CHAPTER 18

INDEFINITE INTEGRALS

18.1 PRIMITIVE OR ANTIDERIVATIVE

DEFINITION *A function $\phi(x)$ is called a primitive (or an antiderivative or an integral) of a function $f(x)$, if $\phi'(x) = f(x)$.*

For example, $\frac{x^4}{4}$ is a primitive of x^3, because $\frac{d}{dx}\left(\frac{x^4}{4}\right) = x^3$.

Let $\phi(x)$ be a primitive of a function $f(x)$ and let C be any constant. Then,

$$\frac{d}{dx}\{\phi(x) + C\} = \phi'(x) = f(x) \qquad [\because \phi'(x) = f(x)]$$

$\therefore$ $\phi(x) + C$ is also a primitive of $f(x)$.

Thus, if a function $f(x)$ possesses a primitive, then it possesses infinitely many primitives which are contained in the expression $\phi(x) + C$, where C is a constant.

For example, $\frac{x^4}{4}, \frac{x^4}{4} + 2, \frac{x^4}{4} - 1$ etc. are primitives of x^3.

18.2 INDEFINITE INTEGRAL

DEFINITION *Let $f(x)$ be a function. Then the family of all its primitives (or antiderivatives) is called the indefinite integral of $f(x)$ and is denoted by $\int f(x)\, dx$.*

The symbol $\int f(x)\, dx$ is read as the indefinite integral of $f(x)$ with respect to x.

Thus, $\frac{d}{dx}\left(\phi(x) + C\right) = f(x) \Leftrightarrow \int f(x)\, dx = \phi(x) + C$ ...(i)

where $\phi(x)$ is primitive of $f(x)$ and C is an arbitrary constant known as the *constant of integration*.

Here, $\int$ is the integral sign, $f(x)$ is the integrand, x is the variable of integration and dx is the element of integration or differential of x.

DEFINITION *The process of finding an indefinite integral of a given function is called integration of the function.*

It follows from the above discussion that integrating a function $f(x)$ means finding a function $\phi(x)$ such that $\frac{d}{dx}\phi(x) = f(x)$.

18.3 FUNDAMENTAL INTEGRATION FORMULAS

We know that

$$\frac{d}{dx}\{\phi(x)\} = f(x) \Leftrightarrow \int f(x)\, dx = \phi(x) + C$$

Based upon this and various standard differentiation formulae, we obtain the following integration formulae:

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right)=x^n, n \neq -1 \quad \Rightarrow \int x^n\,dx=\frac{x^{n+1}}{n+1}+C, n\neq -1$

(ii) $\frac{d}{dx}\left(\log_e x\right)=\frac{1}{x} \quad \Rightarrow \int \frac{1}{x}\,dx=\log_e|x|+C$

(iii) $\frac{d}{dx}(e^x)=e^x \quad \Rightarrow \int e^x\,dx=e^x+C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right)=a^x, a>0, a\neq 1 \quad \Rightarrow \int a^x\,dx=\frac{a^x}{\log_e a}+C$

(v) $\frac{d}{dx}(-\cos x)=\sin x \quad \Rightarrow \int \sin x\,dx=-\cos x+C$

(vi) $\frac{d}{dx}(\sin x)=\cos x \quad \Rightarrow \int \cos x\,dx=\sin x+C$

(vii) $\frac{d}{dx}(\tan x)=\sec^2 x \quad \Rightarrow \int \sec^2 x\,dx=\tan x+C$

(viii) $\frac{d}{dx}(-\cot x)=\operatorname{cosec}^2 x \quad \Rightarrow \int \operatorname{cosec}^2 x\,dx=-\cot x+C$

(ix) $\frac{d}{dx}(\sec x)=\sec x\tan x \quad \Rightarrow \int \sec x\tan x\,dx=\sec x+C$

(x) $\frac{d}{dx}(-\operatorname{cosec} x)=\operatorname{cosec} x\cot x \quad \Rightarrow \int \operatorname{cosec} x\cot x\,dx=-\operatorname{cosec} x+C$

(xi) $\frac{d}{dx}(\log\sin x)=\cot x \quad \Rightarrow \int \cot x\,dx=\log|\sin x|+C$

(xii) $\frac{d}{dx}(-\log\cos x)=\tan x \quad \Rightarrow \int \tan x\,dx=-\log|\cos x|+C$

(xiii) $\frac{d}{dx}\{\log(\sec x+\tan x)\}=\sec x \quad \Rightarrow \int \sec x\,dx=\log|\sec x+\tan x|+C$

(xiv) $\frac{d}{dx}\{\log(\operatorname{cosec} x-\cot x)\}=\operatorname{cosec} x \quad \Rightarrow \int \operatorname{cosec} x\,dx=\log|\operatorname{cosec} x-\cot x|+C$

(xv) $\frac{d}{dx}\left(\sin^{-1}\frac{x}{a}\right)=\frac{1}{\sqrt{a^2-x^2}} \quad \Rightarrow \int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\left(\frac{x}{a}\right)+C$

(xvi) $\frac{d}{dx}\left(\cos^{-1}\frac{x}{a}\right)=-\frac{1}{\sqrt{a^2-x^2}} \quad \Rightarrow \int -\frac{1}{\sqrt{a^2-x^2}}\,dx=\cos^{-1}\left(\frac{x}{a}\right)+C$

(xvii) $\frac{d}{dx}\left(\frac{1}{a}\tan^{-1}\frac{x}{a}\right)=\frac{1}{a^2+x^2} \quad \Rightarrow \int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$

(xviii) $\frac{d}{dx}\left(\frac{1}{a}\cot^{-1}\frac{x}{a}\right)=-\frac{1}{a^2+x^2} \quad \Rightarrow \int -\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right)+C$

(xix) $\frac{d}{dx}\left(\frac{1}{a}\sec^{-1}\frac{x}{a}\right)=\frac{1}{x\sqrt{x^2-a^2}} \quad \Rightarrow \int \frac{1}{x\sqrt{x^2-a^2}}\,dx=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+C$

(xx) $\frac{d}{dx}\left(\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}\right)=-\frac{1}{x\sqrt{x^2-a^2}} \quad \Rightarrow \int -\frac{1}{x\sqrt{x^2-a^2}}\,dx=\frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right)+C$

Let us now discuss evaluation of some integrals based upon the above formulae.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following integrals:*

(i) $\int x^4\,dx$ (ii) $\int \sqrt{x}\,dx$ (iii) $\int \frac{1}{\sqrt{x}}\,dx$ (iv) $\int \frac{1}{x^3}\,dx$ (v) $\int a^{3\log_a x}\,dx$

SOLUTION (i) $\int x^4\,dx = \frac{x^{4+1}}{4+1} + C = \frac{x^5}{5} + C$ [Using formula (i)]

(ii) $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{2}{3}x^{3/2} + C$ [Using formula (i)]

(iii) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2x^{1/2} + C$ [Using formula (i)]

(iv) $\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2x^2} + C$ [Using formula (i)]

(v) $\int a^{3\log_a x}\,dx = \int a^{\log_a x^3}\,dx = \int x^3\,dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$ $[\because a^{\log_a x} = x]$

EXAMPLE 2 *Evaluate:* $\int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}}\,dx$

SOLUTION We know that $e^{a\log_e x} = x^a$

$$\therefore \int \frac{e^{5\log_e x} - e^{4\log_e x}}{e^{3\log_e x} - e^{2\log_e x}} = \int \frac{x^5 - x^4}{x^3 - x^2}\,dx = \int \frac{x^4\,(x-1)}{x^2\,(x-1)}\,dx = \int x^2\,dx = \frac{x^3}{3} + C$$

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{2}{1+\cos 2x}\,dx$ (ii) $\int \frac{2}{1-\cos 2x}\,dx$

SOLUTION We know that $1+\cos 2x = 2\cos^2 x$ and $1-\cos 2x = 2\sin^2 x$. Therefore,

(i) $\int \frac{2}{1+\cos 2x}\,dx = \int \frac{2}{2\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C$

(ii) $\int \frac{2}{1-\cos 2x}\,dx = \int \frac{2}{2\sin^2 x}\,dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + C$

EXAMPLE 4 *Evaluate:*

(i) $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$ [NCERT, CBSE 2018] (ii) $\int \frac{2\cos^2 x - \cos 2x}{\sin^2 x}\,dx$

SOLUTION (i) We know that $1 - \cos 2x = 2\sin^2 x$.

$$\therefore \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x}\,dx$$

$$= \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx = \int \sec^2 x\,dx = \tan x + C$$

(ii) We know that $\cos 2x = 2\cos^2 x - 1$.

$$\therefore \quad \int \frac{2\cos^2 x - \cos 2x}{\sin^2 x}\,dx = \int \frac{2\cos^2 x-(2\cos^2 x-1)}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *If $a > 0$ and $a \neq 1$ evaluate the following integrals:*

(i) $\int e^{x \log_e a}\,dx$ (ii) $\int e^{a \log_e x}\,dx$ (iii) $\int e^x a^x\,dx$ (iv) $\int 2^{\log_e x}\,dx$

SOLUTION (i) We know that $e^{\log_e k} = k$.

(i) $$\int e^{x\log_e a}\,dx = \int e^{\log_e a^x}\,dx = \int a^x\,dx = \frac{a^x}{\log_e a} + C$$

(ii) $$\int e^{a\log_e x}\,dx = \int e^{\log_e x^a}\,dx = \int x^a\,dx = \frac{x^{a+1}}{a+1} + C$$

(iii) $$\int e^x a^x\,dx = \int (ae)^x\,dx = \frac{(ae)^x}{\log(ae)} + C$$

(iv) We know that $x^{\log_a y} = y^{\log_a x}$

$$\therefore \quad \int 2^{\log_e x}\,dx = \int x^{\log_e 2}\,dx = \frac{x^{\log_e 2+1}}{\log_e 2+1} + C = \frac{x^{\log_e (2e)}}{\log_e (2e)} + C$$

EXERCISE 18.1

BASIC

1. *Evaluate each of the following integrals:*

(i) $\int x^4\,dx$ (ii) $\int x^{5/4}\,dx$ (iii) $\int \frac{1}{x^5}\,dx$ (iv) $\int \frac{1}{x^{3/2}}\,dx$

(v) $\int 3^x\,dx$ (vi) $\int \frac{1}{\sqrt[3]{x^2}}\,dx$ (vii) $\int 3^{2\log_3 x}\,dx$ (viii) $\int \log_x x\,dx$

2. *Evaluate each of the following integrals:*

(i) $\int \sqrt{\frac{1+\cos 2x}{2}}\,dx$ (ii) $\int \sqrt{\frac{1-\cos 2x}{2}}\,dx$

(iii) $\int \frac{\cos 2x + 2\sin^2 x}{\sin^2 x}\,dx$ (iv) $\int \frac{2\cos^2 x - \cos 2x}{\cos^2 x}\,dx$

(v) $\int \frac{e^{6\log_e x} - e^{5\log_e x}}{e^{4\log_e x} - e^{3\log_e x}}\,dx$ [NCERT] (vi) $\int \frac{1}{a^x b^x}\,dx$ (vii) $\int \frac{e^{\log\sqrt{x}}}{x}\,dx$

ANSWERS

1. (i) $\frac{x^5}{5} + C$ (ii) $\frac{4}{9}x^{9/4} + C$ (iii) $-\frac{1}{4x^4} + C$ (iv) $-\frac{2}{\sqrt{x}} + C$

(v) $\frac{3^x}{\log 3} + C$ (vi) $3x^{1/3} + C$ (vii) $\frac{x^3}{3} + C$ (viii) $x + C$

2. (i) $\sin x + C$ (ii) $-\cos x + C$ (iii) $-\cot x + C$ (iv) $\tan x + C$

(v) $\frac{x^3}{3} + C$ (vi) $\frac{a^{-x} b^{-x}}{-\log_e (ab)} + C$ (vii) $2\sqrt{x} + C$

HINTS TO SELECTED PROBLEMS

2.(v) $\int \frac{e^{6 \log_e x} - e^{5 \log_e x}}{e^{4 \log_e x} - e^{3 \log_e x}} dx = \int \frac{e^{\log_e x^6} - e^{\log_e x^5}}{e^{\log_e x^4} - e^{\log_e x^3}} dx = \int \frac{x^6 - x^5}{x^4 - x^3} dx = \int x^2 dx = \frac{x^3}{3} + C$

18.4 SOME STANDARD RESULTS ON INTEGRATION

THEOREM (i) $\frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$

i.e., the differentiation of an integral is the integrand itself or differentiation and integration are inverse operations.

(ii) $\int k\, f(x)\, dx = k \int f(x)\, dx$, *where k is a constant*

i.e., the integral of the product of a constant and a function = the constant × integral of the function.

(iii) $\int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

i.e., the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

PROOF (i) Let $\int f(x)\, dx = \phi(x)$. Then, by definition of an integral, we obtain

$$\frac{d}{dx}\left(\phi(x)\right) = f(x) \Rightarrow \frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$$

(ii) Let $\int f(x)\, dx = \phi(x)$. Then, by the definition of an integral, we obtain

$$\frac{d}{dx}\left(\phi(x)\right) = f(x) \quad \text{...(i)}$$

$\therefore \quad \frac{d}{dx}\{k\,\phi(x)\} = k \cdot \frac{d}{dx}(\phi(x)) = k\, f(x)$ [Using (i)]

$\Rightarrow \quad \int k\, f(x)\, dx = k\,\phi(x)$ [By definition of an integral]

$\Rightarrow \quad \int k\, f(x)\, dx = k \int f(x)\, dx$ [$\because \int f(x)\, dx = \phi(x)$]

COROLLARY If $f(x) = 1$, then $\int k \cdot dx = k \int 1 \cdot dx = k \int x^0\, dx = kx + C$

Thus, integration of a constant k with respect to x is kx.

(iii) Let $\int f(x)\, dx = \phi(x)$ and $\int g(x)\, dx = \psi(x)$ ...(i)

Then, $\frac{d}{dx}(\phi(x)) = f(x)$ and $\frac{d}{dx}(\psi(x)) = g(x)$

$\Rightarrow \quad \frac{d}{dx}(\phi(x)) \pm \frac{d}{dx}(\psi(x)) = f(x) \pm g(x)$

$\Rightarrow \quad \frac{d}{dx}\{\phi(x) \pm \psi(x)\} = f(x) \pm g(x)$

$\Rightarrow \quad \int \{f(x) \pm g(x)\}\, dx = \phi(x) \pm \psi(x)$ [By definition of an integral]

$\Rightarrow \quad \int \{f(x) \pm g(x)\}\, dx = \int f(x)\, dx \pm \int g(x)\, dx$ [Using (i)]

GENERALIZATION *The above results can be generalized to the form*

$$\int \{k_1 f(x) \pm k_2 f_2(x) \pm \ldots \pm k_n f_n(x)\}\, dx = k_1 \int f_1(x)\, dx \pm k_2 \int f_2(x)\, dx \pm \ldots \pm k_n \int f_n(x)\, dx$$

i.e., the integration of the linear combination of a finite number of functions is equal to the linear combination of their integrals.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **PROBLEMS BASED ON** $\int k\, f(x)\, dx = k \int f(x)\, dx$

EXAMPLE 1 *Evaluate:*

(i) $\int 4x^5\, dx$ (ii) $\int 2 \sin x\, dx$ (iii) $\int 3^{x+2}\, dx$ (iv) $\int \frac{1}{2} \sec^2 x\, dx$

SOLUTION Using $\int k\, f(x)\, dx = k \int f(x)\, dx$, we obtain

(i) $\int 4x^5\, dx = 4 \int x^5\, dx = 4\left(\frac{x^{5+1}}{5+1}\right) + C = \frac{4}{6} x^6 + C = \frac{2}{3} x^6 + C$

(ii) $\int 2 \sin x\, dx = 2 \int \sin x\, dx = -2 \cos x + C$

(iii) $\int 3^{x+2}\, dx = \int 3^x \cdot 3^2\, dx = 9 \int 3^x\, dx = 9\left(\frac{3^x}{\log 3}\right) + C$

(iv) $\int \frac{1}{2} \sec^2 x\, dx = \frac{1}{2} \int \sec^2 x\, dx = \frac{1}{2} \tan x + C$

Type II **EVALUATION OF INTEGRALS BY USING** $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$ **and** $\int \frac{1}{x}\, dx = \log_e x + C$

EXAMPLE 2 *Evaluate:*

(i) $\int x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\, dx$ (ii) $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2}\, dx$ (iii) $\int (1-x)\sqrt{x}\, dx$

(iv) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$ (v) $\int \left(x^2 + \frac{1}{x^2}\right)^3 dx$ (vi) $\int \frac{(1+x)^2}{\sqrt{x}}\, dx$

(vii) $\int \frac{x^3 - x^2 + x - 1}{x - 1}\, dx$

SOLUTION (i) Let $I = \int x^3 + 5x^2 - 4 + \frac{7}{x} + \frac{2}{\sqrt{x}}\, dx$. Then,

$$I = \int x^3\, dx + \int 5x^2\, dx - \int 4dx + \int \frac{7}{x}\, dx + \int \frac{2}{\sqrt{x}}\, dx$$

$$\Rightarrow \quad I = \int x^3\, dx + 5 \int x^2\, dx - 4 \int 1\, dx + 7 \int \frac{1}{x}\, dx + 2 \int x^{-1/2}\, dx$$

$$\Rightarrow \quad I = \frac{x^4}{4} + 5 \times \frac{x^3}{3} - 4x + 7 \log|x| + 2\left(\frac{x^{1/2}}{1/2}\right) + C = \frac{x^4}{4} + \frac{5}{3} x^3 - 4x + 7 \log|x| + 4\sqrt{x} + C$$

(ii) Let $I = \int \frac{x^3 + 5x^2 + 4x + 1}{x^2}\, dx$. Then,

$$I = \int \left(x + 5 + \frac{4}{x} + \frac{1}{x^2}\right) dx$$ [Dividing each term by x^2]

$$\Rightarrow \quad I = \int x\,dx + \int 5dx + \int \frac{4}{x}\,dx + \int \frac{1}{x^2}\,dx$$

$$\Rightarrow \quad I = \int x\,dx + 5\int 1dx + 4\int \frac{1}{x}\,dx + \int x^{-2}\,dx$$

$$\Rightarrow \quad I = \frac{x^2}{2} + 5x + 4\log|x| + \left(\frac{x^{-1}}{-1}\right) + C = \frac{x^2}{2} + 5x + 4\log|x| - \frac{1}{x} + C$$

(iii) Let $I = \int (1-x)\sqrt{x}\,dx$. Then,

$$I = \int (\sqrt{x} - x\sqrt{x})\,dx = \int \sqrt{x}\,dx - \int x^{3/2}\,dx = \frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} + C = \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$$

(iv) Let $I = \int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$. Then,

$$I = \int \left(x + \frac{1}{x} + 2\right) dx = \int x\,dx + \int \frac{1}{x}\,dx + 2\int 1\cdot dx = \frac{x^2}{2} + \log|x| + 2x + C$$

(v) Let $I = \int \left(x^2 + \frac{1}{x^2}\right)^3 dx$. Then,

$$I = \int \left(x^6 + \frac{1}{x^6} + 3x^2 + \frac{3}{x^2}\right) dx = \int x^6\,dx + \int x^{-6}\,dx + 3\int x^2\,dx + 3\int x^{-2}\,dx$$

$$\Rightarrow \quad I = \frac{x^7}{7} + \left(\frac{x^{-5}}{-5}\right) + 3\left(\frac{x^3}{3}\right) + 3\left(\frac{x^{-1}}{-1}\right) + C = \frac{x^7}{7} - \frac{1}{5x^5} + x^3 - \frac{3}{x} + C$$

(vi) Let $I = \int \frac{(1+x)^2}{\sqrt{x}}\,dx$. Then,

$$I = \int \frac{1 + 2x + x^2}{\sqrt{x}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sqrt{x}} + 2\sqrt{x} + x^{3/2}\,dx = \int x^{-1/2}\,dx + 2\int x^{1/2}\,dx + \int x^{3/2}\,dx$$

$$\Rightarrow \quad I = \frac{x^{1/2}}{1/2} + 2\times\frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} + C = 2x^{1/2} + \frac{4}{3}x^{3/2} + \frac{2}{5}x^{5/2} + C$$

(vii) Let $I = \int \frac{x^3 - x^2 + x - 1}{x-1}\,dx$. Then,

$$I = \int \frac{x^2(x-1) + (x-1)}{x-1}\,dx = \int \frac{(x^2+1)(x-1)}{x-1}\,dx = \int x^2 + 1\,dx = \frac{x^3}{3} + x + C$$

EXAMPLE 3 *Evaluate:* $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1}\,dx$

SOLUTION Let $I = \int \frac{x^4 + x^2 + 1}{x^2 - x + 1}\,dx$. Then,

$$I = \int \frac{(x^2+1)^2 - x^2}{x^2 - x + 1}\,dx = \int \frac{(x^2+1+x)(x^2+1-x)}{(x^2-x+1)}\,dx = \int (x^2+x+1)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C$$

Type III INTEGRATION OF TRIGONOMETRIC FUNCTIONS

EXAMPLE 4 *Evaluate:*

(i) $\int (3\sin x - 2\cos x + 4\sec^2 x - 5\operatorname{cosec}^2 x)\, dx$ (ii) $\int \sqrt{1+\cos 2x}\, dx$

(iii) $\int \sqrt{1-\cos 2x}\, dx$ (iv) $\int \sqrt{1+\sin 2x}\, dx$

(v) $\int \sqrt{1-\sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$ [CBSE 2019] (vi) $\int \frac{\cos x - \cos 2x}{1-\cos x}\, dx$ [NCERT EXEMPLAR]

SOLUTION (i) Let $I = \int (3 \sin x - 2 \cos x + 4 \sec^2 x - 5 \operatorname{cosec}^2 x)\, dx$. Then,

$$I = 3\int \sin x\, dx - 2\int \cos x\, dx + 4\int \sec^2 x\, dx - 5\int \operatorname{cosec}^2 x\, dx$$

$$\Rightarrow \quad I = -3\cos x - 2\sin x + 4\tan x + 5\cot x + C$$

(ii) Let $I = \int \sqrt{1+\cos 2x}\, dx = \int \sqrt{2\cos^2 x}\, dx = \sqrt{2}\int \cos x\, dx = \sqrt{2}\sin x + C$

(iii) Let $I = \int \sqrt{1-\cos 2x}\, dx = \int \sqrt{2\sin^2 x}\, dx = \sqrt{2}\int \sin x\, dx = -\sqrt{2}\cos x + C$

(iv) Let $I = \int \sqrt{1+\sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx$

$$\Rightarrow \quad I = \int (\sin x + \cos x)\, dx = \int \sin x\, dx + \int \cos x\, dx = -\cos x + \sin x + C$$

(v) Let $I = \int \sqrt{1-\sin 2x}\, dx = \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}\, dx$

$$\Rightarrow \quad I = \int \sqrt{\sin^2 x + \cos^2 x - 2\sin x\cos x}\, dx = \int \sqrt{(\sin x - \cos x)^2}\, dx = \int |\sin x - \cos x|\, dx$$

Now, following cases arise:

Case I *When* $\sin x > \cos x$: *In this case* $|\sin x - \cos x| = \sin x - \cos x$

$$\therefore \quad I = \int |\sin x - \cos x|\, dx = \int (\sin x - \cos x)\, dx = -\cos x + \sin x + C$$

Case II *When* $\sin x < \cos x$: *In this case* $|\sin x - \cos x| = -(\sin x - \cos x)$

$$\therefore \quad I = \int |\sin x - \cos x|\, dx = \int -(\sin x - \cos x)\, dx = -(-\cos x - \sin x) + C$$

(vi) Let $I = \int \frac{\cos x - \cos 2x}{1-\cos x}\, dx = \int \frac{\cos x - (2\cos^2 x - 1)}{1-\cos x}\, dx = \int \frac{-(2\cos^2 x - \cos x - 1)}{-(\cos x - 1)}\, dx$

$$\Rightarrow \quad I = \int \frac{(2\cos x + 1)(\cos x - 1)}{(\cos x - 1)}\, dx = \int (2\cos x + 1)\, dx = 2\sin x + x + C$$

EXAMPLE 5 *Evaluate:*

(i) $\int \tan^2 x\, dx$ (ii) $\int \cot^2 x\, dx$ (iii) $\int \frac{1}{\sin^2 x \cos^2 x}\, dx$

(iv) $\int \frac{\cos 2x}{\sin^2 x \cos^2 x}\, dx$ (v) $\int \frac{2+3\cos x}{\sin^2 x}\, dx$ (vi) $\int (2\tan x - 3\cot x)^2\, dx$

(vii) $\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha}\, dx$ [NCERT, CBSE 2013]

SOLUTION (i) Let $I = \int \tan^2 x\, dx$. Then,

$$I = \int (\sec^2 x - 1)\, dx = \int \sec^2 x\, dx - \int 1 \cdot dx = \tan x - x + C$$

(ii) Let $I = \int \cot^2 x\, dx$. Then,

$$I = \int (\operatorname{cosec}^2 x - 1)\, dx = \int \operatorname{cosec}^2 x\, dx - \int 1 \cdot dx = -\cot x - x + C$$

(iii) Let $I = \int \frac{1}{\sin^2 x \cos^2 x} dx$. Then,

$$\Rightarrow \quad I = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} dx = \int \sec^2 x\, dx + \int \operatorname{cosec}^2 x\, dx = \tan x - \cot x + C$$

(iv) Let $I = \int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$. Then,

$$I = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx$$

$$\Rightarrow \quad I = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx = \int \operatorname{cosec}^2 x\, dx - \int \sec^2 x\, dx = -\cot x - \tan x + C$$

(v) Let $I = \int \frac{2 + 3\cos x}{\sin^2 x} dx$. Then,

$$I = \int \frac{2}{\sin^2 x} + \frac{3\cos x}{\sin^2 x} dx = \int (2\operatorname{cosec}^2 x + 3\cot x \operatorname{cosec} x)\, dx$$

$$\Rightarrow \quad I = 2\int \operatorname{cosec}^2 x\, dx + 3\int \operatorname{cosec} x \cot x\, dx = -2\cot x - 3\operatorname{cosec} x + C$$

(vi) Let $I = \int (2\tan x - 3\cot x)^2\, dx$. Then,

$$I = \int (4\tan^2 x + 9\cot^2 x - 12\tan x \cot x)\, dx = \int \{4(\sec^2 x - 1) + 9(\operatorname{cosec}^2 x - 1) - 12\}\, dx$$

$$\Rightarrow \quad I = \int (4\sec^2 x + 9\operatorname{cosec}^2 x - 25)\, dx = 4\tan x - 9\cot x - 25x + C$$

(vii) Let $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$. Then,

$$I = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx = \int \frac{2(\cos^2 x - \cos^2 \alpha)}{\cos x - \cos \alpha} dx$$

$$\Rightarrow \quad I = 2\int (\cos x + \cos \alpha)\, dx = 2\int \cos x\, dx + 2\int \cos \alpha\, dx$$

$$\Rightarrow \quad I = 2\int \cos x\, dx + 2\cos \alpha \int 1 \cdot dx = 2\sin x + 2x\cos \alpha + C$$

EXAMPLE 6 *Evaluate:*

(i) $\int \frac{1}{1+\sin x} dx$ (ii) $\int \frac{1}{1+\cos x} dx$ (iii) $\int \frac{\sin x}{1+\sin x} dx$ (iv) $\int \frac{\sec x}{\sec x + \tan x} dx$

SOLUTION (i) Let $I = \int \frac{1}{1+\sin x} dx$. Then,

$$I = \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx = \int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx$$

$$\Rightarrow \quad I = \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx = \int \sec^2 x\, dx - \int \tan x \sec x\, dx = \tan x - \sec x + C$$

(ii) Let $I = \int \frac{1}{1+\cos x}\, dx$. Then,

$$I = \int \frac{1}{1+\cos x} \times \frac{1-\cos x}{1-\cos x}\, dx = \int \frac{1-\cos x}{1-\cos^2 x}\, dx$$

$$\Rightarrow \quad I = \int \frac{1-\cos x}{\sin^2 x}\, dx = \int \operatorname{cosec}^2 x \; dx - \int \cot x \; \operatorname{cosec} x \; dx = -\cot x + \operatorname{cosec} x + C$$

(iii) Let $I = \int \frac{\sin x}{1+\sin x}\, dx$. Then,

$$I = \int \frac{\sin x\,(1-\sin x)}{(1+\sin x)\,(1-\sin x)}\, dx = \int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\, dx$$

$$\Rightarrow \quad I = \int \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx = \int \frac{\sin x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}\, dx$$

$$\Rightarrow \quad I = \int \tan x \sec x \; dx - \int \tan^2 x \; dx = \int \tan x \; \sec x \; dx - \int (\sec^2 x - 1) \; dx$$

$$\Rightarrow \quad I = \int \sec x \; \tan x \; dx - \int \sec^2 x\, dx + \int 1 \cdot dx = \sec x - \tan x + x + C$$

(iv) Let $I = \int \frac{\sec x}{\sec x + \tan x}\, dx$. Then,

$$I = \int \frac{\sec x\,(\sec x - \tan x)}{(\sec x + \tan x)\,(\sec x - \tan x)}\, dx = \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x}\, dx$$

$$\Rightarrow \quad I = \int \sec^2 x - \sec x \tan x\, dx = \int \sec^2 x \; dx - \int \sec x \; \tan x\, dx = \tan x - \sec x + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Evaluate:*

(i) $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$ [CBSE 2014] (ii) $\int \frac{1+\cos 4x}{\cot x - \tan x}\, dx$

(iii) $\int \frac{1}{\tan x + \cot x + \sec x + \operatorname{cosec} x}\, dx$

SOLUTION (i) Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x}\, dx$. Then,

$$I = \int \frac{(\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x\,(\sin^2 x + \cos^2 x)}{\sin^2 x \cos^2 x}\, dx \quad \left[\begin{array}{l}\text{Using}: a^3 + b^3 \\ = (a+b)^3 - 3ab\,(a+b)\end{array}\right]$$

$$\Rightarrow \quad I = \int \frac{1 - 3\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \left\{\frac{1}{\sin^2 x \cos^2 x} - 3\right\} dx$$

$$\Rightarrow \quad I = \int \left\{\frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} - 3\right\} dx = \int (\sec^2 x + \operatorname{cosec}^2 x - 3)\, dx = \tan x - \cot x - 3x + C$$

(ii) Let $I = \int \frac{1+\cos 4x}{\cot x - \tan x}\, dx$. Then,

$$I = \int \frac{2\cos^2 2x \cos x \sin x}{\cos^2 x - \sin^2 x}\, dx = \int \cos 2x \sin 2x \, dx = \frac{1}{2}\int \sin 4x \, dx = -\frac{1}{8}\cos 4x + C$$

(iii) Let $I = \int \frac{1}{\tan x + \cot x + \sec x + \operatorname{cosec} x} dx$. Then,

$$I = \int \frac{1}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\cos x} + \frac{1}{\sin x}} dx = \int \frac{\sin x \cos x}{\sin^2 x + \cos^2 x + \sin x + \cos x} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{2\sin x \cos x}{1 + \sin x + \cos x} dx = \frac{1}{2}\int \frac{(1 + 2\sin x \cos x) - 1}{1 + \sin x + \cos x} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{(\sin^2 x + \cos^2 x + 2\sin x \cos x) - 1}{1 + \sin x + \cos x} dx = \frac{1}{2}\int \frac{(\sin x + \cos x)^2 - 1^2}{\sin x + \cos x + 1} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{(\sin x + \cos x) + 1)(\sin x + \cos x - 1)}{\sin x + \cos x + 1} dx = \frac{1}{2}\int (\sin x + \cos x - 1)\, dx$$

$$\Rightarrow \quad I = \frac{1}{2}(-\cos x + \sin x - x) + C$$

EXAMPLE 8 *Evaluate:*

(i) $\int \sin^{-1}(\cos x)\, dx, \ 0 \le x \le \pi$ [NCERT] (ii) $\int \tan^{-1}\left\{\sqrt{\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)}\right\} dx, \ 0 < x < \pi/2$

(iii) $\int \tan^{-1}(\sec x + \tan x)\, dx, \ -\pi/2 < x < \pi/2$ [CBSE 2003]

SOLUTION (i) Let $I = \int \sin^{-1}(\cos x)\, dx$. Then,

$$I = \int \sin^{-1}\left\{\sin\left(\frac{\pi}{2} - x\right)\right\} dx = \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}\int 1 \cdot dx - \int x\, dx = \frac{\pi}{2}x - \frac{x^2}{2} + C$$

(ii) Let $I = \int \tan^{-1}\left\{\sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}\right\} dx$. Then,

$$I = \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx = \int \tan^{-1}(\tan x)\, dx = \int x\, dx = \frac{x^2}{2} + C$$

(iii) Let $I = \int \tan^{-1}(\sec x + \tan x)\, dx$. Then,

$$I = \int \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right) dx$$

$$\Rightarrow \quad I = \int \tan^{-1}\left\{\frac{1 - \cos\left(\frac{\pi}{2} + x\right)}{\sin\left(\frac{\pi}{2} + x\right)}\right\} dx = \int \tan^{-1}\left\{\frac{2\sin^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}\right\} dx$$

$$\Rightarrow \quad I = \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\} dx = \int \frac{\pi}{4} + \frac{x}{2}\, dx = \frac{\pi}{4}\int 1 \cdot dx + \frac{1}{2}\int x\, dx = \frac{\pi}{4}x + \frac{x^2}{4} + C$$

EXAMPLE 9 *Evaluate:* $\int \tan^{-1}\left\{\sqrt{\frac{1-\sin x}{1+\sin x}}\right\} dx, -\pi/2 < x < \pi/2$ **[CBSE 2003, 2006]**

SOLUTION Let $I = \int \tan^{-1}\left\{\sqrt{\frac{1-\sin x}{1+\sin x}}\right\} dx$. Then,

$$I = \int \tan^{-1}\left\{\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\right\} dx = \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\right\} dx$$

$$\Rightarrow \quad I = \int \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right\} = \int \left(\frac{\pi}{4}-\frac{x}{2}\right) dx = \frac{\pi}{4}x - \frac{x^2}{4} + C$$

Type IV INTEGRATION OF EXPONENTIAL FUNCTIONS

EXAMPLE 10 *Evaluate:*

(i) $\int e^{x\log a} + e^{a\log x} + e^{a\log a}\ dx$ (ii) $\int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right) dx$

SOLUTION (i) Let $I = \int e^{x\log a} + e^{a\log x} + e^{a\log a}\ dx$. Then,

$$I = \int e^{\log a^x} + e^{\log x^a} + e^{\log a^a}\ dx = \int (a^x + x^a + a^a)\ dx \qquad [\because e^{\log\lambda} = \lambda]$$

$$\Rightarrow \quad I = \int a^x\ dx + \int x^a\ dx + \int a^a\ dx = \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + a^a\ x + C$$

(ii) Let $I = \int \left(\frac{x}{m} + \frac{m}{x} + x^m + m^x\right) dx$. Then,

$$I = \frac{1}{m}\int x\,dx + m\int \frac{1}{x}\,dx + \int x^m\ dx + \int m^x\ dx = \frac{x^2}{2m} + m\log|x| + \frac{x^{m+1}}{m+1} + \frac{m^x}{\log m} + C$$

EXAMPLE 11 *Evaluate:* (i) $\int \frac{2^x + 3^x}{5^x}\,dx$ (ii) $\int \frac{(a^x + b^x)^2}{a^x\,b^x}\,dx$

SOLUTION (i) Let $I = \int \frac{2^x + 3^x}{5^x}\,dx$. Then,

$$I = \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x}\right) dx = \int \left\{\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x\right\} dx = \frac{(2/5)^x}{\log_e(2/5)} + \frac{(3/5)^x}{\log_e(3/5)} + C$$

(ii) Let $I = \int \frac{(a^x + b^x)^2}{a^x\,b^x}\,dx$. Then,

$$I = \int \frac{a^{2x} + b^{2x} + 2a^x\,b^x}{a^x\,b^x}\,dx$$

$$\Rightarrow \quad I = \int \frac{a^x}{b^x} + \frac{b^x}{a^x} + 2\,dx = \int \left\{\left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2\right\} dx = \frac{(a/b)^x}{\log_e(a/b)} + \frac{(b/a)^x}{\log_e(b/a)} + 2x + C$$

Type V MISCELLANEOUS PROBLEMS

EXAMPLE 12 *Evaluate:* (i) $\int (x^4 + x^2 + 1)\, d(x^2)$ (ii) $\int \sin(e^x)\, d(e^x)$

SOLUTION (i) Let $x^2 = t$. Then,

$$I = \int (x^4 + x^2 + 1)\, d(x^2) = \int (t^2 + t + 1)\, dt = \frac{t^3}{3} + \frac{t^2}{2} + t + C = \frac{x^6}{3} + \frac{x^4}{2} + x^2 + C \qquad [\because t = x^2]$$

(ii) Let $e^x = t$. Then, $I = \int \sin(e^x)\, d(e^x) = \int \sin t\, dt = -\cos t + C = -\cos(e^x) + C$

EXAMPLE 13 *If* $f'(x) = 3x^2 - \frac{2}{x^3}$ *and* $f(1) = 0$, *find* $f(x)$.

