CHAPTER 30

PROBABILITY

30.1 INTRODUCTION

There are three approaches to theory of probability, namely Experimental or Empirical approach, Classical approach and Axiomatic approach. In class IX, we have learnt about experimental approach. The classical approach has been discussed in class X. The axiomatic approach, formulated by Russian Mathematician A.N. Kolmogorov (1903-1987), has been discussed in class XI. We have also established the equivalence between the axiomatic theory of probability and the classical theory of probability in case of equally likely outcomes. On the basis of this relationship we obtained probabilities of events associated with discrete sample spaces. We have also studied addition theorem of probability. In continuation of these, we will introduce the concept of conditional probability which will be useful in obtaining multiplication rule of probability. The same will be used to derive a formula for the conditional probability. All these results will be helpful in understanding total probability theorem and Baye's theorem which will be introduced in the end of the chapter.

30.2 RECAPITULATION

Let us recall important terms and concepts which we have studied in earlier classes.

RANDOM EXPERIMENT *If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes, then such an experiment is called a random experiment or a probabilistic experiment.*

ELEMENTARY EVENT *If a random experiment is performed, then each of its outcomes is known as an elementary event.*

SAMPLE SPACE *The set of all possible outcomes of a random experiment called the sample space associated with it.*

EVENT *A subset of the sample space associated with a random experiment is called an event.*

OCCURRENCE OF AN EVENT *An event associated to a random experiment is said to occur if any one of the elementary events belonging to it is an outcome.*

Corresponding to every event A, associated to a random experiment, we define an event "not A denoted by $\overline{A}$" which is said to occur when and only when A does not occur.

CERTAIN (OR SURE) EVENT *An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.*

IMPOSSIBLE EVENT *An event associated with a random experiment is called an impossible event if it never occurs whenever the experiment is performed.*

COMPOUND EVENT *An event associated with a random experiment is a compound event, if it is the disjoint union of two or more elementary events.*

MUTUALLY EXCLUSIVE EVENTS *Two or more events associated with a random experiment are said to be mutually exclusive or impossible events if the occurrence of any one of them prevents the occurrence of all others, i.e. if no two or more of them can occur simultaneously in the same trial.*

EXHAUSTIVE EVENTS *Two or more events associated with a random experiment are exhaustive if their union is the sample space.*

FAVOURABLE ELEMENTARY EVENTS *Let S be the sample space associated with a random experiment and A be an event associated with the experiment. Then, elementary events belonging to A are known as favourble elementary events to the event A.*

Thus, an elementary event E is favourable to an event A, if the occurrence of E ensures the happening or occurrence of event A.

Events associated to a random experiment are generally described verbally and it is very important to have the ability of conversion of verbal description to equivalent set theoretic notations. Following table provides verbal description of some events and their equivalent set theoretic notations for ready reference.

Verbal description of the event	*Equivalent set theoretic notation*
Not A	$\overline{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
B but not A	$\overline{A} \cap B$
Neither A nor B	$\overline{A} \cap \overline{B}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$
Exactly one of A, B and C	$(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

30.2.2 PROBABILITY OF AN EVENT

DEFINITION *If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by P (A) and is defined as the ratio* $\frac{m}{n}$.

Thus, $P(A) = \frac{m}{n}$.

If $P(A) = 1$, then A is called the certain event and A is called an impossible event if $P(A) = 0$.

Also, $P(A) + P(\overline{A}) = 1$

The odds in favour of occurrence of the event A are defined by $m : (n - m)$ i.e., $P(A) : P(\overline{A})$ and the odds against the occurrence of A are defined by $(n - m) : m$ i.e., $P(\overline{A}) : P(A)$.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 1 *Twelve balls are distributed among three boxes, find the probability that the first box will contain three balls.*

SOLUTION Since each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes it 3^{12}.

Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways. Put these three balls in the first box. Now, remaining 9 balls are to be put in the remaining two boxes. This can be done in 2^9 ways.

So, the total number of ways in 3 balls can be put in the first box and the remaining 9 in other two boxes is ${}^{12}C_3 \times 2^9$.

Hence, required probability $= \dfrac{{}^{12}C_3 \times 2^9}{3^{12}}$

EXAMPLE 2 *Find the probability that the birth days of six different persons will fall in exactly two calender months.*

SOLUTION Since each person can have his birth day in any one of the 12 calender months. So, there are 12 options for each person.

$\therefore$ Total number of ways in which 6 persons can have their birth days

$$= 12 \times 12 \times 12 \times 12 \times 12 \times 12 = 12^6.$$

Out of 12 months, 2 months can be chosen in ${}^{12}C_2$ ways.
Now, birth days of six persons can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are two ways when all six birth days fall in one month. So, there are $(2^6 - 2)$ ways in which six birth days fall in the chosen 2 months.
$\therefore$ Number of ways in which six birth days fall in exactly two calender months $= {}^{12}C_2 \times (2^6-2)$

Hence, required probability $= \dfrac{{}^{12}C_2 \times (2^6 - 2)}{12^6} = \dfrac{341}{12^5}$

EXAMPLE 3 *If each element of a second order determinant is either zero or one, what is the probability that the value the determinant is non-negative?*
SOLUTION In a 2×2 determinant there are 4 elements and each element can take 2 values. So, total number of 2×2 determinants with elements 0 and 1 is $2^4 = 16$. Out of these determinants the values of the three determinants $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}$ and $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$ are negative. So, there are 13 determinants having non-negative values.

Hence, required probability $= \dfrac{13}{16}$

EXAMPLE 4 *Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary six faced die. Find the probability that the equation will have real roots.*
SOLUTION Since each of the coefficients a, b, and c can take the values from 1 to 6.
$\therefore$ Total number of equations $= 6 \times 6 \times 6 = 216$.
The roots of the equation $ax^2 + bx + c = 0$ will be real, if $b^2 - 4\,ac \geq 0 \Rightarrow b^2 \geq 4\,ac$.
The favourable number of elementary events can be enumerated as follows:

ac	a	c	$4ac$	b (so that $b^2 \geq 4\,ac$)	Number of ways
1	1	1	4	2, 3, 4, 5, 6	$1 \times 5 = 5$
2	1 2	2 1	8	3, 4, 5, 6	$2 \times 4 = 8$
3	1 3	3 1	12	4, 5, 6	$2 \times 3 = 6$
4	1 4 2	4 1 2	16	4, 5, 6	$3 \times 3 = 9$
5	1 5	5 1	20	5, 6	$2 \times 2 = 4$
6	1 6 2 3	6 1 3 2	24	5, 6	$4 \times 2 = 8$
7	ac is not possible				0
8	2 4	4 2	32	6	$2 \times 1 = 2$
9	3	3	36	6	1
					Total $= 43$

Since $b^2 \geq 4\,ac$ and since the maximum value of b^2 is 36, therefore ac = 10, 11, 12 ... etc. is not possible.

$\therefore$ Total number of favourable elementary events = 43.

Hence, required probability $= \frac{43}{216}$.

EXAMPLE 5 *Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. Find the probability that $x^2 + bx + c > 0$ for all $x \in R$.*

SOLUTION Since b and c both can assume values from 1 to 9. So, total number of ways of choosing b and c is $9 \times 9 = 81$.

Now, $x^2 + bx + c > 0$ for all $x \in R$

$\Rightarrow$ Disc < 0 *i.e.* $b^2 - 4c < 0$

The following table shows the possible values of b and c for which $b^2 - 4c < 0$

c	b	*Total*
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3,	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5
		32

So, favourable number of elementary events = 32.

Hence, required probability $= \frac{32}{81}$.

EXAMPLE 6 *A die is rolled thrice, find the probability of getting a larger number each time than the previous number.*

SOLUTION When a die is rolled thrice there are $6 \times 6 \times 6 = 216$ possible outcomes.

$\therefore$ Total number of elementary events = $6 \times 6 \times 6 = 216$.

Clearly, the second number has to be greater than unity. If the second number is i $(i > 1)$, then the first can be chosen in $(i - 1)$ ways and the third $(6 - i)$ ways. So, three numbers can be chosen in $(i-1) \times 1 \times (6-i)$ ways. But, the second number can vary from 2 to 5. Therefore,

Favourable number of elementary events = $\sum_{i=2}^{5} (i-1)(6-i) = 1 \times 4 + 2 \times 3 + 3 \times 2 + 4 \times 1 = 20.$

Hence, required probability $= \frac{20}{216} = \frac{5}{54}$.

EXAMPLE 7 *In how many ways, can three girls and nine boys be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls should sit together in a back row on adjacent seats? Now, if all the seating arrangements are equally likely, what is the probability of 3 girls sitting together in a back row on adjacent seats?*

SOLUTION Each van has 7 seats. So, there are 14 numbered seats in two vans.

The total number of ways in which 3 girls and 9 boys can sit on these seats is ${}^{14}C_{12} \times 12!$.

So, total number of seating arrangements $= {}^{14}C_{12} \times 12!$

In a van 3 girls can choose adjacent seats in the back row in two ways (1, 2, 3, or 2, 3, 4). So, the number of ways in which 3 girls can sit in the back row on adjacent seats is 2 (3!) ways. The number of ways in which 9 boys can sit on the remaining 11 seats is ${}^{11}C_9 \times 9!$ ways. Therefore, the number of ways in which 3 girls and 9 boys can sit in two vans

$$= 2(3!) \times {}^{11}C_9 \times 9! + 2(3!) \times {}^{11}C_9 \times 9! = 4 \times 3! \times {}^{11}C_9 \times 9!.$$

Hence, required probability $= \dfrac{4 \times 3! \times {}^{11}C_9 \times 9!}{{}^{14}C_{12} \times 12!} = \dfrac{1}{91}$.

EXAMPLE 8 *If the letters of the word 'ATTRACTION' are written down at random, find the probability that* (i) *all the T's occur together,* (ii) *no two T's occur together.*

SOLUTION The total number of arrangements of the letters of the word 'ATTRACTION' is $\dfrac{10!}{3!\,2!}$.

(i) Considering three T's as one letter there are 8 letters consisting of two identical A's. These 8 letters can be arranged in $\dfrac{8!}{2!}$ ways.

Hence, required probability $= \dfrac{\frac{8!}{2!}}{\frac{10!}{3!\,2!}} = \dfrac{3!\,8!}{10!} = \dfrac{1}{15}$

(ii) Other than 3 T's there are 7 letters which can be arranged in $\dfrac{7!}{2!}$ ways. In each arrangement of these seven letters there are 8 places, 6 between the 7 letters and one on extreme left and the other on extreme right. To separate three T's, we arrange them in these 8 places. This can be done in ${}^8C_3$ ways. Therefore,

Number of ways in which no two T's are together $= \dfrac{7!}{2!} \times {}^8C_3$

Hence, required probability $= \dfrac{\frac{7!}{2!} \times {}^8C_3}{\frac{10!}{3!\,2!}} = \dfrac{7}{15}$

EXAMPLE 9 *What is the probability that for S's come consecutively in the word 'MISSISSIPPI'?*

SOLUTION The total number of arrangements of the letters of the word 'MISSISSIPPI' is $\dfrac{11!}{4!\,4!\,2!}$.

Considering 4 S's as one letter, there are 8 letters which can be arranged in a row in $\dfrac{8!}{4!\,2!}$ ways.

So, the number of ways in which 4 S's come together $= \dfrac{8!}{4!\,2!}$

Hence, required probability $= \dfrac{8!/4!\,2!}{11!/4!\,4!\,2!} = \dfrac{8! \times 4!}{11!} = \dfrac{4}{165}$

30.2.2 ADDITION THEOREM

DEFINITION *If A and B are two events associated with a random experiment, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

If A, B, C are three events associated with a random experiment, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

If A, B, C are mutually exclusive events, then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

For any two events A and B, we have

(i) Probability of occurrence of A only $= P(A \cap \overline{B}) = P(A) - P(A \cap B)$

(ii) Probability of occurrence of B only $= P(\overline{A} \cap B) = P(B) - P(A \cap B)$

(iii) Probability of occurrence of exactly one of A and B

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B) = P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B).$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *For any two events A and B associated to a random experiment, prove that :*

(i) $P(A) = P(A \cap B) + P(A \cap \overline{B})$

(ii) $P(B) = P(A \cap B) + P(\overline{A} \cap B)$

(iii) $P(A \cup B) = P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B)$ [NCERT EXEMPLAR]

SOLUTION (i) It is evident from Fig. 30.1 that $A \cap \overline{B}$ and $A \cap B$ are mutually exclusive events such that

$$A = (A \cap \overline{B}) \cup (A \cap B)$$

$\therefore$ $P(A) = P\{(A \cap \overline{B}) \cup (A \cap B)\}$

$\Rightarrow$ $P(A) = P(A \cap \overline{B}) + P(A \cap B)$ [By addition Theorem for mutually exclusive events]

(ii) Proceed as in (i).

(iii) From Fig. 30.1, we observe that $A \cap \overline{B}, A \cap B$ and $\overline{A} \cap B$ are mutually exclusive events such that

$$A \cup B = (A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B)$$

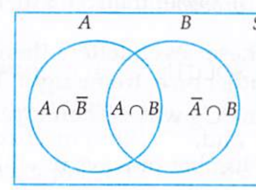

Fig. 30.1

So, by addition theorem for mutually exclusive events

$$P(A \cup B) = P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B)$$

EXAMPLE 2 *A basket contains 20 apples and 10 oranges out of which 5 apples and 3 oranges are defective. If a person takes out 2 at random what is the probability that either both are apples or both are good?*

SOLUTION Out of 30 items, two can be selected in ${}^{30}C_2$ ways.

So, total number of elementary events $= {}^{30}C_2$.

Consider the events: A = Getting two apples ; B = Getting two good items

Required probability $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ...(i)

There are 20 apples, out of which 2 can be drawn in $20C_2$ ways.

$\therefore$ $P(A) = \dfrac{{}^{20}C_2}{{}^{30}C_2}$

There are 8 defective pieces and the remaining 22 are good. Out of 22 good pieces, two can be selected in ${}^{22}C_2$ ways.

$\therefore$ $P(B) = \dfrac{{}^{22}C_2}{{}^{30}C_2}$

Since there are 15 pieces which are good apples out of which 2 can be selected in ${}^{15}C_2$ ways.

$\therefore \quad P(A \cap B)$ = Probability of getting 2 pieces which are good apples = $\dfrac{^{15}C_2}{^{30}C_2}$

Substituting the values of $P(A)$, $P(B)$, $P(B)$ and $P(A \cap B)$ in (i), we get

$$\text{Required probability} = \frac{^{20}C_2}{^{30}C_2} + \frac{^{22}C_2}{^{30}C_2} - \frac{^{15}C_2}{^{30}C_2} = \frac{316}{435}.$$

EXAMPLE 3 *The probability that a person will get an electric contract is $\frac{2}{5}$ and the probability that he will not get plumbing contract is $\frac{4}{7}$. If the probability of getting at least one contract is $\frac{2}{3}$, what is the probability that he will get both?*

SOLUTION Consider the following events:

A = Person gets an electric contract, B = Person gets plumbing contract

We have,

$$P(A) = \frac{2}{5},\ P(\overline{B}) = \frac{4}{7} \text{ and } P(A \cup B) = \frac{2}{3}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow \quad \frac{2}{3} = \frac{2}{5} + \left(1 - \frac{4}{7}\right) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = \frac{2}{5} + \frac{3}{7} - \frac{2}{3} = \frac{17}{105}$$

EXAMPLE 4 *The probability of simultaneous occurrence of at least one of two events A and B is p. If the probability that exactly one of A, B occurs is q, then prove that $P(\overline{A}) + P(\overline{B}) = 2 - 2p + q$.* **[NCERT EXEMPLAR]**

SOLUTION It is given that

P(Simultaneous occurrence of at least one of A and B) $= p$

and, P(Occurrence of exactly one of A and B) $= q$

$\Rightarrow \quad P(A \cup B) = p$ and $p(A \cup B) - P(A \cap B) = q$

$\Rightarrow \quad p - P(A \cap B) = q$

$\Rightarrow \quad P(A \cap B) = p - q$

$\Rightarrow \quad 1 - P(\overline{A \cap B}) = p - q \qquad [\because P(A \cap B) + P(\overline{A \cap B}) = 1]$

$\Rightarrow \quad 1 - P(\overline{A} \cup \overline{B}) = p - q$

$\Rightarrow \quad P(\overline{A} \cup \overline{B}) = 1 - p + q$

$\Rightarrow \quad P(\overline{A}) + P(\overline{B}) - P(\overline{A} \cap \overline{B}) = 1 - p + q \qquad$ [By addition theorem]

$\Rightarrow \quad P(\overline{A}) + P(\overline{B}) = 1 - p + q + P(\overline{A} \cap \overline{B})$

$\Rightarrow \quad P(\overline{A}) + P(\overline{B}) = 1 - p + q + P(\overline{A \cup B})$

$\Rightarrow \quad P(\overline{A}) + P(\overline{B}) = 1 - p + q + 1 - p \qquad [\because P(A \cup B) = p \therefore P(\overline{A \cup B}) = 1 - p]$

$\Rightarrow \quad P(\overline{A}) + P(\overline{B}) = 2 - 2p + q$

EXAMPLE 5 *Let A, B, C be three events. If the probability of occurring exactly one event out of A and B is $1 - x$, out of B and C is $1 - 2x$, out of C and A is $1 - x$, and that of occurring three events simultaneously is x^2, then prove that the probability that atleast one out of A, B, C will occur is greater than 1/2.*

SOLUTION We have,

$P(A) + P(B) - 2P(A \cap B) = 1 - x$ ...(i)

$P(B) + P(C) - 2P(B \cap C) = 1 - 2x$ ...(ii)

$P(C) + P(A) - 2P(C \cap A) = 1 - x$ ...(iii)

and, $P(A \cap B \cap C) = x^2$ ...(iv)

Adding (i), (ii) and (iii), we get

$$P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A) = \frac{3-4x}{2} \qquad ...(v)$$

$\therefore$ Probability that atleast one out of A, B, C will occur

$$= P(A\cup B\cup C)$$
$$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$
$$= \frac{3-4x}{2}+x^2 \qquad \text{[Using (iv) and (v)]}$$
$$= x^2-2x+\frac{3}{2} = (x-1)^2+\frac{1}{2} > \frac{1}{2}$$

EXAMPLE 6 *For the three events A, B and C, P (exactly one of the events A or B occurs) $= P$ (exactly one of the events B or C occurs) $= P$ (exactly one of the events C and A occurs) $= p$ and P (all the three events occur simultaneously) $= p^2$, where $0<p<1/2$. Then, find the probability of occurrence of at least one of the three events A, B, and C.*

SOLUTION We have,

$$P(\text{Exactly one of the events } A \text{ or } B \text{ occurs}) = p \Rightarrow P(A)+P(B)-2P(A\cap B) = p \qquad ...(i)$$
$$P(\text{Exactly one of the events } B \text{ or } C \text{ occurs}) = p \Rightarrow P(B)+P(C)-2P(B\cap C) = p \qquad ...(ii)$$
$$P(\text{Exactly one of the events } C \text{ or } A \text{ occurs}) = p \Rightarrow P(C)+P(A)-2P(A\cap B) = p \qquad ...(iii)$$
$$P(\text{All the three events occur simultaneously}) = p^2 \Rightarrow P(A\cap B\cap C) = p^2 \qquad ...(iv)$$

Adding (i), (ii) and (iii), we get

$$P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C) = \frac{3p}{2} \qquad ...(v)$$

$\therefore$ Required probability $= P(A\cup B\cup C)$

$$= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$$
$$= \frac{3p}{2}+p^2 = \frac{3p+2p^2}{2}$$

EXAMPLE 7 *The probabilities that a student pass in Mathematics, Physics and Chemistry are m, p and c respectively. Of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Find the value of $p+m+c$.*

SOLUTION Let A, B and C be three events given by

A = The student passes in Mathematics
B = The student passes in Physics
C = The student passes in Chemistry

It is given that

$$P(A) = p,\ P(B) = m \text{ and } P(C) = c$$
$$P(A\cup B\cup C) = \frac{75}{100} \qquad ...(i)$$
$$P(A\cap B)+P(B\cap C)+P(C\cap A)-2P(A\cap B\cap C) = \frac{50}{100} \qquad ...(ii)$$
$$P(A\cap B)+P(B\cap C)+P(C\cap A)-3P(A\cap B\cap C) = \frac{40}{100} \qquad ...(iii)$$

Subtracting (iii) from (ii), we get

$$P(A\cap B\cap C) = \frac{1}{10} \qquad ...(iv)$$

Adding (ii) and (iii), we get

$$2\{P(A\cap B)+P(B\cap C)+P(C\cap A)\}-5P(A\cap B\cap C) = \frac{9}{10}$$

$$\Rightarrow \quad 2\{P(A \cap B) + P(B \cap C) + P(C \cap A)\} = \frac{9}{10} + \frac{5}{10} \qquad \text{[Using (iv)]}$$

$$\Rightarrow \quad P(A \cap B) + P(B \cap C) + P(C \cap A) = \frac{7}{10} \qquad \text{...(v)}$$

From (i), we have

$$P(A \cup B \cup C) = \frac{75}{100}$$

$$\Rightarrow \quad P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C) = \frac{3}{4}$$

$$\Rightarrow \quad P(A) + P(B) + P(C) - \{P(A \cap B) + P(B \cap C) + P(C \cap A\} + P(A \cap B \cap C) = \frac{3}{4}$$

$$\Rightarrow \quad p + m + c - \frac{7}{10} + \frac{1}{10} = \frac{3}{4} \qquad \text{[Using (iv) and (v)]}$$

$$\Rightarrow \quad p + m + c = \frac{27}{20}$$

30.3 CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by $P(A/B)$. Thus, we have

$P(A/B)$ = Probability of occurrence of A given that B has already occurred.

Similarly, $P(B/A)$ when $P(A) \neq 0$ is defined as the probability of occurrence of event B when A has already occurred.

In fact, the meanings of symbols $P(A/B)$ and $P(B/A)$ depend on the nature of the events A and B and also on the nature of the random experiment. These two symbols have the following meaning also.

$P(A/B)$ = Probability of occurrence of A when B occurs

OR

Probability of occurrence of A when B is taken as the sample space

OR

Probability of occurrence of A with respect to B.

and, $P(B/A)$ = Probability of occurrence of B when A occurs

OR

Probability of occurrence of B when A is taken as the sample space.

OR

Probability of occurrence of B with respect to A.

In order to understand properly the meaning of conditional probability let us discuss the following illustrations.

ILLUSTRATION 1 *Let there be a bag containing 5 white and 4 red balls. Two balls are drawn from the bag one after the other without replacement. Consider the following events:*

A = Drawing a white ball in the first draw, B = Drawing a red ball in the second draw.

Now,

$P(B/A)$ = Probability of drawing a red ball in second draw given that a white ball has already been drawn in the first draw

$\Rightarrow$ $P(B/A)$ = Probability of drawing a red ball from a bag containing 4 white and 4 red balls

$$\Rightarrow \quad P(B/A) = \frac{4}{8} = \frac{1}{2}$$

For this random experiment $P(A/B)$ is not meaningful because A cannot occur after the occurrence of event B.

NOTE *In the above illustration events A and B were subsets of two different sample spaces as they are outcomes of two different trials which are performed one after the other.*

ILLUSTRATION 2 *Consider the random experiment of throwing a pair of dice and two events associated with it given by*

$A =$ The sum of the numbers on two dice is $8 = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$

$B =$ There is an even number on first die $= \{(2, 1), \ldots, (2, 6), (4, 1), \ldots, (4, 6), (6, 1), \ldots, (6, 6)\}$

In this case, events A and B are the subsets of the same sample space. So, we have the following meanings for $P(A/B)$ and $P(B/A)$.

$P(A/B) =$ Probability of occurrence of A when B occurs

or

Probability of occurrence A when B is taken as the sample space.

$$\therefore \quad P(A/B) = \frac{\text{Number of elementary events in } B \text{ which are favourable to } A}{\text{Number of elementary events in } B}$$

$$\Rightarrow \quad P(A/B) = \frac{\text{Number of elementary events favourable to } A \cap B}{\text{Number of elementary events favourable to } B} = \frac{3}{18}$$

Similarly,

$P(B/A) =$ Probability of occurrence of B when A occurs.

or

Probability of occurrence of B when A is taken as the sample space

$$\Rightarrow \quad P(B/A) = \frac{\text{Number of elementary events in } A \text{ which are favourable to } B}{\text{Number of elementary events in } A}$$

$$\Rightarrow \quad P(B/A) = \frac{\text{Number of elementary events favourable to } A \cap B}{\text{Number of elementary events favourable to } A} = \frac{3}{5}$$

ILLUSTRATION 3 *A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?*

SOLUTION Consider the following events :

$A =$ Number 4 appears at least once, $B =$ The sum of the numbers appearing is 6.

$\therefore$ Required probability $= P(A/B)$

$=$ Probability of occurrence of A when B is taken as the sample space

$$= \frac{\text{Number of elementary events favourable to } A \text{ which are favourable to } B}{\text{Number of elementary events favourable to } B}$$

$$= \frac{\text{Number of elementary events favourable to } (A \cap B)}{\text{Number of elementary events favourable to } B} = \frac{2}{5}$$

It follows from illustrations 2 and 3 that if A and B are two events associated with the same sample space S of a random experiment, then

$$P(A/B) = \frac{\text{Number of elementary events favourableto } A \cap B}{\text{Number of elementary events favourable to } B}$$

$$\Rightarrow \quad P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{\dfrac{n(A \cap B)}{n(S)}}{\dfrac{n(B)}{n(S)}} \qquad \text{[Dividing numerator and denominator by } n(S)\text{]}$$

$$\Rightarrow \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, we have

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

These results have also been derived in the next section by using multiplication theorem on probability.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I EXAMPLES BASED ON $P(A/B) = \frac{n(A \cap B)}{n(B)}$, $P(B/A) = \frac{n(A \cap B)}{n(A)}$

EXAMPLE 1 *A fair dice is rolled. Consider the following events:*

$A = \{1, 3, 5\}$, $B = \{2, 3\}$, *and* $C = \{2, 3, 4, 5\}$. *Find* [NCERT]

(i) $P(A/B)$ *and* $P(B/A)$ (ii) $P(A/C)$ *and* $P(C/A)$

(iii) $P(A \cup B/C)$ *and* $P(A \cap B/C)$ (iv) $P(A \cap B/C)$

SOLUTION We have, $n(A) = 3, n(B) = 2, n(C) = 4$

Clearly,

$A \cap B = \{3\}, A \cap C = \{3, 5\}, A \cup B \cap C = \{2, 3, 5\}$ and $A \cap B \cap C = \{3\}$

$\Rightarrow \quad n(A \cap B) = 1, n(A \cap C) = 2, n(A \cup B \cap C) = 3$ and $n(A \cap B \cap C) = 1$

(i) $P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$ and, $P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{1}{3}$

(ii) $P(A/C) = \frac{n(A \cap C)}{n(C)} = \frac{2}{4} = \frac{1}{2}$ and, $P(C/A) = \frac{n(A \cap C)}{n(A)} = \frac{2}{3}$

(iii) $P(A \cup B/C) = \frac{n(A \cup B \cap C)}{n(C)} = \frac{3}{4}$

(iv) $P(A \cap B/C) = \frac{n(A \cap B \cap C)}{n(C)} = \frac{1}{4}$

EXAMPLE 2 *A coin is tossed three times. Find P (E/F) in each of the following:*

(i) E = *Head on the third toss*, F = *Heads on first two tosses*

(ii) E = *At least two heads*, F = *At most two heads*

(iii) E = *At most two tails*, F = *At least one tail* [NCERT]

SOLUTION The sample space associated to the given random experiment is given by

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

(i) We have,

$E = \{HHH, HTH, THH, TTH\}$, $F = \{HHH, HHT\}$

$\therefore \quad E \cap F = \{HHH\}$

Clearly, $n(E \cap F) = 1$ and $n(F) = 2$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{1}{2}$$

(ii) We have,

$E = \{HHH, HHT, HTH, THH\}$, $F = \{TTT, THT, TTH, HTT, THH, HTH, HHT\}$

$\therefore \quad E \cap F = \{HHT, HTH, THH\}$

Clearly, $n(E \cap F) = 3$ and $n(F) = 7$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{3}{7}$$

(iii) We have,

$$E = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$$
$$F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$$
$$\therefore \quad E \cap F = \{HHT, HTH, THH, HTT, THT, TTH\}$$

Clearly, $n(E \cap F) = 6$ and $n(F) = 7$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{6}{7}$$

EXAMPLE 3 *Two coins are tossed once. Find $P(E/F)$ in each of the following :*

(i) *E = Tail appears on at least one coin, F = One coin shows head*

(ii) *E = No tail appears, F = No head appears.* [NCERT]

SOLUTION The sample space associated to the random experiment of tossing two coins is given by $S = \{HH, HT, TH, TT\}$.

(i) We have,

$$E = \{HT, TH, TT\}, F = \{HT, TH\}$$
$$\therefore \quad E \cap F = \{HT, TH\}$$

Clearly, $n(E \cap F) = 2$ and $n(F) = 2$

$$\therefore \quad P(E/F) = \frac{n(E \cap F)}{n(F)} = \frac{2}{2} = 1$$

(ii) We have,

$$E = \{HH\}, F = \{TT\}$$
$$\therefore \quad E \cap F = \{\ \} = \phi$$

Clearly, $n(E \cap F) = 0$ and $n(F) = 1$

$$\therefore \quad \text{Required probability} = \frac{n(E \cap F)}{n(F)} = \frac{0}{1} = 0$$

EXAMPLE 4 *Mother, father and son line up at random for a family picture. Find $P(A/B)$, if A and B are defined as follows:*

A = Son on one end, B = Father in the middle [NCERT]

SOLUTION Total number of ways in which Mother (M), Father (F) and Son (S) can be lined up at random in one of the following ways:

$$MFS, MSF, FMS, FSM, SFM, SMF$$

We have,

$$A = \{SMF, SFM, MFS, FMS\} \text{ and } B = \{MFS, SFM\}$$
$$\therefore \quad A \cap B = \{MFS, SFM\}$$

Clearly, $n(A \cap B) = 2$ and $n(B) = 2$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{2}{2} = 1$$

EXAMPLE 5 *A and B are two events such that $P(A) \neq 0$. Find $P(B/A)$, if*

(i) *A is a subset of B* (ii) $A \cap B = \phi$

SOLUTION (i) If A is a subset of B, then

$$A \cap B = A \Rightarrow n(A \cap B) = n(A)$$
$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = 1$$

(ii) If $A \cap B = \phi$, then $n(A \cap B) = 0$

$$\therefore \quad P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{0}{n(A)} = 0$$

EXAMPLE 6 *A couple has two children. Find the probability that*

(i) *both the children are boys, if it is known that the older child is a boy.* **[NCERT]**

(ii) *both the children are girls, if it is known that the older child is a girl.*

(iii) *both the children are boys, if it is known that at least one of the children is a boy.*

SOLUTION Let B_i and G_i $(i = 1, 2)$ stand for the event that i^{th} child be a boy and a girl respectively. Then, the sample space associated to the random experiment is $S = \{B_1 B_2, B_1 G_2, G_1 B_2, G_1 G_2\}$.

(i) Consider the following events:

A = Both the children are boys $= \{B_1 B_2\}$, B = The older child is a boy $= \{B_1 B_2, B_1 G_2\}$

$\therefore \quad A \cap B = \{B_1 B_2\}$. Clearly, $n(A \cap B) = 1$ and $n(B) = 2$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$$

(ii) Consider the following events:

A = Both the children are girls $= \{G_1 G_2\}$, B = The older child is a girl $= \{G_1 G_2, G_1 B_2\}$

$\therefore \quad A \cap B = \{G_1 G_2\}$

Clearly, $n(A \cap B) = 1$ and $n(B) = 2$

$$\therefore \quad \text{Required probability} = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$$

(iii) Consider the following events:

A = Both the children are boys $= \{B_1 B_2\}$

B = At least one of the children is a boy $= \{B_1 B_2, B_1 G_2, G_1 B_2\}$

$\therefore \quad A \cap B = \{B_1 B_2\}$

Clearly, $n(A \cap B) = 1$ and $n(B) = 3$

$$\therefore \quad \text{Required probability} = \frac{n(A \cap B)}{n(B)} = \frac{1}{3}$$

EXAMPLE 7 *A pair of dice is thrown. If the two numbers appearing on them are different, find the probability (i) the sum of the numbers is 6 (ii) the sum of the numbers is 4 or less (iii) the sum of the numbers is 4*

SOLUTION Consider the following events:

A = Numbers appearing on two dice are different

B = The sum of the numbers on two dice is 6

C = The sum of the numbers on two dice is 4 or less

D = The sum of the numbers on two dice is 4

Clearly,

$A = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2)(4, 3), (4, 5), (4, 6), (5, 1), (5, 2) (5, 3), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)\}$

$B = \{(1, 5), (5, 1), (2, 4) (4, 2), (3, 3)\}$

$C = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)\}$ and, $D = \{(1, 3), (3, 1), (2, 2)\}$

Clearly, $A \cap B = \{(1, 5), (5, 1), (2, 4), (4, 2)\}$, $A \cap C = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$

and, $A \cap D = \{(1, 3), (3, 1)\}$

$\therefore \quad n(A \cap B) = 4, n(A \cap C) = 4, n(A \cap D) = 2$ and $n(A) = 30$

$$\text{(i)} \quad \text{Required probability} = P(B/A) = \frac{n(A \cap B)}{n(A)} = \frac{4}{30} = \frac{2}{15}$$

$$\text{(ii)} \quad \text{Required probability} = P(D/A) = \frac{n(A \cap D)}{n(A)} = \frac{2}{30} = \frac{1}{15}$$

(iii) Required probability $= P(C/A) = \dfrac{n(A \cap C)}{n(A)} = \dfrac{4}{30} = \dfrac{2}{15}$

EXAMPLE 8 *A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the probability that the number 4 has appeared at least once?* **[NCERT]**

SOLUTION Consider the following events:

$A =$ Sum of the numbers appearing on two dice is $6 = \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$

$B =$ Number 4 has appeared at least once

$= \{(1, 4), (4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4), (4, 5), (5, 4), (4, 6), (6, 4)\}$

$\therefore \quad A \cap B = \{(2, 4)\ (4, 2)\}$

Required probability $= P(B/A) = \dfrac{n(A \cap B)}{n(A)} = \dfrac{2}{5}$

EXAMPLE 9 *A die is thrown three times. Events A and B are defined as below:*

$A =$ *Getting 4 on third die,* $B =$ *Getting 6 on the first* and *5 on the second throw*

Find the probability of A given that B has already occurred. **[NCERT]**

SOLUTION The sample space S associated to the random experiment of throwing three dice has $6 \times 6 \times 6 = 216$ elements.

We have,

$$\begin{aligned} A = \{&(1, 1, 4), (1, 2, 4), (1, 3, 4), (1, 4, 4), (1, 5, 4), (1, 6, 4) \\ &(2, 1, 4), (2, 2, 4), (2, 3, 4), (2, 4, 4), (2, 5, 4), (2, 6, 4) \\ &\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ &(6, 1, 4), (6, 2, 4), (6, 3, 4), (6, 4, 4), (6, 5, 4), (6, 6, 4)\} \end{aligned}$$

$B = \{(6, 5, 1), (6, 5, 2), (6, 5, 3), (6, 5, 4), (6, 5, 5), (6, 5, 6)\}$

$\therefore \quad A \cap B = \{(6, 5, 4)\}$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{1}{6}$

EXAMPLE 10 *A black and a red dice are rolled in order. Find the conditional probability of obtaining*

(i) *a sum grater than 9, given that the black die resulted in a 5.*

(ii) *a sum 8, given that the red die resulted in a number less than 4.* **[NCERT]**

SOLUTION (i) Consider the following events:

$A =$ Getting a sum greater than 9, $B =$ Black die resulted in a 5

Clearly, $A = \{(5, 5), (6, 4), (4, 6), (6, 5), (5, 6), (6, 6)\}$, $B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$\therefore \quad A \cap B = \{(5, 5), (5, 6)\}$

Thus, $n(A \cap B) = 2, n(A) = 6$ and $n(B) = 6$

Hence, Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{6} = \dfrac{1}{3}$

(ii) Consider the following events:

$A =$ Getting 8 as the sum, $B =$ Red die resulted in a number less than 4

Clearly, $A = \{(2, 6), (6, 2) (3, 5), (5, 3), (4, 4)\}$

$B = \{(6, 1), (6, 2), (6, 3), (5, 1), (5, 2), (5, 3), (4, 1), (4, 2), (4, 3), (3, 1), (3, 2), (3, 3)$
$(2, 1), (2, 2), (2, 3), (1, 1), (1, 2), (1, 3)\}$

$\therefore \quad A \cap B = \{(6, 2), (5, 3)\}$

Thus, $n(A \cap B) = 2, n(B) = 18$

Required probability $= P(A/B) = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{18} = \dfrac{1}{9}$

EXAMPLE 11 *Consider the experiment of throwing a die, if a multiple of 3 comes up, throw the die again and if any other number comes, toss a coin. Find the conditional probability of the event 'the coin shows a tail' given that 'at least one die shows a three'.* [NCERT]

SOLUTION The sample space S associated to the given random experiment is given by

$$S = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),$$
$$(1,H),(1,T),(2,H),(2,T),(4,H),(4,T),(5,H),(5,T)\}$$

Consider the following events:

A = The coin shows a tail, B = At least one die shows a three

Clearly, $A = \{(1,T),(2,T),(4,T),(5,T)\}$, $B = \{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(6,3)\}$

$\therefore \quad A \cap B = \phi$

Thus, $n(A \cap B) = 0, n(A) = 4$ and $n(B) = 7$

$$\text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{0}{7} = 0$$

EXAMPLE 12 *In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?* [NCERT]

SOLUTION Let A be the event that a student chosen randomly studies in class XII and B be the event that the randomly chosen student is a girl. We have to find $P(A/B)$.

