CHAPTER 30

DERIVATIVES

30.1 DERIVATIVE AT A POINT

DEFINITION *Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$, iff*

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely.}$$

This limit is called the derivative or differentiation of $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\left\{\frac{d}{dx} f(x)\right\}_{x=c}$.

That is,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}, \text{ provided that the limit exists.}$$

Throughout this chapter it will be assumed that a given function $f(x)$ is differentiable at every point in its domain i.e. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists for all c in its domain.

$$\therefore \quad f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \quad f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \quad \text{or,} \quad f'(c) = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the derivative of $f(x) = k$ at $x = 0$ and $x = 5$.* **[NCERT]**

SOLUTION By definition

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

and, $$f'(5) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

EXAMPLE 2 *Find the derivative of* $\sin x$ *at* $x = 0$. **[NCERT]**

SOLUTION Let $f(x) = \sin x$. Then,

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$$

EXAMPLE 3 *Let f be a real valued function defined by $f(x) = x^2 + 1$. Find $f'(2)$.*

SOLUTION We have, $f(x) = x^2 + 1$

$$\therefore \quad f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{\{(2+h)^2 + 1\} - \{2^2 + 1\}}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{(h^2 + 4h + 5) - 5}{h} = \lim_{h \to 0} \frac{h^2 + 4h}{h} = \lim_{h \to 0} h + 4 = 4$$

EXAMPLE 4 *If f is a real valued function defined by* $f(x) = x^2 + 4x + 3$, *then find* $f'(1)$ *and* $f'(3)$.

SOLUTION We have, $f(x) = x^2 + 4x + 3$

$$\therefore \quad f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{h \to 0} \frac{\{(1+h)^2 + 4(1+h) + 3\} - \{1^2 + 4 \times 1 + 3\}}{h}$$

$$\Rightarrow \quad f'(1) = \lim_{h \to 0} \frac{(h^2 + 6h + 8) - 8}{h} = \lim_{h \to 0} \frac{h^2 + 6h}{h} = \lim_{h \to 0} h + 6 = 6$$

$$\text{and,} \quad f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h \to 0} \frac{\{(3+h)^2 + 4(3+h) + 3\} - \{3^2 + 4 \times 3 + 3\}}{h}$$

$$\Rightarrow \quad f'(3) = \lim_{h \to 0} \frac{(h^2 + 10h + 24) - 24}{h} = \lim_{h \to 0} \frac{h^2 + 10h}{h} = \lim_{h \to 0} h + 10 = 10$$

EXAMPLE 5 *Find the derivative of* $f(x) = 2x^2 + 3x - 5$ *at* $x = -1$. *Also, prove that*

$f'(0) + 3f'(-1) = 0.$ **[NCERT]**

SOLUTION Let us first find the derivatives of $f(x)$ at $x = 0$ and $x = -1$.

By definition

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{(2h^2 + 3h - 5) - \{2 \times (0)^2 + 3 \times (0) - 5\}}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{(2h^2 + 3h - 5) - (-5)}{h}$$

$$\Rightarrow \quad f'(0) = \lim_{h \to 0} \frac{2h^2 + 3h}{h} = \lim_{h \to 0} (2h + 3) = 2 \times 0 + 3 = 3$$

$$\text{and,} \quad f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}$$

$$\Rightarrow \quad f'(-1) = \lim_{h \to 0} \frac{\{2(-1+h)^2 + 3(-1+h) - 5\} - \{2(-1)^2 + 3(-1) - 5\}}{h}$$

$$\Rightarrow \quad f'(-1) = \lim_{h \to 0} \frac{2h^2 - h}{h} = \lim_{h \to 0} (2h - 1) = 2 \times 0 - 1 = -1$$

$$\therefore \quad f'(0) + 3f'(-1) = 3 + 3 \times -1 = 3 - 3 = 0$$

EXERCISE 30.1

LEVEL-1

1. Find the derivative of $f(x) = 3x$ at $x = 2$
2. Find the derivative of $f(x) = x^2 - 2$ at $x = 10$ **[NCERT]**
3. Find the derivative of $f(x) = 99x$ at $x = 100$ **[NCERT]**
4. Find the derivative of $f(x) = x$ at $x = 1$ **[NCERT]**
5. Find the derivative of $f(x) = \cos x$ at $x = 0$
6. Find the derivative of $f(x) = \tan x$ at $x = 0$
7. Find the derivatives of the following functions at the indicated points:

   (i) $\sin x$ at $x = \frac{\pi}{2}$ (ii) x at $x = 1$ (iii) $2\cos x$ at $x = \frac{\pi}{2}$ (iv) $\sin 2x$ at $x = \frac{\pi}{2}$

ANSWERS

1. 3 2. 20 3. 99 4. 1 5. 0 6. 1
7. (i) 0 (ii) 1 (iii) -2 (iv) -2

HINTS TO NCERT & SELECTED PROBLEMS

2. We have, $f(x) = x^2 - 2$.

$$\therefore \quad f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h} = \lim_{h \to 0} \frac{\{(10+h)^2 - 2\} - \{10^2 - 2\}}{h}$$

$$= \lim_{h \to 0} \frac{(10+h)^2 - 10^2}{h} = \lim_{h \to 0} \frac{(20+h)h}{h} = \lim_{h \to 0} (20+h) = 20$$

3. We have, $f(x) = 99x$

$$\therefore \quad f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$

$$= \lim_{h \to 0} \frac{99(100+h) - 99 \times 100}{h}$$

$$= 99 \lim_{h \to 0} \frac{100 + h - 100}{h} = 99 \times \lim_{h \to 0} 1 = 99$$

4. We have, $f(x) = x$

$$\therefore \quad f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h) - 1}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$$

30.1.1 PHYSICAL INTEGPRETATION OF DERIVATIVE AT A POINT

Let a particle be moving in a straight line OX starting from point O towards point X as shown in Fig. 30.1.

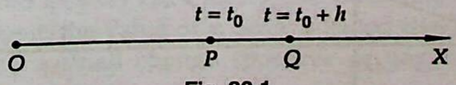

Fig. 30.1

Clearly, the position of the particle at any time t depends upon the time elapsed. In other words, the distance of the particle from O depends upon the time i.e. it is a function f of time t taken by the particle.

Let at any time t_0 i.e. at time $t = t_0$, the particle be at P and after a further time h i.e. at time $t = t_0 + h$, it is at Q.

$\therefore \quad OP = f(t_0)$ and $OQ = f(t_0 + h)$

Distance travelled in time $h = PQ = OQ - OP = f(t_0 + h) - f(t_0)$

Clearly, Average speed of the particle during the journey from P to $Q = \dfrac{PQ}{h} = \dfrac{f(t_0 + h) - f(t_0)}{h}$

As $h \to 0$, we observe that $Q \to P$.

$$\therefore \quad \text{(Instantaneous speed at time } t = t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} = f'(t_0)$$

Thus, if $f(t)$ gives the distance of a moving particle at time t, then $f'(t_0)$ i.e. the derivative of f at $t = t_0$ represents the instantaneous speed of the particle at time $t = t_0$ or, at the point P.

ILLUSTRATION *The distance $f(t)$ in metres moved by a particle travelling in a straight line in t seconds is given by $f(t) = t^2 + 3t + 4$. Find the speed of the particle at the end of 2 seconds.*

SOLUTION We have, $f(t) = t^2 + 3t + 4$.

The speed of the particle at the end of 2 seconds is given by $f'(2)$ i.e. the derivative of $f(t)$ at $t = 2$. Now,

$$f'(2) = \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{\{(2 + h)^2 + 3(2 + h) + 4\} - \{2^2 + 3 \times 2 + 4\}}{h}$$

$$\Rightarrow \quad f'(2) = \lim_{h \to 0} \frac{(h^2 + 7h + 14) - 14}{h} = \lim_{h \to 0} \frac{h^2 + 7h}{h} = \lim_{h \to 0} h + 7 = 7$$

Hence, the speed of the particle at the end of 2 seconds is 7 m/sec.

30.1.2 GEOMETRICAL INTERPETATION OF DERIVATIVE AT A POINT

Let $f(x)$ be a differentiable function. Consider the curve $y = f(x)$. Let $P(c, f(c))$ be a point on the curve $y = f(x)$ as shown in Fig. 30.2 and let $Q(c + h, f(c + h))$ be a neighbouring point on the curve $y = f(x)$. Then,

$$\text{Slope of chord } PQ = \tan \angle QPN = \frac{QN}{PN} = \frac{f(c + h) - f(c)}{h}$$

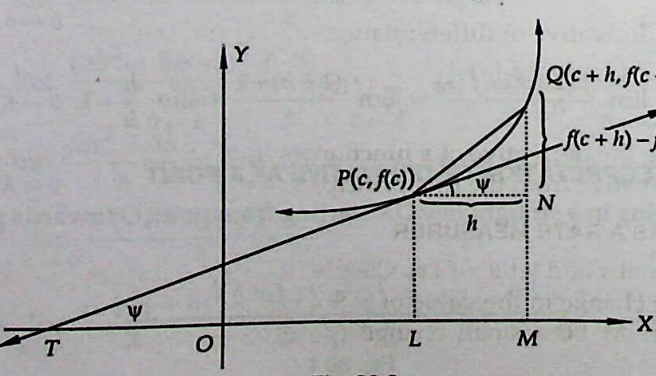

Fig. 30.2

Taking limit as $Q \to P$ i e. $h \to 0$, we obtain

$$\lim_{Q \to P} (\text{Slope of chord } PQ) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \quad \text{...(i)}$$

As $Q \to P$, chord PQ tends to the tangent to $y = f(x)$ at point P. Therefore, from (i), we get

$$\text{Slope of the tangent at } P = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$\Rightarrow$ Slope of the tangent at $P = f'(c)$ i.e, $\tan \psi = f'(c)$,

where ψ is the inclination of the tangent to the curve $y = f(x)$ at point $(c, f(c))$ with the x-axis.

Thus, the derivative of a function $f(x)$ at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.

ILLUSTRATION *Find the slope of the tangent to the curve $y = x^2$ at $(-1/2, 1/4)$.*

SOLUTION Let $f(x) = x^2$. Then, $y = f(x)$ is the given curve. Clearly, slope of the tangent to the curve at $(-1/2, 1/4)$ is equal to $f'(-1/2)$ i.e. the derivative of $f(x)$ at $x = -1/2$.

Now, $$f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{f\left(-\frac{1}{2} + h\right) - f\left(-\frac{1}{2}\right)}{h}$$

$$\Rightarrow \quad f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{\left(-\frac{1}{2} + h\right)^2 - \left(-\frac{1}{2}\right)^2}{h}$$

$$\Rightarrow \quad f'\left(-\frac{1}{2}\right) = \lim_{h \to 0} \frac{\left(\frac{1}{4} - h + h^2\right) - \frac{1}{4}}{h} = \lim_{h \to 0} \frac{h^2 - h}{h} = \lim_{h \to 0} h - 1 = -1$$

Hence, slope of the tangent to the curve $y = x^2$ at point $(-1/2, 1/4)$ is equal to -1. This means that the tangent to the curve at point $(-1/2, 1/4)$ makes $135°$ angle with the positive direction of X-axis.

30.2 DERIVATIVE OF A FUNCTION

In the previous section, we have learnt about the derivative of a function at a point in its domain. Let $f(x)$ be a function differentiable at every point in its domain. Then corresponding to every point c in the domain, we obtain a unique real number equal to the derivative $f'(c)$ of $f(x)$ at $x = c$. Thus, there is one-to-one correspondence between points in the domain of the function and the derivatives at these points. This correspondence induces a function such that the image of any point x in the domain is the value of the derivative of f at x i.e. $f'(x)$ or $\frac{d}{dx}(f(x))$. This function is called the derivative or differentiation of $f(x)$ and is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or,} \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of finding the derivative of a function by using the above formula is known as the differentiation or derivative from the first principles.

30.2.1 DERIVATIVE AS A RATE MEASURER

Let $f(x)$ be a function of x and let $y = f(x)$. Clearly, the value of y depends upon the value of x and it changes with a change in the value of x. So, x is called the *independent variable* and y the *dependent variable*. Let Δx be a small change (positive or negative) in x and let Δy be the corresponding change in $y = f(x)$. Then, the value of x changes from x to $x + \Delta x$ and the value of the $f(x)$ changes from $f(x)$ to $f(x + \Delta x)$. So, change in the value of f is

$$f(x + \Delta x) - f(x) \quad \text{or,} \quad \Delta y = f(x + \Delta x) - f(x) \qquad \text{...(i)}$$

Thus, we observe that due to change Δx in x, there is change Δy in y. Therefore, due to one unit change in x, change in y is equal to $\dfrac{\Delta y}{\Delta x}$. This is known as the average rate of charge of y with respect to x.

As $\Delta x \to 0$, we observe that Δy also tends to zero.