SOLUTION We know that $f(x) = \int f'(x)\, dx$

$$\therefore \quad f(x) = \int \left(3x^2 - \frac{2}{x^3}\right) dx = 3\left(\frac{x^3}{3}\right) - 2\left(\frac{x^{-2}}{-2}\right) + C$$

$$\Rightarrow \quad f(x) = x^3 + \frac{1}{x^2} + C \qquad \text{...(i)}$$

$$\Rightarrow \quad f(1) = 1 + 1 + C \qquad \text{[Replacing } x \text{ by 1]}$$

$$\Rightarrow \quad 0 = C + 2 \Rightarrow C = -2 \qquad [\because f(1) = 0]$$

$$\therefore \quad f(x) = x^3 + \frac{1}{x^2} - 2 \qquad \text{[Putting } C = -2 \text{ in (i)]}$$

EXERCISE 18.2

BASIC

Evaluate the following integrals (1-44):

1. $\int (3x\sqrt{x} + 4\sqrt{x} + 5)\, dx$
2. $\int \left(2^x + \frac{5}{x} - \frac{1}{x^{1/3}}\right) dx$
3. $\int \left\{\sqrt{x}\,(ax^2 + bx + c)\right\} dx$
4. $\int (2 - 3x)(3 + 2x)(1 - 2x)\, dx$
5. $\int \left(\frac{m}{x} + \frac{x}{m} + m^x + x^m + mx\right) dx$
6. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$ [NCERT]
7. $\int \frac{(1+x)^3}{\sqrt{x}}\, dx$
8. $\int \left\{x^2 + e^{\log x} + \left(\frac{e}{2}\right)^x\right\} dx$
9. $\int (x^e + e^x + e^e)\, dx$
10. $\int \sqrt{x}\left(x^3 - \frac{2}{x}\right) dx$
11. $\int \frac{1}{\sqrt{x}}\left(1 + \frac{1}{x}\right) dx$
12. $\int \frac{x^6 + 1}{x^2 + 1}\, dx$
13. $\int \frac{x^{-1/3} + \sqrt{x} + 2}{\sqrt[3]{x}}\, dx$
14. $\int \frac{(1 + \sqrt{x})^2}{\sqrt{x}}\, dx$
15. $\int \sqrt{x}\,(3 - 5x)\, dx$
16. $\int \frac{(x+1)(x-2)}{\sqrt{x}}\, dx$
17. $\int \frac{x^5 + x^{-2} + 2}{x^2}\, dx$
18. $\int (3x + 4)^2\, dx$
19. $\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x}\, dx$
20. $\int \frac{5x^4 + 12x^3 + 7x^2}{x^2 + x}\, dx$
21. $\int \frac{\sin^2 x}{1 + \cos x}\, dx$
22. $\int (\sec^2 x + \operatorname{cosec}^2 x)\, dx$
23. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x \cos^2 x}\, dx$ [NCERT]
24. $\int \frac{5\cos^3 x + 6\sin^3 x}{2\sin^2 x \cos^2 x}\, dx$
25. $\int \frac{1 - \cos 2x}{1 + \cos 2x}\, dx$

26. $\int \frac{1}{1-\cos x}\,dx$

27. $\int \frac{1}{1-\sin x}\,dx$

28. $\int \frac{1}{1+\cos 2x}\,dx$

29. $\int \frac{1}{1-\cos 2x}\,dx$

30. $\int \frac{x^3-3x^2+5x-7+x^2a^x}{2x^2}\,dx$

31. $\int \frac{1-\cos x}{1+\cos x}\,dx$ [NCERT]

32. $\int \left\{3\sin x-4\cos x+\frac{5}{\cos^2 x}-\frac{6}{\sin^2 x}+\tan^2 x-\cot^2 x\right\}dx$

33. If $f'(x)=x-\frac{1}{x^2}$ and $f(1)=\frac{1}{2}$, find $f(x)$.

34. If $f'(x)=x+b$, $f(1)=5$, $f(2)=13$, find $f(x)$.

35. If $f'(x)=8x^3-2x$, $f(2)=8$, find $f(x)$.

36. If $f'(x)=a\sin x+b\cos x$ and $f'(0)=4$, $f(0)=3$, $f\left(\frac{\pi}{2}\right)=5$, find $f(x)$.

37. Write the primitive or anti-derivative of $f(x)=\sqrt{x}+\frac{1}{\sqrt{x}}$.

BASED ON LOTS

38. $\int (\tan x+\cot x)^2\,dx$

39. $\int \frac{\cos x}{1-\cos x}\,dx$ or $\int \frac{\cot x}{\operatorname{cosec} x-\cot x}\,dx$

40. $\int \frac{\cos^2 x-\sin^2 x}{\sqrt{1+\cos 4x}}\,dx$

41. $\int \frac{\tan x}{\sec x+\tan x}\,dx$

42. $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x-\cot x}\,dx$

43. $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)dx$

44. $\int \cos^{-1}(\sin x)\,dx$

45. $\int \cot^{-1}\left(\frac{\sin 2x}{1-\cos 2x}\right)dx$

46. $\int \sin^{-1}\left(\frac{2\tan x}{1+\tan^2 x}\right)dx$

47. $\int \frac{(x^3+8)(x-1)}{x^2-2x+4}\,dx$

48. $\int (a\tan x+b\cot x)^2\,dx$

49. $\int \frac{\cos x}{1+\cos x}\,dx$ [NCERT]

ANSWERS

1. $\frac{6}{5}x^{5/2}+\frac{8}{3}x^{3/2}+5x+C$

2. $\frac{2^x}{\log 2}+5\log x-\frac{3}{2}x^{2/3}+C$

3. $\frac{2a}{7}x^{7/2}+\frac{2}{5}b\,x^{5/2}+\frac{2}{3}c\,x^{3/2}+C$

4. $3x^4+\frac{4}{3}x^3-\frac{17}{2}x^2+6x+C$

5. $m\log|x|+\frac{x^2}{2m}+\frac{m^x}{\log m}+\frac{x^{m+1}}{m+1}+\frac{mx^2}{2}+C$

6. $\frac{x^2}{2}-2x+\log|x|+C$

7. $2\sqrt{x}+2x^{3/2}+\frac{6}{5}x^{5/2}+\frac{2}{7}x^{7/2}+C$

8. $\frac{x^3}{3}+\frac{x^2}{2}+\frac{1}{\log\left(\frac{e}{2}\right)}\left(\frac{e}{2}\right)^x+C$

9. $\frac{x^{e+1}}{e+1}+e^x+e^e\,x+C$

10. $\frac{2}{9}x^{9/2}-4\sqrt{x}+C$

11. $2\sqrt{x}-\frac{2}{\sqrt{x}}+C$

12. $\frac{x^5}{5}-\frac{x^3}{3}+x+C$

13. $3x^{1/3}+\frac{6}{7}x^{7/6}+3x^{2/3}+C$

14. $2\sqrt{x}+2x+\frac{2}{3}x^{3/2}+C$

15. $2x^{3/2}-2x^{5/2}+C$

16. $\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}-4\sqrt{x}+C$

17. $\frac{x^4}{4}-\frac{x^{-3}}{3}-\frac{2}{x}+C$

18. $\frac{1}{9}(3x+4)^3+C$

19. $\frac{2}{3}x^3+\frac{3}{2}x^2+C$

20. $\frac{5}{3}x^3+\frac{7}{2}x^2+C$

21. $x-\sin x+C$

22. $\tan x-\cot x+C$

23. $\sec x+\operatorname{cosec} x+C$

24. $-\frac{5}{2}\operatorname{cosec} x+3\sec x+C$

25. $\tan x-x+C$

26. $-\cot x-\operatorname{cosec} x+C$

27. $\tan x+\sec x+C$

28. $\frac{1}{2}\tan x+C$

29. $-\frac{1}{2}\cot x+C$

30. $\frac{1}{2}\left\{\frac{x^2}{2}-3x+5\log x+\frac{7}{x}+\frac{a^x}{\log a}\right\}+C$

31. $2(\operatorname{cosec} x-\cot x)-x+C$

32. $-3\cos x-4\sin x+6\tan x+7\cot x+C$

33. $\frac{x^2}{2}+\frac{1}{x}-1$

34. $\frac{x^2}{2}+\frac{13}{2}x-2$

35. $2x^4-x^2-20$

36. $f(x)=2\cos x+4\sin x+1$

37. $\frac{2}{3}x^{3/2}+2x^{1/2}+C$

38. $\tan x-\cot x+C$

39. $-\operatorname{cosec} x-\cot x-x+C$

40. $\frac{x}{\sqrt{2}}+C$

41. $\sec x-\tan x+x+C$

42. $-\cot x-\operatorname{cosec} x+C$

43. $\frac{x^2}{2}+C$

44. $\frac{\pi}{2}x-\frac{x^2}{2}+C$

45. $\frac{x^2}{2}+C$

46. x^2+C

47. $\frac{x^3}{3}+\frac{x^2}{2}-2x+C$

48. $a^2\tan x-b^2\cot x-(a^2+b^2-2ab)x+C$

49. $-\operatorname{cosec} x+\cot x+x+C$

18.5 GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL

In order to understand the geometrical meaning of an indefinite integral, let us consider a function f given by $f(x) = -2x$.

Clearly,

$$\int f(x)\,dx = -x^2+C,\text{ where C is the constant of integration.}$$

Let us now consider the family of curves given by $y=\int f(x)\,dx$ or, $y=-x^2+C$.

Clearly, $y=-x^2+C$ represents a family of parabolas having their common axis of symmetry along y-axis as shown in Fig. 18.1.

In other words, each integral of $f(x)=-2x$ represents a parabola with its axis of symmetry along y-axis.

Let us now consider the points of intersection of each of these parabolas with a line parallel to y-axis (a line orthogonal to the axis representing the variable of integration). Let $x = a$ be a line parallel to y-axis which cuts the parabolas $y = -x^2$, $y = -x^2 + 1$, $y = -x^2 + 2$, $y = -x^2 + 3$ etc. respectively at points P_1, P_2, P_3, P_4 etc. At each of these points, we have

$$\frac{dy}{dx} = -2a$$

That is the slopes of the tangents to the parabolas at P_1, P_2, P_3, P_4 etc. are same. Consequently, tangents at P_1, P_2, P_3, P_4 etc. are parallel.

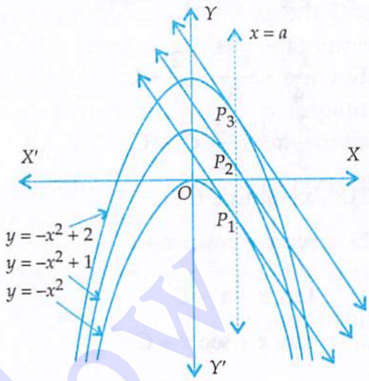

Fig. 18.1

Thus, the indefinite integral of a function may be interpreted geometrically as follows:

The indefinite integral of a function represents geometrically a family of curves having parallel tangents at their points of intersection with the lines orthogonal to the axis representing the variable of integration.

18.6 COMPARISON BETWEEN DIFFERENTIATION AND INTEGRATION

Following points will help us to understand the difference between the differentiation and integration.

(i) The operations of differentiation and integration are defined on functions.

(ii) The derivative of a function, when it exists is a unique function whereas the integral of a function is not unique. In fact there are infinitely many integrals of a function such that any two integrals differ by a constant.

(iii) The derivative of a function at a point (if it exists) is meaningful but the integral of a function at a point does not have any sense.

(iv) Every function is not differentiable. Similarly, every function is not integrable.

(v) The derivative of a function at a point determines, the slope of the tangent to the corresponding curve at that point. The integral of a function represents a family of curves having parallel tangents at the points of intersection of the curves of the family with the lines orthogonal to the axis representing the variable of integration.

(vi) The processes of differentiation and integration are inverse of each other.

(vii) The operations of differentiation and integration are operations on functions.

(viii) Both the operations are linear.

i.e. $$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

and, $$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx$$

(ix) The constant can be taken out side the differential as well as integral sign.

i.e. $\dfrac{d}{dx}(k\, f(x)) = k \dfrac{d}{dx}(f(x))$ and, $\int k\, f(x)\, dx = k \int f(x)\, dx$

(x) Differentiation and integration both are processes involving limits.

18.7 METHODS OF INTEGRATION

We have the following methods of integration:

(i) Integration by substitution (ii) Integration by parts

(iii) Integration of rational algebraic functions by using partial fractions.

We shall now discuss these methods in the sections to follow:

18.8 INTEGRATION BY SUBSTITUTION

In section 18.4, we have considered the problems on integration of functions in standard forms and the problems involving combinations of these functions. Integrals of certain functions cannot be obtained directly if they are not in one of the standard forms given in section 18.4, but they may be reduced to standard forms by proper substitution. The method of evaluating an integral by reducing it to standard form by a proper substitution is called *integration by substitution*.

If $\phi(x)$ is a continuously differentiable function, then to evaluate integrals of the form

$\int f(\phi(x))\ \phi'(x)\, dx$, we substitute $\phi(x) = t$ and $\phi'(x)\, dx = dt$.

This substitution reduces the above integral to $\int f(t)\, dt$. After evaluating this integral we substitute back the value of t.

18.8.1 INTEGRALS OF THE FORM $\int f(ax+b)\, dx$

THEOREM 1 *If* $\int f(x)\, dx = \phi(x)$, *then* $\int f(ax+b)\, dx = \frac{1}{a}\phi(ax+b)$

PROOF Let $I = \int f(ax+b)\, dx$ and let $ax+b = t$. Then, $d(ax+b) = dt \Rightarrow a\, dx = dt \Rightarrow dx = \frac{1}{a} dt$

Substituting $ax+b = t$ and $dx = \frac{1}{a} dt$, we get

$$I = \int f(ax+b)\, dx = \frac{1}{a}\int f(t)\, dt = \frac{1}{a}\phi(t) = \frac{1}{a}\phi(ax+b)$$

Q.E.D.

THEOREM 2 *Prove that* $\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C,\ n \neq -1$

PROOF Let $ax+b = t$. Then, $d(ax+b) = dt \Rightarrow a\, dx = dt \Rightarrow dx = \frac{1}{a} dt$.

Putting $ax+b = t$ and $dx = \frac{1}{a} dt$, we get

$$\int (ax+b)^n\, dx = \frac{1}{a}\int t^n\, dt = \frac{1}{a} \times \frac{t^{n+1}}{n+1} + C = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

Q.E.D.

THEOREM 3 *Prove that* $\int \frac{1}{ax+b}\, dx = \frac{1}{a}\ln|ax+b| + C$

PROOF Let $ax+b = t$. Then, $d(ax+b) = dt \Rightarrow a\, dx = dt \Rightarrow dx = \frac{1}{a} dt$. Putting $ax+b = t$, and $dx = \frac{1}{a} dt$, we get

$$\int \frac{1}{ax+b}\, dx = \frac{1}{a}\int \frac{1}{t}\, dt = \frac{1}{a}\ln|t| + C = \frac{1}{a}\ln|ax+b| + C$$

Q.E.D.

THEOREM 4 *Prove that* $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b) + C$.

PROOF Let $ax+b = t$. Then, $d(ax+b) = dt \Rightarrow a\, dx = dt \Rightarrow dx = \frac{1}{a} dt$. Putting $ax+b = t$ and $dx = \frac{1}{a} dt$, we get

$$\int \sin(ax+b)\,dx = \frac{1}{a}\int \sin t\,dt = -\frac{1}{a}\cos t + C = -\frac{1}{a}\cos(ax+b) + C$$

Q.E.D.

On comparing these three results with

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1,\ \int \frac{1}{x}\,dx = \log|x| + C \text{ and, } \int \sin x\,dx = -\cos x + C$$

respectively we find that if x is replaced by $ax+b$, then the same formula is applicable but we must divide by the coefficient of x or derivative of $(ax+b)$ with respect to x i.e., a.

Thus, in any of the fundamental integration formulas given in section 18.3 if in place of x we have $ax+b$, then the same formula is applicable but we must divide by coefficient of x or derivative of $(ax+b)$ i.e. a.

We give below a list of some of them which are frequently used.

(i) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$ (ii) $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$

(iii) $\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$ (iv) $\int a^{bx+c}\,dx = \frac{1}{b}\cdot\frac{a^{bx+c}}{\ln a} + C, a > 0$ and $a \neq 1$

(v) $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$ (vi) $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$

(vii) $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$ (viii) $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$

(ix) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b) + C$

(x) $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + C$

(xi) $\int \tan(ax+b)\,dx = -\frac{1}{a}\ln|\cos(ax+b)| + C$

(xii) $\int \cot(ax+b)\,dx = \frac{1}{a}\ln|\sin(ax+b)| + C$

(xiii) $\int \sec(ax+b)\,dx = \frac{1}{a}\ln|\sec(ax+b) + \tan(ax+b)| + C$

(xiv) $\int \operatorname{cosec}(ax+b)\,dx = \frac{1}{a}\ln|\operatorname{cosec}(ax+b) - \cot(ax+b)| + C.$

Following examples illustrate the applications of these formulae to evaluate integrals.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **EVALUATION OF INTEGRALS DIRECTLY BASED UPON ABOVE GIVEN FORMULAE**

EXAMPLE 1 *Evaluate:*

(i) $\int \left\{(2x-3)^5 + \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} + \frac{1}{2-3x} + \sqrt{3x+2}\right\} dx$

(ii) $\int e^{2x-3}\,dx$ [NCERT] (iii) $\int a^{3x+2}\,dx$

SOLUTION (i) Let $I = \int \left\{(2x-3)^5 + \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}} + \frac{1}{2-3x} + \sqrt{3x+2}\right\} dx$. Then,

$$I = \int (2x-3)^5\,dx + \int (7x-5)^{-3}\,dx + \int (5x-4)^{-1/2}\,dx + \int \frac{1}{2-3x}\,dx + \int \sqrt{3x+2}\,dx$$

$$\Rightarrow \quad I = \frac{(2x-3)^6}{2\times 6} + \frac{(7x-5)^{-2}}{7\times -2} + \frac{(5x-4)^{1/2}}{5\times \frac{1}{2}} + \left(\frac{1}{-3}\right)\log|2-3x| + \frac{(3x+2)^{3/2}}{3\times\frac{3}{2}} + C$$

$$\Rightarrow \quad I = \frac{1}{12}(2x-3)^6 - \frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4} - \frac{1}{3}\log|2-3x| + \frac{2}{9}(3x+2)^{3/2} + C$$

(ii) $\int e^{2x-3}\,dx = \frac{1}{2}\times e^{2x-3} + C$

(iii) $\int a^{3x+2}\,dx = \frac{1}{3\log a}\times a^{3x+2} + C$

EXAMPLE 2 *Evaluate:*

(i) $\int \sec^2(7-4x)\,dx$ **[NCERT, CBSE 2009]**

(ii) $\int \frac{1}{\sin^2 x\cos^2 x}\,dx$

(iii) $\int \operatorname{cosec}^2(3x+2)\,dx$ **[NCERT]**

(iv) $\int \sin(ax+b)\cos(ax+b)\,dx$ **[NCERT]**

(v) $\int \frac{\sin 4x}{\sin 2x}\,dx$

(vi) $\int \frac{\sin 4x}{\cos 2x}\,dx$

SOLUTION (i) $\int \sec^2(7-4x)\,dx = -\frac{1}{4}\tan(7-4x) + C$

(ii) Let $I = \int \frac{1}{\sin^2 x\cos^2 x}\,dx$. Then,

$$I = \int \frac{4}{(2\sin x\cos x)^2}\,dx$$

$$\Rightarrow \quad I = 4\int \frac{1}{\sin^2 2x}\,dx = 4\int \operatorname{cosec}^2 2x\,dx = -\frac{4}{2}\cot 2x + C = -2\cot 2x + C$$

(iii) $\int \operatorname{cosec}^2(3x+2)\,dx = -\frac{1}{3}\cot(3x+2) + C$

(iv) Let $I = \int \sin(ax+b)\cos(ax+b)\,dx$. Then,

$$I = \frac{1}{2}\int 2\sin(ax+b)\cos(ax+b)\,dx = \frac{1}{2}\int \sin 2(ax+b)\,dx = -\frac{1}{4a}\cos(2ax+2b) + C$$

(v) Let $I = \int \frac{\sin 4x}{\sin 2x}\,dx$. Then,

$$I = \int \frac{2\sin 2x\cos 2x}{\sin 2x}\,dx = 2\int \cos 2x\,dx = \frac{2}{2}\sin 2x + C = \sin 2x + C$$

(vi) Let $I = \int \frac{\sin 4x}{\cos 2x}\,dx$. Then,

$$I = \int \frac{2\sin 2x\cos 2x}{\cos 2x}\,dx = 2\int \sin 2x\,dx = -\cos 2x + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:* $\int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$ **[NCERT]**

SOLUTION Let $I = \int \frac{\sin^8 x - \cos^8 x}{1-2\sin^2 x\cos^2 x}\,dx$. Then,

$$I = \int \frac{(\sin^4 x + \cos^4 x)(\sin^4 x - \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x}\, dx$$

$$\Rightarrow I = \int \frac{(\sin^4 x + \cos^4 x)(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)}{\sin^4 x + \cos^4 x}\, dx = -\int \cos 2x\, dx = -\frac{1}{2}\sin 2x + C$$

EXAMPLE 4 *Evaluate:* $\int \sqrt{1 \mp \sin x}\, dx,\ 0 < x < \pi/2$

SOLUTION Let $I = \int \sqrt{1 \mp \sin x}\, dx$. Then,

$$I = \int \sqrt{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} \mp 2\sin\frac{x}{2}\cos\frac{x}{2}}\, dx$$

$$\Rightarrow \quad I = \int \sqrt{\left(\cos\frac{x}{2} \mp \sin\frac{x}{2}\right)^2}\, dx = \int \left(\cos\frac{x}{2} \mp \sin\frac{x}{2}\right) dx = \int \cos\frac{x}{2}\, dx \mp \int \sin\frac{x}{2}\, dx$$

$$\Rightarrow \quad I = 2\sin\frac{x}{2} \pm 2\cos\frac{x}{2} + C = 2\left(\sin\frac{x}{2} \pm \cos\frac{x}{2}\right) + C$$

EXAMPLE 5 *Evaluate:*

(i) $\int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}\, dx$ (ii) $\int \frac{1}{\sqrt{1-2x} + \sqrt{3-2x}}\, dx$

SOLUTION (i) Let $I = \int \frac{1}{\sqrt{3x+4} - \sqrt{3x+1}}\, dx$. Then,

$$I = \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{\left(\sqrt{3x+4} + \sqrt{3x+1}\right)\left(\sqrt{3x+4} - \sqrt{3x+1}\right)}\, dx = \int \frac{\sqrt{3x+4} + \sqrt{3x+1}}{(3x+4) - (3x+1)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{3}\int \{\sqrt{3x+4} + \sqrt{3x+1}\}\, dx = \frac{1}{3}\int \sqrt{3x+4}\, dx + \frac{1}{3}\int \sqrt{3x+1}\, dx$$

$$\Rightarrow \quad I = \frac{1}{3}\left\{\frac{(3x+4)^{3/2}}{3 \times \frac{3}{2}}\right\} + \frac{1}{3}\left\{\frac{(3x+1)^{3/2}}{3 \times \frac{3}{2}}\right\} + C = \frac{2}{27}\{(3x+4)^{3/2} + (3x+1)^{3/2}\} + C$$

(ii) Let $I = \int \frac{1}{\sqrt{1-2x} + \sqrt{3-2x}}\, dx$. Then,

$$I = \int \frac{\sqrt{1-2x} - \sqrt{3-2x}}{(\sqrt{1-2x} + \sqrt{3-2x})(\sqrt{1-2x} - \sqrt{3-2x})}\, dx = \int \frac{\sqrt{1-2x} - \sqrt{3-2x}}{(1-2x) - (3-2x)}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\int \sqrt{1-2x}\, dx + \frac{1}{2}\int \sqrt{3-2x}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{2}\left\{\frac{(1-2x)^{3/2}}{-2 \times \frac{3}{2}}\right\} + \frac{1}{2}\left\{\frac{(3-2x)^{3/2}}{-2 \times \frac{3}{2}}\right\} + C = \frac{1}{6}(1-2x)^{3/2} - \frac{1}{6}(3-2x)^{3/2} + C$$

EXAMPLE 6 *Evaluate:* $\int \frac{8^{1+x} + 4^{1-x}}{2^x}\, dx$

SOLUTION Let $I = \int \frac{8^{1+x} + 4^{1-x}}{2^x}\, dx$. Then,

$$I = \int \frac{2^{3x+3} + 2^{2-2x}}{2^x}\, dx$$

$$\Rightarrow \quad I = \int 2^{2x+3} + 2^{2-3x}\, dx = \frac{2^{2x+3}}{2 \ln 2} + \frac{2^{2-3x}}{(-3) \ln 2} + C = \frac{2^{2x+2}}{\ln 2} - \frac{2^{2-3x}}{3 \ln 2} + C$$

EXERCISE 18.3

BASIC

1. $\int (2x-3)^5 + \sqrt{3x+2}\, dx$
2. $\int \frac{1}{(7x-5)^3} + \frac{1}{\sqrt{5x-4}}\, dx$
3. $\int \frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}}\, dx$
4. $\int \sin x \sqrt{1+\cos 2x}\, dx$
5. $\int \frac{1+\cos x}{1-\cos x}\, dx$ [CBSE 2000]
6. $\int \frac{1-\cos x}{1+\cos x}\, dx$
7. $\int \frac{1}{1+\cos 3x}\, dx$
8. $\int \tan^2 (2x-3)\, dx$ [NCERT]

BASED ON LOTS

9. $\int \frac{x+3}{(x+1)^4}\, dx$
10. $\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\, dx$
11. $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-3}}\, dx$
12. $\int \frac{2x}{(2x+1)^2}\, dx$
13. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}\, dx$
14. $\int \frac{1}{1-\sin \frac{x}{2}}\, dx$
15. $\int (e^x+1)^2 e^x\, dx$
16. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$
17. $\int \frac{1+\cos 4x}{\cot x - \tan x}\, dx$
18. $\int \frac{1}{\sqrt{x+3}-\sqrt{x+2}}\, dx$ [CBSE 2002]
19. $\int \frac{1}{\cos^2 x (1-\tan x)^2}\, dx$ [NCERT]

ANSWERS

1. $\frac{(2x-3)^6}{12} + \frac{2}{9}(3x+2)^{3/2} + C$
2. $-\frac{1}{14}(7x-5)^{-2} + \frac{2}{5}\sqrt{5x-4} + C$
3. $-\frac{1}{3}\log|2-3x| + \frac{2}{3}\sqrt{3x-2} + C$
4. $-\frac{1}{2\sqrt{2}} \cos 2x + C$
5. $-2\cot\left(\frac{x}{2}\right) - x + C$
6. $2\tan\left(\frac{x}{2}\right) - x + C$
7. $\frac{1-\cos 3x}{3\sin 3x} + C$
8. $\frac{1}{2}\tan(2x-3) - x + C$
9. $-\frac{1}{2(x+1)^2} - \frac{2}{3(x+1)^3} + C$
10. $\frac{2}{3}\left\{(x+1)^{3/2} - x^{3/2}\right\} + C$
11. $\frac{1}{18}\left\{(2x+3)^{3/2} - (2x-3)^{3/2}\right\} + C$
12. $\frac{1}{2}\log|2x+1| + \frac{1}{2(2x+1)} + C$

13. $\dfrac{2}{3(a-b)}\left\{(x+a)^{3/2}-(x+b)^{3/2}\right\}+C$

14. $2\left(\tan\dfrac{x}{2}+\sec\dfrac{x}{2}\right)+C$

15. $\dfrac{1}{3}(e^x+1)^3+C$

16. $\dfrac{1}{2}e^{2x}+2x-\dfrac{1}{2}e^{-2x}+C$

17. $-\dfrac{1}{8}\cos 4x+C$

18. $\dfrac{2}{3}\left\{(x+3)^{3/2}+(x+2)^{3/2}\right\}+C$

19. $\dfrac{1}{2}\tan\left(\dfrac{\pi}{4}+x\right)+C$

18.8.2 EVALUATION OF INTEGRALS OF THE FORM $\dfrac{P(x)}{(ax+b)^n}$, $n \in N$, WHERE $p(x)$ IS A POLYNOMIAL

In order to evaluate this type of Integrals, we may follow the following algorithm.

Step I *Check whether degree of $P(x) \geq$ or $< n$.*

Step II *If degree of $P(x) < n$, express $P(x)$ in the form*

$$A_0 + A_1(ax+b) + A_2(ax+b)^2 + \ldots + A_{n-1}(ax+b)^{n-1}$$

Step III *Write* $\dfrac{P(x)}{(ax+b)^n}$ *as* $\dfrac{A_0}{(ax+b)^n}+\dfrac{A_1}{(ax+b)^{n-1}}+\dfrac{A_2}{(ax+b)^{n-2}}+\ldots+\dfrac{A_{n-1}}{ax+b}$

Step IV *Evaluate*

$$\int \frac{P(x)}{(ax+b)^n}\,dx = A_0\int\frac{1}{(ax+b)^n}\,dx + A_1\int\frac{1}{(ax+b)^{n-1}}\,dx+\ldots+A_{n-1}\int\frac{1}{ax+b}\,dx$$

Step V *If degree of $P(x) \geq n$, then divide $P(x)$ by $(ax+b)^n$ and express* $\dfrac{P(x)}{(ax+b)^n}$ *as*

$Q(x)+\dfrac{R(x)}{(ax+b)^n}$, *where degree of $R(x)$ is less than n.*

Step VI *Use step II and III to evaluate* $\displaystyle\int\frac{R(x)}{(ax+b)^n}\,dx$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 1 *Evaluate:*

(i) $\displaystyle\int\frac{x^3}{(x+2)^4}\,dx$ (ii) $\displaystyle\int\left(\frac{x-1}{x+1}\right)^4 dx$

SOLUTION (i) Let $I = \displaystyle\int\frac{x^3}{(x+2)^4}\,dx$. Then,

$$I=\int\frac{\{(x+2)-2\}^3}{(x+2)^4}\,dx=\int\frac{(x+2)^3-6(x+2)^2+12(x+2)-8}{(x+2)^4}\,dx$$

$$\Rightarrow I=\int\left\{\frac{1}{x+2}-\frac{6}{(x+2)^2}+\frac{12}{(x+2)^3}-\frac{8}{(x+2)^4}\right\}dx=\log|x+2|+\frac{6}{x+2}-\frac{6}{(x+2)^2}+\frac{8}{3(x+2)^3}+C$$

(ii) Let $I = \int \left(\frac{x-1}{x+1}\right)^4 dx$. Then,

$$I = \int \frac{\{(x+1)-2\}^4}{(x+1)^4}\, dx$$

$$\Rightarrow \quad I = \int \frac{(x+1)^4 - {}^4C_1 (x+1)^3 \times 2 + {}^4C_2 (x+1)^2 \times 2^2 - {}^4C_3 (x+1) \times 2^3 + {}^4C_4 (2)^4}{(x+1)^4}\, dx$$

$$\Rightarrow \quad I = \int \frac{(x+1)^4 - 8(x+1)^3 + 24(x+1)^2 - 32(x+1) + 16}{(x+1)^4}\, dx$$

$$\Rightarrow \quad I = \int \left\{1 - \frac{8}{x+1} + \frac{24}{(x+1)^2} - \frac{32}{(x+1)^3} + \frac{16}{(x+1)^4}\right\} dx$$

$$\Rightarrow \quad I = x - 8\log|x+1| - \frac{24}{x+1} + \frac{16}{(x+1)^2} - \frac{16}{3(x+1)^3} + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{ax+b}{(cx+d)^2}\, dx$ (ii) $\int \frac{x+2}{(x+1)^2}\, dx$ (iii) $\int \frac{2+x+x^2}{x^2(2+x)} + \frac{2x-1}{(x+1)^2}\, dx$

SOLUTION (i) Let $I = \int \frac{ax+b}{(cx+d)^2}\, dx$

Let $ax + b = \lambda(cx+d) + \mu$. On equating coefficients of like powers of x, we get

$$a = \lambda c \text{ and } b = \lambda d + \mu \Rightarrow \lambda = \frac{a}{c} \text{ and } \mu = \frac{bc-ad}{c}$$

$$\therefore \quad I = \int \frac{ax+b}{(cd+d)^2}\, dx = \int \frac{\lambda(cx+d)+\mu}{(cx+d)^2}\, dx = \lambda \int \frac{1}{cx+d}\, dx + \mu \int \frac{1}{(cx+d)^2}\, dx$$

$$\Rightarrow \quad I = \frac{\lambda}{c}\log|cx+d| - \frac{\mu}{c(cx+d)} + C = \frac{a}{c^2}\log|cx+d| - \frac{(bc-ad)}{c^2} \times \frac{1}{cx+d} + C$$

ALITER Let

$$I = \int \frac{ax+b}{(cx+d)^2}\, dx. \text{ Then,}$$

$$I = a\int \frac{x + \frac{b}{a}}{(cx+d)^2}\, dx \qquad \text{[Making coefficient of } x \text{ unity in the numerator]}$$

$$\Rightarrow \quad I = \frac{a}{c}\int \frac{cx + \frac{bc}{a}}{(cx+d)^2}\, dx \qquad \text{[Making } c \text{ as the coefficient of } x \text{ in the numerator]}$$

$$\Rightarrow \quad I = \frac{a}{c}\int \frac{(cx+d) + \frac{bc}{a} - d}{(cx+d)^2}\, dx \qquad \text{[Adding and subtracting } d \text{ in the numerator]}$$

$$\Rightarrow \quad I = \frac{a}{c}\int \frac{1}{(cx+d)}\, dx + \frac{(bc-ad)}{c}\int \frac{1}{(cx+d)^2}\, dx \qquad \text{[Separating the integrals]}$$

$$\Rightarrow \quad I = \frac{a}{c^2}\log|cx+d| - \frac{(bc-ad)}{c^2} \times \frac{1}{(cx+d)} + C$$

(ii) Let $x + 2 = \lambda(x+1) + \mu$. On equating the coefficients of like powers of x on both sides, we get

$\lambda = 1$ and $2 = \lambda + \mu \Rightarrow \lambda = 1, \mu = 1$

$\therefore \quad I = \int \frac{x+2}{(x+1)^2}\, dx = \int \frac{\lambda(x+1)+\mu}{(x+1)^2}\, dx = \int \left\{ \frac{\lambda}{x+1} + \frac{\mu}{(x+1)^2} \right\} dx$

$\Rightarrow \quad I = \lambda \int \frac{1}{x+1}\, dx + \mu \int \frac{1}{(x+1)^2}\, dx = \lambda \ln|x+1| - \frac{\mu}{x+1} + C = \ln|x+1| - \frac{1}{x+1} + C$

ALITER $I = \int \frac{x+2}{(x+1)^2}\, dx = \int \frac{(x+1)+1}{(x+1)^2}\, dx = \int \left\{ \frac{1}{x+1} + \frac{1}{(x+1)^2} \right\} dx = \ln|x+1| - \frac{1}{x+1} + C$

(iii) Let $I = \int \left\{ \frac{2+x+x^2}{x^2(2+x)} + \frac{2x-1}{(x+1)^2} \right\} dx$. Then,

$I = \int \left\{ \frac{(2+x)+x^2}{x^2(2+x)} + \frac{2(x+1)-3}{(x+1)^2} \right\} dx = \int \left\{ \frac{1}{x^2} + \frac{1}{2+x} + \frac{2}{x+1} - \frac{3}{(x+1)^2} \right\} dx$

$\Rightarrow \quad I = -\frac{1}{x} + \ln|2+x| + 2\ln|x+1| + \frac{3}{x+1} + C$

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{x^3}{(x+1)^2}\, dx$ (ii) $\int \frac{x^2}{(a+bx)^2}\, dx$ (iii) $\int \frac{x^2+1}{(x+1)^2}\, dx$ **[CBSE 2006]**

SOLUTION (i) Let $I = \int \frac{x^3}{(x+1)^2}\, dx$. Using long division method, we obtain

$$\frac{x^3}{(x+1)^2} = x - 2 + \frac{3x+2}{(x+1)^2} = x - 2 + \frac{3(x+1)-3+2}{(x+1)^2} = x - 2 + \frac{3}{x+1} - \frac{1}{(x+1)^2}$$

$\therefore \quad I = \int \frac{x^3}{(x+1)^2}\, dx = \int \left\{ x - 2 + \frac{3}{x+1} - \frac{1}{(x+1)^2} \right\} dx = \frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1} + C$

