt{x}}\right)^{18} = \left\{9x+\left(\frac{-1}{3\sqrt{x}}\right)\right\}^{18}$$

$$\therefore \quad T_{13} = T_{12+1} = {}^{18}C_{12}\,(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} = {}^{18}C_{12}\,(9x)^6\left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$\Rightarrow \quad T_{13} = {}^{18}C_6 \times 9^6 \times x^6 \times \frac{1}{3^{12}\,x^6} = {}^{18}C_6 = \frac{18!}{12!\,6!} = 18564$$

EXAMPLE 6 *Find the 4th term from the end in the expansion of* $\left(\frac{3}{x^2} - \frac{x^3}{6}\right)^7$.

SOLUTION Clearly, the given expansion contains 8 terms.
So, 4th term from the end $= (8-4+1)$th $= 5$th term from the beginning

$$\therefore \quad \text{Required term} = T_5 = T_{4+1} = {}^7C_4\left(\frac{3}{x^2}\right)^{7-4}\left(-\frac{x^3}{6}\right)^4$$

$$= {}^7C_3\left(\frac{3}{x^2}\right)^3\left(\frac{x^3}{6}\right)^4 = \frac{7\times 6\times 5}{3\times 2\times 1}\left(\frac{3^3}{x^6}\right)\left(\frac{x^{12}}{6^4}\right) = \frac{35}{48}x^6 \quad \left[\because\ {}^7C_4 = {}^7C_3\right]$$

EXAMPLE 7 *Find the 11th term from the end in the expansion of* $\left(2x - \frac{1}{x^2}\right)^{25}$.

SOLUTION Clearly, the given expansion contains 26 terms.
So, 11th term from the end $= (26 - 11 + 1)$th term from the beginning i.e. 16th term from the beginning

$$\therefore \quad \text{Required term} = T_{16} = T_{15+1} = {}^{25}C_{15}\,(2x)^{25-15}\left(-\frac{1}{x^2}\right)^{15}$$

$$= {}^{25}C_{15} \times 2^{10} \times x^{10} \times \frac{(-1)^{15}}{x^{30}} = -\,{}^{25}C_{15} \times \frac{2^{10}}{x^{20}}$$

EXAMPLE 8 *Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of* $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ *is* $\sqrt{6}:1$. **[NCERT]**

SOLUTION Clearly,

Fifth term from the end $= (n+1-5+1)^{th}$ term from the beginning
$= (n-3)^{th}$ term from the beginning

Now, $T_5 = T_{4+1} = {}^nC_4\left\{\sqrt[4]{2}\right\}^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^nC_4 \times 2^{\frac{n-4}{4}} \times \frac{1}{3}$

and, $T_{n-3} = T_{(n-4)+1} = {}^nC_{n-4}\left\{\sqrt[4]{2}\right\}^{n-(n-4)}\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^nC_{n-4} \times 2 \times \frac{1}{3^{\frac{n-4}{4}}}$

It is given that

$$\frac{T_5}{T_{n-3}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad \frac{{}^nC_4 \times 2^{\frac{n-4}{4}} \times \frac{1}{3}}{{}^nC_{n-4} \times 2 \times 3^{-\frac{n-4}{4}}} = \frac{\sqrt{6}}{1}$$

$$\Rightarrow \quad 2^{\frac{n-4}{4}-1} \times 3^{\frac{n-4}{4}-1} = 6^{1/2} \qquad \left[\because {}^nC_4 = {}^nC_{n-4}\right]$$

$$\Rightarrow \quad 2^{\frac{n-8}{4}} \times 3^{\frac{n-8}{4}} = 6^{1/2}$$

$$\Rightarrow \quad (2 \times 3)^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow 6^{\frac{n-8}{4}} = 6^{1/2} \Rightarrow \frac{n-8}{4} = \frac{1}{2} \Rightarrow n-8 = 2 \Rightarrow n = 10$$

EXAMPLE 9 *Find a, if 17th and 18th terms in the expansion of $(2+a)^{50}$ are equal.* **[NCERT]**

SOLUTION We have,

$$T_{17} = T_{16+1} = {}^{50}C_{16}\,(2)^{50-16}\,a^{16} = {}^{50}C_{16} \times 2^{34} \times a^{16}$$

and, $$T_{18} = T_{17+1} = {}^{50}C_{17}\,(2)^{50-17}\,a^{17} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

It is given that 17^{th} and 18^{th} terms are equal.

i.e. $T_{17} = T_{18}$

$$\Rightarrow \quad {}^{50}C_{16} \times 2^{34} \times a^{16} = {}^{50}C_{17} \times 2^{33} \times a^{17}$$

$$\Rightarrow \quad \frac{{}^{50}C_{16}}{{}^{50}C_{17}} \times 2 = \frac{a^{17}}{a^{16}} \Rightarrow a = \frac{50!}{34!\,16!} \times \frac{33!\,17!}{50!} \times 2 = \frac{17}{34} \times 2 = 1$$

Type II ON FINDING THE MIDDLE TERM(S)

EXAMPLE 10 *Find the middle term in the expansion of* $\left(\frac{2}{3}x^2 - \frac{3}{2x}\right)^{20}$.

SOLUTION Here $n = 20$, which is an even number. So, $\left(\frac{20}{2}+1\right)^{th}$ term i.e. 11th term is the middle term.

Hence, the middle term $= T_{11} = T_{10+1} = {}^{20}C_{10}\left(\frac{2}{3}x^2\right)^{20-10}\left(-\frac{3}{2x}\right)^{10} = {}^{20}C_{10}\,x^{10}$

EXAMPLE 11 *Find the middle terms in the expansion of* $\left(3x - \frac{x^3}{6}\right)^7$.

SOLUTION The given expression is $\left(3x - \frac{x^3}{6}\right)^7$. Here $n = 7$, which is an odd number.

So, $\left(\frac{7+1}{2}\right)^{th}$ and $\left(\frac{7+1}{2}+1\right)^{th}$ i.e. 4th and 5th terms are two middle terms.

Now, $T_4 = T_{3+1} = {}^7C_3\,(3x)^{7-3}\left(-\frac{x^3}{6}\right)^3 = (-1)^3\;{}^7C_3\,(3x)^4\left(\frac{x^3}{6}\right)^3 = -\frac{105x^{13}}{8}$

and, $T_5 = T_{4+1} = {}^7C_4\,(3x)^{7-4}\left(-\dfrac{x^3}{6}\right)^4 = {}^7C_4\,(3x)^3\left(-\dfrac{x^3}{6}\right)^4 = \dfrac{35\,x^{15}}{48}$

Hence, the middle terms are $-\dfrac{105\,x^{13}}{8}$ and $\dfrac{35\,x^{15}}{48}$.

EXAMPLE 12 *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is*

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,2^n\cdot x^n$$ **[NCERT]**

SOLUTION The exponent of $(1+x)$ in $(1+x)^{2n}$ is an even number $2n$.

So, $\left(\dfrac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term in the binomial expansion of $(1+x)^{2n}$.

Now, $T_{n+1} = {}^{2n}C_n\,(1)^{2n-n}\,x^n = {}^{2n}C_n\,x^n$

$$= \frac{(2n)!}{(2n-n)!\,n!}\,x^n$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots(2n-3)\,(2n-2)\,(2n-1)\,(2n)}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\,(2n-1)\}\,\{2\cdot 4\cdot 6\ldots(2n-2)\,(2n)\}}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\,(2n-1)\}\,\{1\cdot 2\cdot 3\ldots(n-1)\,(n)\}\,2^n}{n!\,n!}\,x^n$$

$$= \frac{\{1\cdot 3\cdot 5\ldots(2n-3)\,(2n-1)\}\,n!\cdot 2^n\cdot x^n}{n!\,n!} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,2^n\,x^n$$

EXAMPLE 13 *Show that the middle term in the expansion of* $\left(x-\dfrac{1}{x}\right)^{2n}$ *is*

$$\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\,(-2)^n$$ **[NCERT EXEMPLAR]**

SOLUTION The exponent in $\left(x-\dfrac{1}{x}\right)^{2n}$ is an even natural number. So, $\left(\dfrac{2n}{2}+1\right)^{th}$ i.e. $(n+1)^{th}$ term is the middle term and is given by

$$T_{n+1} = {}^{2n}C_n\,(x)^{2n-n}\left(-\frac{1}{x}\right)^n$$

$\Rightarrow$ $$T_{n+1} = \frac{(2n)!}{n!\;n!}\,x^n\times\frac{(-1)^n}{x^n}$$

$\Rightarrow$ $$T_{n+1} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\ldots(2n-1)\,(2n)}{n!\,n!}\times(-1)^n$$

$\Rightarrow$ $$T_{n+1} = \frac{\{1\cdot 3\cdot 5\ldots(2n-1)\}\{2\cdot 4\cdot 6\ldots(2n-2)\,(2n)\}}{n!\,n!}\times(-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{\{1\cdot3\cdot5\ldots(2n-1)\}\{1\cdot2\cdot3\ldots(n-1)\,n\}}{n!\,n!} \times (-1)^n$$

$$\Rightarrow \quad T_{n+1} = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!} \times 2^n \times (-1)^n = \frac{1\cdot3\cdot5\ldots(2n-1)}{n!} \times (-2)^n$$

EXAMPLE 14 *Prove that the coefficient of the middle term in the expansion of* $(1+x)^{2n}$ *is equal to the sum of the coefficients of middle terms in the expansion of* $(1+x)^{2n-1}$. **[NCERT]**

SOLUTION As discussed in the previous example, the middle term in the expansion of $(1+x)^{2n}$ is given by $T_{n+1} = {}^{2n}C_n\, x^n$.

So, the coefficient of the middle term in the expansion of $(1+x)^{2n}$ is ${}^{2n}C_n$.

Now, consider the expansion of $(1+x)^{2n-1}$. Here, the index $(2n-1)$ is odd.

So, $\left(\dfrac{(2n-1)+1}{2}\right)^{th}$ and $\left(\dfrac{(2n-1)+1}{2}+1\right)^{th}$ i.e. n^{th} and $(n+1)^{th}$ terms are middle terms.

Now, $T_n = T_{(n-1)+1} = {}^{2n-1}C_{n-1}\,(1)^{(2n-1)-(n-1)}\,x^{n-1} = {}^{2n-1}C_{n-1}\,x^{n-1}$

and, $T_{n+1} = {}^{2n-1}C_n\,(1)^{(2n-1)-n}\,x^n = {}^{2n-1}C_n\,x^n$

So, the coefficients of two middle terms in the expansion of $(1+x)^{2n-1}$ are ${}^{2n-1}C_{n-1}$ and ${}^{2n-1}C_n$.

$\therefore$ Sum of these coefficients $= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$

$= {}^{(2n-1)+1}C_n$ $\quad [\because\ {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]$

$= {}^{2n}C_n$

$=$ Coefficient of middle term in the expansion of $(1+x)^{2n}$

Type III **ON FINDING THE COEFFICIENT FOR A GIVEN INDEX (EXPONENT) OF THE VARIABLE**

EXAMPLE 15 *Find the coefficient of* x^{10} *in the binomial expansion of* $\left(2x^2 - \dfrac{3}{x}\right)^{11}$, *when* $x \neq 0$.

SOLUTION Suppose $(r+1)$th term contains x^{10} in the binomial expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{11}$.

Now, $T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\dfrac{3}{x}\right)^r = (-1)^r\,{}^{11}C_r\,2^{(11-r)}\,.\,3^r\,.\,x^{22-3r}$ ...(i)

If T_{r+1} contains x^{10}, then

$22 - 3r = 10 \Rightarrow r = 4.$

So, $(4+1)$th i.e. 5th term contains x^{10}.

