CHAPTER 10

SINE AND COSINE FORMULAE AND THEIR APPLICATIONS

10.1 INTRODUCTION

In any triangle the three sides and the three angles are generally called the elements of the triangle. A triangle which does not contain a right angle is called an *oblique triangle.*

In any triangle ABC, the measures of the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$ are denoted by the letters A, B and C respectively. The sides BC, CA and AB opposite to the angles A, B and C respectively are denoted by a, b and c. These six elements of a triangle are not independent and are connected by the relations: (i) $A+B+C=\pi$ (ii) $a+b>c;\ b+c>a;\ c+a>b$. In addition to these relations, the elements of a triangle are connected by some trigonometric relations. We intend to discuss those relations in the sections to follow of this chapter.

10.2 THE LAW OF SINES OR SINE RULE

THEOREM *The sides of a triangle are proportional to the sines of the angles opposite to them i.e. in a* ΔABC, $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

PROOF The following cases arise:

CASE I *When ΔABC is an acute angled triangle:*

Draw AD perpendicular from A to the opposite side BC meeting it in the point D.

In the triangle ABD, we have

$$\sin B = \frac{AD}{AB} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c\sin B \qquad \text{...(i)}$$

In the triangle ACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b\sin C \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$c\sin B = b\sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

In a similar manner, by drawing a perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

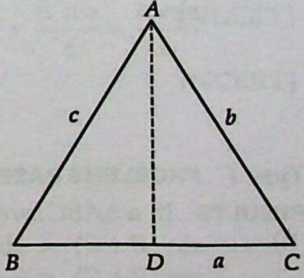

Fig. 10.1

CASE II *When ΔABC is an obtuse angled triangle.*

Draw AD perpendicular from A on CB produced meeting it in D.

In ΔADB, we have

$$\sin\angle ABD = \frac{AD}{AB} \Rightarrow \sin(180-B) = \frac{AD}{c} \Rightarrow \sin B = \frac{AD}{c} \Rightarrow AD = c\sin B \qquad \text{...(i)}$$

In ΔACD, we have

$$\sin C = \frac{AD}{AC} \Rightarrow \sin C = \frac{AD}{b} \Rightarrow AD = b \sin C \quad \text{...(ii)}$$

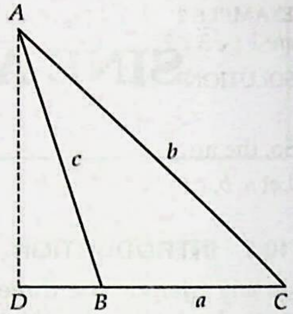

Fig. 10.2

From (i) and (ii), we obtain

$$c \sin B = b \sin C \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Similarly, by drawing perpendicular from B on AC, we obtain

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Hence, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

<u>CASE III</u> *When ΔABC is a right angled triangle:*
In ΔABC, we have

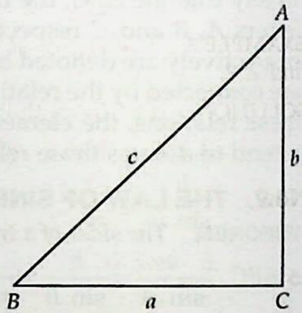

Fig. 10.3

$$\sin C = \sin \frac{\pi}{2} = 1, \ \sin A = \frac{BC}{AB} = \frac{a}{c} \text{ and, } \sin B = \frac{AC}{AB} = \frac{b}{c}$$

$$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = c$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{1}$$

$$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \left[\because \sin C = \sin \frac{\pi}{2} = 1\right]$$

Hence, in all the cases, we obtain

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q.E.D.

<u>REMARK 1</u> *The above rule may also be expressed as* $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

<u>REMARK 2</u> *The sine rule is a very useful tool to express sides of a triangle in terms of the sines of angles and vice-versa in the following manner.*

Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ (say) Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$.

Similarly,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \lambda \text{ (say)} \Rightarrow \sin A = \lambda a, \ \sin B = \lambda b \text{ and } \sin C = \lambda c.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED ON SINE RULE**

RESULTS In a ΔABC, we have

$$\sin(B+C) = \sin A, \ \sin(C+A) = \sin B, \ \sin(A+B) = \sin C$$
$$\cos(B+C) = -\cos A, \ \cos(C+A) = -\cos B, \ \cos(A+B) = -\cos C$$
$$\tan(B+C) = -\tan A, \ \tan(C+A) = -\tan B, \ \tan(A+B) = -\tan C$$

EXAMPLE 1 *In a ΔABC, if $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$, find $\angle B$.*

SOLUTION We have,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \quad \frac{2}{(2/3)}=\frac{3}{\sin B} \Rightarrow 3=\frac{3}{\sin B} \Rightarrow \sin B=1 \Rightarrow \angle B=90^\circ$$

EXAMPLE 2 *If in any triangle the angles be to one another as $1:2:3$, prove that the corresponding sides are $1:\sqrt{3}:2$.*

SOLUTION Let the measures of the angles be x, $2x$ and $3x$. Then,

$$x+2x+3x = 180^\circ \Rightarrow 6x = 180^\circ \Rightarrow x = 30^\circ$$

So, the angles are 30°, 60° and 90°

Let a, b, c be the lengths of the sides apposite to these angles. Then,

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 60^\circ} = \frac{c}{\sin 90^\circ}$$

$$\Rightarrow \quad a:b:c = \sin 30^\circ : \sin 60^\circ : \sin 90^\circ$$

$$\Rightarrow \quad a:b:c: = \frac{1}{2}:\frac{\sqrt{3}}{2}:1 \Rightarrow a:b:c=1:\sqrt{3}:2$$

EXAMPLE 3 *The angles of a triangle ABC are in A. P. and it is being given that $b:c = \sqrt{3}:\sqrt{2}$, find $\angle A$.*

SOLUTION It is given that the angles $\angle A$, $\angle B$, $\angle C$ are in A.P.

$$\therefore \quad 2\angle B = \angle A + \angle C \Rightarrow 3\angle B = \angle A+\angle B+\angle C \Rightarrow 3\angle B=180^\circ \Rightarrow \angle B=60^\circ$$

Now, $\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$\Rightarrow \quad \frac{b}{c}=\frac{\sin B}{\sin C}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sin 60^\circ}{\sin C} \qquad [\because\ b:c=\sqrt{3}:\sqrt{2}]$$

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{2}}=\frac{\sqrt{3}/2}{\sin C}$$

$$\Rightarrow \quad \sin C=\frac{1}{\sqrt{2}} \Rightarrow \angle C = 45^\circ$$

$$\therefore \quad \angle A = 180^\circ-(\angle B+\angle C) = 180^\circ-(60^\circ+45^\circ) = 75^\circ$$

EXAMPLE 4 *In any triangle ABC, prove that:*

(i) $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{b^2-c^2}{a^2}$ **[NCERT]**

(ii) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B) = 0$ **[NCERT]**

(iii) $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B) = 0$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then,

$$a = k\sin A,\ b = k\sin B \text{ and } c = k\sin C. \qquad \text{...(i)}$$

(i) $$\text{RHS} = \frac{b^2-c^2}{a^2} = \frac{k^2\sin^2 B-k^2\sin^2 C}{k^2\sin^2 A} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin^2 B-\sin^2 C}{\sin^2 A} = \frac{\sin(B+C)\sin(B-C)}{\sin^2 A}$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin(\pi-A)\sin(B-C)}{\sin^2 A} \qquad [\because A+B+C=\pi \Rightarrow B+C=\pi-A]$$

$$\Rightarrow \quad \text{RHS} = \frac{\sin A \sin (B-C)}{\sin^2 A} = \frac{\sin (B-C)}{\sin A} = \frac{\sin (B-C)}{\sin (B+C)} = \text{LHS}$$

(ii) $\text{LHS} = a \sin (B-C) + b \sin (C-A) + c \sin (A-B)$

$= k \sin A \sin (B-C) + k \sin B \sin (C-A) + k \sin C \sin (A-B)$ [Using (i)]

$$= k\left\{\sin (B+C) \sin (B-C) + \sin (C+A) \sin (C-A) + \sin (A+B) \sin (A-B)\right\}$$

$$= k\left\{\sin^2 B - \sin^2 C + \sin^2 C - \sin^2 A + \sin^2 A - \sin^2 B\right\} = k\,(0) = \text{RHS}$$

(iii) $\text{LHS} = k^3 \sin^3 A \sin (B-C) + k^3 \sin^3 B \sin (C-A) + k^3 \sin^3 C \sin (A-B)$

$$= k^3\left\{\sin^2 A \sin A \sin (B-C) + \sin^2 B \sin B \sin (C-A) + \sin^2 C \sin C \sin (A-B)\right\}$$

$$= k^3\left\{\sin^2 A \sin (B+C) \sin (B-C) + \sin^2 B \sin (C+A) \sin (C-A) + \sin^2 C \sin (A+B) \sin (A-B)\right\}$$

$$= k^3\left\{\sin^2 A (\sin^2 B - \sin^2 C) + \sin^2 B (\sin^2 C - \sin^2 A) + \sin^2 C (\sin^2 A - \sin^2 B)\right\}$$

$$= k^3\left\{\sin^2 A \sin^2 B - \sin^2 A \sin^2 C + \sin^2 B \sin^2 C - \sin^2 B \sin^2 A + \sin^2 C \sin^2 A - \sin^2 C \sin^2 B\right\}$$

$$= k^3 \times 0 = 0 = \text{RHS}$$

EXAMPLE 5 *In any triangle ABC, prove that:*

$$\frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B} = 0$$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$

Now,

$$\text{LHS} = \frac{a^2 \sin (B-C)}{\sin B + \sin C} + \frac{b^2 \sin (C-A)}{\sin C + \sin A} + \frac{c^2 \sin (A-B)}{\sin A + \sin B} =$$

$$= \frac{k^2 \sin^2 A \sin (B-C)}{\sin B + \sin C} + \frac{k^2 \sin^2 B \sin (C-A)}{\sin C + \sin A} + \frac{k^2 \sin^2 C \sin (A-B)}{\sin A + \sin B}$$

$$= k^2\left\{\frac{\sin A \sin (B+C) \sin (B-C)}{\sin B + \sin C} + \frac{\sin B \sin (C+A) \sin (C-A)}{\sin C + \sin A} + \frac{\sin C \sin (A+B) \sin (A-B)}{\sin A + \sin B}\right\}$$

$$= k^2\left\{\frac{\sin A (\sin^2 B - \sin^2 C)}{\sin B + \sin C} + \frac{\sin B (\sin^2 C - \sin^2 A)}{\sin C + \sin A} + \frac{\sin C (\sin^2 A - \sin^2 B)}{\sin A + \sin B}\right\}$$

$$= k^2 \{\sin A (\sin B - \sin C) + \sin B (\sin C - \sin A) + \sin C (\sin A - \sin B)\}$$

$$= k^2 \times 0 = 0 = \text{RHS}$$

EXAMPLE 6 *In any triangle ABC, prove that:*

(i) $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$

(ii) $\dfrac{b^2-c^2}{\cos B+\cos C}+\dfrac{c^2-a^2}{\cos C+\cos A}+\dfrac{a^2-b^2}{\cos A+\cos B}=0$