(ii) Let $I = \int \frac{x^2}{(a+bx)^2}\, dx$. Using Long division method, we get

$$\frac{x^2}{(a+bx)^2} = \frac{1}{b^2} + \frac{-\frac{2a}{b}x - \frac{a^2}{b^2}}{(bx+a)^2} = \frac{1}{b^2} - \frac{a}{b^2}\,\frac{(2bx+a)}{(bx+a)^2}$$

$\Rightarrow \quad \frac{x^2}{(a+bx)^2} = \frac{1}{b^2} - \frac{a}{b^2}\left\{ \frac{2(bx+a)-a}{(bx+a)^2} \right\} = \frac{1}{b^2} - \frac{2a}{b^2} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2}$

$\therefore \quad I = \int \frac{x^2}{(a+bx)^2}\, dx = \int \left\{ \frac{1}{b^2} - \frac{2a}{b^2} \times \frac{1}{bx+a} + \frac{a^2}{b^2} \times \frac{1}{(bx+a)^2} \right\} dx$

$\Rightarrow \quad I = \frac{1}{b^2} \int 1 \cdot dx - \frac{2a}{b^2} \int \frac{1}{bx+a}\, dx + \frac{a^2}{b^2} \int \frac{1}{(bx+a)^2}\, dx$

$\Rightarrow \quad I = \frac{x}{b^2} - \frac{2a}{b^3} \ln|bx+a| - \frac{a^2}{b^3} \times \frac{1}{bx+a} + C = \frac{1}{b^3}\left\{ bx - 2a\ln|bx+a| - \frac{a^2}{bx+a} \right\} + C$

ALITER We have,

$$I = \int \frac{x^2}{(a+bx)^2}\,dx = \frac{1}{b^2}\int \frac{b^2 x^2}{(a+bx)^2}\,dx = \frac{1}{b^2}\int \frac{(b^2x^2+2abx+a^2)-(2abx+a^2)}{(bx+a)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{b^2}\int \frac{(bx+a)^2-\{2a(bx+a)-2a^2+a^2\}}{(bx+a)^2}\,dx = \frac{1}{b^2}\int \frac{(bx+a)^2-2a(bx+a)+a^2}{(bx+a)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{b^2}\int \left\{1-\frac{2a}{bx+a}+\frac{a^2}{(bx+a)^2}\right\}dx = \frac{1}{b^2}\left\{x-\frac{2a}{b}\ln|bx+a|-\frac{a^2}{b(bx+a)}\right\}+C$$

$$\Rightarrow \quad I = \frac{1}{b^3}\left\{bx-2a\ln|bx+a|-\frac{a^2}{bx+a}\right\}+C$$

(iii) Let $I = \int \frac{x^2+1}{(x+1)^2}\,dx$. Then,

$$I = \int \frac{x^2+1+2x-2x}{(x+1)^2}\,dx = \int \frac{(x+1)^2-2x}{(x+1)^2}\,dx = \int 1-\frac{2x}{(x+1)^2}\,dx$$

$$\Rightarrow \quad I = \int 1\cdot dx - 2\int \frac{x}{(x+1)^2}\,dx = \int 1\cdot dx - 2\int \frac{(x+1)-1}{(x+1)^2}\,dx$$

$$\Rightarrow \quad I = \int 1\cdot dx - 2\int \left\{\frac{1}{x+1}-\frac{1}{(x+1)^2}\right\}dx$$

$$\Rightarrow \quad I = \int 1\cdot dx - 2\int \frac{1}{x+1}\,dx + 2\int \frac{1}{(x+1)^2}\,dx = x-2\ln|x+1|-\frac{2}{x+1}+C$$

EXAMPLE 4 *Evaluate:*

(i) $\int \frac{x^3}{x+2}\,dx$ (ii) $\int \frac{x^7}{x+1}\,dx$ (iii) $\int \frac{x^6}{x-1}\,dx$

SOLUTION (i) Using long division method, we obtain

$$\frac{x^3}{x+2} = x^2-2x+4-\frac{8}{x+2}$$

$$\Rightarrow \quad \int \frac{x^3}{x+2}\,dx = \int \left\{x^2-2x+4-\frac{8}{x+2}\right\}dx = \frac{x^3}{3}-x^2+4x-8\ln|x+2|+C$$

ALITER Let $I = \int \frac{x^3}{x+2}\,dx$. Then,

$$I = \int \frac{(x^3+2^3)-2^3}{x+2}\,dx = \int \left\{\frac{(x+2)(x^2-2x+4)}{x+2}-\frac{8}{x+2}\right\}dx$$

$$\Rightarrow \quad I = \int \left\{x^2-2x+4-\frac{8}{x+2}\right\}dx = \frac{x^3}{3}-x^2+4x-8\ln|x+2|+C$$

(ii) Using long division method, we have

$$\frac{x^7}{x+1} = x^6-x^5+x^4-x^3+x^2-x+1-\frac{1}{x+1}$$

$$\therefore \quad \int \frac{x^7}{x+1}\, dx = \int \left\{ x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 - \frac{1}{x+1} \right\} dx$$

$$\Rightarrow \quad \int \frac{x^7}{x+1}\, dx = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

ALITER Let $I = \int \frac{x^7}{x+1}\, dx$. Then,

$$I = \int \frac{(x^7+1)-1}{x+1}\, dx = \int \frac{x^7+1}{x+1}\, dx - \int \frac{1}{x+1}\, dx = \int \frac{x^7-(-1)^7}{x-(-1)}\, dx - \int \frac{1}{x+1}\, dx$$

$$\Rightarrow \quad I = \int (x^6 - x^5 + x^4 - x^3 + x^2 - x + 1)\, dx - \int \frac{1}{x+1}\, dx \quad \left[\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \ldots + x a^{n-2} + a^{n-1} \right]$$

$$\Rightarrow \quad I = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \log|x+1| + C$$

(iii) Let $I = \int \frac{x^6}{x-1}\, dx = \int \frac{x^6 - 1^6 + 1^6}{x-1}\, dx = \int \frac{x^6-1^6}{x-1} + \frac{1}{x-1}\, dx$

$$\Rightarrow \quad I = \int \left\{ x^5 + x^4 + x^3 + x^2 + x + 1 + \frac{1}{x-1} \right\} dx \quad \left[\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2} a + x^{n-3} a^2 + \ldots + x a^{n-2} + a^{n-1} \right]$$

$$\Rightarrow \quad I = \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \log|x-1| + C$$

EXERCISE 18.4

BASED ON HOTS

1. $\int \frac{x^2+5x+2}{x+2}\, dx$
2. $\int \frac{x^3}{x-2}\, dx$
3. $\int \frac{x^2+x+5}{3x+2}\, dx$
4. $\int \frac{2x+3}{(x-1)^2}\, dx$
5. $\int \frac{x^2+3x-1}{(x+1)^2}\, dx$
6. $\int \frac{2x-1}{(x-1)^2}\, dx$

ANSWERS

1. $\frac{x^2}{2} + 3x - 4\log|x+2| + C$
2. $\frac{x^3}{3} + x^2 + 4x + 8\log|x-2| + C$
3. $\frac{x^2}{6} + \frac{1}{9}x + \frac{43}{27}\log|3x+2| + C$
4. $2\log|x-1| - \frac{5}{x-1} + C$
5. $x + \log|x+1| + \frac{3}{x+1} + C$
6. $-\frac{1}{x-1} + 2\log|x-1| + C$

18.8.3 EVALUATION OF INTEGRALS OF THE FORM $\int (ax+b)\sqrt{cx+d}\, dx$ AND $\int \frac{ax+b}{\sqrt{cx+d}}\, dx$

In order to evaluate this type of integrals, we may use the following algorithm:

ALGORITHM

Step I *Write $(ax+b)$ in terms of $(cx+d)$ as follows:* $(ax+b) = \lambda(cx+d)+\mu$

Step II *Find λ and μ by equating coefficients of like powers of x on both sides.*

Step III *Replace $ax+b$ by $\lambda(cx+d)+\mu$ in the given integral to obtain*

$$\int (ax+b)\sqrt{cx+d}\;dx = \int \left\{\lambda(cx+d)+\mu\right\}\sqrt{cx+d}\;dx$$
$$= \lambda\int (cx+d)^{3/2}\,dx + \mu\int\sqrt{cx+d}\;dx$$
$$= \frac{2\lambda}{5c}(cx+d)^{5/2} + \frac{2\mu}{3c}(cx+d)^{3/2} + C$$
$$\int \frac{ax+b}{\sqrt{cx+d}}\,dx = \int \frac{\lambda(cx+d)+\mu}{\sqrt{cx+d}}\,dx = \lambda\int\sqrt{cx+d}\;dx + \mu\int\frac{1}{\sqrt{cx+d}}\,dx$$
$$= \frac{2\lambda}{3c}(cx+d)^{3/2} + \frac{2\mu}{c}(cx+d)^{1/2} + C$$

Following examples will illustrate the above procedure:

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:* (i) $\int x\sqrt{x+2}\;dx$ (ii) $\int (7x-2)\sqrt{3x+2}\;dx$

SOLUTION (i) Let $I = \int x\sqrt{x+2}\;dx$. Then,

$$I = \int \{(x+2)-2\}\sqrt{x+2}\;dx \qquad [\because x = (x+2)-2]$$

$$\Rightarrow \quad I = \int \{(x+2)^{3/2} - 2(x+2)^{1/2}\}\,dx = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$$

(ii) Let $I = \int (7x-2)\sqrt{3x+2}\;dx$. Let $7x-2 = \lambda(3x+2)+\mu$

On equating the coefficients of like powers of x on both sides, we get

$$3\lambda = 7 \text{ and } -2 = 2\lambda+\mu \Rightarrow \lambda = \frac{7}{3} \text{ and } \mu = -\frac{20}{3}$$

$$\therefore \quad I = \int \{\lambda(3x+2)+\mu\}\sqrt{3x+2}\;dx = \int \{\lambda(3x+2)^{3/2} + \mu(3x+2)^{1/2}\}\,dx$$

$$\Rightarrow \quad I = \lambda\left\{\frac{(3x+2)^{5/2}}{\frac{5}{2}\times 3}\right\} + \mu\left\{\frac{(3x+2)^{3/2}}{3\times\frac{3}{2}}\right\} + C = \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

ALITER Let $I = \int (7x-2)\sqrt{3x+2}\;dx$. Then,

$$I = \int 7\left(x-\frac{2}{7}\right)\sqrt{3x+2}\,dx = \int \frac{7}{3}\left(3x-\frac{6}{7}\right)\sqrt{3x+2}\,dx = \frac{7}{3}\int\left(3x+2-2-\frac{6}{7}\right)\sqrt{3x+2}\,dx$$

$$\Rightarrow I = \frac{7}{3}\int\left\{(3x+2)-\frac{20}{7}\right\}\sqrt{3x+2}\;dx = \frac{7}{3}\int\left\{(3x+2)^{3/2} - \frac{20}{7}\sqrt{3x+2}\right\}dx$$

$$\Rightarrow I = \frac{7}{3}\left\{\frac{(3x+2)^{5/2}}{3\times\frac{5}{2}} - \frac{20}{7}\times\frac{(3x+2)^{3/2}}{\frac{3}{2}\times 3}\right\} + C = \frac{14}{45}(3x+2)^{5/2} - \frac{40}{27}(3x+2)^{3/2} + C$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{x}{\sqrt{x+2}}\,dx$ (ii) $\int \frac{x+1}{\sqrt{2x-1}}\,dx$

SOLUTION (i) Let $I = \int \frac{x}{\sqrt{x+2}}\,dx$. Then,

$$I = \int \frac{(x+2)-2}{\sqrt{x+2}}\,dx = \int \left\{\sqrt{x+2} - \frac{2}{\sqrt{x+2}}\right\} dx$$

$$\Rightarrow \quad I = \left\{\frac{(x+2)^{3/2}}{\frac{3}{2}}\right\} - 2\left\{\frac{(x+2)^{1/2}}{\frac{1}{2}}\right\} + C = \frac{2}{3}(x+2)^{3/2} - 4(x+2)^{1/2} + C$$

(ii) Let $I = \int \frac{x+1}{\sqrt{2x-1}}\,dx$. Then,

$$I = \frac{1}{2}\int \frac{2x+2}{\sqrt{2x-1}}\,dx = \frac{1}{2}\int \frac{(2x-1)+3}{\sqrt{2x-1}}\,dx = \frac{1}{2}\int \left\{\sqrt{2x-1} + \frac{3}{\sqrt{2x-1}}\right\} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \sqrt{2x-1}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{2x-1}}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left\{\frac{(2x-1)^{3/2}}{2\times\frac{3}{2}}\right\} + \frac{3}{2}\left\{\frac{(2x-1)^{1/2}}{2\times\frac{1}{2}}\right\} + C = \frac{1}{6}(2x-1)^{3/2} + \frac{3}{2}(2x-1)^{1/2} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:* $\int \frac{8x+13}{\sqrt{4x+7}}\,dx$

SOLUTION Let $I = \int \frac{8x+13}{\sqrt{4x+7}}\,dx$. Let $8x+13 = \lambda(4x+7)+\mu$

On equating the coefficients of like powers of x on both sides, we get

$$8 = 4\lambda,\ 13 = 7\lambda + \mu \Rightarrow \lambda = 2 \text{ and } \mu = -1$$

Replacing $8x+13$ by $\lambda(4x+7)+\mu$ in the given integral, we obtain

$$I = \int \frac{\lambda(4x+7)+\mu}{\sqrt{4x+7}}\,dx = \int \left\{\lambda\sqrt{4x+7} + \frac{\mu}{\sqrt{4x+7}}\right\} dx$$

$$\Rightarrow \quad I = \lambda \int \sqrt{4x+7}\,dx + \mu \int \frac{1}{\sqrt{4x+7}}\,dx$$

$$\Rightarrow \quad I = \lambda\left\{\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}}\right\} + \mu\left\{\frac{(4x+7)^{1/2}}{4\times\frac{1}{2}}\right\} + C = \frac{1}{3}(4x+7)^{3/2} - \frac{1}{2}(4x+7)^{1/2} + C$$

<u>ALITER</u> Let $I = \int \frac{8x+13}{\sqrt{4x+7}}\,dx$. Then,

$$I = 8\int \frac{x+\frac{13}{8}}{\sqrt{4x+7}}\,dx$$

$$\Rightarrow \quad I = \frac{8}{4}\int \frac{4x+\frac{13}{2}}{\sqrt{4x+7}}\,dx = 2\int \frac{(4x+7)+\left(\frac{13}{2}-7\right)}{\sqrt{4x+7}}\,dx = 2\int \frac{(4x+7)-\frac{1}{2}}{\sqrt{4x+7}}\,dx$$

$$\Rightarrow \quad I = 2\int \left\{\sqrt{4x+7}-\frac{1}{2}\times\frac{1}{\sqrt{4x+7}}\right\}dx$$

$$\Rightarrow \quad I = 2\left\{\frac{(4x+7)^{3/2}}{4\times\frac{3}{2}}\right\}-\frac{1}{2}\left\{\frac{(4x+7)^{1/2}}{4\times\frac{1}{2}}\right\}+C = \frac{1}{3}(4x+7)^{3/2}-\frac{1}{2}(4x+7)^{1/2}+C$$

EXAMPLE 4 *Evaluate:* $\displaystyle\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx.$

SOLUTION Let $I = \displaystyle\int \frac{x}{\sqrt{x+a}+\sqrt{x+b}}\,dx$. Then,

$$I = \int \frac{x\left\{\sqrt{x+a}-\sqrt{x+b}\right\}}{\left\{\sqrt{x+a}+\sqrt{x+b}\right\}\left\{\sqrt{x+a}-\sqrt{x+b}\right\}}\,dx = \int \frac{x\left\{\sqrt{x+a}-\sqrt{x+b}\right\}}{a-b}\,dx$$

$$\Rightarrow \quad I = \frac{1}{a-b}\int\left\{x\sqrt{x+a}-x\sqrt{x+b}\right\}dx$$

$$\Rightarrow \quad I = \frac{1}{a-b}\int\left\{(x+a-a)\sqrt{x+a}-(x+b-b)\sqrt{x+b}\right\}dx$$

$$\Rightarrow \quad I = \frac{1}{a-b}\int\left\{(x+a)^{3/2}-a\sqrt{x+a}-(x+b)^{3/2}+b\sqrt{x+b}\right\}dx$$

$$\Rightarrow \quad I = \frac{1}{a-b}\left\{\frac{2}{5}(x+a)^{5/2}-\frac{2a}{3}(x+a)^{3/2}-\frac{2}{5}(x+b)^{5/2}+\frac{2b}{3}(x+b)^{3/2}\right\}+C$$

EXERCISE 18.5

BASED ON LOTS

1. $\displaystyle\int x\sqrt{x+2}\,dx$
2. $\displaystyle\int \frac{x-1}{\sqrt{x+4}}\,dx$
3. $\displaystyle\int \frac{x}{\sqrt{x+4}}\,dx$
4. $\displaystyle\int \frac{x+1}{\sqrt{2x+3}}\,dx$

BASED ON HOTS

5. $\displaystyle\int (x+2)\sqrt{3x+5}\,dx$
6. $\displaystyle\int \frac{2x+1}{\sqrt{3x+2}}\,dx$
7. $\displaystyle\int \frac{3x+5}{\sqrt{7x+9}}\,dx$
8. $\displaystyle\int \frac{2-3x}{\sqrt{1+3x}}\,dx$
9. $\displaystyle\int (5x+3)\sqrt{2x-1}\,dx$
10. $\displaystyle\int \frac{x}{\sqrt{x+a}-\sqrt{x+b}}\,dx$

ANSWERS

1. $\frac{2}{5}(x+2)^{5/2}-\frac{4}{3}(x+2)^{3/2}+C$
2. $\frac{2}{3}(x+4)^{3/2}-10(x+4)^{1/2}+C$

3. $\frac{2}{3}(x-8)\sqrt{x+4}+C$

4. $\frac{1}{6}(2x+3)^{3/2}-\frac{1}{2}(2x+3)^{1/2}+C$

5. $\frac{2}{135}(9x+20)(3x+5)^{3/2}+C$

6. $\frac{2}{27}(6x+1)\sqrt{3x+2}+C$

7. $\frac{2}{49}\sqrt{7x+9}\,(7x+17)+C$

8. $\frac{2}{9}(8-3x)\sqrt{1+3x}+C$

9. $\frac{1}{3}(3x+4)(2x-1)^{3/2}+C$

10. $\frac{1}{a-b}\left\{\frac{2}{5}(x+a)^{5/2}-\frac{2a}{3}(x+a)^{3/2}+\frac{2}{5}(x+b)^{5/2}-\frac{2b}{3}(x+b)^{3/2}\right\}+C$

18.8.4 EVALUATION OF INTEGRALS OF THE FORM $\int \sin^m x\,dx$, $\int \cos^m x\,dx$, WHERE $m \le 4$, $m \in N$

To evaluate integrals of the form $\int \sin^m x\,dx$, $\int \cos^m x\,dx$ where $m \le 4$, we express $\sin^m x$ and $\cos^m x$ in terms of sines and cosines of multiples of x by using the following trigonometrical identities:

(i) $\sin^2 x = \frac{1-\cos 2x}{2}$

(ii) $\cos^2 x = \frac{1+\cos 2x}{2}$

(iii) $\sin 3x = 3\sin x - 4\sin^3 x$

(v) $\cos 3x = 4\cos^3 x - 3\cos x$.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \sin^2 x\,dx$ (ii) $\int \cos^2 x\,dx$ [NCERT] (iii) $\int \sin^2 x \cos^2 x\,dx$

SOLUTION (i) Let $I = \int \sin^2 x\,dx$. Then,

$$I = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}+C$$

(ii) Let $I = \int \cos^2 x\,dx$. Then,

$$I = \int \frac{1+\cos 2x}{2}\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}+C$$

(iii) Let $I = \int \sin^2 x \cos^2 x\,dx$. Then,

$$I = \frac{1}{4}\int (2\sin x\cos x)^2\,dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int \sin^2 2x\,dx = \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx = \frac{1}{8}\int (1-\cos 4x)\,dx = \frac{1}{8}\left\{x - \frac{\sin 4x}{4}\right\}+C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \sin^3 x\,dx$ [NCERT] (ii) $\int \cos^3 x\,dx$ (iii) $\int \sin^3 x\cos^3 x\,dx$ [NCERT]

SOLUTION (i) Let $I = \int \sin^3 x\,dx$. Then,

$$I = \int \frac{3\sin x - \sin 3x}{4}\,dx = \frac{1}{4}\int (3\sin x - \sin 3x)\,dx = \frac{1}{4}\left\{-3\cos x + \frac{\cos 3x}{3}\right\}+C$$

(ii) Let $I = \int \cos^3 x\, dx$. Then,

$$I = \int \frac{\cos 3x + 3\cos x}{4}\, dx = \frac{1}{4}\int \cos 3x + 3\cos x\, dx = \frac{1}{4}\left\{\frac{\sin 3x}{3} + 3\sin x\right\} + C$$

(iii) Let $I = \int \sin^3 x \cos^3 x\, dx$. Then,

$$I = \frac{1}{8}\int (2\sin x \cos x)^3\, dx = \frac{1}{8}\int \sin^3 2x\, dx = \frac{1}{8}\int \frac{3\sin 2x - \sin 6x}{4}\, dx$$

$$\Rightarrow \quad I = \frac{1}{32}\int (3\sin 2x - \sin 6x)\, dx = \frac{1}{32}\left\{-\frac{3}{2}\cos 2x + \frac{1}{6}\cos 6x\right\} + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:*

(i) $\int \sin^4 x\, dx$ [NCERT, CBSE 2000, 2004] (ii) $\int \cos^4 x\, dx$ [CBSE 2000, 2003]

(iii) $\int \sin^4 x \cos^4 x\, dx$

SOLUTION (i) Let $I = \int \sin^4 x\, dx$. Then,

$$I = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \qquad \left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$\Rightarrow \quad I = \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\, dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int 1 - 2\cos 2x + \frac{1+\cos 4x}{2}\, dx = \frac{1}{8}\int 2 - 4\cos 2x + 1 + \cos 4x\, dx$$

$$\Rightarrow \quad I = \frac{1}{8}\int 3 - 4\cos 2x + \cos 4x\, dx = \frac{1}{8}\left\{3x - 2\sin 2x + \frac{\sin 4x}{4}\right\} + C$$

(ii) Let $I = \int \cos^4 x\, dx$. Then,

$$I = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \qquad \left[\because \cos^2\theta = \frac{1+\cos 2\theta}{2}\right]$$

$$\Rightarrow \quad I = \frac{1}{4}\int 1 + 2\cos 2x + \cos^2 2x\, dx = \frac{1}{4}\int 1 + 2\cos 2x + \frac{1+\cos 4x}{2}\, dx$$

$$\Rightarrow \quad I = \frac{1}{8}\int 3 + 4\cos 2x + \cos 4x\, dx = \frac{1}{8}\left\{3x + 2\sin 2x + \frac{\sin 4x}{4}\right\} + C$$

(iii) Let $I = \int \sin^4 x \cos^4 x\, dx$. Then,

$$I = \frac{1}{16}\int (2\sin x\cos x)^4\, dx = \frac{1}{16}\int (\sin 2x)^4\, dx = \frac{1}{16}\int (\sin^2 2x)^2\, dx$$

$$\Rightarrow \quad I = \frac{1}{16}\int \left(\frac{1-\cos 4x}{2}\right)^2 dx$$

$$\Rightarrow \quad I = \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x)\, dx = \frac{1}{64}\int \left\{1 - 2\cos 4x + \frac{1+\cos 8x}{2}\right\} dx$$

$$\Rightarrow \quad I = \frac{1}{128}\int \left(3 - 4\cos 4x + \cos 8x\right) dx = \frac{1}{128}\left\{3x - \sin 4x + \frac{1}{8}\sin 8x\right\} + C$$

EXERCISE 18.6

BASIC

1. $\int \sin^2(2x+5)\, dx$ [NCERT]
2. $\int \sin^3(2x+1)\, dx$ [NCERT]

3. $\int \sin^2 b\,x\,dx$ 4. $\int \sin^2 \frac{x}{2}\,dx$ 5. $\int \cos^2 \frac{x}{2}\,dx$

6. $\int \cos^2 nx\,dx$ 7. $\int \sin x\sqrt{1-\cos 2x}\,dx$

BASED ON LOTS

8. $\int \cos^4 2x\,dx$ [NCERT] 9. $\int \frac{\sin 3x}{\sin x}\,dx$ [CBSE 2022] 10. $\int \frac{\cos 3x}{\cos x}\,dx$

ANSWERS

1. $\frac{x}{2}-\frac{1}{8}\sin(4x+10)+C$ 2. $-\frac{3}{8}\cos(2x+1)+\frac{1}{24}\cos(6x+3)+C$ 3. $\frac{x}{2}-\frac{\sin 2bx}{4b}+C$

4. $\frac{1}{2}(x-\sin x)+C$ 5. $\frac{1}{2}(x+\sin x)+C$ 6. $\frac{x}{2}+\frac{1}{4n}\sin 2nx+C$ 7. $\frac{1}{\sqrt{2}}x-\frac{\sin 2x}{2\sqrt{2}}+C$

8. $\frac{3x}{8}+\frac{\sin 4x}{8}+\frac{\sin 8x}{64}+C$ 9. $x+\sin 2x+C$ 10. $\sin 2x-x+C$

HINTS TO SELECTED PROBLEMS

1. $\int \sin^2(2x+5)\,dx = \frac{1}{2}\int\left\{1-\cos(4x+10)\right\}dx = \frac{1}{2}\left\{x-\frac{1}{4}\sin(4x+10)\right\}+C$

2. $I=\int \sin^3(2x+1)\,dx = \frac{1}{4}\int\left\{3\sin(2x+1)-\sin 3(2x+1)\right\}dx$

$$\Rightarrow \quad I = \frac{1}{4}\int\left\{3\sin(2x+1)-\sin(6x+3\right\}dx = -\frac{3}{8}\cos(2x+1)+\frac{1}{24}\cos(6x+3)+C$$

8. $I=\int \cos^4 2x\,dx = \int(\cos^2 2x)^2\,dx = \int\left(\frac{1+\cos 4x}{2}\right)^2 dx$

$$= \frac{1}{4}\int(1+2\cos 4x+\cos^2 4x)\,dx = \frac{1}{4}\int 1+2\cos 4x+\frac{1+\cos 8x}{2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{8}\int(3+4\cos 4x+\cos 8x)\,dx = \frac{1}{8}\left(3x+\sin 4x+\frac{1}{8}\sin 8x\right)+C$$

18.8.5 EVALAUTION OF INTEGRALS OF THE FORM $\int \sin mx\cos nx\,dx$, $\int \sin mx\sin nx\,dx$, $\int \cos mx\cos nx\,dx$

To evaluate this type of integrals we use the following trigonometrical identities to express the products into sums.

$$2\sin A\cos B=\sin(A+B)+\sin(A-B);\; 2\cos A\sin B=\sin(A+B)-\sin(A-B)$$
$$2\cos A\cos B=\cos(A+B)+\cos(A-B);\; 2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \sin 4x\cos 3x\,dx$ (ii) $\int \sin 3x\sin 2x\,dx$ [CBSE 2022]

(iii) $\int \sin 3x\cos 4x\,dx$ [NCERT] (iv) $\int \cos 2x\cos 4x\,dx$ [CBSE 2007]

SOLUTION (i) Let $I=\int \sin 4x\cos 3x\,dx$. Then,

$$I=\frac{1}{2}\int 2\sin 4x\cos 3x\,dx = \frac{1}{2}\int(\sin 7x+\sin x)\,dx = \frac{1}{2}\left\{-\frac{\cos 7x}{7}-\cos x\right\}+C$$

(ii) Let $I = \int \sin 3x \sin 2x \, dx$. Then,

$$I = \frac{1}{2}\int 2 \sin 3x \sin 2x \, dx = \frac{1}{2}\int (\cos x - \cos 5x) \, dx = \frac{1}{2}\left\{\sin x - \frac{\sin 5x}{5}\right\} + C$$

(iii) Let $I = \int \sin 3x \cos 4x \, dx$. Then,

$$I = \frac{1}{2}\int 2 \sin 3x \cos 4x \, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \left\{\sin 7x + \sin(-x)\right\} dx = \frac{1}{2}\int (\sin 7x - \sin x) \, dx = \frac{1}{2}\left\{-\frac{\cos 7x}{7} + \cos x\right\} + C$$

(iv) Let $I = \int \cos 2x \cos 4x \, dx$. Then,

$$I = \frac{1}{2}\int 2 \cos 4x \cos 2x \, dx = \frac{1}{2}\int (\cos 6x + \cos 2x) \, dx = \frac{1}{2}\left\{\frac{\sin 6x}{6} + \frac{\sin 2x}{2}\right\} + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Evaluate:*

(i) $\int \cos 2x \cos 4x \cos 6x \, dx$ [NCERT] (ii) $\int \sin x \sin 2x \sin 3x \, dx$ [CBSE 2012, NCERT]

SOLUTION (i) Let $I = \int \cos 2x \cos 4x \cos 6x \, dx$. Then,

$$I = \frac{1}{2}\int (2 \cos 4x \cos 2x) \cos 6x \, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (\cos 6x + \cos 2x) \cos 6x \, dx = \frac{1}{4}\int (2\cos^2 6x + 2 \cos 6x \cos 2x) \, dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int 1 + \cos 12x + \cos 8x + \cos 4x = \frac{1}{4}\left\{x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right\} + C$$

(ii) Let $I = \int \sin x \sin 2x \sin 3x \, dx$. Then,

$$I = \frac{1}{2}\int (2 \sin 2x \sin x) \sin 3x \, dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int (\cos x - \cos 3x) \sin 3x \, dx = \frac{1}{4}\int (2 \sin 3x \cos x - 2 \sin 3x \cos 3x) \, dx$$

$$\Rightarrow \quad I = \frac{1}{4}\int (\sin 4x + \sin 2x - \sin 6x) \, dx = \frac{1}{4}\left\{-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right\} + C$$

EXAMPLE 3 *Evaluate:* $\int \frac{\sin 4x}{\sin x} dx$

SOLUTION Let $I = \int \frac{\sin 4x}{\sin x} dx$. Then,

$$I = \int \frac{2 \sin 2x \cos 2x}{\sin x} dx = \int \frac{4 \sin x \cos x \cos 2x}{\sin x} dx$$

$$\Rightarrow \quad I = 2\int 2 \cos 2x \cos x \, dx = 2\int (\cos 3x + \cos x) \, dx = 2\left\{\frac{\sin 3x}{3} + \sin x\right\} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Evaluate:* $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$ [NCERT EXEMPLAR]

SOLUTION Let $I = \int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\, dx$. Then,

$$\Rightarrow \quad I = \int \frac{\sin 3x(\cos 5x + \cos 4x)}{\sin 3x - 2\sin 3x \cos 3x}\, dx$$

$$\Rightarrow \quad I = \int \frac{\left(2\sin\frac{3x}{2}\cos\frac{3x}{2}\right)\left(2\cos\frac{9x}{2}\cos\frac{x}{2}\right)}{\sin 3x - \sin 6x}\, dx = 4\int \frac{\sin\frac{3x}{2}\cos\frac{3x}{2}\cos\frac{9x}{2}\cos\frac{x}{2}}{-2\cos\frac{9x}{2}\sin\frac{3x}{2}}\, dx$$

$$\Rightarrow \quad I = -\int 2\cos\frac{3x}{2}\cos\frac{x}{2}\, dx = -\int (\cos 2x + \cos x)\, dx = -\left(\frac{\sin 2x}{2} + \sin x\right) + C$$

EXERCISE 18.7

BASIC

Integrate the following integrals:

1. $\int \sin 4x \cos 7x\, dx$ **[CBSE 2007]**
2. $\int \cos 3x \cos 4x\, dx$
3. $\int \cos mx \cos nx\, dx,\ m \neq n$
4. $\int \sin mx \cos nx\, dx,\ m \neq n$

BASED ON LOTS

5. $\int \sin 2x \sin 4x \sin 6x\, dx$
6. $\int \sin x \cos 2x \sin 3x\, dx$

ANSWERS

1. $-\frac{1}{22}\cos 11x + \frac{1}{6}\cos 3x + C$
2. $\frac{1}{14}\sin 7x + \frac{1}{2}\sin x + C$
3. $\frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\} + C$
4. $\frac{1}{2}\left\{-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right\} + C$
5. $-\frac{1}{32}\cos 8x + \frac{1}{48}\cos 12x - \frac{1}{16}\cos 4x + C$
6. $\frac{x}{4} + \frac{1}{16}\sin 4x - \frac{1}{24}\sin 6x - \frac{1}{8}\sin 2x + C$

18.8.6 INTEGRALS OF THE FORM $\int \frac{f'(x)}{f(x)}\, dx$

THEOREM 1 $\int \frac{f'(x)}{f(x)}\, dx = \log\{f(x)\} + C$

PROOF Let $I = \int \frac{f'(x)}{f(x)}\, dx$. Putting $f(x) = t$ and $f'(x)\, dx = dt$, we get

$$I = \int \frac{1}{t}\, dt = \ln t + C = \ln|f(x)| + C$$

Q.E.D.

REMARK *It follows from the above theorem that if the numerator in integrand is exact differential of the denominator then its integral is logarithm of the denominator.*

SOME STANDARD RESULTS

THEOREM 2 *Prove that:* $\int \tan x\, dx = -\log|\cos x| + C$ or, $\int \tan x\, dx = \log|\sec x| + C$

PROOF Let $I = \int \tan x\, dx$. Then, $I = \int \frac{\sin x}{\cos x}\, dx$

Let $\cos x = t$. Then, $d(\cos x) = dt \Rightarrow -\sin x\, dx = dt \Rightarrow dx = -\frac{dt}{\sin x}$

Putting $\cos x = t$, and $dx = -dt/\sin x$, we get

$$I=\int \frac{\sin x}{\cos x}\times-\frac{dt}{\sin x}=-\int \frac{1}{t}\,dt = -\ln|t|+C = -\ln|\cos x|+C$$

Hence, $\int \tan x\,dx=-\ln|\cos x|+C$ or, $\int \tan x\,dx=\ln|\sec x|+C$

Q.E.D.

THEOREM 3 *Prove that:* $\int \cot x\,dx = \ln|\sin x|+C.$

PROOF Let $I=\int \cot x\,dx$. Then, $I = \int \frac{\cos x}{\sin x}\,dx.$

Let $\sin x=t$. Then, $d(\sin x)=dt \Rightarrow \cos x\,dx=dt \Rightarrow dx=\frac{dt}{\cos x}$

Putting $\sin x=t$, and $dx=dt/\cos x$, we get

$$I = \int \frac{\cos x}{t}\cdot\frac{dt}{\cos x} = \int \frac{1}{t}\,dt = \ln|t|+C = \ln|\sin x|+C$$

Hence, $\int \cot x\,dx = \ln|\sin x|+C.$

Q.E.D.

THEOREM 4 *Prove that:* $\int \sec x\,dx = \ln|\sec x+\tan x|+C.$

PROOF Let $I=\int \sec x\,dx$. Then, $I=\int \frac{\sec x(\sec x+\tan x)}{(\sec x+\tan x)}\,dx$

Let $\sec x+\tan x = t$. Then,

$$d(\sec x+\tan x) = dt \Rightarrow (\sec x\tan x+\sec^2 x)\,dx = dt \Rightarrow dx = \frac{dt}{\sec x(\sec x+\tan x)}$$

Putting $\sec x+\tan x=t$ and $dx=\frac{dt}{\sec x(\sec x+\tan x)}$, we get

$$I=\int \frac{\sec x(\sec x+\tan x)}{t}\times\frac{dt}{\sec x(\sec x+\tan x)}=\int \frac{1}{t}\,dt=\ln|t|+C=\ln|\sec x+\tan x|+C$$

Hence, $\int \sec x\,dx = |\sec x+\tan x|+C.$

Q.E.D.

THEOREM 5 *Prove that:* $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x-\cot x|+C.$

PROOF Let $I=\int \operatorname{cosec} x\,dx$. Then, $I=\int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\,dx$

Let $\operatorname{cosec} x-\cot x = t$. Then,

$$d(\operatorname{cosec} x-\cot x)=dt \Rightarrow (-\operatorname{cosec} x\cot x+\operatorname{cosec}^2 x)\,dx=dt \Rightarrow dx=\frac{dt}{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}$$

Putting $\operatorname{cosec} x-\cot x=t$ and, $dx = \frac{dt}{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}$, we get

$$I = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x-\cot x)}{\operatorname{cosec} x-\cot x}\times\frac{dt}{(\operatorname{cosec} x-\cot x)\operatorname{cosec} x}$$

$$\Rightarrow \quad I = \int \frac{1}{t}\,dt = \ln|t|+C = \ln|\operatorname{cosec} x-\cot x|+C$$

Hence, $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x-\cot x|+C$

Q.E.D.

