CHAPTER 17

MAXIMA AND MINIMA

17.1 INTRODUCTION

In the previous chapters, we have learnt about various applications of differentiation. In this chapter, we will use differentiation to find the maximum and minimum values of differentiable functions in their domains. In the end of the chapter, we will discuss applications of maxima and minima in solving some applied problems.

17.2 MAXIMUM AND MINIMUM VALUES OF A FUNCTION IN ITS DOMAIN

MAXIMUM *Let $f(x)$ be a real function defined on an interval $[a, b]$. Then, $f(x)$ is said to have the maximum value in $[a, b]$, if there exists a point c in $[a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$.*

In such a case, the number $f(c)$ is called the maximum value of $f(x)$ in the interval $[a, b]$ and the point c is called a point of maximum value of f in the interval $[a, b]$.

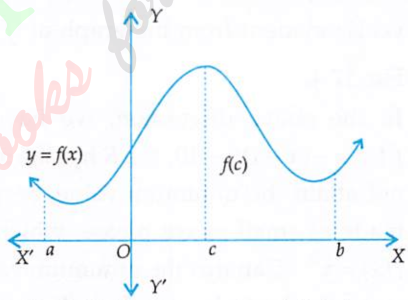

Fig. 17.1 Maximum value of $f(x)$ at $x = c$

Consider the function f given by $f(x) = -(x-1)^2 + 10$.

Clearly, domain $(f) = R = (-\infty, \infty)$.

We observe that

$$-(x-1)^2 \leq 0 \text{ for all } x \in R$$

$$\Rightarrow \quad -(x-1)^2 + 10 \leq 10 \text{ for all } x \in R$$

$$\Rightarrow \quad f(x) \leq 10 \quad \text{for all } x \in R$$

$$\Rightarrow \quad f(x) \leq f(1) \text{ for all } x \in R \qquad [\because f(1) = -(1-1)^2 + 10 = 10]$$

It follows from this expression that $f(1) = 10$ is the maximum value of function f and the point of maximum value of f is $x = 1$. This fact is also evident from the graph of function f as shown in Fig. 17.2.

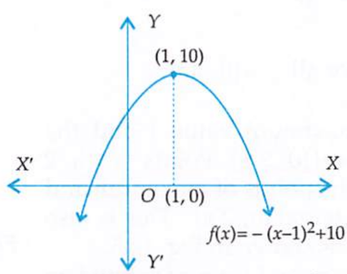

Fig. 17.2 Graph of $f(x) = -(x-1)^2 + 10$

MINIMUM *Let $f(x)$ be a real function defined on an interval $[a, b]$. Then $f(x)$ is said to have the minimum value in interval $[a, b]$, if there exists a point $c \in [a, b]$ such that $f(x) \geq f(c)$ for all $x \in [a, b]$.*

In such a case, the number $f(c)$ is called the minimum value of $f(x)$ in the interval $[a, b]$ and the point c is called a point of minimum value of f in the interval $[a, b]$.

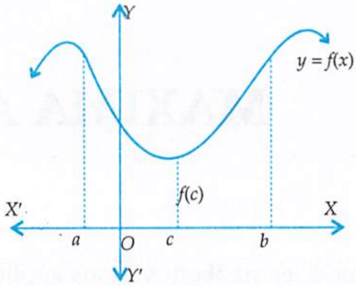

Fig. 17.3 Minimum value of $f(x)$ at $x = c$

Consider the function f given by $f(x) = x^2 + 5$. Clearly, domain $(f) = R = (-\infty, \infty)$.

Now,

$$x^2 \geq 0 \text{ for all } x \in R \Rightarrow x^2 + 5 \geq 5 \text{ for all } x \in R \Rightarrow f(x) \geq 5 \text{ for all } x \in R \Rightarrow f(x) \geq f(0) \text{ for all } x \in R$$

It follows from this expression and the above definition that the minimum value of function $f(x) = x^2 + 5$ defined on R is 5 and the point of minimum value of f is $x = 0$. This observation is also evident from the graph of $f(x) = x^2 + 5$ as shown in Fig. 17.4.

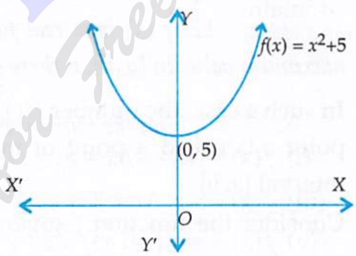

Fig. 17.4 Graph of $f(x) = x^2 + 5$

In the above discussion, we have seen that the function $f(x) = -(x-1)^2 + 10$, $x \in R$ has the maximum value but it does not attain the minimum value, because $-(x-1)^2 + 10$ can be made as small as we please, which is also evident from the graph (Fig. 17.2). The function $f(x) = x^2 + 5$ attains the minimum value 5 at $x = 0$, but it does not attain the maximum value at any point in its domain. In fact, $f(x)$ can be made as large as we please. From the graph of $f(x)$ (Fig. 17.4), we find that the values of $f(x)$ are increasing rapidly. That is why it does not attain the maximum value.

Let us now consider the function $f(x) = \sin x$ defined on the interval $[0, 2\pi]$.

Clearly, $-1 \leq \sin x \leq 1$ for all $x \in [0, 2\pi]$. So, $-1 \leq f(x) \leq 1$ for all $x \in [0, 2\pi]$.

Also, $f\left(\dfrac{\pi}{2}\right) = 1$ and $f\left(\dfrac{3\pi}{2}\right) = -1$.

$$\therefore \qquad f\left(\frac{3\pi}{2}\right) \leq f(x) \leq f\left(\frac{\pi}{2}\right) \text{ for all } x \in [0, 2\pi]$$

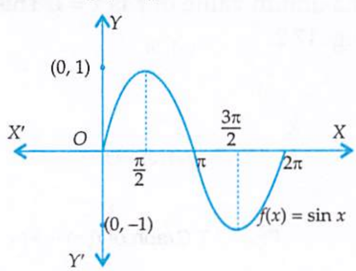

Fig. 17.5 Graph of $f(x) = \sin x, 0 \leq x \leq 2\pi$

Thus, $f(x)$ attains both the maximum value 1 and the minimum value -1 in the interval $[0, 2\pi]$. Points $x = \pi/2$ and $x = 3\pi/2$ are respectively the points of maximum and minimum values of f in the interval $[0, 2\pi]$. This is also evident from the graph of $f(x)$ as shown in Fig. 17.5.

Now, consider the function f given by $f(x) = x^3$ defined on $(-2, 2)$. Clearly, it is an increasing function in the given interval. So, it should have the minimum value at a point closest to -2 on its right and the maximum value at a point closest to 2 on the left. In fact, it is not possible to locate such points as shown in Fig. 17.6. Therefore, $f(x) = x^3$ has neither the maximum value nor the minimum value in the interval $(-2, 2)$.

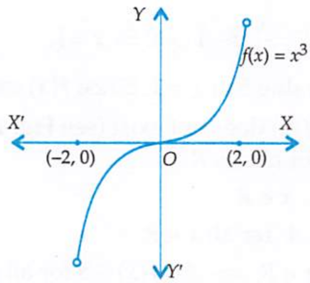

Fig. 17.6 Graph of $f(x) = x^3$

It follows from the above discussion that a function f defined on an interval I.

(i) may attain the maximum value at a point in I but not the minimum value at any point in I.

(ii) may attain the minimum at a point in I but not the maximum value at any point in I.

(iii) may attain both the maximum and minimum values at some points in I.

(iv) may not attain both the maximum and minimum values at any point in I.

Let us now discuss more examples on the maximum and minimum values of functions in their domains.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the maximum and the minimum values, if any, of the following functions*

(i) $f(x) = 3x^2 + 6x + 8, x \in R$ (ii) $f(x) = -|x-1| + 5$ *for all* $x \in R$

(iii) $f(x) = \sin 3x + 4, x \in (-\pi/2, \pi/2)$ (iv) $f(x) = x^3 + 1$ *for all* $x \in R$ **[NCERT]**

(v) $f(x) = \sin(\sin x)$ *for all* $x \in R$ (vi) $f(x) = |x+3|$ *for all* $x \in R$.

SOLUTION (i) We have, $f(x) = 3x^2 + 6x + 8 = 3(x^2 + 2x + 1) + 5 = 3(x+1)^2 + 5$.

Clearly, $3(x+1)^2 \geq 0$ for all $x \in R$

$\Rightarrow$ $3(x+1)^2 + 5 \geq 5$ for all $x \in R \Rightarrow f(x) \geq f(-1)$ for all $x \in R$. $[\because f(-1) = 5]$

Thus, 5 is the minimum value of $f(x)$ which it attains at $x = -1$.

Since $f(x)$ can be made as large as we please. Therefore, the maximum value does not exist which can be observed from Fig. 17.7.

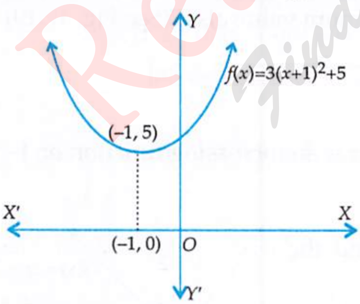

Fig. 17.7 Graph of $f(x) = 3(x+1)^2 + 5$

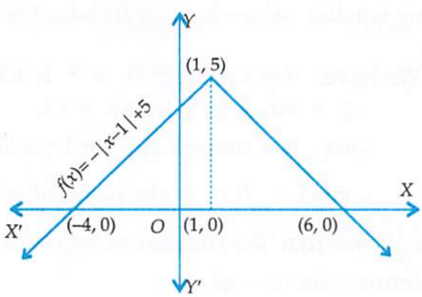

Fig. 17.8 Graph of $f(x) = -|x-1| + 5$

(ii) We have, $f(x) = -|x-1| + 5$ for all $x \in R$.

Clearly, $|x-1| \geq 0$ for all $x \in R$

$\Rightarrow$ $-|x-1| \leq 0$ for all $x \in R \Rightarrow -|x-1| + 5 \leq 5$ for all $x \in R \Rightarrow f(x) \leq 5$ for all $x \in R$.

So, 5 is the maximum value of $f(x)$.

Now, $f(x) = 5 \Rightarrow -|x-1| + 5 = 5 \Rightarrow |x-1| = 0 \Rightarrow x = 1.$

Thus, $f(x)$ attains the maximum value 5 at $x = 1$. Since $f(x)$ can be made as small as we please. Therefore the minimum value of $f(x)$ does not exist (see Fig. 17.8).

(iii) We have, $f(x) = \sin 3x + 4$ for all $x \in R$

Clearly, $-1 \le \sin 3x \le 1$ for all $x \in R$

$\Rightarrow \quad -1 + 4 \le \sin 3x + 4 \le 1 + 4$ for all $x \in R$

$\Rightarrow \quad 3 \le \sin 3x + 4 \le 5$ for all $x \in R \Rightarrow 3 \le f(x) \le 5$ for all $x \in R$.

Thus, the maximum value of $f(x)$ is 5 and the minimum value is 3.

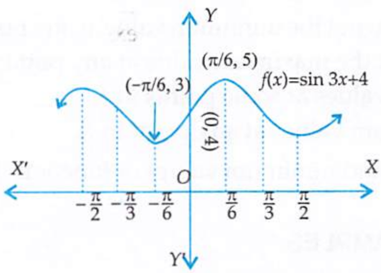

Fig. 17.9 Graph of $f(x) = \sin 3x + 4$

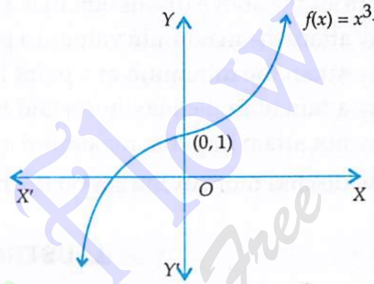

Fig. 17.10 Graph of $f(x) = x^3 + 1$

Now, $f(x) = 5 \Rightarrow \sin 3x + 4 = 5 \Rightarrow \sin 3x = 1 \Rightarrow 3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}.$

So, $f(x)$ attains its maximum value 5 at $x = \frac{\pi}{6}$. (see Fig. 17.9).

Also, $f(x) = 3 \Rightarrow \sin 3x + 4 = 3 \Rightarrow \sin 3x = -1 \Rightarrow 3x = -\frac{\pi}{2} \Rightarrow x = -\frac{\pi}{6}.$

So, $f(x)$ attains the minimum value 3 at $x = -\frac{\pi}{6}$.

(iv) We have, $f(x) = x^3 + 1$, $x \in R$. Here, we observe that the values of $f(x)$ increase when the values of x are increased and $f(x)$ can be made as large as we please by giving large values to x. So, $f(x)$ does not have the maximum value. Similarly, $f(x)$ can be made as small as we please by giving smaller values to x. So $f(x)$ does not have the minimum value also. (See Fig. 17.10).

(v) We have, $f(x) = \sin(\sin x)$, $x \in R$. Clearly,

$-1 \le \sin x \le 1$ for all $x \in R$

$\Rightarrow \quad \sin(-1) \le \sin(\sin x) \le \sin 1$ for all $x \in R$ [$\because \sin x$ is an increasing function on $[-1, 1]$]

$\Rightarrow \quad -\sin 1 \le f(x) \le \sin 1$ for all $x \in R$

This shows that the maximum value of $f(x)$ is $\sin 1$ and the minimum value is $-\sin 1$.

(vi) We have, $f(x) = |x + 3|$ for all $x \in R$

Clearly, $|x + 3| \ge 0$ for all $x \in R \Rightarrow f(x) \ge 0$ for all $x \in R$.

So, the minimum value of $f(x)$ is 0, which it attains at $x = -3$.

Clearly, $f(x) = |x + 3|$ does not have the maximum value. (See Fig. 17.11).

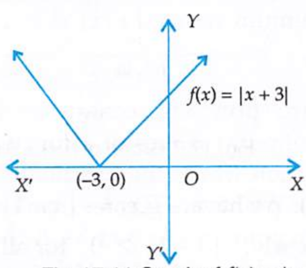

Fig. 17.11 Graph of $f(x) = |x + 3|$

EXERCISE 17.1

BASIC

Find the maximum and the minimum values, if any, without using derivatives of the following functions:

1. $f(x) = 4x^2 - 4x + 4$ on R
2. $f(x) = -(x-1)^2 + 2$ on R [NCERT]
3. $f(x) = |x+2|$ on R
4. $f(x) = \sin 2x + 5$ on R [NCERT]
5. $f(x) = |\sin 4x + 3|$ on R [NCERT]
6. $f(x) = 2x^3 + 5$ on R
7. $f(x) = -|x+1| + 3$ on R [NCERT]
8. $f(x) = 16x^2 - 16x + 28$ on R
9. $f(x) = x^3 - 1$ on R

ANSWERS

1. Min. = 3, Max. dose not exist.
2. Max. = 2, Min. does not exist.
3. Min. = 0, Max. does not exist.
4. Max. = 6, Min. = 4.
5. Max. = 4, Min. = 2.
6. Max and Min. both do not exist.
7. Max. = 3, Min. does not exist.
8. Min. = 24, Max. does not exist
9. Max. and Min. both do not exist.

HINTS TO SELECTED PROBLEMS

2. We have, $f(x) = -(x-1)^2$ for all $x \in R$

Clearly, $-(x-1)^2 \le 0$ for all $x \in R \Rightarrow -(x-1)^2 + 2 \le 2$ for all $x \in R \Rightarrow f(x) \le 2$ for all $x \in R$.

So, $f(x)$ attains maximum value 2 at $x = 1$ and the minimum value does not exist as $f(x)$ can be made as small as we please.

4. We have, $f(x) = \sin 2x + 5, \ x \in R$.

Clearly, $-1 \le \sin 2x \le 1$ for all $x \in R$

$\Rightarrow \ 5 - 1 \le \sin 2x + 5 \le 1 + 5$ for all $x \in R \Rightarrow 4 \le f(x) \le 6$ for all $x \in R$.

So, the minimum and the maximum values of $f(x)$ are 4 and 6 respectively.

5. We have, $f(x) = |\sin 4x + 3|, \ x \in R$. We know that

$-1 \le \sin 4x \le 1$ for all $x \in R$.

$\Rightarrow \ 3 - 1 \le \sin 4x + 3 \le 1 + 3$ for all $x \in R$

$\Rightarrow \ 2 \le \sin 4x + 3 \le 4$ for all $x \in R$

$\Rightarrow \ 2 \le |\sin 4x + 3| \le 4$ for all $x \in R \Rightarrow 2 \le f(x) \le 4$ for all $x \in R$

So, the minimum and the maximum values of $f(x)$ are 2 and 4 respectively.

7. We have, $f(x) = -|x+1| + 3, \ x \in R$. We know that

$-|x+1| \le 0$ for all $x \in R \Rightarrow -|x+1| + 3 \le 3$ for all $x \in R \Rightarrow f(x) \le 3$ for all $x \in R$.

So, the maximum value of $f(x)$ is 3. As $f(x)$ can be made as small as we please. So, the minimum value of $f(x)$ does not exist.

17.3 LOCAL MAXIMA AND LOCAL MINIMA

In the previous section, we have discussed about the greatest (maximum) and the least (minimum) values of a function in its domain. But, there may be points in the domain of a function where the function does not attain the greatest (or the least) value but the values at these points are greater than or less than the values of the function at the neighbouring points. Such points are known as the points of local minimum or local maximum and we will be mainly discussing about the local maximum and local minimum values of a function.

LOCAL MAXIMUM *A function $f(x)$ is said to attain a local maximum at $x = a$ if there exists a neighbourhood $(a - \delta, a + \delta)$ of a such that*

$$f(x) < f(a) \text{ for all } x \in (a - \delta, a + \delta), \ x \ne a.$$

or, $\quad f(x) - f(a) < 0$ *for all* $x \in (a - \delta, a + \delta), \; x \neq a$.

In such a case, $f(a)$ is called the local maximum value of $f(x)$ at $x = a$.

LOCAL MINIMUM *A function* $f(x)$ *is said to attain a local minimum at* $x = a$ *if there exists a neighbourhood* $(a - \delta, a + \delta)$ *of a such that*

$$f(x) > f(a) \text{ for all } x \in (a - \delta, a + \delta), x \neq a$$

or, $$f(x) - f(a) > 0 \text{ for all } x \in (a - \delta, a + \delta), x \neq a.$$

The value of the function at $x = a$ i.e. $f(a)$ is called the local minimum value of $f(x)$ at $x = a$.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$. Thus, a function attains an extreme value at $x = a$ if $f(a)$ is either a local maximum value or a local minimum value. Consequently, at an extreme point 'a', $f(x) - f(a)$ keeps the same sign for all values of x in a deleted neighbourhood of a.

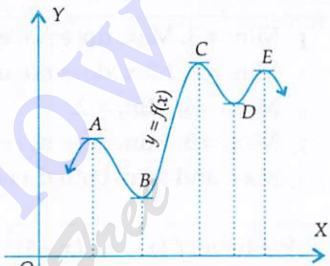

Fig. 17.12

In Fig. 17.12 we observe that the x-coordinates of the points A, C and E are points of local maximum and the values at these points i.e. their y-coordinates are the local maximum values of $f(x)$. The x-coordinates of points B and D are points of local minimum and their y-coordinates are the local minimum values of $f(x)$.

<u>NOTE</u> *By a local maximum (or local minimum) value of a function at a point* $x = a$, *we mean the greatest (or the least) value in the neighbourhood of point* $x = a$ *and not the maximum (or the minimum) in the domain of the function. In fact a function may have any number of points of local maximum (or local minimum) and even a local minimum value may be greater than a local maximum value. In Fig. 17.12 the minimum value at D is greater than the maximum value at A. Thus, a local maximum value may not be the greatest value and a local minimum value may not be the least value of the function in its domain.*

It follows from the above definition that if a is a point of local maximum of a function f, then in the neighbourhood of a the graph of f should be as shown in Fig. 17.13. Clearly, $f(x)$ is increasing in the left neighbourhood $(a - \delta, a)$ of point a and decreasing in the right neighbourhood of $x = a$.

$\therefore \quad f'(x) > 0$ for $x \in (a - \delta, a)$ and, $f'(x) < 0$ for $x \in (a, a + \delta)$

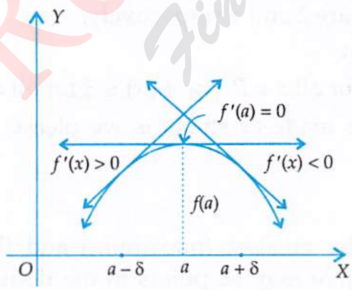

Fig. 17.13 Point of local maximum

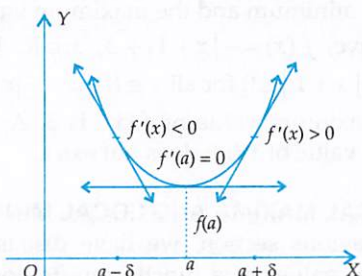

Fig. 17.14 Point of local minimum

This suggests that $f'(a)$ must be zero.

Similarly, if a is a point of local minimum of a function f, then in the neighbourhood of a the graph of f should be as shown in Fig. 17.14. Here, we observe that $f(x)$ is decreasing in the left neighbourhood $(a - \delta, a)$ of a and increasing in the right neighbourhood $(a, a + \delta)$ of a.

$\therefore \quad f'(x) < 0$ for $x \in (a-\delta, a)$ and, $f'(x) > 0$ for $x \in (a, a+\delta)$.

This also suggests that $f'(a)$ must be zero.

In view of the above discussion we state the following theorem (without proof) which is known as the necessary condition for points of local maximum or minimum.

THEOREM *A necessary condition for $f(a)$ to be an extreme value of a function $f(x)$ is that $f'(a) = 0$, in case it exists.*

REMARK 1 *This result states that if the derivative exists, it must be zero at the extreme points. A function may however attain an extreme value at a point without being derivable thereat. For example, the function $f(x) = |x|$ attains the minimum value at the origin even though it is not derivable at $x = 0$.*

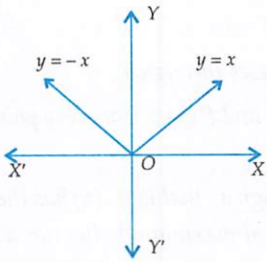

Fig. 17.15 Graph of $f(x) = |x|$

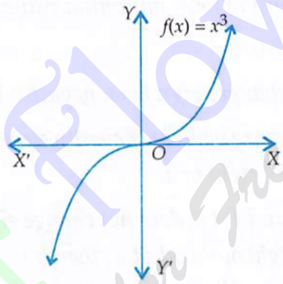

Fig. 17.16 Graph of $f(x) = x^3$

REMARK 2 *This condition is only a necessary condition for the point $x = a$ to be an extreme point. It is not sufficient i.e., $f'(a) = 0$ does not necessarily imply that $x = a$ is an extreme point. There are functions for which the derivatives vanish at a point but do not have an extreme value thereat. For example, for the function $f(x) = x^3$, we have $f'(0) = 0$ but at $x = 0$ the function does not attain an extreme value (See Fig. 17.16). In fact on the left of point $x = 0$, the curve is concave down and on its right the curve is concave up. That is, the concavity of $f(x)$ changes x as increases through O. Such points are called points of inflection. If $x = c$ is a point of inflection of a function $f(x)$, then $f''(c) = 0$ and $f''(x)$ change its sign as x increases through 'c'.*

REMARK 3 *Geometrically the above condition means that the tangent to the curve $y = f(x)$ at a point where the ordinate is maximum or minimum is parallel to the x-axis.*

REMARK 4 *As discussed in Remark 2 that all x, for which $f'(x) = 0$, do not give us the extreme values. The values of x for which $f'(x) = 0$ are called stationary points or turning points and the corresponding values of $f(x)$ are called stationary or turning values of $f(x)$.*

REMARK 5 *The values of x for which $f'(x) = 0$ or, $f'(x)$ does not exist are known as critical points.*

17.4 FIRST DERIVATIVE TEST FOR LOCAL MAXIMA AND MINIMA

In the previous section, we have seen that an extreme point (point of local maximum or minimum) the derivative of the function either does not exist or in case it exists, it must be zero. We have also seen that if a is a point of local maximum value of a function f, then there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that

$f'(x) > 0$ for all $x \in (a-\delta, a)$ [See Fig. 17.13]

and, $f'(x) < 0$ for all $x \in (a, a+\delta)$.

In case, a is a point of local minimum value of function f, then there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that

$f'(x) < 0$ for all $x \in (a-\delta, a)$

and, $f'(x) > 0$ for all $x \in (a, a+\delta)$ [See Fig. 17.14]

In the light of these observations, we state the following theorem (without proof) for finding the points of local maxima or local minima.

THEOREM 1 *(First derivative test) Let f be a differentiable function defined on an interval I and let $a \in I$. Then,*

(a) *$x = a$ is a point of local maximum value of f, if*

(i) $f'(a) = 0$

and, (ii) *$f'(x)$ changes sign from positive to negative as x increases through a*

i.e. $f'(x) > 0$ at every point sufficiently close to and to the left of a, and $f'(x) < 0$ at every point sufficiently close to and to the right of a.

(b) *$x = a$ is a point of local minimum value of f, if*

(i) $f'(a) = 0$

and, (ii) *$f'(x)$ changes sign from negative to positive as x increases through a*

i.e. $f'(x) < 0$ at every point sufficiently close to and to the left of a, and $f'(x) > 0$ at every point sufficiently close to and to the right of a.

(c) *If $f'(a) = 0$ and $f'(x)$ does not change sign as increases through a, that is, $f'(x)$ has the same sign in the complete neighbourhood of a, then a is neither a point of local maximum value nor a point of local minimum value. In fact, such a point is called a point of inflexion.*

The above theorem suggests the following algorithm to find the points to local maxima or local minima of differentiable functions.

ALGORITHM

Step I *Put $y = f(x)$*

Step II *Find $\frac{dy}{dx}$.*

Step III *Put $\frac{dy}{dx} = 0$ and solve this equation for x. Let $c_1, c_2, c_3, \ldots, c_n$ be the roots of this equation. Points $c_1, c_2, c_3, \ldots, c_n$ are critical points (stationary values of x) and these are the possible points where the function can attain a local maximum or a local minimum. So, we test the function at each one of these points.*

Step IV *Consider $x = c_1$.*

If $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through c_1, then the function attains a local maximum at $x = c_1$.

If $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through c_1, then the function attains a local minimum at $x = c_1$.

