CHAPTER 7

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS

7.1 INTRODUCTION

Consider the following system of m linear equations in n unknowns:

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array} \quad \dots(i)$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or $\quad AX = B$, where $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{n \times 1}$

The $m \times n$ matrix A is called the *coefficient matrix* of the system of linear equations.

ILLUSTRATION *Express the following system of simultaneous linear equation as a matrix equation:*

$$\begin{aligned} 2x + 3y - z &= 1 \\ x + y + 2z &= 2 \\ 2x - y + z &= 3 \end{aligned}$$

SOLUTION The given system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ or, } AX = B, \text{ where } A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

SOLUTION *A set of values of the variables $x_1, x_2, \dots x_n$ which simultaneously satisfy all the equations is called a solution of the system of equations.*

For example, $x = 2, y = -3$ is a solution of the system of linear equations

$$\begin{aligned} 3x + y &= 3 \\ 2x + y &= 1 \end{aligned}$$

because $3(2) + (-3) = 3$ and, $2(2) + (-3) = 1$.

CONSISTENT SYSTEM *If the system of equations has one or more solutions, then it is said to be a consistent system of equations, otherwise it is an inconsistent system of equations.*

For example, the system of linear equations

$$2x + 3y = 5$$
$$4x + 6y = 10$$

is consistent, because $x = 1$, $y = 1$ and $x = 2$, $y = 1/3$ are solutions of it.

However, the system of linear equations

$$2x + 3y = 5$$
$$4x + 6y = 9$$

is inconsistent, because there is no set of values of x, y which satisfy the two equations simultaneously.

HOMOGENEOUS AND NON-HOMOGENEOUS SYSTEMS *A system of equations $AX = B$ is called a homogeneous system if $B = O$. Otherwise, it is called a non-homogeneous system of equations.*

For example, the system of equations

$$2x + 3y = 0$$
$$3x - y = 0$$

is a homogeneous system of linear equations whereas the system of equations given by

$$2x + 3y = 1$$
$$3x - y = 5$$

is a non-homogeneous system of linear equations.

7.2 MATRIX METHOD FOR THE SOLUTION OF A NON-HOMOGENEOUS SYSTEM

In the previous section, we have seen that a system of simultaneous linear equations can be expressed as a matrix equation. In this section, we shall discuss about a method for solving a system of non-homogenous simultaneous linear equations in which the number of unknowns is same as the number of equations. In this method, we will use the inverse of the coefficient matrix. So, it is also known as matrix method.

THEOREM 1 *If A is a non-singular matrix, then the system of equations given by $AX = B$ has the unique solution given by $X = A^{-1}B$.*

PROOF We have, $AX = B$, where $|A| \neq 0$. Since $|A| \neq 0$, therefore A^{-1} exists.

Pre-multiplying both sides of $AX = B$ by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B \Rightarrow (A^{-1}A)X = A^{-1}B \Rightarrow IX = A^{-1}B \Rightarrow X = A^{-1}B$$

Thus, the system of equations $AX = B$ has a solution given by $X = A^{-1}B$.

Uniqueness: If possible, let X_1 and X_2 be two solutions of $AX = B$. Then,

$$AX_1 = B \text{ and } AX_2 = B$$
$$\Rightarrow \quad AX_1 = AX_2$$
$$\Rightarrow \quad A^{-1}(AX_1) = A^{-1}(AX_2) \Rightarrow (A^{-1}A)X_1 = (A^{-1}A)X_2 \Rightarrow IX_1 = IX_2 \Rightarrow X_1 = X_2.$$

Hence, the given system of equations has the unique solution given by $X = A^{-1}B$.

Q.E.D.

In the above theorem, we have proved that a non-homogenous system $AX = B$ of n simultaneous linear equations with n-unknowns has the unique solution given by $X = A^{-1}B$, if A is a non-singular matrix. Now, a natural question arises, what happens when A is a singular matrix? In order to answer this, let us consider the following system of equations:

$$2x + y = 3$$
$$4x + 2y = 6$$

This system of equations can be written in matrix form as

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \text{ or } AX = B \text{ where } A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Clearly, $|A| = 0$. Also, the system of equations has infinitely many solutions as the two equations represent coincident lines in xy-plane.

Now, consider the following system of equations:

$$2x + y = 3$$
$$4x + 2y = 5$$

For this system of equations also the determinant of the coefficient matrix A is zero i.e. A is a singular matrix. But, the system has no solution i.e. it is an inconsistent system of equations, as the lines represented by the two equations are non-coincident parallel lines.

It follows from the above discussion that the system of equations $AX = B$ may be inconsistent or it may be consistent with infinitely many solutions when the coefficient matrix A is singular.

We now state and prove the following criterion for the consistency or inconsistency of a non-homogenous system of linear equations.

THEOREM 2 *(Criterion of consistency) Let $AX = B$ be a system of n-linear equations in n unknowns.*

(i) *If $|A| \neq 0$, then the system is consistent and has the unique solution given by $X = A^{-1}B$.*

(ii) *If $|A| = 0$ and $(adj\ A)\ B = O$, then the system is consistent and has infinitely many solutions.*

(iii) *If $|A| = 0$ and $(adj\ A)\ B \neq O$, then the system is inconsistent.*

PROOF (i) See Theorem 1

(ii) We have,

$$AX = B, \text{ where } |A| = 0.$$
$$\Rightarrow \quad (adj\ A)\ (AX) = (adj\ A)\ B$$
$$\Rightarrow \quad ((adj\ A)\ A)X = (adj\ A)\ B$$
$$\Rightarrow \quad (|A|\ I_n)X = (adj\ A)\ B \qquad [\because (adj\ A)\ A = |A|\ I_n]$$
$$\Rightarrow \quad |A|\ X = (adj\ A)\ B$$

If $|A| = 0$ and $(adj\ A)\ B = O$, then $|A|\ X = (adj\ A)\ B$ is true for every value of X.