Putting $r = 4$ in (i), we get

$T_5 = (-1)^4\ {}^{11}C_4\,2^{11-4} \times 3^4 \times x^{10} = {}^{11}C_4 \times 2^7 \times 3^4 \times x^{10}$

$\therefore$ Coefficient of $x^{10} = {}^{11}C_4 \times 2^7 \times 3^4$

EXAMPLE 16 *Find the coefficients of x^{32} and x^{-17} in the expansion of* $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

SOLUTION Suppose $(r+1)$th term involves x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

Now, $T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(-\frac{1}{x^3}\right)^r = (-1)^r\ {}^{15}C_r\,x^{60-7r}$ ...(i)

For this term to contain x^{32}, we must have

$60 - 7r = 32 \Rightarrow r = 4.$

So, $(4+1)$th i.e. 5th term contains x^{32}.

Putting $r = 4$ in (i), we get

$T_5 = (-1)^4\ {}^{15}C_4\,x^{(60-28)} = {}^{15}C_4\,x^{32}.$

$\therefore$ Coefficient of $x^{32} = {}^{15}C_4 = 1365.$

Suppose $(s+1)$th term in the binomial expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ contains x^{-17}.

Now, $T_{s+1} = {}^{15}C_s\,(x^4)^{15-s}\left(-\frac{1}{x^3}\right)^s = (-1)^s\ {}^{15}C_s\,x^{60-7s}$ ...(ii)

If this term contains x^{-17}, we must have

$60 - 7s = -17 \Rightarrow s = 11$

So, $(11+1)$th i.e. 12th term contains x^{-17}.

Putting $s = 11$ in (ii), we get

$T_{12} = (-1)^{11}\ {}^{15}C_{11}\,x^{-17} = -\ {}^{15}C_{11}\,x^{-17} = -\ {}^{15}C_4\,x^{-17}$ $[\because {}^nC_r = {}^nC_{n-r}]$

$\therefore$ Coefficient of $x^{-17} = -\ {}^{15}C_4 = -1365.$

EXAMPLE 17 *Find the coefficient of $x^6\,y^3$ in the expansion of $(x+2y)^9$.* **[NCERT]**

SOLUTION Suppose $x^6\,y^3$ occurs in $(r+1)^{th}$ term of the expansion of $(x+2y)^9$.

Now,

$T_{r+1} = {}^9C_r \times (x)^{9-r} \times (2y)^r = {}^9C_r \times 2^r \times x^{9-r} \times y^r$

This will contain $x^6\,y^3$, if

$9 - r = 6$ and $r = 3 \Rightarrow r = 3$

$\therefore$ Coefficient of $x^6\,y^3 = {}^9C_3 \times 2^3 = \frac{9!}{3!\,6!} \times 2^3 = \frac{9 \times 8 \times 7 \times 6!}{3! \times 6!} \times 8 = 672$

EXAMPLE 18 *Find the coefficient of x^{40} in the expansion of $(1+2x+x^2)^{27}$.*

SOLUTION We have,

$(1+2x+x^2)^{27} = \left\{(1+x)^2\right\}^{27} = (1+x)^{54}$

Suppose x^{40} occurs in $(r+1)^{th}$ term in the expansion of $(1+x)^{54}$.

Now, $T_{r+1} = {}^{54}C_r\,x^r$

For this term to contain x^{40}, we must have $r = 40$.

So, coefficient of $x^{40} = {}^{54}C_{40}$.

ALITER We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$.

$\therefore$ Coefficient of x^{40} in $(1+x)^{54}$ is ${}^{54}C_{40}$.

EXAMPLE 19 *Prove that there is no term involving x^6 in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$, where $r \neq 0$.*

SOLUTION Suppose x^6 occurs in $(r+1)^{\text{th}}$ term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$.

Now, $T_{r+1} = {}^{11}C_r\,(2x^2)^{11-r}\left(-\frac{3}{x}\right)^r = {}^{11}C_r\,(-1)^r\,2^{11-r}\,3^r\,x^{22-3r}$ ...(i)

For this term to contain x^6, we must have

$$22 - 3r = 6 \Rightarrow r = \frac{16}{3}, \text{ which is a fraction.}$$

But, r is a natural number. Hence, there is no term containing x^6.

EXAMPLE 20 *Find the coefficient of x^5 in the expansion of the product $(1+2x)^6\,(1-x)^7$.* **[NCERT]**

SOLUTION We have,

$$(1+2x)^6(1-x)^7 = \left\{1 + {}^6C_1\,(2x) + {}^6C_2\,(2x)^2 + {}^6C_3\,(2x)^3 + {}^6C_4\,(2x)^4 + {}^6C_5\,(2x)^5 + {}^6C_6\,(2x)^6\right\}$$
$$\times\left\{1 - {}^7C_1\,x + {}^7C_2\,x^2 - {}^7C_3\,x^3 + {}^7C_4\,x^4 - {}^7C_5\,x^5 + \ldots\right\}$$
$$= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + \ldots)$$
$$\times(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + \ldots)$$

$\therefore$ Coefficient of x^5 in the product $= 1\times(-21) + 12\times 35 + 60\times(-35) + 160\times 21 + 240\times -7 + 192\times 1$

$$= -21 + 420 - 2100 + 3360 - 1680 + 192 = 171$$

Type IV ON FINDING THE TERM INDEPENDENT OF THE VARIABLE

EXAMPLE 21 *Find the term independent of x in the expansion of $\left(3x^2 - \frac{1}{2x^3}\right)^{10}$.*

SOLUTION Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r\,(3x^2)^{10-r}\left(-\frac{1}{2x^3}\right)^r = {}^{10}C_r\,3^{10-r}\left(-\frac{1}{2}\right)^r x^{20-5r}$ ...(i)

This term will be independent of x, if

$$20 - 5r = 0 \Rightarrow r = 4$$

So, $(4+1)$th i.e. 5th term is independent of x. Putting $r = 4$ in (i), we get

$$T_5 = {}^{10}C_4\,3^6\left(-\frac{1}{2}\right)^4 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\times\frac{729}{16} = \frac{76545}{8}$$

Hence, required term $= \dfrac{76545}{8}$

EXAMPLE 22 *Find the term independent of x in the expansion of*

(i) $\left(x-\frac{1}{x}\right)^{12}$ (ii) $\left(2x-\frac{1}{x}\right)^{10}$

SOLUTION (i) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{12}C_r\, x^{12-r}\left(-\frac{1}{x}\right)^r = {}^{12}C_r\,(-1)^r\, x^{12-2r}$ ...(i)

For this term to be independent of x, we must have

$12-2r=0 \Rightarrow r=6.$

So, $(6+1)$th i.e. 7th term is independent of x. Putting $r=6$ in (i), we get

$T_7 = {}^{12}C_6\,(-1)^6 = {}^{12}C_6$

Hence, required term $= {}^{12}C_6$

(ii) Let $(r+1)$th term be independent of x in the given expression.

Now, $T_{r+1} = {}^{10}C_r\,(2x)^{10-r}\left(-\frac{1}{x}\right)^r = (-1)^r\, {}^{10}C_r\, 2^{10-r}\, x^{10-2r}$ ...(i)

For this term to be independent of x, we must have

$10-2r=0 \Rightarrow r=5$

So, $(5+1)$th i.e. 6th term is independent of x. Putting $r=5$ in (i), we get

$$T_6 = (-1)^5\, {}^{10}C_5 \,.\, 2^{10-5} = -\,{}^{10}C_5 \times 2^5 = -\frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times1}\times 32 = -8064$$

Hence, required term $= -8064$

EXAMPLE 23 *Find the value of a so that the term independent of x in* $\left(\sqrt{x}+\frac{a}{x^2}\right)^{10}$ *is 405.*

SOLUTION Let $(r+1)^{th}$ term in the expansion of $\left(\sqrt{x}+\frac{a}{x^2}\right)^{10}$ be independent of x.

Now,

$$T_{r+1} = {}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{a}{x^2}\right)^r = {}^{10}C_r\, x^{5-\frac{r}{2}-2r}\, a^r \quad ...(i)$$

This will be independent of x, if

$$5-\frac{r}{2}-2r = 0 \Rightarrow 5-\frac{5r}{2} = 0 \Rightarrow 5 = \frac{5r}{2} \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get: $T_3 = {}^{10}C_2\, a^2$

It is given that the term independent of x is equal to 405.

$\therefore \quad {}^{10}C_2\, a^2 = 405 \Rightarrow 45a^2 = 405 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

Type V PROBLEMS RELATING TO COEFFICIENTS IN A BINOMIAL EXPANSION

In solving the problems relating the coefficients in the binomial expansion we generally use the following results:

(i) Coefficient of $(r+1)$th term in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(ii) Coefficient of x^r in the binomial expansion of $(1+x)^n$ is ${}^nC_r$.

(iii) Coefficient of x^r in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

(iv) Coefficient of $(r+1)$th term in the expansion of $(1-x)^n$ is $(-1)^r\ {}^nC_r$.

EXAMPLE 24 *In the binomial expansion of $(1+a)^{m+n}$, prove that the coefficients of a^m and a^n are equal.* **[NCERT]**

SOLUTION Let A and B be the coefficients of a^m and a^n respectively in the expansion of $(1+a)^{m+n}$. Then,

$A =$ Coefficient of a^m in the binomial expansion of $(1+a)^{m+n}$

$$\Rightarrow \quad A = {}^{m+n}C_m = \frac{(m+n)!}{m!\,n!} \quad \text{...(i)}$$

$B =$ Coefficient of a^n in the binomial expansion of $(1+a)^{m+n}$

$$\Rightarrow \quad B = {}^{m+n}C_n = \frac{(m+n)!}{m!\,n!} \quad \text{...(ii)}$$

Clearly, $A = B$ i.e. the coefficients of a^m and a^n in the binomial expansion of $(1+a)^{m+n}$ are equal.

EXAMPLE 25 *Prove that the coefficients of x^n in $(1+x)^{2n}$ is twice the coefficient of x^n in $(1+x)^{2n-1}$.* **[NCERT]**

SOLUTION Let A and B be the coefficients of x^n in the binomial expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively. Then,

$$A = \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{(2n)!}{n!\,n!} = \frac{(2n)\,(2n-1)!}{n\,(n-1)!\,n!} = 2\,\frac{(2n-1)!}{(n-1)!\,n!} \quad \text{...(i)}$$

and,

$$B = \text{Coefficient of } x^n \text{ in } (1+x)^{2n-1} = {}^{2n-1}C_n = \frac{(2n-1)!}{(n-1)!\,n!} \quad \text{...(ii)}$$

From (i) and (ii), we get

$A = 2B$ i.e. Coefficient of x^n in $(1+x)^{2n} = 2 \times$ Coefficient of x^n in $(1+x)^{2n-1}$.

EXAMPLE 26 *In the binomial expansion of $(a+b)^n$, the coefficients of the fourth and thirteenth terms are equal to each other. Find n.*

SOLUTION The coefficients of the fourth and thirteenth terms in the binomial expansion of $(a+b)^n$ are ${}^nC_3$ and ${}^nC_{12}$ respectively. It is given that:

Coefficient of 4th term in $(a+b)^n$ = Coefficient of 13th term in $(a+b)^n$

$$\Rightarrow \quad {}^nC_3 = {}^nC_{12}$$

$$\Rightarrow \quad n = 15 \qquad [\because\ {}^nC_x = {}^nC_y \Rightarrow x = y, \text{ or } x + y = n]$$

EXAMPLE 27 *Find a positive value of m for which the coefficient of x^2 in the expansion of $(1+x)^m$ is 6.* **[NCERT]**

SOLUTION We know that the coefficient of x^r in $(1+x)^n$ is ${}^nC_r$. Therefore, coefficient of x^2 in $(1+x)^m$ is ${}^mC_2$.

