CHAPTER 1

RELATION

1.1 INTRODUCTION

In Class XI, we have introduced the notion of a relation, its domain, co-domain and range. Let us recall that a relation from a set A to a set B is a subset of $A \times B$. If R is a relation from a set A to a set B and $(a, b) \in R$, then we say that a is related to b under relation R and we write as $a\ R\ b$. If $(a, b) \notin R$, then we say that a is not related to b under R and we write as $a \not R\, b$. A relation can be represented in roster form or tabular form. Sometimes, we also describe a relation by describing the common property between the elements of ordered pairs in it. For example, a relation R on the set $A = \{1, 2, 3, 4, 5\}$ defined by $R = \{(a, b) : b = a + 2\}$ can also be expressed as: $a\ R\ b$ if and only if $b = a + 2$. In this chapter, we will study different types of relations.

1.2 RECAPITULATION

In this section, we will recall some definitions learnt in the earlier class.

CARTESIAN PRODUCT OF SETS *Let A and B be two non-empty sets. The set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called the cartesian product of set A with set B and is denoted by $A \times B$.*

Thus, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. Note that $B \times A = \{(b, a) : b \in B \text{ and } a \in A\}$

Also, $A \times B = \phi$, if $A = \phi$ or $B = \phi$

If $A = \{1, 2, 3\}$ and $B = \{x, y\}$, then

$A \times B = \{(1, x), (1, y), (2, x), (2, y), (3, x), (3, y)\}$, $B \times A = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}$

$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

and, $B \times B = \{(x, x), (x, y), (y, x), (y, y)\}$

RELATION *Let A and B be two sets. Then a relation R from set A to set B is a subset of $A \times B$.*

Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

If R is a relation from a non-void set A to a non-void set B and if $(a, b) \in R$, then we write $a\ R\ b$ which is read as "a is related to b by the relation R". If $(a, b) \notin R$, then we write $a \not R\, b$ and we say that a is not related to b by the relation R.

If A and B are finite sets consisting of m and n elements respectively, then $A \times B$ has mn ordered pairs. Therefore, total number of relations from A to B is 2^{mn}.

DOMAIN *Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R.*

Thus, domain of $R = \{a : (a, b) \in R\}$. Clearly, domain of $R \subseteq A$.

If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ is a relation from A to B, then Domain $(R) = \{1, 3, 5\}$.

RANGE *Let R be a relation from a set A to a set B. Then the set of all second components or coordinates of the ordered pairs belonging to R is called the range of R.*

Thus, Range of $R = \{b : (a, b) \in R\}$. Clearly, range of $R \subseteq B$.

If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8, 10\}$ and $R = \{(1, 8), (3, 6), (5, 2), (1, 4)\}$ is a relation from A to B, then Range $(R) = \{8, 6, 2, 4\}$.

RELATION ON A SET *Let A be a non-void set. Then a relation from A to itself i.e. a subset of $A \times A$ is called a relation on set A.*

INVERSE OF A RELATION *Let A, B be two sets and let R be a relation from a set A to a set B. Then, the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$.*

Clearly, $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}$. Also, Domain (R) = Range (R^{-1}) and, Range (R) = Domain (R^{-1})

Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ be two sets and $R = \{(1, a), (1, c), (2, d), (2, c)\}$ be a relation from A to B. Then, $R^{-1} = \{(a, 1), (c, 1), (d, 2), (c, 2)\}$ is a relation from B to A.

1.3 TYPES OF RELATIONS

In this section, we intend to discuss various types of relations on a set A.

1.3.1 VOID, UNIVERSAL AND IDENTITY RELATIONS

VOID RELATION *Let A be a set. Then, $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on set A.*

In other words, a relation R on a set A is called void or empty relation, if no element of A is related to any element of A.

Consider the relation R on the set $A = \{1, 2, 3, 4, 5\}$ defined by $R = \{(a, b) : a - b = 12\}$.

We observe that $a - b \neq 12$ for any two elements of A.

$\therefore$ $(a, b) \notin R$ for any $a, b \in A$.

$\Rightarrow$ R does not contain any element of $A \times A \Rightarrow R$ is empty set $\Rightarrow R$ is the void relation on A.

UNIVERSAL RELATION *Let A be a set. Then, $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.*

In other words, a relation R on a set is called universal relation, if each element of A is related to every element of A.

Consider the relation R on the set $A = \{1, 2, 3, 4, 5, 6\}$ defined by $R = \{(a, b) \in R : |a - b| \geq 0\}$.

We observe that

$|a - b| \geq 0$ for all $a, b \in A$

$\Rightarrow$ $(a, b) \in R$ for all $(a, b) \in A \times A$

$\Rightarrow$ Each element of set A is related to every element of set A

$\Rightarrow$ $R = A \times A$

$\Rightarrow$ R is universal relation on set A

NOTE *It is to note here that the void relation and the universal relation on a set A are respectively the smallest and the largest relations on set A. Both the empty (or void) relation and the universal relation are sometimes called trivial relations.*

ILLUSTRATION *Let A be the set of all students of a boys school. Show that the relation R on A given by $R = \{(a, b) : a$ is sister of $b\}$ is empty relation and $R' = \{(a, b)$: the difference between the heights of a and b is less than 5 meters$\}$ is the universal relation.*

SOLUTION Since the school is boys school. Therefore, no student of the school can be sister of any student of the school. Thus, $(a, b) \notin R$ for any $a, b \in A$.

Hence, $R = \phi$ i.e. R is the empty or void relation on A.

It is obvious that the difference between the heights of any two students of the school has to be less than 5 meters.

$\therefore$ $(a, b) \in R$ for all $a, b \in A \Rightarrow R = A \times A \Rightarrow R$ is the universal relation on set A.

IDENTITY RELATION *Let A be a set. Then, the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.*

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

If $A = \{1, 2, 3\}$, then the relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set A. But, relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to two elements 1 and 3.

1.3.2 REFLEXIVE RELATION

DEFINITION *A relation R on a set A is said to be reflexive, if every element of A is related to itself.*

Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.

A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

ILLUSTRATION 1 *Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But, $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.*

ILLUSTRATION 2 *The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A. For example, the relation $R = \{(a, a), (b, b), (c, c), (a, b)\}$ is a reflexive relation on set $A = \{a, b, c\}$ but it is not the identity relation on A.*

ILLUSTRATION 3 *The universal relation on a non-void set A is reflexive.*

ILLUSTRATION 4 *A relation R on N defined by $(x, y) \in R \Leftrightarrow x \geq y$ is a reflexive relation on N, because every natural number is greater than or equal to itself.*

ILLUSTRATION 5 *Let X be a non-void set and $P(X)$ be the power set of X. A relation R on $P(X)$ defined by $(A, B) \in R \Leftrightarrow A \subseteq B$ is a reflexive relation since every set is subset of itself.*

ILLUSTRATION 6 *Let L be the set of all lines in a plane. Then relation R on L defined by $(l_1, l_2) \in R \Leftrightarrow l_1$ is parallel to l_2 is reflexive, since every line is parallel to itself.*

1.3.3 SYMMETRIC RELATION

DEFINITION *A relation R on a set A is said to be a symmetric relation iff*

$$(a, b) \in R \Rightarrow (b, a) \in R \text{ for all } a, b \in A \text{ i.e. } aRb \Rightarrow bRa \text{ for all } a, b \in A.$$

ILLUSTRATION 1 *The identity and the universal relations on a non-void set are symmetric relations.*

ILLUSTRATION 2 *Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then, R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$ i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.*

ILLUSTRATION 3 *Let S be a non-void set and R be a relation defined on power set $P(S)$ by*

$(A, B) \in R \Leftrightarrow A \subseteq B$ for all $A, B \in P(S)$. Then, R is not a symmetric relation.

NOTE *A relation R on a set A is not a symmetric relation if there are at least two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.*

ILLUSTRATION 4 *Let $A = \{1, 2, 3, 4\}$ and let R_1 and R_2 be relations on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2) (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.*

NOTE *A reflexive relation on a set A is not necessarily symmetric. For example, the relation $R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ is a reflexive relation on set $A = \{1, 2, 3\}$ but it is not symmetric.*

ILLUSTRATION 5 *Prove that a relation R on a set A is symmetric iff $R = R^{-1}$.*

SOLUTION First, let R be a symmetric relation on set A. Then, we have to prove that $R = R^{-1}$. In order to prove this we have to prove that $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$.

Now, $(a, b) \in R$

$\Rightarrow$ $(b, a) \in R$ [$\because$ R is symmetric]

$\Rightarrow$ $(a, b) \in R^{-1}$ [By definition of inverse relation]

Thus, $(a, b) \in R \Rightarrow (a, b) \in R^{-1}$ for all $a, b \in A$.

So, $R \subseteq R^{-1}$ ...(i)

Now, let (x, y) be an arbitrary element of R^{-1}. Then,

$(x, y) \in R^{-1}$

$\Rightarrow$ $(y, x) \in R$ [By definition of inverse relation]

$\Rightarrow$ $(x, y) \in R$ [$\because$ R is symmetric]

Thus, $(x, y) \in R^{-1} \Rightarrow (x, y) \in R$ for all $x, y \in A$.

So, $R^{-1} \subseteq R$ ...(ii)

From (i) and (ii), we obtain: $R = R^{-1}$.

Conversely, let R be a relation on set A such that $R = R^{-1}$. Then we have to prove that R is a symmetric relation on set A. Let $(a, b) \in R$. Then,

$(a, b) \in R$

$\Rightarrow$ $(b, a) \in R^{-1}$ [by definition of inverse relation]

$\Rightarrow$ $(b, a) \in R$ [$\because R = R^{-1}$]

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

So, R is a symmetric relation on A. Hence, R is symmetric iff $R = R^{-1}$.

1.3.4 TRANSITIVE RELATION

DEFINITION *Let A be any set. A relation R on A is said to be a transitive relation iff* $(a, b) \in R$ *and* $(b, c) \in R \Rightarrow (a, c) \in R$ *for all* $a, b, c \in A$. i.e., aRb *and* $bRc \Rightarrow aRc$ *for all* $a, b\, c \in A$.

ILLUSTRATION 1 *The identity and the universal relations on a non-void set are transitive.*

ILLUSTRATION 2 *The relation R on the set N of all natural numbers defined by*

$(x, y) \in R \Leftrightarrow x$ *divides* y, *for all* $x, y \in N$ *is transitive.*

SOLUTION Let $x, y, z \in N$ be such that $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ x divides y and, y divides z

$\Rightarrow$ There exist $p, q \in N$ such that $y = xp$ and $z = yq$

$\Rightarrow$ $z = (xp)\, q$

$\Rightarrow$ $z = x\,(pq)$

$\Rightarrow$ x divides z [$\because$ $pq \in N$]

$\Rightarrow$ $(x, z) \in R$

Thus, $(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in N$. Hence, R is a transitive relation on N.