ALTERATIVE FORMULAE FOR $\int \operatorname{cosec} x\, dx$ AND $\int \sec x\, dx$

Let $I = \int \operatorname{cosec} x\, dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2 \sin x/2 \cos x/2} dx$. Then,

$$I = \int \frac{\sec^2 x/2}{2 \tan x/2} dx \qquad \left[\text{Divide both numerator and denumerator by } \cos^2 \frac{x}{2}\right]$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt \Rightarrow dx = \frac{2\, dt}{\sec^2 \frac{x}{2}}$

Putting $\tan \frac{x}{2} = t$ and $dx = \frac{2dt}{\sec^2 \frac{x}{2}}$, we get

$$I = \int \frac{1}{t} dt = \ln|t| + C = \ln\left|\tan \frac{x}{2}\right| + C$$

Hence, $\int \operatorname{cosec} x\, dx = \ln\left|\tan \frac{x}{2}\right| + C$ ...(i)

$$\int \sec x\, dx = \int \operatorname{cosec}\left(\frac{\pi}{2} + x\right) dx = \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C \qquad \text{[Using (i)]}$$

Hence, $\int \sec x\, dx = \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **PROBLEMS BASED ON** $\int \tan x\, dx, \int \cot x\, dx, \int \sec x\, dx, \int \operatorname{cosec} x\, dx$

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{1}{\sqrt{1 + \cos 2x}} dx$ (ii) $\int \frac{1}{\sqrt{1 - \cos x}} dx$ (iii) $\int \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$ (iv) $\int \sqrt{\frac{1 + \cos x}{1 - \cos x}} dx$

SOLUTION (i) We have,

$$\int \frac{1}{\sqrt{1 + \cos 2x}} dx = \int \frac{1}{\sqrt{2 \cos^2 x}} dx = \frac{1}{\sqrt{2}} \int \sec x\, dx = \frac{1}{\sqrt{2}} \ln|\sec x + \tan x| + C$$

(ii) $$\int \frac{1}{\sqrt{1 - \cos x}} dx = \int \frac{1}{\sqrt{2 \sin^2 \frac{x}{2}}} dx = \frac{1}{\sqrt{2}} \int \operatorname{cosec} \frac{x}{2} dx = \sqrt{2} \ln\left|\operatorname{cosec} \frac{x}{2} - \cot \frac{x}{2}\right| + C$$

(iii) $$\int \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}} dx = \int \sqrt{\frac{2 \sin^2 x}{2 \cos^2 x}} dx = \int \tan x\, dx = \ln|\sec x| + C$$

(iv) $$\int \sqrt{\frac{1 + \cos x}{1 - \cos x}} dx = \int \sqrt{\frac{2 \cos^2 \frac{x}{2}}{2 \sin^2 \frac{x}{2}}} dx = \int \cot \frac{x}{2} dx = 2 \ln\left|\sin \frac{x}{2}\right| + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{1}{\sqrt{1+\sin 2x}}\, dx$ (ii) $\int \frac{1}{\sqrt{1-\sin x}}\, dx$

SOLUTION (i) Let $I = \int \frac{1}{\sqrt{1+\sin 2x}}\, dx$. Then,

$$I = \int \frac{1}{\sqrt{1-\cos\left(\frac{\pi}{2}+2x\right)}}\, dx = \int \frac{1}{\sqrt{2\sin^2\left(\frac{\pi}{4}+x\right)}}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(\frac{\pi}{4}+x\right) dx = \frac{1}{\sqrt{2}} \ln\left|\tan\left(\frac{\pi}{8}+\frac{x}{2}\right)\right| + C$$

(ii) Let $I = \int \frac{1}{\sqrt{1-\sin x}}\, dx$. Then,

$$I = \int \frac{1}{\sqrt{1+\cos\left(\frac{\pi}{2}+x\right)}}\, dx = \int \frac{1}{\sqrt{2\cos^2\left(\frac{\pi}{4}+\frac{x}{2}\right)}}\, dx = \frac{1}{\sqrt{2}} \int \sec\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$$

$$\Rightarrow \quad I = \frac{2}{\sqrt{2}} \ln\left|\tan\left(\frac{\pi}{4}+\frac{\pi}{8}+\frac{x}{4}\right)\right| + C = \sqrt{2}\ln\left|\tan\left(\frac{3\pi}{8}+\frac{x}{4}\right)\right| + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{\sin(x-a)}{\sin x}\, dx$ (ii) $\int \frac{\sin x}{\sin(x-a)}\, dx$ [NCERT, CBSE 2004]

(iii) $\int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$

SOLUTION (i) Let $I = \int \frac{\sin(x-a)}{\sin x}\, dx$. Then,

$$I = \int \frac{\sin x\cos a - \cos x \sin a}{\sin x}\, dx = \int \cos a\, dx - \int \sin a \cot x\, dx$$

$$\Rightarrow \quad I = \cos a \int 1\cdot dx - \sin a \int \cot x\, dx = x\cos a - \sin a \ln|\sin x| + C$$

(ii) Let $I = \int \frac{\sin x}{\sin(x-a)}\, dx$. Then,

$$I = \int \frac{\sin\{(x-a)+a\}}{\sin(x-a)}\, dx$$

$$\Rightarrow \quad I = \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}\, dx = \int \{\cos a + \cot(x-a)\sin a\}\, dx$$

$$\Rightarrow \quad I = \cos a \int 1\cdot dx + \sin a \int \cot(x-a)\, dx = x\cos a + \sin a \ln|\sin(x-a)| + C$$

(iii) Let $I = \int \frac{1}{\sin(x-a)\sin(x-b)}\, dx$. Then,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a)\sin(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \int \{\cot(x-a) - \cot(x-b)\}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \left\{ \ln|\sin(x-a)| - \ln|\sin(x-b)| \right\} + C$$

$$\Rightarrow \quad I = \operatorname{cosec}(a-b) \ln \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + C$$

EXAMPLE 4 *Evaluate:*

(i) $\displaystyle\int \frac{1}{\sin(x-a)\cos(x-b)}\, dx$ (ii) $\displaystyle\int \frac{1}{\cos(x-a)\cos(x-b)}\, dx$ [NCERT]

SOLUTION (i) Let $I = \displaystyle\int \frac{1}{\sin(x-a)\cos(x-b)}\, dx$. Then,

$$I = \frac{1}{\cos(a-b)} \int \frac{\cos(a-b)}{\sin(x-a)\cos(x-b)}\, dx = \frac{1}{\cos(a-b)} \int \frac{\cos\{(x-b)-(x-a)\}}{\sin(x-a)\cos(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\cos(a-b)} \int \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\cos(a-b)} \int \left\{ \cot(x-a) + \tan(x-b) \right\} dx$$

$$\Rightarrow \quad I = \frac{1}{\cos(a-b)} \left\{ \log_e|\sin(x-a)| - \log_e|\cos(x-b)| \right\} + C$$

$$\Rightarrow \quad I = \frac{1}{\cos(a-b)} \log_e \left| \frac{\sin(x-a)}{\cos(x-b)} \right| + C$$

(ii) Let $I = \displaystyle\int \frac{1}{\cos(x-a)\cos(x-b)}\, dx$. Then,

$$I = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}\, dx = \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\cos(x-a)\cos(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \int \left\{ \tan(x-b) - \tan(x-a) \right\} dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \{ -\log_e|\cos(x-b)| + \log_e|\cos(x-a)| \} + C$$

$$\Rightarrow \quad I = \frac{1}{\sin(a-b)} \log_e \left| \frac{\cos(x-a)}{\cos(x-b)} \right| + C$$

EXAMPLE 5 *Evaluate:* $\displaystyle\int \frac{\sin(x+a)}{\sin(x+b)}\, dx$

SOLUTION Let $I = \displaystyle\int \frac{\sin(x+a)}{\sin(x+b)}\, dx$. Then,

$$I = \int \frac{\sin(x+b+a-b)}{\sin(x+b)}\, dx = \int \frac{\sin\{(x+b)+(a-b)\}}{\sin(x+b)}\, dx$$

$$\Rightarrow I = \int \frac{\sin(x+b)\cos(a-b) + \cos(x+b)\sin(a-b)}{\sin(x+b)}$$

$$\Rightarrow I = \int \{\cos(a-b) + \cot(x+b)\sin(a-b)\}\, dx = \cos(a-b)\int 1\cdot dx + \sin(a-b)\int \cot(x+b)\, dx$$

$$\Rightarrow I = x\cos(a-b) + \sin(a-b)\log_e|\sin(x+b)| + C$$

Type II EVALUATION OF INTEGRALS BASED UPON $\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + C$

In order to evaluate this type of integrals, we may use the following algorithm:

ALGORITHM

Step I *Obtain the integral, let it be* $I = \int \frac{f'(x)}{f(x)}\, dx$

Step II *Put* $f(x) = t$ *and replace* $f'(x)\,dx$ *by* dt *to obtain* $I = \int \frac{1}{t}\, dt$

Step III *Evaluate integral obtained in step II to obtain* $I = \log_e|t| + C$

Step IV *Replace* t *by* $f(x)$ *in step III to get* $I = \log_e|f(x)| + C$

Following examples will illustrate the above procedure.

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 6 *Evaluate:*

(i) $\int \frac{2x+5}{x^2+5x-7}dx$ (ii) $\int \frac{1-\tan x}{1+\tan x}dx$ (iii) $\int \frac{\sec^2 x}{3+\tan x}dx$

(iv) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$ [NCERT] (v) $\int e^{3\log x}(x^4+1)^{-1}dx$ [NCERT]

SOLUTION (i) Let $I = \int \frac{2x+5}{x^2+5x-7}dx$

Let $x^2+5x-7 = t$. Then, $d(x^2+5x-7) = dt \Rightarrow (2x+5)\,dx = dt \Rightarrow dx = \frac{dt}{2x+5}$

Putting $x^2+5x-7 = t$ and $dx = \frac{dt}{2x+5}$, we get

$$\therefore \quad I = \int \frac{2x+5}{x^2+5x-7}dx = \int \frac{1}{t}dt = \log_e|t| + C = \log_e|x^2+5x-7| + C$$

(ii) We have, $I = \int \frac{1-\tan x}{1+\tan x}dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}dx$. Let $\cos x + \sin x = t$. Then,

$$d(\cos x + \sin x) = dt \Rightarrow (-\sin x + \cos x)\,dx = dt \Rightarrow dx = \frac{dt}{\cos x - \sin x}$$

Putting $\cos x + \sin x = t$ and $dx = \frac{dt}{\cos x - \sin x}$, we get

$$I = \int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1}{t}dt = \log_e|t| + C = \log_e|\cos x + \sin x| + C$$

(iii) Let $I = \int \frac{\sec^2 x}{3+\tan x}dx$. Let $3 + \tan x = t$. Then,

$d(3+\tan x) = dt \Rightarrow \sec^2 x\, dx = dt \Rightarrow dx = \frac{dt}{\sec^2 x}$. Putting $\tan x = t$ and $dx = \frac{dt}{\sec^2 x}$, we get

$$I = \int \frac{\sec^2 x}{3+t} \times \frac{dt}{\sec^2 x} = \int \frac{1}{3+t}\, dt = \ln|3+t| + C = \ln|3+\tan x| + C$$

(iv) Let $I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx$.

Let $e^x + e^{-x} = t$. Then, $d(e^x + e^{-x}) = dt \Rightarrow (e^x - e^{-x})\, dx = dt \Rightarrow dx = \frac{dt}{e^x - e^{-x}}$

Putting $e^x + e^{-x} = t$ and $dx = \frac{dt}{e^x - e^{-x}}$, we get

$$I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}}\, dx = \int \frac{dt}{t} = \ln|t| + C = \ln|e^x + e^{-x}| + C$$

(v) Let $I = \int e^{3\log x}(x^4+1)^{-1}\, dx = \int \frac{e^{\log x^3}}{x^4+1}\, dx = \int \frac{x^3}{x^4+1}\, dx$

Let $x^4 + 1 = t$. Then, $d(x^4+1) = dt \Rightarrow 4x^3\, dx = dt \Rightarrow dx = \frac{dt}{4x^3}$. Putting $x^4 + 1 = t$ and $dx = \frac{dt}{4x^3}$, we get

$$I = \frac{1}{4}\int \frac{1}{t}\, dt = \frac{1}{4}\ln|t| + C = \frac{1}{4}\ln(x^4+1) + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Evaluate:*

(i) $\int \frac{1}{1+e^{-x}}\, dx$ (ii) $\int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx$ **[CBSE 2005]**

(iii) $\int \frac{\tan x}{a + b\tan^2 x}\, dx$ (iv) $\int \frac{\sin x - x\cos x}{x(x+\sin x)}\, dx$ **[CBSE 2015]**

SOLUTION (i) Let $I = \int \frac{1}{1+e^{-x}}\, dx = \int \frac{1}{1+\frac{1}{e^x}}\, dx = \int \frac{e^x}{e^x+1}\, dx$

Let $e^x + 1 = t$. Then, $d(e^x+1) = dt \Rightarrow e^x dx = dt \Rightarrow dx = \frac{dt}{e^x}$. Putting $1 + e^x = t$ and $dx = \frac{dt}{e^x}$, we get

$$I = \int \frac{e^x}{e^x+1}\, dx = \int \frac{e^x}{t}\frac{dt}{e^x} = \int \frac{1}{t}\, dt = \ln|t| + C = \ln|1+e^x| + C$$

(ii) Let $I = \int \frac{\sin 2x}{a^2 \sin^2 x + b^2 \cos^2 x}\, dx$. Let $a^2 \sin^2 x + b^2 \cos^2 x = t$. Then,

$$d(a^2 \sin^2 x + b^2 \cos^2 x) = dt \Rightarrow (a^2 - b^2) \sin 2x \, dx = dt \Rightarrow dx = \frac{dt}{(a^2 - b^2) \sin 2x}$$

Putting $a^2 \sin^2 x + b^2 \cos^2 x = t$ and $dx = \dfrac{dt}{(a^2 - b^2) \sin 2x}$, we get

$$I = \int \frac{\sin 2x}{t} \times \frac{dt}{2 \sin x \cos x (a^2 - b^2)}$$

$$\Rightarrow \quad I = \frac{1}{(a^2 - b^2)} \int \frac{1}{t} dt = \frac{1}{(a^2 - b^2)} \ln |t| + C = \frac{1}{(a^2 - b^2)} \ln |a^2 \sin^2 x + b^2 \cos^2 x| + C$$

(iii) Let $I = \displaystyle\int \frac{\tan x}{a + b \tan^2 x}$. Then,

$$I = \int \frac{\sin x \cos x}{a + b \dfrac{\sin^2 x}{\cos^2 x}} dx = \int \frac{\sin x \cos x}{a \cos^2 x + b \sin^2 x} dx = \frac{1}{2} \int \frac{\sin 2x}{a \cos^2 x + b \sin^2 x} dx$$

$$\Rightarrow \quad I = \frac{1}{2(b-a)} \ln |a \cos^2 x + b \sin^2 x| + C \qquad \text{[See (ii)]}$$

(iv) Let $I = \displaystyle\int \frac{\sin x - x \cos x}{x(x + \sin x)} dx$. Then,

$$I = \int \frac{(x + \sin x) - x - x \cos x}{x(x + \sin x)} dx = \int \frac{(x + \sin x) - x(1 + \cos x)}{x(x + \sin x)} dx = \int \left\{ \frac{1}{x} - \frac{1 + \cos x}{x + \sin x} \right\} dx$$

$$\Rightarrow \quad I = \ln |x| - \ln (x + \sin x) + C = \ln \left| \frac{x}{x + \sin x} \right| + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *Evaluate:*

(i) $\displaystyle\int \tan x \tan 2x \tan 3x \, dx$ (ii) $\displaystyle\int \tan (x - \theta) \tan (x + \theta) \tan 2x \, dx$

SOLUTION (i) We know that

$$\tan 3x = \tan (2x + x) = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \Rightarrow \tan 3x (1 - \tan 2x \tan x) = \tan 2x + \tan x$$

$$\Rightarrow \tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x \Rightarrow \tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

$$\therefore \ I = \int \tan x \tan 2x \tan 3x \, dx$$

$$\Rightarrow I = \int (\tan 3x - \tan 2x - \tan x) \, dx = -\frac{1}{3} \log_e |\cos 3x| + \frac{1}{2} \log_e |\cos 2x| + \log_e |\cos x| + C$$

(ii) We know that $2x = (x - \theta) + (x + \theta)$

$$\therefore \quad \tan 2x = \tan \{(x - \theta) + (x + \theta)\}$$

$$\Rightarrow \quad \tan 2x = \frac{\tan (x - \theta) + (x + \theta)}{1 - \tan (x - \theta) \tan (x + \theta)}$$

$$\Rightarrow \quad \tan 2x - \tan (x - \theta) \tan (x + \theta) \tan 2x = \tan (x - \theta) + \tan (x + \theta)$$

$$\Rightarrow \quad \tan (x - \theta) \tan (x + \theta) \tan 2x = \tan 2x - \tan (x - \theta) - \tan (x + \theta)$$

$$\therefore \quad I = \int \tan (x - \theta) \tan (x + \theta) \tan 2x \, dx = \int \{\tan 2x - \tan (x - \theta) - \tan (x + \theta)\} \, dx$$

$$\Rightarrow \quad I = -\frac{1}{2} \ln |\cos 2x| + \ln |\cos (x - \theta)| + \ln |\cos (x + \theta)| + C$$

EXAMPLE 9 *Evaluate:* $\int \left\{1 + 2\tan x(\tan x + \sec x)\right\}^{1/2} dx$

SOLUTION Let $I = \int \left\{1 + 2\tan x(\tan x + \sec x)\right\}^{1/2} dx$. Then,

$$I = \int \left\{1 + 2\tan^2 x + 2\tan x \sec x\right\}^{1/2} dx = \int \left\{1 + \tan^2 x + \tan^2 x + 2\tan x \sec x\right\}^{1/2} dx$$

$$\Rightarrow I = \int \left\{\sec^2 x + \tan^2 x + 2\tan x \sec x\right\}^{1/2} dx = \int \left\{(\sec x + \tan x)^2\right\}^{1/2} dx$$

$$\Rightarrow I = \int (\sec x + \tan x)\, dx = \ln|\sec x + \tan x| + \ln|\sec x| + C$$

EXAMPLE 10 *Evaluate:* $\int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)} dx$

SOLUTION Let $I = \int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)} dx$. Then,

$$I = \int \frac{\sin\left\{\left(x - \frac{\pi}{3}\right) + \left(x + \frac{\pi}{3}\right)\right\}}{\sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)} dx$$

$$\Rightarrow \quad I = \int \frac{\sin\left(x - \frac{\pi}{3}\right)\cos\left(x + \frac{\pi}{3}\right) + \cos\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)}{\sin\left(x - \frac{\pi}{3}\right)\sin\left(x + \frac{\pi}{3}\right)}$$

$$\Rightarrow \quad I = \int \left\{\cot\left(x + \frac{\pi}{3}\right) + \cot\left(x - \frac{\pi}{3}\right)\right\} dx = \ln\left|\sin\left(x + \frac{\pi}{3}\right)\right| + \ln\left|\sin\left(x - \frac{\pi}{3}\right)\right| + C$$

EXERCISE 18.8

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{1 - \cos 2x}} dx$
2. $\int \frac{1}{\sqrt{1 + \cos x}} dx$
3. $\int \sqrt{\frac{1 + \cos 2x}{1 - \cos 2x}} dx$
4. $\int \sqrt{\frac{1 - \cos x}{1 + \cos x}} dx$
5. $\int \frac{e^{3x}}{e^{3x} + 1} dx$
6. $\int \frac{\sec x \tan x}{3\sec x + 5} dx$
7. $\int \frac{1}{x(3 + \log x)} dx$
8. $\int \frac{e^x + 1}{e^x + x} dx$
9. $\int \frac{1}{x \log x} dx$
10. $\int \frac{\cos x}{2 + 3\sin x} dx$
11. $\int \frac{1 - \sin x}{x + \cos x} dx$
12. $\int \frac{e^{2x}}{e^{2x} - 2} dx$
13. $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x} dx$ [NCERT]
14. $\int \frac{\cos 2x + x + 1}{x^2 + \sin 2x + 2x} dx$

15. $\int \frac{-\sin x + 2\cos x}{2\sin x + \cos x}\,dx$
16. $\int \frac{\cos 4x - \cos 2x}{\sin 4x - \sin 2x}\,dx$
17. $\int \frac{\sec x}{\log(\sec x + \tan x)}\,dx$

18. $\int \frac{\operatorname{cosec} x}{\log \tan \frac{x}{2}}\,dx$
19. $\int \frac{\operatorname{cosec}^2 x}{1 + \cot x}\,dx$
20. $\int \frac{1 - \sin 2x}{x + \cos^2 x}\,dx$ [CBSE 2000]

21. $\int \frac{1 + \tan x}{x + \log \sec x}\,dx$ [CBSE 2000]
22. $\int \frac{\sin 2x}{a^2 + b^2 \sin^2 x}\,dx$
23. $\int \frac{x+1}{x(x + \log x)}\,dx$

24. $\int \frac{1}{\sqrt{1-x^2}\,(2 + 3\sin^{-1} x)}\,dx$
25. $\int \frac{\sec^2 x}{\tan x + 2}\,dx$
26. $\int \frac{2\cos 2x + \sec^2 x}{\sin 2x + \tan x - 5}\,dx$

BASED ON LOTS

27. $\int \frac{\sec x}{\sec 2x}\,dx$
28. $\int \frac{\cos 2x}{(\cos x + \sin x)^2}\,dx$ [NCERT]

29. $\int \frac{\sin(x-a)}{\sin(x-b)}\,dx$
30. $\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}\,dx$ [CBSE 2006, 2013, 2015]

31. $\int \frac{1 + \tan x}{1 - \tan x}\,dx$
32. $\int \frac{\cos x}{\cos(x-a)}\,dx$
33. $\int \sqrt{\frac{1 - \sin 2x}{1 + \sin 2x}}\,dx$

34. $\int \frac{1 - \cot x}{1 + \cot x}\,dx$
35. $\int \frac{\sec x \operatorname{cosec} x}{\log(\tan x)}\,dx$
36. $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x}\,dx$

37. $\int \frac{a}{b + ce^x}\,dx$
38. $\int \frac{1}{e^x + 1}\,dx$ [CBSE 2003]
39. $\int \frac{\cot x}{\log \sin x}\,dx$

BASED ON HOTS

40. $\int \frac{1}{\cos(x+a)\cos(x+b)}\,dx$ [NCERT]
41. $\int \frac{1}{x \log x \log(\log x)}\,dx$

42. $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}}\,dx$
43. $\int \frac{\sin 2x}{\sin 5x \sin 3x}\,dx$

44. $\int \frac{1 + \cot x}{x + \log \sin x}\,dx$ [CBSE 2000]
45. $\int \frac{\cos(x+a)}{\sin(x+b)}\,dx$ [CBSE 2019]

46. $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)}\,dx$
47. $\int \tan 2x \tan 3x \tan 5x\,dx$

48. $\int \{1 + \tan x \tan(x+\theta)\}\,dx$
49. $\int \frac{\sin 2x}{\sin\left(x - \frac{\pi}{6}\right)\sin\left(x + \frac{\pi}{6}\right)}\,dx$

50. $\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\,dx$
51. $\int \frac{1}{\sin x \cos^2 x}\,dx$
52. $\int \frac{1}{\cos 3x - \cos x}\,dx$

ANSWERS

1. $\frac{1}{\sqrt{2}} \log\left|\tan \frac{x}{2}\right| + C$
2. $\sqrt{2}\log\left|\tan\left(\frac{\pi}{4} + \frac{x}{4}\right)\right| + C$
3. $\log|\sin x| + C$

4. $-2\log\left|\cos\frac{x}{2}\right|+C$

5. $\frac{1}{3}\log|e^{3x}+1|+C$

6. $\frac{1}{3}\log|3\sec x+5|+C$

7. $\log|3+\log x|+C$

8. $\log|e^x+x|+C$

9. $\log|\log x|+C$

10. $\frac{1}{3}\log|2+3\sin x|+C$

11. $\log|x+\cos x|+C$

12. $\frac{1}{2}\log|e^{2x}-2|+C$

13. $\frac{1}{2}\log|2\sin x+3\cos x|+C$

14. $\frac{1}{2}\log|x^2+\sin 2x+2x|+C$

15. $\log|\cos x+2\sin x|+C$

16. $\frac{1}{3}\log|\cos 3x|+C$

17. $\log|\log(\sec x+\tan x)|+C$

18. $\log\left|\log\tan\frac{x}{2}\right|+C$

19. $-\log|1+\cot x|+C$

20. $\log|x+\cos^2 x|+C$

21. $\log|x+\log\sec x|+C$

22. $\frac{1}{b^2}\log(a^2+b^2\sin^2 x)+C$

23. $\log|x+\log x|+C$

24. $\frac{1}{3}\log|2+3\sin^{-1}x|+C$

25. $\log|\tan x+2|+C$

26. $\log|\sin 2x+\tan x-5|+C$

27. $2\sin x-\log|\sec x+\tan x|+C$

28. $\log|\sin x+\cos x|+C$

29. $x\cos(b-a)+\sin(b-a)\log|\sin(x-b)|+C$

30. $x\cos 2\alpha-\sin 2\alpha\log|\sin(x+\alpha)|+C$

31. $-\log|\cos x-\sin x|+C$

32. $(x-a)\cos a-\sin a\log|\sec(x-a)|+C$

33. $\log\left|\cos\left(\frac{\pi}{4}-x\right)\right|+C$

34. $-\log|\cos x+\sin x|+C$

35. $\log(\log\tan x)+C$

36. $\frac{1}{b-a}\log|a\cos^2 x+b\sin^2 x|+C$

37. $-\frac{a}{b}\log|be^{-x}+c|+C$

38. $-\log|1+e^{-x}|+C$

39. $\log|\log\sin x|+C$

40. $\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x+b)}{\cos(x+a)}\right|+C$

41. $\log\{\log(\log x)\}+C$

42. $\log|10^x+x^{10}|+C$

43. $\frac{1}{3}\log|\sin 3x|-\frac{1}{5}\log|\sin 5x|+C$

44. $\log|x+\log\sin x|+C$

45. $\cos(a-b)\log\sin(x+b)-x\sin(a-b)+C$

46. $2\log|\sqrt{x}+1|+C$

47. $\frac{1}{5}\log|\sec 5x|-\frac{1}{2}\log|\sec 2x|-\frac{1}{3}\log|\sec 3x|+C$

48. $\cot\theta\log\left|\frac{\cos x}{\cos(x+\theta)}\right|+C$

49. $\log\left|\sin^2 x-\frac{1}{4}\right|+C$

50. $\frac{1}{e}\log|e^x+x^e|+C$

51. $\sec x+\log\left|\tan\frac{x}{2}\right|+C$

52. $\frac{1}{4}\left\{\operatorname{cosec} x-\log|\sec x+\tan x|\right\}+C$

HINTS TO SELECTED PROBLEMS

27. $\int \frac{\sec x}{\sec 2x} dx = \int \frac{\cos 2x}{\cos x} dx = \int \frac{2\cos^2 x - 1}{\cos x} dx = \int 2\cos x - \sec x \, dx$

$$= 2\sin x - \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right) + C$$

28. Let $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx$. Then, $I = \int \frac{\cos 2x}{1 + \sin 2x} dx$

Let $1 + \sin 2x = t$. Then, $d(1 + \sin 2x) = dt$ or, $2\cos 2x \, dx = dt$

$\therefore \quad I = \frac{1}{2}\int \frac{1}{t} dt = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|1 + \sin 2x| + C$

<u>ALITER</u> $I = \int \frac{\cos 2x}{(\cos x + \sin x)^2} dx = \int \frac{\cos^2 x - \sin^2 x}{(\cos x + \sin x)} dx = \int \frac{\cos x - \sin x}{\cos x + \sin x} dx$

$\Rightarrow \quad I = \ln|\cos x + \sin x| + C$

40. Let $I = \int \frac{1}{\cos(x+a)\cos(x+b)} dx$. Then,

$$I = \frac{1}{\sin(b-a)} \int \frac{\sin\{(x+b)-(x+a)\}}{\cos(x+a)\cos(x+b)} dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(b-a)} \int \frac{\sin(x+b)\cos(x+a) - \cos(x+b)\sin(x+a)}{\cos(x+a)\cos(x+b)} dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(b-a)} \int \left\{\tan(x+b) - \tan(x+a)\right\} dx$$

$$\Rightarrow \quad I = \frac{1}{\sin(b-a)} \left\{-\ln\cos(x+b) + \ln\cos(x+a)\right\} + C = \frac{1}{\sin(a-b)} \ln\frac{\cos(x+b)}{\cos(x+a)} + C$$

50. $I = \int \frac{1}{\sin x \cos^2 x} dx = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} dx = \int (\sec x \tan x + \operatorname{cosec} x) \, dx$

$$= \sec x + \ln\left|\tan\frac{x}{2}\right| + C$$

51. $\int \frac{1}{\cos 3x - \cos x} dx = \int \frac{\sin^2 x + \cos^2 x}{-2\sin 2x \sin x} dx = \int \frac{\sin^2 x + \cos^2 x}{-4\sin^2 x \cos x} dx$

$$= -\frac{1}{4}\int \left(\sec x + \operatorname{cosec} x \cot x\right) dx = -\frac{1}{4}\left\{\log(\sec x + \tan x) - \operatorname{cosec} x\right\} + C$$

18.8.7 INTEGRALS OF THE FORM $\int \{f(x)\}^n f'(x) \, dx$

THEOREM $\int \{f(x)\}^n f'(x) \, dx = \frac{\{f(x)\}^{n+1}}{n+1}, n \neq -1$

<u>PROOF</u> Let $I = \int \{f(x)\}^n f'(x) \, dx$. Putting $f(x) = t$ and $f'(x) \, dx = dt$, we get

$$I = \int \{f(x)\}^n f'(x) \, dx = \int t^n \, dt = \frac{t^{n+1}}{n+1} + C = \frac{\{f(x)\}^{n+1}}{n+1} + C, n \neq -1.$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{3x+1}{(3x^2+2x+1)^3}\,dx$ (ii) $\int \sin^3 x \cos x\,dx$ (iii) $\int \tan^3 x \sec^2 x\,dx$ (iv) $\int \frac{(\log x)^3}{x}\,dx$

SOLUTION (i) Let $I = \int \frac{3x+1}{(3x^2+2x+1)^3}\,dx$. Let $3x^2+2x+1 = t$. Then,

$$d(3x^2+2x+1) = dt \Rightarrow (6x+2)\,dx = dt \Rightarrow dx = \frac{dt}{2(3x+1)}$$

Putting $3x^2+2x+1 = t$ and $dx = \frac{dt}{6x+2}$, we get

$$I = \int \frac{3x+1}{t^3} \times \frac{dt}{2(3x+1)} = \frac{1}{2}\int t^{-3}\,dt = \frac{1}{2}\left(\frac{t^{-2}}{-2}\right) + C = -\frac{1}{4t^2} + C = -\frac{1}{4(3x^2+2x+1)^2} + C$$

(ii) Let $I = \int \sin^3 x \cos x\,dx$. Let $\sin x = t$. Then, $d(\sin x) = dt \Rightarrow \cos x\,dx = dt \Rightarrow dx = \frac{dt}{\cos x}$.

Putting $\sin x = t$ and $dx = \frac{dt}{\cos x}$, we get

$$I = \int \sin^3 x \cos x\,dx = \int t^3 \cos x \times \frac{dt}{\cos x} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C$$

(iii) Let $I = \int \tan^3 x \sec^2 x\,dx$. Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\,dx = dt \Rightarrow dx = \frac{dt}{\sec^2 x}$.

Putting $\tan x = t$ and $dx = dt \sec^2 x$, we get

$$I = \int \tan^3 x \sec^2 x\,dx = \int t^3 \sec^2 x \times \frac{dt}{\sec^2 x} = \int t^3\,dt = \frac{t^4}{4} + C = \frac{\tan^4 x}{4} + C$$

(iv) Let $I = \int \frac{(\log x)^3}{x}\,dx$. Let $\log x = t$. Then, $d(\log x) = dt \Rightarrow \frac{1}{x}\,dx = dt \Rightarrow dx = x\,dt$.

Putting $\log x = t$ and $dx = x\,dt$, we get

$$I = \int \frac{t^3}{x}\,x\,dt = \int t^3\,dt = \int \frac{t^4}{4} + C = \frac{(\log x)^4}{4} + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{4(\sin^{-1} x)^3}{\sqrt{1-x^2}}\,dx$ (ii) $\int \frac{\log\left(\tan \frac{x}{2}\right)}{\sin x}\,dx$ (iii) $\int \frac{\sin x}{\sqrt{3+2\cos x}}\,dx$

SOLUTION (i) Let $I = \int \frac{4(\sin^{-1} x)^3}{\sqrt{1-x^2}}\,dx$. Let $\sin^{-1} x = t$. Then, $d(\sin^{-1} x) = dt$

$\Rightarrow \frac{1}{\sqrt{1-x^2}}\,dx = dt \Rightarrow dx = \sqrt{1-x^2}\,dt$. Putting $\sin^{-1} x = t$ and $dx = \sqrt{1-x^2}\,dt$, we get

$$I = \int \frac{4t^3}{\sqrt{1-x^2}} \sqrt{1-x^2}\, dt = 4\int t^3\, dt = t^4 + C = (\sin^{-1} x)^4 + C$$

(ii) Let $I = \int \frac{\log\left(\tan \frac{x}{2}\right)}{\sin x} dx$. Let $\log \tan \frac{x}{2} = t$. Then,

$$d\left(\log \tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{\tan \frac{x}{2}} \sec^2 \frac{x}{2} \times \frac{1}{2} dx = dt \Rightarrow dx = \sin x\, dt$$

Putting $\log \tan \frac{x}{2} = t$ and $dx = \sin x\, dt$, we get

$$I = \int \frac{t}{\sin x} \sin x\, dt = \int t\, dt = \frac{t^2}{2} + C = \frac{1}{2}\left(\log \tan \frac{x}{2}\right)^2 + C$$

(iii) Let $I = \int \frac{\sin x}{\sqrt{3 + 2\cos x}} dx$. Let $3 + 2\cos x = t$. Then,

$$d(3 + 2\cos x) = dt \Rightarrow -2\sin x\, dx = dt \Rightarrow dx = -\frac{dt}{2\sin x}.$$

Putting $3 + 2\cos x = t$ and $dx = -\frac{dt}{2\sin x}$, we get

$$I = \int \frac{\sin x}{\sqrt{t}} \times -\frac{dt}{2\sin x} = -\frac{1}{2}\int t^{-1/2}\, dt = -\frac{1}{2} \times \frac{t^{1/2}}{1/2} + C = -\sqrt{t} + C = -\sqrt{3 + 2\cos x} + C$$

EXAMPLE 3 *Evaluate:*

(i) $\int \frac{(1 + \log x)^2}{x} dx$ [NCERT, CBSE 2009] (ii) $\int \frac{\sec^2(2\tan^{-1} x)}{1 + x^2} dx$

(iii) $\int \frac{\tan x \sec^2 x}{(a + b\tan^2 x)^2} dx$ (iv) $\int \sec^3 x \tan x\, dx$

SOLUTION (i) Let $\int \frac{(1 + \log x)^2}{x} dx$. Let $1 + \log x = t$. Then,

$d(1 + \log x) = dt \Rightarrow \frac{1}{x} dx = dt \Rightarrow dx = x\, dt$. Putting $1 + \log x = t$ and $dx = = x\, dt$, we get

$$\therefore \quad I = \int \frac{t^2}{x} \times x\, dt = \int t^2\, dt = \frac{t^3}{3} + C = \frac{(1 + \log x)^3}{3} + C$$

(ii) Let $I = \int \frac{\sec^2(2\tan^{-1} x)}{1 + x^2} dx$. Let $2\tan^{-1} x = t$. Then,

$$d(2\tan^{-1} x) = dt \Rightarrow \frac{2}{1 + x^2} dx = dt \Rightarrow dx = \frac{1 + x^2}{2} dt.$$

Putting $2\tan^{-1} x = t$ and $dx = \frac{1 + x^2}{2} dt$, we get

$$I = \int \frac{\sec^2 t}{1 + x^2} \times \frac{1 + x^2}{2} dt = \frac{1}{2}\int \sec^2 t\, dt = \frac{1}{2}\tan t + C = \frac{1}{2}\tan(2\tan^{-1} x) + C$$

(iii) Let $I = \int \frac{\tan x \sec^2 x}{(a + b \tan^2 x)^2} dx$. Let $a + b \tan^2 x = t$. Then,

$$d(a + b \tan^2 x) \; dt \Rightarrow 2b \tan x \sec^2 x \, dx = dt \Rightarrow dx = \frac{dt}{2b \tan x \sec^2 x}$$

Putting $a + b \tan^2 x = t$, and $dx = \frac{dt}{2b \tan x \sec^2 x}$, we get

$$I = \int \frac{\tan x \sec^2 x}{t^2} \times \frac{dt}{2b \tan x \sec^2 x} = \frac{1}{2b} \int \frac{1}{t^2} dt = \frac{1}{2b} \int t^{-2} dt = -\frac{1}{2bt} + C$$

$$\Rightarrow \quad I = -\frac{1}{2b(a + b \tan^2 x)} + C$$

(iv) Let $I = \int \sec^3 x \tan x \, dx = \int \sec^2 x \, (\sec x \tan x) \, dx$

Let $\sec x = t$. Then, $d(\sec x) = dt \Rightarrow \sec x \tan x \, dx = dt \Rightarrow dx = \frac{dt}{\sec x \tan x}$

Putting $\sec x = t$ and $dx = \frac{dt}{\sec x \tan x}$, we get

$$I = \int t^2 \, dt = \frac{t^3}{3} + C = \frac{1}{3} \sec^3 x + C$$

EXAMPLE 4 *Evaluate:*

(i) $\int x^3 \sin x^4 \, dx$ (ii) $\int e^{-x} \operatorname{cosec}^2 (2e^{-x} + 5) \, dx$ (iii) $\int x^2 \frac{\tan^{-1} x^3}{1 + x^6} dx$

SOLUTION (i) Let $I = \int x^3 \sin x^4 \, dx$.

