CHAPTER 4

ALGEBRA OF MATRICES

4.1 MATRIX

DEFINITION *A set of mn numbers (real or imaginary) arranged in the form of a rectangular array of m rows and n columns is called an m × n matrix (to be read as 'm by n' matrix).*

An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In compact form the above matrix is represented by $A = [a_{ij}]_{m \times n}$ or, $A = [a_{ij}]$.

The numbers $a_{11}, a_{12}, \ldots$ etc. are known as the elements of the matrix A. The element a_{ij} belongs to i^{th} row and j^{th} column and is called the $(i, j)^{\text{th}}$ element of the matrix $A = [a_{ij}]$. Thus, in the element a_{ij} the first subscript i always denotes the number of row and the second subscript j, number of column in which the element occurs.

Following are some examples of matrices:

(i) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 2 \end{bmatrix}$ is a matrix having 2 rows and 3 columns and so it is a matrix of order 2×3 such that $a_{11} = 2, a_{12} = 1, a_{13} = -1, a_{21} = 1, a_{22} = 3, a_{23} = 2$.

(ii) $B = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$ is a matrix having 2 rows and 2 columns and so it is a matrix of order 2×2 such that $b_{11} = \sin x, b_{12} = \cos x, b_{21} = \cos x, b_{22} = -\sin x$.

NOTE *It is to note here that to define a matrix we must define its order and its elements either by a general formula (See illustration given below) or separately.*

ILLUSTRATION *Construct a* 3×4 *matrix* $A = [a_{ij}]$ *whose elements are given by*

(i) $a_{ij} = i + j$ (ii) $a_{ij} = i - j$

SOLUTION (i) We have,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \text{ where } a_{ij} = i + j.$$

$\therefore \quad a_{11} = 1 + 1 = 2, \quad a_{12} = 1 + 2 = 3, \quad a_{13} = 1 + 3 = 4, \quad a_{14} = 1 + 4 = 5.$

Similarly, $a_{21} = 3, \; a_{22} = 4, \; a_{23} = 5, \; a_{24} = 6$ and $a_{31} = 4, \; a_{32} = 5, \; a_{33} = 6, \; a_{34} = 7$.

Hence, $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

(ii) Proceeding as above, we obtain

$$A = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}.$$

4.2 TYPES OF MATRICES

ROW MATRIX *A matrix having only one row is called a row-matrix or a row-vector.*

For example, $A = [1 \;\; 2 \; -1 \; -2]$ is a row matrix of order 1×4.

COLUMN MATRIX *A matrix having only one column is called a column matrix or a column-vector.*

For example, $A = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 2 \\ 5 \\ 4 \end{bmatrix}$ are column-matrices of orders 3×1 and 4×1 respectively.

SQUARE MATRIX *A matrix in which the number of rows is equal to the number of columns, say n, is called a square matrix of order n.*

A square matrix of order n is also called a n-rowed square matrix. The elements a_{ij} of a square matrix $A = [a_{ij}]_{n \times n}$ for which $i = j$ i.e. the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal elements and the line along which they lie is called the *principal diagonal* or leading diagonal of the matrix.

For example, the matrix $\begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 5 \\ 1 & 5 & -3 \end{bmatrix}$ is square matrix of order 3 in which the diagonal elements are 2, – 2 and – 3.

DIAGONAL MATRIX *A square matrix $A = [a_{ij}]_{n \times n}$ is called a diagonal matrix if all the elements, except those in the leading diagonal, are zero i.e. $a_{ij} = 0$ for all $i \neq j$.*

A diagonal matrix of order $n \times n$ having $d_1, d_2, \ldots, d_n$ as diagonal elements is denoted by diag $[d_1, d_2, \ldots, d_n]$.

For example, the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix, to be denoted by $A = \text{diag}\,[1, 2, 3]$.

SCALAR MATRIX *A square matrix $A = [a_{ij}]_{n \times n}$ is called a scalar matrix if*

(i) $a_{ij} = 0$ *for all* $i \neq j$ *and,* (ii) $a_{ii} = c$ *for all* i, *where* $c \neq 0$.

In other words, a diagonal matrix in which all the diagonal elements are equal is called the scalar matrix.

For example, the matrices $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1-2i & 0 & 0 \\ 0 & 1-2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$ are scalar matrices of orders 2 and 3 respectively.

IDENTITY OR UNIT MATRIX *A square matrix $A = [a_{ij}]_{n \times n}$ is called an identity or unit matrix if*

(i) $a_{ij} = 0$ *for all* $i \neq j$ *and,* (ii) $a_{ii} = 1$ *for all* i

In other words, a square matrix each of whose diagonal element is unity and each of whose non-diagonal elements is equal to zero is called an identity or unit matrix.

The identity matrix of order n is denoted by I_n.

For example, the matrices $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of orders 2 and 3 respectively.

NULL MATRIX *A matrix whose all elements are zero is called a null matrix or a zero matrix.*

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *are null matrices of orders* 2×2 *and* 2×3 *respectively.*

UPPER TRIANGULAR MATRIX *A square matrix* $A = [a_{ij}]$ *is called an upper triangular matrix if* $a_{ij} = 0$ *for all* $i > j$.

Thus, in an upper triangular matrix, all elements below the main diagonal are zero.

For example, $A = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 5 & 1 & 3 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is an upper triangular matrix.

LOWER TRIANGULAR MATRIX *A square matrix* $A = [a_{ij}]$ *is called a lower triangular matrix if* $a_{ij} = 0$ *for all* $i < j$.

Thus, in a lower triangular matrix, all elements above the main diagonal are zero.

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$ is a lower triangular matrix of order 3.

A triangular matrix $A = [a_{ij}]$ of order $n \times n$ is called a strictly triangular iff $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$.

4.3 EQUALITY OF MATRICES

DEFINITION *Two matrices* $A = [a_{ij}]_{m \times n}$ *and* $B = [b_{ij}]_{r \times s}$ *are equal if*

(i) $m = r$ *i.e. the number of rows in A equals the number of rows in B*

(ii) $n = s$ *i.e. the number of columns in A equals the number of columns in B*

(iii) $a_{ij} = b_{ij}$ *for* $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$.

If two matrices A and B are equal, we write $A = B$, otherwise we write $A \neq B$.

The matrices $A = \begin{bmatrix} 3 & 2 & 1 \\ x & y & 5 \\ 1 & -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & 5 \\ 1 & -1 & z \end{bmatrix}$ are equal if $x = -1$, $y = 0$ and $z = 4$.

Matrices $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are not equal, because their orders are not same.

ILLUSTRATION 1 *If* $\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$, *find* x, y, z, w. **[CBSE 2002 C, 2013]**

SOLUTION Since the corresponding elements of two equal matrices are equal. Therefore,

$$\begin{bmatrix} x-y & 2x+z \\ 2x-y & 3z+w \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix} \Rightarrow x-y=-1,\ 2x+z=5,\ 2x-y=0,\ 3z+w=13.$$

Solving the equations $x - y = -1$ and $2x - y = 0$ as simultaneous linear equations, we get $x = 1$, $y = 2$.

Now putting $x = 1$ in $2x + z = 5$, we obtain $z = 3$. Substituting $z = 3$ in $3z + w = 13$, we obtain $w = 4$.

Thus, $x = 1$, $y = 2$, $z = 3$ and $w = 4$.

ILLUSTRATION 2 *Find the values of x, y, z and a which satisfy the matrix equation*

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

SOLUTION The corresponding elements of two equal matrices are equal.

$$\therefore \quad \begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix} \Rightarrow x+3=0,\ 2y+x=-7,\ z-1=3 \text{ and } 4a-6=2a.$$

Solving these equations, we obtain : $a=3,\ x=-3,\ y=-2,\ z=4$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *A matrix has 12 elements. What are the possible orders it can have?*

SOLUTION We know that if a matrix is of order $m \times n$, then it has mn elements. Therefore, to find all possible orders of a matrix with 12 elements, we will have to find all ordered pairs (a, b) such that a and b are factors of 12. Clearly, all possible ordered pairs of this type are :

$$(1, 12), (12, 1), (3, 4), (4, 3), (2, 6), (6, 2)$$

Hence, possible orders of the matrix are: 1×12, 12×1, 3×4, 4×3, 2×6 and 6×2.

EXAMPLE 2 *If $A=[a_{ij}]$ is a matrix given by $A=[a_{ij}] = \begin{bmatrix} 4 & -2 & 1 & 3 \\ 5 & 7 & 9 & 6 \\ 21 & 15 & 18 & -25 \end{bmatrix}$, write the order of A and find the elements a_{24}, a_{34}. Also, show that $a_{32} = a_{23} + a_{24}$.*

SOLUTION We observe that there are 3 rows and 4 columns in matrix A. Therefore, it is of order 3×4. The element lying at the intersection of 2nd row and fourth column is 6. Therefore, $a_{24}=6$. Similarly, the element lying at the intersection of third row and fourth column is -25. Therefore, $a_{34}=-25$. Similarly, $a_{32}=15$, $a_{23}=9$ and $a_{24}=6$.

Clearly, $a_{32} = 15 = 9+6 = a_{23}+a_{24}$.

EXAMPLE 3 *Construct a 2×2 matrix $A=[a_{ij}]$ whose elements are given by $a_{ij} = \dfrac{(i+2j)^2}{2}$.*

SOLUTION Here $a_{ij} = \dfrac{(i+2j)^2}{2}$, $1 \le i \le 2$ and $1 \le j \le 2$. **[NCERT]**

$$\therefore \quad a_{11} = \frac{(1+2\times 1)^2}{2} = \frac{(1+2)^2}{2} = \frac{9}{2},\quad a_{12} = \frac{(1+2\times 2)^2}{2} = \frac{25}{2}$$

$$a_{21} = \frac{(2+2\times 1)^2}{2} = 8 \text{ and } a_{22} = \frac{(2+2\times 2)^2}{2} = 18$$

$$\text{Hence, } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{9}{2} & \frac{25}{2} \\ 8 & 18 \end{bmatrix}$$

EXAMPLE 4 *Construct a 2×3 matrix $A=[a_{ij}]$ whose elements are given by $a_{ij} = \dfrac{i-j}{i+j}$.*

SOLUTION We have, $a_{ij} = \dfrac{i-j}{i+j}$, $1 \le i \le 2$ and $1 \le j \le 3$. Therefore,

$$a_{11}=0,\quad a_{12} = -\frac{1}{3},\quad a_{13} = -\frac{1}{2},\quad a_{21} = \frac{1}{3},\quad a_{22}=0 \text{ and } a_{23} = -\frac{1}{5}.$$

$$\therefore \quad A=\begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{2} \\ \frac{1}{3} & 0 & -\frac{1}{5} \end{bmatrix}$$

EXAMPLE 5 *Construct a 3×2 matrix $A=[a_{ij}]$ whose elements are given by*

(i) $a_{ij}=e^{ix}\sin jx$ (ii) $a_{ij}=e^{-ix}\cos\left(\frac{\pi}{2}i+jx\right)$

SOLUTION (i) It is given that $A=[a_{ij}]$ is a 3×2 matrix such that $a_{ij}=e^{ix}\sin jx$, $1\le i\le 3$ and $1\le j\le 2$. Therefore, $a_{11}=e^{x}\sin x$, $a_{12}=e^{x}\sin 2x$, $a_{21}=e^{2x}\sin x$, $a_{22}=e^{2x}\sin 2x$, $a_{31}=e^{3x}\sin x$ and $a_{32}=e^{3x}\sin 2x$.

$$\text{Hence,}\quad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}=\begin{bmatrix} e^{x}\sin x & e^{x}\sin 2x \\ e^{2x}\sin x & e^{2x}\sin 2x \\ e^{3x}\sin x & e^{3x}\sin 2x \end{bmatrix}$$

(ii) It is given that $A=[a_{ij}]$ is a 3×2 matrix such that $a_{ij}=e^{-ix}\cos\left(\frac{\pi}{2}i+jx\right)$, $1\le i\le 3$ and $1\le j\le 2$.

$$\therefore \quad a_{11}=e^{-x}\cos\left(\frac{\pi}{2}+x\right)=-e^{-x}\sin x,\ a_{12}=e^{-x}\cos\left(\frac{\pi}{2}+2x\right)=-e^{-x}\sin 2x$$

$$a_{21}=e^{-2x}\cos(\pi+x)=-e^{-2x}\cos x,\ a_{22}=e^{-2x}\cos(\pi+2x)=-e^{-2x}\cos 2x$$

$$a_{31}=e^{-3x}\cos\left(\frac{3\pi}{2}+x\right)=e^{-3x}\sin x,\ a_{32}=e^{-3x}\cos\left(\frac{3\pi}{2}+2x\right)=e^{-3x}\sin 2x$$

$$\text{Hence,}\quad A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}=\begin{bmatrix} -e^{-x}\sin x & -e^{-x}\sin 2x \\ -e^{-2x}\cos x & -e^{-2x}\cos 2x \\ e^{-3x}\sin x & e^{-3x}\sin 2x \end{bmatrix}$$

EXAMPLE 6 *Find x, y, z and w such that* $\begin{bmatrix} x-y & 2z+w \\ 2x-y & 2x+w \end{bmatrix}=\begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}$.

SOLUTION We know that the corresponding elements of two equal matrices are equal.

$$\therefore \quad \begin{bmatrix} x-y & 2z+w \\ 2x-y & 2x+w \end{bmatrix}=\begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix} \Rightarrow x-y=5,\ 2z+w=3,\ 2x-y=12 \text{ and } 2x+w=15$$

Solving $x-y=5$ and $2x-y=12$ as simultaneous linear equations, we get $x=7$, $y=2$. Putting $x=7$ in equation $2x+w=15$, we get $w=1$. Putting $w=1$ in $2z+w=3$, we get $z=1$. Hence, $x=7$, $y=2$, $z=1$ and $w=1$.

EXAMPLE 7 *Consider the following information regarding the number of men and women workers in three factories I, II and III.*

	Men workers	*Women workers*
I	30	25
II	25	31
III	27	26

Represent the above information in the form of 3×2 matrix. What does the entry in the third row and second column represent?

SOLUTION The given information can be represented in the form of a 3×2 matrix as follows:

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array}\begin{array}{c} \begin{array}{cc} \text{Men workers} & \text{Women workers} \end{array} \\ \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix} \end{array}$$

The entry in third row and second column represents the number of women workers in factory III.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *If* $\begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$, *find the values of a and b.*

SOLUTION The corresponding elements of two equal matrices are equal.

$\therefore \quad \begin{bmatrix} a+b & 2 \\ 5 & ab \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix} \Rightarrow a+b=6$ and $ab=8 \Rightarrow a+\dfrac{8}{a}=6 \qquad [\because\ ab=8 \Rightarrow b=8/a]$

$\Rightarrow \quad a^2+8=6a \Rightarrow a^2-6a+8=0 \Rightarrow (a-4)(a-2)=0 \Rightarrow a=2, 4.$

Now, $a=2$ and $ab=8 \Rightarrow b=4$ and, $a=4$ and $ab=8 \Rightarrow b=2$.

Hence, $a=2$ and $b=4$, or $a=4$ and $b=2$.

EXAMPLE 9 *For what values of x and y are the following matrices equal?*

$$A = \begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix}, B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

SOLUTION The corresponding elements of two equal matrices are equal. Therefore,

$$\begin{bmatrix} 2x+1 & 3y \\ 0 & y^2-5y \end{bmatrix} = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

$\Rightarrow \quad 2x+1=x+3,\ 3y=y^2+2$ and $y^2-5y=-6$

$\Rightarrow \quad x=2,\ y^2-3y+2=0$ and $y^2-5y+6=0$

$\Rightarrow \quad x=2, (y-1)(y-2)=0$ and $(y-2)(y=3)=0 \Rightarrow x=2, y=1, 2$ and $y=2, 3$

$\Rightarrow \quad x=2, y=2 \qquad [\because\ y=1, 2$ and $y=2, 3 \Rightarrow y=2]$

EXERCISE 4.1

BASIC

1. If a matrix has 8 elements, what are the possible orders it can have? What if it has 5 elements? [NCERT]

2. If $A=[a_{ij}]=\begin{bmatrix} 2 & 3 & -5 \\ 1 & 4 & 9 \\ 0 & 7 & -2 \end{bmatrix}$ and $B=[b_{ij}]=\begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 1 & 2 \end{bmatrix}$ then find: (i) $a_{22}+b_{21}$ (i) $a_{11}b_{11}+a_{22}b_{22}$

3. Let A be a matrix of order 3×4. If R_1 denotes the first row of A and C_2 denotes its second column, then determine the orders of matrices R_1 and C_2.

4. Construct a 2×3 matrix $A=[a_{ij}]$ whose elements a_{ij} are given by:

   (i) $a_{ij}=i \times j$ (ii) $a_{ij}=2i-j$ (iii) $a_{ij}=i+j$ (iv) $a_{ij}=\dfrac{(i+j)^2}{2}$

5. Construct a 2×2 matrix $A=[a_{ij}]$ whose elements a_{ij} are given by:

   (i) $\dfrac{(i+j)^2}{2}$ [NCERT] (ii) $a_{ij}=\dfrac{(i-j)^2}{2}$

   (iii) $a_{ij}=\dfrac{(i-2j)^2}{2}$ [CBSE 2002, NCERT EXEMPLAR] (iv) $a_{ij}=\dfrac{(2i+j)^2}{2}$ [CBSE 2002]

(v) $a_{ij} = \dfrac{|2i-3j|}{2}$ [NCERT EXEMPLAR] (vi) $a_{ij} = \dfrac{|-3i+j|}{2}$ [NCERT]

(vii) $a_{ij} = e^{2ix}\sin xj$ [NCERT EXEMPLAR]

6. Construct a 3×4 matrix $A=[a_{ij}]$ whose elements a_{ij} are given by:

(i) $a_{ij} = i+j$ (ii) $a_{ij} = i-j$ (iii) $a_{ij} = 2i$

(iv) $a_{ij} = j$ (v) $a_{ij} = \dfrac{1}{2}\,|-3i+j|$ [NCERT]

7. Construct a 4×3 matrix $A=[a_{ij}]$ whose elements a_{ij} are given by:

(i) $a_{ij} = 2i+\dfrac{i}{j}$ (ii) $a_{ij} = \dfrac{i-j}{i+j}$ (iii) $a_{ij} = i$

8. Find x, y, a and b, if $\begin{bmatrix} 3x+4y & 2 & x-2y \\ a+b & 2a-b & -1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 5 & -5 & -1 \end{bmatrix}$.

9. Find x, y, a and b, if $\begin{bmatrix} 2x-3y & a-b & 3 \\ 1 & x+4y & 3a+4b \end{bmatrix} = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 6 & 29 \end{bmatrix}$.

10. Find the values of a, b, c and d, if $\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$. [NCERT]

11. Find x, y and z so that $A=B$, where $A = \begin{bmatrix} x-2 & 3 & 2z \\ 18z & y+2 & 6z \end{bmatrix}$, $B = \begin{bmatrix} y & z & 6 \\ 6y & x & 2y \end{bmatrix}$.

12. If $\begin{bmatrix} x & 3x-y \\ 2x+z & 3y-\omega \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, find x, y, z, ω.

13. If $\begin{bmatrix} x-y & z \\ 2x-y & \omega \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find x, y, z, ω. [CBSE 2014]

14. If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ 4x+6 & a-1 & 0 \\ b-3 & 3b & z+2c \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ 2x & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$ [NCERT]

Obtain the values of a, b, c, x, y and z.

15. If $\begin{bmatrix} 2x+1 & 5x \\ 0 & y^2+1 \end{bmatrix} = \begin{bmatrix} x+3 & 10 \\ 0 & 26 \end{bmatrix}$, find the value of $(x+y)$. [CBSE 2012]

16. If $\begin{bmatrix} xy & 4 \\ z+6 & x+y \end{bmatrix} = \begin{bmatrix} 8 & w \\ 0 & 6 \end{bmatrix}$, then find the values of x, y, z and w. [NCERT EXEMPLAR]

17. Give an example of: (i) a row matrix which is also a column matrix

(ii) a diagonal matrix which is not scalar (iii) a triangular matrix.

BASED ON LOTS

18. The sales figure of two car dealers during January 2013 showed that dealer A sold 5 deluxe, 3 premium and 4 standard cars, while dealer B sold 7 deluxe, 2 premium and 3 standard cars. Total sales over the 2 month period of January-February revealed that dealer A sold 8 deluxe 7 premium and 6 standard cars. In the same 2 month period, dealer B sold 10 deluxe, 5 premium and 7 standard cars. Write 2×3 matrices summarizing sales data for January and 2-month period for each dealer.