Clearly, $n(A \cap B) = 10\%$ of $430 = 430 \times \frac{10}{100} = 43$ and, $n(B) = 430$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{43}{430} = \frac{1}{10}$$

EXAMPLE 13 *An instructor has a question bank consisting of 300 easy True/false questions, 200 difficult True/False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be any easy question given that it is a multiple choice question?* [NCERT]

SOLUTION Let A be the event that selected question is an easy question and B be the event that the question selected is a multiple choice question.

We have,

$$n(A) = 300 + 500 = 800, n(B) = 500 + 400 = 900$$

$A \cap B$ = Selected question is an easy multiple choice question

Clearly, $n(A \cap B) = 500$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{500}{900} = \frac{5}{9}$$

Type II EXAMPLES BASED ON $P(A/B) = \frac{P(A \cap B)}{P(B)}$ AND $P(B/A) = \frac{P(A \cap B)}{P(A)}$

EXAMPLE 14 *Given that A and B are two events such that $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A/B)$ and $P(B/A)$.* [NCERT]

SOLUTION We have,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \text{ and } P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$\Rightarrow \quad P(A/B) = \frac{0.2}{0.3} = \frac{2}{3} \text{ and } P(B/A) = \frac{0.2}{0.6} = \frac{1}{3}$$

EXAMPLE 15 *If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find*

(i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$ [NCERT]

SOLUTION (i) We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$

(ii) $\quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/11}{5/11} = \frac{4}{5}$

(iii) $\quad P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{4/11}{6/11} = \frac{2}{3}$

EXAMPLE 16 *Evaluate $P(A \cup B)$, if $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$* [NCERT]

SOLUTION We have, $2P(A) = P(B) = \frac{5}{13} \Rightarrow P(A) = \frac{5}{26}$ and $P(B) = \frac{5}{13}$

Now, $\quad P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \quad \frac{2}{5} = \frac{P(A \cap B)}{\frac{5}{13}} \Rightarrow P(A \cap B) = \frac{2}{13}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$

Type III PROBLEMS BASED UPON THE MEANING OF CONDITIONAL PROBABILITY

EXAMPLE 17 *Two integers are selected at random from integers 1 to 11. If the sum is even, find the probability that both the numbers are odd.*

SOLUTION Out of integers from 1 to 11, there are 5 even integers and 6 odd integers.

Consider the following events:

A = Both the numbers chosen are odd, B = The sum of the numbers chosen is even,

$\therefore$ Required probability

= $P(A/B)$

= Probability that the two numbers chosen are odd if it is given that the sum of the numbers chosen is even.

$= \frac{{}^6C_2}{{}^5C_2 + {}^6C_2}$ $\quad \left[\begin{array}{l} \because \text{The number of ways of getting the sum as an} \\ \quad \text{even number} = {}^5C_2 + {}^6C_2 \\ \text{The number of ways of selecting two odd numbers} = {}^6C_2 \end{array}\right]$

$= \frac{3}{5}$

EXAMPLE 18 *A die is thrown three times, if the first throw is a four, find the chance of getting 15 as the sum.*

SOLUTION Consider the following events:

A = Getting 15 as the sum in a throw of three dice, B = Getting 4 on the first die.

$\therefore$ Required probability

= $P(A/B)$

= Probability of getting 15 as the sum of the numbers if there is 4 on the first die.

$= \frac{2}{36}$ $\quad \left[\begin{array}{l} \because \text{Total number of ways} = 1 \times 6 \times 6 = 36 \\ \text{There are two favourable elementary events viz. } (4,6,5), (4,5,6). \end{array}\right]$

$= \frac{1}{18}$

EXERCISE 30.1

BASIC

1. Ten cards numbered 1 through 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?
2. Assume that each child born is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (i) the youngest is a girl, (ii) at least one is a girl? [NCERT]
3. Given that the two numbers appearing on throwing two dice are different. Find the probability of the event 'the sum of numbers on the dice is 4'.
4. A coin is tossed three times, if head occurs on first two tosses, find the probability of getting head on third toss.
5. A die is thrown three times, find the probability that 4 appears on the third toss if it is given that 6 and 5 appear respectively on first two tosses.
6. Compute $P(A/B)$, if $P(B) = 0.5$ and $P(A \cap B) = 0.32$
7. If $P(A) = 0.4$, $P(B) = 0.3$ and $P(B/A) = 0.5$, find $P(A \cap B)$ and $P(A/B)$.
8. If A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ and $P(A \cup B) = \frac{11}{30}$, find $P(A/B)$ and $P(B/A)$.
9. A couple has two children. Find the probability that both the children are (i) males, if it is known that at least one of the children is male. (ii) females, if it is known that the elder child is a female. [CBSE 2010]
10. A pair of dice is thrown. It is given that the sum of the numbers appearing on both dice is an even number. Find the probability that the number appearing on at least one die is 3. [CBSE 2022]

ANSWERS

1. $\frac{4}{7}$ 2. (i) $\frac{1}{2}$ (ii) $\frac{1}{3}$ 3. $\frac{1}{15}$ 4. $\frac{1}{2}$ 5. $\frac{1}{6}$

6. $\frac{16}{25}$ 7. $0.2, \frac{2}{3}$ 8. $\frac{5}{6}, \frac{1}{2}$ 9. (i) $\frac{1}{3}$ (ii) $\frac{1}{2}$ 10. $\frac{5}{18}$

HINTS TO SELECTED PROBLEMS

1. The sample space associated to the given random experiment is given by

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

Consider the following events:

A = Number on the card drawn is even = $\{2, 4, 6, 8, 10\}$

B = Number on the card drawn is greater than 3 = $\{4, 5, 6, 7, 8, 9, 10\}$

$$\therefore \quad \text{Required probability} = P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{4}{7}$$

2. Consider the following events:

A = Both children are girls, B = The youngest child is a girl, C = At least one child is a girl.

Clearly, $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$, $A = \{G_1G_2\}$, $B = \{B_1G_2, G_1G_2\}$ and

$C = \{B_1G_2, G_1B_2, G_1G_2\}$

(i) Required probability $= P(A/B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{2}$

(ii) Required probability $= P(A/C) = \frac{n(A \cap C)}{n(C)} = \frac{1}{3}$

30.4 MULTIPLICATION THEOREMS ON PROBABILITY

In this section, we shall discuss some theorems which are helpful in computing the probabilities of simultaneous occurrences of two or more events associated with a random experiment.

THEOREM 1 *If A and B are two events associated with a random experiment, then*

$$P(A \cap B) = P(A)\,P(B/A), \quad \text{if } P(A) \neq 0$$

or, $$P(A \cap B) = P(B)\,P(A/B), \quad \text{if } P(B) \neq 0$$

PROOF *Let S be the sample space associated with the given random experiment. Suppose S contains n elementary events. Let* m_1, m_2 and *m be the number of elementary events favourable to A, B and* $A \cap B$ *respectively. Then,*

$$P(A) = \frac{m_1}{n}, \; P(B) = \frac{m_2}{n} \text{ and } P(A \cap B) = \frac{m}{n}.$$

Since m_1 elementary events are favourable to A out of which m are favourable to B. Therefore, $P(B/A) = \dfrac{m}{m_1}$. Similarly, we have $P(A/B) = \dfrac{m}{m_2}$

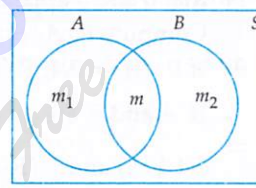

Fig. 30.2

Now, $$P(A \cap B) = \frac{m}{n}$$

$$\Rightarrow \quad P(A \cap B) = \frac{m}{m_1} \times \frac{m_1}{n} = P(B/A) \times P(A) \qquad \text{...(i)}$$

Again, $$P(A \cap B) = \frac{m}{n}$$

$$\Rightarrow \quad P(A \cap B) = \frac{m}{m_2} \times \frac{m_2}{n} = P(A/B)\,P(B) \qquad \text{...(ii)}$$

Q.E.D.

NOTE 1 *From (i) and (ii) in the above theorem, we obtain that*

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{and} \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

REMARK *If A and B are independent events, then* $P(A/B) = P(A)$ *and* $P(B/A) = P(B)$.

$$\therefore \quad P(A \cap B) = P(A)\,P(B)$$

Also,
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A)\,P(B) && [\because P(A \cap B) = P(A)\,P(B)] \\ &= 1 - 1 + P(A) + P(B) - P(A)\,P(B) && [\text{Adding and subtracting } 1] \\ &= 1 - [1 - P(A) - P(B) + P(A)\,P(B)] \\ &= 1 - [\{(1 - P(A)\}\{1 - P(B))\}] = 1 - P(\overline{A})\,P(\overline{B}) \end{aligned}$$

THEOREM 2 *(Extension of multiplication theorem). If* $A_1, A_2, \ldots, A_n$ *are n events associated with a random experiment, then*

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) \ldots\ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}),$$

where $P(A_i/A_1 \cap A_2 \ldots \cap A_{i-1})$ represents the conditional probability of the occurrence of event A_i, given that the events $A_1, A_2, \ldots, A_{i-1}$ have already occurred.

PROOF This theorem can be proved by using the principle of mathematical induction.

PARTICULAR CASE *If A, B, C are three events associated with a random experiment, then*

$$P(A \cap B \cap C) = P(A)\,P(B/A)\,P(C/A \cap B).$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE PROBABILITY OF SIMULTANEOUS OCCURRENCE OF TWO OR MORE EVENTS**

EXAMPLE 1 *A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that first is white and second is black?*

SOLUTION Consider the following events:

A = Getting a white ball in first draw, B = Getting a black ball in second draw.

Required probability

= Probability of getting a white ball in first draw and a black ball in second draw

$= P(A \text{ and } B)$

$= P(A \cap B)$

$= P(A)\,P(B/A)$ [By Multiplication Theorem] ...(i)

Now, $P(A) = \frac{{}^{10}C_1}{{}^{25}C_1} = \frac{10}{25} = \frac{2}{5}$

and, $P(B/A)$ = Probability of getting a black ball in second draw when a white ball has already been drawn in first draw

$\Rightarrow \quad P(B/A) = \frac{{}^{15}C_1}{{}^{24}C_1} = \frac{15}{24} = \frac{5}{8}$ [$\because$ 24 ball are left after drawing a white ball in first-draw out of which 15 are black]

Substituting these values in (i), we obtain

Required probability $= P(A \cap B) = P(A)\,P(B/A) = \frac{2}{5} \times \frac{5}{8} = \frac{1}{4}$

EXAMPLE 2 *Find the probability of drawing a diamond card in each of the two consecutive draws from a well shuffled pack of cards, if the card drawn is not replaced after the first draw.* [CBSE 2002C]

SOLUTION Let A be the event of drawing a diamond card in the first draw and B be the event of drawing a diamond card in the second draw. Then,

$$P(A) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52} = \frac{1}{4}$$

After drawing a diamond card in first draw 51 cards are left out of which 12 cards are diamond cards.

$\therefore$ $P(B/A)$ = Probability of drawing a diamond card in second draw when a diamond card has already been drawn in first draw

$\Rightarrow \quad P(B/A) = \frac{{}^{12}C_1}{{}^{51}C_1} = \frac{12}{51} = \frac{4}{17}$

Required probability $= P(A \cap B) = P(A)\,P(B/A) = \frac{1}{4} \times \frac{4}{17} = \frac{1}{17}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, find the probability of getting all white balls.*

SOLUTION Let A, B, C and D denote events of getting a white ball in first, second, third and fourth draw respectively. Then,

Required probability $= P(A \cap B \cap C \cap D)$

$= P(A)\,P(B/A)\,P(C/A \cap B)\,P(D/A \cap B \cap C)$ ...(i)

Now, $P(A)$ = Probability of drawing a white ball in first draw $= \frac{5}{20} = \frac{1}{4}$

When a white ball is drawn in the first draw there are 19 balls left in the bag, out of which 4 are white.

$$\therefore \quad P(B/A) = \frac{4}{19}$$

Since the ball drawn is not replaced, therefore after drawing a white ball in second draw there are 18 balls left in the bag, out of which 3 are white.

$$\therefore \quad P(C/A \cap B) = \frac{3}{18} = \frac{1}{6}$$

After drawing a white ball in third draw there are 17 balls left in the bag, out of which 2 are white.

$$\therefore \quad P(D/A \cap B \cap C) = \frac{2}{17}$$

Hence, Required probability $= P(A \cap B \cap C \cap D)$

$$= P(A)\, P(B/A)\, P(C/A \cap B)\, P(D/A \cap B \cap C)$$

$$= \frac{1}{4} \times \frac{4}{19} \times \frac{1}{6} \times \frac{2}{17} = \frac{1}{969}$$

EXAMPLE 4 *A bag contains 19 tickets, numbered from 1 to 19. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.*

SOLUTION Let A be the event of drawing an even numbered ticket in first draw and B be the event of drawing an even numbered ticket in the second draw. Then,

$$\text{Required probability} = P(A \cap B) = P(A)\, P(B/A) \quad \text{...(i)}$$

Since there are 19 tickets, numbered 1 to 19, in the bag out of which 9 are even numbered viz. 2, 4, 6, 8, 10, 12, 14, 16, 18.

$$\therefore \quad P(A) = \frac{9}{19}$$

Since the ticket drawn in the first draw is not replaced, therefore second ticket drawn is from the remaining 18 tickets, out of which 8 are even numbered.

$$\therefore \quad P(B/A) = \frac{8}{18} = \frac{4}{9}.$$

Substituting these values in (i), we get

$$\text{Required probability} = P(A \cap B) = P(A)\, P(B/A) = \frac{9}{19} \times \frac{4}{9} = \frac{4}{19}$$

EXAMPLE 5 *An urn contains 5 white and 8 black balls. Two successive drawings of three balls at a time are made such that the balls are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and second draw gives 3 black balls.*

SOLUTION Consider the following events:

A = Drawing 3 white balls in first draw, B = Drawing 3 black balls in the second draw.

$$\text{Required probability} = P(A \cap B) = P(A)\, P(B/A) \quad \text{...(i)}$$

Now, $P(A) = \frac{{}^5C_3}{{}^{13}C_3} = \frac{10}{286} = \frac{5}{143}$

After drawing 3 white balls in first draw 10 balls are left in the bag, out of which 8 are black balls.

$$\therefore \quad P(B/A) = \frac{{}^8C_3}{{}^{10}C_3} = \frac{56}{120} = \frac{7}{15}$$

Substituting these values in (i), we obtain

$$\text{Required probability} = P(A \cap B) = P(A)\, P(B/A) = \frac{5}{143} \times \frac{7}{15} = \frac{7}{429}$$

EXAMPLE 6 *Two balls are drawn from an urn containing 2 white, 3 red and 4 black balls one by one without replacement. What is the probability that at least one ball is red?*

SOLUTION Let R_1 and R_2 denote the events of getting a red ball in first and second draws respectively. Then,

$$\begin{aligned}\text{Required probability} &= P(R_1 \cup R_2)\\ &= 1 - P(\overline{R_1 \cup R_2})\\ &= 1 - P(\overline{R}_1 \cap \overline{R}_2)\\ &= 1 - P(\overline{R}_1)P(\overline{R}_2/\overline{R}_1) \qquad \ldots\text{(i)}\end{aligned}$$

Now,

$$\begin{aligned}P(\overline{R_1}) &= \text{Probability of not getting a red ball in first draw}\\ &= \text{Probability of getting an other colour (white or black) ball in first draw}\\ &= \frac{6}{9} = \frac{2}{3}\end{aligned}$$

When another colour ball is drawn in first draw there are 5 other colour (white and black) balls and 3 red balls, out of which one other colour ball can be drawn in ${}^5C_1$ ways.

$$\therefore \quad P(\overline{R}_2/\overline{R}_1) = \frac{5}{8}$$

Substituting these values in (i), we obtain

$$\text{Required probability} = 1 - P(\overline{R}_1)\, P(\overline{R}_2/\overline{R}_1) = 1 - \frac{2}{3} \times \frac{5}{8} = \frac{7}{12}.$$

EXAMPLE 7 *To test the quality of electric bulbs produced in a factory, two bulbs are randomly selected from a large sample without replacement. If either bulb is defective, the entire lot is rejected. Suppose a sample of 200 bulbs contains 5 defective bulbs. Find the probability that the sample will be rejected.*

SOLUTION Clearly, the sample will be rejected if at least one of the two bulbs is defective. Therefore, if we consider the following events:

A = First bulb is defective, B = Second bulb is defective.

Then,

$$\begin{aligned}\text{Required probability} &= P(A \cup B)\\ &= 1 - P(\overline{A \cup B}) = 1 - P(\overline{A} \cap \overline{B})\\ &= 1 - P(\overline{A})\, P(\overline{B}/\overline{A}) = 1 - \frac{195}{200} \times \frac{194}{199} = 1 - \frac{3783}{3980} = \frac{197}{3980}.\end{aligned}$$

EXAMPLE 8 *A bag contains n white and n red balls. Pairs of balls are drawn without replacement until the bag is empty. Show that the probability that each pair consists of one white and one red ball is* $\dfrac{2^n}{{}^{2n}C_n}$.

SOLUTION Let A_i $(i = 1, 2, \ldots, n)$ denote the event of getting one white and one red ball in ith draw. Then,

$$\begin{aligned}\text{Required probability} &= P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)\\ &= P(A_1)\, P(A_2/A_1)\, P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \qquad \ldots\text{(i)}\end{aligned}$$

Now, $$P(A_1) = \frac{{}^nC_1 \times {}^nC_1}{{}^{2n}C_2} = \frac{n^2}{{}^{2n}C_2}$$

$$P(A_2/A_1) = \frac{{}^{n-1}C_1 \times {}^{n-1}C_1}{{}^{2n-2}C_2} = \frac{(n-1)^2}{{}^{2n-2}C_2}$$

$$P(A_3/A_1 \cap A_2) = \frac{{}^{n-2}C_1 \times {}^{n-2}C_1}{{}^{2n-4}C_2} = \frac{(n-2)^2}{{}^{2n-4}C_2} \quad \text{and so on.}$$

Finally, $$P(A_{n-1}/A_1 \cap A_2 \cap \ldots \cap A_{n-2}) = \frac{{}^2C_1 \times {}^2C_1}{{}^4C_2} = \frac{2^2}{{}^4C_2}$$

and, $$P(A_n/A_1 \cap A_2 \ldots \cap A_{n-1}) = \frac{1}{{}^2C_2}$$

Substituting these values in (i), we get

$$\text{Required probability} = \frac{n^2}{^{2n}C_2} \times \frac{(n-1)^2}{^{2n-2}C_2} \times \frac{(n-2)^2}{^{2n-4}C_2} \times \ldots \times \frac{2^2}{^4C_2} \times \frac{1}{^2C_2}$$

$$= \frac{(n!)^2}{\frac{(2n)(2n-1)}{2} \times \frac{(2n-2)(2n-3)}{2} \times \ldots \times \frac{4\times 3}{2} \times \frac{2\times 1}{2}} = \frac{(n!)^2\, 2^n}{(2n)!} = \frac{2^n}{^{2n}C_n}$$

EXERCISE 30.2

BASED ON LOTS

1. From a pack of 52 cards, two are drawn one by one without replacement. Find the probability that both of them are kings.
2. From a pack of 52 cards, 4 are drawn one by one without replacement. Find the probability that all are aces (or, kings). [NCERT EXEMPLAR]
3. Find the chance of drawing 2 white balls in succession from a bag containing 5 red and 7 white balls, the ball first drawn not being replaced.
4. A bag contains 25 tickets, numbered from 1 to 25. A ticket is drawn and then another ticket is drawn without replacement. Find the probability that both tickets will show even numbers.
5. From a deck of cards, three cards are drawn on by one without replacement. Find the probability that each time it is a card of spade.
6. Two cards are drawn without replacement from a pack of 52 cards. Find the probability that
   (i) both are kings (ii) the first is a king and the second is an ace
   (iii) the first is a heart and second is red.
7. A bag contains 20 tickets, numbered from 1 to 20. Two tickets are drawn without replacement. What is the probability that the first ticket has an even number and the second an odd number.
8. An urn contains 3 white, 4 red and 5 black balls. Two balls are drawn one by one without replacement. What is the probability that at least one ball is black?
9. A bag contains 5 white, 7 red and 3 black balls. If three balls are drawn one by one without replacement, find the probability that none is red. [CBSE 2002C]
10. A card is drawn from a well-shuffled deck of 52 cards and then a second card is drawn. Find the probability that the first card is a heart and the second card is a diamond if the first card is not replaced.
11. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black? [NCERT]
12. Three cards are drawn successively, without replacement from a pack of 52 well shuffled cards. What is the probability that first two cards are kings and third card drawn is an ace? [NCERT]
13. A box of oranges is inspected by examining three randomly selected oranges drawn without replacement. If all the three oranges are good, the box is approved for sale otherwise it is rejected. Find the probability that a box containing 15 oranges out of which 12 are good and 3 are bad ones will be approved for sale.
14. A bag contains 4 white, 7 black and 5 red balls. Three balls are drawn one after the other without replacement. Find the probability that the balls drawn are white, black and red respectively.
15. A bag contains cards numbered 1 to 25. Two cards are drawn at random, one after the other, without replacement. Find the probability that the number on each card is a multiple of 7. [CBSE 2022]

ANSWERS

1. $\frac{1}{221}$ 2. $\frac{1}{270725}$ 3. $\frac{7}{22}$ 4. $\frac{11}{50}$ 5. $\frac{11}{850}$

6. (i) $\frac{1}{221}$ (ii) $\frac{4}{663}$ (iii) $\frac{25}{204}$ 7. $\frac{5}{19}$ 8. $\frac{15}{22}$ 9. $\frac{8}{65}$
10. $\frac{13}{204}$ 11. $\frac{3}{7}$ 12. $\frac{2}{5525}$ 13. $\frac{44}{91}$ 14. $\frac{1}{24}$ 15. $\frac{1}{100}$

HINTS TO SELECTED PROBLEMS

10. Consider the following events:
A = Getting heart card in first draw, B = Getting a diamond card in second draw
Required probability $= P(A \cap B) = P(A)\, P(B/A) = \frac{13}{52} \times \frac{13}{51} = \frac{13}{204}$

11. Let B_i $(i = 1, 2)$ denote the event of getting a black ball in the ith draw.
Required probability $= P(B_1 \cap B_2) = P(B_1)\, P(B_2/B_1) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

12. Consider the following events:
K_1 = Getting a king in first draw, K_2 = Getting a king in second draw
A_3 = Getting an ace in third draw.
Required probability $= P(K_1 \cap K_2 \cap A_3)$
$= P(K_1)\, P(K_2/K_1)\, P(A_3/K_1 \cap K_2) = \frac{4}{52} \times \frac{3}{51} \times \frac{4}{50} = \frac{2}{5525}$

13. Let G_i $(i = 1, 2, 3)$ denote the event of getting a good orange in ith draw.
Required probability $= P(G_1 \cap G_2 \cap G_3)$
$= P(G_1)\, P(G_2/G_1)\, P(G_3/G_1 \cap G_2) = \frac{12}{15} \times \frac{11}{14} \times \frac{10}{13} = \frac{44}{91}$

30.5 MORE ON CONDITIONAL PROBABILITY

In section 30.3, we have introduced the concept of conditional probability which has been used in multiplication theorem of probability. In this section, we will obtain a formula for finding the conditional probability.

If A and B are two events associated with a random experiment, then

$P(A \cap B) = P(A)\, P(B/A)$, if $P(A) \neq 0$ [Multiplication theorem]

or, $P(A \cap B) = P(B)\, P(A/B)$, if $P(B) \neq 0$

$\therefore$ $P(A/B) = \frac{P(A \cap B)}{P(B)}$ and $P(B/A) = \frac{P(A \cap B)}{P(A)}$

30.5.1 PROPERTIES OF CONDITIONAL PROBABILITY

Following are some properties of conditional probability which are stated and proved as theorems.

THEOREM 1 *Let A and B be two events associated with sample space S, then* $0 \leq P(A/B) \leq 1$.

PROOF We know that

$$A \cap B \subset B \Rightarrow P(A \cap B) \leq P(B) \Rightarrow \frac{P(A \cap B)}{P(B)} \leq 1, \text{ if } P(B) \neq 0 \quad [\because P(B) > 0]$$

Also, $P(A \cap B) \geq 0$ and $P(B) > 0$. Therefore, $\frac{P(A \cap B)}{P(B)} \geq 0$

Thus, we obtain: $0 \leq \frac{P(A \cap B)}{P(B)} \leq 1$ or, $0 \leq P(A/B) \leq 1$. Hence, $0 \leq P(A/B) \leq 1$

Q.E.D.

THEOREM 2 *If A is an event associated with the sample space S of a random experiment, then* $P(S/A) = P(A/A) = 1$

PROOF We have,

$$P(S/A) = \frac{P(S \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1. \text{ Also, } P(A/A) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

Hence, $P(S/A) = P(A/A) = 1$ Q.E.D.

THEOREM 3 *Let A and B be two events associated with a random experiment and S be the sample space. If C is an event such that $P(C) \neq 0$, then*

$$P\left((A \cup B)/C\right) = P(A/C) + P(B/C) - P((A \cap B)/C)$$

In particular, if A and B are mutually exclusive events, then

$$P\left((A \cup B)/C\right) = P(A/C) + P(B/C)$$

PROOF We have,

$$P\left((A \cup B)/C\right) = \frac{P\{(A \cup B) \cap C\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P\{(A \cap C) \cup (B \cap C)\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C) + (B \cap C) - P\{(A \cap C) \cap (B \cap C)\}}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad P\left((A \cup B)/C\right) = P(A/C) + P(B/C) - P\left((A \cap B)/C\right)$$

If A and B are mutually exclusive events, then $P\left((A \cap B)/C\right) = 0$

$$\therefore \quad P\left((A \cap B)/C)\right) = P(A/C) + P(B/C)$$

Q.E.D.

THEOREM 4 *If A and B are two events associated with a random experiment, then* $P(\overline{A}/B) = 1 - P(A/B)$.

PROOF We know that

$P(S/B) = 1$ [See Theorem 2]

$\Rightarrow \quad P\left((A \cup \overline{A})/B\right) = 1$

$\Rightarrow \quad P(A/B) + P(\overline{A}/B) = 1$ [$\because$ A and $\overline{A}$ are mutually exclusive events]

$\Rightarrow \quad P(\overline{A}/B) = 1 - P(A/B)$ Q.E.D.

Following examples will illustrate the applications of the above formulae and properties for conditional probability.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If A and B are two events such that $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$, find $P(A/B)$ and $P(B/A)$.*

SOLUTION We have, $P(A) = 0.5$, $P(B) = 0.6$ and $P(A \cup B) = 0.8$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.5 + 0.6 - 0.8 = 0.3$$

$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.6} = \frac{1}{2}$ and, $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.5} = \frac{3}{5}$

EXAMPLE 2 *If A and B are two events such that $P(A) = 0.3$, $P(B) = 0.6$ and $P(B/A) = 0.5$, find $P(A/B)$ and $P(A \cup B)$.*

SOLUTION We have, $P(A) = 0.3$, $P(B) = 0.6$ and $P(B/A) = 0.5$

$\therefore \quad P(A \cap B) = P(A)\,P(B/A) = 0.3 \times 0.5 = 0.15$ and, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.6} = \frac{1}{4}$

Thus, we obtain: $P(A) = 0.3$, $P(B) = 0.6$ and $P(A \cap B) = 0.15$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.6 - 0.15 = 0.75$

EXAMPLE 3 *If P (not A) = 0.7, P (B) = 0.7 and P (B/A) = 0.5, then find P (A/B) and P (A ∪ B).*

SOLUTION We have, $P(\text{not } A) = 0.7$ or, $P(\bar{A}) = 0.7 \Rightarrow 1 - P(A) = 0.7 \Rightarrow P(A) = 0.3$.

Now, $\quad P(B/A) = \frac{P(A \cap B)}{P(A)} \Rightarrow 0.5 = \frac{P(A \cap B)}{0.3} \Rightarrow P(A \cap B) = 0.15$

$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.7} = \frac{3}{14}$

and, $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.7 - 0.15 = 0.85$

EXAMPLE 4 *If A and B are two events associated with a random experiment such that $P(A) = 0.8$, $P(B) = 0.5$, $P(B/A) = 0.4$, find* (i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(A \cup B)$

SOLUTION (i) We have,

$P(B/A) = 0.4$

$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} = 0.4 \qquad \left[\because P(B/A) = \frac{P(B \cap A)}{P(A)}\right]$

$\Rightarrow \quad \frac{P(A \cap B)}{0.8} = 0.4 \Rightarrow P(A \cap B) = 0.32 \qquad [\because P(A) = 0.8 \text{ (given)}]$

(ii) We know that: $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow \quad P(A/B) = \frac{0.32}{0.5} = 0.64 \qquad [\because P(A \cap B) = 0.32 \text{ and } P(B) = 0.5]$

(iii) We know that: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A \cup B) = 0.8 + 0.5 - 0.32 = 0.98 \qquad [\because P(A) = 0.8, P(B) = 0.5 \text{ and } P(A \cap B) = 0.32]$

EXAMPLE 5 *A fair die is rolled. Consider the events $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$. Find*
(i) *$P(A/B)$ and $P(B/A)$* (ii) *$P(A/C)$ and $P(C/A)$* (iii) *$P(A \cup B/C)$ and $P(A \cap B/C)$*

SOLUTION We have, $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, $B = \{2, 3\}$ and $C = \{2, 3, 4, 5\}$

$\Rightarrow \quad n(S) = 6, n(A) = 3, n(B) = 2$ and $n(C) = 4$

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{2}{6} = \frac{1}{3}$, $P(C) = \frac{4}{6} = \frac{2}{3}$, $P(A \cap B) = \frac{1}{6}$, $P(A \cap C) = \frac{2}{6} = \frac{1}{3}$,

$P(B \cap C) = \frac{2}{6} = \frac{1}{3}$, $P(A \cap B \cap C) = \frac{1}{6}$ and $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$

(i) $\quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{1/3} = \frac{1}{2}$ and, $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$

(ii) $\quad P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{1/3}{2/3} = \frac{1}{2}$ and, $P(C/A) = \frac{P(A \cap C)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$

(iii) $\quad P(A \cap B/C) = \frac{P\left((A \cap B) \cap C\right)}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)} = \frac{1/6}{2/3} = \frac{1}{4}$

and, $P(B/C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{1/3}{2/3} = \dfrac{1}{2}$

$\therefore$ $P(A \cup B/C) = P(A/C) + (B/C) - P(A \cap B/C) = \dfrac{1}{2} + \dfrac{1}{2} - \dfrac{1}{4} = \dfrac{3}{4}$

EXAMPLE 6 *Three events A, B and C have probabilities* $\frac{2}{5}, \frac{1}{3}$ *and* $\frac{1}{2}$ *respectively. Given that* $P(A \cap C) = \frac{1}{5}$ *and* $P(B \cap C) = \frac{1}{4}$, *find the values of* $P(C/B)$ *and* $P(\overline{A} \cap \overline{C})$. [NCERT EXEMPLAR]

SOLUTION We have,

$$P(A) = \frac{2}{5}, P(B) = \frac{1}{3}, P(C) = \frac{1}{2}, P(A \cap C) = \frac{1}{5} \text{ and } P(B \cap C) = \frac{1}{4}$$

$\therefore$ $P(C/B) = \dfrac{P(C \cap B)}{P(B)} = \dfrac{1/4}{1/3} = \dfrac{3}{4}$

and, $P(\overline{A} \cap \overline{C}) = P(\overline{A \cup C}) = 1 - \{P(A) + P(C) - P(A \cap C)\} = 1 - \left(\dfrac{2}{5} + \dfrac{1}{2} - \dfrac{1}{5}\right) = \dfrac{3}{10}$

EXAMPLE 7 *If* $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ *and* $P(A \cap B) = \frac{1}{4}$, *find* $P(\overline{A}/\overline{B})$ *and* $P(\overline{B}/\overline{A})$.

SOLUTION We know that $P(\overline{A}/\overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$ and $P(\overline{B}/\overline{A}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{A})}$.

Therefore, to find $P(\overline{A}/\overline{B})$ and $P(\overline{B}/\overline{A})$, we need the values of $P(\overline{A} \cap \overline{B})$, $P(\overline{A})$ and $P(\overline{B})$. So, let us first, compute these probabilities.

Now, $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B})$

$$= 1 - P(A \cup B) = 1 - \{P(A) + P(B) - P(A \cap B)\} = 1 - \left\{\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right\} = \frac{3}{8}$$

$$P(\overline{A}) = 1 - P(A) = \frac{5}{8} \text{ and } P(\overline{B}) = 1 - P(B) = \frac{1}{2}$$

$\therefore$ $P(\overline{A}/\overline{B}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{B})} = \dfrac{3/8}{1/2} = \dfrac{3}{4}$ and $P(\overline{B}/\overline{A}) = \dfrac{P(\overline{A} \cap \overline{B})}{P(\overline{A})} = \dfrac{3/8}{5/8} = \dfrac{3}{5}$

EXAMPLE 8 *A die is rolled twice and the sum of the numbers appearing on them is observed to be 7. What is the conditional probability that the number 2 has appeared at least once?*

SOLUTION Consider the following events:

A = Getting number 2 at least once; B = Getting 7 as the sum of the numbers on two dice.

We have,

$$A = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2) (6, 2)\}$$

and, $B = \{(2, 5), (5, 2), (6, 1), (1, 6), (3, 4), (4 3)\}$

When a die is rolled twice, there are 36 elementary events.

$\therefore$ $P(A) = \dfrac{11}{36}$, $P(B) = \dfrac{6}{36}$ and $P(A \cap B) = \dfrac{2}{36}$

So, Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{2/36}{6/36} = \dfrac{1}{3}$

EXAMPLE 9 *A black and a red die are rolled.*

(i) *Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5.*

(ii) *Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.* [CBSE 2018]

SOLUTION Consider the following events:

A = Getting a sum greater than 9, B = Getting 5 on black die

C = Getting 8 as the sum, D = Getting a number less than 4 on red die.

Clearly,

$A = \{(4, 6), (6, 4), (5, 5), (6, 5), (5, 6), (6, 6)\}$, $B = \{(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$C = \{(2, 6), (6, 2), (4, 4), (3, 5), (5, 3)\}$

and, $D = \{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3)\}$

Clearly, $n(A) = 6, n(B) = 6, n(C) = 5, n(D) = 18, n(A \cap B) = 2$ and $n(C \cap D) = 2$.

$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{6}{36} = \frac{1}{6}, P(C) = \frac{5}{36}, P(D) = \frac{18}{36} = \frac{1}{2}, P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

and, $P(C \cap D) = \frac{2}{36} = \frac{1}{18}$

(i) Required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/18}{1/6} = \frac{1}{3}$

(ii) Required probability $= P(C/D) = \frac{P(C \cap D)}{P(D)} = \frac{1/18}{1/2} = \frac{1}{9}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd.*

SOLUTION Out of integers from 1 to 11, there are 5 even integers and 6 odd integers.

Consider the following events:

A = Both the numbers chosen are odd, B = The sum of the numbers chosen is even

Since the sum of two integers is even if either both are even or both are odd.

$\therefore \quad P(A) = \frac{^6C_2}{^{11}C_2}, \; P(B) = \frac{^6C_2 + {^5C_2}}{^{11}C_2}$ and $P(A \cap B) = \frac{^6C_2}{^{11}C_2}$

Required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{^6C_2 / {^{11}C_2}}{\frac{^6C_2 + {^5C_2}}{^{11}C_2}} = \frac{^6C_2}{^6C_2 + {^5C_2}} = \frac{15}{15 + 10} = \frac{3}{5}$

EXAMPLE 11 *10% of the bulbs produced in a factory are red colour and 2% are red and defective. If one bulb is picked at random, determine the probability of its being defective if it is red.*

SOLUTION Consider the following events: **[NCERT EXEMPLAR]**

A = The bulb produced is red, B = The bulb produced is defective.

It is given that $P(A) = \frac{10}{100} = \frac{1}{10}$ and $P(A \cap B) = \frac{2}{100} = \frac{1}{50}$

$\therefore \quad$ Required probability $= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/50}{1/20} = \frac{1}{5}$

EXAMPLE 12 *A couple has 2 children. Find the probability that both are boys, if it is known that (i) one of the children is a boy (ii) the older child is a boy.* **[CBSE 2010, 2014]**

SOLUTION Let B_i and G_i stand for i^{th} child be a boy and girl respectively. Then the sample space can be expressed as $S = \{B_1 B_2, B_1 G_2, G_1 B_2, G_1 G_2\}$.