Let $x^4 = t$. Then, $d(x^4) = dt \Rightarrow 4x^3 \, dx = dt \Rightarrow dx = \frac{1}{4x^3} dt$

$$\therefore \quad I = \int x^3 \sin t \times \frac{dt}{4x^3} = \frac{1}{4} \int \sin t \, dt = -\frac{1}{4} \cos t + C = -\frac{1}{4} \cos x^4 + C$$

(ii) Let $I = \int e^{-x} \operatorname{cosec}^2 (2e^{-x} + 5) \, dx$.

Let $2e^{-x} + 5 = t$ Then, $d(2e^{-x} + 5) = dt \Rightarrow -2e^{-x} \, dx = dt \Rightarrow dx = -\frac{dt}{2e^{-x}}$

$$\therefore \quad I = \int e^{-x} \operatorname{cosec}^2 t \left(-\frac{dt}{2e^{-x}} \right) = -\frac{1}{2} \int \operatorname{cosec}^2 t \, dt = \frac{1}{2} \cot t + C = \frac{1}{2} \cot (2e^{-x} + 5) + C$$

(iii) Let $I = \int x^2 \frac{\tan^{-1} x^3}{1 + x^6} dx$.

Let $\tan^{-1} x^3 = t$. Then, $d(\tan^{-1} x^3) = dt \Rightarrow \frac{1}{1 + x^6} 3x^2 \, dx = dt \Rightarrow dx = \frac{(1 + x^6)}{3x^2} dt$

$$\therefore \quad I = \int x^2 \times \frac{t}{1 + x^6} \times \frac{1 + x^6}{3x^2} dt = \frac{1}{3} \int t \, dt = \frac{1}{6} t^2 + C = \frac{1}{6} \{\tan^{-1} x^3\}^2 + C$$

EXAMPLE 5 *Evaluate:*

(i) $\int \sqrt{\tan x}\,(1+\tan^2 x)\,dx$ (ii) $\int \{f(ax+b)\}^n f'(ax+b)\,dx, n \neq -1$ [NCERT]

SOLUTION (i) Let $I = \int \sqrt{\tan x}\,(1+\tan^2 x)\,dx = \int \sqrt{\tan x}\,\sec^2 x\,dx$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$I = \int \sqrt{t}\,dt = \frac{t^{3/2}}{3/2} + C = \frac{2}{3}(\tan x)^{3/2} + C$$

(ii) Let $I = \int \{f(ax+b)\}^n f'(ax+b)\,dx$. Putting $f(ax+b) = t$ and $f'(ax+b).\,adx = dt$, we get

$$I = \frac{1}{a}\int t^n\,dt = \frac{1}{a}\left(\frac{t^{n+1}}{n+1}\right) + C = \frac{\{f(ax+b)\}^{n+1}}{a(n+1)} + C, n \neq -1$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Evaluate:*

(i) $\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx$ (ii) $\int 2^{2^{2^x}}\, 2^{2^x}\, 2^x\,dx$

(iii) $\int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}\,dx, \alpha \neq n\pi, n \in Z$ [NCERT] (iv) $\int \frac{(x^4 - x)^{1/4}}{x^5}\,dx$

SOLUTION (i) Let $I = \int \frac{\sin 2x}{(a+b\cos x)^2}\,dx = \int \frac{2\sin x \cos x}{(a+b\cos x)^2}\,dx$

Putting $a + b\cos x = t$ and $-b\sin x\,dx = dt$ or, $dx = -dt/b\sin x$, we get

$$I = \int \frac{2\sin x\cos x}{t^2} \times -\frac{dt}{b\sin x} = -\frac{2}{b}\int \frac{1}{t^2}\cdot\cos x\,dt$$

$$\Rightarrow \quad I = -\frac{2}{b}\int \frac{1}{t^2}\left(\frac{t-a}{b}\right)dt = -\frac{2}{b^2}\int\left(\frac{1}{t} - \frac{a}{t^2}\right)dt \quad \left[\because a+b\cos x = t \therefore \cos x = \frac{t-a}{b}\right]$$

$$\Rightarrow \quad I = -\frac{2}{b^2}\left\{\log|t| + \frac{a}{t}\right\} + C = -\frac{2}{b^2}\left\{\log|a+b\cos x| + \frac{a}{a+b\cos x}\right\} + C$$

(ii) Let $I = \int 2^{2^{2^x}}\, 2^{2^x}\, 2^x\,dx$. Let $2^{2^{2^x}} = t$. Then,

$$d\left(2^{2^{2^x}}\right) = dt \Rightarrow 2^{2^{2^x}}\, 2^{2^x}\, 2^x (\log 2)^3\,dx = dt.$$

Putting $2^{2^{2^x}} = t$ and $2^{2^{2^x}}\, 2^{2^x}\, 2^x (\log 2)^3\,dx = dt$, we get

$$I = \int \frac{1}{(\log 2)^3}\,dt = \frac{1}{(\log 2)^3}\,t + C = \frac{1}{(\log 2)^3}\, 2^{2^{2^x}} + C$$

(iii) We find that

$$\sin^3 x \sin(x+\alpha) = \sin^3 x(\sin x\cos\alpha + \cos x\sin\alpha) = \sin^4 x(\cos\alpha + \cot x\sin\alpha)$$

$$\therefore \quad I = \int \frac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sin^2 x\sqrt{\cos\alpha + \cot x \sin\alpha}}\,dx = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos\alpha + \cot x\sin\alpha}}\,dx$$

Putting $\cos\alpha + \cot x \sin\alpha = t$ and, $-\operatorname{cosec}^2 x \sin\alpha\, dx = dt$, we get

$$I = \int -\frac{1}{\sin\alpha\sqrt{t}}\,dt = -\frac{1}{\sin\alpha}\int t^{-1/2}\,dt = -\frac{1}{\sin\alpha}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad I = -2(\operatorname{cosec}\alpha)\sqrt{t} + C = -2\operatorname{cosec}\alpha\,(\cos\alpha + \cot x\sin\alpha)^{1/2} + C$$

(iv) Let $I = \int \frac{(x^4 - x)^{1/4}}{x^5} = \int \frac{x\left(1 - \frac{1}{x^3}\right)^{1/4}}{x^5}\,dx = \int \frac{1}{x^4}\left(1 - \frac{1}{x^3}\right)^{1/4} dx$

Let $1 - \frac{1}{x^3} = t$. Then, $d\left(1 - \frac{1}{x^3}\right) = dt \Rightarrow \frac{3}{x^4}\,dx = dt \Rightarrow dx = \frac{x^4}{3}\,dt$

Putting $\left(1 - \frac{1}{x^3}\right) = t$ and, $dx = \frac{x^4}{3}\,dt$, we get

$$I = \frac{1}{3}\int \frac{1}{x^4} \times t^{1/4} \times x^4\,dt = \frac{1}{3}\int t^{1/4}\,dt = \frac{4}{15}t^{5/4} + C = \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{5/4} + C$$

EXAMPLE 7 *Evaluate:* (i) $\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$ (ii) $\int \frac{\cos^9 x}{\sin x}\,dx$

SOLUTION (i) Let $I = \int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$. Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$I = \int \frac{\sec^4 x}{\sqrt{t}} \times \frac{dt}{\sec^2 x} = \int \frac{\sec^2 x}{\sqrt{t}}\,dt = \int \frac{1 + \tan^2 x}{\sqrt{t}}\,dt = \int \frac{1 + t^2}{\sqrt{t}}\,dt$$

$$\Rightarrow \quad I = \int (t^{-1/2} + t^{3/2})\,dt = 2t^{1/2} + \frac{2}{5}t^{5/2} + C = 2\sqrt{\tan x} + \frac{2}{5}\tan^{5/2} x + C$$

(ii) Let $I = \int \frac{\cos^9 x}{\sin x}\,dx$. Putting $\sin x = t$ and $\cos x\,dx = dt$, we get

$$I = \int \frac{\cos^9 x}{t} \times \frac{dt}{\cos x} = \int \frac{\cos^8 x}{t}\,dt = \int \frac{(1 - \sin^2 x)^4}{t}\,dt = \int \frac{(1 - t^2)^4}{t}\,dt$$

$$\Rightarrow \quad I = \int \frac{1 - 4t^2 + 6t^4 - 4t^6 + t^8}{t}\,dt = \int \frac{1}{t} - 4t + 6t^3 - 4t^5 + t^7\,dt$$

$$\Rightarrow \quad I = \log|t| - 2t^2 + \frac{3}{2}t^4 - \frac{2}{3}t^6 + \frac{1}{8}t^8 + C$$

$$\Rightarrow \quad I = \log|\sin x| - 2\sin^2 x + \frac{3}{2}\sin^4 x - \frac{2}{3}\sin^6 x + \frac{1}{8}\sin^8 x + C$$

EXAMPLE 8 *Evaluate:* $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}\,dx$

SOLUTION Let $\int \frac{1}{\sqrt[4]{(x-1)^3(x+2)^5}}\,dx$. Then,

$$I=\int \frac{1}{\sqrt[4]{\left(\frac{x-1}{x+2}\right)^3(x+2)^8}}\,dx=\int \frac{1}{(x+2)^2\left(\frac{x-1}{x+2}\right)^{3/4}}\,dx=\int \left(\frac{x-1}{x+2}\right)^{-3/4}\times\frac{1}{(x+2)^2}\,dx$$

Let $\frac{x-1}{x+2}=t$ or, $1-\frac{3}{x+2}=t$. Then, $d\left(1-\frac{3}{x+2}\right)=dt \Rightarrow \frac{3}{(x+2)^2}\,dx=dt \Rightarrow \frac{1}{(x+2)^2}\,dx=\frac{1}{3}dt$.

Putting $\frac{x-1}{x+2}=t$ and $\frac{1}{(x+2)^2}dx=\frac{1}{3}dt$, we obtain

$$I=\frac{1}{3}\int t^{-3/4}\,dt=\frac{4}{3}t^{1/4}+C=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}+C$$

EXAMPLE 9 *Evaluate:* (i) $\int x^x(1+\log x)\,dx$ (ii) $\int x^{2x}(1+\log x)\,dx$

SOLUTION (i) Let $x^x=t$. Then,

$$d(x^x)=dt \Rightarrow d(e^{x\log x})=dt \Rightarrow e^{x\log x}(\log x+1)\,dx=dt \Rightarrow x^x(1+\log x)\,dx=dt$$

$$\therefore \quad I=\int x^x(1+\log)\,dx=\int dt=t+C=x^x+C$$

(ii) Let $I=\int x^{2x}(1+\log x)\,dx$. Putting $x^x=t$ and $x^x(1+\log x)\,dx=dt$, we obtain

$$I=\int x^x\,x^x(1+\log x)\,dx=\int t\,dt=\frac{1}{2}t^2+C=\frac{1}{2}(x^x)^2+C=\frac{1}{2}x^{2x}+C$$

EXAMPLE 10 *Evaluate:* $\int \frac{x}{x-\sqrt{x^2-1}}\,dx$

SOLUTION Let $I=\int \frac{x}{x-\sqrt{x^2-1}}\,dx$. Then,

$$I=\int \frac{x}{x-\sqrt{x^2-1}}\times\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}\,dx=\int \frac{x(x+\sqrt{x^2-1})}{x^2-(x^2-1)}\,dx=\int x^2+x\sqrt{x^2-1}\,dx$$

$$\Rightarrow \quad I=\int x^2dx+\int \sqrt{x^2-1}\,x\,dx=\frac{x^3}{3}+\frac{1}{2}\int \sqrt{t}\,dt, \text{ where } t=x^2-1$$

$$\Rightarrow \quad I=\frac{x^3}{3}+\frac{1}{3}t^{3/2}+C=\frac{x^3}{3}+\frac{1}{3}(x^2-1)^{3/2}+C$$

EXAMPLE 11 *Evaluate:* $\int \frac{\cos^3 x}{\sin^2 x+\sin x}\,dx$

SOLUTION Let $I=\int \frac{\cos^3 x}{\sin^2 x+\sin x}\,dx$. Then,

$$I=\int \frac{\cos^2 x}{\sin x\,(\sin x+1)}\cos x\,dx=\int \frac{(1-\sin^2 x)}{\sin x\,(1+\sin x)}\cos x\,dx$$

Let $\sin x = t$. Then, $d(\sin x)=dt$ or, $\cos x\,dx = dt$, we obtain

$$\therefore \quad I=\int \frac{1-t^2}{t\,(1+t)}dt=\int \frac{1-t}{t}\,dt=\int\left(\frac{1}{t}-1\right)dt=\log|t|-t+C = \log|\sin x|-\sin x+C$$

EXERCISE 18.9

Evaluate the following integrals:

1. $\int \frac{\log x}{x}\,dx$
2. $\int \sqrt{1+e^x}\,e^x\,dx$
3. $\int \sqrt[3]{\cos^2 x}\,\sin x\,dx$
4. $\int \frac{e^x}{(1+e^x)^2}\,dx$
5. $\int \cot^3 x\,\mathrm{cosec}^2 x\,dx$
6. $\int \frac{\{e^{\sin^{-1} x}\}^2}{\sqrt{1-x^2}}\,dx$
7. $\int \frac{1+\sin x}{\sqrt{x-\cos x}}\,dx$
8. $\int \frac{1}{\sqrt{1-x^2}\,(\sin^{-1} x)^2}\,dx$
9. $\int \frac{1}{\sqrt{\tan^{-1} x}\,(1+x^2)}\,dx$
10. $\int \frac{1}{x}(\log x)^2\,dx$
11. $\int \sin^5 x\cos x\,dx$
12. $\int \tan^{3/2} x\sec^2 x\,dx$
13. $\int \frac{1+\cos x}{(x+\sin x)^3}\,dx$
14. $\int \frac{\cos x-\sin x}{1+\sin 2x}\,dx$ [NCERT]
15. $\int \frac{\sin 2x}{(a+b\cos 2x)^2}\,dx$
16. $\int \frac{\log x^2}{x}\,dx$
17. $\int \frac{\sin x}{(1+\cos x)^2}\,dx$
18. $\int x^3\cos x^4\,dx$
19. $\int \frac{x\sin^{-1} x^2}{\sqrt{1-x^4}}\,dx$
20. $\int x^3\sin(x^4+1)\,dx$
21. $\int x^2 e^{x^3}\cos(e^{x^3})\,dx$
22. $\int 2x\sec^3(x^2+3)\tan(x^2+3)\,dx$
23. $\int\left(\frac{x+1}{x}\right)(x+\log x)^2\,dx$ [NCERT, CBSE 2002C]
24. $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$
25. $\int \frac{e^{\sqrt{x}}\cos(e^{\sqrt{x}})}{\sqrt{x}}\,dx$
26. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ [NCERT, CBSE 2009]
27. $\int \frac{e^m\sin^{-1} x}{\sqrt{1-x^2}}\,dx$
28. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ [NCERT, CBSE 2009]
29. $\int \frac{\sin(\log x)}{x}\,dx$
30. $\int \frac{e^{m\tan^{-1} x}}{1+x^2}\,dx$ [NCERT]
31. $\int \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\,dx$
32. $\int \frac{\sin(2+3\log x)}{x}\,dx$
33. $\int x\,e^{x^2}\,dx$
34. $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}\,dx$ [CBSE 2000]
35. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}}\,dx$

BASED ON LOTS

36. $\int \frac{\log\left(1+\frac{1}{x}\right)}{x(1+x)}dx$

37. $\int \frac{\cot x}{\sqrt{\sin x}}\, dx$

38. $\int \frac{\tan x}{\sqrt{\cos x}}dx$

39. $\int \frac{\cos^3 x}{\sqrt{\sin x}}dx$

40. $\int \frac{\sin^3 x}{\sqrt{\cos x}}dx$

41. (i) $\int \frac{\sqrt{\tan x}}{\sin x \cos x}dx$ [NCERT]

(ii) $\int \frac{\sqrt{\cot x}}{\sin x \cos x}\, dx$ [CBSE 2022]

42. $\int \frac{x^3}{(x^2+1)^3}\, dx$

43. $\int (4x+2)\sqrt{x^2+x+1}\, dx$ [NCERT]

44. $\int \frac{4x+3}{\sqrt{2x^2+3x+1}}\, dx$

45. $\int e^{\cos^2 x} \sin 2x\, dx$

46. $\int \cot x \log \sin x\, dx$ [NCERT]

47. $\int \sec x \ln(\sec x + \tan x)\, dx$ [NCERT]

48. $\int \operatorname{cosec} x \log(\operatorname{cosec} x - \cot x)\, dx$

49. $\int \frac{2x}{\sqrt[3]{x^2-1}}dx$ [CBSE 2020]

50. $\int \frac{(x+1)\, e^x}{\cos^2(xe^x)}\, dx$

51. $\int \frac{(x+1)\, e^x}{\sin^2(xe^x)}\, dx$

52. $\int \frac{x \tan^{-1} x^2}{1+x^4}\, dx$

53. $\int \frac{1}{1+\sqrt{x}}\, dx$

54. $\int \tan x \sec^2 x \sqrt{1-\tan^2 x}\, dx$

55. $\int \log x \frac{\sin\{1+(\log x)^2\}}{x}dx$

56. $\int \sec^4 x \tan x\, dx$

57. $\int \frac{\cos^5 x}{\sin x}\, dx$ [CBSE 2005]

58. $\int 5^{x+\tan^{-1} x}\left(\frac{x^2+2}{x^2+1}\right)dx$

59. $\int \frac{\sin(\tan^{-1} x)}{1+x^2}\, dx$ [CBSE 2002]

60. $\int \frac{x}{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}\, dx$

61. $\int \frac{e^{2x}}{1+e^x}\, dx$

62. $\int \tan^3 2x \sec 2x\, dx$ [NCERT]

63. $\int \frac{x+\sqrt{x+1}}{x+2}\, dx$

64. $\int 5^{5^{5^x}}\, 5^{5^x}\, 5^x\, dx$

65. $\int \frac{1}{x\sqrt{x^4-1}}\, dx$ [NCERT EXEMPLAR]

66. $\int \sqrt{e^x-1}\, dx$

67. $\int \frac{1}{(x+1)(x^2+2x+2)}\, dx$

68. $\int \frac{x^5}{\sqrt{1+x^3}}\, dx$

69. $\int 4x^3\sqrt{5-x^2}\, dx$

70. $\int \frac{1}{\sqrt{x}+x}\, dx$

71. $\int \frac{1}{x^2(x^4+1)^{3/4}}\, dx$ [NCERT]

72. $\int \frac{\sin^5 x}{\cos^4 x}\, dx$

ANSWERS

1. $\frac{(\log x)^2}{2}+C$

2. $\frac{2}{3}(1+e^x)^{3/2}+C$

3. $-\frac{3}{5}\cos^{5/3} x + C$

4. $-\dfrac{1}{(1+e^x)}+C$

5. $-\dfrac{1}{4}\cot^4 x+C$

6. $\dfrac{1}{2}\{e^{\sin^{-1}x}\}^2+C$

7. $2\sqrt{x-\cos x}+C$

8. $-\dfrac{1}{\sin^{-1}x}+C$

9. $2\sqrt{\tan^{-1}x}+C$

10. $\dfrac{1}{3}(\log x)^3+C$

11. $\dfrac{1}{6}\sin^6 x+C$

12. $\dfrac{2}{5}\tan^{5/2}x+C$

13. $\dfrac{-1}{2(x+\sin x)^2}+C$

14. $-\dfrac{1}{(\sin x+\cos x)}+C$

15. $\dfrac{1}{2b(a+b\cos 2x)}+C$

16. $(\log x)^2+C$

17. $\dfrac{1}{1+\cos x}+C$

18. $\dfrac{1}{4}\sin x^4+C$

19. $\dfrac{1}{4}(\sin^{-1}x^2)^2+C$

20. $-\dfrac{1}{4}\cos(x^4+1)+C$

21. $\dfrac{1}{3}\sin(e^{x^3})+C$

22. $\dfrac{1}{3}\sec^3(x^2+3)+C$

23. $\dfrac{1}{3}(x+\log x)^3+C$

24. $-\dfrac{1}{2}\left(\dfrac{1}{x}\right)-\dfrac{1}{4}\sin\left(\dfrac{2}{x}\right)+C$

25. $2\sin(e^{\sqrt{x}})+C$

26. $-2\cos\sqrt{x}+C$

27. $\dfrac{1}{m}e^{m\sin^{-1}x}+C$

28. $2\sin\sqrt{x}+C$

29. $-\cos(\log x)+C$

30. $\dfrac{1}{m}e^{m\tan^{-1}x}+C$

31. $\dfrac{1}{4}(\sin^{-1}x)^4+C$

32. $-\dfrac{1}{3}\cos(2+3\ln x)+C$

33. $\dfrac{1}{2}e^{x^2}+C$

34. $2\tan\sqrt{x}+C$

35. $\dfrac{2}{3}(1+\sqrt{x})^3+C$

36. $-\dfrac{1}{2}\left\{\ln\left(1+\dfrac{1}{x}\right)\right\}^2+C$

37. $-\dfrac{2}{\sqrt{\sin x}}+C$

38. $\dfrac{2}{\sqrt{\cos x}}+C$

39. $2\sqrt{\sin x}-\dfrac{2}{5}(\sin x)^{5/2}+C$

40. $-2\sqrt{\cos x}+\dfrac{2}{5}\cos^{5/2}x+C$

41. (i) $2\sqrt{\tan x}+C$ (ii) $-2\sqrt{\cot x}+C$

42. $-\dfrac{(1+2x^2)}{4(x^2+1)^2}+C$

43. $\dfrac{4}{3}(x^2+x+1)^{3/2}+C$

44. $2\sqrt{2x^2+3x+1}+C$

45. $-e^{\cos^2 x}+C$

46. $\dfrac{1}{2}\left\{\log|\sin x|\right\}^2+C$

47. $\dfrac{1}{2}\left\{\log|\sec x+\tan x|\right\}^2+C$

48. $\dfrac{1}{2}\left\{\log|\operatorname{cosec} x-\cot x|\right\}^2+C$

49. $\dfrac{3}{2}(x^2-1)^{2/3}+C$

50. $\tan(xe^x)+C$

51. $-\cot(xe^x)+C$

52. $\dfrac{1}{4}(\tan^{-1}x^2)^2+C$

53. $2\sqrt{x}-2\ln|1+\sqrt{x}|+C$

54. $-\dfrac{1}{3}(1-\tan^2 x)^{3/2}+C$

55. $-\dfrac{1}{2}\cos\{1+(\ln x)^2\}+C$

56. $\dfrac{1}{2}\tan^2 x+\dfrac{1}{4}\tan^4 x+C$

57. $\dfrac{1}{4}\sin^4 x-\sin^2 x+\log|\sin x|+C$

58. $\dfrac{1}{\ln 5}(5^{x+\tan^{-1}x})+C$

59. $-\cos(\tan^{-1}x)+C$

60. $\dfrac{1}{6a^2}\{(x^2+a^2)^{3/2}-(x^2-a^2)^{3/2}\}+C$

61. $e^x-\log(1+e^x)+C$

62. $\frac{1}{6}\sec^3 2x - \frac{1}{2}\sec 2x + C$ 63. $(x+1) + 2\sqrt{x+1} - 2\log|x+2| - 2\tan^{-1}\sqrt{x+1} + C$

64. $\frac{5^{5^{5^x}}}{(\log_e 5)^3} + C$ 65. $\frac{1}{2}\sec^{-1}(x^2) + C$ 66. $2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$

67. $\log\left|\frac{x+1}{\sqrt{x^2+2x+2}}\right| + C$ 68. $\frac{2}{9}(1+x^3)^{3/2} - \frac{2}{3}(1+x^3)^{1/2} + C$

69. $\frac{4}{5}(5-x^2)^{5/2} - \frac{20}{3}(5-x^2)^{3/2} + C$ 70. $2\log|1+\sqrt{x}| + C$

71. $-\left(1+\frac{1}{x^4}\right)^{1/4} + C$ 72. $-\cos x - \frac{2}{\cos x} + \frac{1}{3\cos^3 x} + C$

18.8.8 INTEGRALS OF THE FORM $\int (ax+b)^n P(x)\,dx$, $\int \frac{P(x)}{(ax+b)^n}\,dx$, WHERE $P(x)$ IS A POLYNOMIAL AND n IS A POSITIVE RATIONAL NUMBER

In order to evaluate this type of integrals, we may follow the following algorithm.

ALGORITHM

Step I *Substitute* $ax + b = t$ *or*, $x = \frac{t-b}{a}$ *and* $dx = \frac{1}{a}\,dt$

Step II *Simplify the integrand in terms of t and integrate with respect to t by using* $\int t^n dt = \frac{t^{n+1}}{n+1} + C$.

Step III *Replace t by* $ax + b$

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\int \frac{x^2}{\sqrt{x+2}}\,dx$

SOLUTION Let $I = \int \frac{x^2}{\sqrt{x+2}}\,dx$. Substituting $x + 2 = t$ and $dx = dt$, we get

$$I = \int \frac{(t-2)^2}{\sqrt{t}}\,dt = \int \frac{t^2 - 4t + 4}{\sqrt{t}}\,dt = \int (t^{3/2} - 4t^{1/2} + 4t^{-1/2})\,dt$$

$$\Rightarrow \quad I = \frac{2}{5}t^{5/2} - \frac{8}{3}t^{3/2} + 8t^{1/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{8}{3}(x+2)^{3/2} + 8\sqrt{x+2} + C$$

EXAMPLE 2 *Evaluate:* $\int x^2\sqrt{1+x}\,dx$

SOLUTION Let $I = \int x^2\sqrt{1+x}\,dx$. Substituting $1 + x = t$ and $dx = dt$, we get

$$I = \int (t-1)^2\sqrt{t}\,dt = \int (t^2 - 2t + 1)\sqrt{t}\,dt = \int (t^{5/2} - 2t^{3/2} + t^{1/2})\,dt$$

$$\Rightarrow \quad I = \frac{2}{7}t^{7/2} - \frac{4}{5}t^{5/2} + \frac{2}{3}t^{3/2} + C = \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C$$

EXAMPLE 3 *Evaluate:* $\int x(1-x)^n \, dx$

SOLUTION Let $I = \int x(1-x)^n \, dx$. Substituting $1-x = t$ and $dx = -dt$, we get

$$I = -\int (1-t)\, t^n \, dt = -\int (t^n - t^{n+1})\, dt$$

$$\Rightarrow \quad I = -\frac{t^{n+1}}{n+1} + \frac{t^{n+2}}{n+2} + C = -\frac{1}{n+1}(1-x)^{n+1} + \frac{1}{n+2}(1-x)^{n+2} + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Evaluate:* $\int \frac{x^5}{x+1} \, dx$

SOLUTION Let $I = \int \frac{x^5}{x+1} \, dx$. Substituting $x+1 = t$ and $dx = dt$, we get

$$I = \int \frac{(t-1)^5}{t} \, dt = \int \frac{1}{t} \left({}^5C_0 \, t^5 - {}^5C_1 \, t^4 + {}^5C_2 \, t^3 - {}^5C_3 \, t^2 + {}^5C_4 \, t - {}^5C_5) \, dt \right) dt$$

$$\Rightarrow \quad I = \int \frac{1}{t}(t^5 - 5t^4 + 10t^3 - 10t^2 + 5t - 1)\, dt$$

$$\Rightarrow \quad I = \int \left(t^4 - 5t^3 + 10t^2 - 10t + 5 - \frac{1}{t} \right) dt = \frac{t^5}{5} - \frac{5}{4}t^4 + \frac{10}{3}t^3 - 5t^2 + 5t - \log|t| + C$$

$$\Rightarrow \quad I = \frac{1}{5}(x+1)^5 - \frac{5}{4}(x+1)^4 + \frac{10}{3}(x+1)^3 - 5(x+1)^2 + 5(x+1) - \log|x+1| + C$$

EXAMPLE 5 *Evaluate:* $\int \frac{x^2}{(a+bx)^2} \, dx$

SOLUTION Let $I = \int \frac{x^2}{(a+bx)^2} \, dx$. Substituting $a+bx = t$ and $d(a+bx) = dt$ or, $b\,dx = dt$, we get

$$I = \int \frac{(t-a)^2}{b^2 t^2} \times \frac{1}{b} \, dt = \frac{1}{b^3} \int \frac{t^2 - 2at + a^2}{t^2} \, dt = \frac{1}{b^3} \int \left(1 - \frac{2a}{t} + \frac{a^2}{t^2} \right) dt$$

$$\Rightarrow \quad I = \frac{1}{b^3} \left\{ t - 2a \log|t| - \frac{a^2}{t} \right\} + C = \frac{1}{b^3} \left\{ (a+bx) - 2a \log|a+bx| - \frac{a^2}{a+bx} \right\} + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Evaluate:* $\int \frac{1}{x^{1/2} + x^{1/3}} \, dx$ **[NCERT, CBSE 2022]**

SOLUTION Here, the exponents of x are $\frac{1}{2}$ and $\frac{1}{3}$ and the LCM of their denominators is 6.

So, to remove fractional exponents, we substitute $x = t^6$ and $dx = 6t^5 \, dt$.

$$\therefore \quad I = \int \frac{1}{x^{1/2} + x^{1/3}} \, dx = \int \frac{6t^5}{t^3 + t^2} \, dt = 6 \int \frac{t^3}{t+1} \, dt = 6 \int \frac{(t^3+1) - 1}{t+1} \, dt$$

$$\Rightarrow \quad I = 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt$$

$\Rightarrow \qquad I = 6\left\{\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right\} + C = 2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6} + 1| + C$

EXAMPLE 7 *Evaluate:* $\int \frac{x^{1/2}}{1 + x^{3/4}}\, dx$

SOLUTION Here, the LCM of the denominators 2 and 4 of the exponents $\frac{1}{2}$ and $\frac{3}{4}$ is 4. So, to remove fractional exponents, we substitute $x = t^4$ and $dx = 4t^3 dt$.

$\therefore \qquad I = \int \frac{x^{1/2}}{1+x^{3/4}}\, dx = \int \frac{t^2}{1+t^3} 4t^3 dt = 4\int \frac{t^5}{t^3+1}\, dt = 4\int \frac{t^3}{t^3+1} t^2\, dt$

Let $t^3 + 1 = u$. Then $3t^2 dt = du$ or, $t^2 dt = \frac{1}{3} du$.

$\therefore \qquad I = 4\int \frac{u-1}{u} \times \frac{1}{3}\, du = \frac{4}{3}\int\left(1 - \frac{1}{u}\right) du = \frac{4}{3}(u - \ln u) + C = \frac{4}{3}\left\{(t^3+1) - \ln(t^3+1)\right\} + C$

$\Rightarrow \qquad I = \frac{4}{3}\left\{\left(x^{3/4}+1\right) - \ln\left(x^{3/4}+1\right)\right\} + C$

EXAMPLE 8 *Evaluate :* $\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}}\, dx$

SOLUTION Let $I = \int \frac{x^{1/2}}{x^{1/2} - x^{1/3}}\, dx$. Clearly, the LCM of 2 and 3 is 6. So, by putting $x = t^6$ and $dx = 6t^5\, dt$, we get

$$I = \int \frac{t^3}{t^3 - t^2} \cdot 6t^5\, dt = 6\int \frac{t^6}{t-1}\, dt = 6\int \frac{t^6 - 1 + 1}{t-1}\, dt = 6\int \frac{t^6 - 1^6}{t-1} + \frac{1}{t-1}\, dt$$

$\Rightarrow \qquad I = 6\int t^5 + t^4 + t^2 + t + 1 + \frac{1}{t-1}\, dt = 6\left\{\frac{t^6}{6} + \frac{t^5}{5} + \frac{t^3}{3} + \frac{t^2}{2} + t + \log(t-1)\right\} + C$

$\Rightarrow \qquad I = 6\left\{\frac{x}{6} + \frac{x^{5/6}}{5} + \frac{x^{1/2}}{3} + \frac{x^{1/3}}{2} + x^{1/6} + \log(x^{1/6} - 1)\right\} + C$

EXAMPLE 9 *Evaluate :* $\int \frac{1}{\sqrt[3]{x+1} + \sqrt{x+1}}\, dx.$

SOLUTION Let $\int \frac{1}{\sqrt[3]{x+1} + \sqrt{x+1}}\, dx$. Here the exponents of $(1+x)$ are $\frac{1}{2}$ and $\frac{1}{3}$ and the LCM of their denominators is 6. So, we substitute $x + 1 = t^6$ and $dx = 6t^5 dt$.

$\therefore \qquad I = \int \frac{1}{t^2 + t^3} 6t^5\, dt = 6\int \frac{t^3}{t+1}\, dt = 6\int \frac{t^3 + 1 - 1}{t+1}\, dt$

$\Rightarrow \qquad I = 6\int \frac{t^3+1}{t+1}\, dt - 6\int \frac{1}{t+1}\, dt = 6\int (t^2 - t + 1)\, dt - 6\int \frac{1}{t+1}\, dt$

$\Rightarrow \qquad I = 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t\right) - 6\log|t+1| + C = 2t^3 - 3t^2 + 6t - 6\log|t+1| + C$

$\Rightarrow \qquad I = 2(x+1)^{1/2} - 3(x+1)^{1/3} + 6(x+1)^{1/6} - 6\log|(x+1)^{16} + 1| + C$

EXERCISE 18.10

BASIC

1. $\int x^2 \sqrt{x+2}\ dx$
2. $\int \frac{x^2}{\sqrt{x-1}}\ dx$
3. $\int \frac{x^2}{\sqrt{3x+4}}\ dx$

BASED ON LOTS

4. $\int \frac{2x-1}{(x-1)^2}\ dx$
5. $\int (2x^2+3)\sqrt{x+2}\ dx$
6. $\int \frac{x^2+3x+1}{(x+1)^2}\ dx$
7. $\int \frac{x^2}{\sqrt{1-x}}\ dx$
8. $\int x(1-x)^{23}\ dx$

BASED ON HOTS

9. $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\ dx$
10. $\int \frac{1}{x^{1/3}(x^{1/3}-1)}\ dx$

ANSWERS

1. $\frac{2}{7}(x+2)^{7/2}-\frac{8}{5}(x+2)^{5/2}+\frac{8}{3}(x+2)^{3/2}+C$
2. $\frac{2}{15}(3x^2+4x+8)\sqrt{x-1}+C$
3. $\frac{2}{135}(3x+4)^{5/2}-\frac{16}{81}(3x+4)^{3/2}+\frac{32}{27}(3x+4)^{1/2}+C$
4. $-\frac{1}{x-1}+2\log|x-1|+C$
5. $\frac{4}{7}(x+2)^{7/2}-\frac{16}{5}(x+2)^{5/2}+\frac{22}{3}(x+2)^{3/2}+C$
6. $x+\frac{1}{x+1}+\log|x+1|+C$
7. $\frac{2}{15}(3x^2+4x+8)\sqrt{1-x}+C$
8. $-\frac{1}{600}(1-x)^{24}(1+24x)+C$
9. $2\sqrt{x}-4x^{1/4}+4\log|1+x^4|+C$
10. $3x^{1/3}+3\log(x^{1/3}-1)+C$

18.8.9 INTEGRALS OF THE FORM $\int \tan^m x \sec^{2n} x\, dx, \int \cot^m x \operatorname{cosec}^{2n} x\, dx;\ m, n \in N$

In order to evaluate this type of integrals. We may follow the following algorithm.