It is given that the coefficient of x^2 in $(1+x)^m$ is 6.

$$\therefore \quad {}^mC_2 = 6$$

$$\Rightarrow \quad \frac{m\,(m-1)}{2!} = 6$$

$$\Rightarrow \quad m^2 - m = 12$$

$\Rightarrow \quad m^2 - m - 12 = 0$

$\Rightarrow \quad (m-4)(m+3) = 0$

$\Rightarrow \quad m - 4 = 0$ $\quad [\because m + 3 \neq 0]$

$\Rightarrow \quad m = 4.$

EXAMPLE 28 *If the coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are equal, find r.* **[NCERT]**

SOLUTION We know that the coefficient of r^{th} term in the expansion of $(1+x)^n$ is ${}^nC_{r-1}$.

Therefore,

Coefficients of $(r-5)^{th}$ and $(2r-1)^{th}$ terms in the expansion of $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$ respectively.

It is given that these coefficients are equal

$\therefore \quad {}^{34}C_{r-6} = {}^{34}C_{2r-2}$

$\Rightarrow \quad r - 6 = 2r - 2$ or, $r - 6 + 2r - 2 = 34$ $\quad \left[\because {}^nC_r = {}^nC_s \Rightarrow r = s \text{ or, } r + s = n\right]$

$\Rightarrow \quad 3r - 8 = 34$ $\quad [\because r - 6 = 2r - 2 \Rightarrow r = -4,$ which is not possible$]$

$\Rightarrow \quad 3r = 42 \Rightarrow r = 14$

Type VI **PROBLEMS BASED ON CONSECUTIVE TERMS OR CONSECUTIVE COEFFICIENTS**

If consecutive terms or coefficients of consecutive terms in the expansion of $(x+a)^n$ are given, we assume that the consecutive terms are r^{th}, $(r+1)^{\text{th}}$ and $(r+2)^{\text{th}}$ i.e. T_r, T_{r+1} and T_{r+2}.

In case of consecutive terms, we find $\dfrac{T_{r+1}}{T_r}$ and $\dfrac{T_r}{T_{r-1}}$.

It should be noted that $\dfrac{T_{r+1}}{T_r} = \dfrac{n-r+1}{r}\,\dfrac{a}{x}$

In case of consecutive coefficients, we find the ratios $\dfrac{r^{\text{th}} \text{ coefficient}}{(r+1)^{\text{th}} \text{ coefficient}}$ and $\dfrac{(r+1)^{th} \text{ coefficient}}{(r+2)^{\text{th}} \text{ coefficient}}$ etc. to get equations and solve them.

In computing these ratios, we may use the following results:

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} \quad \text{and} \quad \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1}$$

EXAMPLE 29 *The coefficients of three consecutive terms in the expansion of $(1+x)^n$ are in the ratio 1 : 7 : 42. Find n.* **[NCERT]**

SOLUTION Let the three consecutive terms be rth, $(r+1)$th and $(r+2)$th terms. Their coefficients in the expansion of $(1+x)^n$ are ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ respectively. It is given that,

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42.$$

Now, $\quad \dfrac{{}^nC_{r-1}}{{}^nC_r} = \dfrac{1}{7}$

$\Rightarrow \quad \dfrac{r}{n-r+1} = \dfrac{1}{7}$ $\quad \left[\because \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$\Rightarrow \quad n - 8r + 1 = 0$ ...(i)

and, $\quad \dfrac{^nC_r}{^nC_{r+1}} = \dfrac{7}{42}$

$\Rightarrow \quad \dfrac{r+1}{n-r} = \dfrac{1}{6}$ $\left[\because \dfrac{^nC_{r+1}}{^nC_r} = \dfrac{n-r}{r+1}\right]$

$\Rightarrow \quad n - 7r - 6 = 0$ ...(ii)

Solving (i) and (ii), we get $r = 7$ and $n = 55$.

EXAMPLE 30 *In the binomial expansion of $(1 + x)^n$, the coefficients of the fifth, sixth and seventh terms are in A.P. Find all values of n for which this can happen.*

SOLUTION The coefficients of fifth, sixth and seventh terms in the binomial expansion of $(1 + x)^n$ are $^nC_4$, $^nC_5$ and $^nC_6$ respectively. We are given that $^nC_4$, $^nC_5$ and $^nC_6$ are in A.P.

$\therefore \quad 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$\Rightarrow \quad 2 = \dfrac{^nC_4}{^nC_5} + \dfrac{^nC_6}{^nC_5}$ [Dividing both sides by $^nC_5$]

$\Rightarrow \quad 2 = \dfrac{5}{n-4} + \dfrac{n-5}{6}$ $\left[\because \dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$\Rightarrow \quad 2 = \dfrac{30 + (n-4)(n-5)}{6(n-4)}$

$\Rightarrow \quad 12n - 48 = 30 + n^2 - 9n + 20 \Rightarrow n^2 - 21n + 98 = 0 \Rightarrow (n-14)(n-7) = 0 \Rightarrow n = 7, 14.$

EXAMPLE 31 *If the coefficients of a^{r-1}, a^r, a^{r+1} in the binomial expansion of $(1 + a)^n$ are in A.P., prove that $n^2 - n(4r + 1) + 4r^2 - 2 = 0$.* [NCERT]

SOLUTION The coefficients of a^{r-1}, a^r and a^{r+1} in the binomial expansion of $(1 + a)^n$ are $^nC_{r-1}$, ^nC_r and $^nC_{r+1}$ respectively. It is given that $^nC_{r-1}$, ^nC_r and $^nC_{r+1}$ are in A.P.

$\therefore \quad 2\,{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$

$\Rightarrow \quad 2 = \dfrac{^nC_{r-1}}{^nC_r} + \dfrac{^nC_{r+1}}{^nC_r}$

$\Rightarrow \quad 2 = \dfrac{r}{n-r+1} + \dfrac{n-r}{r+1}$ $\left[\because \dfrac{^nC_r}{^nC_{r-1}} = \dfrac{n-r+1}{r}\right]$

$\Rightarrow \quad 2 = \dfrac{r(r+1) + (n-r)(n-r+1)}{(r+1)(n-r+1)}$

$\Rightarrow \quad 2\left\{(n-r+1)(r+1)\right\} = r(r+1) + (n-r)(n-r+1)$

$\Rightarrow \quad 2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + r^2 + n - r$

$\Rightarrow \quad n^2 - 4nr - n + 4r^2 - 2 = 0 \Rightarrow n^2 - n(4r + 1) + 4r^2 - 2 = 0$

EXAMPLE 32 *The coefficients of $(r-1)^{th}$, rth and $(r+1)^{th}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r.* **[NCERT]**

SOLUTION We know that the coefficient of rth term in the expansion of $(x+1)^n$ is ${}^nC_{r-1}$.
Therefore, coefficients of $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms are ${}^nC_{r-2}$, ${}^nC_{r-1}$ and ${}^nC_r$ respectively.

It is given that

$$ {}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r = 1:3:5 $$

$$ \Rightarrow \quad \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{5}{3} \text{ and } \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = \frac{3}{1} $$

$$ \Rightarrow \quad \frac{n-r+1}{r} = \frac{5}{3} \text{ and } \frac{n-r+2}{r-1} = \frac{3}{1} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right] $$

$$ \Rightarrow \quad 3n-8r+3=0 \text{ and } n-4r+5=0 \Rightarrow n=7 \text{ and } r=3 $$

LEVEL-2

Type VII **ON FINDING THE UNKNOWN WHEN THE VALUE OF A TERM IS GIVEN**

EXAMPLE 33 *If the third term in the expansion of $\left(\frac{1}{x} + x^{\log_{10} x}\right)^5$ is 1000, then find x.*

SOLUTION We have,

$$ T_3 = 1000 $$

$$ \Rightarrow \quad T_{2+1} = 1000 $$

$$ \Rightarrow \quad {}^5C_2 \left(\frac{1}{x}\right)^{5-2} (x^{\log x})^2 = 1000 $$

$$ \Rightarrow \quad 10\,(x^{\log_{10} x})^2 \times x^{-3} = 1000 $$

$$ \Rightarrow \quad x^{2\log_{10} x} \times x^{-3} = 100 $$

$$ \Rightarrow \quad x^{2\log_{10} x - 3} = 10^2 $$

$$ \Rightarrow \quad 2\log_{10} x - 3 = \log_x 10^2 $$

$$ \Rightarrow \quad 2\log_{10} x - 3 = \frac{2}{\log_{10} x} $$

$$ \Rightarrow \quad 2y - 3 = \frac{2}{y}, \text{ where } y = \log_{10} x $$

$$ \Rightarrow \quad 2y^2 - 3y - 2 = 0 $$

$$ \Rightarrow \quad (2y+1)(y-2) = 0 $$

$$ \Rightarrow \quad y = 2 \text{ or } y = -\frac{1}{2} $$

$$ \Rightarrow \quad \log_{10} x = 2 \text{ or, } \log_{10} x = -\frac{1}{2} \Rightarrow x = 10^2 = 100 \text{ or, } x = 10^{-1/2} = \frac{1}{\sqrt{10}}. $$

EXAMPLE 34 *If the fourth term in the expansion of* $\left\{\sqrt{x^{\frac{1}{\log x+1}}}+x^{\frac{1}{12}}\right\}^{6}$ *is equal to 200 and* $x>1$, *then find* x.

SOLUTION It is given that

$$T_4 = 200$$

$$\Rightarrow \quad T_{3+1} = 200$$

$$\Rightarrow \quad {}^6C_3\left\{\sqrt{x^{\frac{1}{\log x+1}}}\right\}^{6-3}(x^{1/12})^3 = 200$$

$$\Rightarrow \quad 20\left(x^{\frac{1}{\log x+1}}\right)^{3/2} x^{1/4} = 200$$

$$\Rightarrow \quad x^{\frac{3}{2}\left(\frac{1}{\log x+1}\right)+\frac{1}{4}} = 10$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{1}{\log x+1}\right)+\frac{1}{4} = \log_x 10$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{1}{\log_{10} x+1}\right)+\frac{1}{4} = \frac{1}{\log_{10} x}$$

$$\Rightarrow \quad \frac{3}{2(y+1)}+\frac{1}{4}=\frac{1}{y}, \text{ where } y=\log_{10} x$$

$$\Rightarrow \quad \frac{6+y+1}{4(y+1)} = \frac{1}{y}$$

$$\Rightarrow \quad y^2+3y-4 = 0$$

$$\Rightarrow \quad (y+4)(y-1) = 0$$

$$\Rightarrow \quad y = 1, -4$$

$$\Rightarrow \quad \log_{10} x = 1, -4$$

$$\Rightarrow \quad x=10 \text{ or, } x=10^{-4} \Rightarrow x=10 \qquad [\because x>1]$$

EXAMPLE 35 *For what value of* x *is the ninth term in the expansion of*

$$\left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{(-1/8)\log_3(5^{x-1}+1)}\right\}^{10}$$ *is equal to 180?*

SOLUTION We know that $a^{\log_a N}=N$.

$$\therefore \quad \left\{3^{\log_3\sqrt{25^{x-1}+7}}+3^{(-1/8)\log_3(5^{x-1}+1)}\right\}^{10} = \left\{\sqrt{25^{x-1}+7}+(5^{x-1}+1)^{-1/8}\right\}^{10}$$

Let T_9 be the 9th term in the above expansion. Then,

$$T_9 = 180$$

$$\Rightarrow \quad {}^{10}C_8 \left\{\sqrt{25^{x-1}+7}\right\}^{10-8} \left\{(5^{x-1}+1)^{-1/8}\right\}^8 = 180$$

$$\Rightarrow \quad {}^{10}C_8\ (25^{x-1}+7)\ (5^{x-1}+1)^{-1} = 180$$

$$\Rightarrow \quad \frac{45\,(25^{x-1}+7)}{5^{x-1}+1} = 180$$

$$\Rightarrow \quad \frac{25^{x-1}+7}{5^{x-1}+1} = 4$$

$$\Rightarrow \quad \frac{y^2+7}{y+1} = 4, \text{ where } y = 5^{x-1}$$

$$\Rightarrow \quad y^2 - 4y + 3 = 0$$

$$\Rightarrow \quad (y-3)(y-1) = 0$$

$$\Rightarrow \quad y = 3, -1$$

$$\Rightarrow \quad 5^{x-1} = 3 \text{ or, } 5^{x-1} = 1 \Rightarrow 5^x = 15 \text{ or, } 5^x = 5 \Rightarrow x = \log_5 15 \text{ or, } x = 1.$$

EXAMPLE 36 *If the fourth term in the expansion of* $\left(ax + \frac{1}{x}\right)^n$ *is* $\frac{5}{2}$, *then find the values of a and n.*

SOLUTION It is given that

$$T_4 = \frac{5}{2}$$

$$\Rightarrow \quad T_{3+1} = \frac{5}{2}$$

$$\Rightarrow \quad {}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^nC_3\,a^{n-3}\,x^{n-6} = \frac{5}{2} \qquad \text{...(i)}$$

Clearly, RHS of the above equality is independent of x.