19. For what values of x and y are the following matrices equal?

$$A = \begin{bmatrix} 2x+1 & 2y \\ 0 & y^2-5y \end{bmatrix}, \; B = \begin{bmatrix} x+3 & y^2+2 \\ 0 & -6 \end{bmatrix}$$

20. Find the values of x and y if $\begin{bmatrix} x+10 & y^2+2y \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3x+4 & 3 \\ 0 & y^2-5y \end{bmatrix}$

21. Find the values of a and b if $A = B$, where $A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-10 \end{bmatrix}$

[NCERT EXEMPLAR]

ANSWERS

1. (i) $1\times 8, 8\times 1, 2\times 4, 4\times 2$ (ii) $1\times 5, 5\times 1$
2. (i) 1 (ii) 20
3. $1\times 4, 3\times 1$
4. (i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 9/2 & 8 \\ 9/2 & 8 & 25/2 \end{bmatrix}$
5. (i) $\begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1/2 & 9/2 \\ 0 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 9/2 & 8 \\ 25/2 & 18 \end{bmatrix}$

   (v) $\begin{bmatrix} 1/2 & 2 \\ 1/2 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 1 & 1/2 \\ 5/2 & 2 \end{bmatrix}$
6. (i) $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 \\ 6 & 6 & 6 & 6 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$

   (v) $\begin{bmatrix} 1 & 1/2 & 0 & 1/2 \\ 5/2 & 2 & 3/2 & 1 \\ 4 & 7/2 & 3 & 5/2 \end{bmatrix}$
7. (i) $\begin{bmatrix} 3 & 5/2 & 7/3 \\ 6 & 5 & 14/3 \\ 9 & 15/2 & 7 \\ 12 & 10 & 28/3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -1/3 & -1/2 \\ 1/3 & 0 & -1/5 \\ 1/2 & 1/5 & 0 \\ 3/5 & 1/3 & 1/7 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}$
8. $x=2, y=-1, a=0, b=5$
9. $x=2, y=1, a=3, b=5$
10. $a=1, b=2, c=3, d=4$
11. $x=11, y=9, z=3$
12. $x=3, y=7, z=-2, \omega=14$
13. $x=1, y=2, z=4, \omega=5$
14. $a=-2, b=-7, c=-1, x=-3, y=-5, z=2$
15. $7, -3$
16. $x=2, y=4, z=-6, w=4$ or $x=4, y=2, z=-6, w=4$
17. (i) [5] (ii) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 4 & 3 & 5 \\ 0 & 7 & 1 \\ 0 & 0 & 2 \end{bmatrix}$
18. $\begin{array}{c} \\ \text{Dealer } A \\ \text{Dealer } B \end{array} \begin{array}{c} \begin{array}{ccc} \text{Deluxe} & \text{Premium} & \text{Standard} \end{array} \\ \begin{bmatrix} 5 & 3 & 4 \\ 7 & 2 & 3 \end{bmatrix} \end{array}$, $\begin{array}{c} \\ \text{Dealer } A \\ \text{Dealer } B \end{array} \begin{array}{c} \begin{array}{ccc} \text{Deluxe} & \text{Premium} & \text{Standard} \end{array} \\ \begin{bmatrix} 8 & 7 & 6 \\ 10 & 5 & 7 \end{bmatrix} \end{array}$
19. A and B cannot be equal for any value of y.
20. $x=3, y=1$
21. $a=2, b=2$

4.4 ADDITION OF MATRICES

DEFINITION *Let A, B be two matrices, each of order $m \times n$. Then their sum $A + B$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.*

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order, their sum $A + B$ is defined to be the matrix of order $m \times n$ such that

$$(A + B)_{ij} = a_{ij} + b_{ij} \quad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad j = 1, 2, \ldots, n$$

NOTE *The sum of two matrices is defined only when they are of the same order.*

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$, then $A + B = \begin{bmatrix} 1+6 & 2+5 & 3+4 \\ 4+3 & 5+2 & 6+1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 5 & -2 \end{bmatrix}$, then $A + B$ is not defined, because A and B are not of the same order.

For the following pairs of matrices $A + B$ is not defined because they are of different orders:

(i) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 0 & 0 & 5 \\ 1 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 5 \end{bmatrix}$

4.4.1 PROPERTIES OF MATRIX ADDITION

THEOREM 1 *(Commutativity) If A and B are two $m \times n$ matrices, then $A + B = B + A$. i.e. matrix addition is commutative.*

PROOF Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two $m \times n$ matrices. Then, $A + B$ and $B + A$ both are $m \times n$ matrices such that

$$\begin{aligned} (A + B)_{ij} &= a_{ij} + b_{ij} && \text{[By definition of addition]} \\ &= b_{ij} + a_{ij} && \text{[By commutativity of addition of numbers]} \\ &= (B + A)_{ij} && \text{[By definition of addition]} \\ &= (B + A)_{ij} && \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \end{aligned}$$

Thus, $A + B$ and $B + A$ are two matrices such that their orders are same and the corresponding elements are equal. Hence, $A + B = B + A$.

Q.E.D.

NOTE *To prove that two matrices are equal it is required to prove that their orders are same and the corresponding elements are equal.*

THEOREM 2 *(Associativity) If A, B, C are three matrices of the same order, then =*

$(A + B) + C = A + (B + C)$ i.e. matrix addition is associative.

PROOF Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ be three $m \times n$ matrices. Then, $(A + B) + C$ and $A + (B + C)$ are $m \times n$ matrices such that

$$\begin{aligned} ((A + B) + C)_{ij} &= (A + B)_{ij} + (C)_{ij} && \text{[By definition of addition]} \\ &= (a_{ij} + b_{ij}) + c_{ij} && \text{[By definition of addition]} \\ &= a_{ij} + (b_{ij} + c_{ij}) && \text{[By associativity of addition of numbers]} \\ &= (A)_{ij} + (B + C)_{ij} && \text{[By definition of addition]} \\ &= (A + (B + C))_{ij} && \text{[By definition of addition]} \\ &= (A + (B + C))_{ij} && \text{for } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n \end{aligned}$$

Thus, $(A + B) + C$ and $A + (B + C)$ are two matrices such that their orders are same and the corresponding elements are equal. Hence, $(A + B) + C = A + (B + C)$.

Q.E.D.

THEOREM 3 *(Existence of Identity) The null matrix is the identity element for matrix addition, i.e.* $A + O = A = O + A$.

PROOF Let $A = [a_{ij}]$ be any matrix of order $m \times n$ and O be a null matrix of order $m \times n$. Then, $A + O$ and $O + A$ are $m \times n$ matrices such that

$$(A+O)_{ij} = a_{ij} + 0 = a_{ij} = (A)_{ij} \text{ and, } (O+A)_{ij} = 0 + a_{ij} = a_{ij} = (A)_{ij} \text{ for all } i, j$$

Hence, $A + O = A = O + A$. **Q.E.D.**

THEOREM 4 *(Existence of Inverse) For every matrix $A = [a_{ij}]_{m\times n}$ there exists a matrix $[-a_{ij}]_{m\times n}$, denoted by $-A$, such that $A + (-A) = O = (-A) + A$.*

PROOF We have,

$$(A+(-A))_{ij} = a_{ij} + (-a_{ij}) = a_{ij} - a_{ij} = 0 \text{ and, } ((-A)+A)_{ij} = (-a_{ij}) + a_{ij} = -a_{ij} + a_{ij} = 0 \text{ for all } i, j.$$

Hence, $A + (-A) = O = (-A) + A$.

Q.E.D.

The matrix $-A = [-a_{ij}]_{m\times n}$ is called the additive inverse of the matrix $A = [a_{ij}]_{m\times n}$.

If $A = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 2 & 5 & 7 & -4 \end{bmatrix}$, then $(-A) = \begin{bmatrix} -1 & 2 & -4 & -3 \\ -2 & -5 & -7 & 4 \end{bmatrix}$.

THEOREM 5 *(Cancellation laws) If A, B, C are matrices of the same order, then*

$A + B = A + C \Rightarrow B = C$ *[Left cancellation law]*

and, $B + A = C + A \Rightarrow B = C$ *[Right cancellation law]*

PROOF We have,

$A + B = A + C$

$\Rightarrow (-A) + (A + B) = (-A) + (A + C)$ [Adding $(-A)$ on both sides]

$\Rightarrow (-A + A) + B = (-A + A) + C$ [By associativity of addition]

$\Rightarrow O + B = O + C$ [$\because -A + A = O$]

$\Rightarrow B = C$ [$\because$ O is the additive identity]

Similarly, we can prove that : $B + A = C + A \Rightarrow B = C$.

Q.E.D.

4.5 MULTIPLICATION OF A MATRIX BY A SCALAR (SCALAR MULTIPLICATION)

DEFINITION *Let $A = [a_{ij}]$ be an $m \times n$ matrix and k be any number called a scalar. Then the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.*

Thus, $kA = [k\, a_{ij}]_{m\times n}$

For example, if $A = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 3 & 4 \\ 1 & 2 & -1 \end{bmatrix}$, then $3A = \begin{bmatrix} 3 & 6 & 15 \\ -6 & 9 & 12 \\ 3 & 6 & -3 \end{bmatrix}$ and $\frac{1}{2}A = \begin{bmatrix} \frac{1}{2} & 1 & \frac{5}{2} \\ -1 & \frac{3}{2} & 2 \\ \frac{1}{2} & 1 & \frac{-1}{2} \end{bmatrix}$.

4.5.1 PROPERTIES OF SCALAR MULTIPLICATION

Various properties of scalar multiplication are stated and proved in the following theorem.

THEOREM *If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ are two matrices and k, l are scalars, then*

(i) $k(A + B) = kA + kB$ (ii) $(k + l)A = kA + lA$

(iii) $(kl)A = k(lA) = l(kA)$ (iv) $(-k)A = -(kA) = k(-A)$

(v) $1A = A$ (vi) $(-1)A = -A$

PROOF (i) Since A and B are matrices of the same order $m \times n$, $A + B$ is also a matrix of order $m \times n$. Therefore, $k(A+B)$ is also of order $m \times n$. Further, kA and kB are of order $m \times n$. Therefore, $kA + kB$ is also of order $m \times n$. Thus, $k(A+B)$ and $kA + kB$ are matrices of the same order such that

$$\begin{aligned}
(k(A+B))_{ij} &= k(A+B)_{ij} && \text{[By definition of scalar multiplication]}\\
&= k(a_{ij}+b_{ij}) && \text{[By definition of addition of matrices]}\\
&= k\,a_{ij} + k\,b_{ij} && \text{[By distributivity of multiplication over addition]}\\
&= (kA)_{ij} + (kB)_{ij} && \text{[By definition of scalar multiplication]}\\
&= (kA+kB)_{ij} && \text{[By definition of matrix addition]}\\
&= (kA+kB)_{ij} && \text{for } i=1,2,\dots,m \text{ and } j=1,2,\dots,n
\end{aligned}$$

Hence, $k(A+B) = kA + kB$ [By definition of equality of two matrices]

(ii) Since k and l are scalars, $k + l$ is also a scalar. Therefore, $(k+l)A$ is a matrix of order $m \times n$. Also, kA and lA are $m \times n$ matrices. Therefore, $kA + lA$ is also an $m \times n$ matrix.

Thus, $(k+l)A$ and $kA + lA$ are two matrices of the same order $m \times n$ such that

$$\begin{aligned}
((k+l)A)_{ij} &= (k+l)\,a_{ij} && \text{[By definition of scalar multiplication]}\\
&= ka_{ij} + la_{ij} && \text{[By distributivity of multiplication over addition]}\\
&= (kA)_{ij} + (lA)_{ij} && \text{[By definition of scalar multiplication]}\\
&= (kA+lA)_{ij} && \text{[By definition of addition of matrices]}\\
&= (kA+lA)_{ij} && \text{for } i=1,2,\dots,m \text{ and } j=1,2,\dots,n
\end{aligned}$$

Hence, $(k+l)A = kA + lA$. [By definition of equality of two matrices]

(iii) Since k and l are scalars, kl is also a scalar. Therefore, $(kl)A$ is an $m \times n$ matrix. Also, note that lA and kA are matrices of order $m \times n$. Therefore, $k(lA)$ and $l(kA)$ are matrices of order $m \times n$. Thus, $(kl)A$ and $k(lA)$ are two matrices of the same order $m \times n$ such that

$$\begin{aligned}
((kl)A)_{ij} &= (kl)\,a_{ij} && \text{[By definition of scalar multiplication]}\\
&= k(la_{ij}) && \text{[By association of multiplication]}\\
&= k(lA)_{ij} && \text{[By definition of scalar multiplication]}\\
&= (k(lA))_{ij} && \text{[By definition of scalar multiplication]}\\
&= (k(lA))_{ij} && \text{for } i=1,2,\dots,m \text{ and } j=1,2,\dots,n
\end{aligned}$$

Hence, $(kl)A = k(lA)$. [By definition of equality of two matrices]

Similarly, it can be proved that $(kl)A = l(kA)$. Hence, $(kl)A = k(lA) = l(kA)$.

(iv) Putting $l = -1$ in (iii), we obtain: $(-k)A = k(-A) = -(kA)$

(v) Putting $k = -1$ in (iv), we obtain: $1A = A$.

(vi) Putting $k = 1$ in (iv), we obtain: $(-1)A = -A$. **Q.E.D.**

4.6 SUBTRACTION OF MATRICES

DEFINITION *For two matrices A and B of the same order, the subtraction of matrix B from matrix A is denoted by $A - B$ and is defined as $A - B = A + (-B)$.*

For example, if $A = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & -2 \\ -1 & 4 & -2 \end{bmatrix}$, then

$$A - B = A + (-B) = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -4 & 7 \end{bmatrix} + \begin{bmatrix} -3 & -5 & 2 \\ 1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -3 & 3 \\ 2 & -8 & 9 \end{bmatrix}$$

ILLUSTRATION *If* $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & 6 \\ 5 & 8 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 & 5 \\ 5 & 3 & 2 \\ 0 & 4 & 7 \end{bmatrix}$, *find* $3A - 2B$.

SOLUTION We find that $3A - 2B = 3A + (-2)B$

$$\therefore \quad 3A - 2B = \begin{bmatrix} 6 & 9 & 12 \\ 0 & 12 & 18 \\ 15 & 24 & 27 \end{bmatrix} + \begin{bmatrix} -6 & 0 & -10 \\ -10 & -6 & -4 \\ 0 & -8 & -14 \end{bmatrix} = \begin{bmatrix} 0 & 9 & 2 \\ -10 & 6 & 14 \\ 15 & 16 & 13 \end{bmatrix}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If* $A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 5 & 1 \\ -2 & 7 & 3 \end{bmatrix}$, *find* $A + B$ *and* $A - B$.

SOLUTION Clearly, A and B both are matrices of the same order 2×3. So, $A + B$ and $A - B$ both are defined.

Now,

$$A + B = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 1 \\ -2 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 2+0 & 3+5 & -5+1 \\ 1-2 & 2+7 & -1+3 \end{bmatrix} = \begin{bmatrix} 2 & 8 & -4 \\ -1 & 9 & 2 \end{bmatrix}$$

and, $$A - B = A + (-B) = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 5 & 1 \\ -2 & 7 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -5 & -1 \\ 2 & -7 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 3-5 & -5-1 \\ 1+2 & 2-7 & -1-3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -6 \\ 3 & -5 & -4 \end{bmatrix}$$

EXAMPLE 2 *If* $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$, *find* $3A - 2B$.

SOLUTION Clearly,

$$3A = \begin{bmatrix} 6 & -3 \\ 9 & 3 \end{bmatrix} \text{ and, } (-2)B = \begin{bmatrix} -2 & -8 \\ -14 & -4 \end{bmatrix}$$

$$\therefore \quad 3A - 2B = 3A + (-2)B = \begin{bmatrix} 6 & -3 \\ 9 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -8 \\ -14 & -4 \end{bmatrix} = \begin{bmatrix} 6+(-2) & -3+(-8) \\ 9+(-14) & 3+(-4) \end{bmatrix} = \begin{bmatrix} 4 & -11 \\ -5 & -1 \end{bmatrix}$$

EXAMPLE 3 *If* $A = diag\,(1 \; -1 \;\; 2)$ *and* $B = diag\,(2 \;\; 3 \; -1)$, *find* $A + B$, $3A + 4B$.

SOLUTION We have,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and, } B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\therefore \quad A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{diag}\,(3 \;\; 2 \;\; 1)$$

and, $$3A + 4B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \text{diag}\,(11 \;\; 9 \;\; 2)$$

REMARK *It is evident from the above example that if* $A = \text{diag}(a_1 \; a_2 \; a_3 \ldots a_n)$ *and* $B = \text{diag}(b_1 \; b_2 \; b_3 \ldots b_n)$ *Then,* $A + B = \text{diag}(a_1 + b_1 \; a_2 + b_2 \; a_3 + b_3 \ldots a_n + b_n)$.

EXAMPLE 4 *Simplify:* $\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

[CBSE 2012, NCERT]

SOLUTION We have,

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

EXAMPLE 5 *Find a matrix X such that* $2A + B + X = O$, *where* $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$ *and,* $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$.

[CBSE 2000]

SOLUTION We have,

$$2A + B + X = O$$

$$\Rightarrow \quad X = -2A - B$$

$$\Rightarrow \quad X = -2\begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -6 & -8 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 2-3 & -4+2 \\ -6-1 & -8-5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$$

EXAMPLE 6 *Find a matrix A such that* $2A - 3B + 5C = O$, *where* $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ *and* $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$.

[CBSE 2019]

SOLUTION We have,

$$2A - 3B + 5C = O$$

$$\Rightarrow \quad 2A = 3B - 5C$$

$$\Rightarrow \quad 2A = 3\begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix} + \begin{bmatrix} -10 & 0 & 10 \\ -35 & -5 & -30 \end{bmatrix}$$

$$\Rightarrow \quad 2A = \begin{bmatrix} -6-10 & 6+0 & 0+10 \\ 9-35 & 3-5 & 12-30 \end{bmatrix} = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}$$

$$\Rightarrow \quad A = \frac{1}{2}\begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix} = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Find X and Y, if* $X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ *and* $X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

[NCERT]

SOLUTION We have,

$$X + Y = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} \quad \text{and,} \quad X - Y = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\therefore \quad (X+Y) + (X-Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}$$

$$\Rightarrow \quad 2X = \begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow X = \frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

and, $(X+Y)-(X-Y) = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 7-3 & 0+0 \\ 2+0 & 5-3 \end{bmatrix}$

$\Rightarrow \quad 2Y = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow Y = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Hence, $X = \begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

EXAMPLE 8 *Find a matrix A, if* $A + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$.

SOLUTION Let $B = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$. Then, the given matrix equation is $A + B = C$.

Now, $A + B = C$

$\Rightarrow \quad (A+B) + (-B) = C + (-B)$ [Adding $-B$ on both sides]

$\Rightarrow \quad A + (B + (-B)) = C + (-B)$ [Using associativity of matrix addition on LHS]

$\Rightarrow \quad A + O = C - B$

$\Rightarrow \quad A = C - B.$

$\therefore \quad A = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 3-2 & -6-3 \\ -3+1 & 8-4 \end{bmatrix} = \begin{bmatrix} 1 & -9 \\ -2 & 4 \end{bmatrix}$

EXAMPLE 9 *Find x, y, z, t if* $2\begin{bmatrix} x & z \\ y & t \end{bmatrix} + 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$.

SOLUTION The given matrix equation can be written as **[NCERT]**

$$\begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x+3 & 2z-3 \\ 2y & 2t+6 \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix}$$

$\Rightarrow \quad 2x + 3 = 9,\ 2z - 3 = 15,\ 2y = 12$ and $2t + 6 = 18$ [By definition of equality of addition]

$\Rightarrow \quad x = 3,\ z = 9,\ y = 6$ and $t = 6.$

EXAMPLE 10 *Find non-zero values of x satisfying the matrix equation:*

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}$$

SOLUTION We have, **[NCERT EXEMPLAR]**

$$x\begin{bmatrix} 2x & 2 \\ 3 & x \end{bmatrix} + 2\begin{bmatrix} 8 & 5x \\ 4 & 4x \end{bmatrix} = 2\begin{bmatrix} x^2+8 & 24 \\ 10 & 6x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2 & 2x \\ 3x & x^2 \end{bmatrix} + \begin{bmatrix} 16 & 10x \\ 8 & 8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x^2+16 & 2x+10x \\ 3x+8 & x^2+8x \end{bmatrix} = \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$$

$\Rightarrow \quad 2x^2+16 = 2x^2+16,\ 2x+10x = 48,\ 3x+8 = 20$ and $x^2+8x = 12x$

$\Rightarrow \quad 12x = 48,\ 3x = 12$ and $x^2 - 4x = 0 \Rightarrow x = 4$ and, $x(x-4) = 0 \Rightarrow x = 4$ and $x = 0, 4 \Rightarrow x = 4$

EXAMPLE 11 *Solve the matrix equation* $\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$.

SOLUTION We have,

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - 3\begin{bmatrix} x \\ 2y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2 \\ y^2 \end{bmatrix} - \begin{bmatrix} 3x \\ 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix} \Rightarrow \begin{bmatrix} x^2 - 3x \\ y^2 - 6y \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \quad x^2 - 3x = -2 \text{ and } y^2 - 6y = 9 \Rightarrow x^2 - 3x + 2 = 0 \text{ and } y^2 - 6y - 9 = 0$$

$$\Rightarrow \quad (x-1)(x-2) = 0 \text{ and } y = \frac{6 \pm \sqrt{36+36}}{2} \Rightarrow x = 1, 2 \text{ and } y = 3 \pm 3\sqrt{2}$$

EXAMPLE 12 *If A, B and C are three matrices of the same order, then prove that*

$$A = B \Rightarrow A + C = B + C.$$

SOLUTION Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ and $C = [c_{ij}]_{m \times n}$ be three matrices of the same order $m \times n$. Then, $A + C$ and $B + C$ are also of the same order $m \times n$.

Now, $A = B$

$\Rightarrow \quad a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$

$\Rightarrow \quad a_{ij} + c_{ij} = b_{ij} + c_{ij}$ for all $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$ [Adding c_{ij} on both sides]

$\Rightarrow \quad (A + C)_{ij} = (B + C)_{ij}$ for all $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$

$\Rightarrow \quad A + C = B + C.$

EXAMPLE 13 *Two farmers Ram Kishan and Gurcharan Singh cultivate only three varities of rice namely Basmati, Permal and Naura. The sale (in ₹) of these varities of rice by both the farmers in the month of September and October are given by the following matrices A and B*

September sales (in ₹)

$$A = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10{,}000 & 20{,}000 & 30{,}000 \\ 50{,}000 & 30{,}000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

October sales (in ₹)

$$B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5{,}000 & 10{,}000 & 6{,}000 \\ 20{,}000 & 10{,}000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

Find:

(i) *What were the combined sales in September and October for each farmer in each variety.*

(ii) *What was the change in sales from September to October?*

(iii) *If both farmers receive 2% profit on gross rupees sales, compute the profit for each farmer and for each variety sold in October.*

SOLUTION (i) The combined sales in September and October is given by $A + B$.