ALGORITHM

Step I *Write the given integral as* $I=\int \tan^m x(\sec^2 x)^{(n-1)}\sec^2 x\ dx$

Step II *Put* $\tan x = t$ *and* $\sec^2 x\, dx = dt$ *and write the integral as*

$$I=\int \tan^m x(\sec^2 x)^{n-1}\sec^2 x\, dx=\int \tan^m x(1+\tan^2 x)^{n-1}\sec^2 x\, dx=\int t^m(1+t^2)^{n-1}dt$$

Step III *Expand* $(1+t^2)^{n-1}$ *by binomial theorem in step II and integrate.*

Step IV *Replace* t *by* $\tan x$ *in step III.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \tan^n x \sec^2 x\, dx$ (ii) $\int \tan^2 x \sec^4 x\, dx$ (iii) $\int \sec^4 x\, dx$

SOLUTION (i) Let $I=\int \tan^n x \sec^2 x\, dx$. Substituting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$I=\int t^n\, dt=\frac{t^{n+1}}{n+1}+C=\frac{1}{n+1}\tan^{n+1} x+C$$

(ii) Let $I = \int \tan^2 x \sec^4 x \ dx$. Then,

$$I = \int \tan^2 x \sec^2 x \sec^2 x \, dx = \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$$

Substituting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int t^2 (1 + t^2) \, dt = \int (t^2 + t^4) \, dt = \frac{t^3}{3} + \frac{t^5}{5} + C = \frac{1}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C$$

(iii) Let $I = \int \sec^4 x \ dx$. Then,

$$I = \int \sec^2 x \sec^2 x \ dx = \int (1 + \tan^2 x) \sec^2 x \, dx$$

Putting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int (1 + t^2) \ dt = t + \frac{t^3}{3} + C = \tan x + \frac{1}{3} \tan^3 x + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \cot^2 x \operatorname{cosec}^4 x \ dx$ (ii) $\int \operatorname{cosec}^4 x \ dx$

SOLUTION (i) Let $I = \int \cot^2 x \operatorname{cosec}^4 x \ dx$. Then,

$$I = \int \cot^2 x \operatorname{cosec}^2 x \operatorname{cosec}^2 x \ dx$$

$$\Rightarrow \quad I = \int \cot^2 x (\cot^2 x + 1) \operatorname{cosec}^2 x \, dx = \int (\cot^4 x + \cot^2 x) \ \operatorname{cosec}^2 x \ dx$$

Substituting $\cot x = t$ and $-\operatorname{cosec}^2 x \, dx = dt$, we get

$$I = -\int (t^4 + t^2) \ dt = -\frac{t^5}{5} - \frac{t^3}{3} + C = -\frac{1}{5} \cot^5 x - \frac{1}{3} \cot^3 x + C$$

(ii) Let $I = \int \operatorname{cosec}^4 x \ dx$. Then,

$$I = \int \operatorname{cosec}^2 x \operatorname{cosec}^2 x \, dx = \int (1 + \cot^2 x) \operatorname{cosec}^2 x \, dx$$

Substituting $\cot x = t$ and $-\operatorname{cosec}^2 x \, dx = dt$, we get

$$I = -\int (1 + t^2) \ dt = -t - \frac{t^3}{3} + C = -\cot x - \frac{1}{3} \cot^3 x + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:* $\int \tan^8 x \sec^4 x \, dx$ **[NCERT EXEMPLAR]**

SOLUTION Let $I = \int \tan^8 x \sec^4 x \ dx$. Then,

$$I = \int \tan^8 x \sec^2 x \sec^2 x \, dx = \int \tan^8 x (1 + \tan^2 x) \sec^2 x \, dx$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x \, dx = dt$.

$$I = \int t^8 (1 + t^2) \, dt = \int t^8 + t^{10} \ dt = \frac{t^9}{9} + \frac{t^{11}}{11} + C = \frac{1}{9} \tan^9 x + \frac{1}{11} \tan^{11} x + C$$

18.8.10 INTEGRALS OF THE FORM $\int \tan^{2m+1} x \sec^{2n+1} x\,dx$, WHERE m, n ARE NON-NEGATIVE INTEGERS

In order to evaluate this type of integrals, we may follow the following algorithm.

ALGORITHM

Step I *Write the given integral as* $I = \int (\tan^2 x)^m (\sec x)^{2n} \sec x \tan x\ dx$

Step II *Substitute* $\sec x = t$ *and* $\sec x \tan x\ dx = dt$ *and write the integrals as*

$$I = \int (\sec^2 x - 1)^m\ (\sec x)^{2n} \sec x \tan x\,dx = \int (t^2 - 1)^m\, t^n\ dt$$

Step III *Expand* $(t^2 - 1)^m$ *by binomial theorem in step II and integrate.*

Step IV *Replace* t *by* $\sec x$ *in step III.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\int \tan^3 x \sec^3 x\ dx$ [CBSE 2020]

SOLUTION Let $I = \int \tan^3 x \sec^3 x\ dx$. Then,

$$I = \int \tan^2 x \sec^2 x (\sec x \tan x)\ dx = \int (\sec^2 x - 1)\ \sec^2 x (\sec x \tan x)\ dx$$

Substituting $\sec x = t$ and $\sec x \tan x\ dx = dt$, we get

$$I = \int (t^2 - 1)\, t^2\ dt = \int (t^4 - t^2)\ dt = \frac{t^5}{5} - \frac{t^3}{3} + C = \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

EXAMPLE 2 *Evaluate:* $\int \sec^n x \tan x\,dx$

SOLUTION Let $I = \int \sec^n x \tan x\ dx$. Then,

$$I = \int \sec^{n-1} x (\sec x \tan x)\ dx$$

Substituting $\sec x = t$ and $\sec x \tan x\ dx = dt$, we get

$$I = \int t^{n-1}\ dt = \frac{t^n}{n} + C = \frac{1}{n} \sec^n x + C$$

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

18.8.11 INTEGRALS OF THE FORM $\int \tan^n x\,dx$, $\int \cot^n x\,dx$

EXAMPLE 1 *Evaluate:*

(i) $\int \tan^3 x\,dx$ (ii) $\int \tan^4 x\ dx$ [NCERT] (iii) $\int \cot^3 x\ dx$ (iv) $\int \cot^4 x\ dx$

SOLUTION (i) Let $I = \int \tan^3 x\ dx$. Then,

$$I = \int \tan^2 x\ \tan x\ dx = \int (\sec^2 x - 1) \tan x\ dx = \int \tan x \sec^2 x\ dx - \int \tan x\ dx$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$ in first integral, we get

$$I = \int t\ dt - \int \tan x\ dx = \frac{t^2}{2} + \log|\cos x| + C = \frac{1}{2} \tan^2 x + \log|\cos x| + C$$

(ii) Let $I = \int \tan^4 x\ dx$. Then,

$$I = \int \tan^2 x \times \tan^2 x \ dx = \int \tan^2 x \ (\sec^2 x - 1) \ dx = \int (\tan^2 x \sec^2 x - \tan^2 x) \ dx$$

$$\Rightarrow \quad I = \int \tan^2 x \sec^2 x \ dx - \int \tan^2 x \ dx = \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \ dx$$

Putting $\tan x = t$ and $\sec^2 dx = dt$ in first integral, we get

$$I = \int t^2 \, dt - \int (\sec^2 x - 1) \ dt = \frac{t^3}{3} - (\tan x - x) + C = \frac{\tan^3 x}{3} - \tan x + x + C$$

(iii) Let $I = \int \cot^3 x \ dx$. Then,

$$I = \int \cot^2 x \cot x \, dx = \int (\operatorname{cosec}^2 x - 1) \cot x \ dx = \int (\cot x \operatorname{cosec}^2 x - \cot x) \ dx$$

$$\Rightarrow \quad I = \int \cot x \operatorname{cosec}^2 x \ dx - \int \cot x \ dx$$

Substituting $\cot x = t$ and $-\operatorname{cosec}^2 x \, dx = dt$ in first integral, we get

$$I = -\int t \ dt - \int \cot x \, dx = -\frac{t^2}{2} - \log|\sin x| + C = -\frac{1}{2}\cot^2 x - \log|\sin x| + C$$

(iv) Let $I = \int \cot^4 x \ dx$. Then,

$$I = \int \cot^2 x \cot^2 x \ dx = \int (\operatorname{cosec}^2 x - 1) \cot^2 x \, dx = \int (\cot^2 x \operatorname{cosec}^2 x - \cot^2 x) \ dx$$

$$\Rightarrow \quad I = \int \cot^2 x \operatorname{cosec}^2 x \, dx - \int \cot^2 x \ dx = \int \cot^2 x \operatorname{cosec}^2 x \, dx - \int (\operatorname{cosec}^2 x - 1) \, dx$$

Substituting $\cot x = t$ and $-\operatorname{cosec}^2 x \, dx = dt$ in the first integral, we get

$$I = -\int t^2 dt - \int (\operatorname{cosec}^2 x - 1) \, dx = -\frac{t^3}{3} - (-\cot x - x) + C = -\frac{1}{3}\cot^3 x + \cot x + x + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 2 *Prove that:* $\int \tan^n x \ dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \ dx$

SOLUTION Let $I_n = \int \tan^n x \ dx$. Then,

$$I_n = \int \tan^{n-2} x \tan^2 x \ dx = \int \tan^{n-2} x \, (\sec^2 x - 1) \ dx$$

$$\Rightarrow \quad I_n = \int \tan^{n-2} x \sec^2 x \ dx - \int \tan^{n-2} x \ dx$$

Substituting $\tan x = t$, $\sec^2 x \, dx = dt$ in the first integral on the right hand side, we get

$$I_n = \int t^{n-2} \, dt - \int \tan^{n-2} x \, dx = \frac{t^{n-1}}{n-1} - \int \tan^{n-2} x \ dx$$

$$\Rightarrow \quad \int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx$$

EXERCISE 18.11

BASED ON LOTS

Evaluate the following integrals (1-12):

1. $\int \tan^3 x \ \sec^2 x \ dx$
2. $\int \tan x \sec^4 x \ dx$
3. $\int \tan^5 x \sec^4 x \, dx$
4. $\int \sec^6 x \tan x \ dx$
5. $\int \tan^5 x \ dx$
6. $\int \sqrt{\tan x} \ \sec^4 x \ dx$
7. $\int \sec^4 2x \ dx$
8. $\int \operatorname{cosec}^4 3x \ dx$
9. $\int \cot^n x \operatorname{cosec}^2 x \ dx, n \neq -1$

10. $\int \cot^5 x \operatorname{cosec}^4 x \; dx$ 11. $\int \cot^5 x \; dx$ 12. $\int \cot^6 x \, dx$

13. If $\frac{d}{dx}(F(x)) = \frac{\sec^4 x}{\operatorname{cosec}^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find $F(x)$. [CBSE 2022]

ANSWERS

1. $\frac{1}{4}\tan^4 x + C$
2. $\frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$
3. $\frac{1}{6}\tan^6 x + \frac{1}{8}\tan^8 x + C$
4. $\frac{1}{6}\sec^6 x + C$
5. $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C$
6. $\frac{2}{3}\tan^{3/2} x + \frac{2}{7}\tan^{7/2} x + C$
7. $\frac{1}{2}\tan 2x + \frac{1}{6}\tan^3 2x + C$
8. $-\frac{1}{3}\cot 3x - \frac{1}{9}\cot^3 3x + C$
9. $-\frac{1}{n+1}\cot^{n+1} x + C$
10. $-\frac{1}{6}\cot^6 x - \frac{1}{8}\cot^8 x + C$
11. $-\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \log|\sin x| + C$
12. $-\frac{1}{5}\cot^5 x + \frac{1}{3}\cot^3 x - \cot x - x + C$
13. $F(x) = \frac{1}{3}\tan^3 x - \tan x + x + \frac{2}{3}$

18.8.12 INTEGRALS OF THE FORM $\int \sin^m x \cos^n x \, dx, m, n \in N$

In order to evaluate the integrals of the form $\int \sin^m x \cos^n x \, dx$, we may use the following algorithm.

ALGORITHM

Step I *Obtain the integral, say,* $\int \sin^m x \cos^n x \, dx$.

Step II *Check the exponents of* $\sin x$ *and* $\cos x$.

Step III *If the exponent of* $\sin x$ *is an odd positive integer put* $\cos x = t$.

If the exponent of $\cos x$ *is an odd positive integer put* $\sin x = t$.

If the exponents of $\sin x$ *and* $\cos x$ *both are odd positive integers put either* $\sin x = t$ *or,* $\cos x = t$.

If the exponents of $\sin x$ *and* $\cos x$ *both are even positive integers, then express* $\sin^m x \cos^n x$ *in terms of sines and cosines of multiples of* x *by using trigonometric results or De' Moivere's theorem.*

Step IV *Evaluate the integral obtained in step III.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:*

(i) $\int \sin^3 x \cos^4 x \, dx$ (ii) $\int \sin^2 x \cos^5 x \, dx$ (iii) $\int \sin^3 x \cos^5 x \, dx$

SOLUTION (i) Let $I = \int \sin^3 x \cos^4 x \, dx$. Here, power of $\sin x$ is odd, so we substitute

$$\cos x = t \Rightarrow -\sin x \, dx = dt \Rightarrow dx = -\frac{dt}{\sin x}$$

$$\therefore \quad I = \int \sin^3 x \; t^4 \left(-\frac{dt}{\sin x}\right) = -\int \sin^2 x \, t^4 \, dt = -\int (1-t^2)\, t^4 \, dt = -\int (t^4 - t^6)\, dt$$

$$\Rightarrow \quad I = -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$$

(ii) Let $I = \int \sin^2 x \cos^5 x \, dx$. Here, power of $\cos x$ is odd, so we substitute

$$\sin x = t \Rightarrow \cos x \, dx = dt \Rightarrow dx = \frac{dt}{\cos x}$$

$$\therefore \quad I = \int t^2 \cos^5 x \frac{dt}{\cos x} = \int t^2 (1 - \sin^2 x)^2 \, dt = \int t^2 (1 - t^2)^2 \, dt$$

$$\Rightarrow \quad I = \int (t^2 - 2t^4 + t^6) \, dt = \frac{t^3}{3} - \frac{2}{5} t^5 + \frac{t^7}{7} + C = \frac{\sin^3 x}{3} - \frac{2}{5} \sin^5 x + \frac{\sin^7 x}{7} + C$$

(iii) Let $I = \int \sin^3 x \cos^5 x \, dx$. Here, powers of both $\sin x$ and $\cos x$ are odd. So we can substitute either $\sin x = t$ or, $\cos x = t$. Putting $\cos x = t$ and $-\sin x \, dx = dt$ or, $dx = -\frac{dt}{\sin x}$, we get

$$I = \int \sin^3 x \, t^5 \times -\frac{dt}{\sin x} = -\int t^5 \sin^2 x \, dt = -\int t^5 (1 - t^2) \, dt = -\int (t^5 - t^7) \, dt$$

$$\Rightarrow \quad I = -\frac{t^6}{6} + \frac{t^8}{8} + C = -\frac{\cos^6 x}{6} + \frac{\cos^8 x}{8} + C$$

EXAMPLE 2 *Evaluate:* $\int \cos^3 x \, e^{\log \sin x} \, dx$.

SOLUTION We have, $I = \int \cos^3 x \, e^{\log \sin x} \, dx = \int \cos^3 x \sin x \, dx$

Putting $\cos x = t$ and $-\sin x \, dx = dt$ or, $\sin x \, dx = -dt$, we get

$$I = -\int t^3 \, dt = -\frac{t^4}{4} + C = -\frac{\cos^4 x}{4} + C$$

18.8.13 TO EVALUATE INTEGRALS OF THE FORM $\int \sin^m x \cos^n x \, dx$, WHERE $m, n \in Q$ SUCH THAT $m + n$ IS A NEGATIVE EVEN INTEGER

ALGORITHM

Step I *Change the integrand in terms of* $\tan x$ and $\sec^2 x$ *by, dividing numerator and denominator by* $\cos^k x$, *where* $k = -(m+n)$.

Step II *Substitute* $\tan x = t$.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Evaluate:* (i) $\int \frac{\sin^4 x}{\cos^8 x} \, dx$ (ii) $\int \frac{1}{\sqrt{\sin^3 x \cos^5 x}} \, dx$

SOLUTION (i) Let $I = \int \frac{\sin^4 x}{\cos^8 x} \, dx$. Here $k = -(m+n) = -(4-8) = 4$. Dividing numerator and denominator by $\cos^4 x$, we obtain

$$I = \int \frac{\frac{\sin^4 x}{\cos^4 x}}{\frac{\cos^8 x}{\cos^4 x}} dx = \int \tan^4 x \sec^4 x \, dx = \int \tan^4 x (1+\tan^2 x) \sec^2 x \, dx$$

$$\Rightarrow \quad I = \int \tan^4 x (1+\tan^2 x) \sec^2 x \, dx$$

Putting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int t^4 (1+t^2) \, dt = \frac{t^5}{5} + \frac{t^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

(ii) Let $I = \int \frac{1}{\sqrt{\sin^3 x \cos^5 x}} dx$. Then, $I = \int \frac{1}{\sin^{3/2} x \cos^{5/2} x} dx = \int \sin^{-3/2} x \cos^{-5/2} x \, dx$

Here $k = -(m+n) = -\left(-\frac{3}{2} - \frac{5}{2}\right) = 4$. Dividing numerator and denominator by $\cos^4 x$, we obtain

$$I = \int \frac{\sec^4 x}{\tan^{3/2} x} dx = \int \frac{(1+\tan^2 x)}{\tan^{3/2} x} \sec^2 x \, dx$$

Putting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int \frac{1+t^2}{t^{3/2}} dt = \int (t^{-3/2} + t^{1/2}) \, dt = \frac{-2}{\sqrt{t}} + \frac{t^{3/2}}{3/2} + C = -\frac{2}{\sqrt{\tan x}} + \frac{2}{3} (\tan x)^{3/2} + C$$

EXAMPLE 2 *Evaluate:* $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x \, dx$

SOLUTION Let $I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x \, dx$. Then,

$$I = \int \frac{1}{\cos^{4/3} x \sin^{8/3} x} dx = \int \cos^{-4/3} x \sin^{-8/3} x \, dx$$

Here $k = -(m+n) = -\left(\frac{4}{3} - \frac{8}{3}\right) = 4$. So, we divide both numerator and denominator by $\cos^4 x$.

$\therefore \quad I = \int \frac{\sec^4 x}{\tan^{8/3} x} dx = \int \frac{(1+\tan^2 x)}{\tan^{8/3} x} \sec^2 x \, dx$. Putting $\tan x = t$ and $\sec^2 x \, dx = dt$, we get

$$I = \int \frac{1+t^2}{t^{8/3}} dt = \int (t^{-8/3} + t^{-2/3}) \, dt = -\frac{3}{5} t^{-5/3} + 3 t^{1/3} + C$$

$$\Rightarrow I = -\frac{3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:* $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}} dx$

SOLUTION Let $I = \int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}} dx = \int \sin^{2/3} x \cos^{-14/3} x \, dx$

Here, the sum of the exponents of sin x and cos x in -4, which is a negative even integer. So, we divide and multiply by $\cos^4 x$ to get

$$I = \int \sin^{2/3} x \cos^{-14/3} x \cos^4 x \sec^4 x \, dx = \int \frac{\sin^{2/3} x}{\cos^{2/3} x} \sec^4 x \, dx$$

$$\Rightarrow \quad I = \int \tan^{2/3} x \,(1 + \tan^2 x) \sec^2 x \, dx$$

Putting $\tan x = t$, and $\sec^2 x \, dx = dt$, we get

$$I = \int (t^{2/3} + t^{8/3}) \, dt = \frac{3}{5} t^{5/3} + \frac{3}{11} t^{11/3} + C = \frac{3}{5} \tan^{5/3} x + \frac{3}{11} \tan^{11/3} x + C$$

EXERCISE 18.12

BASIC

Evaluate the following integrals:

1. $\int \sin^4 x \cos^3 x \, dx$
2. $\int \sin^5 x \, dx$
3. $\int \cos^5 x \, dx$
4. $\int \sin^5 x \cos x \, dx$
5. $\int \sin^3 x \cos^6 x \, dx$
6. $\int \sin^3 x \cos^5 x \, dx$

BASED ON LOTS

7. $\int \cos^7 x \, dx$
8. $\int x \cos^3 x^2 \sin x^2 \, dx$
9. $\int \sin^7 x \, dx$

BASED ON HOTS

10. $\int \frac{1}{\sin^4 x \cos^2 x} \, dx$
11. $\int \frac{1}{\sin^3 x \cos^5 x} \, dx$
12. $\int \frac{1}{\sin^3 x \cos x} \, dx$
13. $\int \frac{1}{\sin x \cos^3 x} \, dx$ [NCERT]

ANSWERS

1. $\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$
2. $-\left\{ \cos x - \frac{2}{3} \cos^3 x + \frac{1}{5} \cos^5 x \right\} + C$
3. $\left\{ \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x \right\} + C$
4. $\frac{\sin^6 x}{6} + C$
5. $-\left\{ \frac{\cos^7 x}{7} - \frac{\cos^9 x}{9} \right\} + C$
6. $-\frac{1}{6} \cos^6 x + \frac{1}{8} \cos^8 x + C$
7. $\sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C$
8. $-\frac{1}{8} \cos^4 x^2 + C$
9. $-\cos x + \cos^3 x - \frac{3}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C$
10. $-\frac{1}{3} \cot^3 x - 2 \cot x + \tan x + C$
11. $-\frac{1}{2} (\tan x)^{-2} + 3 \log |\tan x| + \frac{3}{2} \tan^2 x + \frac{1}{4} \tan^4 x + C$
12. $\log |\tan x| - \frac{1}{2 \tan^2 x} + C$
13. $\frac{1}{2} \tan^2 x + \log |\tan x| + C$

HINTS TO SELECTED PROBLEMS

13. $I = \int \frac{1}{\sin x \cos^3 x}\, dx = \int \frac{\sec^4 x}{\tan x}\, dx$ [Diving numerator and denominator by $\cos^4 x$]

$$\Rightarrow \quad I = \int \frac{(1+\tan^2 x)}{\tan x} \sec^2 x\, dx = \int \left(\frac{1+t^2}{t}\right) dt, \text{ where } t = \tan x \text{ and } dt = \sec^2 x\, dx$$

$$\Rightarrow \quad I = \int \left(\frac{1}{t} + t\right) dt = \frac{t^2}{2} + \log t + C = \frac{1}{2} \tan^2 x + \log \tan x + C$$

18.9 EVALUATION OF INTEGRALS BY USING TRIGONOMETRIC SUBSTITUTIONS

In this section, we will discuss evaluation of integrals by using trigonometric substitutions. Following are some substitutions useful in evaluating integrals.

Expression	*Substitution*	*Expression*	*Substitution*
(i) $a^2 + x^2$	$x = a \tan \theta$ or $a \cot \theta$	(iv) $\sqrt{\frac{a-x}{a+x}}$ or, $\sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(ii) $a^2 - x^2$	$x = a \sin \theta$ or $a \cos \theta$	(v) $\sqrt{\frac{x-\alpha}{\beta-x}}$ or, $\sqrt{(x-\alpha)(x-\beta)}$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$.
(iii) $x^2 - a^2$	$x = a \sec \theta$ or $a \operatorname{cosec} \theta$		

Let us discuss some problems on evaluation of integrals by making above substitutions.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I **EVALUATION OF INTEGRALS BY MAKING SUBSTITUTION** $x = a \sin \theta$ **or,** $x = a \sin^2 \theta$

EXAMPLE 1 *Evaluate:* $\int \frac{1}{(a^2 - x^2)^{3/2}}\, dx$.

SOLUTION Let $I = \int \frac{1}{(a^2 - x^2)^{3/2}}\, dx$ and $x = a \sin \theta$. Then, $dx = d\,(a \sin \theta) \Rightarrow dx = a \cos \theta\, d\theta$.

$$\therefore \quad I = \int \frac{1}{(a^2 - a^2 \sin^2 \theta)^{3/2}}\, a \cos \theta\, d\theta = \int \frac{a \cos \theta}{a^3 \cos^3 \theta}\, d\theta = \frac{1}{a^2} \int \sec^2 \theta\, d\theta$$

$$\Rightarrow \quad I = \frac{1}{a^2} \tan \theta + C = \frac{1}{a^2} \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}} + C = \frac{x}{a^3 \sqrt{1 - \frac{x^2}{a^2}}} + C = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C$$

EXAMPLE 2 *Evaluate:* $\int \frac{x^2}{\sqrt{1-x^2}}\, dx$.

SOLUTION Let $I = \int \frac{x^2}{\sqrt{1-x^2}}\, dx$ and $x = \sin \theta$. Then, $dx = d\,(\sin \theta) = \cos \theta\, d\theta$.

$$I = \int \frac{\sin^2 \theta}{\sqrt{1 - \sin^2 \theta}} \cos \theta\, d\theta = \int \sin^2 \theta\, d\theta = \frac{1}{2} \int (1 - \cos 2\theta)\, d\theta$$

$$\Rightarrow \quad I = \frac{1}{2}\left(\theta - \frac{1}{2} \sin 2\theta\right) + C = \frac{1}{2}\theta - \frac{1}{2} \sin \theta \cos \theta + C = \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C$$

EXAMPLE 3 *Evaluate:* $\int \frac{x^2 - 3x + 1}{\sqrt{1 - x^2}}\, dx.$ **[CBSE 2015]**

SOLUTION Let $I = \int \frac{x^2 - 3x + 1}{\sqrt{1 - x^2}}\, dx$ and $x = \sin\theta$. Then, $dx = d(\sin\theta) = \cos\theta\, d\theta$.

$$\therefore \quad I = \int \frac{\sin^2\theta - 3\sin\theta + 1}{\sqrt{1 - \sin^2\theta}} \cos\theta\, d\theta = \int (\sin^2\theta - 3\sin\theta + 1)\, d\theta$$

$$\Rightarrow \quad I = \int \left(\frac{1 - \cos 2\theta}{2} - 3\sin\theta + 1 \right) d\theta = \frac{1}{2} \int (3 - 6\sin\theta - \cos 2\theta)\, d\theta$$

$$\Rightarrow \quad I = \frac{1}{2}\left(3\theta + 6\cos\theta - \frac{1}{2}\sin 2\theta \right) + C = \frac{1}{2}\left\{ 3\theta + 6\sqrt{1 - \sin^2\theta} - \sin\theta\sqrt{1 - \sin^2\theta} \right\} + C$$

$$\Rightarrow \quad I = \frac{1}{2}\left\{ 3\sin^{-1} x + 6\sqrt{1 - x^2} - x\sqrt{1 - x^2} \right\} + C$$

EXAMPLE 4 *Evaluate:* $\int \frac{x^2}{\sqrt{1 - x}}\, dx$

SOLUTION Let $I = \int \frac{x^2}{\sqrt{1 - x}}\, dx = \int \frac{x^2}{\sqrt{1 - (\sqrt{x})^2}}\, dx.$

Let $\sqrt{x} = \sin\theta$ or, $x = \sin^2\theta$. Then, $dx = d(\sin^2\theta) = 2\sin\theta\cos\theta\, d\theta$.

$$\therefore \quad I = \int \frac{(\sin^2\theta)^2}{\sqrt{1 - \sin^2\theta}} 2\sin\theta\cos\theta\, d\theta = 2\int \sin^5\theta\, d\theta = 2\int (1 - \cos^2\theta)^2 \sin\theta\, d\theta.$$

Let $\cos\theta = u$. Then, $d(\cos\theta) = du$ or, $-\sin\theta\, d\theta = du$.

$$I = -2\int (1 - u^2)^2\, du = -2\int (1 - 2u^2 + u^4)\, du = -2\left(u - \frac{2}{3}u^3 + \frac{u^5}{5} \right) + C$$

$$\Rightarrow \quad I = -\frac{2}{15} u(15 - 10u^2 + 3u^4) + C = -\frac{2}{15}(15 - 10\cos^2\theta + 3\cos^4\theta)\cos\theta + C$$

$$\Rightarrow \quad I = -\frac{2}{15}\left\{ 15 - 10(1 - \sin^2\theta) + 3(1 - \sin^2\theta)^2 \right\}\sqrt{1 - \sin^2\theta} + C$$

$$\Rightarrow \quad I = -\frac{2}{15}\left\{ 8 + 4\sin^2\theta + 3\sin^4\theta \right\}\sqrt{1 - \sin^2\theta} + C$$

$$\Rightarrow \quad I = -\frac{2}{15}\left\{ 8 + 4\sin^2\theta + 3\sin^4\theta \right\}\sqrt{1 - \sin^2\theta} + C = -\frac{2}{15}(8 + 4x + 3x^2)\sqrt{1 - x} + C$$

Type II INTEGRALS BASED UPON THE SUBSTITUTION $x = a\tan\theta$ OR, $x = a\tan^2\theta$

EXAMPLE 5 *Evaluate :* $\int \frac{1}{(a^2 + x^2)^2}\, dx$

SOLUTION Let $I = \int \frac{1}{(a^2 + x^2)^2}\, dx$ and $x = a\tan\theta$. Then, $dx = d(a\tan\theta) = a\sec^2\theta\, d\theta$.

$$\therefore \quad I = \int \frac{1}{(a^2 + a^2\tan^2\theta)^2} a\sec^2\theta\, d\theta = \frac{1}{a^3}\int \cos^2\theta\, d\theta = \frac{1}{2a^3}\int (1 + \cos 2\theta)\, d\theta$$

$$I = \frac{1}{2a^3}\left(\theta + \frac{1}{2}\sin 2\theta \right) + C = \frac{1}{2a^3}\left(\theta + \frac{\tan\theta}{1 + \tan^2\theta} \right) + C = \frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a} + \frac{ax}{a^2 + x^2} \right) + C$$

EXAMPLE 6 *Evaluate* : $\int \frac{1}{x^4+x^6}\,dx$

SOLUTION Let $I=\int \frac{1}{x^4+x^6}\,dx$. Then, $I=\int \frac{1}{x^4+x^6}\,dx=\int \frac{1}{x^4(1+x^2)}\,dx$

Let $x=\tan\theta$. Then, $dx=\sec^2\theta\,d\theta$.

$$\therefore \quad I=\int \frac{1}{x^4(1+x^2)}\,dx=\int \frac{1}{\tan^4\theta\,(1+\tan^2\theta)}\sec^2\theta\,d\theta=\int \cot^4\theta\,d\theta$$

$$\Rightarrow \quad I=\int \cot^2\theta\,(\operatorname{cosec}^2\theta-1)\,d\theta=\int \cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int \cot^2\theta\,d\theta$$

$$\Rightarrow \quad I=\int \cot^2\theta\operatorname{cosec}^2\theta\,d\theta-\int(\operatorname{cosec}^2\theta-1)\,d\theta$$

$$\Rightarrow \quad I=-\int t^2dt-\int(\operatorname{cosec}^2\theta-1)\,d\theta,\ \text{where } t=\cot\theta$$

$$\Rightarrow \quad I=-\frac{1}{3}t^3-(-\cot\theta-\theta)+C=-\frac{1}{3}\cot^3\theta+\cot\theta+\theta+C=-\frac{1}{3x^3}+\frac{1}{x}+\tan^{-1}x+C$$

Type III EVALUATION OF INTEGRALS BY MAKING SUBSTITUTION $x=a\sec\theta$ OR, $x=a\sec^2\theta$

EXAMPLE 7 *Evaluate* : $\int \frac{1}{x^3\sqrt{x^2-a^2}}\,dx$

SOLUTION Let $I=\int \frac{1}{x^3\sqrt{x^2-a^2}}\,dx$ and $x=a\sec\theta$. Then, $dx=a\sec\theta\tan\theta\,d\theta$.

$$\therefore \quad I=\int \frac{1}{(a\sec\theta)^3\sqrt{a^2\sec^2\theta-a^2}}\,a\sec\theta\tan\theta\,d\theta=\frac{1}{a^3}\int \cos^2\theta\,d\theta=\frac{1}{2a^3}\int(1+\cos 2\theta)\,d\theta$$

$$\Rightarrow \quad I=\frac{1}{2a^3}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C=\frac{1}{2a^3}(\theta+\sin\theta\cos\theta)+C=\frac{1}{2a^3}\left(\sec^{-1}\frac{x}{a}+\frac{a}{x^2}\sqrt{x^2-a^2}\right)+C$$

EXAMPLE 8 *Evaluate* : $\int \frac{1}{x\sqrt{x^4-1}}\,dx$

SOLUTION Let $\int \frac{1}{x\sqrt{x^4-1}}\,dx$. Then, $I=\int \frac{1}{x\sqrt{(x^2)^2-1}}\,dx=\int \frac{1}{x^2\sqrt{(x^2)^2-1}}\,x\,dx$

Let $x^2=\sec\theta$. Then, $d(x^2)=d(\sec\theta)$ or, $2x\,dx=\sec\theta\tan\theta\,d\theta$ or, $dx=\frac{\sec\theta\tan\theta}{2x}d\theta$.

$$\therefore \quad I=\int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}\times\frac{1}{2}\sec\theta\tan\theta\,d\theta=\frac{1}{2}\int 1\cdot d\theta=\frac{1}{2}\theta+C=\frac{1}{2}\sec^{-1}x^2+C$$

ALITER $I=\int \frac{1}{x\sqrt{(x^2)^2-1}}\,x\,dx=\frac{1}{2}\int \frac{1}{t\sqrt{t^2-1}}\,dt$, where $t=x^2$

$$\Rightarrow \quad I=\frac{1}{2}\sec^{-1}t+C=\frac{1}{2}\sec^{-1}x^2+C$$

Type I INTEGRALS BASED ON THE SUBSTITUTION $x=a\sin^2\theta$ or $x=a\sin\theta$

EXAMPLE 9 *Evaluate:* $\int \frac{x}{(1-x^4)^{3/2}}\,dx$.

SOLUTION Let $I=\int \frac{x}{(1-x^4)^{3/2}}\,dx=\int \frac{x}{\left\{1-(x^2)^2\right\}^{3/2}}\,dx$.

Let $x^2 = \sin\theta$. Then, $d(x^2) = d(\sin\theta) \Rightarrow 2x\,dx = \cos\theta\,d\theta \Rightarrow dx = \frac{\cos\theta}{2x}dx$

$$\therefore\quad I = \int \frac{x}{(1-\sin^2\theta)^{3/2}}\frac{\cos\theta}{2x}dx = \frac{1}{2}\int \sec^2\theta\,d\theta = \frac{1}{2}\tan\theta + C = \frac{1}{2}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}} + C = \frac{1}{2}\frac{x^2}{\sqrt{1-x^4}} + C$$

EXAMPLE 10 *Evaluate:* $\int \frac{x^7}{(1-x^2)^5}dx.$

SOLUTION Let $x = \sin\theta$. Then, $dx = d(\sin\theta) = \cos\theta\,d\theta$.

$$\therefore\quad I = \int \frac{x^7}{(1-x^2)^5}dx = \int \frac{\sin^7\theta}{(1-\sin^2\theta)^5}\cos\theta\,d\theta = \int \tan^7\theta\sec^2\theta\,d\theta$$

Let $\tan\theta = u$. Then, $\sec^2\theta\,d\theta = du$ or, $d\theta = \frac{du}{\sec^2\theta}$

$$\therefore\quad I = \int u^7\,du = \frac{u^8}{8} + C = \frac{1}{8}\tan^8\theta + C = \frac{1}{8}\frac{\sin^8\theta}{(1-\sin^2\theta)^4} + C = \frac{1}{8}\frac{x^8}{(1-x^2)^4} + C$$

EXAMPLE 11 *Evaluate:* $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx.$

SOLUTION Putting $x = \sin^2 t$ and $dx = 2\sin t\cos t\,dt$, we get

$$I = \int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx = \int \frac{1}{(1+\sin t)\sqrt{\sin^2 t - \sin^4 t}}\,2\sin t\cos t\,dt$$

$$\Rightarrow\quad I = 2\int \frac{1}{1+\sin t}dt = 2\int \frac{1-\sin t}{\cos^2 t}dt = 2\int (\sec^2 t - \tan t\sec t)\,dt$$

$$\Rightarrow\quad I = 2(\tan t - \sec t) + C = 2\left(\frac{\sin t}{\sqrt{1-\sin^2 t}} - \frac{1}{\sqrt{1-\sin^2 t}}\right) + C = 2\sqrt{\frac{x}{1-x}} - \frac{2}{\sqrt{1-x}} + C$$

Type II INTEGRALS BASED ON THE SUBSTITUTION $x = a\tan\theta$ OR $x = a\tan^2\theta$

EXAMPLE 12 *Evaluate:* $\int \frac{1}{(x^2+2x+2)^2}dx$

SOLUTION Let $I = \int \frac{1}{(x^2+2x+2)^2}dx$. Then, $I = \int \frac{1}{\left\{(x+1)^2+1^2\right\}^2}dx$

Let $x+1 = \tan\theta$. Then, $d(x+1) = d(\tan\theta) \Rightarrow dx = \sec^2\theta\,d\theta$.

$$\therefore\quad I = \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta = \int \cos^2\theta\,d\theta = \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$$

$$\Rightarrow\quad I = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{2}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C = \frac{1}{2}\left\{\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right\} + C$$

EXAMPLE 13 *Evaluate:* $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}dx$

SOLUTION Let $I = \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}}dx$. Then,

$$I = \int \frac{\frac{1+x^2}{x^2}}{\left(\frac{1}{x}-x\right)\sqrt{\frac{1+x^2+x^4}{x^2}}}\,dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } x^2]$$

$$\Rightarrow \quad I = -\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}+1}}\,dx = -\int \frac{1}{\left(x-\frac{1}{x}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+3}}\left(1+\frac{1}{x^2}\right)dx$$

Let $x-\frac{1}{x}=t$. Then, $d\left(x-\frac{1}{x}\right)=dt \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt.$

$$\therefore \quad I = -\int \frac{dt}{t\sqrt{t^2+3}} = -\int \frac{u\,du}{(u^2-3)\sqrt{u^2}}, \text{ where } t^2+3=u^2 \text{ and } 2t\,dt = 2u\,du$$

$$\Rightarrow \quad I = -\int \frac{1}{u^2-3}\,du = -\frac{1}{2\sqrt{3}}\log\left|\frac{u-\sqrt{3}}{u+\sqrt{3}}\right| + C$$

$$\Rightarrow \quad I = -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right| + C = -\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{x^2+\frac{1}{x^2}+1}-\sqrt{3}}{\sqrt{x^2+\frac{1}{x^2}+1}+\sqrt{3}}\right| + C$$

EXAMPLE 14 *Evaluate* : $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,dx.$

SOLUTION Let $I = \int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}\,dx.$

$$\Rightarrow \quad I = \int \frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}\,dx \qquad \left[\text{Multiplying the } N^r \text{ and } D^r \text{ by } (x+1)\right]$$

$$\Rightarrow \quad I = \int \frac{(x^2-1)}{(x^2+2x+1)\sqrt{x^3+x^2+x}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}\,dx \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } x^2]$$

Let $x+\frac{1}{x}+1=t^2$. Then, $d\left(x+\frac{1}{x}+1\right)=d(t^2) \Rightarrow \left(1-\frac{1}{x^2}\right)dx = 2t\,dt$

$$\Rightarrow \quad I = \int \frac{2t\,dt}{(t^2+1)\sqrt{t^2}} = 2\int \frac{1}{t^2+1}\,dt = 2\tan^{-1}(t)+C = 2\tan^{-1}\sqrt{x+\frac{1}{x}+1}+C$$

EXERCISE 18.13

BASED ON HOTS

Evaluate the following integrals:

1. $\int \frac{x^2}{(a^2-x^2)^{3/2}} dx$
2. $\int \frac{x^7}{(a^2-x^2)^5} dx$
3. $\int \cos\left\{2\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right\} dx$
4. $\int \frac{\sqrt{1+x^2}}{x^4} dx$
5. $\int \frac{1}{(x^2+2x+10)^2} dx$

ANSWERS

1. $\frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$
2. $\frac{1}{8a^2}\frac{x^8}{(a^2-x^2)^4} + C$
3. $\frac{x^2}{2} + C$
4. $-\frac{1}{3}\frac{(x^2+1)^{3/2}}{x^3} + C$
5. $\frac{1}{54}\left\{\tan^{-1}\frac{x+1}{3} + \frac{3(x+1)}{x^2+2x+10}\right\} + C$

18.10 SOME SPECIAL INTEGRALS

Let us discuss problems on evaluation of integrals by making above substitutions.