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a=k\sin A, b=k\sin B, c=k\sin C$

(i) $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{k\sin A\sin(B-C)}{k^2\sin^2 B-k^2\sin^2 C}=\frac{k\sin(B+C)\sin(B-C)}{k^2(\sin^2 B-\sin^2 C)}$$

$$=\frac{k(\sin^2 B-\sin^2 C)}{k^2(\sin^2 B-\sin^2 C)}=\frac{1}{k}$$

and, $$\frac{b\sin(C-A)}{c^2-a^2}=\frac{k\sin B\sin(C-A)}{k^2\sin^2 C-k^2\sin^2 A}=\frac{k\sin(C+A)\sin(C-A)}{k^2(\sin^2 C-\sin^2 A)}$$

$$=\frac{k(\sin^2 C-\sin^2 A)}{k^2(\sin^2 C-\sin^2 A)}=\frac{1}{k}$$

Similarly, it can be shown that $\dfrac{c\sin(A-B)}{a^2-b^2}=\dfrac{1}{k}$

Hence, $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$

(ii) $$\text{LHS}=\frac{k^2(\sin^2 B-\sin^2 C)}{\cos B+\cos C}+\frac{k^2(\sin^2 C-\sin^2 A)}{\cos C+\cos A}+\frac{k^2(\sin^2 A-\sin^2 B)}{\cos A+\cos B}$$ [Using (i)]

$$=\frac{k^2\{(1-\cos^2 B)-(1-\cos^2 C)\}}{\cos B+\cos C}+\frac{k^2\{(1-\cos^2 C)-(1-\cos^2 A)\}}{\cos C+\cos A}+\frac{k^2\{(1-\cos^2 A)-(1-\cos^2 B)\}}{\cos A+\cos B}$$

$$=k^2\left\{\frac{(\cos^2 C-\cos^2 B)}{\cos B+\cos C}+\frac{(\cos^2 A-\cos^2 C)}{\cos C+\cos A}+\frac{(\cos^2 B-\cos^2 A)}{\cos A+\cos B}\right\}$$

$$=k^2\{(\cos C-\cos B)+(\cos A-\cos C)+(\cos B-\cos A)\}=k\times 0=0=\text{RHS}$$

EXAMPLE 7 *In any triangle ABC, prove that:*

(i) $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}$

(ii) $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$ **[NCERT]**

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a=k\sin A, b=k\sin B, c=k\sin C$

(i) $$\text{LHS}=\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}=\frac{1-(\cos^2 A-\sin^2 B)}{1-(\cos^2 A-\sin^2 C)}$$

$$=\frac{1-\cos^2 A+\sin^2 B}{1-\cos^2 A+\sin^2 C}=\frac{\sin^2 A+\sin^2 B}{\sin^2 A+\sin^2 C}=\frac{a^2/k^2+b^2/k^2}{a^2/k^2+c^2/k^2}=\frac{a^2+b^2}{a^2+c^2}=\text{RHS}$$

(ii) $$\begin{aligned}\text{LHS} &= a\cos A + b\cos B + c\cos C\\ &= k\sin A\cos A + k\sin B\cos B + k\sin C\cos C\\ &= \frac{k}{2}\{\sin 2A + \sin 2B + \sin 2C\} = \frac{k}{2}(4\sin A\sin B\sin C)\\ &= 2k\sin A\sin B\sin C = 2a\sin B\sin C = \text{RHS} \qquad [\because k\sin A = a]\end{aligned}$$

EXAMPLE 8 *In any triangle ABC, prove that:*

(i) $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right)\cos\frac{A}{2}$ [NCERT] (ii) $a\cos\left(\frac{B-C}{2}\right) = (b+c)\sin\frac{A}{2}$ [NCERT]

(iii) $$\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$$

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) $$\text{RHS} = \left(\frac{b-c}{a}\right)\cos\frac{A}{2} = \left\{\frac{k\sin B - k\sin C}{k\sin A}\right\}\cos\frac{A}{2} = \left\{\frac{\sin B - \sin C}{\sin A}\right\}\cos\frac{A}{2}$$

$$= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}\cos\frac{A}{2} = \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi-A}{2}\right)}{\sin\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)}{\sin\frac{A}{2}} = \frac{\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{\sin\frac{A}{2}} = \sin\left(\frac{B-C}{2}\right) = \text{LHS}$$

(ii) $$\text{RHS} = (b+c)\sin\frac{A}{2} = (k\sin B + k\sin C)\sin\frac{A}{2} = k(\sin B + \sin C)\sin\frac{A}{2}$$

$$= k\times 2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = 2k\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2}$$

$$= 2k\cos\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\sin\frac{A}{2} = k\left(2\sin\frac{A}{2}\cos\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$= k\sin A\cos\left(\frac{B-C}{2}\right) = a\cos\left(\frac{B-C}{2}\right) = \text{LHS} \qquad [\because k\sin A = a]$$

(iii) $$\text{LHS} = \frac{b-c}{b+c} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)} = \text{RHS}$$

EXAMPLE 9 *In a triangle ABC, if $a\cos A = b\cos B$, show that the triangle is either isosceles or right angled.*

SOLUTION Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k\sin A$, $b = k\sin B$, $c = k\sin C$

Now, $a\cos A = b\cos B$

$$\Rightarrow \quad k \sin A \cos A = k \sin B \cos B$$
$$\Rightarrow \quad 2 \sin A \cos A = 2 \sin B \cos B$$
$$\Rightarrow \quad \sin 2A = \sin 2B$$
$$\Rightarrow \quad 2A = 2B \text{ or, } 2A = \pi - 2B$$
$$\Rightarrow \quad A = B \text{ or, } A + B = \pi/2$$
$$\Rightarrow \quad A = B \text{ or, } C = \frac{\pi}{2}$$
$$\Rightarrow \quad BC = CA \text{ or, } C = \frac{\pi}{2}$$

$\Rightarrow$ ΔABC is either isosceles or right angled.

EXAMPLE 10 *If in a ΔABC, $\dfrac{\sin A}{\sin C} = \dfrac{\sin (A-B)}{\sin (B-C)}$, prove that a^2, b^2, c^2 are in A.P.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$, $\sin C = ck$

Now, $\dfrac{\sin A}{\sin C} = \dfrac{\sin (A-B)}{\sin (B-C)}$

$$\Rightarrow \quad \frac{\sin (B+C)}{\sin (A+B)} = \frac{\sin (A-B)}{\sin (B-C)} \qquad [\because \sin A = \sin (B+C) \text{ and } \sin C = \sin (A+B)]$$
$$\Rightarrow \quad \sin (B+C) \sin (B-C) = \sin (A+B) \sin (A-B)$$
$$\Rightarrow \quad \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$
$$\Rightarrow \quad k^2 b^2 - k^2 c^2 = k^2 a^2 - k^2 b^2$$
$$\Rightarrow \quad b^2 - c^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2 + c^2 \Rightarrow a^2, b^2, c^2 \text{ are in A.P.}$$

Type II APPLICATIONS OF SINE FORMULA IN PROBLEMS ON HEIGHTS AND DISTANCES

EXAMPLE 11 *A tree stands vertically on a hill side which makes an angle of 15° with the horizontal. From a point on the ground 35 m down the hill from the base of the tree, the angle of elevation of the top of the tree is 60°. Find the height of the tree.* **[NCERT]**

SOLUTION Let PQ be the tree on the hill which makes an angel of 15° with the horizontal AR, where A is a point on the ground 35 m down the hill from the base P of the tree.

In $\Delta\, ARQ$, we have

$$\angle RAQ = 60° \text{ and } \angle ARQ = 90°$$
$$\therefore \quad \angle AQP = 30°$$

In $\Delta\, APQ$, we have

$$\angle PAQ = 45° \text{ and } \angle AQP = 30°$$

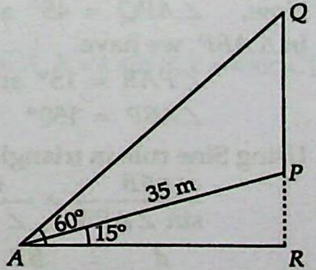

Fig. 10.4

Using Sine rule in ΔAPQ, we get

$$\frac{AP}{\sin \angle AQP} = \frac{PQ}{\sin \angle PAQ}$$

$$\Rightarrow \quad \frac{35}{\sin 30°} = \frac{PQ}{\sin 45°} \Rightarrow \frac{35}{1/2} = \frac{PQ}{1/\sqrt{2}} \Rightarrow PQ = \frac{70}{\sqrt{2}} \Rightarrow PQ = 35\sqrt{2} \text{ m}.$$

Hence, height of the tree is $35\sqrt{2}$ m.

EXAMPLE 12 *A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°, when he retreats 20 m from the bank, he finds the angle to be 30°. Find the height of the tree and the breadth of the river.*

SOLUTION Let AB be the tree at one bank of the river and let the P be the position of the person at the other bank of the river. After retreating 20 m from the bank, let the man be at Q. It is given that $\angle APB = 60°$ and $\angle AQB = 30°$.