$\therefore \quad n - 6 = 0 \Rightarrow n = 6.$

Putting $n = 6$ in (i), we get

$$^6C_3\,a^3 = \frac{5}{2} \Rightarrow \frac{6\times5\times4}{3\times2\times1}a^3 = \frac{5}{2} \Rightarrow a^3 = \frac{1}{8} \Rightarrow a = \frac{1}{2}$$

Hence, $a = \frac{1}{2}$ and $n = 6$.

Type VIII ON MIDDLE TERM (S) IN A BINOMIAL EXPENSION

EXAMPLE 37 *Find the value of* α *for which the coefficients of the middle terms in the expansions of* $(1+\alpha x)^4$ *and* $(1-\alpha x)^6$ *are equal, find* α.

SOLUTION In the expansion of $(1+\alpha x)^4$.

Middle term $= {}^4C_2\ (\alpha x)^2 = 6\alpha^2 x^2$

In the expansion of $(1-\alpha x)^6$.

Middle term $= {}^6C_3\,(-\alpha x)^3 = -20\,\alpha^3 x^3$

It is given that:

Coefficient of the middle term in $(1+\alpha x)^4$ = Coefficient of the middle term in $(1-\alpha x)^6$

$\Rightarrow 6\alpha^2 = -20\alpha^3 \Rightarrow \alpha = 0, \alpha = -\frac{3}{10}$

EXAMPLE 38 *If the middle term in the binomial expansion of* $\left(\frac{1}{x}+x\sin x\right)^{10}$ *is equal to* $\frac{63}{8}$, *find the value of x.* **[NCERT EXEMPLAR]**

SOLUTION In the binomial expansion of $\left(\frac{1}{x}+x\sin x\right)^{10}$, $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6th term is the middle term.

It is given that

$$T_6=\frac{63}{8}$$

$$\Rightarrow \quad {}^{10}C_5\left(\frac{1}{x}\right)^{10-5}(x\sin x)^5=\frac{63}{8}$$

$$\Rightarrow \quad \frac{10!}{5!5!}(\sin x)^5=\frac{63}{8}$$

$$\Rightarrow \quad (\sin x)^5=\left(\frac{1}{2}\right)^5$$

$$\Rightarrow \quad \sin x=\frac{1}{2}=\sin\frac{\pi}{6}$$

$$\Rightarrow \quad x=n\pi+(-1)^n\frac{\pi}{6}, n\in Z$$

Type IX ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION

EXAMPLE 39 *The sum of the coefficients of first three terms in the expansion of* $\left(x-\frac{3}{x^2}\right)^m$, $x\neq 0$, *m being a natural number, is 559. Find the term of the expansion containing* x^3. **[NCERT]**

SOLUTION We have,

$$\left(x-\frac{3}{x^2}\right)^m={}^mC_0\,x^m+{}^mC_1\,x^{m-1}\left(-\frac{3}{x^2}\right)+{}^mC_2\,x^{m-2}\left(-\frac{3}{x^2}\right)^2+\dots+{}^mC_m\,x^0\left(-\frac{3}{x^2}\right)^m$$

$$\Rightarrow \left(x-\frac{3}{x^2}\right)^m={}^mC_0\,x^m+(-3\times{}^mC_1)\,x^{m-3}+(9\times{}^mC_2)\,x^{m-6}+\dots+{}^mC_m\,(-3)^m\times x^{-2m}$$

Clearly, the coefficients of first three terms are: ${}^mC_0$, $-3\times{}^mC_1$ and $9\times{}^mC_2$

It is given that the sum of these coefficients is 559.

$$\therefore \quad {}^mC_0-3\times{}^mC_1+9\times{}^mC_2=559$$

$$\Rightarrow \quad 1-3m+\frac{9m(m-1)}{2}=559$$

$$\Rightarrow \quad 2-6m+9m(m-1)=1118$$

$$\Rightarrow \quad 9m^2-15m-1116=0$$

$$\Rightarrow \quad 3m^2-5m-372=0$$

$$\Rightarrow \quad 3m^2-36m+31m-372=0$$

$$\Rightarrow \quad 3m(m-12)+31(m-12)=0$$

$$\Rightarrow \quad (m-12)(3m+31)=0$$

$$\Rightarrow \quad m=12 \qquad [\because m\in N \therefore 3m+31\neq 0]$$

Suppose $(r+1)^{th}$ term contains x^3.

Now,

$$T_{r+1} = {}^mC_r\,(x)^{m-r}\left(-\frac{3}{x^2}\right)^r = {}^mC_r\,(-3)^r\,x^{m-3r} = {}^{12}C_r\,(-3)^r\,x^{12-3r} \qquad [\because m = 12]$$

This will contain x^3, if $12 - 3r = 3$ i.e. $r = 3$.
Putting $r = 3$ in T_{r+1}, we get
Required term $= T_5 = {}^{12}C_3\,(-3)^3\,x^{12-9} = -5940x^3$

EXAMPLE 40 *Find the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ and find the relation between a and b so that these coefficients are equal.*

SOLUTION Suppose x^7 occurs in $(r+1)^{th}$ term of the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$.

Now,

$$T_{r+1} = {}^{11}C_r\,(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r\,a^{11-r}\,b^{-r}\,x^{22-3r} \qquad \text{...(i)}$$

This will contain x^7, if

$22 - 3r = 7 \Rightarrow 3r = 15 \Rightarrow r = 5.$

Putting $r = 5$ in (i), we obtain that

Coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,a^6\,b^{-5}$.

Suppose x^{-7} occurs in $(r+1)^{th}$ term of the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$.

Now, $$T_{r+1} = {}^{11}C_r\,(ax)^{11-r}\left(-\frac{1}{bx^2}\right)^r = {}^{11}C_r\,a^{11-r}\,(-1)^r\,b^{-r}\,x^{11-3r} \qquad \text{... (ii)}$$

This will contain x^{-7}, if

$11 - 3r = -7 \Rightarrow 3r = 18 \Rightarrow r = 6.$

Putting $r = 6$ in (ii), we obtain that

Coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6\,a^5\,b^{-6}\,(-1)^6$.

If the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is equal to the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$, then

$${}^{11}C_5\,a^6\,b^{-5} = {}^{11}C_6\,a^5\,b^{-6}\,(-1)^6 \Rightarrow {}^{11}C_5\,ab = {}^{11}C_6 \Rightarrow ab = 1 \qquad \left[\because {}^{11}C_5 = {}^{11}C_6\right]$$

EXAMPLE 41 *If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is*

$$\left\{\frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}\right\}.$$

SOLUTION Suppose x^p occurs in $(r+1)^{th}$ term in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$.

Now, $T_{r+1} = {}^{2n}C_r\,(x^2)^{2n-r}\left(\frac{1}{x}\right)^r = {}^{2n}C_r\,x^{4n-3r}$ ...(i)

For this term to contain x^p, we must have

$$4n-3r = p \Rightarrow r = \frac{4n-p}{3}$$

$\therefore$ Coefficient of $x^p = {}^{2n}C_r$ where $r = \frac{4n-p}{3}$

$\Rightarrow$ Coefficient of $x^p = \frac{(2n)!}{(2n-r)!\,r!}$, where $r = \frac{4n-p}{3}$

$\Rightarrow$ Coefficient of $x^p = \frac{(2n)!}{\left\{2n-\left(\frac{4n-p}{3}\right)\right\}!\left(\frac{4n-p}{3}\right)!} = \frac{(2n)!}{\left(\frac{2n+p}{3}\right)!\left(\frac{4n-p}{3}\right)!}$

EXAMPLE 42 *Find the coefficient of x^n in the expansion of $(1+x)(1-x)^n$.*

SOLUTION Coefficient of x^n in $(1+x)(1-x)^n$

$= $ Coefficient of x^n in $(1-x)^n$ + Coefficient of x^{n-1} in $(1-x)^n$

$= (-1)^n\ {}^nC_n + (-1)^{n-1}\,{}^nC_{n-1}$

$= (-1)^n\ (1-n)$

EXAMPLE 43 *Find the coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$.*

[NCERT EXEMPLAR]

SOLUTION $(1+x+x^2+x^3)^{11} = \{(1+x)+x^2(1+x)\}^{11} = \{(1+x)(1+x^2)^{11}\} = (1+x)^{11}(1+x^2)^{11}$

$= \left({}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2\,x^2 + {}^{11}C_3\,x^3 + {}^{11}C_4\,x^4 + {}^{11}C_5\,x^5 + \ldots\right) \times$

$\left({}^{11}C_0 + {}^{11}C_1 x^2 + {}^{11}C_2 (x^2)^2 + {}^{11}C_3 (x^2)^3 + \ldots\right)$

$\therefore$ Coefficient of x^4 in $(1+x+x^2+x^3)^4 = {}^{11}C_0 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_1 + {}^{11}C_4 \times {}^{11}C_0$

$= 55 + 55 \times 11 + 330 = 990$

EXAMPLE 44 *If the coefficients of x and x^2 in the expansion of $(1+x)^m (1-x)^n$ are 3 and -6 respectively. Find the values of m and n.*

SOLUTION We have,

$(1+x)^m (1-x)^n$

$= \left\{{}^mC_0 + {}^mC_1\,x + {}^mC_2\,x^2 + \ldots + {}^mC_m\,x^m\right\} \times \left\{{}^nC_0 - {}^nC_1\,x + {}^nC_2\,x^2 \ldots + (-1)^n\,{}^nC_n\,x^n\right\}$

$= {}^mC_0\,{}^nC_0 - \left({}^mC_0\,{}^nC_1 - {}^nC_0\,{}^mC_1\right)x + \left({}^mC_0\,{}^nC_2 + {}^nC_0\,{}^mC_2 - {}^mC_1\,{}^nC_1\right)x^2 + \ldots$

It is given that the coefficients of x and x^2 in the expansion of $(1+x)^m (1-x)^n$ are 3 and -6 respectively.

