CHAPTER 33

PROBABILITY

33.1 INTRODUCTION

In earlier classes, we have learnt about two approaches to the theory of probability, namely, (i) Statistical approach and (ii) Classical approach. The statistical approach has been discussed in class IX. It is also known as repeated experiments and observed frequency approach. In this approach, we have defined the probability of an event as the ratio of observed frequency to the total frequency. The classical approach has been discussed in class X. In this approach, we define the probability of occurrence of an event as the ratio of favourable number of outcomes to the total number of equally likely outcomes. These equally likely outcomes are also known as elementary events associated to the experiment. Both the theories have some serious deficiencies and limitations. For instance, these approaches cannot be applied to the experiments which have large number of outcomes. The classical definition of probability cannot be applied whenever it is not possible to make a simple enumeration of cases which can be considered equally likely. For instance, how does it apply to probability of rain? What are the possible outcomes? We might think that there are two cases 'rain' and 'no rain'. But at any given locality it will not usually be agreed that they are equally like. The classical approach also fails to answer questions like "what is the probability that a male will die before the age of 60", "what is the probability that a bulb will burn in less than 2000 hours? etc. In fact, the classical definition is difficult to apply as soon as we deviate from the experiments pertaining to coins, dice, cards and other simple games of chance.

The statistical definition has difficulties from a mathematical point of view because an actual limiting number may not really exist. For this reason, modern probability theory has been developed axiomatically. This theory of probability was developed by A.N. Kolmogrov (1903-1987) a Russian Mathematician in 1933. He laid down certain axioms to interpret probability, in his book 'Foundation of Probability' published in 1933. The axiomatic definition of probability includes 'both' the classical and statistical approaches as particular cases and overcomes the deficiencies of each of them. In order to understand this approach we must know about some basic terms viz. random experiment, elementary events, sample space, compound events etc. So, let us begin with the term random experiment as discussed in the following section.

33.2 RANDOM EXPERIMENTS

The word experiment means an operation which can produce some well-defined outcome(s). There are two types of experiments viz. (i) Deterministic experiments and (ii) Random or Probability experiments.

DETERMINISTIC EXPERIMENTS In our day-to-day life, we perform many activities/experiments which have a fixed outcome or result no matter any number of times they are repeated. Such experiments are known as deterministic experiments. For example, from the set of all triangles in a plane if a triangle is choosen, then even without knowing the three angles, we can definitely say that the sum of the measures of the angles is 180°. In fact, when experiments in science and engineering are repeated under identical conditions, we get the same result every time.

RANDOM OR PROBABILISTIC EXPERIMENTS If an experiment, when repeated under identical conditions, do not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes then such an experiment is known as a probabilistic experiment or a random experiment. In other words, an experiment whose outcomes cannot be predicted or determined in advance is called a random experiment.

For example, in tossing of a coin one is not sure if a head or a tail will be obtained so it is a random experiment. Similarly, rolling an unbiased die and drawing a card from a well shuffled pack of playing cards are examples of random experiments.

33.3 SAMPLE SPACES

In the previous section, we have learnt about random experiments. Throughout this chapter the term experiment will mean random experiment. Associated to every random experiment there are two basic terms viz. outcomes (or elementary events) and sample space. In this section, we will discuss about these two for different random experiments.

ELEMENTARY EVENT *If a random experiment is performed, then each of its outcomes is known as an elementary event.*

In other words, outcomes of a random experiment are known as elementary events associated to it. Elementary events are also known as simple events.

SAMPLE SPACE *The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S.*

If $E_1, E_2, E_3, \dots, E_n$ are the possible outcomes (or elementary events) of a random experiment, then $S = \{E_1, E_2, \dots, E_n\}$ is the sample space associated to it.

ILLUSTRATION 1 *Consider the random experiment of tossing of a coin. The possible outcomes of this experiment are H and T. Thus, if we define*

E_1 = Getting head (H) on the upper face and, E_2 = Getting tail (T) on the upper face.

Then, E_1 and E_2 are elementary events associated to the random experiment of tossing of a coin. The sample space associated to this experiment is given by $S = \{E_1, E_2\}$.

E_1 and E_2 are generally denoted by H and T respectively. Thus, we have $S = \{H, T\}$.

ILLUSTRATION 2 *Consider the experiment of throwing a die. Let the six faces of a die be marked as 1, 2, 3, 4, 5 and 6. If the die is thrown, then any one of the six faces may come upward. So, there are six possible outcomes of this experiment, namely, 1, 2, 3, 4, 5, 6. Thus, if we define*

E_i = *Getting a face marked with number i*, where $i = 1, 2, 3, 4, 5, 6$

Then, $E_1, E_2, \dots, E_6$ are six elementary events associated to this experiment. The sample space associated to this experiment is $S = \{E_1, E_2, \dots E_6\}$.

In this experiment, elementary even E_i is denoted by i, where $i = 1, 2, \dots, 6$. Thus, we have

$$S = \{1, 2, 3, 4, 5, 6\}.$$

ILLUSTRATION 3 *Consider the experiment of tossing two coins together or a coin twice. In this experiment the possible outcomes are:*

Head on first and Head on second,
Head on first and Tail on second,
Tail on first and Head on second,
Tail on first and Tail on second.

If we define

HH = Getting head on both coins,
HT = Getting head on first and tail on second,
TH = Getting tail on first and head on second,
TT = Getting tail on both coins.

Then,

HH, HT, TH and TT are elementary events associated to the random experiment of tossing of two coins. The sample space associated to this experiment is given by $S = \{HH, HT, TH, TT\}$.

Similarly, the sample space associated to the random experiment of tossing three coins simultaneously or tossing a coin three times is given by

$$S = \{HHH, HHT, HTH, THH, TTH, HTT, THT, TTT\}$$

ILLUSTRATION 4 *Consider the random experiment in which two dice are tossed together or a die is tossed twice. If we define*

E_{ij} = Getting number i on the upper face of first die and number j on the upper face of second die,

where $i = 1, 2, ..., 6$ and $j = 1, 2, ..., 6$.

Then, E_{ij} are elementary events associated to this experiment and are generally denoted by (i, j).

Thus, $(1, 1), (1, 2), ..., (1, 6), (2, 1) ..., (2, 6), (3, 1), ..., (3, 6), (4, 1)..., (4, 6), (5, 1), ..., (5, 6)$ and $(6, 1), ..., (6, 6)$ are 36 elementary events associated to the random experiment of tossing two dice and the sample space associated to it is given by

$$S = \{(1, 1), ..., (1, 6), (2, 1), ..., (2, 6), (3, 1), ..., (3, 6), ..., (6, 1), ..., (6, 6)\}.$$

ILLUSTRATION 5 *Let there be a bag containing 3 white and 2 black balls. Let the white balls be denoted by W_1, W_2 W_3 and black balls be denoted by B_1, B_2. If we draw two balls from the bag, then there are ${}^5C_2$ elementary events associated to this experiment. These elementary events are:*

$B_1 W_1$, $B_1 W_2$, $B_1 W_3$, $B_2 W_1$, $B_2 W_2$, $B_2 W_3$, $W_1 W_2$, $W_1 W_3$, $W_2 W_3$, and $B_1 B_2$. The set of all these elementary events is the sample space associated to the experiment.

ILLUSTRATION 6 *A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red and 4 black balls ; if it shows a tail, we throw a die. If we denote three red balls as R_1, R_2 and R_3 and four black balls as B_1, B_2, B_3 and B_4. Then the elementary events associated to this experiment are :*

$$HR_1, HR_2, HR_3, HB_1, HB_2, HB_3, HB_4, T1, T2, T3, T4, T5 \text{ and } T6.$$

The set of these elementary events is the sample space associated to the given random experiment.

<u>REMARK 1</u> *Elementary events associated to a random experiment are also known as indecomposable events.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *From a group of 2 boys and 3 girls, two children are selected. Find the sample space associated to this random experiment.*

SOLUTION Let the two boys be taken as B_1 and B_2 and the three girls be taken as G_1, G_2 and G_3. Clearly, there are 5 children, out of which two children can be chosen in ${}^5C_2$ ways. So, there are ${}^5C_2 = 10$ elementary events associated to this experiments and are given by

$$B_1 B_2, B_1 G_1, B_1 G_2, B_1 G_3, B_2 G_1, B_2 G_2, B_2 G_3, G_1 G_2, G_1 G_3 \text{ and } G_2 G_3$$

Consequently, the sample space S associated to this random experiment is given by

$$S = \{B_1 B_2, B_1 G_1, B_1 G_2, B_1 G_3, B_2 G_1, B_2 G_2, B_2 G_3, G_1 G_2, G_1 G_3, G_2 G_3\}.$$

EXAMPLE 2 *A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red and 4 black balls; if it shows tail, we throw a die. What is the sample associated to this experiment?* **[NCERT]**

SOLUTION Let the three red balls be taken as R_1, R_2, R_3 and four black balls be taken as B_1, B_2, B_3 and B_4.

If the coin shows head, we draw a ball which can be any one of the 7 balls. So, possible outcomes are $(H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4)$.

If the coin shows tail, then we through a die which may produce any one of the six numbers on its upper face. In this case, possible outcomes are $(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$.

Thus, all elementary events associated to the experiment are:

$(H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$.

Consequently, the sample space S is given by

$$S = \{(H, R_1), (H, R_2), (H, R_3), (H, B_1), (H, B_2), (H, B_3), (H, B_4), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

EXAMPLE 3 *An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment.* **[NCERT]**

SOLUTION If the die is rolled and we get an even number (2 or 4 or 6) on its upper face, then we toss a coin which may result in head (H) or tail (T). So the possible outcomes in this case are :

$(2, H), (4, H), (6, H), (2, T), (4, T), (6, T)$

If the die is rolled and we get an odd number (1 or 3 or 5) on its upper face, then the coin is tossed twice which may result in one of the following ways: HH, HT, TH, TT. So, the possible outcomes, in this case, are

$(1, HH), (3, HH), (5, HH), (1, HT), (3, HT), (5, HT), (1, TH), (3, TH), (5, TH), (1, TT), (3, TT), (5, TT)$.

Thus, all elementary events associated to the experiment are:

$(2, H), (4, H), (6, H), (2, T), (4, T), (6, T), (1, HH), (1, HT), (1, TH), (1, TT), (3, HH), (3, HT), (3, TH), (3, TT), (5, HH), (5, HT), (5, TH), (5, TT)$.

So, the sample space associated to the random experiment is

$$S = \{(2, H), (4, H), (6, H), (2, T), (4, T), (6, T), (1, HH), (1, HT), (1, TH), (1, TT), (3, HH), (3, HT), (3, TH), (3, TT), (5, HH), (5, HT), (5, TH), (5, TT)\}.$$

EXAMPLE 4 *The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space for the experiment.* **[NCERT]**

SOLUTION It is given that two slips are drawn from the box one after the other without replacement.

If the slip drawn in first draw bears number 1, then the slip drawn in second draw may bear any one of the remaining 3 numbers viz. 2, 3 and 4. Possible outcomes in this case are $(1, 2), (1, 3)$ and $(1, 4)$.

If the slip drawn in first draw bears number 2, then the slip drawn in second draw may bear any one of the remaining three numbers viz. 1, 3 and 4.

Thus, possible outcomes, in this case, are $(2, 1), (2, 3)$ and $(2, 4)$.

Similarly, possible outcomes when the slip drawn in first draw bears number 3 and 4 are respectively $(3, 1), (3, 2), (3, 4)$ and $(4, 1), (4, 2), (4, 3)$.

Thus, all elementary events associated to the random experiment are $(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2)$ and $(4, 3)$.

The set of all these elementary events is the required sample space.

EXAMPLE 5 *A coin is tossed. If the result is a head, a die is thrown. If the die shows up an even number, the die is thrown again. What is the sample space for this experiment.* **[NCERT]**

SOLUTION A coin is tossed, if the outcome is tail (T). The experiment is over. If the outcome is head (H), a die is thrown and if the die shows up an odd number the experiment is stopped. Possible outcomes in this case are:

$(H, 1), (H, 3), (H, 5)$.

If the die shows up an even number it is thrown again. In this case, possible outcomes are:

$(H, 2, 1), (H, 2, 2), (H, 2, 3), (H, 2, 4)$,
$(H, 2, 5), (H, 2, 6)$
$(H, 4, 1), (H, 4, 2), (H, 4, 3), (H, 4, 4)$,
$(H, 4, 5), (H, 4, 6)$
$(H, 6, 1), (H, 6, 2), (H, 6, 3), (H, 6, 4)$,
$(H, 6, 5), (H, 6, 6)$

So, all elementary events associated to the given experiment are

$T, (H, 1), (H, 3), (H, 5), (H, 2, 1), (H, 2, 2)$,
$(H, 2, 3), (H, 2, 4), (H, 2, 5), (H, 2, 6)$
$(H, 4, 1), (H, 4, 2), (H, 4, 3), (H, 4, 4)$,
$(H, 4, 5), (H, 4, 6)$
$(H, 6, 1), (H, 6, 2), (H, 6, 3), (H, 6, 4)$,
$(H, 6, 5), (H, 6, 6)$

The set of all these elementary events is the required sample space.