So, the system of equations $AX = B$ is consistent and it has infinitely many solutions.

(iii) If $|A| = 0$ and $(adj\ A)B \neq O$, then the equation $|A|\ X = (adj\ A)\ B$ is not true because its LHS is always a null matrix whereas the RHS is non-null matrix. So, the system is inconsistent.

Q.E.D.

The above discussion suggests the following algorithm to solve a system of simultaneous linear equations.

ALGORITHM

Step I *Obtain the system of equations and express it in the matrix equation from $AX = B$.*

Step II *Find $|A|$.*

Step III *If $|A| \neq 0$, then the given system of equations is consistent with unique solution. To obtain the solution compute A^{-1} by using $A^{-1} = \frac{1}{|A|}$ adj A and use the formula $X = A^{-1}\ B$.*

Step IV *If $|A| = 0$, then the given system of equations is either inconsistent or it has infinitely many solutions. To distinguish these two proceed as follows:*

Compute $(adj\ A)\ B$.

If $(adj\ A)\ B \neq O$, then the given system of equations is inconsistent i.e. it has no solution.

If $(adj\ A)\ B = O$, then the given system of equations is consistent with infinitely many solutions.

In order to find these infinitely many solutions, replace one of the variables by some real number. This will reduce the number of variables by one. Now, take any two out of the three equations and solve them by matrix method.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I SOLVING THE GIVEN SYSTEM OF LINEAR EQUATIONS WHEN THE COEFFICIENT MATRIX IS NON-SINGULAR

EXAMPLE 1 *Use matrix method to solve the following system of equations :*

$$5x-7y = 2, 7x-5y = 3$$

SOLUTION The given system of equations can be written as

$$5x-7y = 2$$
$$7x-5y = 3$$

or, $\begin{bmatrix}5 & -7\\ 7 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}2\\ 3\end{bmatrix}$ or, $AX = B$ where $A=\begin{bmatrix}5 & -7\\ 7 & -5\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}2\\ 3\end{bmatrix}$

Now, $|A| = \begin{vmatrix}5 & -7\\ 7 & -5\end{vmatrix} = -25+49=24 \neq 0$. So, the given system has a unique solution given by $X=A^{-1}B$.

Let C_{ij} be the cofactors of elements a_{ij} in $A=[a_{ij}]$. Then,

$C_{11}=(-1)^{1+1}(-5)=-5,\ C_{12}=(-1)^{1+2}7=-7,\ C_{21}=(-1)^{2+1}(-7)=7, C_{22}=(-1)^{2+2}5=5.$

$$\therefore \quad \text{adj } A = \begin{bmatrix}-5 & -7\\ 7 & 5\end{bmatrix}^T = \begin{bmatrix}-5 & 7\\ -7 & 5\end{bmatrix} \text{ and, } A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{24}\begin{bmatrix}-5 & 7\\ -7 & 5\end{bmatrix}$$

$$\therefore \quad X = A^{-1}B = \frac{1}{24}\begin{bmatrix}-5 & 7\\ -7 & 5\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix} = \frac{1}{24}\begin{bmatrix}-10+21\\ -14+15\end{bmatrix} = \begin{bmatrix}11/24\\ 1/24\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}11/24\\ 1/24\end{bmatrix} \Rightarrow x = \frac{11}{24} \text{ and } y = \frac{1}{24}$$

Hence, $x=11/24$ and $y=1/24$ is the required solution.

EXAMPLE 2 *Use matrix method to solve the following system of equations:*

$$x-2y-4 = 0, -3x+5y+7 = 0$$

SOLUTION The given system of equations can be written as

$$x-2y = 4$$
$$-3x+5y = -7$$

or, $\begin{bmatrix}1 & -2\\ -3 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}4\\ -7\end{bmatrix}$ or, $AX=B$ where $A=\begin{bmatrix}1 & -2\\ -3 & 5\end{bmatrix}$, $X=\begin{bmatrix}x\\ y\end{bmatrix}$ and $B=\begin{bmatrix}4\\ -7\end{bmatrix}$

Now, $|A| = \begin{vmatrix}1 & -2\\ -3 & 5\end{vmatrix} = 5-6 = -1 \neq 0$

So, the given system has a unique solution given by $X=A^{-1}B$. Let C_{ij} be the co-factors of elements a_{ij} in $A=[a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}5 = 5, C_{12} = (-1)^{1+2}(-3) = 3, C_{21}=(-1)^{2+1}(-2)=2$$

and $C_{22} = (-1)^{2+2}1 = 1.$

$$\therefore \quad \text{adj}\,A = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} \text{ and, } A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{(-1)}\begin{bmatrix} 5 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B = \begin{bmatrix} -5 & -2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -7 \end{bmatrix} = \begin{bmatrix} -20 & +14 \\ -12 & +7 \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

$$\Rightarrow \quad x = -6 \text{ and } y = -5$$

Hence, $x = -6$ and $y = -5$ is the required solution.