$\therefore$ Instantaneous rate of change of y with respect to $x = \displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

If we use the phrase rate of change instead of instantaneous rate of change, we have

$$\begin{aligned}
\text{Rate of change in } y \text{ with respect to } x &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} && \text{[Using (i)]} \\
&= \frac{d}{dx}(f(x)) && \text{[Using def. of derivative]} \\
&= \frac{dy}{dx}
\end{aligned}$$

Thus, $\dfrac{dy}{dx}$ or, $\dfrac{d}{dx}(f(x))$ measures the rate of change of $y = f(x)$ with respect to x.

i.e., $\dfrac{dy}{dx} = \displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

REMARK 1 *The meaning of the term "rate of change of y with respect to x" is that if x is increased by an additional unit the change in y is given by* $\dfrac{dy}{dx}$. *For example, the rate of change of displacement of a particle is defined as its velocity, so if we say that a particle is moving with the velocity v km/hr then it means that when time is increased by one hour the displacement changes by v km.*

REMARK 2 *Some authors also define* $\dfrac{dy}{dx}$ *as* $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ *or,* $\displaystyle\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ *which are exactly identical to the definition given in this book.*

REMARK 3 *We have seen that* $\dfrac{dy}{dx}$ *or,* $\dfrac{d}{dx}(f(x))$ *is the derivative or differentiation of* $y = f(x)$. *Also,* $\dfrac{dy}{dx}$ *or,* $\dfrac{d}{dx}(f(x))$ *measures the rate of change of y with respect to x. So, we can say that the derivative of a function* $y = f(x)$ *is same as the rate of change of* $f(x)$ *with respect to x. Consequently, phrases such as "differentiation of a function* $f(x)$*" and "differentiation of a function* $f(x)$ *with respect to x" convey the same meaning and are used invariably.*

30.3 DIFFERENTIATION FROM FIRST PRINCIPLES

In the previous sections, we have learnt that the derivative of a function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

The process of finding the derivative of a function by using the above definition is called the differentiation from first principles or by *ab*-initio method or, by delta method.

In this section, we will find the derivatives of some standard functions viz. $x^n, e^x, a^x,$

$\log x$, $\sin x$, $\cos x$, $\tan x$, $\cot x$, cosec x and sec x by first principles. Following results will be very helpful in finding the same.

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (iv) $\tan A \pm \tan B = \dfrac{\sin(A \pm B)}{\cos A \cos B}$

(v) $\tan A - \tan B = \tan(A - B)\{1 + \tan A \tan B\}$

(vi) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(vii) $\sin C - \sin D = 2 \sin \dfrac{C-D}{2} \cos \dfrac{C+D}{2}$

(viii) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

(ix) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2} = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$

(x) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(xi) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

(xii) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(xiii) $2 \sin A \sin B = \cos(A - B) - \cos(A - B)$

(xiv) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(xv) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$, $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$

(xvi) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$, $\lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$

(xvii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a,\ a > 0,\ a \neq 1$

(xviii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

(xix) $\lim\limits_{x \to 0} \dfrac{\log_e(1+x)}{x} = 1$

THEOREM 1 *If $f(x) = x^n$, where $n \in R$, then, the differentiation of x^n with respect to x is nx^{n-1}.*

i.e. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ **[NCERT]**

PROOF Let $f(x) = x^n$. Then, $f(x+h) = (x+h)^n$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{(x+h) - x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{z \to x} \frac{z^n - x^n}{z - x}, \text{ where } z = x + h \text{ and } z \to x \text{ as } h \to 0$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = nx^{n-1} \qquad \left[\text{Using: } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

Hence, $\frac{d}{dx}(x^n) = nx^{n-1}$

Q.E.D.

ILLUSTRATIONS *Using the above formula, we obtain*

(i) $\frac{d}{dx}(x^5) = 5x^{5-1} = 5x^4$

(ii) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = \frac{-3}{x^4}$

(iii) $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-1/2-1} = -\frac{1}{2}x^{\frac{-3}{2}}$

(v) $\frac{d}{dx}(x) = \frac{d}{dx}(x^1) = 1 \times x^{1-1} = 1 \times x^0 = 1$

(vi) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \times x^{-1-1} = -\frac{1}{x^2}$

Q.E.D.

THEOREM 2 *The differentiation of e^x with respect to x is e^x.*

i.e. $\frac{d}{dx}(e^x) = e^x$

<u>PROOF</u> Let $f(x) = e^x$. Then, $f(x+h) = e^{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x}{h} = \lim_{h \to 0} e^x \left(\frac{e^h - 1}{h}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^x \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = e^x \times 1 = e^x \qquad \left[\because \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) = 1\right]$$

Hence, $\frac{d}{dx}(e^x) = e^x$

Q.E.D.

THEOREM 3 *The differentiation of a^x $(a > 0, a \neq 1)$ with respect to x is $a^x \log_e a$.*

i.e. $\frac{d}{dx}(a^x) = a^x \log_e a$

<u>PROOF</u> Let $f(x) = a^x$. Then, $f(x+h) = a^{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = a^x \lim_{h \to 0}\left(\frac{a^h - 1}{h}\right) = a^x \log_e a \qquad \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a\right]$$

Q.E.D.

Hence, $\frac{d}{dx}(a^x) = a^x \log_e a$

ILLUSTRATIONS *Using the above formula, we get*

(i) $\frac{d}{dx}(5^x) = 5^x \log_e 5$

(ii) $\frac{d}{dx}(10^x) = 10^x \log_e 10.$

(iii) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}\left((e^2)^x\right) = (e^2)^x \log e^2 = e^{2x} \cdot 2 \log e = 2e^{2x}$

Q.E.D.

THEOREM 4 *The differentiation of* $\log_e x$, $x > 0$ is $\frac{1}{x}$.

i.e. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

PROOF Let $f(x) = \log_e x$. Then, $f(x+h) = \log_e (x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_e (x+h) - \log_e x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_e (1 + h/x)}{h} = \lim_{h \to 0} \frac{\log_e (1 + h/x)}{h/x} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{x} \qquad \left[\because \lim_{x \to 0} \frac{\log_e (1+x)}{x} = 1\right]$$

Hence, $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Q.E.D.

THEOREM 5 *The differentiation of* $\log_a x$ $(a > 0, a \neq 1)$ *with respect to x is* $\frac{1}{x \log_e a}$

i.e. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

PROOF Let $f(x) = \log_a x$. Then, $f(x+h) = \log_a (x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h} = \lim_{h \to 0} \frac{\log_a \left(\frac{x+h}{x}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\log_a (1 + h/x)}{h} = \lim_{h \to 0} \frac{\log_e (1 + h/x)}{(\log_e a) \cdot h} \qquad \left[\because \log_a \lambda = \frac{\log_e \lambda}{\log_e a}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{\log_e a} \lim_{h \to 0} \frac{\log_e (1 + h/x)}{x\,(h/x)} = \frac{1}{x \log_e a} \qquad \left[\because \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} = 1\right]$$

Hence, $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

Q.E.D.

ILLUSTRATIONS *We have,*

(i) $\frac{d}{dx}(\log_3 x) = \frac{1}{x \log_e 3}$

(ii) $\frac{d}{dx}\left(\frac{1}{\log_x 5}\right) = \frac{d}{dx}(\log_5 x) = \frac{1}{x \log_e 5}$

THEOREM 6 *The differentiation of* sin *x with respect to x* is cos *x*.

i.e. $\frac{d}{dx}(\sin x) = \cos x$ **[NCERT]**

PROOF Let $f(x) = \sin x$. Then, $f(x+h) = \sin(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h} \quad \left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{(2\sin h/2)\cos(x+h/2)}{2(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \cos\left(x+\frac{h}{2}\right) \times \lim_{h\to 0} \frac{\sin(h/2)}{(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (\cos x)\times 1 = \cos x \quad \left[\because \lim_{h\to 0}\frac{\sin(h/2)}{(h/2)} = 1\right]$$

Hence, $\frac{d}{dx}(\sin x) = \cos x$ **Q.E.D.**

THEOREM 7 *The differentiation of* cos *x with respect to x is* – sin *x*. **[NCERT]**

i.e. $\frac{d}{dx}(\cos x) = -\sin x$

PROOF Let $f(x) = \cos x$. Then, $f(x+h) = \cos(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}\times\sin\frac{C-D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)} = (-\sin x)\times 1 \quad \left[\because \lim_{h\to 0}\frac{\sin(h/2)}{(h/2)} = 1\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\sin x$$

Hence, $\frac{d}{dx}(\cos x) = -\sin x$

THEOREM 8 *The differentiation of* tan *x with respect to x is* $\sec^2 x$. **Q.E.D.**

i.e. $\frac{d}{dx}(\tan x) = \sec^2 x$ **[NCERT]**

PROOF Let $f(x) = \tan x$. Then, $f(x+h) = \tan(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin h}{h} \times \frac{1}{\cos x \cos(x+h)} \quad [\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{1}{\cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \frac{1}{\cos x \cos x} = \sec^2 x \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Hence, $\frac{d}{dx}(\tan x) = \sec^2 x$ Q.E.D.

THEOREM 9 The differentiation of cot x with respect to x is $-\operatorname{cosec}^2 x$.

i.e. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

PROOF Let $f(x) = \cot x$. Then, $f(x+h) = \cot(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cot(x+h) - \cot x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x-(x+h))}{h \sin x \sin(x+h)} \quad [\because \sin A \cos B - \cos A \sin B = \sin(A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin h}{h} \times \lim_{h \to 0} \frac{1}{\sin x \sin(x+h)} \quad [\because \sin(-h) = -\sin h]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-1)\frac{1}{\sin x \sin x} = -\operatorname{cosec}^2 x \qquad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Hence, $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

THEOREM 10 *The differentiation of* sec x *with respect to* x *is* sec x tan x. Q.E.D.

i.e. $\frac{d}{dx}(\sec x) = \sec x \tan x$

PROOF Let $f(x) = \sec x$. Then, $f(x+h) = \sec(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{\cos(x+h)} - \frac{1}{\cos x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x+h-x}{2}\right)}{h \cos x \cos(x+h)}$$

$$\left[\because \cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{h \cos x \cos(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \times \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{\sin x}{\cos x \cos x} \times 1 = \tan x \sec x. \qquad \left[\because \lim_{h \to 0} \frac{\sin(h/2)}{(h/2)} = 1\right]$$

Hence, $\frac{d}{dx}(\sec x) = \sec x \tan x$. **Q.E.D.**

THEOREM 11 *The differentiation of* cosec x *with respect to* x *is* $-$ cosec x cot x.

i.e. $\quad \frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$

<u>PROOF</u> *Let* $f(x) = \text{cosec } x$. *Then,* $f(x+h) = \text{cosec }(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\text{cosec}(x+h) - \text{cosec } x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{\sin(x+h)} - \frac{1}{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin x - \sin(x+h)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin\left(\frac{x-x-h}{2}\right) \cos\left(\frac{x+x+h}{2}\right)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2 \sin(-h/2) \cos(x+h/2)}{h \sin x \sin(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin(h/2)}{h/2} \times \lim_{h \to 0} \frac{\cos(x + h/2)}{\sin x \ \sin(x+h)} \qquad \left[\because \sin\left(-\frac{h}{2}\right) = -\sin\frac{h}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-1) \times \frac{\cos x}{\sin x \sin x} = -\cot x \operatorname{cosec} x.$$

Hence, $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x.$ **Q.E.D.**

The above results can be summarized as under:

(i) $\frac{d}{dx}(x^n) = nx^{n-1}$ (ii) $\frac{d}{dx}(e^x) = e^x$ (iii) $\frac{d}{dx}(a^x) = a^x \log_e a.$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$ (v) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$ (vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$ (viii) $\frac{d}{dx}(\tan x) = \sec^2 x$ (ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the derivatives of the following functions from first principles:*

(i) $x^3 - 27$ (ii) $(x-1)(x-2)$ (iii) $\frac{1}{x^2}$ **[NCERT]**

SOLUTION (i) Let $f(x) = x^3 - 27$. Then, $f(x+h) = (x+h)^3 - 27$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{(x+h)^3 - 27\right\} - \left(x^3 - 27\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x^3 + 3x^2 h + 3xh^2 + h^3 - 27) - (x^3 - 27)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2 + 3x \times 0 + 0 = 3x^2$$

(ii) Let $f(x) = (x-1)(x-2)$. Then, $f(x+h) = (x+h-1)(x+h-2)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\{(x-1)+h\}\{(x-2)+h\} - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x-1)(x-2) + h(x-1) + h(x-2) + h^2 - (x-1)(x-2)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h(x-1) + h(x-2) + h^2}{h} = \lim_{h \to 0} \{(x-1) + (x-2)\} + h = 2x - 3$$

(iii) Let $f(x) = \frac{1}{x^2}$. Then, $f(x+h) = \frac{1}{(x+h)^2}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{x^2 - (x+h)^2}{h\, x^2 (x+h)^2}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-2hx - h^2}{hx^2 (x+h)^2} = \lim_{h \to 0} \frac{-2x - h}{x^2 (x+h)^2} = \frac{-2x - 0}{x^2 \times x^2} = \frac{-2}{x^3}$$

EXAMPLE 2 *Differentiate the following functions with respect to x from first principles:*

(i) $\sqrt{x}$ (ii) $\sqrt{ax+b}$ (iii) $\frac{1}{x}$ (iv) $\frac{1}{ax+b}$

SOLUTION (i) Let $f(x) = \sqrt{x}$. Then, $f(x+h) = \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left(\sqrt{x+h} + \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

(ii) Let $f(x) = \sqrt{ax+b}$. Then, $f(x+h) = \sqrt{a(x+h)+b} = \sqrt{(ax+b)+ah}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{(ax+b)+ah} - \sqrt{ax+b}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{(ax+b)+ah} - \sqrt{ax+b}}{h} \times \frac{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}{\sqrt{(ax+b)+ah} + \sqrt{ax+b}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(ax+b)+ah-(ax+b)}{h\left\{\sqrt{(ax+b)+ah}+\sqrt{ax+b}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{ah}{h\left\{\sqrt{(ax+b)+ah}+\sqrt{ax+b}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a}{\sqrt{(ax+b)+ah}+\sqrt{ax+b}} = \frac{a}{\sqrt{ax+b}+\sqrt{ax+b}}$$

Hence, $\frac{d}{dx}(\sqrt{ax+b}) = \frac{a}{2\sqrt{ax+b}}$

(iii) Let $f(x) = \frac{1}{x}$. Then, $f(x+h) = \frac{1}{x+h}$.