Clearly,

$$A + B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10{,}000 + 5000 & 20{,}000 + 10{,}000 & 30{,}000 + 6000 \\ 50{,}000 + 20{,}000 & 30{,}000 + 10{,}000 & 10{,}000 + 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

$$\Rightarrow \quad A + B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 15{,}000 & 30{,}000 & 36{,}000 \\ 70{,}000 & 40{,}000 & 20{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ram Kishan} \\ \text{Gurcharan Singh} \end{array}$$

(ii) The change in sales from September to October is given by $A - B$.

Clearly,

$$A-B=\begin{array}{c}\begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array}\\ \begin{bmatrix}10{,}000-5{,}000 & 20{,}000-10{,}000 & 30{,}000-6000\\ 50{,}000-20{,}000 & 30{,}000-10{,}000 & 10{,}000-10{,}000\end{bmatrix}\end{array}\begin{array}{l}\text{Ram Kishan}\\ \text{Gurcharan Singh}\end{array}$$

$$A-B=\begin{array}{c}\begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array}\\ \begin{bmatrix}5{,}000 & 10{,}000 & 24{,}000\\ 30{,}000 & 20{,}000 & 0\end{bmatrix}\end{array}\begin{array}{l}\text{Ram Kishan}\\ \text{Gurcharan Singh}\end{array}$$

(iii) The profit for each farmer and for each variety sold in October at the rate of 2% of gross sale is given by

$$2\% \text{ of } B=\frac{2}{100}\times B=0.02\,B=0.02\begin{array}{c}\begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array}\\ \begin{bmatrix}5{,}000 & 10{,}000 & 6{,}000\\ 20{,}000 & 10{,}000 & 10{,}000\end{bmatrix}\end{array}\begin{array}{l}\text{Ram Kishan}\\ \text{Gurcharan Singh}\end{array}$$

$$=\begin{array}{c}\begin{array}{ccc}\text{Basmati} & \text{Permal} & \text{Naura}\end{array}\\ \begin{bmatrix}100 & 200 & 120\\ 400 & 200 & 200\end{bmatrix}\end{array}\begin{array}{l}\text{Ram Kishan}\\ \text{Gurcharan Singh}\end{array}$$

Thus, in October Ram Kishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice respectively, and Gurcharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in each variety of rice respectively.

EXERCISE 4.2

BASIC

1. Compute the following sums:

(i) $\begin{bmatrix}3 & -2\\ 1 & 4\end{bmatrix}+\begin{bmatrix}-2 & 4\\ 1 & 3\end{bmatrix}$ (ii) $\begin{bmatrix}2 & 1 & 3\\ 0 & 3 & 5\\ -1 & 2 & 5\end{bmatrix}+\begin{bmatrix}1 & -2 & 3\\ 2 & 6 & 1\\ 0 & -3 & 1\end{bmatrix}$

2. Let $A=\begin{bmatrix}2 & 4\\ 3 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 3\\ -2 & 5\end{bmatrix}$ and $C=\begin{bmatrix}-2 & 5\\ 3 & 4\end{bmatrix}$. Find each of the following:

(i) $2A-3B$ (ii) $B-4C$ (iii) $3A-C$ (iv) $3A-2B+3C$

3. If $A=\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}$, $B=\begin{bmatrix}-1 & 0 & 2\\ 3 & 4 & 1\end{bmatrix}$, $C=\begin{bmatrix}-1 & 2 & 3\\ 2 & 1 & 0\end{bmatrix}$, find

(i) $A+B$ and $B+C$ (ii) $2B+3A$ and $3C-4B$.

4. Let $A=\begin{bmatrix}-1 & 0 & 2\\ 3 & 1 & 4\end{bmatrix}$, $B=\begin{bmatrix}0 & -2 & 5\\ 1 & -3 & 1\end{bmatrix}$ and $C=\begin{bmatrix}1 & -5 & 2\\ 6 & 0 & -4\end{bmatrix}$. Compute $2A-3B+4C$.

5. If $A=\text{diag}\,(2\quad -5\quad 9)$, $B=\text{diag}\,(1\quad 1\quad -4)$ and $C=\text{diag}\,(-6\quad 3\quad 4)$, find

(i) $A-2B$ (ii) $B+C-2A$ (iii) $2A+3B-5C$

6. Given the matrices

$$A=\begin{bmatrix}2 & 1 & 1\\ 3 & -1 & 0\\ 0 & 2 & 4\end{bmatrix},\quad B=\begin{bmatrix}9 & 7 & -1\\ 3 & 5 & 4\\ 2 & 1 & 6\end{bmatrix}\quad\text{and}\quad C=\begin{bmatrix}2 & -4 & 3\\ 1 & -1 & 0\\ 9 & 4 & 5\end{bmatrix}$$

Verify that $(A+B)+C=A+(B+C)$.

BASED ON LOTS

7. Find matrices X and Y, if $X+Y=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}$ and $X-Y=\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$. [NCERT]

8. Find X, if $Y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$. **[NCERT]**

9. Find matrices X and Y, if $2X-Y=\begin{bmatrix}6 & -6 & 0\\ -4 & 2 & 1\end{bmatrix}$ and $X+2Y=\begin{bmatrix}3 & 2 & 5\\ -2 & 1 & -7\end{bmatrix}$.

10. If $X-Y=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ and $X+Y=\begin{bmatrix}3 & 5 & 1\\ -1 & 1 & 4\\ 11 & 8 & 0\end{bmatrix}$, find X and Y.

11. Find matrix A, if $\begin{bmatrix}1 & 2 & -1\\ 0 & 4 & 9\end{bmatrix}+A=\begin{bmatrix}9 & -1 & 4\\ -2 & 1 & 3\end{bmatrix}$.

12. If $A=\begin{bmatrix}9 & 1\\ 7 & 8\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 7 & 12\end{bmatrix}$, find matrix C such that $5A+3B+2C$ is a null matrix.

13. If $A=\begin{bmatrix}2 & -2\\ 4 & 2\\ -5 & 1\end{bmatrix}$, $B=\begin{bmatrix}8 & 0\\ 4 & -2\\ 3 & 6\end{bmatrix}$, find matrix X such that $2A+3X=5B$.

14. If $A=\begin{bmatrix}1 & -3 & 2\\ 2 & 0 & 2\end{bmatrix}$ and, $B=\begin{bmatrix}2 & -1 & -1\\ 1 & 0 & -1\end{bmatrix}$, find the matrix C such that $A+B+C$ is zero matrix.

15. Find x, y satisfying the matrix equations

(i) $\begin{bmatrix}x-y & 2 & -2\\ 4 & x & 6\end{bmatrix}+\begin{bmatrix}3 & -2 & 2\\ 1 & 0 & -1\end{bmatrix}=\begin{bmatrix}6 & 0 & 0\\ 5 & 2x+y & 5\end{bmatrix}$

(ii) $[x \;\; y+2 \;\; z-3]+[y \;\; 4 \;\; 5]=[4 \;\; 9 \;\; 12]$

(iii) $x\begin{bmatrix}2\\ 1\end{bmatrix}+y\begin{bmatrix}3\\ 5\end{bmatrix}+\begin{bmatrix}-8\\ -11\end{bmatrix}=O$ **[NCERT EXEMPLAR]**

16. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$, find x and y.

17. Find the value of λ, a non-zero scalar, if $\lambda\begin{bmatrix}1 & 0 & 2\\ 3 & 4 & 5\end{bmatrix}+2\begin{bmatrix}1 & 2 & 3\\ -1 & -3 & 2\end{bmatrix}=\begin{bmatrix}4 & 4 & 10\\ 4 & 2 & 14\end{bmatrix}$.

18. (i) Find a matrix X such that $2A+B+X=O$, where $A=\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$ **[CBSE 2000]**

(ii) If $A=\begin{bmatrix}8 & 0\\ 4 & -2\\ 3 & 6\end{bmatrix}$ and $B=\begin{bmatrix}2 & -2\\ 4 & 2\\ -5 & 1\end{bmatrix}$, then find the matrix X of order 3×2 such that $2A+3X=5B$. **[NCERT]**

19. Find x, y, z and t, if

(i) $3\begin{bmatrix}x & y\\ z & t\end{bmatrix}=\begin{bmatrix}x & 6\\ -1 & 2t\end{bmatrix}+\begin{bmatrix}4 & x+y\\ z+t & 3\end{bmatrix}$ **[NCERT]**

(ii) $2\begin{bmatrix}x & 5\\ 7 & y-3\end{bmatrix}+\begin{bmatrix}3 & 4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 14\\ 15 & 14\end{bmatrix}$ **[NCERT EXEMPLAR, CBSE 2002, 2012]**

20. If X and Y are 2×2 matrices, then solve the following matrix equations for X and Y.

$2X+3Y=\begin{bmatrix}2 & 3\\ 4 & 0\end{bmatrix}$, $3X+2Y=\begin{bmatrix}-2 & 2\\ 1 & -5\end{bmatrix}$ **[NCERT EXEMPLAR]**

21. In a certain city there are 30 colleges. Each college has 15 peons, 6 clerks, 1 typist and 1 section officer. Express the given information as a column matrix. Using scalar multiplication, find the total number of posts of each kind in all the colleges.

22. The monthly incomes of Aryan and Babban are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15000 per month, find their monthly incomes using matrix method. This problem reflects which value? [CBSE 2016]

ANSWERS

1. (i) $\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & -1 & 6 \\ 2 & 9 & 6 \\ -1 & -1 & 6 \end{bmatrix}$

2. (i) $\begin{bmatrix} 1 & -1 \\ 12 & -11 \end{bmatrix}$ (ii) $\begin{bmatrix} 9 & -17 \\ -14 & -11 \end{bmatrix}$ (iii) $\begin{bmatrix} 8 & 7 \\ 6 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} -2 & 21 \\ 22 & 8 \end{bmatrix}$

3. (i) $A+B$ does not exist, $B+C=\begin{bmatrix} -2 & 2 & 5 \\ 5 & 5 & 1 \end{bmatrix}$.

(ii) $2B+3A$ does not exit, $3C-4B=\begin{bmatrix} 1 & 6 & 1 \\ -6 & -13 & -4 \end{bmatrix}$ 4. $\begin{bmatrix} 2 & -14 & -3 \\ 27 & 11 & -11 \end{bmatrix}$

5. (i) diag $(0 \quad -7 \quad 17)$ (ii) diag $(-9 \quad 14 \quad -18)$ (iii) diag $(37 \quad -22 \quad -14)$

7. $X=\begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}, Y=\begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$ 8. $\begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$

9. $X=\begin{bmatrix} 3 & -2 & 1 \\ -2 & 1 & -1 \end{bmatrix}, Y=\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$ 10. $X=\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}, Y=\begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 5 & 4 & 0 \end{bmatrix}$

11. $\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$ 12. $\begin{bmatrix} -24 & -10 \\ -28 & -38 \end{bmatrix}$ 13. $\begin{bmatrix} 12 & 4/3 \\ 4 & -14/3 \\ 25/3 & 28/3 \end{bmatrix}$

14. $\begin{bmatrix} -3 & 4 & -1 \\ -3 & 0 & -1 \end{bmatrix}$ 15. (i) $x=\frac{3}{2}, y=-\frac{3}{2}$ (ii) $x=1, y=3, z=10$ (iii) $x=1, y=2$

16. $x=2, y=-8$ 17. $\lambda=2$ 18. (i) $\begin{bmatrix} -1 & -2 \\ -7 & -13 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -10/3 \\ 4 & 14/3 \\ -31/3 & -7/3 \end{bmatrix}$

19. (i) $x=2, y=4, t=3, z=1$ (ii) $x=2, y=9$ 20. $X=\begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}, Y=\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

21. $A=\begin{bmatrix} 15 \\ 6 \\ 1 \\ 1 \end{bmatrix}, 30A=\begin{bmatrix} 450 \\ 180 \\ 30 \\ 30 \end{bmatrix}$ 22. ₹ 90,000, ₹ 120,000

4.7 MULTIPLICATION OF MATRICES

Let us first define the product of a row matrix and a column matrix.

Let $A=[a_1\ a_2 \ldots a_n]$ be a row matrix and $B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column matrix. Then, we define

$$AB = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n .$$

For example, if $A=[1\ \ -2\ \ 3\ \ 4]$ and $B=\begin{bmatrix}5\\-4\\1\\-2\end{bmatrix}$. Then,

$$AB=[1\ \ -2\ \ 3\ \ 4]\begin{bmatrix}5\\-4\\1\\-2\end{bmatrix}=1\times 5+(-2)(-4)+3\times 1+4\times(-2)=5+8+3-8=8$$

Using the product of a row matrix and a column matrix, let us now define the multiplication of any two matrices.

DEFINITION *Two matrices A and B are conformable for the product AB if the number of columns in A (pre-multiplier) is same as the number of rows in B (post-multiplier).*

Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ are two matrices of orders $m\times n$ and $n\times p$ respectively, then their product AB is of orders $m\times p$ and is defined as

$$(AB)_{ij}=(i^{\text{th}}\text{ row of }A)\,(j^{\text{th}}\text{ column of }B)\quad\text{for all } i=1,2,\ldots,m \text{ and } j=1,2,\ldots,p.$$

$$\Rightarrow\quad (AB)_{ij}=[a_{i1}\ a_{i2}\ \ldots\ a_{in}]\begin{bmatrix}b_{1j}\\b_{2j}\\\vdots\\b_{nj}\end{bmatrix}$$

$$\Rightarrow\quad (AB)_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\ldots+a_{ir}b_{rj}+\ldots+a_{in}b_{nj}=\sum_{r=1}^{n}a_{ir}b_{rj}$$

NOTE *If A and B are two matrices such that AB exists, then BA may or may not exist.*

ILLUSTRATION 1 *If* $A=\begin{bmatrix}2&1&3\\3&-2&1\\-1&0&1\end{bmatrix}$ *and* $B=\begin{bmatrix}1&-2\\2&1\\4&-3\end{bmatrix}$, *then A is a 3×3 matrix and B is a 3×2 matrix. Therefore, A and B are conformable for the product AB and it is of order 3×2 such that*

$$(AB)_{11}=(\text{First row of }A)\,(\text{First column of }B)=[2\ 1\ 3]\begin{bmatrix}1\\2\\4\end{bmatrix}=2\times 1+1\times 2+3\times 4=16$$

$$(AB)_{12}=(\text{First row of }A)\,(\text{Second column of }B)=[2\ 1\ 3]\begin{bmatrix}-2\\1\\-3\end{bmatrix}=2\times -2+1\times 1+3\times -3=-12$$

$$(AB)_{21}=(\text{Second row of }A)\,(\text{First column of }B)=[3\ -2\ 1]\begin{bmatrix}1\\2\\4\end{bmatrix}=3\times 1+(-2)\times 2+1\times 4=3$$

Similarly, we obtain: $(AB)_{22}=-11, (AB)_{31}=3$ and $(AB)_{32}=-1$.

$$\therefore\quad AB=\begin{bmatrix}16&-12\\3&-11\\3&-1\end{bmatrix}$$

NOTE *In this case BA does not exist, because the number of columns in B is not same as the number of rows in A.*

ILLUSTRATION 2 *Let* $A=\begin{bmatrix}1 & -2 & 3\\ 3 & 2 & -1\end{bmatrix}$ *and* $B=\begin{bmatrix}2 & 3\\ -1 & 2\\ 4 & -5\end{bmatrix}$. *Find AB and BA and show that* $AB \neq BA$.

SOLUTION Here, A is a 2×3 matrix and B is a 3×2 matrix. So, AB exists and it is of order 2×2.

$$\therefore \quad AB = \begin{bmatrix}1 & -2 & 3\\ 3 & 2 & -1\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 2\\ 4 & -5\end{bmatrix} = \begin{bmatrix}2+2+12 & 3-4-15\\ 6-2-4 & 9+4+5\end{bmatrix} = \begin{bmatrix}16 & -16\\ 0 & 18\end{bmatrix}$$

Again, B is a 3×2 matrix and A is a 2×3 matrix. So, BA exists and it is or order 3×3.

$$\therefore \quad BA = \begin{bmatrix}2 & 3\\ -1 & 2\\ 4 & -5\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ 3 & 2 & -1\end{bmatrix} = \begin{bmatrix}2+9 & -4+6 & 6-3\\ -1+6 & 2+4 & -3-2\\ 4-15 & -8-10 & 12+5\end{bmatrix} = \begin{bmatrix}11 & 2 & 3\\ 5 & 6 & -5\\ -11 & -18 & 17\end{bmatrix}$$

Clearly, $AB \neq BA$.

4.7.1 PROPERTIES OF MATRIX MULTIPLICATION

THEOREM 1 *Matrix multiplication is not commutative in general.*

PROOF Let A and B be two matrices such that AB exists. Then it is quite possible that BA may not exist. For example, if A is a 3×3 matrix and B is a 3×2 matrix, then AB exists but BA does not exist. Similarly, if BA exists, then AB may not exist. Further, if AB and BA both exist, then they may not be equal as shown in illustration 2 (given above). Hence, in general, $AB \neq BA$.

Q.E.D.

THEOREM 2 *Matrix multiplication is associative i.e.* $(AB)\,C = A\,(BC)$, *whenever both sides are defined.*

PROOF Let $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{n\times p}$ and $C=[c_{ij}]_{p\times q}$. Then AB is an $m\times p$ matrix and so $(AB)\,C$ is a $m\times q$ matrix. Clearly, BC is of order $n\times q$ and so $A\,(BC)$ is of order $m\times q$. Thus, $(AB)\,C$ and $A\,(BC)$ are of the same order.

$$\text{Now,}\quad ((AB)\,C)_{ij} = \sum_{r=1}^{p} (AB)_{ir}\,(C)_{rj}$$

$$= \sum_{r=1}^{p}\left(\sum_{s=1}^{n} a_{is}\,b_{sr}\right)c_{rj} = \sum_{r=1}^{p}\sum_{s=1}^{n}(a_{is}\,b_{sr})\,c_{rj}$$

$$= \sum_{r=1}^{p}\sum_{s=1}^{n} a_{is}\,(b_{sr}\,c_{rj}) \qquad \text{[By associativity of multiplication of numbers]}$$

$$= \sum_{s=1}^{n} a_{is}\left(\sum_{r=1}^{p}(b_{sr}\,c_{rj})\right) = \sum_{s=1}^{n} a_{is}\,(BC)_{sj} = (A\,(BC))_{ij} \quad \text{for all } i, j$$

Thus, $(AB)\,C$ and $A\,(BC)$ are matrices of the same order such that their corresponding elements are equal. Hence, $(AB)\,C = A\,(BC)$.

Q.E.D.

THEOREM 3 *Matrix multiplication is distributive over matrix addition.*

i.e. (i) $A\,(B+C) = AB + AC$ (ii) $(A+B)\,C = AC + BC$

whenever both sides of equality are defined.

PROOF Let $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{n\times p}$ and $C=[c_{ij}]_{n\times p}$ be three matrices. Then, $B+C$ is of order $n\times p$ and so $A\,(B+C)$ is of order $m\times p$. Since AB and AC both are of the same order $m\times p$. Therefore, $AB+AC$ is of order $m\times p$. Thus, $A\,(B+C)$ and $AB+AC$ are of the same order $m\times p$ such that

$$(A(B+C))_{ij} = \sum_{r=1}^{n} a_{ir}(B+C)_{rj} = \sum_{r=1}^{n} a_{ir}(b_{rj}+c_{rj}) = \sum_{r=1}^{n} a_{ir} b_{rj} + \sum_{r=1}^{n} a_{ir} c_{rj}$$

$$= (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij} \quad \text{for all } i, j$$

Thus, $A(B+C)$ and $AB+AC$ are two matrices of the same order such that their corresponding elements are equal. Hence, $A(B+C) = AB + AC$.

Similarly, $(A+B)C = AC + BC$.

Q.E.D.

THEOREM 4 *If A is an $m \times n$ matrix, then $I_m A = A = A I_n$.*

PROOF Let $A = [a_{ij}]_{m \times n}$. Then, $I_m A$ and AI_n are of the same order $m \times n$ such that

$$(I_m A)_{ij} = \sum_{r=1}^{m} (I_m)_{ir} (A)_{rj} = \sum_{r=1}^{m} (I_m)_{ir} a_{rj} = (I_m)_{i1} a_{1j} + (I_m)_{i2} a_{2j} + \ldots + (I_m)_{ii} a_{ij} + \ldots + (I_m)_{im} a_{mj}$$

$$= a_{ij} \text{ for all } i, j \qquad \left[\because (I_m)_{ir} = \begin{cases} 0 & \text{for } r \neq i \\ 1 & \text{for } r = i \end{cases}\right]$$

$\therefore \quad I_m A = A.$

Now, $$(A I_n)_{ij} = \sum_{r=1}^{n} (A)_{ir} (I_n)_{rj} = \sum_{r=1}^{n} a_{ir} (I_n)_{rj} = a_{i1}(I_n)_{1j} + a_{i2}(I_n)_{2j} + \ldots + a_{ij}(I_n)_{jj} + \ldots + a_{in}(I_n)_{nj}$$

$$= a_{ij} \text{ for all } i, j \qquad [\because (I_n)_{jj} = 1 \text{ and } (I_n)_{rj} = 0 \text{ for } r \neq j]$$

Thus, AI_n and A are matrices of the same order such that their corresponding elements are equal. So, $AI_n = A$. Hence, $I_m A = A = AI_n$. Q.E.D.

REMARK 1 *The product of two matrices can be the null matrix while neither of them is the null matrix.*

For example, if $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, *then* $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *while neither A nor B is the null matrix.*

THEOREM 5 *If A is $m \times n$ matrix and O is a null matrix, then*

(i) $A_{m \times n} O_{n \times p} = O_{m \times p}$ (ii) $O_{p \times m} A_{m \times n} = O_{p \times n}$

i.e. the product of the matrix with a null matrix is always a null matrix.

PROOF (i) Let $A = [a_{ij}]_{m \times n}$ and $O_{n \times p} = [b_{ij}]_{n \times p}$, where $b_{ij} = 0$ for all i, j. Then, $A_{m \times n} O_{n \times p}$ is an $m \times p$ matrix such that

$$(A_{m \times n} O_{n \times p})_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = 0 \quad \text{for all } i, j \qquad [b_{ij} = 0 \text{ for all } i, j]$$

Thus, $A_{m \times n} O_{n \times p}$ and $O_{m \times p}$ are two matrices of the same order such that their corresponding elements are equal. Hence, $A_{m \times n} \cdot O_{n \times p} = O_{n \times p}$.