EXAMPLE 3 *Solve the following system of equations, using matrix method:*

$$x + 2y + z = 7,\; x + 3z = 11,\; 2x - 3y = 1$$

[CBSE 2002, 2003, 2005]

SOLUTION The given system of equations is

$$\begin{aligned} x + 2y + z &= 7 \\ x + 0y + 3z &= 11 \\ 2x - 3y + 0z &= 1 \end{aligned} \quad \text{or,} \quad \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\text{or,} \quad AX = B, \text{ where } A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix},\; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\text{Now, } \quad |A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix} = 1\,(0+9) - 2\,(0-6) + 1\,(-3-0) = 9 + 12 - 3 = 18 \neq 0$$

So, the given system of equations has a unique solution given by $X = A^{-1}B$. Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 3 \\ -3 & 0 \end{vmatrix} = 9, \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} = -3, \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix} = -3,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = -2, \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix} = 7,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} = 6, \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2,$$

$$\text{and,} \quad C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = -2$$

$$\therefore \quad \text{adj}\,A = \begin{bmatrix} 9 & 6 & -3 \\ -3 & -2 & 7 \\ 6 & -2 & -2 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \text{ and, } A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\text{Now,} \quad X = A^{-1}B$$

$$\Rightarrow \quad X = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 63 & -33 & +6 \\ 42 & -22 & -2 \\ -21 & +77 & -2 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \Rightarrow x = 2,\; y = 1 \text{ and } z = 3$$

Hence, $x = 2$, $y = 1$ and $z = 3$ is the required solution.

Type II SOLVING THE GIVEN SYSTEM OF EQUATIONS WHEN THE COEFFICIENT MATRIX IS SINGULAR

EXAMPLE 4 *Use matrix method to examine the following system of equations for consistency or inconsistency:*

$$4x - 2y = 3, \quad 6x - 3y = 5$$

SOLUTION The given system of equations can be written as

$$AX = B, \text{ where } A = \begin{bmatrix} 4 & -2 \\ 6 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

Now, $|A| = \begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix} = -12 + 12 = 0$

So, the given system of equations is inconsistent or it has infinitely many solutions according as $(\text{adj } A)\, B \neq O$ or, $(\text{adj } A)\, B = O$ respectively. Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}(-3) = -3, C_{12} = (-1)^{1+2}\, 6 = -6, C_{21} = (-1)^{2+1}(-2) = 2, C_{22} = (-1)^{2+2}(4) = 4$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}$$

So, $(\text{adj } A)\, B = \begin{bmatrix} -3 & 2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 & +10 \\ -18 & +20 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq O$

Hence, the given system of equations is inconsistent.

EXAMPLE 5 *Show that the following system of equations is consistent.*

$$2x - y + 3z = 5, \; 3x + 2y - z = 7, \; 4x + 5y - 5z = 9$$

Also, find the solution.

SOLUTION The given system of equation can be written in matrix form as

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

or, $AX = B$, where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix} = 2(-10 + 5) + 1(-15 + 4) + 3(15 - 8) = 0$

So, A is singular. Thus, the given system of equations is either inconsistent or it is consistent with infinitely many solutions according as $(\text{adj } A)\, B \neq O$ or, $(\text{adj } A)\, B = O$ respectively.

Let C_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 5 & -5 \end{vmatrix} = -5, \qquad C_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ 4 & -5 \end{vmatrix} = 11,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 7, \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 3 \\ 5 & -5 \end{vmatrix} = 10,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -22, \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} = -14,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = -5, \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = 11$$

and, $C_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 7,$

$$\therefore \quad (\text{adj } A) = \begin{bmatrix} -5 & 11 & 7 \\ 10 & -22 & -14 \\ -5 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$$

$$\Rightarrow \quad (\text{adj } A)\, B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} -25 & +70 & -45 \\ 55 & -154 & +99 \\ 35 & -98 & +63 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

Thus, $AX = B$ has infinitely many solutions. To find these solutions, we put $z = k$ in the first two equations and write them as follows:

$$2x - y = 5 - 3k \quad \text{and} \quad 3x + 2y = 7 + k$$

or, $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 - 3k \\ 7 + k \end{bmatrix}$

or, $AX = B$, where $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 - 3k \\ 7 + k \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 + 3 = 7 \neq 0$ and $\text{adj } A = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

Now, $X = A^{-1} B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 & -3k \\ 7 & +k \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 10 - 6k + 7 + k \\ -15 + 9k + 2k + 14 \end{bmatrix} = \begin{bmatrix} \dfrac{17 - 5k}{7} \\ \dfrac{11k - 1}{7} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{17 - 5k}{7}, \; y = \frac{11k - 1}{7}$$

These values of x, y and $z = k$ also satisfy the third equation. Hence, $x = \dfrac{17 - 5k}{7}$, $y = \dfrac{11k - 1}{7}$ and $z = k$, where k is any real number satisfy the given system of equations.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type III SOLVING A SYSTEM OF LINEAR EQUATIONS WHEN THE INVERSE OF THE COEFFICIENT MATRIX IS OBTAINED FROM SOME GIVEN RELATION

EXAMPLE 6 *If* $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, *find* A^{-1} *and hence solve the system of linear equations*

$$x + 2y + z = 4, \quad -x + y + z = 0, \quad x - 3y + z = 2.$$

SOLUTION We have, $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 1(1 + 3) + 1(2 + 3) + 1(2 - 1) = 10 \neq 0$$

So, A is invertible.

Let C_{ij} be the co-factors of elements a_{ij} in $A\,[a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 4, \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1, \qquad C_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0, \qquad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = -2$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 2, \qquad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5,$$

and, $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and, } A^{-1} = \frac{1}{|A|}\text{adj}\, A = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \quad \ldots(i)$$

The given system of equations is expressible as

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \text{ or, } A^T X = B \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

Now, $|A^T| = |A| = 10 \neq 0$. So, the given system of equations is consistent with a unique solution given by

$$X = (A^T)^{-1} B = (A^{-1})^T B \qquad [\because (A^T)^{-1} = (A^{-1})^T]$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}^T \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \qquad [\text{Using } (i)]$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+2 \\ 8+0-4 \\ 8+0+6 \end{bmatrix} = \begin{bmatrix} 9/5 \\ 2/5 \\ 7/5 \end{bmatrix}$$

$$\Rightarrow x = 9/5,\ y = 2/5 \text{ and } z = 7/5$$

Hence, $x = 9/5, y = 2/5, z = 7/5$ is the required solution.