If $\frac{dy}{dx}$ does not change sign as x increases through c_1, then $x = c_1$ is neither a point of local maximum nor a point of local minimum. In this case $x = c_1$ is a point of inflexion.

Similarly, we may deal with other values of x.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find all the points of local maxima and minima of the function* $f(x) = x^3 - 6x^2 + 9x - 8$.

[NCERT]

SOLUTION Let $y = f(x) = x^3 - 6x^2 + 9x - 8$. Then, $\frac{dy}{dx} = f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$.

The critical points of $f(x)$ are given by $f'(x) = 0$ or, $\frac{dy}{dx} = 0$.

Now, $\frac{dy}{dx} = 0 \Rightarrow 3(x^2 - 4x + 3) = 0 \Rightarrow x = 1, 3$.

We have to examine whether these points are points of local maxima or local minima or neither of them.

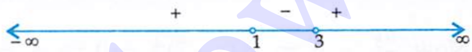

Fig. 17.17 Signs of $\frac{dy}{dx}$ for different values of x.

We have, $\frac{dy}{dx} = 3(x-1)(x-3)$

The changes in signs of $\frac{dy}{dx}$ for different values of x are shown in Fig. 17.17. Clearly, $\frac{dy}{dx}$ changes sign from positive to negative as increases through 1. So, $x = 1$ is a point of local maximum. Also, $\frac{dy}{dx}$ changes sign from negative to positive as x increases through 3. So $x = 3$ is a point of local minimum.

EXAMPLE 2 *Find all the points of local maxima and local minima as well as the corresponding local maximum and local minimum values for the function* $f(x) = (x-1)^3 (x+1)^2$.

SOLUTION Let $y = f(x) = (x-1)^3 (x+1)^2$. Then,

$$\frac{dy}{dx} = 3(x-1)^2 (x+1)^2 + 2(x+1)(x-1)^3 = (x-1)^2 (x+1)\{3(x+1) + 2(x-1)\}$$

$$\Rightarrow \quad \frac{dy}{dx} = (x-1)^2 (x+1)(5x+1).$$

At points of local maxima or local minima, we must have

$$\frac{dy}{dx} = 0 \Rightarrow (x-1)^2 (x+1)(5x+1) = 0 \Rightarrow x = 1 \text{ or, } x = -1 \text{ or, } x = -\frac{1}{5}$$

Now, we have to examine whether these points are points of local maximum or local minimum or neither of them.

Since $(x-1)^2$ is always positive, therefore the sign of $\frac{dy}{dx}$ is same as that the $(x+1)(5x+1)$. The changes in signs of $\frac{dy}{dx}$ for different values of x are shown in Fig. 17.18.

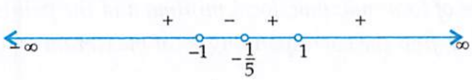

Fig. 17.18 Signs of $\frac{dy}{dx}$ for different values of x.

Clearly, $\frac{dy}{dx}$ does not change its sign as x passes through 1. So $x = 1$ is neither a point of local maximum nor a point of local minimum. In fact, $x = 1$ is a point of inflexion.

Clearly, $\frac{dy}{dx}$ changes sign from positive to negative as x passes through –1. So, $x = -1$ is a point of local maximum.

The local maximum value of $f(x)$ at $x=-1$ is $f(-1)=(-2)^3(-1+1)^2 = 0$.

It is evident from Fig. 17.18 that $\frac{dy}{dx}$ changes sign from negative to positive as x passes through $-1/5$. So, $x=-1/5$ is a point of local minimum.

The local minimum value of $f(x)$ at $x=-\frac{1}{5}$ is $f\left(-\frac{1}{5}\right) = \left(-\frac{1}{5}-1\right)^3\left(-\frac{1}{5}+1\right)^2 = -\frac{3456}{3125}$.

EXAMPLE 3 *Find all the points of local maxima and local minima of the function* $f(x)=x^3-6x^2+12x-8$.

SOLUTION Let $y=f(x)=x^3-6x^2+12x-8$. Then, $\frac{dy}{dx} = 3x^2-12x+12 = 3(x-2)^2$.

The critical points of $y=f(x)$ are given by $\frac{dy}{dx}=0$.

Fig. 17.19 Signs of $\frac{dy}{dx}$ for different values of x.

Now, $\frac{dy}{dx} = 0 \Rightarrow 3(x-2)^2 = 0 \Rightarrow x = 2$.

We observe that $\frac{dy}{dx}=3(x-2)^2>0$ for all $x\neq 2$. Thus, $\frac{dy}{dx}$ does not change sign as x increases through $x=2$.

Hence, $x=2$ is neither a point of local maximum nor a point of local minimum. In fact, it is a point of inflection.

EXAMPLE 4 *Show that the function* $f(x)=4x^3-18x^2+27x-7$ *has neither maxima nor minima.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$y=f(x)=4x^3-18x^2+27x-7 \Rightarrow \frac{dy}{dx}=12x^2-36x+27=3(4x^2-12x+9)=3(2x-3)^2$$

The critical points of $y=f(x)$ are given by $\frac{dy}{dx}=0$.

Now, $\frac{dy}{dx}=0 \Rightarrow 3(2x-3)^2=0 \Rightarrow 2x-3=0 \Rightarrow x=\frac{3}{2}$. Clearly, $\frac{dy}{dx}=3(2x-3)^2>0$ for all $x\neq\frac{3}{2}$

Thus, $\frac{dy}{dx}$ does not change its sign as x increases through $x=3/2$ as shown in Fig. 17.20. Hence, $x=3/2$ is neither a point of local maximum nor a point of local minimum. In fact, it is a point of inflection.

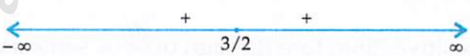

Fig. 17.20 Signs of $\frac{dy}{dx}$ for different values of x.

EXAMPLE 5 *Find the points of local maxima, local minima and the points of inflection of the function* $f(x)=x^5-5x^4+5x^3-1$. *Also, find the corresponding local maximum and local minimum values* **[NCERT EXEMPLAR]**

SOLUTION Let $y=f(x)=x^5-5x^4+5x^3-1$. Then,

$$\frac{dy}{dx}=5x^4-20x^3+15x^2=5x^2(x^2-4x+3)=5x^2(x-1)(x-3)$$

The critical points of $y=f(x)$ are given by $\frac{dy}{dx}=0$.

Now, $\frac{dy}{dx}=0 \Rightarrow 5x^2(x-1)(x-3)=0 \Rightarrow x=0, x=1, x=3$.

Clearly, $\frac{dy}{dx}$ does not change its sign as x increases through 0. So, $x = 0$ is a point of inflection.

It is evident from Fig. 17.21 that $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through 1. So, $x = 1$ is a point of local maximum. The local maximum value of $f(x)$ is $f(1) = 1 - 5 + 5 - 1 = 0$.

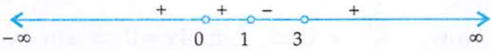

Fig. 17.21 Signs of $\frac{dy}{dx}$ for different values of x.

We observe, from Fig. 17.21, that $\frac{dy}{dx}$ changes its sign from negative to positive as x increases through 3. So, $x = 3$ is a point of local minimum. The local minimum value of $f(x)$ is

$$f(3) = 3^5 - 5 \times 3^4 + 5 \times 3^3 - 1 = -28.$$

EXAMPLE 6 *Find the local maxima or local minima, if any, of the function $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$ using the first derivative test.* **[NCERT]**

SOLUTION We have, $y = f(x) = \sin x + \cos x \Rightarrow \frac{dy}{dx} = \cos x - \sin x$

The critical points of $y = f(x)$ are given by $\frac{dy}{dx} = 0$.

Now, $\frac{dy}{dx} = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ $\left[\because 0 < x < \frac{\pi}{2}\right]$

Now, we will see whether $x = \frac{\pi}{4}$ is a point of local maximum or a point of local minimum or none of these.

In the left neighbourhood $x = \frac{\pi}{4}$, we have

$$x < \frac{\pi}{4} \Rightarrow \cos x > \sin x \Rightarrow \cos x - \sin x > 0 \Rightarrow \frac{dy}{dx} > 0$$

In the right neighbourhood of $x = \frac{\pi}{4}$, we have

$$x > \frac{\pi}{4} \Rightarrow \cos x < \sin x \Rightarrow \cos x - \sin x < 0 \Rightarrow \frac{dy}{dx} < 0$$

Thus, $\frac{dy}{dx}$ changes its sign from positive to negative as x increases through $\frac{\pi}{4}$. So, $f(x)$ attains a local maximum at $x = \frac{\pi}{4}$.

EXAMPLE 7 *Find the local maximum or local minimum, if any, of the function $f(x) = \sin^4 x + \cos^4 x$, $0 < x < \frac{\pi}{2}$ using the first derivative test.*

SOLUTION We have,

$$y = f(x) = \sin^4 x + \cos^4 x$$

$$\Rightarrow \frac{dy}{dx} = 4\sin^3 x \cos x - 4\cos^3 x \sin x = -4\cos x \sin x (\cos^2 x - \sin^2 x)$$

$$\Rightarrow \frac{dy}{dx} = -2\sin 2x \cos 2x = -\sin 4x$$

The critical points of $y = f(x)$ are given by $\frac{dy}{dx} = 0$.

Now, $\frac{dy}{dx} = 0 \Rightarrow -\sin 4x = 0 \Rightarrow \sin 4x = 0 \Rightarrow 4x = \pi \Rightarrow x = \frac{\pi}{4}$ $\left[\because 0 < x < \frac{\pi}{2} \therefore 0 < 4x < 2\pi\right]$

In the left neighbourhood of $x = \frac{\pi}{4}$, we have

$$x < \frac{\pi}{4} \Rightarrow 4x < \pi \Rightarrow \sin 4x > 0 \Rightarrow -\sin 4x < 0 \Rightarrow \frac{dy}{dx} < 0$$

In the right neighbourhood of $x = \frac{\pi}{4}$, we have

$$x > \frac{\pi}{4} \Rightarrow 4x > \pi \Rightarrow \sin 4x < 0 \Rightarrow -4\sin 4x > 0 \Rightarrow \frac{dy}{dx} > 0$$

Thus, $\frac{dy}{dx}$ changes sign from negative to positive as x increases through $\frac{\pi}{4}$. So, $x = \frac{\pi}{4}$ is a point of local minimum. The local minimum value of $f(x)$ at $x = \frac{\pi}{4}$ is

$$f\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^4 + \left(\cos\frac{\pi}{4}\right)^4 = \frac{1}{2}.$$

EXAMPLE 8 *Find the points at which the function f given by* $f(x) = (x-2)^4 (x+1)^3$ has

(i) *local maxima* (ii) *local minima* (iii) *points of inflexion* [NCERT]

SOLUTION We have,

$$f(x) = (x-2)^4 (x+1)^3$$

$$\Rightarrow \quad f'(x) = 4(x-2)^3 (x+1)^3 + 3(x-2)^4 (x+1)^2 = (x-2)^3 (x+1)^2 (7x-2)$$

$$\Rightarrow \quad f'(x) = (x-2)^2 (x+1)^2 (x-2)(7x-2)$$

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow x = 2, -1, \frac{2}{7}$

Since $(x-2)^2 (x+1)^2$ is always positive. So, sign of $f'(x)$ depends upon the sign of $(x-2)(7x-2)$.

The changes in signs of $f'(x)$ as x increases through $\frac{2}{7}$ and 2 are shown in Fig. 17.22.

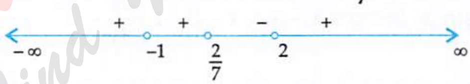

Fig. 17.22 Signs of $f'(x)$ for different values of x.

Clearly, $f'(x)$ changes its sign from positive to negative as increases through $\frac{2}{7}$. So, $x = \frac{2}{7}$ is a point of local maximum.

We observe that $f'(x)$ changes its sign from negative to positive as x increases through 2. So, $x = 2$ is a point of local minimum.

There is no change in the sign of $f'(x)$ as increases through -1. So, $x = -1$ is a point of inflexion.

EXERCISE 17.2

BASIC

Find the points of local maxima or local minima, if any, of the following functions, using the first derivative test. Also, find the local maximum or local minimum values, as the case may be:

1. $f(x) = (x-5)^4$
2. $f(x) = x^3 - 3x$ [NCERT]
3. $f(x) = x^3 (x-1)^2$
4. $f(x) = (x-1)(x+2)^2$

5. $f(x) = \dfrac{1}{x^2+2}$ [NCERT]

6. $f(x) = x^3 - 6x^2 + 9x + 15$

7. $f(x) = \sin 2x,\ 0 < x < \pi$

8. $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$ [NCERT]

9. $f(x) = \cos x,\ 0 < x < \pi$

10. $f(x) = \sin 2x - x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

11. $f(x) = 2\sin x - x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

12. $f(x) = x\sqrt{1-x},\ 0 < x < 1$ [NCERT]

13. $f(x) = x^3(2x-1)^3$

14. $f(x) = \dfrac{x}{2} + \dfrac{2}{x},\ x > 0$

ANSWERS

1. $x = 5$ is a point of local minimum, local minimum value = 0.
2. $x = -1$ is a point of local maximum, local maximum value = 2
   $x = 1$ is a point of local minimum, local minimum value = -2.
3. $x = 1$ is a point of local minimum, local minimum value = 0
   $x = \dfrac{3}{5}$ is a point of local maximum, local maximum value = $-\dfrac{108}{3125}$.
4. $x = 0$ is a point of local minimum, local minimum value = -4
   $x = -2$ is a point of local maximum, local maximum value = 0.
5. Local maximum at $x = 0$, Local maximum value $= \dfrac{1}{2}$
6. $x = 1$ is a point of local maximum, local maximum value = 19
   $x = 3$ is a point of local minimum, local minimum value = 15.
7. $x = \dfrac{\pi}{4}$ is a point of local maximum local maximum value = 1
   $x = \dfrac{3\pi}{4}$ is a point of local minimum local minimum value = -1.
8. $x = \dfrac{3\pi}{4}$ is a point of local maximum, local maximum value = $\sqrt{2}$
   $x = \dfrac{7\pi}{4}$ is a point of local minimum, local minimum value = $-\sqrt{2}$.
9. None in the interval $(0, \pi)$
10. $x = \dfrac{\pi}{6}$ is a point of local maximum, local maximum value = $\dfrac{\sqrt{3}}{2} - \dfrac{\pi}{6}$
    $x = -\dfrac{\pi}{6}$ is a point of local minimum, local minimum value = $-\dfrac{\sqrt{3}}{2} + \dfrac{\pi}{6}$.
11. $x = \dfrac{\pi}{3}$ is a point of local maximum, local maximum value = $\sqrt{3} - \dfrac{\pi}{3}$
    $x = \dfrac{-\pi}{3}$ is a point of local minimum, local minimum value = $-\sqrt{3} + \dfrac{\pi}{3}$.
12. Local maximum at $x = \dfrac{2}{3}$, Local Maximum value $= \dfrac{2\sqrt{3}}{9}$
13. Minimum at $x = \dfrac{1}{4}$, Local Minimum value = $-\dfrac{1}{512}$
14. Minimum at $x = 2$, Local Minimum value = 2

HINTS TO SELECTED PROBLEMS

2. We have, $f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 3(x-1)\,(x+1) = 0 \Rightarrow x = 1, -1$.

The changes in signs of $f'(x)$ for different value of x are as shown in Fig. 17.23.

Fig. 17.23 Signs of $f'(x)$ for different values of x.

Clearly, $f'(x)$ changes its sign from positive to negative as x increases through -1. So, $x = -1$ is a point of local maximum with the local maximum value given by $f(-1) = (-1)^3 - 3(-1) = 2$.

As $f'(x)$ changes its sign from negative to positive as x increases through 1. So, $x = 1$ is a point of local minimum with the local minimum value $f(1) = 1 - 3 = -2$.

5. We have, $f(x) = \dfrac{1}{x^2+2} \Rightarrow f'(x) = \dfrac{-2x}{(x^2+2)^2}$

Fig. 17.24 Signs of $f'(x)$ for different values of x.

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow \dfrac{-2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$

The signs of $f'(x)$ for different values of x are shown in Fig. 17.24. Clearly, $f'(x)$ changes its sign from positive to negative as increases through 0. So, $x = 0$ is a point of local maximum with the local maximum value $f(0) = 1/2$.

8. We have, $f(x) = \sin x - \cos x,\ 0 < x < 2\pi \Rightarrow f'(x) = \cos x + \sin x$

For a local maximum or minimum, we must have

$$f'(x) = 0 \Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

Now, $f'(x) = \cos x + \sin x = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos x + \dfrac{1}{\sqrt{2}}\sin x\right) = \sqrt{2}\sin\left(x + \dfrac{\pi}{4}\right)$

In the left neighbourhood of $x = 3\pi/4$, we have

$$x < \frac{3\pi}{4} \Rightarrow x + \frac{\pi}{4} < \pi \Rightarrow \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) > 0 \Rightarrow f'(x) > 0$$

In the right neighbourhood of $x = \dfrac{3\pi}{4}$, we have

$$x > \frac{3\pi}{4} \Rightarrow x + \frac{\pi}{4} > \pi \Rightarrow \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) < 0 \Rightarrow f'(x) < 0$$

Thus, $f'(x)$ changes its sign from positive to negative as x increases through $\dfrac{3\pi}{4}$. So, $x = \dfrac{3\pi}{4}$ is a point of local maximum with the local maximum value given by

$$f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \sqrt{2}$$

In the left neighbourhood of $x = \dfrac{7\pi}{4}$, we have

$$x < \frac{7\pi}{4} \Rightarrow x + \frac{\pi}{4} < 2\pi \Rightarrow \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) < 0 \Rightarrow f'(x) < 0$$

In the right neighbourhood of $x = \dfrac{7\pi}{4}$, we have

$$x > \frac{7\pi}{4} \Rightarrow x + \frac{\pi}{4} > 2\pi \Rightarrow \sqrt{2}\sin\left(x + \frac{\pi}{4}\right) > 0 \Rightarrow f'(x) > 0$$

Thus, $f'(x)$ changes its sign from negative to positive as x increases through $\dfrac{7\pi}{4}$.

So, $x = \dfrac{7\pi}{4}$ is a point of local minimum with local minimum value given by

$$f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\sqrt{2}.$$

12. We have

$$f(x) = x\sqrt{1-x},\ x > 0 \Rightarrow f'(x) = \sqrt{1-x} - \frac{x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$$

At critical points of $f(x)$, we must have $f'(x) = 0 \Rightarrow \dfrac{2-3x}{2\sqrt{1-x}} = 0 \Rightarrow x = \dfrac{2}{3}$

Now, $f'(x) = \dfrac{2-3x}{2\sqrt{1-x}} = -3\left(\dfrac{x - \frac{2}{3}}{2\sqrt{1-x}}\right)$

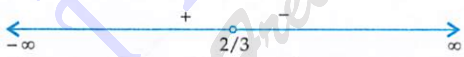

Fig. 17.25 Signs of $f'(x)$ for different values of x.

The changes in signs of $f'(x)$ as x increases through $2/3$ are shown below:

Clearly, $f'(x)$ changes its sign from positive to negative as x increases through $2/3$. So, $x = 2/3$ is a point of local maximum with the local maximum value $f\left(\dfrac{2}{3}\right) = \dfrac{2}{3\sqrt{3}}$

17.5 HIGHER ORDER DERIVATIVE TEST

As we have seen in the previous section that finding the local maximum or local minimum by first derivative test is very time consuming and of course tedious for beginners because it is slightly difficult to determine the change in the sign of $f'(x)$ as x increases through the points given by $f'(x) = 0$. We have another test known as the Higher order derivative test which enables us to find the points of local maxima or local minima more easily and more quickly.

THEOREM *(Higher Order Derivative Test) Let f be a differentiable function defined on an interval I and let c be an interior point of I such that*

(i) $f'(c) = f''(c) = f'''(c) = \dots = f^{n-1}(c) = 0$ *and,* *(ii)* $f^{n}(c)$ *exists and is non-zero.*

Then,

if n is even and $f^{n}(c) < 0 \Rightarrow x = c$ *is a point of local maximum*

if n is even and $f^{n}(c) > 0 \Rightarrow x = c$ *is a point of local minimum*

if n is odd, $x = c$ *is neither a point of local maximum nor a point of local minimum.*

This theorem suggests the following algorithm to find the points of local maximum and local minimum.

ALGORITHM

Step I *Find* $f'(x)$

Step II *Put* $f'(x) = 0$ *and solve this equation for x. Let* $c_1, c_2, \dots, c_n$ *be the roots of this equation. Points* $c_1, c_2, \dots, c_n$ *are stationary values or critical points of* $f(x)$ *and these are the possible points where the function can attain a local maximum or a local minimum. So, we test the function at each one of these points.*

Step III *Find $f''(x)$. Consider $x = c_1$.*

If $f''(c_1) < 0$, then $x = c_1$ is a point of local maximum.

If $f''(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f''(c_1) = 0$, we must find $f'''(x)$ and substitute in it c_1 for x.

If $f'''(c_1) \neq 0$, then $x = c_1$ is neither a point of local maximum nor a point of local minimum and is called the point of inflection.

If $f'''(c_1) = 0$, we must find $f^{IV}(x)$ and substitute in it c_1 for x.

If $f^{IV}(c_1) < 0$, then $x = c_1$ is a point of local maximum and if $f^{IV}(c_1) > 0$, then $x = c_1$ is a point of local minimum.

If $f^{IV}(c_1) = 0$, we must find $f^{V}(x)$, and so on. Similarly, the points $c_2, c_3, \ldots,$ may be tested.

POINT OF INFLECTION *An arc of a curve $y = f(x)$ is called concave upward if, at each of its points, the arc lies above the tangent at the point (see Fig. 17.26).*

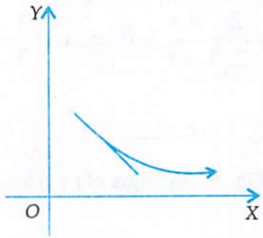

Fig. 17.26 Concave upward curve

Fig. 17.27 Concave upward curve

If $y = f(x)$ is a concave upward curve, then as x increases, $f'(x)$ either is of the same sign and increasing (see Fig. 17.26) or changes sign from negative to positive (see Fig. 17.27). In either case $f'(x)$ is increasing and so $f''(x) > 0$. Thus, for a concave upward curve $f''(x) > 0$.

An arc of a curve $y = f(x)$ is called *concave downward* if, at each of its points, the arc lies below the tangent at the point.

If an arc of a curve $y = f(x)$ is *concave downward*, then as x increases, $f'(x)$ either is of the same sign and decreasing (see Fig. 17.28) or changes sign from positive to negative (see Fig. 17.29). In either case $f'(x)$ is decreasing and so $f''(x) < 0$. Thus, for a concave downward curve $f''(x) < 0$.

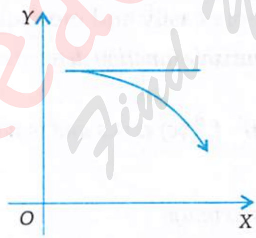

Fig. 17.28 Concave downward curve

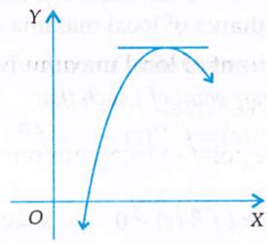

Fig. 17.29 Concave downward curve

POINT OF INFLEXION *A point of inflection is a point at which a curve is changing concave upward to concave downward, or vice-versa.*

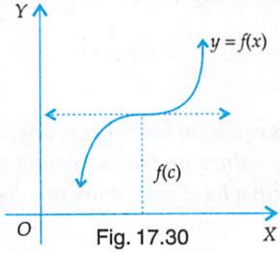

Fig. 17.30

A curve $y = f(x)$ has one of its points $x = c$ as an inflection point, if

(i) $f''(c) = 0$ or is not defined and

(ii) $f''(x)$ changes sign as x increases through $x = c$.

The later condition may be replaced by $f'''(c) \neq 0$ when $f'''(c)$ exists.

Thus, $x = c$ is a point of inflection if $f''(c) = 0$ and $f'''(c) \neq 0$.

PROPERTIES OF MAXIMA AND MINIMA

(i) If $f(x)$ is continuous function in its domain, then at least one maximum or one minimum must lie between two equal values of $f(x)$.

(ii) Maxima and Minima occur alternately, that is, between two maxima there is one minimum and vice-versa.

(iii) If $f(x) \to \infty$ as $x \to a$ or b and $f'(x) = 0$ only for one value of x (say c) between a and b, then $f(c)$ is necessarily the minimum and the least value.

If $f(x) \to -\infty$ as $x \to a$ or b, then $f(c)$ is necessarily the maximum and the greatest value.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function* $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$.

SOLUTION We have, [NCERT EXEMPLAR]

$$f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105 \Rightarrow f'(x) = -3x^3 - 24x^2 - 45x = -3x(x^2 + 8x + 15)$$

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow -3x(x^2 + 8x + 15) = 0 \Rightarrow -3x(x+3)(x+5) = 0 \Rightarrow x = 0, -3, -5$

Thus, $x = 0$, $x = -3$ and $x = -5$ are the possible points of local maxima or minima.

Let us now test the function at each of these points. We have,

$$f'(x) = -3x^3 - 24x^2 - 45x \Rightarrow f''(x) = -9x^2 - 48x - 45$$

At $x = 0$: We have, $f''(0) = -45 < 0$.

So, $x = 0$ is a point of local maximum. The local maximum value of $f(x)$ at $x = 0$ is $f(0) = 105$.

At $x = -3$: We have, $f''(-3) = -9(-3)^2 - 48(-3) - 45 = 18 > 0$

So, $x = -3$ is a point of local minimum. The local minimum value of $f(x)$ at $x = -3$ is

$$f(-3) = -\frac{3}{4}(-3)^4 - 8(-3)^3 \frac{45}{2}(-3)^2 + 105 = \frac{231}{4}$$

At $x = -5$: We have, $f''(-5) = -9(-5)^2 - 48(-5) - 45 = -30 < 0$

So, $x = -5$ is a point of local maximum. The local maximum value of $f(x)$ at $x = -5$ is

$$f(-5) = -\frac{3}{4}(-5)^4 - 8(-5)^3 - \frac{45}{2}(-5)^2 + 105 = \frac{295}{4}$$

EXAMPLE 2 *Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function* $f(x) = 2x^3 - 21x^2 + 36x - 20$.