(ii) Proceed as in (i). Q.E.D.

REMARK 2 *In the case of matrix multiplication if $AB = O$, then it does not necessarily imply that $BA = O$.*

For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then, $AB = O$. But,* $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq O.$

Thus, $AB = O$ while $BA \neq O$.

4.7.2 POSITIVE INTEGRAL POWERS OF A SQUARE MATRIX

For any square matrix, we define

(i) $A^1 = A$ *and,* (ii) $A^{n+1} = A^n \cdot A$, where $n \in N$.

It is evident from this definition that $A^2 = AA$, $A^3 = A^2 A = (AA)A$. etc.

It can be easily shown that

(i) $A^m A^n = A^{m+n}$ and, (ii) $(A^m)^n = A^{mn}$ for all $m, n \in N$.

MATRIX POLYNOMIAL *Let $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n$ be a polynomial and let A be a square matrix of order n. Then,*

$$f(A) = a_0 A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_{n-1} A + a_n I_n$$

is called a matrix polynomial.

For example, if $f(x) = x^2 - 3x + 2$ is a polynomial and A is a square matrix, then $f(A) = A^2 - 3A + 2I$ is a matrix polynomial.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON MULTIPLICATION OF MATRICES**

EXAMPLE 1 *If A, B, C are three matrices such that $A = [x\ y\ z]$, $B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, find ABC.*

SOLUTION Since the product of matrices is associative. Therefore, we can find ABC either by computing $(AB)C$ or by computing $A(BC)$. Let us compute $A(BC)$.

Since B is a 3×3 matrix and C is 3×1 matrix. Therefore, BC is of order 3×1 and is given by

$$BC = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix}$$

Clearly, A is of order 1×3 and BC is of order 3×1. Therefore, $A(BC)$ is of order 1×1 and is given by

$$A(BC) = [x\ y\ z] \begin{bmatrix} ax + hy + gz \\ hx + by + fz \\ gx + fy + cz \end{bmatrix} = x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz)$$

$$= ax^2 + 2hxy + by^2 + cz^2 + 2fyz + 2gzx$$

EXAMPLE 2 *If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, prove that $(A+B)^2 \neq A^2 + 2AB + B^2$.*

SOLUTION We have,

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix},$$

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 2 \end{bmatrix} \Rightarrow 2AB = \begin{bmatrix} 2 & 2 \\ 14 & 4 \end{bmatrix}$$

and, $$B^2 = BB = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^2 + 2AB + B^2 = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 14 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 24 & 12 \end{bmatrix} \quad \ldots(i)$$

Now, $$A + B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$$

$$\Rightarrow \quad (A+B)^2 = (A+B)(A+B) = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 18 & 9 \end{bmatrix} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain that $(A+B)^2 \neq A^2 + 2AB + B^2$.

EXAMPLE 3 *If* $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ *and* $(A+B)^2 = A^2 + B^2$, *find a and b.*

SOLUTION We have, **[CBSE 2015]**

$$(A+B)^2 = A^2 + B^2$$

$$\Rightarrow \quad (A+B)(A+B) = A^2 + B^2$$

$$\Rightarrow \quad (A+B)A + (A+B)B = A^2 + B^2 \qquad \text{[By distributive law]}$$

$$\Rightarrow \quad A^2 + BA + AB + B^2 = A^2 + B^2 \qquad \text{[By distributive law]}$$

$$\Rightarrow \quad BA + AB = O$$

$$\Rightarrow \quad \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 2a-b+2=0,\ -a+1=0,\ 2a-2=0 \text{ and } -b+4=0 \Rightarrow a=1, b=4$$

EXAMPLE 4 *If* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, *find x and y such that* $(xI + yA)^2 = A$.

SOLUTION We have, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\therefore \quad xI + yA = x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

Given that $(xI + yA)^2 = A$

$$\Rightarrow \quad \begin{bmatrix} x & y \\ -y & x \end{bmatrix}\begin{bmatrix} x & y \\ -y & x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x^2-y^2 & 2xy \\ -2xy & x^2-y^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad x^2-y^2 = 0,\ 2xy = 1,\ -2xy = -1 \text{ and } x^2-y^2 = 0$$

$$\Rightarrow \quad x^2-y^2 = 0 \text{ and } 2xy = 1 \Rightarrow x = \pm y \text{ and } 2xy = 1$$

Now two cases arise.

Case I *When* $x = y$ *and* $2xy = 1$: In this case, we have $x = y$ and $2xy = 1 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$.

$$\therefore \quad \left(x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}\right) \text{ or, } \left(x = -\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}}\right)$$

Case II *When* $x = -y$ *and* $2xy = 1$: In this case, we have $x = -y$ and $2xy = 1 \Rightarrow -2x^2 = 1 \Rightarrow x = \pm \frac{i}{\sqrt{2}}$.

$$\therefore \quad \left(x = \frac{i}{\sqrt{2}} \text{ and } y = -\frac{i}{\sqrt{2}}\right) \text{ or, } \left(x = -\frac{i}{\sqrt{2}} \text{ and } y = \frac{i}{\sqrt{2}}\right)$$

EXAMPLE 5 *If* $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$, *find the values of* α *for which* $A^2 = B$.

SOLUTION Given that

$$A^2 = B$$

$$\Rightarrow \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha^2+0 & 0+0 \\ \alpha+1 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha^2 & 0 \\ \alpha+1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \quad \alpha^2 = 1$ and $\alpha + 1 = 5 \Rightarrow \alpha = \pm 1$ and $\alpha = 4$, which is not possible.

Hence, there is no value of α for which $A^2 = B$ is true.

EXAMPLE 6 *Let* $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. *Find a matrix D such that* $CD - AB = O$.

[NCERT, CBSE 2017]

SOLUTION Let $D = \begin{bmatrix} a & b \\ x & y \end{bmatrix}$. Then,

$$CD - AB = O$$

$$\Rightarrow CD = AB$$

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ x & y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5x & 2b+5y \\ 3a+8x & 3b+8y \end{bmatrix} = \begin{bmatrix} 10-7 & 4-4 \\ 15+28 & 6+16 \end{bmatrix} \Rightarrow \begin{bmatrix} 2a+5x & 2b+5y \\ 3a+8x & 3b+8y \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

$\Rightarrow \quad 2a+5x = 3,\ 3a+8x = 43,\ 2b+5y = 0$ and $3b+8y = 22$

Solving $2a+5x=3$ and $3a+8x=43$, we get: $a = -191$ and $x = 77$.

Solving $2b+5y=0$ and $3b+8y=22$, we get: $b = -110$ and $y = 44$.

$$\therefore \quad D = \begin{bmatrix} a & b \\ x & y \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

EXAMPLE 7 *Find the value of x such that* $[1 \ \ x \ \ 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0.$

SOLUTION We have, **[CBSE 2006, NCERT EXEMPLAR]**

$$[1 \ \ x \ \ 1]\begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0 \Rightarrow [1 \ \ x \ \ 1]\begin{bmatrix} 7+2x \\ 12+x \\ 21+2x \end{bmatrix} = 0$$

$\Rightarrow 7+2x+12x+x^2+21+2x = 0 \Rightarrow x^2+16x+28 = 0 \Rightarrow (x+14)(x+2) = 0 \Rightarrow x = -2$ or -14.

EXAMPLE 8 *If* $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ *and* $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, *then find k so that* $A^2 = 8A + kI$.

SOLUTION We have, $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ **[CBSE 2005]**

$$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$$

and, $$8A + kI = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ -8 & 56 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$$

$$\therefore \quad A^2 = 8A + kI \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix} \Rightarrow 1 = 8+k \text{ and } 56+k = 49 \Rightarrow k = -7.$$

EXAMPLE 9 *If* $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$, *find A.* **[NCERT EXEMPLAR]**

SOLUTION Since the product matrix is a 3×3 matrix and the premultiplier of A is a 3×2 matrix. Therefore, A is 2×3 matrix. Let $A = \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix}$. Then, the given equation becomes

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x & y & z \\ a & b & c \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x - a & 2y - b & 2z - c \\ x & y & z \\ -3x + 4a & -3y + 4b & -3z + 4c \end{bmatrix} = \begin{bmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{bmatrix}$$

$\Rightarrow$ $2x - a = -1,\ x = 1,\ -3x + 4a = 9,\ 2y - b = -8,\ y = -2, -3y + 4b = 22,\ 2z - c = -0,$
$z = -5,\ -3z + 4c = 15$

$\Rightarrow$ $x = 1, a = 3, y = -2, b = 4, z = -5$ and $c = 0$

Hence, $A = \begin{bmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{bmatrix}$

Type II ON MATRIX POLYNOMIALS AND MATRIX POLYNOMIAL EQUATIONS

EXAMPLE 10 *Let* $f(x) = x^2 - 5x + 6$. *Find* $f(A)$, *if* $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

SOLUTION First, we note that by $f(A)$ we mean the matrix polynomial $A^2 - 5A + 6I_3$. That is, to obtain $f(A)$, x is replaced by A and the constant term is multiplied by the identity matrix of order same as that of A.

Now, $A^2 = AA = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

$-5A = \begin{bmatrix} (-5)\times 2 & (-5)\times 0 & (-5)\times 1 \\ (-5)\times 2 & (-5)\times 1 & (-5)\times 3 \\ (-5)\times 1 & (-5)\times(-1) & (-5)\times 0 \end{bmatrix} = \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix}$ and, $6I_3 = 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore\ f(A) = A^2 - 5A + 6I_3 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} + \begin{bmatrix} -10 & 0 & -5 \\ -10 & -5 & -15 \\ -5 & 5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$.

EXAMPLE 11 *If* $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, *show that* $A^2 - 5A + 7I_2 = O$.

SOLUTION We have, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore\quad A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$-5A = \begin{bmatrix} (-5)\times 3 & (-5)\times 1 \\ (-5)\times(-1) & (-5)\times 2 \end{bmatrix} = \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix}$ and, $7I_2 = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$$\therefore \qquad A^2 - 5A + 7I_2 = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} -15 & -5 \\ 5 & -10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Type III ON PRINCIPLE OF MATHEMATICAL INDUCTION

The Principle of Mathematical Induction:

Let $P(n)$ be a statement involving positive integer n such that

(i) $P(1)$ is true i.e. the statement is true for $n = 1$, and

(ii) $P(m+1)$ is true whenever $P(m)$ is true i.e. the truth of $P(m)$ implies the truth of $P(m+1)$.

Then, $P(n)$ is true for all positive integer n.

EXAMPLE 12 *Prove the following by the principle of mathematical induction:*

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, *then* $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ *for every positive integer n.* [NCERT]

SOLUTION We shall prove the result by mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we have

$$A^1 = A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1+2(1) & -4(1) \\ 1 & 1-2(1) \end{bmatrix}$$

So, the result is true for n = 1.

Step 2 Let the result be true for n = m. Then,

$$A^m = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix} \qquad \text{...(i)}$$

Now, we will show that the result is true for $n = m+1$ i.e. $A^{m+1} = \begin{bmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}$

By the definition of integral powers of a square matrix, we obtain

$$A^{m+1} = A^m A$$

$$\Rightarrow A^{m+1} = \begin{bmatrix} 1+2m & -4m \\ m & 1-2m \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \qquad \text{[By supposition (i)]}$$

$$\Rightarrow A^{m+1} = \begin{bmatrix} 3+6m-4m & -4-8m+4m \\ 3m+1-2m & -4m-1+2m \end{bmatrix} = \begin{bmatrix} 3+2m & -4-4m \\ m+1 & -1-2m \end{bmatrix} = \begin{bmatrix} 1+2(m+1) & -4(m+1) \\ m+1 & 1-2(m+1) \end{bmatrix}$$

This shows that the result is true for $n = m+1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is valid for any positive integer n.

EXAMPLE 13 *If* $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, *then prove that*

(i) $A_\alpha A_\beta = A_{\alpha+\beta}$ (ii) $(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ *for every positive integer n.*

[NCERT, CBSE 2004]

SOLUTION (i) We find that

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ -\sin\alpha\cos\beta - \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$= A_{\alpha+\beta}.$$

(ii) We shall prove the result by mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we obtain

$$(A_\alpha)^1 = A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(1.\alpha) & \sin(1.\alpha) \\ -\sin(1.\alpha) & \cos(1.\alpha) \end{bmatrix}$$

So, the result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then,

$$(A_\alpha)^m = \begin{bmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{bmatrix} \quad \text{...(i)}$$

Now, we will show that the result is true for $n = m+1$ i.e. $(A_\alpha)^{m+1} = \begin{bmatrix} \cos(m+1)\alpha & \sin(m+1)\alpha \\ -\sin(m+1)\alpha & \cos(m+1)\alpha \end{bmatrix}$

By the definition of integral powers of a square matrix, we have

$$(A_\alpha)^{m+1} = (A_\alpha)^m A_\alpha$$

$$\Rightarrow \quad (A_\alpha)^{m+1} = \begin{bmatrix} \cos m\alpha & \sin m\alpha \\ -\sin m\alpha & \cos m\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad \text{[By assumption (i)]}$$

$$\Rightarrow \quad (A_\alpha)^{m+1} = \begin{bmatrix} \cos m\alpha\cos\alpha - \sin m\alpha\sin\alpha & \cos m\alpha\sin\alpha + \sin m\alpha\cos\alpha \\ -\sin m\alpha\cos\alpha - \cos m\alpha\sin\alpha & -\sin m\alpha\sin\alpha + \cos m\alpha\cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad (A_\alpha)^{m+1} = \begin{bmatrix} \cos(m\alpha+\alpha) & \sin(m\alpha+\alpha) \\ -\sin(m\alpha+\alpha) & \cos(m\alpha+\alpha) \end{bmatrix} = \begin{bmatrix} \cos(m+1)\alpha & \sin(m+1)\alpha \\ -\sin(m+1)\alpha & \cos(m+1)\alpha \end{bmatrix}$$

This shows that the result is true for $n = m+1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is valid for any positive integer n.

EXAMPLE 14 *If a is a non-zero real or complex number. Use the principle of mathematical induction to prove that:*

If $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$, then $A^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$ for every positive integer n.

SOLUTION We have, $A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we have

$$A^1 = A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^1 & 1(a^{1-1}) \\ 0 & a^1 \end{bmatrix}.$$ So, the result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then, $A^m = \begin{bmatrix} a^m & ma^{m-1} \\ 0 & a^m \end{bmatrix}$. ...(i)

Now we will show that the result is true for $n = m+1$. i.e. $A^{m+1} = \begin{bmatrix} a^{m+1} & (m+1)a^m \\ 0 & a^{m+1} \end{bmatrix}$.

By using the definition of integral powers of a square matrix, we obtain

$$A^{m+1} = A^m A = \begin{bmatrix} a^m & ma^{m-1} \\ 0 & a^m \end{bmatrix}\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^m \times a + 0 \times ma^{m-1} & a^m \times 1 + ma^{m-1} \times a^1 \\ a \times 0 + 0 \times a^m & 0 \times 1 + a^m \times a \end{bmatrix}$$

$$\Rightarrow \quad A^{m+1} = \begin{bmatrix} a^{m+1} & a^m + ma^m \\ 0 & a^{m+1} \end{bmatrix} = \begin{bmatrix} a^{m+1} & (m+1)a^m \\ 0 & a^{m+1} \end{bmatrix}$$

This shows that the result is true for $n = m + 1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for any positive integer n.

EXAMPLE 15 *If* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, *then prove that* $A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}$ *for every positive integer n.* [NCERT]

SOLUTION We shall prove the result by the principle of mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we obtain

$$A^1 = A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \\ 3^{1-1} & 3^{1-1} & 3^{1-1} \end{bmatrix}.$$ So, the result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then, $A^m = \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix}$ ...(i)

Now we shall show that the result is true for $n = m + 1$ i.e. $A^{m+1} = \begin{bmatrix} 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \end{bmatrix}$.

By using the definition of integral powers of a matrix, we obtain

$$A^{m+1} = A^m A$$

$$\Rightarrow \quad A^{m+1} = \begin{bmatrix} 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \\ 3^{m-1} & 3^{m-1} & 3^{m-1} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad A^{m+1} = \begin{bmatrix} 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \\ 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \\ 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} & 3^{m-1}+3^{m-1}+3^{m-1} \end{bmatrix}.$$

$$\Rightarrow \quad A^{m+1} = \begin{bmatrix} 3\times 3^{m-1} & 3\times 3^{m-1} & 3\times 3^{m-1} \\ 3\times 3^{m-1} & 3\times 3^{m-1} & 3\times 3^{m-1} \\ 3\times 3^{m-1} & 3\times 3^{m-1} & 3\times 3^{m-1} \end{bmatrix} = \begin{bmatrix} 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \\ 3^m & 3^m & 3^m \end{bmatrix}$$

This shows that the result is true for $n = m + 1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is valid for any positive integer n.

Type IV MISCELLANEOUS PROBLEMS

EXAMPLE 16 *Under what conditions is the matrix equation* $A^2 - B^2 = (A - B)(A + B)$ *is true?*

SOLUTION We have,

$$A^2 - B^2 = (A - B)(A + B)$$

$$\Leftrightarrow \quad A^2 - B^2 = (A - B) A + (A - B) B \quad \left[\begin{array}{l}\text{By disributivity of matrix muliplication} \\ \text{over matrix addition}\end{array}\right]$$

$\Leftrightarrow \quad A^2 - B^2 = A^2 - BA + AB - B^2$ [By disributivity of matrix muliplication over matrix addition]

$\Leftrightarrow \quad O = A^2 - BA + AB - B^2 - A^2 + B^2 \Leftrightarrow O = -\ BA + AB \Leftrightarrow AB = BA$

Thus the given matrix equation is true if the matrices A and B commute with each other.

EXAMPLE 17 *If A is any $m \times n$ matrix such that AB and BA are both defined show that B is an $n \times m$ matrix.*

SOLUTION Since A is an $m \times n$ matrix such that AB exists. Therefore, the number of rows in B should be equal to the number of columns in A. Thus, B has n rows. Further, BA exists, therefore the number of columns in B should be equal to the number of rows in A. So B has m columns.Hence, B is an $n \times m$ matrix.

EXAMPLE 18 *A, B are two matrices such that AB and $A + B$ are both defined; show that A, B are square matrices of the same order.*

SOLUTION Let A be an $m \times n$ matrix. Since $A + B$ is defined, therefore B is also an $m \times n$ matrix. Further since AB exists, therefore the number of columns in A is same as the number of rows in B i.e. $n = m$. Hence, A and B are square matrices of the same order.

EXAMPLE 19 *If A and B are square matrices of order n, then prove that A and B will commute iff $A - \lambda I$ and $B - \lambda I$ commute for every scalar λ.*

SOLUTION $A - \lambda I$ and $B - \lambda I$ commute

$\Leftrightarrow \quad (A - \lambda I)(B - \lambda I) = (B - \lambda I)(A - \lambda I)$

$\Leftrightarrow \quad AB - \lambda IA - \lambda IB + \lambda^2 I^2 = BA - \lambda BI - \lambda IA + \lambda^2 I^2$

$\Leftrightarrow \quad AB - \lambda A - \lambda B + \lambda^2 I = BA - \lambda B - \lambda A + \lambda^2 I \Leftrightarrow AB = BA \Leftrightarrow$ A and B commute.

EXAMPLE 20 *If $AB = A$ and $BA = B$, then show that $A^2 = A$, $B^2 = B$.*

SOLUTION We have, $AB = A$ and $BA = B$

Now, $\quad AB = A$

$(AB)A = AA$ [Multiplying both sides on right by A]

$\Rightarrow \quad A(BA) = A^2$ [By associativity of matrix multiplication]

$\Rightarrow \quad AB = A^2$ [$\because BA = B$]

$\Rightarrow \quad A = A^2$ [$\because AB = A$]

Similarly, it can be proved that $B^2 = B$.

EXAMPLE 21 *Give an example of two matrices* A and B *such that*

(i) $A \neq O,\ B \neq O,\ AB = O$ and $BA \neq O$ [NCERT EXEMPLAR]

(ii) $A \neq O,\ B \neq O,\ AB = BA = O$.

SOLUTION (i) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$, then $A \neq O,\ B \neq O$.

But, $\quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$ and, $BA = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \neq O$

(ii) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then $A \neq O,\ B \neq O$.

But, $\quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & -1+1 \\ 1-1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

and, $\quad BA = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ -1+1 & -1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

EXAMPLE 22 *Give an example of three matrices A, B, C such that $AB = AC$ but $B \neq C$.*

[NCERT EXEMPLAR]

SOLUTION Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, it can be easily verified that $AB = AC = O$. But $B \neq C$.

Type V ON APPLICATIONS OF MATRICES

EXAMPLE 23 *There are two families A and B. There are 4 men, 6 women and 2 children in family A and 2 men, 2 women and 4 children in family B. The recommended daily allowance for calories is : Man : 2400, woman : 1900, child : 1800 and for proteins is : Man : 55 gm, woman : 45 gm and child : 33 gm. Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.*

SOLUTION The members of the two families can be represented by a 2×3 matrix F given below.

$$F = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{ccc} M & W & C \end{array} \\ \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \end{array}$$

and the recommended daily allowance of calories and proteins for each member can be represented by 3×2 matrix R as given below.

$$R = \begin{array}{c} \\ M \\ W \\ C \end{array} \begin{array}{c} \begin{array}{cc} \text{Calories} & \text{Proteins} \end{array} \\ \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix} \end{array}$$

The total requirement of calories and proteins for each of the two families is given by the matrix multiplication FR as given below.

$$FR = \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 2400 & 55 \\ 1900 & 45 \\ 1800 & 33 \end{bmatrix} = \begin{array}{c} A \\ B \end{array} \begin{bmatrix} 24600 & 556 \\ 15800 & 332 \end{bmatrix}$$

Hence, family A requires 24600 calories and 556 gm proteins and family B requires 15,800 calories and 332 gm proteins.