EXAMPLE 7 *Determine the product* $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ *and use it to solve the system of equations:*

$$x - y + z = 4,\ x - 2y - 2z = 9,\ 2x + y + 3z = 1.$$

SOLUTION Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$. Then the given product is

$$CA = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\Rightarrow CA = \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 8I_3$$

$$\Rightarrow \frac{1}{8}CA = I_3 \Rightarrow \left(\frac{1}{8}C\right)A = I_3 \Rightarrow A^{-1} = \frac{1}{8}C \qquad [\text{By definition of inverse}]$$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix} \quad \text{...(i)}$$

The given system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$$

or, $AX = B$, where $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

The solution of this system of equations is given by

$$X = A^{-1}B = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix} \quad \text{[Using (i)]}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} \Rightarrow x = 3,\ y = -2 \text{ and } z = -1$$

EXAMPLE 8 *Find A^{-1}, where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$. Hence solve the system of equations*

$$x + 2y - 3z = -4,\ 2x + 3y + 2z = 2,\ 3x - 3y - 4z = 11.$$

[CBSE 2019, 2020]

SOLUTION We have,

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix} = -6 + 28 + 45 = 67 \neq 0$$

So, A is invertible. Let C_{ij} be the co-factors of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 2 \\ -3 & -4 \end{vmatrix} = -6, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 \\ 3 & -4 \end{vmatrix} = 14,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ 3 & -3 \end{vmatrix} = -15, \quad C_{21} = (-1)^{2+1}\begin{vmatrix} 2 & -3 \\ -3 & -4 \end{vmatrix} = 17,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ 3 & -4 \end{vmatrix} = 5, \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & -3 \end{vmatrix} = 9$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13, \quad C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} = -8,$$

and, $C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T = \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$\text{So,} \quad A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \quad \text{...(i)}$$

The given system of equations is

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = -2$$
$$3x - 3y - 4z = 11$$

or, $\quad AX = B$, where $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

As discussed above A is non-singular and so invertible. The inverse of A is given by (i).

The solution of the given system of equations is given by

$$X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67} \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix} = \frac{1}{67} \begin{bmatrix} 24 & +34 & +143 \\ -56 & +10 & -88 \\ 60 & +18 & -11 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 3,\ y = -2$ and $z = 1$ is the required solution.

Type IV ON APPLICATIONS OF SIMULTANEOUS LINEAR EQUATIONS

EXAMPLE 9 *The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number, we get 12. Using matrices find the numbers.* **[CBSE 2019]**

SOLUTION Let the three numbers be x, y and z respectively. Then,

$x + y + z = 6$ [Given]

Also, $\quad x + 2z = 7$

and, $\quad 3x + y + z = 12$

Thus, we obtain the following system of simultaneous linear equations:

$$x + y + z = 6$$
$$x + 0y + 2z = 7$$
$$3x + y + z = 12$$

The above system of equations can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$$

or, $\quad AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1(0-2) - (1-6) + 1(1-0) = -2 + 5 + 1 = 4 \neq 0$

So, the above system of equations has a unique solution given by $X = A^{-1} B$.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = -2, C_{12} = 5, C_{13} = 1, C_{21} = 0, C_{22} = -2, C_{23} = 2, C_{31} = 2, C_{32} = -1 \text{ and } C_{33} = -1$$

$$\therefore \text{ adj } A = \begin{bmatrix} -2 & 5 & 1 \\ 0 & -2 & 2 \\ 2 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \text{ and, } A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

Now, $\quad X = A^{-1} B$

$$\Rightarrow \quad X = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -12 & +0 & +24 \\ 30 & -14 & -12 \\ 6 & +14 & -12 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \Rightarrow x = 3, y = 1 \text{ and } z = 2.$$

Hence, the three numbers are 3, 1 and 2 respectively.

EXAMPLE 10 *An amount of ₹ 5000 is put into three investments at the rate of interest of 6%, 7% and 8% per annum respectively. The total annual income is ₹ 358. If the combined income from the first two investments is ₹ 70 more than the income from the third, find the amount of each investment by matrix method.*

SOLUTION Let x, y and z ₹ be the investments at the rates of interest of 6%, 7% and 8% per annum respectively. Then,

Total investment = ₹ 5000

$$\Rightarrow \quad x + y + z = 5000.$$

Now, Income from first investment of ₹ x = ₹ $\frac{6x}{100}$

Income from second investment of ₹ y = ₹ $\frac{7y}{100}$

Income from third investment of ₹ z = ₹ $\frac{8z}{100}$.

$\therefore$ Total annual income = ₹ $\left(\frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100}\right)$

$$\Rightarrow \quad \frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100} = 358 \qquad [\because \text{Total annual income} = ₹\ 358]$$

$$\Rightarrow \quad 6x + 7y + 8z = 35800.$$

It is given that the combined income from the first two investments is ₹ 70 more than the income from the third.

$$\therefore \quad \frac{6x}{100} + \frac{7y}{100} = 70 + \frac{8z}{100} \Rightarrow 6x + 7y - 8z = 7000.$$

Thus, we obtain the following system of simultaneous linear equations:

$$x + y + z = 5000$$
$$6x + 7y + 8z = 35800$$
$$6x + 7y - 8z = 7000$$

This system of equations can be written in matrix form as follows:

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix}$$

or, $AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{vmatrix} = 1(-56 - 56) - (-48 - 48) + (42 - 42) = -16 \neq 0.$