SOLUTION We have, $f(x) = 2x^3 - 21x^2 + 36x - 20 \Rightarrow f'(x) = 6x^2 - 42x + 36$

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0 \Rightarrow 6x^2 - 42x + 36 = 0 \Rightarrow (x-1)(x-6) = 0 \Rightarrow x = 1, 6.$

Thus, $x=1$ and $x=6$ are the possible points of local maxima or minima.

Now, we test the function at each of these points.

We have, $f''(x) = 12x - 42$.

At $x=1$: We have, $f''(1) = 12 - 42 = -30 < 0$.

So, $x=1$ is a point of local maximum. The local maximum value is $f(1) = 2 - 21 + 36 - 20 = -3$.

At $x=6$: We have, $f''(6) = 12(6) - 42 = 30 > 0$

So $x=6$ is a point of local minimum. The local minimum value is

$$f(6) = 2(6)^3 - 21(6)^2 + 36 \times 6 - 20 = -128.$$

EXAMPLE 3 *Find the points of local maxima and local minima, if any, of each of the following functions. Find also the local maximum and local minimum values, as the case may be:*

(i) $f(x) = \sin 2x - x$, *where* $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (ii) $f(x) = \sin x + \frac{1}{2}\cos 2x$, *where* $0 < x < \frac{\pi}{2}$

(iii) $f(x) = \sin^4 x + \cos^4 x,\ 0 < x < \frac{\pi}{2}$

SOLUTION (i) We have, $f(x) = \sin 2x - x \Rightarrow f'(x) = 2\cos 2x - 1$

The critical points of $f(x)$ are given by $f'(x) = 0$.

$\therefore\ f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x = -\frac{\pi}{3}$ or, $2x = \frac{\pi}{3}$ $\left[\because -\frac{\pi}{2} < x < \frac{\pi}{2} \therefore -\pi < 2x < \pi\right]$

$\Rightarrow x = -\frac{\pi}{6}$ or, $x = \frac{\pi}{6}$

Thus, $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$ are possible points of local maxima or minima.

Now, we test the function at each of these points.

Clearly, $f''(x) = -4\sin 2x$.

At $x = -\pi/6$: We have, $f''\left(-\frac{\pi}{6}\right) = -4\sin\left(-\frac{\pi}{3}\right) = -4 \times \frac{-\sqrt{3}}{2} = 2\sqrt{3} > 0$. So, $x = -\frac{\pi}{6}$ is a point of local minimum. The local minimum value is $f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$.

At $x = \pi/6$: We have, $f''\left(\frac{\pi}{6}\right) = -4\sin\frac{\pi}{3} = -4\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} < 0$.

So, $x = \frac{\pi}{6}$ is a point of local maximum. The local maximum value is $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

(ii) We have, $f(x) = \sin x + \frac{1}{2}\cos 2x$, where $0 < x < \frac{\pi}{2} \Rightarrow f'(x) = \cos x - \sin 2x$.

The critical points of $f(x)$ are given by $f'(x) = 0$.

$\therefore\ f'(x) = 0 \Rightarrow \cos x - \sin 2x = 0 \Rightarrow \cos x - 2\sin x\cos x = 0 \Rightarrow \cos x(1 - 2\sin x) = 0$

$\Rightarrow \cos x = 0$ or, $1 - 2\sin x = 0 \Rightarrow \cos x = 0$ or, $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$ $\left[\because 0 < x < \frac{\pi}{2}\right]$

Thus, $x = \frac{\pi}{6}$ is a point of local maximum or local minimum. Let us now test the function at this point.

Clearly, $f''(x) = -\sin x - 2\cos 2x \Rightarrow f''\left(\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} - 2\cos\frac{\pi}{3} = -\frac{1}{2} - 2 \times \frac{1}{2} = -\frac{3}{2} < 0$

So, $x = \frac{\pi}{6}$ is the point of local maximum.

The local maximum value is $f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{3}{4}$

(iii) We have, $f(x) = \sin^4 x + \cos^4 x$, where $0 < x < \frac{\pi}{2}$.

$\therefore\ f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x = -4\cos x \sin x(\cos^2 x - \sin^2 x) = -2\sin 2x \cos 2x$

$\Rightarrow f'(x) = -\sin 4x.$

At points of local maximum and minimum, we must have

$$f'(x) = 0 \Rightarrow -\sin 4x = 0 \Rightarrow 4x = \pi \Rightarrow x = \frac{\pi}{4} \qquad \left[\because\ 0 < x < \frac{\pi}{2} \therefore 0 < 4x < 2\pi\right]$$

Now, $f''(x) = -4\cos 4x \Rightarrow f''\left(\frac{\pi}{4}\right) = -4\cos\pi = (-4)(-1) = 4 > 0$

So, $x = \frac{\pi}{4}$ is a point of local minimum and the local minimum value is

$$f\left(\frac{\pi}{4}\right) = \sin^4\frac{\pi}{4} + \cos^4\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^4 + \left(\frac{1}{\sqrt{2}}\right)^4 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

EXAMPLE 4 *Find the points of local maxima or local minima, if any, of the following functions. Find also the local maximum or local minimum values, as the case may be:*

(i) $f(x) = \sin x + \cos x$, *where* $0 < x < \frac{\pi}{2}$ (ii) $f(x) = \sin x - \cos x$, *where* $0 < x < 2\pi$

[CBSE 2015]

(iii) $f(x) = \sin 2x$, *where* $0 < x < \pi$ (iv) $f(x) = 2\cos x + x$, *where* $0 < x < \pi$.

(v) $f(x) = 2\sin x - x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

SOLUTION (i) We have, $f(x) = \sin x + \cos x$, where $0 < x < \frac{\pi}{2}$.

$\therefore\quad f'(x) = \cos x - \sin x.$

The critical points of $f(x)$ are given $f'(x) = 0$.

$\therefore\quad f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4} \quad \left[\because 0 < x < \frac{\pi}{2}\right]$

Thus, $x = \frac{\pi}{4}$ is a point of local maximum or minimum.

Now, $f''(x) = -\sin x - \cos x \Rightarrow f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4} = -\sqrt{2} < 0.$

So, $x = \frac{\pi}{4}$ is a point of local maximum.

The local maximum value is $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$

(ii) We have,

$$f(x) = \sin x - \cos x, \text{ where } 0 < x < 2\pi \Rightarrow f'(x) = \cos x + \sin x$$

At points of local maximum and local minimum, we must have

$$f'(x) = 0$$

$$\Rightarrow \quad \cos x + \sin x = 0 \Rightarrow \sin x = -\cos x \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4} \text{ or, } x = \frac{7\pi}{4} \quad [\because\ 0 < x < 2\pi]$$

Thus, $x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$ are possible points of local maximum or minimum. Let us now test the function at each of these points. Clearly, $f''(x) = -\sin x + \cos x$.

At $x = 3\pi/4$: We have,

$f''\left(\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0$. So, $x = \frac{3\pi}{4}$ is the point of local maximum.

The local maximum value is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{4} - \cos\frac{3\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$.

At $x = \frac{7\pi}{4}$: We have, $f''\left(\frac{7\pi}{4}\right) = -\sin\frac{7\pi}{4} + \cos\frac{7\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} > 0$.

So, the function attains a local minimum at $x = \frac{7\pi}{4}$.

The local minimum value is $f\left(\frac{7\pi}{4}\right) = \sin\frac{7\pi}{4} - \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$.

(iii) We have,

$$f(x) = \sin 2x, \text{ where } 0 < x < \pi \Rightarrow f'(x) = 2\cos 2x.$$

At points of local maximum or local minimum, we must have

$$f'(x) = 0 \Rightarrow 2\cos 2x = 0 \Rightarrow \cos 2x = 0 \Rightarrow 2x = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4} \quad [\because 0 < x < \pi \therefore 0 < 2x < 2\pi]$$

Thus, $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$ are possible points of local maximum or local minimum. Let us now test the function at these points. Clearly, $f''(x) = -4\sin 2x$.

At $x = \frac{\pi}{4}$: We have, $f''\left(\frac{\pi}{4}\right) = -4\sin\frac{\pi}{2} = -4 < 0$. So, $x = \frac{\pi}{4}$ is a point of local maximum.

The local maximum value of $f(x)$ is $f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{2} = 1$.

At $x = \frac{3\pi}{4}$: We have, $f''\left(\frac{3\pi}{4}\right) = -4\sin\frac{3\pi}{2} = 4 > 0$. So, $x = \frac{3\pi}{4}$, is a point of local minimum.

The local minimum value of $f(x)$ is $f\left(\frac{3\pi}{4}\right) = \sin\frac{3\pi}{2} = -1$.

(iv) We have, $f(x) = 2\cos x + x$, where $0 < x < \pi \Rightarrow f'(x) = -2\sin x + 1$

At points of local maximum and minimum, we must have

$$f'(x) = 0 \Rightarrow -2\sin x + 1 = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad [\because 0 < x < \pi]$$

Thus, $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are possible points of local maximum or minimum. Let us now test the function at these points.

Clearly, $f''(x) = -2\cos x$

At $x = \frac{\pi}{6}$: We have, $f''\left(\frac{\pi}{6}\right) = -2\cos\frac{\pi}{6} = -\sqrt{3} < 0$. So, $x = \frac{\pi}{6}$ is a point of local maximum.

The local maximum value of $f(x)$ is $f\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6} + \frac{\pi}{6} = \sqrt{3} + \frac{\pi}{6}$

At $x=\frac{5\pi}{6}$: We have, $f''\left(\frac{5\pi}{6}\right)=-2\cos\frac{5\pi}{6}=\sqrt{3}>0$. So, $x=\frac{5\pi}{6}$ is a point of local minimum.

The local minimum value of $f(x)$ is $f\left(\frac{5\pi}{6}\right)=2\cos\frac{5\pi}{6}+\frac{5\pi}{6}=-\sqrt{3}+\frac{5\pi}{6}$.

(v) We have, $f(x) = 2\sin x - x$, where $-\pi/2 < x < \pi/2 \Rightarrow f'(x) = 2\cos x - 1$

The critical points of $f(x)$ are given by $f'(x)=0$.

$\therefore \quad f'(x) = 0 \Rightarrow 2\cos x - 1 = 0 \Rightarrow \cos x = 1/2 \Rightarrow x = \pm\pi/3 \qquad [\because -\pi/2 < x < \pi/2]$

Thus, $x=\pm\pi/3$ are points of local maximum or minimum. Clearly, $f''(x)=-2\sin x$.

At $x=-\pi/3$: We have,

$$f''(-\pi/3)=-2\sin(-\pi/3)=2\sin\pi/3=2\sqrt{3}/2=\sqrt{3}>0.$$

So, $x=-\pi/3$ is a point of local minimum.

The local minimum value is $f(-\pi/3) = 2\sin(-\pi/3)-(-\pi/3) = -\sqrt{3}+\pi/3$.

At $x=\pi/3$: We have, $f''(\pi/3)=-2\sin\pi/3=-\sqrt{3}<0$. So, $x=\pi/3$ is a point of local maximum.

The local maximum value is $f(\pi/3)=2\sin\pi/3-\pi/3=\sqrt{3}-\pi/3$.

EXAMPLE 5 *Find the local maximum and local minimum values of*

$$f(x)=\sec x+\log\cos^2 x,\ 0<x<2\pi.$$

[NCERT EXEMPLAR]

SOLUTION We have,

$$f(x)=\sec x+2\log\cos x \Rightarrow f'(x)=\sec x\tan x-2\tan x=\tan x(\sec x-2)$$

The critical points of $f(x)$ are given by $f'(x)=0$.

$\therefore \quad f'(x)=0 \Rightarrow \tan x(\sec x-2)=0 \Rightarrow \tan x=0$ or $\sec x=2 \Rightarrow \tan x=0$ or $\cos x=\frac{1}{2}$

$\Rightarrow \quad x=\pi,\ x=\frac{\pi}{3},\frac{5\pi}{3} \qquad [\because 0<x<2\pi]$

Thus, $x=\pi,\ x=\frac{\pi}{3},\frac{5\pi}{3}$ are possible points of local maximum and local minimum.

Now, $\quad f'(x)=\tan x(\sec x-2)$

$\Rightarrow \quad f''(x)=\sec^2 x(\sec x-2)+\tan^2 x\sec x=\sec^2 x(\sec x-2)+\sec x(\sec^2 x-1)$

$\Rightarrow \quad f''(x)=2\sec^3 x-2\sec^2 x-\sec x$

Let us now investigate critical points for points of local maximum and local minimum.

At $x=\frac{\pi}{3}$: We obtain, $f''\left(\frac{\pi}{3}\right)=2\sec^3\frac{\pi}{3}-2\sec^2\frac{\pi}{3}-\sec\frac{\pi}{3}=2\times8-2\times4-2=6>0$

Thus, $x=\frac{\pi}{3}$ is a point of local minimum. The local minimum value of $f(x)$ is given by

$$f\left(\frac{\pi}{3}\right)=\sec\frac{\pi}{3}+\log\cos^2\frac{\pi}{3}=2+\log\frac{1}{4}=2-2\log 2$$

At $x=\pi$: We obtain, $f''(\pi)=2\sec^3\pi-2\sec^2\pi-\sec\pi=-2-2+1=-3<0$

Thus, $x=\pi$ is a point of local maximum. The local maximum value of $f(x)$ is given by

$$f(\pi)=\sec\pi+\log\cos^2\pi=-1+\log 1=-1.$$

At $x=\frac{5\pi}{3}$: We obtain, $f''\left(\frac{5\pi}{3}\right)=2\sec^3\frac{5\pi}{3}-2\sec^2\frac{5\pi}{3}-\sec\frac{5\pi}{3}=2(2)^3-2(2)^2+2=10>0$

Thus, $x=\frac{5\pi}{3}$ is a point of local minimum.

The local minimum value is given by $f\left(\frac{5\pi}{3}\right) = \sec\frac{5\pi}{3} + \log\cos^2\frac{5\pi}{3} = 2 + \log\frac{1}{4} = 2 - 2\log 2.$

EXAMPLE 6 *Show that none of the following functions has a local maximum or a local minimum:*

(i) $x^3 + x^2 + x + 1$ (ii) e^x (iii) $\log x$ (iv) $\cos x,\ 0 < x < \pi$ **[NCERT]**

SOLUTION (i) Let $f(x) = x^3 + x^2 + x + 1$. Then, $f'(x) = 3x^2 + 2x + 1$.

At points of local maximum or minimum, we have

$$f'(x) = 0 \Rightarrow 3x^2 + 2x + 1 = 0$$

But, this equation gives imaginary values of x. So, $f'(x) \neq 0$ for any real value of x. Hence, $f(x)$ does not have a maximum or minimum.

(ii) Let $f(x) = e^x$. Then, $f'(x) = e^x$. Clearly, $f'(x) \neq 0$ for any value of x. So, $f(x) = e^x$ does not have a maximum or a minimum.

(iii) Let $f(x) = \log x$. Then, $f'(x) = \frac{1}{x}$. Clearly, $f'(x) \neq 0$ for any value of $x \in$ Domain (f).

So, $f(x) = \log x$ does not have a maximum or a minimum.

(iv) Let $f(x) = \cos x$. Then, $f'(x) = -\sin x$. Clearly, $f'(x) \neq 0$ for any $x \in (0, \pi)$.

So, $f(x) = \cos x$ does not have a maximum or minimum on $(0, \pi)$.

EXAMPLE 7 *Find the maximum profit that a company can make, if the profit function is given* $P(x) = 41 + 24x - 18x^2$. **[NCERT]**

SOLUTION We have,

$$P(x) = 41 + 24x - 18x^2 \Rightarrow \frac{d}{dx}\left(P(x)\right) = 24 - 36x \text{ and } \frac{d^2}{dx^2}\left(P(x)\right) = -36$$

For maximum or minimum, we must have

$$\frac{d}{dx}\left(P(x)\right) = 0 \Rightarrow 24 - 36x = 0 \Rightarrow x = \frac{2}{3}$$

Also, $\left\{\frac{d^2}{dx^2}\left(P(x)\right)\right\}_{x=2/3} = -36 < 0$. So, profit is maximum when $x = \frac{2}{3}$.

$$\text{Maximum profit} = (\text{Value of } P(x) \text{ at } x = 2/3) = 41 + 24 \times (2/3) - 18\,(2/3)^2 = 49$$

EXAMPLE 8 *At what points, the slope of the curve* $y = -x^3 + 3x^2 + 9x - 27$ *is maximum? Also, find the maximum slope.* **[NCERT EXEMPLAR]**

SOLUTION The slope m of the curve $y = -x^3 + 3x^2 + 9x - 27$ at point (x, y) is given by

$$m = \frac{dy}{dx} = -3x^2 + 6x + 9 \qquad \ldots(i)$$

$$\therefore \quad \frac{dm}{dx} = -6x + 6 \text{ and } \frac{d^2m}{dx^2} = -6$$

For maximum or minimum values of m, we must have

$$\frac{dm}{dx} = 0 \Rightarrow -6x + 6 = 0 \Rightarrow x = 1$$

Clearly, $\frac{d^2m}{dx^2} = -6 < 0$ for all x. So, m is maximum at $x = 1$. Putting $x = 1$ in (i), we obtain $m = 12$.

Putting $x = 1$ in the equation $y = -x^3 + 3x^2 + 9x - 27$, we obtain $y = -16$. Hence, the slope of the given curve is maximum at the point $(1, -16)$ and the maximum value of the slope is 12.

EXAMPLE 9 *If $f(x) = a \log|x| + bx^2 + x$ has extreme values at $x = -1$ and at $x = 2$, then find a and b.*

SOLUTION We observe that $f(x)$ is defined for all $x \neq 0$.

Now, $f(x) = a \log|x| + bx^2 + x \Rightarrow f'(x) = \frac{a}{x} + 2bx + 1$

It is given that $f(x)$ has extreme values at $x = -1$ and $x = 2$.

$\therefore \quad f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow \quad -a - 2b + 1 = 0$ and $\frac{a}{2} + 4b + 1 = 0 \Rightarrow a + 2b = 1$ and $a + 8b = -2$

Solving these equations, we get: $a = 2$ and $b = -1/2$.

EXAMPLE 10 *It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval $[0, 2]$. Find the value of a.* **[NCERT]**

SOLUTION Let $f(x) = x^4 - 62x^2 + ax + 9$. Then, $f'(x) = 4x^3 - 124x + a$.

It is given that $f(x)$ attains its maximum at $x = 1$.

$\therefore \quad f'(1) = 0 \Rightarrow 4 - 124 + a = 0 \Rightarrow a = 120$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 11 *If $y = \frac{ax - b}{(x-1)(x-4)}$ has a turning point $P(2, -1)$, find the values of a and b and show that y is maximum at P.*

SOLUTION We have,

$$y = \frac{ax - b}{(x-1)(x-4)} = \frac{ax - b}{x^2 - 5x + 4} \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2 - 5x + 4)a - (ax - b)(2x - 5)}{(x^2 - 5x + 4)^2} \quad \text{...(ii)}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_P = \frac{(4 - 10 + 4)a - (2a - b)(4 - 5)}{(4 - 10 + 4)^2} = -\frac{b}{4}$$

Since P is a turning point of the curve (i). Therefore,

$$\left(\frac{dy}{dx}\right)_P = 0 \Rightarrow -\frac{b}{4} = 0 \Rightarrow b = 0 \quad \text{...(iii)}$$

Since $P(2, -1)$ lies on $y = \frac{ax - b}{(x-1)(x-4)}$. Therefore,

$$-1 = \frac{2a - b}{(2-1)(2-4)} \Rightarrow -1 = \frac{2a - b}{-2} \Rightarrow 2a - b = 2 \quad \text{...(iv)}$$

From (iii) and (iv), we get $a = 1, b = 0$. Substituting the values of a and b in (ii), we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 4) - x(2x - 5)}{(x^2 - 5x + 4)^2} = \frac{-x^2 + 4}{(x^2 - 5x + 4)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(x^2 - 5x + 4)^2(-2x) - (-x^2 + 4)\, 2(x^2 - 5x + 4)(2x - 5)}{(x^2 - 5x + 4)^4}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-2x(x^2 - 5x + 4) + 2(x^2 - 4)(2x - 5)}{(x^2 - 5x + 4)^3}$$

Now, $\left(\frac{dy}{dx}\right)_{(2,-1)} = 0$ and, $\left(\frac{d^2y}{dx^2}\right)_{(2,-1)} = \frac{(-2)(-4)}{(-2)^3} = -1 < 0$

So, y is maximum at P when $a = 1$ and $b = 0$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *Show that the maximum value of* $\left(\frac{1}{x}\right)^x$ *is* $e^{1/e}$.

SOLUTION Let $y = \left(\frac{1}{x}\right)^x = x^{-x} = e^{-x \ln x}$. Differentiating with respect to x, we obtain

$$\frac{dy}{dx} = e^{-x \ln x} \frac{d}{dx}(-x \ln x)$$

$$\Rightarrow \quad \frac{dy}{dx} = -e^{-x \log x}(1 + \ln x) = -y(1 + \ln x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{dy}{dx}(1 + \ln x) - \frac{y}{x} \qquad \text{[Differentiating with respect to } x]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^{-x}(1 + \ln x)^2 - \frac{x^{-x}}{x} = x^{-x}(1 + \ln x)^2 - x^{-x-1}$$

At points of local maximum and local minimum, we must have

$$\frac{dy}{dx} = 0 \Rightarrow -y(1 + \ln x) = 0 \Rightarrow 1 + \ln x = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$$

Now,

$$\left(\frac{d^2y}{dx^2}\right)_{x=1/e} = \left(\frac{1}{e}\right)^{-1/e}\left(1 + \ln\frac{1}{e}\right)^2 - \left(\frac{1}{e}\right)^{-1/e-1} = -\left(\frac{1}{e}\right)^{-1/e} \times 0 - e^{e^{-1}+1} = -e^{e^{-1}+1} < 0$$

So, $x = 1/e$ is a point of local maximum. The local maximum value of y is obtained by putting $x = 1/e$ in y and is equal to $e^{1/e}$.

EXAMPLE 13 *Show that* $\sin^p \theta \cos^q \theta$ *attains a maximum, when* $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

SOLUTION Let $y = \sin^p \theta \cos^q \theta$. Then,

$$\frac{dy}{d\theta} = p \sin^{p-1}\theta \cos\theta \cos^q\theta + \sin^p\theta \, q \cos^{q-1}\theta(-\sin\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = p \sin^{p-1}\theta \cos^{q+1}\theta - q \sin^{p+1}\theta \cos^{q-1}\theta = \sin^{p-1}\theta \cos^{q-1}\theta (p\cos^2\theta - q\sin^2\theta)$$

$$\Rightarrow \quad \frac{dy}{d\theta} = \sin^p\theta\cos^q\theta\left(\frac{p\cos^2\theta - q\sin^2\theta}{\sin\theta\cos\theta}\right) = \sin^p\theta\cos^q\theta(p\cot\theta - q\tan\theta)$$

For maximum or minimum, we must have

$$\frac{dy}{d\theta} = 0$$

$$\Rightarrow \quad \sin^p\theta\cos^q\theta(p\cot\theta - q\tan\theta) = 0$$

$$\Rightarrow \quad \sin^p\theta = 0 \text{ or, } \cos^q\theta = 0 \text{ or, } p\cot\theta - q\tan\theta = 0$$

$$\Rightarrow \quad \sin^p\theta = 0 \text{ or, } \cos^q\theta = 0 \text{ or, } \tan\theta = \sqrt{\frac{p}{q}} \Rightarrow \theta = 0 \text{ or, } \theta = \frac{\pi}{2} \text{ or, } \theta = \tan^{-1}\sqrt{\frac{p}{q}} = \alpha \text{ (say)}$$

Now, $\frac{dy}{d\theta} = \sin^p \theta \cos^q \theta\,(p \cot \theta - q \tan \theta) \Rightarrow \frac{dy}{d\theta} = y\,(p \cot \theta - q \tan \theta)$

$\therefore \quad \frac{d^2y}{d\theta^2} = \frac{dy}{d\theta}(p \cot \theta - q \tan \theta) + y\,(-p \operatorname{cosec}^2 \theta - q \sec^2 \theta)$

$\Rightarrow \quad \left(\frac{d^2y}{d\theta^2}\right)_{\theta=\alpha} = \left(\frac{dy}{d\theta}\right)_{\theta=\alpha}\left(p\sqrt{\frac{q}{p}} - q\sqrt{\frac{p}{q}}\right) + \sin^p \theta \cos^q \theta \left(-p \operatorname{cosec}^2 \theta - q \sec^2 \theta\right)$

$\Rightarrow \quad \left(\frac{d^2y}{d\theta^2}\right)_{\theta=\alpha} = 0 - \sin^p \theta \cos^q \theta\,(p \operatorname{cosec}^2 \theta + q \sec^2 \theta) < 0$

Hence, y is maximum when $\theta = \alpha = \tan^{-1}\sqrt{\frac{p}{q}}$.

EXERCISE 17.3

BASIC

1. Find the points of local maxima or local minima and corresponding local maximum and local minimum values of each of the following functions. Also, find the points of inflection, if any:
   (i) $f(x) = x^4 - 62x^2 + 120x + 9$
   (ii) $f(x) = x^3 - 6x^2 + 9x + 15$
   (iii) $f(x) = (x-1)(x+2)^2$
   (iv) $f(x) = 2/x - 2/x^2,\ x > 0$
   (v) $f(x) = x\,e^x$
   (vi) $f(x) = x/2 + 2/x,\ x > 0$
   (vii) $f(x) = (x+1)(x+2)^{1/3},\ x \geq -2$
   (viii) $f(x) = x\sqrt{32 - x^2},\ -5 \leq x \leq 5$
   (ix) $f(x) = x^3 - 2ax^2 + a^2 x,\ a > 0,\ x \in R$
   (x) $f(x) = x + \frac{a^2}{x},\ a > 0,\ x \neq 0$
   (xi) $f(x) = x\sqrt{2 - x^2}\ -\sqrt{2} \leq x \leq \sqrt{2}$
   (xii) $f(x) = x + \sqrt{1 - x},\ x \leq 1$
2. Find the local extremum values of the following functions :
   (i) $f(x) = (x-1)(x-2)^2$
   (ii) $f(x) = x\sqrt{1-x},\ x \leq 1$
   (iii) $f(x) = -(x-1)^3 (x+1)^2$

BASED ON LOTS

3. The function $y = a \log x + bx^2 + x$ has extreme values at $x = 1$ and $x = 2$. Find a and b.
4. Show that $\frac{\log x}{x}$ has a maximum value at $x = e$. [NCERT]
5. Find the maximum and minimum values of the function $f(x) = \frac{4}{x+2} + x$.
6. Find the maximum and minimum values of $f(x) = \tan x - 2x$.
7. If $f(x) = x^3 + ax^2 + bx + c$ has a maximum at $x = -1$ and minimum at $x = 3$. Determine a, b and c.
8. Prove that $f(x) = \sin x + \sqrt{3} \cos x$ has maximum value at $x = \frac{\pi}{6}$. [NCERT EXEMPLAR]

ANSWERS

1. (i) Local Max. at $x = 1$, Local Max. value = 68
   Local Min. at $x = 5, -6$; Local Min. values are -316 and -1647.