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{x+h-x}{h\left\{\sqrt{x+h}+\sqrt{x}\right\}} = \lim_{h \to 0} \frac{x-(x+h)}{hx(x+h)} = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

Hence, $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(iv) Let $f(x) = \frac{1}{ax+b}$. Then, $f(x+h) = \frac{1}{a(x+h)+b} = \frac{1}{(ax+b)+ah}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{1}{(ax+b)+ah}-\frac{1}{ax+b}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\{ax+b\}-\{(ax+b)+ah\}}{h(ax+b)\{(ax+b)+ah\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-ah}{h(ax+b)\{(ax+b)+ah\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-a}{(ax+b)\{(ax+b)+ah\}} = -\frac{a}{(ax+b)^2}$$

Hence, $\frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{-a}{(ax+b)^2}$

EXAMPLE 3 *Differentiate the following functions with respect to x from first principles:*

(i) $\sqrt{2x+3}$ (ii) $\sqrt{4-x}$ (iii) $ax^2+\frac{b}{x}$ (iv) $\frac{2x+3}{3x+2}$ (v) $x^{-3/2}$

SOLUTION (i) Let $f(x)=\sqrt{2x+3}$. Then, $f(x+h)=\sqrt{2(x+h)+3}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sqrt{2(x+h)+3}-\sqrt{2x+3}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\sqrt{2(x+h)+3}-\sqrt{2x+3}\right\}\left\{\sqrt{2(x+h)+3}+\sqrt{2x+3}\right\}}{h\left\{\sqrt{2(x+h)+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{(2x+2h+3-2x-3)}{h}\times\frac{1}{\left\{\sqrt{2x+2h+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{2h}{h}\times\lim_{h\to 0}\frac{1}{\left\{\sqrt{2x+2h+3}+\sqrt{2x+3}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2\times\frac{1}{\sqrt{2x+3}+\sqrt{2x+3}}=\frac{2}{2(\sqrt{2x+3})}=\frac{1}{\sqrt{2x+3}}$$

(ii) Let $f(x)=\sqrt{4-x}$. Then, $f(x+h) = \sqrt{4-(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sqrt{4-(x+h)}-\sqrt{4-x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\sqrt{4-(x+h)}-\sqrt{4-x}\right\}\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{4-(x+h)-(4-x)}{h\left\{\sqrt{4-(x+h)}+\sqrt{4-x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{-h}{h\left\{\sqrt{4-x-h}+\sqrt{4-x}\right\}}=\frac{-1}{2\sqrt{4-x}}$$

(iii) Let $f(x)=ax^2+\frac{b}{x}$. Then, $f(x+h) = a(x+h)^2+\frac{b}{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{a(x+h)^2+\frac{b}{x+h}\right\}-\left\{ax^2+\frac{b}{x}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a\{(x+h)^2 - x^2\} + b\left\{\frac{1}{x+h} - \frac{1}{x}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{a(2hx + h^2) + b\left\{\frac{x-x-h}{x(x+h)}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{\frac{a(2hx+h^2)}{h} + \frac{b(-h)}{hx(x+h)}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{a(2x+h) - \frac{b}{x(x+h)}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2ax - \frac{b}{x^2}$$

(iv) Let $f(x) = \dfrac{2x+3}{3x+2}$. Then, $f(x+h) = \dfrac{2(x+h)+3}{3(x+h)+2} = \dfrac{2x+3+2h}{3x+2+3h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\frac{2x+3+2h}{3x+2+3h} - \frac{2x+3}{3x+2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2x+3+2h)(3x+2) - (2x+3)(3x+2+3h)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(2x+3)(3x+2) + 2h(3x+2) - (2x+3)(3x+2) - 3h(2x+3)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{h(6x+4-6x-9)}{h(3x+2)(3x+2+3h)}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-5}{(3x+2)(3x+2+3h)} = -\frac{5}{(3x+2)^2}.$$

(v) Let $f(x) = x^{-3/2}$. Then, $f(x+h) = (x+h)^{-3/2}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^{-3/2} - x^{-3/2}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)^{-3/2} - x^{-3/2}}{(x+h) - x}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{z \to x} \frac{z^{-3/2} - x^{-3/2}}{z - x}, \text{ where } z = x+h \text{ and } z \to x \text{ as } h \to 0$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (-3/2)\, x^{-3/2-1} \qquad \left[\text{Using: } \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{3}{2} x^{-5/2}$$

EXAMPLE 4 *Differentiate the following functions with respect to x from first principles:*

(i) $\sin 2x$ (ii) $\sin 2x$ (iii) $\sin x^2$ (iv) $\sin (x^2+1)$

SOLUTION (i) Let $f(x) = \sin 2x$. Then, $f(x+h) = \sin 2(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin 2(x+h) - \sin 2x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{2\sin h \cos(2x+h)}{h} \quad \left[\because \sin C - \sin D = 2\sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 2\lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0} \cos(2x+h) = 2(\cos 2x)(1) = 2\cos 2x$$

$$\therefore \quad \frac{d}{dx}(\sin 2x) = 2\cos 2x$$

(ii) Let $f(x) = \sin^2 x$. Then, $f(x+h) = \sin^2(x+h)$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin^2(x+h) - \sin^2 x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin(x+h+x)\sin(x+h-x)}{h}$$

$$[\because \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin h}{h} \times \lim_{h\to 0}\sin(2x+h) = 1(\sin 2x) = \sin 2x$$

$$\therefore \quad \frac{d}{dx}(\sin^2 x) = \sin 2x$$

(iii) Let $f(x) = \sin x^2$. Then, $f(x+h) = \sin (x+h)^2$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin(x+h)^2 - \sin x^2}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{2\sin\left(\frac{2hx+h^2}{2}\right)\cos\left(\frac{2x^2+2hx+h^2}{2}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{2\sin\left(\frac{2hx+h^2}{2}\right)}{h\left(\frac{2x+h}{2}\right)}\left(\frac{2x+h}{2}\right)\cos\left(\frac{2x^2+2hx+h^2}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin\left(\frac{2hx+h^2}{2}\right)}{\left(\frac{2hx+h^2}{2}\right)} \times \lim_{h\to 0}(2x+h) \times \lim_{h\to 0}\cos\left(\frac{2x^2+2hx+h^2}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{h \to 0}(2x+h) \times \lim_{h \to 0} \cos\left(\frac{2x^2 + 2hx + h^2}{2}\right),$$

where $\theta = \dfrac{2hx + h^2}{2}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = (1) \times (2x)\cos x^2 = 2x \cos x^2$$

$$\therefore \quad \frac{d}{dx}(\sin x^2) = 2x \cos x^2$$

(iv) Let $f(x) = \sin(x^2 + 1)$. Then, $f(x+h) = \sin\left\{(x+h)^2 + 1\right\}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left\{(x+h)^2 + 1\right\} - \sin(x^2+1)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left(\frac{2hx + h^2}{2}\right)\cos\left\{\frac{(x+h)^2 + 1 + x^2 + 1}{2}\right\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left(\frac{2hx + h^2}{2}\right)}{h\left(\frac{2hx + h^2}{2}\right)} \times \left(\frac{2hx + h^2}{2}\right) \times \cos\left\{\frac{(x+h)^2 + 1 + x^2 + 1}{2}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\frac{2hx + h^2}{2}\right)}{\left(\frac{2hx + h^2}{2}\right)} \times (2x+h) \times \cos\left\{\frac{(x+h)^2 + 1 + x^2 + 1}{2}\right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{h \to 0}(2x+h) \times \lim_{h \to 0} \cos\left\{\frac{(x+h)^2 + 1 + x^2 + 1}{2}\right\},$$

where $\theta = \dfrac{2hx + h^2}{2}$. Clearly $\theta \to 0$, as $h \to 0$.

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times (2x) \times \cos(x^2+1) = 2x\cos(x^2+1)$$

LEVEL-2

EXAMPLE 5 *Differentiate xe^x from first principles.*

SOLUTION Let $f(x) = xe^x$. Then, $f(x+h) = (x+h)\,e^{(x+h)}$

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(x+h)\,e^{x+h} - xe^x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(xe^{x+h} - xe^x) + he^{x+h}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \left\{ xe^x \left(\frac{e^h - 1}{h} \right) + e^{x+h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = xe^x \lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) + \lim_{h \to 0} e^{x+h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = xe^x + e^x = (x+1)e^x$$

EXAMPLE 6 *Differentiate the following functions with respect to x from first principles:*

(i) $\tan \sqrt{x}$ (ii) $\cot \sqrt{x}$ (iii) $\sqrt{\sin x}$ (iv) $\sin \sqrt{x}$

SOLUTION (i) Let $f(x) = \tan \sqrt{x}$. Then, $f(x+h) = \tan \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\tan \sqrt{x+h} - \tan \sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{h \cos \sqrt{x+h} \cos \sqrt{x}} \qquad \left[\because \tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{(x+h-x) \cos \sqrt{x} \cos \sqrt{x+h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) \cos \sqrt{x} \cos \sqrt{x+h}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)} \cdot \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right) \cos \sqrt{x+h} \cos \sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1 \times \frac{1}{2\sqrt{x} \cos \sqrt{x} \cos \sqrt{x}} \qquad \left[\because \lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)} = 1\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{1}{2\sqrt{x}} \sec^2 \sqrt{x}$$

(ii) Let $f(x) = \cot \sqrt{x}$. Then, $f(x+h) = \cot \sqrt{x+h}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\cot\sqrt{x+h} - \cot\sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{h \sin\sqrt{x+h}\,\sin\sqrt{x}} \qquad \left[\because \cot A - \cot B = \frac{-\sin(A-B)}{\sin A \sin B}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{-\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\{(x+h) - x\} \sin\sqrt{x+h}\,\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right) \sin\sqrt{x+h}\,\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\lim_{h \to 0} \frac{\sin\left(\sqrt{x+h} - \sqrt{x}\right)}{\sqrt{x+h} - \sqrt{x}} \times \lim_{h \to 0} \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right) \sin\sqrt{x+h}\,\sin\sqrt{x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -\frac{1}{2\sqrt{x}\,\sin\sqrt{x}\,\sin\sqrt{x}} = \frac{-\operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}$$

(iii) Let $f(x) = \sqrt{\sin x}$. Then, $f(x+h) = \sqrt{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{2\sin\left\{\frac{h}{2}\right\}\cos\left\{\frac{2x+h}{2}\right\}}{h\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}} \qquad \left[\because \sin C - \sin D = 2\sin\frac{C-D}{2}\cos\frac{C+D}{2}\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{(\sin h/2)}{(h/2)} \times \lim_{h \to 0} \frac{\cos(x + h/2)}{\left\{\sqrt{\sin(x+h)} + \sqrt{\sin x}\right\}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{\cos x}{\sqrt{\sin x} + \sqrt{\sin x}} = \frac{\cos x}{2\sqrt{\sin x}}$$

(iv) Let $f(x) = \sin\sqrt{x}$. Then, $f(x+h) = \sin\sqrt{x+h}$.

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin\sqrt{x+h}-\sin\sqrt{x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{2\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}\times\frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)h}\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}\times\lim_{h\to 0}\frac{x+h-x}{(\sqrt{x+h}+\sqrt{x})\,h}\times\lim_{h\to 0}\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\times\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}\times\lim_{h\to 0}\cos\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right),$$

where $\theta = \dfrac{\sqrt{x+h}-\sqrt{x}}{2}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = 1\times\frac{1}{2\sqrt{x}}\times(\cos\sqrt{x}) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

$$\therefore \quad \frac{d}{dx}(\sin\sqrt{x}) = \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

EXAMPLE 7 *Differentiate $x^2\cos x$ from first principles.*

SOLUTION Let $f(x) = x^2\cos x$. Then, $f(x+h) = (x+h)^2\cos(x+h)$

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{(x+h)^2\cos(x+h)-x^2\cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{(x^2+2hx+h^2)\cos(x+h)-x^2\cos x}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{x^2\cos(x+h)-x^2\cos x\right\}+2hx\cos(x+h)+h^2\cos(x+h)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\left[x^2\left\{\frac{\cos(x+h)-\cos x}{h}\right\}+2x\cos(x+h)+h\cos(x+h)\right]$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0} -2x^2\,\frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}+\lim_{h\to 0}2x\cos(x+h)+\lim_{h\to 0}h\cos(x+h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -x^2 \lim_{h\to 0} \frac{\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h/2} + \lim_{h\to 0} 2x\cos(x+h) + \lim_{h\to 0} h\cos(x+h)$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = -x^2 \sin x + 2x\cos x + 0\times\cos x = -x^2\sin x + 2x\cos x.$$

EXAMPLE 8 *Using first principles, prove that* $\frac{d}{dx}\left\{\frac{1}{f(x)}\right\} = -\frac{f'(x)}{\{f(x)\}^2}$.