So, A^{-1} exists and the solution of the given system of equations is given by $X = A^{-1} B$.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$C_{11} = -112, C_{12} = 96, C_{13} = 0, C_{21} = 15, C_{22} = -14,$
$C_{23} = -1, C_{31} = 1, C_{32} = -2$ and $C_{33} = 1.$

$$\therefore \quad \text{adj}\, A = \begin{bmatrix} -112 & 96 & 0 \\ 15 & -14 & -1 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

So, $$A^{-1} = \frac{1}{|A|} (\text{adj}\, A) = -\frac{1}{16} \begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Hence, the solution is given by

$$X = A^{-1} B = -\frac{1}{16} \begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix} = -\frac{1}{16} \begin{bmatrix} -560000 & +537000 & +7000 \\ 480000 & -501200 & -14000 \\ 0 & -35800 & +7000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 2200 \\ 1800 \end{bmatrix} \Rightarrow x = 1000, y = 2200 \text{ and } z = 1800$$

Hence, three investments are of ₹ 1000, ₹ 2200 and ₹ 1800 respectively.

EXAMPLE 11 *A mixture is to be made of three foods A, B, C. The three foods A, B, C contain nutrients P, Q, R as shown below:*

Food	*Ounces per pound of Nutrient*		
	P	*Q*	*R*
A	1	2	5
B	3	1	1
C	4	2	1

How to form a mixture which will have 8 ounces of P, 5 ounces of Q and 7 ounces of R?

SOLUTION Let x pounds of food A, y pounds of food B and z pounds of food C be needed to form the mixture.

Since one pound of food A contains 1 ounce of nutrient P. So, x pounds of food A will contain x ounces of nutrient P. Similarly, the amount of nutrient P in y pounds of food B and z pounds of food C are $3y$ and $4z$ ounces respectively. Therefore,

Total quantity of nutrient P in x pounds of food A, y pounds of food B and z pounds of food C is $x + 3y + 4z$ ounces.

$\therefore \quad x + 3y + 4z = 8$

Similarly, $2x + y + 2z = 5$ [For nutrient Q]

and $5x + y + z = 7$ [For nutrient R]

The above system of simultaneous linear equations can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

or, $$AX = B, \text{ where } A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{vmatrix} = 1(1-2) - 3(2-10) + 4(2-5) = -1 + 24 - 12 = 11 \neq 0$$

So, A^{-1} exists.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = -1, C_{12} = 8, C_{13} = -3, C_{21} = 1, C_{22} = -19,$$
$$C_{23} = 14,\ C_{31} = 22, C_{32} = 6 \text{ and } C_{33} = -5$$

$$\therefore \quad \operatorname{adj} A = \begin{bmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix}$$

Thus, the solution of the system of equations is given by

$$X = A^{-1} B = \frac{1}{11} \begin{bmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -8 & +5 & +14 \\ 64 & -95 & +42 \\ -24 & +70 & -35 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 \\ 11 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow x = 1, y = 1 \text{ and } z = 1.$$

Hence, the mixture is formed by mixing one pound of each of the foods A, B and C.

EXERCISE 7.1

BASIC

1. Solve the following system of equations by matrix method:

(i) $5x + 7y + 2 = 0$, $4x + 6y + 3 = 0$

(ii) $5x + 2y = 3$, $3x + 2y = 5$

(iii) $3x + 4y - 5 = 0$, $x - y + 3 = 0$

(iv) $3x + y = 19$, $3x - y = 23$

(v) $3x + 7y = 4$, $x + 2y = -1$

(vi) $3x + y = 7$, $5x + 3y = 12$

2. Solve the following system of equations by matrix method:

(i) $x + y - z = 3$, $2x + 3y + z = 10$, $3x - y - 7z = 1$

(ii) $x + y + z = 3$, $2x - y + z = -1$, $2x + y - 3z = -9$

(iii) $6x - 12y + 25z = 4$, $4x + 15y - 20z = 3$, $2x + 18y + 15z = 10$

[CBSE 2004, 2005]

(iv) $3x + 4y + 7z = 14$, $2x - y + 3z = 4$, $x + 2y - 3z = 0$

(v) $\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10$, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$, $\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$

(vi) $5x + 3y + z = 16$, $2x + y + 3z = 19$, $x + 2y + 4z = 25$

[CBSE 2005, 07]

(vii) $3x + 4y + 2z = 8$, $2y - 3z = 3$, $x - 2y + 6z = -2$

(viii) $2x + y + z = 2$, $x + 3y - z = 5$, $3x + y - 2z = 6$ [CBSE 2008]

(ix) $2x + 6y = 2$, $3x - z = -8$, $2x - y + z = -3$ [CBSE 2003]

(x) $x - y + z = 2$, $2x - y = 0$, $2y - z = 1$ [CBSE 2003]

(xi) $8x + 4y + 3z = 18$, $2x + y + z = 5$, $x + 2y + z = 5$ [CBSE 2008]

(xii) $x + y + z = 6$, $x + 2z = 7$, $3x + y + z = 12$ [CBSE 2009]

(xiii) $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4,$
$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,$
$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2; x, y, z \neq 0$ **[CBSE 2011]**