ILLUSTRATION 3 *On the set N of natural numbers, the relation R defined by $xRy \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$,* we find that

$x < y$ *and* $y < z \Rightarrow x < z$ *i.e.,* xRy *and* $yRz \Rightarrow xRz$

ILLUSTRATION 4 *Let S be a non-void set and R be a relation defined on power set $P(S)$ by $(A, B) \in R \Leftrightarrow A \subseteq B$ for all $A, B \in P(S)$. Then, R is a transitive relation on $P(S)$, because for any $A, B, C \in P(S)$*

$(A, B) \in R$ and $(B, C) \in R \Rightarrow A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C \Rightarrow (A, C) \in R$

ILLUSTRATION 5 *Let L be the set of all straight lines in a plane. Then the relation "is parallel to" on L is a transitive relation, because for any $l_1, l_2, l_3 \in L$, we find that: $l_1 \| l_2$ and $l_2 \| l_3 \Rightarrow l_1 \| l_3$.*

ILLUSTRATION 6 *The relation "is congruent to" on the set T of all triangles in a plane is a transitive relation.*

1.3.5 ANTISYMMETRIC RELATION

DEFINITION *A relation R on set A is said to be an antisymmetric relation iff*

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b \text{ for all } a, b \in A$$

NOTE *It follows from this definition that if $(a, b) \in R$ but $(b, a) \notin R$, then also R is an antisymmetric relation.*

ILLUSTRATION 1 *The identity relation on a set A is an antisymmetric relation.*

ILLUSTRATION 2 *The universal relation on a set A containing at least two elements is not antisymmetric, because if $a \neq b$ are in A, then a is related to b and b is related to a under the universal relation will imply that $a = b$ but $a \neq b$.*

ILLUSTRATION 3 *Let R be a relation on the set N of natural numbers defined by*

$$xRy \Leftrightarrow \text{'x divides y' for all } x, y \in N$$

This relation is an antisymmetric relation on N. Since for any two numbers $a, b \in N$, we find that

$$a \mid b \text{ and } b \mid a \Rightarrow a = b \text{ i.e. } aRb \text{ and } bRa \Rightarrow a = b$$

It should be noted that this relation is not antisymmetric on the set Z of integers, because we find that for any non-zero integer a, $aR\,(-a)$ and $(-a)\ R\ a$ but $a \neq -a$.

ILLUSTRATION 4 *Let S be a non-void set and R be a relation on the power set P(S) defined by*

$$(A, B) \in R \Leftrightarrow A \subseteq B \text{ for all } A, B \in P(S)$$

Then, R is an antisymmetric relation on P(S), because

$$(A, B) \in R \text{ and } (B, A) \in R \Rightarrow A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B$$

ILLUSTRATION 5 *The relation $\leq$ ("less than or equal to") on the set R of real numbers is antisymmetric, because $a \leq b$ and $b \leq a \Rightarrow a = b$ for all $a, b \in R$.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Three relations R_1, R_2 and R_3 are defined on set $A = \{a, b, c\}$ as follows:*

(i) $R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$,

(ii) $R_2 = \{(a, b), (b, a), (a, c), (c, a)\}$

(iii) $R_3 = \{(a, b), (b, c), (c, a)\}$.

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

SOLUTION (i) *Reflexive:* Clearly $(a, a), (b, b), (c, c) \in R_1$. So, R_1 is reflexive on A.

Symmetric: We observe that $(a, b) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not a symmetric relation on A.

Transitive: We find that $(b, c) \in R_1$ and $(c, a) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not a transitive relation on A.

(ii) *Reflexive:* Since $(a, a), (b, b)$ and (c, c) are not in R_2. So, it is not a reflexive relation on A.

Symmetric: We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive: Clearly $(a, b) \in R_2$ and $(b, a) \in R_2$ but $(a, a) \notin R_2$. So, it is not a transitive relation on A.

(iii) *Reflexive:* Since none of $(a, a), (b, b)$ and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric: Clearly, $(b, c) \in R_3$ but $(c, b) \notin R_3$. So, R_3 is not a symmetric relation on A.

Transitive: Clearly, $(a, b) \in R_3$ and $(b, c) \in R_3$ but $(a, c) \notin R_3$. So, R_3 is not a transitive relation on A.

EXAMPLE 2 *Show that the relation R on the set $A = \{1, 2, 3\}$ given by $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$ is reflexive but neither symmetric nor transitive.* **[NCERT]**

SOLUTION Since $1, 2, 3 \in A$ and $(1, 1), (2, 2), (3, 3) \in R$ i.e. for each $a \in A$, $(a, a) \in R$. So, R is reflexive.

We observe that $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric.

Also, $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. So, R is not transitive.

EXAMPLE 3 *Show that the relation R on the set $A = \{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ is symmetric but neither reflexive nor transitive.* **[NCERT]**

SOLUTION We observe that $(1, 1), (2, 2)$ and $(3, 3)$ do not belong to R. So, R is not reflexive.

Clearly, $(1, 2) \in R$ and $(2, 1) \in R$. So, R is symmetric.

As $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$. So, R is not transitive.

EXAMPLE 4 *Check the following relations R and S for reflexivity, symmetry and transitivity:*

(i) *aRb iff b is divisible by a, where $a, b \in N$*

(ii) *$l_1 \, S \, l_2$ iff $l_1 \perp l_2$, where l_1 and l_2 are straight lines in a plane.* **[NCERT]**

SOLUTION (i) We have, $aRb \Leftrightarrow a \mid b$ for all $a, b \in N$.

Reflexivity: For any $a \in N$, we find that $a \mid a \Rightarrow aRa$. Thus, aRa for all $a \in N$.

So, R is reflexive on N.

Symmetry: R is not symmetric because if $a \mid b$, then b may not divide a. For example, $2 \mid 6$ but $6 \nmid 2$.

Transitivity: Let $a, b, c \in N$ such that aRb and bRc. Then,

$$aRb \text{ and } bRc \Rightarrow a \mid b \text{ and } b \mid c \Rightarrow a \mid c \Rightarrow aRc.$$

So, R is a transitive relation on N.

(ii) Let L be the set of all lines in a plane. We are given that $l_1 \, S \, l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in L$.

Reflexivity: S is not reflexive because a line cannot be perpendicular to itself i.e. $l \perp l$ is not true.

Symmetry: Let $l_1, l_2 \in L$ such that $l_1 \, S \, l_2$. Then, $l_1 \, S \, l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_1 \Rightarrow l_2 \, S \, l_1$.

So, S is symmetric on L.

Transitive: S is not transitive, because $l_1 \perp l_2$ and $l_2 \perp l_3$ does not imply that $l_1 \perp l_3$.

EXAMPLE 5 *Let a relation R_1 on the set R of real numbers be defined as $(a, b) \in R_1 \Leftrightarrow 1 + ab > 0$ for all $a, b \in R$. Show that R_1 is reflexive and symmetric but not transitive.*

SOLUTION We observe the following properties of relation R_1:

Reflexivity: Let a be an arbitrary element of R. Then,

$$a \in R$$

$$\Rightarrow \quad 1 + a \cdot a = 1 + a^2 > 0 \qquad [\because a^2 > 0 \text{ for all } a \in R]$$

$$\Rightarrow \quad (a, a) \in R_1 \qquad [\text{By definition of } R_1]$$

Thus, $(a, a) \in R_1$ for all $a \in R$. So, R_1 is reflexive on R.

Symmetry: Let $(a, b) \in R_1$. Then,

$$(a, b) \in R_1$$

$$\Rightarrow \quad 1 + ab > 0$$

$$\Rightarrow \quad 1 + ba > 0 \qquad [\because ab = ba \text{ for all } a, b \in R]$$

$$\Rightarrow \quad (b, a) \in R_1 \qquad [\text{By definition of } R_1]$$

Thus, $(a, b) \in R_1 \Rightarrow (b, a) \in R_1$ for all $a, b \in R$. So, R_1 is symmetric on R.

Transitivity: We observe that $(1, 1/2) \in R_1$ and $(1/2, -1) \in R_1$ but $(1, -1) \notin R_1$ because $1 + 1 \times (-1) = 0 \not> 0$. So, R_1 is not transitive on R.

EXAMPLE 6 *Determine whether each of the following relations are reflexive, symmetric and transitive:*

(i) *Relation R on the set $A = \{1, 2, 3, \dots, 13, 14\}$ defined as $R = \{(x, y) : 3x - y = 0\}$*

(ii) *Relation R on the set N of all natural numbers defined as* $R = \{(x, y) : y = x + 5 \text{ and } x < 4\}$

(iii) *Relation R on the set* $A = \{1, 2, 3, 4, 5, 6\}$ *defined as* $R = \{(x, y) : y \text{ is divisible by } x\}$

(iv) *Relation R on the set Z of all integer defined as* $R = \{(x, y) : x - y \text{ is an integer}\}$ **[NCERT]**

SOLUTION (i) We have, $R = \{(x, y) : 3x - y = 0\}$, where $x, y \in A = \{1, 2, 3, \ldots, 13, 14\}$

i.e., $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Reflexivity: Clearly, $(1, 1) \notin R$. So, R is not a reflexive relation on A.

Symmetry: We observe that $(1, 3) \in R$ but $(3, 1) \notin R$. So, R is not a symmetric relation A.

Transitivity: We observe that $(1, 3) \in R$ and $(3, 9) \in R$ but $(1, 9) \notin R$. So, R is not a transitive relation A.

(ii) We have, $R = \{(x, y) : y = x + 5 \text{ and } x < 4, \text{ where } x, y \in N\}$ i.e. $R = \{(1, 6), (2, 7), (3, 8)\}$

Reflexivity: Clearly, $(1, 1), (2, 2)$ etc. are not in R. So, R is not reflexive.

Symmetry: We find that $(1, 6) \in R$ but $(6, 1) \notin R$. So, R is not symmetric.

Transitivity: Since $(1, 6) \in R$ and there is no order pair in R which has 6 as the first element. Same is the case for $(2, 7)$ and $(3, 8)$. So, R is transitive.

(iii) $R = \{(x, y) : y \text{ is divisible by } x\}$, where $x, y \in A = \{1, 2, 3, 4, 5, 6\}$.

Reflexivity: We know that

x is divisible by x for all $x \in A \Rightarrow (x, x) \in R$ for all $x \in A \Rightarrow R$ is reflexive on set A.

Symmetry: We observe that 6 is divisible by 2 but 2 is not divisible by 6. This means that $(2, 6) \in R$ but $(6, 2) \notin R$. So, R is not symmetric on set A.

Transitivity: Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R$ and $(y, z) \in R$.

$\Rightarrow$ y is divisible by x and, z is divisible by $y \Rightarrow z$ is divisible by $x \Rightarrow (x, z) \in R$

So, R is transitive relation on A.

(iv) We have, $R = \{(x, y) : x - y \text{ is an integer, where } x, y \in Z\}$

Reflexivity: We find that

$x - x = 0$, which is an integer for all $x \in Z \Rightarrow (x, x) \in R$ for all $x \in Z \Rightarrow R$ is reflexive on Z.

Symmetry: Let $(x, y) \in R$. Then,

$(x, y) \in R$

$\Rightarrow$ $x - y$ is an integer, say, λ

$\Rightarrow$ $y - x = -\lambda$

$\Rightarrow$ $y - x$ is an integer [$\because \lambda \in Z \Rightarrow -\lambda \in Z$]

$\Rightarrow$ $(y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$. So, R is symmetric on Z.

Transivity: Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $x - y$ and $y - z$ are integers

$\Rightarrow$ $(x - y) + (y - z)$ is an integer [$\because$ Sum of two integers is an integer]

$\Rightarrow$ $x - z$ is an integer $\Rightarrow (x, z) \in R$

So, R is transitive on Z.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Show that the relation R on* $\mathbf{R}$ *defined as* $R = \{(a, b) : a \le b\}$, *is reflexive and transitive but not symmetric.* **[NCERT, CBSE 2019]**

SOLUTION We have, $R = \{(a, b) : a \le b\}$, where $a, b \in \mathbf{R}$.

Reflexivity: For any $a \in \mathbf{R}$, we find that

$a \le a \Rightarrow (a, a) \in R$ for all $a \in \mathbf{R} \Rightarrow R$ is reflexive.

Symmetry: We observe that $(2, 3) \in R$ but $(3, 2) \notin R$. So, R is not symmetric.

Transitivity: Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$$(a, b) \in R \text{ and } (b, c) \in R \Rightarrow a \leq b \text{ and } b \leq c \Rightarrow a \leq c \Rightarrow (a, c) \in R$$

So, R is transitive.

EXAMPLE 8 *Let S be the set of all points in a plane and R be a relation on S defined as*

$$R = \{(P, Q): \text{Distance between } P \text{ and } Q \text{ is less than 2 units}\}.$$

Show that R is reflexive and symmetric but not transitive.