Consider the following events:

A = Both the children are boys; B = One of the children is a boy; C = The older child is a boy.

Then, $A=\{B_1 B_2\}$, $B=\{B_1 G_2, B_1 B_2, G_1 B_2\}$ and $C=\{B_1 B_2, B_1 G_2\}$

$\therefore$ $A \cap B = \{B_1 B_2\}$ and $A \cap C = \{B_1 B_2\}$

(i) Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/4}{3/4} = \dfrac{1}{3}$

(ii) Required probability $= P(A/C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{1/4}{2/4} = \dfrac{1}{2}$

EXAMPLE 13 *Consider a random experiment in which a coin is tossed and if the coin shows head it is tossed again but if it shows a tail then a die is tossed. If 8 possible outcomes are equally likely, find the probability that the die shows a number greater than 4 if it is known that the first throw of the coin results in a tail.* **[CBSE 2014]**

SOLUTION The sample space S associated with the given random experiment is

$$S=\{(H,H),(H,T),(T,1),(T,2),(T,3)(T,4),(T,5),(T,6)\}.$$

Let A be the event that the die shows a number greater than 4 and B be the event that the first throw of the coin results in a tail. Then,

$$A = \{(T,5)(T,6)\} \text{ and } B = \{(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

$\therefore$ Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{n(A \cap B)}{n(B)} = \dfrac{2}{6} = \dfrac{1}{3}$

EXAMPLE 14 *A coin is tossed twice and the four possible outcomes are assumed to be equally likely. If A is the event, 'both head and tail have appeared', and B be the event, 'at most one tail is observed', find P (A), P (B), P (A/B) and P (B/A).*

SOLUTION Here, $S=\{HH,HT,TH,TT\}$, $A=\{HT,TH\}$ and $B=\{HH,HT,TH\}$.

$\therefore$ $A \cap B = \{HT,TH\}$.

Now, $P(A) = \dfrac{n(A)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$, $P(B) = \dfrac{n(B)}{n(S)} = \dfrac{3}{4}$ and, $P(A \cap B) = \dfrac{n(A \cap B)}{n(S)} = \dfrac{2}{4} = \dfrac{1}{2}$

$\therefore$ $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/2}{3/4} = \dfrac{2}{3}$ and $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/2}{1/2} = 1.$

EXAMPLE 15 *A bag contains 3 red and 4 black balls and another bag has 4 red and 2 black balls. One bag is selected at random and from the selected bag a ball is drawn. Let A be the event that the first bag is selected, B be the event that the second bag is selected and C be the event that the ball drawn is red. Find P(A), P(B), P (C/A) and P (C/B).*

SOLUTION There are two bags. Therefore, $P(A) = \dfrac{1}{2}$ and $P(B) = \dfrac{1}{2}$

Now, $P(C/A)$ = Probability of drawing a red ball when first is selected

= Probability of drawing a red ball from first bag $= \dfrac{3}{7}$

and, $P(C/B)$ = Probability of drawing a red ball from second bag $= \dfrac{4}{6} = \dfrac{2}{3}$

EXAMPLE 16 *A coin is tossed, then a die is thrown. Find the probability of obtaining a '6' given that head came up.*

SOLUTION The sample space S associated to the given random experiment is given by

$$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}$$

Consider the events: A = Getting head on the coin, B = Getting 6 on the dice.

Clearly, $A = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)\}$ and $B = \{(H,6),(T,6)\}$

$\therefore$ Required probability $= P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/12}{6/12} = \dfrac{1}{6}$

EXAMPLE 17 *A committee of 4 students is selected at random from a group consisting of 8 boys and 4 girls. Given that there is at least one girl in the committee, calculate the probability that there are exactly 2 girls in the committee.* [NCERT EXEMPLAR]

SOLUTION Consider the following events:

A = There is at least one girl on the committee, B = There are exactly 2 girls on the committee.

We have to find $P(B/A)$. We know that: $P(B/A) = \frac{P(A \cap B)}{P(A)}$.

Now,

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{{}^8C_4}{{}^{12}C_4} = 1 - \frac{70}{495} = \frac{85}{99}$$

$$P(A \cap B) = P(\text{Selecting 2 girls and 2 boys out 8 boys and 4 girls}) = \frac{{}^4C_2 \times {}^8C_4}{{}^{12}C_4} = \frac{6 \times 28}{495} = \frac{56}{165}$$

$$\therefore \ P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{56}{165} \div \frac{85}{99} = \frac{168}{425}$$

EXAMPLE 18 *Two coins are tossed. What is the probability of coming up two heads if it is known that at least one head comes up.*

SOLUTION Consider the events: A = Getting at least one head, B = Getting two heads.

Clearly, $A = \{HT, TH, HH\}$, $B = \{HH\}$ and so $A \cap B = \{H\,H\}$

$$\therefore \quad P(A) = \frac{3}{4}, P(A \cap B) = \frac{1}{4} \qquad [\because \ S = \{HH, HT, TH, TT\}]$$

$$\text{Required probability} = P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

EXAMPLE 19 *An instructor has a test bank consisting of 300 easy True/False questions, 200 difficult True/False questions, 500 easy multiple choice questions (MCQ) and 400 difficult multiple choice questions. If a questions is selected at random from the test bank, what is the probability that it will be an easy question given that it is a multiple choice question.*

SOLUTION Consider the following events:

E = The question selected is an easy question,

D = The question selected is a difficult question

T = The question selected is a True/False question,

M = The question selected is a multiple choice question.

$\because$ Total number of questions $= 300 + 200 + 500 + 400 = 1400$

$$\therefore \quad P(E) = \frac{800}{1400} = \frac{4}{7}, P(D) = \frac{600}{1400} = \frac{3}{7}, P(T) = \frac{500}{1400} = \frac{5}{14},$$

$$P(M) = \frac{900}{1400} = \frac{9}{14} \text{ and } P(E \cap M) = \frac{500}{1400} = \frac{5}{14}$$

$$\text{Required probability} = P(E/M) = \frac{P(E \cap M)}{P(M)} = \frac{5/14}{9/14} = \frac{5}{9}$$

EXAMPLE 20 *A die is thrown three times. Events A and B are defined as follows:*

A : 4 on the third throw, B : 6 on the first and 5 on the second throw.

Find the probability of A given that B has already occurred.

SOLUTION There are $6 \times 6 \times 6 = 216$ elementary events associated with the random experiment.

Clearly,

$$A = \{(1,1,4),(1,2,4),(1,3,4),(1,4,4),(1,5,4),(1,6,4),(2,1,4),(2,2,4),(2,3,4),$$
$$(2,4,4),(2,5,4),(2,6,4),(3,1,4),(3,2,4),(3,3,4),(3,4,4),(3,5,4),(3,6,4)$$
$$(4,1,4),(4,2,4),(4,3,4),(4,4,4),(4,5,4),(4,6,4),(5,1,4),(5,2,4),(5,3,4),$$
$$(5,4,4),(5,5,4),(5,6,4),(6,1,4),(6,2,4),(6,3,4),(6,4,4),(6,5,4),(6,6,4)\}$$

$$B = \{(6,5,1),(6,5,2),(6,5,3),(6,5,4),(6,5,5),(6,5,6)\}$$

and, $A \cap B = \{(6,5,4)\}$.

We observe that $n(A) = 36, n(B) = 6$ and $n(A \cap B) = 1$

$\therefore$ $P(A \cap B) = \frac{1}{216}$ and $P(B) = \frac{6}{216}$

Required probability $= \frac{P(A \cap B)}{P(B)} = \frac{1/216}{6/216} = \frac{1}{6}$

EXAMPLE 21 *Three dice are thrown at the same time. Find the probability of getting three two's if it is known that the sum of the numbers on the dice was a six.* **[NCERT EXEMPLAR]**

SOLUTION Associated to the random experiment of throwing three dice there are $6 \times 6 \times 6 = 216$ elementary events.

Consider the following events: A = Sum of the numbers on the dice is six, B = Getting three twos

We have to find $P(B/A)$. We observe that

$$A = \{(1,2,3),(1,3,2),(2,3,1),(2,1,3),(3,1,2),(3,2,1),(1,1,4),(1,4,1),(4,1,1),(2,2,2)\}$$

and $B = \{(2,2,2)\}$

$\therefore$ $P(A) = \frac{10}{216}$, $P(B) = \frac{1}{216}$ and $P(A \cap B) = \frac{1}{216}$

Hence, Required probability $= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/216}{10/216} = \frac{1}{10}$

EXAMPLE 22 *In a hostel 60% of the students read Hindi newspaper, 40% read English newspaper and 20 % read both Hindi and English newspapers. A student is selected at random.*

(i) *Find the probability that she reads neither Hindi nor English news papers.*

(ii) *If she reads Hindi newspaper, find the probability that she reads English newspaper.*

(iii) *If she reads English newspaper, find the probability that she reads Hindi newspaper.*

SOLUTION Consider the following events:

H = Student reads Hindi newspaper, E = Student reads English newspaper.

We find that: $P(H) = \frac{60}{100} = \frac{3}{5}$, $P(E) = \frac{40}{100} = \frac{2}{5}$ and $P(H \cap E) = \frac{20}{100} = \frac{1}{5}$

(i) Required probability $= P(\overline{H} \cap \overline{E}) = P(\overline{H \cup E}) = 1 - P(H \cup E)$

$$= 1 - \{P(H) + P(E) - P(H \cap E)\} = 1 - \left\{\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right\} = 1 - \frac{4}{5} = \frac{1}{5}$$

(ii) Required probability $= P(E/H) = \frac{P(H \cap E)}{P(H)} = \frac{1/5}{3/5} = \frac{1}{3}$

(iii) Required probability $= P(H/E) = \frac{P(H \cap E)}{P(E)} = \frac{1/5}{2/5} = \frac{1}{2}$

EXAMPLE 23 *An electronic assembly consists of two sub-systems say A and B. From previous testing procedures, the following probabilities are assumed to be known.*

P (*A fails*) $= 0.2$, P (*B fails alone*) $= 0.15$, P (*A and B fail*) $= 0.15$.

Evaluate the following probabilities: (i) P(A fails/B has failed) (ii) P (A fails alone)

SOLUTION Consider the following events: $E = A$ fails, $F = B$ fails. It is given that

$P(A \text{ fails}) = 0.2$, $P(A \text{ and } B \text{ fails}) = 0.15$ and, $P(B \text{ fails alone}) = 0.15$

$\Rightarrow \quad P(E) = 0.2,\ P(E \cap F) = 0.15$ and $P(\overline{E} \cap F) = 0.15$

Now, $\quad P(\overline{E} \cap F) = 0.15$

$\Rightarrow \quad P(F) - P(E \cap F) = 0.15$

$\Rightarrow \quad P(F) = P(E \cap F) + 0.15 = 0.15 + 0.15 = 0.30 \qquad [\because P(E \cap F) = 0.15]$

(i) $\quad P(A \text{ fails}/B \text{ has failed}) = P(E/F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{0.15}{0.30} = \dfrac{1}{2}$

(ii) $\quad P(A \text{ fails alone}) = P(E \cap \overline{F}) = P(E) - P(E \cap F) = 0.2 - 0.15 = 0.05$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 24 *Three distinguishable balls are distributed in three cells. Find the conditional probability that all the three occupy the same cell, given that at least two of them are in the same cell.*

SOLUTION Since each ball can be placed in a cell in three ways. Therefore, three distinct balls can be placed in three cells in $3 \times 3 \times 3 = 27$ ways.

Consider the following events:

E = All balls are in the same cell, F = At least two balls are in the same cell.

All balls can be placed in the same cell in three ways.

$\therefore \quad P(E) = \dfrac{3}{27}$

Now, $P(F) = P$ (At least two balls are in the same cell) $= 1 - P$ (Balls are placed in distinct cells)

$\Rightarrow \quad P(F) = 1 - \dfrac{3!}{27} = 1 - \dfrac{6}{27} = \dfrac{21}{27}$

Clearly, $E \subset F \Rightarrow E \cap F = E \Rightarrow P(E \cap F) = P(E) = \dfrac{3}{27}$

Required probability $= P(E/F) = \dfrac{P(E \cap F)}{P(F)} = \dfrac{3/27}{21/27} = \dfrac{1}{7}$

EXAMPLE 25 *Consider the experiment of tossing a coin. If the coin shows head toss it again but if it shows tail then throw a die. Find the conditional probability of the event 'the die shows a number greater than 4, given that 'there is at least one tail'.* **[NCERT]**

SOLUTION The outcomes of the experiment can be represented in the following tree diagram.

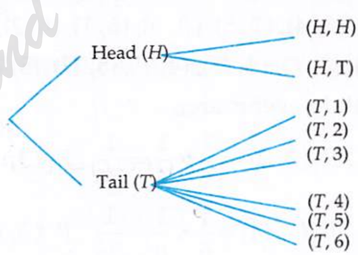

Fig. 30.3 Outcomes of the random experiment

The sample space S of the experiment is given as

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

The probabilities of these elementary events are:

$P\{(H, H)\} = \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$, $P\{(H, T)\} = \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4}$, $P\{(T, 1)\} = \dfrac{1}{2} \times \dfrac{1}{6} = \dfrac{1}{12}$,

$P\{(T, 2\} = \dfrac{1}{2} \times \dfrac{1}{6} = \dfrac{1}{12}$, $P\{(T, 3)\} = \dfrac{1}{2} \times \dfrac{1}{6} = \dfrac{1}{12}$, $P\{(T, 4\} = \dfrac{1}{2} \times \dfrac{1}{6} = 12$,

$$P\{(T,5)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} \text{ and, } P\{(T,6)\} = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}.$$

Consider the following events:

A = The die shows a number greater than 4, B = There is at least one tail.

Clearly, $A = \{(T,5), (T,6)\}$, $B = \{(H,T), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$

and, $A \cap B = \{(T,5)\ (T,6)\}$

$\therefore$ $P(B) = P\{(H,T)\} + P\{(T,1)\} + P\{(T,2)\} + P\{(T,3)\} + P\{(T,4)\} + P\{(T,5)\} + P\{(T,6)\}$

$\Rightarrow$ $P(B) = \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{3}{4}$ [See Fig. 30.4]

and, $P(A \cap B) = P\{(T,5)\} + P\{(T,6)\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$

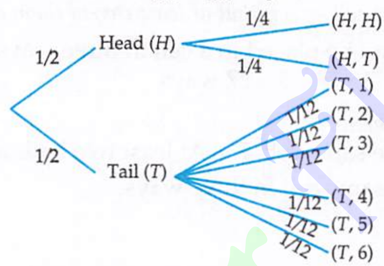

Fig. 30.4 Computation of probabilities of the outcomes of the experiement

$\therefore$ Required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/4} = \frac{4}{18} = \frac{2}{9}$

REMARK *Here, the elementary events are not equally likely. So, we cannot say that*

$$P(B) = \frac{7}{8},\ P(A \cap B) = \frac{2}{8} \text{ and so } P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/8}{7/8} = \frac{2}{7}$$

EXAMPLE 26 *Consider the experiment of throwing a die, if a multiple of 3 comes up throw the die again and if any other number comes toss a coin. Find the conditional probability of the event 'the coin shows a tail', given that 'at least one die shows a 2'.*

SOLUTION The sample space of the experiment is given by

$S = \{(3,1), (3,2)\ (3,3), (3,4), (3,5), (3,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)$
$(1,H), (1,T), (2,H), (2,T), (4,H), (4,T), (5,H), (5,T)\}$

The probabilities of the elementary events are:

$$P\{(3,1)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(3,2)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(3,3)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(3,4)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(3,5)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(3,6)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(6,1)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(6,2)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(6,3)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(6,4)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(6,5)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},\ P\{(6,6)\} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36},$$

$$P\{(1,H)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},\ P\{(1,T)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},\ P\{(2,H)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},$$

$$P\{(2,T)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},\ P\{(4,H)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},\ P\{(4,T)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12},$$

$$P\{(5, H)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}, P\{(5, T)\} = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

Clearly, the elementary events are not equally likely.

Consider the following events:

A = The coin shows a tail, B = At least one die shows a 2.

Clearly,

$A = \{(1, T), (2, T), (4, T), (5, T)\}$, $B = \{(3, 2), (6, 2), (2, H), (2, T)\}$ and, $A \cap B = \{(2, T)\}$

$\therefore \quad P(B) = P\{(3, 2)\} + P\{(6, 2)\} + \quad = \frac{1}{36} + \frac{1}{36} + \frac{1}{12} + \frac{1}{12} = \frac{2}{9}$

and, $\quad P(A \cap B) = P\{(2, T)\} = \frac{1}{12} = \frac{1}{12}$

Hence, Required probability $= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{12}}{\frac{2}{9}} = \frac{9}{24} = \frac{3}{8}$

REMARK *As the elementary events are not equally likely. Therefore, we cannot say that* $P(B) = \frac{4}{20}$, $P(A \cap B) = \frac{1}{20}$ *and so* $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/20}{4/20} = \frac{1}{4}$.

EXERCISE 30.3

BASIC

1. If $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, find $P(A/B)$.
2. If A and B are events such that $P(A) = 0.6$, $P(B) = 0.3$ and $P(A \cap B) = 0.2$, find $P(A/B)$ and $P(B/A)$.
3. If A and B are two events such that $P(A \cap B) = 0.32$ and $P(B) = 0.5$, find $P(A/B)$.
4. If $P(A) = 0.4$, $P(B) = 0.8$, $P(B/A) = 0.6$. Find $P(A/B)$ and $P(A \cup B)$.
5. If A and B are two events such that
   (i) $P(A) = 1/3$, $P(B) = 1/4$ and $P(A \cup B) = 5/12$, find $P(A/B)$ and $P(B/A)$.
   (ii) $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find $P(A \cap B)$, $P(A/B)$, $P(B/A)$
   (iii) $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, find $P(\overline{A}/B)$.
   (iv) $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$, find $P(A/B)$, $P(B/A)$, $P(\overline{A}/B)$ and $P(\overline{A}/\overline{B})$.
   [NCERT EXEMPLAR]
6. If A and B are two events such that $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$, find $P(A \cup B)$.
7. If $P(A) = \frac{6}{11}$, $P(B) = \frac{5}{11}$ and $P(A \cup B) = \frac{7}{11}$, find
   (i) $P(A \cap B)$ (ii) $P(A/B)$ (iii) $P(B/A)$
8. A coin is tossed three times. Find $P(A/B)$ in each of the following:
   (i) A = Heads on third toss, B = Heads on first two tosses
   (ii) A = At least two heads, B = At most two heads
   (iii) A = At most two tails, B = At least one tail.
9. Two coins are tossed once. Find $P(A/B)$ in each of the following:
   (i) A = Tail appears on one coin, B = One coin shows head.
   (ii) A = No tail appears, B = No head appears.

10. A die is thrown three times. Find $P(A/B)$ and $P(B/A)$, if
$A = 4$ appears on the third toss, $B = 6$ and 5 appear respectively on first two tosses.
11. Mother, father and son line up at random for a family picture. If A and B are two events given by A = Son on one end, B = Father in the middle, find $P(A/B)$ and $P(B/A)$.

BASED ON LOTS

12. A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?
13. Two dice are thrown. Find the probability that the numbers appeared has the sum 8, if it is known that the second die always exhibits 4.
14. A pair of dice is thrown. Find the probability of getting 7 as the sum, if it is known that the second die always exhibits an odd number.
15. A pair of dice is thrown. Find the probability of getting 7 as the sum if it is known that the second die always exhibits a prime number.
16. A die is rolled. If the outcome is an odd number, what is the probability that it is prime?
17. A pair of dice is thrown. Find the probability of getting the sum 8 or more, if 4 appears on the first die.
18. Find the probability that the sum of the numbers showing on two dice is 8, given that at least one die does not show five.
19. Two numbers are selected at random from integers 1 through 9. If the sum is even, find the probability that both the numbers are odd.
20. A die is thrown twice and the sum of the numbers appearing is observed to be 8. What is the conditional probability that the number 5 has appeared at least once? **[CBSE 2003]**
21. Two dice are thrown and it is known that the first die shows a 6. Find the probability that the sum of the numbers showing on two dice is 7.
22. A pair of dice is thrown. Let E be the event that the sum is greater than or equal to 10 and F be the event "5 appears on the first-die". Find $P(E/F)$. If F is the event "5 appears on at least one die", find $P(E/F)$.
23. The probability that a student selected at random from a class will pass in Mathematics is 4/5, and the probability that he/she passes in Mathematics and Computer Science is 1/2. What is the probability that he/she will pass in Computer Science if it is known that he/she has passed in Mathematics?
24. The probability that a certain person will buy a shirt is 0.2, the probability that he will buy a trouser is 0.3, and the probability that he will buy a shirt given that he buys a trouser is 0.4. Find the probability that he will buy both a shirt and a trouser. Find also the probability that he will buy a trouser given that he buys a shirt.
25. In a school there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in class XII given that the chosen student is a girl?
26. Ten cards numbered 1 through 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the drawn card is more than 3, what is the probability that it is an even number?
27. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the constitutional probability that both are girls? Given that
(i) the youngest is a girl (ii) at least one is girl. **[CBSE 2014]**

ANSWERS

1. $\frac{4}{9}$ 2. $\frac{2}{3}, \frac{1}{3}$ 3. 0.64 4. 0.3, 0.96
5. (i) $\frac{2}{3}, \frac{1}{2}$ (ii) $\frac{4}{11}, \frac{4}{5}, \frac{2}{3}$ (iii) $\frac{5}{9}$ (iv) $\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{5}{8}$ 6. $\frac{11}{26}$

7. (i) $\frac{4}{11}$ (ii) $\frac{4}{5}$ (iii) $\frac{2}{3}$ **8.** (i) $\frac{1}{2}$ (ii) $\frac{3}{7}$ (iii) $\frac{6}{7}$ **9.** (i) 1 (ii) 0 **10.** $\frac{1}{6}, \frac{1}{36}$

11. $1, \frac{1}{2}$ **12.** $\frac{2}{5}$ **13.** $\frac{1}{6}$ **14.** $\frac{1}{6}$ **15.** $\frac{1}{6}$ **16.** $\frac{2}{3}$

17. $\frac{1}{2}$ **18.** $\frac{3}{25}$ **19.** $\frac{5}{8}$ **20.** $\frac{2}{5}$ **21.** $\frac{1}{6}$ **22.** $\frac{1}{3}, \frac{3}{11}$

23. $\frac{5}{8}$ **24.** 0.12, 0.6 **25.** $\frac{1}{10}$ **26.** $\frac{4}{7}$ **27.** (i) $\frac{1}{2}$ (iii) $\frac{1}{3}$

HINTS TO SELECTED PROBLEMS

6. $P(A \cap B) = P(B)\, P(A/B) = \frac{5}{13} \times \frac{2}{5} = \frac{2}{13}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{26} + \frac{5}{13} - \frac{2}{13} = \frac{11}{26}$

7. (i) $P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{6}{11} + \frac{5}{11} - \frac{7}{11} = \frac{4}{11}$

(ii) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{4/11}{5/11} = \frac{4}{5}$

(iii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{4/11}{6/11} = \frac{2}{3}$

8. (i) We have, $A = \{HHH, HTH, THH, TTH\}$; $B = \{HHH, HHT\}$

$\therefore \quad P(A/B) = \frac{1}{2}$

(ii) We have,

$A = \{HHH, HTH, THH, HHT\}$, $B = \{TTT, TTH, HTT, THT, HHT, THH, HTH\}$

$\therefore \quad P(A/B) = \frac{3}{7}$

(iii) $A = \{HHH, HTH, THH, HHT, THT, HTT, TTH\}$

$B = \{THH, HTH, HHT, TTH, THT, HTT, TTT\}$

$\therefore \quad P(A/B) = \frac{6}{7}$

9. (i) We have, $A = \{TH, HT\}$, $B = \{HT, TH\}$

$\therefore \quad P(A/B) = 1$

(ii) We have, $A = \{HH\}$, $B = \{TT\}$

$\therefore \quad P(A/B) = 0$

10. We have,

$A = \{(1,1,4), (1,2,4), (1,3,4), (1,4,4), (1,5,4), (1,6,4), (2,1,4), (2,2,4), (2,3,4), (2,4,4), (2,5,4), (2,6,4), (3,1,4), (3,2,4), (3,3,4), (3,4,4), (3,5,4), (3,6,4), (4,1,4), (4,2,4), (4,3,4), (4,4,4), (4,5,4), (4,6,4), (5,1,4), (5,2,4), (5,3,4), (5,4,4), (5,5,4), (5,6,4), (6,1,4), (6,2,4), (6,3,4), (6,4,4), (6,5,4), (6,6,4)\}$

$B = \{(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)\}$

We observe that $n(A) = 36$, $n(B) = 6$ and $n(A \cap B) = 1$

$\therefore \quad P(A \cap B) = \frac{1}{216}$, $P(A) = \frac{36}{216} = \frac{1}{6}$ and $P(B) = \frac{6}{216} = \frac{1}{36}$

Hence, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/36}{1/6} = \frac{1}{6}$, $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/216}{1/36} = \frac{1}{6}$

11. The sample space S is given by $S = \{MFS, MSF, FSM, FMS, SMF, SFM\}$

Clearly, $A = \{MFS, FMS\ SMF, SFM\}$, $B = \{MFS, SFM\}$ and so $A \cap B = \{MFS, SFM\}$

$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{2/6}{2/6} = 1$ and, $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{4/6} = \frac{1}{2}$

19. Let A = Getting two odd numbers, B = Getting the sum as an even number.

Required probability $= P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{{}^5C_2/{}^9C_2}{({}^4C_2 + {}^5C_2)/{}^9C_2} = \dfrac{{}^5C_2}{{}^4C_2 + {}^5C_2} = \dfrac{10}{16}$

25. A = Student chosen randomly studies in class XII, B = Randomly chosen student is a girl.

$$P(B) = \frac{430}{1000} \text{ and } P(A \cap B) = \frac{43}{1000}$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = 0.1$$

30.6 INDEPENDENT EVENTS

DEFINITION *Events are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence or non-occurrence of the other.*

Suppose a bag contains 6 white and 3 red balls. Two balls are drawn from the bag one after the other. Consider the events

A = Drawing a white ball in first draw; B = Drawing a red ball in second draw.

If the ball drawn in the first draw is not replaced back in the bag, then events A and B are dependent events because $P(B)$ is increased or decreased according as the first draw results as a white or a red ball. If the ball drawn in first draw is replaced back in the bag, then A and B are independent events because $P(B)$ remains same whether we get a white ball or a red ball in first draw i.e. $P(B) = P(B/A)$ and $P(B) = P(B/\overline{A})$.

It is evident from the above discussion that if A and B are two independent events associated with a random experiment, then

$$P(A/B) = P(A) \text{ and } P(B/A) = P(B) \text{ and vice-versa.}$$

THEOREM 1 *If A and B are independent events associated with a random experiment, then*

$$P(A \cap B) = P(A)\,P(B)$$

i.e., the probability of simultaneous occurrence of two independent events is equal to the product of their probabilities.

PROOF By multiplication theorem, we have

$$P(A \cap B) = P(A)\,P(B/A)$$

Since A and B are independent events, therefore $P(B/A) = P(B)$.

Hence, $P(A \cap B) = P(A)\,P(B)$. **Q.E.D.**

THEOREM 2 *If $A_1, A_2, \ldots, A_n$ are independent events associated with a random experiment, then*

$$P(A_1 \cap A_2 \cap A_3 \ldots \cap A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$$

PROOF By multiplication theorem, we have

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) \ldots$$
$$\ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Since $A_1, A_2, \ldots, A_{n-1}, A_n$ are independent events. Therefore,

$$P(A_2/A_1) = P(A_2),\; P(A_3/A_1 \cap A_2) = P(A_3), \ldots, P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1}) = P(A_n)$$

Hence, $P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\,P(A_2) \ldots P(A_n)$. **Q.E.D.**

PAIRWISE INDEPENDENT EVENTS *Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be pairwise independent, if*

$$P(A_i \cap A_j) = P(A_i)\,P(A_j) \text{ for } i \neq j;\; i, j = 1, 2, \ldots, n$$

MUTUALLY INDEPENDENT EVENTS *Let $A_1, A_2, \ldots, A_n$ be n events associated to a random experiment. These events are said to be mutually independent if the probability of the simultaneous occurrence of any finite number of them is equal to the product of their separate probabilities.*

i.e. $\quad P(A_i \cap A_j) = P(A_i)\,P(A_j)$, for $i \neq j; i, j = 1, 2, \ldots, n$

$$P(A_i \cap A_j \cap A_k) = P(A_i)\,P(A_j)\,P(A_k), \quad \text{for } i \neq j \neq k;\, i, j, k = 1, 2, \dots, n$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$P(A_1 \cap A_2 \dots \cap A_n) = P(A_1)\,P(A_2) \dots P(A_n)$$

REMARK 1 *If $A_1, A_2 \dots A_n$ are pairwise independent events, then the total number of conditions for their pairwise independence is ${}^nC_2$ whereas for their mutual independences there must be ${}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - n - 1$ conditions.*

REMARK 2 *It follows from the above definitions that mutually independent events are always pairwise independent but the converse need not be true as illustrated below:*

ILLUSTRATION 1 *A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as*

A : "the first bulb is defective",

B : "the second bulb is non-defective",

C : "the two bulbs are both defective or both non-defective."

Determine whether (i) A, B, C are pairwise independent (ii) A, B, C are mutually independent.

SOLUTION We have,

$$P(A) = \frac{1}{2},\ P(B) = \frac{1}{2},\ P(C) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$P(A \cap B)$ = Probability that the first is defective and the second is non-defective

$$\Rightarrow \quad P(A \cap B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A)\,P(B)$$

$P(B \cap C)$ = Probability that both the bulbs are non-defective

$$\Rightarrow \quad P(B \cap C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(B)\,P(C)$$

and, $P(A \cap C)$ = Probability that both the bulbs are defective

$$\Rightarrow \quad P(A \cap C) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A)\,P(C)$$

Hence, A, B, C are pairwise independent.

Now, $P(A \cap B \cap C)$ = Probability that the first bulb is defective and the second is non-defective and the first and second are both defective or both non-defective

$= 0$

and, $P(A)\,P(B)\,P(C) = \frac{1}{8}$

Clearly, $P(A \cap B \cap C) \neq P(A)\,P(B)\,P(C)$. Thus, A, B, C are not mutually independent.

REMARK 3 *In case of two events only associated to a random experiment, there is no distinction between their mutual independence and pairwise independence.*

THEOREM 3 *If A and B are independent events associated with a random experiment, then prove that*

(i) *$\overline{A}$ and B are independent events* (ii) *A and $\overline{B}$ are independent events* **[CBSE 2017]**

(iii) *$\overline{A}$ and $\overline{B}$ are also independent events.* **[NCERT]**

SOLUTION Since A and B are independent events. Therefore,

$$P(A \cap B) = P(A)\,P(B) \qquad \dots(i)$$

(i) It is evident from the Venn-diagram (Fig. 30.5) that $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive events such that $(A \cap B) \cup (\bar{A} \cap B) = B$. Therefore, by addition theorem on probability, we have

$$P(A \cap B) + P(\bar{A} \cap B) = P(B)$$

$$\Rightarrow \quad P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= P(B) - P(A)\,P(B) \quad \text{[Using (i)]}$$

$$= P(B)\{1 - P(A)\}$$

$$= P(B)\,P(\bar{A}) = P(\bar{A})\,P(B)$$

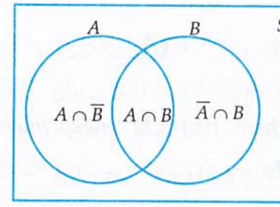

Fig. 30.5

Thus, $P(\bar{A} \cap B) = P(\bar{A})\,P(B)$. Hence, $\bar{A}$ and B are independent events.

(ii) It is clear from the Venn-diagram (see Fig. 30.5) that $A \cap \bar{B}$ and $A \cap B$ are mutually exclusive events such that $(A \cap \bar{B}) \cup (A \cap B) = A$. So, by addition theorem on probability, we have

$$P(A \cap \bar{B}) + P(A \cap B) = P(A)$$

$$\Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A)\,P(B) \quad \text{[Using (i)]}$$

$$= P(A)\{1 - P(B)\} = P(A)\,P(\bar{B})$$

Thus, $P(A \cap \bar{B}) = P(A)\,P(\bar{B})$. Hence, A and $\bar{B}$ are independent events.

(iii) We have to show that $\bar{A}$ and $\bar{B}$ are independent events if A and B are independent events. For this it is sufficient to show that $P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})$.

Now,

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) \quad [\because \bar{A} \cap \bar{B} = \overline{A \cap B}]$$

$$= 1 - [P(A) + P(B) - P(A \cap B)] \quad \text{[By Addition Theorem]}$$

$$= 1 - [P(A) + P(B) - P(A)\,P(B)] \quad \text{[Using (i)]}$$

$$= \{1 - P(A)\} - P(B)\{1 - P(A)\}$$

$$= \{1 - P(A)\}\{1 - P(B)\} = P(\bar{A})\,P(\bar{B})$$

Hence, $\bar{A}$ and $\bar{B}$ are independent events. **Q.E.D.**

REMARK 4 *In what follows the term independent events will mean mutually independent events.*

REMARK 5 *If A and B are independent events associated to a random experiment, then*

Probability of occurrence of at least one $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)\,P(B)$$

$$= 1 - \{1 - P(A) - P(B) + P(A)\,P(B)\}$$

$$= 1 - \{1 - P(A)\}\{1 - P(B)\} = 1 - P(\bar{A})\,P(\bar{B})$$

REMARK 6 *If $A_1, A_2, \ldots A_n$ are independent events associated with a random experiment, then*

Probability of occurrence of at least one $= P(A_1 \cup A_2 \cup \ldots \cup A_n)$

$$= 1 - P(\overline{A_1 \cup A_2 \cup \ldots \cup A_n})$$

$$= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n)$$

$$= 1 - P(\bar{A}_1)\,P(\bar{A}_2) \ldots P(\bar{A}_n)$$

INDEPENDENT EXPERIMENTS *Two random experiments are independent if for every pair of events A and B where A is associated with the first experiment and B with the second experiment, we have*

$$P(A \cap B) = P(A)\,P(B)$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I PROBLEMS ON PROVING THE INDEPENDENCE OR DEPENDENCE OF EVENTS

EXAMPLE 1 *A coin is tossed thrice and all eight outcomes are equally likely.*

E : "The first throw results in head" F : "The last throw results in tail"

Prove that events E and F are independent.

SOLUTION Let S be the sample space associated with the given random experiment. Then,

$$S = \{HHH, HHT, THH, HTH, TTH, HTT, THT, TTT\},\ E = \{HHT, HTH, HTT, HHH\},$$

$$F = \{HHT, HTT, THT, TTT\},\quad E \cap F = \{HHT, HTT\}$$

$$\therefore \quad P(E) = \frac{n(E)}{n(S)} = \frac{4}{8} = \frac{1}{2}, P(F) = \frac{4}{8} = \frac{1}{2} \text{ and } P(E \cap F) = \frac{2}{8} = \frac{1}{4}.$$

Clearly, $P(E \cap F) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(E)\,P(F)$.

Hence, E and F are independent events.

EXAMPLE 2 *An unbiased die is thrown twice. Let the event A be 'odd number on the first throw' and B the event 'odd number on the second throw'. Check the independence of events A and B.* **[NCERT]**

SOLUTION Clearly, Total number of elementary events = 36

An odd number on the first throw means an odd number on first throw and any number on second throw. Therefore, favourable number of elementary events to event A is $3 \times 6 = 18$.

$$\therefore \quad P(A) = \frac{18}{36} = \frac{1}{2}. \text{ Similarly, } P(B) = \frac{18}{36} = \frac{1}{2}$$

and, $P(A \cap B) = P$ (Getting an odd number on both throws) $= \frac{9}{36} = \frac{1}{4}$

Clearly, $P(A \cap B) = \frac{1}{4} = \frac{1}{2} \times \frac{1}{2} = P(A)\,P(B)$. Hence, A and B are independent events.

EXAMPLE 3 *Three coins are tossed. Consider the events : E = three heads or three tails, F = At least two heads and G = At most two heads. Of the pairs (E, F), (E, G) and (F, G) which are independent? Which are dependent?* **[NCERT]**

SOLUTION The sample space S associated with the experiment is given by

$$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$$

Clearly, $E = \{HHH, TTT\}$, $F = \{HHH, HHT, HTH, THH\}$

and, $G = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$

Also, $E \cap F = \{HHH\}, E \cap G = \{TTT\}, F \cap G = \{HHT, HTH, THH\}$

$$\therefore \quad P(E) = \frac{2}{8} = \frac{1}{4}, P(F) = \frac{4}{8} = \frac{1}{2}, P(G) = \frac{7}{8}, P(E \cap F) = \frac{1}{8}, P(E \cap G) = \frac{1}{8} \text{ and } P(F \cap G) = \frac{3}{8}$$

Clearly, $P(E \cap F) = P(E)\,P(F)$, but $P(E \cap G) \neq P(E)\,P(G)$ and $P(F \cap G) \neq P(F)\,P(G)$

So, E and F are independents, E and G are dependent events and F and G are also dependent events.