SOLUTION Let $\phi(x) = \frac{1}{f(x)}$. Then, $\phi(x+h) = \frac{1}{f(x+h)}$

$$\therefore \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h\to 0}\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h\to 0}\frac{f(x)-f(x+h)}{hf(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = \lim_{h\to 0}\frac{f(x)-f(x+h)}{h}\times\lim_{h\to 0}\frac{1}{f(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\times\lim\frac{1}{f(x)\,f(x+h)}$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -f'(x)\times\frac{1}{f(x)\,f(x)} \qquad \left[\begin{array}{l}\because f(x) \text{ is differentiable}\\ \therefore f(x) \text{ is continuous} \Rightarrow \lim\limits_{h\to 0} f(x+h) = f(x)\end{array}\right]$$

$$\Rightarrow \quad \frac{d}{dx}\{\phi(x)\} = -\frac{f'(x)}{\{f(x)\}^2}$$

EXAMPLE 9 *Find the derivative of* $\sqrt[3]{\sin x}$ *from first principles.*

SOLUTION Let $f(x) = \sqrt[3]{\sin x}$. Then, $f(x+h) = \sqrt[3]{\sin(x+h)}$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sqrt[3]{\sin(x+h)}-\sqrt[3]{\sin x}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\left\{\sqrt[3]{\sin(x+h)}\right\}^3-\left\{\sqrt[3]{\sin x}\right\}^3}{h\left[\sin^{2/3}(x+h)+\sin^{1/3}(x+h)\sin^{1/3}x+\sin^{2/3}x\right]}$$

$$\left[\because a-b = \frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\times\frac{1}{\sin^{2/3}(x+h)+\sin^{1/3}(x+h)\sin^{1/3}x+\sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h}\times\frac{1}{\sin^{2/3}(x+h)+\sin^{1/3}(x+h)\sin^{1/3}x+\sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} \times \lim_{h\to 0} \frac{1}{\sin^{2/3}(x+h)+\sin^{1/3}(x+h)\sin^{1/3}x+\sin^{2/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \cos x \times \frac{1}{\sin^{2/3}x+\sin^{2/3}x+\sin^{1/3}x\sin^{1/3}x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \frac{\cos x}{3\sin^{2/3}x}$$

EXAMPLE 10 *Differentiate* log sin x *from first principles.*

SOLUTION Let $f(x) = \log \sin x$. Then, $f(x+h) = \log \sin (x+h)$

$$\therefore \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log \sin (x+h) - \log \sin x}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{\frac{\sin(x+h)}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)}{\sin x}-1\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{\sin x}\right\}}{h\left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}} \times \left\{\frac{\sin(x+h)-\sin x}{\sin x}\right\}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{h}\right\}} \times \frac{\sin(x+h)-\sin x}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{h}\right\}} \times \lim_{h\to 0} \frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{\log\left\{1+\frac{\sin(x+h)-\sin x}{h}\right\}}{\left\{\frac{\sin(x+h)-\sin x}{h}\right\}} \times \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{\frac{h}{2}} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{d}{dx}(f(x)) = 1 \times \cos x \times \frac{1}{\sin x} = \cot x.$$

EXAMPLE 11 *Differentiate $e^{\sqrt{\tan x}}$ from first principles.*

SOLUTION Let $f(x) = e^{\sqrt{\tan x}}$. Then, $f(x+h) = e^{\sqrt{\tan(x+h)}}$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{e^{\sqrt{\tan(x+h)}} - e^{\sqrt{\tan x}}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} e^{\sqrt{\tan x}} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \lim_{h \to 0} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \times \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h} \right\}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \left\{ \frac{e^{\sqrt{\tan(x+h)} - \sqrt{\tan x}} - 1}{\sqrt{\tan(x+h)} - \sqrt{\tan x}} \right\} \times \lim_{h \to 0} \frac{\sqrt{\tan(x+h)} - \sqrt{\tan x}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times 1 \times \lim_{h \to 0} \frac{\tan(x+h) - \tan x}{h} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \lim_{h \to 0} \frac{\sin h}{h \cos(x+h) \cos x} \times \frac{1}{\sqrt{\tan(x+h)} + \sqrt{\tan x}}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = e^{\sqrt{\tan x}} \times \frac{1}{\cos^2 x} \times \frac{1}{2\sqrt{\tan x}} = \frac{e^{\sqrt{\tan x}}}{2\sqrt{\tan x}} \sec^2 x$$

EXERCISE 30.2

LEVEL-1

Differentiate each of the following from first principles:

1. (i) $\frac{2}{x}$ (ii) $\frac{1}{\sqrt{x}}$ (iii) $\frac{1}{x^3}$

   (iv) $\frac{x^2+1}{x}$ [NCERT] (v) $\frac{x^2-1}{x}$ (vi) $\frac{x+1}{x+2}$

   (vii) $\frac{x+2}{3x+5}$ (viii) $k\, x^n$ (ix) $\frac{1}{\sqrt{3-x}}$

   (x) $x^2 + x + 3$ (xi) $(x+2)^3$ (xii) $x^3 + 4x^2 + 3x + 2$

   (xiii) $(x^2+1)(x-5)$ (xiv) $\sqrt{2x^2+1}$ (xv) $\frac{2x+3}{x-2}$

2. (i) e^{-x} (ii) e^{3x} (iii) e^{ax+b}

   (iv) $x\, e^x$ (v) $-x$ [NCERT] (vi) $(-x)^{-1}$ [NCERT]

   (vii) $\sin(x+1)$ [NCERT] (viii) $\cos\left(x - \frac{\pi}{8}\right)$ [NCERT] (ix) $x \sin x$

   (x) $x \cos x$ (xi) $\sin(2x-3)$

LEVEL-2

3. (i) $\sqrt{\sin 2x}$ (ii) $\dfrac{\sin x}{x}$ (iii) $\dfrac{\cos x}{x}$
(iv) $x^2 \sin x$ (v) $\sqrt{\sin(3x+1)}$ (vi) $\sin x + \cos x$
(vii) $x^2 e^x$ (viii) e^{x^2+1} (ix) $e^{\sqrt{2x}}$
(x) $e^{\sqrt{ax+b}}$ (xi) $a^{\sqrt{x}}$ (xii) 3^{x^2}

4. (i) $\tan^2 x$ (ii) $\tan(2x+1)$ (iii) $\tan 2x$ (iv) $\sqrt{\tan x}$

5. (i) $\sin\sqrt{2x}$ (ii) $\cos\sqrt{x}$ (iii) $\tan\sqrt{x}$ (iv) $\tan x^2$

ANSWERS

1. (i) $-2x^{-2}$ (ii) $-\dfrac{1}{2}x^{-3/2}$ (iii) $-3x^{-4}$ (iv) $1-\dfrac{1}{x^2}$
(v) $1+\dfrac{1}{x^2}$ (vi) $\dfrac{1}{(x+2)^2}$ (vii) $\dfrac{-1}{(3x+5)^2}$ (viii) nkx^{n-1}
(ix) $\dfrac{1}{2(3-x)^{3/2}}$ (x) $2x+1$ (xi) $3(x+2)^2$ (xii) $3x^2+8x+3$
(xiii) $3x^2-10x+1$ (xiv) $\dfrac{2x}{\sqrt{2x^2+1}}$ (xv) $\dfrac{-7}{(x-2)^2}$

2. (i) $-e^{-x}$ (ii) $3e^{3x}$ (iii) $a\,e^{ax+b}$ (iv) $(x+1)e^x$
(v) -1 (vi) $\dfrac{1}{x^2}$ (vii) $\cos(x+1)$ (viii) $-\sin\left(x-\dfrac{\pi}{8}\right)$
(ix) $\sin x + x\cos x$ (x) $\cos x - x\sin x$ (xi) $2\cos(2x-3)$

3. (i) $\dfrac{\cos 2x}{\sqrt{\sin 2x}}$ (ii) $\dfrac{x\cos x-\sin x}{x^2}$ (iii) $\dfrac{-x\sin x-\cos x}{x^2}$ (iv) $x^2\cos x + 2x\sin x$
(v) $\dfrac{3\cos(3x+1)}{2\sqrt{\sin(3x+1)}}$ (vi) $\cos x-\sin x$ (vii) $(x^2+2x)e^x$ (viii) $2x\,e^{x^2+1}$
(ix) $\dfrac{e^{\sqrt{2x}}}{\sqrt{2x}}$ (x) $\dfrac{a\,e^{\sqrt{ax+b}}}{2\sqrt{ax+b}}$ (xi) $\dfrac{1}{2\sqrt{x}}a^{\sqrt{x}}\log_e a$ (xii) $2x\,3^{x^2}\log 3$

4. (i) $2\tan x\sec^2 x$ (ii) $2\sec^2(2x+1)$ (iii) $2\sec^2 2x$ (iv) $\dfrac{\sec^2 x}{2\sqrt{\tan x}}$

5. (i) $\dfrac{\cos\sqrt{2x}}{\sqrt{2x}}$ (ii) $-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$ (iii) $\dfrac{\sec^2\sqrt{x}}{2\sqrt{x}}$ (iv) $2x\sec^2 x^2$

HINTS TO NCERT & SELECTED PROBLEMS

1. (iv) Let $f(x)=\dfrac{x^2+1}{x}=x+\dfrac{1}{x}$. Then, $f(x+h)=(x+h)+\dfrac{1}{x+h}$

$$\therefore \quad f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \quad f'(x)=\lim_{h\to 0}\frac{\left\{(x+h)+\dfrac{1}{x+h}\right\}-\left\{x+\dfrac{1}{x}\right\}}{h}=\lim_{h\to 0}\frac{\left\{(x+h)-x\right\}+\left\{\dfrac{1}{x+h}-\dfrac{1}{x}\right\}}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{h + \dfrac{x - x - h}{x(x+h)}}{h} = \lim_{h \to 0} \frac{h - \dfrac{h}{x(x+h)}}{h} = \lim_{h \to 0} 1 - \frac{1}{x(x+h)} = 1 - \frac{1}{x^2}$$

2. (v) Let $f(x) = -x$. Then, $f(x+h) = -(x+h)$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-(x+h) + x}{h} = \lim_{h \to 0} \frac{-h}{h} = -1.$$

(vi) Let $f(x) = -\dfrac{1}{x}$. Then, $f(x+h) = -\dfrac{1}{x+h}$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{-\dfrac{1}{x+h} + \dfrac{1}{x}}{h} = \lim_{h \to 0} \frac{-x + (x+h)}{x(x+h)h} = \lim_{h \to 0} \frac{h}{hx(x+h)} = \lim_{h \to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}$$

(vii) Let $f(x) = \sin(x+1)$. Then, $f(x+h) = \sin(x+h+1) = \sin\{(x+1)+h\}$

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\sin((x+1)+h) - \sin(x+1)}{h} = \lim_{h \to 0} \frac{2\sin\left(\dfrac{h}{2}\right)\cos\left\{(x+1) + \dfrac{h}{2}\right\}}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\sin\left(\dfrac{h}{2}\right)\cos\left\{(x+1) + \dfrac{h}{2}\right\}}{\left(\dfrac{h}{2}\right)} = \cos(x+1)$$

(viii) We have, $f(x) = \cos\left(x - \dfrac{\pi}{8}\right)$

$$\therefore \quad f(x+h) = \cos\left(x + h - \frac{\pi}{8}\right) = \cos\left\{\left(x - \frac{\pi}{8}\right) + h\right\}$$

So, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{\cos\left\{\left(x - \dfrac{\pi}{8}\right) + h\right\} - \cos\left(x - \dfrac{\pi}{8}\right)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \to 0} \frac{-2\sin\left\{\left(x - \dfrac{\pi}{8}\right) + \dfrac{h}{2}\right\}\sin\left(\dfrac{h}{2}\right)}{h}$$

$$\Rightarrow \quad f'(x) = -\lim_{h \to 0} \sin\left\{\left(x - \frac{\pi}{8}\right) + \frac{h}{2}\right\} \frac{\sin\left(\dfrac{h}{2}\right)}{\left(\dfrac{h}{2}\right)} = -\sin\left(x - \frac{\pi}{8}\right)$$

30.4 FUNDAMENTAL RULES FOR DIFFERENTIATION

In the previous section, we have used the definition of derivative to find derivatives. This section is mainly devoted to develop several rules that allow us to find derivatives without using definition directly.

THEOREM 1 *Differentiation of a constant function is zero i.e.,* $\frac{d}{dx}(c) = 0$.

PROOF Let $f(x) = c$ be a constant function. Then,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

Hence, $\frac{d}{dx}(c) = 0$, *where c is a constant.*

REMARK *Geometrically the graph of a constant function is a straight line parallel is x-axis. So, tangent at every point is parallel to x-axis. Consequently, the slope of the tangent at every point is zero, i.e.* $\frac{dy}{dx} = 0$.

THEOREM 2 *Let $f(x)$ be a differentiable function and let c be a constant. Then, $c f(x)$ is also differentiable such that* $\frac{d}{dx}\{c f(x)\} = c \frac{d}{dx}(f(x))$.

i.e. the derivative of a constant times a function is the constant times the derivative of the function.

PROOF Since $f(x)$ is differentiable. Therefore,

$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists finitely and is equal to $\frac{d}{dx}(f(x))$.

i.e. $$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{...(i)}$$

Let $g(x) = c f(x)$. Then,

$$\Rightarrow \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{c f(x+h) - c f(x)}{h}$$

$$\Rightarrow \frac{d}{dx}(g(x)) = c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c \frac{d}{dx}(f(x)) \quad \text{[Using (i)]}$$

Hence, $g(x) = c f(x)$ is differentiable such that $\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\log_x x$ (ii) $e^{3 \log x}$ (iii) $2^{\log_2 x}$ (iv) $5(2^{3 \log_2 x})$ (v) $5 e^x$ (vi) $9(3^x)$

SOLUTION (i) We know that $\log_x x = 1$.

$$\therefore \quad \frac{d}{dx}(\log_x x) = \frac{d}{dx}(1) = 0.$$

(ii) We know that $e^{\log k} = k$.

$$\therefore \quad \frac{d}{dx}(e^{3 \log x}) = \frac{d}{dx}(e^{\log x^3}) = \frac{d}{dx}(x^3) = 3x^2$$

(iii) We know that $a^{\log_a n} = n$

$$\therefore \quad \frac{d}{dx}(2^{\log_2 x}) = \frac{d}{dx}(x) = 1.$$

(iv) Clearly,

$$\frac{d}{dx}(5\cdot 2^{3\log_2 x}) = 5\frac{d}{dx}(2^{3\log_2 x}) = 5\frac{d}{dx}(2^{\log_2 x^3}) = 5\frac{d}{dx}(x^3) \qquad [\because a^{\log_a n} = n]$$

$$= 5(3x^2) = 15x^2$$

(v) Clearly,

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx}(e^x) = 5e^x$$

(vi) Clearly,

$$\frac{d}{dx}(9.3^x) = 9\frac{d}{dx}(3^x) = 9(3^x \log_e 3).$$

THEOREM 3 *If $f(x)$ and $g(x)$ are differentiable functions, then show that $f(x) \pm g(x)$ are also differentiable such that*

$$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$$

i.e. the derivative of the sum or difference of two functions is the sum or difference of their derivatives.