EXAMPLE 24 *Use matrix multiplication to divide ₹ 30,000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts ₹ 3060.*

SOLUTION Let the two parts be ₹ x and ₹ $(30000 - x)$ respectively. Let A be the 1×2 matrix representing these two parts

i.e. $$A = \begin{array}{cc} \text{Part I} & \text{Part II} \\ [x & 30000 - x] \end{array}$$

Let R denote the 2×1 matrix representing the annual interest rates of interest on two parts i.e.

$$R = \begin{array}{c} \text{Part I} \\ \text{Part II} \end{array} \begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix}$$

The total annual interest on the two parts is given by the matrix multiplication AR.

$\therefore \quad AR = 3060$

$\Rightarrow \quad [x \quad 30000 - x] \begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix} = 3060$

$\Rightarrow \quad 0.09\,x + 0.11\,(30000 - x) = 3060$

$\Rightarrow \quad \frac{9}{100}x + \frac{11}{100}(30000 - x) = 3060 \Rightarrow 9x + 330000 - 11\,x = 306000 \Rightarrow x = 12{,}000$

Hence two parts of ₹ 30,000 are ₹ 12,000 and ₹ 18,000 respectively.

EXAMPLE 25 *Three schools A, B and C organised a mela for collecting funds for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recyled material at a cost of ₹ 25, ₹ 100 and ₹ 50 each. The number of articles sold are given below:*

School / Article	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the funds collected by each school separately by sailing the above articles. Also, find the total funds collected for the purpose. **[CBSE 2015]**

SOLUTION Three items sold by three schools are represented by the following 3×3 matrix Q as given below.

$$Q = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} \text{Hand-fans} & \text{Mats} & \text{Plates} \end{array} \\ \begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix} \end{array}$$

The price matrix representing price of of three articles in ₹ is a 3×1 matrix given by

$$P = \begin{array}{c} \text{Hand-fan} \\ \text{Mat} \\ \text{Plate} \end{array}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

The funds collected by schools A, B and C separately by selling three articles are given by the product matrix QP.

$$\therefore \quad QP = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} \text{Hand-fans} & \text{Mats} & \text{Plates} \end{array} \\ \begin{bmatrix} 40 & 50 & 20 \\ 25 & 40 & 30 \\ 35 & 50 & 40 \end{bmatrix} \end{array}\begin{bmatrix} 25 \\ 100 \\ 50 \end{bmatrix}$$

$$\Rightarrow \quad QP = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 40 \times 25 + 50 \times 100 + 20 \times 50 \\ 25 \times 25 + 40 \times 100 + 30 \times 50 \\ 35 \times 25 + 50 \times 100 + 40 \times 50 \end{bmatrix} = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} ₹7000 \\ ₹6125 \\ ₹7875 \end{bmatrix}$$

Hence, the funds collected by schools A, B and C are ₹ 7000, ₹ 6125 and ₹ 7875 respectively. The total funds collected = ₹ (7000 + 6125 + 7875) = ₹21000.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON MULTIPLICATION OF MATRICES

EXAMPLE 26 *Prove that the product of matrices*

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is the null matrix, when θ *and* ϕ *differ by an odd multiple of* $\frac{\pi}{2}$.

SOLUTION We have,

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta\cos^2\phi + \cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi + \cos\theta\sin\theta\sin^2\phi \\ \cos^2\phi\cos\theta\sin\theta + \sin^2\theta\cos\phi\sin\phi & \cos\theta\sin\theta\cos\phi\sin\phi + \sin^2\theta\sin^2\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \sin\theta\cos\phi\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \left[\because \theta-\phi=(2n+1)\frac{\pi}{2}, n\in Z \therefore \cos(\theta-\phi)=\cos(2n+1)\frac{\pi}{2}=0\right]$$

EXAMPLE 27 *Let* $A=\begin{bmatrix} 0 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 0 \end{bmatrix}$ *and I be the identity matrix of order 2. Show that* $I+A=(I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. **[NCERT]**

SOLUTION We have,

$$I+A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

and, $$I-A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}.$$

$$\therefore \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & -\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \\ \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} & \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} \end{bmatrix}$$

$$\Rightarrow \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} \dfrac{1-t^2}{1+t^2} & -\dfrac{2t}{1+t^2} \\ \dfrac{2t}{1+t^2} & \dfrac{1-t^2}{1+t^2} \end{bmatrix}, \text{ where } t=\tan\frac{\alpha}{2}.$$

$$\Rightarrow \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \dfrac{1-t^2+2t^2}{1+t^2} & \dfrac{-2t+t-t^3}{1+t^2} \\ \dfrac{-t+t^3+2t}{1+t^2} & \dfrac{2t^2+1-t^2}{1+t^2} \end{bmatrix}$$

$$\Rightarrow \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \dfrac{1+t^2}{1+t^2} & \dfrac{-t(1+t^2)}{1+t^2} \\ \dfrac{t(1+t^2)}{1+t^2} & \dfrac{1+t^2}{1+t^2} \end{bmatrix} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}$$

$$\Rightarrow \quad (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} = I + A$$

EXAMPLE 28 *Let* $F(x)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}$. *Show that* $F(x)\,F(y)=F(x+y)$. **[NCERT]**

SOLUTION We have,

$$F(x)\cdot F(y)=\begin{bmatrix}\cos x & -\sin x & 0\\ \sin x & \cos x & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos y & -\sin y & 0\\ \sin y & \cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad F(x)\cdot F(y)=\begin{bmatrix}\cos x\cos y-\sin x\sin y & -\sin y\cos x-\sin x\cos y & 0\\ \sin x\cos y+\cos x\sin y & -\sin x\sin y+\cos x\cos y & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad F(x)\cdot F(y)=\begin{bmatrix}\cos(x+y) & -\sin(x+y) & 0\\ \sin(x+y) & \cos(x+y) & 0\\ 0 & 0 & 1\end{bmatrix}=F(x+y)$$

Type II ON MATRIX POLYNOMIALS AND MATRIX POLYNOMIAL EQUATIONS

EXAMPLE 29 *Let* $A=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ *and* $f(x)=x^2-4x+7$. *Show that* $f(A)=O$. *Use this result to find* A^5. **[NCERT EXEMPLAR]**

SOLUTION We have, $f(x)=x^2-4x+7$. Therefore, $f(A)=A^2-4A+7I_2$.

Now, $A^2=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}4-3 & 6+6\\ -2-2 & -3+4\end{bmatrix}=\begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix}$,

$-4A=\begin{bmatrix}-8 & -12\\ 4 & -8\end{bmatrix}$ and, $7I_2=\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$

$\therefore \quad f(A)=A^2-4A+7I_2$

$\Rightarrow \quad f(A)=\begin{bmatrix}1 & 12\\ -4 & 1\end{bmatrix}+\begin{bmatrix}-8 & -12\\ 4 & -8\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}=\begin{bmatrix}1-8+7 & 12-12+0\\ -4+4+0 & 1-8+7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

Now, $f(A)=O$

$\Rightarrow \quad A^2-4A+7I_2=O$

$\Rightarrow \quad A^2=4A-7I_2$

$\Rightarrow \quad A^3=A^2A=(4A-7I_2)A=4A^2-7I_2A$

$\Rightarrow \quad A^3=4(4A-7I_2)-7A$ [Using : $A^2=4A-7I_2$]

$\Rightarrow \quad A^3=9A-28I_2$

$\Rightarrow \quad A^4=A^3A=(9A-28I_2)A$

$\Rightarrow \quad A^4=9A^2-28A=9(4A-7I_2)-28A$ [Using : $A^2=4A-7I_2$]

$\Rightarrow \quad A^4=36A-63I_2-28A=8A-63I_2$

$\Rightarrow \quad A^5=A^4A=(8A-63I_2)A=8A^2-63I_2A$

$\Rightarrow \quad A^5=8(4A-7I_2)-63A=-31A-56I_2$ [Using : $A^2=4A-7I_2$]

$\Rightarrow \quad A^5=-31\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}-56\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-62 & -93\\ 31 & -62\end{bmatrix}+\begin{bmatrix}-56 & 0\\ 0 & -56\end{bmatrix}=\begin{bmatrix}-118 & -93\\ 31 & -118\end{bmatrix}$.

Type III ON PRINCIPLE OF MATHEMATICAL INDUCTION

EXAMPLE 30 *If* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, *prove that* $(aI + bA)^n = a^n I + na^{n-1} bA$ *where I is a unit matrix of order 2 and n is a positive integer.* **[NCERT]**

SOLUTION We shall prove the result by mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we have

$$(aI + bA)^1 = aI + bA = a^1 I + 1\, a^0\, bA = a^1 I + 1a^{1-1} bA$$

So, the result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then, $(aI + bA)^m = a^m I + ma^{m-1} bA$ ...(i)

Now we shall show that the result is true for $n = m + 1$.

i.e. $(aI + bA)^{m+1} = a^{m+1} I + (m+1)\, a^m\, bA$

By using the definition of integral powers of a matrix, we obtain

$$\begin{aligned} (aI + bA)^{m+1} &= (aI + bA)^m (aI + bA) \\ &= (a^m I + ma^{m-1} bA)(aI + bA) && \text{[Using (i)]} \\ &= (a^m I)(aI) + (a^m I)(bA) + (ma^{m-1} bA)(aI) + (ma^{m-1} bA)(bA) \\ &= (a^m a)(I \,.\, I) + a^m b\,(IA) + ma^m b\,(AI) + ma^{m-1} b^2 (AA) \\ &= a^{m+1} I + a^m bA + ma^m bA + ma^{m-1} b^2 A^2 && [\because IA = AI = A, I \,.\, I = I] \\ &= a^{m+1} I + (ma^m b + a^m b) A + ma^{m-1} b^2 A^2 \\ &= a^{m+1} I + (m+1)\, a^m bA + ma^{m-1} b^2 O && [\because A^2 = O] \\ &= a^{m+1} I + (m+1)\, a^m bA \end{aligned}$$

This shows that the result is true for $n = m + 1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is valid for any positive integer n.

EXAMPLE 31 *Let A, B be two matrices such that they commute. Show that for any positive integer n*

(i) $AB^n = B^n A$ (ii) $(AB)^n = A^n B^n$ **[NCERT EXEMPLAR]**

SOLUTION (i) We shall prove the result by the principle of mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we have

$$\begin{aligned} AB^1 &= AB = BA && [\because AB = BA \text{ (given)}] \\ &= B^1 A && [\because B^1 = B] \end{aligned}$$

So, that result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then, $AB^m = B^m A$. ...(i)

We shall now show that the result is true for $n = m + 1$ i.e. $AB^{m+1} = B^{m+1} A$.

By the definition of integral powers of a matrix, we obtain

$$\begin{aligned} AB^{m+1} &= A\,(B^m B) = A\,(BB^m) \\ &= (AB)\, B^m && \text{[By associativity of matrix multiplication]} \\ &= (BA)\, B^m && [\because AB = BA \text{ (given)}] \\ &= B\,(AB^m) && \text{[By associativity of matrix multiplication]} \\ &= B\,(B^m A) && \text{[Using induction assumption (i)]} \end{aligned}$$

$$= (BB^m)\,A \qquad \text{[By associativity of matrix multiplication]}$$
$$= B^{m+1}\,A \qquad \text{[By definition of integral powers]}$$

This shows that the result is true for $n = m + 1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for any positive integer n.

(ii) We shall prove this result also by the principle of mathematical induction on n.

Step 1 When $n = 1$, by the definition of integral powers of a matrix, we have

$$(AB)^1 = AB = A^1 B^1 \qquad [\because\ A^1 = A,\ B^1 = B]$$

So, the result is true for $n = 1$.

Step 2 Let the result be true for $n = m$. Then, $(AB)^m = A^m B^m$ ...(i)

Now we shall show that the result is true for $n = m + 1$ i.e. $(AB)^{m+1} = A^{m+1} B^{m+1}$.

By using the definition of integral powers of a matrix, we obtain

$$(AB)^{m+1} = (AB)^m\,(AB)$$
$$= (A^m B^m)\,(AB) \qquad \text{[By induction assumption (i)]}$$
$$= A^m\,(B^m\,(AB)) \qquad \text{[By associativity of matrix multiplication]}$$
$$= A^m\,(B^m\,(BA)) \qquad [\because\ AB = BA \text{ (given)}]$$
$$= A^m\,((B^m B)\,A) \qquad \text{[By associativity of matrix multiplication]}$$
$$= A^m\,(B^{m+1}\,A) \qquad \text{[By definition of integral powers]}$$
$$= A^m\,(AB^{m+1}) \qquad [\because AB^n = B^n A \text{ for all } n \in N \text{ (proved in (i))}]$$
$$= (A^m\,A)\,B^{m+1} \qquad \text{[By associativity of matrix multiplication]}$$
$$= A^{m+1}\,B^{m+1}$$

This shows that the result is true for $n = m + 1$, whenever it is true for $n = m$.

Hence, by the principle of mathematical induction, the result is true for every positive integer n.

EXAMPLE 32 *If A is a square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.*

[NCERT EXEMPLAR]

SOLUTION Using matrix multiplication, we obtain

$$(I + A)^2 = (I + A)\,(I + A)$$
$$= I\,(I + A) + A\,(I + A) \qquad \text{[By distributivity of multiplication over addition]}$$
$$= I^2 + IA + AI + A^2$$
$$= I + A + A + A^2 \qquad [\because\ IA = AI = A]$$
$$= I + 2A + A^2 = I + 2A + A = I + 3A \qquad [\because A^2 = A]$$

$$\therefore \quad (I + A)^3 = (I + A)^2\,(I + A)$$
$$= (I + 3A)\,(I + A)$$
$$= I\,(I + A) + 3A\,(I + A) = I^2 + IA + 3(AI) + 3\,(AA)$$
$$= I + A + 3A + 3A^2 \qquad [\because IA = AI = A]$$
$$= I + A + 3A + 3A = I + 7A \qquad [\because A^2 = A]$$

EXAMPLE 33 *If A is a square matrix such that $A^2 = I$, then find the simplified value of $(A - I)^3 + (A + I)^3 - 7A$.* **[CBSE 2016]**

SOLUTION We have, $A^2 = I$

$\therefore \quad A^3 = A^2A = IA = A$ ...(i)

We know that

$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3 \text{ and, } (A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

provided that $AB = BA$.

Since $AI = IA = A$.

$\therefore \quad (A+I)^3 = A^3 + 3A^2I + 3AI^2 + I^3$ and $(A-I)^3 = A^3 - 3A^2I + 3AI^2 - I^3$

$\Rightarrow \quad (A+I)^3 = A^3 + 3A^2 + 3A + I$ and $(A-I)^3 = A^3 - 3A^2 + 3A - I$

$\therefore \quad (A+I)^3 + (A-I)^3 = 2(A^3 + 3A) = 2(A+3A) = 8A$ [Using (i)]

Hence, $(A-I)^3 + (A+I)^3 - 7A = 8A - 7A = A$.

EXAMPLE 34 *If $A = [3 \quad 5]$, $B = [7 \quad 3]$, then find a non-zero matrix C such that $AC = BC$.* **[NCERT EXEMPLAR]**

SOLUTION Clearly, A and B are 1×2 matrix. Therefore, products AC and BC exist if C is of order $2 \times n$, where $n \in N$.

Now, following cases arise.

Case I *When $n = 1$:* In this case, matrix C is a 2×1 matrix. So, let $C = \begin{bmatrix} a \\ b \end{bmatrix}$. Then,

$$AC = BC$$

$$\Rightarrow \quad [3 \quad 5]\begin{bmatrix} a \\ b \end{bmatrix} = [7 \quad 3]\begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow 3a + 5b = 7a + 3b \Rightarrow 4a = 2b \Rightarrow 2a = b$$

$$\therefore \quad C = \begin{bmatrix} a \\ 2a \end{bmatrix}, \text{ where } a \in R.$$

Case II *When $n = 2$:* In this case, matrix C is a 2×2 matrix. So, let $C = \begin{bmatrix} a & x \\ b & y \end{bmatrix}$. Then,

$$AC = BC$$

$$\Rightarrow \quad [3 \quad 5]\begin{bmatrix} a & x \\ b & y \end{bmatrix} = [7 \quad 3]\begin{bmatrix} a & x \\ b & y \end{bmatrix}$$

$$\Rightarrow \quad [3a+5b \quad 3x+5y] = [7a+3b \quad 7x+3y] \Rightarrow 3a+5b = 7a+3b \text{ and } 3x+5y = 7x+3y$$

$$\Rightarrow \quad b = 2a \text{ and } y = 2x.$$

$$\therefore \quad C = \begin{bmatrix} a & x \\ 2a & 2x \end{bmatrix}, \text{ where } a, x \in R.$$

Similarly, if $n = 3$

$$C = \begin{bmatrix} a & x & y \\ 2a & 2x & 2y \end{bmatrix}, \text{ where } a, x, y \in R \text{ and so on.}$$

EXERCISE 4.3

BASIC

1. Compute the indicated products:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$

2. Show that $AB \neq BA$ in each of the following cases:

(i) $A=\begin{bmatrix}5 & -1\\ 6 & 7\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}$ (ii) $A=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 2 & 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}$

(iii) $A=\begin{bmatrix}1 & 3 & 0\\ 1 & 1 & 0\\ 4 & 1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 5 & 1\end{bmatrix}$

3. Compute the products AB and BA whichever exists in each of the following cases:

(i) $A=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{bmatrix}$ (ii) $A=\begin{bmatrix}3 & 2\\ -1 & 0\\ -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & 5 & 6\\ 0 & 1 & 2\end{bmatrix}$

(iii) $A=[1\ \ -1\ \ 2\ \ 3]$ and $B=\begin{bmatrix}0\\ 1\\ 3\\ 2\end{bmatrix}$ (iv) $[a\ b]\begin{bmatrix}c\\ d\end{bmatrix}+[a\ b\ c\ d]\begin{bmatrix}a\\ b\\ c\\ d\end{bmatrix}$

4. Show that $AB \neq BA$ in each of the following cases:

(i) $A=\begin{bmatrix}1 & 3 & -1\\ 2 & -1 & -1\\ 3 & 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 3 & -1\\ -1 & 2 & -1\\ -6 & 9 & -4\end{bmatrix}$ (ii) $A=\begin{bmatrix}10 & -4 & -1\\ -11 & 5 & 0\\ 9 & -5 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & 2\\ 3 & 4 & 1\\ 1 & 3 & 2\end{bmatrix}$

5. Evaluate the following:

(i) $\left(\begin{bmatrix}1 & 3\\ -1 & -4\end{bmatrix}+\begin{bmatrix}3 & -2\\ -1 & 1\end{bmatrix}\right)\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix}$ (ii) $[1\ 2\ 3]\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 4\\ 6\end{bmatrix}$

(iii) $\begin{bmatrix}1 & -1\\ 0 & 2\\ 2 & 3\end{bmatrix}\left(\begin{bmatrix}1 & 0 & 2\\ 2 & 0 & 1\end{bmatrix}-\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 2\end{bmatrix}\right)$

6. If $A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ and $C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then show that $A^2=B^2=C^2=I_2$.

7. If $A=\begin{bmatrix}2 & -1\\ 3 & 2\end{bmatrix}$ and $B=\begin{bmatrix}0 & 4\\ -1 & 7\end{bmatrix}$, find $3A^2-2B+I$. [CBSE 2005]

8. If $A=\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$, prove that $(A-2I)(A-3I)=O$.

9. If $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, show that $A^2=\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}$ and $A^3=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}$.

10. If $A=\begin{bmatrix}ab & b^2\\ -a^2 & -ab\end{bmatrix}$, show that $A^2=O$.

11. If $A=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$, find A^2. [CBSE 2000C]

12. If $A=\begin{bmatrix}2 & -3 & -5\\ -1 & 4 & 5\\ 1 & -3 & -4\end{bmatrix}$ and $B=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$, show that $AB=BA=O_{3\times 3}$.

13. If $A=\begin{bmatrix}0 & c & -b\\ -c & 0 & a\\ b & -a & 0\end{bmatrix}$ and $B=\begin{bmatrix}a^2 & ab & ac\\ ab & b^2 & bc\\ ac & bc & c^2\end{bmatrix}$, show that $AB=BA=O_{3\times 3}$.

14. If $A=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}$ and $B=\begin{bmatrix}2&-2&-4\\-1&3&4\\1&-2&-3\end{bmatrix}$, show that $AB=A$ and $BA=B$.

15. Let $A=\begin{bmatrix}-1&1&-1\\3&-3&3\\5&5&5\end{bmatrix}$ and $B=\begin{bmatrix}0&4&3\\1&-3&-3\\-1&4&4\end{bmatrix}$, compute A^2-B^2.

16. For the following matrices verify the associativity of matrix multiplication i.e. $(AB)C=A(BC)$.

(i) $A=\begin{bmatrix}1&2&0\\-1&0&1\end{bmatrix}$, $B=\begin{bmatrix}1&0\\-1&2\\0&3\end{bmatrix}$ and $C=\begin{bmatrix}1\\-1\end{bmatrix}$

(ii) $A=\begin{bmatrix}4&2&3\\1&1&2\\3&0&1\end{bmatrix}$, $B=\begin{bmatrix}1&-1&1\\0&1&2\\2&-1&1\end{bmatrix}$ and $C=\begin{bmatrix}1&2&-1\\3&0&1\\0&0&1\end{bmatrix}$.

17. For the following matrices verify the distributivity of matrix multiplication over matrix addition i.e. $A(B+C)=AB+AC$.

(i) $A=\begin{bmatrix}1&-1\\0&2\end{bmatrix}$, $B=\begin{bmatrix}-1&0\\2&1\end{bmatrix}$ and $C=\begin{bmatrix}0&1\\1&-1\end{bmatrix}$

(ii) $A=\begin{bmatrix}2&-1\\1&1\\-1&2\end{bmatrix}$, $B=\begin{bmatrix}0&1\\1&1\end{bmatrix}$ and $C=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$.