EXAMPLE 4 *A fair coin and an unbiased die are tossed. Let A be the event 'head appears on the coin' and B be the event '3 on the die.'. Check whether A and B are independent event or not.* **[NCERT]**

SOLUTION The sample space related to the experiment is given by

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5)\ (T, 6)\}$$

We have,

$$A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\},\ B = \{(H, 3), (T, 3)\} \text{ and so } A \cap B) = \{(H, 3)\}.$$

$\therefore \quad P(A) = \frac{6}{12} = \frac{1}{2}$, $P(B) = \frac{2}{12} = \frac{1}{6}$ and $P(A \cap B) = \frac{1}{12}$

Clearly, $P(A \cap B) = P(A)\,P(B)$. Hence, A and B are independent events.

EXAMPLE 5 *A die is marked 1, 2, 3 in red and 4, 5, 6 in green is toosed. Let A be the event 'number is even' and B be the event 'number is red'. Are A and B independent?* **[NCERT, CBSE 2019]**

SOLUTION We have, $A = \{2, 4, 6\}$, $B = \{1, 2, 3\}$ and $A \cap B = \{2\}$

$\therefore \quad P(A) = \frac{3}{6} = \frac{1}{2}$, $P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{6}$

Clearly, $P(A \cap B) \neq P(A)\,P(B)$. So, A and B are not independent events.

EXAMPLE 6 *Events A and B are such that* $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ *and P (not A or not B)* $= \frac{1}{4}$. *State whether A and B are independent?* **[NCERT]**

SOLUTION We have, $P(\text{not } A \text{ or not } B) = \frac{1}{4}$

$$\Rightarrow \quad P(\overline{A} \cup \overline{B}) = \frac{1}{4} \Rightarrow P(\overline{A \cap B}) = \frac{1}{4} \Rightarrow 1 - P(A \cap B) = \frac{1}{4} \Rightarrow P(A \cap B) = \frac{3}{4}$$

Thus, we obtain: $P(A \cap B = \frac{3}{4}$ and $P(A)\,P(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24}$

$\therefore \quad P(A \cap B) \neq P(A)\,P(B)$. So, A and B are not independent events.

EXAMPLE 7 *An urn contains four tickets with numbers 112, 121, 211, 222 and one ticket is drawn. Let* A_i $(i = 1, 2, 3)$ *be the event that the* i^{th} *digit of the number on ticket drawn is 1. Discuss the independence of the events* A_1, A_2, A_3.

SOLUTION We have,

$P(A_1)$ = Probability that the first digit of the number on the drawn ticket is 1.

$\Rightarrow \quad P(A_1) = \frac{2}{4} = \frac{1}{2}$

$P(A_2)$ = Probability that the second digit of the number on the drawn ticket is 1.

$\Rightarrow \quad P(A_2) = \frac{2}{4} = \frac{1}{2}$

$P(A_3)$ = Probability that the third digit of the number on the drawn ticket is 1.

$\Rightarrow \quad P(A_3) = \frac{2}{4} = \frac{1}{2}$

$P(A_1 \cap A_2)$ = Probability that first and second digits of the number on the drawn ticket are each equal to 1.

$\Rightarrow \quad P(A_1 \cap A_2) = \frac{1}{4}$

Similarly, we obtain

$$P(A_2 \cap A_3) = \frac{1}{4},\ P(A_1 \cap A_3) = \frac{1}{4}$$

and, $P(A_1 \cap A_2 \cap A_3)$ = Probability that all the digits of the number on the drawn ticket are unity

$\Rightarrow \quad P(A_1 \cap A_2 \cap A_3) = 0$

We observe that

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2),\ P(A_2 \cap A_3) = P(A_2)\,P(A_3),\ P(A_3 \cap A_1) = P(A_3)\,P(A_1)$$

But, $\quad P(A_1 \cap A_2 \cap A_3) \neq P(A_1)\,P(A_2)\,P(A_3)$.

Hence, A_1, A_2 and A_3 are pairwise independent but not mutually independent.

EXAMPLE 8 *A die is thrown once. If A is the event "the number appearing is a multiple of 3" and B is the event "the number appearing is even". Are the events A and B independent?* **[NCERT]**

SOLUTION It is given that : $P(A) = \frac{2}{6} = \frac{1}{3}$, $P(B) = \frac{3}{6} = \frac{1}{2}$

and, $P(A \cap B) = P$ (Number appearing is even and a multiple of 3)

$= P$ (Number appearing is 6) $= \frac{1}{6}$

Clearly, $P(A \cap B) = P(A) \times P(B)$. Hence, A and B are independent events.

EXAMPLE 9 *Two dice are thrown together. Let A be the event "getting 6 on the first die" and B be the event "getting 2 on the second die". Are the events A and B independent?* **[NCERT EXEMPALR]**

SOLUTION The elementary events favorable to A are: (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6) and that to B are: (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2).

So, there is only one elementary event viz. (6, 2) favourable to $(A \cap B)$.

$\therefore \quad P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{6}{36} = \frac{1}{6}$ and $P(A \cap B) = \frac{1}{36}$

Clearly, $P(A \cap B) = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6} = P(A)\ P(B)$. Hence, A and B are independent events.

EXAMPLE 10 *For a loaded die, the probabilities of outcomes are given as under:*

$$P(1) = P(2) = \frac{2}{10},\ P(3) = P(5) = P(6) = \frac{1}{10} \text{ and } P(4) = \frac{3}{10}$$

The die is thrown two times. Let A and B be the events as defined below

A= Getting same number each time, B = Getting a total score of 10 or more.

Determine whether or not A and B are independent events. **[NCERT EXEMPALR]**

SOLUTION We have,

$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$ and, $B = \{(4, 6), (6, 4), (5, 5), (6, 5), (5, 6), (6, 6)\}$

So, $A \cap B = \{(6, 6), (5, 5)\}$.

$$\therefore \quad P(A) = P(1, 1) + P(2, 2) + P(3, 3) + P(4, 4) + P(5, 5) + P(6, 6)$$

$$= P(1)\ P(1) + P(2)\ P(2) + P(3)\ P(3) + P(4)\ P(4) + P(5)\ P(5) + P(6)\ P(6)$$

$$= \frac{2}{10} \times \frac{2}{10} + \frac{2}{10} \times \frac{2}{10} + \frac{1}{10} \times \frac{1}{10} + \frac{3}{10} \times \frac{3}{10} + \frac{1}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{1}{10} = \frac{1}{5}$$

$$P(B) = P(4, 6) + P(6, 4) + P(5, 5) + P(6, 5) + P(5, 6) + P(6, 6)$$

$$= P(4)\ P(6) + P(6)\ P(4) + P(5)\ P(5) + P(6)\ P(5) + P(5)\ P(6) + P(6)\ P(6)$$

$$= \frac{3}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{3}{10} + \frac{1}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{1}{10} = \frac{1}{10}$$

and, $P(A \cap B) = P(5, 5) + P(6, 6) = P(5)\ P(5) + P(6)\ P(6) = \frac{1}{10} \times \frac{1}{10} + \frac{1}{10} \times \frac{1}{10} = \frac{1}{50}$

Clearly, $P(A \cap B) = P(A)\ P(B)$. Hence, A and B are independent events.

EXAMPLE 11 *In the above example, if the die were fair, determine whether or not the events A and B are independent.*

SOLUTION If the die is fair, then $P(A) = \frac{6}{36} = \frac{1}{6}$, $P(B) = \frac{6}{36} = \frac{1}{6}$ and $P(A \cap B) = \frac{2}{36} = \frac{1}{18}$

Clearly, $P(A \cap B) = \frac{1}{18} \neq \frac{1}{6} \times \frac{1}{6} = P(A)\ P(B)$. So, events A and B are not independent events.

EXAMPLE 12 *In the two dice experiment, if A is the event of getting the sum of the numbers on dice as 11 and B is the event of getting a number other than 5 on the first die, find P (A and B). Are A and B independent events?*

OR

Two dice are tossed. Find whether the following two events A and B are independent:

$A = \{(x, y) : x + y = 11\}$, $B = \{(x, y) : x \neq 5\}$, *where* (x, y) *denote a typical sample point.*

SOLUTION We have, Total number of elementary events = 36

Number of elementary events favourable to $A = 2$

Number of elementary events favourable to $B = 30$

$\therefore \quad P(A) = \frac{2}{36} = \frac{1}{18}, P(B) = \frac{30}{36} = \frac{5}{6}$

Now,

$P(A \cap B) = P$ (Getting the sum of the numbers on dice as 11 when 5 does not occur on first die)

$= \frac{1}{36}$

Clearly, $P(A \cap B) = \frac{1}{36} \neq \frac{1}{18} \times \frac{5}{6} = P(A)\,P(B)$. So, A and B are not independent events.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 13 *Given that the events A and B are such that* $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ *and* $P(B) = p$. *Find p, if they are (i) mutually exclusive, (ii) independent.* [NCERT]

SOLUTION (i) If A and B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B) \Rightarrow \frac{3}{5} = \frac{1}{2} + p \Rightarrow p = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$$

(ii) If A, B are independent events, then $P(A \cap B) = P(A)\,P(B) = \frac{1}{2}p$

$$\therefore \quad P(A \cup B) = \frac{3}{5} \Rightarrow P(A) + P(B) - P(A \cap B) = \frac{3}{5} \Rightarrow \frac{1}{2} + p - \frac{p}{2} = \frac{3}{5} \Rightarrow \frac{p}{2} = \frac{3}{5} - \frac{1}{2} \Rightarrow p = \frac{1}{5}$$

EXAMPLE 14 *If A and B are two events such that* $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ *and* $P(A \cap B) = \frac{1}{8}$, *find P (not A and not B).* [NCERT]

SOLUTION We observe that: $P(A \cap B) = \frac{1}{8} = \frac{1}{4} \times \frac{1}{2} = P(A)\,P(B)$. So, A and B are independent events.

Now,

$$P(\text{not } A \text{ and not } B) = P(\overline{A} \cap \overline{B}) = P(\overline{A})\,P(\overline{B}) \quad [\because \overline{A} \text{ and } \overline{B} \text{ are independent events}]$$

$$= \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{2}\right) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$$

Type II BASED UPON THE FORMULA $P(A \cap B) = P(A)\,P(B)$ FOR INDEPENDENT EVENTS

EXAMPLE 15 *Events E and F are independent. Find P (F), if P (E) = 0.35 and* $P(E \cup F) = 0.6$.

SOLUTION We have,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F) - P(E)\,P(F) \quad [\because E \text{ and } F \text{ are independent}]$$

$$\Rightarrow \quad P(E \cup F) = P(E) + P(F)\Big(1 - P(E)\Big)$$

$$\Rightarrow \quad 0.6 = 0.35 + P(F)(1 - 0.35) \quad [\text{Substituting the values of } P(E) \text{ and } P(E \cup F)]$$

$$\Rightarrow \quad 0.25 = (0.65)\,P(F) \Rightarrow P(F) = \frac{0.25}{0.65} = \frac{5}{13}$$

EXAMPLE 16 *If* $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ *and A and B are given to be independent events, find the value of p.*

SOLUTION Since A and B are independent events. Therefore, $P(A \cap B) = P(A)\,P(B)$.

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A)\,P(B) \quad [\because P(A \cap B) = P(A)\,P(B)]$$

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B)\left(1 - P(A)\right)$$

$$\Rightarrow \quad 0.6 = 0.4 + p(1 - 0.4) \qquad [\because P(A) = 0.4,\ P(A \cup B) = 0.6 \text{ and } P(B) = p]$$

$$\Rightarrow \quad 0.2 = 0.6p \Rightarrow p = 1/3.$$

EXAMPLE 17 *Let A and B be two independent events. The probability of their simultaneous occurrence is 1/8 and the probability that neither occurs is 3/8. Find P (A) and P (B).*

SOLUTION Let $P(A) = x$ and $P(B) = y$. We have,

$$P(A \cap B) = 1/8 \text{ and } P(\overline{A} \cap \overline{B}) = 3/8$$

Now, $P(A \cap B) = 1/8 \Rightarrow P(A)\,P(B) = 1/8 \Rightarrow xy = 1/8$ ...(i)

Since A and B are independent events. Therefore, so are $\overline{A}$ and $\overline{B}$.

Thus, $P(\overline{A} \cap \overline{B}) = \dfrac{3}{8} \Rightarrow P(\overline{A})\,P(\overline{B}) = \dfrac{3}{8} \Rightarrow (1 - x)(1 - y) = \dfrac{3}{8}$

$$\Rightarrow \quad 1 - x - y + xy = \frac{3}{8} \Rightarrow x + y - xy = \frac{5}{8} \Rightarrow x + y - \frac{1}{8} = \frac{5}{8} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad x + y = \frac{3}{4} \qquad \text{...(ii)}$$

Now, $(x - y)^2 = (x + y)^2 - 4xy$

$$\Rightarrow \quad (x - y)^2 = \frac{9}{16} - 4 \times \frac{1}{8} = \frac{1}{16} \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad x - y = \pm\frac{1}{4}$$

Case I *When* $x - y = \dfrac{1}{4}$: In this case, we have

$$x - y = \frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{2} \text{ and } y = \frac{1}{4} \Rightarrow P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{4}.$$

Case II *When* $x - y = -\dfrac{1}{4}$: In this case, we have

$$x - y = -\frac{1}{4} \text{ and } x + y = \frac{3}{4} \Rightarrow x = \frac{1}{4} \text{ and } y = \frac{1}{2} \Rightarrow P(A) = \frac{1}{4} \text{ and } P(B) = \frac{1}{2}$$

Hence, $P(A) = \dfrac{1}{2}$ and $P(B) = \dfrac{1}{4}$ or, $P(A) = \dfrac{1}{4}$ and $P(B) = \dfrac{1}{2}$.

EXAMPLE 18 *If A and B are two independent events such that* $P(\overline{A} \cap B) = \dfrac{2}{15}$ *and* $P(A \cap \overline{B}) = \dfrac{1}{6}$, *then find P(A) and P(B).* **[CBSE 2015]**

SOLUTION Let $P(A) = x$ and $P(B) = y$. It is given that A and B are independent events such that

$$P(\overline{A} \cap B) = \frac{2}{15} \text{ and } P(A \cap \overline{B}) = \frac{1}{6}$$

$$\Rightarrow \quad P(\overline{A})\,P(B) = \frac{2}{15} \text{ and } P(A)\,P(\overline{B}) = \frac{1}{6}$$

$$\Rightarrow \quad \left(1 - P(A)\right)P(B) = \frac{2}{15} \text{ and } P(A)\left(1 - P(B)\right) = \frac{1}{6}$$

$$\Rightarrow \quad (1 - x)\,y = \frac{2}{15} \text{ and } x(1 - y) = \frac{1}{6}$$

$$\Rightarrow \quad y - xy = \frac{2}{15} \quad \text{...(i)} \qquad \text{and,} \qquad x - xy = \frac{1}{6} \quad \text{...(ii)}$$

Subtracting (i) from (ii), we obtain

$$x-y=\frac{1}{6}-\frac{2}{15} \Rightarrow x-y=\frac{1}{30} \Rightarrow x=y+\frac{1}{30} \qquad \ldots(iii)$$

Putting $x=y+\frac{1}{30}$ in (i), we obtain

$$y-\left(y+\frac{1}{30}\right)y=\frac{2}{15} \Rightarrow y-y^2-\frac{1}{30}y=\frac{2}{15} \Rightarrow y^2-\frac{29}{30}y+\frac{2}{15}=0 \Rightarrow 30y^2-29y+4=0$$

$$\Rightarrow 30y^2-24y-5y+4=0 \Rightarrow 6y(5y-4)-1(5y-4)=0 \Rightarrow (6y-1)(5y-4)=0 \Rightarrow y=\frac{1}{6} \text{ or } y=\frac{4}{5}$$

Case I *When* $y=\frac{1}{6}$: Putting $y=\frac{1}{6}$ in (iii), we obtain $x=\frac{1}{5}$.

Case II *When* $y=\frac{4}{5}$: Putting $y=\frac{4}{5}$ in (iii), we obtain $x=\frac{5}{6}$.

Thus, $P(A)=\frac{1}{5}$ and $P(B)=\frac{1}{6}$ or $P(A)=\frac{5}{6}$ and $P(B)=\frac{4}{5}$

EXAMPLE 19 *A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.* [NCERT EXEMPLAR]

SOLUTION Let E and F be the events defined by

E = Candidate A is selected, F = Candidate B is selected

Clearly, E and F are independent events.

It is given that $P(E)=0.7$ and $P\left((E\cap\bar{F})\cup(\bar{E}\cap F)\right)=0.6$

Now,

$$P\left((E\cap\bar{F})\cup(\bar{E}\cap F)\right)=0.6$$

$$\Rightarrow \quad P(E)+P(F)-2P(E\cap F)=0.6$$

$$\Rightarrow \quad P(E)+P(F)-2\,P(E)\,P(F)=0.6 \qquad [\because E \text{ and } F \text{ are independent events}]$$

$$\Rightarrow \quad 0.7+P(F)-2\times 0.7\times P(F)=0.6$$

$$\Rightarrow \quad 0.7+P(F)-1.4\,P(F)=0.6 \Rightarrow 0.4\,P(F)=0.7-0.6 \Rightarrow P(F)=\frac{1}{4}$$

Hence, the probability that B is selected is $\frac{1}{4}$.

Type III ON FINDING THE PROBABILITY OF SIMULTANEOUS OCCURRENCE OF INDEPENDENT EVENTS

EXAMPLE 20 *A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white?*

SOLUTION Let A, B, C and D denote the events of not getting a white ball in first, second, third and fourth draw respectively. Since the balls are drawn with replacement. Therefore, A, B, C and D are independent events such that

$$P(A)=P(B)=P(C)=P(D)$$

There are 16 balls out of which 11 are not white. Therefore, $P(A)=11/16=P(B)=P(C)=P(D)$.

$$\text{Required probability}=P(A\cap B\cap C\cap D)=P(A)\,P(B)\,P(C)\,P(D)=\left(\frac{11}{16}\right)^4$$

EXAMPLE 21 *A class consists of 80 students; 25 of them are girls and 55 boys; 10 of them are rich and the remaining poor; 20 of them are fair complexioned. What is the probability of selecting a fair complexioned rich girl?*

SOLUTION Consider the following events:

A = Selecting a fair complexioned student; B = Selecting a rich student;
C = Selecting a girl.

Clearly, $P(A) = \frac{20}{80} = \frac{1}{4}$, $P(B) = \frac{10}{80} = \frac{1}{8}$ and $P(C) = \frac{25}{80} = \frac{5}{16}$

$$\begin{aligned}\text{Required probability} &= P(A \cap B \cap C)\\ &= P(A)\,P(B)\,P(C) \qquad [\because A, B, C \text{ are independent events}]\\ &= \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512}\end{aligned}$$

EXAMPLE 22 *A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is draw from each bag. Find the probability that both are (i) red (ii) black.*

SOLUTION (i) Let A be the event that a red ball is drawn from first bag and B be the event that a red ball is drawn from the second bag. Then, A and B are independent events such that

$$P(A) = \frac{3}{8} \text{ and } P(B) = \frac{6}{10}.$$

$\therefore$ Required probability $= P(A \cap B) = P(A)\,P(B) = \frac{3}{8} \times \frac{6}{10} = \frac{9}{40}$.

(ii) Let A and B be the events of drawing a black ball from first and second bag respectively. Then, A and B are independent events such that $P(A) = 5/8$ and $P(B) = 4/10$.

$\therefore$ Required probability $= P(A \cap B) = P(A)\,P(B) = \frac{5}{8} \times \frac{4}{10} = \frac{1}{4}$.

EXAMPLE 23 *A police-man fires four bullets on a dacoit. The probability that the dacoit will be killed by one bullet is 0.6. What is the probability that the dacoit is still alive?*

SOLUTION Let A_i ; $i = 1, 2, 3, 4$ be the event that the dacoit is not killed by the i^{th} bullet. Then, $P(A_i) = 1 - 0.6 = 0.4$. If the dacoit is alive after four shots, then non of the four shots hits the dacoit. As all 4 shots are independent.

$$\begin{aligned}\therefore \text{ Probability that the dacoit is still alive} &= P(A_1 \cap A_2 \cap A_3 \cap A_4)\\ &= P(A_1)\,P(A_2)\,P(A_3)\,P(A_4) = (0.4)^4 = 0.0256.\end{aligned}$$

EXAMPLE 24 *Two dice are thrown. Find the probability of getting an odd number on the first die and a multiple of 3 on the other.*

SOLUTION Consider the following events:

A = Getting an odd number on first die, B = Getting a multiple of 3 on the second die.

Clearly, $A = \{1, 3, 5\}$ and $B = \{3, 6\}$. Therefore, $P(A) = \frac{3}{6} = \frac{1}{2}$ and $P(B) = \frac{2}{6} = \frac{1}{3}$.

$$\begin{aligned}\text{Required probability} &= P(A \cap B)\\ &= P(A)\,P(B) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6} \qquad [A \text{ and } B \text{ are independent events}]\end{aligned}$$

ALITER There are 36 elementary events associated with the given experiment. An odd number on the first die and a multiple of 3 on the other can be obtained in one of the following 6 ways:

$(1, 3), (1, 6), (3, 1), (3, 6), (5, 1), (5, 6)$

$\therefore$ Required probability $= \frac{6}{36} = \frac{1}{6}$

Type IV ON FINDING THE PROBABILITY OF OCCURRENCE OF AT LEAST ONE EVENT FOR INDEPENDENT EVENTS

EXAMPLE 25 *A bag contains 5 white, 7 red and 8 black balls. Four balls are drawn one by one with replacement, what is the probability that at least one is white?*

SOLUTION Let A_i be the event that ball drawn in ith draw is white $1 \leq i \leq 4$.

Since the balls are drawn one by one with replacement. Therefore, A_1, A_2, A_3, A_4 are independent events such that

$$P(A_i) = \frac{5}{20} = \frac{1}{4}, \ i = 1, 2, 3, 4.$$

$$\text{Required probability} = P(A_1 \cup A_2 \cup A_3 \cup A_4)$$
$$= 1 - P(\overline{A}_1)\, P(\overline{A}_2)\, P(\overline{A}_3)\, P(\overline{A}_4)$$
$$= 1 - \left(\frac{3}{4}\right)^4 \qquad [\because A_1, A_2, A_3, A_4 \text{ are independent}]$$

EXAMPLE 26 *A problem in mathematics is given to 3 students whose chances of solving it are* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. *What is the probability that the problem is solved?*

SOLUTION Let A, B, C be the respective events of solving the problem. Then, $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(C) = \frac{1}{4}$.

Clearly A, B, C are independent events and the problem is solved if at least one student solves it.

$\therefore$ Required probability $= P(A \cup B \cup C)$

$$= 1 - P(\overline{A})\, P(\overline{B})\, P(\overline{C}) = 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

EXAMPLE 27 *A can solve 90% of the problems given in a book and B can solve 70%. What is the probability that at least one of them will solve the problem, selected at random from the book?*

SOLUTION Let E and F be the events defined as follows:

E = A solves the problem, F = B solves the problem.

Clearly, E and F are independent events such that $P(E) = \frac{90}{100} = \frac{9}{10}$ and $P(F) = \frac{70}{100} = \frac{7}{10}$.

$$\text{Required probability} = P(E \cup F) = 1 - P(\overline{E})\, P(\overline{F}) \qquad [\because E \text{ and } F \text{ independent events}]$$

$$= 1 - \left(1 - \frac{9}{10}\right)\left(1 - \frac{7}{10}\right) = 1 - \frac{1}{10} \times \frac{3}{10} = 0.97$$

EXAMPLE 28 *The odds against A solving a certain problem are 4 to 3 and the odds in favour of B solving the same problem are 7 to 5. Find the probability that the problem will be solved.*

SOLUTION We are given that the odds against A are 4 to 3. Therefore, $P(A) = \frac{3}{4+3} = \frac{3}{7}$.

It is also given that the odds in favour of B are 7 to 5. Therefore, $P(B) = \frac{7}{7+5} = \frac{7}{12}$.

The problem will be solved if at least one of them solves the problem. So, we have to find $P(A \cup B)$. Since A and B are independent events.

$$\therefore \quad P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B}) = 1 - \left(1 - \frac{3}{7}\right)\left(1 - \frac{7}{12}\right) = \frac{16}{21}$$

Hence, required probability is 16/21.

EXAMPLE 29 *The probability that a teacher will give an un-announced test during any class meeting is 1/5. If a student is absent twice, what is the probability that he will miss at least one test?*

SOLUTION Let E_i be the event that the student misses i^{th} test (i = 1, 2). Then E_1 and E_2 are independent events such that $P(E_1) = \frac{1}{5} = P(E_2)$.

$\therefore$ Required probability $= P(E_1 \cup E_2) = 1 - P(\bar{E}_1)\, P(\bar{E}_2)$ [$\because$ E_1, E_2 are independent]

$$= 1 - \left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{5}\right) = \frac{9}{25}.$$

EXAMPLE 30 *A machine operates if all of its three components function. The probability that the first component fails during the year is 0.14, the second component fails is 0.10 and the third component fails is 0.05. What is the probability that the machine will fail during the year?*

SOLUTION Consider the following events:

A = First component of the machine fails during the year

B = Second component of the machine fails during the year

C = Third component of the machine fails during the year

We have, $P(A) = 0.14$, $P(B) = 0.10$ and $P(C) = 0.05$

Clearly, the machine will fail if at least one of its three components fails during the year.

$\therefore$ Required probability $= P(A \cup B \cup C)$

$$= 1 - P(\overline{A \cup B \cup C})$$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$$

$$= 1 - P(\bar{A})\, P(\bar{B})\, P(\bar{C}) \quad [\because A, B, C \text{ are independent events}]$$

$$= 1 - (1 - 0.14)(1 - 0.10)(1 - 0.05)$$

$$= 1 - (0.86)(0.90)(0.95) = 0.2647$$

EXAMPLE 31 *If two switches S_1 and S_2 have respectively 90% and 80% chances of working. Find the probabilities that each of the following circuits will work.*

SOLUTION Consider the events: A = Switch S_1 works, B = Switch S_2 works.

We have, $P(A) = \frac{90}{100} = \frac{9}{10}$ and $P(B) = \frac{80}{100} = \frac{8}{10}$

(i) The circuit will work if the current flows in the circuit. This is possible only when both the switches work together (see Fig. 30.6).

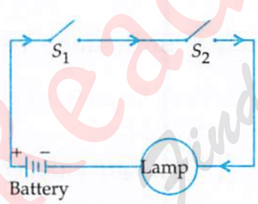

Fig. 30.6

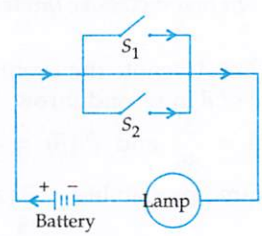

Fig. 30.7

$\therefore$ Required probability $= P(A \cap B) = P(A)\, P(B)$ [$\because$ A and B are independent events]

$$= \frac{9}{10} \times \frac{8}{10} = \frac{72}{100} = \frac{18}{25}$$

(ii) The circuit will work if the current flows in the circuit. This is possible only when at least one of the two switches S_1, S_2 works (See Fig. 30.7).

$\therefore$ Required Probability $= P(A \cup B) = 1 - P(\bar{A})\, P(\bar{B})$ [$\because$ A, B are independent events]

$$= 1 - \left(1 - \frac{9}{10}\right)\left(1 - \frac{8}{10}\right) = 1 - \frac{1}{10} \times \frac{2}{10} = \frac{49}{50}$$

EXAMPLE 32 *What is the probability that series circuit in Fig. 30.7 with three switches S_1, S_2 and S_3 with probabilities $\frac{1}{3}, \frac{1}{2}$ and $\frac{3}{4}$ respectively, of functioning will work?*

SOLUTION Consider the events: A = Switch S_1 works, B = Switch S_2 works, and C = Switch S_3 works.

We have, $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$ and $P(C) = \frac{3}{4}$

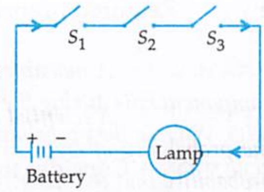

Fig. 30.8

Clearly, current flows through the circuit if switches S_1, S_2 and S_3 work together.

$\therefore$ Required probability $= P(A \cap B \cap C)$

$= P(A)\,P(B)\,P(C)$ $\left[\because A, B, C \text{ are independent events}\right]$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{3}{4} = \frac{1}{8}$

EXAMPLE 33 *A coin is tossed and a die is thrown. Find the probability that the outcome will be a head or a number greater than 4, or both.*

SOLUTION Let A be the event of getting head in a single toss of a coin and B be the event of getting a number greater than 4 in a throw of a die. Then, $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{6} = \frac{1}{3}$.

$\therefore$ Required probability $= P(A \cup B) = 1 - P(\overline{A})\,P(\overline{B})$ [$\because A$ and B are independent events]

$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$

ALITER In the above solution events A and B are subsets of distinct sample spaces associated with independent experiments. If tossing a coin and throwing a die is considered as one experiment, then the sample space S associated with it is

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$$

The elementary events favourable to the event "Getting head or a number greater than 4 of both" are $(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 5), (T, 6)$.

$\therefore$ Favourable number of elementary events = 8

Hence, required probability = 8/12 = 2/3

EXAMPLE 34 *In two successive throws of a pair of dice, determine the probability of getting a total of 8 each time.*

SOLUTION Let A denote the event of getting a total of 8 in first throw and B be the event of getting a total of 8 in second throw. Then,

$P(A) = \frac{5}{36}$ and $P(B) = \frac{5}{36}$

$\therefore$ Required probability $= P(A \cap B) = P(A)\,P(B)$ [$\because A, B$ are independent events]

$= \frac{5}{36} \times \frac{5}{36} = \frac{25}{1296}$

EXAMPLE 35 *Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that*

(i) *the problem is solved* **[CBSE 2019, 2022]**

(ii) *exactly one of them solves the problem.* **[CBSE 2011, NCERT]**

SOLUTION Let E be the event that the problem is solved by A and F be the event that the problem is solved by B. It is given that $P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{3}$.

(i) The problem is solved if at least one of A and B solves the problem. Therefore,

Required probability $= P(E \cup F)$

$= 1 - P(\overline{E})\,P(\overline{F})$ [$\because A, B$ are independent events]

$$= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{2}{3}$$

(ii) Required probability $= P(E) + P(F) - 2\,P(E \cap F)$

$$= P(E) + P(F) - 2\,P(E)\,P(F) = \frac{1}{2} + \frac{1}{3} - 2 \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{2}$$

EXAMPLE 36 *A scientist has to make a decision on each of the two independent events I and II. Suppose the probability of error in making decision on event I is 0.02 and that on event II is 0.05. Find the probability that the scientist will make the correct decision on (i) both the events (ii) only one event*

SOLUTION Consider the following events:

A = Scientist will make the correct decision on event I

B = Scientist will make the correct decision on event II

We have, $P(A) = 1 - 0.02 = 0.98$ and $P(B) = 1 - 0.05 = 0.95$

(i) Required probability $= P(A \cap B)$

$= P(A)\,P(B)$ [$\because$ A and B are independent events]

$= 0.98 \times 0.95 = 0.931$

(ii) Required probability = Probability of occurrence of exactly one of A and B

$= P(A) + P(B) - 2P(A \cap B)$

$= P(A) + P(B) - 2P(A)\,P(B)$ [$\because$ A & B are ndependent events]

$= 0.98 + 0.95 - 2 \times 0.98 \times 0.95 = 0.068$

EXAMPLE 37 *A town has two fire extinguishing engines functioning independently. The probability of availability of each engine, when needed, is 0.95. What is the probability that*

(i) *neither of them is available when needed?* (ii) *an engine is available when needed?*

(iii) *exactly one engine is available when needed?*

SOLUTION Let A denote the event that first engine is available when needed and B, the event that second engine is available when needed. Then, $P(A) = P(B) = 0.95$.

(i) Required probability $= P(\overline{A} \cap \overline{B})$

$= P(\overline{A})\,P(\overline{B}) = (0.05) \times (0.05) = 0.0025$ [$\because$ A, B are independent]

(ii) Required probability $= P(A \cup B)$

$= 1 - P(\overline{A})\,P(\overline{B}) = 1 - (0.05)(0.05) = 0.9975$ [$\because$ A, B are independent]

(iii) Required probability $= P(A) + P(B) - 2P(A \cap B)$

$= P(A) + P(B) - 2\,P(A) \times P(B) = 0.95 + 0.95 - 2 \times 0.95 \times 0.95 = 0.095$

EXAMPLE 38 *A company has estimated that the probabilities of success for three products introduced in the market are $\frac{1}{3}, \frac{2}{5}$ and $\frac{2}{3}$ respectively. Assuming independence, find the probability that*

(i) *the three products are successful.* (ii) *none of the products is successful.*

SOLUTION Consider the following events:

A = First product is successful, and B = Second product is successful,

C = Third product is successful

We have, $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(C) = \frac{2}{3}$.

(i) Required probability = P (All three products are successful)

$= P(A \cap B \cap C)$

$= P(A)\,P(B)\,P(C)$ [$\because$ A, B, C are independent events]

$$= \frac{1}{3} \times \frac{2}{5} \times \frac{2}{3} = \frac{4}{45}$$

(ii) Required probability = P (None of the products is successful)

$$= P(\overline{A} \cap \overline{B} \cap \overline{C})$$

$$= P(\overline{A})\,P(\overline{B})\,P(\overline{C}) \qquad [\because A, B, C \text{ are independent events}]$$

$$= \frac{2}{3} \times \frac{3}{5} \times \frac{1}{3} = \frac{2}{15}$$

EXAMPLE 39 *A can hit a target 4 times in 5 shots, B 3 times in 4 shots, and C 2 times in 3 shots. Calculate the probability that*

(i) *A, B, C all may hit*

(ii) *B, C may hit and A may not.*

(iii) *any two of A, B and C will hit the target* **[CBSE 2005]**

(iv) *none of them will hit the target.* **[CBSE 2005]**

SOLUTION Consider the events: $E = A$ hits the target, $F = B$ hits the target, and $G = C$ hits the target

We have, $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{4}$ and $P(G) = \frac{2}{3}$

(i) Required probability = P (A, B, C all may hit)

$$= P(E \cap F \cap G)$$

$$= P(E)\,P(F)\,P(G) \qquad [\because E, F, G \text{ are independent events}]$$

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{2}{5}$$

(ii) Required probability = P (B, C may hit and A may not)

$$= P(\overline{E} \cap F \cap G)$$

$$= P(\overline{E})\,P(F)\,P(G) \qquad [\because E, F, G \text{ are independent events}]$$

$$= \left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$$

(iii) Required probability = P (Any two of A, B and C will hit the target)

$$= P\left((E \cap F \cap \overline{G}) \cup (\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G)\right)$$

$$= P(E \cap F \cap \overline{G}) + P(\overline{E} \cap F \cap G) + P(E \cap \overline{F} \cap G)$$

$$= P(E)\,P(F)\,P(\overline{G}) + P(\overline{E})\,P(F)\,P(G)\,P(G) + P(E)\,P(\overline{F})\,P(G)$$

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} = \frac{13}{30}$$

(iv) Required probability = P (None of A, B and C will hit the target)

$$= P(\overline{E} \cap \overline{F} \cap \overline{G}) = P(\overline{E})\,P(\overline{F})\,P(\overline{G}) = \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{60}$$

EXAMPLE 40 *A combination lock on a suitcase has 3 wheels each labelled with nine digits from 1 to 9. If an opening combination is a particular sequence of three digits with no repeats, what is the probability of a person guessing the right combination?*

SOLUTION Let A_i, $i = 1, 2, 3$ be the event that the digit on i^{th} wheel occupies the correct position. Then,

Required probability = $P(A_1 \cap A_2 \cap A_3)$

$$= P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) = \frac{1}{9} \times \frac{1}{8} \times \frac{1}{7} = \frac{1}{504}.$$

EXERCISE 30.4

BASIC

1. A coin is tossed thrice and all the eight outcomes are assumed equally likely. In which of the following cases are the following events A and B are independent?
   (i) A = the first throw results in head, B = the last throw results in tail
   (ii) A = the number of heads is odd, B = the number of tails is odd
   (iii) A = the number of heads is two, B = the last throw results in head
2. Prove that in throwing a pair of dice, the occurrence of the number 4 on the first die is independent of the occurrence of 5 on the second die.
3. A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. In which of the following cases are the events A and B independent?
   (i) A = the card drawn is a king or queen, B = the card drawn is a queen or jack
   (ii) A = the card drawn is black, B = the card drawn is a king
   (iii) B = the card drawn is a spade, B = the card drawn in an ace **[NCERT]**
4. A coin is tossed three times. Let the events A, B and C be defined as follows:
   A = first toss is head, B = second toss is head, and C = exactly two heads are tossed in a row.
   Check the independence of (i) A and B (ii) B and C and (iii) C and A
5. If A and B be two events such that $P(A) = 1/4$, $P(B) = 1/3$ and $P(A \cup B) = 1/2$, show that A and B are independent events.
6. Given two independent events A and B such that $P(A) = 0.3$ and $P(B) = 0.6$. Find **[CBSE 2020]**
   (i) $P(A \cap B)$ (ii) $P(A \cap \overline{B})$ (iii) $P(\overline{A} \cap B)$
   (iv) $P(\overline{A} \cap \overline{B})$ (v) $P(A \cup B)$ (vi) $P(A/B)$ (vii) $P(B/A)$
7. If P (not B) = 0.65, $P(A \cup B) = 0.85$, and A and B are independent events, then find $P(A)$.
8. If A and B are two independent events such that $P(\overline{A} \cap B) = 2/15$ and $P(A \cap \overline{B}) = 1/6$, then find $P(B)$. **[CBSE 2015]**
9. A and B are two independent events. The probability that A and B occur is $1/6$ and the probability that neither of them occurs is $1/3$. Find the probability of occurrence of two events.
10. If A and B are two independent events such that $P(A \cup B) = 0.60$ and $P(A) = 0.2$, find $P(B)$.
11. A die is tossed twice. Find the probability of getting a number greater than 3 on each toss.
12. Given the probability that A can solve a problem is $2/3$ and the probability that B can solve the same problem is $3/5$. Find the probability that none of the two will be able to solve the problem.
13. An unbiased die is tossed twice. Find the probability of getting 4, 5, or 6 on the first toss and 1, 2, 3 or 4 on the second toss.