PROOF Since $f(x)$ and $g(x)$ both are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \text{ and } \frac{d}{dx}(g(x)) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \text{ both exist.} \qquad \text{...(i)}$$

Now,

$$\frac{d}{dx}\left\{f(x)+g(x)\right\} = \lim_{h\to 0}\frac{\{f(x+h)\pm g(x+h)\}-\{f(x)\pm g(x)\}}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x+h)-f(x)\}\pm\{g(x+h)-g(x)\}}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f(x+h)-f(x)}{h}\right\} \pm \lim_{h\to 0}\left\{\frac{g(x+h)-g(x)}{h}\right\}$$

$$= \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\} \qquad \text{[Using (i)]}$$

Hence, $f(x) \pm g(x)$ is differentiable and $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x)\} \pm \frac{d}{dx}\{g(x)\}$.

REMARK *The above result can be extended to a finite number of differentiable functions. Thus, we have*

$$\frac{d}{dx}\{f_1(x) \pm f_2(x) + ... \pm f_n(x)\} = \frac{d}{dx}\{f_1(x)\} \pm \frac{d}{dx}\{f_2(x)\} \pm ... \pm \frac{d}{dx}\{f_n(x)\}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x.*

(i) $x^2 + \sin x + \frac{1}{x^2}$ (ii) $\frac{ax^2+bx+c}{\sqrt{x}}$ (iii) $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$

SOLUTION (i) Clearly,

$$\frac{d}{dx}\left\{x^2 + \sin x + \frac{1}{x^2}\right\}$$

$$= \frac{d}{dx}(x^2 + \sin x + x^{-2})$$

$$= \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin x) + \frac{d}{dx}(x^{-2}) = 2x + \cos x + (-2)x^{-3} = 2x + \cos x - \frac{2}{x^3}$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\frac{ax^2+bx+c}{\sqrt{x}}\right\}$$

$$= \frac{d}{dx}\left\{\frac{ax^2}{\sqrt{x}}+\frac{bx}{\sqrt{x}}+\frac{c}{\sqrt{x}}\right\}$$

$$= \frac{d}{dx}\left\{ax^{3/2}+bx^{1/2}+cx^{-1/2}\right\}$$

$$= \frac{d}{dx}\left\{ax^{3/2}\right\}+\frac{d}{dx}\left\{bx^{1/2}\right\}+\frac{d}{dx}\left\{cx^{-1/2}\right\}$$

$$= a\frac{d}{dx}\left\{x^{3/2}\right\}+b\frac{d}{dx}\left\{x^{1/2}\right\}+c\frac{d}{dx}\left\{x^{-1/2}\right\}$$

$$= a\left(\frac{3}{2}x^{1/2}\right)+b\left(\frac{1}{2}x^{-1/2}\right)+c\left(-\frac{1}{2}x^{-3/2}\right) = \frac{3a}{2}x^{1/2}+\frac{b}{2}x^{-1/2}-\frac{c}{2}x^{-3/2}$$

(iii) We have,

$$\frac{d}{dx}\left\{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2\right\}$$

$$= \frac{d}{dx}\left\{x+\frac{1}{x}+2\right\} = \frac{d}{dx}(x)+\frac{d}{dx}(x^{-1})+\frac{d}{dx}(2) = 1+(-1)\,x^{-2}+0 = 1-\frac{1}{x^2}$$

EXAMPLE 2 *If* $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$, *show that* $\frac{dy}{dx}=y$.

SOLUTION We have,

$$y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(1)+\frac{d}{dx}\left(\frac{x}{1!}\right)+\frac{d}{dx}\left(\frac{x^2}{2!}\right)+\frac{d}{dx}\left(\frac{x^3}{3!}\right)+\ldots$$

or, $$\frac{dy}{dx} = \frac{d}{dx}(1)+\frac{1}{1!}\frac{d}{dx}(x)+\frac{1}{2!}\frac{d}{dx}(x^2)+\frac{1}{3!}\frac{d}{dx}(x^3)+\ldots$$

or, $$\frac{dy}{dx}=0+1+\frac{1}{2!}(2x)+\frac{1}{3!}(3x^2)+\ldots$$

or, $$\frac{dy}{dx} = 1+x+\frac{x^2}{2!}+\ldots$$

or, $$\frac{dy}{dx}=y.$$

<u>ALITER</u> We have,

$$y=e^x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x = y.$$

EXAMPLE 3 *If* $y=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^n}{n!}$, *show that* $\frac{dy}{dx}-y+\frac{x^n}{n!}=0$.

SOLUTION We have,

$$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}$$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}\left(\frac{x}{1!}\right) + \frac{d}{dx}\left(\frac{x^2}{2!}\right) + \frac{d}{dx}\left(\frac{x^3}{3!}\right) + \ldots + \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(1) + \frac{1}{1!}\frac{d}{dx}(x) + \frac{1}{2!}\frac{d}{dx}(x^2) + \frac{1}{3!}\frac{d}{dx}(x^3) + \ldots + \frac{1}{n!}\frac{d}{dx}(x^n)$$

$$\Rightarrow \quad \frac{dy}{dx} = 0 + \frac{1}{1!} + \frac{1}{2!}(2x) + \frac{1}{3!}(3x^2) + \ldots + \frac{1}{n!}(n\, x^{n-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}.$$

$$\Rightarrow \quad \frac{dy}{dx} = \left\{1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!}\right\} - \frac{x^n}{n!}$$

$$\Rightarrow \quad \frac{dy}{dx} = y - \frac{x^n}{n!}$$

$$\Rightarrow \quad \frac{dy}{dx} - y + \frac{x^n}{n!} = 0.$$

EXAMPLE 4 *If* $y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$, $x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, *then find* $\frac{dy}{dx}$.

SOLUTION We have,

$$y = \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \sqrt{\frac{2\sin^2 x}{2\cos^2 x}} = \sqrt{\tan^2 x}$$

$$\Rightarrow \quad y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \quad y = \begin{cases} \tan x, & x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, & x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\Rightarrow \quad \frac{dy}{dx} = \begin{cases} \sec^2 x, & \text{if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, & \text{if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

EXAMPLE 5 *Differentiate the following functions with respect to x:*

(i) $(x^2 - 3x + 2)(x + 2)$ (ii) $\left(x^2 + \frac{1}{x^2}\right)^3$

SOLUTION (i) Clearly,

$$\frac{d}{dx}\left\{(x^2 - 3x + 2)(x + 2)\right\}$$

$$= \frac{d}{dx}(x^3 - x^2 - 4x + 4)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) - \frac{d}{dx}(4x) + \frac{d}{dx}(4)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) - 4\frac{d}{dx}(x) + \frac{d}{dx}(4) = 3x^2 - 2x - 4 + 0 = 3x^2 - 2x - 4$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\left(x^2 + \frac{1}{x^2}\right)^3\right\}$$

$$= \frac{d}{dx}\left\{x^6 + 3x^2 + \frac{3}{x^2} + \frac{1}{x^6}\right\}$$

$$= \frac{d}{dx}(x^6) + \frac{d}{dx}(3x^2) + \frac{d}{dx}(3x^{-2}) + \frac{d}{dx}(x^{-6})$$

$$= \frac{d}{dx}(x^6) + 3\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(x^{-6})$$

$$= 6x^5 + 6x + 3(-2)x^{-3} + (-6)x^{-7} = 6x^5 + 6x - \frac{6}{x^3} - \frac{6}{x^7}$$

EXAMPLE 6 *If $f(x) = \alpha x^n$, prove that $\alpha = \frac{f'(1)}{n}$.*

SOLUTION We have, $f(x) = \alpha x^n$

Differentiating both sides with respect to x, we obtain

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\alpha x^n)$$

$$\Rightarrow \quad f'(x) = \alpha \frac{d}{dx}(x^n)$$

$$\Rightarrow \quad f'(x) = \alpha n x^{n-1}$$

Putting $x = 1$ on both sides, we get

$$f'(1) = \alpha n \Rightarrow \alpha = \frac{f'(1)}{n}$$

EXAMPLE 7 *If $f(x) = x^n$ and if $f'(1) = 10$, find the value of n.*

SOLUTION We have, $f(x) = x^n$.

Differentiating both sides with respect to x, we get $f'(x) = nx^{n-1}$.

Putting $x = 1$, we get

$$f'(1) = n \Rightarrow 10 = n \qquad [\because f'(1) = 10]$$

EXAMPLE 8 *If $f(x) = mx + c$ and $f(0) = f'(0) = 1$. What is $f(2)$?*

SOLUTION We have,

$$f(x) = mx + c \qquad \text{...(i)}$$

Differentiating with respect to we get

$$f'(x) = m \cdot 1 + 0 \Rightarrow f'(x) = m \qquad \text{...(ii)}$$

Putting $x = 0$ in (i) and (ii), we get

$$f(0) = c \text{ and } f'(0) = m$$

$$\Rightarrow \quad 1 = c \text{ and } 1 = m \qquad [\because f(0) = f'(0) = 1]$$

Putting the values of m and c in $f(x) = mx + c$, we get $f(x) = x + 1$.

$$\therefore \quad f(2) = 2 + 1 = 3. \qquad [\text{Putting } x = 2 \text{ in } f(x) = x + 1]$$

EXAMPLE 9 *Find* $\frac{dy}{dx}$, *when* $y = 3 \tan x + 5 \log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$.

SOLUTION We have,

$$y = 3 \tan x + 5 \log_a x + \sqrt{x} - 3e^x + \frac{1}{x}$$

$$\Rightarrow \quad y = 3 \tan x + 5 \log_a x + x^{1/2} - 3e^x + x^{-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}(3 \tan x) + \frac{d}{dx}(5 \log_a x) + \frac{d}{dx}(x^{1/2}) - \frac{d}{dx}(3e^x) + \frac{d}{dx}(x^{-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 3\frac{d}{dx}(\tan x) + 5\frac{d}{dx}(\log_a x) + \frac{d}{dx}(x^{1/2}) - 3\frac{d}{dx}(e^x) + \frac{d}{dx}(x^{-1})$$

$$\Rightarrow \quad \frac{dy}{dx} = 3 \sec^2 x + \frac{5}{x \log_e a} + \frac{1}{2}x^{-1/2} - 3e^x + (-1)x^{-2}.$$

EXAMPLE 10 *Differentiate the following functions with respect to x:*

(i) $\sin(x + a)$ **[NCERT]** (ii) $\frac{\sin(x+a)}{\cos x}$ **[NCERT]**

SOLUTION (i) Clearly,

$$\frac{d}{dx}\{\sin(x+a)\}$$

$$= \frac{d}{dx}\{\sin x \cos a + \cos x \sin a\}$$

$$= \frac{d}{dx}(\sin x \cos a) + \frac{d}{dx}(\cos x \sin a)$$

$$= \cos a \frac{d}{dx}(\sin x) + \sin a \frac{d}{dx}(\cos x)$$

$$= \cos a \cos x + \sin a(-\sin x) = \cos x \cos a - \sin x \sin a = \cos(x+a)$$

(ii) Clearly,

$$\frac{d}{dx}\left\{\frac{\sin(x+a)}{\cos x}\right\}$$

$$= \frac{d}{dx}\left\{\frac{\sin x \cos a + \cos x \sin a}{\cos x}\right\}$$

$$= \frac{d}{dx}\{\tan x \cos x + \sin a\}$$

$$= \frac{d}{dx}(\tan x \cos a) + \frac{d}{dx}(\sin a)$$

$$= \cos a \frac{d}{dx}(\tan x) + \frac{d}{dx}(\sin a) = \cos a \times \sec^2 x + 0 = \sec^2 x \cos a$$

EXERCISE 30.3

LEVEL-1

Differentiate the following functions with respect to x: (1-18)

1. $x^4 - 2 \sin x + 3 \cos x$
2. $3^x + x^3 + 3^3$
3. $\frac{x^3}{3} - 2\sqrt{x} + \frac{5}{x^2}$
4. $e^{x \log a} + e^{a \log x} + e^{a \log a}$
5. $(2x^2 + 1)(3x + 2)$
6. $\log_3 x + 3 \log_e x + 2 \tan x$

7. $\left(x+\frac{1}{x}\right)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$
8. $\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^3$
9. $\frac{2x^2+3x+4}{x}$

10. $\frac{(x^3+1)(x-2)}{x^2}$
11. $\frac{a\cos x+b\sin x+c}{\sin x}$
12. $2\sec x+3\cot x-4\tan x$

13. $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$.
14. $\frac{1}{\sin x}+2^{x+3}+\frac{4}{\log_x 3}$

15. $\frac{(x+5)(2x^2-1)}{x}$
16. $\log\left(\frac{1}{\sqrt{x}}\right)+5x^a-3a^x+3\sqrt{x^2}+6\,{}^4\sqrt{x^{-3}}$

17. $\cos(x+a)$
18. $\frac{\cos(x-2)}{\sin x}$

19. If $y=\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2$, find $\frac{dy}{dx}$ at $x=\frac{\pi}{6}$.

20. If $y=\left(\frac{2-3\cos x}{\sin x}\right)$, find $\frac{dy}{dx}$ at $x=\frac{\pi}{4}$

21. Find the slope of the tangent to the curve $f(x)=2x^6+x^4-1$ at $x=1$.

22. If $y=\sqrt{\frac{x}{a}}+\sqrt{\frac{a}{x}}$, prove that $2xy\frac{dy}{dx}=\left(\frac{x}{a}-\frac{a}{x}\right)$

23. Find the rate at which the function $f(x)=x^4-2x^3+3x^2+x+5$ changes with respect to x.

24. If $y=\frac{2x^9}{3}-\frac{5}{7}x^7+6x^3-x$, find $\frac{dy}{dx}$ at $x=1$.

25. If for $f(x)=\lambda x^2+\mu x+12$, $f'(4)=15$ and $f'(2)=11$, then find λ and μ.

26. For the function $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\ldots+\frac{x^2}{2}+x+1$. Prove that $f'(1)=100\,f'(0)$. [NCERT]