REMARK *There are three stages in the above experiment. Possible outcomes at various stages can be depicted as shown in Fig. 33.1.*

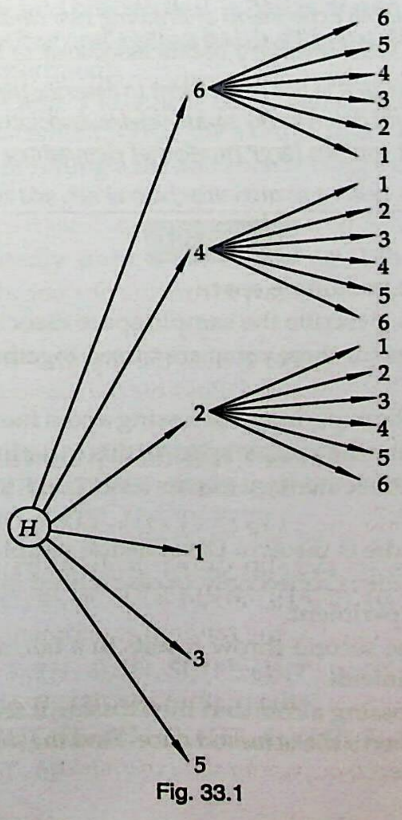

Fig. 33.1

EXAMPLE 6 *A coin is tossed repeatedly until a head comes for the first time. Describe the sample space.*

SOLUTION In this experiment, a coin is tossed. If the outcome is head the experiment is over. Otherwise, the coin is tossed again. In the second toss also if the outcome is head the experiment is over. Otherwise, the coin is tossed again. In the third toss, if the outcome is head the experiment is over, otherwise the coin is tossed again. This process continues indefinitely.

Possible outcomes in various tosses may be exhibited as follows:

Hence, the sample space S associated to this random experiment is

$$S = \{H, TH, TTH, TTTH, TTTTH, ...\}$$

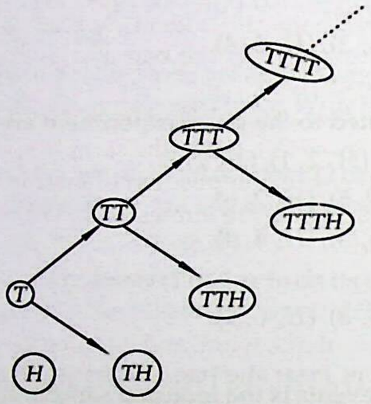

Fig. 33.2

REMARK 1 *In the above example, the sample space is an infinite set.*

REMARK 2 *Let us consider the random experiment of drawing two cards from a well shuffled pack of 52 playing cards. There are ${}^{52}C_2 = 1326$ elementary events associated to this experiment. So, the sample space consists of 1326 elements. Clearly, it is not convenient to describe the sample space completely. In the remaining part of this chapter, we will describe the sample space associated to a given random experiment only if it is convenient and does not contain large number of elementary events.*

EXERCISE 33.1

LEVEL-1

1. A coin is tossed once. Write its sample space
2. If a coin is tossed two times, describe the sample space associated to this experiment.
3. If a coin is tossed three times (or three coins are tossed together), then describe the sample space for this experiment. [NCERT]
4. Write the sample space for the experiment of tossing a coin four times. [NCERT]
5. Two dice are thrown. Describe the sample space of this experiment. [NCERT]
6. What is the total number of elementary events associated to the random experiment of throwing three dice together?
7. A coin is tossed and then a die is thrown. Describe the sample space for this experiment.
8. A coin is tossed and then a die is rolled only in case a head is shown on the coin. Describe the sample space for this experiment.
9. A coin is tossed twice. If the second throw results in a tail, a die is thrown. Describe the sample space for this experiment.
10. An experiment consists of tossing a coin and then tossing it second time if head occurs. If a tail occurs on the first toss, then a die is tossed once. Find the sample space. [NCERT]

11. A coin is tossed. If it shows tail, we draw a ball from a box which contains 2 red 3 black balls; if it shows head, we throw a die. Find the sample space of this experiment.
12. A coin is tossed repeatedly until a tail comes up for the first time. Write the sample space for this experiment.
13. A box contains 1 red and 3 black balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment. **[NCERT]**
14. A pair of dice is rolled. If the outcome is a doublet, a coin is tossed. Determine the total number of elementary events associated to this experiment.
15. A coin is tossed twice. If the second draw results in a head, a die is rolled. Write the sample space for this experiment.
16. A bag contains 4 identical red balls and 3 identical black balls. The experiment consists of drawing one ball, then putting it into the bag and again drawing a ball. What are the possible outcomes of the experiment?
17. In a random sampling three items are selected from a lot. Each item is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.
18. An experiment consists of boy-girl composition of families with 2 children.
    (i) What is the sample space if we are interested in knowing whether it is a boy or girl in the order of their births?
    (ii) What is the sample space if we are interested in the number of boys in a family? **[NCERT]**
19. There are three coloured dice of red, white and black colour. These dice are placed in a bag. One die is drawn at random from the bag and rolled, its colour and the number on its uppermost face is noted. Describe the sample space for this experiment. **[NCERT]**
20. 2 boys and 2 girls are in room P and 1 boy 3 girls are in room Q. Write the sample space for the experiment in which a room is selected and then a person. **[NCERT]**
21. A bag contains one white and one red ball. A ball is drawn from the bag. If the ball drawn is white it is replaced in the bag and again a ball is drawn. Otherwise, a die is tossed. Write the sample space for this experiment.
22. A box contains 1 white and 3 identical black balls. Two balls are drawn at random in succession without replacement. Write the sample space for this experiment.
23. An experiment consists of rolling a die and then tossing a coin once if the number on the die is even. If the number on the die is odd, the coin is tossed twice. Write the sample space for this experiment. **[NCERT]**
24. A die is thrown repeatedly until a six comes up. What is the sample space for this experiment. **[NCERT]**

ANSWERS

1. $S = \{H, T\}$
2. $S = \{HH, HT, TH, TT\}$
3. $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
4. $S = \{HHHH, HHHT, HTHH, THHH, HHTH, HHTT, HTTH, TTHH, THHT, HTHT, THTH, TTTH, TTHT, THTT, HTTT, TTTT\}$
5. $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$ $(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$ $(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$
6. 216
7. $S = \{(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)\}$
8. $S = \{T, (H,1), (H,2), (H,3), (H,4), (H,5), (H,6)\}$
9. $S = \{HH, TH, (HT,1), (HT,2), (HT,3), (HT,4), (HT,5), (HT,6), (TT,1), (TT,2), (TT,3), (TT,4), (TT,5), (TT,6)\}$

10. $S = \{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6), (H, H), (H, T)\}$
11. $S = \{(T, R_1), (T, R_2), (T, B_1), (T, B_2), (T, B_3), (H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6)\}$
12. $S = \{T, HT, HHT, HHHT, HHHHT, \ldots\}$
13. $S = \{(R, B_1), (R, B_2), (R, B_3), (B_1, R), (B_1, B_2), (B_1, B_3), (B_2, B_1), (B_2, B_3), (B_2, R), (B_3, R), (B_3, B_1), (B_3, B_2)\}$
14. 42
15. $\{TT, HT, (TH, 1), (TH, 2), (TH, 3), (TH, 4), (TH, 5), (TH, 6), (HH, 1), (HH, 2), (HH, 3), (HH, 4), (HH, 5), (HH, 6)\}$
16. RR, RB, BR, BB
17. $S = \{DDD, DDN, DND, NDD, DNN, NDN, NND, NNN\}$
18. (i) $S = \{(B_1, B_2), (B_1, G_2), (G_1, B_2), (G_1, G_2)\}$ (ii) $S = \{0, 1, 2\}$
19. $S = \{(R, 1), (R, 2), (R, 3), (R, 4), (R, 5), (R, 6), (B, 1), (B, 2), (B, 3), (B, 4), (B, 5), (B, 6) (W, 1), (W, 2), (W, 3), (W, 4), (W, 5), (W, 6)\}$
20. $S = \{(P, B_1), (P, B_2), (P, G_1), (P, G_2), (Q, B_3), (Q, G_3), (Q, G_4), (Q, G_5)\}$
21. $S = \{(W, W), (W, R), (R, 1), (R, 2), (R, 3), (R, 4), (R, 5), (R, 6)\}$
22. $S = \{WB, BW, BB\}$
23. $S = \{(2, H), (2, T), (4, H), (4, T), (6, H), (6, T), (1, HH), (1, HT), (1, TH), (1, TT), (3, HH), (3, HT), (3, TH), (3, TT), (5, HH), (5, HT), (5, TH), (5, TT)\}$
24. $S = \{6, (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (1, 1, 6), (1, 2, 6), (1, 3, 6), (1, 4, 6), (1, 5, 6), (2, 1, 6), (2, 2, 6), (2, 3, 6), \ldots\}$

33.4 EVENT

In the previous section, we have learnt about sample spaces associated with several random experiments. In this section, we will introduce an important term associated with a random experiment.

EVENT *A subset of the sample space associated with a random experiment is called an event.*

Consider the random experiment of throwing a die. The sample space associated with this experiment is $S = \{1, 2, 3, 4, 5, 6\}$. Clearly, S has $2^6 = 64$ subsets.

Each one of these 64 subsets is an event associated with the random experiment of throwing a die.

For Example, $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{3, 4, 5, 6\}$, $D = \{1, 2, 6\}$ etc. are events as they are subsets of S.

These events A, B and C can also be described in words as follows:

A = Getting an even number, B = Getting an odd number,
C = Getting a number greater than 2

However, there is no general description in words for the event D. Thus, we find that some events associated with a random experiment may be described in words. However, it is not possible for every event.

Consider the experiment of tossing three coins at a time. The sample space S associated with this experiment is $S = \{HHH, HHT, THH, HTH, TTH, THT, HTT, TTT\}$. Let

$A = \{HHT, HTH, THH\}$, $B = \{HHH, HHT, HTH, THH\}$
$C = \{HHH, HHT, HTH, THH, TTH, HTT, THT\}$ and, $D = \{HHH, TTT, HTH\}$

Clearly, A, B, C and D, being subsets of S, are events associated with the random experiment of tossing three coins (or tossing a coin three times). These events can also be described in words as follows:

A = Getting two heads, B = Number of heads exceeds the number of tails,
C = Getting at least one head.

But, event D cannot be described in words.

REMARK *Single element subsets of sample space associated with a random experiment define elementary events associated with the random experiment.*

OCCURRENCE OF AN EVENT *An event A associated to a random experiment is said to occur if any one of the elementary events associated to it is an outcome.*

Thus, if an elementary event E is an outcome of a random experiment and A is an event such that $E \in A$, then we say that the event A has occurred.

Consider the random experiment of throwing an unbiased die. Let A be an event of getting an even number. Then, $A = \{2, 4, 6\}$. Suppose in a trial the outcome is 4. Since $4 \in A$, so we say that the event A has occurred. In another trial, let the outcome be 3, since $3 \notin A$, so we say that in this trial the event A has not occurred.

Suppose a die is thrown and the outcome of the trial is 4. Then, we can say that each of the following events have occurred:

(i) Getting a number greater than or equal to 2, represented by the set $\{2, 3, 4, 5, 6\}$
(ii) Getting a number less than or equal to 5, represented by the set $\{1, 2, 3, 4, 5\}$.

On the basis of the same outcome, we can also say that the following events have not occurred:

(i) Getting an odd number represented by the set $\{1, 3, 5\}$
(ii) Getting a multiple of 3, represented by the set $\{3, 6\}$.

Let us now consider the random experiment of throwing a pair of dice. If (2, 6) is an outcome of a trial, then we can say that each of the following events has occurred:

(i) Getting an even number on first die. (ii) Getting even numbers on both dice.
(iii) Getting 8 as the sum of the numbers on two dice.

However, on the basis of the same outcome, one can also say that following events have not occurred:

(i) Getting a multiple of 3 on first die. (ii) Getting an odd number on first die.
(iii) Getting a doublet.

33.5 ALGEBRA OF EVENTS

In this section, we shall see how new events can be constructed by combining two or more events associated to a random experiment.