SOLUTION We observe the following properties of relation R:

Reflexivity: For any point P in set S, we find that

Distance between P and itself is 0 which is less than 2 units $\Rightarrow (P, P) \in R$

Thus, $(P, P) \in R$ for all $P \in S$. So, R is reflexive on S.

Symmetry: Let P and Q be two points in S such that $(P, Q) \in R$. Then,

$(P, Q) \in R$

$\Rightarrow$ Distance between P and Q is less than 2 units.

$\Rightarrow$ Distance between Q and P is less than 2 units $\Rightarrow (Q, P) \in R$

So, R is symmetric on S.

Transitivity: Consider points P, Q and R having coordinates (0, 0), (1.5, 0) and (3.2, 0). We observe that the distance between P and Q is 1.5 units which is less than 2 units and the distance between Q and R is 1.7 units which is also less than 2 units. But, the distance between P and R is 3.2 which is not less than 2 units. This means that $(P, Q) \in R$ and $(Q, R) \in R$ but $(P, R) \notin R$. So, R is not transitive on S.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Let* $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. *Let* R_1 *be a relation on X given by* $R_1 = \{(x, y) : x - y$ *is divisible by 3\} and* R_2 *be another relation on X given by* $R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ *or* $\{x, y\} \subset \{2, 5, 8\}$ *or* $\{x, y\} \subset \{3, 6, 9\}\}$. *Show that* $R_1 = R_2$.

[NCERT]

SOLUTION Clearly, R_1 and R_2 are subsets of $X \times X$. In order to prove that $R_1 = R_2$, it is sufficient to show that $R_1 \subset R_2$ and $R_2 \subset R_1$.

We observe that the difference between any two elements of each of the sets $\{1, 4, 7\}$, $\{2, 5, 8\}$ and $\{3, 6, 9\}$ is a multiple of 3.

Let (x, y) be an arbitrary element of R_1. Then,

$(x, y) \in R_1$

$\Rightarrow$ $x - y$ is divisible by 3.

$\Rightarrow$ $x - y$ is a multiple of 3.

$\Rightarrow$ $\{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\} \Rightarrow (x, y) \in R_2$

Thus, $(x, y) \in R_1 \Rightarrow (x, y) \in R_2$. So, $R_1 \subset R_2$ ...(i)

Now, let (a, b) be an arbitrary element of R_2. Then,

$(a, b) \in R_2$

$\Rightarrow$ $\{a, b\} \subset \{1, 4, 7\}$ or $\{a, b\} \subset \{2, 5, 8\}$ or $\{a, b\} \subset \{3, 6, 9\}$

$\Rightarrow$ $a - b$ is divisible by 3 $\Rightarrow (a, b) \in R_1$

Thus, $(a, b) \in R_2 \Rightarrow (a, b) \in R_1$. So, $R_2 \subset R_1$ ...(ii)

From (i) and (ii), we get: $R_1 = R_2$

EXAMPLE 10 *Show that the relation R on the set* **R** *of all real numbers, defined as* $R = \{(a, b) : a \leq b^2\}$ *is neither reflexive nor symmetric nor transitive.*

[NCERT]

SOLUTION We have, $R = \{(a, b) : a \leq b^2\}$, where $a, b \in R$.

Reflexivity: We observe that $\frac{1}{2} \leq \left(\frac{1}{2}\right)^2$ is not true. Therefore, $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$. So, R is not reflexive.

Symmetry: We observe that $-1 \leq 3^2$ but $3 \not\leq (-1)^2$ i.e. $(-1, 3) \in R$ but $(3, -1) \notin R$. So, R is not symmetric.

Transivity: We observe that

$$2 \leq (-3)^2 \text{ and } -3 \leq 1^2 \text{ but } 2 \nleq 1^2 \text{ i.e. } (2, -3) \in R \text{ and } (-3, -1) \in R \text{ but } (2, 1) \notin R.$$

So, R is not transitive.

EXAMPLE 11 *Let $A = \{1, 2, 3\}$. Then, show that the number of relations containing $(1, 2)$ and $(2, 3)$ which are reflexive and transitive but not symmetric is three.* **[NCERT]**

SOLUTION The smallest reflexive relation on set A containing $(1, 2)$ and $(2, 3)$ is

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$$

Since $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. So, R is not transitive. To make it transitive we have to include $(1, 3)$ in R. Including $(1, 3)$ in R, we get: $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$.

This is reflexive and transitive but not symmetric as $(1, 3) \in R_1$ but $(3, 1) \notin R_1$.

Now, if we add the pair $(2, 1)$ to R_1 to get: $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (2, 1)\}$.

The relation R_2 is reflexive and transitive but not symmetric. Similarly, by adding $(3, 2)$ and $(3, 1)$ respectively to R_1, we get

$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 2)\}$$

$$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3), (3, 1), (3, 2)\}$$

These relations are reflexive and transitive but not symmetric.

We observe that out of ordered pairs $(2, 1)$, $(3, 2)$ and $(3, 1)$ at a time if we add any two ordered pairs at a time to R_1, then to maintain the transitivity we will be forced to add the remaining third pair and in this process the relation will become symmetric also which is not required. Hence, the total number of reflexive, transitive but not symmetric relations containing $(1, 2)$ and $(2, 3)$ is three.

EXERCISE 1.1

BASIC

1. Let A be the set of all human beings in a town at a particular time. Determine whether each of the following relations are reflexive, symmetric and transitive:
   (i) $R = \{(x, y) : x$ and y work at the same place$\}$
   (ii) $R = \{(x, y) : x$ and y live in the same locality$\}$
   (iii) $R = \{(x, y) : x$ is wife of $y\}$ (iv) $R = \{(x, y) : x$ is father of $y\}$ **[NCERT]**
2. Relations R_1, R_2, R_3 and R_4 are defined on a set $A = \{a, b, c\}$ as follows:
   $R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)\}$, $R_2 = \{(a, a)\}$, $R_3 = \{(b, c)\}$,
   $R_4 = \{(a, b), (b, c), (c, a)\}$.
   Find whether or not each of the relations R_1, R_2, R_3, R_4 on A is (i) reflexive (ii) symmetric (iii) transitive.
3. Test whether the following relations R_1, R_2, and R_3 are (i) reflexive (ii) symmetric and (iii) transitive:
   (i) R_1 on Q_0 defined by $(a, b) \in R_1 \Leftrightarrow a = 1/b$
   (ii) R_2 on Z defined by $(a, b) \in R_2 \Leftrightarrow |a - b| \leq 5$
   (iii) R_3 on R defined by $(a, b) \in R_3 \Leftrightarrow a^2 - 4ab + 3b^2 = 0$.

4. Let $A = \{1, 2, 3\}$, and let $R_1 = \{(1, 1), (1, 3), (3, 1), (2, 2), (2, 1), (3, 3)\}$, $R_2 = \{(2,2), (3, 1), (1, 3)\}$, $R_3 = \{(1, 3), (3, 3)\}$. Find whether or not each of the relations R_1, R_2, R_3 on A is (i) reflexive (ii) symmetric (iii) transitive.

BASED ON LOTS

5. Each of the following defines a relation on N:
 (i) $x > y, x, y \in N$ (ii) $x + y = 10, x, y \in N$
 (iii) xy is square of an integer $x, y \in N$ (iv) $x + 4y = 10, x, y \in N$
 Determine which of the above relations are reflexive, symmetric and transitive. [NCERT EXEMPLAR]
6. The following relations are defined on the set of real numbers:
 (i) aRb iff $a - b > 0$ (ii) aRb iff $1 + ab > 0$ (iii) aRb iff $|a| \le b$.
 Find whether these relations are reflexive, symmetric or transitive.
7. Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive. [NCERT]
8. (i) Check whether the relation R on $\mathbf{R}$ defined by $R = \{(a, b) : a \le b^3\}$ is reflexive, symmetric or transitive. [NCERT, CBSE 2010]
 (ii) Check if the relation R in the set of all real numbers defined as $R = \{(a, b) : a < b\}$ is
 (i) symmetric (ii) transitive [CBSE 2020]
9. Prove that every identity relation on a set is reflexive, but the converse is not necessarily true.
10. If $A = \{1, 2, 3, 4\}$, define relations on A which have properties of being
 (i) reflexive, transitive but not symmetric.
 (ii) symmetric but neither reflexive nor transitive.
 (iii) reflexive, symmetric and transitive. [NCERT EXEMPLAR]
11. Let R be a relation defined on the set of natural numbers N as
 $R = \{(x, y) : x, y \in N, 2x + y = 41\}$
 Find the domain and range of R. Also, verify whether R is (i) reflexive, (ii) symmetric (iii) transitive. [CBSE 2014]
12. An integer m is said to be related to another integer n if m is a multiple of n. Check if the relation is symmetric, reflexive and transitive.
13. Show that the relation "$\ge$" on the set R of all real numbers is reflexive and transitive but not symmetric.
14. Give an example of a relation which is
 (i) reflexive and symmetric but not transitive.
 (ii) reflexive and transitive but not symmetric.
 (iii) symmetric and transitive but not reflexive.
 (iv) symmetric but neither reflexive nor transitive.
 (v) transitive but neither reflexive nor symmetric. [NCERT]
15. Let W denote the set of words in the English dictionary. Define the relation R by $R = \{(x, y) \in W \times W$ such that x and y have at least one letter in common$\}$. Show that this relation R is reflexive and symmetric but not transitive. [CBSE 2020]

BASED ON HOTS

16. Is it true that every relation which is symmetric and transitive is also reflexive? Give reasons.
17. Given the relation $R = \{(1, 2), (2, 3)\}$ on the set $A = \{1, 2, 3\}$, add a minimum number of ordered pairs so that the enlarged relation is symmetric, transitive and reflexive.
18. Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (1, 1), (2, 3)\}$ be a relation on A. What minimum number of ordered pairs may be added to R so that it may become a transitive relation on A.
19. Let $A = \{a, b, c\}$ and the relation R be defined on A as follows: $R = \{(a, a), (b, c), (a, b)\}$. Then, write minimum number of ordered pairs to be added in R to make it reflexive and transitive. [NCERT EXEMPLAR]

ANSWERS

1. (i) Reflexive, symmetric and transitive
   (ii) Reflexive, symmetric and transitive
   (iii) Neither reflexive, nor symmetric but transitive
   (iv) neither reflexive nor symmetric nor transitive
2. R_1 is reflexive but neither symmetric nor transitive.
   R_2 is symmetric and transitive but not reflexive.
   R_3 is transitive but neither reflexive nor symmetric.
   R_4 is neither reflexive nor symmetric nor transitive.
3. (i) R_1 is symmetric but it is neither reflexive nor transitive
   (ii) R_2 is reflexive and symmetric but it is not transitive
   (iii) R_3 is reflexive but it is neither symmetric nor transitive.
4. R_1 is reflexive but neither symmetric nor transitive
   R_2 is symmetric but neither reflexive nor transitive.
   R_3 is transitive but neither reflexive nor symmetric.
5. (i) transitive (ii) symmetric (iii) reflexive, symmetric and transitive (iv) transitive.
6. (i) Transitive (ii) Reflexive and symmetric but not transitive (iii) Transitive neither reflexive nor symmetric.
7. Neither reflexive nor symmetric nor transitive
8. (i) Neither reflexive nor symmetric nor transitive. (ii) Transitive but not symmetric.
10. (i) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2)\}$ (ii) $R = \{(1, 2), (2, 1)\}$
    (iii) $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$
11. Domain $R = \{1, 2, 3, ..., 19, 20\}$, Range $R = \{39, 37, 35, ..., 7, 5, 3, 1\}$.
    R is neither reflexive nor symmetric and is not transitive.
12. Reflexive and transitive but not symmetric.
14. (i) $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\ (2, 3), (3, 2)\}$ on $A = \{1, 2, 3\}$
    (ii) $R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ on $A = \{1, 2, 3\}$
    (iii) $R = \{(1, 3), (3, 1), (1, 1), (3, 3)\}$ on $A = \{1, 2, 3\}$
    (iv) $R = \{(1, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$ on $A = \{1, 2, 3\}$ (v) $R = \{(1, 1)\}$ on $A = \{1, 2, 3\}$
16. No. Relation $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ on $A = \{1, 2, 3\}$ is symmetric and transitive but not reflexive.
17. $(1, 1), (2, 2), (3, 3), (1, 3), (2, 1), (3, 2), (3, 1)$ 18. $(1, 3)$, One 19. $(b, b), (c, c), (a, c)$