ANSWERS

1. $4x^3-2\cos x-3\sin x$.
2. $3^x\log 3+3x^2$
3. $x^2-x^{-1/2}-10x^{-3}$
4. $a^x\log a+ax^{a-1}$
5. $18x^2+8x+3$
6. $\frac{1}{x\log 3}+\frac{3}{x}+2\sec^2 x$
7. $\frac{3}{2}x^{1/2}+\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}-\frac{3}{2}x^{-5/2}$
8. $\frac{3}{2}x^{1/2}-\frac{3}{2}x^{-5/2}+\frac{3}{2}x^{-1/2}-\frac{3}{2}x^{-3/2}$
9. $2-\frac{4}{x^2}$
10. $2x-2-\frac{1}{x^2}+\frac{4}{x^3}$
11. $-a\,\text{cosec}^2 x-c\,\text{cosec}\,x\,\cot x$.
12. $2\sec x\tan x-3\,\text{cosec}^2 x-4\sec^2 x$
13. $n a_0 x^{n-1}+(n-1)a_1 x^{n-2}+\ldots+a_{n-1}$
14. $-\text{cosec}\,x\,\cot x+2^{x+3}\log 2+\frac{4}{x\log 3}$
15. $4x+10+\frac{5}{x^2}$
16. $-\frac{1}{2x}+5ax^{a-1}-3a^x\log a+\frac{2}{3}x^{-1/3}-\frac{9}{2}x^{-7/4}$
17. $-\sin(x+a)$
18. $-\text{cosec}^2 x\cos 2$
19. $\frac{\sqrt{3}}{2}$
20. $6-2\sqrt{2}$
21. 16
23. $4x^3-6x^2+6x+1$
24. 18
25. $\lambda=1$, $\mu=7$

HINTS TO NCERT & SELECTED PROBLEMS

26. We have,

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

$$\Rightarrow \quad f'(x) = x^{99} + x^{98} + \dots + x + 1$$

$$\therefore \quad f'(1) = 1^{99} + 1^{98} + \dots + 1^1 + 1 = 100 \text{ and } f'(0) = 1.$$

30.4.1 PRODUCT RULE FOR DIFFERENTIATION

THEOREM 1 *If $f(x)$ and $g(x)$ are two differentiable functions, show that $f(x)\, g(x)$ is also differentiable such that*

$$\frac{d}{dx}[f(x)\, g(x)] = f(x) \frac{d}{dx}\{g(x)\} + g(x) \frac{d}{dx}\{f(x)\}$$

i.e. Derivative of the product of two functions

= [(First function) × (Derivative of 2nd function) + (Second function) × (Derivative of first function)]

PROOF Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ and } \frac{d}{dx}(g(x)) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \quad \text{...(i)}$$

Let $\phi(x) = f(x)\, g(x)$. Then,

$$\frac{d}{dx}(\phi(x)) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x)\, g(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h \to 0} \frac{f(x+h)\, g(x+h) - f(x+h)\, g(x) + f(x+h)\, g(x) - f(x)\, g(x)}{h}$$

[Adding and subtracting $f(x+h)\, g(x)$ in numerator]

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h \to 0} \frac{f(x+h)\{g(x+h) - g(x)\} + g(x)\{f(x+h) - f(x)\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h \to 0} \frac{f(x+h)\{g(x+h) - g(x)\}}{h} + \lim_{h \to 0} \frac{g(x)\{f(x+h) - f(x)\}}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(\phi(x)) = f(x) \frac{d}{dx}\{g(x)\} + g(x) \frac{d}{dx}\{f(x)\}$$

$$\left[\begin{array}{l} \because f(x) \text{ is differentiable.} \\ \therefore \text{ It is continuous, and hence } \lim_{h \to 0} f(x+h) = f(x) \end{array}\right]$$

Hence, $\phi(x) = f(x)\, g(x)$ is differentiable and $\frac{d}{dx}\{f(x)\, g(x)\} = f(x) \frac{d}{dx}\{g(x)\} + g(x) \frac{d}{dx}\{f(x)\}$.

REMARK The above result may also be expressed as

$$(fg)' = f'g + fg' \text{ or, } (fg)' = (fg)\left(\frac{f'}{f} + \frac{g'}{g}\right)$$

It can be generalized for the derivative of the product of more than two functions as given below

$$(fgh)' = (fgh)\left(\frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}\right)$$

THEOREM 2 *(Generalization of the product rule) Let $f(x)$, $g(x)$, $h(x)$ be three differentiable functions. Then,*

$$\frac{d}{dx}\{f(x)\,g(x)\,h(x)\}$$

$$= \left\{\frac{d}{dx}(f(x))\right\} g(x)\,h(x) + f(x)\left\{\frac{d}{dx}(g(x))\right\} h(x) + f(x)\,g(x)\left\{\frac{d}{dx}(h(x))\right\}$$

PROOF We have,

$$\frac{d}{dx}\{f(x)\,g(x)\,h(x)\} = \frac{d}{dx}[\{f(x)\,g(x)\}\,h(x)]$$

$$= \{f(x)\,g(x)\}\frac{d}{dx}\{h(x)\} + h(x)\frac{d}{dx}\{f(x)\,g(x)\}$$

$$= \{f(x)\,g(x)\}\frac{d}{dx}\{h(x)\} + h(x)\left[f(x)\frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}\right]$$

$$= \left\{\frac{d}{dx}(f(x))\right\} g(x)\,h(x) + f(x)\left\{\frac{d}{dx}(g(x))\right\} h(x) + f(x)\,g(x)\left\{\frac{d}{dx}(h(x))\right\}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $x \sin x$ (ii) $\dfrac{x^3 \sin x}{\cos x}$

(iii) $e^x \sin x + x^n \cos x$ (iv) $e^x (x + \log x)$

(v) $(x + \sec x)(x - \tan x)$ [NCERT] (vi) $(x + \cos x)(x - \tan x)$ [NCERT]

(vii) $(x^2 + 1)\cos x$ [NCERT] (viii) $(ax^2 + \sin x)(p + q\cos x)$ [NCERT]

SOLUTION (i) $\dfrac{d}{dx}(x \sin x)$

$$= x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x) = x\cos x + \sin x \cdot 1 = x\cos x + \sin x.$$

(ii) We have,

$$\frac{d}{dx}\left(x^3\,\frac{\sin x}{\cos x}\right)$$

$$= \frac{d}{dx}(x^3 \tan x)$$

$$= x^3\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x^3) = x^3\sec^2 x + (\tan x)\,3x^2 = x^3\sec^2 x + 3x^2\tan x.$$

(iii) We have,

$$\frac{d}{dx}(e^x \sin x + x^n \cos x)$$

$$= \frac{d}{dx}(e^x \sin x) + \frac{d}{dx}(x^n \cos x)$$

$$= e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x) + x^n \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^n)$$
$$= e^x \cos x + (\sin x)\, e^x + x^n (-\sin x) + \cos x\,(nx^{n-1})$$
$$= e^x \cos x + e^x \sin x - x^n \sin x + nx^{n-1} \cos x$$

(iv) We have,

$$\frac{d}{dx}\{e^x (x + \log x)\}$$
$$= e^x \frac{d}{dx}(x + \log x) + (x + \log x)\frac{d}{dx}(e^x)$$
$$= e^x \left\{ \frac{d}{dx}(x) + \frac{d}{dx}(\log x) \right\} + (x + \log x)\, e^x$$
$$= e^x \left\{1 + \frac{1}{x}\right\} + (x + \log x)\, e^x = e^x \left(1 + \frac{1}{x} + x + \log x\right)$$

(v) We have,

$$\frac{d}{dx}\{(x + \sec x)(x - \tan x)\}$$
$$= \frac{d}{dx}(x + \sec x) \times (x - \tan x) + (x + \sec x)\frac{d}{dx}(x - \tan x)$$
$$= \left\{\frac{d}{dx}(x) + \frac{d}{dx}(\sec x)\right\}(x - \tan x) + (x + \sec x)\left\{\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right\}$$
$$= (1 + \sec x \tan x)(x - \tan x) + (x + \sec x)(1 - \sec^2 x)$$

(vi) We have,

$$\frac{d}{dx}\left\{(x + \cos x)\ (x - \tan x)\right\}$$
$$= (x - \tan x)\frac{d}{dx}(x + \cos x) + (x + \cos x)\frac{d}{dx}(x - \tan x)$$
$$= (x - \tan x)\left\{\frac{d}{dx}(x) + \frac{d}{dx}(\cos x)\right\} + (x + \cos x)\left\{\frac{d}{dx}(x) - \frac{d}{dx}(\tan x)\right\}$$
$$= (x - \tan x)(1 - \sin x) + (x + \cos x)(1 - \sec^2 x)$$

(vii) We have,

$$\frac{d}{dx}\left\{(x^2 + 1)\cos x\right\}$$
$$= \cos x \frac{d}{dx}(x^2 + 1) + (x^2 + 1)\frac{d}{dx}(\cos x)$$
$$= (2x + 0)\cos x + (x^2 + 1)(-\sin x)$$
$$= 2x \cos x + (x^2 + 1)(-\sin x) = 2x \cos x - (x^2 + 1)\sin x$$

(viii) We have,

$$\frac{d}{dx}\left\{(ax^2 + \sin x)(p + q\cos x)\right\}$$
$$= (p + q\cos x)\frac{d}{dx}(ax^2 + \sin x) + (ax^2 + \sin x)\frac{d}{dx}(p + q\cos x)$$

$$=(p+q\cos x)\left\{\frac{d}{dx}(ax^2)+\frac{d}{dx}(\sin x)\right\}+(ax^2+\sin x)\left\{\frac{d}{dx}(p)+\frac{d}{dx}(q\cos x)\right\}$$

$$=(p+q\cos x)\{a(2x)+\cos x\}+(ax^2+\sin x)(0-q\sin x)$$

$$=(p+q\cos x)(2ax+\cos x)-(ax^2+\sin x)\,q\sin x.$$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $x^3 e^x \sin x$ (ii) $x \sin x \log x$ (iii) $x^n \log_a x\, e^x$

SOLUTION (i) We have,

$$\frac{d}{dx}(x^3 e^x \sin x)$$

$$=\left\{\frac{d}{dx}(x^3)\right\}e^x \sin x + x^3\left\{\frac{d}{dx}(e^x)\right\}\sin x + x^3 e^x\left\{\frac{d}{dx}(\sin x)\right\}$$

$$=3x^2 e^x \sin x + x^3 e^x \sin x + x^3 e^x \cos x = x^2 e^x (3\sin x + x\sin x + x\cos x)$$

(ii) We have,

$$\frac{d}{dx}(x\sin x\log x)$$

$$=\left\{\frac{d}{dx}(x)\right\}\sin x\log x + x\frac{d}{dx}(\sin x)\log x + x\sin x\frac{d}{dx}(\log x)$$

$$=1(\sin x)\cdot\log x + x\cdot(\cos x)\cdot\log x + x\cdot(\sin x)\frac{1}{x} = \sin x\log x + x\cos x\cdot\log x + \sin x.$$

(iii) We have,

$$\frac{d}{dx}\left(x^n\cdot\log_a x\cdot e^x\right)$$

$$=\left\{\frac{d}{dx}(x^n)\right\}(\log_a x)\,e^x + x^n\left\{\frac{d}{dx}(\log_a x)\right\}e^x + x^n\log_a x\left\{\frac{d}{dx}(e^x)\right\}$$

$$=n\,x^{n-1}\log_a x\,e^x + x^n\left(\frac{1}{x\log_e a}\right)e^x + x^n(\log_a x)\,e^x$$

$$=e^x x^{n-1}\left\{n\log_a x + \frac{1}{\log_e a} + x\log_a x\right\}$$

EXAMPLE 3 *Using mathematical induction prove that:* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for all* $n\in N$

SOLUTION Let $P(n)$ be the statement given by

$$P(n):\frac{d}{dx}(x^n) = nx^{n-1}$$

<u>STEP I</u> *We have,*

$$\frac{d}{dx}(x^1)=\frac{d}{dx}(x)=1=1\times x^{1-1}$$

$\therefore$ $P(1)$ is true.

<u>STEP II</u> *Let the statement be true for* $n=m$. *Then,*

$$\frac{d}{dx}(x^m) = mx^{m-1} \qquad \text{...(i)}$$

Now, $\frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x \times x^m)$

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = x^m \frac{d}{dx}(x) + x \frac{d}{dx}(x^m)$ [Using product rule]

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = x^m + x \times m x^{m-1}$ [Using (i)]

$\Rightarrow \quad \frac{d}{dx}(x^{m+1}) = (m+1)\, x^m$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$

i.e. $\frac{d}{dx}(x^n) = nx^{n-1}$ for all $n \in N$.