BASED ON LOTS

14. A bag contains 3 red and 2 black balls. One ball is drawn from it at random. Its colour is noted and then it is put back in the bag. A second draw is made and the same procedure is repeated. Find the probability of drawing (i) two red balls, (ii) two black balls, (iii) first red and second black ball.
15. Three cards are drawn with replacement from a well shuffled pack of cards. Find the probability that the cards drawn are king, queen and jack.
16. An article manufactured by a company consists of two parts X and Y. In the process of manufacture of the part X, 9 out of 100 parts may be defective. Similarly, 5 out of 100 are likely to be defective in the manufacture of part Y. Calculate the probability that the assembled product will not be defective.
17. The probability that A hits a target is $1/3$ and the probability that B hits it, is $2/5$. What is the probability that the target will be hit, if each one of A and B shoots at the target?

18. An anti-aircraft gun can take a maximum of 4 shots at an enemy plane moving away from it. The probabilities of hitting the plane at the first, second, third and fourth shot are 0.4, 0.3, 0.2 and 0.1 respectively. What is the probability that the gun hits the plane?

19. The odds against a certain event are 5 to 2 and the odds in favour of another event, independent to the former are 6 to 5. Find the probability that (i) at least one of the events will occur, and (ii) none of the events will occur.

20. A die is thrown thrice. Find the probability of getting an odd number at least once.

21. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (i) both balls are red, (ii) first ball is black and second is red, (iii) one of them is black and other is red. **[NCERT]**

22. An urn contains 4 red and 7 black balls. Two balls are drawn at random with replacement. Find the probability of getting (i) 2 red balls, (ii) 2 black balls, (iii) one red and one black ball. **[CBSE 2007]**

23. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time. **[CBSE 2013]**

24. Two dice are thrown together and the total score is noted. The event E, F and G are "a total 4", "a total of 9 or more", and "a total divisible by 5", respectively. Calculate $P(E)$, $P(F)$ and $P(G)$ and decide which pairs of events, if any, are independent. **[NCERT EXEMPLAR]**

25. Let A and B be two independent events such that $P(A) = p_1$ and $P(B) = p_2$. Describe in words the events whose probabilities are:

(i) $p_1 p_2$ (ii) $(1-p_1)\, p_2$ (iii) $1-(1-p_1)(1-p_2)$ (iv) $p_1 + p_2 = 2p_1 p_2$ **[NCERT EXEMPLAR]**

26. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three? **[CBSE 2020]**

ANSWERS

1. (i) (iii) independent

3. (ii) and (iii)

4. (i) independent (ii) dependent

6. (i) 0.18 (ii) 0.12 (iii) 0.42 (iv) 0.28 (v) 0.72 (vi) 0.3 (vii) 0.6

7. 0.77

8. $\frac{1}{6}$ or $\frac{4}{5}$

9. $P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$ or $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$

10. 0.5

11. $\frac{1}{4}$

12. $\frac{2}{15}$

13. $\frac{1}{3}$

14. (i) $\frac{9}{25}$ (ii) $\frac{4}{25}$ (iii) $\frac{6}{25}$

15. $\frac{6}{2197}$

16. 0.8645

17. $\frac{3}{5}$

18. 0.696

19. (i) $\frac{52}{77}$ (ii) $\frac{25}{77}$

20. $\frac{7}{8}$

21. (i) $\frac{16}{81}$ (ii) $\frac{20}{81}$ (iii) $\frac{40}{81}$

22. (i) $\frac{16}{121}$ (ii) $\frac{49}{121}$ (iii) $\frac{56}{121}$

23. $\frac{26}{49}$

24. $P(E) = \frac{1}{12}, P(F) = \frac{5}{18}, P(G) = \frac{7}{36}$, No pair is independent

25. (i) A and B occur (ii) A does not occur, but B occurs (iii) At least one of A and B occurs (iv) Exactly one of A and B occurs.

26. $\frac{126}{1000}$

HINTS TO SELECTED PROBLEMS

2. Here, $n(S) = 36$, $A = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$ and $B = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)\}$.
   So, $A \cap B = \{(4,5)\}$. Now, show that $P(A \cap B) = P(A)\,P(B)$

3. (iii) We have, $P(A) = \frac{13}{52} = \frac{1}{4}$, $P(B) = \frac{4}{52} = \frac{1}{13}$ and $P(A \cap B) = \frac{1}{52}$
   Clearly, $P(A \cap B) = P(A)\,P(B)$. So, A and B are independent events.

12. Required probability $= P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B})$

16. Let, A = Part X is not defective, B = Part Y is not defective.
    Required probability $= P(A \cap B) = P(A)\,P(B) = \frac{91}{100} \times \frac{95}{100}$

17. Required probability $= P(A \cup B) = 1 - P(\bar{A})\,P(\bar{B})$

18. The gun hits the plane, if it hits the plane in at least one shot.
    $\therefore$ Required probability $= 1 - (1-0.4)(1-0.3)(1-0.2)(1-0.1)$

19. Let A and B be the events. Then, $P(A) = \frac{2}{2+5} = \frac{2}{7}$ and $P(B) = \frac{6}{6+5} = \frac{6}{11}$
    (i) Required probability $= P(A \cup B) = 1 - P(\bar{A})\,P(\bar{B}) = 1 - \frac{5}{11} \times \frac{5}{7} = \frac{52}{77}$
    (ii) Required probability $= P(\bar{A} \cap \bar{B}) = P(\bar{A})\,P(\bar{B}) = \frac{5}{11} \times \frac{5}{7} = \frac{25}{77}$

20. Let A_i denote the event of getting an odd number in i^{th} throw, $i = 1, 2, 3$. Then,
    $$P(A_i) = \frac{3}{6} = \frac{1}{2};\quad i = 1, 2, 3.$$
    $$\text{Required probability} = P(A_1 \cup A_2 \cup A_3)$$
    $$= 1 - P(\bar{A}_1)\,P(\bar{A}_2)\,P(\bar{A}_3) = 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{7}{8}$$

21. Consider the following events:
    $R_i = i^{th}$ ball drawn is red, $B_i = i^{th}$ ball drawn is black, where $i = 1, 2$
    (i) Required probability $= P(R_1 \cap R_2) = P(R_1)\,P(R_2) = \frac{8}{18} \times \frac{8}{18} = \frac{16}{81}$
    (ii) Required probability $= P(B_1 \cap R_2) = P(B_1)\,P(R_2) = \frac{10}{18} \times \frac{8}{18} = \frac{20}{81}$
    (iii) Required probability $= P((R_1 \cap B_2) \cup (B_1 \cap R_2)) = P(R_1 \cap B_2) + P(B_1 \cap R_2)$
    $$= P(R_1)\,P(B_2) + P(B_1)\,P(R_2) = \frac{8}{18} \times \frac{10}{18} + \frac{10}{18} \times \frac{8}{18} = \frac{40}{81}$$

30.7 MORE ON THEOREMS OF PROBABILITY

In the previous sections, we have discussed those problems based on addition and multiplication on theorems which require the use of only one of the two theorems. In this section, we will discuss problems based upon the use of both the theorems. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *A bag contains 4 white and 2 black balls. Another contains 3 white and 5 black balls. If one ball is drawn from each bag, find the probability that*

(i) *both are white;* (ii) *both are black* (iii) *one is white and one is black.*

SOLUTION Consider the following events:

W_1 = Drawing a white ball from first bag, W_2 = Drawing a white ball from second bag.

B_1 = Drawing a black ball from first bag, B_2 = Drawing a black ball from second bag.

Clearly, $P(W_1) = 4/6$, $P(B_1) = 2/6$, $P(W_2) = 3/8$ and $P(B_2) = 5/8$.

(i) P (both balls are white) = P [(white ball from 1st bag) and (white ball from 2nd bag)]

$$= P(W_1 \cap W_2)$$
$$= P(W_1)\,P(W_2) \qquad [\because W_1 \text{ and } W_2 \text{ are independent events})]$$
$$= \frac{4}{6} \times \frac{3}{8} = \frac{1}{4}$$

(ii) P (both balls are black) = P [(black ball from 1st bag) and (black ball from 2nd bag)]

$$= P(B_1 \cap B_2)$$
$$= P(B_1)\,P(B_2) = \frac{2}{6} \times \frac{5}{8} = \frac{5}{24} \quad [\because B_1 \text{ and } B_2 \text{ are independent events}]$$

(iii) P (one white ball and one black ball)

= P [(black from 1st and white from 2nd) or (white from 1st and black from 2nd)]

$$= P[(B_1 \cap W_2) \cup (W_1 \cap B_2)]$$
$$= P(B_1 \cap W_2) + P(W_1 \cap B_2) \qquad \text{[By addition theorem for mutually exclusive events]}$$
$$= P(B_1)\,P(W_2) + P(W_1)\,P(B_2) \quad [\because B_1 \text{ and } W_2;\ B_2\ \&\ W_1 \text{ are pairs of independent events}]$$
$$= \frac{2}{6} \times \frac{3}{8} + \frac{4}{6} \times \frac{5}{8} = \frac{13}{24}$$

EXAMPLE 2 *A box contains 3 red and 5 blue balls. Two balls are drawn one by one at a time at random without replacement. Find the probability of getting 1 red and 1 blue ball.*

SOLUTION Consider the following events:

R_1 = Getting a red ball in first draw, R_2 = Getting a red ball in second draw,

B_1 = Getting a blue ball in first draw, B_2 = Getting a blue ball in second draw.

Now, P (one red and one blue ball)

= P [(red ball in first draw and blue ball in second draw) or (blue ball in first draw and red ball in second draw)]

$$= P[(R_1 \cap B_2) \cup (B_1 \cap R_2)]$$
$$= P(R_1 \cap B_2) + P(B_1 \cap R_2) \qquad \text{[By addition Theorem for mutually exclusive events]}$$
$$= P(R_1)\,P(B_2/R_1) + P(B_1)\,P(R_2/B_1) = \frac{3}{8} \times \frac{5}{7} + \frac{5}{8} \times \frac{3}{7} = \frac{15}{28}$$

EXAMPLE 3 *Two cards are drawn from a well shuffled pack of 52 cards without replacement. What is the probability that one is a red queen and the other is a king of black colour?*

SOLUTION Consider the following events :

R_i = Getting a red queen in i^{th} draw ; $i = 1, 2$.

K_i = Getting a black king in i^{th} draw ; $i = 1, 2$

$$\text{Required probability} = P\left((R_1 \cap K_2) \cup (K_1 \cap R_2)\right)$$
$$= P(R_1 \cap K_2) + P(K_1 \cap R_2)$$
$$= P(R_1)\,P(K_2/R_1) + P(K_1)\,P(R_2/K_1)$$
$$= \frac{{}^2C_1}{{}^{52}C_1} \times \frac{{}^2C_1}{{}^{51}C_1} + \frac{{}^2C_1}{{}^{52}C_1} \times \frac{{}^2C_1}{{}^{51}C_1} = \left(\frac{2}{52} \times \frac{2}{51}\right) + \left(\frac{2}{52} \times \frac{2}{51}\right) = \frac{2}{663}$$

EXAMPLE 4 *Two cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that one is a spade and other is a queen of red colour.*

SOLUTION Consider the following events:

S_i = Getting a spade card in i^{th} draw; $i = 1, 2$

Q_i = Getting a red queen in i^{th} draw; $i = 1, 2$

Required probability = $P\left((S_1 \cap Q_2) \cup (Q_1 \cap S_2)\right)$

$$= P(S_1 \cap Q_2) + P(Q_1 \cap S_2) \quad \text{[By addition Theorem]}$$

$$= P(S_1)\, P(Q_2/S_1) + P(Q_1)\, P(S_2/Q_1)$$

$$= \frac{{}^{13}C_1}{{}^{52}C_1} \times \frac{{}^{2}C_1}{{}^{51}C_1} + \frac{{}^{2}C_1}{{}^{52}C_1} \times \frac{{}^{13}C_1}{{}^{51}C_1} = 2\left(\frac{13}{52} \times \frac{2}{51}\right) = \frac{1}{51}.$$

EXAMPLE 5 *A bag contains 5 white and 3 black balls. Four balls are successively drawn out without replacement. What is the probability that they are alternately of different colours?*

SOLUTION Let W_i denote the event of drawing a white ball in i^{th} drawn and B_i denote the event of drawing a black ball in i^{th} draw, where $i = 1, 2, 3, 4$.

Required probability = $P[(W_1 \cap B_2 \cap W_3 \cap B_4) \cup (B_1 \cap W_2 \cap B_3 \cap W_4)]$

$$= P(W_1 \cap B_2 \cap W_3 \cap B_4) + P(B_1 \cap W_2 \cap B_3 \cap W_4) \quad \text{[By addition Theorem]}$$

$$= P(W_1)\, P(B_2/W_1)\, P(W_3/W_1 \cap B_2)\, P(B_4/W_1 \cap B_2 \cap W_3) + P(B_1)\, P(W_2/B_1)\, P(B_3/B_1 \cap W_2)\, P(W_4/B_1 \cap W_2 \cap B_3)$$

[By Multiplication Theorem]

$$= \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} + \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} \times \frac{4}{5} = \frac{1}{7}$$

EXAMPLE 6 *Cards are numbered* 1 *to* 25. *Two cards are drawn one after the other. Find the probability that the number on one card is multiple of* 7 *and on the other it is a multiple of* 11.

SOLUTION Two cards can be drawn in the following mutually exclusive ways:

(i) First card bears a multiple of 7 and second bears a multiple of 11

(ii) First card bears a multiple of 11 and second bears a multiple of 7.

Thus, if we define the following events:

A_1 = First card drawn bears a multiple of 7, A_2 = Second card drawn bears a multiple of 7,
B_1 = First card drawn bears a multiple of 11, B_2 = Second card drawn bears a multiple of 11.

Then,

Required probability $= P[(A_1 \cap B_2) \cup (B_1 \cap A_2)]$

$$= P(A_1 \cap B_2) + P(B_1 \cap A_2) \quad \text{[By addition Theorem]}$$

$$= P(A_1)\, P(B_2/A_1) + P(B_1)\, P(A_2/B_1) \quad \text{...(i)}$$

Between 1 and 25, there are three multiples of 7 viz. 7, 14, 21 and 2 multiples of 11 viz. 11, 22.

$$\therefore \quad P(A_1) = \frac{3}{25},\ P(B_2/A_1) = \frac{2}{24},\ P(B_1) = \frac{2}{25},\ P(A_2/B_1) = \frac{3}{24}$$

Substituting these values in (i), we obtain

$$\text{Required probability} = \frac{3}{25} \times \frac{2}{24} + \frac{2}{25} \times \frac{3}{24} = \frac{1}{50}$$

EXAMPLE 7 *Bag A contains 4 red and 5 black balls and bag B contains 3 red and 7 black balls. One ball is drawn from bag A and two from bag B. Find the probability that out of 3 balls drawn, two are black and one is red.*

SOLUTION Two black and one red ball can be drawn from two bags in two mutually exclusive ways:

(I) Drawing one black ball from bag A and two balls from bag B out of which one is black and other is red

(II) Drawing one red ball from bag A two black balls from bag B.

Thus, if we define the following events:

$E_1 =$ Drawing a black ball from bag A, $E_2 =$ Drawing one red and one black ball from bag B
$E_3 =$ Drawing one red ball from bag A, $E_4 =$ Drawing two black balls from bag B.

Then, $P(E_1) = \frac{5}{9}$, $P(E_2) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{21}{45}$, $P(E_3) = \frac{4}{9}$, $P(E_4) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}$

Now, P (Two black balls and one red ball)

$= P$ [(1 black from bag A and one red and one black from bag B) or (one red from bag A and 2 black from bag B)]

$= P[(E_1 \cap E_2) \cup (E_3 \cap E_4)]$

$= P(E_1 \cap E_2) + P(E_3 \cap E_4)$ [By addition Theorem]

$= P(E_1)\, P(E_2) + P(E_3)\, P(E_4)$ [By multiplication Theorem for independent events]

$$= \frac{5}{9} \times \frac{21}{45} + \frac{4}{9} \times \frac{7}{15} = \frac{63}{135} = \frac{7}{15}$$

EXAMPLE 8 *A bag contains 5 red marbles and 3 black marbels. Three marbels are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be bloack, if the first marbel is red?* **[NCERT EXEMPLAR]**

SOLUTION For $i = 1, 2, 3$, let R_i denote the event "Getting a red marbel in i^{th} draw" and B_i denote the event "Getting a black marbel in i^{th} draw". Then,

$$\text{Required probability} = P\{(R_1 \cap R_2 \cap B_3) \cup (R_1 \cap B_2 \cap R_3) \cup (R_1 \cap B_2 \cap B_3)\}$$
$$= P(R_1 \cap R_2 \cap B_3) + P(R_1 \cap B_2 \cap R_3) + P(R_1 \cap B_2 \cap B_3)$$
$$= P(R_1)\, P(R_2/R_1)\, P(B_3/R_1 \cap R_2) + P(R_1)\, P(B_2/R_1)\, P(R_3/R_1 \cap B_2) + P(R_1)\, P(B_2/R_1)\, P(B_3/R_1 \cap B_2)$$
$$= \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{2}{6} = \frac{150}{316} = \frac{25}{56}$$

ALITER Consider the following events:

$A =$ Getting a red marbel in first draw

$B =$ Getting at least one black marbel in the last two draws

Required probability $= P(A \cap B) = P(A)\, P(B/A) = P(A)\{1 - P(\overline{B}/A)\}$ ...(i)

Now,

$$P(A) = \frac{5}{8}$$

and, $P(\overline{B}/A) = P$ (Getting no black marbel in second and third draws when a red marbel has already been drawn in firat draw)

$= P$ (Getting red marbels in second and third draw from the bag when a red marbel has already been drawn in first draw)

$$= \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$$

Substituting $P(A) = \frac{5}{8}$ and $P(\overline{B}/A) = \frac{2}{7}$ in (i), we obtain

$$\text{Required probability} = \frac{5}{8}\left(1 - \frac{2}{7}\right) = \frac{25}{36}$$

EXAMPLE 9 *Three groups of children contain 3 girls and 1 boy; 2 girls and 2 boys; 1 girl and 3 boys respectively. One child is selected at random from each group. Find the chance that the three selected comprise one girl and 2 boys.*

SOLUTION One girl and 2 boys can be selected in the following mutually exclusive ways:

	Group 1	Group 2	Group 3
(I)	Girl	Boy	Boy
(II)	Boy	Girl	Boy
(III)	Boy	Boy	Girl

Thus, if we define G_1, G_2, G_3 as the events of selecting a girl from first, second and third group respectively and B_1, B_2, B_3 as the events of selecting a boy from first, second and third group respectively. Then $B_1, B_2, B_3, G_1, G_2, G_3$ are independent events such that

$$P(G_1) = \frac{3}{4},\ P(G_2) = \frac{2}{4},\ P(G_3) = \frac{1}{4},\ P(B_1) = \frac{1}{4},\ P(B_2) = \frac{2}{4},\ P(B_3) = \frac{3}{4}$$

Required probability $= P$ (Selecting 1 girl and 2 boys)

$= $ (I or II or III)

$= P(\text{I} \cup \text{II} \cup \text{III})$

$= P[(G_1 \cap B_2 \cap B_3) \cup (B_1 \cap G_2 \cap B_3) \cup (B_1 \cap B_2 \cap G_3)]$

$= P(G_1 \cap B_2 \cap B_3) + P(B_1 \cap G_2 \cap B_3) + P(B_1 \cap B_2 \cap G_3)$

$= P(G_1)\,P(B_2)\,P(B_3) + P(B_1)\,P(G_2)\,P(B_3) + P(B_1)\,P(B_2)\,P(G_3)$

$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$

EXAMPLE 10 *The probabilities of A, B and C solving a problem are 1/3, 2/7 and 3/8 respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them can solve it.*

SOLUTION Let E_1, E_2 and E_3 be the events that the problem is solved by A, B and C respectively. Then,

$$P(E_1) = \frac{1}{3},\ P(E_2) = \frac{2}{7} \text{ and } P(E_3) = \frac{3}{8}.$$

Exactly one of A, B and C can solve the problem in the following mutually exclusive ways:

(I) A solves but B and C do not solve i.e. $E_1 \cap \overline{E}_2 \cap \overline{E}_3$

(II) B solves but A and C do not solve i.e. $\overline{E}_1 \cap E_2 \cap \overline{E}_3$

(III) C solves but A and B do not solve i.e. $\overline{E}_1 \cap \overline{E}_2 \cap E_3$

$\therefore$ Required probability $= P$ (I or II or III)

$= P[(E_1 \cap \overline{E}_2 \cap \overline{E}_3) \cup (\overline{E}_1 \cap E_2 \cap \overline{E}_3) \cup (\overline{E}_1 \cap \overline{E}_2 \cap E_3]$

$= P(E_1 \cap \overline{E}_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap E_2 \cap \overline{E}_3) + P(\overline{E}_1 \cap \overline{E}_2 \cap E_3)$

$= P(E_1)\,P(\overline{E}_2)\,P(\overline{E}_3) + P(\overline{E}_1)\,P(E_2)\,P(\overline{E}_3) + P(\overline{E}_1)\,P(\overline{E}_2)\,P(E_3)$

$= \frac{1}{3}\left(1-\frac{2}{7}\right)\left(1-\frac{3}{8}\right) + \left(1-\frac{1}{3}\right)\left(\frac{2}{7}\right)\left(1-\frac{3}{8}\right) + \left(1-\frac{1}{3}\right)\left(1-\frac{2}{7}\right)\left(\frac{3}{8}\right)$

$= \frac{1}{3} \times \frac{5}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{2}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{5}{7} \times \frac{3}{8} = \frac{25}{168} + \frac{5}{42} + \frac{5}{28} = \frac{25}{56}$

EXAMPLE 11 *Three critics review a book. Odds in favour of the book are 5 : 2, 4 : 3 and 3 : 4 respectively for three critics. Find the probability that the majority are in favour of the book.*

SOLUTION Let A, B and C denote the events that the book will be reviewed favourably by the first, the second and the third critic respectively. Then A, B, C are independent events and we are given that

$$P(A) = \frac{5}{5+2} = \frac{5}{7},\ P(B) = \frac{4}{4+3} = \frac{4}{7} \text{ and } P(C) = \frac{3}{3+4} = \frac{3}{7}.$$

The book will be favourably reviewed by the majority of the reviewers if at least two (out of three) review it favourably. This happens in any one of the following mutually exclusive ways :

(I) 1st favours, 2nd favours and third does not favour i.e. $A \cap B \cap \overline{C}$
(II) 1st favours, 2nd does not favour and third favours i.e. $A \cap \overline{B} \cap C$
(III) 1st does not favour, 2nd favours and third favours i.e. $\overline{A} \cap B \cap C$
(IV) 1st favours, 2nd favours and third also favours i.e. $A \cap B \cap C$

Required probability $= P$ (I $\cup$ II $\cup$ III $\cup$ IV)

$$= P\,(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)$$

$$= P\,(A \cap B \cap \overline{C}) + P\,(A \cap \overline{B} \cap C) + P\,(\overline{A} \cap B \cap C) + P\,(A \cap B \cap C)$$

[$\because$ Four events are mutually exclusive]

$$= P\,(A)\,P\,(B)\,P\,(\overline{C}) + P\,(A)\,P\,(\overline{B})\,P\,(C) + P\,(\overline{A})\,P\,(B)\,P\,(C) + P\,(A)\,P\,(B)\,P\,(C)$$

[$\because$ A, B and C are independent]

$$= \frac{5}{7}\times\frac{4}{7}\times\frac{4}{7} + \frac{5}{7}\times\frac{3}{7}\times\frac{3}{7} + \frac{2}{7}\times\frac{4}{7}\times\frac{3}{7} + \frac{5}{7}\times\frac{4}{7}\times\frac{3}{7} = \frac{80+45+24+60}{343} = \frac{209}{343}$$

EXAMPLE 12 *A, B and C shot to hit a target. If A hits the target 4 times in 5 trials; B hits it 3 times in 4 trials and C hits 2 times in 3 trials; what is the probability that the target is hit by a least 2 persons?*

SOLUTION Let E_1, E_2 and E_3 be the events that A hits the target, B hits the target and C hits the target respectively. Then, E_1, E_2, E_3 are independent events such that

$$P\,(E_1) = \frac{4}{5},\ P\,(E_2) = \frac{3}{4} \text{ and } P\,(E_3) = \frac{2}{3}.$$

The target is hit by at least two persons in the following mutually exclusive ways:

(I) A hits, B hits and C does not hit i.e. $E_1 \cap E_2 \cap \overline{E}_3$
(II) A hits, B does not hit and C hits i.e. $E_1 \cap \overline{E}_2 \cap E_3$
(III) A does not hit, B hits and C hits i.e. $\overline{E}_1 \cap E_2 \cap E_3$
(IV) A hits, B hits and C hits i.e. $E_1 \cap E_2 \cap E_3$

$\therefore$ Required probability

$= P$ (I $\cup$ II $\cup$ III $\cup$ IV)

$$= P\,[(E_1 \cap E_2 \cap \overline{E}_3) \cup (E_1 \cap \overline{E}_2 \cap E_3) \cup (\overline{E}_1 \cap E_2 \cap E_3) \cup (E_1 \cap E_2 \cap E_3)]$$

$$= P\,(E_1 \cap E_2 \cap \overline{E}_3) + P\,(E_1 \cap \overline{E}_2 \cap E_3) + P\,(\overline{E}_1 \cap \overline{E}_2 \cap E_3) + P\,(E_1 \cap E_2 \cap E_3)$$

[$\because$ Four events are mutually exclusive]

$$= P\,(E_1)\,P\,(E_2)\,P\,(\overline{E}_3) + P\,(E_1)\,P\,(\overline{E}_2)\,P\,(E_3) + P\,(\overline{E}_1)\,P\,(E_2)\,P\,(E_3) + P\,(E_1)\,P(E_2)\,P\,(E_3)$$

[$\because$ E_1, E_2 and E_3 are independent]

$$= \frac{4}{5}\times\frac{3}{4}\times\frac{1}{3} + \frac{4}{5}\times\frac{1}{4}\times\frac{2}{3} + \frac{1}{5}\times\frac{3}{4}\times\frac{2}{3} + \frac{4}{5}\times\frac{3}{4}\times\frac{2}{3} = \frac{1}{5} + \frac{2}{15} + \frac{1}{10} + \frac{2}{5} = \frac{5}{6}$$

EXAMPLE 13 *A speaks truth in 60% of the cases and B in 90% of the cases. In what percentage of cases are they likely to (i) contradict each other in stating the same fact? (ii) agree in stating the same fact?*

[CBSE 2003, 2013]

SOLUTION Let E be the event that A speaks truth and F be the event that B speaks truth. Then, E and F are independent events such that

$$P\,(E) = \frac{60}{100} = \frac{3}{5} \text{ and } P\,(F) = \frac{90}{100} = \frac{9}{10}$$

(i) A and B will contradict each other in narrating the same fact in the following mutually exclusive ways:

(I) A speaks truth and B tells a lie i.e. $E \cap \overline{F}$ (II) A tells a lie and B speaks truth i.e. $\overline{E} \cap F$

$\therefore$ P (A and B contradict each other)

$= P(\text{I or II}) = P(\text{I} \cup \text{II})$
$= P[(E \cap \bar{F}) \cup (\bar{E} \cap F)]$
$= P(E \cap \bar{F}) + P(\bar{E} \cap F)$ [$\because E \cap \bar{F}$ and $\bar{E} \cap F$ are mutually exclusive]
$= P(E)\,P(\bar{F}) + P(\bar{E})\,P(F)$ [$\because E$ and F are in dependent]
$= \frac{3}{5} \times \left(1 - \frac{9}{10}\right) + \left(1 - \frac{3}{5}\right) \times \frac{9}{10} = \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{42}{100}$

Hence, in 42% cases A and B are likely to contradict each other.

(ii) A and B will agree in stating the same fact in the following mutually exclusive ways:
(I) A and B both speak truth (II) A and B both tell a lie.

$\therefore$ $P(A \text{ and } B \text{ agree}) = P((E \cap F) \cup (\bar{E} \cap \bar{F}))$
$= P(E \cap F) + P(\bar{E} \cap \bar{F})$
$= P(E)\,P(F) + P(\bar{E})\,P(\bar{F}) = \frac{3}{5} \times \frac{9}{10} + \frac{2}{5} \times \frac{1}{10} = \frac{29}{50} = \frac{58}{100}$

Hence, A and B will agree in 58% cases.

EXAMPLE 14 *The odds against a husband who is 45 years old, living till he is 70 are 7 : 5 and the odds against his wife who is now 36, living till she is 61 are 5 : 3. Find the probability that*
(i) *the couple will be alive 25 years hence,*
(ii) *exactly one of them will be alive 25 years hence,*
(iii) *none of them will be alive 25 years hence,*
(iv) *at least one of them will be alive 25 years hence.*

SOLUTION Let A be the event that the husband will be alive 25 years hence and B be the event that the wife will be alive 25 years hence. Then, A and B are independent events such that

$$P(A) = \frac{5}{7+5} = \frac{5}{12} \text{ and } P(B) = \frac{3}{5+3} = \frac{3}{8}.$$

(i) $P(\text{couple will be alive 25 years hence}) = P(A \text{ and } B) = P(A \cap B)$
$= P(A)\,P(B)$ [$\because A$ and B are independent events]
$= \frac{5}{12} \times \frac{3}{8} = \frac{5}{32}$

(ii) Exactly one of them will be alive 25 years hence in two mutually exclusive ways:
(I) Husband will be alive 25 years hence and wife will not i.e. $A \cap \bar{B}$
(II) Wife will be alive 25 years hence and husband will not i.e. $\bar{A} \cap B$

$\therefore$ $P(\text{Exactly one will be alive 25 years hence})$
$= P(\text{I or II})$
$= P[(A \cap \bar{B}) \cup (\bar{A} \cap B)]$
$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$ [$\because A \cap \bar{B}$ and $\bar{A} \cap B$ are mutually exclusive]
$= P(A)\,P(\bar{B}) + P(\bar{A})\,P(B)$ [$\because A, B$ are independent events]
$= \frac{5}{12} \times \left(1 - \frac{3}{8}\right) + \left(1 - \frac{5}{12}\right) \times \frac{3}{8} = \frac{5}{12} \times \frac{5}{8} + \frac{7}{12} \times \frac{3}{8} = \frac{46}{96} = \frac{23}{48}$

(iii) $P(\text{None of them will be alive 25 years hence}) = P(\bar{A} \cap \bar{B})$
$= P(\bar{A})\,P(\bar{B})$ [$\because A, B$ are independent events]
$= \left(1 - \frac{5}{12}\right) \times \left(1 - \frac{3}{8}\right) = \frac{7}{12} \times \frac{5}{8} = \frac{35}{96}$

(iv) P (At least one of them will be alive 25 years hence) $= 1 - P(\bar{A})\, P(\bar{B})$

$$= 1 - \left(1 - \frac{5}{12}\right)\left(1 - \frac{3}{8}\right) = 1 - \frac{35}{96} = \frac{61}{96}$$

EXAMPLE 15 *A bag contains 3 white, 3 black and 2 red balls. One by one, three balls are drawn without replacing them. Find the probability that the third ball is red.*

SOLUTION Let O_i be the event of drawing a ball other than a red ball in ith draw and R_i be the event of drawing a red ball in ith draw $(1 \le i \le 3)$.

A red ball can be drawn in third draw in the following mutually exclusive ways:

(I) First draw gives an other colour ball, second draw gives an other colour ball and the third draw gives a red ball i.e. $O_1 \cap O_2 \cap R_3$.

(II) First draw gives a red ball, second draw gives other colour ball and the third draw gives a red ball i.e. $R_1 \cap O_2 \cap R_3$

(III) First draw gives an other colour ball, second draw gives a red ball and the third draw gives a red ball i.e. $O_1 \cap R_2 \cap R_3$

$\therefore$ P (Third ball is red)

$= P$ (I or II or III) $= P(\mathrm{I} \cup \mathrm{II} \cup \mathrm{III})$

$= P[(O_1 \cap O_2 \cap R_3) \cup (R_1 \cap O_2 \cap R_3) \cup (O_1 \cap R_2 \cap R_3)]$

$= P(O_1 \cap O_2 \cap R_3) + P(R_1 \cap O_2 \cap R_3) + P(O_1 \cap R_2 \cap R_3)$

[$\because$ Events are mutually exclusive]

$= P(O_1)\, P(O_2/O_1)\, P(R_3/O_1 \cap O_2) + P(R_1)\, P(O_2/R_1)\, P(R_3/R_1 \cap O_2)$
$+ P(O_1)\, P(R_2/O_1)\, P(R_3/O_1 \cap R_2)$

$$= \frac{6}{8} \times \frac{5}{7} \times \frac{2}{6} + \frac{2}{8} \times \frac{6}{7} \times \frac{1}{6} + \frac{6}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{5}{28} + \frac{1}{28} + \frac{1}{28} = \frac{1}{4}$$

EXAMPLE 16 *The probability of student A passing an examination is 3/7 and of student B passing is 5/7. Assuming the two events "A passes", "B passes", as independent, find the probability of :*

(i) *Only A passing the examination* (ii) *Only one of them passing the examination*

SOLUTION Consider the following events:

$E_1 = A$ passes the examination, $E_2 = B$ passes the examination.