HINTS TO SELECTED PROBLEMS

1. (iv) The relation R on the set A of all human beings in a town is given by
   $(x, y) \in R$ iff x is father of y.
   Reflexivity: Since a person x cannot be father of himself. So, $(x, x) \notin A$. Consequently, R is not reflexive.
   Symmetry: Let $x, y \in A$ be such that $(x, y) \in R$. Then,
   $$(x, y) \in R \Rightarrow x \text{ is father of } y \Rightarrow y \text{ cannot be father of } x \Rightarrow (y, x) \notin R$$
   So, R is not symmetric.
   Transitivity: Let $x, y, z \in A$ be such that $(x, y) \in R$ and $(y, z) \in R$. Then,
   $$(x, y) \in R \text{ and } (y, z) \in R$$
   $$\Rightarrow x \text{ is father of } y \text{ and } y \text{ is father of } z \Rightarrow x \text{ is grandfather of } z \Rightarrow (x, z) \notin R$$
7. The relation R on set $A = \{1, 2, 3, 4, 5, 6\}$ is defined as $(a, b) \in R$ iff $b = a + 1$. Therefore, $R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}$. Clearly, $(a, a) \notin R$ for any $a \in A$. So, R is not reflective on A. We observe that $(1, 2) \in R$ but $(2, 1) \notin R$. So, R is not symmetric.
   We also observe that $(1, 2) \in R$ and $(2, 3) \in R$ but $(1, 3) \notin R$. So, R is not transitive.

8. The relation R on **R** is defined by $R = \{(a, b) : a \le b^3\}$. We observe that $(-2) \in R$ is such that $(-2) \le (-2)^3$ is not true. So, R is not reflexive.
Since $1 \le (3^{1/3})^3$ but $3^{1/3} \not\le 1$ i.e. $(1, 3^{1/3}) \in R$ but $(3^{1/3}, 1) \notin R$. So, R is not symmetric.
R is not transitive because $(5, 2) \in R$ and $(2, 2^{1/3}) \in R$ but $(5, 2^{1/3}) \notin R$.

9. Let I be the identity relation on a set A. Then, $(a, a) \in I$ for all $a \in A \Rightarrow I$ is reflexive.
Converse: The relation $\{(1, 1), (2, 2), (3, 3), (1, 3)\}$ is a reflexive relation on set $A = \{1, 2, 3\}$ but it is not the identity relation on A.

16. A relation R on the set Z of integers defined by $(a, b) \in R \Leftrightarrow a$ and b are both odd, is symmetric and transitive but it is not reflexive. Because no even integer is related to itself.

17. For reflexivity, we must add (1, 1), (2, 2) and (3, 3). For symmetry and transitivity we must add (2, 1), (3, 2), (1, 3), (3, 1) in R.

1.3.5 EQUIVALENCE RELATION AND EQUIVALENCE CLASSES

DEFINITION *A relation R on a set A is said to be an equivalence relation on A iff it is*

(i) reflexive i.e. $(a, a) \in R$ for all $a \in A$.

(ii) symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$.

and, (iii) transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Consider the relation R defined on the set Z of integers by the rule

$$(x, y) \in R \Leftrightarrow x - y \text{ is divisible by } 2.$$

This relation R has the following properties:

Reflexivity: For any $x \in Z$, we find that

$$x - x = 0 \Rightarrow x - x = 0 \times 2 \Rightarrow x - x \text{ is divisible by } 2 \Rightarrow (x, x) \in R$$

Thus, $(x, x) \in R$ for all $x \in Z$. So, R is a reflexive relation on Z.

Symmetry: For any $x, y \in Z$, we find that

$$(x, y) \in R \Rightarrow x - y \text{ is divisible by } 2$$

$$\Rightarrow \quad x - y = 2\lambda \text{ for some } \lambda \in Z \Rightarrow y - x = 2(-\lambda) \Rightarrow y - x \text{ is divisible by } 2 \Rightarrow (y, x) \in R$$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$. So, R is a symmetric relation on Z.

Transitivity: For any $x, y, z \in Z$, we find that

$$(x, y) \in R \text{ and } (y, z) \in R$$

$$\Rightarrow \quad x - y \text{ is divisible by 2 and } y - z \text{ is divisible by } 2$$

$$\Rightarrow \quad x - y = 2p \text{ and } y - z = 2q \text{ for some } p, q \in Z$$

$$\Rightarrow \quad (x - y) + (y - z) = 2p + 2q \Rightarrow x - z = 2(p + q) \Rightarrow x - z \text{ is divisible by } 2 \Rightarrow (x, z) \in R$$

Thus, $(x\ y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in Z$.

So, R is a transitive relation on Z. Hence, R is an equivalence relation on Z.

Let us now find the sets of integers realted to various integers in Z. For any integer a, let $[a]$ denote the set of integers related to a under relation R i.e. $[a] = \{x \in Z : (x, a) \in R\}$. Then, we find that

$$[0] = \{x \in Z : (x, 0) \in R\} = \{x \in Z : x - 0 \text{ is divisible } 2\} = \{x \in Z : x \text{ is divisible by } 2\}$$
$$= \{0, \pm 2, \pm 4, \pm 6, \ldots\}$$

$$[1] = \{x \in Z : (x, 1) \in R\} = \{x \in Z : x - 1 \text{ is divisible by } 2\} = \{x \in Z : x - 1 = 2\lambda, \lambda \in Z\}$$
$$= \{x \in Z : x = 2\lambda + 1, \lambda \in Z\} = \{\pm 1, \pm 3, \pm 5, \pm 5, \pm 7, \ldots\}$$

$$[2] = \{x \in Z : (x, 2) \in R\} = \{x \in Z : x - 2 \text{ is divisible by } 2\}$$
$$= \{x \in Z : x - 2 = 2\lambda, \lambda \in Z\} = \{x \in Z : x = 2 + 2\lambda, \lambda \in Z\}$$
$$= \{0, \pm 2, \pm 4, \pm 6, \ldots\}, \text{ which is same as the set } [0]$$

$$[3] = \{x \in Z : (x, 3) \in R\} = \{x \in Z : x - 3 \text{ is divisible by } 2\}$$
$$= \{x \in Z : x - 3 = 2\lambda, \lambda \in Z\} = \{x \in Z : x = 3 + 2\lambda, \lambda \in Z\}$$

$=\{\pm 1, \pm 3, \pm 5 \pm 7, \ldots\}$, which is sam as the set [1]

Continuing in this manner, we find that

$[0]=[2]=[4]=[6]=\ldots;\ [1]=[3]=[5]=[7]=\ldots$ and, $[0]\cap[1]=\phi$. Also, $Z=[0]\cup[1]$.

Thus, R partitions the set Z into two pairwise disjoint sets known as equivalence classes.

Similarly, the relation R on Z given by

$(x, y)\in R \Leftrightarrow x-y$ is divisible by 3

partitions Z into 3 pairwise disjoints sets i.e. equivalence classes given by

$[0]=\{-6, -3, 0, 3, 6, 9, \ldots\}$, $[1]=\{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$, $[2]=\{\ldots, 7, -4, -1, 2, 5, 8, 11, \ldots\}$ such that $Z=[0]\cup[1]\cup[2]$.

It is evident from the above discussion that an equivalence relation R defined on a set A partitions the set A into pairwise disjoint subsets. These subsets are called equivalence classes determined by relation R. The set of all elements of A related to an element $a\in A$ is denoted by $[a]$ i.e. $[a]=\{x\in A : (x, a)\in R\}$. This is an equivalence class. Corresponding to every element in A there is an equivalence class. Any two equivalence classes are either identical or disjoint. The collection of all equivalence classes forms a partition of set A.

ILLUSTRATION *Let R be the equivalence relation in the set $A=\{0, 1, 2, 3, 4, 5\}$ given by $R=\{(a, b) : 2$ divides $(a-b)\}$. Write the equivalence class $[0]$.* **[CBSE 2014]**

SOLUTION Clearly, the equivalence class [0] is the set of those elements in A which are related to 0 under the relation R. i.e. $[0]=\{(a, 0)\in R : a\in A\}$.

Now, $(a, 0)\in R$

$\Rightarrow$ $a-0$ is divisible by 2 and $a\in A \Rightarrow a\in A$ such that 2 divides a $\Rightarrow a=0, 2, 4$

Thus, $[0]=\{0, 2, 4\}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2)\in R \Leftrightarrow$ line l_1 is parallel to line l_2. Show that R is an equivalence relation.*

SOLUTION Let L be the given set of all lines in a plane. Then, we observe the following properties.

Reflexive: For each line $l\in L$, we find that

$l \| l \Rightarrow (l, l)\in R$ for all $l\in L \Rightarrow R$ is reflexive

Symmetric: Let $l_1, l_2\in L$ such that $(l_1, l_2)\in R$. Then,

$(l_1, l_2)\in R \Rightarrow l_1 \| l_2 \Rightarrow l_2 \| l_1 \Rightarrow (l_2, l_1)\in R$. So, R is symmetric on L.

Transitive: Let $l_1, l_2, l_3\in L$ such that $(l_1, l_2)\in R$ and $(l_2, l_3)\in R$. Then,

$(l_1, l_2)\in R$ and $(l_2, l_3)\in R \Rightarrow l_1 \| l_2$ and $l_2 \| l_3 \Rightarrow l_1 \| l_3 \Rightarrow (l_1, l_3)\in R$. So, R is transitive on L.

Hence, R being reflexive, symmetric and transitive is an equivalence relation on L.

EXAMPLE 2 *Show that the relation 'is congruent to' on the set of all triangles in a plane is an equivalence relation.*

SOLUTION Let S be the set of all triangles in a plane and let R be the relation on S defined by

$(\Delta_1, \Delta_2)\in R \Leftrightarrow$ triangle Δ_1 is congruent to triangle Δ_2.

We observe the following properties of relation R:

Reflexivity: For each triangle $\Delta\in S$, we find that

$\Delta\cong\Delta \Rightarrow (\Delta, \Delta)\in R$ for all $\Delta\in S \Rightarrow R$ is reflexive on S

Symmetry: Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then,

$(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow (\Delta_2, \Delta_1) \in R$. So, R is symmetric on S

Transitivity: Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

EXAMPLE 3 *Show that the relation R defined on the set A of all triangles in a plane as $R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$ is an equivalence relation.*

Consider three right angled triangles T_1 with sides 3, 4, 5; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related? [NCERT]

SOLUTION We observe the following properties of relation R.

Reflexivity: We know that every triangle is similar to itself.

$\therefore$ $(T, T) \in R$ for all $T \in A \Rightarrow R$ is reflexive.

Symmetry: Let $T_1, T_2 \in A$ such that $(T_1, T_2) \in R$. Then,

$(T_1, T_2) \in R \Rightarrow T_1$ is similar to $T_2 \Rightarrow T_2$ is similar to $T_1 \Rightarrow (T_2, T_1) \in R$

So, R is symmetric.