18. If $A=\begin{bmatrix}1&0&-2\\3&-1&0\\-2&1&1\end{bmatrix}$, $B=\begin{bmatrix}0&5&-4\\-2&1&3\\-1&0&2\end{bmatrix}$ and $C=\begin{bmatrix}1&5&2\\-1&1&0\\0&-1&1\end{bmatrix}$, verify that $A(B-C)=AB-AC$.

19. Compute the elements a_{43} and a_{22} of the matrix:

$$A=\begin{bmatrix}0&1&0\\2&0&2\\0&3&2\\4&0&4\end{bmatrix}\begin{bmatrix}2&-1\\-3&2\\4&3\end{bmatrix}\begin{bmatrix}0&1&-1&2&-2\\3&-3&4&-4&0\end{bmatrix}$$

20. If $A=\begin{bmatrix}0&1&0\\0&0&1\\p&q&r\end{bmatrix}$ and I is the identity matrix of order 3, show that $A^3=pI+qA+rA^2$.

21. If w is a complex cube root of unity, show that

$$\left(\begin{bmatrix}1&w&w^2\\w&w^2&1\\w^2&1&w\end{bmatrix}+\begin{bmatrix}w&w^2&1\\w^2&1&w\\w&w^2&1\end{bmatrix}\right)\begin{bmatrix}1\\w\\w^2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

22. If $A=\begin{bmatrix}2&-3&-5\\-1&4&5\\1&-3&-4\end{bmatrix}$, show that $A^2=A$.

23. If $A=\begin{bmatrix}4&-1&-4\\3&0&-4\\3&-1&-3\end{bmatrix}$, show that $A^2=I_3$.

24. (i) If $[1 \quad 1 \quad x]\begin{bmatrix}1 & 0 & 2\\0 & 2 & 1\\2 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = 0$, find x.

(ii) If $\begin{bmatrix}2 & 3\\5 & 7\end{bmatrix}\begin{bmatrix}1 & -3\\-2 & 4\end{bmatrix}=\begin{bmatrix}-4 & 6\\-9 & x\end{bmatrix}$, find x. **[CBSE 2012]**

25. If $[x \quad 4 \quad 1]\begin{bmatrix}2 & 1 & 2\\1 & 0 & 2\\0 & 2 & -4\end{bmatrix}\begin{bmatrix}x\\4\\-1\end{bmatrix} = 0$, find x.

26. If $[1 \quad -1 \quad x]\begin{bmatrix}0 & 1 & -1\\2 & 1 & 3\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = 0$, find x.

27. If $A=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then prove that $A^2-A+2I=O$.

28. If $A=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then find λ so that $A^2=5A+\lambda I$.

29. If $A=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, show that $A^2-5A+7I_2=O$. **[CBSE 2003, 2007]**

30. If A is a square matrix such that $A^2=A$, then find $(2+A)^3-19A$. **[CBSE 2020]**

31. Show that the matrix $A=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$ satisfies the equation $A^3-4A^2+A=O$. **[CBSE 2005]**

32. Show that the matrix $A=\begin{bmatrix}5 & 3\\12 & 7\end{bmatrix}$ is a root of the equation $A^2-12A-I=O$.

33. If $A=\begin{bmatrix}3 & -5\\-4 & 2\end{bmatrix}$, find $A^2-5A-14I$. **[CBSE 2004]**

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34. If $A=\begin{bmatrix}3 & 1\\-1 & 2\end{bmatrix}$, show that $A^2-5A+7I=O$. Use this to find A^4. **[NCERT]**

35. If $A=\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix}$, find k such that $A^2=kA-2I_2$. **[NCERT, CBSE 2003]**

36. If $A=\begin{bmatrix}1 & 0\\-1 & 7\end{bmatrix}$, find k such that $A^2-8A+kI=O$. **[CBSE 2005]**

37. If $A=\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$ and $f(x)=x^2-2x-3$, show that $f(A)=O$. **[CBSE 2005]**

38. If $A=\begin{bmatrix}2 & 3\\1 & 2\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then find λ, μ so that $A^2=\lambda A+\mu I$

39. Find the value of x for which the matrix product $\begin{bmatrix}2 & 0 & 7\\0 & 1 & 0\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}-x & 14x & 7x\\0 & 1 & 0\\x & -4x & -2x\end{bmatrix}$ equal to an identity matrix.

40. Solve the matrix equations:

(i) $[x \quad 1]\begin{bmatrix}1 & 0\\ -2 & -3\end{bmatrix}\begin{bmatrix}x\\ 5\end{bmatrix}=0$ (ii) $[1 \quad 2 \quad 1]\begin{bmatrix}1 & 2 & 0\\ 2 & 0 & 1\\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0\\ 2\\ x\end{bmatrix}=0$ [NCERT]

(iii) $[x \; -5 \; -1]\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}\begin{bmatrix}x\\ 4\\ 1\end{bmatrix}=0$ [NCERT] (iv) $[2x \quad 3]\begin{bmatrix}1 & 2\\ -3 & 0\end{bmatrix}\begin{bmatrix}x\\ 8\end{bmatrix}=0$

41. If $A=\begin{bmatrix}1 & 2 & 0\\ 3 & -4 & 5\\ 0 & -1 & 3\end{bmatrix}$, compute $A^2-4A+3I_3$.

42. If $f(x)=x^2-2x$, find $f(A)$, where $A=\begin{bmatrix}0 & 1 & 2\\ 4 & 5 & 0\\ 0 & 2 & 3\end{bmatrix}$.

43. If $f(x)=x^3+4x^2-x$, find $f(A)$, where $A=\begin{bmatrix}0 & 1 & 2\\ 2 & -3 & 0\\ 1 & -1 & 0\end{bmatrix}$.

44. If $A=\begin{bmatrix}1 & 0 & 2\\ 0 & 2 & 1\\ 2 & 0 & 3\end{bmatrix}$, then show that A is a root of the polynomial $f(x)=x^3-6x^2+7x+2$. [NCERT]

45. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$, then prove that $A^2-4A-5I=O$. [CBSE 2008]

46. If $A=\begin{bmatrix}3 & 2 & 0\\ 1 & 4 & 0\\ 0 & 0 & 5\end{bmatrix}$, show that $A^2-7A+10I_3=O$.

47. Without using the concept of inverse of a matrix, find the matrix $\begin{bmatrix}x & y\\ z & u\end{bmatrix}$ such that

$\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}\begin{bmatrix}x & y\\ z & u\end{bmatrix}=\begin{bmatrix}-16 & -6\\ 7 & 2\end{bmatrix}$.

48. Find the matrix A such that

(i) $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}A=\begin{bmatrix}3 & 3 & 5\\ 1 & 0 & 1\end{bmatrix}$ (ii) $A\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{bmatrix}$ [NCERT]

(iii) $\begin{bmatrix}4\\ 1\\ 3\end{bmatrix}A=\begin{bmatrix}-4 & 8 & 4\\ -1 & 2 & 1\\ -3 & 6 & 3\end{bmatrix}$ (iv) $[2 \quad 1 \quad 3]\begin{bmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}=A$

[NCERT EXEMPLAR] [NCERT EXEMPLAR]

(v) $\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}A=\begin{bmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{bmatrix}$ (vi) $A\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}=\begin{bmatrix}-7 & -8 & -9\\ 2 & 4 & 6\\ 11 & 10 & 9\end{bmatrix}$

[CBSE 2017] [CBSE 2017]

49. Find a 2×2 matrix A such that $A\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix} = 6I_2$.

50. If $A = \begin{bmatrix}-3 & 2\\ 1 & -1\end{bmatrix}$ and $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find scalar k so that $A^2 + I = kA$. [CBSE 2020]

51. If $A = \begin{bmatrix}0 & -x\\ x & 0\end{bmatrix}$, $B = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ and $x^2 = -1$, then show that $(A+B)^2 = A^2 + B^2$.

[NCERT EXEMPLAR]

52. If $A = \begin{bmatrix}1 & 0 & -3\\ 2 & 1 & 3\\ 0 & 1 & 1\end{bmatrix}$, then verify that $A^2 + A = A\,(A + I)$, where I is the identity matrix.

[NCERT EXEMPLAR]

53. If $A = \begin{bmatrix}3 & -5\\ -4 & 2\end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, obtain A^3. [NCERT EXEMPLAR]

54. (i) If $P(x) = \begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$, then show that $P(x)\, P(y) = P(x+y) = P(y)\, P(x)$.

(ii) If $P = \begin{bmatrix}x & 0 & 0\\ 0 & y & 0\\ 0 & 0 & z\end{bmatrix}$ and $Q = \begin{bmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{bmatrix}$, prove that $PQ = \begin{bmatrix}xa & 0 & 0\\ 0 & yb & 0\\ 0 & 0 & zc\end{bmatrix} = QP$

[NCERT EXEMPLAR]

55. If $A = \begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$, find $A^2 - 5A + 4I$ and hence find a matrix X such that $A^2 - 5A + 4I + X = O$. [CBSE 2015]

56. If $A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, prove that $A^n = \begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$ for all positive integers n.

57. If $A = \begin{bmatrix}a & b\\ 0 & 1\end{bmatrix}$, prove that $A^n = \begin{bmatrix}a^n & b\left(\dfrac{a^n - 1}{a - 1}\right)\\ 0 & 1\end{bmatrix}$ for every positive integer n.

58. If $A = \begin{bmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{bmatrix}$, then prove by principle of mathematical induction that

$$A^n = \begin{bmatrix}\cos n\theta & i\sin n\theta\\ i\sin n\theta & \cos n\theta\end{bmatrix} \text{ for all } n \in N.$$ [CBSE 2005]

59. If $A = \begin{bmatrix}\cos\alpha + \sin\alpha & \sqrt{2}\sin\alpha\\ -\sqrt{2}\sin\alpha & \cos\alpha - \sin\alpha\end{bmatrix}$, prove that $A^n = \begin{bmatrix}\cos n\alpha + \sin n\alpha & \sqrt{2}\sin n\alpha\\ -\sqrt{2}\sin n\alpha & \cos n\alpha - \sin n\alpha\end{bmatrix}$ for all $n \in N$.

60. If $A = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$, then use the principle of mathematical induction to show that

$$A^n = \begin{bmatrix}1 & n & n(n+1)/2\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix} \text{ for every positive integer } n.$$

61. If B, C are n rowed square matrices and if $A = B + C$, $BC = CB$, $C^2 = O$, then show that for every $n \in N$, $A^{n+1} = B^n\,(B + (n+1)\,C)$.

62. If $A = \text{diag}(a \ \ b \ \ c)$, show that $A^n = \text{diag}(a^n \ \ b^n \ \ c^n)$ for all positive integer n.

63. If A is a square matrix, using mathematical induction prove that $(A^T)^n = (A^n)^T$ for all $n \in N$. **[NCERT EXEMPLAR]**

64. A matrix X has $a + b$ rows and $a + 2$ columns while the matrix Y has $b + 1$ rows and $a + 3$ columns. Both matrices XY and YX exist. Find a and b. Can you say XY and YX are of the same type? Are they equal.

65. Give examples of matrices
   (i) A and B such that $AB \neq BA$.
   (ii) A and B such that $AB = O$ but $A \neq O, B \neq O$.
   (iii) A and B such that $AB = O$ but $BA \neq O$.
   (iv) A, B and C such that $AB = AC$ but $B \neq C, A \neq O$.

66. Let A and B be square matrices of the same order. Does $(A + B)^2 = A^2 + 2AB + B^2$ hold? If not, why?

67. If A and B are square matrices of the same order, explain, why in general
   (i) $(A + B)^2 \neq A^2 + 2\,AB + B^2$ (ii) $(A - B)^2 \neq A^2 - 2\,AB + B^2$
   (iii) $(A + B)(A - B) \neq A^2 - B^2$.

68. Let A and B be square matrices of the other 3×3. Is $(AB)^2 = A^2 B^2$? Give reasons. **[NCERT EXEMPLAR]**

69. If A and B are square matrices of the same order such that $AB = BA$, then show that $(A+B)^2 = A^2 + 2AB + B^2$. **[NCERT EXEMPLAR]**

70. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \\ -2 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 2 \\ -3 & 5 \\ 5 & 0 \end{bmatrix}$.

   Verify that $AB = AC$ though $B \neq C, A \neq O$.

71. Three shopkeepers A, B and C go to a store to buy stationary. A purchases 12 dozen notebooks, 5 dozen pens and 6 dozen pencils. B purchases 10 dozen notebooks, 6 dozen pens and 7 dozen pencils. C purchases 11 dozen notebooks, 13 dozen pens and 8 dozen pencils. A notebook costs 40 paise, a pen costs ₹ 1.25 and a pencil costs 35 paise. Use matrix multiplication to calculate each individual's bill.

72. The cooperative stores of a particular school has 10 dozen physics books, 8 dozen chemistry books and 5 dozen mathematics books. Their selling prices are ₹ 8.30, ₹ 3.45 and ₹ 4.50 each respectively. Find the total amount the store will receive from selling all the items.

73. In a legislative assembly election, a political group hired a public relations firm to promote its candidates in three ways: telephone, house calls and letters. The cost per contact (in paise) is given matrix A as

$$\begin{array}{cc} & \text{Cost per contact} \\ A = & \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{House call} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given in matrix B as

$$\begin{array}{cc} & \begin{array}{ccc} \text{Telephone} & \text{House call} & \text{Letter} \end{array} \\ B = & \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$

Find the total amount spent by the group in the two cities X and Y.

74. A trust fund has ₹ 30000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide ₹ 30000 among the two types of bonds. If the trust fund must obtain an annual total interest of (i) ₹ 1800 (ii) ₹ 2000. [NCERT]

75. To promote making of toilets for women, an organisation tried to generate awarness through (i) house calls (ii) letters, and (iii) announcements. The cost for each mode per attempt is given below:

(i) ₹ 50 (ii) ₹ 20 (iii) ₹ 40

The number of attempts made in three villages X, Y and Z are given below:

	(i)	(ii)	(iii)
X	400	300	100
Y	300	250	75
Z	500	400	150

Find the total cost incurred by the organisation for three villages separately, using matrices. [CBSE 2015]

76. There are 2 families A and B. There are 4 men, 6 women and 2 children in family A, and 2 men, 2 women and 4 children in family B. The recommend daily amount of calories is 2400 for men, 1900 for women, 1800 for children and 45 grams of proteins for men, 55 grams for women and 33 grams for children. Represent the above information using matrix. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families. What awareness can you create among people about the planned diet from this question? [CBSE 2015]

77. In a parliament election, a political party hired a public relations firm to promote its candidates in three ways — telephone, house calls and letters. The cost per contact (in paisa) is given in matrix A as

$$A = \begin{array}{c} \begin{bmatrix} 140 \\ 200 \\ 150 \end{bmatrix} \end{array} \begin{array}{l} \text{Telephone} \\ \text{House calls} \\ \text{Letters} \end{array}$$

The number of contacts of each type made in two cities X and Y is given in the matrix B as

$$\begin{array}{ccc} \text{Telephone} & \text{House call} & \text{Letters} \end{array}$$
$$B = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{array}{l} \text{City } X \\ \text{City } Y \end{array}$$

Find the total amount spent by the party in the two cities.

What should one consider before casting his/her vote — party's promotional activity or their social activities? [CBSE 2015]

78. The monthly incomes of Aryan and Babbar are in the ratio 3 : 4 and their monthly expenditures are in the ratio 5 : 7. If each saves ₹ 15000 per month, find their monthly incomes using matrix method. This problem reflects which value? [CBSE 2016]

79. A trust invested some money in two type of bonds. The first bond pays 10% interest and second bond pays 12% interest. The trust received ₹ 2800 as interest. However, if trust had interchanged money in bonds, they would have got ₹ 100 less as interest. Using matrix method, find the amount invested by the trust. [CBSE 2016]

ANSWERS

1. (i) $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$ (ii) $\begin{bmatrix} 7 & -2 & 5 \\ -7 & 10 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 14 & 0 & 42 \\ 18 & -1 & 56 \\ 22 & -2 & 70 \end{bmatrix}$

3. (i) $AB=\begin{bmatrix}-3 & -4 & 1\\ 8 & 13 & 9\end{bmatrix}$, BA does not exist (ii) $AB=\begin{bmatrix}12 & 17 & 22\\ -4 & -5 & -6\\ -4 & -4 & -4\end{bmatrix}$, $BA=\begin{bmatrix}1 & 14\\ -3 & 2\end{bmatrix}$

(iii) $AB = [11]$, $BA = \begin{bmatrix}0 & 0 & 0 & 0\\ 1 & -1 & 2 & 3\\ 3 & -3 & 6 & 9\\ 2 & -2 & 4 & 6\end{bmatrix}$ (iv) $[a^2+b^2+c^2+d^2+ac+bd]$

5. (i) $\begin{bmatrix}6 & 16 & 26\\ -8 & -18 & -28\end{bmatrix}$ (ii) $[82]$ (iii) $\begin{bmatrix}0 & -1 & 1\\ 2 & 0 & -2\\ 5 & -2 & -3\end{bmatrix}$ **7.** $\begin{bmatrix}4 & -20\\ 38 & -10\end{bmatrix}$

11. $\begin{bmatrix}\cos 4\theta & \sin 4\theta\\ -\sin 4\theta & \cos 4\theta\end{bmatrix}$ **15.** $\begin{bmatrix}-2 & -9 & -1\\ 3 & 26 & 3\\ 35 & 15 & 34\end{bmatrix}$ **19.** $a_{43}=8,\ a_{22}=0$

24. (i) $x=-2$ (ii) $x=13$ **25.** $x=-2,-1$ **26.** 2

28. -7 **30.** $8I$ **33.** $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ **34.** (ii) $\begin{bmatrix}39 & 55\\ -55 & -16\end{bmatrix}$ **35.** $k=1$

36. $k=7$ **38.** $\lambda=4,\ \mu=-1$ **39.** $\frac{1}{5}$

40. (i) $x=-3, 5$ (ii) $x=-1$ (iii) $x=4\sqrt{3}$ (iv) $x=0,\ -\frac{23}{2}$

41. $\begin{bmatrix}6 & -14 & 10\\ -21 & 36 & -25\\ -3 & 5 & -5\end{bmatrix}$ **42.** $\begin{bmatrix}4 & 7 & 2\\ 12 & 19 & 8\\ 8 & 12 & 3\end{bmatrix}$ **43.** $\begin{bmatrix}6 & -2 & 6\\ 0 & 4 & 4\\ 1 & 1 & 4\end{bmatrix}$

47. $\begin{bmatrix}1 & -4\\ 3 & -2\end{bmatrix}$ **48.** (i) $\begin{bmatrix}2 & 3 & 4\\ 1 & 0 & 1\end{bmatrix}$ (ii) $\begin{bmatrix}1 & -2\\ 2 & 0\end{bmatrix}$ (iii) $[-1\ \ 2\ \ 1]$ (iv) $[-4]$ (v) $\begin{bmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{bmatrix}$

(vi) $\begin{bmatrix}1 & -2\\ 2 & 0\\ -5 & 4\end{bmatrix}$ **49.** $\begin{bmatrix}4 & 2\\ -1 & 1\end{bmatrix}$ **50.** -4 **53.** $\begin{bmatrix}187 & -195\\ -156 & 148\end{bmatrix}$

55. $\begin{bmatrix}-1 & -1 & -3\\ -1 & -3 & -10\\ -5 & 4 & 2\end{bmatrix}, \begin{bmatrix}1 & 1 & 3\\ 1 & 3 & 10\\ 5 & -4 & -2\end{bmatrix}$ **64.** $a=2, b=3$, No **68.** True when $AB=BA$

71. ₹ 157.80, ₹ 167.40, ₹ 281.40 **72.** ₹1597.20 **73.** ₹ 3400, ₹ 7200

74. (i) ₹ 15000 each (ii) ₹ 5000, ₹ 25000

75. X : ₹ 30,000
Y : ₹ 23,000
Z : ₹ 29,000

76.

	Calories	Proteins
Family A :	24600	576
Family B :	15800	332

77. X : ₹ 9900
Y : ₹ 21200

78. ₹ 90,000, ₹ 120,000 **79.** ₹ 10,000, ₹ 15,000

HINTS TO SELECTED PROBLEMS

34. We have,

$$A = \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$\Rightarrow\quad A^2 = AA = \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix} = \begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$\therefore\quad A^2-5A+7I = \begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix} + \begin{bmatrix}-15 & -5\\ 5 & -10\end{bmatrix} + \begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix} = \begin{bmatrix}8-15+7 & 5-5+0\\ -5+5+0 & 3-10+7\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} = O$$

$\Rightarrow \quad A^2 = 5A - 7I$

$\therefore \quad A^4 = A^2 A^2 = (5A-7I)(5A-7I) = 5A(5A-7I) - 7I(5A-7I)$

$= 25A^2 - 35AI - 35IA + 49II = 25A^2 - 35A - 35A + 49I$

$= 25A^2 - 70A + 49I = 25(5A-7I) - 70A + 49I$

$= 125A - 175I - 70A + 49I = 55A - 126I$

$= 55\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - 126\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 165 & 55 \\ -55 & 110 \end{bmatrix} + \begin{bmatrix} -126 & 0 \\ 0 & -126 \end{bmatrix} = \begin{bmatrix} 39 & 55 \\ -55 & -16 \end{bmatrix}$

35. We have, $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$

It is given that $A^2 = kA - 2I_2$

$\therefore \quad \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$

$\Rightarrow \quad 3k - 2 = 1,\ 4k = 4,\ -2k = -2$ and $-2k - 2 = -4 \Rightarrow k = 1$

40. (ii) $[1 \quad 2 \quad 1]\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = 0 \Rightarrow [1 \quad 2 \quad 1]\begin{bmatrix} 4 \\ x \\ 2x \end{bmatrix} = 0 \Rightarrow 4 + 2x + 2x = 0 \Rightarrow x = -1$

44. We have, $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$ and $f(x) = x^3 - 6x^2 + 7x + 2$. Therefore, $f(A) = A^3 - 6A^2 + 7A + 2I_3$.