(ii) Local Max. at $x = 1$, Local Max. value = 19, Local Min. at $x = 3$, Local Min. value = 15

(iii) Local Max. at $x = -2$, Local Max. value = 0, Local Min at $x = 0$, Local Min value = -4

(iv) Local Max. at $x = 2$, Local Max. value = $1/2$

(v) Local Min. at $x = -1$, Local Min. value = $-1/e$

(vi) Local Min. at $x = 2$, Local Min. value = 2

(vii) Local Min. at $x = -7/4$, Local Min. value $= -\dfrac{3}{4^{4/3}}$

(viii) Local Max. at $x = 4$; Local Max. value = 16, Local Min. at $x = -4$; Local Min. value = -16

(ix) Local Max. at $x = a/3$, Local Max. value $= \dfrac{4a^3}{27}$, Local Min. at $x = a$, Local Min. value = 0

(x) Local Max. at $x = -a$, Local Max value = $-2a$, Local Min. at $x = a$, Local Min. value = $2a$

(xi) Local Max. at $x = 1$, Local Max. value = 1, Local Min. at $x = -1$, Local Min. value = -1

(xii) Local Max. at $x = 3/4$, Local Max. value = $5/4$

2. (i) Local Max. value = $4/27$ at $x = 4/3$, Local Min. value = 0 at $x = 2$

(ii) Local Max. value $= \dfrac{2}{3\sqrt{3}}$ at $x = 2/3$

(iii) Local Max. value $3456/3125$ at $x = -1/5$; Local Min. value = 0 at $x = -1$

3. $a = -2/3, b = -1/6$

5. Local Max. value = -6 at $x = -4$; Local Min. value = 2 at $x = 0$.

6. Local Max. value $= -1 - 3\pi/2$ at $x = 3\pi/4$; Local Min. value $= 1 - \pi/2$ at $x = \pi/4$.

7. $a = -3, b = -a, c \in R$

17.6 MAXIMUM AND MINIMUM VALUES IN A CLOSED INTERVAL

Let $y = f(x)$ be a function defined on $[a, b]$. By a local maximum (or local minimum) value of a function at a point $c \in [a, b]$ we mean the greatest (or the least) value in the immediate neighbourhood of $x = c$. It does not mean the greatest or the maximum (or the least or the minimum) of $f(x)$ in the interval $[a, b]$. A function may have a number of local maxima or local minima in a given interval and even a local minimum may be greater than a local maximum.

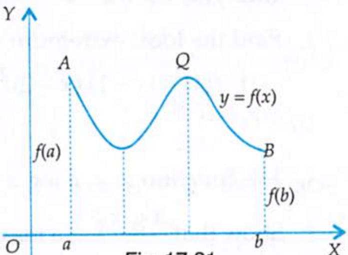

Fig. 17.31

Thus, a local maximum value may not be the greatest (the maximum) value and a local minimum value may not be the least (the minimum) value of the function in any given interval as shown in Fig. 17.31.

However, if a function $f(x)$ is differentiable and consequently continuous on a closed interval $[a, b]$, then it attains the absolute maximum (absolute minimum) at stationary points (points where $f'(x) = 0$) or at the end points of the interval $[a, b]$. Thus, to find the absolute maximum (absolute minimum) value of the function, we choose the largest and the smallest amongst the numbers $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$ where $x = c_1, c_2, \ldots, c_n$ are the stationary points.

We may use the following algorithm for finding the maximum (absolute maximum) and the minimum (absolute minimum) of a function f defined on a closed interval $[a, b]$.

ALGORITHM

Step I *Find $f'(x)$*

Step II *Put $f'(x) = 0$ and find values of x. Let $c_1, c_2, \ldots, c_n$ be the values of x.*

Step III *Take the maximum and minimum values out of the values $f(a), f(c_1), f(c_2), \ldots, f(c_n), f(b)$.*

The maximum and minimum values obtained in step III are respectively the largest (or absolute maximum) and the smallest (or absolute minimum) values of the function.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the maximum and minimum values of* $f(x)=2x^3-24x+107$ *in the interval* $[1, 3]$. **[NCERT]**

SOLUTION We have,

$$f(x)=2x^3-24x+107 \Rightarrow f'(x)=6x^2-24$$

Now, $f'(x)=0 \Rightarrow 6x^2-24=0 \Rightarrow x=\pm 2$

But, $x=-2 \notin [1, 3]$. So $x=2$ is the only stationary point. Let us now compute the values of $f(x)$ at $x=1, 2, 3$. We find that

$$f(1)=2-24+107=85,\ f(2)=2(2)^3-24(2)+107=75$$

and, $f(3)=2(3)^3-24\times 3+107=89$

Clearly, largest of these values is 89 and the least is 75. Hence, the maximum value of $f(x)$ is 89 which it attains at $x=3$ and the minimum value is 75 which is attained at $x=2$.

EXAMPLE 2 *Find the maximum and minimum values of* $f(x)=\sin x$ *in the interval* $[\pi, 2\pi]$.

SOLUTION We have, $f(x)=\sin x$. Therefore, $f'(x)=\cos x$.

At stationary points, we must have

$$f'(x)=0 \Rightarrow \cos x=0 \Rightarrow x=\frac{3\pi}{2} \qquad [\because x\in[\pi, 2\pi]]$$

Let us now compute the values of $f(x)$ at $x=\pi, \frac{3\pi}{2}, 2\pi$. We find that

$$f(\pi)=\sin\pi=0,\ f\left(\frac{3\pi}{2}\right)=\sin\frac{3\pi}{2}=-1 \text{ and } f(2\pi)=\sin 2\pi=0.$$

The greatest and the least of these values are 0 and –1 respectively. Hence, the maximum value of $f(x)$ is 0 which it attains at $x=\pi$ and 2π, and the minimum value is – 1 which it attains at $x=3\pi/2$.

EXAMPLE 3 *Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:*

(i) $f(x)=\left(\frac{1}{2}-x\right)^2+x^3$ in $[-2, 2.5]$ (ii) $f(x)=\sin x+\cos x$ in $[0, \pi]$ **[NCERT]**

SOLUTION (i) We have,

$$f(x)=\left(\frac{1}{2}-x\right)^2+x^3, \text{ where } x\in[-2, 2.5] \Rightarrow f'(x)=-2(1/2-x)+3x^2=-1+2x+3x^2$$

At the points of local maximum and local maximum, we must have

$$f'(x)=0 \Rightarrow 3x^2+2x-1=0 \Rightarrow (3x-1)(x+1)=0 \Rightarrow x=1/3, -1$$

The values of $f(x)$ at these points and also at the end-points of the interval are computed as given below.

$$f(-2)=\left(\frac{1}{2}+2\right)^2+(-2)^3=\frac{25}{4}-8=-\frac{7}{4},\ f\left(\frac{1}{3}\right)=\left(\frac{1}{2}-\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3=\frac{1}{36}+\frac{1}{27}=\frac{7}{108},$$

$$f(-1)=\left(\frac{1}{2}+1\right)^2+(-1)^3=\frac{5}{4} \text{ and, } f(2.5)=\left(\frac{1}{2}-2.5\right)^2+(2.5)^3=\frac{157}{8}$$

Of these values, the maximum value of $f(x)$ is $\frac{157}{8}$ and the minimum value is $-\frac{7}{4}$.

Thus, the absolute maximum $= \frac{157}{8}$ and, the absolute minimum $= -\frac{7}{4}$.

(ii) We have,

$$f(x) = \sin x + \cos x, \text{ where } x \in [0, \pi] \Rightarrow f'(x) = \cos x - \sin x$$

The critical points of $f(x)$ are given by

$$f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$$

Let us now calculate the values of $f(x)$ at the critical points and the end-points of the interval.

$$f(0) = \sin 0 + \cos 0 = 1,\ f(\pi/4) = \sin \pi/4 + \cos \pi/4 = \sqrt{2} \text{ and, } f(\pi) = \sin \pi + \cos \pi = -1.$$

Of these values, the maximum and minimum values of $f(x)$ are $\sqrt{2}$ and -1 respectively.

So, absolute maximum $= \sqrt{2}$ and, absolute minimum $= -1$.

EXAMPLE 4 *Find both the maximum and the minimum value of* $3x^4 - 8x^3 + 12x^2 - 48x + 1$ *on the interval* $[1, 4]$.

SOLUTION Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 1$. Then, $f'(x) = 12x^3 - 24x^2 + 24x - 48$.

The critical points of $f(x)$ are given by $f'(x) = 0$.

Now, $f'(x) = 0$

$\Rightarrow 12x^3 - 24x^2 + 24x - 48 = 0$

$\Rightarrow x^3 - 2x^2 + 2x - 4 = 0 \Rightarrow x^2(x-2) + 2(x-2) = 0 \Rightarrow (x-2)(x^2+2) = 0 \Rightarrow x = 2 \quad [\because x^2 + 2 \neq 0]$

The values of $f(x)$ at critical points and at the end-points of the interval are computed as follows:

$$f(2) = -59,\quad f(1) = -40 \text{ and } f(4) = 257.$$

Of these values the largest and the smallest values are $f(4) = 257$ and $f(2) = -59$.

So, the minimum and maximum values of $f(x)$ on $[1, 4]$ are -59 and 257 respectively.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Find the maximum and minimum values of* $f(x) = \sin x + \frac{1}{2}\cos 2x$ *in* $[0, \pi/2]$.

SOLUTION We have, $f(x) = \sin x + \frac{1}{2}\cos 2x$. Therefore, $f'(x) = \cos x - \sin 2x$.

At stationary points, we have

$$f'(x) = 0 \Rightarrow \cos x - 2\sin x \cos x = 0 \Rightarrow \cos x = 0 \text{ or, } \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{2} \text{ and } x = \frac{\pi}{6} \quad \left[\because 0 \leq x \leq \frac{\pi}{2}\right]$$

Let us now calculate the values of $f(x)$ at these points and at the end-points of the interval. We find that

$$f(0) = \sin 0 + \frac{1}{2}\cos 0 = \frac{1}{2},\ f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3} = \frac{3}{4} \text{ and, } f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\cos \pi = \frac{1}{2}$$

Of these values, the leagest value is $\frac{3}{4}$ and the smallest value is $\frac{1}{2}$. Thus, the maximum and minimum values of $f(x)$ are $\frac{3}{4}$ and $\frac{1}{2}$ respectively which it attains at $x = \frac{\pi}{6}$ and $x = 0$, $x = \frac{\pi}{2}$ respectively.

EXAMPLE 6 *Find the maximum and minimum values of* $f(x) = x^{50} - x^{20}$ *in the interval* $[0, 1]$.

SOLUTION Let $f(x) = x^{50} - x^{20}$. Then, $f'(x) = 50x^{49} - 20x^{19}$.

At stationary points, we must have

$$f'(x) = 0 \Rightarrow 50x^{49} - 20x^{19} = 0 \Rightarrow x^{19}(50x^{30} - 20) = 0$$

$$\Rightarrow \quad x = 0 \text{ or, } 50x^{30} = 20 \Rightarrow x = 0 \text{ or, } x = \left(\frac{2}{5}\right)^{1/30}.$$

The values of $f(x)$ at these points and at the end-points of the interval $[0, 1]$ are as given below.

Now, $f(0) = 0, f\left(\frac{2}{5}\right)^{1/30} = \left(\frac{2}{5}\right)^{50/30} - \left(\frac{2}{5}\right)^{20/30} = \left(\frac{2}{5}\right)^{2/3}\left(\frac{2}{5} - 1\right) = -\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}$ and, $f(1) = 1 - 1 = 0$.

Of these values, the maximum value is 0 and the minimum value is $-\frac{3}{5}\left(\frac{2}{5}\right)^{1/3}$.

Thus, the maximum value of $f(x)$ in $[0, 1]$ is 0 and the minimum value of $f(x)$ in $[0, 1]$ is $-\frac{3}{5}\left(\frac{2}{5}\right)^{2/3}$.

EXAMPLE 7 *Find the maximum and minimum values of* $f(x) = x + \sin 2x$ *in the interval* $[0, 2\pi]$. **[NCERT]**

SOLUTION We have, $f(x) = x + \sin 2x$. Therefore, $f'(x) = 1 + 2\cos 2x$.

At stationary points, we have

$$f'(x) = 0 \Rightarrow 1 + 2\cos 2x = 0 \Rightarrow \cos 2x = -\frac{1}{2}$$

$$\Rightarrow \quad 2x = \frac{2\pi}{3} \text{ or, } 2x = \frac{4\pi}{3} \Rightarrow x = \frac{\pi}{3} \text{ or, } x = \frac{2\pi}{3} \qquad [\because 0 \le x \le 2\pi \therefore 0 \le 2x \le 4\pi]$$

Let us now compute the values of $f(x)$ at these stationary points and at the end-points of the interval $[0, 2\pi]$.

Now, $f(0) = 0 + \sin 0 = 0$, $f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$, $f\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + \sin\frac{4\pi}{3} = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

and, $f(2\pi) = 2\pi + \sin 4\pi = 2\pi + 0 = 2\pi$.

Of these values, the maximum value is 2π and the minimum value is 0. Thus, the maximum value of $f(x)$ is 2π and the minimum value is 0.

EXAMPLE 8 *Find the difference between the greatest and least values of the function* $f(x) = \sin 2x - x$ *on* $[-\pi/2, \pi/2]$. **[NCERT EXEMPLAR]**

SOLUTION We have, $f(x) = \sin 2x - x$. Therefore, $f'(x) = 2\cos 2x - 1$.

At stationary points, we must have

$$f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x = -\frac{\pi}{3}, \frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}, \frac{\pi}{6}$$

Let us now compute the values of $f(x)$ at these stationary points and also at the end-points of the interval $[-\pi/2, \pi/2]$.

Now, $f(x) = \sin 2x - x$

$$\Rightarrow \quad f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2}, \quad f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6} \text{ and, } f\left(\frac{\pi}{2}\right) = \sin\pi - \frac{\pi}{2} = -\frac{\pi}{2}$$

Of these values, the largest is $\frac{\pi}{2}$ and the least is $-\frac{\pi}{2}$. So, the greatest and the least values of $f(x)$ on $[-\pi/2, \pi/2]$ are $\pi/2$ and $-\pi/2$ respectively. Hence, required difference $= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

EXAMPLE 9 *Show that* $f(x)=\sin x\,(1+\cos x)$ *is maximum at* $x=\dfrac{\pi}{3}$ *in the interval* $[0, \pi]$.

SOLUTION We have,

$$f(x) = \sin x\,(1+\cos x).$$

$$\Rightarrow \quad f'(x) = \cos x\,(1+\cos x) - \sin^2 x = \cos x + \cos^2 x - (1-\cos^2 x)$$

$$\Rightarrow \quad f'(x) = 2\cos^2 x + \cos x - 1 = (2\cos x - 1)(\cos x + 1).$$

At stationary points, we have

$$f'(x)=0 \Rightarrow (2\cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \frac{1}{2} \text{ or, } \cos x = -1 \Rightarrow x = \frac{\pi}{3} \text{ or, } x=\pi.$$

Let us now compute the values of x at these stationary points and at the end-points of the interval. We find that

$$f(0)=0,\ f\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}\left(1+\cos\frac{\pi}{3}\right)=\frac{3\sqrt{3}}{4} \text{ and } f(\pi)=0.$$

Of these values, the maximum value is $\dfrac{3\sqrt{3}}{4}$. Hence, $f(x)$ attains the maximum value $\dfrac{3\sqrt{3}}{4}$ at $x=\pi/3$.

EXERCISE 17.4

BASIC

1. Find the absolute maximum and the absolute minimum values of the following functions in the given intervals:

   (i) $f(x)=4x-\dfrac{x^2}{2}$ in $[-2, 4.5]$ [NCERT] (ii) $f(x)=(x-1)^2+3$ in $[-3, 1]$ [NCERT]

   (iii) $f(x)=3x^4-8x^3+12x^2-48x+25$ in $[0, 3]$ [NCERT]

   (iv) $f(x)=(x-2)\sqrt{x-1}$ in $[1, 9]$

2. Find the maximum value of $2x^3-24x+107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$. [NCERT]

3. Find absolute maximum and minimum values of a function f given by $f(x)=12x^{4/3}-6x^{1/3}, x\in[-1, 1]$. [NCERT]

4. Find the absolute maximum and minimum values of a function f given by $f(x)=2x^3-15x^2+36x+1$ on the interval $[1, 5]$. [NCERT]

BASED ON LOTS

5. Find the absolute maximum and minimum values of the function f given by $f(x)=\cos^2 x+\sin x, x\in[0, \pi]$. [NCERT]

ANSWERS

1. (i) Absolute Maximum = 8 at $x=4$, Absolute Minimum = -10 at $x=-2$

   (ii) Absolute Maximum = 19 at $x=-3$, Absolute Minimum = 3 at $x=1$

   (iii) Absolute Maximum = 25 at $x=0$, Absolute Minimum = -39 at $x=2$

   (iv) Absolute Maximum = $14\sqrt{2}$ at $x=9$, Absolute Minimum $=-\dfrac{2}{3\sqrt{3}}$ at $x=\dfrac{4}{3}$

2. Maximum value = 89 at $x=3$ in $[1, 3]$, Maximum value = 139 at $x=-2$ in $[-3, -1]$

3. Absolute Maximum $= 5/4$, Absolute Minimum $= 1$
4. Absolute Minimum. value $= -\frac{9}{4}$ at $x = \frac{1}{8}$, Absolute Maximum value $=$ 18 at $x = -1$
5. Absolute Maximum value $= 56$ at $x = 5$, Absolute Minimum value $= 24$ at $x = 1$

17.7 APPLIED PROBLEMS ON MAXIMA AND MINIMA

In this section, we will discuss some applied problems on maxima and minima for which following results will be very useful.

(i) For a square of side x: Area $= x^2$, Perimeter $= 4x$.

(ii) For a rectangle of sides x *and* y: Area $= xy$, Perimeter $= 2(x + y)$.

(iii) For a trapezium: Area $= \frac{1}{2}$ (Sum of parallel sides) $\times$ (Distance between them).

(iv) For a circle of radius r: Area $= \pi r^2$, Circumference $= 2\pi r$.

(v) For a sphere of radius r: Volume $= \frac{4}{3}\pi r^3$, Surface Area $= 4\pi r^2$.

(vi) For a right circular cylinder of base radius r and height h:
Volume $= \pi r^2 h$, Surface $= 2\pi rh + 2\pi r^2$, Curved surface $= 2\pi rh$.

(vii) For a right circular cone of height h, slant height l and radius of the base r:
Volume $= \left(\frac{1}{3}\right)\pi r^2 h$, Curved surface $= \pi rl$, Total surface $= \pi r^2 + \pi rl$.

(viii) For a cuboid of edges of lengths x, y *and* z: Volume $= xyz$, Surface $= 2(xy + yz + zx)$.

(ix) For a cube of edge length x: Volume $= x^3$, Surface Area $= 6x^2$.

(x) Area of an equilateral triangle $= \frac{\sqrt{3}}{4}(Side)^2$.

REMARK *If k is a positive constant, then a function of the form* $k f(x)$, $k + f(x)$, $\{f(x)\}^k$, $\{f(x)\}^{1/k}$, $\log f(x)$ *will be maximum or minimum according as* $f(x)$ *is maximum or minimum provided that* $f(x) > 0$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find two numbers whose sum is 24 and whose product is as large as possible.* [NCERT]

SOLUTION Let the numbers be x and y. Then,

$$x + y = 24 \text{ (given)} \quad \text{...(i)}$$

Let P be the product of these numbers. Then,

$$P = xy = x(24 - x) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad P = 24x - x^2 \Rightarrow \frac{dP}{dx} = 24 - 2x \text{ and } \frac{d^2P}{dx^2} = -2$$

The critical points of P are given by $\frac{dP}{dx} = 0$.

$$\therefore \quad \frac{dP}{dx} = 0 \Rightarrow 24 - 2x = 0 \Rightarrow x = 12$$

Also, $\left(\frac{d^2P}{dx^2}\right)_{x=12} = -2 < 0$. So, P is maximum when $x = 12$.

Putting $x=12$ in (i), we obtain $y=12$. Hence, the required numbers are both equal to 12.

EXAMPLE 2 *Find two positive numbers x and y such that $x+y=60$ and xy^3 is maximum.* [NCERT]

SOLUTION Let $P=xy^3$. It is given that $x+y=60$. Therefore, $x=60-y$.

Now, $P=xy^3$

$$\Rightarrow \quad P=(60-y)\,y^3=60y^3-y^4 \Rightarrow \frac{dP}{dy}=180y^2-4y^3 \text{ and } \frac{d^2P}{dy^2}=360y-12y^2$$

The critical points of P are given by $\frac{dP}{dy}=0$.

$$\therefore \quad \frac{dP}{dy}=0.$$

$$\Rightarrow \quad 180y^2-4y^3=0 \Rightarrow 4y^2(45-y)=0 \Rightarrow y=0, y=45 \Rightarrow y=45 \quad [\because y=0 \text{ is not possible}]$$

Now, $$\left(\frac{d^2P}{dy^2}\right)_{y=45} = 360\times 45-12(45)^2=12\times 45\,(30-45)=-8100<0$$

So, P is maximum when $y=45$. Putting $y=45$ in $x+y=60$, we obtain $x=15$.
Hence, xy^3 is maximum when $x=15$ and $y=45$.

EXAMPLE 3 *Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is maximum.* [NCERT]

SOLUTION Let $P=x^2y^5$. It is given that $x+y=35 \Rightarrow x=35-y$ ...(i)

Putting $x=35-y$ in $P=x^2y^5$, we get

$$P=(35-y)^2y^5$$

$$\Rightarrow \quad \frac{dP}{dy}=-2(35-y)\,y^5+5(35-y)^2y^4=(35-y)\,y^4\{-2y+5(35-y)\}=y^4(35-y)\,(175-7y)$$

$$\Rightarrow \quad \frac{dP}{dy}=7y^4(35-y)(25-y)$$

The critical points of P are given by $\frac{dP}{dy}=0$.

$$\therefore \quad \frac{dP}{dy}=0 \Rightarrow 7y^4(35-y)(25-y)=0 \Rightarrow y=0, 25, 35$$

But, $y=0$ and $y=35$ are not possible. So, $y=25$.

Now, $$\frac{d^2P}{dy^2}=28y^3(35-y)(25-y)-7y^4(25-y)-7y^4(35-y)$$

$$\therefore \quad \left(\frac{d^2P}{dy^2}\right)_{y=25}=-7\,(25)^4(35-20)=-7\,(25)^4(10)<0$$

Thus, P has maximum when $y=25$. Putting $y=25$ in (i), we obtain $x=10$. Hence, x^2y^5 is maximum when $x=10$, and $y=25$.

EXAMPLE 4 *Amongst all pairs of positive numbers with product 256, find those whose sum is the least.*

SOLUTION Let the required numbers be x and y. Then, $xy=256$ (given) ...(i)

Let $S=x+y$. Then,

$$S=x+\frac{256}{x} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dx} = 1 - \frac{256}{x^2} \text{ and, } \frac{d^2 S}{dx^2} = \frac{512}{x^3}$$

The critical points of S are given by $\frac{dS}{dx} = 0$.

$$\therefore \quad \frac{dS}{dx} = 0 \Rightarrow 1 - \frac{256}{x^2} = 0 \Rightarrow x^2 = 256 \Rightarrow x = 16$$

Now, $\left(\frac{d^2 S}{dx^2}\right)_{x=16} = \frac{512}{(16)^3} = \frac{1}{8} > 0$. Thus, S is minimum when $x = 16$.

Putting $x = 16$ in (i) we get $y = 16$. Hence, the required numbers are both equal to 16.

EXAMPLE 5 *Find two positive numbers whose sum is 14 and the sum of whose squares is minimum.*

SOLUTION Let the numbers be x and y. Then,

$$x + y = 14 \qquad \text{...(i)}$$

Let S be the sum of the squares of x and y. Then,

$$S = x^2 + y^2$$

$$\Rightarrow \quad S = x^2 + (14 - x)^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad S = 2x^2 - 28x + 196 \Rightarrow \frac{dS}{dx} = 4x - 28 \text{ and } \frac{d^2S}{dx^2} = 4$$

The critical points of S are given by $\frac{dS}{dx} = 0$.

$$\therefore \quad \frac{dS}{dx} = 0 \Rightarrow 4x - 28 = 0 \Rightarrow x = 7$$

Clearly, $\frac{d^2S}{dx^2} = 4 > 0$. Thus, S is minimum when $x = 7$.

Putting $x = 7$ in (i), we obtain $y = 7$. Hence, the required numbers are both equal to 7.

EXAMPLE 6 *The combined resistance R of two resistors R_1 and $R_2 (R_1, R_2 > 0)$ is given by* $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

If $R_1 + R_2 = C$ (a constant), show that the maximum resistance R is obtained by choosing $R_1 = R_2$.

SOLUTION We have,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \text{ and } R_1 + R_2 = C$$

$$\Rightarrow \quad \frac{1}{R} = \frac{R_1 + R_2}{R_1 R_2} = \frac{C}{R_1 R_2} = \frac{C}{R_1 (C - R_1)} \qquad [\because R_2 = C - R_1]$$

$$\Rightarrow \quad R = \frac{R_1 C - {R_1}^2}{C} = R_1 - \frac{{R_1}^2}{C} \Rightarrow \frac{dR}{dR_1} = 1 - \frac{2R_1}{C} \text{ and } \frac{d^2R}{d{R_1}^2} = -\frac{2}{C}$$

The critical numbers of R are given by $\frac{dR}{dR_1} = 0$.

$$\therefore \quad \frac{dR}{dR_1} = 0 \Rightarrow 1 - \frac{2R_1}{C} = 0 \Rightarrow R_1 = \frac{C}{2}$$

Clearly, $\frac{d^2R}{dR_1^2} = -\frac{2}{C} < 0$ for all values of R_1. Thus, R is maximum when $R_1 = \frac{C}{2}$.