$\therefore \quad -({}^mC_0\ {}^nC_1 - {}^nC_0\ {}^mC_1) = 3$ and, ${}^mC_0\ {}^nC_2 + {}^nC_0\ {}^mC_2 - {}^mC_1\ {}^nC_1 = -6$

$\Rightarrow \quad m - n = 3$ and $n(n-1) + m(m-1) - 2mn = -12$

$\Rightarrow \quad m - n = 3$ and $(m-n)^2 - (m+n) = -12$

$\Rightarrow \quad m - n = 3$ and $m + n = 21$

$\Rightarrow \quad m = 12, n = 9$

Type X **ON FINDING THE TERM INDEPENDENT OF THE VARIABLE**

EXAMPLE 45 *Find the coefficient of the term independent of x in the expansion of*

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}.$$

SOLUTION We have,

$$\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}$$

$$=\frac{(x^{1/3})^3+1^3}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x^{1/2}\,(x^{1/2}-1)}$$

$$=\frac{(x^{1/3}+1)\,(x^{2/3}-x^{1/3}+1)}{x^{2/3}-x^{1/3}+1}-\frac{\left(x^{1/2}+1\right)\left(x^{1/2}-1\right)}{x^{1/2}\left(x^{1/2}-1\right)}$$

$$=\left(x^{1/3}+1\right)-\left(\frac{x^{1/2}+1}{x^{1/2}}\right)=x^{1/3}+1-1-x^{-1/2}=x^{1/3}-x^{-1/2}$$

$$\therefore \quad \left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}=(x^{1/3}-x^{-1/2})^{10}$$

Let T_{r+1} be the general term in $\left(x^{1/3}-x^{-1/2}\right)^{10}.$ Then,

$$T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(-1)^r\,(x^{-1/2})^r = (-1)^r\ {}^{10}C_r\,x^{\frac{10-r}{3}-\frac{r}{2}}$$

For this term to be independent of x, we must have

$$\frac{10-r}{3}-\frac{r}{2}=0 \Rightarrow 20-2r-3r=0 \Rightarrow r=4$$

So, required coefficient $= {}^{10}C_4\,(-1)^4 = 210.$

EXAMPLE 46 *Find the greatest value of the term independent of x in the expansion of* $\left(x\sin\alpha+\frac{\cos\alpha}{x}\right)^{10}$, *where* $\alpha \in R$.

SOLUTION Let $(r+1)^{th}$ term be independent of x.

Now, $T_{r+1} = {}^{10}C_r\,(x\sin\alpha)^{10-r}\left(\frac{\cos\alpha}{x}\right)^r = {}^{10}C_r\,x^{10-2r}\,(\sin\alpha)^{10-r}\,(\cos\alpha)^r$

If it is independent of x, then $r=5$.

$\therefore$ Term independent of $x = T_6 = {}^{10}C_5\,(\sin\alpha\cos\alpha)^5 = {}^{10}C_5 \times 2^{-5}\,(\sin 2\alpha)^5$

Clearly, it is greatest when $2\alpha = \pi/2$ and its greatest value is ${}^{10}C_5 \times 2^{-5} = \dfrac{10!}{2^5\,(5!)^2}$

Type XI ON COEFFICIENTS OF TERMS IN A BINOMIAL EXPANSION

EXAMPLE 47 *Find the coefficient of x^5 in the expansion of $(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$.*

SOLUTION We have,

$$(1+x)^{21}+(1+x)^{22}+\ldots+(1+x)^{30}$$

$$=(1+x)^{21}\left\{\frac{(1+x)^{10}-1}{(1+x)-1}\right\}=\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}$$

$\therefore$ Coefficient of x^5 in the given expression $=$ Coefficient of x^5 in $\left[\frac{1}{x}\left\{(1+x)^{31}-(1+x)^{21}\right\}\right]$

$$= \text{Coefficient of } x^6 \text{ in } \left\{(1+x)^{31}-(1+x)^{21}\right\}$$

$$= {}^{31}C_6 - {}^{21}C_6$$

EXAMPLE 48 *Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$.*

SOLUTION Let $S=(1+x)^{1000}+x(1+x)^{999}+x^2(1+x)^{998}+\ldots+x^{1000}$. Clearly, it is a G.P. consisting of 1001 terms with first term $(1+x)^{1000}$ and common ratio $\frac{x}{1+x}$.

$$\therefore \quad S=(1+x)^{1000}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{1-\left(\frac{x}{1+x}\right)}\right\}$$

$$\Rightarrow \quad S=(1+x)^{1000}\left\{\frac{(1+x)^{1001}-x^{1001}}{(1+x)^{1000}}\right\}=(1+x)^{1001}-x^{1001}$$

$\therefore$ Coefficient of x^{50} in $S=$ Coefficient of x^{50} in $\left\{(1+x)^{1001}-x^{1001}\right\}$

$$= \text{Coefficient of } x^{50} \text{ in } (1+x)^{1001}$$

$$= {}^{1001}C_{50}.$$

EXAMPLE 49 *If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$.*

SOLUTION Clearly, **[NCERT EXEMPLAR]**

$$(1+x)^n\left(1+\frac{1}{x}\right)^n=\frac{(1+x)^n(1+x)^n}{x^n}=\frac{(1+x)^{2n}}{x^n}$$

$\therefore$ Coefficient of x^{-1} in $(1+x)^n\left(1+\frac{1}{x}\right)^n=$ Coefficient of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$

$$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^{2n}$$

$$= {}^{2n}C_{n-1}$$

EXAMPLE 50 *If in the expansion of $(1-x)^{2n-1}$, the coefficient of x^r is denoted by a_r, then prove that $a_{r-1}+a_{2n-r}=0$.*

SOLUTION We have,

$$a_{r-1} = \text{Coefficient of } x^{r-1} \text{ in } (1-x)^{2n-1} = (-1)^{r-1}\ {}^{2n-1}C_{r-1}$$

$$a_{2n-r} = \text{Coefficient of } x^{2n-r} \text{ in } (1-x)^{2n-1} = (-1)^{2n-r}\ {}^{2n-1}C_{2n-r}$$

$$\therefore\ a_{r-1} + a_{2n-r} = (-1)^{r-1}\ {}^{2n-1}C_{r-1} + (-1)^{2n-r}\ {}^{2n-1}C_{2n-r}$$

$$= (-1)^{r-1}\ {}^{2n-1}C_{(2n-1)-(r-1)} + (-1)^{2n}(-1)^{-r}\ {}^{2n-1}C_{2n-r} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$$= (-1)^{r-1}\ {}^{2n-1}C_{2n-r} + (-1)^{-r}\ {}^{2n-1}C_{2n-r} \qquad [\because (-1)^{2n} = 1]$$

$$= [(-1)^{r-1} + (-1)^{-r}]\ {}^{2n-1}C_{2n-r}$$

$$= \left\{(-1)^{r-1} + \frac{1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r}$$

$$= \left\{\frac{(-1)^{2r-1}+1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r} = \left\{\frac{-1+1}{(-1)^r}\right\}\ {}^{2n-1}C_{2n-r} = 0 \qquad [\because (-1)^{2r-1} = -1]$$

Type XII ON CONSECUTIVE TERMS AND THEIR COEFFICIENTS

EXAMPLE 51 *If a_1, a_2, a_3, a_4 be the coefficients of four consecutive terms in the expansion of $(1+x)^n$, then prove that:* $\dfrac{a_1}{a_1+a_2} + \dfrac{a_3}{a_3+a_4} = \dfrac{2a_2}{a_2+a_3}$. **[NCERT EXEMPLAR]**

SOLUTION Let a_1, a_2, a_3, a_4 be the coefficients of 4 consecutive terms viz. the rth, the $(r+1)$th, the $(r+2)$th and the $(r+3)$th terms. Then,

$$a_1 = {}^nC_{r-1},\ a_2 = {}^nC_r,\ a_3 = {}^nC_{r+1} \text{ and } a_4 = {}^nC_{r+2}$$

Now, $a_1 + a_2 = {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$, $a_2 + a_3 = {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$

and, $a_3 + a_4 = {}^nC_{r+1} + {}^nC_{r+2} = {}^{n+1}C_{r+2}$

$$\therefore \quad \frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{{}^nC_{r-1}}{{}^{n+1}C_r} + \frac{{}^nC_{r+1}}{{}^{n+1}C_{r+2}}$$

$$= \frac{{}^nC_{r-1}}{\left(\frac{n+1}{r}\right){}^nC_{r-1}} + \frac{{}^nC_{r+1}}{\left(\frac{n+1}{r+2}\right){}^nC_{r+1}} \qquad \left[\because {}^nC_r = \frac{n}{r}\cdot{}^{n-1}C_{r-1}\right]$$

$$= \frac{r}{n+1} + \frac{r+2}{n+1} = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(i)}$$

$$\text{and, } 2\frac{a_2}{a_2+a_3} = 2\frac{{}^nC_r}{{}^{n+1}C_{r+1}} = 2\left(\frac{{}^nC_r}{\frac{n+1}{r+1}\cdot{}^nC_r}\right) = 2\left(\frac{r+1}{n+1}\right) \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

EXAMPLE 52 *The 3^{rd}, 4^{th} and 5^{th} terms in the expansion of $(x+a)^n$ are respectively 84, 280 and 560, find the values of x, a and n.*

SOLUTION It is given that: $T_3 = 84$, $T_4 = 280$ and $T_5 = 560$

We have,

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, x^{n-r}\, a^r}{{}^nC_{r-1}\, x^{n-r+1}\, a^{r-1}} = \frac{n-r+1}{r}\cdot\frac{a}{x}$$

$$\therefore \quad \frac{T_4}{T_3} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{T_5}{T_4} = \frac{n-3}{4}\cdot\frac{a}{x}$$

$$\Rightarrow \quad \frac{280}{84} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{560}{280} = \frac{n-3}{4}\cdot\frac{a}{x} \qquad [\because T_3 = 84, T_4 = 280 \text{ and } T_5 = 560]$$

$$\Rightarrow \quad \frac{10}{3} = \frac{n-2}{3}\cdot\frac{a}{x} \text{ and } \frac{2}{1} = \frac{n-3}{4}\cdot\frac{a}{x}$$

$$\Rightarrow \quad \frac{a}{x} = \frac{10}{n-2} \text{ and } \frac{a}{x} = \frac{8}{n-3}$$

$$\Rightarrow \quad \frac{10}{n-2} = \frac{8}{n-3} \Rightarrow 5n-15 = 4n-8 \Rightarrow n = 7$$

Putting $n = 7$ in $\frac{a}{x} = \frac{10}{n-2}$, we get

$$\frac{a}{x} = \frac{10}{5} \Rightarrow 2x = a$$

Now, $T_3 = 84$

$$\Rightarrow \quad {}^nC_2\, x^{n-2}\, a^2 = 84$$

$$\Rightarrow \quad {}^7C_2\, x^5\, (2x)^2 = 84 \qquad [\because a = 2x \text{ and } n = 7]$$

$$\Rightarrow \quad 21 \times 2^4 \times x^7 = 84 \Rightarrow x^7 = 1 \Rightarrow x = 1$$

$$\therefore \quad a = 2x = 2 \times 1 = 2$$

Hence, $n = 7$, $a = 2$ and $x = 1$.

Type XIII **ON APPLICATIONS OF BINOMIAL THEOREM**

EXAMPLE 53 *How many terms are free from radical signs in the expansion of $(x^{1/5} + y^{1/10})^{55}$.*

SOLUTION The general term in the expansion of $(x^{1/5} + y^{1/10})^{55}$ is given by

$$T_{r+1} = {}^{55}C_r \left(x^{1/5}\right)^{55-r} \left(y^{1/10}\right)^r \Rightarrow T_{r+1} = {}^{55}C_r\, x^{11-r/5}\, y^{r/10}$$

Clearly, T_{r+1} will be free from radical signs, if $\frac{r}{5}$ and $\frac{r}{10}$ are integers for $0 \le r \le 55$

$$\therefore \quad r = 0, 10, 20, 30, 40, 50.$$

Hence, there are 6 terms in the expansion of $(x^{1/5} + y^{1/10})^{55}$ which are independent of radical signs.