EXERCISE 30.4

LEVEL-1

Differentiate the following functions with respect to x: (1-22)

1. $x^3 \sin x$
2. $x^3 e^x$
3. $x^2 e^x \log x$
4. $x^n \tan x$
5. $x^n \log_a x$
6. $(x^3 + x^2 + 1) \sin x$
7. $\sin x \cos x$
8. $\frac{2^x \cot x}{\sqrt{x}}$
9. $x^2 \sin x \log x$
10. $x^5 e^x + x^6 \log x$
11. $(x \sin x + \cos x)(x \cos x - \sin x)$
12. $(x \sin x + \cos x)(e^x + x^2 \log x)$
13. $(1 - 2 \tan x)(5 + 4 \sin x)$
14. $(1 + x^2) \cos x$
15. $\sin^2 x$
16. $\log_{x^2} x$
17. $e^x \log \sqrt{x} \tan x$
18. $x^3 e^x \cos x$
19. $\frac{x^2 \cos \frac{\pi}{4}}{\sin x}$ [NCERT]
20. $x^4 (5 \sin x - 3 \cos x)$ [NCERT]
21. $(2x^2 - 3) \sin x$
22. $x^5 (3 - 6x^{-9})$
23. $x^{-4} (3 - 4x^{-5})$
24. $x^{-3} (5 + 3x)$
25. $(ax + b)/(cx + d)$
26. $(ax + b)^n (cx + d)^m$
27. Differentiate in two ways, using product rule and otherwise, the function $(1 + 2 \tan x)(5 + 4 \cos x)$. Verify that the answers are the same.
28. Differentiate each of the following functions by the product rule and the other method and verify that answer from both the methods is the same.

(i) $(3x^2 + 2)^2$ (ii) $(x + 2)(x + 3)$ (iii) $(3 \sec x - 4 \operatorname{cosec} x)(-2 \sin x + 5 \cos x)$

ANSWERS

1. $x^2 (x \cos x + 3 \sin x)$
2. $x^2 e^x (3 + x)$
3. $xe^x (1 + x \log x + 2 \log x)$
4. $x^{n-1} (n \tan x + x \sec^2 x)$
5. $x^{n-1} \left(n \log_a x + \frac{1}{\log a} \right)$

6. $(x^3 + x^2 + 1)\cos x + (3x^2 + 2x)\sin x$ 7. $\cos 2x$

8. $\frac{2^x}{\sqrt{x}}\left\{\log 2 \cdot \cot x - \operatorname{cosec}^2 x - \frac{\cot x}{2x}\right\}$ 9. $2x \sin x \log x + x^2 \cos x \cdot \log x + x \sin x$

10. $x^4 (5e^x + xe^x + x + 6x \log x)$ 11. $x\{x \cos 2x - \sin 2x\}$

12. $x \cos x \{e^x + x^2 \log x\} + (x \sin x + \cos x)(e^x + x + 2x \log x)$

13. $4(\cos x - 2 \sin x - 2 \tan x \sec x - 5/2 \sec^2 x)$

14. $2x \cos x - (1 + x^2)\sin x$ 15. $\sin 2x$ 16. 0

17. $\frac{1}{2}e^x\left\{\log x . \tan x + \frac{\tan x}{x} + \log x . \sec^2 x\right\}$

18. $x^2 e^x (x \cos x + 3 \cos x - x \sin x)$ 19. $\left(\frac{2x}{\sin x} - \frac{x^2 \cot x}{\sin x}\right)\cos\frac{\pi}{4}$

20. $20x^3 \sin x + 5x^4 \cos x - 12x^3 \cos x + 3x^4 \sin x$

21. $4x \sin x + (2x^2 - 3)\cos x$ 22. $15x^4 + 24x^{-5}$ 23. $-12x^{-5} + 36x^{-10}$

24. $-15x^{-4} - 6x^{-3}$ 25. $\frac{ad - bc}{(cx + d)^2}$

26. $(ax + b)^{n-1}(cx + d)^{m-1}\{mc(ax + b) + na(cx + d)\}$

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19. $$\frac{d}{dx}\left(\frac{x^2 \cos\frac{\pi}{4}}{\sin x}\right) = \frac{1}{\sqrt{2}}\frac{d}{dx}(x^2 \operatorname{cosec} x)$$

$$= \frac{1}{\sqrt{2}}\left\{\operatorname{cosec} x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\operatorname{cosec} x)\right\} = \frac{1}{\sqrt{2}}(2x \operatorname{cosec} x - x^2 \operatorname{cosec} x \cot x)$$

20. $$\frac{d}{dx}\left\{x^4 (5 \sin x - 3 \cos x)\right\} = (5 \sin x - 3 \cos x)\frac{d}{dx}(x^4) + x^4 \frac{d}{dx}(5 \sin x - 3 \cos x)$$

$$= 4x^3 (5 \sin x - 3 \cos x) + x^4(5 \cos x + 3 \sin x)$$

30.4.2 QUOTIENT RULE FOR DIFFERENTIATION

THEOREM *If $f(x)$ and $g(x)$ are two differentiable functions and $g(x) \neq 0$, then show that $\frac{f(x)}{g(x)}$ is also differentiable and*

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}\{f(x)\} - f(x)\frac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

PROOF Since $f(x)$ and $g(x)$ are differentiable functions. Therefore,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0}\frac{f(x+h) - f(x)}{h} \text{ and, } \frac{d}{dx}(g(x)) = \lim_{h \to 0}\frac{g(x+h) - g(x)}{h} \quad ...(i)$$

Let $\phi(x) = \dfrac{f(x)}{g(x)}$. Then,

$$\frac{d}{dx}(\phi(x))$$

$$= \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x)\,g(x+h)\,h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{h\,g(x)\,g(x+h)}$$

[On subtracting and adding $f(x)\,g(x)$ in numerator]

$$= \lim_{h \to 0} \frac{g(x)\,[f(x+h) - f(x)] - f(x)\,[g(x+h) - g(x)]}{h\,g(x)\,g(x+h)}$$

$$= \left[\lim_{h \to 0} g(x) \left\{\frac{f(x+h) - f(x)}{h}\right\} - \lim_{h \to 0} f(x) \left\{\frac{g(x+h) - g(x)}{h}\right\}\right] \times \left[\lim_{h \to 0} \frac{1}{g(x)\,g(x+h)}\right]$$

$$= \left[g(x) \lim_{h \to 0} \left\{\frac{f(x+h) - f(x)}{h}\right\} - f(x) \lim_{h \to 0} \left\{\frac{g(x+h) - g(x)}{h}\right\}\right] \times \left[\lim_{h \to 0} \frac{1}{g(x)\,g(x+h)}\right]$$

$$= \left[g(x) \frac{d}{dx}\{f(x)\} - f(x) \frac{d}{dx}\{g(x)\}\right] \times \frac{1}{[g(x)]^2}$$

$\left[\begin{array}{l} \because\ g(x) \text{ is differentiable.} \\ \therefore\ \text{It is continuous, and hence} \\ \displaystyle\lim_{h \to 0} g(x+h) = g(x) \end{array}\right]$

$$= \frac{g(x) \dfrac{d}{dx}\{f(x)\} - f(x) \dfrac{d}{dx}\{g(x)\}}{[g(x)]^2}$$

Hence, $\dfrac{f(x)}{g(x)}$ is differentiable and $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{g(x) \dfrac{d}{dx}\{f(x)\} - f(x) \dfrac{d}{dx}\{g(x)\}}{[g(x)]^2}$

<u>REMARK</u> *It is advisable to remember this result in the following form:*

$$\frac{d}{dx}\left\{\frac{N^r}{D^r}\right\} = \frac{D^r \dfrac{d}{dx}(N^r) - N^r \dfrac{d}{dx}(D^r)}{(D^r)^2}, \text{ where } N^r = \text{Numerator}, D^r = \text{Denominator.}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Differentiate the following functions with respect to x:*

(i) $\dfrac{e^x}{1 + \sin x}$ (ii) $\dfrac{x + \sin x}{x + \cos x}$

(iii) $\dfrac{\sin x + \cos x}{\sin x - \cos x}$ [NCERT] (iv) $\dfrac{\sec x - 1}{\sec x + 1}$ [NCERT]

SOLUTION (i) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{e^x}{1+\sin x}\right)$$

$$= \frac{(1+\sin x)\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(1+\sin x)}{(1+\sin x)^2}$$

$$= \frac{(1+\sin x)\,e^x - e^x\,(0+\cos x)}{(1+\sin x)^2} = \frac{e^x\,(1+\sin x-\cos x)}{(1+\sin x)^2}.$$

(ii) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{x+\sin x}{x+\cos x}\right)$$

$$= \frac{(x+\cos x)\frac{d}{dx}(x+\sin x) - (x+\sin x)\frac{d}{dx}(x+\cos x)}{(x+\cos x)^2}$$

$$= \frac{(x+\cos x)(1+\cos x) - (x+\sin x)(1-\sin x)}{(x+\cos x)^2}$$

$$= \frac{x+\cos x + x\cos x + \cos^2 x - x - \sin x + x\sin x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x\cos x + x\sin x + \cos^2 x + \sin^2 x}{(x+\cos x)^2}$$

$$= \frac{\cos x - \sin x + x(\cos x + \sin x) + 1}{(x+\cos x)^2}.$$

(iii) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)$$

$$= \frac{(\sin x-\cos x)\frac{d}{dx}(\sin x+\cos x) - (\sin x+\cos x)\frac{d}{dx}(\sin x-\cos x)}{(\sin x-\cos x)^2}$$

$$= \frac{(\sin x-\cos x)(\cos x-\sin x) - (\sin x+\cos x)(\cos x+\sin x)}{(\sin x-\cos x)^2}$$

$$= \frac{-(\sin x-\cos x)^2 - (\sin x+\cos x)^2}{(\sin x-\cos x)^2}$$

$$= \frac{-\left\{(\sin x-\cos x)^2 + (\sin x+\cos x)^2\right\}}{(\sin x-\cos x)^2} = \frac{-2\left(\sin^2 x+\cos^2 x\right)}{\sin^2 x+\cos^2 x-2\sin x\cos x} = \frac{-2}{1-\sin 2x}$$

(iv) We have,

$$\frac{d}{dx}\left(\frac{\sec x-1}{\sec x+1}\right) = \frac{(\sec x+1)\frac{d}{dx}(\sec x-1) - (\sec x-1)\frac{d}{dx}(\sec x+1)}{(\sec x+1)^2}$$

$$= \frac{(\sec x + 1)(\sec x \tan x - 0) - (\sec x - 1)(\sec x \tan x + 0)}{(\sec x + 1)^2}$$

$$= \frac{\sec^2 x \tan x + \sec x \tan x - \sec^2 x \tan x + \sec x \tan x}{(\sec x + 1)^2}$$

$$= \frac{2 \sec x \tan x}{(\sec x + 1)^2} = \frac{2 \sin x}{(1 + \cos x)^2}$$

EXAMPLE 2 *Differentiate the following functions with respect to x:*

(i) $\dfrac{2x + 3}{x^2 - 5}$ (ii) $\dfrac{x + 3}{x^2 + 1}$ (iii) $\dfrac{1 + \tan x}{1 - \tan x}$ (iv) $\dfrac{\sec x + \tan x}{\sec x - \tan x}$

SOLUTION (i) Using quotient rule, we have

$$\frac{d}{dx}\left(\frac{2x + 3}{x^2 - 5}\right) = \frac{(x^2 - 5)\frac{d}{dx}(2x + 3) - (2x + 3)\frac{d}{dx}(x^2 - 5)}{(x^2 - 5)^2}$$

$$= \frac{(x^2 - 5)(2) - (2x + 3)(2x)}{(x^2 - 5)^2} = \frac{-2(x^2 + 3x + 5)}{(x^2 - 5)^2}.$$

(ii) $$\frac{d}{dx}\left(\frac{x + 3}{x^2 + 1}\right) = \frac{(x^2 + 1)\frac{d}{dx}(x + 3) - (x + 3)\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)\,1 - (x + 3)(2x)}{(x^2 + 1)^2} = \frac{-x^2 - 6x + 1}{(x^2 + 1)^2} = \frac{1 - 6x - x^2}{(x^2 + 1)^2}.$$

(iii) $$\frac{d}{dx}\left(\frac{1 + \tan x}{1 - \tan x}\right) = \frac{(1 - \tan x)\frac{d}{dx}(1 + \tan x) - (1 + \tan x)\frac{d}{dx}(1 - \tan x)}{(1 - \tan x)^2}$$

$$= \frac{(1 - \tan x)(0 + \sec^2 x) - (1 + \tan x)(0 - \sec^2 x)}{(1 - \tan x)^2}$$

$$= \frac{2 \sec^2 x}{(1 - \tan x)^2} = \frac{2}{(\cos x - \sin x)^2} = \frac{2}{1 - \sin 2x}.$$

(iv) We have,

$$\frac{\sec x + \tan x}{\sec x - \tan x} = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x}}{\frac{1}{\cos x} - \frac{\sin x}{\cos x}} = \frac{1 + \sin x}{1 - \sin x}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{\sec x + \tan x}{\sec x - \tan x}\right) = \frac{d}{dx}\left(\frac{1 + \sin x}{1 - \sin x}\right)$$

$$= \frac{(1 - \sin x)\frac{d}{dx}(1 + \sin x) - (1 + \sin x)\frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$

$$= \frac{(1 - \sin x)(0 + \cos x) - (1 + \sin x)(0 - \cos x)}{(1 - \sin x)^2} = \frac{2 \cos x}{(1 - \sin x)^2}$$

EXERCISE 30.5

LEVEL-1

Differentiate the following functions with respect to x :