Now, $\angle APB = 60° \Rightarrow \angle BPQ = 120°$ and $\angle PBA = 30°$

In ΔBPQ, we have

$$\angle PQB = 30°, \ \angle BPQ = 120° \text{ and, } \angle PBQ = 30°$$

Using Sine rule in triangles BPQ and PAB, we get

$$\frac{BP}{\sin \angle PQB} = \frac{PQ}{\sin \angle PBQ} \text{ and } \frac{BP}{\sin \angle PAB} = \frac{AP}{\sin \angle ABP} = \frac{AB}{\sin \angle APB}$$

$$\Rightarrow \quad \frac{BP}{\sin 30°} = \frac{PQ}{\sin 30°} \text{ and } \frac{BP}{\sin 90°} = \frac{AP}{\sin 30°} = \frac{AB}{\sin 60°}$$

$$\Rightarrow \quad \frac{BP}{1/2} = \frac{20}{1/2} \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad BP = 20 \text{ and } BP = 2AP = \frac{2AB}{\sqrt{3}} \qquad [\because PQ = 20 \text{ m}]$$

$$\Rightarrow \quad 20 = 2AP = \frac{2AB}{\sqrt{3}}$$

$$\Rightarrow \quad AP = 10 \text{ and } AB = 10\sqrt{3}$$

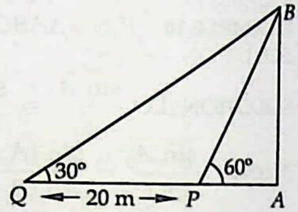

Fig. 10.5

Hence, the breadth of the river is 10 m and height of the tree is $10\sqrt{3}$ m.

EXAMPLE 13 *The angle of elevation of the top point P of the vertical tower PQ of heigth h from a point A is 45° and from a point B, the angle of elevation is 60°, where B is a point at a distance d from the point A measured along the line AB which makes an angle 30° with AQ. Prove that $d = (\sqrt{3} - 1)\, h$.* **[NCERT]**

SOLUTION It is given that $\angle PAQ = 45°$ and $\angle BAQ = 30°$. Therefore, $\angle BAP = 15°$.

In ΔAQP, we have

$$\angle PAQ = 45° \text{ and } \angle PQA = 90°$$

$$\therefore \quad \angle APQ = 45°$$

In ΔBRP, we have

$$\angle PBR = 60° \text{ and } \angle PRB = 90°$$

$$\therefore \quad \angle BPR = 30°$$

Now, $\angle APQ = 45°$ and $\angle BPR = 30° \Rightarrow \angle BPA = 15°$

In ΔABP, we have

$$\angle PAB = 15° \text{ and } \angle BPA = 15°.$$

$$\therefore \quad \angle ABP = 150°$$

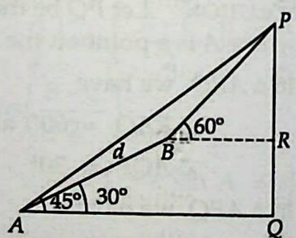

Fig. 10.6

Using Sine rule in triangle ABP, we get

$$\frac{AB}{\sin \angle APB} = \frac{BP}{\angle PAB} = \frac{AP}{\sin \angle ABP}$$

$$\Rightarrow \quad \frac{d}{\sin 15°} = \frac{BP}{\sin 15°} = \frac{AP}{\sin 150°}$$

$$\Rightarrow \quad \frac{d}{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{AP}{\frac{1}{2}} \Rightarrow AP = \frac{\sqrt{2}\, d}{\sqrt{3}-1} \qquad \text{...(i)}$$

Using Sine rule in ΔAQP, we get

$$\frac{AP}{\sin \angle AQP} = \frac{AQ}{\sin \angle APQ} = \frac{PQ}{\sin \angle PAQ}$$

$$\Rightarrow \quad \frac{AP}{\sin 90°} = \frac{PQ}{\sin 45°} \Rightarrow AP = \sqrt{2}\ PQ \Rightarrow AP = \sqrt{2}\ h \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\sqrt{2}h = \frac{\sqrt{2}\ d}{\sqrt{3}-1} \Rightarrow d = (\sqrt{3}-1)\ h$$

LEVEL-2

EXAMPLE 14 *If in a* $\Delta\ ABC$, $\dfrac{a^2-b^2}{a^2+b^2} = \dfrac{\sin(A-B)}{\sin(A+B)}$, *prove that it is either a right angled or an isosceles triangle.*

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$, (say). Then, $a = k\sin A, b = k\sin B, c = k\sin C$

$$\therefore \quad \frac{a^2-b^2}{a^2+b^2} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{k^2\sin^2 A - k^2\sin^2 B}{k^2\sin^2 A + k^2\sin^2 B} = \frac{\sin(A-B)}{\sin(A+B)}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin(\pi-C)} = \frac{\sin(A+B)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin(\pi-C)\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \frac{\sin(A-B)}{\sin C} = \frac{\sin C\sin(A-B)}{\sin^2 A + \sin^2 B}$$

$$\Rightarrow \quad \sin(A-B)\left\{\frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B}\right\} = 0$$

$$\Rightarrow \quad \text{either } \sin(A-B) = 0 \quad \text{or,} \quad \frac{1}{\sin C} - \frac{\sin C}{\sin^2 A + \sin^2 B} = 0$$

$$\Rightarrow \quad \text{either } A - B = 0 \quad \text{or,} \quad \sin^2 A + \sin^2 B - \sin^2 C = 0$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad \frac{a^2}{k^2} + \frac{b^2}{k^2} - \frac{c^2}{k^2} = 0 \qquad [\because\ a = k\sin A, b = k\sin B, c = k\sin C]$$

$$\Rightarrow \quad \text{either } A = B \quad \text{or,} \quad a^2 + b^2 = c^2$$

$\Rightarrow$ either the triangle is isosceles or it is right angled.

EXAMPLE 15 *In any triangle ABC, prove that:*

$$(b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2} = 0$$

SOLUTION We have,

$$\text{LHS} = (b-c)\cot\frac{A}{2} + (c-a)\cot\frac{B}{2} + (a-b)\cot\frac{C}{2}$$

$$= k(\sin B - \sin C)\cot\frac{A}{2} + k(\sin C - \sin A)\cot\frac{B}{2} + k(\sin A - \sin B)\cot\frac{C}{2}$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\cos\left(\frac{C+A}{2}\right)\cot\frac{B}{2}\right.$$
$$\left. + 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\cot\frac{C}{2}\right]$$

$$= k\left[2\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}\cot\frac{A}{2} + 2\sin\left(\frac{C-A}{2}\right)\sin\frac{B}{2}\cot\frac{B}{2} + 2\sin\left(\frac{A-B}{2}\right)\sin\frac{C}{2}\cot\frac{C}{2}\right]$$

$$= k\left[2\cos\frac{A}{2}\sin\left(\frac{B-C}{2}\right) + 2\cos\frac{B}{2}\sin\left(\frac{C-A}{2}\right) + 2\cos\frac{C}{2}\sin\left(\frac{A-B}{2}\right)\right]$$

$$= 2k\left[\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) + \sin\left(\frac{C+A}{2}\right)\sin\left(\frac{C-A}{2}\right) + \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\right]$$

$$\left[\because \cos\frac{A}{2} = \sin\left(\frac{B+C}{2}\right) \text{ etc.}\right]$$

$$= 2k\left\{\sin^2\frac{B}{2} - \sin^2\frac{C}{2} + \sin^2\frac{C}{2} - \sin^2\frac{A}{2} + \sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right\} = 2k \times 0 = 0 = \text{RHS}$$

EXAMPLE 16 *Let O be a point inside a triangle ABC such that $\angle OAB = \angle OBC = \angle OCA = \omega$, then show that:*

(i) $\cot\omega = \cot A + \cot B + \cot C$ (ii) $\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

SOLUTION (i) In ΔOBC,

$$\angle OCB = \angle C - \omega \text{ and, } \angle BOC = 180 - \omega - (C - \omega) = 180° - C$$

Similarly, we obtain $\angle AOB = 180° - B$

Applying sine rule in ΔOAB, we obtain

$$\frac{OB}{\sin\angle OAB} = \frac{AB}{\sin\angle AOB}$$

$$\frac{OB}{\sin\omega} = \frac{AB}{\sin(180° - B)}$$

$$\Rightarrow \quad \frac{OB}{\sin\omega} = \frac{c}{\sin B}$$

$$\Rightarrow \quad OB = \frac{c\sin\omega}{\sin B} \quad \text{...(i)}$$

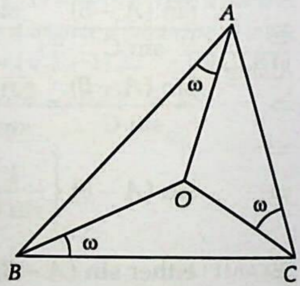

Fig. 10.7

Applying sine rule in ΔOBC, we get

$$\frac{OB}{\sin\angle BCO} = \frac{BC}{\sin\angle BOC}$$

$$\Rightarrow \quad \frac{OB}{\sin(C-\omega)} = \frac{BC}{\sin(180° - C)} \Rightarrow \frac{OB}{\sin(C-\omega)} = \frac{a}{\sin C} \Rightarrow OB = \frac{a\sin(C-\omega)}{\sin C} \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{c\sin\omega}{\sin B} = \frac{a\sin(C-\omega)}{\sin C}$$

$$\Rightarrow \quad \frac{k\sin C\sin\omega}{\sin B} = \frac{k\sin A\sin(C-\omega)}{\sin C} \quad \text{[Using sine rule]}$$

$$\Rightarrow \quad \sin^2 C\sin\omega = \sin A\sin B\sin(C-\omega)$$

$$\Rightarrow \quad \sin C\sin(A+B)\sin\omega = \sin A\sin B\sin(C-\omega) \quad [\because \sin C = \sin(\pi - (A+B)) = \sin(A+B)]$$

$$\Rightarrow \quad \frac{\sin(A+B)}{\sin A\sin B} = \frac{\sin(C-\omega)}{\sin C\sin\omega}$$

$$\Rightarrow \quad \frac{\sin A\cos B + \cos A\sin B}{\sin A\sin B} = \frac{\sin C\cos\omega - \cos C\sin\omega}{\sin C\sin\omega}$$

$$\Rightarrow \quad \cot B + \cot A = \cot\omega - \cot C \Rightarrow \cot\omega = \cot A + \cot B + \cot C$$

(ii) From (i), we have

$$\cot \omega = \cot A + \cot B + \cot C$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2(\cot A \cot B + \cot B \cot C + \cot C \cot A)$$

$$\Rightarrow \quad \cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2 \quad [\because \cot A \cot B + \cot B \cot C + \cot C \cot A = 1]$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega - 1 = (\operatorname{cosec}^2 A - 1) + (\operatorname{cosec}^2 B - 1) + (\operatorname{cosec}^2 C - 1) + 2$$

$$\Rightarrow \quad \operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

EXAMPLE 17 *The angle of elevation of the top of a tower from a point A due South of the tower is α and from B due East of the tower is β. If $AB = d$, show that the height of the tower is $\dfrac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$.*

SOLUTION Let OP be the tower and let A and B be two points due South and East respectively of the tower such that $\angle OAP = \alpha$ and $\angle OBP = \beta$. Then,

$$\angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta.$$

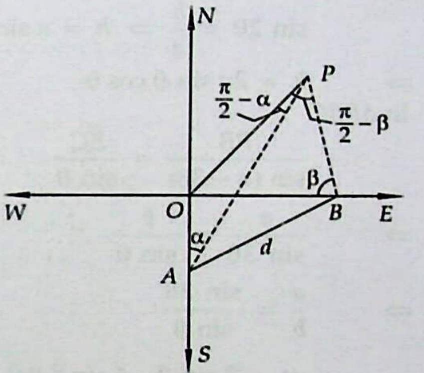

Fig. 10.8

Using sine rule in Δ's OAP and OBP, we have

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and, } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad \frac{OA}{\cos \alpha} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\cos \beta} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Using Pythagoras theorem in $\Delta\, AOB$, we get

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad d^2 = OP^2 \cot^2 \alpha + OP^2 \cot^2 \beta$$

$$\Rightarrow \quad OP = \frac{d}{\sqrt{\cot^2 \alpha + \cot^2 \beta}}$$

EXAMPLE 18 *The elevation of a tower at a station A due North of it is α and at a station B due West of A is β. Prove that the height of the tower is $\dfrac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$.*

SOLUTION Let OP be the tower and let A be a point due north of the tower OP and let B be the point due west of A such that $\angle OAP = \alpha$ and $\angle OBP = \beta$.