Transitivity: Let $T_1, T_2, T_3 \in A$ such that $(T_1, T_2) \in R$ and $(T_2, T_3) \in R$. Then,

$(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow$ T_1 is similar to T_2 and T_2 is similar to $T_3 \Rightarrow T_1$ is similar to $T_3 \Rightarrow (T_1, T_3) \in R$

So, R is transitive. Hence, R is an equivalence relation on set A.

In triangles T_1 and T_3, we observe that the corresponding angles are equal and the corresponding sides are proportional i.e. $\frac{3}{6} = \frac{4}{8} = \frac{5}{10}$. Hence, T_1 and T_3 are related.

EXAMPLE 4 *Let n be a positive integer. Prove that the relation R on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n, is an equivalence relation on Z.* [NCERT EXEMPLAR]

SOLUTION We observe the following properties of relation R.

Reflexivity: For any $a \in Z$

$a - a = 0 = 0 \times n \Rightarrow a - a$ is divisible by $n \Rightarrow (a, a) \in R$

Thus, $(a, a) \in R$ for all $a \in Z$. So, R is reflexive on Z

Symmetry: Let $(a, b) \in R$. Then,

$(a, b) \in R$

$\Rightarrow$ $(a - b)$ is divisible by n

$\Rightarrow$ $(a - b) = np$ for some $p \in Z$

$\Rightarrow$ $b - a = n(-p)$

$\Rightarrow$ $b - a$ is divisible by n $[\because p \in Z \Rightarrow -p \in Z]$

$\Rightarrow$ $(b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in Z$. So, R is symmetric on Z.

Transitivity: Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by $n \Rightarrow a - b = np$ for some $p \in Z$

and, $(b, c) \in R \Rightarrow (b - c)$ is divisible by $n \Rightarrow b - c = nq$ for some $q \in Z$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R \Rightarrow a - b = np$ and $b - c = nq \Rightarrow (a - b) + (b - c) = np + nq$

$\Rightarrow$ $a - c = n(p + q) \Rightarrow a - c$ is divisible by n $[\because p, q \in Z \Rightarrow p + q \in Z]$

$\Rightarrow$ $(a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in Z$. So, R is transitive relation on Z.

Thus, R being reflexive, symmetric and transitive, is an equivalence relation on Z.

EXAMPLE 5 *Show that the relation R on the set A of all the books in a library of a college given by $R = \{(x, y) : x$ and y have the same number of pages$\}$, is an equivalence relation.* **[NCERT]**

SOLUTION We observe the following properties of relation R.

Reflexivity: For any book x in set A, we observe that x and x have the same number of pages.

$\therefore \quad (x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in A$. So, R is reflexive.

Symmetry: Let $(x, y) \in R$. Then,

$(x, y) \in R$

$\Rightarrow$ x and y have the same number of pages $\Rightarrow$ y and x have the same number of pages

$\Rightarrow \quad (y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$. So, R is symmetric.

Transitivity: Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ (x and y have the same number of pages) and (y and z have the same number of pages)

$\Rightarrow$ x and z have the same number of pages $\Rightarrow (x, z) \in R$

So, R is transitive.

Thus, R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

REMARK *Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b \pmod{m}$.*

Thus, $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m.

For example, $18 \equiv 3 \pmod{5}$ because $18 - 3 = 15$ which is divisible by 5. Similarly, $3 \equiv 13 \pmod{2}$ because $3 - 13 = -10$, which is divisible by 2. But, $25 \not\equiv 2 \pmod{4}$ because 4 is not a divisor of $25 - 2 = 23$.

EXAMPLE 6 *Prove that the relation 'congruence modulo m' on the set Z of all integers is an equivalence relation.*

SOLUTION We observe the following properties of the given relation.

Reflexivity: Let a be an arbitrary integer. Then,

$a - a = 0 = 0 \times m \Rightarrow a - a$ is divisible by $m \Rightarrow a \equiv a \pmod{m}$

Thus, $a \equiv a$ (mod m) for all $a \in Z$. So, "congruence modulo m" is reflexive.

Symmetry: Let $a, b \in Z$ such that $a \equiv b \pmod{m}$. Then,

$a \equiv b \pmod{m}$

$\Rightarrow \quad a - b$ is divisible by m

$\Rightarrow \quad a - b = \lambda m$ for $\lambda \in Z$

$\Rightarrow \quad b - a = (-\lambda) m$

$\Rightarrow \quad b - a$ is divisible by m $\quad [\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\Rightarrow \quad b \equiv a \pmod{m}$

So, "congruence modulo m" is symmetric on Z.

Transitivity: Let $a, b, c \in Z$ such that $a \equiv b$ (mod m) and $b \equiv c \pmod{m}$. Then,

$a \equiv b$ (mod m) $\Rightarrow a - b$ is divisible by $m \Rightarrow a - b = \lambda_1 m$ for some $\lambda_1 \in Z$

$b \equiv c$ (mod m) $\Rightarrow b - c$ is divisible by $m \Rightarrow b - c = \lambda_2$ m for some $\lambda_2 \in Z$

$\therefore \quad (a - b) + (b - c) = \lambda_1 m + \lambda_2 m = (\lambda_1 + \lambda_2) m$

$\Rightarrow \quad a - c = \lambda_3 m$, where $\lambda_3 = \lambda_1 + \lambda_2 \in Z \Rightarrow a \equiv c \pmod{m}$

Thus, $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m} \Rightarrow a \equiv c \pmod{m}$. So, "congruence modulo m" is transitive on Z.

Hence, "congruence modulo m" is an equivalence relation on Z.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Given a non-empty set X, consider P (X) which is the set of all subsets of X. Define a relation in P (X) as follows:*

For subsets A, B in P (X), A R B if $A \subset B$. Is R an equivalence relation on P (X)? Justify your answer. **[NCERT]**

SOLUTION It is given that for any A, B in $P(X)$: $ARB \Leftrightarrow A \subset B$

We observe the following properties of R.

Reflexivity: For any A in $P(X)$, we find that : $A \subset A \Rightarrow ARA$. So, R is reflexive on $P(X)$.

Symmetry: Let A, B in $P(X)$ be such that ARB. Then, $ARB \Rightarrow A \subset B$.

This need not imply that $B \subset A$. In fact it is possible only when $A = B$.

Also, we know that $\{1, 2\} \subset \{1, 2, 3\}$, but $\{1, 2, 3\} \not\subset \{1, 2\}$. So, R is not a symmetric relation on $P(X)$.

Transitivity: Let A, B, C be in $P(X)$ such that

$$A\,R\,B \text{ and } B\,R\,C \Rightarrow A \subset B \text{ and } B \subset C \Rightarrow A \subset C \Rightarrow A\,R\,C$$

So, R is a transitive relation on $P(X)$.

Thus, R is reflexive and transitive relation on $P(X)$ but it is not symmetric.

Hence, R is not an equivalence relation on $P(X)$.

EXAMPLE 8 *Show that the relation R on the set $A = \{1, 2, 3, 4, 5\}$, given by $R = \{(a, b) : |a - b| \text{ is even}\}$, is* an equivalence relation.

Show that all the elements of $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other. But, no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

[NCERT, CBSE 2009]

SOLUTION We have,

$$R = \{(a, b) : |a - b| \text{ is even}\}, \text{ where } a, b \in A = \{1, 2, 3, 4, 5\}.$$

We observe the following properties of relation R.

Reflexivity: For any $a \in A$, we find that: $|a - a| = 0$, which is even

$\therefore \quad (a, a) \in R$ for all $a \in A$. So, R is reflexive.

Symmetry: Let $(a, b) \in R$. Then,

$$(a, b) \in R \Rightarrow |a - b| \text{ is even} \Rightarrow |b - a| \text{ is even} \Rightarrow (b, a) \in R$$

Thus, $\quad (a, b) \in R \Rightarrow (b, a) \in R$. So, R is symmetric.

Transitivity: Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a - b|$ is even and $|b - c|$ is even

$\Rightarrow \quad$ (a and b both are even or both are odd) and (b and c both are even or both are odd)

Now two cases arise:

Case I *When b is even:* In this case,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow \quad |a - b|$ is even and $|b - c|$ is even

$\Rightarrow \quad a$ is even and c is even $\quad [\because b \text{ is even}]$

$\Rightarrow \quad |a - c|$ is even $\Rightarrow (a, c) \in R$

Case II *When b is odd*: In this case,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a-b|$ is even and $|b-c|$ is even

$\Rightarrow$ a is odd and c is odd [$\because$ b is odd]

$\Rightarrow$ $|a-c|$ is even $\Rightarrow (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. So, R is transitive.

Hence, R is an equivalence relation.

We know that the difference of any two odd (even) natural numbers is always an even natural number. Therefore, all the elements of set $\{1, 3, 5\}$ are related to each other and all the elements of $\{2, 4\}$ are related to each other.

We know that the difference of an even natural number and an odd natural number is an odd natural number. Therefore, no element of $\{1, 3, 5\}$ is related to any element of $\{2, 4\}$.

EXAMPLE 9 *Show that the relation R on the set* $A = \{x \in Z : 0 \le x \le 12\}$, *given by* $R = \{(a, b) : |a-b|$ *is a multiple of 4*$\}$ *is an equivalence relation. Find the set of all elements related to 1 i.e. equivalence class* [1]. **[NCERT, CBSE 2010, 2018, 2019]**

SOLUTION We have,

$$R = \{(a, b) : |a-b| \text{ is a multiple of } 4\}, \text{ where } a, b \in A = \{x \in Z : 0 \le x \le 12\} = \{0, 1, 2, \ldots, 12\}.$$

We observe the following properties of relation R.

Reflexivity: For any $a \in A$, we find that

$|a-a| = 0$, which is a multiple of 4 $\Rightarrow (a, a) \in R$

Thus, $(a, a) \in R$ for all $a \in A$. So, R is reflexive.

Symmetry: Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow |a-b|$ is a multiple of 4 $\Rightarrow |a-b| = 4\lambda$ for some $\lambda \in N$

$\Rightarrow$ $|b-a| = 4\lambda$ for some $\lambda \in N$ [$\because$ $|a-b| = |b-a|$]

$\Rightarrow$ $(b, a) \in R$

So, R is symmetric.

Transitivity: Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $|a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4

$\Rightarrow$ $|a-b| = 4\lambda$ and $|b-c| = 4\mu$ for some $\lambda, \mu \in N$

$\Rightarrow$ $a-b = \pm 4\lambda$ and $b-c = \pm 4\mu$

$\Rightarrow$ $a-c = \pm 4\lambda \pm 4\mu \Rightarrow a-c$ is a multiple of 4 $\Rightarrow |a-c|$ is a multiple of 4 $\Rightarrow (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

So, R is transitive. Hence, R is an equivalence relation.

Let x be an element of A such that $(x, 1) \in R$. Then,

$|x-1|$ is a multiple of 4 $\Rightarrow |x-1| = 0, 4, 8, 12 \Rightarrow x-1 = 0, 4, 8, 12 \Rightarrow x = 1, 5, 9$ [$\because 13 \notin A$]

Hence, the set of all elements of A which are related to 1 is $\{1, 5, 9\}$ i.e. $[1] = \{1, 5, 9\}$.

EXAMPLE 10 *Show that the relation R on the set A of points in a plane, given by*

$R = \{(P, Q)$: *Distance of the point P from the origin is same as the distance of the point Q from the origin*$\}$,

is an equivalence relation. Further show that the set of all points related to a point $P \ne (0, 0)$ *is the circle passing through P with origin as centre.* **[NCERT]**

SOLUTION Let O denote the origin in the given plane. Then, $R = \{(P, Q) : OP = OQ\}$.

We observe the following properties of relation R.

Reflexivity: For any point P in set A, we find that

$OP = OP \Rightarrow (P, P) \in R$

Thus, $(P, P) \in R$ for all $P \in A$. So, R is reflexive.