Putting $R_1 = \frac{C}{2}$ in $R_1 + R_2 = C$, we get: $R_2 = C - \frac{C}{2} = \frac{C}{2}$. Hence, R is maximum when $R_1 = R_2 = C/2$.

EXAMPLE 7 *A beam of length l is supported at two ends and is uniformly loaded. If W is the uniform load per unit length, the bending moment M at a distance x from one end is given by* $M = \frac{W}{2} lx - \frac{1}{2} Wx^2$. *Find the point on the beam at which the bending moment has the maximum value.*

SOLUTION We have,

$$M = \frac{Wlx}{2} - \frac{Wx^2}{2} \Rightarrow \frac{dM}{dx} = \frac{Wl}{2} - Wx \text{ and } \frac{d^2M}{dx^2} = -W$$

The critical numbers of M are given by $\frac{dM}{dx} = 0$.

Now, $\frac{dM}{dx} = 0 \Rightarrow \frac{Wl}{2} - Wx = 0 \Rightarrow x = \frac{l}{2}$

Clearly, $\frac{d^2M}{dx^2} = -W < 0$ for all values of x. Thus, M is maximum when $x = l/2$.

Hence, the required point is at a distance of $l/2$ from the supporting end.

EXAMPLE 8 *Find the minimum value of* $ax + by$, *where* $xy = c^2$ *and* a, b, c *are positive.* **[CBSE 2015, 2020]**

SOLUTION Let $z = ax + by$, where $xy = c^2$. Then,

$$z = ax + \frac{bc^2}{x} \qquad \left[\because xy = c^2 \Rightarrow y = \frac{c^2}{x}\right] \qquad \ldots(i)$$

$$\Rightarrow \quad \frac{dz}{dx} = a - \frac{bc^2}{x^2} \text{ and } \frac{d^2z}{dx^2} = \frac{2bc^2}{x^3}$$

The critical points of z are given by $\frac{dz}{dx} = 0$.

$$\therefore \quad \frac{dz}{dx} = 0 \Rightarrow a - \frac{bc^2}{x^2} = 0 \Rightarrow x^2 = \frac{bc^2}{a} \Rightarrow x = \pm\sqrt{\frac{b}{a}}\, c$$

At $x = \sqrt{\frac{b}{a}}\, c$: We find that $\frac{d^2z}{dx^2} = 2bc^2 \left(\sqrt{\frac{a}{b}} \times \frac{1}{c}\right)^3 = 2\,\frac{a}{c}\sqrt{\frac{a}{b}} > 0$. So, z is minimum at $x = c\sqrt{\frac{b}{a}}$.

The minimum value of z is given by $z = a\sqrt{\frac{b}{a}}\, c + \frac{bc^2}{c}\sqrt{\frac{a}{b}} = 2\sqrt{ab}\, c$ $\quad \left[\text{Putting } x = \sqrt{\frac{b}{a}}\, c \text{ in (i)}\right]$

At $x = -\sqrt{\frac{b}{a}}\, c$: We find that $\frac{d^2z}{dx^2} = 2bc^2 \left(-\frac{a}{bc^3}\sqrt{\frac{a}{b}}\right) = -2\,\frac{a}{c}\sqrt{\frac{a}{b}} < 0$. So, z is maximum at $x = -\sqrt{\frac{b}{a}}\, c$.

EXAMPLE 9 *Show that all the rectangles with a given perimeter, the square has the largest area.*

SOLUTION Let x and y be the lengths of two sides of the rectangle of fixed parameter P and let A be its area. Then,

$$P = 2(x + y) \quad \ldots(i) \qquad \text{and,} \qquad A = xy \quad \ldots(ii)$$

Now, $P = 2(x + y) \Rightarrow y = \frac{P}{2} - x$

$$\therefore \quad A = xy = x\left(\frac{P}{2} - x\right) = \frac{Px}{2} - x^2 \Rightarrow \frac{dA}{dx} = \frac{P}{2} - 2x \text{ and } \frac{d^2A}{dx^2} = -2$$

The critical points of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow \frac{P}{2} - 2x = 0 \Rightarrow P = 4x \Rightarrow 2x + 2y = 4x \Rightarrow 2x = 2y \Rightarrow x = y$$

Clearly, $\left(\frac{d^2 A}{dx^2}\right)_{x=y} = -2 < 0$. Hence, A is maximum when $x = y$ i.e. the rectangle is a square.

EXAMPLE 10 *Show that of all the rectangles of given area, the square has the smallest perimeter.*

[CBSE 2011]

SOLUTION Let x and y be the lengths of two sides of a rectangle of given area A, and let P be the perimeter. Then,

$$A = xy \quad \text{...(i)} \qquad \text{and,} \qquad P = 2(x + y) \quad \text{...(ii)}$$

Now, $A = xy \Rightarrow y = \frac{A}{x}$

$$\therefore \quad P = 2(x + y) = 2\left(x + \frac{A}{x}\right) \Rightarrow \frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right) \text{ and } \frac{d^2P}{dx^2} = \frac{4A}{x^3}$$

The critical points of P are given by $\frac{dP}{dx} = 0$.

$$\therefore \quad \frac{dP}{dx} = 0 \Rightarrow 2\left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow 1 - \frac{A}{x^2} = 0 \Rightarrow x^2 = A \Rightarrow x^2 = xy \Rightarrow x = y.$$

Clearly, $\frac{d^2 P}{dx^2} = \frac{4A}{x^3} > 0$ for all positive values of x. Hence, P is minimum when $x = y$ i.e the rectangle is a square.

EXAMPLE 11 *Show that of all the rectangles inscribed in a given circle, the square has the maximum area.*

[NCERT, CBSE 2002, 2006, 2008, 2011, 2013]

SOLUTION Let $ABCD$ be a rectangle inscribed in a given circle with centre at O and radius a. Let $AB = 2x$ and $BC = 2y$. Applying Pythagoras theorem in right triangle OAM, we obtain

$$OA^2 = AM^2 + OM^2 \Rightarrow a^2 = x^2 + y^2 \Rightarrow y = \sqrt{a^2 - x^2} \quad \text{...(i)}$$

Let A be the area of the rectangle $ABCD$. Then,

$$A = 4xy = 4x\sqrt{a^2 - x^2} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dA}{dx} = 4\left\{\sqrt{a^2 - x^2} - \frac{x^2}{\sqrt{a^2 - x^2}}\right\} = 4\left\{\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right\}$$

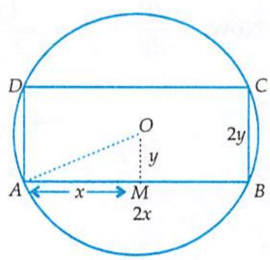

Fig. 17.32

The critical points of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0$$

$$\Rightarrow \quad 4\left\{\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right\} = 0 \Rightarrow a^2 - 2x^2 = 0 \Rightarrow x = \frac{a}{\sqrt{2}}$$

Now, $\frac{dA}{dx} = 4\left\{\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}\right\}$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4\frac{d}{dx}\left\{(a^2 - 2x^2)(a^2 - x^2)^{-1/2}\right\}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4\left\{-4x(a^2 - x^2)^{-1/2} + (a^2 - 2x^2)(-1/2)(a^2 - x^2)^{-3/2}(-2x)\right\}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4\left\{\frac{-4x}{\sqrt{a^2 - x^2}} + \frac{x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}\right\}$$

Clearly, $\left(\frac{d^2A}{dx^2}\right)_{x = a/\sqrt{2}} = -16 < 0$. Thus, A is maximum when $x = \frac{a}{\sqrt{2}}$. Putting $x = \frac{a}{\sqrt{2}}$ in (i), we get $y = \frac{a}{\sqrt{2}}$.

Now, $\quad x = y = a/\sqrt{2} \Rightarrow 2x = 2y = \sqrt{2}a \Rightarrow AB = BC \Rightarrow ABCD$ is a square.

Hence, area A is maximum when the rectangle is a square.

EXAMPLE 12 *Show that the rectangle of maximum perimeter which can be inscribed in a circle of radius a is a square of side $\sqrt{2}a$.* **[CBSE 2002]**

SOLUTION Let $ABCD$ be a rectangle in a given circle of radius a with centre at O. Let $AB = 2x$ and $AD = 2y$ be the sides of the rectangle. Applying Pythagoras theorem in ΔOAM, we get

$$AM^2 + OM^2 = OA^2 \Rightarrow x^2 + y^2 = a^2 \Rightarrow y = \sqrt{a^2 - x^2} \quad \text{...(i)}$$

Let P be the perimeter of the rectangle $ABCD$. Then,

$$P = 4x + 4y$$

$$\Rightarrow \quad P = 4x + 4\sqrt{a^2 - x^2} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{a^2 - x^2}}$$

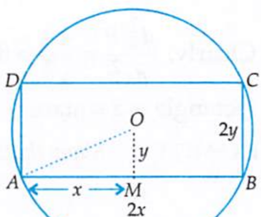

Fig. 17.33

The critical points of P are given by $\frac{dP}{dx} = 0$.

$$\therefore \quad \frac{dP}{dx} = 0$$

$$\Rightarrow \quad 4 - \frac{4x}{\sqrt{a^2 - x^2}} = 0$$

$$\Rightarrow \quad 4 = \frac{4x}{\sqrt{a^2 - x^2}} \Rightarrow \sqrt{a^2 - x^2} = x \Rightarrow a^2 - x^2 = x^2 \Rightarrow 2x^2 = a^2 \Rightarrow x = \frac{a}{\sqrt{2}}$$

Now, $\quad \frac{dP}{dx} = 4 - \frac{4x}{\sqrt{a^2 - x^2}}$

$$\Rightarrow \quad \frac{d^2P}{dx^2} = \frac{-4\left\{\sqrt{a^2 - x^2} - \frac{x(-x)}{\sqrt{a^2 - x^2}}\right\}}{\left\{\sqrt{a^2 - x^2}\right\}^2} = \frac{-4a^2}{(a^2 - x^2)^{3/2}}$$

$$\therefore \quad \left(\frac{d^2P}{dx^2}\right)_{x = a/\sqrt{2}} = \frac{-4a^2}{\left(a^2 - \frac{a^2}{2}\right)^{3/2}} = \frac{-8\sqrt{2}}{a} < 0.$$ Thus, P is maximum when $x = \frac{a}{\sqrt{2}}$.

Putting $x = \frac{a}{\sqrt{2}}$ in (i), we obtain $y = \frac{a}{\sqrt{2}}$.

Now, $x = y = a\sqrt{2} \Rightarrow 2x = 2y \Rightarrow AB = BC \Rightarrow ABCD$ is a square.

Hence, P is maximum when the rectangle is square of side $2x = \frac{2a}{\sqrt{2}} = \sqrt{2}\,a$.

EXAMPLE 13 *AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.* **[NCERT EXEMPLAR]**

SOLUTION Let $AB = 2a$, $AC = x$ and $CB = y$. Since AB is a diameter of the circle having centre O and C is a point on the semi-circle ACB. Therefore, $\angle ACB = \frac{\pi}{2}$.

Applying Pythagoras theorem in ΔACB, we obtain

$$AB^2 = AC^2 + CB^2 \Rightarrow (2a)^2 = x^2 + y^2 \Rightarrow y = \sqrt{4a^2 - x^2} \qquad \ldots(i)$$

Let A be the area of ΔACB. Then,

$$A = \frac{1}{2} AC \times CB = \frac{1}{2} xy$$

$$\Rightarrow \quad A = \frac{1}{2} x \sqrt{4a^2 - x^2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left\{\sqrt{4a^2 - x^2} - \frac{x^2}{\sqrt{4a^2 - x^2}}\right\} = \frac{2a^2 - x^2}{\sqrt{4a^2 - x^2}}$$

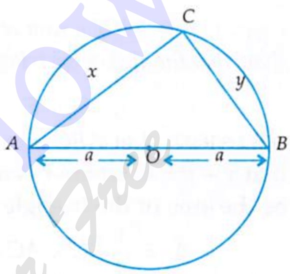

Fig. 17.34

The stationary values of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow \frac{2a^2 - x^2}{\sqrt{4a^2 - x^2}} = 0 \Rightarrow 2a^2 = x^2 \Rightarrow x = \sqrt{2}a$$

$$\text{Now,} \quad \frac{dA}{dx} = \frac{2a^2 - x^2}{\sqrt{4a^2 - x^2}} \Rightarrow \frac{d^2A}{dx^2} = \frac{\sqrt{4a^2 - x^2} \times -2x - (2a^2 - x^2) \times \dfrac{-x}{\sqrt{4a^2 - x^2}}}{\left(\sqrt{4a^2 - x^2}\right)^2} = -\frac{x(6a^2 - x^2)}{(4a^2 - x^2)^{3/2}}$$

Clearly, $\left(\frac{d^2A}{dx^2}\right)_{x = \sqrt{2}a} = -2 < 0$. Thus, A is maximum when $x = \sqrt{2}a$ and $y = \sqrt{2}a$.

Hence, the area of ΔABC is maximum when it is isosceles.

EXAMPLE 14 *Tangent to the circle $x^2 + y^2 = a^2$ at any point on it in the first quadrant makes intercepts OA and OB on x and y axes respectively, O being the centre of the circle. Find the minimum value of OA + OB.* **[CBSE 2015]**

SOLUTION Let $P(a\cos\theta, a\sin\theta)$ be an arbitrary point on the circle $x^2 + y^2 = a^2$. If P lies in the first quadrant, then $0 \le \theta \le \pi/2$. The equation of tangent to $x^2 + y^2 = a^2$ at $P(a\cos\theta, a\sin\theta)$ is

$$x\cos\theta + y\sin\theta = a \qquad \left[\begin{array}{l}\text{The tangent to } x^2 + y^2 = a^2 \\ \text{at } (x_1, y_1) \text{ is } xx_1 + yy_1 = a^2\end{array}\right]$$

This cuts x and y-axis at $A\,(a\sec\theta, 0)$ and $B\,(0, a\,\text{cosec}\,\theta)$ respectively.

$$\therefore \quad OA = a\sec\theta \text{ and } OB = a\,\text{cosec}\,\theta$$

Let $S = OA + OB$. Then, $S = a(\sec\theta + \text{cosec}\,\theta)$.

$$\therefore \quad \frac{dS}{d\theta} = a(\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta)$$

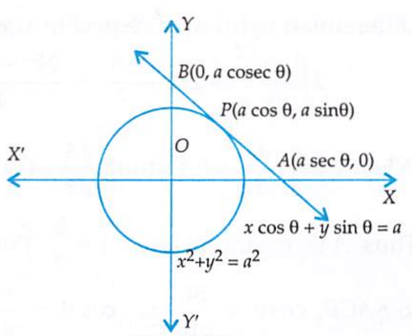

Fig. 17.35

and, $$\frac{d^2S}{d\theta^2} = a\,(\sec^3\theta + \sec\theta\tan^2\theta + \text{cosec}^3\,\theta + \text{cosec}\,\theta\cot^2\theta)$$

For maximum or minimum values of S, we must have

$$\frac{dS}{d\theta} = 0 \Rightarrow a\,(\sec\theta\tan\theta - \text{cosec}\,\theta\cot\theta) = 0 \Rightarrow \tan^3\theta = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$$

At $\theta = \frac{\pi}{4}$, we obtain : $\frac{d^2S}{d\theta^2} = a\,(2\sqrt{2} + \sqrt{2} + 2\sqrt{2} + \sqrt{2}) = 6\sqrt{2}a > 0$

Hence, S is minimum at $\theta = \frac{\pi}{4}$ and the minimum value of S is given by

$$S = a\left(\sec\frac{\pi}{4} + \text{cosec}\,\frac{\pi}{4}\right) = 2\sqrt{2}\,a$$

EXAMPLE 15 *If the sum of the lengths of the hypotenues and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\pi/3$.*

[CBSE 2009, 2014, 2016, 2017]

SOLUTION Let ABC be a right angled triangle with base $BC = x$ and hypotenues $AC = y$ such that $x + y = k$, where k is a constant. Let θ be the angle between the base and hypotenues. Let A be the area of the triangle. Then,

$$A = \frac{1}{2}BC \times AC = \frac{1}{2}x\sqrt{y^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}(y^2 - x^2)$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\{(k - x)^2 - x^2\} \qquad [\because x + y = k]$$

$$\Rightarrow \quad A^2 = \frac{k^2x^2 - 2kx^3}{4} \qquad \text{...(i)}$$

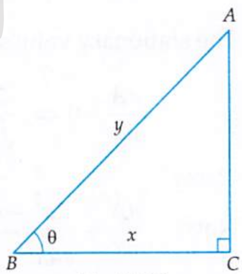

Fig. 17.36

Differentiating with respect to x, we get

$$2A\frac{dA}{dx} = \frac{2k^2x - 6kx^2}{4} \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{k^2x - 3kx^2}{4A}$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

Now, $\frac{dA}{dx} = 0 \Rightarrow \frac{k^2x - 3kx^2}{4A} = 0 \Rightarrow x = \frac{k}{3}$.

Differentiating (ii) with respect to x, we get

$$2\left(\frac{dA}{dx}\right)^2 + 2A\frac{d^2A}{dx^2} = \frac{2k^2 - 12kx}{4} \qquad \text{...(iii)}$$

When $x = \frac{k}{3}$, $\frac{dA}{dx} = 0$. Putting $\frac{dA}{dx} = 0$ and $x = \frac{k}{3}$ in (iii), we get: $\frac{d^2A}{dx^2} = \frac{-k^2}{4A} < 0$.

Thus, A is maximum when $x = \frac{k}{3}$. Putting $x = \frac{k}{3}$ in $x + y = k$, we obtain $y = \frac{2k}{3}$.

In ΔACB, $\cos\theta = \frac{BC}{AB} \Rightarrow \cos\theta = \frac{x}{y} \Rightarrow \cos\theta = \frac{k/3}{2k/3} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.

Thus, area of triangle ABC is maximum, when angle θ between base BC and hypotenuse AB is $\pi/3$.

EXAMPLE 16 *Prove that the area of right-angled triangle of given hypotenues is maximum when the triangle is isosceles.*

SOLUTION Let h be the hypotenues of the right-angled triangle, and let x be its altitude. Then,

Base of the triangle $=\sqrt{h^2-x^2}$. Let A be the area of the triangle. Then, ...(i)

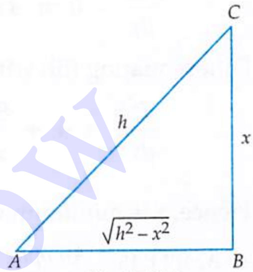

Fig. 17.37

$$A = \frac{1}{2}x\sqrt{h^2-x^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{2}\left\{\sqrt{h^2-x^2} + x\,\frac{1}{2}(h^2-x^2)^{-1/2}\frac{d}{dx}(h^2-x^2)\right\}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{2}\left\{\sqrt{h^2-x^2} - \frac{x^2}{\sqrt{h^2-x^2}}\right\} = \frac{1}{2}\left\{\frac{h^2-2x^2}{\sqrt{h^2-x^2}}\right\}$$

The critical numbers of x are given by $\frac{dA}{dx}=0$.

$$\therefore \frac{dA}{dx} = 0 \Rightarrow \frac{1}{2}\left\{\frac{h^2-2x^2}{\sqrt{h^2-x^2}}\right\} = 0 \Rightarrow h^2 = 2x^2 \Rightarrow x = \frac{h}{\sqrt{2}}$$

Now, $\frac{dA}{dx}=\frac{1}{2}\left\{\frac{h^2-2x^2}{\sqrt{h^2-x^2}}\right\}$

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{2}\left\{(-4x)\frac{1}{\sqrt{h^2-x^2}} + (h^2-2x^2)\left(-\frac{1}{2}\right)(h^2-x^2)^{-3/2}\frac{d}{dx}(h^2-x^2)\right\}$$

$$\Rightarrow \frac{d^2A}{dx^2} = \frac{1}{2}\left\{\frac{-4x}{\sqrt{h^2-x^2}} + \frac{x(h^2-2x^2)}{(h^2-x^2)^{3/2}}\right\} \Rightarrow \left(\frac{d^2A}{dx^2}\right)_{x=\frac{h}{\sqrt{2}}} = -2<0.$$

Thus, A is maximum when $x=\frac{h}{\sqrt{2}}$. Putting $x=\frac{h}{\sqrt{2}}$ in (i), we get: Base $=\sqrt{h^2-\left(\frac{h^2}{2}\right)}=\frac{h}{\sqrt{2}}$.

Therefore, $AB=BC=\frac{h}{\sqrt{2}}$. Hence, A is maximum when the triangle is isosceles.

EXAMPLE 17 *Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.*

SOLUTION Let V be the fixed volume of a closed cuboid with length x, breadth x and height y. Let S be the surface area of the cuboid. Then,

$$V = x^2y \qquad ...(i)$$

and, $S = 2(x^2+xy+xy) = 2x^2+4xy$ ...(ii)

Now, $S = 2x^2+4xy$

$$\Rightarrow \quad S = 2x^2 + 4x\left(\frac{V}{x^2}\right) \qquad \left[\because V = x^2 y \therefore y = \frac{V}{x^2}\right]$$

$$\Rightarrow \quad S = 2x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 4x - \frac{4V}{x^2} \qquad \text{...(iii)}$$

The critical numbers of S are given by $\frac{dS}{dx} = 0$.

$$\therefore \quad \frac{dS}{dx} = 0 \Rightarrow 4x - \frac{4V}{x^2} = 0 \Rightarrow V = x^3 \Rightarrow x^2 y = x^3 \Rightarrow x = y. \qquad [\because V = x^2 y]$$

Differentiating (iii) with respect to x, we get

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3} = 4 + \frac{8x^2y}{x^3} = 4 + \frac{8y}{x} \Rightarrow \left(\frac{d^2S}{dx^2}\right)_{y=x} = 12 > 0.$$

Hence, S is minimum when length $= x$, breadth $= x$ and height $= x$ i.e., when it is a cube.

EXAMPLE 18 *An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when depth of the tank is half of its width.* **[CBSE 2007, 2010, 2018]**

SOLUTION Let the length, width and height of the open tank be x, x and y units respectively. Then, its volume is $x^2\, y$ and the total surface area is $x^2 + 4xy$. It is given that the tank can hold a given quantity of water. This means that its volume is constant. Let it be V. Then,

$$V = x^2 y \qquad \text{...(i)}$$

The cost of the material will be least if the total surface area is least. Let S denote the total surface area. Then,

$$S = x^2 + 4xy \qquad \text{...(ii)}$$

We have to minimize S subject to the condition that the volume V is constant.

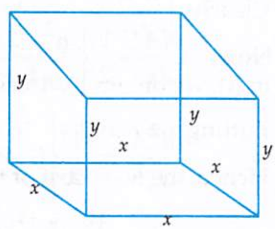

Fig. 17.38

Now,

$$S = x^2 + 4xy$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2} \text{ and } \frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3}$$

The critical numbers of S are given by $\frac{dS}{dx} = 0$.

Now, $\frac{dS}{dx} = 0 \Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow 2x^3 = 4V \Rightarrow 2x^3 = 4x^2\, y \Rightarrow x = 2y \quad [\because V = x^2 y]$

Clearly, $\frac{d^2 S}{dx^2} = 2 + \frac{8V}{x^3} > 0$ for all x. Hence, S is minimum when $x = 2y$ i.e. the depth (height) of the tank is half of its width.

EXAMPLE 19 *A metal box with a square base and vertical sides is to contain 1024 cm*^{3} *of water. the material for the top and bottom costs ₹ 5 per cm*^{2} *and the material for the sides costs ₹ 2.50 per cm*^{2}*. Find the least cost of the box.* **[NCERT EXEMPALR]**

SOLUTION Let the length, breadth and height of the metal box be x cm, x cm and y cm respectively. It is given that the box can contain 1024 cm^3 of water.

$\therefore \quad 1024 = x^2 y \Rightarrow y = \dfrac{1024}{x^2} \quad \ldots$(i)

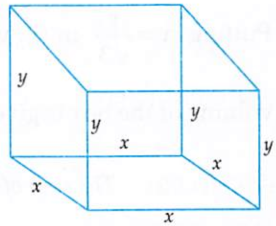

Fig. 17.39

Let C be the total cost in ₹ of material used to construct the box.
Then,

$$C = 5x^2 + 5x^2 + \frac{5}{2} \times 4xy \Rightarrow C = 10x^2 + 10xy$$

We have to find the least value of C.
Now,

$$C = 10x^2 + 10xy$$

$$\Rightarrow \quad C = 10x^2 + 10x \times \frac{1024}{x^2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad C = 10x^2 + \frac{10240}{x} \Rightarrow \frac{dC}{dx} = 20x - \frac{10240}{x^2} \text{ and } \frac{d^2C}{dx^2} = 20 + \frac{20480}{x^3}$$

The critical numbers for C are given by $\dfrac{dC}{dx} = 0$.

Now, $\dfrac{dC}{dx} = 0 \Rightarrow 20x - \dfrac{10240}{x^2} = 0 \Rightarrow x^3 = 512 \Rightarrow x^3 = 8^3 \Rightarrow x = 8$

Clearly, $\left(\dfrac{d^2C}{dx^2}\right)_{x=8} = 20 + \dfrac{20480}{8^3} > 0.$ Thus, the cost of the box is least when $x = 8$. Putting $x = 8$ in (i), we obtain $y = 16$. So, the dimensions of the box are $8 \times 8 \times 16$.

Putting $x = 8$ and $y = 16$ in $C = 10x^2 + 10xy$, we obtain $C = 1920$.

Hence, the least cost of the box is ₹1920.

EXAMPLE 20 *An open box with a square base is to be made out of a given quantity of card board of area c^2 square units. Show that the maximum volume of the box is $\dfrac{c^3}{6\sqrt{3}}$ cubic units.*

[CBSE 2001C, 05, 2012, NCERT EXEMPLAR]

SOLUTION Let the length, breadth and height of the box x, x and y units respectively. It is given that the area of the card board is c^2 sq. units.

$$\therefore \quad x^2 + 4xy = c^2 \qquad \ldots\text{(i)}$$

Let V be the volume of the box. Then,

$$V = x^2 y \qquad \ldots\text{(ii)}$$

$$\Rightarrow \quad V = x^2 \left(\frac{c^2 - x^2}{4x}\right) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad V = \frac{c^2}{4} x - \frac{x^3}{4} \Rightarrow \frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4} \text{ and } \frac{d^2V}{dx^2} = -\frac{3x}{2}$$

The critical points of V are given by $\dfrac{dV}{dx} = 0$.