Clearly, triangles AOP and BOP are right triangles right angled at O.

$$\therefore \quad \angle OPA = \frac{\pi}{2} - \alpha \text{ and } \angle OPB = \frac{\pi}{2} - \beta$$

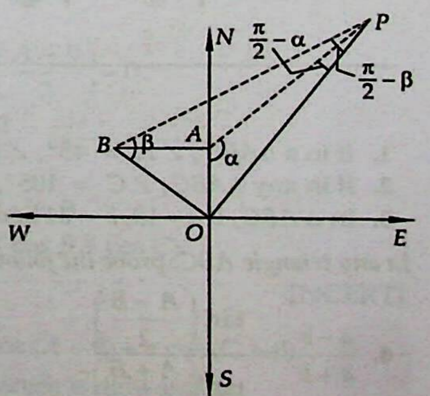

Fig. 10.9

Using sine rule in triangles AOP and BOP, we get

$$\frac{OA}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{OP}{\sin \alpha} \text{ and } \frac{OB}{\sin\left(\frac{\pi}{2} - \beta\right)} = \frac{OP}{\sin \beta}$$

$$\Rightarrow \quad OA = OP \cot \alpha \text{ and } OB = OP \cot \beta$$

Applying Pythagoras theorem in ΔOAB, we get

$$OB^2 = OA^2 + AB^2$$

$$\Rightarrow \quad OB^2 - OA^2 = AB^2$$

$$\Rightarrow \quad OP^2 \cot^2 \beta - OP^2 \cot \alpha^2 = AB^2$$

$$\Rightarrow \quad OP^2 (\cot^2 \beta - \cot^2 \alpha) = AB^2$$

$$\Rightarrow \quad OP^2 (\operatorname{cosec}^2 \beta - \operatorname{cosec}^2 \alpha) = AB^2$$

$$\Rightarrow \quad OP^2 \frac{(\sin^2 \alpha - \sin^2 \beta)}{\sin^2 \alpha \sin^2 \beta} = AB^2 \Rightarrow OP = \frac{AB \sin \alpha \sin \beta}{\sqrt{\sin^2 \alpha - \sin^2 \beta}}$$

EXAMPLE 19 *An object is observed from three points A, B, C in the same horizontal line passing through the base of the object. The angle of elevation at B is twice and at C thrice that at A. If AB = a, BC = b prove that the height of the object is* $\frac{a}{2b}\sqrt{(a+b)(3b-a)}$.

SOLUTION Let the object be at P at a height h from OA. Let the object when observed from A, B and C the angles of elevation are θ, 2θ and 3θ respectively.

In ΔPAB, we have

$$2\theta = \theta + \angle APB \Rightarrow \angle APB = \theta$$

$$\therefore \quad \angle PAB = \angle APB = \theta \Rightarrow AB = BP = a$$

Similarly, in triangle BPC, $\angle BPC = \theta$.

In ΔOPB

$$\sin 2\theta = \frac{h}{a} \Rightarrow h = a \sin 2\theta$$

$$\Rightarrow \quad h = 2a \sin \theta \cos \theta \qquad \text{...(i)}$$

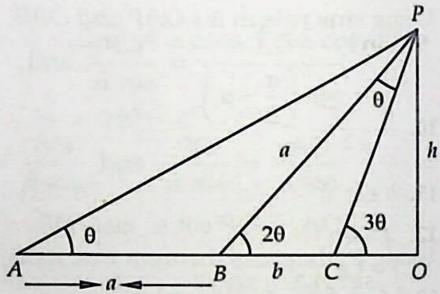

Fig. 10.10

In ΔPBC

$$\frac{PB}{\sin(\pi - 3\theta)} = \frac{BC}{\sin \theta} \qquad \text{[Using sine rule]}$$

$$\Rightarrow \quad \frac{a}{\sin 3\theta} = \frac{b}{\sin \theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{\sin 3\theta}{\sin \theta}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{3 \sin \theta - 4 \sin^3 \theta}{\sin \theta}$$

$$\Rightarrow \quad \frac{a}{b} = 3 - 4\sin^2\theta \Rightarrow 4\sin^2\theta = 3 - \frac{a}{b} \Rightarrow \sin^2\theta = \frac{3b-a}{4b} \Rightarrow \sin\theta = \sqrt{\frac{3b-a}{4b}}$$

$$\therefore \quad \cos^2\theta = 1 - \sin^2\theta \Rightarrow \cos^2\theta = 1 - \frac{3b-a}{4b} = \frac{a+b}{4b} \Rightarrow \cos\theta = \sqrt{\frac{a+b}{4b}}$$

Substituting the values of $\sin\theta$ and $\cos\theta$ in (i), we get

$$h = 2a\sqrt{\frac{3b-a}{4b}} \times \sqrt{\frac{a+b}{4b}} = \frac{a}{2b}\sqrt{(a+b)(3b-a)}$$

EXERCISE 10.1

LEVEL-1

1. If in a ΔABC, $\angle A = 45°$, $\angle B = 60°$, and $\angle C = 75°$; find the ratio of its sides.
2. If in any ΔABC, $\angle C = 105°$, $\angle B = 45°$, $a = 2$, then find b.
3. In ΔABC, if $a = 18$, $b = 24$ and $C = 30$ and $\angle C = 90°$, find $\sin A$, $\sin B$ and $\sin C$.

In any triangle ABC, prove the following: (4-24)

4. $\dfrac{a-b}{a+b} = \dfrac{\tan\left(\dfrac{A-B}{2}\right)}{\tan\left(\dfrac{A+B}{2}\right)}$

5. $(a-b)\cos\frac{C}{2} = c\sin\left(\frac{A-B}{2}\right)$ **[NCERT]**

6. $\frac{c}{a-b} = \frac{\tan\left(\frac{A}{2}\right)+\tan\left(\frac{B}{2}\right)}{\tan\left(\frac{A}{2}\right)-\tan\left(\frac{B}{2}\right)}$

7. $\frac{c}{a+b} = \frac{1-\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}{1+\tan\left(\frac{A}{2}\right)\tan\left(\frac{B}{2}\right)}$

8. $\frac{a+b}{c} = \frac{\cos\left(\frac{A-B}{2}\right)}{\sin\frac{C}{2}}$ **[NCERT]**

9. $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{a}\cos\frac{A}{2}$

10. $\frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

11. $b\sin B - c\sin C = a\sin(B-C)$

12. $a^2\sin(B-C) = (b^2-c^2)\sin A$

13. $\frac{\sqrt{\sin A}-\sqrt{\sin B}}{\sqrt{\sin A}+\sqrt{\sin B}} = \frac{a+b-2\sqrt{ab}}{a-b}$

14. $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B) = 0$

15. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C} = 0$

16. $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)+c^2(\cos^2 A-\cos^2 B) = 0$

17. $b\cos B+c\cos C = a\cos(B-C)$

18. $\frac{\cos 2A}{a^2}-\frac{\cos 2B}{b^2} = \frac{1}{a^2}-\frac{1}{b^2}$

19. $\frac{\cos^2 B-\cos^2 C}{b+c}+\frac{\cos^2 C-\cos^2 A}{c+a}+\frac{\cos^2 A-\cos^2 B}{a+b} = 0$

20. $a\sin\frac{A}{2}\sin\left(\frac{B-C}{2}\right)+b\sin\frac{B}{2}\sin\left(\frac{C-A}{2}\right)+c\sin\frac{C}{2}\sin\left(\frac{A-B}{2}\right)=0.$

21. $\frac{b\sec B+c\sec C}{\tan B+\tan C} = \frac{c\sec C+a\sec A}{\tan C+\tan A} = \frac{a\sec A+b\sec B}{\tan A+\tan B}.$

22. $a\cos A+b\cos B+c\cos C = 2b\sin A\sin C = 2c\sin A\sin B$

23. $a(\cos B\cos C+\cos A)=b(\cos C\cos A+\cos B)=c(\cos A\cos B+\cos C).$

24. $a(\cos C-\cos B)=2(b-c)\cos^2\frac{A}{2}.$ **[NCERT]**

25. In ΔABC prove that, if θ be any angle, then $b\cos\theta = c\cos(A-\theta)+a\cos(C+\theta)$.

26. In a ΔABC, if $\sin^2 A+\sin^2 B = \sin^2 C$, show that the triangle is right angled.

LEVEL-2

27. In any ΔABC, if a^2, b^2, c^2 are in A.P., prove that $\cot A$, $\cot B$ and $\cot C$ are also in A.P.

28. The upper part of a tree broken over by the wind makes an angle of 30° with the ground and the distance from the root to the point where the top of the tree touches the ground is 15 m. Using sine rule, find the height of the tree.

29. At the foot of a mountain the elevation of its summit is 45°; after ascending 1000 m towards the mountain up a slope of 30° inclination, the elevation is found to be 60°. Find the height of the mountain.

30. A person observes the angle of elevation of the peak of a hill from a station to be α. He walks c metres along a slope inclined at the angle β and finds the angle of elevation of the peak of the hill to be γ. Show that the height of the peak above the ground is $\dfrac{c \sin \alpha \sin (\gamma - \beta)}{(\sin \gamma - \alpha)}$.