Let A and B be two events associated to a random experiment with sample space S. We define the event "A or B" which is said to occur if an elementary event favourable to either A or B or both is an outcome. In other words, the event "A or B" occurs if either A or B or both occur i.e. at least one of A and B occurs. Thus, "A or B" is represented by the subset $A \cup B$ of the sample space S.

For example, in a single throw of a die consider the following events:

A = Getting an even number, B = Getting a multiple of 3.

These two events are described by the sets $\{2, 4, 6\}$ and $\{3, 6\}$ respectively.

Clearly,

$A \cup B$ = Getting a number which is either even or a multiple of 3 or both $= \{2, 3, 4, 6\}$.

Similarly, if A, B and C are three events associated to a random experiment, then $A \cup B \cup C$ denotes the occurrence of at least one of the three events.

The event "A and B" is said to occur if an elementary event favourable to both A and B is an outcome. In other words, the event "A and B" occurs if A and B both occur. The event A and B is denoted by $A \cap B$.

For example, in a single throw of a pair of dice if we define

A = Getting an even number on first-die and, B = Getting 8 as the sum of the numbers on two dice,

Then,

$A \cap B$ = Getting an even number on first die such that the sum of the numbers is 8
$= \{(2, 6), (6, 2), (4, 4)\}$.

NEGATION OF AN EVENT *Corresponding to every event A associated to a random experiment, we define an event " not A" which is said to occur when and only when A does not occur.*

For example, in a single throw of a die if A denotes the event that the outcome is an odd number. Then $A = \{1, 3, 5\}$ and A does not occur if the outcome is any one of the outcomes 2, 4, 6. Thus, the event "not A" is represented by the set $\overline{A}$ and is called the *complementary event of A or negation of A*.

Sometimes the occurrence of one event implies the occurrence of other. For example, in a single throw of a die if A denotes the event that the outcome is 2 or 4 and B denotes the event that the outcome is even. Then, $A = \{2, 4\}$ and $B = \{2, 4, 6\}$. Clearly, the occurrence of A implies the occurrence of B. For if 2 or 4 occurs, we say that the outcome is an even number.

Thus, if the occurrence of an event A implies the occurrence of event B, then we say that "A implies B". Clearly, if A implies B, then we have $A \subset B$.

Verbal description of the event	*Equivalent set theoretic notation*
Not A	$\overline{A}$
A or B (at least one of A or B)	$A \cup B$
A and B	$A \cap B$
A but not B	$A \cap \overline{B}$
Neither A nor B	$\overline{A} \cap \overline{B}$
At least one of A, B or C	$A \cup B \cup C$
Exactly one of A and B	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$
All three of A, B and C	$A \cap B \cap C$
Exactly two of A, B and C	$(A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (\overline{A} \cap B \cap C)$

In the above discussion and in the previous sections, we have seen that the events associated to a random experiment are generally described verbally, and it is very important to have the ability of conversion of verbal description to equivalent set theoretical notations. In the following table, we give verbal descriptions of some events and their equivalent set theoretic notations for ready reference.

ILLUSTRATION *If A, B and C are three arbitrary events. Find the expression for the events noted below, in the context of A, B and C.*

(i) *Only A occurs*
(ii) *Both A and B, but not C occur*
(iii) *All the three events occur*
(iv) *At least one occurs*
(v) *At least two occur*
(vi) *One and no more occurs*
(vii) *Two and no more occur*
(viii) *None occurs*
(ix) *Not more than two occur.*

SOLUTION (i) $A \cap \overline{B} \cap \overline{C}$ (ii) $A \cap B \cap \overline{C}$ (iii) $A \cap B \cap C$ (iv) $A \cup B \cup C$

(v) $(A \cap B) \cup (B \cap C) \cup (A \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

(vii) $(A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$

(viii) $\overline{A} \cap \overline{B} \cap \overline{C} = \overline{A \cup B \cup C}$

(ix) $(A \cap \overline{B} \cap \overline{C}) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C) \cup (A \cap B \cap \overline{C}) \cup (\overline{A} \cap B \cap C) \cup (A \cap \overline{B} \cap C)$.

33.6 TYPES OF EVENTS

Let there be n elementary events associated with a random experiment. Then the corresponding sample space has n elements and hence 2^n subsets. Each subset of S is an event associated to the random experiment and the sample space is the universal set of these events. These 2^n events are divided into different types on the basis of their nature of occurrence. In this section, we shall learn about such types.

CERTAIN (OR SURE) EVENT *An event associated with a random experiment is called a certain event if it always occurs whenever the experiment is performed.*

For example, associated with the random experiment of rolling a die, the event A "Getting an even number or an odd number" is a certain event. Clearly, this event is represented by the set $\{1, 2, 3, 4, 5, 6\}$ which is the sample space of the experiment.

If S is the sample space associated with a random experiment. Then, S, being subset of itself, defines an event. Also, every outcome of the experiment is an element of S, so the event represented by S always occurs whenever we perform the experiment. Consequently, the event represented by S is a certain event.

Thus, the sample space associated with a random experiment defines a certain event.

IMPOSSIBLE EVENT *An event associated with a random experiment is called an impossible event if it never occurs whenever the experiment is performed.*

Consider the experiment of rolling a die. Let A be the event "The number turns up is divisible by 7". Clearly, none of the possible outcomes 1, 2, 3, 4, 5, 6 is divisible by 7. So, the event A cannot occur at all. In other words, there is no outcome belonging to set representing event A. So, the set A is the null set.

If S is the sample space associated with a random experiment, then the null (empty) set ϕ is a subset of S and no outcome of the experiment is a member of ϕ. So, the event represented by ϕ is an impossible event.

COMPOUND EVENT *An event associated with a random experiment is a compound event, if it is the disjoint union of two or more elementary events.*

In other words, an event having more than one sample point is called a compound event.

In fact, other than elementary events and impossible events associated with a random experiment, all events are compound events as they are obtained by combining two or more elementary events.

For example, in a single throw of an ordinary die there are, 6 elementary events and the total number of events is $2^6 = 64$. So, $2^6 - (6+1) = 57$ is the total number of compound events.

<u>REMARK</u> *If there are n elementary events associated to a random experiment, then the sample space associated to it has n elements and so there are 2^n subsets of it. Out of these 2^n subsets there are n single element subsets. These single element subsets define n elementary events and the remaining $2^n - (n+1)$ subsets (excluding null set) define compound events. Some of these compound events can be described in words whereas for others there may not be any general description.*

If a pair of dice is thrown together, then there are 36 elementary events associated to this experiment. The sample space associated to this experiment is:

$$S = \{(1,1), (1,2) \dots, (1,6), (2,1), (2,2) \dots, (2,6), \dots, (6,1), (6,2), \dots, (6,6)\}$$

If we define the event A as "Getting a doublet". i.e. $A = \{(1,1), (2,2), \dots, (6,6)\}$

Clearly, it is a compound event obtained by combining 6 elementary events.

Similarly, the event B given by "Getting 8 as the sum" can be written as

$$B = \{(2, 6), (6, 2), (4, 4), (3, 5), (5, 3)\}$$

It is also a compound event obtained by combining 5 elementary events.

MUTUALLY EXCLUSIVE EVENTS *Two or more events associated with a random experiment are said to be mutually exclusive or incompitable events if the occurrence of any one of them prevents the occurrence of all others i.e., if no two or more of them can occur simultaneously in the same trial.*

Clearly, elementary events associated with a random experiment are always mutually exclusive, because elementary events are outcomes (results) of an experiment when it is performed and at a time only one outcome is possible.

Consider the random experiment of rolling a die. Let A, B, C be three events associated with the experiment as given below:

A = Getting an even number, B = Getting an odd number, C = Getting a multiple of 3.

These events in set theoretical notations are: $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$ and $C = \{3, 6\}$.

Clearly, $A \cap B = \phi$, $A \cap C \neq \phi$, $B \cap C \neq \phi$ and $A \cap B \cap C = \phi$.

So, A and B are mutually exclusive events but A and C as well as B and C are not mutually exclusive. However, A, B and C taken all the three together are mutually exclusive events.

In the experiment of throwing a pair of dice events A = Getting 8 as the sum and B = Getting an even number on first die are not mutually exclusive, because $A \cap B = \{(2, 6), (6, 2), (4, 4)\} \neq \phi$

Let two cards be drawn from a well-shuffled pack of 52 cards. Consider the following events:

A = Getting both red cards, B = Getting both black cards.

Clearly, A and B are mutually exclusive events because two cards drawn cannot be both red and black at the same time.

EXHAUSTIVE EVENTS *Two or more events associated with a random experiment are exhaustive if their union is the sample space i.e. events $A_1, A_2, \ldots, A_n$ associated with a random experiment with sample space S are exhaustive if $A_1 \cup A_2 \cup \ldots \cup A_n = S$.*

Thus, a set of events associated with a random experiment is an exhaustive set of events if one of them necessarily occurs whenever the experiment is performed.

It is evident from the above definition that all elementary events associated with a random experiment form a set of exhaustive events.

Consider the experiment of drawing a card from a well shuffled deck of playing cards. Let A be the event " card is red", B be the event "card is black." Clearly, A and B are exhaustive events because $A \cup B = S$.

In a single throw of an ordinary die, let us consider the following events:

A_1 = Getting an even number = $\{2, 4, 6\}$, A_2 = Getting an odd number = $\{1, 3, 5\}$,

A_3 = Getting a multiple of 3 = $\{3, 6\}$, A_4 = Getting a number greater than 3 = $\{4, 5, 6\}$

We observe that $A_1 \cup A_2 = S$. Also, $A_1 \cup A_2 \cup A_3 \cup A_4 = S$. But, $A_1 \cup A_3 \neq S$. So, A_1 and A_2 are exhaustive events. Also, A_1, A_2, A_3, A_4 are exhaustive events but A_1 and A_3 are not exhaustive events.

MUTUALLY EXCLUSIVE AND EXHAUSTIVE SYSTEM OF EVENTS *Let S be the sample space associated with a random experiment. A set of events $A_1, A_2, \ldots, A_n$ is said to form a set of mutually exclusive and exhaustive system of events if*

(i) $A_1 \cup A_2 \ldots \cup A_n = S$ *i.e. events $A_1, A_2, \ldots, A_n$ form an exhaustive set of events.*

(ii) $A_i \cap A_j = \phi$ for $i \neq j$ *i.e. events $A_1, A_2, \ldots, A_n$ are mutually exclusive.*

Clearly, elementary events associated with a random experiment always form a system of mutually exclusive and exhaustive events.

In a single throw of a die, the events A = Getting an even number and, B = Getting an odd number are mutually exclusive and exhaustive events.

Consider the experiment of drawing a card from a well-shuffled deck of 52 playing cards. Let A_1, A_2, A_3, A_4 be four events defined as follows:

A_1 = Card drawn is spades, A_2 = Card drawn is clubs,

A_3 = Card drawn is hearts, A_4 = Card drawn is diamonds

Since the card drawn is one of the four types of cards, so one of these events surely occurs whenever the experiment is performed. Also, if one of these events occurs, the others cannot occur. So, A_1, A_2, A_3 and A_4 form a mutually exclusive and exhaustive system of events.

Suppose a die is thrown once. Let A be the event "Getting a number greater than 3", B be the event "Getting a number less than 5". Then, $A = \{4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$. Clearly, $A \cup B = S$ and $A \cap B = \{4\} \neq \phi$. So, events A and B are exhaustive but not mutually exclusive.

FAVOURABLE ELEMENTARY EVENTS *Let S be the sample space associated with a random experiment and A be an event associated with the experiment. Then, elementary events belonging to A are known as favourable elementary events to the event A.*

Thus, an elementary event E is favourable to an event A if the occurrence of E ensures the happening or occurrence of event A.

In a single throw of an ordinary die, let A be the event "Getting a multiple of 3". Clearly, $A = \{3, 6\}$. So, there are two elementary events favourable to A.

Consider the random experiment of throwing a pair of dice. Let A be the event "Getting 8 as the sum". Then, $A = \{(2, 6), (6, 2), (4, 4), (5, 3), (3, 5)\}$. Clearly, A occurs if any one of the elementary events $(2, 6), (6, 2), (4, 4), (5, 3)$, and $(3, 5)$ is an outcome of the experiment. So, all these elementary events are favourable to event A.

Consider a random experiment of drawing 4 cards from a well-shuffled deck of 52 playing cards. There are ${}^{52}C_4$ elementary events associated with this experiment as 4 cards can be drawn out of 52 cards in ${}^{52}C_4$ ways. Let A be the event "Getting all red cards". There are ${}^{26}C_4$ elementary events favourable to A, because 4 red cards can be chosen out of 26 red cards in $26C_4$ ways. In this case, it is not convenient to list all favourable elementary events.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *An experiment consists of rolling die until a 2 appears.*

(i) How many elements of the sample space correspond to the event that the 2 appears on the k^{th} roll of the die?