Now, $A^2 = AA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$

$\Rightarrow \quad A^3 = A^2 A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$

$\therefore \quad f(A) = A^3 - 6A^2 + 7A + 2I_3$

$\Rightarrow \quad f(A) = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$

Hence, A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

48. (ii) We have to find a matrix A satisfying the equation $A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$.

Clearly, the product of A with a 2×3 matrix is a 2×3 matrix. Therefore, A is a 2×2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$A\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix} = \begin{bmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{bmatrix}$

$\Rightarrow \quad a+4b = -7,\ 2a+5b = -8,\ 3a+6b = -9, c+4d = 2,\ 2c+5d = 4,\ 3c+6d = 6$

$\Rightarrow \quad a = 1,\ b = -2, c = 2, d = 0$

$\therefore \quad A = \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

65. (ii) $A = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

74. Let ₹ x be invested in first bond and ₹ y be invested in second bond. Let A be the investment matrix and B be the interest per rupee matrix. Then,

$$A = [x \quad y] \text{ and } B = \begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix}. \text{ Total annual interest } = AB = [x \quad y]\begin{bmatrix} \frac{5}{100} \\ \frac{7}{100} \end{bmatrix} = \frac{5x}{100} + \frac{7y}{100}$$

Also, $x + y = 30000$ ...(i)

(i) If total interest is ₹ 1800. Then, $\frac{5x}{100} + \frac{7y}{100} = 1800 \Rightarrow 5x + 7y = 180000$ ...(ii)

Solving (i) and (ii), we get : $x = y = 15000$.

(ii) If total interest is ₹ 2000. Then, $\frac{5x}{100} + \frac{7y}{100} = 2000 \Rightarrow 5x + 7y = 200000$ ...(iii)

Solving (i) and (iii), we get : $x = 5000$ and $y = 25000$

76. Let F be the family matrix and R be the requirement matrix. Then,

$$F = \begin{array}{c} \\ \text{Family } A \\ \text{Family } B \end{array}\begin{array}{c} \begin{array}{ccc} \text{Men} & \text{Women} & \text{Children} \end{array} \\ \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \end{array} \text{ and, } R = \begin{array}{c} \\ \text{Men} \\ \text{Women} \\ \text{Children} \end{array}\begin{array}{c} \begin{array}{cc} \text{Calories} & \text{Proteins} \end{array} \\ \begin{bmatrix} 2400 & 45 \\ 1900 & 55 \\ 1800 & 33 \end{bmatrix} \end{array}$$

Total requirement of calories and proteins of each of the two families is given by the matrix product

$$FR = \begin{array}{c} \\ \text{Family } A \\ \text{Family } B \end{array}\begin{array}{c} \begin{array}{ccc} \text{Men} & \text{Women} & \text{Children} \end{array} \\ \begin{bmatrix} 4 & 6 & 2 \\ 2 & 2 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ \text{Men} \\ \text{Women} \\ \text{Children} \end{array}\begin{array}{c} \begin{array}{cc} \text{Calories} & \text{Proteins} \end{array} \\ \begin{bmatrix} 2400 & 45 \\ 1900 & 55 \\ 1800 & 33 \end{bmatrix} \end{array}$$

$$\Rightarrow \quad FR = \begin{array}{c} \\ \text{Family } A \\ \text{Family } B \end{array}\begin{array}{c} \begin{array}{cc} \text{Calories} & \text{Proteins} \end{array} \\ \begin{bmatrix} 24600 & 576 \\ 15800 & 332 \end{bmatrix} \end{array}$$

4.8 TRANSPOSE OF A MATRIX

DEFINITION *Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, the transpose of A, denoted by A^T or A', is an $n \times m$ matrix such that*

$$(A^T)_{ij} = a_{ji} \text{ for all } i = 1, 2, \ldots, n; j = 1, 2, \ldots, m.$$

Thus, A^T is obtained from A by changing its rows into columns and columns into rows.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 4 & 1 \\ 4 & 1 & 4 \end{bmatrix}$.

The first row of A^T is the first column of A. The second row of A^T is the second column of A and so on.

4.8.1 PROPERTIES OF TRANSPOSE

We shall now state and prove some properties of transpose of a matrix as theorems given below.

THEOREM 1 *For any matrix A, $(A^T)^T = A$.*

PROOF Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, A^T is an $n \times m$ matrix and so $(A^T)^T$ is an $m \times n$ matrix. Thus, the matrices A and $(A^T)^T$ are of the same order such that

$$\left((A^T)^T\right)_{ij} = (A^T)_{ji} \qquad \text{[By the definition of transpose]}$$

$$\Rightarrow \left((A^T)^T\right)_{ij} = (A)_{ij} \text{ for all } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Hence, by the definition of equality of two matrices, we obtain

$$(A^T)^T = A. \qquad \textbf{Q.E.D.}$$

THEOREM 2 *For any two matrices A and B of the same order, $(A+B)^T = A^T + B^T$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$. Then, $A + B$ will be a matrix of the order $m \times n$ and so $(A+B)^T$ will be a matrix of order $n \times m$. Since A^T and B^T are both $n \times m$ matrices. Therefore, $A^T + B^T$ will be a matrix of the order $n \times m$. Thus, the matrices $(A+B)^T$ and $A^T + B^T$ are of the same order such that

$$\begin{aligned} ((A+B)^T)_{ij} &= (A+B)_{ji} && \text{[By the definition of transpose]} \\ &= a_{ji} + b_{ji} && \text{[By the definition of addition]} \\ &= (A^T)_{ij} + (B^T)_{ij} \\ &= (A^T + B^T)_{ij} \text{ for all } i, j && \text{[By the definition of addition]} \end{aligned}$$

Hence, by the definition of equality of two matrices, we obtain: $(A+B)^T = A^T + B^T$. **Q.E.D.**

THEOREM 3 *If A is a matrix and k is a scalar, then $(kA)^T = k(A^T)$.*

PROOF Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then, for any scalar k, kA is also an $m \times n$ matrix and so $(kA)^T$ is an $n \times m$ matrix. Again A^T is an $n \times m$ matrix and so kA^T is an $n \times m$ matrix. Thus, the two matrices $(kA)^T$ and kA^T are of the same order such that

$$\begin{aligned} ((kA)^T)_{ij} &= (kA)_{ji} && \text{[By the definition of transpose]} \\ &= k\,a_{ji} && \text{[By the definition of scalar multiplication]} \\ &= k\,(A^T)_{ij} && \text{[By the definition of transpose]} \\ &= (kA^T)_{ij} && \text{[By the definition of scalar multiplication]} \end{aligned}$$

Hence, by the definition of equality of two matrices, we obtain : $(kA)^T = k\,A^T$. **Q.E.D.**

THEOREM 4 *If A and B are two matrices such that AB is defined, then $(AB)^T = B^T A^T$.*

PROOF Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices. Then AB is an $m \times p$ matrix and therefore $(AB)^T$ is a $p \times m$ matrix. Since A^T and B^T are $n \times m$ and $p \times n$ matrices, therefore $B^T A^T$ is a $p \times m$ matrix. Thus, the two matrices $(AB)^T$ and $B^T A^T$ are of the same order such that

$$((AB)^T)_{ij} = (AB)_{ji} \quad \text{[By the definition of transpose]}$$

$$= \sum_{r=1}^{n} a_{jr} b_{ri} \quad \text{[By the definition of matrix multiplication]}$$

$$= \sum_{r=1}^{n} b_{ri} a_{jr} \quad \text{[By commutativity of multiplication of numbers]}$$

$$= \sum_{r=1}^{n} (B^T)_{ir} (A^T)_{rj} \quad \text{[By definition of transpose]}$$

$$= (B^T A^T)_{ij} \quad \text{[By definition of multiplication of matrices]}$$

Hence, by the definition of equality of two matrices, we obtain $(AB)^T = B^T A^T$.

Q.E.D.

GENERALISATION *If A, B, C are three matrices confirmable for the products (AB) C and A (BC), then* $(ABC)^T = C^T B^T A^T$.

REMARK *The above law is called the reversal law for transposes i.e. the transpose of the product is the product of the transposes taken in the reverse order.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If* $A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ *and* $B = [-2 \ \ -1 \ \ -4]$, *verify that* $(AB)^T = B^T A^T$.

SOLUTION We have, [CBSE 2002, 2005]

$$A = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \text{ and } B = [-2 \ \ -1 \ \ -4]$$

$$\therefore \quad AB = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} [-2 \ \ -1 \ \ -4] = \begin{bmatrix} 2 & 1 & 4 \\ -4 & -2 & -8 \\ -6 & -3 & -12 \end{bmatrix} \Rightarrow (AB)^T = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \text{...(i)}$$

Also, $$B^T A^T = [-2 \ \ -1 \ \ -4]^T \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}^T = \begin{bmatrix} -2 \\ -1 \\ -4 \end{bmatrix} [-1 \ \ 2 \ \ 3] = \begin{bmatrix} 2 & -4 & -6 \\ 1 & -2 & -3 \\ 4 & -8 & -12 \end{bmatrix} \quad \text{...(ii)}$$

From (i) and (ii), we observe that $(AB)^T = B^T A^T$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *If* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, *then find the values of* θ *satisfying the equation* $A^T + A = I_2$.

SOLUTION We have,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Now, $A^T + A = I_2$

$$\Rightarrow \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = 2n\pi \pm \frac{\pi}{3},\ n \in Z$$

EXAMPLE 3 *If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA^T = 9I_3$, then find the values of a and b.*

SOLUTION We have,

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$$

$$\therefore \quad AA^T = 9I_3$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 9 & 0 & a+2b+4 \\ 0 & 9 & 2a+2-2b \\ a+2b+4 & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \quad a+2b+4 = 0,\ 2a+2-2b = 0 \text{ and } a^2+4+b^2 = 9$$

$$\Rightarrow \quad a+2b+4 = 0,\ a-b+1 = 0 \text{ and } a^2+b^2 = 5$$

Solving $a+2b+4 = 0$ and $a-b+1 = 0$, we get: $a = -2$ and $b = -1$.

EXAMPLE 4 *Find the values of x, y, z if the matrix $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $A^T A = I_3$.* **[NCERT]**

SOLUTION We have,

$$A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix}$$

It is given that

$$A^T A = I_3$$

$$\Rightarrow \quad \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2x^2 = 1,\ 6y^2 = 1,\ 3z^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{6}},\ z = \pm\frac{1}{\sqrt{3}}$$

EXERCISE 4.4

BASIC

1. Let $A = \begin{bmatrix} 2 & -3 \\ -7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & -4 \end{bmatrix}$, verify that

(i) $(2A)^T = 2A^T$ (ii) $(A+B)^T = A^T + B^T$ (iii) $(A-B)^T = A^T - B^T$ (iv) $(AB)^T = B^T A^T$

2. If $A = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$ and $B = [1 \quad 0 \quad 4]$, verify that $(AB)^T = B^T A^T$. [CBSE 2002]

3. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. Find A^T, B^T and verify that

(i) $(A+B)^T = A^T + B^T$ (ii) $(AB)^T = B^T A^T$ (iii) $(2A)^T = 2A^T$

4. If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$, $B = [1 \; 3 \quad -6]$, verify that $(AB)^T = B^T A^T$.

5. If $A = \begin{bmatrix} 2 & 4 & -1 \\ -1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 2 & 1 \end{bmatrix}$, find $(AB)^T$.

6. (i) For two matrices A and B, $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$ verify that $(AB)^T = B^T A^T$.

(ii) For the matrices A and B, verify that $(AB)^T = B^T A^T$, where $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$

7. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, find $A^T - B^T$. [CBSE 2012]

BASED ON LOTS

8. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that $A^T A = I_2$. [NCERT]

9. If $A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, verify that $A^T A = I_2$. [NCERT]

BASED ON HOTS

10. If l_i, m_i, n_i ; $i = 1, 2, 3$ denote the direction cosines of three mutually perpendicular vectors in space, prove that $AA^T = I$, where $A = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$.

ANSWERS

5. $\begin{bmatrix} 0 & 1 \\ 15 & -2 \end{bmatrix}$ 7. $\begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

4.9 SYMMETRIC AND SKEW-SYMMETRIC MATRICES

SYMMETRIC MATRIX *A square matrix $A = [a_{ij}]$ is called a symmetric matrix, if $a_{ij} = a_{ji}$ for all i, j.*

For example, the matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 2 & 5 \\ 1 & 5 & -2 \end{bmatrix}$ is symmetric, because $a_{12} = -1 = a_{21}$, $a_{13} = 1 = a_{31}$, $a_{23} = 5 = a_{32}$ i.e. $a_{ij} = a_{ji}$ for all i, j.

It follows from the definition of a symmetric matrix that A is symmetric, iff

$$a_{ij} = a_{ji} \text{ for all } i, j \Leftrightarrow (A)_{ij} = (A^T)_{ij} \text{ for all } i, j \Leftrightarrow A = A^T.$$

Thus, a square matrix A is a symmetric matrix iff $A^T = A$.

Matrices $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $B = \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$ are symmetric matrices, because $A^T = A$ and $B^T = B$.

SKEW-SYMMETRIC MATRIX *A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j.*

For example, the matrix $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$ is skew-symmetric, because

$$a_{12} = 2,\ a_{21} = -2 \Rightarrow a_{12} = -a_{21};\ a_{13} = -3,\ a_{31} = 3 \Rightarrow a_{13} = -a_{31};$$

and, $a_{23} = 5,\ a_{32} = -5 \Rightarrow a_{23} = -a_{32}$

It follows from the definition of a skew-symmetric matrix that A is skew-symmetric iff

$$\Leftrightarrow a_{ij} = -a_{ji} \text{ for all } i, j \Leftrightarrow (A)_{ij} = -(A^T)_{ij} \text{ for all } i, j \Leftrightarrow A = -A^T \Leftrightarrow A^T = -A.$$

Thus, a square matrix A is a skew-symmetric matrix iff $A^T = -A$.

Matrices $A = \begin{bmatrix} 0 & 2i & 3 \\ -2i & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -3 & 5 \\ 3 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix}$ are skew-symmetric matrices because $A^T = -A$ and $B^T = -B$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that the elements on the main diagonal of a skew-symmetric matrix are all zero.* **[CBSE 2017]**

SOLUTION Let $A = [a_{ij}]$ be a skew-symmetric matrix. Then,

$a_{ij} = -a_{ji}$ for all i, j [By definition]

$\Rightarrow$ $a_{ii} = -a_{ii}$ for all values of i

$\Rightarrow$ $2a_{ii} = 0 \Rightarrow a_{ii} = 0$ for all values of $i \Rightarrow a_{11} = a_{22} = a_{33} = \dots = a_{nn} = 0$.

EXAMPLE 2 *If the matrix $A = \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew-symmetric, find the values of a, b and c.*

[NCERT EXEMPLAR]

SOLUTION For a skew-symmetric $A=[a_{ij}]$, we have $a_{ij}=-a_{ji}$ for all $i \neq j$ and $a_{ii}=0$ for all i.

Thus, if $A=\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ is skew-symmetric, then

$$a_{22}=0,\ a_{12}=-a_{21} \text{ and } a_{31}=-a_{13} \Rightarrow b=0,\ a=-2 \text{ and } c=-3$$

ALITER *If* $A=\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ *is skew-symmetric, then*

$$A^T=-A$$

$$\Rightarrow \begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}^T=-\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & -a & -3 \\ -2 & -b & 1 \\ -c & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad 2=-a,\ c=-3 \text{ and } b=-b \Rightarrow a=-2,\ c=-3 \text{ and } 2b=0 \Rightarrow a=-2,\ b=0 \text{ and } c=-3$$

EXAMPLE 3 *Let A be a square matrix. Then,*

(i) $A+A^T$ *is a symmetric matrix* [NCERT] (ii) $A-A^T$ *is a skew-symmetric matrix.* [NCERT]

(iii) AA^T *and* $A^T A$ *are symmetric matrices.*

SOLUTION (i) Let $P=A+A^T$. Then,

$$P^T=(A+A^T)^T=A^T+(A^T)^T \qquad [\because (A+B)^T=A^T+B^T]$$

$$\Rightarrow \quad P^T=A^T+A \qquad [\because (A^T)^T=A]$$

$$\Rightarrow \quad P^T=A+A^T=P \qquad \text{[By commutativity of matrix addition]}$$

$\therefore$ P is a symmetric matrix.

(ii) Let $Q=A-A^T$. Then,

$$Q^T=(A-A^T)^T=A^T-(A^T)^T \qquad [\because (A+B)^T=A^T+B^T]$$

$$\Rightarrow \quad Q^T=A^T-A=-(A-A^T)=-Q \qquad [\because (A^T)^T=A]$$

$\Rightarrow$ Q is skew-symmetric

(iii) We find that

$$(AA^T)^T=(A^T)^T A^T \qquad \text{[By reversal law]}$$

$$\Rightarrow \quad (AA^T)^T=AA^T \qquad [\because (A^T)^T=A]$$

$\Rightarrow$ AA^T is symmetric

Similarly, *it can be proved that* $A^T A$ is symmetric.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Prove that every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.* [NCERT]

SOLUTION Let A be a square matrix. Then,

$$A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)=P+Q \text{ (say), where } P=\frac{1}{2}(A+A^T) \text{ and } Q=\frac{1}{2}(A-A^T).$$

$$\text{Now,} \quad P^T=\left(\frac{1}{2}(A+A^T)\right)^T=\frac{1}{2}(A+A^T)^T \qquad [\because (kA)^T=k\,A^T]$$

$\Rightarrow \quad P^T = \frac{1}{2}(A^T + (A^T)^T)$ $\quad [\because (A+B)^T = A^T + B^T]$

$\Rightarrow \quad P^T = \frac{1}{2}(A^T + A)$ $\quad [\because (A^T)^T = A]$

$\Rightarrow \quad P^T = \frac{1}{2}(A + A^T) = P$ $\quad$ [By commutativity of matrix addition]

$\therefore \quad P$ is a symmetric matrix.

And, $Q^T = \left(\frac{1}{2}(A - A^T)\right)^T = \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -Q$

$\therefore \quad Q$ is a skew-symmetric matrix.

Thus, $A = P + Q$, where P is a symmetric matrix and Q is a skew-symmetric matrix.

Hence, A is expressible as the sum of a symmetric and a skew-symmetric matrix.

Uniqueness: If possible, let $A = R + S$, where R is symmetric and S is skew-symmetric. Then,

$A^T = (R+S)^T = R^T + S^T = R - S$ $\quad [\because R^T = R$ and $S^T = -S]$

Now, $\quad A = R + S$ and $A^T = R - S \Rightarrow R = \frac{1}{2}(A + A^T) = P, S = \frac{1}{2}(A - A^T) = Q.$

Hence, A is uniquely expressible as the sum of a symmetric and a skew-symmetric matrix.

EXAMPLE 5 *If A and B are symmetric matrices, then show that AB is symmetric iff AB = BA i.e. A and B commute.* [NCERT]

SOLUTION $\quad AB$ is symmetric

$\Leftrightarrow \quad (AB)^T = AB$

$\Leftrightarrow \quad B^T A^T = AB$ $\quad [\because (AB)^T = B^T A^T]$

$\Leftrightarrow \quad BA = AB$ $\quad [\because A$ and B are symmetric matrices $\therefore A^T = A, B^T = B]$

EXAMPLE 6 *Show that the matrix $B^T AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.* [NCERT EXEMPLAR]

SOLUTION **Case I** Let A be a symmetric matrix. Then, $A^T = A$.

Now, $\quad (B^T AB)^T = B^T A^T (B^T)^T$ $\quad$ [By reversal law]

$= B^T A^T B$ $\quad [\because (B^T)^T = B]$

$= B^T AB$ $\quad [\because A^T = A]$

$\therefore \quad B^T AB$ is a symmetric matrix.

Case II Let A be a skew-symmetric matrix. Then, $A^T = -A$.

Now,

$(B^T AB)^T = B^T A^T (B^T)^T$ $\quad$ [By reversal law]

$= B^T A^T B$ $\quad [\because (B^T)^T = B]$

$= B^T (-A) B = -B^T AB$ $\quad [\because A^T = -A]$

$\therefore \quad B^T AB$ is a skew-symmetric matrix.

EXAMPLE 7 *Let A and B be symmetric matrices of the same order. Then, show that*

(i) *$A + B$ is a symmetric matrix.* $\quad$ (ii) *$AB - BA$ is a skew-symmetric matrix.* $\quad$ [NCERT]

(iii) *$AB + BA$ is a symmetric matrix.*

SOLUTION Since A and B are symmetric matrices. Therefore, $A^T = A$ and $B^T = B$.

(i) We find that: $(A+B)^T = A^T + B^T = A + B$ $[\because A^T = A, B^T = B]$

$\therefore$ $A + B$ is symmetric

(ii) We find that

$$(AB - BA)^T = (AB)^T - (BA)^T = B^T A^T - A^T B^T \quad \text{[By reversal law]}$$

$$= BA - AB = -(AB - BA) \quad [\because B^T = B, A^T = A]$$

$\therefore$ $AB - BA$ is skew-symmetric.

(iii) We find that

$$(AB + BA)^T = (AB)^T + (BA)^T$$

$$= B^T A^T + A^T B^T \quad \text{[By reversal law]}$$

$$= BA + AB = AB + BA \quad [\because A^T = A, B^T = B]$$

$\therefore$ $AB + BA$ is symmetric matrix.