EXAMPLE 54 *Find the number of integral terms in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.*

SOLUTION The general term T_{r+1} in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$ is given by

$$T_{r+1} = {}^{1024}C_r \left(5^{1/2}\right)^{1024-r} \left(7^{1/8}\right)^r$$

$$\Rightarrow \quad T_{r+1} = {}^{1024}C_r \; 5^{512-\frac{r}{2}} \; 7^{r/8}$$

$$\Rightarrow \quad T_{r+1} = \left\{ {}^{1024}C_r \; 5^{512-r} \right\} \times 5^{r/2} \times 7^{r/8}$$

$$\Rightarrow \quad T_{r+1} = \left\{ {}^{1024}C_r \; 5^{512-r} \right\} \times \left(5^4 \times 7\right)^{r/8}$$

Clearly, T_{r+1} will be an integer, iff

$\frac{r}{8}$ is an integer such that $0 \leq r \leq 1024$

$\Rightarrow$ r is a multiple of 8 satisfying $0 \leq r \leq 1024$

$\Rightarrow$ $r = 0, 8, 16, 24, \ldots, 1024$

$\Rightarrow$ r can assume 129 values.

Hence, there are 129 integral terms in the expansion of $\left(5^{1/2} + 7^{1/8}\right)^{1024}$.

EXERCISE 18.2

LEVEL-1

1. Find the 11th term from the beginning and the 11th term from the end in the expansion of $\left(2x - \frac{1}{x^2}\right)^{25}$.

2. Find the 7th term in the expansion of $\left(3x^2 - \frac{1}{x^3}\right)^{10}$.

3. Find the 5th term from the end in the expansion of $\left(3x - \frac{1}{x^2}\right)^{10}$.

4. Find the 8th term in the expansion of $(x^{3/2} y^{1/2} - x^{1/2} y^{3/2})^{10}$.

5. Find the 7th term in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x}\right)^8$.

6. Find the 4th term from the beginning and 4th term from the end in the expansion of $\left(x + \frac{2}{x}\right)^9$.

7. Find the 4th term from the end in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$.

8. Find the 7th term from the end in the expansion of $\left(2x^2 - \frac{3}{2x}\right)^8$.

9. Find the coefficient of:

   (i) x^{10} in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$.

(ii) x^7 in the expansion of $\left(x - \frac{1}{x^2}\right)^{40}$.

(iii) x^{-15} in the expansion of $\left(3x^2 - \frac{a}{3x^3}\right)^{10}$.

(iv) x^9 in the expansion of $\left(x^2 - \frac{1}{3x}\right)^9$.

(v) x^m in the expansion of $\left(x + \frac{1}{x}\right)^n$.

(vi) x in the expansion of $(1 - 2x^3 + 3x^5)\left(1 + \frac{1}{x}\right)^8$.

(vii) $a^5 b^7$ in the expansion of $(a - 2b)^{12}$. [NCERT]

(viii) x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$. [NCERT EXEMPLAR]

10. Which term in the expansion of $\left\{\left(\frac{x}{\sqrt{y}}\right)^{1/3} + \left(\frac{y}{x^{1/3}}\right)^{1/2}\right\}^{21}$ contains x and y to one and the same power?

11. Does the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$ contain any term involving x^9?

12. Show that the expansion of $\left(x^2 + \frac{1}{x}\right)^{12}$ does not contain any term involving x^{-1}.

13. Find the middle term in the expansion of:

(i) $\left(\frac{2}{3}x - \frac{3}{2x}\right)^{20}$

(ii) $\left(\frac{a}{x} + bx\right)^{12}$

(iii) $\left(x^2 - \frac{2}{x}\right)^{10}$

(iv) $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$

14. Find the middle terms in the expansion of:

(i) $\left(3x - \frac{x^3}{6}\right)^9$ [NCERT EXEMPLAR]

(ii) $\left(2x^2 - \frac{1}{x}\right)^7$

(iii) $\left(3x - \frac{2}{x^2}\right)^{15}$

(iv) $\left(x^4 - \frac{1}{x^3}\right)^{11}$

15. Find the middle term(s) in the expansion of:

(i) $\left(x - \frac{1}{x}\right)^{10}$

(ii) $(1 - 2x + x^2)^n$

(iii) $(1 + 3x + 3x^2 + x^3)^{2n}$

(iv) $\left(2x - \frac{x^2}{4}\right)^9$

(v) $\left(x-\frac{1}{x}\right)^{2n+1}$ (vi) $\left(\frac{x}{3}+9y\right)^{10}$ [NCERT]

(vii) $\left(3-\frac{x^3}{6}\right)^7$ (viii) $\left(2ax-\frac{b}{x^2}\right)^{12}$ [NCERT EXEMPLAR]

(ix) $\left(\frac{p}{x}+\frac{x}{p}\right)^9$ [NCERT EXEMPLAR] (x) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ [NCERT EXEMPLAR]

16. Find the term independent of x in the expansion of the following expressions:

(i) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ (ii) $\left(2x+\frac{1}{3x^2}\right)^9$

(iii) $\left(2x^2-\frac{3}{x^3}\right)^{25}$ (iv) $\left(3x-\frac{2}{x^2}\right)^{15}$ [NCERT EXEMPLAR]

(v) $\left(\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$ [NCERT EXEMPLAR] (vi) $\left(x-\frac{1}{x^2}\right)^{3n}$

(vii) $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$ (viii) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

[NCERT EXEMPLAR]

(ix) $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}$, $x>2$ (x) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^6$ [NCERT]

17. If the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal, find r. [NCERT EXEMPLAR]

18. If the coefficients of $(2r+1)$th term and $(r+2)$th term in the expansion of $(1+x)^{43}$ are equal, find r.

19. Prove that the coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of rth and $(r+1)$th terms in the expansion of $(1+x)^n$.

20. Prove that the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ is $\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\cdot 2^n$.

21. The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A.P., find n.

22. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in A.P., show that $2n^2-9n+7=0$. [NCERT EXEMPLAR]

23. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^n$ are in A.P., then find the value of n.

24. If in the expansion of $(1+x)^n$, the coefficients of pth and qth terms are equal, prove that $p+q=n+2$, where $p\neq q$.

25. Find a, if the coefficients of x^2 and x^3 in the expansion of $(3+ax)^9$ are equal. [NCERT]

26. Find the coefficient of a^4 in the product $(1+2a)^4(2-a)^5$ using binomial theorem.

[NCERT]

LEVEL-2

27. In the expansion of $(1+x)^n$ the binomial coefficients of three consecutive terms are respectively 220, 495 and 792, find the value of n.
28. If in the expansion of $(1+x)^n$, the coefficients of three consecutive terms are 56, 70 and 56, then find n and the position of the terms of these coefficients.
29. If 3rd, 4th, 5th and 6th terms in the expansion of $(x+\alpha)^n$ be respectively a, b, c and d, prove that $\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$.
30. If a, b, c and d in any binomial expansion be the 6th, 7th, 8th and 9th terms respectively, then prove that $\frac{b^2-ac}{c^2-bd}=\frac{4a}{3c}$.
31. If the coefficients of three consecutive terms in the expansion of $(1+x)^n$ be 76, 95 and 76, find n.
32. If the 6th, 7th and 8th terms in the expansion of $(x+a)^n$ are respectively 112, 7 and 1/4, find x, a, n.
33. If the 2nd, 3rd and 4th terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, find x, a, n. **[NCERT]**
34. Find a, b and n in the expansion of $(a+b)^n$, if the first three terms in the expansion are 729, 7290 and 30375 respectively. **[NCERT]**
35. If the term free from x in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$ is 405, find the value of k. **[NCERT EXEMPLAR]**
36. Find the sixth term in the expansion $\left(y^{1/2}+x^{1/3}\right)^n$, if the binomial coefficient of the third term from the end is 45. **[NCERT EXEMPLAR]**
37. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2}+2\right)^8$ is 1120, find p. **[NCERT EXEMPLAR]**
38. Find n in the binomial $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$. **[NCERT EXEMPLAR]**
39. If the seventh term from the beginning and end in the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$ are equal, find n. **[NCERT EXEMPLAR]**

ANSWERS

1. ${}^{25}C_{10}\left(\frac{2^{15}}{x^5}\right), -{}^{25}C_{15}\left(\frac{2^{10}}{x^{20}}\right)$
2. $\frac{17010}{x^{10}}$
3. $\frac{17010}{x^8}$

4. $-120\,x^8 y^{12}$ 5. $\dfrac{4375}{x^4}$ 6. $672\,x^3, \dfrac{5376}{x^3}$ 7. $\dfrac{10500}{x^3}$

8. $4032\,x^{10}$ 9. (i) ${}^{20}C_{10}\cdot 2^{10}$ (ii) $-{}^{40}C_{11}$ (iii) $-\dfrac{40}{27}a^7$

(iv) $-\dfrac{28}{9}$ (v) $\dfrac{n!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$ (vi) 154 (vii) -101376 (viii) -19

10. 10th 11. No

13. (i) ${}^{20}C_{10}$ (ii) $924\,a^6 b^6$ (iii) $-8064\,x^5$ (iv) -252

14. (i) $\dfrac{189}{8}x^{17}, -\dfrac{21}{16}x^{19}$ (ii) $-560\,x^5, 280\,x^2$

(iii) $\dfrac{-6435\times 3^8\times 2^7}{x^6}, \dfrac{6437\times 3^7\times 2^8}{x^9}$ (iv) $-462\,x^9,\ 462\,x^2$

15. (i) -252 (ii) $\dfrac{(2n)!}{(n!)^2}(-1)^n x^n$ (iii) $\dfrac{(6n)!}{[(3n!]^2}\,x^{3n}$

(iv) $\dfrac{63}{4}x^{13}, -\dfrac{63}{32}x^{14}$ (v) $(-1)^n\,.\,{}^{2n+1}C_n\,x,\ (-1)^{n+1}\,.\,{}^{2n+1}C_n\,\dfrac{1}{x}$

(vi) $61236\,x^5 y^5$ (vii) $-\dfrac{105}{8}x^9, \dfrac{35}{48}x^{12}$ (viii) $\dfrac{59\,136\,a^6 b^6}{x^6}$

(ix) $\dfrac{126x}{p}$ (x) -252

16. (i) $\dfrac{7}{18}$ (ii) $\dfrac{64}{27}\times {}^9C_3$ (iii) ${}^{25}C_{10}\,(2^{15}\times 3^{10})$

(iv) $-3003\times 3^{10}\times 2^5$ (v) $\dfrac{5}{12}$ (vi) $(-1)^n\ {}^{3n}C_n$

(vii) 7 (viii) $\dfrac{17}{54}$ (ix) $\dfrac{{}^{18}C_9}{2^9}$ (x) $\dfrac{5}{12}$

17. 6 18. 14 21. 7 or 14 23. 7

25. $\dfrac{8}{7}$ 26. -438 27. 12 28. $n=8$, 4th, 5th, 6th

31. 8 32. $n=8,\ x=4,\ a=\dfrac{1}{2}$ 33. $n=5, x=2, a=3$

34. $a=3, b=5, n=6$ 35. $k=\pm 3$ 36. $252\,y^{5/2}x^{5/3}$ 37. $p=\pm 2$

38. $n=9$ 39. $n=12$

HINTS TO NCERT & SELECTED PROBLEMS

9. (vii) Let T_{r+1} be the $(r+1)^{\text{th}}$ term in the expansion of $(a-2b)^{12}$. Then,

$$T_{r+1} = {}^{12}C_r\, a^{12-r}\,(-2b)^r = {}^{12}C_r\,(-1)^r\,2^r\,a^{12-r}b^r$$

If $a^5\,b^7$ appears in $(r+1)^{\text{th}}$ term, then

$$12-r=5 \text{ and } r=7 \Rightarrow r=7$$

Thus, $a^5\,b^7$ appears in 8^{th} term given by $T_8 = {}^{12}C_7\,(-1)^7\,2^7\,a^5b^7 = -101376a^5b^7$

Hence, Coefficient of $a^5b^7 = -101376$

(viii) $(1-3x+7x^2)(1-x)^{16} = (1-3x+7x^2)\,({}^{16}C_0 - {}^{16}C_1\,x{}^{16}C_2\,x^2 - {}^{16}C_3\,x^3 + \ldots)$

$\therefore$ Coefficient of x in $(1-3x+7x^2)(1-x)^{16} = 1\times -{}^{16}C_1 - 3\times{}^{16}C_0 = -16-3 = -19$

15. (vi) In the expansion of $\left(\frac{x}{3}+9y\right)^{10}$ there are 11 terms. So, $\left(\frac{10}{2}+1\right)^{th}$ i.e. 6th term is the middle term.

$$\text{Now, } T_6 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = 61236x^5y^5$$

16. (x) Let $(r+1)^{th}$ term in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$ be independent of x. Then the exponent of x in $(r+1)^{th}$ term must be zero.

$$\text{Now, } T_{r+1} = {}^6C_r\left(\frac{3}{2}x^2\right)^{6-r}\left(-\frac{1}{3x}\right)^r$$

$$\Rightarrow \quad T_{r+1} = {}^6C_r\left(\frac{3}{2}\right)^{6-r}\left(-\frac{1}{3}\right)^r x^{12-3r} \qquad \ldots(i)$$

For T_{r+1} to be independent of x, we must have

$12-3r=0 \Rightarrow r=4$

Hence, 5^{th} term is independent of x.