(ii) How many elements of the sample space correspond to the event that 2 appears not later than the k^{th} roll of the die? **[NCERT EXEMPLAR]**

SOLUTION (i) 2 appears on k^{th} roll of the die means that each one of the first $(k-1)$ rolls have 5 outcomes (1, 3, 4, 5, 6) and k^{th} roll results in 1 outcome i.e. 2.

$\therefore$ Number of elements of the sample space in the event $= \underbrace{5 \times 5 \times \ldots \times 5}_{(k-1) \text{ times}} \times 1 = 5^{k-1}$

(ii) 2 appears not later than k^{th} roll means that 2 may appear in the first roll or in second roll or in third roll, ..., or in k^{th} roll.

From (i), the number of elements of the sample space corresponding to the event that 2 appears on the k^{th} roll of the die is 5^{k-1}.

Hence, The number of elements of the sample space corresponding to the event that 2 appears not later than the k^{th} roll of the die

$$=5^{1-1}+5^{2-1}+5^{3-1}+\ldots+5^{k-1}=1+5+5^2+\ldots+5^{k-1}=\left(\frac{5^k-1}{5-1}\right)=\frac{5^k-1}{4}.$$

EXAMPLE 2 *An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events.* **[NCERT]**

A = the sum is greater than 8, $B = 2$ occurs on either die., C = the sum is at least 7 and a multiple of 3.

Also, find $A \cap B$, $B \cap C$ and $A \cap C$.

Are: (i) *A and B mutually exclusive?* (ii) *B and C mutually exclusive?*

(iii) *A and C mutually exclusive?*

SOLUTION The sample space associated with the given random experiment is given by

$$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

We have,

A = The sum is greater than 8

$\Rightarrow$ $A = \{(3,6),(4,5),(5,4),(6,3),(4,6),(6,4),(5,5),(6,5),(5,6),(6,6)\}$

$B = 2$ occurs on either die

$\Rightarrow$ $B = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)\}$.

and, C = The sum is atleast 7 and a multiple of 3

$\Rightarrow$ C = The sum is 9 or, 12. $= \{(3,6),(6,3),(4,5),(5,4),(6,6)\}$.

(i) Clearly, $A \cap B = \phi$. So, A and B are mutually exclusive events.

(ii) Clearly, $B \cap C = \phi$. So, B and C are mutually exclusive events.

(iii) Clearly, $A \cap C = \{(3,6),(6,3),(4,5),(5,4),(6,6)\} \neq \phi$. So, A and C are not mutually exclusive events.

EXAMPLE 3 *From a group of 2 boys and 3 girls, two children are selected at random. Describe the events.*

(i) *A = both selected children are girls.* (ii) *B = the selected group consists of one boy and one girl.*

(iii) *C = at least one boy is selected.*

Which pair (s) of events is (are) mutually exclusive?

SOLUTION Let B_1, B_2 be two boys and G_1, G_2, G_3 be three girls. Then, the sample space associated with the random experiment is

$$S = \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3, G_2G_3\}$$

(i) We have,

A = Both selected children are girls $= \{G_1G_2, G_1G_3, G_2G_3\}$

(ii) We have,

B = The selected group consists of one boy and one girl.

$\Rightarrow$ $B = \{B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$

(iii) We have,

C = At least one boy is selected $= \{B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3\}$

Clearly, $A \cap B = \phi$ and $A \cap C = \phi$. So, A and B, A and C are two pairs of mutually exclusive events.

EXAMPLE 4 *Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events:* **[NCERT]**

A = *The sum is even*, B = *The sum is multiple of* 3, C = *The sum is less than* 4,

D = *The sum is greater than* 11

Which pairs of these events are mutually exclusive?

SOLUTION The sample space associated with the random experiment is given in example 1. We have,

A = The sum is even

= The sum is either 2 or 4 or 6, or 8 or 10 or 12

$= \{(1,1),(2,2),(1,3),(1,5),(2,4),(2,6),(3,1),(3,3),(3,5),(4,2),(4,4),(4,6),(5,1),(5,3),(5,5),(6,2),(6,4),(6,6)\}$

B = The sum is a multiple of 3

= The sum is either 3 *or* 6 *or* 9 *or* 12

$= \{(1,2),(2,1),(1,5),(5,1),(2,4),(4,2),(3,3),(3,6),(6,3),(4,5),(5,4),(6,6)\}$

C = The sum is less than 4 = (The sum is 2 or 3 = $\{(1,1),(1,2),(2,1)\}$

D = The sum is greater than 11 = The sum is 12 = $\{(6,6)\}$

We observe that $A \cap B = \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,6)\} \neq \phi$. So, A and B are not mutually exclusive events.

Similarly, we observe that $A \cap C \neq \phi$, $A \cap D \neq \phi$, $B \cap C \neq \phi$, $B \cap D \neq \phi$ and, $C \cap D = \phi$. Hence, C and D are mutually exclusive events.

EXERCISE 33.2

LEVEL-1

1. A coin is tossed. Find the total number of elementary events and also the total number events associated with the random experiment.
2. List all events associated with the random experiment of tossing of two coins. How many of them are elementary events?
3. Three coins are tossed once. Describe the following events associated with this random experiment: **[NCERT]**

   A = Getting three heads, B = Getting two heads and one tail,

   C = Getting three tails, D = Getting a head on the first coin.

   (i) Which pairs of events are mutually exclusive?

   (ii) Which events are elementary events?

   (iii) Which events are compound events?
4. In a single throw of a die describe the following events:

   (i) A = Getting a number less than 7 (ii) B = Getting a number greater than 7

   (iii) C = Getting a multiple of 3 (iv) D = Getting a number less than 4

   (v) E = Getting an even number greater than 4 (vi) F = Getting a number not less than 3.

   Also, find $A \cup B$, $A \cap B$, $B \cap C$, $E \cap F$, $D \cap F$ and $\overline{F}$.
5. Three coins are tossed. Describe

   (i) two events A and B which are mutually exclusive. **[NCERT]**

   (ii) three events A, B and C which are mutually exclusive and exhaustive.

   (iii) two events A and B which are not mutually exclusive.

   (iv) two events A and B which are mutually exclusive but not exhaustive.

6. A die is thrown twice. Each time the number appearing on it is recorded. Describe the following events:
   (i) A = Both numbers are odd. (ii) B = Both numbers are even.
   (iii) C = sum of the numbers is less than 6
   Also, find $A \cup B$, $A \cap B$, $A \cup C$, $A \cap C$. Which pairs of events are mutually exclusive?
7. Two dice are thrown. The events A, B, C, D, E and F are described as follows:
   A = Getting an even number on the first die.
   B = Getting an odd number on the first die.
   C = Getting at most 5 as sum of the numbers on the two dice.
   D = Getting the sum of the numbers on the dice greater than 5 but less than 10.
   E = Getting at least 10 as the sum of the numbers on the dice.
   F = Getting an odd number on one of the dice.
   (i) Describe the following events: A and B, B or C, B and C, A and E, A or F, A and F
   (ii) State true or false:
   (a) A and B are mutually exclusive.
   (b) A and B are mutually exclusive and exhaustive events.
   (c) A and C are mutually exclusive events.
   (d) C and D are mutually exclusive and exhaustive events.
   (e) C, D and E are mutually exclusive and exhaustive events.
   (f) A' and B' are mutually exclusive events.
   (g) A, B, F are mutually exclusive and exhaustive events.
8. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the following events:
   A = The number on the first slip is larger than the one on the second slip.
   B = The number on the second slip is greater than 2
   C = The sum of the numbers on the two slips is 6 or 7
   D = The number on the second slips is twice that on the first slip.
   Which pair(s) of events is (are) mutually exclusive?
9. A card is picked up from a deck of 52 playing cards.
   (i) What is the sample space of the experiment?
   (ii) What is the event that the chosen card is black faced card?

ANSWERS

1. 2, 4
2. $\{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, HT, TT\}, \{HH, TH, TT\}, \{HT, TH, TT\}, \{HH, HT, TH, TT\}$ .4
3. $A = \{HHH\}, B = \{HHT, THH, HTH\}, C = \{TTT\}, D = \{HHH, HHT, HTH, HTT\}$
   (i) $A, B;\ A, C;\ B, C;\ C, D$ (ii) A and C (iii) B and D
4. (i) $A = \{1, 2, 3, 4, 5, 6\}$ (ii) ϕ (iii) $C = \{3, 6\}$
   (iv) $D = \{1, 2, 3\}$ (v) $E = \{6\}$ (vi) $F = \{3, 4, 5, 6\}$
   $A \cup B = \{1, 2, 3, 4, 5, 6\}, A \cap B = \phi, B \cap C = \phi, E \cap F = \{6\}, D \cap F = \{3\}, \overline{F} = \{1, 2\}$
5. (i) A = Getting at least two heads, B = Getting at least two tails.
   (ii) A = Getting at most one head, B = Getting exactly two heads,
   C = Getting exactly three heads.
   (iii) A = Getting at most two tails, B = Getting exactly two heads
   (iv) A = Getting exactly one head, B = Getting exactly two heads.

6. (i) $A = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5)\}$

(ii) $B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

(iii) $C = \{(1,1),(1,2),(2,1),(1,3),(3,1),(2,2),(1,4),(4,1),(2,3),(3,2)\}$

$A \cup B = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(2,2),(2,4),(2,6),$
$(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$A \cap B = \phi$

$A \cup C = \{(1,1),(1,3),(1,5),(3,1),(3,3),(3,5),(5,1),(5,3),(5,5),(1,2),(1,4),(2,1),$
$(2,2),(2,3),(3,2),(4,1)\}$

$A \cap C = \{(1,1),(1,3),(3,1)\}$ and $B \cap C = \phi$.

A and B, B and C are pairs of mutually exclusive events.

7. (i) $A \cap B = \phi$

$B \cup C = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(3,1),(3,2)$
$(3,3),(3,4),(3,5),(3,6),(4,1),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\}$

$B \cap C = \{(1,1),(1,2),(1,3),(1,4)\ (3,1),(3,2)\}$

$A \cap E = \{(4,6),(6,4),(6,5),(6,6)\}$

$A \cap F = \{(1,2),(1,4),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(3,4),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,2),(5,4),(5,6),(6,1),(6,2),(6,3),$
$(6,4),(6,5),(6,6)\}$

$A \cap F = \{(2,1),(2,3),(2,5),(4,1),(4,3),(4,5),(6,1),(6,3),(6,5)\}$

(ii) (a) True, (b) True (c) False (d) False (e) True (f) True (g) False

8. $A = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$, $B = \{(1,3),(2,3),(1,4),(2,4),(3,4),(4,3)\}$

$C = \{(2,4),(3,4),(4,2),(4,3)\}$, $D = \{(1,2),(2,4)\}$

A and D form a pair of mutually exclusive events.

9. (i) The sample space is the set of 52 cards.

(ii) Required event is the set of jack, king and queen of spades and clubs.

HINTS TO NCERT & SELECTED PROBLEMS

3. We have,

$A = \{HHH\}$, $B = \{HHT, HTH, THH\}$, $C = \{TTT\}$, $D = \{HHH, HHT, HTH, HTT\}$

(i) We observe that $A \cap B = \phi$, $A \cap C = \phi$ but $A \cap D \neq \phi$

So, A, B and A, C are pair of simutally exclusive events.

Also, $B \cap C = \phi$ but $B \cap D \neq \phi$. So, B, C is a pair of mutually exclusive events.

Finally, $C \cap D = \phi$. So, C and D form a pair of mutally exclusive events.

(ii) Clearly, HHH and TTT may be outcomes of the random experiment of tossing three coins. So, A and C are elementary events.

(iii) We observe that events B and D are obtained by combining more than one elementary events. So, B and D are compound events.

5. The smple space associated to the random experiment of tossing three coins is

$S = \{HHH, HHT, THH, HTH, HTT, THT, TTH, TTT\}$

(i) Clearly, $A = \{HHT, THH, HHT\}$ and $B = \{TTH, THT, HTT\}$ are mutually exclusive events.

(ii) We observe that $A = \{HHH, TTT\}$, $B = \{HHT, HTH, THH\}$ and $C = \{HTT, THT, TTH\}$ are exhaustive and mutually exclusive events. Because, $A \cap B = \phi = B \cap C = C \cap A$ and $A \cup B \cup C = S$.