31. If the sides a, b, c of a ΔABC are in H.P., prove that $\sin^2 \dfrac{A}{2}, \sin^2 \dfrac{B}{2}, \sin^2 \dfrac{C}{2}$ are in H.P.

ANSWERS

1. $2 : \sqrt{6} : \sqrt{3} + 1$ 2. $2\sqrt{2}$ 3. $\sin A = \dfrac{3}{5}$, $\sin B = \dfrac{4}{5}$

28. $15\sqrt{3}$ m 29. $500(\sqrt{3} + 1)$ metres

10.3 THE LAW OF COSINES

THEOREM *In any ΔABC, we have:*

(i) $a^2 = b^2 + c^2 - 2bc \cos A$ or, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(ii) $b^2 = c^2 + a^2 - 2ac \cos B$ or, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$

(iii) $c^2 = a^2 + b^2 - 2ab \cos C$ or, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

PROOF The following cases may arise:

CASE I *When ΔABC is an acute angled triangle:*

Draw perpendicular AD from A on BC.

In ΔABD, we have

$$\cos B = \frac{BD}{c} \Rightarrow BD = c \cos B \quad \text{...(i)}$$

In ΔACD, we have

$$\cos C = \frac{CD}{b} \Rightarrow CD = b \cos C$$

In ΔACD, using Pythagoras theorem, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow AC^2 = AD^2 + (BC - BD)^2$$

$$\Rightarrow AC^2 = AD^2 + BC^2 + BD^2 - 2\,BC\,.\,BD$$

$$\Rightarrow AC^2 = BC^2 + (AD^2 + BD^2) - 2BC\,.\,BD$$

$$\Rightarrow AC^2 = BC^2 + AB^2 - 2\,BC\,.\,BD \quad [\because AB^2 = BD^2 + AD^2]$$

$$\Rightarrow b^2 = a^2 + c^2 - 2ac \cos B \quad \text{[Using (i)]}$$

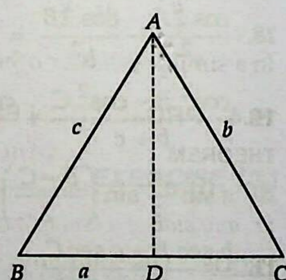

Fig. 10.11

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ca \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

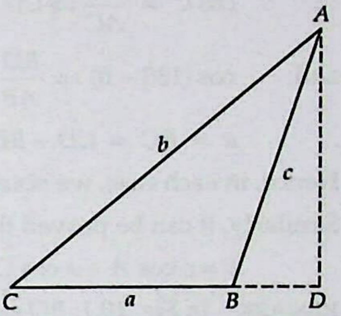

Fig. 10.12

CASE II *When ΔABC is an obtuse angled triangle:*

Draw perpendicular AD from A on CB produced.

In ΔABD, we have

$$\cos(180 - B) = \frac{BD}{AB} \Rightarrow BD = -AB \cos B = -c \cos B \quad \text{...(i)}$$

Using Pythagoras theorem in ΔACD, we have

$$AC^2 = AD^2 + CD^2$$

$$\Rightarrow \quad AC^2 = AD^2 + (CB + BD)^2$$

$$\Rightarrow \quad AC^2 = AD^2 + CB^2 + BD^2 + 2CB \cdot BD$$

$$\Rightarrow \quad AC^2 = BC^2 + (BD^2 + AD^2) + 2BC \cdot BD$$

$$\Rightarrow \quad AC^2 = BC^2 + AB^2 + 2BC \cdot BD \qquad [\text{In } \Delta ABD,\ AB^2 = AD^2 + BD^2]$$

$$\Rightarrow \quad b^2 = a^2 + c^2 + 2a(-c \cos B) \qquad [\text{Using (i)}]$$

$$\Rightarrow \quad b^2 = c^2 + a^2 - 2ac \cos B$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

CASE III When ΔABC is a right angled triangle:

Let ΔABC be a right angled triangle with right angle at B. Then, by Pythagoras theorem, we obtain

$$b^2 = a^2 + c^2$$

$$\Rightarrow \quad b^2 = a^2 + c^2 - 2ac \cos B \qquad \left[\because B = \frac{\pi}{2} \therefore \cos B = 0\right]$$

$$\Rightarrow \quad \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

Hence, in all the cases, we have

$$b^2 = c^2 + a^2 - 2ac \cos B \Rightarrow \cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

In a similar manner other results can be proved.

10.4 PROJECTION FORMULAE

THEOREM *In any ΔABC, we have*

(i) $a = b \cos C + c \cos B$ (ii) $b = c \cos A + a \cos C$ (iii) $c = a \cos B + b \cos A$

i.e. any side of a triangle is equal to the sum of the projections of other two sides on it.

PROOF The following cases arise:

CASE I When ΔABC is an acute angled triangle:

In Fig. 10.1, we have

$$\cos B = \frac{BD}{AB} \Rightarrow BD = AB \cos B \Rightarrow BD = c \cos B$$

and, $$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

Hence, $a = BC = BD + CD \Rightarrow a = c \cos B + b \cos C$

CASE II *When ΔABC is an obtuse angled triangle*

In Fig. 10.2, we have

$$\cos C = \frac{CD}{AC} \Rightarrow CD = AC \cos C \Rightarrow CD = b \cos C$$

and, $$\cos (180 - B) = \frac{BD}{AB} \Rightarrow BD = AB \cos (180 - B) \Rightarrow BD = -c \cos B$$

$\therefore$ $$a = BC = CD - BD \Rightarrow a = b \cos C + c \cos B$$

Hence, in each case, we obtain $a = b \cos C + c \cos B$

Similarly, it can be proved that

$$b = c \cos A + a \cos C \text{ and } c = a \cos B + b \cos A$$ Q.E.D.

REMARK *In Fig. 10.1, BD and CD are the projections of AB and AC respectively on BC.*

10.5 NAPIER'S ANALOGY (LAW OF TANGENTS)

THEOREM *In any ΔABC, we have*

(i) $\tan\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{b+c}\right)\cot\frac{A}{2}$ (ii) $\tan\left(\frac{A-B}{2}\right) = \left(\frac{a-b}{a+b}\right)\cot\frac{C}{2}$

(iii) $\tan\left(\frac{C-A}{2}\right) = \left(\frac{c-a}{c+a}\right)\cot\frac{B}{2}$

PROOF Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$. Then, $a = k \sin A, b = k \sin B, c = k \sin C$ ...(i)

(i) $$\text{RHS} = \frac{b-c}{b+c}\cot\frac{A}{2} = \frac{k\sin B - k\sin C}{k\sin B + k\sin C}\cot\frac{A}{2}$$ [Using (i)]

$$= \left(\frac{\sin B - \sin C}{\sin B + \sin C}\right)\cot\frac{A}{2} = \left\{\frac{2\sin\left(\frac{B-C}{2}\right)\cos\left(\frac{B+C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}\right\}$$

$$= \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{B+C}{2}\right)\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right)\cot\left(\frac{\pi}{2}-\frac{A}{2}\right)\cot\frac{A}{2}$$

$$= \tan\left(\frac{B-C}{2}\right)\tan\frac{A}{2}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right) = \text{LHS}$$

Similarly, (ii) and (iii) can be proved.

10.6 AREA OF A TRIANGLE

THEOREM *Prove that the area of ΔABC is given by*

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \frac{1}{2} ab \sin C$$

PROOF Let ABC be a triangle. Then the following cases arise :

CASE I *When ΔABC is an acute angled triangle:*

In Fig. 10.1, we have

$$\sin B = \frac{AD}{AB} \Rightarrow AD = AB \sin B = c \sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a c \sin B$

CASE II *When ΔABC is an obtuse angled triangle:*

In Fig. 10.2, we have

$$\sin(180 - B) = \frac{AD}{AB} \Rightarrow AD = AB \sin B = c \sin B$$

$\therefore \quad \Delta = \text{Area of } \Delta ABC = \frac{1}{2} BC \times AD = \frac{1}{2} a c \sin B$

Thus, in each case, we have $\Delta = \frac{1}{2} ac \sin B$

Similarly, it can be proved that $\Delta = \frac{1}{2} ab \sin C$ and $\Delta = \frac{1}{2} bc \sin A$

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS ON APPLICATIONS OF COSINE FORMULA AND SINE RULE**

EXAMPLE 1 *In a ΔABC, if $a = 3, b = 5$ and $c = 7$, find $\cos A$, $\cos B$ and $\cos C$.*

SOLUTION We have, $a = 3, b = 5$ and $c = 7$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{25 + 49 - 9}{2 \times 5 \times 7} = \frac{65}{70} = \frac{13}{14}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2bc} = \frac{49 + 9 - 25}{2 \times 3 \times 7} = \frac{33}{42} = \frac{11}{14}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 25 - 49}{2 \times 3 \times 5} = -\frac{15}{30} = -\frac{1}{2}$$

EXAMPLE 2 *If the sides of a $\Delta\, ABC$ are $a = 4, b = 6$ and $c = 8$, show that $4 \cos B + 3 \cos C = 2$.*

SOLUTION We have, $a = 4, b = 6$ and, $c = 8$

$$\therefore \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{16 + 64 - 36}{2 \times 4 \times 8} = \frac{44}{64} = \frac{11}{16}$$

$$\text{and,} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{16 + 36 - 64}{2 \times 4 \times 6} = -\frac{12}{48} = -\frac{1}{4}$$

$$\therefore \quad 4 \cos B + 3 \cos C = 4 \times \frac{11}{16} - \frac{3}{4} = \frac{11}{4} - \frac{3}{4} = 2$$

EXAMPLE 3 *In any ΔABC, prove that:*

(i) $a(b \cos C - c \cos B) = b^2 - c^2$ **[NCERT]**

(ii) $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$ **[NCERT]**

(iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

SOLUTION (i) LHS $= a(b \cos C - c \cos B) = ab \cos C - ac \cos B$

$$= ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right) - ac\left(\frac{a^2 + c^2 - b^2}{2ac}\right)$$

$$= \frac{1}{2}\left\{(a^2 + b^2 - c^2) - (a^2 + c^2 - b^2)\right\} = b^2 - c^2 = \text{RHS}$$

(ii) $$\text{LHS} = \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2acb} + \frac{a^2 + b^2 - c^2}{2abc} = \frac{a^2 + b^2 + c^2}{2abc} = \text{RHS}$$