Puting $r=4$ in (i), we get

$$T_5 = {}^6C_4\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^4 = 15\times\frac{1}{4\times 9} = \frac{5}{12}$$

18. ${}^{18}C_{2r+3} = {}^{18}C_{r-3} \Rightarrow (2r+3)+(r-3)=18 \Rightarrow r=6$

25. We have,

$$(3+ax)^9 = {}^9C_0\times 3^9 + {}^9C_1\times 3^8\times(ax)^1 + {}^9C_2\times 3^7\times(ax)^2 + {}^9C_3\times 3^6\times(ax)^3 + \ldots + {}^9C_9(ax)^9$$

$\therefore$ Coefficient of $x^2 = {}^9C_2\times 3^7\times a^2$ and, Coefficient of $x^3 = {}^9C_3\times 3^6\times a^3$

Now, Coefficient of x^2 = Coefficient of x^3

$$\Rightarrow \quad {}^9C_2\times 3^7\times a^2 = {}^9C_3\times 3^6\times a^3 \Rightarrow 36\times 3^7\times a^2 = 84\times 3^6\times a^3 \Rightarrow a = \frac{36\times 3^7}{84\times 3^6} = \frac{9}{7}$$

26. $$(1+2a)^4(2-a)^5 = \left\{{}^4C_0 + {}^4C_1(2a) + {}^4C_2(2a)^2 + {}^4C_3(2a)^3 + {}^4C_4(2a)^4\right\}$$
$$\times\left\{{}^5C_0\,2^5 - {}^5C_1\,2^4a + {}^5C_2\,2^3a^2 - {}^5C_3\,2^2a^3 + {}^5C_4(2)\,a^4 - {}^5C_5\,a^5\right\}$$

$$\therefore \text{ Coefficient of } a^4 = {}^4C_0\times\left({}^5C_4\times 2\right) + \left({}^4C_1\times 2\right)\times\left(-{}^5C_3\times 2^2\right) + \left({}^4C_2\times 2^2\right)\times\left({}^5C_2\times 2^3\right)$$
$$+\left({}^4C_3\times 2^3\right)\times\left(-{}^5C_1\times 2^4\right) + \left({}^4C_4\times 2^4\right)\times\left({}^5C_0\times 2^5\right)$$

$$= 10 + 8 \times (-40) + 24 \times 80 + (4 \times 8)(-80) + (16 \times 32)$$
$$= 10 - 320 + 1920 - 2560 + 512 = -438$$

33. It is given that in the expansion of $(x+a)^n$

$$T_2 = 240, T_3 = 720 \text{ and } T_4 = 1080$$

$$\Rightarrow \quad \frac{T_3}{T_2} = 3 \text{ and } \frac{T_4}{T_3} = \frac{3}{2}$$

$$\Rightarrow \quad \frac{n-2+1}{2}\frac{a}{x} = 3 \text{ and } \frac{n-3+1}{3}\frac{a}{x} = \frac{3}{2} \qquad \left[\because \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{a}{x}\right]$$

$$\Rightarrow \quad (n-1)\frac{a}{x} = 6 \text{ and } (n-2)\frac{a}{x} = \frac{9}{2}$$

$$\therefore \quad \frac{n-1}{n-2} = \frac{6}{9} \times 2 \Rightarrow \frac{n-1}{n-2} = \frac{4}{3} \Rightarrow 4n - 8 = 3n - 3 \Rightarrow n = 5$$

Putting $n = 5$ in $(n-1)\frac{a}{x} = 6$, we get $2a = 3x$.

Now, $T_2 = 240$

$$\Rightarrow \quad {}^nC_1 x^{n-1} a = 240$$

$$\Rightarrow \quad n x^{n-1} a = 240$$

$$\Rightarrow \quad 5x^4 a = 240 \qquad [\because n = 5]$$

$$\Rightarrow \quad x^4 \times \frac{3x}{2} = 48 \qquad \left[\because a = \frac{3x}{2}\right]$$

$$\Rightarrow \quad x^5 = 32 \Rightarrow x^5 = (2)^5 \Rightarrow x = 2$$

$$\therefore \quad 2a = 3x \Rightarrow a = 3$$

Hence, $x = 2$, $a = 3$ and $n = 5$.

34. We have,

$${}^nC_0\, a^n b^0 = 729, {}^nC_1\, a^{n-1} b = 7290 \text{ and } {}^nC_2\, a^{n-2} b^2 = 30375$$

$$\Rightarrow \quad a^n = 729, n\, a^{n-1} b = 7290 \text{ and } n(n-1)\, a^{n-2} b^2 = 60750$$

$$\therefore \quad \frac{n\, a^{n-1} b}{a^n} = \frac{7290}{729} \text{ and } \frac{n(n-1)\, a^{n-2} b^2}{n\, a^{n-1} b} = \frac{60750}{7290}$$

$$\Rightarrow \quad \frac{nb}{a} = 10 \text{ and } \frac{(n-1)\,b}{a} = \frac{25}{3}$$

$$\Rightarrow \quad \frac{(n-1)\frac{b}{a}}{n\frac{b}{a}} = \frac{25}{30} \Rightarrow \frac{n-1}{n} = \frac{5}{6} \Rightarrow n = 6$$

Now, $a^n = 729 \Rightarrow a^6 = 3^6 \Rightarrow a = 3$

$$\therefore \quad \frac{nb}{a} = 10 \Rightarrow \frac{6 \times b}{3} = 10 \Rightarrow b = 5$$

35. Let $(r+1)^{th}$ term, in the expansion of $\left(\sqrt{x}-\frac{k}{x^2}\right)^{10}$, be free from x and be equal to T_{r+1}. Then,

$$T_{r+1} = {}^{10}C_r\,(\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r\,x^{5-\frac{5r}{2}}\,(-k)^r \quad \text{...(i)}$$

If T_{r+1} is independent of x, then

$5-\frac{5r}{2}=0 \Rightarrow r=2$

Putting $r=2$ in (i), we obtain

$T_3 = {}^{10}C_r\,(-k)^2 = 45k^2$

But, it is given that the value of term free from x is 405.

$\therefore\quad 45k^2 = 405 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

36. In the binomial expansion of $(y^{1/2}+x^{1/3})^n$, there are $(n+1)$ terms. The third term from the end is $((n+1)-3+1)^{th}$ i.e. $(n-1)^{th}$ term from the beginning.

$\therefore\quad$ The binomial coefficient of 3rd term from the end

$=$ The binomial coefficient of $(n-1)^{th}$ term from the beginning $= {}^nC_{n-2} = {}^nC_2$

It is given that the binomial coefficient of the third term from the end is 45.

$\therefore\quad {}^nC_2 = 45 \Rightarrow \frac{n(n-1)}{2} = 45 \Rightarrow n^2-n-90=0 \Rightarrow (n-10)(n+9)=0 \Rightarrow n=10.$

Let T_6 be the sixth term in the binomial expansion of $(y^{1/2}+x^{1/3})^n$. Then,

$$T_6 = {}^nC_5\,(y^{1/2})^{n-5}\,(x^{1/3})^5 = {}^{10}C_5\,y^{5/2}\,x^{5/3} = 252\,y^{5/2}x^{5/3} \quad [\because n=10]$$

37. In the expansion of $\left(\frac{p}{2}+2\right)^8$, we observe that $\left(\frac{8}{2}+1\right)^{th}$ i.e. 5th term is the middle term. It is given that the middle term is 1120.

$\therefore\quad T_5 = 1120$

$\Rightarrow\quad {}^8C_4\left(\frac{p}{2}\right)^{8-4}(2)^4 = 1120 \Rightarrow p^4 = 16 \Rightarrow p = \pm 2$

38. In the binomial expansion of $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)^n$, $\left((n+1)-7+1\right)^{th}$ i.e. $(n-5)^{th}$ term from the beginning is the 7th term from the end.

Now,

$$T_7 = {}^nC_6\,(\sqrt[3]{2})^{n-6}\left(\frac{1}{\sqrt[3]{3}}\right)^6 = {}^nC_6 \times 2^{\frac{n}{3}-2} \times \frac{1}{3^2}$$

and, $T_{n-5} = {}^nC_{n-6}\,(\sqrt[3]{2})^6\left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} = {}^nC_6 \times 2^2 \times \frac{1}{3^{n/3-2}}$

It is given that

$$\frac{T_7}{T_{n-5}} = \frac{1}{6}$$

$$\Rightarrow \frac{{}^nC_6 \times 2^{(n/3)-2} \times \frac{1}{3^2}}{{}^nC_6 \times 2^2 \times \frac{1}{3^{(n/3)-2}}} = \frac{1}{6}$$

$$\Rightarrow 2^{(n/3)-4} \times 3^{(n/3)-4} = 6^{-1}$$

$$\Rightarrow 6^{(n/3)-4} = 6^{-1} \Rightarrow \frac{n}{3} - 4 = -1 \Rightarrow n = 9$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the number of terms in the expansion of $(2+\sqrt{3}x)^{10} + (2-\sqrt{3}x)^{10}$.
2. Write the sum of the coefficients in the expansion of $(1-3x+x^2)^{111}$.
3. Write the number of terms in the expansion of $(1-3x+3x^2-x^3)^8$.
4. Write the middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$.
5. Which term is independent of x, in the expansion of $\left(x-\frac{1}{3x^2}\right)^9$?
6. If a and b denote respectively the coefficients of x^m and x^n in the expansion of $(1+x)^{m+n}$, then write the relation between a and b.
7. If a and b are coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then write the relation between a and b.
8. Write the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.
9. If a and b denote the sum of the coefficients in the expansions of $(1-3x+10x^2)^n$ and $(1+x^2)^n$ respectively, then write the relation between a and b.
10. Write the coefficient of the middle term in the expansion of $(1+x)^{2n}$.
11. Write the number of terms in the expansion of $\{(2x+y^3)^4\}^7$.
12. Find the sum of the coefficients of two middle terms in the binomial expansion of $(1+x)^{2n-1}$.
13. Find the ratio of the coefficients of x^p and x^q in the expansion of $(1+x)^{p+q}$.
14. Write last two digits of the number 3^{400}.
15. Find the number of terms in the expansion of $(a+b+c)^n$.
16. If a and b are the coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, find $\frac{a}{b}$.