(xiv) $x-y+2z=7$
$3x+4y-5z=-5$
$2x-y+3z=12$ **[CBSE 2012]**

3. Show that each of the following systems of linear equations is consistent and also find their solutions:

(i) $6x+4y = 2$
$9x+6y = 3$

(ii) $2x+3y = 5$
$6x+9y = 15$

(iii) $5x+3y+7z = 4$
$3x+26y+2z = 9$
$7x+2y+10z = 5$

(iv) $x-y+z = 3$
$2x+y-z = 2$
$-x-2y+2z = 1$

(v) $x+y+z = 6$
$x+2y+3z = 14$
$x+4y+7z = 30$

(vi) $2x+2y-2z = 1$
$4x+4y-z = 2$
$6x+6y+2z = 3$

4. Show that each one of the following systems of linear equations is inconsistent:

(i) $2x+5y = 7$
$6x+15y = 13$

(ii) $2x+3y = 5$
$6x+9y = 10$

(iii) $4x-2y = 3$
$6x-3y = 5$

(iv) $4x-5y-2z = 2$
$5x-4y+2z = -2$
$2x+2y+8z = -1$

(v) $3x-y-2z = 2$
$2y-z = -1$
$3x-5y = 3$

(vi) $x+y-2z = 5$
$x-2y+z = -2$
$-2x+y+z = 4$

BASED ON LOTS

5. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ are two square matrices, find AB and hence solve the system of linear equations:
$x-y = 3,\ 2x+3y+4z=17,\ y+2z=7$ **[CBSE 2010, 2012]**

6. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} and hence solve the system of linear equations:
$2x-3y+5z=11,\ 3x+2y-4z=-5,\ x+y-2z=-3$ **[CBSE 2007, 2009, 2012, 2018, 2020]**

7. Find A^{-1}, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$. Hence, solve the following system of linear equations:
$x+2y+5z = 10,\ x-y-z = -2,\ 2x+3y-z = -11$ **[CBSE 2010, 2012]**

8. (i) If $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of linear equations:
$x-2y = 10,\ 2x+y+3z = 8,\ -2y+z = 7$

(ii) $A = \begin{bmatrix} 3 & -4 & 2 \\ 2 & 3 & 5 \\ 1 & 0 & 1 \end{bmatrix}$, find A^{-1} and hence solve the following system of equations:
$3x-4y+2z=-1,\ 2x+3y+5z=7,\ x+z=2$ **[CBSE 2011]**

(iii) $A=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{bmatrix}$, find AB. Hence, solve the system of equations:

$x-2y=10,\ 2x+y+3z=8$ and $-2y+z=7$ **[CBSE 2011]**

(iv) If $A=\begin{bmatrix}1 & 2 & 0\\ -2 & -1 & -2\\ 0 & -1 & 1\end{bmatrix}$, find A^{-1}. Using A^{-1}, solve the system of linear equations

$x-2y=10,\ 2x-y-z=8,\ -2y+z=7$ **[NCERT EXEMPLAR]**

(v) Given $A=\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}$, find BA and use this to solve the system of equations $y+2z=7,\ x-y=3,\ 2x+3y+4z=17$ **[NCERT EXEMPLAR]**

(vi) If $A=\begin{bmatrix}2 & 3 & 1\\ 1 & 2 & 2\\ -3 & 1 & -1\end{bmatrix}$, find A^{-1} and hence solve the system of equations

$2x+y-3z=13,\ 3x+2y+z=4,\ x+2y-z=8.$ **[CBSE 2017]**

(vii) Use the product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ to solve the system of equations

$x+3z=-9,-x+2y-2z=4,2x-3y+4z=-3.$ **[CBSE 2017]**

9. The sum of three numbers is 2. If twice the second number is added to the sum of first and third, the sum is 1. By adding second and third number to five times the first number, we get 6. Find the three numbers by using matrices.

10. An amount of ₹ 10,000 is put into three investments at the rate of 10, 12 and 15% per annum. The combined income is ₹ 1310 and the combined income of first and second investment is ₹ 190 short of the income from the third. Find the investment in each using matrix method.

11. A company produces three products every day. Their production on a certain day is 45 tons. It is found that the production of third product exceeds the production of first product by 8 tons while the total production of first and third product is twice the production of second product. Determine the production level of each product using matrix method.

12. The prices of three commodities P, Q and R are ₹ x, y and z per unit respectively. A purchases 4 units of R and sells 3 units of P and 5 units of Q. B purchases 3 units of Q and sells 2 units of P and 1 unit of R. C purchases 1 unit of P and sells 4 units of Q and 6 units of R. In the process A, B and C earn ₹ 6000, ₹ 5000 and ₹ 13000 respectively. If selling the units is positive earning and buying the units is negative earnings, find the price per unit of three commodities by using matrix method.

13. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management must include for awards. **[CBSE 2013]**

14. A school wants to award its students for the values of Honesty, Regularity and Hardwork with a total cash award of ₹ 6000. Three times the award money for Hardwork added to that given for honesty amounts to ₹ 11000. The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hardwork, suggest one more value which the school must include for awards. [CBSE 2013]

15. Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of ₹ x, ₹ y and ₹ z respectively per person. The first institution decided to award respectively 4, 3 and 2 employees with a total prize money of ₹ 37000 and the second institution decided to award respectively 5, 3 and 4 employees with a total prize money of ₹ 47000. If all the three prizes per person together amount to ₹ 12000, then using matrix method find the value of x, y and z. What values are described in this equations? [CBSE 2013]

16. Two factories decided to award their employees for three values of (a) adaptable to new techniques, (b) careful and alert in difficult situations and (c) keeping calm in tense situations, at the rate of ₹ x, ₹ y and ₹ z per person respectively. The first factory decided to honour respectively 2, 4 and 3 employees with a total prize money of ₹ 29000. The second factory decided to honour respectively 5, 2 and 3 employees with the prize money of ₹ 30500. If the three prizes per person together cost ₹ 9500, then
    (i) represent the above situation by a matrix equation and form linear equations using matrix multiplication.
    (ii) Solve these equations using matrices.
    (iii) Which values are reflected in the questions? [CBSE 2013]

17. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 16,00. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. [CBSE 2014]

18. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each , ₹ y each and ₹ z each for the three respectively values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value.
    Apart from the above three values, suggest one more value for awards. [CBSE 2014]