Symmetry: Let P and Q be two points in set A such that

$(P, Q) \in R \Rightarrow OP = OQ \Rightarrow OQ = OP \Rightarrow (Q, P) \in R$

Thus, $(P, Q) \in R \Rightarrow (Q, P) \in R$ for all $P, Q \in A$. So, R is symmetric.

Transitivity: Let P, Q and S be three points in set A such that

$(P, Q) \in R$ and $(Q, S) \in R \Rightarrow OP = OQ$ and $OQ = OS \Rightarrow OP = OS \Rightarrow (P, S) \in R$

Thus, $(P, Q) \in R$ and $(Q, S) \in R \Rightarrow (P, S) \in R$ for all $P, Q, S \in A$. So, R is transitive.

Hence, R is an equivalence relation.

Let P be a fixed point in set A and Q be any point in set A such that $(P, Q) \in R$. Then,

$(P, Q) \in R$

$\Rightarrow$ $OP = OQ$

$\Rightarrow$ Q moves in the plane in such a way that its distance from the origin O (0, 0) is always same and is equal to OP.

$\Rightarrow$ Locus of Q is a circle with centre at the origin and radius OP.

Hence, the set of all points related to P is the circle passing through P with origin O as centre.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *Prove that the relation R on the set $N \times N$ defined by*

$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$ *for all* $(a, b), (c, d) \in N \times N$ *is an equivalence relation.*

Also, find the equivalence classes $[(2, 3)]$ *and* $[(1, 3)]$. **[CBSE 2010]**

SOLUTION We observe the following properties of relation R.

Reflexivity: Let (a, b) be an arbitrary element of $N \times N$. Then,

$(a, b) \in N \times N$

$\Rightarrow$ $a, b \in N$

$\Rightarrow$ $a + b = b + a$ [By commutativity of addition on N]

$\Rightarrow$ $(a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$. So, R is reflexive on $N \times N$.

Symmetry: Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$. Then,

$(a, b)\ R\ (c, d)$

$\Rightarrow$ $a + d = b + c$

$\Rightarrow$ $c + b = d + a$ [By commutativity of addition on N]

$\Rightarrow$ $(c, d)\ R\ (a, b)$ [By definition of R]

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$. So, R is symmetric on $N \times N$.

Transitivity: Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$\left.\begin{aligned}(a, b)\ R\ (c, d) \Rightarrow a + d &= b + c \\ (c, d)\ R\ (e, f) \Rightarrow c + f &= d + e\end{aligned}\right\} \Rightarrow (a + d) + (c + f) = (b + c) + (d + e)$$

$$\Rightarrow a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.

So, R is transitive on $N \times N$.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

$$\begin{aligned}[(2, 3)] &= \{(x, y) \in N \times N : (x, y)\ R\ (2, 3)\} = \{(x, y) \in N \times N : x + 3 = y + 2\} \\ &= \{(x, y) \in N \times N : x - y = 1\} = \{(x, y) \in N \times N : y = x + 1\} \\ &= \{(x, x + 1) : x \in N\} = \{(1, 2), (2, 3), (3, 4), (4, 5), \ldots\}\end{aligned}$$

$$[(7, 3)] = \{(x, y) \in N \times N : (x, y)\ R\ (7, 3)\} = \{(x, y) \in N \times N : x + 3 = y + 7\}$$
$$= \{(x, y) \in N \times N : y = x - 4\} = \{(x, x - 4) \in N \times N : x \in N\}$$
$$= \{(5, 1), (6, 2), (7, 3), (8, 4), (9, 5), \ldots\}$$

EXAMPLE 12 *Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation on $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for all $(a, b), (c, d) \in A \times A$. Prove that R is an equivalence relation and also obtain the equivalence class $[(2, 5)]$.* [NCERT EXEMPLAR, CBSE 2014]

SOLUTION We observe the following properties of relation R.

Reflexivity: Let (a, b) be an arbitrary element of $A \times A$. Then,

$(a, b) \in A \times A \Rightarrow a, b \in A \Rightarrow a + b = b + a$ [By commutativity of addition on N]

$\Rightarrow (a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in A \times A$. So, R is reflexive on $A \times A$.

Symmetry: Let $(a, b), (c, d) \in A \times A$ be such that $(a, b)\ R\ (c, d)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a$ [By commutativity of addition on N]

$\Rightarrow (c, d)\ R\ (a, b)$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in A \times A$. So, R is symmetric on $A \times A$.

Transitivity: Let $(a, b), (c, d), (e, f) \in A \times A$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$\left.\begin{array}{l}(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \\ (c, d)\ R\ (e, f) \Rightarrow c + f = d + e\end{array}\right\} \Rightarrow (a + d) + (c + f) = (b + c) + (d + e) \Rightarrow a + f = b + e \Rightarrow (a, b) R (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in A \times A$.

So, R is a transitive relation on $A \times A$. Hence, R is an equivalence relation on $A \times A$.

Now,

$$[(2, 5)] = \{(x, y) \in A \times A : (x, y)\ R\ (2, 5)\} = \{(x, y) \in A \times A : x + 5 = y + 2\} = \{(x, y) \in A \times A : y = x + 3\}$$
$$= \{(x, x + 3) : x \in A \text{ and } x + 3 \in A\} = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$$

EXAMPLE 13 *Let N be the set of all natural numbers and let R be a relation on $N \times N$, defined by*

$$(a, b)\ R\ (c, d) \Leftrightarrow ad = bc \text{ for all } (a, b), (c, d) \in N \times N.$$

Show that R is an equivalence relation on $N \times N$. Also, find the equivalence class $[(2, 6)]$.

SOLUTION We observe the following properties of relation R.

Reflexivity: Let (a, b) be an arbitrary element of $N \times N$. Then,

$(a, b) \in N \times N \Rightarrow a, b \in N \Rightarrow ab = ba$ [By commutativity of multiplication on N]

$\Rightarrow (a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$. So, R is reflexive on $N \times N$.

Symmetry: Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$. Then,

$(a, b)\ R\ (c, d) \Rightarrow ad = bc \Rightarrow cb = da$ [By commutativity of multiplication on N]

$\Rightarrow (c, d)\ R\ (a, b)$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$. So, R is symmetric on $N \times N$.

Transitivity: Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$\left.\begin{array}{l}(a, b)\ R\ (c, d) \Rightarrow ad = bc \\ (c, d)\ R\ (e, f) \Rightarrow cf = de\end{array}\right\} \Rightarrow (ad)\ (cf) = (bc)\ (de) \Rightarrow af = be \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.

So, R is transitive on $N \times N$.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

$$[(2, 6)] = \{(x, y) \in N \times N : (x, y)\ R\ (2, 6)\} = \{(x, y) \in N \times N : 3x = y\} = \{(x, 3x) : x \in N\}$$
$$= \{(1, 3), (2, 6), (3, 9), (4, 12), \ldots\}$$

EXAMPLE 14 *Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b)\ R\ (c, d) \Leftrightarrow ad\,(b + c) = bc\,(a + d)$. Check whether R is an equivalence relation on $N \times N$.* **[CBSE 2015]**

SOLUTION We observe the following properties of relation R.

Reflexivity: Let (a, b) be an arbitrary element of $N \times N$. Then,

$$(a, b) \in N \times N$$
$$\Rightarrow \quad a, b \in N$$
$$\Rightarrow \quad ab\,(b + a) = ba\,(a + b) \quad \text{[By commutativity of addition and multiplication on } N]$$
$$\Rightarrow \quad (a, b)\ R\ (a, b)$$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$. So, R is reflexive on $N \times N$.

Symmetry: Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$. Then,

$$(a, b)\ R\ (c, d)$$
$$\Rightarrow \quad ad\,(b + c) = bc(a + d)$$
$$\Rightarrow \quad cb\,(d + a) = da(c + b) \quad \text{[By commutativity of addition and multiplication on N]}$$
$$\Rightarrow \quad (c, d)\ R\ (a, b)$$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.

So, R is symmetric on $N \times N$.

Transitivity: Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow ad(b + c) = bc(a + d) \Rightarrow \frac{b + c}{bc} = \frac{a + d}{ad} \Rightarrow \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \quad \text{...(i)}$$

and, $$(c, d)\ R\ (e, f) \Rightarrow cf(d + e) = de(c + f) \Rightarrow \frac{d + e}{de} = \frac{c + f}{cf} \Rightarrow \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \quad \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f} \Rightarrow \frac{b + e}{be} = \frac{a + f}{af} \Rightarrow af\,(b + e) = be(a + f) \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.

So, R is transitive on $N \times N$.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

EXAMPLE 15 *Show that the number of equivalence relations on the set $\{1, 2, 3\}$ containing $(1, 2)$ and $(2, 1)$ is two.* **[NCERT]**

SOLUTION The smallest equivalence relation R_1 containing $(1, 2)$ and $(2, 1)$ is

$$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$$

Now, we are left with four ordered pairs namely $(2, 3), (3, 2), (1, 3)$ and $(3, 1)$. If we add any one, say $(2, 3)$ to R_1, then for symmetry we must add $(3, 2)$ and then for transitivity we are forced to add $(1, 3)$ and $(3, 1)$. Thus, the only equivalence relation other than R_1 is the universal relation. Hence, the total number of equivalence relations containing $(1, 2)$ and $(2, 1)$ is two.

EXAMPLE 16 *On the set N of all natural numbers, a relation R is defined as follows:*
$nRm \Leftrightarrow$ Each of the natural numbers n and m leaves the same remainder less than 5 when divided by 5. Show that R is an equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

SOLUTION We observe the following properties of relation R.

Reflexivity: Let a be an arbitrary element of N. Then, either a is less than 5 and if $a \geq 5$, then on dividing a by 5 we obtain a remainder as one of the numbers 0, 1, 2, 3, 4.

Thus, aRa for all $a \in N$. So, R is reflexive on N.

Symmetry: Let $a, b \in N$ such that aRb. Then,

$aRb \Rightarrow$ Each of a and b leaves the same remainder less than 5 when divided by 5

$\Rightarrow$ Each of b and a leave the same remainder less than 5 when divided by 5 $\Rightarrow bRa$

Thus, $aRb \Rightarrow bRa$ for all $a, b \in N$. So, R is symmetric.

Transitivity: Let $a, b, c \in N$ be such that aRb and bRc. Then,

$aRb \Rightarrow$ Each of a and b leaves the same remainder less than 5 when divided by 5

$bRc \Rightarrow$ Each of b and c leaves the same remainder less than 5 when divided by 5

$\therefore$ Each of a and c leaves the same remainder less than 5 when divided by 5

$\Rightarrow aRc$

Thus, aRb and $bRc \Rightarrow aRc$ for all $a, b, c \in N$. So, R is a transitive relation on N.

Hence, R is an equivalence relation on N. Let us now find the equivalence classes.