(iii) $$\text{LHS} = 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$= 2bc\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + 2ca\left(\frac{c^2 + a^2 - b^2}{2ac}\right) + 2ab\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= (b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + (a^2 + b^2 - c^2) = a^2 + b^2 + c^2 = \text{RHS}$$

EXAMPLE 4 *In any ΔABC, prove that:* $(a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2} = c^2$

PROOF We have,

$$\text{LHS} = (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2}$$

$$= a^2\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) + b^2\left(\cos^2 \frac{C}{2} + \sin^2 \frac{C}{2}\right) - 2ab\left(\cos^2 \frac{C}{2} - \sin^2 \frac{C}{2}\right)$$

$$= a^2 + b^2 - 2ab \cos C = c^2 = \text{RHS}$$

EXAMPLE 5 *In a ΔABC, prove that:*

(i) $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0$ **[NCERT]**

(ii) $\left(\frac{b^2 - c^2}{a^2}\right) \sin 2A + \left(\frac{c^2 - a^2}{b^2}\right) \sin 2B + \left(\frac{a^2 - b^2}{c^2}\right) \sin 2C = 0$ **[NCERT]**

SOLUTION Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$. Then, $\sin A = ak$, $\sin B = bk$ and $\sin C = ck$

(i) $$\text{LHS} = (b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C$$

$$= (b^2 - c^2) \frac{\cos A}{\sin A} + (c^2 - a^2) \frac{\cos B}{\sin B} + (a^2 - b^2) \frac{\cos C}{\sin C}$$

$$= \frac{(b^2 - c^2)}{ka}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{(c^2 - a^2)}{kb}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + \frac{(a^2 - b^2)}{kc}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2 - a^2) + (c^2 - a^2)(a^2 + c^2 - b^2) + (a^2 - b^2)(a^2 + b^2 - c^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2) - a^2(b^2 - c^2) + (c^2 - a^2)(c^2 + a^2) - b^2(c^2 - a^2) + (a^2 - b^2)(a^2 + b^2) - c^2(a^2 - b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^2 - c^2)(b^2 + c^2) + (c^2 - a^2)(c^2 + a^2) + (a^2 - b^2)(a^2 + b^2) - a^2(b^2 - c^2) - b^2(c^2 - a^2) - c^2(a^2 - b^2)\right\}$$

$$= \frac{1}{2kabc}\left\{(b^4 - c^4) + (c^4 - a^4) + (a^4 - b^4) - (a^2b^2 - a^2c^2) - (b^2c^2 - b^2a^2) - (c^2a^2 - c^2b^2)\right\}$$

$$= \frac{1}{2kabc} \times 0 = 0 = \text{RHS}$$

(ii) LHS $= \left(\dfrac{b^2-c^2}{a^2}\right)\sin 2A + \left(\dfrac{c^2-a^2}{b^2}\right)\sin 2B + \left(\dfrac{a^2-b^2}{c^2}\right)\sin 2C$

$= \left(\dfrac{b^2-c^2}{a^2}\right) 2\sin A\cos A + \left(\dfrac{c^2-a^2}{b^2}\right) 2\sin B\cos B + \left(\dfrac{a^2-b^2}{c^2}\right) 2\sin C\cos C$

$= \left(\dfrac{b^2-c^2}{a^2}\right) 2ka\left(\dfrac{b^2+c^2-a^2}{2bc}\right) + \left(\dfrac{c^2-a^2}{b^2}\right) 2kb\left(\dfrac{a^2+c^2-b^2}{2ac}\right) + \left(\dfrac{a^2-b^2}{c^2}\right) 2kc\left(\dfrac{a^2+b^2-c^2}{2ab}\right)$

$= \dfrac{k}{abc}\left\{(b^2-c^2)(b^2+c^2-a^2)+(c^2-a^2)(c^2+a^2-b^2)+(a^2-b^2)(a^2+b^2-c^2)\right\}$

$= \dfrac{k}{abc}\times 0 = 0 = \text{RHS}$

Type II PROBLEMS BASED ON COSINE, SINE AND PROJECTION FORMULAE

EXAMPLE 6 *In any ΔABC, prove that:*

(i) $\dfrac{\sin B}{\sin C} = \dfrac{c-a\cos B}{b-a\cos C}$

(ii) $2\left\{a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right\} = a+c-b$

(iii) $2\left\{b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right\} = a+b+c$

(iv) $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C = a+b+c$

SOLUTION Let $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$. Then, $a = k\sin A, b = k\sin B, c = k\sin C$

(i) RHS $= \dfrac{c-a\cos B}{b-a\cos C} = \dfrac{(a\cos B+b\cos A)-a\cos B}{(a\cos C+c\cos A)-a\cos C} = \dfrac{b\cos A}{c\cos A} = \dfrac{b}{c} = \dfrac{k\sin B}{k\sin C} = \dfrac{\sin B}{\sin C} = \text{LHS}$

(ii) LHS $= 2\left(a\sin^2\dfrac{C}{2}+c\sin^2\dfrac{A}{2}\right) = \left\{a(1-\cos C) + c(1-\cos A)\right\}$

$= a+c-(a\cos C+c\cos A) = a+c-b = \text{RHS}$

(iii) LHS $= 2\left(b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right) = \left\{b(1+\cos C) + c(1+\cos B)\right\}$

$= (b+c+b\cos C+c\cos B) = b+c+a = a+b+c = \text{RHS}$

(iv) LHS $= (b+c)\cos A+(c+a)\cos B+(a+b)\cos C$

$= (b\cos A+a\cos B)+(c\cos A+a\cos C)+(b\cos C+c\cos B)$

$= c+b+a = a+b+c = \text{RHS}$

EXAMPLE 7 *In any ΔABC, prove that:*

$$\frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A} = \frac{a^2+b^2+c^2}{2abc}$$

SOLUTION We have,

$$\text{LHS} = \frac{\cos A}{b\cos C+c\cos B}+\frac{\cos B}{c\cos A+a\cos C}+\frac{\cos C}{a\cos B+b\cos A}$$

$$= \frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{2abc}+\frac{c^2+a-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc} = \frac{a^2+b^2+c^2}{2abc} = \text{RHS}$$

EXAMPLE 8 *In a triangle ABC, if* $\cos A = \dfrac{\sin B}{2 \sin C}$, *show that the triangle is isosceles.*

SOLUTION Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k$. Then, $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now, $\cos A = \dfrac{\sin B}{2 \sin C}$

$\Rightarrow \quad 2 \cos A \sin C = \sin B$

$\Rightarrow \quad 2\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right) kc = kb$

$\Rightarrow \quad b^2 + c^2 - a^2 = b^2$

$\Rightarrow \quad c^2 = a^2 \Rightarrow c = a$

$\Rightarrow \quad \Delta ABC$ is isosceles.

EXAMPLE 9 *If in a triangle ABC,* $\dfrac{2 \cos A}{a} + \dfrac{\cos B}{b} + \dfrac{2 \cos C}{c} = \dfrac{a}{bc} + \dfrac{b}{ca}$, *then prove that the triangle is right angled.*

SOLUTION We have,

$$\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$

$\Rightarrow \quad 2\left(\dfrac{b^2 + c^2 - a^2}{2abc}\right) + \left(\dfrac{c^2 + a^2 - b^2}{2abc}\right) + 2\left(\dfrac{a^2 + b^2 - c^2}{2abc}\right) = \dfrac{a}{bc} + \dfrac{b}{ca}$

$\Rightarrow \quad 2(b^2 + c^2 - a^2) + (c^2 + a^2 - b^2) + 2(a^2 + b^2 - c^2) = 2a^2 + 2b^2$

$\Rightarrow \quad b^2 + c^2 = a^2$

$\Rightarrow \quad \Delta ABC$ is a right angled triangle

Type III ON FINDING THE AREA OF A TRIANGLE WHEN ITS PARTS ARE GIVEN

EXAMPLE 10 *Find the area of a triangle ABC in which* $\angle A = 60°$, $b = 4$ *cm and* $c = \sqrt{3}$ *cm.*

SOLUTION The area Δ of triangle ABC is given by

$$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \times 4\sqrt{3} \times \sin 60° = 2\sqrt{3} \times \frac{\sqrt{3}}{2} = 3 \text{ sq. cm.}$$

EXAMPLE 11 *In any triangle ABC, prove that:* $\Delta = \dfrac{b^2 + c^2 - a^2}{4 \cot A}$.

SOLUTION We have,

$$\text{RHS} = \frac{b^2 + c^2 - a^2}{4 \cot A} = \frac{b^2 + c^2 - a^2}{4 \cos A} \sin A = \frac{b^2 + c^2 - a^2}{4(b^2 + c^2 - a^2)} \times 2bc \sin A$$

$$= \frac{1}{2} bc \sin A = \Delta = \text{LHS}$$

EXAMPLE 12 *In any* ΔABC, *prove that:* $\Delta = \dfrac{a^2 - b^2}{2} \cdot \dfrac{\sin A \sin B}{\sin (A - B)}$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

$$\text{RHS} = \frac{a^2 - b^2}{2} \times \frac{\sin A \sin B}{\sin (A - B)} = \frac{k^2 \sin^2 A - k^2 \sin^2 B}{2} \times \frac{\sin A \sin B}{\sin (A - B)}$$

$$= \frac{k^2}{2}(\sin^2 A - \sin^2 B) \times \frac{\sin A \sin B}{\sin (A - B)} = \frac{k^2}{2} \sin (A + B) \sin (A - B) \frac{\sin A \sin B}{\sin (A - B)}$$

$$= \frac{1}{2} k^2 \sin (A + B) \sin A \sin B = \frac{1}{2} (k \sin A)(k \sin B) \sin (\pi - C)$$

$$= \frac{1}{2} ab \sin C = \Delta = \text{LHS}.$$

EXAMPLE 13 *In any* ΔABC, *prove that:* $a \cos A + b \cos B + c \cos C = \dfrac{8\Delta^2}{abc}$. **[NCERT]**

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k \sin A,\ b = k \sin B,\ c = k \sin C$

Now, $a \cos A + b \cos B + c \cos C$

$$= k \sin A \cos A + k \sin B \cos B + k \sin C \cos C$$

$$= \frac{k}{2}(\sin 2A + \sin 2B + \sin 2C)$$

$$= \frac{k}{2}(4 \sin A \sin B \sin C) = 2k \sin A \sin B \sin C = 2a \sin B \sin C$$

$$= 2a \times \frac{2\Delta}{ac} \times \frac{2\Delta}{ab} \qquad \left[\because \Delta = \frac{1}{2} ab \sin C = \frac{1}{2} ac \sin B \therefore \sin B = \frac{2\Delta}{ac}, \sin C = \frac{2\Delta}{ab}\right]$$

$$= \frac{8\Delta^2}{abc} = \text{RHS}$$

EXAMPLE 14 *Two ships leave a port at the same time. One goes 24 km per hour in the direction N 45° E and other travels 32 km per hour in the direction S 75° E. Find the distance between the ships at the end of 3 hours.* **[NCERT]**