We have, $P(E_1) = \dfrac{3}{7}$ and $P(E_2) = \dfrac{5}{7}$

(i) Required probability $= P(E_1 \cap \bar{E}_2)$

$= P(E_1)\, P(\bar{E}_2)$ $\left[\begin{array}{l}\because E_1 \text{ and } E_2 \text{ are independent} \\ \therefore E_1 \text{ and } \bar{E}_2 \text{ are also independent}\end{array}\right]$

$$= \frac{3}{7}\left(1 - \frac{5}{7}\right) = \frac{6}{49}$$

(ii) Required probability

$= P[(E_1 \cap \bar{E}_2) \cup (\bar{E}_1 \cap E_2)]$

$= P(E_1 \cap \bar{E}_2) + P(\bar{E}_1 \cap E_2)$ [$\because$ $E_1 \cap \bar{E}_2$ and $\bar{E}_1 \cap E_2$ are mutually exclusive events]

$= P(E_1)\, P(\bar{E}_2) + P(\bar{E}_1)\, P(E_2)$ [$\because$ E_1 and $\bar{E}_2$; $\bar{E}_1$ & E_2 are pairs of independent events]

$$= \frac{3}{7}\left(1 - \frac{5}{7}\right) + \left(1 - \frac{3}{7}\right)\frac{5}{7} = \frac{26}{49}$$

EXAMPLE 17 *There are three urns A, B and C. Urn A contains 4 white balls and 5 blue balls. Urn B contains 4 white balls and 3 blue balls. Urn C contains 2 white balls and 4 blue balls. One ball is drawn from each of these urns. What is the probability that out of these three balls drawn, two are white balls and one is a blue ball?*

SOLUTION Consider the following events:

E_1 = Ball drawn from urn A is white, E_2 = Ball drawn from urn B is white,

E_3 = Ball drawn from urn C is white

Then, $P(E_1) = \frac{4}{9}$, $P(E_2) = \frac{4}{7}$ and $P(E_3) = \frac{2}{6} = \frac{1}{3}$

$\therefore$ $P(\bar{E}_1)$ = Ball drawn from urn A is blue $= 1 - P(E_1) = 1 - \frac{4}{9} = \frac{5}{9}$

$P(\bar{E}_2)$ = Ball drawn from urn B is blue $= 1 - P(E_2) = 1 - \frac{4}{7} = \frac{3}{7}$

and, $P(\bar{E}_3)$ = Ball drawn from urn C is blue $= 1 - P(E_3) = 1 - \frac{1}{3} = \frac{2}{3}$

Two white balls and one blue ball can be drawn in the following mutually exclusive ways:

(I) White from urn$A$, white from urn B and blue from urn C i.e. $E_1 \cap E_2 \cap \bar{E}_3$

(II) White from urn A, blue from urn B and white from urn C i.e. $E_1 \cap \bar{E}_2 \cap E_3$

(III) Blue from urn A, white from urn B and white from urn C i.e. $\bar{E}_1 \cap E_2 \cap E_3$

$$\begin{aligned}
\therefore \quad \text{Required probability} &= P(\text{I}) + P(\text{II}) + P(\text{III}) \\
&= P(\text{I} \cup \text{II} \cup \text{III}) \\
&= P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) \\
&= P(E_1)\, P(E_2)\, P(\bar{E}_3) + P(E_1)\, P(\bar{E}_2)\, P(E_3) + P(\bar{E}_1)\, P(E_2)\, P(E_3) \\
&\qquad [\because E_1, E_2, E_3 \text{ are independent events}] \\
&= \frac{4}{9} \times \frac{4}{7} \times \frac{2}{3} + \frac{4}{9} \times \frac{3}{7} \times \frac{1}{3} + \frac{5}{9} \times \frac{4}{7} \times \frac{1}{3} = \frac{64}{189}.
\end{aligned}$$

EXAMPLE 18 *A certain team wins with probability* 0.7, *loses with probability* 0.2 *and ties with probability* 0.1 *the team plays three games. Find the probability that the team wins at least two of the games, but not lose.*

SOLUTION Let W_i, L_i and D_i; $i = 1, 2, 3$ denote respectively the events that the team wins, loses and ties the i^{th} game. Then,

$P(W_i) = 0.7$, $P(L_i) = 0.2$ and $P(D_i) = 0.1$; $i = 1, 2, 3$

Required probability

$= P$ (Team wins at least two games and does not lose any game)

$= [(W_1 \cap W_2 \cap D_3) \cup (W_1 \cap D_2 \cap W_3) \cup (D_1 \cap W_2 \cap W_3) \cup (W_1 \cap W_2 \cap W_3)]$

$= P(W_1 \cap W_2 \cap D_3) + P(W_1 \cap D_2 \cap W_3) + P(D_1 \cap W_2 \cap W_3) + P(W_1 \cap W_2 \cap W_3)$

$= P(W_1)\, P(W_2)\, P(D_3) + P(W_1)\, P(D_2)\, P(W_3) + P(D_1)\, P(W_2)\, P(W_3) + P(W_1)\, P(W_2)\, P(W_3)$

$= (0.7)(0.7) \times 0.1 + (0.7) \times (0.1) \times (0.7) + (0.1) \times (0.7) \times (0.7) + (0.7)(0.7)(0.7)$

$= (0.049) \times 3 + 0.343 = 0.49$

EXAMPLE 19 *A clerk was asked to mail three report cards to three students. He addresses three envelopes but unfortunately paid no attention to which report card be put in which envelope. What is the probability that exactly one of the students received his or her own card?*

SOLUTION Consider the following events:

A = First report card is put in the correct envelope.

B = Second report card is put in the correct envelope.

C = Third report card is put in the correct envelope.

We have,

$$P(A) = P(B) = P(C) = \frac{1}{3}.$$

Required probability

$= P$ [Exactly one of the report cards is put in the correct envelope]

$= P(A) + P(B) + P(C) - 2[P(A \cap B) + P(B \cap C) + P(C \cap A)] + 3P(A \cap B \cap C)$

$= 3P(A) - 2[P(A)\,P(B/A) + P(B)\,P(C/B) + P(A)\,P(C/A)] + 3P(A)\,P(B/A)\,P(C/A \cap B)$

$= 3 \times \frac{1}{3} - 2\left[\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}\right] + 3 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} \times 1 = 1 - 2 \times \frac{1}{2} + \frac{1}{2} = \frac{1}{2}$

ALITER Required probability

$= P[(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)]$

$= P(A \cap \overline{B} \cap \overline{C}) + (\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap \overline{B} \cap C)]$

$= P(A)\,P(\overline{B}/A)\,P(\overline{C}/A \cap \overline{B}) + P(B)\,P(\overline{A}/B)P(\overline{C}/B \cap \overline{A}) + P(C)\,P(\overline{A}/C)P(\overline{B}/\overline{A} \cap C)$

$= \frac{1}{3} \times \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{1}{2} \times 1 + \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{3}{6} = \frac{1}{2}.$

EXAMPLE 20 *Neelam is taking up subjects Mathematics, Physics and Chemistry. She estimates that her probabilities of receiving grade A in these courses are 0.2, 0.3 and 0.9 respectively. If the grades can be regarded as independent events, find the probabilities that she receives.*

(i) All A's *(ii) No A's* *(iii) Exactly two A's*

SOLUTION Consider the following events:

E = Neelam receives grade A in Mathematics, F = Neelam receives grade A in Physics

G = Neelam receives grade A in Chemistry

Then, $P(E) = 0.2$, $P(F) = 0.3$ and $P(G) = 0.9$

(i) Required probability $= P(E \cap F \cap G)$

$= P(E)\,P(F)\,P(G)$ [$\because$ E, F, G, are independent events]

$= 0.2 \times 0.3 \times 0.9 = 0.054$

(ii) Required probability $= P(\overline{E} \cap \overline{F} \cap \overline{G})$

$= P(\overline{E})\,P(\overline{F})\,P(\overline{G})$ [$\because$ E, F, G, are independent events]

$= 0.8 \times 0.7 \times 0.1 = 0.056$

(iii) Required probability $= P[(E \cap F \cap \overline{G}) \cup (\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G)]$

$= P(E \cap F \cap \overline{G}) + P(\overline{E} \cap F \cap G) + P(E \cap \overline{F} \cap G)$

$= P(E)\,P(F)\,P(\overline{G}) + P(\overline{E})\,P(F)\,P(G) + P(E)\,P(\overline{F})\,P(G)$

$= 0.2 \times 0.3 \times 0.1 + 0.8 \times 0.3 \times 0.9 + 0.2 \times 0.7 \times 0.9$

$= 0.006 + 0.216 + 0.126 = 0.348.$

EXAMPLE 21 *A doctor claims that 60% of the patients he examines are allergic to some type of weed. What is the probability that (i) exactly 3 of his next 4 patients are allergic to weeds? (ii) none of his next 4 patients is allergic to weeds?*

SOLUTION Consider the following events:

A = First patient is allergic to weeds, B = Second patient is allergic to weeds

C = Third patient is allergic to weeds, D = Fourth patient is allergic to weeds

Clearly, A, B, C, D are independent events such that

$$P(A) = P(B) = P(C) = P(D) = \frac{60}{100} = \frac{3}{5}$$

(i) Required probability

$= P[(A \cap B \cap C \cap \overline{D}) \cup (A \cap B \cap \overline{C} \cap D) \cup (A \cap \overline{B} \cap C \cap D) \cup (\overline{A} \cap B \cap C \cap D)]$

$= P(A \cap B \cap C \cap \overline{D}) + P(A \cap B \cap \overline{C} \cap D) + P(A \cap \overline{B} \cap C \cap D) + P(\overline{A} \cap B \cap C \cap D)$

$= P(A)\,P(B)\,P(C)\,P(\overline{D}) + P(A)\,P(B)\,P(\overline{C})\,P(D) + P(A)\,P(\overline{B})\,P(C)\,P(D)$
$+ P(\overline{A})\,P(B)\,P(C)\,P(D)$

$= \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} + \frac{2}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{216}{625}$

(ii) Required probability $= P(\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D})$

$$= P(\overline{A})\, P(\overline{B})\, P(\overline{C})\, P(\overline{D}) = \left(\frac{2}{5}\right)^4 = \frac{16}{625}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 22 *If each element of a second order determinant is either zero or one, what is the probability that the value of the determinant is positive? (Assuming that the individual entries of the determinant are chosen independently, each value being assumed with probability 1/2).*

SOLUTION Let the given determinant be $\Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, where $a_{ij} = 0$ or $1; i, j = 1, 2$

We observe that $\Delta \le 0$, if $a_{11} = 0$ or, $a_{22} = 0$.

$\therefore$ neither $a_{11} = 0$ nor $a_{22} = 0 \Rightarrow a_{11} = a_{22} = 1$

Also, when $a_{11} = a_{22} = 1$, we observe that $\Delta = 0$, if $a_{12} = a_{21} = 1$.

Thus. we must have

$$a_{11} = a_{22} = 1 \text{ and } a_{12} \ne 1, a_{21} \ne 1$$

So, we have the following possibilities:

$$a_{11} = a_{22} = 1,\ a_{12} = 1,\ a_{21} = 0$$
$$a_{11} = a_{22} = 1,\ a_{12} = 0,\ a_{21} = 1$$
$$a_{11} = a_{22} = 1,\ a_{12} = 0,\ a_{21} = 0$$

$\therefore$ Required probability $= P(a_{11} = a_{22} = 1, a_{12} = 1, a_{21} = 0)$

$$+ P(a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = 1) + P(a_{11} = a_{22} = 1, a_{12} = a_{21} = 0)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{16}$$

EXAMPLE 23 *An electric system has open-closed switches S_1, S_2 and S_3 as shown in Fig. 30.9. The switches operate independently of one another and the current will flow from A to B either if S_1 is closed or if both S_2 and S_3 are closed. If $P(S_1) = P(S_2) = P(S_3) = \frac{1}{2}$, find the probability that the circuit will work.*

SOLUTION Required probability

$$= P(S_1 \cup (S_2 \cap S_3))$$
$$= P(S_1) + P(S_2 \cap S_3) - P(S_1 \cap (S_2 \cap S_3))$$
$$= P(S_1) + P(S_2 \cap S_3) - P(S_1 \cap S_2 \cap S_3)$$
$$= P(S_1) + P(S_2)\, P(S_3) - P(S_1)\, P(S_2)\, P(S_3)$$
$$= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$$

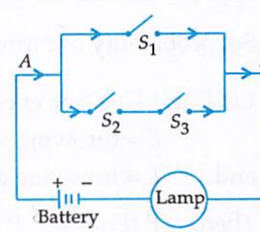

Fig. 30.9

EXAMPLE 24 *Two persons A and B throw a die alternately till one of them gets a 'three' and wins the game. Find their respectively probabilities of winning, if A begins.*

SOLUTION We define the following events.

E = Person A gets a three, F = Person B gets a three.

Clearly, $P(E) = \frac{1}{6}$, $P(F) = \frac{1}{6}$, $P(\overline{E}) = 1 - \frac{1}{6} = \frac{5}{6}$ and $P(\overline{F}) = 1 - \frac{1}{6} = \frac{5}{6}$

A wins if he throws a 'three' in 1st or 3rd or 5th ... throws.

His probability of throwing a 'three' in first throw $= P(E) = \frac{1}{6}$

A will get third throw if he fails in first and B fails in second throw.

$\therefore$ Probability of winning of A in third throw $= P(\overline{E} \cap \overline{F} \cap E)$

$$= P(\overline{E})\, P(\overline{F})\, P(E) = \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \left(\frac{5}{6}\right)^2 \times \frac{1}{6}$$

Similarly, we have

Probability of winning of A in fifth throw $= P(\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E)$

$$= P(\overline{E})\, P(\overline{F})\, P(\overline{E})\, P(\overline{F})\, P(E)$$

$$= (P(\overline{E}))^2\, (P(\overline{F}))^2\, P(E) = \left(\frac{5}{6}\right)^4 \times \frac{1}{6}$$

and so on.

Hence, probability of winning of $A = P[E \cup (\overline{E} \cap \overline{F} \cap E) \cup (\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E) \cup \ldots]$

$$= P(E) + P(\overline{E} \cap \overline{F} \cap E) + P(\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E) + \ldots$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \ldots$$

$$= \frac{16}{1-(5/6)^2} = \frac{6}{11} \qquad \left[\because a + ar + ar^2 + \ldots = \frac{a}{1-r}\right]$$

Thus, probability of winning of $B = 1 -$ Probability of winning of $A = 1 - \frac{6}{11} = \frac{5}{11}$

EXAMPLE 25 *A and B throw alternately a pair of dice. A wins if he throws 6 before B throws 7 and B wins if the throws 7 before A throws 6. Find their respective chance of winning, if A begins.*

[NCERT EXEMPLAR]

SOLUTION 6 can be thrown with a pair of dice in the following ways: (1, 5), (5, 1), (4, 2), (2, 4), (3, 3).

So, probability of throwing a '6' $= \frac{5}{36}$ and, probability of not throwing a '6' $= 1 - \frac{5}{36} = \frac{31}{36}$.

Now, 7 can be thrown with a pair of dice in 6 ways, viz. (1, 6), (6, 1), (2,5), (5, 2), (4, 3), (3, 4).

So, probability of throwing a '7' $= \frac{6}{36} = \frac{1}{6}$ and, probability of not throwing a '7' $= 1 - \frac{1}{6} = \frac{5}{6}$

Let E and F be two events defined as:

$E =$ throwing a '6' in a single throw of a pair of dice,

and, $F =$ throwing a '7' in a single throw of a pair of dice

Then, $P(E) = \frac{5}{36}$, $P(\overline{E}) = \frac{31}{36}$, $P(F) = \frac{1}{6}$ and $P(\overline{F}) = \frac{5}{6}$

A wins if he throws '6' in 1st or 3rd or 5th ... throws.

Probability of A throwing a '6' in first throw $= P(E) = \frac{5}{36}$

A will get third throw if he fails in first and B fails in second throw.

Probability of A throwing a '6' in third throw $P(\overline{E} \cap \overline{F} \cap E) = P(\overline{E})\, P(\overline{F})\, P(E) = \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36}$

Similarly,

Probability of A throwing a '6' in fifth throw $= P(\overline{E} \cap \overline{F} \cap \overline{E} \cap \overline{F} \cap E)$

$$= P(\overline{E})\, P(\overline{F})\, P(\overline{E})\, P(\overline{F})\, P(E)$$

$$= \left(\frac{31}{36}\right)^2 \times \left(\frac{5}{6}\right)^2 \times \frac{5}{36}$$

and so on

Hence,

$$\text{Probability of winning of } A = P\,(E \cup (\bar{E} \cap \bar{F} \cap E) \cup (\bar{E} \cap \bar{F} \cap \bar{E} \cap \bar{F} \cap E) \cup \ldots)$$
$$= P\,(E) + P\,(\bar{E} \cap \bar{F} \cap E) + P\,(\bar{E} \cap \bar{F} \cap \bar{E} \cap \bar{F} \cap E) + \ldots$$
$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right) \times \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \ldots$$
$$= \frac{5/36}{1-(31/36)\times(5/6)} = \frac{30}{61}.$$

Thus, probability of winning of $B = 1 - \frac{30}{61} = \frac{31}{61}$.

EXAMPLE 26 *Three persons A, B, C throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning, if A begins.*

SOLUTION Let E be the event of 'getting a six' in a single throw of an unbiased die. Then,

$$P\,(E) = \frac{1}{6} \text{ and } P\,(\bar{E}) = 1 - \frac{1}{6} = \frac{5}{6}.$$

A wins if he gets a 'six' in 1st or 4th or 7th ... throw. His probability of getting a 'six' in first throw is $P\,(E) = \frac{1}{6}$.

A will get fourth throw if he fails in first, B fails in second and C fails in third throw.

$\therefore$ Probability of winning of A in fourth throw

$$= P\,(\bar{E} \cap \bar{E} \cap \bar{E} \cap E) = P\,(\bar{E})\,P\,(\bar{E})\,P\,(\bar{E})\,P\,(E) = \left(\frac{5}{6}\right)^3 \times \frac{1}{6}$$

Similarly,

$$\text{Probability of winning of } A \text{ in 7th throw} = \left(\frac{5}{3}\right)^6 \times \frac{1}{6} \quad \text{and so on.}$$

$$\text{Hence, Probability of winning of } A = \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^6 \frac{1}{6} + \ldots = \frac{1/6}{1-(5/6)^3} = \frac{36}{91}$$

B wins if he gets a 'six' in 2nd throw or 5th throw or 8th throw ...

$$\therefore \quad \text{Probability of winning of } B = \left(\frac{5}{6}\right)\frac{1}{6} + \left(\frac{5}{6}\right)^4 \frac{1}{6} + \left(\frac{5}{6}\right)^7 \frac{1}{6} \ldots = \frac{\left(\frac{5}{6}\right)\frac{1}{6}}{1-\left(\frac{5}{6}\right)^3} = \frac{30}{91}$$

Hence, probability of winning of $C = 1 - \left(\frac{36}{91} + \frac{30}{91}\right) = \frac{25}{91}$

EXERCISE 30.5

BASIC

1. A bag contains 6 black and 3 white balls. Another bag contains 5 black and 4 white balls. If one ball is drawn from each bag, find the probability that these two balls are of the same colour.
2. A bag contains 3 red and 5 black balls and a second bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that one is red and the other is black.
3. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (i) both the balls are red. (ii) first ball is black and second is red. (iii) one of them is black and other is red. [CBSE 2005]
4. Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Find the probability of exactly one ace.

5. A speaks truth in 75% and B in 80% of the cases. In what percentage of cases are they likely to contradict each other in narrating the same incident?
6. Kamal and Monica appeared for an interview for two vacancies. The probability of Kamal's selection is 1/3 and that of Monika's selection is 1/5. Find the probability that
   (i) both of them will be selected (ii) none of them will be selected
   (iii) at least one of them will be selected (iv) only one of them will be selected.
7. A bag contains 3 white, 4 red and 5 black balls. Two balls are drawn one after the other, without replacement. What is the probability that one is white and the other is black?
8. A bag contains 8 red and 6 green balls. Three balls are drawn one after another without replacement. Find the probability that at least two balls drawn are green.
9. Arun and Tarun appeared for an interview for two vacancies. The probability of Arun's selection is 1/4 and that of Tarun's rejection is 2/3. Find the probability that at least one of them will be selected.

BASED ON LOTS

10. A and B toss a coin alternately till one of them gets a head and wins the game. If A starts the game, find the probability that B will win the game.
11. Two cards are drawn from a well shuffled pack of 52 cards, one after another without replacement. Find the probability that one of these is red card and the other a black card? **[CBSE 2020]**
12. Tickets are numbered from 1 to 10. Two tickets are drawn one after the other at random. Find the probability that the number on one of the tickets is a multiple of 5 and on the other a multiple of 4.
13. In a family, the husband tells a lie in 30% cases and the wife in 35% cases. Find the probability that both contradict each other on the same fact.
14. A husband and wife appear in an interview for two vacancies for the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that
   (i) both of them will be selected? (ii) only one of them will be selected?
   (iii) none of them will be selected?
15. A bag contains 7 white, 5 black and 4 red balls. Four balls are drawn without replacement. Find the probability that at least three balls are black.
16. A, B, and C are independent witness of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses?
17. A bag contains 4 white balls and 2 black balls. Another contains 3 white balls and 5 black balls. If one ball is drawn from each bag, find the probability that
   (i) both are white (ii) both are black (iii) one is white and one is black
18. A bag contains 4 white, 7 black and 5 red balls. 4 balls are drawn with replacement. What is the probability that at least two are white?
19. Three cards are drawn with replacement from a well shuffled pack of 52 cards. Find the probability that the cards are a king, a queen and a jack.
20. A bag contains 4 red and 5 black balls, a second bag contains 3 red and 7 black balls. One ball is drawn at random from each bag, find the probability that the (i) balls are of different colours (ii) balls are of the same colour.
21. A can hit a target 3 times in 6 shots, B : 2 times in 6 shots and C : 4 times in 4 shots. They fix a volley. What is the probability that at least 2 shots hit?
22. The probability of student A passing an examination is 2/9 and of student B passing is 5/9. Assuming the two events : 'A passes', 'B passes' as independent, find the probability of : (i) only A passing the examination (ii) only one of them passing the examination.

23. There are three urns A, B, and C. Urn A contains 4 red balls and 3 black balls. Urn B contains 5 red balls and 4 black balls. Urn C contains 4 red and 4 black balls. One ball is drawn from each of these urns. What is the probability that 3 balls drawn consist of 2 red balls and a black ball?

24. X is taking up subjects – Mathematics, Physics and Chemistry in the examination. His probabilities of getting grade A in these subjects are 0.2, 0.3 and 0.5 respectively. Find the probability that he gets
   (i) Grade A in all subjects (ii) Grade A in no subject
   (iii) Grade A in two subjects. **[CBSE 2005]**

25. A and B take turns in throwing two dice, the first to throw 9 being awarded the prize. Show that their chance of winning are in the ratio 9:8.

26. A, B and C in order toss a coin. The one to throw a head wins. What are their respective chances of winning assuming that the game may continue indefinitely?

27. Three persons A, B, C throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning.

28. A and B take turns in throwing two dice, the first to throw 10 being awarded the prize, show that if A has the first throw, their chance of winning are in the ratio 12 : 11.

29. There are 3 red and 5 black balls in bag 'A'; and 2 red and 3 black balls in bag 'B'. One ball is drawn from bag 'A' and two from bag 'B'. Find the probability that out of the 3 balls drawn one is red and 2 are black.

30. Fatima and John appear in an interview for two vacancies for the same post. The probability of Fatima's selection is $\frac{1}{7}$ and that of John's selection is $\frac{1}{5}$. What is the probability that (i) both of them will be selected? (ii) only one of them will be selected?
   (iii) none of them will be selected?

31. A bag contains 8 marbles of which 3 are blue and 5 are red. One marble is drawn at random, its colour is noted and the marble is replaced in the bag. A marble is again drawn from the bag and its colour is noted. Find the probability that the marble will be
   (i) blue followed by red. (ii) blue and red in any order. (iii) of the same colour.

32. An urn contains 7 red and 4 blue balls. Two balls are drawn at random with replacement. Find the probability of getting
   (i) 2 red balls (ii) 2 blue balls (iii) One red and one blue ball.

33. A card is drawn from a well-shuffled deck of 52 cards. The outcome is noted, the card is replaced and the deck reshuffled. Another card is then drawn from the deck.
   (i) What is the probability that both the cards are of the same suit?
   (ii) What is the probability that the first card is an ace and the second card is a red queen?

34. Out of 100 students, two sections of 40 and 60 are formed. If you and your friend are among 100 students, what is the probability that: (i) you both enter the same section? (ii) you both enter the different sections? **[CBSE 2008]**

35. In a hockey match, both teams A and B scored same number of goals upto the end of the game, so to decide the winner, the refree asked both the captains to throw a die alternately and decide that the team, whose captain gets a first six, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the refree was fair or not. **[CBSE 2013]**

36. A and B throw a pair of dice alternately. A wins the game if he gets a total of 7 and B wins the game if he gets a total of 10. If A starts the game, then find the probability that B wins. **[CBSE 2016]**

37. A and B throw a pair of dice alternatively till one of then gets the sum of the numbers as multiples of 6 and wins the game. If A starts first, find the probability of B winning the game. **[CBSE 2020]**

ANSWERS

1. $\frac{14}{27}$ 2. $\frac{21}{40}$ 3. (i) $\frac{16}{81}$ (ii) $\frac{20}{81}$ (iii) $\frac{40}{81}$ 4. $\frac{32}{221}$

5. 35% 6. (i) $\frac{1}{15}$ (ii) $\frac{8}{15}$ (iii) $\frac{7}{15}$ (iv) $\frac{2}{5}$

7. $\frac{5}{22}$ 8. $\frac{5}{13}$ 9. $\frac{1}{2}$ 10. $\frac{1}{3}$ 11. $\frac{26}{51}$

12. $\frac{4}{45}$ 13. 0.44 14. (i) $\frac{1}{35}$ (ii) $\frac{2}{7}$ (iii) $\frac{24}{35}$

15. $\frac{23}{364}$ 16. $\frac{107}{120}$ 17. (i) $\frac{1}{4}$ (ii) $\frac{5}{24}$ (iii) $\frac{13}{24}$

18. $\frac{67}{256}$ 19. $\frac{6}{2197}$ 20. (i) $\frac{43}{90}$ (ii) $\frac{47}{90}$ 21. $\frac{2}{3}$

22. (i) $\frac{8}{81}$ (ii) $43/81$ 23. $\frac{17}{42}$ 24. (i) 0.03 (ii) 0.28

(iii) 0.22 26. $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$ 27. $\frac{36}{91}, \frac{30}{91}, \frac{25}{91}$ 29. $\frac{39}{80}$ 30. (i) $\frac{1}{35}$

(ii) $\frac{2}{7}$ (iii) $\frac{24}{35}$ 31. (i) $\frac{15}{64}$ (ii) $\frac{15}{32}$ (iii) $\frac{17}{32}$

32. (i) $\frac{49}{121}$ (ii) $\frac{16}{121}$ (iii) $\frac{56}{121}$ 33. (i) $\frac{1}{4}$ (ii) $\frac{1}{338}$ 34. (i) $\frac{17}{33}$ (ii) $\frac{16}{33}$

35. Team A: $\frac{6}{11}$, Team B: $\frac{5}{11}$; The decision was fair as the two probabilities are almost equal.

36. $\frac{5}{17}$ 37. $\frac{5}{11}$

HINTS TO SELECTED PROBLEMS

1. For $i = 1, 2$, let us define the following events:

B_i = Drawing a black ball from i^{th} bag, W_i = Drawing a white ball from i^{th} bag

So, required probability $= P((B_1 \cap B_2)) \cup ((W_1 \cap W_2))$

$$= P(B_1 \cap B_2) + P(W_1 \cap W_2) = P(B_1)\, P(B_2) + P(W_1)\, P(W_2)$$

6. Consider the following events : K = Kamal is selected, M = Monica is selected. Then, $P(K) = 1/3$, $P(M) = 1/5$.

(i) Required probability $= P(K \cap M) = P(K)\, P(M) = \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$

(ii) Required probability $= P(\overline{K} \cap \overline{M}) = P(\overline{K})\, P(\overline{M}) = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$

(iii) Required probability $= 1 - P(\overline{K})\, P(\overline{M}) = 1 - \frac{2}{3} \times \frac{4}{5} = \frac{7}{15}$

(iv) Required probability $= P(K \cap \overline{M}) + P(\overline{K} \cap M) = P(K)\, P(\overline{M}) + P(\overline{K})\, P(M)$

9. Required probability $= 1 - P(\overline{A})\, P(\overline{T}) = 1 - \left(1 - \frac{1}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{2}$

11. Required probability $= P[(R \text{ and } B) \text{ or } (B \text{ and } R)] = P(R \cap B) + P(B \cap R)$

$$= P(R)\, P(B/R) + P(B)\, P(R/B) = \frac{26}{52} \times \frac{26}{51} + \frac{26}{52} \times \frac{26}{51} = \frac{26}{51}$$

12. Let, A_i = Getting a multiple of 5 in i^{th} draw, B_i = Getting a multiple of 4 in i^{th} draw, where $i = 1, 2$.

Required probability $= P[(A_1 \cap B_2) \cup P(B_1 \cap A_2)] = P(A_1 \cap B_2) + P(B_1 \cap A_2)$

$$= P(A_1)\, P(B_2/A_1) + P(B_1)\, P(A_2/B_1) = \frac{2}{10} \times \frac{2}{9} + \frac{2}{10} \times \frac{2}{9}$$

20. (i) Required probability $= P((R_1 \cap B_2) \cup (B_1 \cap R_2)) = P(R_1 \cap B_2) + P(B_1 \cap R_2)$

$$= P(R_1)\,P(B_2) + P(R_2)\,P(B_1) = \frac{4}{9} \times \frac{7}{10} + \frac{5}{9} \times \frac{3}{10}$$

(ii) Required probability $= P((R_1 \cap R_2) \cup (B_1 \cap B_2)) = P(R_1 \cap R_2) + P(B_1 \cap B_2)$

$$= P(R_1)\,P(R_2) + P(B_1)\,P(B_2) = \frac{4}{9} \times \frac{3}{10} + \frac{5}{9} \times \frac{7}{10}$$

21. Required probability

$= P[(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C) \cup (A \cap B \cap C)]$

$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$

$= P(A)\,P(B)\,P(\overline{C}) + P(A)\,P(\overline{B})\,P(C) + P(\overline{A})\,P(B)\,P(C) + P(A)\,P(B)\,P(C)$

30.8 THE LAW OF TOTAL PROBABILITY

THEOREM *(Law of Total Probability) Let S be the sample space and let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then*

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + \ldots + P(E_n)\,P(A/E_n).$$

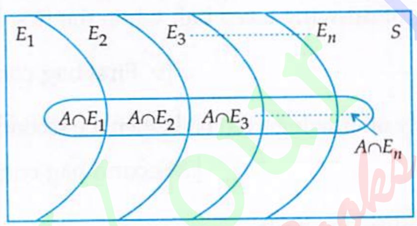

Fig. 30.10

PROOF Since $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events. Therefore,

$$S = E_1 \cup E_2 \cup E_3 \ldots \cup E_n, \text{ where } E_i \cap E_j = \phi \text{ for } i \neq j.$$

Clearly, $A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)$

$\therefore \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) \ldots + P(A \cap E_n)$ [By addition theorem]

But, $\quad P(A \cap E_i) = P(E_i)\,P(A/E_i)$ for $i = 1, 2, \ldots, n$

Hence, $P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + \ldots + P(E_n)\,P(A/E_n)$

Q.E.D.

The law of total probability as stated and proved above says that if an event A can occur in n mutually exclusive ways, then the probability of occurrence of A is the sum of the probabilities of all mutually exclusive ways as shown in the following tree diagram.

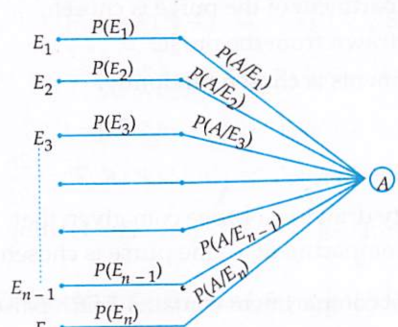

Fig. 30.11

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.*

SOLUTION A red ball can be drawn in two mutually exclusive ways.

(I) Selecting bag I and then drawing a red ball from it.

(II) Selecting bag II and then drawing a red ball from it.

Let E_1, E_2 and A denote the events defined as follows:

E_1 = Selecting bag I,

E_2 = Selecting bag II,

A = Drawing a red ball

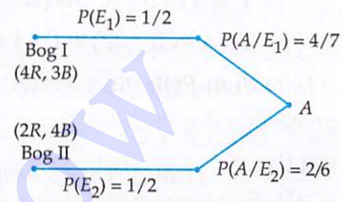

Fig. 30.12

Since one of the two bags is selected randomly.

$\therefore \quad P(E_1) = \dfrac{1}{2}$ and $P(E_2) = \dfrac{1}{2}$.

Now, $P(A/E_1)$ = Probability of drawing a red ball when the first bag has been chosen.

$= \dfrac{4}{7}$ [$\because$ First bag contains 4 red and 3 black balls]

and, $P(A/E_2)$ = Probability of drawing a red ball when the second bag has been selected

$= \dfrac{2}{6}$ [$\because$ Second bag contains 2 red and 4 black balls]

Using the law of total probability, we have

$$\text{Required probability} = P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$
$$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{2}{6} = \frac{19}{42}$$

EXAMPLE 2 *Find the probability of drawing a one-rupee coin from a purse with two compartments one of which contains 3 fifty-paise coins and 2 one-rupee coins and other contains 2 fifty-paise coins and 3 one-rupee coins.*

SOLUTION A one rupee coin can be drawn in two mutually exclusive ways:

(I) Selecting compartment I and then drawing a rupee coin from it.

(II) Selecting compartment II and then drawing a rupee coin from it.

Let E_1, E_2 and A be the events defined as follows:

E_1 = the first compartment of the purse is chosen,

E_2 = the second compartment of the purse is chosen,

A = a rupee coin is drawn from the purse.

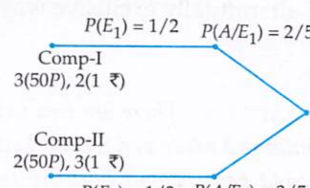

Fig. 30.13

Since one of the two compartments is chosen randomly.

$\therefore \quad P(E_1) = \dfrac{1}{2} = P(E_2)$

Also,

$P(A/E_1)$ = Probability drawing a rupee coin given that the first compartment of the purse is chosen

$= \dfrac{2}{5}$ [$\because$ First compartment contains 3 fifth paise coins and 2 one rupee coins] and, $P(A/E_2)$ = Probability of drawing a rupee coin given that the second compartment of the purse is chosen

$= \frac{3}{5}$ [$\because$ Second compartment contains 2 fifth paise coins and 3 one rupee coins]

By the law of total probability

P (Drawing a one rupee coin) $= P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{3}{5} = \frac{1}{2}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.* **[NCERT EXEMPLAR, CBSE 2022]**

SOLUTION A white ball can be drawn from the second bag in two mutually exclusive ways:

(I) By transferring a white ball from first bag to the second bag and then drawing a white ball from it.

(II) By transferring a black ball from first bag to the second bag and then drawing a white ball from it.

E_1 = a white ball is transferred from the first bag to the second bag

E_2 = a black ball is transferred from the first bag to the second bag

A = a white ball is drawn from the second bag

Since the first bag contains 4 white and 5 black balls.

$\therefore \quad P(E_1) = \frac{4}{9}$ and $P(E_2) = \frac{5}{9}$

If E_1 has already occurred, that is a white ball has already been transferred from first bag to the second bag, then the second bag contains 7 white and 7 black balls.

So, $P(A/E_1) = \frac{7}{14}$

If E_2 has already occurred, that is a black ball has been transferred from first bag to the second bag, then the second bag contains 6 white and 8 black balls.

So, $P(A/E_2) = \frac{6}{14}$

Using law of total probability, we obtain

$$P(\text{Getting a white ball}) = P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$$

$$= \frac{4}{9} \times \frac{7}{14} + \frac{5}{9} \times \frac{6}{14} = \frac{58}{126} = \frac{29}{63}$$

EXAMPLE 4 *There are two bags. The first bag contains 5 white and 3 black balls and the second bag contains 3 white and 5 black balls. Two balls are drawn at random from the first bag and are put into the second bag without noticing their colours. Then two balls are drawn from the second bag. Find the probability that the balls are white and black.*

SOLUTION A white and a black ball can be drawn from the second bag in the following mutually exclusive ways:

(I) By transferring 2 black balls from first bag to the second bag and then drawing a white and a black ball from it.

(II) By transferring 2 white balls from first bag to the second bag and then drawing a white and a black ball from it.

(III) By transferring one white and one black ball from first bag to the second bag and then drawing a white and a black ball from it.

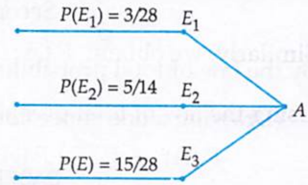

Fig. 30.14

Let E_1, E_2, E_3 and A be the events as defined below:

E_1 = Two black balls are drawn from the first bag,

E_2 = Two white balls are drawn from the first bag,

E_3 = One white and one black ball is drawn from the first bag,

A = Two balls drawn from the second bag are white and black.

We have, $P(E_1) = \dfrac{{}^3C_2}{{}^8C_2} = \dfrac{3}{28}$, $P(E_2) = \dfrac{{}^5C_2}{{}^8C_2} = \dfrac{5}{14}$, and $P(E_3) = \dfrac{{}^3C_1 \times {}^5C_1}{{}^8C_2} = \dfrac{15}{28}$

If E_1 has already occurred, that is, if two black balls have been transferred from the first bag to the second bag, then the second bag will contain 3 white and 7 black balls.

$\therefore$ Probability of drawing a white and a black ball from the second bag is $\dfrac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2}$

$$\therefore \quad P(A/E_1) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{7}{15}$$

Similarly, we obtain: $P(A/E_2) = \dfrac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} = \dfrac{5}{9}$ and $P(A/E_3) = \dfrac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \dfrac{8}{15}$

Using the law of total probability, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= \frac{3}{28} \times \frac{7}{15} + \frac{5}{14} \times \frac{5}{9} + \frac{15}{28} \times \frac{8}{15} = \frac{673}{1260}$$

EXAMPLE 5 *A bag contains 6 red and 5 blue balls and another bag contains 5 red and 8 blue balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag. A ball is then drawn from the second bag. Find the probability that the ball drawn is blue in colour.*

SOLUTION A blue colour ball can be drawn from the second bag in the following mutually exclusive ways:

(I) By transferring a blue ball from first bag to the second bag and then drawing a blue ball from the second bag.

(II) By transferring a red ball from first bag to the second bag and then drawing a blue ball from the second bag.