(iii) We observe that the events $A = \{HHH, HHT, HTH, THH\}$ and $B = \{HHT, THH, HTH, HTT, THT, TTH, TTT\}$ are not mutuually exclusive, because $A \cap B \neq \phi$.

(iv) Events $A = \{HHT, HTH, THH\}$ $B = \{TTT, TTH, HTT, THT\}$ are mutually exclusive but not exhausive as $A \cap B = \phi$ but $A \cup B \neq S$.

33.7 AXIOMATIC APPROACH TO PROBABILITY

The axiomatic approach to probability is deduced from the mathematical concepts laid down in the previous sections. It is based upon certain axioms. In this approach, for a given sample space associated to a random experiment, the probability is considered as a function which assigns a non-negative real number $P(A)$ to every event A. This non-negative real number is called the probability of the event A.

PROBABILITY FUNCTION *Let $S = \{w_1, w_2, w_3, ..., w_n\}$ be the sample space associated to a random experiment. Then a function P which assigns every event $A \subset S$ to a unique non-negative real number $P(A)$ is called the probability function, if the following axioms hold:*

Axiom 1: $0 \leq P(w_i) \leq 1$ for all $w_i \in S$

Axiom 2: $P(S) = 1$ i.e., $P(w_1) + P(w_2) + ... + P(w_n) = 1$.

Axiom 3: For any event $A \subset S$, $P(A) = \sum_{w_k \in A} P(w_k)$, the number $P(w_k)$ is called the probability of elementary event w_k.

Consider the experiment 'tossing a coin'. The sample space associated to this random experiment is $S = \{H, T\}$. If we assign the number $\frac{1}{2}$ to each of the outcomes (elementary events) H and T i.e. $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$. Then this assignment satisfies first two axioms i.e. $0 \leq P(H) \leq 1$, $0 \leq P(T) \leq 1$, and $P(H) + P(T) = \frac{1}{2} + \frac{1}{2} = 1$. So, P is the probability function on S and we can say that the probability of getting head is $\frac{1}{2}$ and the probability of getting tail is also $\frac{1}{2}$. If we assign the number $\frac{1}{4}$ to H and $\frac{3}{4}$ to T i.e. $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. This assignment also defines a probability function on $S = \{H, T\}$. However, if we take $P(H) = \frac{1}{8}$ and $P(T) = \frac{3}{8}$. Then this assignment is not a probability function on $S = \{H, T\}$.

ILLUSTRATION *Let $S = \{w_1, w_2\, w_3\, w_4, w_5, w_6\}$ be a sample space. Which of the following assignments of probability to each outcome are valid?* **[NCERT]**

Outcomes or Elementary events:		w_1	w_2	w_3	w_4	w_5	w_6
Probabilities:	(i)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
	(ii)	1	0	0	0	0	0
	(iii)	$\frac{1}{8}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{4}$
	(iv)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{3}{2}$
	(v)	0.1	0.2	0.3	0.4	0.5	0.6

SOLUTION (i) We have, $P(w_i) = \frac{1}{6}$, $i = 1, 2, 3, 4, 5, 6$.

$\therefore$ $0 \leq P(w_i) \leq 1$ for all $w_i \in S$

and, $P(w_1) + P(w_2) + ... + P(w_6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

Hence, the assignment of the probabilities is valid.

(ii) We have,

$$P(w_1) = 1, P(w_2) = P(w_3) = P(w_4) = P(w_5) = P(w_6) = 0$$

$\therefore$ $0 \le P(w_i) \le 1$ for all $w_i \in S$

Also, $P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_5) + P(w_6) = 1$

Hence, the assignment of the probabilities is valid.

(iii) We have,

$$P(w_5) = -\frac{1}{4} < 0 \text{ and } P(w_6) = -\frac{1}{3} < 0.$$

So, axiom 1 is not satisfied. Hence, the assignment of the probabilities is not valid.

(iv) We have, $P(w_6) = \frac{3}{2} > 1.$

So, axiom 1 is not satisfied. Hence, the assignment of the probabilities is not valid.

(v) We observe that axiom 1 is satisfied. But,

$$P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_5) + P(w_6) = 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 = 2.1 \ne 1.$$

So, axiom 2 is not satisfied. Hence, the assignment is not valid.

Let $S = \{w_1, w_2, \dots, w_n\}$ be the sample space associated to a random experiment such that all the outcomes (elementary events) $w_1, w_2, w_3, \dots, w_n$ are equally likely to occur i.e. the chance of occurrence of each elementary event is same. i.e. $P(w_i) = p$ for all $w_i \in S$ where $0 \le p \le 1$.

Using axiom 2, we have

$$\sum_{i=1}^{n} P(w_i) = 1 \Rightarrow \underbrace{p + p + \dots + p}_{(n\text{-times})} = 1 \Rightarrow np = 1 \Rightarrow p = \frac{1}{n}$$

If A is an event such that m elementary events are favourable to A. Then,

$$P(A) = \sum_{w_k \in A} P(w_k)$$

$$\Rightarrow \quad P(A) = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{m\text{-times}}$$

$$\Rightarrow \quad P(A) = \frac{m}{n} = \frac{\text{Favourable number of elementary events}}{\text{Total number of elementary events}}$$

Thus, we have the following definition of probability of an event when all the elementary events are equally likely to occur.

PROBABILITY OF AN EVENT *If there are n elementary events associated with a random experiment and m of them are favourable to an event A, then the probability of happening or occurrence of A is denoted by P (A) and is defined as the ratio* $\frac{m}{n}$.

Thus, $P(A) = \frac{m}{n}$

Clearly, $0 \le m \le n$. Therefore,

$$0 \le \frac{m}{n} \le 1 \Rightarrow 0 \le P(A) \le 1$$

If $P(A) = 1$, then A is called certain event and A is called an impossible event, if $P(A) = 0$.

The number of elementary events which will ensure the non-occurrence of A i.e. which ensure the occurrence of $\overline{A}$ is $(n - m)$. Therefore,

$$P(\overline{A}) = \frac{n-m}{n}$$

$\Rightarrow \quad P(\overline{A}) = 1 - \dfrac{m}{n}$

$\Rightarrow \quad P(\overline{A}) = 1 - P(A)$

$\Rightarrow \quad P(A) + P(\overline{A}) = 1$

The odds in favour of occurrence of the event A are defined by $m : (n - m)$ i.e. $P(A) : P(\overline{A})$ and the odds against the occurrence of A are defined by $n - m : m$ i.e. $P(\overline{A}) : P(A)$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED ON CONSTRUCTION OF SAMPLE SPACE**

EXAMPLE 1 *Find the probability of getting a head in a toss of an unbiased coin.*

SOLUTION The sample space associated with the random experiment is $S = \{H, T\}$.

$\therefore$ Total number of elementary events = 2.

We observe that there are two elementary events viz. H, T associated to the given random experiment. Out of these two elementary events only one is favourable i.e. H.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \dfrac{1}{2}$.

EXAMPLE 2 *In a simultaneous toss of two coins, find the probability of getting:*

(i) *2 heads* (ii) *exactly one head* (iii) *exactly 2 tails*
(iv) *exactly one tail* (v) *no tails.*

SOLUTION The sample space associated to the given random experiment is given by

$S = \{HH, HT, TH, TT\}$

Clearly, there are 4 elements in S.

$\therefore$ Total number of elementary events = 4.

(i) There is only one elementary event i.e. HH favourable to the given event

So, required probability $= \dfrac{1}{4}$.

(ii) We observe that exactly one head can be obtained in two ways: HT or, TH.

So, favourable number of elementary events = 2.

Hence, required probability $= \dfrac{2}{4} = \dfrac{1}{2}$.

(iii) Exactly 2 tails can be obtained in one way i.e. TT. So, favourable number of elementary events = 1.

Hence, required probability $= \dfrac{1}{4}$.

(iv) Exactly one tail can be obtained in one of the following two ways: HT, TH

$\therefore$ Favourable number of elementary events = 2.

Hence, required probability $= \dfrac{2}{4} = \dfrac{1}{2}$.

(v) There is only one elementary event viz. HH favourable to the event "getting no tails".

So, required probability $= \dfrac{1}{4}$.

EXAMPLE 3 *Three coins are tossed once. Find the probability of getting:*

(i) *all heads* (ii) *at least two heads* (iii) *at most two heads*
(iv) *no heads* (v) *exactly one tail* (vi) *exactly 2 tails*
(vii) *a head on first coin.*

SOLUTION Let S be the sample space associated with the random experiment of tossing three coins. Then, $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Clearly, there are 8 elements in S.

$\therefore$ Total number of elementary events = 8.

(i) There is only one elementary event, namely HHH, favourable to the given event.

$\therefore$ Required probability $= \frac{1}{8}$.

(ii) At least two heads can be obtained if we obtain one of the following elementary events as an outcome: HHH, HHT, HTH, THH

$\therefore$ Favourable number of elementary events = 4.

Hence, required probability $= \frac{4}{8} = \frac{1}{2}$.

(iii) At most two heads can be obtained in any one of the following ways:

$HHT, THH, HTH, HTT, THT, TTH, TTT$

$\therefore$ Favourable number of elementary events $= \frac{7}{8}$.

(iv) "Getting no heads" means "Getting all tails". So, there is only one elementary event viz. TTT favourable to the given event.

Hence, required probability $= \frac{1}{8}$.

(v) Elementary events favourable to "Getting exactly one tail" are: HHT, THH, HTH.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$.

(vi) Elementary events favourable to "Exactly 2 tails" are: HTT, THT, TTH.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$.

(vii) A head on first coin can be obtained in one of the following ways: HTT, HHH, HTH, HHT.

$\therefore$ Favourable number of elementary events = 4.

Hence, required probability $= \frac{4}{8} = \frac{1}{2}$.

EXAMPLE 4 *A die is thrown. Find the probability of getting:*

(i) *an even number* (ii) *a prime number*

(iii) *a number greater than or equal to 3* (iv) *a number less than or equal to 4*

(v) *a number more than 6* (vi) *a number less than or equal to 6.*

SOLUTION The sample space associated with the random experiment of rolling a die is given by $S = \{1, 2, 3, 4, 5, 6\}$. Clearly, there are 6 elements in S.

$\therefore$ Total number of elementary events = 6.

(i) An even number is obtained, if we obtain any one of 2, 4, 6 as an outcome.

So, favourable number of elementary events = 3.

Hence, required probability $= \frac{3}{6} = \frac{1}{2}$.

(ii) A prime number is obtained, if we get any one of 2, 3, 5 as an outcome.

So, favourable number of elementary events = 3.

Hence, required probability $= \frac{3}{6} = \frac{1}{2}$.

(iii) A number greater than or equal to 3 is obtained, if we get any one of 3, 4, 5, 6 as an outcome.

So, favourable number of elementary events = 4

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$.

(iv) A number less than or equal to 4 is obtained, if we get any one of 1, 2, 3, 4 as an outcome.

So, favourable number of elementary events = 4

Hence, required probability $= \frac{4}{6} = \frac{2}{3}$.

(v) Since no face of the die is marked with a number greater than 6.

So, favourable number of elementary events = 0

Hence, required probability $= \frac{0}{6} = 0$.

In fact, the given event is an impossible event. So, probability of its occurrence is zero.

(vi) Since every face of a die is marked with a number less than or equal to 6.

So, favourable number of elementary events = 6.

Hence, required probability $= \frac{6}{6} = 1$.

In fact, the given event is a certain event. So, probability of its occurrence is 1.

EXAMPLE 5 *Two dice are thrown simultaneously. Find the probability of getting:*

(i) *an even number as the sum* (ii) *the sum as a prime number*

(iii) *a total of at least 10* (iv) *a doublet of even number*

(v) *a multiple of 2 on one dice and a multiple of 3 on the other dice*

(vi) *same number on both dice* (vii) *a multiple of 3 as the sum*

SOLUTION When two dice are thrown together the sample space S associated with the random experiment is given by

$$S = \{(1,1), (1,2), \ldots, (1,6), (2,1), (2,2), \ldots, (2,6), (3,1), (3,2), \ldots, (3,6), (4,1), (4,2), \ldots, (4,6), (5,1), (5,2), \ldots, (5,6), (6,1), (6,2), \ldots, (6,6)\}$$

Clearly, total number of elementary events = 36.

(i) Let A be the event "getting an even number as the sum" i.e., 2, 4, 6, 8, 10, 12 as the sum. Then,

$$A = \{(1,1), (1,3), (3,1), (2,2), (1,5), (5,1), (3,3), (2,4), (4,2), (3,5), (5,3), (4,4), (6,2), (2,6), (5,5), (6,4), (4,6), (6,6)\}$$

$\therefore$ Favourable number of elementary events = 18

So, required probability $= \frac{18}{36} = \frac{1}{2}$.