Now, $\dfrac{dV}{dx} = 0 \Rightarrow \dfrac{c^2}{4} - \dfrac{3x^2}{4} = 0 \Rightarrow x = \dfrac{c}{\sqrt{3}}$

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x=c/\sqrt{3}} = \frac{-3c}{2\sqrt{3}} < 0$. Thus, V is maximum when $x = \frac{c}{\sqrt{3}}$.

Putting $x = \frac{c}{\sqrt{3}}$ in (i), we obtain $y = \frac{c}{2\sqrt{3}}$. Putting $x = \frac{c}{\sqrt{3}}$ and $y = \frac{c}{2\sqrt{3}}$ in (ii) the maximum volume of the box is given by $V = \frac{c^2}{3} \times \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}}$ cubic units.

EXAMPLE 21 *The sum of the surface areas of a rectangular parallelopiped with sides x, 2x and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of the volumes is minimum, if x is equal to three times the radius of the sphere. Also, find the minimum value of the sum of their volumes.*

[NCERT EXEMPLAR, CBSE 2016]

SOLUTION Let y be the radius of the sphere and let S be the constant value of the sum of the surface areas of the parallelopiped and the sphere. Then,

$$S = 2\left(x \times 2x + 2x \times \frac{x}{3} + \frac{x}{3} \times x\right) + 4\pi y^2 \Rightarrow S = 6x^2 + 4\pi y^2 \qquad \text{...(i)}$$

Let V be the sum of the volumes of the sphere and the parallelepiped. Then,

$$V = \frac{4}{3}\pi y^3 + x \times 2x \times \frac{x}{3}$$

$$\Rightarrow \quad V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3 = \frac{4}{3}\pi\left(\frac{S-6x^2}{4\pi}\right)^{3/2} + \frac{2}{3}x^3 \qquad \left[\because S = 6x^2 + 4\pi y^2 \Rightarrow y^2 = \frac{S-6x^2}{4\pi}\right]$$

$$\Rightarrow \quad V = \frac{1}{6\sqrt{\pi}}(S-6x^2)^{3/2} + \frac{2}{3}x^3$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{1}{6\sqrt{\pi}} \times \frac{3}{2}(S-6x^2)^{1/2}(-12x) + \frac{2}{3} \times 3x^2 \Rightarrow \frac{dV}{dx} = -\frac{3}{\sqrt{\pi}}(S-6x^2)^{1/2}x + 2x^2 \qquad \text{...(ii)}$$

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

Now,

$$\frac{dV}{dx} = 0 \Rightarrow -\frac{3}{\sqrt{\pi}}(S-6x^2)^{1/2}x + 2x^2 = 0 \Rightarrow \frac{3x}{\sqrt{\pi}}(S-6x^2)^{1/2} = 2x^2$$

$$\Rightarrow \quad \frac{3}{\sqrt{\pi}}(S-6x^2)^{1/2} = 2x \Rightarrow 9(S-6x^2) = 4\pi x^2 \qquad \text{[Squaring both sides]}$$

$$\Rightarrow \quad 9(4\pi y^2) = 4\pi x^2 \Rightarrow 9y^2 = x^2 \Rightarrow x = 3y \qquad \text{[Using (i)]}$$

Putting $x = 3y$ or, $y = \frac{x}{3}$ in (i), we obtain $S = 6x^2 + \frac{4\pi x^2}{9}$.

Differentiating (ii), we obtain: $\frac{d^2V}{dx^2} = -\frac{3}{\sqrt{\pi}}(S-6x^2)^{1/2} + \frac{18x^2}{\sqrt{\pi}\sqrt{S-6x^2}} + 4x$

When $x = 3y$ or, $y = \frac{x}{3}$, we obtain

$$\frac{d^2V}{dx^2} = \frac{-3}{\sqrt{\pi}}\left(\frac{4\pi x^2}{9}\right)^{1/2} + \frac{18x^2}{\sqrt{\pi}\left(\frac{2}{3}\sqrt{\pi}x\right)} + 4x = -2x + \frac{27x}{\pi} + 4x = \frac{27x}{\pi} + 2x > 0$$

So, V is minimum when $x = 3y$. Putting $x = 3y$ or, $y = \frac{x}{3}$ in $V = \frac{4}{3}\pi y^3 + \frac{2}{3}x^3$, we obtain

$$V = \frac{4}{3}\pi\left(\frac{x}{3}\right)^3 + \frac{2}{3}x^3 = \frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$$

Hence, the sum of the volume is minimum when $x = 3y$ i.e. x is equal to three times the radius of the sphere and the maximum value of the sum of the volumes is $V = \frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)$.

EXAMPLE 22 *Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.*

SOLUTION Let ABC be a triangle inscribed in a given circle with centre O and radius r.

The area of the triangle will be maximum if its vertex A opposite to the base BC is at a maximum distance from the base BC. This is possible only when A lies on the diameter perpendicular to BC. Thus, $AD \perp BC$. So, triangle ABC must be an isosceles triangle. Let $OD = x$.

Applying Pythagoras theorem in right triangle ODB, we get

$$OB^2 = OD^2 + BD^2 \Rightarrow r^2 = x^2 + BD^2 \Rightarrow BD = \sqrt{r^2 - x^2}$$

$$\therefore \quad BC = 2BD = 2\sqrt{r^2 - x^2}$$

Also, $AD = AO + OD = r + x$. Let A denote the area of ΔABC. Then,

$$A = \frac{1}{2}(BC \times AD)$$

$$\Rightarrow \quad A = \frac{1}{2} \times 2\sqrt{r^2 - x^2} \times (r + x)$$

$$\Rightarrow \quad A = (r + x)\sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \sqrt{r^2 - x^2} - \frac{x(r + x)}{\sqrt{r^2 - x^2}} \quad \text{[Differentiating with respect to } x\text{]}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$$

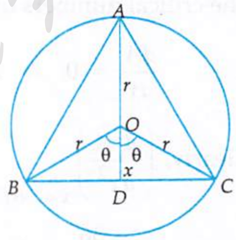

Fig. 17.40

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}} = 0 \Rightarrow (r - 2x)(r + x) = 0 \Rightarrow r - 2x = 0 \Rightarrow x = \frac{r}{2} \quad [\because\ r + x \neq 0]$$

Now, $\frac{dA}{dx} = \frac{r^2 - rx - 2x^2}{\sqrt{r^2 - x^2}}$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{(-r - 4x)}{\sqrt{r^2 - x^2}} + \frac{(r^2 - rx - 2x^2)\,x}{(r^2 - x^2)^{3/2}} \quad \text{[Differentiating both sides with respect to } x\text{]}$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x = r/2} = -2\sqrt{3} < 0$$

Thus, A is maximum when $x = \frac{r}{2}$. Therefore, $BD = \sqrt{r^2 - x^2} \Rightarrow BD = \frac{\sqrt{3}r}{2}$.

In ΔODB, we obtain

$$\tan\theta = \frac{BD}{OD} \Rightarrow \tan\theta = \frac{\sqrt{3}r/2}{r/2} = \sqrt{3} \Rightarrow \theta = 60° \Rightarrow \angle BAC = \theta = 60°$$

But, $AB = AC$. Therefore, $\angle B = \angle C = 60°$. Thus, we obtain $\angle A = \angle B = \angle C = 60°$.

Hence, A is maximum when $\Delta\, ABC$ is equilateral.

EXAMPLE 23 *A wire of length 36 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces, so that the combined area of the square and the circle is minimum?*

SOLUTION Let the length of a side of the square be x metres and the radius of the circle be y metres. It is given that the length of the wire is 36 m.

$$\therefore \quad 4x + 2\pi y = 36 \Rightarrow 2x + \pi y = 18 \qquad \text{...(i)}$$

Let A be the combined area of the square and the circle. Then,

$$A = x^2 + \pi y^2 \qquad \text{...(ii)}$$

$$\Rightarrow \quad A = x^2 + \pi\left(\frac{18-2x}{\pi}\right)^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad A = x^2 + \frac{1}{\pi}(18-2x)^2$$

$$\Rightarrow \quad \frac{dA}{dx} = 2x + \frac{2}{\pi}(18-2x)(-2) = 2x - \frac{4}{\pi}(18-2x) \text{ and, } \frac{d^2A}{dx^2} = 2 - \frac{4}{\pi}(-2) = 2 + \frac{8}{\pi}$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow 2x - \frac{4}{\pi}(18-2x) = 0 \Rightarrow x = \frac{36}{\pi+4}$$

Clearly, $\left(\frac{d^2A}{dx^2}\right)_{x = 36/\pi+4} = 2 + \frac{8}{\pi} > 0$. Thus, A is minimum when $x = \frac{36}{\pi+4}$.

Putting $x = \frac{36}{\pi+4}$ in (i), we obtain $y = \frac{18}{\pi+4}$. So, lengths of the two pieces of wire are

$$4x = 4 \times \frac{36}{\pi+4} = \frac{144}{\pi+4} \text{ m and } 2\pi y = 2\pi \times \frac{18}{\pi+4} = \frac{36\pi}{\pi+4} \text{ m}$$

Hence, the combined area of the square and the circle is minimum when the lengths of two pieces are $\frac{144}{\pi+4}$ metres and $\frac{36\pi}{\pi+4}$ metres.

EXAMPLE 24 *A figure consists of a semi-circle with a rectangle on its diameter. Given the perimeter of the figure, find its dimensions in order that the area may be maximum.* **[CBSE 2002]**

SOLUTION Let $ABCD$ be a rectangle and let the semi-circle be described on side AB as diameter. Let $AB = 2x$ and $AD = 2y$. Let P be the perimeter and A be the area of the figure. Then,

$$P = 2x + 4y + \pi x \quad \text{...(i)} \qquad \text{and,} \quad A = (2x)(2y) + \frac{\pi x^2}{2} \qquad \text{...(ii)}$$

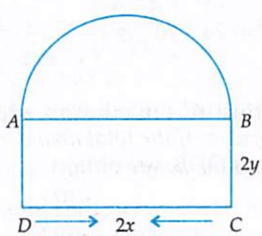

Fig. 17.41

Now $\quad A = 4xy + \frac{\pi x^2}{2}$

$$\Rightarrow \quad A = x(P - 2x - \pi x) + \frac{\pi x^2}{2} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad A = Px - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$\Rightarrow \quad A = Px - 2x^2 - \frac{\pi x^2}{2} \Rightarrow \frac{dA}{dx} = P - 4x - \pi x \text{ and } \frac{d^2A}{dx^2} = -4 - \pi$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow P - 4x - \pi x = 0 \Rightarrow x = \frac{P}{\pi + 4}$

Clearly, $\frac{d^2A}{dx^2} = -4 - \pi < 0$ for all values of x. Thus, A is maximum when $x = \frac{P}{\pi + 4}$.

Putting $x = \frac{P}{\pi + 4}$ in (i) we get $y = \frac{P}{2(\pi + 4)}$. So, area of the figure is maximum when dimensions of the figure are: Length $= 2x = \frac{2P}{\pi + 4}$ and Breadth $= 2y = \frac{P}{\pi + 4}$.

EXAMPLE 25 *A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume.*

SOLUTION Let x cm be the length of a side of the square which is cut-off from each corner of the plate. Then, dimensions of the box as shown in Fig. 17.42 are Length $= 24 - 2x$, Breadth $= 24 - 2x$ and height $= x$.

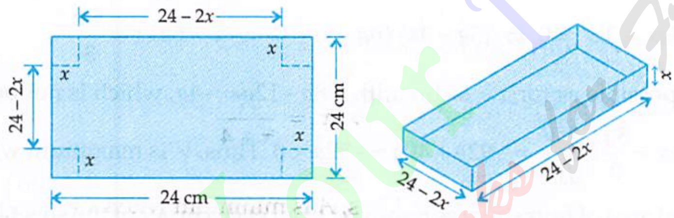

Fig. 17.42

Let V be the volume of the box. Then,

$$V = (24 - 2x)^2 x = 4x^3 - 96x^2 + 576x \Rightarrow \frac{dV}{dx} = 12x^2 - 192x + 576 \text{ and } \frac{d^2V}{dx^2} = 24x - 192$$

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow 12x^2 - 192x + 576 = 0$

$\Rightarrow \quad x^2 - 16x + 48 = 0 \Rightarrow (x - 12)(x - 4) = 0 \Rightarrow x = 12, 4$

But, $x = 12$ is not possible. Therefore, $x = 4$.

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x=4} = 24 \times 4 - 192 < 0$. Thus, V is maximum when $x = 4$.

Hence, the volume of the box is maximum when the side of the square is 4 cm.

Putting $x = 4$ in $V = (24 - 2x)^2 x$, we obtain that the maximum volume of the box is given by $V = (24 - 8)^2 \times 4 = 1024 \text{ cm}^3$.

EXAMPLE 26 *A rectangular sheet of fix perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed square is 100 square units, the resulting box has maximum volume. Find the length of the sides of the rectangular sheet.*

SOLUTION Let the sides of rectangular sheet be $8a$ and $15a$ units respectively. Let the length of each side of the squares of same size removed from each corner of the sheet be x units. Then, the dimensions of the open box, formed by folding up the flaps, are:

Length $= 15a - 2x$, breadth $= 8a - 2x$, height $= x$

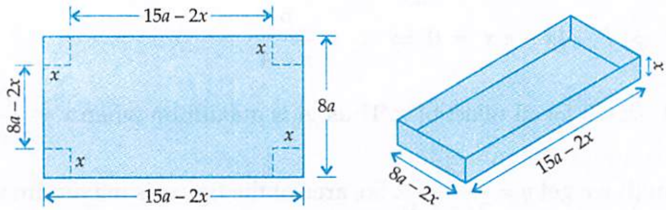

Fig. 17.43

Let V be the volume of the box formed. Then,

$$V = (15a - 2x)(8a - 2x)\,x$$

$$\Rightarrow \quad V = 120a^2 x - 46ax^2 + 4x^3, \ \frac{dV}{dx} = 120a^2 - 92ax + 12x^2 \text{ and } \frac{d^2V}{dx^2} = -92a + 24x$$

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

$$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow 120a^2 - 92ax + 12x^2 = 0$$

$$\Rightarrow \quad 30a^2 - 23ax + 3x^2 = 0 \Rightarrow (5a - 3x)(6a - x) = 0 \Rightarrow x = 6a,\ x = \frac{5a}{3}$$

But, $x = 6a$ is not possible as for $x = 6a$ breadth $= 8a - 12a = -4a$, which is not possible.

So, $x = \frac{5a}{3}$. When $x = \frac{5a}{3}$, $\frac{d^2V}{dx^2} = -92a + 40a = -52a < 0$. Thus, V is maximum when $x = \frac{5a}{3}$.

It is given that total area of four squares removed from each corner of the sheet is 100 sq. units.

$$\therefore \quad 4x^2 = 100 \Rightarrow x^2 = 25 \Rightarrow \frac{25a^2}{9} = 25 \Rightarrow a^2 = 9 \Rightarrow a = 3$$

Hence, the dimensions of the sheet are $15a = 45$ and $8a = 24$.

BASED ON LOTS

EXAMPLE 27 *Find the volume of the largest cylinder that can be inscribed in a sphere of radius r cm.* **[CBSE 2009, 2012]**

SOLUTION Let h be the height and R be the radius of the base of the inscribed cylinder. Let V be the volume of the cylinder. Then,

$$V = \pi R^2 h \qquad \text{...(i)}$$

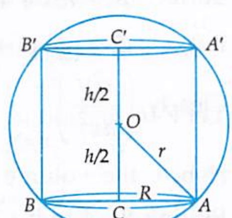

Fig. 17.44

Applying Pythagoras Theorem in ΔOCA, we get

$$\therefore \quad OA^2 = OC^2 + CA^2 \Rightarrow r^2 = \left(\frac{h}{2}\right)^2 + R^2 \Rightarrow R^2 = r^2 - \frac{h^2}{4}$$

Substituting the value of R^2 in (i), we get

$$V = \pi\left(r^2 - \frac{h^2}{4}\right)h$$

$$\Rightarrow \quad V = \pi r^2 h - \frac{\pi}{4}h^3, \ \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4} \text{ and } \frac{d^2V}{dh^2} = -\frac{3\pi h}{2}$$

The critical numbers of V are given by $\frac{dV}{dh} = 0$.

$$\therefore \quad \frac{dV}{dh} = 0 \Rightarrow \pi r^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h^2 = \frac{4r^2}{3} \Rightarrow h = \frac{2}{\sqrt{3}}r$$

Clearly, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{2r}{\sqrt{3}}} = -\sqrt{3}\pi r < 0$. Thus, V is maximum when $h = \frac{2r}{\sqrt{3}}$.

Putting $h = \frac{2r}{\sqrt{3}}$ in $R^2 = r^2 - \frac{h^2}{4}$, we obtain $R = \sqrt{\frac{2}{3}}\, r$. Substituting the values of R^2 and h in (i), we find that the maximum volume of the cylinder is given by

$$V = \pi R^2 h = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}$$

EXAMPLE 28 *Show that a cylinder of a given volume which is open at the top, has minimum total surface area, provided its height is equal to the radius of its base.* [CBSE 2011, 2014]

SOLUTION Let r be the radius and h be the height of a cylinder of given volume V. Then,

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2} \quad \text{...(i)}$$

Let S be the total surface area of the cylinder which is open at the top. Then,

$$S = 2\pi rh + \pi r^2$$

$$\Rightarrow \quad S = 2\pi r \times \frac{V}{\pi r^2} + \pi r^2 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad S = \frac{2V}{r} + \pi r^2,\ \frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r \text{ and } \frac{d^2s}{dr^2} = \frac{4V}{r^3} + 2\pi$$

The critical numbers of S are given by $\frac{dS}{dr} = 0$.

$$\therefore \quad \frac{dS}{dr} = 0 \Rightarrow -\frac{2V}{r^2} + 2\pi r = 0 \Rightarrow V = \pi r^3 \Rightarrow \pi r^2 h = \pi r^3 \Rightarrow h = r \quad [\because V = \pi r^2 h]$$

Clearly, $\left(\frac{d^2S}{dr^2}\right)_{r=h} = \frac{4V}{h^3} + 2\pi > 0$. Hence, S is minimum when $h = r$ i.e., when the height of the cylinder is equal to the radius of the base.

EXAMPLE 29 *Show that the height of the closed cylinder of given surface and maximum volume, is equal to the diameter of its base.* [NCERT, CBSE 2012]

SOLUTION Let r be the radius of the base and h be the height of a closed cylinder of given surface area S. Then,

$$S = 2\pi r^2 + 2\pi rh \Rightarrow h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2\left(\frac{S - 2\pi r^2}{2\pi r}\right) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad V = \frac{rS - 2\pi r^3}{2} = \frac{rS}{2} - \pi r^3,\ \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2 \text{ and } \frac{d^2V}{dr^2} = -6\pi r$$

The critical numbers of V are given by $\frac{dV}{dr} = 0$.

Now, $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} - 3\pi r^2 = 0 \Rightarrow S = 6\pi r^2 \Rightarrow 2\pi r^2 + 2\pi rh = 6\pi r^2 \Rightarrow h = 2r.$

Clearly, $\dfrac{d^2V}{dr^2} = -6\pi r < 0$ for all r. Hence, V is maximum when $h = 2r$ i.e., when the height of the cylinder is equal to the diameter of the base.

EXAMPLE 30 *Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base.* **[CBSE 2004, 2010]**

SOLUTION Let r be the radius and h be the height of a cylinder of given surface S. Then,

$$S = \pi r^2 + 2\pi rh \Rightarrow h = \frac{S - \pi r^2}{2\pi r} \qquad \text{...(i)}$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad V = \frac{Sr - \pi r^3}{2} = \frac{Sr}{2} - \frac{\pi r^3}{2} \Rightarrow \frac{dV}{dr} = \frac{S}{2} - \frac{3}{2}\pi r^2 \text{ and } \frac{d^2V}{dr^2} = -3\pi r$$

The critical numbers of V are given by $\dfrac{dV}{dr} = 0$.

$$\therefore \quad \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} - \frac{3}{2}\pi r^2 = 0 \Rightarrow S = 3\pi r^2 \Rightarrow \pi r^2 + 2\pi rh = 3\pi r^2 \Rightarrow r = h.$$

Clearly, $\dfrac{d^2V}{dr^2} = -3\pi r < 0$. Hence, V is maximum when $r = h$ i.e., when the height of the cylinder is equal to the radius of its base.

EXAMPLE 31 *Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is* $\dfrac{2a}{\sqrt{3}}$. **[NCERT, CBSE 2001, 2012, 2013, 2014]**

SOLUTION Let r be the radius of the base and h be the height of the cylinder $ABCD$ which is inscribed in a sphere of radius a. It is obvious that for maximum volume the axis of the cylinder must be along the diameter of the sphere. Let O be the centre of the sphere such that $OL = x$. By symmetry, O is the mid-points of LM. Applying Pythagoras Theorem in ΔALO, we get

$$OA^2 = OL^2 + AL^2 \Rightarrow a^2 = x^2 + AL^2 \Rightarrow AL = \sqrt{a^2 - x^2}$$

Let V be the volume of the cylinder. Then,

$$V = \pi (AL)^2 \times LM$$

$$\Rightarrow \quad V = \pi (AL)^2 \times 2(OL) = \pi (a^2 - x^2) \times 2x = 2\pi (a^2 x - x^3)$$

$$\Rightarrow \quad \frac{dV}{dx} = 2\pi (a^2 - 3x^2) \text{ and } \frac{d^2V}{dx^2} = -12\pi x$$

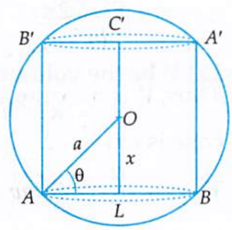

Fig. 17.45

The critical numbers of V are given by $\dfrac{dV}{dx} = 0$.

$$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow 2\pi (a^2 - 3x^2) = 0 \Rightarrow x = \frac{a}{\sqrt{3}}$$

Clearly, $\left(\dfrac{d^2V}{dx^2}\right)_{x = a/\sqrt{3}} = -12\pi \times \dfrac{a}{\sqrt{3}} < 0$. Hence, V is maximum when $x = \dfrac{a}{\sqrt{3}}$ and hence $LM = 2x = \dfrac{2a}{\sqrt{3}}$. In otherwords, the height of the cyclic of maximum volume is $2a/\sqrt{3}$.

EXAMPLE 32 *Show that the semi-vertical angle of a cone of maximum volume and given slant height is* $\tan^{-1}\sqrt{2}$ *or* $\cos^{-1}\dfrac{1}{\sqrt{3}}$. **[NCERT, CBSE 2011, 2014]**

SOLUTION Let α be the semi-vertical angle of a cone VAB of given slant height l.

In $\Delta\, AOV$,

$$\cos\alpha = \frac{VO}{VA} \text{ and } \sin\alpha = \frac{OA}{VA} \Rightarrow \cos\alpha = \frac{VO}{l} \text{ and } \sin\alpha = \frac{OA}{l} \Rightarrow VO = l\cos\alpha,\ OA = l\sin\alpha$$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi(OA)^2(VO)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi(l\sin\alpha)^2(l\cos\alpha)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi l^3\sin^2\alpha\cos\alpha \Rightarrow \frac{dV}{d\alpha} = \frac{\pi}{3}l^3\left(-\sin^3\alpha + 2\sin\alpha\cos^2\alpha\right)$$

$$\Rightarrow \quad \frac{dV}{d\alpha} = \frac{\pi l^3}{3}\sin\alpha\left(-\sin^2\alpha + 2\cos^2\alpha\right) \qquad \text{...(i)}$$

The critical points of V are given by $\dfrac{dV}{d\alpha} = 0$.

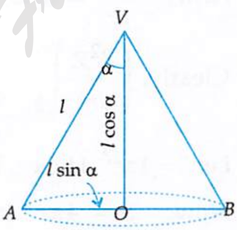

Fig. 17.46

$$\therefore \quad \frac{dV}{d\alpha} = 0$$

$$\Rightarrow \quad \frac{\pi l^3}{3}\sin\alpha\left(-\sin^2\alpha + 2\cos^2\alpha\right) = 0$$

$$\Rightarrow \quad 2\cos^2\alpha = \sin^2\alpha \Rightarrow \tan^2\alpha = 2 \Rightarrow \tan\alpha = \sqrt{2} \qquad [\because\ \alpha \text{ is acute} \therefore \sin\alpha \neq 0]$$

$$\therefore \quad \cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{1}{\sqrt{3}} \qquad [\because\ \tan\alpha = \sqrt{2}]$$

Differentiating (i) with respect to α, we get

$$\frac{d^2V}{d\alpha^2} = \frac{\pi l^3}{3}(-3\sin^2\alpha\cos\alpha + 2\cos^3\alpha - 4\sin^2\alpha\cos\alpha) = \frac{\pi l^3}{3}\cos^3\alpha\,(2 - 7\tan^2\alpha)$$

$$\therefore \quad \left(\frac{d^2V}{d\alpha^2}\right)_{\tan\alpha = \sqrt{2}} = \frac{1}{3}\pi l^3\left(\frac{1}{\sqrt{3}}\right)^3(2 - 7\times 2) = \frac{-4\pi l^3}{3\sqrt{3}} < 0.$$

Thus, V is maximum, when $\tan\alpha = \sqrt{2}$ or, $\alpha = \tan^{-1}\sqrt{2}$ i.e. when the semi-vertical angle of the cone is $\tan^{-1}\sqrt{2}$.

EXAMPLE 33 *Show that the semi-vertical angle of a right circular cone of given surface area and maximum volume is* $\sin^{-1}\left(\dfrac{1}{3}\right)$. **[NCERT]**

SOLUTION Let r be radius, l be the slant height and h be the height of the cone VAB of given surface area S. Then,

$$S = \pi r^2 + \pi r l \Rightarrow l = \frac{S - \pi r^2}{\pi r} \qquad \text{...(i)}$$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi r^2 h$$

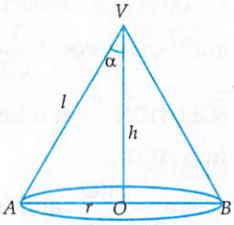

Fig. 17.47

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \qquad [\because l^2 = r^2 + h^2]$$

$$\Rightarrow \quad V^2 = \frac{\pi^2}{9} r^4 \left\{ \left(\frac{S - \pi r^2}{\pi r} \right)^2 - r^2 \right\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad V^2 = \frac{\pi^2 r^4}{9} \left\{ \frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2} \right\} = \frac{1}{9} S (S r^2 - 2\pi r^4)$$

Let $Z = V^2$. Then, V is maximum or minimum according as Z is maximum or minimum.