THEOREM *For any constant a, prove the following:*

(i) $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

(ii) $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(iii) $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$

(iv) $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

(v) $\int \frac{1}{\sqrt{a^2+x^2}} dx = \log|x+\sqrt{a^2+x^2}| + C$

(vi) $\int \frac{1}{\sqrt{x^2-a^2}} dx = \log|x+\sqrt{x^2-a^2}| + C$

PROOF (i) Let $I = \int \frac{1}{x^2+a^2} dx$. Putting $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$, we get

$$I = \int \frac{a\sec^2\theta\, d\theta}{a^2+a^2\tan^2\theta} = \frac{1}{a}\int 1\cdot d\theta = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad \left[\because \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\frac{x}{a}\right]$$

Similarly by making substitution $x = a\cot\theta$, we obtain: $\int -\frac{1}{a^2+x^2} dx = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + C$

(ii) We find that

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{1}{2a}\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$

$$\therefore \quad I = \int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\int\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\} dx = \frac{1}{2a}\left\{\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx\right\}$$

$$\Rightarrow \quad I = \frac{1}{2a}\left\{\log|x-a| - \log|x+a|\right\} + C = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(iii) Clearly, $\dfrac{1}{a^2-x^2} = \dfrac{1}{(a-x)(a+x)} = \dfrac{1}{2a}\left\{\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\}$

$$\therefore \quad I = \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\left\{\int \frac{1}{a+x}\,dx + \int \frac{1}{a-x}\,dx\right\}$$

$$\Rightarrow \quad I = \frac{1}{2a}\left\{\log|a+x| - \log|a-x|\right\} + C = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

(iv) Let $I = \displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx$. Putting $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$, we get

$$I = \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int 1\cdot d\theta = \theta + C$$

$$\Rightarrow \quad I = \sin^{-1}\left(\frac{x}{a}\right) + C \qquad \left[\because x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\frac{x}{a}\right]$$

Similarly, by making substitution $x = a\cos\theta$, we obtain

$$\int -\frac{1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C$$

(v) Let $I = \displaystyle\int \frac{1}{\sqrt{a^2+x^2}}\,dx$. Putting $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$, we get

$$I = \int \frac{1}{\sqrt{a^2+a^2\tan^2\theta}}\,a\sec^2\theta\,d\theta = \int \sec\theta\,d\theta$$

$$\Rightarrow \quad I = \log|\sec\theta + \tan\theta| + C = \log|\tan\theta + \sqrt{1+\tan^2\theta}| + C$$

$$\Rightarrow \quad I = \log\left|\frac{x}{a} + \sqrt{1+\frac{x^2}{a^2}}\right| + C = \log|x + \sqrt{a^2+x^2}| - \log a + C \qquad \left[\because \tan\theta = \frac{x}{a}\right]$$

$$\Rightarrow \quad I = \log|x + \sqrt{a^2+x^2}| + C_1, \text{ where } C_1 = C - \log a$$

(vi) Let $I = \displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx$. Putting $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$, we get

$$I = \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}}\,a\sec\theta\tan\theta\,d\theta = \int \sec\theta\,d\theta = \log|\sec\theta + \tan\theta| + C$$

$$\Rightarrow \quad I = \log|\sec\theta + \sqrt{\sec^2\theta - 1}| + C = \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right| + C \qquad \left[\because \sec\theta = \frac{x}{a}\right]$$

$$\Rightarrow \quad I = \log|x + \sqrt{x^2-a^2}| - \log a + C = \log|x + \sqrt{x^2-a^2}| + C_1, \text{ where } C_1 = C - \log a.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\displaystyle\int \frac{1}{4+9x^2}\,dx$ (ii) $\displaystyle\int \frac{1}{9x^2-4}\,dx$ (iii) $\displaystyle\int \frac{1}{16-9x^2}\,dx$

SOLUTION (i) Let $I = \int \frac{1}{4+9x^2}\,dx$. Then,

$$I = \frac{1}{9}\int \frac{1}{\frac{4}{9}+x^2}\,dx = \frac{1}{9}\int \frac{1}{(2/3)^2+x^2}\,dx = \frac{1}{9}\times\frac{1}{(2/3)}\tan^{-1}\left(\frac{x}{2/3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+C$$

(ii) Let $I = \int \frac{1}{9x^2-4}\,dx$. Then,

$$I = \frac{1}{9}\int \frac{1}{x^2-(2/3)^2}\,dx = \frac{1}{9}\times\frac{1}{2\times\frac{2}{3}}\log\left|\frac{x-\frac{2}{3}}{x+\frac{2}{3}}\right|+C = \frac{1}{12}\log\left|\frac{3x-2}{3x+2}\right|+C$$

(iii) Let $I = \int \frac{1}{16-9x^2}\,dx$. Then,

$$I = \frac{1}{9}\int \frac{1}{\frac{16}{9}-x^2}\,dx = \frac{1}{9}\int \frac{1}{\left(\frac{4}{3}\right)^2-x^2}\,dx = \frac{1}{9}\times\frac{1}{2\left(\frac{4}{3}\right)}\times\log\left|\frac{\frac{4}{3}+x}{\frac{4}{3}-x}\right|+C = \frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right|+C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{1}{\sqrt{9-25x^2}}\,dx$ **[NCERT, CBSE 2020]** (ii) $\int \frac{1}{\sqrt{16x^2+25}}\,dx$ (iii) $\int \frac{1}{\sqrt{4x^2-9}}\,dx$

SOLUTION (i) Let $I = \int \frac{1}{\sqrt{9-25x^2}}\,dx$. Then,

$$I = \frac{1}{5}\int \frac{1}{\sqrt{\frac{9}{25}-x^2}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{\left(\frac{3}{5}\right)^2-x^2}}\,dx = \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right)+C = \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right)+C$$

(ii) Let $I = \int \frac{1}{\sqrt{16x^2+25}}\,dx$. Then,

$$I = \frac{1}{4}\int \frac{1}{\sqrt{x^2+\left(\frac{5}{4}\right)^2}}\,dx = \frac{1}{4}\log\left|x+\sqrt{x^2+\left(\frac{5}{4}\right)^2}\right|+C = \frac{1}{4}\log\left|\frac{4x+\sqrt{16x^2+25}}{4}\right|+C$$

$$\Rightarrow \quad I = \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right|-\frac{1}{4}\log 4+C = \frac{1}{4}\log\left|4x+\sqrt{16x^2+25}\right|+C_1,$$

where $C_1 = -\frac{1}{4}\log 4+C$

(iii) Let $I = \int \frac{1}{\sqrt{4x^2-9}}\,dx$. Then,

$$I = \frac{1}{2}\int \frac{1}{\sqrt{x^2-\frac{9}{4}}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}\,dx$$

$$\Rightarrow I = \frac{1}{2}\log\left|x+\sqrt{x^2-\left(\frac{3}{2}\right)^2}\right|+C = \frac{1}{2}\log\left|x+\sqrt{x^2-\frac{9}{4}}\right|+C = \frac{1}{2}\log\left|\frac{2x+\sqrt{4x^2+9}}{2}\right|+C$$

$$\Rightarrow I = \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right|-\frac{1}{2}\log 2+C = \frac{1}{2}\log\left|2x+\sqrt{4x^2-9}\right|+C_1, \text{ where } C_1 = -\frac{1}{2}\log 2+C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:* $\int \frac{x^4}{x^2+1}\,dx$

SOLUTION Let $I = \int \frac{x^4}{x^2+1}\,dx$. Then,

$$I = \int \frac{x^4-1+1}{x^2+1}\,dx$$

$$\Rightarrow I = \int \frac{x^4-1}{x^2+1}+\frac{1}{x^2+1}\,dx = \int (x^2-1)\,dx + \int \frac{1}{x^2+1}\,dx = \frac{x^3}{3}-x+\tan^{-1}x+C$$

EXERCISE 18.14

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{a^2-b^2x^2}\,dx$
2. $\int \frac{1}{a^2x^2-b^2}\,dx$
3. $\int \frac{1}{a^2x^2+b^2}\,dx$
4. $\int \frac{x^2-1}{x^2+4}\,dx$
5. $\int \frac{1}{\sqrt{1+4x^2}}\,dx$ [NCERT]
6. $\int \frac{1}{\sqrt{a^2+b^2x^2}}\,dx$
7. $\int \frac{1}{\sqrt{a^2-b^2x^2}}\,dx$
8. $\int \frac{1}{\sqrt{(2-x)^2+1}}\,dx$
9. $\int \frac{1}{\sqrt{(2-x)^2-1}}\,dx$

BASED ON LOTS

10. $\int \frac{x^4+1}{x^2+1}\,dx$ [CBSE 2002 C]

ANSWERS

1. $\frac{1}{2ab}\log\left|\frac{a+bx}{a-bx}\right|+C$
2. $\frac{1}{2\,ab}\log\left|\frac{ax-b}{ax+b}\right|+C$
3. $\frac{1}{ab}\tan^{-1}\left(\frac{ax}{b}\right)+C$
4. $x-\frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right)+C$
5. $\frac{1}{2}\log|2x+\sqrt{4x^2+1}|+C$
6. $\frac{1}{b}\log\left|bx+\sqrt{a^2+b^2x^2}\right|+C$
7. $\frac{1}{b}\sin^{-1}\left(\frac{bx}{a}\right)+C$
8. $-\log\left|(2-x)+\sqrt{(2-x)^2+1}\right|+C$
9. $-\log\left|2-x+\sqrt{(2-x)^2-1}\right|+C$
10. $\frac{x^3}{3}-x+2\tan^{-1}x+C$

HINTS TO SELECTED PROBLEMS

4. $\int \frac{x^2-1}{x^2+4} dx = \int 1 - \frac{5}{x^2+4} dx = \int 1 \cdot dx - 5\int \frac{1}{x^2+2^2} dx = x - \frac{5}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

5. $\int \frac{1}{\sqrt{1+4x^2}} dx = \int \frac{1}{\sqrt{1+(2x)^2}} dx = \frac{1}{2}\log\left|2x+\sqrt{1+4x^2}\right| + C$

8. $\int \frac{1}{\sqrt{(2-x)^2+1}} dx = -\int \frac{1}{\sqrt{(2-x)^2+1^2}} d(2-x) = -\log\left|(2-x)+\sqrt{(2-x)^2+1}\right| + C$

9. $\int \frac{1}{\sqrt{(2-x)^2-1^2}} dx = -\int \frac{1}{\sqrt{(2-x)^2-1^2}} d(2-x) = -\log\left|2-x+\sqrt{(2-x)^2-1}\right| + C$

18.10.1 EVALUATION OF INTEGRALS OF THE TYPE $\int \frac{1}{ax^2+bx+c} dx$

To evaluate this type of integrals we express ax^2+bx+c as the sum or difference of two squares by using the following algorithm.

ALGORITHM

Step I *Make the coefficient of x^2 unity, if it is not, by multiplying and dividing by it.*

Step II *Add and subtract the square of the half of coefficient of x to express ax^2+bx+c in the form a*

$$\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$$

Step III *Use the suitable formula from the following formulas:*

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C, \quad \int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+C$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+C.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate*

(i) $\int \frac{1}{x^2-x+1} dx$ (ii) $\int \frac{1}{2x^2+x-1} dx$ (iii) $\int \frac{1}{3+2x-x^2} dx$

SOLUTION (i) Let $I = \int \frac{1}{x^2-x+1} dx$. Then,

$$I = \int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1} dx = \int \frac{1}{(x-1/2)^2+3/4} dx$$

$$\Rightarrow \quad I = \int \frac{1}{(x-1/2)^2+(\sqrt{3}/2)^2} dx = \frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x-12}{\sqrt{3}/2}\right)+C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$$

(ii) Let $I = \int \frac{1}{2x^2+x-1} dx$. Then,

$$I = \frac{1}{2}\int \frac{1}{x^2 + \frac{x}{2} - \frac{1}{2}}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{1}{x^2 + x/2 + (1/4)^2 - (1/4)^2 - 1/2}\,dx = \frac{1}{2}\int \frac{1}{(x+1/4)^2 - (3/4)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2} \times \frac{1}{2(3/4)} \log\left|\frac{x + 1/4 - 3/4}{x + 1/4 + 3/4}\right| + C = \frac{1}{3}\log\left|\frac{x - 1/2}{x+1}\right| + C = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + C$$

(iii) Let $I = \int \frac{1}{3 + 2x - x^2}\,dx$. Then,

$$I = \int \frac{1}{-(x^2 - 2x - 3)}\,dx = \int \frac{1}{-(x^2 - 2x + 1 - 1 - 3)}\,dx = \int \frac{1}{-\left\{(x-1)^2 - 2^2\right\}}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{2^2 - (x-1)^2}\,dx = \frac{1}{2(2)}\log\left|\frac{2 + (x-1)}{2 - (x-1)}\right| + C = \frac{1}{4}\log\left|\frac{x+1}{3-x}\right| + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{1}{3x^2 + 13x - 10}\,dx$ **[NCERT]** (ii) $\int \frac{1}{4x^2 - 4x + 3}\,dx$

(iii) $\int \frac{1}{x^2 + 4x + 8}$ **[CBSE 2002, 2017]** (iv) $\int \frac{1}{9x^2 + 6x + 10}$

SOLUTION (i) Let $I = \int \frac{1}{3x^2 + 13x - 10}\,dx$. Then,

$$I = \frac{1}{3}\int \frac{1}{x^2 + \frac{13}{3}x - \frac{10}{3}}\,dx$$

$$\Rightarrow \quad I = \frac{1}{3}\int \frac{1}{x^2 + \frac{13}{3}x + \left(\frac{13}{6}\right)^2 - \left(\frac{13}{6}\right)^2 - \left(\frac{10}{3}\right)}\,dx = \frac{1}{3}\int \frac{1}{\left(x + \frac{13}{6}\right)^2 - \left(\frac{17}{6}\right)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{3} \times \frac{1}{2\left(\frac{17}{6}\right)} \log\left|\frac{x + \frac{13}{6} - \frac{17}{6}}{x + \frac{13}{6} + \frac{17}{6}}\right| + C = \frac{1}{17}\log\left|\frac{x - 4/6}{x+5}\right| + C = \frac{1}{17}\log\left|\frac{3x-2}{3(x+5)}\right| + C$$

(ii) Let $I = \int \frac{1}{4x^2 - 4x + 3}\,dx$. Then,

$$I = \frac{1}{4}\int \frac{1}{x^2 - x + 3/4}\,dx = \frac{1}{4}\int \frac{1}{x^2 - x + \frac{1}{4} - \frac{1}{4} + \frac{3}{4}}\,dx = \frac{1}{4}\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{4} \times \frac{1}{(1/\sqrt{2})} \tan^{-1}\left(\frac{x - 1/2}{1/\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + C$$

(iii) Let $I = \int \frac{1}{x^2 + 4x + 8}\,dx$. Then,

$$I = \int \frac{1}{x^2 + 4x + 4 + 4}\,dx = \int \frac{1}{(x+2)^2 + 2^2}\,dx = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + C$$

(iv) Let $I = \int \frac{1}{9x^2 + 6x + 10}\, dx$. Then,

$$I = \frac{1}{9} \int \frac{1}{x^2 + \frac{2}{3}x + \frac{10}{9}}\, dx = \frac{1}{9} \int \frac{1}{x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9} + \frac{10}{9}}\, dx$$

$$\Rightarrow \quad I = \frac{1}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + 1^2}\, dx = \frac{1}{9} \times \frac{1}{1} \tan^{-1}\left(\frac{x + \frac{1}{3}}{1}\right) + C = \frac{1}{9} \tan^{-1}\left(\frac{3x+1}{3}\right) + C$$

EXERCISE 18.15

Evaluate the following integrals:

1. $\int \frac{1}{4x^2 + 12x + 5}\, dx$
2. $\int \frac{1}{x^2 - 10x + 34}\, dx$
3. $\int \frac{1}{1 + x - x^2}\, dx$
4. $\int \frac{1}{2x^2 - x - 1}\, dx$
5. $\int \frac{1}{x^2 + 6x + 13}\, dx$ [NCERT]
6. $\int \frac{x+1}{(x+2)(x+3)}\, dx$ [CBSE 2020]

ANSWERS

1. $\frac{1}{8} \log\left|\frac{2x+1}{2x+5}\right| + C$
2. $\frac{1}{3} \tan^{-1}\left(\frac{x-5}{3}\right) + C$
3. $\frac{1}{\sqrt{5}} \log\left|\frac{\sqrt{5} - 1 + 2x}{\sqrt{5} + 1 - 2x}\right| + C$
4. $\frac{1}{3} \log\left|\frac{x-1}{2x+1}\right| + C$
5. $\frac{1}{2} \tan^{-1}\left(\frac{x+3}{2}\right) + C$
6. $\log\left|\frac{(x+3)^2}{x+2}\right| + C$

18.10.2 INTEGRALS REDUCIBLE TO THE FORM $\int \frac{1}{ax^2 + bx + c}\, dx$

Following examples will illustrate the procedure of evaluating the above type of integrals.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{x}{x^4 + x^2 + 1}\, dx$ (ii) $\int \frac{e^x}{e^{2x} + 6e^x + 5}\, dx$ (iii) $\int \frac{\sin x}{1 + \cos^2 x}\, dx$ (iv) $\int \frac{2x^3}{4 + x^8}\, dx$

SOLUTION (i) Let $I = \int \frac{x}{x^4 + x^2 + 1}\, dx = \int \frac{x}{(x^2)^2 + x^2 + 1}\, dx$

Let $x^2 = t$. Then, $d(x^2) = dt \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{dt}{2x}$

$$\therefore \quad I = \int \frac{x}{t^2 + t + 1} \times \frac{dt}{2x} = \frac{1}{2} \int \frac{1}{t^2 + t + 1}\, dt = \frac{1}{2} \int \frac{1}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2} \times \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1}\left(\frac{t + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + C = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + C$$

(ii) Let $I = \int \frac{e^x}{e^{2x} + 6e^x + 5}\, dx = \int \frac{e^x}{(e^x)^2 + 6e^x + 5}\, dx$. Let $e^x = t$. Then, $d(e^x) = dt \Rightarrow e^x dx = dt$

$$\therefore \quad I = \int \frac{dt}{t^2 + 6t + 5} = \int \frac{1}{(t+3)^2 - 2^2}\, dt = \frac{1}{2 \times 2} \log\left|\frac{t+3-2}{t+3+2}\right| + C = \frac{1}{4} \log\left|\frac{e^x + 1}{e^x + 5}\right| + C$$

(iii) Let $I = \int \frac{\sin x}{1 + \cos^2 x}\, dx$. Let $\cos x = t$ Then, $d(\cos x) = dt \Rightarrow -\sin x\, dx = dt \Rightarrow dx = -\, dt \sin x$

$$\therefore \quad I = \int \frac{\sin x}{1+t^2} \times -\frac{dt}{\sin x} = -\int \frac{1}{1+t^2}\, dt = -\tan^{-1}(t) + C = -\tan^{-1}(\cos x) + C$$

(iv) $I = \int \frac{2x^3}{4 + x^8}\, dx = \int \frac{2x^3}{2^2 + (x^4)^2}\, dx.$

Let $x^4 = t$. Then, $d(x^4) = dt \Rightarrow 4x^3\, dx = dt \Rightarrow dx = \frac{dt}{4x^3}$

$$\therefore \quad I = \int \frac{2x^3}{4 + t^2} \times \frac{dt}{4x^3} = \frac{1}{2}\int \frac{1}{2^2 + t^2}\, dt = \frac{1}{2} \times \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) + C = \frac{1}{4}\tan^{-1}\left(\frac{x^4}{2}\right) + C$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}}\, dx$ (ii) $\int \frac{e^{-x}}{16 + 9e^{-2x}}\, dx$

SOLUTION (i) Let $I = \int \frac{1}{x\{6(\log x)^2 + 7\log x + 2\}}\, dx$

Let $\log x = t$. Then, $d(\log x) = dt \Rightarrow \frac{1}{x}\, dx = dt \Rightarrow dx = x\, dt$

$$\therefore \quad I = \int \frac{1}{6t^2 + 7t + 2}\, dt$$

$$I = \frac{1}{6}\int \frac{1}{t^2 + \frac{7}{6}t + \frac{1}{3}}\, dt = \frac{1}{6}\int \frac{1}{\left(t + \frac{7}{12}\right)^2 + \frac{1}{3} - \frac{49}{144}}\, dt = \frac{1}{6}\int \frac{1}{\left(t + \frac{7}{12}\right)^2 - \left(\frac{1}{12}\right)^2}\, dt$$

$$\Rightarrow \quad I = \frac{1}{6} \times \frac{1}{2\left(\frac{1}{12}\right)} \log\left|\frac{t + \frac{7}{12} - \frac{1}{12}}{t + \frac{7}{12} + \frac{1}{12}}\right| + C = \log\left|\frac{2t+1}{3t+2}\right| + C = \log\left|\frac{2\log x + 1}{3\log x + 2}\right| + C$$

(ii) Let $I = \int \frac{e^{-x}}{16 + 9e^{-2x}}\, dx = \int \frac{e^{-x}}{4^2 + (3e^{-x})^2}\, dx$. Let $3e^{-x} = t$. Then,

$d(3e^{-x}) = dt \Rightarrow -3e^{-x} dx = dt \Rightarrow dx = -\frac{dt}{3e^{-x}}$. Putting $3e^{-x} = t$ and $dx = -\frac{dt}{3e^{-x}}$, we get

$$\therefore \quad I = \int \frac{e^{-x}}{16 + t^2}\left(-\frac{dt}{3e^{-x}}\right) = -\frac{1}{3}\int \frac{dt}{16 + t^2} = -\frac{1}{3}\int \frac{1}{(4)^2 + t^2}\, dt$$

$$\Rightarrow I = -\frac{1}{3} \times \frac{1}{4}\tan^{-1}\left(\frac{t}{4}\right) + C = -\frac{1}{12}\tan^{-1}\left(\frac{3t}{4}\right) + C = -\frac{1}{12}\tan^{-1}\left(\frac{3e^{-x}}{4}\right) + C$$

EXAMPLE 3 *Evaluate:* (i) $\int \frac{1}{x(x^n+1)}\,dx$ [CBSE 2000C] (ii) $\int \frac{1}{x(x^5+1)}\,dx$

SOLUTION We have, $I = \int \frac{1}{x(x^n+1)}\,dx = \int \frac{x^{n-1}}{x^n(x^n+1)}\,dx$

Let $x^n+1 = t$. Then, $d(x^n+1) = dt \Rightarrow n\,x^{n-1}\,dx = dt \Rightarrow dx = \frac{dt}{n\,x^{n-1}}$

$\therefore \quad I = \int \frac{1}{n\,x^n\,t}\,dt = \frac{1}{n}\int \frac{1}{(t-1)\,t}\,dt \qquad [\because\ x^n+1=t \ \therefore\ x^n=t-1]$

$\Rightarrow \quad I = \frac{1}{n}\int \frac{1}{t^2-t}\,dt = \frac{1}{n}\int \frac{dt}{t^2-t+1/4-1/4} = \frac{1}{n}\int \frac{1}{(t-1/2)^2-(1/2)^2}\,dt$

$\Rightarrow \quad I = \frac{1}{n}\times\frac{1}{2(1/2)}\log\left|\frac{t-1/2-1/2}{t-1/2+1/2}\right| + C = \frac{1}{n}\log\left|\frac{t-1}{t}\right| + C = \frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$

(ii) $I = \int \frac{1}{x(x^5+1)}\,dx = \int \frac{x^4}{x^5(x^5+1)}\,dx$

Let $x^5+1 = t$. Then, $d(x^5+1) = dt \Rightarrow 5x^4\,dx = dt \Rightarrow dx = \frac{dt}{5x^4}$

$\therefore \quad I = \frac{1}{5}\int \frac{1}{tx^5}\,dt = \frac{1}{5}\int \frac{1}{t(t-1)}\,dt = \frac{1}{5}\int \frac{1}{t^2-t}\,dt$

$\Rightarrow \quad I = \frac{1}{5}\int \frac{1}{t^2-t+1/4-1/4}\,dt = \frac{1}{5}\int \frac{1}{(t-1/2)^2-(1/2)^2}$

$\Rightarrow \quad I = \frac{1}{5}\times\frac{1}{2(1/2)}\log\left|\frac{t-1/2-1/2}{t-1/2+1/2}\right| + C = \frac{1}{5}\log\left|\frac{t-1}{t}\right| + C = \frac{1}{5}\log\left|\frac{x^5}{x^5+1}\right| + C$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Evaluate:* $\int \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

SOLUTION We observe that $\sin x+\cos x$ occurs in the derivative of $-\cos x+\sin x$. So, we express $9+16\sin 2x$ in terms of $-\cos x+\sin x$ as follows.

$\because \quad (-\cos x+\sin x)^2 = 1-\sin 2x$ or, $\sin 2x = 1-(-\cos x+\sin x)^2$

$\therefore \quad 9+16\sin 2x = 9+16\left\{1-(\cos x+\sin x)^2\right\} = 25-\left\{4(-\cos x+\sin x)\right\}^2$

Thus, $I = \int \frac{\sin x+\cos x}{9+16\sin 2x}\,dx = \int \frac{\sin x+\cos x}{25-\left\{4(-\cos x+\sin x)\right\}^2}\,dx$

Let $4(-\cos x+\sin x) = t$. Then,

$d\{4(-\cos x+\sin x)\} = dt$ or, $4(\sin x+\cos x)\,dx = dt$ or, $dx = \frac{dt}{4(\sin x+\cos x)}$

$\therefore \quad I = \int \frac{\sin x+\cos x}{25-t^2}\times\frac{dt}{4(\sin x+\cos x)}$

$\Rightarrow \quad I = \frac{1}{4}\int \frac{1}{25-t^2}\,dt = \frac{1}{4}\times\frac{1}{10}\log\left|\frac{5+t}{5-t}\right| + C = \frac{1}{40}\log\left|\frac{5+4(-\cos x+\sin x)}{5-4(-\cos x+\sin x)}\right| + C$

EXAMPLE 5 *Evaluate :* $\int \frac{1}{\sin x + \sec x}\,dx$

SOLUTION Let $I = \int \frac{1}{\sin x + \sec x}\,dx$. Then,

$$I = \int \frac{\cos x}{1 + \sin x \cos x}\,dx = \int \frac{2\cos x}{2 + 2\sin x \cos x}\,dx = \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + 2\sin x \cos x}\,dx$$

$$\Rightarrow \quad I = \int \frac{\cos x + \sin x}{2 + 2\sin x \cos x}\,dx + \int \frac{\cos x - \sin x}{2 + 2\sin x \cos x}\,dx$$

$$\Rightarrow \quad I = \int \frac{\cos x + \sin x}{3 - (1 - 2\sin x \cos x)}\,dx + \int \frac{\cos x - \sin x}{1 + (1 + 2\sin x \cos x)}\,dx$$

$$\Rightarrow \quad I = \int \frac{(\cos x + \sin x)}{3 - (\sin x - \cos x)^2}\,dx + \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}\,dx$$

$$\Rightarrow \quad I = \int \frac{1}{(\sqrt{3})^2 - u^2}\,du + \int \frac{1}{1 + v^2}\,dv, \text{ where } u = \sin x - \cos x \text{ and } v = \sin x + \cos x$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + u}{\sqrt{3} - u} \right| + \tan^{-1} v + C = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + (\sin x - \cos x)}{\sqrt{3} - (\sin x - \cos x)} \right| + \tan^{-1} (\sin x + \cos x) + C$$

EXERCISE 18.16

BASIC

Evaluate the following integrals:

1. $\int \frac{\sec^2 x}{1 - \tan^2 x}\,dx$
2. $\int \frac{e^x}{1 + e^{2x}}\,dx$
3. $\int \frac{\cos x}{\sin^2 x + 4\sin x + 5}\,dx$
4. $\int \frac{e^x}{e^{2x} + 5e^x + 6}\,dx$
5. $\int \frac{e^{3x}}{4e^{6x} - 9}\,dx$
6. $\int \frac{1}{e^x + e^{-x}}\,dx$
7. $\int \frac{x}{x^4 + 2x^2 + 3}\,dx$
8. $\int \frac{3x^5}{1 + x^{12}}\,dx$
9. $\int \frac{x^2}{x^6 - a^6}\,dx$
10. $\int \frac{x^2}{x^6 + a^6}\,dx$
11. $\int \frac{1}{x(x^6 + 1)}\,dx$
12. $\int \frac{x}{x^4 - x^2 + 1}\,dx$ [CBSE 2007]
13. $\int \frac{x}{3x^4 - 18x^2 + 11}\,dx$
14. $\int \frac{e^x}{(1 + e^x)(2 + e^x)}\,dx$ [CBSE 2022]
15. $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x}\,dx$ [CBSE 2019]

BASED ON LOTS

16. $\int \frac{1}{\cos x + \operatorname{cosec} x}\,dx$

ANSWERS

1. $\frac{1}{2} \log \left| \frac{1 + \tan x}{1 - \tan x} \right| + C$
2. $\tan^{-1} \left(\frac{e^x}{1} \right) + C$
3. $\tan^{-1} (\sin x + 2) + C$
4. $\log \left| \frac{e^x + 2}{e^x + 3} \right| + C$
5. $\frac{1}{36} \log \left| \frac{2e^{3x} - 3}{2e^{3x} + 3} \right| + C$
6. $\tan^{-1} (e^x) + C$
7. $\frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 + 1}{\sqrt{2}} \right) + C$
8. $\frac{1}{2} \tan^{-1} (x^6) + C$
9. $\frac{1}{6a^3} \log \left| \frac{x^3 - a^3}{x^3 + a^3} \right| + C$

10. $\frac{1}{3a^3}\tan^{-1}\left(\frac{x^3}{a^3}\right)+C$ **11.** $\frac{1}{6}\log\left|\frac{x^6}{x^6+1}\right|+C$ **12.** $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2-1}{\sqrt{3}}\right)+C$

13. $\frac{\sqrt{3}}{48}\log\left|\frac{x^2-3-\frac{4}{\sqrt{3}}}{x^2-3+\frac{4}{\sqrt{3}}}\right|+C$ **14.** $\log\left|\frac{1+e^x}{2+e^x}\right|+C$ **15.** $\frac{1}{6}\log\left|\frac{1+\tan^3 x}{1-\tan^3 x}\right|+C$

16. $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\sin x-\cos x}{\sqrt{3}-\sin x+\cos x}\right|-\tan^{-1}(\sin x+\cos x)+C$

18.10.3 INTEGRALS OF THE TYPE $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

In order to evaluate this type of integrals, we may use the following algorithm.