1. $\dfrac{x^2+1}{x+1}$
2. $\dfrac{2x-1}{x^2+1}$
3. $\dfrac{x+e^x}{1+\log x}$
4. $\dfrac{e^x-\tan x}{\cot x - x^n}$
5. $\dfrac{ax^2+bx+c}{px^2+qx+r}$
6. $\dfrac{x}{1+\tan x}$ [NCERT]
7. $\dfrac{1}{ax^2+bx+c}$
8. $\dfrac{e^x}{1+x^2}$
9. $\dfrac{e^x+\sin x}{1+\log x}$
10. $\dfrac{x\tan x}{\sec x+\tan x}$
11. $\dfrac{x\sin x}{1+\cos x}$
12. $\dfrac{2^x\cot x}{\sqrt{x}}$
13. $\dfrac{\sin x - x\cos x}{x\sin x+\cos x}$
14. $\dfrac{x^2-x+1}{x^2+x+1}$
15. $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$
16. $\dfrac{a+\sin x}{1+a\sin x}$
17. $\dfrac{10^x}{\sin x}$
18. $\dfrac{1+3^x}{1-3^x}$
19. $\dfrac{3^x}{x+\tan x}$
20. $\dfrac{1+\log x}{1-\log x}$
21. $\dfrac{4x+5\sin x}{3x+7\cos x}$ [NCERT]
22. $\dfrac{x}{1+\tan x}$ [NCERT]
23. $\dfrac{a+b\sin x}{c+d\cos x}$ [NCERT]
24. $\dfrac{px^2+qx+r}{ax+b}$ [NCERT]
25. $\dfrac{\sec x-1}{\sec x+1}$ [NCERT]
26. $\dfrac{x^5-\cos x}{\sin x}$ [NCERT]
27. $\dfrac{x+\cos x}{\tan x}$ [NCERT]
28. $\dfrac{x^n}{\sin x}$ [NCERT]
29. $\dfrac{ax+b}{px^2+qx+r}$ [NCERT]
30. $\dfrac{1}{ax^2+bx+c}$ [NCERT]

ANSWERS

1. $\dfrac{x^2+2x-1}{(x+1)^2}$
2. $\dfrac{2(1+x-x^2)}{(1+x^2)^2}$
3. $\dfrac{x\log x\cdot(1+e^x)-e^x(1-x)}{x(1+\log x)^2}$
4. $\dfrac{(\cot x - x^n)(e^x-\sec^2 x)+(e^x-\tan x)(\operatorname{cosec}^2 x+nx^{n-1})}{(\cot x-x^n)^2}$
5. $\dfrac{(aq-bp)x^2+2(ar-cp)x+br-cq}{(px^2+qx+r)^2}$
6. $\dfrac{1+\tan x - x\sec^2 x}{(1+\tan x)^2}$
7. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$
8. $\dfrac{e^x(1-x)^2}{(1+x^2)^2}$
9. $\dfrac{x(1+\log x)(e^x+\cos x)-(e^x+\sin x)}{x(1+\log x)^2}$
10. $\dfrac{x\sec x(\sec x-\tan x)+\tan x}{(\sec x+\tan x)}$
11. $\dfrac{(x+\sin x)}{(1+\cos x)}$
12. $\dfrac{2^x\left[-x\operatorname{cosec}^2 x+x\cot x\cdot\log 2-\left(\frac{1}{2}\right)\cot x\right]}{x^{3/2}}$

13. $\dfrac{x^2}{(x\sin x+\cos x)^2}$ 14. $\dfrac{2(x^2-1)}{(x^2+x+1)^2}$ 15. $\dfrac{\sqrt{a}}{\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$ 16. $\dfrac{(1-a^2)\cos x}{(1+a\sin x)^2}$

17. $10^x \operatorname{cosec} x\,[\log 10-\cot x]$ 18. $\dfrac{2\cdot 3^x\log 3}{(1-3^x)^2}$

19. $\dfrac{3^x\{(x+\tan x)\log 3-(1+\sec^2 x)\}}{(x+\tan x)^2}$ 20. $\dfrac{2}{x(1-\log x)^2}$

21. $\dfrac{15x\cos x+28x\sin x+28\cos x-15\sin x+35}{(3x+7\cos x)^2}$

22. $\dfrac{1+\tan x-x\sec^2 x}{(1+\tan x)^2}$ 23. $\dfrac{bc\cos x+ad\sin x+bd}{(c+d\cos x)^2}$

24. $\dfrac{apx^2+2bpx+bq-ar}{(ax+b)^2}$ 25. $\dfrac{2\sec x\tan x}{(\sec x+1)^2}$

26. $\dfrac{-x^5\cos x+5x^4\sin x-1}{\sin^2 x}$ 27. $\dfrac{(1-\sin x)\tan x-(x+\cos x)\sec^2 x}{\tan^2 x}$

28. $\dfrac{nx^{n-1}\sin x-x^n\cos x}{\sin^2 x}$ 29. $\dfrac{-apx^2-2bpx+ar-bq}{(px^2+qx+r)^2}$ 30. $\dfrac{-(2ax+b)}{(ax^2+bx+c)^2}$

HINTS TO NCERT & SELECTED PROBLEMS

6. $$\frac{d}{dx}\left(\frac{x}{1+\tan x}\right)=\frac{(1+\tan x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}=\frac{(1+\tan x)-x\sec^2 x}{(1+\tan x)^2}$$

21. $$\frac{d}{dx}\left(\frac{4x+5\sin x}{3x+7\cos x}\right)=\frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x)-(4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2}$$

$$=\frac{(3x+7\cos x)(4+5\cos x)-(4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2}$$

$$=\frac{15x\cos x+28x\sin x+28\cos x-15\sin x+35}{(3x+7\cos x)^2}$$

22. $$\frac{d}{dx}\left(\frac{1}{ax^2+bx+c}\right)=\frac{(ax^2+bx+c)\frac{d}{dx}(1)-\frac{d}{dx}(ax^2+bx+c)\times 1}{(ax^2+bx+c)^2}$$

$$=\frac{(ax^2+bx+c)\times 0-(2ax+b)}{(ax^2+bx+c)^2}=\frac{-(2ax+b)}{(ax^2+bx+c)^2}$$

23. $$\frac{d}{dx}\left(\frac{a+b\sin x}{c+d\cos x}\right)=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

$$=\frac{(c+d\cos x)(0+b\cos x)-(a+b\sin x)(0-d\sin x)}{(c+d\cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

24. $$\frac{d}{dx}\left(\frac{px^2 + qx + r}{ax + b}\right) = \frac{(ax + b)\frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r)\frac{d}{dx}(ax + b)}{(ax + b)^2}$$

$$= \frac{(ax + b)(2px + q) - a(px^2 + qx + r)}{(ax + b)^2} = \frac{apx^2 + 2bpx + bq - ar}{(ax + b)^2}$$

25. $$\frac{d}{dx}\left(\frac{\sec x - 1}{\sec x + 1}\right) = \frac{(\sec x + 1)\frac{d}{dx}(\sec x - 1) - (\sec x - 1)\frac{d}{dx}(\sec x + 1)}{(\sec x + 1)^2}$$

$$= \frac{(\sec x + 1)\sec x \tan x - (\sec x - 1)(\sec x \tan x)}{(\sec x + 1)^2} = \frac{2 \sec x \tan x}{(\sec x + 1)^2}$$

26. $$\frac{d}{dx}\left(\frac{x^5 - \cos x}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x)\frac{d}{dx}(\sin x)}{(\sin x)^2}$$

$$= \frac{\sin x(5x^4 + \sin x) - (x^5 - \cos x)\cos x}{\sin^2 x} = \frac{-x^5 \cos x + 5x^4 \sin x + 1}{\sin^2 x}$$

27. $$\frac{d}{dx}\left(\frac{x + \cos x}{\tan x}\right) = \frac{\tan x \frac{d}{dx}(x + \cos x) - (x + \cos x)\frac{d}{dx}(\tan x)}{\tan^2 x}$$

$$= \frac{\tan x(1 - \sin x) - (x + \cos x)\sec^2 x}{\tan^2 x}$$

28. $$\frac{d}{dx}\left(\frac{x^n}{\sin x}\right) = \frac{\sin x \frac{d}{dx}(x^n) - x^n \frac{d}{dx}(\sin x)}{(\sin x)^2} = \frac{(\sin x)(nx^{n-1}) - x^n \cos x}{\sin^2 x}$$

29. $$\frac{d}{dx}\left(\frac{ax + b}{px^2 + qx + r}\right) = \frac{(px^2 + qx + r)\frac{d}{dx}(ax + b) - (ax + b)\frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2}$$

$$= \frac{a(px^2 + qx + r) - (ax + b)(2px + q)}{(px^2 + qx + r)^2} = \frac{-apx^2 - 2bpx + ar - bq}{(px^2 + qx + r)^2}$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$.

2. Write the value of $\lim_{x \to a} \frac{x f(a) - a f(x)}{x - a}$.

3. If $x < 2$, then write the value of $\frac{d}{dx}(\sqrt{x^2 - 4x + 4})$.

4. If $\frac{\pi}{2} < x < \pi$, then find $\frac{d}{dx}\left(\sqrt{\frac{1 + \cos 2x}{2}}\right)$.

5. Write the value of $\frac{d}{dx}(x|x|)$.

6. Write the value of $\frac{d}{dx}\{(x + |x|)|x|\}$.

7. If $f(x) = |x| + |x - 1|$, write the value of $\frac{d}{dx}(f(x))$.

8. Write the value of the derivative of $f(x) = |x - 1| + |x - 3|$ at $x = 2$.

9. If $f(x) = \frac{x^2}{|x|}$, write $\frac{d}{dx}(f(x))$.

10. Write the value of $\frac{d}{dx}(\log|x|)$.

11. If $f(1) = 1, f'(1) = 2$, then write the value of $\lim_{x \to 1} \frac{\sqrt{f(x)} - 1}{\sqrt{x} - 1}$.

12. Write the derivative of $f(x) = 3|2 + x|$ at $x = -3$.

13. If $|x| < 1$ and $y = 1 + x + x^2 + x^3 + \dots$, then write the value of $\frac{dy}{dx}$.

14. If $f(x) = \log_{x^2} x^3$, write the value of $f'(x)$.

ANSWERS

1. $f'(c)$
2. $f(a) - a f'(a)$
3. -1
4. $\sin x$
5. $\begin{cases} 2x, & x > 0 \\ -2x, & x < 0 \end{cases}$
6. $\begin{cases} 0, & x < 0 \\ 4x, & x > 0 \end{cases}$
7. $\frac{d}{dx}\{f(x)\} = \begin{cases} 2, & x > 1 \\ 0, & 0 < x < 1 \\ -2, & x < 1 \end{cases}$
8. 0
9. $\frac{d}{dx}(f(x)) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$
10. $\frac{1}{|x|}, x \neq 0$
11. 2
12. -3
13. $\frac{1}{(1 - x)^2}$
14. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. Let $f(x) = x - [x]$, $x \in R$, then $f'\left(\frac{1}{2}\right)$ is

(a) $\frac{3}{2}$ (b) 1 (c) 0 (d) -1

2. If $f(x) = \frac{x - 4}{2\sqrt{x}}$, then $f'(1)$ is

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$ (c) 1 (d) 0

3. If $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, then $\frac{dy}{dx} =$

(a) $y + 1$ (b) $y - 1$ (c) y (d) y^2

4. If $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{99} + x^{100}$, then $f'(1)$ equals

(a) 150 (b) –50 (c) –150 (d) 50

5. If $y = \dfrac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\dfrac{dy}{dx} =$

(a) $-\dfrac{4x}{(x^2-1)^2}$ (b) $-\dfrac{4x}{x^2-1}$ (c) $\dfrac{1-x^2}{4x}$ (d) $\dfrac{4x}{x^2-1}$

6. If $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, then $\dfrac{dy}{dx}$ at $x = 1$ is

(a) 1 (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{2}}$ (d) 0

7. If $f(x) = x^{100} + x^{99} + \ldots + x + 1$, then $f'(1)$ is equal to

(a) 5050 (b) 5049 (c) 5051 (d) 50051

8. If $f(x) = 1 + x + \dfrac{x^2}{2} + \ldots + \dfrac{x^{100}}{100}$, then $f'(1)$ is equal to

(a) $\dfrac{1}{100}$ (b) 100 (c) 50 (d) 0

9. If $y = \dfrac{\sin x + \cos x}{\sin x - \cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) –2 (b) 0 (c) 1/2 (d) does not exist

10. If $y = \dfrac{\sin(x+9)}{\cos x}$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(a) cos 9 (b) sin 9 (c) 0 (d) 1

11. If $f(x) = \dfrac{x^n - a^n}{x - a}$, then $f'(a)$ is

(a) 1 (b) 0 (c) $\dfrac{1}{2}$ (d) does not exist

12. If $f(x) = x \sin x$, then $f'(\pi/2) =$

(a) 0 (b) 1 (c) – 1 (d) $\dfrac{1}{2}$

ANSWERS

1. (b) 2. (a) 3. (c) 4. (d) 5. (a) 6. (d) 7. (a) 8. (b)
9. (a) 10. (a) 11. (d) 12. (b)

SUMMARY

1. A function $f(x)$ is differentiable at $x = c$ iff $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists finitely.

   This limit is called the derivative or differentiation of $f(x)$ at $x = c$ and is denoted by $f'(c)$.
2. Geometrically the derivative of a function $f(x)$ at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.
3. If $f(x)$ is a differentiable function, then $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ is called the differentiation of $f(x)$ or differentiation of $f(x)$ with respect to x.

4. Mechanically, $\frac{d}{dx}(f(x))$ measures the rate of change of $f(x)$ with respect to x.

5. Following are some standard derivatives:

(i) $\frac{d}{dx}(x^n) = n\,x^{n-1}$

(ii) $\frac{d}{dx}(a^x) = a^x \log_e a,\ a > 0,\ a \neq 1$

(iii) $\frac{d}{dx}(e^x) = e^x$

(iv) $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

(v) $\frac{d}{dx}(\sin x) = \cos x$

(vi) $\frac{d}{dx}(\cos x) = -\sin x$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(viii) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(x) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

6. Following are the fundamental rules for differentiation:

(i) Differentiation of a constant function is zero i.e., $\frac{d}{dx}(c) = 0$

(ii) Differentiation of a constant and a function is equal to constant times the differentiation of the function.

(iii) If $f(x)$ and $g(x)$ are differentiable functions, then

(a) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

(b) $\frac{d}{dx}\{f(x) \times g(x)\} = g(x) \times \frac{d}{dx}(f(x)) + f(x) \times \frac{d}{dx}(g(x))$

(c) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \times \frac{d}{dx}(f(x)) - f(x) \times \frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$