(ii) Let A be the event "getting the sum as a prime number.. i.e., 2, 3, 5, 7, 11 as the sum. Then,

$$A = \{(1,1), (1,2), (2,1), (1,4), (4,1), (2,3), (3,2), (1,6), (6,1), (2,5), (5,2), (3,4), (4,3), (6,5), (5,6)\}$$

$\therefore$ Favourable number of elementary events = 15

So, required probability $= \frac{15}{36} = \frac{5}{12}$.

(iii) Let A be the event "getting a total of at least 10" *i.e.*, 10, 11, 12 as the sum. Then,

$$A = \{(6,4), (4,6), (5,5), (6,5), (5,6), (6,6)\}$$

$\therefore$ Favourable number of elementary events = 6

So, required probability $= \dfrac{6}{36} = \dfrac{1}{6}$.

(iv) Let A be the event "getting a doublet of even number". Then, $A = \{(2, 2), (4, 4), (6, 6)\}$

$\therefore$ Favourable number of elementary events = 3

So, required probability $= \dfrac{3}{36} = \dfrac{1}{12}$.

(v) Let A be the event "getting a multiple of 2 on one die and a multiple of 3 on the other". Then,

$$A = \{(2, 3), (2, 6), (4, 3), (4, 6), (6, 3), (6, 6), (3, 2), (3, 4), (3, 6), (6, 2), (6, 4)\}$$

$\therefore$ Favourable number of elementary events = 11

So, required probability $= \dfrac{11}{36}$.

(vi) Let A be the event "getting the same number on both the dice". Then,

$$A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$\therefore$ Favourable number of elementary events = 6

So, required probability $= \dfrac{6}{36} = \dfrac{1}{6}$.

(vii) Let A be the event " getting a multiple of 3 as the sum" 3, 6, 9, 12 as the sum. Then,

$$A = \{(1, 2), (2, 1), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (3, 6), (6, 3), (5, 4), (4, 5), (6, 6)\}$$

$\therefore$ Favourable number of elementary events = 12

So, required probability $= \dfrac{12}{36} = \dfrac{1}{3}$.

EXAMPLE 6 *A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed, find the probability that the sum of numbers that turn up is (i) 3 (ii) 12.* **[NCERT]**

SOLUTION The sample space S associated to the given random experiment is given by

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$\therefore$ Total number of elementary events = 12.

(i) Let A be the event that the sum of the number is 3. Then, $A = \{(1, 2)\}$

$\therefore$ Favourable number of elementary events = 1

Hence, required probability = $P(A) = \dfrac{1}{12}$

(ii) Let B denote the event that the sum of the numbers is 12. Then, $B = \{(6,6)\}$.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability = $\dfrac{1}{12}$

EXAMPLE 7 *Suppose each child born is equally likely to be a boy or a girl. Consider the family with exactly three children.*

(a) List the eight elements in the sample space whose outcomes are all possible gender of three children.

(b) Write each of the following events as a set and find its probability:

(i) The event that exactly one child is a girl. (ii) *The event that at least two children are girls.*

(iii) The event that no child is a girl.

SOLUTION (a) All possible genders of three children are:

BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG

So the sample space S is given by $S = \{$ *BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG* $\}$.

(b) (i) Let A denote the event "Exactly one child is a girl". Then, $A = \{BBG, BGB, GBB\}$

Clearly, favourable number of elementary events to A is 3.

Hence, $P(A) = \frac{3}{8}$.

(ii) Let B denote the event that at least two children are girls. Then, $B = \{BGG, GBG, GGB, GGG\}$

Clearly, favourable number of elementary events to B is 4.

Hence, $P(B) = \frac{4}{8} = \frac{1}{2}$

(iii) Let C denote the event: "No child is a girl". Then,

$C = \{BBB\}$ and favourable number of elementary events to C is 1.

Hence, $P(C) = \frac{1}{8}$.

EXAMPLE 8 *Find the probability that a leap year, selected at random, will contain 53 Sundays.*

SOLUTION In a leap year there are 366 days.

366 days = 52 weeks and 2 days.

Thus, a leap year has always 52 Sundays. The remaining 2 days can be:

(i) Sunday and Monday, (ii) Monday and Tuesday, (iii) Tuesday and Wednesday, (iv) Wednesday and Thursday, (v) Thursday and Friday, (vi) Friday and Saturday, (vii) Saturday and Sunday.

If S is the sample space associated with this experiment, then S consists of the above seven points.

$\therefore$ Total number of elementary events = 7.

Let A be the event that a leap year has 53 Sundays. In order that a leap year, selected at random, should contain 53 Sundays, one of the 'over' days must be a Sunday. This can be in any one of the following two ways:

(i) Sunday and Monday (ii) Saturday and Sunday

$\therefore$ Favourable number of elementary events = 2.

Hence, required probability = $\frac{2}{7}$.

EXAMPLE 9 *Three dice are thrown together. Find the probability of getting a total of at least 6.*

SOLUTION Since one die can be thrown in six ways to obtain any one of the six numbers marked on its six faces. Therefore, if three dice are thrown, the total number of elementary events $= 6 \times 6 \times 6 = 216$.

Let A be the event of getting a total of at least 6. Then, $\overline{A}$ denotes the event of getting a total of less than 6 i.e., 3, 4, 5.

$\therefore\ \overline{A} = \{(1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)\}$

So, favourable number of elementary events = 10

$$\therefore \quad P(\overline{A}) = \frac{10}{216}$$

$$\Rightarrow \quad 1 - P(A) = \frac{10}{216} \Rightarrow P(A) = 1 - \frac{10}{216} = \frac{103}{108}.$$

EXAMPLE 10 *What is the probability that a number selected from the numbers 1, 2, 3, ..., 25, is prime number, when each of the given numbers is equally likely to be selected?*

SOLUTION Let S be the sample space associated with the given experiment and A be the event "selecting a prime number". Then, $S = \{1, 2, 3, ..., 25\}$ and $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$

$\therefore$ Total number of elementary events = 25 and, Favourable number of elementary events = 9

Hence, required probability $= \frac{9}{25}$.

EXAMPLE 11 *Tickets numbered from 1 to 20 are mixed up together and then a ticket is drawn at random. What is the probability that the ticket has a number which is a multiple of 3 or 7?*

SOLUTION Let S be the sample space associated with the given random experiment and A denote the event "getting a ticket bearing a number which is a multiple of 3 or 7". Then,

$S = \{1, 2, ..., 20\}$ and $A = \{3, 6, 9, 12, 15, 18, 7, 14\}$

$\therefore$ Total number of elementary events = 20

Favourable number of elementary events = 8

Hence, required probability $= 8/20 = 2/5$

EXAMPLE 12 *A coin is tossed. If head comes up, a die is thrown but if tail comes up, the coin is tossed again. Find the probability of obtaining:*

(i) *two tails* (ii) *head and number 6* (iii) *head and an even number,*

SOLUTION The sample space S associated with the given random experiment is

$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, H), (T, T)\}$

Clearly, it has 8 elements.

$\therefore$ Total number of elementary events = 8

(i) If the outcome is (T, T), then we say that two tails are obtained.

$\therefore$ Favourable number of elementary events = 1

Hence, required probability $= \frac{1}{8}$

(ii) Head and the number 6 is obtain in only one way i.e. when the outcome is $(H, 6)$

$\therefore$ Favourable number elementary events = 1

Hence, required probability $= \frac{1}{8}$

(iii) Head and an even number can be obtained in any one of the following ways:

$(H, 2), (H, 4), (H, 6)$.

$\therefore$ Favourable number of elementary events = 3

Hence, required probability $= \frac{3}{8}$.

EXAMPLE 13 *One urn contains two black balls (labelled B1 and B2) and one white ball. A second urn contains one black ball and two white balls (labelled W1 and W2). Suppose the following experiment is performed. One of the two urns is chosen at random. Next a ball is randomly chosen from the urn. Then a second ball is chosen at random from the same urn without replacing the first ball.*

(i) Write the sample space showing all possible outcomes.

(ii) What is the probability that two black balls are chosen?

(iii) What is the probability that two balls of opposite colour are chosen? **[NCERT EXEMPLAR]**

SOLUTION (i) Let the contents of first urn be W, $B1$, $B2$, and that of second urn be B, $W1$, $W2$.

When two balls are drawn in succession from first urn, we may get any one of the following outcomes as an outcome:

$WB1, WB2, B1, B2, B1W, B2W, B2B1$

Similarly, when we draw two balls in succession from the second urn, we may obtain any one of the following as an outcome:

$BW1, BW2, W1B, W1W2, W2W1, W2B$

Thus, the sample space S is

$S = \{WB1, WB2, B1B2, B1W, B2W, B2B1, BW1, BW2, W1BW1W2, W2W1, W2B\}$

(ii) We obtain two black balls if the outcome is one of the following outcomes: $B1B2$, $B2B1$.

$\therefore$ Favourable number of elementary events = 2

Hence, required probability $= \frac{2}{12} = \frac{1}{6}$.

(iii) Two balls of opposite colour can be drawn in any one of the following outcomes:

$WB1, WB2, B1W, B2W, BW1, BW2, W1B, W2B$

$\therefore$ Favourable number of elementary events = 8

Hence, required probability $= \frac{8}{12} = \frac{2}{3}$.

<u>NOTE</u> *Consider an experiment of drawing 2 cards from a pack of 52 cards. The sample space associated with this experiment consists of ${}^{52}C_2 = 1426$ points and therefore it is not easy to list all the elements of the sample space. So, in future we will not be writing the sample space associated with the given random experiment.*

EXAMPLE 14 *On her vacations Veena visits four cities A, B, C and D in a random order. What is the probability that she visits:*

(i) *A before B?* (ii) *A before B and B berfore C?*
(iii) *A first and B last?* (iv) *A either first or second?*
(v) *A just before B?* **[NCERT]**

SOLUTION Veena can visit four cities A, B, C and D in any one of the following orders:

$ABCD, ABDC, ACDB, ACBD, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBDA, CBAD, CDAB, CDBA, DABC, DACB, DBCA, DBAC, DCAB, DCBA$

$\therefore$ Total number of arrangements (orders) in which Veena can visit four cities A, B, C and D is $4! = 24$.

(i) Out of these 24 ordered arrangements Veena can visit city A before city B in the following arrangements:

$ABCD, ABDC, ACDB, ACBD, ADBC, ADCB, CABD, CADB, CDAB, DABC, DACB, DCAB$

So, there are 12 ways in which Veena can visit city A before city B.

$\therefore$ Required probability $= \frac{12}{24} = \frac{1}{2}$

(ii) Veena can visit A before B and B before C in any one of the following four ways:

$ABCD, ABDC, DABC, ADBC$

$\therefore$ Required probability $= \frac{4}{24} = \frac{1}{6}$

(iii) Veena can visit city A first and city B last in any one of the following two ways:

$ACDB, ADCB$

$\therefore$ Required probability $= \frac{2}{24} = \frac{1}{12}$

(iv) Veena can visit city A first in 3! ways or city A second in 3! ways.

$\therefore$ Number of ways in which Veena can visity city A either first or second $= 3! + 3! = 12$

$\therefore$ Required probability $= \frac{12}{24} = \frac{1}{2}$

(v) Taking AB together A, B, C, D can be arranged in 3! ways.

So, required probability $= \frac{3!}{24} = \frac{1}{4}$

EXAMPLE 15 *A die has two faces each with number '1', three faces each with number '2' and one face with number '3'. If die is rolled once determine:* **[NCERT]**

(i) $P(2)$ (ii) $P(1 \text{ or } 3)$ (iii) $P(\text{not } 3)$

SOLUTION Total number of elementary events = 6.

(i) Out of six faces, 3 faces are marked with number 2.

$\therefore$ $P(2) = \frac{3}{6} = \frac{1}{2}$

(ii) Two faces are marked with number 1 and one face with number 3. Therefore, a face marked with 1 or 3 can be chosen in 3 ways.

$\therefore$ $P(1 \text{ or } 3) = \frac{3}{6} = \frac{1}{2}$

(iii) There is only one face marked with number 3. Therefore, $P(3) = \frac{1}{6}$.

Hence, $P(\text{not } 3) = 1 - P(3) = 1 - \frac{1}{6} = \frac{5}{6}$

EXAMPLE 16 *What is the probability that a randomly chosen two-digit integer is a multiple of 3?*

SOLUTION 2-digit positive integers are 10, 11, 12, ..., 98, 99. These are 90 numbers out of which one number can be chosen in 90 ways.

$\therefore$ Total number of elementary events = 90.

Out of these 90 numbers, 30 numbers (12, 15, 18, ..., 96, 99) are multiples of 3. One number out of these 30 numbers can be chosen in 30 ways.