$[1] = \{x \in N : x\,R\,1\}$

$= \{x \in N : x$ and 1 leave the remainder less than 5 when divided by 5$\}$

$= \{x \in N : x$ leaves the remainder 1 when divided by 5$\} = \{1, 6, 11, 16, 21, \dots\}$

$[2] = \{x \in N : x\,R\,2\}$

$= \{x \in N :$ Each of x and 2 leave the remainder less than 5 when divided by 5$\}$

$= \{x \in N : x$ leaves the remainder 2 when divided by 5$\} = \{2, 7, 12, 17, 22, \dots\}$

$[3] = \{x \in N : x\,R\,3\}$

$= \{x \in N :$ Each of x and 3 leave the remainder less than 5 when divided by 5$\}$

$= \{x \in N : x$ leaves the remainder 3 when divided by 5$\} = \{3, 8, 13, 18, 23, \dots\}$

$[4] = \{x \in N : x\,R\,4\}$

$= \{x \in N :$ Each of x and 4 leave the remainder less than 5 when divided by 5$\}$

$= \{x \in N : x$ leaves the remainder 4 when divided by 5$\} = \{4, 9, 14, 19, \dots\}$

$[5] = \{x \in N : x\,R\,5\}$

$= \{x \in N :$ Each of x and 5 leave the remainder less than 5 when divided by 5$\}$

$= \{x \in N : x$ leaves the remainder 0 when divided by 5$\} = \{5, 10, 15, \dots\}$

Proceeding in this manner we find that

$[1] = [6] = [11] \dots; \; [2] = [7] = [12] \dots; \; [3] = [8] = [13] \dots; \; [4] = [9] = [14] \dots$ and, $[5] = [10] = [15] = \dots$

Thus, we obtain the disjoint equivalence classes: $[1], [2], [3], [4], [5]$ such that $N = [1] \cup [2] \cup [3] \cup [4] \cup [5]$

EXERCISE 1.2

BASIC

1. Show that the relation R defined by $R = \{(a, b) : a - b$ is divisible by 3; $a, b \in Z\}$ is an equivalence relation. [CBSE 2008]
2. Show that the relation R on the set Z of integers, given by $R = \{(a, b) : 2$ divides $a - b\}$, is an equivalence relation. [NCERT]
3. Prove that the relation R on Z defined by $(a, b) \in R \Leftrightarrow a - b$ is divisible by 5 is an equivalence relation on Z. [CBSE 2010]
4. Let n be a fixed positive integer. Define a relation R on Z as follows:

   $(a, b) \in R \Leftrightarrow a - b$ is divisible by n.

   Show that R is an equivalence relation on Z.
5. Let Z be the set of integers. Show that the relation $R = \{(a, b) : a, b \in Z$ and $a + b$ is even$\}$ is an equivalence relation on Z.
6. m is said to be related to n if m and n are integers and $m - n$ is divisible by 13. Does this define an equivalence relation?

BASED ON LOTS

7. Let R be a relation on the set A of ordered pairs of non-zero integers defined by $(x, y)\ R\ (u, v)$ iff $xv = yu$. Show that R is an equivalence relation. [NCERT]
8. Show that the relation R on the set $A = \{x \in Z\ ;\ 0 \le x \le 12\}$, given by $R = \{(a, b) : a = b\}$, is an equivalence relation. Find the set of all elements related to 1.
9. Let L be the set of all lines in XY-plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.
10. Show that the relation R, defined on the set A of all polygons as
$R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$,
is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5? [NCERT]
11. Let R be the relation defined on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ by $R = \{(a, b)$: both a and b are either odd or even$\}$. Show that R is an equivalence relation. Further, show that all the elements of the subset $\{1, 3, 5, 7\}$ are related to each other and all the elements of the subset $\{2, 4, 6\}$ are related to each other, but no element of the subset $\{1, 3, 5, 7\}$ is related to any element of the subset $\{2, 4, 6\}$. [NCERT]
12. Check whether the relation R on the set N of natural numbers given by $R = \{(a, b) : a$ is divisor of $b\}$ is reflexive, symmetric or transitive. Also, determine whether R is an equivalence relation. [CBSE 2020]

BASED ON HOTS

13. Let S be a relation on the set R of all real numbers defined by $S = \{(a, b) \in R \times R : a^2 + b^2 = 1\}$.
Prove that S is not an equivalence relation on R.
14. Let Z be the set of all integers and Z_0 be the set of all non-zero integers. Let a relation R on $Z \times Z_0$ be defined as : $(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$ for all $(a, b), (c, d) \in Z \times Z_0$
Prove that R is an equivalence relation on $Z \times Z_0$.
15. Let C be the set of all complex numbers and C_0 be the set of all non-zero complex numbers. Let a relation R on C_0 be defined as $z_1\ R\ z_2 \Leftrightarrow \dfrac{z_1 - z_2}{z_1 + z_2}$ is real for all $z_1, z_2 \in C_0$. Show that R is an equivalence relation.

ANSWERS

8. $\{1\}$ 9. $\{y = 2x + c : c \in R\}$ 10. Set of all triangles 12. Equivalence

HINTS TO SELECTED PROBLEMS

2. The relation R on Z is given by $R = \{(a, b) : 2$ divides $a - b\}$. We observe the following properties of relation R.

Reflexivity: For any $a \in Z$, we find that
$$a - a = 0 = 0 \times 2 \Rightarrow 2 \text{ divides } a - a \Rightarrow (a, a) \in R$$
So, R is a reflexive relation on Z.

Symmetry: Let $a, b \in Z$ be such that
$$(a, b) \in R$$
$\Rightarrow$ 2 divides $a - b \Rightarrow a - b = 2\lambda$ for some $\lambda \in Z \Rightarrow b - a = 2(-\lambda)$, where $-\lambda \in Z$
$\Rightarrow$ 2 divides $b - a \Rightarrow (b, a) \in R$
Thus, $(a, b) \in R \Rightarrow (b, a) \in R$. So, R is a symmetric relation on Z.

Transitivity: Let $a, b, c \in Z$ be such that $(a, b) \in R$ and $(b, c) \in R$. Then,
$(a, b) \in R \Rightarrow$ 2 divides $b - a \Rightarrow b - a = 2\lambda$ for some $\lambda \in Z$
and, $(b, c) \in R \Rightarrow$ 2 divides $c - b \Rightarrow c - b = 2\mu$ for some $\mu \in Z$
$\therefore$ $b - a + c - b = 2(\lambda + \mu) \Rightarrow c - a = 2(\lambda + \mu)$, where $\lambda + \mu \in Z$
$\Rightarrow$ 2 divides $c - a \Rightarrow (a, c) \in R$
Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$. So, R is a transitive relation on Z.

Hence, R is an equivalence relation on Z.

7. The relation R on $Z \times Z$ is defined by $(x, y)\ R\ (u, v) \Leftrightarrow xv = yu$ for all $(x, y), (u, v) \in Z \times Z$

We observe the following properties of R on $Z \times Z$.

Reflexivity: For any $(x, y) \in Z \times Z$

$xy = yx \Rightarrow (x, y)\ R\ (x, y)$ [$\because$ Multiplication is commutative on Z]

Thus, $(x, y)\ R\ (x, y)$ for all $(x, y) \in Z \times Z$. So, R is a reflexive relation on $Z \times Z$.

Symmetry: Let $(x, y), (u, v) \in Z \times Z$ such that $(x, y)\ R\ ((u, v)$. Then,

$(x, y)\ R\ (u, v) \Rightarrow xv = yu \Rightarrow uy = vx \Rightarrow (u, v)\ R\ (x, y)$

Thus, $(x, y)\ R\ (u, v) \Rightarrow (u, v)\ R\ (x, y)$ for all $(x, y), (u, v) \in Z \times Z$.

So, R is a symmetric relation on Z.

Transitivity: Let $(x, y), (u, v), (a, b) \in Z \times Z$ be such that $(x, y)\ R\ (u, v)$ and $(u, v)\ R\ (a, b)$. Then,

$$\left.\begin{array}{r} (x, y)\ R\ (u, v) \Rightarrow xv = yu \\ \text{and, } (u, v)\ R\ (a, b) \Rightarrow ub = va \end{array}\right\} \Rightarrow (xv)(ub) = (yv)(va) \Rightarrow xb = ya \Rightarrow (x, y)\ R\ (a, b)$$

So, R is a transitive relation on $Z \times Z$.

Hence, R is an equivalence relation on $Z \times Z$.

10. The relation R on the set of A of all polygons is defined as $R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$. We observe the following properties of R on A.

Reflexivity: Let P be any polygon in A. Then,

P and P have same number of sides $\Rightarrow (P, P) \in R$

Thus, $(P, P) \in R$ for all $P \in A$. So, R is a reflexive relation on A.

Symmetry: Let P_1, P_2 be two polygons in A such that $(P_1, P_2) \in R$. Then,

$(P_1, P_2) \in R \Rightarrow P_1$ and P_2 have same number of sides

$\Rightarrow$ P_2 and P_1 have same number of sides $\Rightarrow (P_2, P_1) \in R$

So, R is symmetric on A.

Transitivity: Let P_1, P_2, P_3 be three polygons in A such that $(P_1, P_2) \in R$ and $(P_2, P_3) \in R$. Then,

$(P_1, P_2) \in R \Rightarrow P_1$ and P_2 have same number of sides

and, $(P_2, P_3) \in R \Rightarrow P_2$ and P_3 have same number of sides

$\therefore$ P_1 and P_3 have same number of sides $\Rightarrow (P_1, P_3) \in R$

Thus, $(P_1, P_2) \in R$ and $(P_2, P_3) \in R \Rightarrow (P_1, P_3) \in R$. So, R is a transitive relation on A.

Hence, R is an equivalence relation on A.

Let P be a polygon in A such that $(P, T) \in R$, where T is a right angled triangle with sides 3, 4 and 5. Then,

$(P, T) \in R \Rightarrow$ Polygon P and triangle T have same number of sides $\Rightarrow P$ is any triangle in A

Hence, the set of all elements in A related to T is the set of all triangles in A.

11. The relation R on set $A = \{1, 2, 3, 4, 5, 6, 7\}$ is defined by $R = \{(a, b) :$ both a and b are either odd or even$\}$. We observe the following properties of R on A:

Reflexivity: Clearly, $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7) \in R$. So, R is a reflexive relation on A.

Symmetry: Let $a, b \in A$ be such that $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow$ Both a and b are either odd or even

$\Rightarrow$ Both b and a are either odd or even $\Rightarrow (b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$. So, R is a symmetric relation on A.

Transitivity: Let $a, b, c \in Z$ be such that $(a, b) \in R$, $(b, c) \in R$. Then,

$(a, b) \in R \Rightarrow$ Both a and b are either odd or even

$(b, c) \in R \Rightarrow$ Both b and c are either odd or even

If both a and b are even, then $(b, c) \in R \Rightarrow$ Both b and c are even

$\therefore$ Both a and c are even

If both a and b are odd, then $(b, c) \in R \Rightarrow$ Both b and c are odd

$\therefore$ Both a and c are odd

Thus, both a and c are even or odd. Therefore, $(a, c) \in R$.

So, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$.

Consequently, R is a transitive relation on A. Hence, R is an equivalence relation on A.

We observe that two numbers in A are related if both are odd or both are even. Since $\{1, 3, 5, 7\}$ has all odd numbers of A. So, all the numbers of $\{1, 3, 5, 7\}$ are related to each other. Similarly, all the numbers of $\{2, 4, 6\}$ are related to each other as it contains all even numbers of set A. An even odd number in A is related to an even (odd) number in A. So, no number of the subset $\{1, 3, 5, 7\}$ is related to any number of the subset $\{2, 4, 6\}$.

1.4 SOME USEFUL RESULTS ON RELATIONS

In this section, we shall discuss some useful results on relations as theorems.

THEOREM 1 *If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.*

OR

The intersection of two equivalence relations on a set is an equivalence relation on the set. **[NCERT]**

PROOF It is given that R and S are relations on set A.

$\therefore \quad R \subseteq A \times A$ and $S \subseteq A \times A \Rightarrow R \cap S \subseteq A \times A \Rightarrow R \cap S$ is also a relation on A.

Now, we shall show that it is an equivalence relation on A. We observe the following properties of relation $R \cap S$.

Reflexivity: Let a be an arbitrary element of A. Then,

$a \in A \Rightarrow (a, a) \in R$ and $(a, a) \in S$ $\quad [\because R$ and S are reflexive$]$

$\Rightarrow \quad (a, a) \in R \cap S$

Thus, $(a, a) \in R \cap S$ for all $a \in A$. So, $R \cap S$ is a reflexive relation on A.