19. Two schools P and Q want to award their selected students on the values of Tolerance, Kindness and Leadership. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 2,200. School Q wants to spend ₹ 3,100 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as school P). If the total amount of award for one prize on each values is ₹ 1,200, using matrices, find the award money for each value.
    Apart from these three values, suggest one more value which should be considered for award. [CBSE 2014]

20. A total amount of ₹ 7000 is deposited in three different saving bank accounts with annual interest rates 5%, 8% and $8\frac{1}{2}\%$ respectively. The total annual interest from these three accounts is ₹ 550. Equal amounts have been deposited in the 5% and 8% savings accounts. Find the amount deposited in each of the three accounts, with the help of matrices. [CBSE 2014]

21. A shopkeeper has 3 varieties of pens 'A', 'B' and 'C'. Meenu purchased 1 pen of each variety for a total of ₹ 21. Jeen purchased 4 pens of 'A' variety, 3 pens of 'B' variety and 2 pens of 'C' variety for ₹ 60. While Shikha purchased 6 pens of 'A' variety, 2 pens of 'B' variety and 3 pens of 'C' variety for ₹ 70. Using matrix method find the cost of each pen. [CBSE 2016]

ANSWERS

1. (i) $x=\frac{9}{2},\ y=-\frac{7}{2}$ (ii) $x=-1,\ y=4$ (iii) $x=-1,\ y=2$
   (iv) $x=7,\ y=-2$ (v) $x=-15,\ y=7$ (vi) $x=\frac{9}{4},\ y=\frac{1}{4}$
2. (i) $x=3,\ y=1,\ z=1$ (ii) $x=-\frac{8}{7},\ y=\frac{10}{7},\ z=\frac{19}{7}$ (iii) $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{5}$
   (iv) $x=1,\ y=1,\ z=1$ (v) $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{5}$ (vi) $x=1,\ y=2,\ z=5$
   (vii) $x=-2,\ y=3,\ z=1$ (viii) $x=1,\ y=1,\ z=-1$ (ix) $x=-2,\ y=1,\ z=2$
   (x) $x=1,\ y=2,\ z=3$ (xi) $x=1,\ y=1,\ z=2$ (xii) $x=3,\ y=2,\ z=1$
   (xiii) $x=2,\ y=3,\ z=5$ (xiv) $x=2,\ y=1,\ z=3$
3. (i) $x=\frac{1-2k}{3},\ y=k$ (ii) $x=\frac{5-3k}{2},\ y=k$ (xii) $x=3,\ y=1,\ z=2$
   (iii) $x=\frac{7-16k}{11},\ y=\frac{k+3}{11},\ z=k$ (iv) $x=\frac{5}{3},\ y=\frac{-4}{3}+k,\ z=k$
   (v) $x=k-2,\ y=8-2k,\ z=k$ (vi) $x=\frac{1}{2}-k,\ y=k,\ z=0$
5. $x=2,\ y=-1,\ z=4$ 6. $x=1,\ y=2,\ z=3$ 7. $x=-1,\ y=-2,\ z=3$
8. (i) $x=4,\ y=-3,\ z=1$ (ii) $x=3,\ y=2,\ z=-1$ (iii) $x=4,\ y=-3,\ z=1$
   (iv) $x=0,\ y=-5,\ z=-3$ (v) $x=2,\ y=-1,\ z=4$ (vi) $z=1\ y=2,\ z=-3$
   (viii) $x=36,\ y=5,\ z=-15$ 9. 1, −1, 2 10. ₹ 2000, ₹ 3000, ₹ 5000
11. 11, 15, 19 12. $x=3000,\ y=1000,\ z=2000$
13. $x=3,\ y=4,\ z=5$ 14. $x=500,\ y=2000,\ z=3500$ 15. $x=4000,\ y=5000,\ z=3000$
16. $x=2500,\ y=3000,\ z=4000$ 17. $x=200,\ y=300,\ z=400$ 18. $x=100,\ y=200,\ z=300$
19. $x=300,\ y=400,\ z=500$ 20. ₹ 1125, ₹ 1125, ₹ 4750
21. Variety A : ₹ 5, Variety B : ₹ 8, Variety C : ₹ 8

HINTS TO SELECTED PROBLEMS

13. The given data suggests the following equations:
    $x+y+z=12,\ x-2y+z=0,\ 2x+3y+3z=33$
14. Let the award money for Honesty, Regularity and Hardwork be ₹ x, y and z respectively. Then, $x+y+z=6000,\ x+3z=11000$ and $x-2y+z=0$.
15. $4x+3y+2z=37000,\quad 5x+3y+4z=47000,\quad x+y+z=12000$
16. $2x+4y+3z=29000,\quad 5x+2y+3z=30500,\quad x+y+z=9500$

7.3 SOLUTION OF HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

In chapter 6, we have learnt about determinant method to solve a homogeneous system of linear equations. In this section, we shall discuss matrix method to solve the same.

Let $AX=O$ be a homogeneous system of n linear equations with n unknowns.

Let us now discuss two cases:

Case I *When $|A|\neq 0$ i.e. matrix A is non-singular.*

If $|A|\neq 0$, then A^{-1} exists.

$$\therefore \quad AX = O$$
$$\Rightarrow \quad A^{-1}(AX) = A^{-1}O$$
$$\Rightarrow \quad (A^{-1}A)X = O \Rightarrow I_n X = O \Rightarrow X = O$$
$$\Rightarrow \quad x_1 = x_2 = \ldots = x_n = 0$$

Thus, if the coefficient matrix A is a non-singular, then the homogeneous system of equations has the unique solution $X = O$ i.e. $x_1 = x_2 = \ldots = x_n = 0$.

This solution is known as the trivial solution.