$\therefore$ Favourable number of elementary events = 30.

Hence, required probability $= \frac{30}{90} = \frac{1}{3}$.

Type II PROBLEMS BASED UPON COMBINATIONS OR SELECTIONS

EXAMPLE 17 *One card is drawn from a pack of 52 cards, each of the 52 cards being equally likely to be drawn. Find the probability that the card drawn is:*

(i) an ace (ii) red (iii) either red or king (iv) red and a king.

SOLUTION Out of 52 cards, one card can be drawn in ${}^{52}C_1$ ways.

$\therefore$ Total number of elementary events $= {}^{52}C_1 = 52$.

(i) There are four aces in a pack of 52 cards, out of which one ace can be drawn in ${}^{4}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_1 = 4$.

So, required probability $= \frac{4}{52} = \frac{1}{13}$.

(ii) There are 26 red cards, out of which one red card can be drawn in ${}^{26}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{26}C_1 = 26$.

So, required probability $= \frac{26}{52} = \frac{1}{2}$.

(iii) There are 26 red cards including 2 red kings and there are 2 more kings. Therefore, there are 28 cards which are either red or king, out of 28 cards, one can be drawn in ${}^{28}C_1$ ways.

$\therefore$ Favourable number of elementary events = ${}^{28}C_1 = 28$.

So, required probability = $\frac{28}{52} = \frac{7}{13}$.

(iv) There are 2 cards which are red and king i.e., red kings.

$\therefore$ Favourable number of elementary events = ${}^{2}C_1 = 2$

So, required probability = $\frac{2}{52} = \frac{1}{26}$.

EXAMPLE 18 *An urn contains 9 red, 7 white and 4 black balls. If two balls are drawn at random, find the probability that:*

(i) *both the balls are red,* (ii) *one ball is white*

(iii) *the balls are of the same colour* (iv) *one is white and other red.*

SOLUTION There are 20 balls in the bag out of which 2 balls can be drawn in ${}^{20}C_2$ ways.

So, total number of elementary events = ${}^{20}C_2 = 190$.

(i) There are 9 red balls out of which 2 balls can be drawn in ${}^{9}C_2$ ways.

$\therefore$ Favourable number of elementary events = ${}^{9}C_2 = 36$.

So, required probability = $\frac{36}{190} = \frac{18}{95}$.

(ii) There are 7 white balls out of which one white can be drawn in ${}^{7}C_1$ ways. One ball from the remaining 13 balls can be drawn in ${}^{13}C_1$ ways. Therefore, one white and one other colour ball can be drawn in ${}^{7}C_1 \times {}^{13}C_1$ ways.

So, favourable number of elementary events = ${}^{7}C_1 \times {}^{13}C_1$.

Hence, required probability = $\frac{{}^{7}C_1 \times {}^{13}C_1}{{}^{20}C_2} = \frac{91}{190}$.

(iii) Two balls drawn are of the same colour means that either both are red or both are white or both are black. Out of 9 red balls two red balls can be drawn in ${}^{9}C_2$ ways. Similarly, two white balls can be drawn from 7 white balls in ${}^{7}C_2$ ways and two black balls from 4 black balls in ${}^{4}C_2$ ways. Therefore,

The number of ways of drawing 2 balls of the same colour = ${}^{9}C_2 + {}^{7}C_2 + {}^{4}C_2 = 36 + 21 + 6 = 63$

$\therefore$ Favourable number of elementary events = 63.

So, required probability = $\frac{63}{190}$.

(iv) Out of 7 white balls one white ball can be drawn in ${}^{7}C_1$ ways and out of 9 red balls one red ball can be drawn in ${}^{9}C_1$ ways. Therefore,

One white and one red ball can be drawn in ${}^{7}C_1 \times {}^{9}C_1$ ways.

So, favourable number of elementary events = ${}^{7}C_1 \times {}^{9}C_1 = 63$.

So, required probability = 63/190.

EXAMPLE 19 *A box contains 10 red marbles, 20 blue marbles and 30 green marbles. Five marbles are drawn from the box, what is the probability that (i) all will be blue? (ii) at least one will be green?* **[NCERT]**

SOLUTION Out of 60 marbles, 5 marbles can be drawn in ${}^{60}C_5$ ways.

$\therefore$ Total number of elementary events = ${}^{60}C_5$

(i) Out of 20 blue marbles, 5 blue marbles can be chosen in ${}^{20}C_5$ ways.

$\therefore$ Favourable number of ways $= {}^{20}C_5$

Hence, required probability $= \dfrac{{}^{20}C_5}{{}^{60}C_5}$

(ii) Clearly,

Required probability $= 1 -$ Probability that no ball is green

$= 1 -$ Probability that 5 balls drawn are red or blue. $= 1 - \dfrac{{}^{30}C_5}{{}^{60}C_5}$

EXAMPLE 20 *In a lottery 10,000 tickets are sold and ten equal prizes are awarded. What is the probability of not getting a prize if you buy (i) 1 ticket (ii) two tickets (iii) 10 tickets.* **[NCERT]**

SOLUTION (i) Out of 10,000 tickets, one ticket can be chosen in ${}^{10000}C_1 = 10000$ ways.

There are 9990 tickets not containing a prize. Out of these 9990 tickets one can be chosen in ${}^{9990}C_1$ ways.

$\therefore$ Probability of not getting a prize $= \dfrac{9990}{10000} = \dfrac{999}{1000}$

(ii) Out of 10,000 tickets, two tickets can be chosen in ${}^{10000}C_2$ ways. As there are 9990 tickets without any prize. Therefore, two drawn tickets will not contain any prize, if they are chosen from the remaining 9990 tickets. This can be done in ${}^{9990}C_2$ ways.

So, required probability $= \dfrac{{}^{9990}C_2}{{}^{10000}C_2}$

(iii) 10 tickets can be drawn out of 10,000 tickets in ${}^{10000}C_{10}$ ways. There are 9990 tickets without any prize. Out of these tickets 10 tickets can be chosen in ${}^{9990}C_{10}$ ways. So, 10 drawn tickets will not contain any prize, if they are chosen from the remaining 9990 tickets.

Hence, required probability $= \dfrac{{}^{9990}C_{10}}{{}^{10000}C_{10}}$

EXAMPLE 21 *The number lock of a suitcase has 4 wheels, each labelled with ten digits i.e. from 0 to 9. The lock opens with a sequence of four digits with no repeats. What is the probability of a person getting the right sequence to open the suitcase.* **[NCERT]**

SOLUTION There are ${}^{10}C_4 \times 4! = 5040$ sequences of 4 distinct digits out of which there is only one sequence in which the lock opens.

$\therefore$ Required probability $= \dfrac{1}{5040}$

EXAMPLE 22 *Out of 100 students, two sections of 40 and 60 students are formed. If you and your friends are among the 100 students, what is the probability that* **[NCERT]**

(i) *you both enter the same section?* (ii) *you both enter the different sections?*

SOLUTION Out of 100 students, two sections of 40 and 60 students can be formed in ${}^{100}C_{40} \times {}^{60}C_{60} = \dfrac{100!}{40!\,60!}$ ways.

(i) You and your friend can be in the same section in ${}^{98}C_{38} \times {}^{60}C_{60} + {}^{98}C_{58} \times {}^{40}C_{40}$

$$= \left(\frac{98!}{60!\,38!} + \frac{98!}{58!\,40!}\right) \text{ways}$$

$\therefore$ Probability that you and your friend enter the same section $= \dfrac{\dfrac{98!}{60!\,38!} + \dfrac{98!}{58!\,40!}}{\dfrac{100!}{60!\,40!}}$

$$= \frac{40 \times 39}{100 \times 99} + \frac{60 \times 59}{100 \times 99} = \frac{17}{33}$$

(ii) Required probability $= 1 -$ Probability that you and your friend enter the same section

$$= 1 - \frac{17}{33} = \frac{16}{33}$$

EXAMPLE 23 *Four cards are drawn at random from a pack of 52 playing cards. Find the probability of getting*

(i) *all the four cards of the same suit* (ii) *all the four cards of the same number*
(iii) *one card from each suit* (iv) *two red cards and two black cards*
(v) *all cards of the same colour* (vi) *all face cards.*

SOLUTION Four cards can be drawn from a pack of 52 cards in ${}^{52}C_4$ ways.

So, total number of elementary events $= {}^{52}C_4$

(i) There are four suits viz. club, spade, heart and diamond, each of 13 cards. All the four cards are of the same suit means that either four cards drawn are club cards or spade cards or heart cards or diamond cards. So, the total number of ways of getting all the four cards of the same suit is ${}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4\,({}^{13}C_4)$

So, required probability $= \dfrac{4\,({}^{13}C_4)}{{}^{52}C_4} = \dfrac{198}{20825}$

(ii) Four cards drawn can be of the same number in any one of the following ways:

(1, 1, 1, 1), (2, 2, 2, 2), (3, 3, 3, 3), ..., (13, 13, 13, 13)

$\therefore$ Favourable number of elementary events = 13.

So, required probability $= \dfrac{13}{{}^{52}C_4} = \dfrac{13}{270725}$

(iii) There are four suits each of 13 cards. One card from each suit means that there is one diamond card, one club card, one spade card and one heart card. There are 13 diamond cards, out of which one can be selected in ${}^{13}C_1$ ways. Similarly, one club, one spade and one heart, each can be selected in ${}^{13}C_1$ ways.

$\therefore$ The number of ways of selecting 4 cards, one from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$
$= ({}^{13}C_1)^4$

So, required probability $= \dfrac{({}^{13}C_1)^4}{{}^{52}C_4} = \dfrac{2197}{20825}$

(iv) There are 26 red cards and 26 black cards. Out of 26 red cards, 2 cards can be drawn in ${}^{26}C_2$ ways. Similarly, 2 black cards can be drawn in ${}^{26}C_2$ ways. Therefore, 2 red and 2 black cards can be drawn in ${}^{26}C_2 \times {}^{26}C_2$ ways.

So, required probability $= \dfrac{{}^{26}C_2 \times {}^{26}C_2}{{}^{52}C_4}$

(v) There are two colours viz. red and black. Out of 26 red colour cards, 4 cards can be drawn in ${}^{26}C_4$ ways. 4 black cards can be drawn in ${}^{26}C_4$ ways. Therefore, 4 red or 4 black cards can be drawn in ${}^{26}C_4 + {}^{26}C_4 = 2\,({}^{26}C_4)$ ways.

So, required probability $= \dfrac{2\,({}^{26}C_4)}{{}^{52}C_4}$

(vi) There are 12 face cards (4 kings, 4 queens and 4 jacks). Out of these 12 face cards, 4 cards can be selected in ${}^{12}C_4$ ways.

$\therefore$ Favourable number of elementary events $= {}^{12}C_4$

So, required probability $= \dfrac{{}^{12}C_4}{{}^{52}C_4}$

EXAMPLE 24 *In a lottery of 50 tickets numbered 1 to 50, two tickets are drawn simultaneously. Find the probability that:*

(i) *both the tickets drawn have prime numbers,* (ii) *none of the tickets drawn has prime number,* (iii) *one ticket has prime number.*

SOLUTION Out of 50 tickets 2 tickets can be drawn in ${}^{50}C_2$ ways.

So, total number of elementary events $= {}^{50}C_2 = 1225$

(i) There are 15 prime numbers between 1 and 50 viz. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47. Out of these 15 prime numbers 2 numbers can be selected in ${}^{15}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{15}C_2 = 105$

So, required probability $= \dfrac{105}{1225} = \dfrac{21}{245}$

(ii) Number of non-primes from 1 to 50 = 50 − 15 = 35. Out of these 35 numbers 2 can be selected in ${}^{35}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{35}C_2 = 595$

So, required probability $= \dfrac{595}{1225} = \dfrac{17}{35}$

(iii) Out of 15 primes from 1 to 50, one prime number can be selected in ${}^{15}C_1$ ways. Therefore, one prime and one non-prime can be selected in ${}^{15}C_1 \times {}^{35}C_1$ ways.

$\therefore$ Favourable number of elementary events $= {}^{15}C_1 \times {}^{35}C_1 = 525$

So, required probability $= \dfrac{525}{1225} = \dfrac{3}{7}$

EXAMPLE 25 *Four persons are to be chosen at random from a group of 3 men, 2 women and 4 children. Find the probability of selecting:*

(i) 1 man, 1 woman and 2 children (ii) exactly 2 children (iii) 2 women

SOLUTION There are 9 persons viz. 3 men, 2 women and 4 children. Out of these 9 persons 4 persons can be selected in ${}^{9}C_4 = 126$ ways.