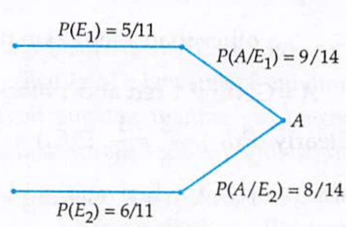

Fig. 30.15

Let E_1, E_2 and A be the events defined as follows:

E_1 = Ball drawn from first bag is blue

E_2 = Ball drawn from first bag is red

A = Ball drawn from the second bag is blue

Since first bag contains 6 red and 5 blue balls. Therefore,

$$P(E_1) = \frac{5}{11} \text{ and } P(E_2) = \frac{6}{11}$$

If E_1 has already occurred, that is, if a blue ball is transferred from the first bag to the second bag, then the second bag contains 5 red and 9 blue balls, therefore the probability of drawing a blue ball from the second bag is $\frac{9}{14}$. Therefore, $P(A/E_1) = \frac{9}{14}$.

Similarly, we obtain: $P(A/E_2) = \frac{8}{14}$

Using the law of total probability, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{5}{11} \times \frac{9}{14} + \frac{6}{11} \times \frac{8}{14} = \frac{93}{154}.$$

EXAMPLE 6 *There are two bags, one of which contains 3 black and 4 white, balls, while the other contains 4 black and 3 white balls. A fair die is cast, if the face 1 or 3 turns up, a ball is taken from the first bag, and if any other face turns up a ball is chosen from the second bag. Find the probability of choosing a black ball.* **[NCERT EXEMPLAR]**

SOLUTION Let E_1, E_2 and A be the events defined as follows:

E_1 = The die shows 1 or 3, E_2 = The die shows 2, 4, 5 or 6, and A = The ball drawn is black.

We find that, $P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$.

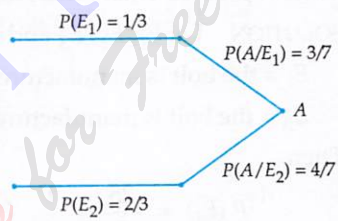

Fig. 30.16

If E_1 occurs, then the first bag is chosen and the probability of drawing a black ball from it is $\frac{3}{7}$.

$$\therefore \quad P(A/E_1) = \frac{3}{7}$$

If E_2 occurs, then the second bag is chosen and the probability of drawing a black ball from it is $\frac{4}{7}$.

$$\therefore \quad P(A/E_2) = \frac{4}{7}$$

Using the law of total probability, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{1}{3} \times \frac{3}{7} + \frac{2}{3} \times \frac{4}{7} = \frac{11}{21}.$$

EXAMPLE 7 *A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag B. If two balls are drawn at random (without replacement) from the selected bag, find the probability of one of them being red and another black.* **[CBSE 2015]**

SOLUTION Consider the following events:

E_1 = Getting 1 or 2 on the die, E_2 = Getting any number other than 1 and 2 on the die

A = Getting 1 red and 1 black ball from the bag.

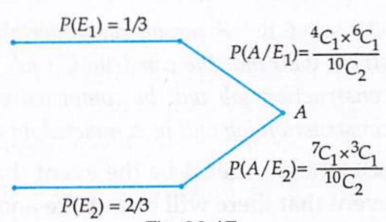

Fig. 30.17

Clearly, $P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$

$$P(A/E_1) = P\text{ (Getting 1 red and 1 black ball from bag } A) = \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \frac{8}{15}$$

$$P(A/E_2) = P\text{ (Getting 1 red and 1 black ball from bag } B) = \frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7}{15}$$

$\therefore$ Required probability $= P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) = \frac{1}{3} \times \frac{8}{15} + \frac{2}{3} \times \frac{7}{15} = \frac{22}{45}$

EXAMPLE 8 *Two thirds of the students in a class are boys and the rest girls. It is known that the probability of a girl getting a first class is 0.25 and that of a boy getting a first class is 0.28. Find the probability that a student chosen at random will get first class marks in the subject.*

SOLUTION Let E_1, E_2 and A be the events defined as follows:

E_1 = a boy is chosen from the class, E_2 = a girl is chosen from the class,

and, A = the students gets first class marks.

Then, $P(E_1) = 2/3$, $P(E_2) = 1/3$, $P(A/E_1) = 0.28$ and $P(A/E_2) = 0.25$

Using the law of total probability, we obtain

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) = \frac{2}{3} \times 0.28 + \frac{1}{3} \times 0.25 = 0.27$$

EXAMPLE 9 *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. What is the probability that the bolt drawn is defective?*

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = the bolt is manufactured by machine A; E_2 = the bolt is manufactured by machine B;

E_3 = the bolt is manufactured by machine C, and, A = the bolt is defective.

Then,

$$P(E_1) = \frac{25}{100} = \frac{1}{4},\ P(E_2) = \frac{35}{100},\ P(E_3) = \frac{40}{100}.$$

$P(A/E_1)$ = Probability that the bolt drawn is defective given the condition that it is manufactured by machine A

$= 5/100$

Similarly, we obtain

$$P(A/E_2) = \frac{4}{100} \text{ and } P(A/E_3) = \frac{2}{100}$$

Using the law of total probability, we obtain

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)$$

$$= \frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100} = 0.0345$$

EXAMPLE 10 *A person has undertaken a construction job. The probabilities are 0.65 that there will be strike, 0.80 that the construction job will be completed on time if there is no strike, and 0.32 that the construction job will be completed on time if there is a strike. Determine the probability that the construction job will be completed on time.*

SOLUTION Let A be the event that the construction job will be completed on time, E_1 be the event that there will be a strike and E_2 be the event that there will be no strike.

We have,

$$P(E_1) = 0.65,\ P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35,\ P(A/E_1) = 0.32 \text{ and } P(A/E_2) = 0.80$$

By total probability theorem, we obtain

$$\text{Required probability} = P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$$

$$= 0.65 \times 0.32 + 0.35 \times 0.80 = 0.208 + 0.28 = 0.488$$

EXAMPLE 11 *A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?* **[NCERT EXEMPLAR]**

SOLUTION First ball drawn from the box may be blue or red. So, let us define the following events:

E_1 = First ball drawn from the bag is blue,

E_2 = First ball drawn from the bag is red

A = Second ball drawn from the bag is blue.

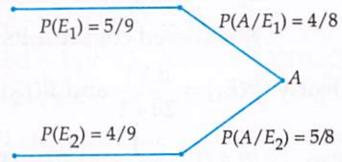

Fig. 30.18

Clearly, $P(E_1) = \frac{5}{9}$, $P(E_2) = \frac{4}{9}$, $P(A/E_1) = \frac{4}{8}$ and $P(A/E_2) = \frac{5}{8}$.

By total probability theorem, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{5}{9} \times \frac{4}{8} + \frac{4}{9} \times \frac{5}{8} = \frac{5}{9}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.* **[NCERT EXEMPLAR]**

SOLUTION First ball drawn from the urn may be white or black. So, let us define the following events.

E_1 = First ball drawn in white, E_2 = First ball drawn in black.

Clearly, $P(E_1) = \frac{m}{m+n}$ and $P(E_2) = \frac{n}{m+n}$

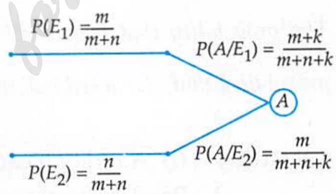

Fig. 30.19

If first ball drawn is white, it is put back in the urn along with k white balls and then a ball is drawn from the urn. The probability getting a white ball now is $\frac{m+k}{m+n+k}$.

If the first ball drawn in black, it is also put back in the urn along with k black balls and then a ball is drawn from the urn. The probability of drawing a white ball now is $\frac{m}{m+n+k}$.

Thus, if A denotes the event "Getting a white ball from the urn when ball drawn in first draw is put back along with k balls of the same colour". Then,

$$P(A/E_1) = \frac{m+k}{m+n+k} \text{ and } P(A/E_2) = \frac{m}{m+n+k}$$

By using total probability theorem, we obtain

$$\text{Required probability} = P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$

$$= \frac{m}{m+n} \times \frac{m+k}{m+n+k} + \frac{n}{m+n} \times \frac{m}{m+n+k}$$

$$= \frac{m(m+k+n)}{(m+n)\,(m+n+k)} = \frac{m}{m+n}, \text{ which is independent of } k.$$

EXAMPLE 13 *A bag contains $(2n+1)$ coins. It is known that n of these coins have a head on both sides whereas the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.* **[NCERT EXEMPLAR]**

SOLUTION A coin picked up at random from the bag may be a fair coin or a coin having head on both sides. The tossed coin may result in a head in the following mutually exclusive ways.

(i) A fair coin is taken out of the bag and then it is tossed to get a head.

(ii) A two headed coin is taken out of the bag and then it is tossed to get a head.

So, let us define the following events.

E_1 = Taking out a fair coin from the bag.

E_2 = Taking out a coin having head on both sides

A = Tossed coin results in a head.

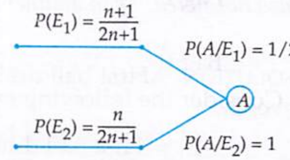

Fig. 30.20

Clearly, $P(E_1) = \dfrac{n+1}{2n+1}$ and $P(E_2) = \dfrac{n}{2n+1}$.

Also, $P(A/E_1) = \dfrac{1}{2}$ and $P(A/E_2) = 1$

By using total probability theorem, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{n+1}{2n+1} \times \frac{1}{2} + \frac{n}{2n+1} \times 1 = \frac{3n+1}{2\,(2n+1)}$$

But, it is given that $P(A) = \dfrac{31}{42}$.

$$\therefore \quad \frac{31}{42} = \frac{3n+1}{2(2n+1)} \Rightarrow 62n + 31 = 63n + 21 \Rightarrow n = 10$$

EXAMPLE 14 *Three bags contain a number of red and white balls as follows:*

Bag I : 3 red balls ; Bag II : 2 red balls and 1 white ball ; Bag III : 3 white balls

The probability that bag i will be chosen and a ball is selected from it is $\dfrac{i}{6}, i = 1, 2, 3$. *What is the probability that (i) a red ball will be selected ? (ii) a white ball will be selected?*

[NCERT EXEMPLAR]

SOLUTION (i) A red ball can be selected in the following mutually exclusive ways:

I Bag I is chosen and a red ball is drawn from it.

II Bag II is chosen and a red ball is drawn from it.

III Bag III is chosen and a red ball is drawn from it.

So, we define the following events.

E_i = Selecting i^{th} bag, $i = 1, 2, 3$.

A = Getting a white ball.

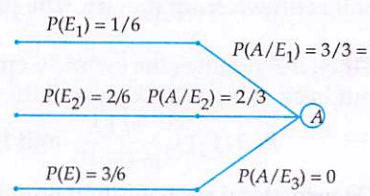

Fig. 30.21

It is given that $P(E_i) = \dfrac{i}{6}, i = 1, 2, 3$.

Clearly, $P(A/E_1) = \dfrac{3}{3} = 1$, $P(A/E_2) = \dfrac{2}{3}$ and $P(A/E_3) = \dfrac{0}{3} = 0$

$$\text{Required probability} = P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$$

$$= \frac{1}{6} \times 1 + \frac{2}{6} \times \frac{2}{3} + \frac{3}{6} \times 0 = \frac{7}{18}$$

(ii) Required probability $= P(\overline{A}) = 1 - P(A) = 1 - \dfrac{7}{18} = \dfrac{11}{18}$

EXAMPLE 15 *A bag contains 3 white and 2 black balls and another bag contains 2 white and some black balls. One bag is chosen at random. From the selected bag, one ball is drawn. If the probability of getting a black ball from the selected bag is* $\dfrac{8}{15}$, *find the number of black balls in the second bag.*

SOLUTION Let there be x black balls in the second bag. Then, the contents of two bags are:

	W	B
Bag I:	3	2
Bag II:	2	x

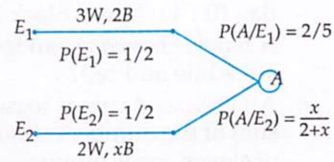

Fig. 30.22

Consider the following events:

E_1 = Selecting I bag, E_2 = Selecting II bag

A = Getting a black ball from the selected bag.

We observe that:

$$P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2},\ P(A/E_1) = \frac{2}{5},\ P(A/E_2) = \frac{x}{2+x}$$

$$\therefore \quad P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$

$$\Rightarrow \quad P(A) = \frac{1}{2} \times \frac{2}{5} + \frac{1}{2} \times \frac{x}{2+x}$$

$$\Rightarrow \quad \frac{8}{15} = \frac{1}{5} + \frac{1}{2} \times \frac{x}{2+x} \Rightarrow \frac{1}{3} = \frac{1}{2} \times \frac{x}{2+x} \Rightarrow 4 + 2x = 3x \Rightarrow x = 4 \quad \left[\because P(A) = \frac{8}{15} \text{ (given)}\right]$$

Hence, there are 4 black balls in the second bag.

EXAMPLE 16 *A bag I contains 3 red and n black balls and another bag II contains 5 red and 12 black balls. A ball is drawn at random from bag I and transferred to bag II. If the probability of drawing a red ball from bag I, after transfer, is 1/3, then find the value of n.*

SOLUTION A ball transferred from bag I to bag II may be red or black. So, let us define the following events.

E_1 = Ball transferred from bag I to bag II is red.

E_2 = Ball transferred from bag I to bag II is black.

A = Drawing a red ball from bag I after transferring a ball to bag II

Clearly $P(E_1) = \dfrac{3}{n+3}$, $P(E_2) = \dfrac{n}{n+3}$, $P(A/E_1) = \dfrac{2}{n+2}$ and $P(A/E_2) = \dfrac{3}{n+2}$

Using total probability theorem, we obtain

$$P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$$

$$\Rightarrow \quad \frac{1}{3} = \frac{3}{n+3} \times \frac{2}{n+2} + \frac{n}{n+3} \times \frac{3}{n+2} \quad \left[\because P(A) = \frac{1}{3} \text{ (given)}\right]$$

$$\Rightarrow \quad \frac{1}{3} = \frac{3(n+2)}{(n+2)(n+3)} \Rightarrow n + 3 = 9 \Rightarrow n = 6$$

Hence, there are 6 black balls in bag I.

EXERCISE 30.6

BASIC

1. A bag A contains 5 white and 6 black balls. Another bag B contains 4 white and 3 black balls. A ball is transferred from bag A to the bag B and then a ball is taken out of the second bag. Find the probability of this ball being black.
2. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled at random from one of the two purses, what is the probability that it is a silver coin? **[CBSE 2020]**
3. One bag contains 4 yellow and 5 red balls. Another bag contains 6 yellow and 3 red balls. A ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. Find the probability that ball drawn is yellow. **[CBSE 2002]**
4. A bag contains 3 white and 2 black balls and another bag contains 2 white and 4 black balls. One bag is chosen at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is white. **[NCERT EXEMPLAR]**

BASED ON LOTS

5. The contents of three bags I, II and III are as follows:
Bag I : 1 white, 2 black and 3 red balls, Bag II : 2 white, 1 black and 1 red ball;
Bag III : 4 white, 5 black and 3 red balls.
A bag is chosen at random and two balls are drawn. What is the probability that the balls are white and red?
6. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the sum of the numbers obtained is noted. If the result is a tail, a card from a well shuffled pack of eleven cards numbered 2, 3, 4, ..., 12 is picked and the number on the card is noted. What is the probability that the noted number is either 7 or 8?
7. A factory has two machines A and B. Past records show that the machine A produced 60% of the items of output and machine B produced 40% of the items. Further 2% of the items produced by machine A were defective and 1% produced by machine B were defective. If an item is drawn at random, what is the probability that it is defective?
8. The bag A contains 8 white and 7 black balls while the bag B contains 5 white and 4 black balls. One ball is randomly picked up from the bag A and mixed up with the balls in bag B. Then a ball is randomly drawn out from it. Find the probability that ball drawn is white. **[CBSE 2007]**
9. A bag contains 4 white and 5 black balls and another bag contains 3 white and 4 black balls. A ball is taken out from the first bag and without seeing its colour is put in the second bag. A ball is taken out from the latter. Find the probability that the ball drawn is white.
10. One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is transferred from first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.
11. An urn contains 10 white and 3 black balls. Another urn contains 3 white and 5 black balls. Two are drawn from first urn and put into the second urn and then a ball is drawn from the latter. Find the probability that it is a white ball.
12. A bag contains 6 red and 8 black balls and another bag contains 8 red and 6 black balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag. A ball is drawn from the second bag. Find the probability that the ball drawn is red in colour.
13. Three machines E_1, E_2, E_3 in a certain factory produce 50%, 25% and 25% respectively, of the total daily output of electric bulbs. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective, and that 5% of those produced on E_3 are defective. If one tube is picked up at random from a day's production, calculate the probability that it is defective. **[NCERT EXEMPLAR, CBSE 2015]**

ANSWERS

1. $\frac{39}{88}$ 2. $\frac{19}{42}$ 3. $\frac{29}{45}$ 4. $\frac{7}{15}$ 5. $\frac{118}{495}$
6. $\frac{193}{792}$ 7. 0.016 8. $\frac{83}{150}$ 9. $\frac{31}{72}$ 10. $\frac{29}{63}$
11. $\frac{59}{130}$ 12. $\frac{59}{105}$ 13. $\frac{17}{400}$

HINTS TO SELECTED PROBLEMS

2. Consider the following events:
E_1 = Selecting first purse, E_2 = Selecting second purse, A = Coin drawn is silver coin.
We have, $P(E_1) = P(E_2) = \frac{1}{2}, P(A/E_1) = \frac{2}{6}, P(A/E_2) = \frac{4}{7}$.
Required probability $= P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$.

4. Consider the following events:

E_1 = Selecting first bag, E_2 = selecting second bag, A = ball drawn is white.

Then, $P(E_1) = P(E_2) = 1/2$, $P(A/E_1) = 3/5$, $P(A/E_2) = 2/6$.

Required probability $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) = \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{6} = \frac{7}{15}$

5. Let E_1 = bag I is selected, E_2 = bag II is selected, E_3 = bag III is selected

and, A = Two balls drawn from the chosen bag are white and red.

Then, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$,

$$P(A/E_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2},\ P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} \text{ and } P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2}.$$

$\therefore$ Required probability $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)$

6. Let E_1 = The coin shows a head, E_1 = The coin shows a tail, A = The noted number is 7 or 8.

Then, $P(E_1) = 1/2$, $P(E_2) = 1/2$, $P(A/E_1) = 11/36$ and $P(A/E_2) = 2/11$.

Required probability $= P(A) = P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)$

30.9 BAYE'S THEOREM

THEOREM *(Baye's Theorem) Let S be the sample space and let $E_1, E_2 \ldots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with E_1 or E_2 or ... or E_n, then*

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)},\ i = 1, 2, \ldots, n$$

PROOF Events $E_1, E_2, \ldots, E_n$ are n mutually exclusive and exhaustive events such that

$$S = E_1 \cup E_2 \cup \ldots \cup E_n, \text{ where } E_i \cap E_j = \phi \text{ for } i \neq j$$

$$\therefore \quad A = A \cap S$$

$$\Rightarrow \quad A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n) \qquad \text{[By addition theorem]}$$

$$\Rightarrow \quad P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

$$\Rightarrow \quad P(A) = \sum_{i=1}^{n} P(E_i)\,P(A/E_i) \qquad [\because P(A \cap E_i) = P(E_i)\,P(A/E_i)] \qquad \ldots\text{(i)}$$

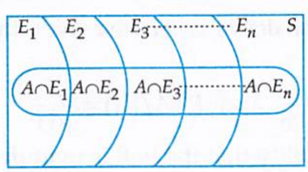

Fig. 30.23

Using multiplication theorem of probability, we obtain

$$P(A \cap E_i) = P(A)\,P(E_i/A) \quad \text{for } i = 1, 2, \ldots, n$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(A \cap E_i)}{P(A)}$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{P(A)} \qquad [\because P(A \cap E_i) = P(E_i)\,P(A/E_i)]$$

$$\Rightarrow \quad P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)} \qquad \text{[Using (i)]}$$

Hence, $$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\,P(A/E_i)},\ i = 1, 2, \ldots, n$$

Q.E.D.

NOTE 1 *The events $E_1, E_2, \ldots, E_n$ are usually referred to as 'hypothesis' and the probabilities $P(E_1)$, $P(E_2), \ldots, P(E_n)$ are known as the 'priori' probabilities as they exist before we obtain any information from the experiment.*

NOTE 2 *The probabilities $P(A/E_i)$; $=1, 2, \ldots, n$ are called the 'likelyhood probabilities' as they tell us how likely the event A under consideration occurs, given each and every priori probabilities.*

NOTE 3 *The probabilities $P(E_i/A)$; $i = 1, 2, \ldots, n$ are called the 'posterior probabilities' as they are determined after the results of the experiment are known.*

The significance of Baye's theorem may be understood in the following manner :

An experiment can be performed in n mutually exclusive and exhaustive ways $E_1, E_2, \ldots, E_n$. The probability $P(E_i)$ of the occurrence of event E_i; $i = 1, 2, \ldots, n$ is known. The experiment is performed and we are told that the event A has occurred. With this information the probability $P(E_i)$ is changed to $P(E_i/A)$. Baye's theorem enables us to evaluate $P(E_i/A)$ if all the $P(E_i)$ (priori probabilities) and $P(A/E_i)$ (likelyhood probabilities) are known as explained in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product. If the bolt drawn is found to be defective, what is the probability that it is manufactured by the machine B?* **[NCERT, CBSE 2008, 2015]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = Bolt is manufactured by machine A, E_2 = Bolt is manufactured by machine B,

E_3 = Bolt is manufactured by machine C, A = Bolt is defective.

Then, $P(E_1)$ = (Probability that the bolt drawn is manufactured by machine A) = 25/100,

$P(E_2)$ = (Probability that the bolt drawn is manufactured by machine B) = 35/100,

$P(E_3)$ = (Probability that the bolt drawn is manufactured by machine C) = 40/100.

$P(A/E_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5/100

Similarly, we obtain: $P(A/E_2) = \dfrac{4}{100}$ and $P(A/E_3) = \dfrac{2}{100}$

Required probability = Probability that the bolt is manufactured by machine B given that the bolt drawn is defective

$$= P(E_2/A)$$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{35}{100} \times \frac{4}{100}}{\frac{25}{100} \times \frac{5}{100} + \frac{35}{100} \times \frac{4}{100} + \frac{40}{100} \times \frac{2}{100}} = \frac{140}{125 + 140 + 80} = \frac{140}{345} = \frac{28}{69}$$

EXAMPLE 2 *Three urns contain 6 red, 4 black; 4 red, 6 black, and 5 red, 5 black balls respectively. One of the urns is selected at random and a ball is drawn from it. If the ball drawn is red, find the probability that it is drawn from the first urn.*

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = First urn is chosen, E_2 = Second urn is chosen,

E_3 = Third urn is chosen, and A = Ball drawn is red.

Since there are three urns and one of the three urns is chosen at random. Therefore,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}.$$

If E_1 has already occurred, then first urn has been chosen which contains 6 red and 4 black balls. The probability of drawing a red ball from it is 6/10.

So, $P(A/E_1) = \frac{6}{10}$. Similarly, we obtain: $P(A/E_2) = \frac{4}{10}$ and $P(A/E_3) = \frac{5}{10}$.

We have to find $P(E_1/A)$, i.e. given that the ball drawn is red, what is the probability that it is drawn from the first urn.

By Baye's theorem, obtain

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{6}{10}}{\frac{1}{3} \times \frac{6}{10} + \frac{1}{3} \times \frac{4}{10} + \frac{1}{3} \times \frac{5}{10}} = \frac{6}{15} = \frac{2}{5}$$

EXAMPLE 3 *A company has two plants to manufacture scooters. Plant I manufactures 70% of the scooters and Plant II manufactures 30%. At Plant I, 80% of the scooters are rated as of standard quality and at Plant II, 90% of the scooters are rated as of standard quality. A scooter is chosen at random and is found to be of standard quality. What is the probability that it has come from Plant II?* **[CBSE 2000, 04, 05, NCERT EXEMPLAR]**

SOLUTION Let E_1, E_2 and A be the following events:

E_1 = Plant I is chosen, E_2 = Plant II is chosen, and A = Scooter is of standard quality.

Then, $P(E_1) = \frac{70}{100}$, $P(E_2) = \frac{30}{100}$, $P(A/E_1) = \frac{80}{100}$ and $P(A/E_2) = \frac{90}{100}$.

We have to find $P(E_2/A)$.

By using Baye's theorem, we obtain

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{30}{100} \times \frac{90}{100}}{\frac{70}{100} \times \frac{80}{100} + \frac{30}{100} \times \frac{90}{100}} = \frac{27}{56 + 27} = \frac{27}{83}$$

EXAMPLE 4 *An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident involving a scooter driver, car driver and a truck driver are 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver?* **[CBSE 2000, 02, 08, 12, 14, 2019]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = Person chosen is a scooter driver, E_2 = Person chosen is a car driver,

E_3 = Person chosen is a truck driver, and A = Person meets with an accident.

Since there are 12000 drivers out of which scooter, car and truck drivers are 2000, 4000 and 6000 respectively.

$\therefore$ $P(E_1) = \frac{2000}{12000} = \frac{1}{6}$ and $P(E_2) = \frac{4000}{12000} = \frac{1}{3}$ and $P(E_2) = \frac{6000}{12000} = \frac{1}{2}$

It is given that

$P(A/E_1)$ = Probability that a person meets with an accident given that he is a scooter driver = 0.01.

Similarly, $P(A/E_2) = 0.03$ and $P(A/E_3) = 0.15$.

We are required to find $P(E_1/A)$, i.e. given that the person meets with an accident, what is the probability that he was a scooter driver?

By using Baye's rule, we obtain

$$P(E_1/A) = \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{1}{6} \times 0.01}{\frac{1}{6} \times 0.01 + \frac{1}{3} \times 0.03 + \frac{1}{2} \times 0.15} = \frac{1}{1+6+45} = \frac{1}{52}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Urn A contains 2 white, 1 black and 3 red balls, urn B contains 3 white, 2 black and 4 red balls and urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn at random from the urn. If the chosen balls happen to be red and black, what is the probability that both balls come from urn B?*

SOLUTION Let E_1, E_2, E_3 and A denote the following events.

E_1 = Urn A is chosen, E_2 = Urn B is chosen, E_3 = Urn C is chosen, and A = two balls drawn at random are red and black.

Since one of the urns is chosen at random.

$$\therefore \quad P(E_1) = \frac{1}{3} = P(E_2) = P(E_3)$$

If E_1 has already occurred, then urn A has been chosen. The urn A contains 2 white, 1 black and 3 red balls. Therefore, the probability of drawing a red and a black ball is $\frac{{}^3C_1 \times {}^1C_1}{{}^6C_2}$

i.e. $$P(A/E_1) = \frac{{}^3C_1 \times {}^1C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

Similarly, $P(A/E_2) = \frac{{}^4C_1 \times {}^2C_1}{{}^9C_2} = \frac{2}{9}$, and $P(A/E_3) = \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} = \frac{1}{6}$

We are required to find $P(E_2/A)$. By Baye's theorem

$$P(E_2/A) = \frac{P(E_2)\, P(A/E_2)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_2/A) = \frac{\frac{1}{3} \times \frac{2}{9}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{2}{9} + \frac{1}{3} \times \frac{1}{6}} = \frac{\frac{2}{9}}{\frac{1}{5} + \frac{2}{9} + \frac{1}{6}} = \frac{20}{53}$$

EXAMPLE 6 *There are 3 bags, each containing 5 white balls and 3 black balls. Also there are 2 bags, each containing 2 white balls and 4 black balls. A white ball is drawn at random. Find the probability that this white ball is from a bag of the first group.*

SOLUTION Let E_1, E_2 and A be the events defined as follows:

E_1 = Selecting a bag from the first group, E_2 = Selecting a bag from the second group

and, A = Ball drawn is white

Since there are 5 bags out of which 3 bags belong to first group and 2 bags to second group.

$\therefore \quad P(E_1) = \frac{3}{5}, \; P(E_2) = \frac{2}{5}$

If E_1 has already occurred, then a bag from the first group is chosen. The bag chosen contains 5 white balls and 3 black balls. Therefore, the probability of drawing a white ball from it is 5/8.

$\therefore \quad P(A/E_1) = 5/8$

Similarly, $P(A/E_2) = 2/6 = 1/3$.

We have to find $P(E_1/A)$, i.e. given that the ball drawn is white, what is the probability that it is drawn from a bag of the first group.

By using Baye's theorem, we obtain

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{3}{5} \times \frac{5}{8}}{\frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{1}{3}} = \frac{45}{61}$$

EXAMPLE 7 *A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be hearts. Find the probability of the missing card to be a heart.* [CBSE 2000, 2010]

SOLUTION Let E_1, E_2, E_3, E_4 and A be the events as defined below:

E_1 = Missing card is a heart card, E_2 = Missing card is a spade card,

E_3 = Missing card is a club card, E_4 = Missing card is a diamond card

and, A = Drawing two heart cards from the remaining cards.

Then,

$$P(E_1) = \frac{13}{52} = \frac{1}{4},\; P(E_2) = \frac{13}{52} = \frac{1}{4},\; P(E_3) = \frac{13}{52} = \frac{1}{4},\; P(E_4) = \frac{13}{52} = \frac{1}{4}.$$

$P(A/E_1)$ = Probability of drawing two heart cards given that one heart card is missing

$$\Rightarrow \quad P(A/E_1) = \frac{{}^{12}C_2}{{}^{51}C_2}$$

$P(A/E_2)$ = Probability of drawing two heart cards given that one spade card is missing

$$\Rightarrow \quad P(A/E_2) = \frac{{}^{13}C_2}{{}^{51}C_2}$$

Similarly, $P(A/E_3) = \frac{{}^{13}C_2}{{}^{51}C_2}$ and $P(A/E_4) = \frac{{}^{13}C_2}{{}^{51}C_2}$

By using Baye's Theorem, we obtain

Required probability $= P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(AE_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$= \frac{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4} \times \frac{{}^{12}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2} + \frac{1}{4} \times \frac{{}^{13}C_2}{{}^{51}C_2}}$$

$$= \frac{{}^{12}C_2}{{}^{12}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2} = \frac{66}{66 + 78 + 78 + 78} = \frac{11}{50}$$

EXAMPLE 8 *Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets a 1, 2, 3, or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head; what is the probability that she threw a 1, 2, 3, or 4 with the die?* **[CBSE 2012, NCERT, 2018]**

SOLUTION Consider the following events:

E_1 = Getting 5 or 6 in a single throw of a die

E_2 = Getting 1, 2, 3, or 4 in a single throw of a die.

A = Getting exactly one head.

Clearly, $P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$

$P(A/E_1)$ = Probability of getting exactly one head when a coin is tossed three times

$$= {}^3C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

$P(A/E_2)$ = Probability of getting exactly one head when a coin is tossed once only

$$= \frac{1}{2}$$

$\therefore$ Required probability $= P(E_2/A) = \dfrac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$

$$= \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{1}{2} \times \frac{2}{3}} = \frac{8}{11}$$

EXAMPLE 9 *Given three identical boxes I, II and III, each containing two coins. In box I both coins are gold coins, in box II both are silver coins and in box III there is one gold and one sliver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?* **[CBSE 2011, NCERT]**

SOLUTION Consider the following events:

E_1 = Box I is choosen, E_2 = Box II is choosen, E_3 = Box III is choosen.

A = The coin drawn is of gold.

Clearly, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(A/E_1)$ = Probability of drawing a gold coin from box I $= \frac{2}{2} = 1$

$P(A/E_2)$ = Probability of drawing a gold coin from box II $= 0$

$P(A/E_3)$ = Probability of drawing a gold coin from box III $= \frac{1}{2}$

Probability that the other coin in the box is of gold

= Probability that gold coin is drawn from the box I

= $P(E_1/A)$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)} = \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}$$

EXAMPLE 10 *Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.* **[CBSE 2012]**

SOLUTION Consider the following events:

E_1 = Ball transferred from Bag I to Bag II is red

E_2 = Ball transferred from Bag I to Bag II is black

A = Ball drawn from Bag II is red in colour.

Clearly,

$$P(E_1) = \frac{3}{7}, P(E_2) = \frac{4}{7}, P(A/E_1) = \frac{5}{10} = \frac{1}{2} \text{ and } P(A/E_2) = \frac{4}{10} = \frac{2}{5}$$

$$\text{Required probability} = P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{4}{7}\times\frac{2}{5}}{\frac{3}{7}\times\frac{1}{2}+\frac{4}{7}\times\frac{2}{5}} = \frac{16}{31}.$$

EXAMPLE 11 *Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.* **[CBSE 2011, NCERT]**

SOLUTION Consider the following events:

E_1 = Person selected is male, E_2 = Person selected is female.

A = Person selected is grey haired.

Clearly,

$$P(E_1) = P(E_2) = \frac{1}{2}, P(A/E_1) = \frac{5}{100} \text{ and } P(A/E_2) = \frac{1}{400}$$

$$\therefore \quad \text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{100}}{\frac{1}{2}\times\frac{5}{100}+\frac{1}{2}\times\frac{1}{400}} = \frac{20}{21}$$

EXAMPLE 12 *A bag contains 4 balls. Two balls are drawn at random without replacement and are found to be white. What is the probability that all balls are white?* **[CBSE 2010, 2016]**

SOLUTION Since two balls drawn are white. So, we have the following possibilities:

(i) The bag contains two white balls and 2 balls of other colour.

(ii) The bag contains 3 white balls and one ball of other colour.

(iii) The bag contains all white balls.

Consider the following events:

E_1 = There are two white and two other colour balls in the bag,

E_2 = There are three white and one other colour ball in the bag,

E_3 = There are all white balls in the bag , A = Drawing 2 white balls from the bag

Clearly, $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$$P(A/E_1) = \frac{{}^2C_2}{{}^4C_2} = \frac{1}{6}, P(A/E_2) = \frac{{}^3C_2}{{}^4C_2} = \frac{1}{2}, P(A/E_3) = \frac{{}^4C_2}{{}^4C_2} = 1$$

$$\therefore \quad \text{Required probability} = P(E_3/A) = \frac{P(E_3)\,P(A/E_3)}{\sum_{i=1}^{3} P(E_i)\,P(A/E_i)} = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times\frac{1}{6}+\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times 1} = \frac{3}{5}$$

EXAMPLE 13 *A bag contains 3 red and 7 black balls. Two balls are selected at random one-by-one without replacement. If the second selected ball happens to be red, what is the probability that the first selected ball is also red?* **[CBSE 2014]**

SOLUTION Consider the following events:

E_1 = First ball drawn is red and second is of any colour

E_2 = First ball drawn is black and second is of any colour, A = Second ball drawn is red.

Clearly, $P(E_1) = \frac{3}{10}$, $P(E_2) = \frac{7}{10}$, $P(A/E_1) = \frac{2}{9}$ and $P(A/E_2) = \frac{3}{9}$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{3}{10}\times\frac{2}{9}}{\frac{3}{10}\times\frac{2}{9}+\frac{7}{10}\times\frac{3}{9}} = \frac{2}{9}$$

EXAMPLE 14 *Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed. 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?* **[NCERT EXEMPLAR]**

SOLUTION Consider the following events:

E_1 = A person with blood group O is selected

E_2 = A person with other blood groups is selected.

A = A left handed person is selected.

We have,

$$P(E_1) = \frac{30}{100} = \frac{3}{10},\ P(E_2) = \frac{70}{100} = \frac{7}{10},\ P(A/E_1) = \frac{6}{100} \text{ and } P(A/E_2) = \frac{10}{100}$$

$$\therefore \quad \text{Required probability} = P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{3}{10}\times\frac{6}{100}}{\frac{3}{10}\times\frac{6}{100}+\frac{7}{10}\times\frac{10}{100}} = \frac{18}{88} = \frac{9}{44}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.* **[NCERT, CBSE 2005, 2011, 2014, 2017]**

SOLUTION Let E_1, E_2 and A be the events defined as follows:

E_1 = Six occurs, E_2 = Six does not occur, and A = The man reports that it is a six.

Clearly, $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$.

Now, $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die

= Probability the man speaks truth = 3/4

and, $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die

= Probability that the man does not speak truth $= 1 - \frac{3}{4} = \frac{1}{4}$

We have to find $P(E_1/A)$ i.e., the probability that there is six on the die given that the man has reported that there is six.

By Baye's theorem

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{3}{8}$$

EXAMPLE 16 *In a test, an examinee either guesses or copies or knows the answer to a multiple choice question with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copied it, is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.* **[NCERT]**

SOLUTION Let E_1, E_2, E_3 and A be the events defined as follows:

E_1 = Examinee guesses the answer, E_2 = Examinee copies the answer, E_3 = Examinee knows the answer, and A = Examinee answers correctly.

Clearly, $P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{1}{6}$. Since E_1, E_2, E_3 are mutually exclusive and exhaustive events.

$$\therefore \quad P(E_1) + P(E_2) + P(E_3) = 1 \Rightarrow P(E_3) = 1 - (P(E_1) + P(E_2)) = 1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}.$$

If E_1 has already occurred, the examinee guesses, then there are four choices out of which only one is correct. Therefore, the probability that he answers correctly given that he has made a guess is $1/4$ i.e. $P(A/E_1) = 1/4$. It is given that $P(A/E_2) = 1/8$.

and, $\quad P(A/E_3)$ = Probability that he answers correctly given that he knew the answer $= 1$

By Baye's Theorem

$$\text{Required probability} = P(E_3/A)$$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}$$

EXAMPLE 17 *A letter is known to have come either from TATANAGAR or CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter has come from (i) Calcutta (ii) Tatanagar?* **[NCERT EXEMPLAR]**

SOLUTION Let E_1 be the event that the letter came from Calcutta and E_2 be the event that the letter came from Tatanagar. Let A denote the event that two consecutive letters visible on the envelope are TA.