Now, $Z = \frac{1}{9} S (S r^2 - 2\pi r^4) \Rightarrow \frac{dZ}{dr} = \frac{1}{9} S (2Sr - 8\pi r^3)$ and $\frac{d^2 Z}{dr^2} = \frac{S}{9}(2S - 24\pi r^2)$

The critical numbers of Z are given by $\frac{dZ}{dr} = 0$.

Now, $\frac{dZ}{dr} = 0 \Rightarrow 2Sr - 8\pi r^3 = 0 \Rightarrow S = 4\pi r^2 \Rightarrow r^2 = \frac{S}{4\pi}$ ...(iii)

Clearly, $\left(\frac{d^2 Z}{dr^2} \right)_{r^2 = \frac{S}{4\pi}} = \frac{S}{9}\left(2S - 24\pi \times \frac{S}{4\pi} \right) = -\frac{4S^2}{9} < 0$. Thus, Z is maximum when $r^2 = \frac{S}{4\pi}$

i.e. $S = 4\pi r^2$. Hence, V is maximum when $S = 4\pi r^2$.

Now, $S = 4\pi r^2 \Rightarrow \pi r l + \pi r^2 = 4\pi r^2 \Rightarrow l = 3r$.

$\therefore \quad \sin\alpha = \frac{r}{l} = \frac{r}{3r} = \frac{1}{3}$. Hence, V is maximum when $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 34 *Show that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the sphere.* **[NCERT, CBSE 2008, 2010 C, 2012, 2013, 2014, 2016]**

SOLUTION Let VAB be a cone of greatest volume inscribed in a sphere of radius R. It is obvious that for maximum volume the axis of the cone must be along a diameter of the sphere. Let VC be the axis of the cone and O be the centre of the sphere such that $OC = x$. Then,

$$VC = VO + OC = R + x = \text{height of the cone.}$$

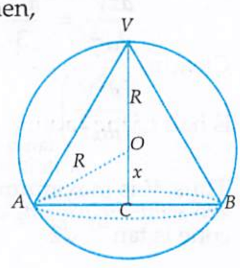

Fig. 17.48

Applying Pythagoras Theorem in ΔACO, we get

$$OA^2 = AC^2 + OC^2 \Rightarrow AC^2 = OA^2 - OC^2 = R^2 - x^2$$

Let V be the volume of the cone. Then,

$$V = \frac{1}{3}\pi (AC)^2 (VC)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi (R^2 - x^2)(R + x) \qquad \ldots\text{(i)}$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{1}{3}\pi \left\{ R^2 - x^2 - 2x(R + x) \right\}$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{1}{3}\pi (R^2 - 2Rx - 3x^2) \text{ and } \frac{d^2 V}{dx^2} = \frac{1}{3}\pi(-2R - 6x)$$

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

$$\therefore \quad \frac{dV}{dx} = 0$$

$$\Rightarrow \quad R^2 - 2Rx - 3x^2 = 0 \Rightarrow (R - 3x)(R + x) = 0 \Rightarrow R - 3x = 0 \Rightarrow x = \frac{R}{3} \quad [\because R + x \neq 0]$$

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x = R/3} = -\frac{4}{3} R\pi < 0$. Thus, V is maximum when $x = \frac{R}{3}$.

Putting $x = \frac{R}{3}$ in (i), we obtain

$$V = \text{Maximum volume of the cone} = \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(R + \frac{R}{3}\right) = \frac{32\pi R^3}{81}$$

$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right) = \frac{8}{27} \text{(Volume of the sphere)}.$$

EXAMPLE 35 *Prove that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half of that of the cone.* **[CBSE 2010 C, 2012, 2013, 2020]**

SOLUTION Let VAB be the cone of base radius $r = 0A$ and height $h = VO$. Let a cylinder of base radius $OC = x$ and height $= OO'$ be inscribed in the cone. Clearly, $\Delta VOB \sim \Delta B'DB$.

$$\therefore \quad \frac{VO}{B'D} = \frac{OB}{DB} \Rightarrow \frac{h}{B'D} = \frac{r}{r - x} \Rightarrow B'D = \frac{h(r - x)}{r}$$

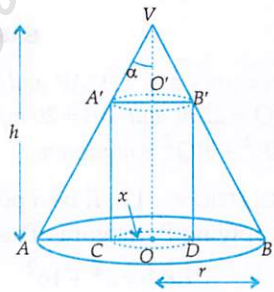

Fig. 17.49

Let S be the curved surface area of the cylinder. Then,

$$S = 2\pi(OC)(B'D)$$

$$\Rightarrow \quad S = 2\pi x \frac{h(r - x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

$$\Rightarrow \quad \frac{dS}{dx} = \frac{2\pi h}{r}(r - 2x) \text{ and } \frac{d^2S}{dx^2} = -\frac{4\pi h}{r}$$

The critical numbers of S are given by $\frac{dS}{dx} = 0$.

$$\therefore \quad \frac{dS}{dx} = 0 \Rightarrow \frac{2\pi h}{r}(r - 2x) = 0 \Rightarrow x = \frac{r}{2}$$

Clearly, $\frac{d^2S}{dx^2} = \frac{-4\pi h}{r} < 0$ for all x. Hence, S is maximum when $x = \frac{r}{2}$ i.e. radius of the cylinder is half of the radius of the cone.

EXAMPLE 36 *Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2\alpha$. Also, show that height of the cylinder is $\frac{h}{3}$.*

[NCERT, CBSE 2001C, 2007, 2008, 2010, 2017, 2020]

SOLUTION Let VAB be a given cone of height h, semi-vertical angle α and let x be the radius of the base of the cylinder $A'B'DC$ which is inscribed in the cone VAB.

In $\Delta VO'A'$

$$\tan\alpha = \frac{O'A'}{VO'} = \frac{x}{VO'} \Rightarrow VO' = x\cot\alpha \Rightarrow OO' = VO - VO' = h - x\cot\alpha \quad ...(i)$$

Let V be the volume of the cylinder. Then,

$$V = \pi(O'B')^2(OO')$$

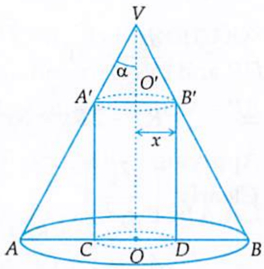

Fig. 17.50

$$\Rightarrow \quad V = \pi x^2 (h - x \cot \alpha) \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dV}{dx} = 2\pi xh - 3\pi x^2 \cot \alpha \text{ and } \frac{d^2V}{dx^2} = 2\pi h - 6\pi x \cot \alpha$$

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

$$\therefore \quad \frac{dV}{dx} = 0 \Rightarrow 2\pi xh - 3\pi x^2 \cot \alpha = 0 \Rightarrow x = \frac{2h}{3} \tan \alpha \quad [\because x \neq 0]$$

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x=\frac{2h}{3}\tan\alpha} = \pi(2h - 4h) = -2\pi h < 0$. Hence, V is maximum when $x = \frac{2h}{3} \tan \alpha$.

Putting $x = \frac{2h}{3} \tan \alpha$ in (ii), the maximum volume of the cylinder is given by

$$V = \pi \left(\frac{2h}{3} \tan \alpha\right)^2 \left(h - \frac{2h}{3}\right) = \frac{4}{27} \pi h^3 \tan^2 \alpha.$$

Putting $x = \frac{2h}{3}$ $\tan \alpha$ in (i), we get: $OO' = h - x \cot \alpha = h - \frac{2h}{3} = \frac{h}{3}$.

Hence, height of the cylinder $= OO' = \frac{h}{3}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 37 *Let AP and BQ be two vertical poles at points A and B respectively. If $AP = 16$ m, $BQ = 22$ m and $AB = 20$ m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.* **[NCERT, CBSE 2010]**

SOLUTION Let R be a point on AB such that $AR = x$ m. Then, $RB = (20 - x)$ m.
Applying Pythagoras Theorem in Δ's RAP and RBQ, we get

$$PR^2 = x^2 + 16^2 \quad \text{...(i)}$$

and, $$RQ^2 = 22^2 + (20 - x)^2 \quad \text{...(ii)}$$

$$\therefore \quad PR^2 + RQ^2 = x^2 + 16^2 + 22^2 + (20 - x)^2 = 2x^2 - 40x + 1140$$

Let $Z = RP^2 + RQ^2$. Then,

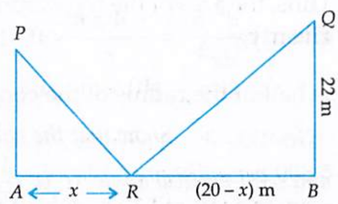

Fig. 17.51

$$Z = 2x^2 - 40x + 1140 \Rightarrow \frac{dZ}{dx} = 4x - 40 \text{ and } \frac{d^2Z}{dx^2} = 4$$

The critical numbers of Z are given by $\frac{dZ}{dx} = 0$.

$$\therefore \quad \frac{dZ}{dx} = 0 \Rightarrow 4x - 40 = 0 \Rightarrow x = 10$$

Clearly, $\frac{d^2Z}{dx^2} = 4 > 0$ for all x. So, Z is minimum when $x = 10$. Thus, $RP^2 + RQ^2$ is minimum when, the distance of R from A is 10 m.

EXAMPLE 38 *If the length of three sides of a trapezium other than base are equal to 10 cm, then find the area of trapezium when it is maximum.* **[NCERT, CBSE 2010, 2013]**

SOLUTION Let $ABCD$ be the given trapezium such that $AD = DC = BC = 10$ cm. Draw DP and CQ perpendiculars from D and C respectively on AB. Clearly, $\Delta APD \cong \Delta BQC$. Let $AP = x$ cm. Then, $BQ = x$ cm.

Applying Pythagoras Theorem in ΔAPD and ΔBQC, we obtain: $DP = QC = \sqrt{100 - x^2}$.

Let A be the area of trapezium $ABCD$. Then,

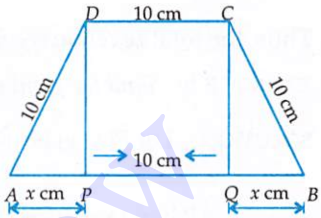

Fig. 17.52

$$A = \frac{1}{2}(AB + CD) \times DP$$

$$\Rightarrow \quad A = \frac{1}{2}(10 + 10 + 2x) \times \sqrt{100 - x^2}$$

$$\Rightarrow \quad A = (10 + x)\sqrt{100 - x^2} \quad \ldots \text{(i)}$$

$$\Rightarrow \quad \frac{dA}{dx} = \sqrt{100 - x^2} - \frac{x(10 + x)}{\sqrt{100 - x^2}} = \frac{100 - 10x - 2x^2}{\sqrt{100 - x^2}}$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow \frac{100 - 10x - 2x^2}{\sqrt{100 - x^2}} = 0$$

$$\Rightarrow 100 - 10x - 2x^2 = 0 \Rightarrow x^2 + 5x - 50 = 0 \Rightarrow (x + 10)(x - 5) = 0 \Rightarrow x = 5 \quad [\because x > 0 \therefore x + 10 \neq 0]$$

Now, $$\frac{dA}{dx} = \frac{100 - 10x - 2x^2}{\sqrt{100 - x^2}}$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{\sqrt{100 - x^2}\,(-10 - 4x) + \dfrac{(100 - 10x - 2x^2)\,x}{\sqrt{100 - x^2}}}{100 - x^2} = \frac{2x^3 - 300x - 1000}{(100 - x^2)^{3/2}}$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x=5} = \frac{-30}{\sqrt{75}} < 0$$

Thus, the area of the trapezium is maximum when $x = 5$. Putting $x = 5$ in (i), the maximum area is given by

$$A = \frac{1}{2}(10 + 5)\sqrt{100 - 25} = \frac{75\sqrt{3}}{2} \text{ cm}^2.$$

EXAMPLE 39 *A telephone company in a town has 500 subscribers on its list and collects fixed charges of ₹ 300 per subscriber. The company proposes to increase the annual subscription and it is believed that every increase of ₹ 1 one subscriber will discontinue the service. Find what increase will bring maximum revenue?* **[NCERT EXEMPLAR]**

SOLUTION Let the increase of ₹ x in annual subscription of ₹ 300 maximize the profit of the company. Due to this increase of ₹ x, x subscribers will discontinue the service. Therefore,

Number of subscriber using the service $= 500 - x$

Annual subscription of each subscriber $=$ ₹ $(300 + x)$

Let R be the total annual revenue of the company. Then,

$$R = (500 - x)(300 + x) \Rightarrow R = 150000 + 200x - x^2 \Rightarrow \frac{dR}{dx} = 200 - 2x \text{ and } \frac{d^2R}{dx^2} = -2$$

The stationary values of R are given by $\frac{dR}{dx} = 0$.

$$\therefore \quad \frac{dR}{dx} = 0 \Rightarrow 200 - 2x = 0 \Rightarrow x = 100$$

Clearly, $\frac{d^2R}{dx^2} = -2 < 0$ for all x. So, R is maximum when $x = 100$.

Thus, the total revenue received will be maximum if annual subscription is increased by ₹100.

EXAMPLE 40 *Find the point on the curve $y^2 = 4x$ which is nearest to the point $(2, 1)$.*

SOLUTION Let $P(x, y)$ be a point on $y^2 = 4x$ and $A(2, 1)$ be the given point. Then,

$$AP^2 = (x-2)^2 + (y-1)^2 \Rightarrow AP^2 = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 \qquad \left[\because y^2 = 4x \therefore x = y^2/4\right]$$

Let $Z = AP^2$. Then, Z is maximum or minimum according as AP is maximum or minimum.

$$\text{Now,} \quad Z = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 \Rightarrow \frac{dZ}{dy} = 2\left(\frac{y^2}{4} - 2\right)\left(\frac{2y}{4}\right) + 2(y-1) = \frac{y^3}{4} - 2 \text{ and, } \frac{d^2Z}{dy^2} = \frac{3y^2}{4}$$

The critical numbers of Z are given by $\frac{dZ}{dy} = 0$.

$$\therefore \quad \frac{dZ}{dy} = 0 \Rightarrow \frac{y^3}{4} - 2 = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2$$

Clearly, $\left(\frac{d^2Z}{dy^2}\right)_{y=2} = \frac{3(2)^2}{4} = 3 > 0$. Thus, Z is minimum when $y = 2$.

Putting $y = 2$ in $y^2 = 4x$, we obtain $x = 1$. So, the coordinates of P are $(1, 2)$. Hence, the point $(1, 2)$ on $y^2 = 4x$ is nearest to the point $(2, 1)$.

EXAMPLE 41 *A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point $(3, 2)$. What is the shortest distance between the soldier and the jet?*

SOLUTION Let $P(x, y)$ be the position of jet and the soldier is placed at $A(3, 2)$. Then, the distance between the soldier and jet is given by

$$AP = \sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x-3)^2 + x^4} \qquad \left[\because y = x^2 + 2\right]$$

Let $\quad Z = AP^2$. Then, $Z = (x-3)^2 + x^4$

Clearly AP is maximum or minimum according as Z is maximum or minimum.

$$\text{Now,} \quad Z = (x-3)^2 + x^4 \Rightarrow \frac{dZ}{dx} = 2(x-3) + 4x^3 \text{ and } \frac{d^2Z}{dx^2} = 12x^2 + 2$$

The critical numbers of Z are given by $\frac{dZ}{dx} = 0$.

$$\therefore \quad \frac{dZ}{dx} = 0 \Rightarrow 2(x-3) + 4x^3 = 0 \Rightarrow 2x^3 + x - 3 = 0$$

$$\Rightarrow \quad (x-1)(2x^2 + 2x + 3) = 0 \Rightarrow x = 1 \qquad \left[\because 2x^2 + 2x + 3 = 0 \text{ gives imaginary values of } x\right]$$

Clearly, $\left(\frac{d^2Z}{dx^2}\right)_{x=1} = 12+2 = 14 > 0$. Thus, Z is minimum when $x = 1$. Putting $x=1$ in $y = x^2 + 2$, we obtain $y = 3$. So, the coordinates of P are $(1, 3)$. Hence, AP is minimum when jet is at the point $(1, 3)$ on the curve.

Putting $x=1$ and $y=3$ in $AP = \sqrt{(x-3)^2+(y-2)^2}$, we obtain: $AP = \sqrt{(1-3)^2+1^2} = \sqrt{5}$.

Hence, the shortest distance $= \sqrt{5}$.

EXAMPLE 42 *Find the shortest distance between the line $y - x = 1$ and the curve $x = y^2$.*

SOLUTION Let $P(t^2, t)$ be any point on the curve $x = y^2$. The distance S of P from the given line is

$$S = \left|\frac{t - t^2 - 1}{\sqrt{2}}\right| = \left|\frac{t^2 - t + 1}{\sqrt{2}}\right| = \frac{t^2 - t + 1}{\sqrt{2}} \qquad [\because\ t^2 - t + 1 > 0 \text{ for all } t \in R]$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{2t-1}{\sqrt{2}} \text{ and } \frac{d^2S}{dt^2} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

The critical numbers of S are given by $\frac{dS}{dt} = 0$.

$$\therefore \quad \frac{dS}{dt} = 0 \Rightarrow 2t - 1 = 0 \Rightarrow t = \frac{1}{2}$$

Clearly, $\frac{d^2S}{dt^2} = \sqrt{2} > 0$ for all t. So, S is minimum when $t = \frac{1}{2}$.

Putting $t = \frac{1}{2}$ in $S = \frac{t^2 - t + 1}{\sqrt{2}}$, the minimum value of S is $S = \frac{\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 1}{\sqrt{2}} = \frac{3\sqrt{2}}{8}$.

EXAMPLE 43 *Find the shortest distance of the point $(0, c)$ from the parabola $y = x^2$, where $0 \le c \le 5$.* **[NCERT]**

SOLUTION Let $P(x, y)$ be any point on the parabola and $Q(0, c)$ be the given point. Then,

$$PQ^2 = x^2 + (y-c)^2 = x^2 + (x^2 - c)^2 \qquad [\because y = x^2]$$

$$\Rightarrow \quad PQ^2 = x^4 - x^2(2c-1) + c^2$$

Clearly, PQ will be minimum when PQ^2 is minimum. Let $Z = PQ^2$. Then,

$$Z = x^4 - x^2(2c-1) + c^2$$

$$\Rightarrow \quad \frac{dZ}{dx} = 4x^3 - 2x(2c-1) \text{ and } \frac{d^2Z}{dx^2} = 12x^2 - 2(2c-1)$$

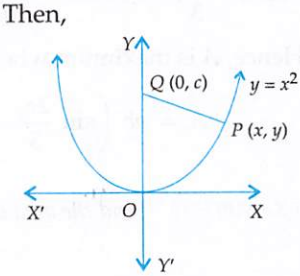

Fig. 17.53

The critical numbers of Z are given by $\frac{dZ}{dx} = 0$.

$$\therefore \quad \frac{dZ}{dx} = 0$$

$$\Rightarrow \quad 4x^3 - 2x(2c-1) = 0$$

$$\Rightarrow \quad 2x\{2x^2 - (2c-1)\} = 0 \Rightarrow x = 0, x = \pm\sqrt{\frac{2c-1}{2}} \Rightarrow x = 0, x = \pm\alpha, \text{ where } \alpha = \sqrt{\frac{2c-1}{2}}$$

Clearly, $\left(\dfrac{d^2Z}{dx^2}\right)_{x=\pm\alpha} = 12\alpha^2 - 4\alpha^2 = 8\alpha^2 > 0$. So, Z is minimum at $x = \pm\alpha$.

Hence, PQ is minimum at $x = \pm\alpha$. Putting $x = \pm\alpha$ in $PQ^2 = x^2 + (x^2 - c)^2$, the minimum value of PQ is given by

$$PQ^2 = \alpha^2 + (\alpha^2 - c)^2 = \frac{2c-1}{2} + \left(\frac{2c-1}{2} - c\right)^2 = \frac{2c-1}{2} + \frac{1}{4} = \frac{4c-1}{4} \Rightarrow PQ = \frac{\sqrt{4c-1}}{2}$$

Hence, the minimum distance is $\dfrac{\sqrt{4c-1}}{2}$.

EXAMPLE 44 *Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.* **[NCERT, CBSE 2010]**

SOLUTION Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Let APQ be an isosceles triangle having one vertex at $A\,(a, 0)$. Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Then the coordinates of Q are $(a\cos\theta, -b\sin\theta)$. Let A be the area of $\Delta\, APQ$. Then,

$$A = \frac{1}{2}\,(PQ)\,(AM) = \frac{1}{2}\,(2b\sin\theta)\,(a - a\cos\theta) = ab\,(\sin\theta - \sin\theta\cos\theta)$$

$$\Rightarrow \quad \frac{dA}{d\theta} = ab\,(\cos\theta - \cos^2\theta + \sin^2\theta)$$

$$\Rightarrow \quad \frac{dA}{d\theta} = ab\,(\cos\theta - \cos 2\theta)$$

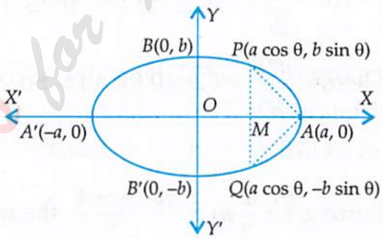

Fig. 17.54

The critical numbers of A are given by $\dfrac{dA}{d\theta} = 0$.

$$\therefore \quad \frac{dA}{d\theta} = 0$$

$$\Rightarrow \quad ab\,(\cos\theta - \cos 2\theta) = 0 \Rightarrow \cos\theta = \cos 2\theta \Rightarrow \theta = 2\pi - 2\theta \Rightarrow \theta = \frac{2\pi}{3}$$

Now, $\dfrac{dA}{d\theta} = ab\,(\cos\theta - \cos 2\theta) \Rightarrow \dfrac{d^2A}{d\theta^2} = ab\,(-\sin\theta + 2\sin 2\theta)$

For $\theta = \dfrac{2\pi}{3}$, we obtain: $\dfrac{d^2A}{d\theta^2} = ab\left(-\sin\dfrac{2\pi}{3} + 2\sin\dfrac{4\pi}{3}\right) = ab\left(-\dfrac{\sqrt{3}}{2} - 2\times\dfrac{\sqrt{3}}{2}\right) < 0$

Hence, A is maximum when $\theta = 2\pi/3$. The maximum area A is given by

$$A = ab\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{3}\cos\frac{2\pi}{3}\right) = ab\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\times\frac{1}{2}\right) = \frac{3\sqrt{3}}{4}\,ab.$$

EXAMPLE 45 *Find the area of the greatest rectangle that can be inscribed in an ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. **[CBSE 2013]**

SOLUTION Let $PQRS$ be a rectangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Let the coordinate of P be $(a\cos\theta, b\sin\theta)$. Then, the coordinates of Q, R and S are $(-a\cos\theta, b\sin\theta)$, $(-a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$ respectively. Let A be the area of rectangle $PQRS$. Then,

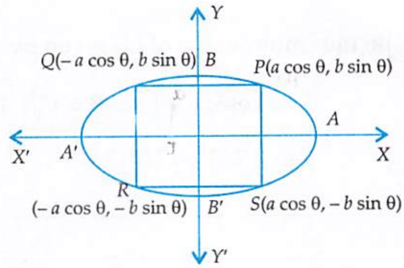

Fig. 17.55

$$A = PQ \times PS = 2a\cos\theta \times 2b\sin\theta = 2ab\sin 2\theta \quad \ldots \text{(i)}$$

$$\Rightarrow \frac{dA}{d\theta} = 4ab\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -8ab\sin 2\theta$$

The critical numbers of A are given by $\frac{dA}{d\theta} = 0$.

$$\therefore \frac{dA}{d\theta} = 0$$

$$\Rightarrow 4ab\cos 2\theta = 0 \Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2} \text{ or, } \frac{3\pi}{2} \Rightarrow \theta = \frac{\pi}{4} \text{ or, } \theta = \frac{3\pi}{4}$$

Clearly, $\left(\frac{d^2A}{d\theta^2}\right)_{\theta=\frac{\pi}{4}} = -8ab\sin\frac{\pi}{2} = -8ab < 0$. So, A is maximum when $\theta = \frac{\pi}{4}$. Putting $\theta = \frac{\pi}{4}$ in (i), the maximum value of A is given by $A = 2ab\sin\frac{\pi}{2} = 2ab$. Hence, the area of the greatest rectangle is $2ab$ sq. units.

EXAMPLE 46 *A point on the hypotenuse of a right triangle is at distances a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is* $(a^{2/3} + b^{2/3})^{3/2}$.

[NCERT, CBSE 2008]

SOLUTION Let AOB be a right triangle with hypotenuse AB such that a point P on AB is at distances a and b from OA and OB respectively. i.e. $PL = a$ and $PM = b$.

Let $\angle OAB = \theta$. In Δ's ALP and PMB, we obtain

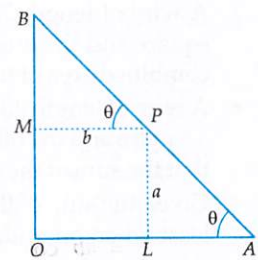

Fig. 17.56

$$\sin\theta = \frac{PL}{AP} \text{ and } \cos\theta = \frac{PM}{BP}$$

$$\Rightarrow \sin\theta = \frac{a}{AP} \text{ and } \cos\theta = \frac{b}{BP} \Rightarrow AP = a\,\text{cosec}\,\theta \text{ and } BP = b\sec\theta$$

Let l be the length of the hypotenuse AB. Then,

$$l = AP + BP$$

$$\Rightarrow l = a\,\text{cosec}\,\theta + b\sec\theta$$

$$\Rightarrow \frac{dl}{d\theta} = -a\,\text{cosec}\,\theta\cot\theta + b\sec\theta\tan\theta$$

$$\text{and, } \frac{d^2l}{d\theta^2} = a\,\text{cosec}^3\theta + a\,\text{cosec}\,\theta\cot^2\theta + b\sec^3\theta + b\sec\theta\tan^2\theta$$

The critical numbers of l are given by $\frac{dl}{d\theta} = 0$.

$$\therefore \frac{dl}{d\theta} = 0$$

$$\Rightarrow -a\,\text{cosec}\,\theta\cot\theta + b\sec\theta\tan\theta = 0 \Rightarrow -\frac{a\cos\theta}{\sin^2\theta} + \frac{b\sin\theta}{\cos^2\theta} = 0 \Rightarrow \tan^3\theta = \frac{a}{b}$$

$$\Rightarrow \tan\theta = \left(\frac{a}{b}\right)^{1/3} \Rightarrow \sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}} \text{ and, } \cos\theta = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

Clearly, $\frac{d^2l}{d\theta^2} > 0$ for $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$. Thus, l is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}$.