Case II *When $|A| = 0$ i.e. matrix A singular.*

If $|A| = 0$, then $(adj\ A)\ B = (adj\ A)\ O = O$

i.e. the condition of consistency is always satisfied. So, the given system of equations is consistent and it has infinitely many solutions which can be obtained by giving any real value to one of the variables and then solving the remaining equations by matrix method.

In order to solve a homogeneous system of the three linear equations with 3 unknowns x, y, z, we may use the following algorithm.

ALGORITHM

Step I *Obtain the system of equations and express it in the matrix equation of the form $AX = O$.*

Step II *Find $|A|$.*

Step III *If $|A| \neq 0$, then $x = y = z = 0$ is the only solution of the homogeneous system. So, write $x = 0$, $y = 0$, $z = 0$ as the solution.*

Step IV *If $|A| = 0$, then the system has infinitely many solutions. In order to find these solutions put $z = k$ (any real number) and solve any two equations for x and y by the matrix method. The values of x and y so obtained with $z = k$ give a solution of the system.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I WHEN THE DETERMINANT OF THE COEFFICIENT MATRIX IS NON-SINGULAR

EXAMPLE 1 *Solve the following system of homogeneous equations:*

$$2x + 3y - z = 0$$
$$x - y - 2z = 0$$
$$3x + y + 3z = 0$$

SOLUTION The given system of homogeneous equations can be written as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $\quad AX = O$, where $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $\quad |A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = -2 - 27 - 4 = -33 \neq 0.$

Thus $|A| \neq 0$. So, the given system has only the trivial solution given by $x = y = z = 0$.

Type II WHEN THE DETERMINANT OF THE COEFFICIENT MATRIX IS SINGULAR

EXAMPLE 2 *Show that the homogeneous system of equations*

$$x - 2y + z = 0, \quad x + y - z = 0, \quad 3x + 6y - 5z = 0$$

has a non-trivial solution. Also, find the solution.

SOLUTION The given system of homogeneous equations can be written in matrix form as

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, $AX = O$, where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{vmatrix} = 1(-5+6) + 2(-5+3) + 1(6-3) = 0.$

So, the given system of equations has a non-trivial solution. To find these solutions, we put $z = k$ in the first two equations and write them as follows:

$$x - 2y = -k \text{ and } x + y = k.$$

or, $\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k \\ k \end{bmatrix}$ or, $AX = B$ where $A = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} -k \\ k \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \neq 0$. So, A^{-1} exists.

Clearly, $\operatorname{adj} A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\therefore$ $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -k \\ k \end{bmatrix} = \begin{bmatrix} k/3 \\ 2k/3 \end{bmatrix} \Rightarrow x = k/3,\ y = 2k/3.$

These values of x, y and z also satisfy the third equation.

Hence $x = k/3$, $y = 2k/3$ and $z = k$, where k is any real number satisfy the given system of equations.

EXERCISE 7.2

BASIC

Solve the following systems of homogeneous linear equations by matrix method:

1. $2x - y + z = 0$, $3x + 2y - z = 0$, $x + 4y + 3z = 0$
2. $2x - y + 2z = 0$, $5x + 3y - z = 0$, $x + 5y - 5z = 0$
3. $3x - y + 2z = 0$, $4x + 3y + 3z = 0$, $5x + 7y + 4z = 0$
4. $x + y - 6z = 0$, $x - y + 2z = 0$, $-3x + y + 2z = 0$
5. $x + y + z = 0$, $x - y - 5z = 0$, $x + 2y + 4z = 0$
6. $x + y - z = 0$, $x - 2y + z = 0$, $3x + 6y - 5z = 0$
7. $3x + y - 2z = 0$, $x + y + z = 0$, $x - 2y + z = 0$
8. $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$

ANSWERS

1. $x = y = z = 0$
2. $x = \frac{-5k}{11}$, $y = \frac{12k}{11}$, $z = k$
3. $x = \frac{-9k}{13}$, $y = \frac{-k}{13}$, $z = k$
4. $x = 2k$, $y = 4k$, $z = k$
5. $x = 2k$, $y = -3k$, $z = k$
6. $x = k$, $y = 2k$, $z = 3k$
7. $x = y = z = 0$
8. $x = y = z = 0$

5. $x = 2, y = 3, z = -1$

6. 2

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If the system of equations $x + ay = 0$, $az + y = 0$, $ax + z = 0$ has infinitely many solutions, then $a =$ ....
2. If the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = 12$ is inconsistent, then $\lambda =$ ....
3. The number of solutions of the system of equations $x + 2y + z = 3$, $2x + 3y + z = 3$, $3x + 5y + 2z = 1$ is ....
4. If the system of equations $2x - y - z = 12$, $x - 2y + z = -4$, $x + y + \lambda z = 4$ has no solution, then $\lambda =$ ....
5. If the system of equations $x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$ has a non-zero solution, then the values of k are ....
6. The real value of λ for which the system of equations $\lambda x + y + z = 0$, $-x + \lambda y + z = 0$, $-x - y + \lambda z = 0$ has a non-zero solution, is ....
7. The set of values of k for which the system of equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$ has a unique solution, is ....

ANSWERS

1. -1 2. 3 3. 0 4. $\lambda = -2$ 5. $1, -1$ 6. 0 7. $R - \{0\}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, find x, y and z.
2. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, find x, y and z.
3. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ -1 \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, find x, y and z.
4. Solve $\begin{bmatrix} 3 & -4 \\ 9 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$ for x and y.
5. If $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, find x, y, z.
6. If $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} n \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ and $AX = B$, then find n.

ANSWERS

1. $x = 1, y = -1, z = 0$ 2. $x = 1, y = 0, z = -1$ 3. $x = 1, y = 0, z = 1$
4. $x = \frac{2}{3}, y = -2$ 5. $x = 2, y = 3, z = -1$ 6. 2