Since the letters have come either from Calcutta or Tatanagar.

$$\therefore \quad P(E_1) = \frac{1}{2} = P(E_2)$$

If E_1 has occurred, then it means that the letter came from Calcutta. In the word CALCUTTA there are 8 letters in which TA occurs in the end. Considering TA as one letter there are seven letters out of which one can be in 7 ways.

$$\therefore \quad P(A/E_1) = \frac{1}{7}$$

If E_2 has occurred, then the letter came from Tatanagar. In the word TATANAGAR there are 9 letters in which TA occurs twice. Considering one of the two TA's as one letter there are 8 letters.

$$\therefore \quad P(A/E_2) = \frac{2}{8}$$

We have to find $P(E_1/A)$ and $P(E_2/A)$.

(i) $$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{1}{7}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{4}{11}$$

(ii) $$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{1}{2}\times\frac{2}{8}}{\frac{1}{2}\times\frac{1}{7}+\frac{1}{2}\times\frac{2}{8}} = \frac{7}{11}$$

EXAMPLE 18 *A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively* $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ *and* $\frac{2}{5}$. *The probability that he will be late are* $\frac{1}{4}, \frac{1}{3}$ *and* $\frac{1}{12}$, *if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?* **[NCERT]**

SOLUTION Let E_1, E_2, E_3, E_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively. It is given that

$$P(E_1) = \frac{3}{10}, P(E_2) = \frac{1}{5}, P(E_3) = \frac{1}{10} \text{ and } (P(E_4) = \frac{2}{5}$$

Let A denote the event that the doctor visits the patient late. It is given that

$P(A/E_1)$ = Probability that the doctor will be late if he comes by train = $\frac{1}{4}$

$P(A/E_2)$ = Probability that the doctor will be late if he comes by bus = $\frac{1}{3}$

$P(A/E_3)$ = Probability that the doctor will be late if he comes by scootor = $\frac{1}{12}$

and, $P(A/E_4)$ = Probability that the doctor will be late if he comes by other means of transport = 0

We have to find $P(E_1/A)$.

By using Baye's theorem, we obtain

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3) + P(E_4)\,P(A/E_4)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, required probability is $\frac{1}{2}$.

EXAMPLE 19 *Let d_1, d_2, d_3 be three mutually exclusive diseases. Let $S = \{s_1, s_2, s_3 ..., s_6\}$ be the set of observable symptoms of these diseases. For example, s_1 is the shortness of breath, s_2 is loss of weight, s_3 is fatigue etc. Suppose a random sample of 10,000 patients contains 3200 patients with disease d_1, 3500 with disease d_2 and 3300 with disease d_3. Also, 3100 patients with disease d_1, 3300 with disease d_2 and 3000 with disease d_3 show the symptom S. Knowing that the patient has symptom S, the doctor wishes to determine the patient's illness. On the basis of this information, what should the doctor conclude?*

SOLUTION Let E_i denote the event that the patient has disease d_i ; $i = 1, 2, 3$ and A be the event that the patient has symptom S. Then,

$$P(E_1) = \frac{3200}{10000} = \frac{32}{100}, P(E_2) = \frac{3500}{10000} = \frac{35}{100}, P(E_3) = \frac{3300}{10000} = \frac{33}{100}$$

$$P(A \cap E_1) = \frac{3100}{10000} = \frac{31}{100}, P(A \cap E_2) = \frac{3300}{10000} = \frac{33}{100} \text{ and, } P(A \cap E_3) = \frac{30000}{10000} = \frac{30}{100}$$

$$\therefore \quad P(A/E_1) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{31/100}{32/100} = \frac{31}{32}, \quad P(A/E_2) = \frac{P(A \cap E_2)}{P(E_2)} = \frac{33/100}{35/100} = \frac{33}{35}$$

$$P(A/E_3) = \frac{P(A \cap E_3)}{P(E_3)} = \frac{30/100}{33/100} = \frac{30}{33}$$

Using Baye's theorem, we obtain

$$P(E_1/A) = \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_1/A) = \frac{\frac{32}{100} \times \frac{31}{32}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{31}{94}$$

$$P(E_2/A) = \frac{P(E_2)\, P(A/E_2)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_2/A) = \frac{\frac{35}{100} \times \frac{33}{35}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{33}{94}$$

and,

$$P(E_3/A) = \frac{P(E_3)\, (A/E_3)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + P(E_3)\, P(A/E_3)}$$

$$\Rightarrow \quad P(E_3/A) = \frac{\frac{33}{100} \times \frac{30}{33}}{\frac{32}{100} \times \frac{31}{32} + \frac{35}{100} \times \frac{33}{35} + \frac{33}{100} \times \frac{30}{33}} = \frac{30}{94}$$

Clearly, $P(E_3/A) < P(E_1/A) < P(E_2/A)$ i.e. $P(E_2/A)$ is largest.

Thus, the doctor should conclude that the patient is most likely to have disease d_2.

EXAMPLE 20 *Suppose that the reliability of a HIV test is specified as follows:*

Of people having HIV, 90% of the test detect the disease but 10% go undetected. Of people free of HIV, 99% of the test are judged HIV–ive but 1% are diagnosed as showning HIV+ive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV +ive. What is the probability that the person actually has HIV? **[NCERT]**

SOLUTION Consider the following events:

E_1 = The person selected is actually having HIV

E_2 = The person selected is not having HIV

A = The person's HIV test is diagnosed as positive

We have, $P(E_1) = 0.1\% = \frac{0.1}{100} = 0.001$, $P(E_2) = 1 - P(E_1) = 1 - 0.001 = 0.999$

$P(A/E_1)$ = Probability that the person tested as HIV +ive given that he/she is actually having HIV.

$$= \frac{90}{100} = 0.9$$

and,

$P(A/E_2)$ = Probability that the person tested as HIV +ive given that he/she is actually not having HIV

$$= \frac{1}{100} = 0.01$$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{90}{1089}$$

EXAMPLE 21 *Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?* **[NCERT EXEMPLAR]**

SOLUTION Consider the following events:

E_1 = Taking out a fair coin from the pocket.

E_2 =Taking out a two-headed coin from the pocket

A =Getting a head on the coin when it is tossed.

We have, $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$, $P(A/E_1) = \frac{1}{2}$ and $P(A/E_2) = 1$

By using Baye's theorem, we obtain

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times 1} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

EXAMPLE 22 *Three bags contain a number of red and white balls are as follows:*

Bag I : 3 red balls; Bag II : 2 red balls and 1 white ball; Bag III : 3 white balls

The probability that bag i will be chosen and a ball is selected from it is $\frac{i}{6}$, $i = 1, 2, 3$. If a white ball is selected, what is the probability that it came from (i) Bag II (ii) Bag III **[NCERT EXEMPLAR]**

SOLUTION Consider the following events:

E_1 = Bag I is selected, E_2 = Bag II is selected, E_3 = Bag III is selected

We have,

$P(E_i) = \frac{i}{6}$, $i = 1, 2, 3$

$P(A/E_1)$ = Probability of drawing a white ball when bag I is selected = 0

$P(A/E_2) = \frac{1}{3}$ and $P(A/E_3) = \frac{3}{3} = 1$

(i) $$\text{Required probability} = P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{2}{6}\times\frac{1}{3}}{\frac{1}{6}\times 0 + \frac{2}{6}\times\frac{1}{3} + \frac{3}{6}\times 1} = \frac{1/9}{11/18} = \frac{2}{11}$$

(ii) $$\text{Required probability} = P(E_3/A) = \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{3}{6} \times 1}{\frac{1}{6} \times 0 + \frac{2}{6} \times \frac{1}{3} + \frac{3}{6} \times 1} = \frac{1/2}{11/18} = \frac{9}{11}$$

EXAMPLE 23 *A shopkeeper sells three types of seeds A_1, A_2 and A_3. They are sold as a mixture where the proportions are 4 : 4 : 2 respectively. The germination rates of three types of seeds are 45%, 60% and 35%. Calculate the probability*

(i) *that it will not germinate given that the seed is of type A_3*

(ii) *of a randomly chosen seed to germinate.*

(iii) *that it is of type A_2 given that a randomly chosen seed does not germinate.*

SOLUTION Consider the following events: **[NCERT EXEMPLAR]**

E_i = Seed chosen is of type A_i, $i = 1, 2, 3$

A = Seed chosen garminates.

E_3 = Bag III is selected

We have,

$$P(E_1) = \frac{4}{10}, P(E_2) = \frac{4}{10} \text{ and } P(E_3) = \frac{2}{10}$$

$$P(A/E_1) = \frac{45}{100}, P(A/E_2) = \frac{60}{100}, P(A/E_3) = \frac{35}{100}$$

(i) Required probability $= P(\overline{A}/E_3) = 1 - P(A/E_3) = 1 - \frac{35}{100} = \frac{65}{100} = 0.65$

(ii) Required probability $= P(A) = P(E_1) P(A/E_1) + P(E_2) P(A/E_2) + P(E_3) P(A/E_3)$

$$= \frac{4}{10} \times \frac{45}{100} + \frac{4}{10} \times \frac{60}{100} + \frac{2}{10} \times \frac{35}{100} = \frac{490}{1000} = 0.49$$

(iii) Required probability $= P(E_2/\overline{A})$

$$= \frac{P(E_2 \cap \overline{A})}{P(\overline{A})} = \frac{P(E_2)\, P(\overline{A}/E_2)}{P(\overline{A})} = \frac{P(E_2)\,(1 - P(A/E_2))}{1 - P(A)}$$

$$= \frac{\frac{4}{10} \times \left(1 - \frac{60}{100}\right)}{1 - \frac{49}{100}} = \frac{\frac{4}{10} \times \frac{40}{100}}{\frac{51}{100}} = \frac{16}{51}$$

EXAMPLE 24 *There are two boxes, namely, box-I and box-II. Box-I contains 3 red and 6 black balls. Box-II contains 5 red and n black balls. One of the two boxes, box-I and box-II is selected at random and a ball is drawn at random. The ball drawn is found to be red. If the probability that this red ball comes out from box-II is 1/3, find the value of n.*

SOLUTION Consider the following events.

E_1 = Selecting box-I, E_2 = Selecting box-II, A = Getting a red ball from selected box.

Clearly, $P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$, $P(A/E_1) = \frac{3}{9} = \frac{1}{3}$, $P(A/E_2) = \frac{5}{n+5}$

It is given that $P(E_2/A) = \frac{1}{3}$

Using Baye's theorem, we obtain: $P(E_2/A) = \dfrac{P(E_2)\, P(A/E_2)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)}$

$$\Rightarrow \quad \frac{1}{3} = \frac{\frac{1}{2} \times \frac{5}{n+5}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{5}{n+5}} \Rightarrow \frac{1}{3} = \frac{15}{n+20} \Rightarrow n + 20 = 45 \Rightarrow n = 25$$

EXERCISE 30.7

BASIC

1. The contents of urns I, II, III are as follows:
Urn I : 1 white, 2 black and 3 red balls
Urn II : 2 white, 1 black and 1 red balls
Urn III : 4 white, 5 black and 3 red balls.
One urn is chosen at random and two balls are drawn. They happen to be white and red. What is the probability that they come from Urns I, II, III? **[CBSE 2003]**
2. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag B. **[CBSE 2007, 2010, 2011]**
3. Three urns contain 2 white and 3 black balls; 3 white and 2 black balls and 4 white and 1 black ball respectively. One ball is drawn from an urn chosen at random and it was found to be white. Find the probability that it was drawn from the first urn. **[NCERT EXEMPLAR]**
4. The contents of three urns are as follows:
Urn 1 : 7 white, 3 black balls, Urn 2 : 4 white, 6 black balls, and Urn 3 : 2 white, 8 black balls. One of these urns is chosen at random with probabilities 0.20, 0.60 and 0.20 respectively. From the chosen urn two balls are drawn at random without replacement. If both these balls are white, what is the probability that these came from urn 3?
5. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die? **[CBSE 2015]**
6. Two groups are competing for the positions of the Board of Directors of a Corporation. The probabilities that the first and the second groups will win are 0.6 and 0.4 respectively. Further, if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group. **[NCERT, CBSE 2009]**
7. (i) Suppose 5 men out of 100 and 25 women out of 1000 are good orators. An orator is chosen at random. Find the probability that a male person is selected. Assume that there are equal number of men and women.
(ii) A card from a pack of 52 playing cards is lost. From the remaining cards, 2 cards are drawn at random without replacement, and are found to be both aces. Find the probability that lost card being an ace. **[CBSE 2022]**

BASED ON LOTS

8. A letter is known to have come either from LONDON or CLIFTON. On the envelope just two consecutive letters ON are visible. What is the probability that the letter has come from (i) LONDON (ii) CLIFTON?
9. In a class, 5% of the boys and 10% of the girls have an IQ of more than 150. In this class, 60% of the students are boys. If a student is selected at random and found to have an IQ of more than 150, find the probability that the student is a boy.
10. A factory has three machines X, Y and Z producing 1000, 2000 and 3000 bolts per day respectively. The machine X produces 1% defective bolts, Y produces 1.5% and Z produces 2% defective bolts. At the end of a day, a bolt is drawn at random and is found to be defective. What is the probability that this defective bolt has been produced by machine X? **[CBSE 2002]**
11. An insurance company insured 3000 scooters, 4000 cars and 5000 trucks. The probabilities of the accident involving a scooter, a car and a truck are 0.02, 0.03 and 0.04 respectively. One of the insured vehicles meet with an accident. Find the probability that it is a (a) scooter (ii) car (iii) truck. **[NCERT, CBSE 2001C]**

12. Suppose we have four boxes A, B, C, D containing coloured marbles as given below:

Marble Colour / *Box*	*Red*	*White*	*Black*
A	1	6	3
B	6	2	2
C	8	1	1
D	0	6	4

One of the boxes has been selected at random and a single marble is drawn from it. If the marble is red, what is the probability that it was drawn from box A? box B? box C? **[NCERT]**

13. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job for 30% of the time and C on the job for 20% of the time. A defective item is produced. What is the probability that it was produced by A? **[NCERT, CBSE 2019]**

14. An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufacture on machine A, 30% on B and 20% on C. 2% of the items produced on A and 2% of items produced on B are defective and 3% of these produced on C are defective. All the items stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A? **[NCERT EXEMPLAR]**

15. There are three coins. One is two-headed coin (having head on both faces), another is biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tail 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin? **[CBSE 2014]**

16. In a factory, machine A produces 30% of the total output, machine B produces 25% and the machine C produces the remaining output. If defective items produced by machines A, B and C are 1%, 1.2%, 2% respectively. Three machines working together produce 10000 items in a day. An item is drawn at random from a day's output and found to be defective. Find the probability that it was produced by machine B?

17. A company has two plants to manufacture bicycles. The first plant manufactures 60% of the bicycles and the second plant 40%. Out of that 80% of the bicycles are rated of standard quality at the first plant and 90% of standard quality at the second plant. A bicycle is picked up at random and found to be standard quality. Find the probability that it comes from the second plant. **[CBSE 2003]**

18. Three urns A, B and C contain 6 red and 4 white; 2 red and 6 white; and 1 red and 5 white balls respectively. An urn is chosen at random and a ball is drawn. If the ball drawn is found to be red, find the probability that the ball was drawn from urn A. **[CBSE 2004]**

19. In a group of 400 people, 160 are smokers and non-vegetarian, 100 are smokers and vegetarian and the remaining are non-smokers and vegetarian. The probabilities of getting a special chest disease are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the disease. What is the probability that the selected person is a smoker and non-vegetarian? **[CBSE 2013]**

20. A factory has three machines A, B and C, which produce 100, 200 and 300 items of a particular type daily. The machines produce 2%, 3% and 5% defective items respectively. One day when the production was over, an item was picked up randomly and it was found to be defective. Find the probability that it was produced by machine A. **[CBSE 2004]**

21. A bag contains 1 white and 6 red balls, and a second bag contains 4 white and 3 red balls. One of the bags is picked up at random and a ball is randomly drawn from it, and is found to be white in colour. Find the probability that the drawn ball was from the first bag. **[CBSE 2005]**

22. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Further more, 60% of the students in the colleges are girls. A student selected at random from the college is found to be taller than 1.75 metres. Find the probability that the selected students is girl. [CBSE 2012]

23. For A, B and C the chances of being selected as the manager of a firm are in the ratio 4:1:2 respectively. The respective probabilities for them to introduce a radical change in marketing strategy are 0.3, 0.8 and 0.5. If the change does take place, find the probability that it is due to the appointment of B or C. [CBSE 2005]

24. Three persons A, B and C apply for a job of Manager in a private company. Chances of their selections (A, B and C) are in the ratio 1 : 2 : 4. The probabilities that A, B and C can introduce changes to improve profits of the company are 0.8, 0.5 and 0.3 respectively. If the changes do not take place, find the probability that it is due to the appointment of C. [CBSE 2016]

25. An insurance company insured 2000 scooters and 3000 motorcycles. The probability of an accident involving a scooter is 0.01 and that of a motorcycle is 0.02. An insured vehicle met with an accident. Find the probability that the accidented vehicle was a motorcycle. [CBSE 2005]

26. Of the students in a college, it is known that 60% reside in a hostel and 40% do not reside in hostel. Previous year results report that 30% of students residing in hostel attain A grade and 20% of ones not residing in hostel attain A grade in their annual examination. At the end of the year, one students is chosen at random from the college and he has an A grade. What is the probability that the selected student is a hosteler? [CBSE 2011, 2012]

27. There are three coins. One is two headed coin, another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is choosen at random and tossed, it shows heads, what is the probability that it was the two headed coin? [NCERT, CBSE 2009]

28. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options and patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga? [NCERT, CBSE 2013]

29. Coloured balls are distributed in four boxes as shown in the following table:

Box	*Colour*			
	Black	*White*	*Red*	*Blue*
I	3	4	5	6
II	2	2	2	2
III	1	2	3	1
IV	4	3	1	5

A box is selected at random and then a ball is randomly drawn from the selected box. The colour of the ball is black, what is the probability that ball drawn is from the box III.

30. If a machine is correctly set up it produces 90% acceptable items. If it is incorrectly set up it produces only 40% acceptable items. Past experience shows that 80% of the setups are correctly done. If after a certain set up, the machine produces 2 acceptable items, find the probability that the machine is correctly set up. [NCERT]

31. Bag A contains 3 red and 5 black balls, white bag B contains 4 red and 4 black balls. Two balls are transferred at random from bag A to bag B and then a ball is drawn from bag B at random. If the ball drawn from bag B is found to be red, find the probability that two red balls were transferred from bag A to bag B. [CBSE 2016]

BASED ON HOTS

32. By examining the chest X-ray, probability that T.B is detected when a person is actually suffering is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.001. In a certain city 1 in 1000 persons suffers from T.B. A person is selected at random is diagnosed to have T.B. What is the chance that he actually has T.B.? [NCERT EXEMPLAR]

33. A test for detection of a particular disease is not fool proof. The test will correctly detect the disease 90% of the time, but will incorrectly detect the disease 1% of the time. For a large population of which an estimated 0.2% have the disease, a person is selected at random, given the test, and told that he has the disease. What are the chances that the person actually have the disease?

34. Let d_1, d_2, d_3 be three mutually exclusive diseases. Let S be the set of observable symptoms of these diseases. A doctor has the following information from a random sample of 5000 patients: 1800 had disease d_1, 2100 has disease d_2 and the others had disease d_3.
1500 patients with disease d_1, 1200 patients with disease d_2 and 900 patients with disease d_3 showed the symptom. Which of the diseases is the patient most likely to have?

35. A is known to speak truth 3 times out of 5 times. He throws a die and reports that it is one. Find the probability that it is actually one. [CBSE 2004]

36. (i) A speaks the truth 8 times out of 10 times. A die is tossed. He reports that it was 5. What is the probability that it was actually 5? [CBSE 2005]
(ii) A man is known to speak truth 7 out of 10 times. He throws a pair of dice and reports that doublet appeared. Find the probability that it was actually a doublet. [CBSE 2022]

37. In answering a question on a multiple choice test a student either knows the answer or guesses. Let $3/4$ be the probability that he knows the answer and $1/4$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $1/4$. What is the probability that a student knows the answer given that he answered it correctly?

38. A laboratory blood test is 99% effective in detecting a certain disease when its infection is present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then, with probability 0.005, the test will imply he has the disease). If 0.1% of the population actually has the disease, what is the probability that a person has the disease given that his test result is positive? [NCERT]

39. There are three categories of students in a class of 60 students:
A : Very hardworking ; B : Regular but not so hardworking; C : Careless and irregular
10 students are in category A, 30 in category B and rest in category C. It is found that the probability of students of category A, unable to get good marks in the final year examination is 0.002, of category B it is 0.02 and of category C, this probability is 0.20. A student selected at random was found to be one who could not get good marks in the examination. Find the probability that this student is of category C. [CBSE 2017]

ANSWERS

1. $\frac{33}{118}, \frac{55}{118}, \frac{30}{118}$ 2. $\frac{25}{52}$ 3. $\frac{2}{9}$ 4. $\frac{1}{40}$ 5. $\frac{8}{11}$

6. $\frac{2}{9}$ 7. (i) $\frac{2}{3}$ (ii) $\frac{1}{25}$ 8. (i) $\frac{12}{17}$ (ii) $\frac{5}{17}$ 9. $\frac{3}{7}$

10. 0.1 11. (i) $\frac{3}{19}$ (ii) $\frac{6}{19}$ (iii) $\frac{10}{19}$ 12. $\frac{1}{15}, \frac{2}{5}, \frac{8}{15}$ 13. $\frac{5}{34}$

14. $\frac{5}{11}$ 15. $\frac{20}{47}$ 16. 0.2 17. $\frac{3}{7}$ 18. $\frac{36}{61}$

19. $\frac{28}{45}$ 20. $\frac{2}{23}$ 21. $\frac{1}{5}$ 22. $\frac{3}{11}$ 23. $\frac{3}{5}$

24. $\frac{7}{10}$ 25. $\frac{3}{4}$ 26. $\frac{9}{13}$ 27. $\frac{4}{9}$ 28. $\frac{14}{29}$ 29. $\frac{156}{947}$

30. $\frac{81}{85}$ 31. $\frac{18}{133}$ 32. $\frac{110}{221}$ 33. $\frac{90}{589}$ 34. d_1 35. $\frac{3}{13}$

36. $\frac{4}{9}$ 37. (i) $\frac{12}{13}$ (ii) $\frac{7}{22}$ 38. $\frac{22}{133}$ 39. $\frac{20}{231}$

HINTS TO SELECTED PROBLEMS

1. Let us define the following events E_1 = Urn I is chosen, E_2 = Urn II is chosen, E_3 = Urn III is chosen, and A = Two balls drawn are white and red.

Clearly, $P(E_1) = 1/3 = P(E_2) = P(E_3)$,

$$P(A/E_1) = \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2},\ P(A/E_2) = \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} \text{ and, } P(A/E_3) = \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2}$$

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

2. Let E_1 = Bag A is chosen, E_2 = Bag B is chosen, and A = Ball drawn is red.

$P(E_1) = P(E_2) = 1/2,\ P(A/E_1) = 3/5$ and $P(A/E_2) = 5/9$.

5. Consider the following events:

E_1 = Getting 1 or 2 in a throw of a die, E_2 = Getting 3, 4, 5 or 6 in a throw of a die

A = Getting exactly one tail.

Clearly, $P(E_1) = \frac{2}{6} = \frac{1}{3},\ P(E_2) = \frac{4}{6} = \frac{2}{3},\ P(A/E_1) = \frac{3}{8},\ P(A/E_2) = \frac{1}{2}$

$$\text{Required probability} = P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{2}} = \frac{8}{11}$$

6. Consider the following events:

E_1 = First group wins, E_2 = Second group wins, A = New product is introduced.

It is given that

$P(E_1) = 0.6,\ P(E_2) = 0.4,\ P(A/E_1) = 0.7,\ P(A/E_2) = 0.3$

$$\text{Required probability} = P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{0.4 \times 0.3}{0.6 \times 0.7 + 0.4 \times 0.3} = \frac{12}{54} = \frac{2}{9}$$

7. Let E_1 = Person chosen is a man, E_2 = Person chosen is a woman, A = Person is a good orator.

$$P(E_1) = P(E_2) = \frac{1}{2},\ P(A/E_1) = \frac{5}{100} \text{ and } P(A/E_2) = \frac{25}{1000}$$

Now, find $P(E_1 A)$ by using Bay's theorem.

12. Let E_1, E_2, E_3, E_4 and A be the events as defined below.

E_1 = Box A is selected, E_2 = Box B is selected, E_3 = Box C is selected, E_4 = Box D is selected.

A = Marble drawn is of red colour.

Since there are four boxes and one of them is selected at random. Therefore,

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}.$$

Clearly, $P(A/E_1)=\frac{1}{10}$, $P(A/E_2)=\frac{6}{10}$, $P(A/E_3)=\frac{8}{10}$, and $P(A/E_4)=\frac{0}{10}$

$$\therefore\ P(E_1/A)=\frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)+P(E_4)\,P(A/E_4)}$$

$$=\frac{\frac{1}{4}\times\frac{1}{10}}{\frac{1}{4}\times\frac{1}{10}+\frac{1}{4}\times\frac{6}{10}+\frac{1}{4}\times\frac{8}{10}+\frac{1}{4}\times\frac{0}{10}}=\frac{1}{15}$$

$$P(E_2/A)=\frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)+P(E_4)\,P(A/E_4)}$$

$$=\frac{\frac{1}{4}\times\frac{6}{10}}{\frac{1}{4}\times\frac{1}{10}+\frac{1}{4}\times\frac{6}{10}+\frac{1}{4}\times\frac{8}{10}+\frac{1}{4}\times\frac{0}{10}}=\frac{6}{15}=\frac{2}{5}$$

$$P(E_3/A)=\frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)+P(E_4)\,P(A/E_4)}=\frac{8}{15}$$

13. Let E_1, E_2, E_3 and A be the following events:

E_1 = Operator A performs the job, E_2 = Operator B performs the job,

E_3 = Operator C performs the job, A = Item produced is defective

Clearly, $P(E_1)=\frac{50}{100}=\frac{50}{10}$, $P(E_2)=\frac{30}{100}=\frac{30}{10}$, $P(E_3)=\frac{20}{100}=\frac{2}{10}$

$P(A/E_1)=\frac{1}{100}$, $P(A/E_2)=\frac{5}{100}$ and $P(A/E_3)=\frac{7}{100}$

Required probability $=P(E_1/A)=\frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)}$

$$=\frac{\frac{5}{10}\times\frac{1}{100}}{\frac{5}{10}\times\frac{1}{100}+\frac{3}{10}\times\frac{5}{100}+\frac{2}{10}\times\frac{7}{100}}=\frac{5}{5+15+14}=\frac{5}{34}$$

16. Let E_1, E_2, E_3 be the events as defined below:

E_1 = Item is produced by machine A, E_2 = Item is produced by machine B

E_3 = Item is produced by machine C and, A = The event that the item is defective.

It is given that, $P(E_1)=\frac{30}{100}$, $P(E_2)=\frac{25}{100}$, $P(E_3)=\frac{45}{100}$,

$P(A/E_1)=\frac{1}{100}$, $P(A/E_2)=\frac{1.2}{100}$ and $P(A/E_3)=\frac{2}{100}$.

Required probability $=P(E_2/A)=\frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)+P(E_3)\,P(A/E_3)}$.

27. Consider the following events:

E_1 = Selecting two headed coin, E_2 = Selecting biased coin,

E_3 = Selecting unbiased coin, A = Getting head on the coin.

Clearly, $P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}$, $P(A/E_1)=1$, $P(A/E_2)=\frac{75}{100}=\frac{3}{4}$, $P(A/E_3)=\frac{1}{2}$

$\therefore$ Required probability $=P(E_1/A)=\frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$

28. Consider the following events:

E_1 = The patient follows a course of meditation and yoga,

E_2 = The patient takes a certain drug, A = The patient suffers a heart attack.

Clearly,

$$P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2},\ P(A/E_1) = \frac{70}{100} \times \frac{40}{100} \text{ and } P(A/E_2) = \frac{75}{100} \times \frac{40}{100}$$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{70}{100} \times \frac{40}{100}}{\frac{1}{2} \times \frac{70}{100} \times \frac{40}{100} + \frac{1}{2} \times \frac{75}{100} \times \frac{40}{100}} = \frac{14}{29}$$

30. Let A be the event that the machine produces 2 acceptable items. Let E_1 represent the event of correct setup and E_2 represent the event of incorrect setup.

It is given that

$$P(E_1) = 0.8,\ P(E_2) = 0.2,\ P(A/E_1) = 0.9 \times 0.9,\ P(A/E_2) = 0.4 \times 0.4$$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

32. Let E_1 = The person selected is suffering from T.B

E_2 = The person selected is not suffering from T.B, A = The doctor diagnoses correctly.

Then, $P(E_1) = 1/1000$, $P(E_2) = 999/1000$, $P(A/E_1) = 0.99$ and $P(A/E_2) = 0.001$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

33. Let E_1, E_2 and A be the following events:

E_1 = The person selected has disease, E_2 = The person selected does not have disease,

A = Test is positive.

It is given that $P(E_1) = 0.002$, $P(E_2) = 0.998$, $P(A/E_1) = 0.90$, $P(A/E_2) = 0.01$.

$$\therefore \text{ Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

34. Let E_i denote the event that the patient has disease d_i ; i = 1, 2, 3 and A be the event that the patient showed the symptom S.

$$\text{Clearly, } P(E_1) = \frac{1800}{5000},\ P(E_2) = \frac{2100}{5000},\ P(E_3) = \frac{1100}{5000}$$

$$P(A/E_1) = \frac{P(A \cap E_1)}{P(E_1)} = \frac{1500/5000}{1800/5000} = \frac{15}{18},\ P(A/E_2) = \frac{12}{21} \text{ and } P(A/F_3) = \frac{9}{11}.$$

Compute $P(E_1/A)$, $P(E_2/A)$ and $P(E_3/A)$ and find the greatest of these.

37. Consider the following events:

E_1 = Student knows the answer, E_2 = Student guesses the answer,

A = Student answers correctly.

$$\text{Clearly, } P(E_1) = \frac{3}{4},\ P(E_2) = \frac{1}{4},\ P(A/E_2) = \frac{1}{4},\ P(A/E_1) = 1$$

$$\text{Required probability} = P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)} = \frac{\frac{3}{4} \times 1}{\frac{3}{4} \times 1 + \frac{1}{4} \times \frac{1}{4}} = \frac{12}{13}$$

38. Consider the following events:

E_1 = Person selected has the disease, E_2 = Person selected does not have the disease,

A = Test result is positive

Clearly, $P(E_1)=\frac{1}{1000}$, $P(E_2)=\frac{999}{1000}$, $P(A/E_1)=\frac{99}{100}$ and $P(A/E_2)=\frac{5}{1000}$

Required probability $= P(E_1/A)=\frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1)+P(E_2)\,P(A/E_2)}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If A and $\overline{B}$ are independent events, then $P(\overline{A}\cup B)=1-x$, where $x=$ ....
2. If A and B are independent events such that $P(A)=p$, $P(B)=2p$ and P(Exactly one of $A, B)=\frac{5}{9}$, then $p=$ ....
3. If A and B are two events such that $P(\overline{A}\cup\overline{B})=\frac{2}{3}$ and $P(A\cup B)=\frac{5}{9}$, then $P(\overline{A})+P(\overline{B})=$ ....
4. If A and B are two events such that $P(A/B)=p$, $P(A)=p$, $P(B)=\frac{1}{3}$ and $P(A\cup B)=\frac{5}{9}$, then $p=$ ....
5. Let A and B be two events. If $P(A/B)=P(A)$, then A is .... of B.
6. Let A and B be two events such that $P(A)\neq 0$, $P(B)\neq 1$ and $P(\overline{A}/\overline{B})=\frac{1-k}{P(\overline{B})}$, then $k=$.....
7. If two events A and B are mutually exclusive, then $P(A/B)=$.......................... .
8. If A and B are two events such that $A\subseteq B$, then $P(B/A)=$ ....
9. If $4\,P(A)=6\,P(B)=10\,P(A\cap B)=1$, then $P(B/A)=$.......................... .
10. If A and B are two events, then $P(\overline{A}\cap B)=$ ....
11. If A and B are two events, then the probability of occurrence of A only is equal to ....
12. If A and B are two events, then the probability of occurrence of exactly one of A and B is equal to ....
13. For two event A and B, if $P(A)=P(A/B)=\frac{1}{4}$ and $P(B/A)=\frac{1}{2}$, then A and B are .... events.
14. Let A and B be two events for which $P(A)=a$, $P(B)=b$, $P(A\cap B)=c$, then $P(\overline{A}\cap B)=$ ....
15. Let A, B, C be pairwise independent events with $P(C)>0$ and $P(A\cap B\cap C)=0$. If $P((\overline{A}\cap\overline{B})/C)=1-x$, then $x=$.......................... .
16. Let A, B, C be three events such that $P(A\cap B\cap C)=0$, P (Exactly one of A and B occurs) $=x$, P (exactly one of B and C occurs) $=y$, P(Exactly one of A and C occurs) $=z$. Then $P(A\cup B\cup C)=$ ....
17. If A and B are two events such that $P(A\cup B)=(A\cap B)$, then P (exactly one of A and B occurs) $=$ ....

ANSWERS

1. $P(A)\,P(\overline{B})$
2. $\frac{1}{3}, \frac{5}{12}$
3. $\frac{10}{9}$
4. $\frac{1}{3}$
5. independent
6. $P(A\cup B)$
7. 0
8. 1
9. $\frac{2}{5}$
10. $P(B)-P(A\cap B)$
11. $P(A)-P(A\cap B)$
12. $P(A)+P(B)-2P(A\cap B)$
13. independent

14. $b-c$ **15.** $P(A)+P(B)$ **16.** $\frac{x+y+z}{2}$ **17.** 0

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. A four digit number is formed using the digits 1, 2, 3, 5 with no repetitions. Write the probability that the number is divisible by 5.
2. When three dice are thrown, write the probability of getting 4 or 5 on each of the dice simultaneously.
3. Three digit numbers are formed with the digits 0, 2, 4, 6 and 8. Write the probability of forming a three digit number with the same digits.
4. A ordinary cube has four plane faces, one face marked 2 and another face marked 3, find the probability of getting a total of 7 in 5 throws.
5. Three numbers are chosen from 1 to 20. Find the probability that they are consecutive.
6. 6 boys and 6 girls sit in a row at random. Find the probability that all the girls sit together.
7. If A and B are two independent events such that $P(A)=0.3$ and $P(A \cup \overline{B})=0.8$. Find $P(B)$.
8. An unbiased die with face marked 1, 2, 3, 4, 5, 6 is rolled four times. Out of 4 face values obtained, find the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5.
9. If A and B are two events write the expression for the probability of occurrence of exactly one of two events.
10. Write the probability that a number selected at random from the set of first 100 natural numbers is a cube.
11. In a competition A, B and C are participating. The probability that A wins is twice that of B, the probability that B wins is twice that of C. Find the probability that A losses.
12. If A, B, C are mutually exclusive and exhaustive events associated to a random experiment, then write the value of $P(A)+P(B)+P(C)$.
13. If two events A and B are such that $P(\overline{A})=0.3$, $P(B)=0.4$ and $P(A \cap \overline{B})=0.5$, find $P(B/\overline{A} \cap \overline{B})$
14. If A and B are two independent events, then write $P(A \cap \overline{B})$ in terms of $P(A)$ and $P(B)$.
15. If $P(A)=0.3$, $P(B)=0.6$, $P(B/A)=0.5$, find $P(A \cup B)$.
16. If A, B and C are independent events such that $P(A)=P(B)=P(C)=p$, then find the probability of occurrence of at least two of A, B and C.
17. If A and B are independent events then write expression for P(exactly one of A, B occurs).
18. If A and B are independent events such that $P(A)=p$, $P(B)=2p$ and P (Exactly one of A and B occurs) $=\frac{5}{9}$, find the value of p.
19. Let A and B be two events such that $P(A)=\frac{5}{8}$, $P(B)=\frac{1}{2}$ and $P(A/B)=\frac{3}{4}$. Find the value of $P(B/A)$. **[CBSE 2022]**

ANSWERS

1. 1/4 **2.** 1/27 **3.** 1/25 **4.** $5/6^4$ **5.** $\frac{18}{{}^{20}C_3}$ **6.** 1/132 **7.** 2/7 **8.** 16/81
9. $P(A)+P(B)-2P(A \cap B)$ **10.** 1/25 **11.** 3/7 **12.** 1 **13.** 1/4 **15.** 0.75
16. $2p^2-3p^3$ **17.** $P(A)\,P(\overline{B})+P(B)\,P(\overline{A})$ **18.** $1/3, 5/12$ **19.** 3/5