The minimum value of l is given by

$$l = a \operatorname{cosec} \theta + b \sec \theta = a\sqrt{1+\cot^2\theta} + b\sqrt{1+\tan^2\theta} = a\sqrt{1+\left(\frac{b}{a}\right)^{2/3}} + b\sqrt{1+\left(\frac{a}{b}\right)^{2/3}}$$

$$\Rightarrow \quad l = (a^{2/3} + b^{2/3})^{3/2}.$$

EXERCISE 17.5

BASIC

1. Determine two positive numbers whose sum is 15 and the sum of whose squares is minimum.
2. Divide 64 into two parts such that the sum of the cubes of two parts is minimum.
3. How should we choose two numbers, each greater than or equal to -2, whose sum is $1/2$ so that the sum of the first and the cube of the second is minimum?
4. Divide 15 into two parts such that the square of one multiplied with the cube of the other is minimum.
5. (i) Of all the closed cylindrical cans (right circular), which enclose a given volume of 100 cm^3, which has the minimum surface area? [NCERT, CBSE 2014]

   (ii) Amongst all open (from the top) right circular cylindrical boxes of volume $125\pi\ \text{cm}^3$, find the dimensions of the box which has the least surface area. [CBSE 2020]
6. A beam is supported at the two ends and is uniformly loaded. The bending moment M at a distance x from one end is given by

   (i) $M = \frac{WL}{2}x - \frac{W}{2}x^2$ (ii) $M = \frac{Wx}{3} - \frac{W}{3}\frac{x^3}{L^2}$

   Find the point at which M is maximum in each case.
7. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the lengths of the two pieces so that the combined area of the circle and the square is minimum? [NCERT, CBSE 2007, 2010]
8. A wire of length 20 m is to be cut into two pieces. One of the pieces will be bent into shape of a square and the other into shape of an equilateral triangle. Where the wire should be cut so that the sum of the areas of the square and triangle is minimum? [CBSE 2005]
9. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when one side of the square is equal to diameter of the circle. [NCERT, CBSE 2005, 2011, 2014]
10. Find the largest possible area of a right angled triangle whose hypotenuse is 5 cm long. [CBSE 2000]
11. Two sides of a triangle have lengths 'a' and 'b' and the angle between them is θ. What value of θ will maximize the area of the triangle? Find the maximum area of the triangle also. [CBSE 2002 C]
12. A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum? Also, find this maximum volume. [NCERT]
13. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off squares from each corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum possible? [NCERT]
14. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square matre for sides, what is the cost of least expensive tank? [NCERT, CBSE 2009, 2019]

15. A window in the form of a rectangle is surmounted by a semi-circular opening. The total perimeter of the window is 10 m. Find the dimensions of the rectangular part of the window to admit maximum light through the whole opening.
[NCERT, CBSE 2000, 2002, 2011, 2014]

16. A large window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 metres find the dimensions of the rectangle that will produce the largest area of the window. [CBSE 2011]

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. [NCERT]

18. A rectangle is inscribed in a semi-circle of radius r with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also the area.

BASED ON LOTS

19. Prove that a conical tent of given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. [NCERT, CBSE 2007, 2011, 2013]

20. Show that the cone of the greatest volume which can be inscribed in a given sphere has an altitude equal to 2/3 of the diameter of the sphere.

21. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface is $\cot^{-1}(\sqrt{2})$. [CBSE 2014]

22. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius a. Show that the area of the triangle is maximum when $\theta = \frac{\pi}{6}$. [NCERT EXEMPLAR]

23. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. [CBSE 2016]

24. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved about one of its sides. [NCERT EXEMPLAR]

25. Show that the height of the cone of maximum volume that can be inscribed in a sphere of radius 12 cm is 16 cm. [CBSE 2005]

26. A closed cylinder has volume 2156 cm^3. What will be the radius of its base so that its total surface area is minimum? [CBSE 2000C]

27. Show that the maximum volume of the cylinder which can be inscribed in a sphere of radius $5\sqrt{3}$ cm is $500\,\pi\,cm^3$. [CBSE 2004]

28. Show that among all positive numbers x and y with $x^2 + y^2 = r^2$, the sum $x + y$ is largest when $x = y = r/\sqrt{2}$.

29. Determine the points on the curve $x^2 = 4y$ which are nearest to the point (0, 5).

30. Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, – 8).

31. Find the point on the curve $x^2 = 8y$ which is nearest to the point (2, 4). [CBSE 2007]

32. Find the point on the parabolas $x^2 = 2y$ which is closest to the point (0, 5).

33. Find the coordinates of a point on the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$. [CBSE 2015]

34. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point (1, 4).
[CBSE 2011]

35. Find the maximum slope of the curve $y = -x^3 + 3x^2 + 2x - 27$.

36. The total cost of producing x radio sets per day is ₹ $\left(\frac{x^2}{4} + 35x + 25\right)$ and the price per set at which they may be sold is ₹ $\left(50 - \frac{x}{2}\right)$. Find the daily output to maximize the total profit.

37. Manufacturer can sell x items at a price of ₹ $\left(5 - \frac{x}{100}\right)$ each. The cost price is ₹ $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to earn maximum profit. **[NCERT, CBSE 2009]**

38. An open tank is to be constructed with a square base and vertical sides so as to contain a given quantity of water. Show that the expenses of lining with lead will be least, if depth is made half of width.

39. A box of constant volume c is to be twice as long as it is wide. The material on the top and four sides cost three times as much per square metre as that in the bottom. What are the most economic dimensions?

40. The sum of the surface areas of a sphere and a cube is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

BASED ON HOTS

41. A given quantity of metal is to be cast into a half cylinder with a rectangular base and semicircular ends. Show that in order that the total surface area may be minimum, the ratio of the length of the cylinder to the diameter of its semi-circular ends is $\pi : (\pi + 2)$.

42. The strength of a beam varies as the product of its breadth and square of its depth. Find the dimensions of the strongest beam which can be cut from a circular log of radius a.

43. A straight line is drawn through a given point $P(1, 4)$. Determine the least value of the sum of the intercepts on the coordinate axes.

44. The total area of a page is 150 cm^2. The combined width of the margin at the top and bottom is 3 cm and the side 2 cm. What must be the dimensions of the page in order that the area of the printed matter may be maximum?

45. The space s described in time t by a particle moving in a straight line is given by $s = t^5 - 40t^3 + 30t^2 + 80t - 250$. Find the minimum value of acceleration.

46. A particle is moving in a straight line such that its distance s at any time t is given by $s = \frac{t^4}{4} - 2t^3 + 4t^2 - 7$. Find when its velocity is maximum and acceleration minimum.

ANSWERS

1. 15/2, 15/2
2. 32, 32
3. $\left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right), \frac{1}{\sqrt{3}}$
4. 6, 9
5. (i) The cylinder with radius $\left(\frac{50}{\pi}\right)^{1/3}$ (ii) Radius = Height = 5 cm
6. (i) $x = \frac{L}{2}$ (ii) $x = \frac{L}{\sqrt{3}}$
7. $\frac{28\pi}{\pi + 4}$ m, $\frac{112}{\pi + 4}$ m
8. $\frac{80\sqrt{3}}{9 + 4\sqrt{3}}, \frac{180}{9 + 4\sqrt{3}}$
10. $\frac{25}{4}$ cm^2
11. $\frac{\pi}{2}$, Area $= \frac{1}{2}ab$
12. 3 cm, 432 cm^3
13. 5 cm
14. ₹ 1000
15. Length $= \frac{20}{\pi + 4}$, Breadth $= \frac{10}{\pi + 4}$

16. $\dfrac{12}{6-\sqrt{3}}, \dfrac{18-6\sqrt{3}}{6-\sqrt{3}}$ 18. $\dfrac{r}{\sqrt{2}}, \sqrt{2}r$, Area $= r^2$ 24. 12 cm, 6 cm

26. 7 cm 29. $(\pm 2\sqrt{3}, 3)$ 30. $(4, -4)$

31. $(4, 2)$ 32. $(\pm 2\sqrt{2}, 4)$ 33. $(-2, -8)$

34. $(2, 2)$ 35. 5 at $(1, -23)$ 36. 10 units 37. 240

38. $a = -3, b = -9, c \in R$ 39. Length $= 2\left(\dfrac{9c}{16}\right)^{1/3}$, Breadth $= \left(\dfrac{9c}{16}\right)^{1/3}$, Height $= \left(\dfrac{32c}{81}\right)^{1/3}$

42. Breadth $= \dfrac{2a}{\sqrt{3}}$, Depth $= 2a\sqrt{\dfrac{2}{3}}$ 43. 9

44. Length = 15 cm, Width = 10 cm 45. $a = -260$ at $t = 2$

46. Velocity is max. at $t = 2 - \dfrac{2}{\sqrt{3}}$, Acceleration is min. at $t = 2$

HINTS TO SELECTED PROBLEMS

5. Let r be the radius and h be the height of the closed cylindrical cane of volume 100 cm^3. Then,

$$\pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2} \qquad \text{...(i)}$$

Let S be the surface area of the can. Then,

$$S = 2\pi r h + 2\pi r^2 \Rightarrow S = \frac{200}{r} + 2\pi r^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \frac{dS}{dr} = -\frac{200}{r^2} + 4\pi r \text{ and } \frac{d^2S}{dr^2} = \frac{400}{r^3} + 4\pi$$

The critical numbers of S are given by $\dfrac{dS}{dr} = 0$.

$$\therefore \frac{dS}{dr} = 0 \Rightarrow -\frac{200}{r^2} + 4\pi r = 0 \Rightarrow 4\pi r^3 = 200 \Rightarrow r = \left(\frac{50}{\pi}\right)^{1/3}$$

Clearly, $\dfrac{d^2S}{dr^2} > 0$ for all r. Hence, S is minimum when $r = \left(\dfrac{50}{\pi}\right)^{1/3}$.

7. Let r be the radius of the circle and x meter be the length of each side of the square. Then,

$$2\pi r + 4x = 28 \Rightarrow \pi r + 2x = 14 \Rightarrow r = \frac{14-2x}{\pi} \qquad \text{...(i)}$$

Let A be the combined area of the circle and the square. Then,

$$A = \pi r^2 + x^2 = \pi\left(\frac{14-2x}{\pi}\right)^2 + x^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow A = \frac{1}{\pi}(14-2x)^2 + x^2 = \frac{4}{\pi}(7-x)^2 + x^2 \Rightarrow \frac{dA}{dx} = -\frac{8}{\pi}(7-x) + 2x \text{ and } \frac{d^2A}{dx^2} = \frac{8}{\pi}$$

The critical numbers of A are given by $\dfrac{dA}{dx} = 0$.

$$\therefore \frac{dA}{dx} = 0 \Rightarrow -\frac{8}{\pi}(7-x) + 2x = 0 \Rightarrow x = \frac{28}{\pi + 4}$$

Clearly, $\frac{d^2A}{dx^2} = \frac{8}{\pi} > 0$ for all x. Hence, A is minimum when $x = \frac{28}{\pi+4}$.

The lengths of two partions are $4x = \frac{112}{\pi+4}$ meter and, $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ m respectively.

9. Let x be the length of the each side of the square and y be the radius of the circle. Let S be the sum of their perimeters. Then,

$$S = 4x + 2\pi y \Rightarrow y = \frac{S-4x}{2\pi} \quad \text{...(i)}$$

Let A be the sum of the areas of the square and the circle. Then,

$$A = x^2 + \pi y^2 = x^2 + \frac{1}{4\pi}(S-4x)^2 \quad \text{[Using (i)]}$$

$$\Rightarrow \frac{dA}{dx} = 2x - \frac{2}{\pi}(S-4x) \text{ and } \frac{d^2A}{dx^2} = 2 + \frac{8}{\pi}$$

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \frac{dA}{dx} = 0 \Rightarrow 2x - \frac{2}{\pi}(S-4x) = 0 \Rightarrow \pi x - S + 4x = 0 \Rightarrow x = \frac{S}{\pi+4}$$

Clearly, $\frac{d^2A}{dx^2} = 2 + \frac{8}{\pi} > 0$ for all x. So, A is minimum when $x = \frac{S}{\pi+4}$ and for this value of x the value of y is given by $y = \frac{1}{2\pi}(S-4x) = \frac{1}{2\pi}\left(S - \frac{4S}{\pi+4}\right) = \frac{S}{2(\pi+4)}$.

Clearly, $x = 2y$ i.e. side of the square is equal to the diameter of the circle.
Hence, A is minimum when side of the square is equal to the diameter of the circle.

12. Let the length of the each side of the square which is cut from each corner of the tin sheet be x cm. By folding up the flaps, a cuboidal box is formed whose length, breadth and height are $18-2x$, $18-2x$ and x respectively. Then, its volume V is given by

$$V = (18-2x)(18-2x)x = 324x - 72x^2 + 4x^3$$

$$\Rightarrow \frac{dV}{dx} = 324 - 144x + 12x^2 \text{ and } \frac{d^2V}{dx^2} = -144 + 24x$$

The critical numbers of V are given by

$$\frac{dV}{dx} = 0 \Rightarrow 324 - 144x + 12x^2 = 0 \Rightarrow x^2 - 12x + 27 = 0 \Rightarrow x = 3, 9.$$

But, $x = 9$ is not possible. Therefore, $x = 3$. Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x=3} = -144 + 72 = -72 < 0$.

So, V is maximum when $x = 3$ i.e. the length of each side of the square to be cut is 3 cm.

13. Let the length of a side of the square be x cm and let V be the volume of the box. Then, $V = (45-2x)(24-2x)x$. Now, proceed as in Q. No. 10.

14. Let the length and breadth of the tank be x and y meters respectively. It is given that the volume of the tank is 8m^3 and height is 2m.

$$\therefore 2xy = 8 \Rightarrow xy = 4 \Rightarrow y = \frac{4}{x} \quad \text{...(i)}$$

Let C be the cost of the tank. Then,

$$C = 70xy + 45(2 \times 2y + 2 \times 2x) = 70xy + 180y + 180x$$

$$\Rightarrow C = 280 + \frac{720}{x} + 180x \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dC}{dx} = -\frac{720}{x^2} + 180 \text{ and } \frac{d^2C}{dx^2} = \frac{1440}{x^3}$$

The critical numbers of C are given by $\frac{dC}{dx} = 0$.

$$\therefore \quad \frac{dC}{dx} = 0 \Rightarrow -\frac{720}{x^2} + 180 = 0 \Rightarrow x = 2$$

Clearly, $\left(\frac{d^2C}{dx^2}\right)_{x=2} = 180 > 0$. So, C is minimum when $x = 2$.

Putting $x = 1$ in $C = 280 + \frac{720}{x} + 180x$, we get $C = 1000$. Hence, the cost of least expensive tank is ₹ 1000.

15. Let the width and height of window be $2x$ m and y m respectively. It is given that the perimeter of the window is 10 m.

$$\therefore \quad 2x + 2y + \pi x = 10 \Rightarrow y = 5 - \frac{x}{2}(\pi + 2) \qquad \ldots\text{(i)}$$

Let A be the area of the window. Then

$$A = 2xy + \frac{\pi}{2}x^2$$

$$\Rightarrow \quad A = 10x - (\pi + 2)x^2 + \frac{\pi}{2}x^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \frac{dA}{dx} = 10 - 2x(\pi + 2) + \pi x \text{ and, } \frac{d^2A}{dx^2} = -2(\pi + 2) + \pi = -\pi - 4$$

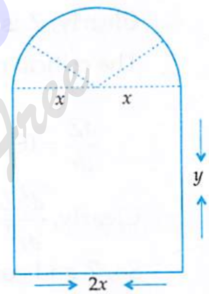

Fig. 19.57

The critical numbers of A are given by $\frac{dA}{dx} = 0$.

$$\therefore \quad \frac{dA}{dx} = 0 \Rightarrow 10 - 2x(\pi + 2) + \pi x = 0 \Rightarrow x = \frac{10}{\pi + 4}$$

Clearly, $\frac{d^2A}{dx^2} = -\pi - 4 < 0$ for all x. So, A is maximum when $x = \frac{10}{\pi + 4}$ and $y = 5 - \frac{5(\pi + 2)}{(\pi + 4)} = \frac{10}{\pi + 4}$. Hence, the dimensions of the window are $2x = \frac{20}{\pi + 4}$ and $y = \frac{10}{\pi + 4}$.

17. Let $OC = OG = x$. Then, $AC = \sqrt{R^2 - x^2}$.

Let V be the volume of the cylinder. Then.

$$V = \pi\left\{\sqrt{(R^2 - x^2)}\right\}^2 2x = 2\pi(R^2 x - x^3)$$

$$\Rightarrow \quad \frac{dV}{dx} = 2\pi(R^2 - 3x^2) \text{ and } \frac{d^2V}{dx^2} = -12\pi x$$

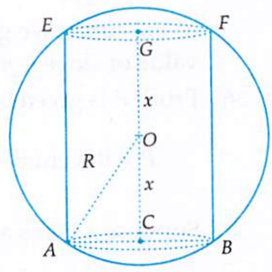

Fig. 19.58

The critical numbers of V are given by $\frac{dV}{dx} = 0$.

Now, $\frac{dV}{dx} = 0 \Rightarrow R^2 - 3x^2 = 0 \Rightarrow x = \frac{R}{\sqrt{3}}$

Clearly, $\left(\frac{d^2V}{dx^2}\right)_{x = \frac{R}{\sqrt{3}}} = \frac{-12\pi R}{\sqrt{3}} < 0$. So, V is maximum when $x = \frac{R}{\sqrt{3}}$ and height of the cylinder $= 2x = \frac{2R}{\sqrt{3}}$.

19. Let r be the radius of the base h be the height and l be the slant height of the conical tent of volume V and surface area S. Then,

$$V = \frac{1}{3}\pi r^2 h \text{ and } S = \pi r l$$

Now, $S = \pi r l$

$$\Rightarrow S^2 = \pi^2 r^2 l^2 \Rightarrow Z = \pi^2 r^2 (r^2 + h^2), \text{ where } Z = S^2$$

$$\Rightarrow Z = \pi^2 r^4 + \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4}\right) \left[\because V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi^2 r^4}\right]$$

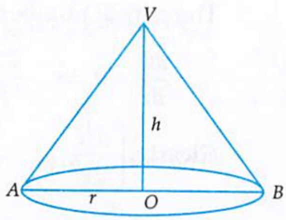

Fig. 19.59

$$\Rightarrow Z = \pi^2 r^4 + \frac{9V^2}{r^2} \Rightarrow \frac{dZ}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} \text{ and } \frac{d^2Z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

Clearly, Z is maximum or minimum according as S is maximum or minimum.

The critical numbers of Z are given by $\frac{dZ}{dr} = 0$.

$$\frac{dZ}{dr} = 0 \Rightarrow 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \Rightarrow 4\pi^2 r^6 = 18V^2 \Rightarrow 2\pi^2 r^6 = \pi^2 r^4 h^2 \Rightarrow h = \sqrt{2}\, r$$

Clearly, $\frac{d^2Z}{dr^2} = 12\pi^2 r^2 + 54\frac{V^2}{r^4} > 0$ for all values of V and r.

So, Z and consequently S is minimum when $h = \sqrt{2}\, r$.

35. Slope m of the curve is given by $m = \frac{dy}{dx} = -3x^2 + 6x + 2$.

Now, $m = -3x^2 + 6x + 2 \Rightarrow \frac{dm}{dx} = -6x + 6$ and $\frac{d^2m}{dx^2} = -6$

The critical numbers of m are given by $\frac{dm}{dx} = 0$. Now, $\frac{dm}{dx} = 0 \Rightarrow -6x + 6 = 0 \Rightarrow x = 1$.

Clearly, $\frac{d^2m}{dx^2} = -6 < 0$ for all x. So, m is maximum when $x = 1$. Putting $x = 1$ in the equation of the curve, we get $y = -23$. Thus, slope is maximum at the point $(1, -23)$. The maximum value of slope is $m = 5$.

36. Profit P is given by

$$P = \text{Revenue} - \text{Cost} = ₹\left\{\left(50 - \frac{x}{2}\right)x - \left(\frac{x^2}{4} + 35x + 25\right)\right\} = ₹\left(-\frac{3}{4}x^2 + 15x - 25\right)$$

37. Suppose x items are sold to maximize the profit P. Then P = Revenue − Cost

$$\Rightarrow P = x\left(5 - \frac{x}{100}\right) - \left(\frac{x}{5} + 500\right) \Rightarrow P = \frac{24}{5}x - \frac{x^2}{100} - 500 \Rightarrow \frac{dP}{dx} = \frac{24}{5} - \frac{x}{50} \text{ and } \frac{d^2P}{dx^2} = -\frac{1}{50}$$

The critical numbers of P are given by $\frac{dP}{dx} = 0$.

$$\therefore \frac{dP}{dx} = 0 \Rightarrow \frac{24}{5} - \frac{x}{50} = 0 \Rightarrow x = 240$$

Clearly, $\frac{d^2P}{dx^2} = -\frac{1}{50} < 0$ for all x. Hence, profit P is maximum when 240 items are sold.

43. The equation of a line passing through $P(1, 4)$ is $y - 4 = m(x-1)$, where $m < 0$. Its intercepts on the axes are $\frac{m-4}{m}$ and $-(m-4)$ respectively.

Let S be the sum of the intercepts. Then,

$$S = \frac{m-4}{m} - (m-4) = -m + 5 - \frac{4}{m} \Rightarrow \frac{dS}{dm} = -1 + \frac{4}{m^2} \text{ and } \frac{d^2 S}{dm^2} = -\frac{8}{m^3}$$

The critical numbers of S are given by $\frac{dS}{dm} = 0$.

Now, $\frac{dS}{dm} = 0 \Rightarrow -1 + \frac{4}{m^2} = 0 \Rightarrow m^2 = 4 \Rightarrow m = -2$ $[\because m < 0]$

For $m = -2$, $\frac{d^2 S}{dm^2} = 1 > 0$. So, S is minimum when $m = -2$.

For $m = -2$, The sum of the intercepts is given by $S = 2 + 5 + 2 = 9$.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $f(x) = \frac{1}{4x^2 + 2x + 1}$, then its maximum value is ....
2. The minimum value of $f(x) = \sin x$ in $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ is ....
3. The maximum value of $f(x) = \sin x + \cos x$ is ....
4. If $f(x)$ has the second order derivative at $x = c$ such that $f'(c) = 0$ and $f''(c) > 0$, then c is a point of ....
5. If $f'(x)$ changes its sign from positive to negative as x increases through c in the interval $(c-h, c+h)$, then $x = c$ is a point of ....
6. If $f'(x)$ changes its sign from negative to positive as x increases through c in the interval $(c-h, c+h)$, then $x = c$ is a point of ....
7. The positive real number x when added to its reciprocal gives the minimum value of the sum when, $x =$ ....
8. The real number which must exceeds its cube is ....
9. The function $f(x) = ax + \frac{b}{x}$, $a, b, x > 0$ takes on the least value at x equal to ....
10. If $y = a \log x + bx^2 + x$ has its extreme values at $x = 1$ and $x = 2$, then $(a,b) =$ ....
11. If the function $f(x) = a \sin x + \frac{1}{3} \sin 3x$ has an extremum at $x = \frac{\pi}{3}$, then $a =$ ....
12. The maximum value of $f(x) = x\, e^{-x}$ is ....
13. If the function $f(x) = x^4 - 62x^2 + ax + 9$ attains a local maximum at $x = 1$, then $a =$ ....
14. If the sum of two non-zero numbers is 4, then the minimum value of the sum of their reciprocals is ....
15. If x and y are two real numbers such that $x > 0$ and $xy = 1$. Then the minimum value of $x + y$ is ....
16. The number that exceeds its square by the greatest amount is ....

17. If m and M respectively denote the minimum and maximum values of $f(x)=(x-1)^2+3$ in the interval $[-3,1]$, then the ordered pair $(m, M)=$ ....
18. The minimum value of $f(x)=x^2+\frac{250}{x}$ is ....
19. The maximum slope of the curve $y=-x^3+3x^2+9x-27$ is ....
20. The function $f(x)=\frac{x}{2}+\frac{2}{x}$ has a local minimum at $x=$ ....
21. The least value of the function $f(x)=ax+\frac{b}{x}\ (a>0, b>0, x>0)$ is ....

[CBSE 2020, NCERT EXEMPLAR]

ANSWERS

1. $\frac{4}{3}$ 2. -1 3. $\sqrt{2}$ 4. Local minimum 5. Local maximum
6. Local minimum 7. 1 8. $\frac{1}{\sqrt{3}}$ 9. $\sqrt{\frac{b}{a}}$ 10. $\left(-\frac{2}{3},-\frac{1}{6}\right)$ 11. 2
12. $\frac{1}{e}$ 13. 120 14. 1 15. 2 16. $\frac{1}{2}$ 17. (3, 19) 18. 75
19. 12 20. 2 21. $2\sqrt{ab}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write necessary condition for a point $x=c$ to be an extreme point of the function $f(x)$.
2. Write sufficient conditions for a point $x=c$ to be a point of local maximum.
3. If $f(x)$ attains a local minimum at $x=c$, then write the values of $f'(c)$ and $f''(c)$.
4. Write the minimum value of $f(x)=x+\frac{1}{x},\ x>0$.
5. Write the maximum value of $f(x)=x+\frac{1}{x},\ x<0$.
6. Write the point where $f(x)=x\log_e x$ attains minimum value.
7. Find the least value of $f(x)=ax+\frac{b}{x}$, where $a>0, b>0$ and $x>0$.
8. Write the minimum value of $f(x)=x^x$.
9. Write the maximum value of $f(x)=x^{1/x}$.
10. Write the maximum value of $f(x)=\frac{\log x}{x}$, if it exists.

ANSWERS

1. $f'(c)=0$ 2. $f'(c)=0$ and $f''(c)<0$ 3. $f'(c)=0$ and $f''(c)>0$
4. 2 5. -2 6. $\left(\frac{1}{e},-\frac{1}{e}\right)$
7. $2\sqrt{ab}$ 8. $e^{-1/e}$ 9. $e^{1/e}$ 10. $\frac{1}{e}$