17. Write the total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$.

18. If $(1-x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$, find the value of $a_0 + a_2 + a_4 + \ldots + a_{2n}$.

ANSWERS

1. 6 2. -1 3. 25 4. 252 5. 4^{th} term 6. $a = b$ 7. $a = 2b$
8. ${}^{10}C_5$ 9. $a = b^3$ 10. ${}^{2n}C_n$ 11. 29 12. ${}^{2n}C_n$ 13. 1 14. 01
15. $\frac{n(n+1)}{2}$ 16. 2 17. 51 18. $\frac{3^n+1}{2}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in the expansion of $(1+x)^{20}$, the coefficients of rth and $(r+4)$th terms are equal, then r is equal to
   (a) 7 (b) 8 (c) 9 (d) 10

2. The term without x in the expansion of $\left(2x - \frac{1}{2x^2}\right)^{12}$ is
   (a) 495 (b) -495 (c) -7920 (d) 7920

3. If rth term in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$ is without x, then r is equal to
   (a) 8 (b) 7 (c) 9 (d) 10

4. If in the expansion of $(a+b)^n$ and $(a+b)^{n+3}$, the ratio of the coefficients of second and third terms, and third and fourth terms respectively are equal, then n is
   (a) 3 (b) 4 (c) 5 (d) 6

5. If A and B are the sums of odd and even terms respectively in the expansion of $(x+a)^n$, then $(x+a)^{2n} - (x-a)^{2n}$ is equal to
   (a) $4(A+B)$ (b) $4(A-B)$ (c) AB (d) $4AB$

6. The number of irrational terms in the expansion of $\left(4^{1/5} + 7^{1/10}\right)^{45}$ is
   (a) 40 (b) 5 (c) 41 (d) none of these

7. The coefficient of x^{-17} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ is
   (a) 1365 (b) -1365 (c) 3003 (d) -3003

8. In the expansion of $\left(x^2 - \frac{1}{3x}\right)^9$, the term without x is equal to
   (a) $\frac{28}{81}$ (b) $\frac{-28}{243}$ (c) $\frac{28}{243}$ (d) none of these

9. If in the expansion of $(1+x)^{15}$, the coefficients of $(2r+3)^{th}$ and $(r-1)^{th}$ terms are equal, then the value of r is
   (a) 5 (b) 6 (c) 4 (d) 3

10. The middle term in the expansion of $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$ is

(a) 251 (b) 252 (c) 250 (d) none of these

11. If in the expansion of $\left(x^4-\frac{1}{x^3}\right)^{15}$, x^{-17} occurs in rth term, then

(a) $r=10$ (b) $r=11$ (c) $r=12$ (d) $r=13$

12. In the expansion of $\left(x-\frac{1}{3x^2}\right)^9$, the term independent of x is

(a) T_3 (b) T_4 (c) T_5 (d) none of these

13. If in the expansion of $(1+y)^n$, the coefficients of 5th, 6th and 7th terms are in A.P., then n is equal to

(a) 7, 11 (b) 7, 14 (c) 8, 16 (d) none of these

14. In the expansion of $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$, the term independent of x is

(a) T_5 (b) T_6 (c) T_7 (d) T_8

15. If the sum of odd numbered terms and the sum of even numbered terms in the expansion of $(x+a)^n$ are A and B respectively, then the value of $(x^2-a^2)^n$ is

(a) A^2-B^2 (b) A^2+B^2 (c) $4AB$ (d) none of these

16. If the coefficient of x in $\left(x^2+\frac{\lambda}{x}\right)^5$ is 270, then $\lambda=$

(a) 3 (b) 4 (c) 5 (d) none of these

17. The coefficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is

(a) $\frac{405}{256}$ (b) $\frac{504}{259}$ (c) $\frac{450}{263}$ (d) none of these

18. The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplification is

(a) 202 (b) 51 (c) 50 (d) none of these

19. If T_2/T_3 in the expansion of $(a+b)^n$ and T_3/T_4 in the expansion of $(a+b)^{n+3}$ are equal, then $n=$

(a) 3 (b) 4 (c) 5 (d) 6

20. The coefficient of $\frac{1}{x}$ in the expansion of $(1+x)^n\left(1+\frac{1}{x}\right)^n$ is

(a) $\frac{n!}{\{(n-1)!(n+1)!\}}$ (b) $\frac{(2n)!}{[(n-1)!(n+1)!]}$

(c) $\frac{(2n)!}{(2n-1)!(2n+1)!}$ (d) none of these

21. If the sum of the binomial coefficients of the expansion $\left(2x+\frac{1}{x}\right)^n$ is equal to 256, then the term independent of x is

(a) 1120 (b) 1020 (c) 512 (d) none of these

22. If the fifth term of the expansion $(a^{2/3} + a^{-1})^n$ does not contain 'a'. Then n is equal to

(a) 2 (b) 5 (c) 10 (d) none of these

23. The coefficient of x^{-3} in the expansion of $\left(x - \frac{m}{x}\right)^{11}$ is

(a) $-924\, m^7$ (b) $-792\, m^5$ (c) $-792\, m^6$ (d) $-330\, m^7$

24. The coefficient of the term independent of x in the expansion of $\left(ax + \frac{b}{x}\right)^{14}$ is

(a) $14!\, a^7 b^7$ (b) $\frac{14!}{7!} a^7 b^7$ (c) $\frac{14!}{(7!)^2} a^7 b^7$ (d) $\frac{14!}{(7!)^3} a^7 b^7$

25. The coefficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \ldots + (1 + x)^{30}$ is

(a) ${}^{51}C_5$ (b) ${}^{9}C_5$ (c) ${}^{31}C_6 - {}^{21}C_6$ (d) ${}^{30}C_5 + {}^{20}C_5$

26. The coefficient of $x^8 y^{10}$ in the expansion of $(x + y)^{18}$ is

(a) ${}^{18}C_8$ (b) ${}^{18}P_{10}$ (c) 2^{18} (d) none of these

27. If the coefficients of the $(n + 1)^{th}$ term and the $(n + 3)^{th}$ term in the expansion of $(1 + x)^{20}$ are equal, then the value of n is

(a) 10 (b) 8 (c) 9 (d) none of these

28. If the coefficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$, $n \in N$ are in A.P., then $n =$

(a) 7 (b) 14 (c) 2 (d) none of these

29. The middle term in the expansion of $\left(\frac{2x}{3} - \frac{3}{2x^2}\right)^{2n}$ is

(a) ${}^{2n}C_n$ (b) $(-1)^n\ {}^{2n}C_n\, x^{-n}$ (c) ${}^{2n}C_n\, x^{-n}$ (d) none of these

30. If r^{th} term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$, then $(r + 3)^{th}$ term is

(a) ${}^{20}C_{14}\left(\frac{x}{2^{14}}\right)$ (b) ${}^{20}C_{12}\, x^2\, 2^{-12}$ (c) $-{}^{20}C_7\, x \,.\, 2^{-13}$ (d) none of these

31. The number of terms with integral coefficients in the expansion of $\left(17^{1/3} + 35^{1/2} x\right)^{600}$ is

(a) 100 (b) 50 (c) 150 (d) 101

32. Constant term in the expansion of $\left(x - \frac{1}{x}\right)^{10}$ is

(a) 152 (b) -152 (c) -252 (d) 252

33. If the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are the same, then the value of a is

(a) $-\frac{7}{9}$ (b) $-\frac{9}{7}$ (c) $\frac{7}{9}$ (d) $\frac{9}{7}$

ANSWERS

1. (c)	2. (d)	3. (c)	4. (c)	5. (d)	6. (c)	7. (b)	8. (c)
9. (a)	10. (b)	11. (c)	12. (b)	13. (b)	14. (b)	15. (a)	16. (a)
17. (a)	18. (b)	19. (c)	20. (b)	21. (a)	22. (c)	23. (d)	24. (c)
25. (b)	26. (a)	27. (c)	28. (a)	29. (b)	30. (c)	31. (d)	32. (c)
33. (d)							

SUMMARY

1. (*Binomial theorem*) If x and a are real numbers, then for all $n \in N$, we have

$$(x+a)^n = {}^nC_0\, x^n a^0 + {}^nC_1\, x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 + \dots\dots\dots + {}^nC_r\, x^{n-r} a^r + \dots + {}^nC_{n-1}\, x^1 a^{n-1} + {}^nC_n\, x^0 a^n$$

i.e., $$(x+a)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} a^r$$

This expansion has the following properties:

(i) It has $(n+1)$ terms.

(ii) The sum of the indices of x and a in each term is n.

(iii) The coefficients of terms equidistant from the beginning and the end are equal.

(vi) General term is given by $T_{r+1} = {}^nC_r\; x^{n-r}\; a^r$

(v) $(x+a)^n = \sum_{r=0}^{n} {}^nC_r\; x^{n-r}\, a^r$ can also be expressed as $(x+a)^n = \sum_{r+s=n} \frac{n!}{r!\,s!}\, x^r\, a^s$

(vi) Replacing a by $-a$ in the expansion of $(x+a)^n$, we get

$$(x-a)^n = {}^nC_0\; x^n a^0 - {}^nC_1\; x^{n-1} a^1 + {}^nC_2\, x^{n-2} a^2 - {}^nC_3\, x^{n-3} a^3 + \dots + (-1)^r\; {}^nC_r\; x^{n-r}\; a^r + \dots + (-1)^n\; {}^nC_n\; x^0\; a^n$$

The general term in the expansion of $(x-a)^n$ is $T_{r+1} = (-1)^r\; {}^nC_r\; x^{n-r}\; a^r$

(vii) Putting $x = 1$ and replacing a by x in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1\, x + {}^nC_2\, x^2 + \dots + {}^nC_n\, x^n = \sum_{r=0}^{n} {}^nC_r\; x^r$$

This is expansion of $(1+x)^n$ is ascending powers of x. In this case, $T_{r+1} = {}^nC_r\, x^r$

(viii) Putting $a = 1$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1} + {}^nC_2\, x^{n-2} + \dots + {}^nC_n\, x^0 = \sum_{r=0}^{n} {}^nC_r\; x^{n-r}$$

This is the expansion of $(1+x)^n$ in descending powers of x. In this case, $T_{r+1} = {}^nC_r\; x^{n-r}$

(ix) $$(x+a)^n + (x-a)^n = 2\left\{ {}^nC_0\, x^n a^0 + {}^nC_2\, x^{n-2} a^2 + \dots \right\}$$

$$= 2\,\{\text{Sum of the odd terms in the expansion of } (x+a)^n\}$$

$$(x+a)^n - (x-a)^n = 2\left\{ {}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \dots \right\}$$

$$= 2\,\{\text{Sum of the even terms in the expansion of } (x+a)^n\}$$

If n is odd, then $\left\{(x+a)^n+(x-a)^n\right\}$ and $\left\{(x+a)^n-(x-a)^n\right\}$ both have $\left(\frac{n+1}{2}\right)$ terms.

If n is even, then $\left\{(x+a)^n+(x-a)^n\right\}$ has $\left(\frac{n}{2}+1\right)$ terms whereas $\left\{(x+a)^n-(x-a)^n\right\}$ has $\left(\frac{n}{2}\right)$ terms.

(x) If O and E denote respectively the sums of odd terms and even terms in the expansion of $(x+a)^n$, then

(a) $(x+a)^n = O+E$ and $(x-a)^n = O-E$ (b) $(x^2-a^2)^n = O^2-E^2$

(c) $4\,OE = (x-a)^{2n}-(n-a)^{2n}$ (d) $(x+a)^{2n}+(x-a)^{2n} = 2\,(O^2+E^2)$

(xi) If n is even, then $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

If n is odd, then $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+3}{2}\right)$ are middle terms.