Symmetry: Let $a, b \in A$ such that $(a, b) \in R \cap S$. Then,

$(a, b) \in R \cap S$

$\Rightarrow \quad (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow \quad (b, a) \in R$ and $(b, a) \in S$ $\quad [\because R$ and S are symmetric$]$

$\Rightarrow \quad (b, a) \in R \cap S$

Thus, $(a, b) \in R \cap S \Rightarrow (b, a) \in R \cap S$ for all $(a, b) \in R \cap S$. So, $R \cap S$ is symmetric on A.

Transitivity: Let $a, b, c \in A$ such that $(a, b) \in R \cap S$ and $(b, c) \in R \cap S$. Then,

$(a, b) \in R \cap S$ and $(b, c) \in R \cap S$

$\Rightarrow \quad \{(a, b) \in R$ and $(a, b) \in S)\}$ and $\{(b, c) \in R$ and $(b, c) \in S)\}$

$\Rightarrow \quad \{(a, b) \in R, (b, c) \in R\}$ and $\{(a, b) \in S, (b, c) \in S\}$

$\Rightarrow \quad (a, c) \in R$ and $(a, c) \in S$ $\quad \left[\begin{array}{l} \because R \text{ and } S \text{ are transitive.} \\ \therefore (a, b) \in R \text{ and } (b, c) \in R \Rightarrow (a, c) \in R \\ \quad (a, b) \in S \text{ and } (b, c) \in S \Rightarrow (a, c) \in S \end{array}\right]$

$\Rightarrow \quad (a, c) \in R \cap S$

Thus, $(a, b) \in R \cap S$ and $(b, c) \in R \cap S \Rightarrow (a, c) \in R \cap S$.

So, $R \cap S$ is transitive on A.

Hence, $R \cap S$ is an equivalence relation on A.

THEOREM 2 *The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.*

PROOF Let $A = \{a, b, c\}$ and let R and S be two relations on A, given by

$$R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\} \quad \text{and,} \quad S = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$$

It can be easily seen that each one of R and S is an equivalence relation on A. But, $R \cup S$ is not transitive, because $(a, b) \in R \cup S$ and $(b, c) \in R \cup S$ but $(a, c) \notin R \cup S$.

Hence, $R \cup S$ is not an equivalence relation on A.

THEOREM 3 *If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.*

OR

The inverse of an equivalence relation is an equivalence relation.

PROOF Since R is a relation on A.

$\therefore \quad R \subseteq A \times A \Rightarrow R^{-1} \subseteq A \times A \Rightarrow R^{-1}$ is also a relation on A.

Now, we shall show that R^{-1} is an equivalence relation on A.

We observe the following properties of relation R^{-1}.

Reflexivity: Let a be an arbitrary element of A. Then,

$a \in A$

$\Rightarrow \quad (a, a) \in R$ [$\because R$ is reflexive]

$\Rightarrow \quad (a, a) \in R^{-1}$ [By definition of R^{-1}]

Thus, $(a, a) \in R^{-1}$ for all $a \in A$. So, R^{-1} is reflexive on A.

Symmetry: Let $(a, b) \in R^{-1}$. Then,

$(a, b) \in R^{-1}$

$\Rightarrow \quad (b, a) \in R$ [By definition of R^{-1}]

$\Rightarrow \quad (a, b) \in R$ [$\because R$ is symmetric]

$\Rightarrow \quad (b, a) \in R^{-1}$ [By definition of R^{-1}]

Thus, $(a, b) \in R^{-1} \Rightarrow (b, a) \in R^{-1}$ for all $a, b \in A$. So, R^{-1} is symmetric on A.

Transitivity: Let $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$. Then,

$(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$

$\Rightarrow \quad (b, a) \in R$ and $(c, b) \in R$ [By definition of R^{-1}]

$\Rightarrow \quad (c, b) \in R$ and $(b, a) \in R$

$\Rightarrow \quad (c, a) \in R$ [$\because R$ is transitive]

$\Rightarrow \quad (a, c) \in R^{-1}$ [By definition of R^{-1}]

Thus, $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1} \Rightarrow (a, c) \in R^{-1}$ for all $a, b, c \in A$. So, R^{-1} is transitive on A.

Hence, R^{-1} is an equivalence relation on A.

EXERCISE 1.3

BASIC

1. If R and S are relations on a set A, then prove the following:
   (i) R and S are symmetric $\Rightarrow R \cap S$ and $R \cup S$ are symmetric
   (ii) R is reflexive and S is any relation $\Rightarrow R \cup S$ is reflexive.
2. If R and S are transitive relations on a set A, then prove that $R \cup S$ may not be a transitive relation on A.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $R = \left\{(x, y) : x^2 + y^2 \leq 4, x, y \in Z\right\}$ is a relation in Z, then the domain of R is ....
2. Let R be a relation in N defined by $R = \{(x, y) : x + 2y = 8\}$, then the range of R is ....
3. The number of relations on a finite set having 5 elements is ....
4. Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $\{(a, b) : a, b \in A, a \times b$ is an even number$\}$, then the range of R is ....
5. Let $A = \{1, 2, 3, 4, 5\}$. The domain of the relation on A defined by $R = \{(x, y) : y = 2x - 1\}$, is ....
6. If R s a relation defined on set $A = \{1, 2, 3\}$ by the rule $(a, b) \in R \Leftrightarrow |a^2 - b^2| \leq 5$, then $R^{-1} =$ ....
7. If R is a relation from $A = \{11, 12, 13\}$ to $B = \{8, 10, 12\}$ defined by $y = x - 3$, then $R^{-1} =$ ....
8. The smallest equivalence relation on the set $A = \{a, b, c, d\}$ is ....
9. The largest equivalence relation on the set $A = \{1, 2, 3\}$ is ....
10. Let R be the equivalence relation on the set Z of integers given by $R = \{(a, b) : 3$ divides $a - b\}$. Then the equivalence class $[0]$ is equal to ....
11. Let R be a relation on the set Z of all integers defined as $(x, y) \in R \Leftrightarrow x - y$ is divisible by 2. Then, the equivalence class $[1]$ is ....
12. The relation $R = \{(1, 2), (1, 3)\}$ on set $A = \{1, 2, 3\}$ is .... only.
13. A relation in a set A is called ....relation, if each element of A is related to itself. **[CBSE 2020]**
14. A relation R on a set A is called a .... relation, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$ for all $a_1, a_2 \in A$. **[CBSE 2020]**

ANSWERS

1. $\{-2, -1, 0, 1, 2\}$ 2. $\{1, 2, 3\}$ 3. 2^{25} 4. $\{2, 4\}$ 5. $\{1, 2, 3\}$
6. $\{(1,1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\}$ 7. $\{(8, 11), (10, 13)\}$
8. $\{(a, a), (b, b), (c, c), (d, d)\}$ 9. $A \times A$ 10. $\{0, \pm 3, \pm 6, \pm 9, \ldots\ldots\}$
11. $\{\pm 1, \pm 3, \pm 5, \pm 7, \ldots\ldots\}$ 12. transitive 13. reflexive 14. symmetric

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the domain of the relation R defined on the set Z of integers as follows:
   $(a, b) \in R \Leftrightarrow a^2 + b^2 = 25$
2. If $R = \{(x, y) : x^2 + y^2 \leq 4 ; x, y \in Z\}$ is a relation on Z, write the domain of R.

3. Write the identity relation on set $A = \{a, b, c\}$.
4. Write the smallest reflexive relation on set $A = \{1, 2, 3, 4\}$.
5. If $R = \{(x, y) : x + 2y = 8\}$ is a relation on N, then write the range of R. [CBSE 2014]
6. If R is a symmetric relation on a set A, then write a relation between R and R^{-1}.
7. Let $R = \{(x, y) : |x^2 - y^2| < 1\}$ be a relation on set $A = \{1, 2, 3, 4, 5\}$. Write R as a set of ordered pairs.
8. If $A = \{2, 3, 4\}$, $B = \{1, 3, 7\}$ and $R = \{(x, y) : x \in A, y \in B \text{ and } x < y\}$ is a relation from A to B, then write R^{-1}.
9. Let $A = \{3, 5, 7\}$, $B = \{2, 6, 10\}$ and R be a relation from A to B defined by $R = \{(x, y) : x$ and y are relatively prime$\}$. Then, write R and R^{-1}.
10. Define a reflexive relation.
11. Define a symmetric relation.
12. Define a transitive relation.
13. Define an equivalence relation.
14. If $A = \{3, 5, 7\}$ and $B = \{2, 4, 9\}$ and R is a relation given by "is less than", write R as a set ordered pairs.
15. $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and if $R = \{(x, y) : y$ is one half of $x;\ x, y \in A\}$ is a relation on A, then write R as a set of ordered pairs.
16. Let $A = \{2, 3, 4, 5\}$ and $B = \{1, 3, 4\}$. If R is the relation from A to B given by $a\,R\,b$ iff "a is a divisor of b". Write R as a set of ordered pairs.
17. State the reason for the relation R on the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1)\}$ not to be transitive. [CBSE 2011]
18. Let $R = \{(a, a^3) : a$ is a prime number less than $5\}$ be a relation. Find the range of R. [CBSE 2014]
19. Let R be the equivalence relation on the set Z of integers given by $R = \{(a, b) : 2 \text{ divides } a - b\}$. Write the equivalence class $[0]$. [NCERT EXEMPLAR]
20. For the set $A = \{1, 2, 3\}$, define a relation R on the set A as follows:
$$R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$$
Write the ordered pairs to be added to R to make the smallest equivalence relation.
21. Let $A = \{0, 1, 2, 3\}$ and R be a relation on A defined as
$$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$$
Is R reflexive? symmetric" transitive?
22. Let the relation R be defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(a, b) : |a^2 - b^2| < 8\}$. Write R as a set of ordered pairs.
23. Let the relation R be defined on N by aRb iff $2a + 3b = 30$. Then write R as a set of ordered pairs.
24. Write the smallest equivalence relation on the set $A = \{1, 2, 3\}$.

ANSWERS

1. $\{0, \pm 3, \pm 4, \pm 5\}$
2. $\{0, \pm 1, \pm 2\}$
3. $\{(a, a), (b, b), (c, c)\}$
4. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
5. $\{1, 2, 3\}$
6. $R = R^{-1}$
7. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$
8. $R^{-1} = \{(3, 2), (7, 2), (7, 3), (7, 4)\}$

9. $R = \{(3, 2), (3, 10), (5, 2), (5, 6), (7, 2), (7, 6), (7, 10)\}$
$R^{-1} = \{(2, 3), (10, 3), (2, 5), (6, 5), (2, 7), (6, 7), (10, 7)\}$

14. $R = \{(3, 4), (3, 9), (5, 9), (7, 9)\}$
15. $R = \{(2, 1), (4, 2), (6, 3), (8, 4)\}$
16. $\{(2, 4), (4, 4), (3, 3)\}$
17. $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$
18. $\{8, 27\}$
19. $[0] = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$
20. $(3, 1)$
21. Reflexive and symmetric
22. $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$
23. $R = \{(3, 8), (6, 6), (9, 4), (12, 2)\}$
24. $\{(1, 1), (2, 2), (3, 3)\}$

HINTS TO SELECTED PROBLEMS

17. We observe that $(1, 2) \in R$ and $(2, 1) \in R$ but $(1, 1) \notin R$. Hence, R is not transitive.

18. We have,

$$R = \{(a, a^3) : a \text{ is prime less than } 5\} = \{(a, a^3) : a = 2, 3\} = \{(2, 8), (3, 27)\}$$

$\therefore$ Range $(R) = \{8, 27\}$

19. We have, $R = \{(a, b) : 2 \text{ divides } a - b\}$

For any $a \in Z$, $[a] = \{x : (x, a) \in R\} = \{x : 2 \text{ divides } x - a\}$

$\therefore$ $[0] = \{x \in Z : 2 \text{ divides } x - 0\} = \{x \in Z : 2 \text{ divides } x\} = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$