EXAMPLE 8 *Express the matrix* $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix}$ *as the sum of a symmetric and a skew-symmetric matrix.*

SOLUTION We have,

$$A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix}$$

$$\therefore \quad A + A^T = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 5 \\ 6 & 10 & 7 \\ 5 & 7 & 10 \end{bmatrix}$$

$$\text{and,} \quad A - A^T = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Let $P = \dfrac{1}{2}(A + A^T) = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix}$ and, $Q = \dfrac{1}{2}(A - A^T) = \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}$.

$$\text{Then,} \quad P^T = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 7/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} = P$$

$$\text{and,} \quad Q^T = \begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1/2 \\ -1 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 & 1/2 \\ 1 & 0 & -1/2 \\ -1/2 & 1/2 & 0 \end{bmatrix} = -Q$$

Thus, P is symmetric and Q is skew-symmetric.

$$\text{Also,} \quad P + Q = \begin{bmatrix} 3 & 3 & 5/2 \\ 3 & 5 & 5/2 \\ 5/2 & 7/2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 5/2 \\ 1 & 0 & 5/2 \\ -1/2 & 5/2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 5 & 3 \\ 2 & 4 & 5 \end{bmatrix} = A$$

Thus, A is expressible as the sum of a symmetric matrix P and a skew-symmetric matrix Q.

EXAMPLE 9 *If A is a symmetric matrix and B is skew-symmetric matrix such that* $A+B=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}$, *then find AB.*

SOLUTION It is given that A is a symmetric matrix and B is a skew-symmetric matrix. Therefore, $A^T=A$ and $B^T=-B$.

Now $\quad A+B=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}$ ...(i)

$\Rightarrow \quad (A+B)^T=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}^T \Rightarrow A^T+B^T=\begin{bmatrix}2 & 5\\ 3 & -1\end{bmatrix} \Rightarrow A-B=\begin{bmatrix}2 & 5\\ 3 & -1\end{bmatrix}$ ...(ii)

Adding (i) and (ii), we obtain

$$2A=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}+\begin{bmatrix}2 & 5\\ 3 & -1\end{bmatrix}=\begin{bmatrix}2+2 & 3+5\\ 5+3 & -1-1\end{bmatrix}=\begin{bmatrix}4 & 8\\ 8 & -2\end{bmatrix} \Rightarrow A=\frac{1}{2}\begin{bmatrix}4 & 8\\ 8 & -2\end{bmatrix}=\begin{bmatrix}2 & 4\\ 4 & -1\end{bmatrix} \quad ...(iii)$$

From (i) and (iii), we obtain

$$\begin{bmatrix}2 & 4\\ 4 & -1\end{bmatrix}+B\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix} \Rightarrow B=\begin{bmatrix}2 & 3\\ 5 & -1\end{bmatrix}-\begin{bmatrix}2 & 4\\ 4 & -1\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\therefore \quad AB=\begin{bmatrix}2 & 4\\ 4 & -1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0+4 & -2+0\\ 0-1 & -4+0\end{bmatrix}=\begin{bmatrix}4 & -2\\ -1 & -4\end{bmatrix}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 10 *Show that all positive integral powers of a symmetric matrix are symmetric.*

SOLUTION Let A be a symmetric matrix and $n \in N$. Then,

$$(A^n)^T = (AAA \ldots A \text{ upto } n\text{-times})^T$$

$$= (A^T A^T A^T \ldots A^T \text{ upto } n\text{-times}) \quad \text{[By reversal law]}$$

$$= (A^T)^n = A^n \quad [\because A^T = A]$$

Hence, A^n is also a symmetric matrix.

EXAMPLE 11 *Show that positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.*

SOLUTION Let A be a skew-symmetric matrix. Then, $A^T = -A$.

We have, $(A^n)^T=(A^T)^n$ for all $n \in N$. [See Example 9]

$$\therefore \quad (A^n)^T = (-A)^n \quad [\because A^T=-A]$$

$$= (-1)^n A^n = \begin{cases} A^n, & \text{if } n \text{ is even} \\ -A^n, & \text{if } n \text{ is odd} \end{cases}$$

Hence, A^n is symmetric if n is even and skew-symmetric if n is odd.

EXAMPLE 12 *A matrix which is both symmetric as well as skew-symmetric is a null matrix.*

[NCERT EXEMPLAR, CBSE 2020]

SOLUTION Let $A=[a_{ij}]$ a matrix which is both symmetric and skew-symmetric.

Now $A=[a_{ij}]$ is a symmetric matrix $\Rightarrow a_{ij}=a_{ji}$ for all i, j ...(i)

Also, $A=[a_{ij}]$ is a skew-symmetric matrix.

$\therefore$ $a_{ij}=-a_{ji}$ for all $i, j \Rightarrow a_{ji}=-a_{ij}$ for all i, j ...(ii)

From (i) and (ii), we obtain

$$a_{ij}=-a_{ij} \text{ for all } i, j \Rightarrow 2a_{ij}=0 \text{ for all } i, j \Rightarrow a_{ij}=0 \text{ for all } i, j \Rightarrow A=[a_{ij}] \text{ is a null matrix.}$$

EXERCISE 4.5

BASIC

1. If $A=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, prove that $A-A^T$ is a skew-symmetric matrix. **[CBSE 2001]**
2. For the matrix $A=\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, find $A+A^T$ and verify that it is a symmetric matrix. **[CBSE 2019]**
3. If $\begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is a symmetric matrix, then find the value of x. **[CBSE 2020]**
4. If the matrix $A=\begin{bmatrix} 5 & 2 & x \\ y & z & -3 \\ 4 & t & -7 \end{bmatrix}$ is a symmetric matrix, find x, y, z and t.

BASED ON LOTS

5. Express the matrix $A=\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.
6. Let $A=\begin{bmatrix} 3 & 2 & 7 \\ 1 & 4 & 3 \\ -2 & 5 & 8 \end{bmatrix}$. Find matrices X and Y such that $X+Y=A$, where X is a symmetric and Y is a skew-symmetric matrix.
7. Express the matrix $\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix and verify your result. **[CBSE 2010]**

ANSWERS

2. $A=\begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix}$ 3. 5 4. $x=4, y=2, z\in C, t=-3$

5. Symmetric matrix $=\begin{bmatrix} 3 & -3/2 \\ -3/2 & -1 \end{bmatrix}$, Skew-symmetric matrix $=\begin{bmatrix} 0 & -5/2 \\ 5/2 & 0 \end{bmatrix}$

6. $X=\begin{bmatrix} 3 & 3/2 & 5/2 \\ 3/2 & 4 & 4 \\ 5/2 & 4 & 8 \end{bmatrix}$, $Y=\begin{bmatrix} 0 & 1/2 & 9/2 \\ -1/2 & 0 & -1 \\ -9/2 & 1 & 0 \end{bmatrix}$

7. Symmetric matrix $=\begin{bmatrix} 3 & 1/2 & -5/2 \\ 1/2 & -2 & -2 \\ -5/2 & -2 & 2 \end{bmatrix}$, Skew-symmetric matrix $=\begin{bmatrix} 0 & -5/2 & -3/2 \\ 5/2 & 0 & -3 \\ 3/2 & 3 & 0 \end{bmatrix}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If A and B are two matrices of orders $a \times 3$ and $3 \times b$ respectively such that AB exists and is of order 2×4. Then, $(a, b) =$ ....
2. If P and Q are two matrices of orders $3 \times n$ and $n \times p$ respectively then the order of the matrix PQ is ....
3. If $A = \begin{bmatrix} -1 & 2 & 3x \\ 2y & 4 & -1 \\ 6 & 5 & 0 \end{bmatrix}$ is a symmetric matrix, then the value of $2x + y$ is ....
4. If a, b are positive integers such that $a < b$ and $[a \quad b]\begin{bmatrix} a \\ b \end{bmatrix} = 25$, then $(a, b) =$ ....
5. If $A = \begin{bmatrix} \frac{1}{3} & 2 \\ 0 & 2x-3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$ and $AB = I$, then $x =$ ....
6. If $A = \begin{bmatrix} x & 1 \\ -1 & -x \end{bmatrix}$ satisfies the equation $A^2 = O$, then $x =$ ....
7. If A is an $m \times n$ matrix and B is a matrix such that both AB and BA are defined, then the order of B is ....
8. If $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ -2 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & -4 & 0 \end{bmatrix}$, then $(AB)_{33} =$ ....
9. If $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, then $A^4 =$ ....
10. If $A = \text{diag}\,(2, -1, 3)$, $B = \text{diag}\,(-1, 3, 2)$, then $A^2B =$ ....
11. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [2 \quad 1 \quad -1] =$ ....
12. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then $x =$ ....
13. If $e\begin{bmatrix} e^x & e^y \\ e^y & e^x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $x =$ ...., $y =$ ....
14. If $A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and $kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$, then $(k, a, b) =$ ....
15. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $AA^T =$ ....
16. If A is 3×4 matrix and B is a matrix such that A^TB and BA^T are both defined. Then the order of B is ....
17. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $AA^T =$ ....

18. If $f(x)=\begin{bmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{bmatrix}$ and $f(x)\,f(y)=f(z)$, then $z=$ ....

19. If A, B and C are $m\times n, n\times p$ and $p\times q$ matrices respectively such that $(BC)\,A$ is defined, then $m=$ ....

20. If $A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$ such that $A^5=\lambda A$, then $\lambda=$ ....

21. If the matrices $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ commute with each other, then $C=$ ....

22. If $A=\begin{bmatrix}4 & x+2\\ 2x-3 & x+1\end{bmatrix}$ is a symmetric matrix, then $x=$ ....

23. If A and B are two skew-symmetric matrices of same order, then AB is symmetric if........................ . [NCERT EXEMPLAR]

24. If A and B are matrices of the same order, then $(3A-2B)^T$ is equal to ....

25. Addition of matrices is defined if order of the matrices is ....

26. If A and B are symmetric matrices of the same order, then AB is symmetric iff........................ . [NCERT EXEMPLAR]

27. If A is symmetric matrix, then $B^T AB$ is .... [NCERT EXEMPLAR]

28. If A is a skew-symmetric matrix, then A^2 is a .... matrix. [NCERT EXEMPLAR]

29. If A is a symmetric matrix, then A^3 is a .... matrix. [NCERT EXEMPLAR]

30. If A is a skew-symmetric matrix, then kA is a ....(k is any scalar). [NCERT EXEMPLAR]

31. If A and B are symmetric matrices of the same order, then
(i) $AB-BA$ is a ....
(ii) $BA-2AB$ is a .... [NCERT EXEMPLAR]

32. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeroes in one or more row, then A^{-1} ....

33. The product of any matrix by the scalar .... is the null matrix.

34. A matrix which is not a square matrix is called .... matrix.

35. The sum of two skew-symmetric matrices is always .... matrix.

36. A and B are square matrices of the same order, then ....
(i) $(AB)^T=$ .... (ii) $(kA)^T=$.......... (iii) $(k\,(A-B))^T=$ .... where k is any scalar.

37. .... matrix is both symmetric and skew-symmetric matrix. [CBSE 2020]

38. Matrix multiplication is .... over matrix addition.

39. The negative of a matrix is obtained by multiplying it by ....

40. If A is a non-singular matrix, then $(A^T)^{-1}=$ ....

41. If $A+B=\begin{bmatrix}1&0\\1&1\end{bmatrix}$ and $A-2B=\begin{bmatrix}-1&1\\0&-1\end{bmatrix}$, then $A=$........................ . [CBSE 2020]

42. Given a skew-symmetric matrix $A=\begin{bmatrix}0&a&1\\-1&b&1\\-1&c&0\end{bmatrix}$ the value of $(a+b+c)^2$ is .... [CBSE 2020]

43. If $[3 \quad -2 \quad 0]\begin{bmatrix} 2 \\ k \\ -5 \end{bmatrix} = 0$, then the value of k is .... **[CBSE 2020]**

ANSWERS

1. $(2, 4)$ 2. $3 \times p$ 3. 5 4. $(3, 4)$ 5. 1
6. ± 1 7. $n \times m$ 8. 4 9. $3^4 I_3$
10. $\text{diag}\,(-4, 3, 18)$ 11. $\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$ 12. 0 13. $x = -1, y = -1$
14. $(-6, -4, -9)$ 15. I 16. 3×4 17. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$
18. $x + y$ 19. q 20. 16 21. 0 22. 5
23. $AB = BA$ 24. $3A^T - 2B^T$ 25. same 26. $AB = BA$
27. symmetric 28. symmetric 29. symmetric 30. skew-symmetric
31. (i) skew-symmetric (ii) neither symmetric nor skew-symmetric
32. does not exist 33. zero 34. rectangular 35. skew-symmetric
36. (i) $B^T A^T$ (ii) $k\,A^T$ (iii) $k\,(A^T - B^T)$ 37. null matrix
38. distributive 39. -1 40. $(A^{-1})^T$ 41. $A = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}$
42. 0 43. 3

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If A is an $m \times n$ matrix and B is $n \times p$ matrix does AB exist? If yes, write its order.
2. If $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$. Write the orders of AB and BA.
3. If $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$, write AB.
4. If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, write AA^T. **[CBSE 2009]**
5. Give an example of two non-zero 2×2 matrices A and B such that $AB = O$.
6. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, find $A + A^T$.
7. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, write A^2.
8. If $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$, find x satisfying $0 < x < \frac{\pi}{2}$ when $A + A^T = I$
9. If $A = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$, find AA^T
10. If $\begin{bmatrix} 1 & 0 \\ y & 5 \end{bmatrix} + 2\begin{bmatrix} x & 0 \\ 1 & -2 \end{bmatrix} = I$, where I is 2×2 unit matrix. Find x and y.

11. If $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, satisfies the matrix equation $A^2 = kA$, write the value of k.

12. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ satisfies $A^4 = \lambda A$, then write the value of λ.

13. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, find A^2.

14. If $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, find A^3.

15. If $A = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$, find A^4.

16. If $[x \quad 2] \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2$, find x

17. If $A = [a_{ij}]$ is a 2×2 matrix such that $a_{ij} = i + 2j$, write A. **[CBSE 2008]**

18. Write matrix A satisfying $A + \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ -3 & 8 \end{bmatrix}$.

19. If $A = [a_{ij}]$ is a square matrix such that $a_{ij} = i^2 - j^2$, then write whether A is symmetric or skew-symmetric.

20. For any square matrix write whether AA^T is symmetric or skew-symmetric.

21. If $A = [a_{ij}]$ is a skew-symmetric matrix, then write the value of $\sum_i a_{ii}$.

22. If $A = [a_{ij}]$ is a skew-symmetric matrix, then write the value of $\sum_i \sum_j a_{ij}$.

23. If A and B are symmetric matrices, then write the condition for which AB is also symmetric.

24. If B is a skew-symmetric matrix, write whether the matrix $AB\,A^T$ is symmetric or skew-symmetric.

25. If B is a symmetric matrix, write whether the matrix $AB\,A^T$ is symmetric or skew-symmetric.

26. If A is a skew-symmetric and $n \in N$ such that $(A^n)^T = \lambda\, A^n$, write the value of λ.

27. If A is a symmetric matrix and $n \in N$, write whether A^n is symmetric or skew-symmetric or neither of these two.

28. If A is a skew-symmetric matrix and n is an even natural number, write whether A^n is symmetric or skew-symmetric or neither of these two.

29. If A is a skew-symmetric matrix and n is an odd natural number, write whether A^n is symmetric or skew-symmetric or neither of the two.

30. If A and B are symmetric matrices of the same order, write whether $AB - BA$ is symmetric or skew-symmetric or neither of the two.

31. Write a square matrix which is both symmetric as well as skew-symmetric.

32. Find the values of x and y, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$. **[CBSE 2008, 2019]**

33. If $\begin{bmatrix} x+3 & 4 \\ y-4 & x+y \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 9 \end{bmatrix}$, find x and y. **[CBSE 2008]**

34. Find the value of x from the following: $\begin{bmatrix} 2x-y & 5 \\ 3 & y \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 3 & -2 \end{bmatrix}$. **[CBSE 2009]**

35. Find the value of y, if $\begin{bmatrix} x-y & 2 \\ x & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$. [CBSE 2009]

36. Find the value of x, if $\begin{bmatrix} 3x+y & -y \\ 2y-x & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -5 & 3 \end{bmatrix}$. [CBSE 2009]

37. If matrix $A = [1 \;\; 2 \;\; 3]$, write AA^T. [CBSE 2009]

38. If $\begin{bmatrix} 2x+y & 3y \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 6 & 4 \end{bmatrix}$, then find x. [CBSE 2010]

39. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find $A + A^T$. [CBSE 2010]

40. If $\begin{bmatrix} a+b & 2 \\ 5 & b \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$, then find a.

41. If A is a matrix of order 3×4 and B is a matrix of order 4×3, find the order of the matrix of AB. [CBSE 2010]

42. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ is identity matrix, then write the value of α. [CBSE 2010]

43. If $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$, then write the value of k. [CBSE 2010]

44. If I is the identity matrix and A is a square matrix such that $A^2 = A$, then what is the value of $(I + A)^2 - 3A$?

45. If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is written as $B + C$, where B is a symmetric matrix and C is a skew-symmetric matrix, then find B.

46. If A is 2×3 matrix and B is a matrix such that $A^T B$ and BA^T both are defined, then what is the order of B?

47. What is the total number of 2×2 matrices with each entry 0 or 1?

48. If $\begin{bmatrix} x & x-y \\ 2x+y & 7 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 8 & 7 \end{bmatrix}$, then find the value of y. [CBSE 2011]

49. If a matrix has 5 elements, write all possible orders it can have. [CBSE 2011]

50. For a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \dfrac{i}{j}$, write the value of a_{12}. [CBSE 2011]

51. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, find the value of x. [CBSE 2012]

52. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find matrix A. [CBSE 2013]

53. If $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$, find the value of b. [CBSE 2013]

54. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix? [CBSE 2013]

55. If matrix $A=\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$ and $A^2=pA$, then write the value of p. **[CBSE 2013]**

56. If A is a square matrix such that $A^2=A$, then write the value of $7A-(I+A)^3$, where I is the identity matrix. **[CBSE 2014]**

57. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$, find $x-y$. **[CBSE 2014]**

58. If $[x\ \ 1]\begin{bmatrix}1 & 0\\ -2 & 0\end{bmatrix}=O$, find x. **[CBSE 2014]**

59. If $\begin{bmatrix}a+4 & 3b\\ 8 & -6\end{bmatrix}=\begin{bmatrix}2a+2 & b+2\\ 8 & a-8b\end{bmatrix}$, write the value of $a-2b$. **[CBSE 2014]**

60. Write a 2×2 matrix which is both symmetric and skew-symmetric. **[CBSE 2014]**

61. If $\begin{bmatrix}xy & 4\\ z+6 & x+y\end{bmatrix}=\begin{bmatrix}8 & w\\ 0 & 6\end{bmatrix}$, write the value of $(x+y+z)$. **[CBSE 2014]**

62. Construct a 2×2 matrix $A=[a_{ij}]$ whose elements a_{ij} are given by $a_{ij}=\begin{cases}\dfrac{|-3i+j|}{2}, & \text{if } i\neq j\\ (I+j)^2, & \text{if } i=j\end{cases}$

63. If $\begin{bmatrix}x+y\\ x-y\end{bmatrix}=\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix}\begin{bmatrix}1\\ -2\end{bmatrix}$, then write the value of (x, y).

64. Matrix $A=\begin{bmatrix}0 & 2b & -2\\ 3 & 1 & 3\\ 3a & 3 & -1\end{bmatrix}$ is given to be symmetric, find the values of a and b. **[CBSE 2016]**

65. Write the number of all possible matrices of order 2×2 with each entry 1, 2 or 3. **[CBSE 2016]**

66. If $[2\ \ 1\ \ 3]\begin{bmatrix}-1 & 0 & -1\\ -1 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}=A$, then write the order of matrix A. **[CBSE 2016]**

67. If $A=\begin{bmatrix}3 & 5\\ 7 & 9\end{bmatrix}$ is written as $A=P+Q$, where as $A=P+Q$, where P is symmetric and Q is skew-symmetric matrix, then write the matrix P. **[CBSE 2016]**

68. Let A and B be matrices of orders 3×2 and 2×4 respectively. Write the order of matrix AB. **[CBSE 2017]**

69. If the matrix $A=\begin{bmatrix}0 & a & -3\\ 2 & 0 & -1\\ b & 1 & 0\end{bmatrix}$ is skew-symmetric, find the values of 'a' and 'b'. **[CBSE 2018]**

ANSWERS

1. Yes, $m\times p$ **2.** 2×2 and 3×3 **3.** $\begin{bmatrix}-7\\ 2\end{bmatrix}$ **4.** $\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix}$

5. $A=\begin{bmatrix}2 & 0\\ 3 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 0\\ 2 & -1\end{bmatrix}$ **6.** $\begin{bmatrix}4 & 8\\ 8 & 14\end{bmatrix}$ **7.** $\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

8. $\frac{\pi}{3}$

9. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

10. $x = 0, y = -2$

11. 2

12. 8

13. $-A$ or, I_3

14. A

15. $\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix}$

16. -2

17. $\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$

18. $\begin{bmatrix} 1 & -9 \\ -2 & 4 \end{bmatrix}$

19. skew-symmetric

20. symmetric

21. 0

22. 0

23. $AB = BA$

24. skew-symmetric

25. symmetric

26. $(-1)^n$

27. symmetric

28. symmetric

29. skew-symmetric

30. skew-symmetric

31. null matrix

32. $x = 3, y = 3$

33. $x = 2, y = 7$

34. $x = 2$

35. $y = 1$

36. $x = 1$

37. 14

38. $x = 3, y = 0$

39. $\begin{bmatrix} 2 & 5 \\ 5 & 8 \end{bmatrix}$

40. 4

41. 3×3

42. $\alpha = 0$

43. 17

44. I

45. $\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

46. 2×3

47. 16

48. 2

49. $1 \times 5, 5 \times 1$

50. 1/2

51. $x = 3$

52. $\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

53. $b = 2$

54. $x = 2$

55. 4

56. $-I$

57. $x = 2, y = -8$

58. $x = 2$

59. 0

60. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

61. 0

62. $\begin{bmatrix} 4 & 1/2 \\ 5/2 & 16 \end{bmatrix}$

63. $(-1, 1)$

64. $a = -\frac{2}{3}, b = \frac{3}{2}$

65. $3^4 = 81$

66. 1×1

67. $\begin{bmatrix} 3 & 6 \\ 6 & 9 \end{bmatrix}$

68. 3×4

69. $a = -2, b = 3$