$\therefore$ Total number of elementary events = 126

(i) 1 man, 1 woman and 2 children can be selected in ${}^{3}C_1 \times {}^{2}C_1 \times {}^{4}C_2 = 36$ ways.

$\therefore$ Favourable number of elementary events = 36

So, required probability $= \dfrac{36}{126} = \dfrac{2}{7}$

(ii) Exactly 2 children means: 2 children out of 4 children and 2 persons from 5 persons consisting of 3 men and 2 women. This can be done in ${}^{4}C_2 \times {}^{5}C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^{4}C_2 \times {}^{5}C_2 = 60$

So, required probability $= \frac{60}{126} = \frac{10}{21}$

(iii) We have to select 4 persons of which 2 are women and the remaining 2 are chosen from 7 persons consisting of 3 men and 4 children. This can be done in ${}^2C_2 \times {}^7C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^2C_2 \times {}^7C_2 = 21$

So, required probability $= \frac{21}{126} = \frac{1}{6}$

EXAMPLE 26 *A box contains 10 bulbs, of which just three are defective. If a random sample of five bulbs is drawn, find the probabilities that the sample contains:*

(i) *exactly one defective bulb,* (ii) *exactlytwo defective bulbs,* (iii) *no defective bulbs.*

SOLUTION Out of 10 bulbs 5 can be chosen in ${}^{10}C_5$ ways.

So, total number of elementary events $= {}^{10}C_5$

(i) There are 3 defective and 7 non-defective bulbs. The number of ways of selecting one defective bulb out of 3 and 4 non-defective out of 7 is ${}^3C_1 \times {}^7C_4$.

$\therefore$ Favourable number of elementary events $= {}^3C_1 \times {}^7C_4$

So, required probability $= \frac{{}^3C_1 \times {}^7C_4}{{}^{10}C_5} = \frac{5}{12}$

(ii) The number of ways of selecting 2 defective bulbs out of 3 defective bulbs and 3 non-defective bulbs out of 7 non defective bulbs is ${}^3C_2 \times {}^7C_3$.

$\therefore$ Favourable number of elementary events $= {}^3C_2 \times {}^7C_3$

So, required probability $= \frac{{}^3C_2 \times {}^7C_3}{{}^{10}C_5} = \frac{5}{12}$

(iii) No defective bulbs means all non-defective bulbs. The number of ways of selecting all 5 non-defective bulbs out of 7 is ${}^7C_5$.

$\therefore$ Favourable number of elementary events $= {}^7C_5$

So, required probability $= \frac{{}^7C_5}{{}^{10}C_5} = \frac{1}{12}$

EXAMPLE 27 *Five marbles are drawn from a bag which contains 7 blue marbles and 4 black marbles. What is the probability that: (i) all will be blue? (ii) 3 will be blue and 2 black?*

SOLUTION There are $7 + 4 = 11$ marbles in the bag out of which 5 marbles can be drawn in ${}^{11}C_5$ ways.

$\therefore$ Total number of elementary events $= {}^{11}C_5$.

(i) There are 7 blue marbles out of which 5 blue marbles can be drawn in ${}^7C_5$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_5$

Hence, required probability $= \frac{{}^7C_5}{{}^{11}C_5} = \frac{7!}{2!5!} \times \frac{5!6!}{11!} = \frac{1}{22}$

(ii) Three blue out of 7 blue balls and 2 black out of 4 black balls can be drawn in ${}^7C_3 \times {}^4C_2$ ways.

$\therefore$ Favourable number of elementary events $= {}^7C_3 \times {}^4C_2$

Hence, required probability $= \frac{{}^7C_3 \times {}^4C_2}{{}^{11}C_5} = \frac{7!}{3!4!} \times \frac{4!}{2!2!} \times \frac{5! \times 6!}{11!} = \frac{5}{11}$

EXAMPLE 28 *Find the probability that when a hand of 7 cards is dealt from a well-shuffled deck of 52 cards, it contains: (i) all 4 kings (ii) exactly 3 kings (iii) at least 3 kings.*

SOLUTION Out of 52 cards from a deck of 52 playing cards, 7 cards can be drawn in ${}^{52}C_7$ ways.

$\therefore$ Total number of elementary events = ${}^{52}C_7$

(i) There are 4 kings. Therefore, 4 kings out of 4 kings and 3 other cards from the remaining 48 cards can be chosen in ${}^4C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events = ${}^4C_4 \times {}^{48}C_3$

Hence, required probability = $\dfrac{{}^4C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{1}{7735}$

(ii) Three kings out of 4 kings and 4 other cards out of remaining 48 cards can be chosen in ${}^4C_3 \times {}^{48}C_4$ ways.

$\therefore$ Favourable number of elementary events = ${}^4C_3 \times {}^{48}C_4$

Hence, required probability = $\dfrac{{}^4C_3 \times {}^{48}C_4}{{}^{52}C_7} = \dfrac{9}{1547}$

(iii) When 7 cards are drawn from a deck of 52 playing cards, then getting at least 3 kings means: getting 3 kings and 4 other cards or getting 4 kings and 3 other cards. This can be done in ${}^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3$ ways.

$\therefore$ Favourable number of elementary events = ${}^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3$

Hence, required probability = $\dfrac{{}^4C_3 \times {}^{48}C_4 + {}^4C_4 \times {}^{48}C_3}{{}^{52}C_7} = \dfrac{46}{7735}$

EXAMPLE 29 *In a single throw of three dice, determine the probability of getting*
(i) *a total of 5* (ii) *a total of at most 5* (iii) *a total of at least 5.*

SOLUTION Total number of elementary events associated with the random experiment of throwing three dice simultaneously is $6 \times 6 \times 6 = 216$.

(i) A total of 5 can be obtain in one of the following ways:

$(1, 1, 3), (3, 1, 1), (1, 3, 1), (2, 2, 1), (1, 2, 2), (2, 1, 2)$

$\therefore$ Favourable number of elementary events = 6

Hence, required probability $= \dfrac{6}{216} = \dfrac{1}{36}$

(ii) A total of at most 5 can be obtained in any one of the following ways:

$(1, 1, 1), (1, 1, 2), (2, 1, 1), (1, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)$

$\therefore$ Favourable number of elementary events = 10

Hence, required probability = $\dfrac{10}{216} = \dfrac{5}{108}$

(iii) Let A be the event "getting a total of at least 5". Then,

$P(A) = 1 - P(\overline{A}) = 1 - P$ (Getting a total of at most 4)

A total of at most 4 can be obtained in any one of the following ways:

$(1, 1, 1), (1, 1, 2), (2, 1, 1), (1, 2, 1)$.

So, Favourable number of elementary events to $\overline{A}$ is 4.

$\therefore \quad P(\overline{A}) = \dfrac{4}{216} \Rightarrow 1 - P(A) = \dfrac{4}{216} \Rightarrow P(A) = 1 - \dfrac{4}{216} = \dfrac{212}{216} = \dfrac{53}{54}$

Type II PROBLEMS BASED UPON PERMUTATIONS OR ARRANGEMENTS

EXAMPLE 30 *If the letters of the word ALGORITHM are arranged at random in a row what is the probability that the letters GOR must remain together as a unit?* **[NCERT EXEMPLAR]**

SOLUTION There are 9 letters in the word ALGORITHM. These 9 letters can be arranged in a row in 9! ways.

$\therefore$ Total number of elementary events = 9!

Considering GOR as one letter there are 7 letters which can be arranged in a row in 7! ways.

$\therefore$ Favourable number of elementary events = 7!

Hence, required probability $= \frac{7!}{9!} = \frac{1}{72}$

EXAMPLE 31 *If the letters of the word 'ATTRACTION' are written down at random, find the probability that* (i) *all the T's occur together* (ii) *no two T's occur together.*

SOLUTION The total number of arrangements of the letters of the word 'ATTRACTION' is $\frac{10!}{3!\,2!}$.

(i) Considering three T's as one letter there are 8 letters consisting of two identical A's. These 8 letters can be arranged in $\frac{8!}{2!}$ ways.

$$\text{Hence, required probability} = \frac{\frac{8!}{2!}}{\frac{10!}{3!\,2!}} = \frac{3!\,8!}{10!} = \frac{1}{15}$$

(ii) Other than 3 T's there are 7 letters which can be arranged in $\frac{7!}{2!}$ ways. There are 8 places, 6 between the 7 letters and one on extreme left and the other on extreme right. To separate three T's, we arrange them in these 8 places. This can be done in ${}^8C_3$ ways. Therefore,

Number of ways in which no two T's are together $= \frac{7!}{2!} \times {}^8C_3$

$$\text{Hence, required probability} = \frac{\frac{7!}{2!} \times {}^8C_3}{\frac{10!}{3!\,2!}} = \frac{7}{15}$$

EXAMPLE 32 *A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number is divisible by 4.*

SOLUTION Total number of five digit numbers formed by the digits 1, 2, 3, 4, 5 is 5!.

$\therefore$ Total number of elementary events = 5! = 120.

We know that a number is divisible by 4 if the number formed by last two digits is divisible by 4. Therefore last two digits can be 12, 24, 32, 52 that is, last two digits can be filled in 4 ways. But corresponding to each of these ways there are 3! = 6 ways of filling the remaining three places. Therefore the total number of five digit numbers formed by the digits 1, 2, 3, 4, 5 and divisible by 4 is $4 \times 6 = 24$

$\therefore$ Favourable number of elementary events = 24

So, required probability $= \frac{24}{120} = \frac{1}{5}$

Type III PROBLEMS BASED UPON COMBINATIONS OR SELECTIONS

EXAMPLE 33 *Out of 9 outstanding students in a college, there are 4 boys and 5 girls. A team of four students is to be selected for a quiz programme. Find the probability that two are boys and two are girls.*

SOLUTION Out of 9 students 4 students can be selected in ${}^9C_4$ ways.

So, total number of elementary events $= {}^9C_4$.

There are 4 boys and 5 girls out of which 2 boys and 2 girls can be selected in ${}^4C_2 \times {}^5C_2$ ways.

So, favourable number of elementary events $= {}^4C_2 \times {}^5C_2$

Hence, required probability $= \dfrac{{}^4C_2 \times {}^5C_2}{{}^9C_4} = \dfrac{10}{21}$

EXAMPLE 34 *In a lot of 12 Microwave ovens, there are 3 defective units. A person has ordered 4 of these units and since each is identically packed, the selection will be random. What is the probability that (i) all 4 units are good. (ii) exactly 3 units are good (iii) at least 2 units are good.* [NCERT]

SOLUTION Out of 12 Microwave ovens, 4 can be chosen in ${}^{12}C_4$ ways.

$\therefore$ Total number of elementary events $= {}^{12}C_4$

(i) There are 9 good units out of which 4 can be chosen in ${}^9C_4$ ways.

$\therefore$ Favourable number of elementary events $= {}^9C_4$

Hence, required probability $= \dfrac{{}^9C_4}{{}^{12}C_4} = \dfrac{14}{55}$

(ii) Exactly 3 good units can be chosen in ${}^9C_3 \times {}^3C_1$ ways.

$\therefore$ Required probability $= \dfrac{{}^9C_3 \times {}^3C_1}{{}^{12}C_4} = \dfrac{28}{55}$

(iii) Required probability $= 1 - P$ (At most one unit is good) $= 1 - \dfrac{{}^9C_1 \times {}^3C_3}{{}^{12}C_4} = 1 - \dfrac{1}{55} = \dfrac{54}{55}$

EXAMPLE 35 *In a relay race there are five teams A, B, C, D and E.*

(i) *What is the probability that A, B and C finish first, second and third respectively.*

(ii) *What is the probability that A, B and C are first three to finish (in any order).* [NCERT]

SOLUTION Out of 5 teams first three positions can be occupied by 3 teams in any order in ${}^5C_3 \times 3!$ ways.

So, total number of elementary events $= {}^5C_3 \times 3! = 60$

(i) Teams A, B and C can finish first, second and third in only one way, because there is only one finishing order.

$\therefore$ Favourable number of elementary events = 1

So, required probability $= \dfrac{1}{60}$

(ii) Teams A, B and C finish at first three places in any order in 3! ways.

$\therefore$ Favourable number of elementary events $= 3! = 6$

So, required probability $= \dfrac{6}{60} = \dfrac{1}{10}$

Type IV **MISCELLANEOUS PROBLEMS**

EXAMPLE 36 *A card is drawn from an ordinary pack of 52 cards and a gambler bets that, it is a spade or an ace. What are the odds against his winning this bet?*

SOLUTION Let A be the event of getting a spade or an ace from a pack of 52 cards. Then,

Total number of elementary events $= {}^{52}C_1 = 52$