SOLUTION Let P and Q be the positions of two ships at the end of 3 hours. Then,

$$OP = 3 \times 24 = 72 \text{ km and } OQ = 3 \times 32 = 96 \text{ km}$$

Using cosine formula in ΔOPQ, we get

$$PQ^2 = OP^2 + OQ^2 - 2\, OP \times OQ \cos 60°$$

$$\Rightarrow \quad PQ^2 = 72^2 + 96^2 - 2 \times 72 \times 96 \times \frac{1}{2}$$

$$\Rightarrow \quad PQ^2 = 5184 + 9216 - 6912 = 7488$$

$$\Rightarrow \quad PQ = \sqrt{7488} \text{ km} = 86{\cdot}533 \text{ km}$$

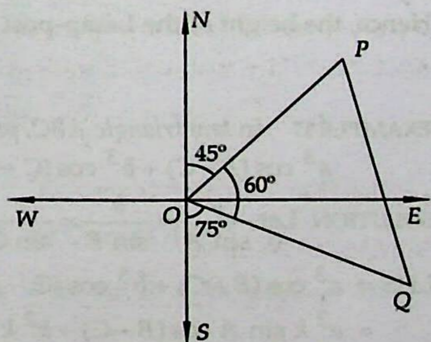

Fig. 10.13

EXAMPLE 15 *Two boats leave a place at the same time. One travels 56 km in the direction N 50° E, while other travels 48 km in the direction S 80° E. What is the distance between the two positions of the boats?*

SOLUTION Let A and B be the position of the boats such that $AB = x$.

Clearly, $\angle AOB = 180° - (50° + 80°) = 50°$

Using cosine formula, we have

$$AB^2 = OA^2 + OB^2 - 2\,OA\,OB\cos\angle AOB$$

$$\Rightarrow \quad x^2 = (56)^2 + (48)^2 - 2\times 56\times 48\cos 50°$$

$$\Rightarrow \quad x^2 = 3136 + 2304 - 2\times 56\times 48\times 0.6428$$

$$\Rightarrow \quad x^2 = 5440 - 3455.69 = 1984.31$$

$$\Rightarrow \quad x = \sqrt{1984.31} = 44.54 \text{ m}$$

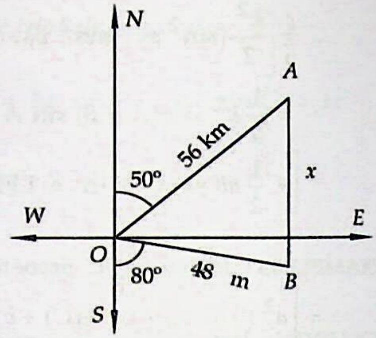

Fig. 10.14

Hence, the distance between the boats is 44.54 km.

EXAMPLE 16 *A lamp-post is situated at the middle point M of the side AC of a triangular plot ABC with BC = 7 m, CA = 8 m and AB = 9 m. Lamp-post subtends an angle of 15° at the point B. Determine the height of the lamp-post.* **[NCERT]**

SOLUTION Using cosine formula in $\Delta\, ABC$, we get

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49+64-81}{2\times 7\times 8} = \frac{32}{112} = \frac{2}{7}$$

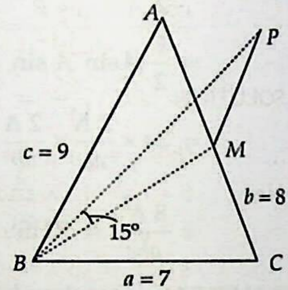

Fig. 10.15

Using cosine formula in $\Delta\, BMC$, we get

$$BM^2 = BC^2 + CM^2 - 2BC\,CM\cos C$$

$$\Rightarrow \quad BM^2 = 49 + 16 - 2\times 7\times 4\times\frac{2}{7} \qquad \left[\because CM = \frac{1}{2}AC = 4\right]$$

$$\Rightarrow \quad BM^2 = 49 \Rightarrow BM = 7$$

In right triangle BMP, we have

$$\tan 15° = \frac{PM}{BM} \Rightarrow \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{PM}{7} \Rightarrow PM = 7\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) = 7\,(2-\sqrt{3}) \text{ m}$$

Hence, the height of the Lamp-post is $7\,(2-\sqrt{3})$ m.

LEVEL-2

EXAMPLE 17 *In any triangle ABC, prove that:*

$$a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B) = 3abc$$

SOLUTION Let $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$. Then, $a = k\sin A,\ b = k\sin B,\ c = k\sin C$

$$\text{LHS} = a^3\cos(B-C) + b^3\cos(C-A) + c^3\cos(A-B)$$

$$= a^2 k\sin A\cos(B-C) + b^2 k\sin B\cos(C-A) + c^2 k\sin C\cos(A-B)$$

$$= \frac{k}{2}\left[a^2\{2\sin A\cos(B-C)\} + b^2\{2\sin B\cos(C-A)\} + c^2\{2\sin C\cos(A-B)\}\right]$$

$$= \frac{k}{2}\left[a^2\{2\sin(B+C)\cos(B-C)\} + b^2\{2\sin(C+A)\cos(C-A)\} + c^2\{2\sin(A+B)\cos(A-B)\}\right]$$

$$= \frac{k}{2}\left[a^2(\sin 2B + \sin 2C) + b^2(\sin 2C + \sin 2A) + c^2(\sin 2A + \sin 2B)\right]$$

$$= \frac{k}{2}\Big[2a^2 (\sin B \cos B + \sin C \cos C) + 2b^2 (\sin C \cos C + \sin A \cos A) + 2c^2 (\sin A \cos A + \sin B \cos B)\Big]$$

$$= \Big[a^2 (k \sin B \cos B + k \sin C \cos C) + b^2 (k \sin C \cos C + k \sin A \cos A) + c^2 (k \sin A \cos A + k \sin B \cos B)\Big]$$

$$= \Big[a^2 (b \cos B + c \cos C) + b^2 (c \cos C + a \cos A) + c^2 (a \cos A + b \cos B)\Big]$$

$$[\because\ k \sin A = a,\ k \sin B = b,\ k \sin C = c]$$

$$= ab (a \cos B + b \cos A) + bc (b \cos C + c \cos B) + ca (a \cos C + c \cos A)$$

$$= abc + bca + cab = 3abc$$

EXAMPLE 18 *With usual notations, if in a triangle ABC* $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$, *then prove that:*

$$\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$$

SOLUTION Let $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \lambda$ (say) Then, $b + c = 11\lambda$, $c + a = 12\lambda$, $a + b = 13\lambda$

$\therefore \quad (b + c + c + a + a + b) = 11\lambda + 12\lambda + 13\lambda \Rightarrow 2(a + b + c) = 36\lambda \Rightarrow a + b + c = 18\lambda$

Now, $\quad b + c = 11\lambda$ and $a + b + c = 18\lambda \Rightarrow a = 7\lambda$

$c + a = 12\lambda$ and $a + b + c = 18\lambda \Rightarrow b = 6\lambda$

$a + b = 13\lambda$ and $a + b + c = 18\lambda \Rightarrow c = 5\lambda$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{36\lambda^2 + 25\lambda^2 - 49\lambda^2}{60\lambda^2} = \frac{12}{60} = \frac{1}{5}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{25\lambda^2 + 49\lambda^2 - 36\lambda^2}{70\lambda^2} = \frac{38}{70} = \frac{19}{35}$$

and, $$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{49\lambda^2 + 36\lambda^2 - 25\lambda^2}{84\lambda^2} = \frac{60}{84} = \frac{5}{7}$$

$$\therefore \quad \cos A : \cos B : \cos C = \frac{1}{5} : \frac{19}{35} : \frac{5}{7} = 7 : 19 : 25$$

EXAMPLE 19 *If* a^2, b^2, c^2 *are in A.P., prove that* $\cot A$, $\cot B$, $\cot C$ *are in A.P.*

SOLUTION Let $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$. Then, $\sin A = ka$, $\sin B = kb$, $\sin C = kc$.

Now, $\cot A$, $\cot B$, $\cot C$ will be in A.P.

$\Leftrightarrow \quad 2 \cot B = \cot A + \cot C$

$$\Leftrightarrow \quad \frac{2 \cos B}{\sin B} = \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C}$$

$$\Leftrightarrow \quad \frac{2 \cos B}{kb} = \frac{\cos A}{ka} + \frac{\cos C}{kc}$$

$$\Leftrightarrow \quad 2\left(\frac{a^2 + c^2 - b^2}{2abc}\right) = \left(\frac{b^2 + c^2 - a^2}{2abc}\right) + \left(\frac{a^2 + b^2 - c^2}{2abc}\right)$$

$\Leftrightarrow \quad 2(a^2 + c^2 - b^2) = (b^2 + c^2 - a^2) + (a^2 + b^2 - c^2) \Leftrightarrow a^2 + c^2 = 2b^2 \Leftrightarrow a^2, b^2, c^2$ are in A.P.

ALITER It is given that a^2, b^2, c^2 are in A.P.

$\therefore$ $-2a^2, -2b^2, -2c^2$ are in A.P.

$\Rightarrow$ $(a^2+b^2+c^2)-2a^2, (a^2+b^2+c^2)-2b^2, (a^2+b^2+c^2)-2c^2$ are in A.P.

$\Rightarrow$ $b^2+c^2-a^2,\ c^2+a^2-b^2,\ a^2+b^2-c^2$ are in A.P.

$\Rightarrow$ $\dfrac{b^2+c^2-a^2}{2abc}, \dfrac{c^2+a^2-b^2}{2abc}, \dfrac{a^2+b^2-c^2}{2abc}$ are in A.P.

$\Rightarrow$ $\dfrac{1}{a}\left(\dfrac{b^2+c^2-a^2}{2bc}\right), \dfrac{1}{b}\left(\dfrac{c^2+a^2-b^2}{2ac}\right), \dfrac{1}{c}\left(\dfrac{a^2+b^2+c^2}{2ab}\right)$ are in A.P.

$\Rightarrow$ $\dfrac{\cos A}{k\sin A}, \dfrac{\cos B}{k\sin B}, \dfrac{\cos C}{k\sin C}$ are in A.P. $\quad [\because\ a = k\sin A, b = k\sin B, c = k\sin C]$

$\Rightarrow$ cot A, cot B, cot C are in A.P.

EXAMPLE 20 *If in a triangle ABC, $\cos A + 2\cos B + \cos C = 2$ prove that the sides of the triangle are in A.P.*

SOLUTION We have,

$$\cos A + 2\cos B + \cos C = 2$$

$\Rightarrow$ $\cos A + \cos C = 2 - 2\cos B$

$\Rightarrow$ $\cos A + \cos C = 2(1-\cos B)$

$\Rightarrow$ $2\cos\left(\dfrac{A+C}{2}\right)\cos\left(\dfrac{A-C}{2}\right) = 2\left(2\sin^2\dfrac{B}{2}\right)$

$\Rightarrow$ $2\sin\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin^2\dfrac{B}{2}$ $\quad \left[\because\ \cos\dfrac{A+C}{2} = \cos\left(\dfrac{\pi}{2}-\dfrac{B}{2}\right) = \sin\dfrac{B}{2}\right]$

$\Rightarrow$ $\cos\left(\dfrac{A-C}{2}\right) = 2\sin\dfrac{B}{2}$ $\quad \left[\because\ 2\sin\dfrac{B}{2} \neq 0\right]$

$\Rightarrow$ $2\cos\dfrac{B}{2}\cos\left(\dfrac{A-C}{2}\right) = 4\sin\dfrac{B}{2}\cos\dfrac{B}{2}$ $\quad$ [Multiplying both sides by $2\cos\dfrac{B}{2}$]

$\Rightarrow$ $2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2} = 2\left(2\sin\dfrac{B}{2}\cos\dfrac{B}{2}\right)$ $\quad \left[\because\ \cos\dfrac{B}{2} = \sin\dfrac{A+C}{2}\right]$

$\Rightarrow$ $\sin A + \sin C = 2\sin B$

$\Rightarrow$ $ka + kc = 2kb$ $\quad \left[\because\ \dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = k\right]$

$\Rightarrow$ $a + c = 2b \Rightarrow a, b, c$ are in A.P.

EXAMPLE 21 *In a triangle ABC, $\angle C = 60°$, then prove that :* $\dfrac{1}{a+c} + \dfrac{1}{b+c} = \dfrac{3}{a+b+c}$.

SOLUTION We have, $\angle C = 60°$

$\Rightarrow$ $\cos C = \dfrac{1}{2} \Rightarrow \dfrac{a^2+b^2-c^2}{2ab} = \dfrac{1}{2} \Rightarrow a^2+b^2-c^2 = ab \Rightarrow a^2+b^2-ab = c^2$ ...(i)

Now, $$\frac{1}{a+c}+\frac{1}{b+c} = \frac{3}{a+b+c}$$

if $$\frac{a+b+2c}{(a+c)(b+c)} = \frac{3}{a+b+c}$$

i.e. if $(a+b+2c)(a+b+c) = 3(a+c)(b+c)$

i.e. if $(a+b)^2 + 2c^2 + 3c(a+b) = 3(ab+ac+bc+c^2)$

i.e. if $a^2+b^2+2ab+2c^2+3ac+3bc = 3ab+3ac+3bc+3c^2$

i.e. if $a^2+b^2-ab = c^2$, which is given [see (i)]

EXAMPLE 22 *Two trees, A and B are on the same side of a river. From a point C in the river the distance of trees A and B are 250 m and 300 m respectively. If the angle C is 45°, find the distance between the trees (Use $\sqrt{2} = 1.44$).*

SOLUTION Using cosine formula in ΔABC, we have

$$AB^2 = AC^2 + BC^2 - 2AC \cdot BC \cos \frac{\pi}{4}$$

$$\Rightarrow \quad AB = \sqrt{(250)^2 + (300)^2 - 2 \times 250 \times 300 \times \frac{1}{\sqrt{2}}}$$

$$\Rightarrow \quad AB = \sqrt{62500 + 90000 - 75000\sqrt{2}}$$

$$\Rightarrow \quad AB = \sqrt{152500 - 75000 \times 1.44}$$

$$\Rightarrow \quad AB = \sqrt{152500 - 108000} = \sqrt{44500} = 210.95 \text{ m}$$

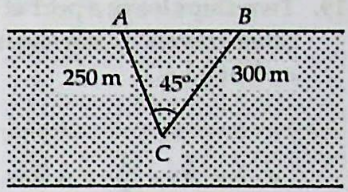

Fig. 10.16

EXERCISE 10.2

LEVEL-1

In any ΔABC, prove the following : (1-13)

1. In a ΔABC, if $a = 5$, $b = 6$ and $C = 60°$, show that its area is $\frac{15\sqrt{3}}{2}$ sq. units.
2. In a ΔABC, if $a = \sqrt{2}$, $b = \sqrt{3}$ and $c = \sqrt{5}$, show that its area is $\frac{1}{2}\sqrt{6}$ sq. units.
3. The sides of a triangle are $a = 5$, $b = 6$ and $c = 8$, show that: $8 \cos A + 16 \cos B + 4 \cos C = 17$.
4. In a ΔABC, if $a = 18$, $b = 24$, $c = 30$, find $\cos A$, $\cos B$ and $\cos C$.
5. $b(c \cos A - a \cos C) = c^2 - a^2$
6. $c(a \cos B - b \cos A) = a^2 - b^2$
7. $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$
8. $(c^2 - a^2 + b^2) \tan A = (a^2 - b^2 + c^2) \tan B = (b^2 - c^2 + a^2) \tan C$
9. $\frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}$
10. $a(\cos B + \cos C - 1) + b(\cos C + \cos A - 1) + c(\cos A + \cos B - 1) = 0$
11. $a \cos A + b \cos B + c \cos C = 2b \sin A \sin C$
12. $a^2 = (b + c)^2 - 4bc \cos^2 \frac{A}{2}$
13. $4\left(bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2}\right) = (a + b + c)^2$

LEVEL-2

14. In a ΔABC, prove that
    $\sin^3 A \cos(B - C) + \sin^3 B \cos(C - A) + \sin^3 C \cos(A - B) = 3 \sin A \sin B \sin C$
15. In in any ΔABC, $\frac{b+c}{12} = \frac{c+a}{13} = \frac{a+b}{15}$, then prove that $\frac{\cos A}{2} = \frac{\cos B}{7} = \frac{\cos C}{11}$.
16. In a ΔABC, if $\angle B = 60°$, prove that $(a + b + c)(a - b + c) = 3ca$
17. If in a ΔABC, $\cos^2 A + \cos^2 B + \cos^2 C = 1$, prove that the triangle is right angled.

18. In a Δ ABC, if $\cos C = \dfrac{\sin A}{2 \sin B}$, prove that the triangle is isosceles.

19. Two ships leave a port at the same time. One goes 24 km/hr in the direction $N\,38°\,E$ and other travels 32 km/hr in the direction $S\,52°\,E$. Find the distance between the ships at the end of 3 hrs.

ANSWERS

4. $\cos A = \dfrac{4}{5}$, $\cos B = \dfrac{3}{5}$, $\cos C = 0$ 19. 120 km

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Find the area of the triangle ΔABC in which $a = 1, b = 2$ and $\angle c = 60°$.
2. In a ΔABC, if $b = \sqrt{3}$, $c = 1$ and $\angle A = 30°$, find a.
3. In a ΔABC, if $\cos A = \dfrac{\sin B}{2 \sin C}$, then show that $c = a$.
4. In a ΔABC, if $b = 20$, $c = 21$ and $\sin A = \dfrac{3}{5}$, find a.
5. In a ΔABC, if $\sin A$ and $\sin B$ are the roots of the equation $c^2x^2 - c\,(a+b)\,x + ab = 0$, then find $\angle C$.
6. In ΔABC, if $a = 8, b = 10, c = 12$ and $C = \lambda A$, find the value of λ.
7. If the sides of a triangle are proportional to $2, \sqrt{6}$ and $\sqrt{3} - 1$, find the measure of its greatest angle.
8. If in a ΔABC, $\dfrac{\cos A}{a} = \dfrac{\cos B}{b} = \dfrac{\cos C}{c}$, then find the measures of angles A, B, C.
9. In any triangle ABC, find the value of $a \sin (B - C) + b \sin (C - A) + c \sin (A - B)$.
10. In any ΔABC, find the value of $\Sigma\, a\,(\sin B - \sin C)$

ANSWERS

1. $\sqrt{3}$ sq. units 2. 1 4. 13 5. 90° 6. 2 7. 120° 8. $A = B = C = 60°$ 9. 0 10. 0

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. In any ΔABC, $\sum a^2 (\sin B - \sin C) =$

   (a) $a^2 + b^2 + c^2$ (b) a^2 (c) b^2 (d) 0

2. In a ΔABC, if $a = 2$, $\angle B = 60°$ and $\angle C = 75°$, then $b =$

   (a) $\sqrt{3}$ (b) $\sqrt{6}$ (c) $\sqrt{9}$ (d) $1 + \sqrt{2}$

3. In the sides of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the measure of its greatest angle is

   (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{3}$ (c) $\dfrac{\pi}{2}$ (d) $\dfrac{2\pi}{3}$

4. In any Δ ABC, $2\,(bc \cos A + ca \cos B + ab \cos C) =$

   (a) abc (b) $a + b + c$ (c) $a^2 + b^2 + c^2$ (d) $\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{c^2}$

5. In a triangle ABC, $a = 4$, $b = 3$, $\angle A = 60°$ then c is a root of the equation

(a) $c^2 - 3c - 7 = 0$ (b) $c^2 + 3c + 7 = 0$ (c) $c^2 - 3c + 7 = 0$ (d) $c^2 + 3c - 7 = 0$

6. In a ΔABC, if $(c + a + b)(a + b - c) = ab$, then the measure of angle C is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{2}$

7. In any ΔABC, the value of $2ac \sin\left(\frac{A - B + C}{2}\right)$ is

(a) $a^2 + b^2 - c^2$ (b) $c^2 + a^2 - b^2$ (c) $b^2 - c^2 - a^2$ (d) $c^2 - a^2 - b^2$

8. In any ΔABC, $a\,(b \cos C - c \cos B) =$

(a) a^2 (b) $b^2 - c^2$ (c) 0 (d) $b^2 + c^2$

ANSWERS

1. (d) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (c) 8. (c)