ALGORITHM

Step I *Make the coefficient of x^2 unity, if it is not.*

Step II *Find half of the coefficient of x.*

Step III *Add and subtract $\left(\frac{1}{2}\text{ Coeff. of } x\right)^2$ inside the square root to express the quantity inside the square root in the form $\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$ or, $\frac{4ac-b^2}{4a^2}-\left(x+\frac{b}{2a}\right)^2$.*

Step IV *Use the suitable formula from the following formulas:*

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left|x+\sqrt{a^2+x^2}\right|+C,\quad \int\frac{1}{\sqrt{x^2-a^2}}\,dx=\log\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)+C$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int\frac{1}{\sqrt{(x-1)(x-2)}}\,dx$ [NCERT] (ii) $\int\frac{1}{\sqrt{9+8x-x^2}}\,dx$ (iii) $\int\frac{1}{\sqrt{x(1-2x)}}\,dx$ (iv) $\int\frac{1}{\sqrt{2x^2+3x-2}}\,dx$

SOLUTION (i) Let $I=\int\frac{1}{\sqrt{(x-1)(x-2)}}\,dx$. Then,

$$I=\int\frac{1}{\sqrt{x^2-3x+2}}\,dx=\int\frac{1}{\sqrt{x^2-3x+\frac{9}{4}-\frac{9}{4}+2}}\,dx=\int\frac{1}{\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}}\,dx$$

$$\Rightarrow\quad I=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{\left(x-\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right|+C=\log\left|\left(x-\frac{3}{2}\right)+\sqrt{x^2-3x+2}\right|+C$$

(ii) Let $I=\int\frac{1}{\sqrt{9+8x-x^2}}\,dx$. Then,

$$I = \int \frac{1}{\sqrt{-\left(x^2 - 8x - 9\right)}}\, dx = \int \frac{1}{\sqrt{-\left(x^2 - 8x + 16 - 25\right)}}\, dx$$

$$\Rightarrow \quad I = \int \frac{1}{\sqrt{-\{(x-4)^2 - 5^2\}}}\, dx = \int \frac{1}{\sqrt{5^2 - (x-4)^2}}\, dx = \sin^{-1}\left(\frac{x-4}{5}\right) + C.$$

(iii) Let $I = \int \frac{1}{\sqrt{x(1-2x)}}\, dx$. Then,

$$I = \int \frac{1}{\sqrt{x - 2x^2}}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left\{x^2 - \frac{x}{2} + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}}}\, dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{-\left\{\left(x - \frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}}}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}}\, dx = \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{x - 1/4}{1/4}\right) + C = \frac{1}{\sqrt{2}} \sin^{-1}(4x - 1) + C.$$

(iv) Let $I = \int \frac{1}{\sqrt{2x^2 + 3x - 2}}\, dx$. Then,

$$I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{x^2 + \frac{3}{2}x - 1}}\, dx = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \frac{9}{16} - 1}}\, dx$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\left(x + \frac{3}{4}\right)^2 - \left(\frac{5}{4}\right)^2}}\, dx = \frac{1}{\sqrt{2}} \log\left|\left(x + \frac{3}{4}\right) + \sqrt{x^2 + \frac{3}{2}x - 1}\right| + C.$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx$ **[CBSE 2001C]** (ii) $\int \frac{1}{\sqrt{x^2 - 4x + 2}}\, dx$

SOLUTION (i) Let $I = \int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx$. Then,

$$I = \int \frac{1}{\sqrt{x^2 - x(a+b) + ab}}\, dx$$

$$\Rightarrow I = \int \frac{1}{\sqrt{x^2 - x(a+b) + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab}}\, dx = \int \frac{1}{\sqrt{\left\{x - \left(\frac{a+b}{2}\right)\right\}^2 - \left(\frac{a-b}{2}\right)^2}}\, dx$$

$$\Rightarrow I = \log\left|\left\{x - \left(\frac{a+b}{2}\right)\right\} + \sqrt{\left\{x - \left(\frac{a+b}{2}\right)\right\}^2 - \left(\frac{a-b}{2}\right)^2}\right| + C$$

$$\Rightarrow I = \log\left|\left(\frac{2x - a - b}{2}\right) + \sqrt{(x-a)(x-b)}\right| + C = \log\left|\frac{(x-a) + (x-b) + 2\sqrt{(x-a)(x-b)}}{2}\right| + C$$

$$\Rightarrow I = \log\left|\left(\sqrt{x-a}+\sqrt{x-b}\right)^2\right| - \log 2 + C = 2\log\left|\sqrt{x-a}+\sqrt{x-b}\right| + C, \text{ where } C_1 = C - \log 2$$

(ii) Let $I = \int \frac{1}{\sqrt{x^2-4x+2}}\,dx$. Then,

$$I = \int \frac{1}{\sqrt{x^2-4x+4-4+2}}\,dx = \int \frac{1}{\sqrt{(x-2)^2-(\sqrt{2})^2}}\,dx$$

$$\Rightarrow I = \log\left|(x-2)+\sqrt{(x-2)^2-(\sqrt{2})^2}\right| + C = \log\left|x-2+\sqrt{x^2-4x+2}\right| + C$$

EXERCISE 18.17

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{\sqrt{2x-x^2}}\,dx$

2. $\int \frac{1}{\sqrt{8+3x-x^2}}\,dx$ [NCERT]

3. $\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx$ [CBSE 2009, 2019]

4. $\int \frac{1}{\sqrt{3x^2+5x+7}}\,dx$

5. $\int \frac{1}{\sqrt{(x-\alpha)(\beta-x)}}\,dx, (\beta > \alpha)$

6. $\int \frac{1}{\sqrt{7-3x-2x^2}}\,dx$

7. $\int \frac{1}{\sqrt{16-6x-x^2}}\,dx$

8. $\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$ [NCERT, CBSE 2002]

9. $\int \frac{1}{\sqrt{5x^2-2x}}\,dx$ [NCERT]

ANSWERS

1. $\sin^{-1}(x-1)+C$

2. $\sin^{-1}\left(\frac{2x-3}{\sqrt{41}}\right)+C$

3. $\frac{1}{\sqrt{2}}\sin^{-1}\left\{\sqrt{\frac{2}{7}}(x+1)\right\}+C$

4. $\frac{1}{\sqrt{3}}\log\left|x+\frac{5}{6}+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right|+C$

5. $2\sin^{-1}\left(\sqrt{\frac{x-\alpha}{\beta-\alpha}}\right)+C$

6. $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{4x+3}{\sqrt{65}}\right)+C$

7. $\sin^{-1}\left(\frac{x+3}{5}\right)+C$

8. $\sin^{-1}\left(\frac{x+3}{4}\right)+C$

9. $\frac{1}{\sqrt{5}}\log\left|\frac{5x-1}{5}+\frac{\sqrt{5x^2-2x}}{\sqrt{5}}\right|+C$

18.10.4 INTEGRALS REDUCIBLE TO THE FORM $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

Following examples will illustrate the procedure of evaluating this type of integrals:

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$ (ii) $\int \frac{x^2}{\sqrt{1-x^6}}\,dx$

(iii) $\int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx$ [NCERT] (iv) $\int \frac{1}{x\sqrt{(\log x)^2-5}}\,dx$

SOLUTION (i) Let $I = \int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx = \int \frac{e^x}{\sqrt{2^2-(e^x)^2}}\,dx$

Let $e^x = t$. Then, $d(e^x) = dt \Rightarrow e^x\,dx = dt \Rightarrow dx = \frac{dt}{e^x}$

$$\therefore \quad I = \int \frac{dt}{\sqrt{4-t^2}} = \int \frac{dt}{\sqrt{2^2-t^2}} = \sin^{-1}\left(\frac{t}{2}\right)+C = \sin^{-1}\left(\frac{e^x}{2}\right)+C$$

(ii) Let $I = \int \frac{x^2}{\sqrt{1-x^6}}\,dx = \int \frac{x^2}{\sqrt{1^2-(x^3)^2}}\,dx$

Let $x^3 = t$. Then, $d(x^3) = dt \Rightarrow 3x^2\,dx = dt \Rightarrow dx = \frac{dt}{3x^2}$

$$\therefore \quad I = \frac{1}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\sin^{-1}(t)+C = \frac{1}{3}\sin^{-1}(x^3)+C$$

(iii) Let, $I = \int \frac{\sec^2 x}{\sqrt{16+\tan^2 x}}\,dx = \int \frac{\sec^2 x}{\sqrt{4^2+\tan^2 x}}\,dx.$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\,dx = dt \Rightarrow dx = \frac{dt}{\sec^2 x}$

$$\therefore \quad I = \int \frac{dt}{\sqrt{16+t^2}} = \int \frac{dt}{\sqrt{4^2+t^2}} = \log\left|t+\sqrt{4^2+t^2}\right|+C = \log\left|\tan x+\sqrt{16+\tan^2 x}\right|+C$$

(iv) Let $I = \int \frac{1}{x\sqrt{(\log x)^2-5}}\,dx$. Let $\log x = t$. Then, $d(\log x) = dt \Rightarrow \frac{1}{x}\,dx = dt \Rightarrow dx = x\,dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{t^2-(\sqrt{5})^2}} = \log|t+\sqrt{t^2-5}|+C = \log\left|\log x+\sqrt{(\log x)^2-5}\right|+C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{a^x}{\sqrt{1-a^{2x}}}\,dx$ (ii) $\int \frac{2x}{\sqrt{1-x^2-x^4}}\,dx$ [CBSE 2005]

(iii) $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$ [CBSE 2009] (iv) $\int \frac{\cos x}{\sqrt{\sin^2 x-2\sin x-3}}\,dx$

(v) $\int \sqrt{\frac{x}{a^3 - x^3}}\, dx$ [CBSE 2016] (vi) $\int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}\, dx$

SOLUTION (i) Let $I = \int \frac{a^x}{\sqrt{1 - a^{2x}}}\, dx = \int \frac{a^x}{\sqrt{1^2 - (a^x)^2}}\, dx.$

Let $a^x = t$. Then, $d(a^x) = dt \Rightarrow a^x \log_e a\, dx = dt \Rightarrow dx = \frac{dt}{a^x \log_e a}$

$$\therefore \quad I = \int \frac{a^x}{\sqrt{1^2 - t^2}} \cdot \frac{dt}{a^x \log a} = \frac{1}{\log a} \int \frac{dt}{\sqrt{1^2 - t^2}} = \frac{1}{\log a} \times \sin^{-1}(t) + C = \frac{1}{\log a} \sin^{-1}(a^x) + C$$

(ii) Let $I = \int \frac{2x}{\sqrt{1 - x^2 - x^4}}\, dx = \int \frac{2x}{\sqrt{1 - x^2 - (x^2)^2}}\, dx.$

Let $x^2 = t$. Then, $d(x^2) = dt \Rightarrow 2x\, dx = dt \Rightarrow dx = \frac{dt}{2x}$

$$\therefore \quad I = \int \frac{1}{\sqrt{1 - t - t^2}}\, dt = \int \frac{1}{\sqrt{-(t^2 + t - 1)}}\, dt = \int \frac{1}{\sqrt{-\left\{t^2 + t + \frac{1}{4} - \frac{1}{4} - 1\right\}}}\, dt$$

$$\Rightarrow \quad I = \int \frac{1}{\sqrt{-\left\{\left(t + \frac{1}{2}\right)^2 - \frac{5}{4}\right\}}}\, dt = \int \frac{1}{\sqrt{\frac{5}{4} - \left(t + \frac{1}{2}\right)^2}}\, dt = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(t + \frac{1}{2}\right)^2}}\, dt$$

$$\Rightarrow \quad I = \sin^{-1}\left(\frac{t + 1/2}{\sqrt{5}/2}\right) + C = \sin^{-1}\left(\frac{2t + 1}{\sqrt{5}}\right) + C = \sin^{-1}\left(\frac{2x^2 + 1}{\sqrt{5}}\right) + C$$

(iii) Let $I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\, dx = \int \frac{e^x}{\sqrt{5 - 4e^x - (e^x)^2}}\, dx.$

Let $e^x = t$. Then, $d(e^x) = dt \Rightarrow e^x\, dx = dt \Rightarrow dx = \frac{dt}{e^x}$

$$\therefore \quad I = \int \frac{1}{\sqrt{5 - 4t - t^2}}\, dt = \int \frac{1}{\sqrt{-(t^2 + 4t - 5)}}\, dt = \int \frac{1}{\sqrt{-\{(t + 2)^2 - 3^2\}}}\, dt$$

$$\Rightarrow \quad I = \int \frac{1}{\sqrt{3^2 - (t + 2)^2}}\, dt = \sin^{-1}\left(\frac{t + 2}{3}\right) + C = \sin^{-1}\left(\frac{e^x + 2}{3}\right) + C$$

(iv) Let $I = \int \frac{\cos x}{\sqrt{\sin^2 x - 2 \sin x - 3}}\, dx$. Let $\sin x = t$. Then,

$$d(\sin x) = dt \Rightarrow \cos x\, dx = dt \Rightarrow dx = \frac{dt}{\cos x}$$

$$\therefore \quad I = \int \frac{dt}{\sqrt{t^2 - 2t - 3}} = \int \frac{dt}{\sqrt{t^2 - 2t + 1 - 1 - 3}} = \int \frac{dt}{\sqrt{(t - 1)^2 - 2^2}}$$

$$\Rightarrow \quad I = \log\left|(t - 1) + \sqrt{(t - 1)^2 - 2^2}\right| + C$$

$$\Rightarrow \quad I = \log\left|t - 1 + \sqrt{t^2 - 2t - 3}\right| + C = \log\left|(\sin x - 1) + \sqrt{\sin^2 x - 2 \sin x - 3}\right| + C$$

(v) Let $I = \int \sqrt{\frac{x}{a^3 - x^3}}\, dx = \int \frac{\sqrt{x}}{\sqrt{\left(a^{3/2}\right)^2 - \left(x^{3/2}\right)^2}}\, dx.$

Let $x^{3/2} = t$. Then, $d\left(x^{3/2}\right) = dt \Rightarrow \frac{3}{2} x^{1/2}\, dx = dt \Rightarrow dx = \frac{2}{3\sqrt{x}}\, dt$

$$\therefore \quad I = \frac{2}{3} \int \frac{1}{\sqrt{\left(a^{3/2}\right)^2 - t^2}}\, dt = \frac{2}{3} \sin^{-1}\left(\frac{t}{a^{3/2}}\right) + C = \frac{2}{3} \sin^{-1}\left(\frac{x^{3/2}}{a^{3/2}}\right) + C$$

(vi) Let $I = \int \frac{\sin 2x \cos 2x}{\sqrt{9 - \cos^4 2x}}\, dx$. Then, $I = \int \frac{\sin 2x \cos 2x}{\sqrt{3^2 - (\cos^2 2x)^2}}\, dx$

Putting $\cos^2 2x = t$ and $-4 \sin 2x \cos 2x\, dx = dt$, we get

$$I = -\frac{1}{4} \int \frac{1}{\sqrt{3^2 - t^2}}\, dt = -\frac{1}{4} \sin^{-1}\left(\frac{t}{3}\right) + C = -\frac{1}{4} \sin^{-1}\left(\frac{\cos^2 2x}{3}\right) + C$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:* (i) $\int \sqrt{\sec x - 1}\, dx$ (ii) $\int \frac{1}{\sqrt{1 - e^{2x}}}\, dx$

SOLUTION (i) Let $I = \int \sqrt{\sec x - 1}\, dx = \int \sqrt{\frac{1 - \cos x}{\cos x}}\, dx$. Then,

$$\Rightarrow \quad I = \int \sqrt{\frac{(1 - \cos x)}{\cos x} \times \frac{(1 + \cos x)}{(1 + \cos x)}}\, dx = \int \sqrt{\frac{1 - \cos^2 x}{\cos x + \cos^2 x}}\, dx = \int \frac{\sin x}{\sqrt{\cos^2 x + \cos x}}\, dx$$

Let $\cos x = t$. Then, $d(\cos x) = dt \Rightarrow -\sin x\, dx = dt \Rightarrow dx = -\frac{dt}{\sin x}$

$$\therefore \quad I = \int \frac{-dt}{\sqrt{t^2 + t}} = -\int \frac{dt}{\sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = -\log\left|\left(t + \frac{1}{2}\right) + \sqrt{\left(t + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$\Rightarrow \quad I = -\log\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right| + C = -\log\left|\left(\cos x + \frac{1}{2}\right) + \sqrt{\cos^2 x + \cos x}\right| + C$$

(ii) Let $I = \int \frac{1}{\sqrt{1 - e^{2x}}}\, dx = \int \frac{1}{\sqrt{1 - \frac{1}{e^{-2x}}}}\, dx = \int \frac{e^{-x}}{\sqrt{e^{-2x} - 1}}\, dx = \int \frac{e^{-x}}{\sqrt{(e^{-x})^2 - 1^2}}\, dx$

Let $e^{-x} = t$. Then, $d(e^{-x}) = dt \Rightarrow -e^{-x}\, dx = dt \Rightarrow dx = -\frac{dt}{e^{-x}}$

$$\therefore \quad I = -\int \frac{dt}{\sqrt{t^2 - 1^2}} = -\log\left|t + \sqrt{t^2 - 1}\right| + C = -\log\left|e^{-x} + \sqrt{e^{-2x} - 1}\right| + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Evaluate:* $\int \sqrt{\frac{\sin (x - \alpha)}{\sin (x + \alpha)}}\, dx$

SOLUTION Let $I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)}}\, dx$

$$\Rightarrow I = \int \sqrt{\frac{\sin(x-\alpha)}{\sin(x+\alpha)} \times \frac{\sin(x-\alpha)}{\sin(x-\alpha)}}\, dx = \int \frac{\sin(x-\alpha)}{\sqrt{\sin^2 x - \sin^2 \alpha}}\, dx = \int \frac{\sin x \cos\alpha - \cos x \sin\alpha}{\sqrt{\sin^2 x - \sin^2\alpha}}\, dx$$

$$\Rightarrow I = \cos\alpha \int \frac{\sin x}{\sqrt{1-\cos^2 x - 1 + \cos^2\alpha}}\, dx - \sin\alpha \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\, dx$$

$$\Rightarrow \quad I = \cos\alpha \int \frac{\sin x}{\sqrt{\cos^2\alpha - \cos^2 x}}\, dx - \sin\alpha \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2\alpha}}\, dx$$

In the first integral we put $\cos x = t$, so that $-\sin x\, dx = dt$ and in the second integral we put $\sin x = u$, so that $\cos x\, dx = du$.

$$\therefore \quad I = -\cos\alpha \int \frac{dt}{\sqrt{\cos^2\alpha - t^2}} - \sin\alpha \int \frac{du}{\sqrt{u^2 - \sin^2\alpha}}$$

$$\Rightarrow \quad I = -\cos\alpha \sin^{-1}\left(\frac{t}{\cos\alpha}\right) - \sin\alpha \log\left|u + \sqrt{u^2 - \sin^2\alpha}\right| + C$$

$$\Rightarrow \quad I = -\cos\alpha \sin^{-1}\left(\frac{\cos x}{\cos\alpha}\right) - \sin\alpha \log\left|\sin x + \sqrt{\sin^2 x - \sin^2\alpha}\right| + C$$

EXAMPLE 5 *Evaluate* : $\int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\, dx$ **[NCERT EXAMPLAR]**

SOLUTION Let $I = \int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}\, dx$. Here, the integration of the numerator of the integrand is $-\cos x + \sin x$. That is the numerator of the integrand occurs in the derivative of $-\cos x + \sin x$. So, we express $1 + \sin 2x$ in terms of $-\cos x + \sin x$. We observe that $(-\cos x + \sin x)^2 = 1 - \sin 2x$. Therefore, we write $1 + \sin 2x = 2 - (1 - \sin 2x) = 2 - (\sin x - \cos x)^2$.

$$\therefore \quad I = \int \frac{\sin x + \cos x}{\sqrt{2 - (\sin x - \cos x)^2}}\, dx$$

Let $\sin x - \cos x = t$. Then, $d(\sin x - \cos x) = dt$ or, $(\cos x + \sin x)\, dx = dt$

$$\therefore \quad I = \int \frac{1}{\sqrt{2 - t^2}}\, dt = \int \frac{1}{\sqrt{(\sqrt{2})^2 - t^2}}\, dt = \sin^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$\Rightarrow \quad I = \sin^{-1}\left\{\frac{1}{\sqrt{2}}(\sin x - \cos x)\right\} + C = \sin^{-1}\left\{\sin\left(x - \frac{\pi}{4}\right)\right\} + C$$

EXERCISE 18.18

BASIC

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^4 + a^4}}\, dx$
2. $\int \frac{\sec^2 x}{\sqrt{4 + \tan^2 x}}\, dx$ [NCERT, CBSE 2019]
3. $\int \frac{e^x}{\sqrt{16 - e^{2x}}}\, dx$

4. $\int \frac{\cos x}{\sqrt{4+\sin^2 x}}\, dx$

5. $\int \frac{\sin x}{\sqrt{4\cos^2 x - 1}}\, dx$

6. $\int \frac{x}{\sqrt{4-x^4}}\, dx$

7. $\int \frac{1}{x\sqrt{4-9(\log x)^2}}\, dx$

8. $\int \frac{\sin 8x}{\sqrt{9+\sin^4 4x}}\, dx$

9. $\int \frac{\cos 2x}{\sqrt{\sin^2 2x+8}}\, dx$

10. $\int \frac{\sin 2x}{\sqrt{\sin^4 x + 4\sin^2 x - 2}}\, dx$

11. $\int \frac{\sin 2x}{\sqrt{\cos^4 x - \sin^2 x + 2}}\, dx$

12. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\, dx$

13. $\int \frac{1}{x^{2/3}\sqrt{x^{2/3}-4}}\, dx$

14. $\int \frac{1}{\sqrt{(1-x^2)\{9+(\sin^{-1} x)^2\}}}\, dx$

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15. $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}}\, dx$

16. $\int \sqrt{\operatorname{cosec} x - 1}\, dx$

17. $\int \frac{\sin x - \cos x}{\sqrt{\sin 2x}}\, dx$ [CBSE 2011]

18. $\int \frac{\cos x - \sin x}{\sqrt{8-\sin 2x}}\, dx$

ANSWERS

1. $\frac{1}{2}\log\left|x^2+\sqrt{x^4+a^4}\right|+C$

2. $\log\left|\tan x+\sqrt{4+\tan^2 x}\right|+C$

3. $\sin^{-1}\left(\frac{e^x}{4}\right)+C$

4. $\log\left|\sin x+\sqrt{4+\sin^2 x}\right|+C$

5. $-\frac{1}{2}\log\left|2\cos x+\sqrt{4\cos^2 x-1}\right|+C$

6. $\frac{1}{2}\sin^{-1}\left(\frac{x^2}{2}\right)+C$

7. $\frac{1}{3}\sin^{-1}\left(\frac{3\log x}{2}\right)+C$

8. $\frac{1}{4}\log\left|\sin^2 4x+\sqrt{9+\sin^4 4x}\right|+C$

9. $\frac{1}{2}\log\left|\sin 2x+\sqrt{\sin^2 2x+8}\right|+C$

10. $\log\left|\sin^2 x+2+\sqrt{\sin^4 x+4\sin^2 x-2}\right|+C$

11. $-\log\left|\left(\cos^2 x+\frac{1}{2}\right)+\sqrt{\cos^4 x+\cos^2 x+1}\right|+C$

12. $\sin^{-1}\left(\frac{\sin x}{2}\right)+C$

13. $3\log\left|x^{1/3}+\sqrt{x^{2/3}-4}\right|+C$

14. $\log\left|\sin^{-1} x+\sqrt{9+(\sin^{-1} x)^2}\right|+C$

15. $\log\left|(\sin x-1)+\sqrt{\sin^2 x-2\sin x-3}\right|+C$

16. $\log\left|\left(\sin x+\frac{1}{2}\right)+\sqrt{\sin^2 x+\sin x}\right|+C$

17. $-\log\left|(\sin x+\cos x)+\sqrt{\sin 2x}\right|+C$

18. $\sin^{-1}\left\{\frac{1}{3}(\sin x+\cos x)\right\}+C$

18.10.5 INTEGRALS OF THE FORM $\int \frac{px+q}{ax^2+bx+c}\,dx$

To evaluate this type of integrals, we use the following algorithm.

ALGORITHM

Step I *Write the numerator $px+q$ in the following form:*

$$px+q = \lambda\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+\mu \text{ i.e. } px+q = \lambda(2ax+b)+\mu$$

Step II *Obtain the values of λ and μ by equating the coefficients of like powers of x on both sides.*

Step III *Replace $px+q$ by $\lambda(2ax+b)+\mu$ in the given integral to get*

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \lambda\int \frac{2ax+b}{ax^2+bx+c}\,dx+\mu\int \frac{1}{ax^2+bx+c}\,dx$$

Step IV *Integrate RHS in step III and put the values of λ and μ obtained in step II.*

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{x}{x^2+x+1}\,dx$ (ii) $\int \frac{4x+1}{x^2+3x+2}\,dx$ (iii) $\int \frac{2x-3}{x^2+3x-18}\,dx$

SOLUTION (i) Let $x=\lambda\frac{d}{dx}(x^2+x+1)+\mu$. Then, $x=\lambda(2x+1)+\mu$

Comparing the coefficients of like powers of x, we get

$$1 = 2\lambda \text{ and } \lambda+\mu = 0 \Rightarrow \lambda = \frac{1}{2} \text{ and } \mu = -\lambda = -\frac{1}{2}$$

$$\therefore \quad I = \int \frac{x}{x^2+x+1}\,dx = \int \frac{1/2\,(2x+1)-1/2}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx-\frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\log|x^2+x+1|-\frac{1}{2}\times\frac{1}{(\sqrt{3/2})}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3/2}}\right)+C$$

$$\Rightarrow \quad I = \frac{1}{2}\log|x^2+x+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)+C$$

(ii) Let $4x+1=\lambda\frac{d}{dx}(x^2+3x+2)+\mu$. Then, $4x+1=\lambda(2x+3)+\mu$.

Comparing coefficients of like powers of x, we get: $2\lambda=4$ and $3\lambda+\mu=1 \Rightarrow \lambda=2$ and $\mu=-5$

$$\therefore \quad I = \int \frac{4x+1}{x^2+3x+2}\,dx = \int \frac{2(2x+3)-5}{x^2+3x+2}\,dx = 2\int \frac{2x+3}{x^2+3x+2}\,dx - 5\int \frac{1}{x^2+3x+2}\,dx$$

$$\Rightarrow \quad I = 2\log|x^2+3x+2| - 5\int \frac{1}{x^2+3x+(9/4)-(9/4)+2}\,dx$$

$$\Rightarrow \quad I = 2\log|x^2+3x+2| - 5\int \frac{1}{(x+3/2)^2-(1/2)^2}\,dx$$

$$\Rightarrow \quad I = 2\log|x^2+3x+2| - 5\times\frac{1}{2(1/2)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$\Rightarrow \quad I = 2\log|x^2+3x+2| - 5\log\left|\frac{x+1}{x+2}\right| + C$$

(iii) Let $2x-3 = \lambda\frac{d}{dx}(x^2+3x-18)+\mu$. Then, $2x-3 = \lambda(2x+3)+\mu$.

Comparing coefficients of like powers of x, we get

$$2\lambda = 2 \text{ and } 3\lambda+\mu = -3 \Rightarrow \lambda = 1 \text{ and } \mu = -6$$

$$\therefore \quad I = \int \frac{2x-3}{x^2+3x-18}\,dx = \int \frac{2x+3-6}{x^2+3x-18}\,dx = \int \frac{2x+3}{x^2+3x-18}\,dx - 6\int \frac{1}{x^2+3x-18}\,dx$$

$$\Rightarrow \quad I = \log|x^2+3x-18| - 6\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}-18}\,dx$$

$$\Rightarrow \quad I = \log|x^2+3x-18| - 6\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I = \log|x^2+3x-18| - 6\times\frac{1}{2\left(\frac{9}{2}\right)}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right| + C$$

$$\Rightarrow \quad I = \log|x^2+3x-18| - \frac{2}{3}\log\left|\frac{x-3}{x+6}\right| + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi - 4\sin\phi}\,d\phi$ (ii) $\int \frac{x^3+x}{x^4-9}\,dx$ [CBSE 2022] (iii) $\int \frac{1}{2e^{2x}+3e^x+1}\,dx$

SOLUTION (i) Let $I = \int \frac{2\sin 2\phi - \cos\phi}{6-\cos^2\phi-4\sin\phi}\,d\phi$. Then,

$$I = \int \frac{(4\sin\phi-1)\cos\phi}{6-(1-\sin^2\phi)-4\sin\phi}\,d\phi = \int \frac{(4\sin\phi-1)\cos\phi}{\sin^2\phi-4\sin\phi+5}\,d\phi$$

Putting $\sin\phi = t$ and $\cos\phi\, d\phi = dt$, we get

$$I = \int \frac{4t-1}{t^2-4t+5}\,dt$$

Let $(4t-1) = \lambda\frac{d}{dt}(t^2-4t+5)+\mu \Rightarrow (4t-1) = \lambda(2t-4)+\mu$.

Comparing coefficients of like powers of t, we get: $2\lambda = 4,\ -4\lambda+\mu = -1 \Rightarrow \lambda = 2,\ \mu = 7$

$$\therefore \quad I = \int \frac{\lambda(2t-4)+\mu}{t^2-4t+5}\,dt = 2\int \frac{2t-4}{t^2-4t+5}\,dt + 7\int \frac{1}{t^2-4t+5}\,dt$$

$$\Rightarrow \quad I = 2\log|t^2-4t+5| + 7\int \frac{1}{(t-2)^2+1^2}\,dt = 2\log|t^2-4t+5| + 7\tan^{-1}(t-2) + C$$

$$\Rightarrow \quad I = 2\log|\sin^2\phi - 4\sin\phi + 5| + 7\tan^{-1}(\sin\phi - 2) + C$$

(ii) Let $I = \int \frac{x^3+x}{x^4-9}\,dx$. Then,

$$I = \int \frac{x^3}{x^4-9}\,dx + \int \frac{x}{x^4-9}\,dx = I_1 + I_2 \text{ (say), where } I_1 = \int \frac{x^3}{x^4-9}\,dx \text{ and } I_2 = \int \frac{x}{x^4-9}\,dx.$$

Putting $x^4-9=t$ and $4x^3\,dx = dt$, we get

$$I_1 = \int \frac{x^3}{t} \times \frac{dt}{4x^3} = \frac{1}{4}\int \frac{1}{t}\,dt = \frac{1}{4}\log|t| = \frac{1}{4}\log|x^4-9|$$

Putting $x^2=t$ and $2x\,dx = dt$, we get

$$I_2 = \int \frac{x}{x^4-9}\,dx = \int \frac{x}{(x^2)^2-3^2}\,dx = \frac{1}{2}\int \frac{dt}{t^2-3^2} = \frac{1}{2}\times\frac{1}{2\times 3}\log\left|\frac{t-3}{t+3}\right| = \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right|$$

Hence, $I = I_1 + I_2 = \frac{1}{4}\log|x^4-9| + \frac{1}{12}\log\left|\frac{x^2-3}{x^2+3}\right| + C$

(iii) We have,

$$I = \int \frac{1}{2e^{2x}+3e^x+1}\,dx = \int \frac{1}{\frac{2}{e^{-2x}}+\frac{3}{e^{-x}}+1}\,dx = \int \frac{e^{-2x}}{2+3e^{-x}+e^{-2x}}\,dx$$

Let $e^{-x}=t$. Then, $d(e^{-x}) = dt \Rightarrow -e^{-x}\,dx = dt \Rightarrow dx = -\frac{dt}{e^{-x}}$

$$\therefore \quad I = \int \frac{-t\,dt}{2+3t+t^2} = -\int \frac{t}{t^2+3t+2}\,dt$$

Let $t = \lambda(2t+3)+\mu$. Comparing the coefficients of like powers of t, we get

$$2\lambda = 1,\ 3\lambda+\mu = 0 \Rightarrow \lambda = 1/2,\ \mu = -3/2$$

$$\therefore \quad I = -\int \frac{\lambda(2t+3)+\mu}{t^2+3t+2}\,dt = -\lambda\int \frac{2t+3}{t^2+3t+2}\,dt - \mu\int \frac{1}{t^2+3t+2}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{2}\int \frac{2t+3}{t^2+3t+2}\,dt + \frac{3}{2}\int \frac{1}{(t+3/2)^2-(1/2)^2}\,dt$$

$$\Rightarrow \quad I = -\frac{1}{2}\log\left|t^2+3t+2\right| + \frac{3}{2}\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$\Rightarrow \quad I = -\frac{1}{2}\log|e^{-2x}+3e^{-x}+2| + \frac{3}{2}\log\left|\frac{e^{-x}+1}{e^{-x}+2}\right| + C$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:* $\int \frac{x^3}{x^4+3x^2+2}\,dx$ [NCERT EXEMPLAR]

SOLUTION (i) Let $I = \int \frac{x^3}{x^4+3x^2+2}\,dx$. Then, $I = \frac{1}{2}\int \frac{x^2}{(x^2)^2+3x^2+2}\,2x\,dx.$

Let $x^2 = t$. Then, $d(x^2) = dt$ or, $2x\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \frac{t}{t^2+3t+2}\,dt = \frac{1}{4}\int \frac{(2t+3)-3}{t^2+3t+2}\,dt = \frac{1}{4}\int \frac{2t+3}{t^2+3t+2}\,dt - \frac{3}{4}\int \frac{1}{t^2+3+2}\,dt$$

$$\Rightarrow \quad I = \frac{1}{4}\int \frac{2t+3}{t^2+3t+2}\,dt - \frac{3}{4}\int \frac{1}{\left(t+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dt$$

$$\Rightarrow \quad I = \frac{1}{4}\log|t^2+3t+2| - \frac{3}{4}\log\left|\frac{t+\frac{3}{2}-\frac{1}{2}}{t+\frac{3}{2}+\frac{1}{2}}\right| + C = \frac{1}{4}\log|x^4+3x^2+2| - \frac{3}{4}\log\left|\frac{x^2+1}{x^2+2}\right| + C$$

EXERCISE 18.19

BASIC

Evaluate the following integrals:

1. $\int \frac{x}{x^2+3x+2}\,dx$ [CBSE 2022]
2. $\int \frac{x+1}{x^2+x+3}\,dx$
3. $\int \frac{x-3}{x^2+2x-4}\,dx$
4. $\int \frac{2x-3}{x^2+6x+13}\,dx$
5. $\int \frac{x-1}{3x^2-4x+3}\,dx$
6. $\int \frac{2x}{2+x-x^2}\,dx$
7. $\int \frac{1-3x}{3x^2+4x+2}\,dx$
8. $\int \frac{2x+5}{x^2-x-2}\,dx$
9. $\int \frac{ax^3+bx}{x^4+c^2}\,dx$
10. $\int \frac{(3\sin x-2)\cos x}{5-\cos^2 x-4\sin x}\,dx$ [CBSE 2013, 2016]
11. $\int \frac{x+2}{2x^2+6x+5}\,dx$ [CBSE 2007]
12. $\int \frac{5x-2}{1+2x+3x^2}\,dx$ [CBSE 2013, 2014]
13. $\int \frac{x+5}{3x^2+13x-10}\,dx$ [CBSE 2017]
14. $\int \frac{(3\sin x-2)\cos x}{13-\cos^2 x-7\sin x}\,dx$ [CBSE 2017]
15. $\int \frac{x+7}{3x^2+25x+28}\,dx$ [CBSE 2017]
16. $\int \frac{3x+5}{x^2+3x-18}\,dx$ [CBSE 2019]

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17. $\int \frac{x^3}{x^4+x^2+1}\,dx$
18. $\int \frac{x^3-3x}{x^4+2x^2-4}\,dx$

ANSWERS

1. $\frac{1}{2}\log|x^2+3x+2| - \frac{3}{2}\log\left|\frac{x+1}{x+2}\right| + C$
2. $\frac{1}{2}\log|x^2+x+3| + \frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{2x+1}{\sqrt{11}}\right) + C$
3. $\frac{1}{2}\log|x^2+2x-4| - \frac{2}{\sqrt{5}}\log\left|\frac{x+1-\sqrt{5}}{x+1+\sqrt{5}}\right| + C$
4. $\log|x^2+6x+13| - \frac{9}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + C$

5. $\frac{1}{6}\log|3x^2-4x+3|-\frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{3x-2}{\sqrt{5}}\right)+C$ 6. $-\log|2+x-x^2|+\frac{1}{3}\log\left|\frac{1+x}{2-x}\right|+C$

7. $-\frac{1}{2}\log|3x^2+4x+2|+\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+2}{\sqrt{2}}\right)+C$ 8. $\log|x^2-x-2|+2\log\left|\frac{x-2}{x+1}\right|+C$

9. $\frac{a}{4}\log|x^4+c^2|+\frac{b}{2c}\tan^{-1}\left(\frac{x^2}{c}\right)+C$ 10. $3\log|2-\sin x|+\frac{4}{2-\sin x}+C$

11. $\frac{1}{4}\log(2x^2+6x+5)+\frac{1}{2}\tan^{-1}(2x+3)+C$

12. $\frac{5}{6}\log\left|3x^2+2x+1\right|-\frac{11}{3\sqrt{2}}\tan^{-1}\frac{3x+1}{\sqrt{2}}+C$ 13. $\frac{1}{6}\log|3x^2+13x-10|+\frac{1}{6}\log\left|\frac{3x-2}{3(x+5)}\right|+C$

14. $10\log(4-\sin x)-7\log(3-\sin x)+C$ 15. $\frac{1}{3}\log|3x+4|+C$

16. $\frac{3}{2}\log|x^2+3x-18|+\frac{1}{18}\log\left|\frac{x-3}{x+6}\right|+C$ 17. $\frac{1}{2}\log|x^4+x^2+1|-\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$

18. $\frac{1}{2}\log|x^4+2x^2-4|-\frac{2}{\sqrt{5}}\log\left|\frac{x^2+1-\sqrt{5}}{x^2+1+\sqrt{5}}\right|+C$

18.10.6 INTEGRALS OF THE FORM $\int\frac{P(x)}{ax^2+bx+c}dx$, WHERE $P(x)$ IS A POLYNOMIAL OF DEGREE TWO OR MORE

To evaluate this type of integrals we divide the numerator by the denominator and express the integrand as

$$Q(x)+\frac{R(x)}{ax^2+bx+c}, \text{ where } R(x) \text{ is a linear function of } x.$$

$$\therefore \quad \int\frac{P(x)}{ax^2+bx+c}dx=\int Q(x)\,dx+\int\frac{R(x)}{ax^2+bx+c}dx$$

Now to evaluate the second integral on RHS apply the method discussed earlier.
Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* (i) $\int\frac{x^3+x+1}{x^2-1}dx$ (ii) $\int\frac{x^2+5x+3}{x^2+3x+2}dx$

SOLUTION (i) Let $I=\int\frac{x^3+x+1}{x^2-1}dx$. Then,

$$I=\int x+\frac{2x+1}{x^2-1}dx$$

$$\Rightarrow \quad I=\int x\,dx+\int\frac{2x}{x^2-1}dx+\int\frac{1}{x^2-1}dx=\frac{x^2}{2}+\log|x^2-1|+\frac{1}{2}\log\left|\frac{x-1}{x+1}\right|+C$$

(ii) Let $I = \int \frac{x^2+5x+3}{x^2+3x+2}\,dx$. Then,

$$I = \int\left(1+\frac{2x+1}{x^2+3x+2}\right)dx = \int 1\cdot dx + \int \frac{2x+3-2}{x^2+3x+2}\,dx$$

$$\Rightarrow \quad I = \int 1\cdot dx + \int \frac{2x+3}{x^2+3x+2}\,dx - 2\int \frac{1}{x^2+3x+2}\,dx$$

$$\Rightarrow \quad I = x + \log|x^2+3x+2| - 2\int \frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2}\,dx$$

$$\Rightarrow \quad I = x + \log|x^2+3x+2| - 2\times\frac{1}{2\left(\frac{1}{2}\right)}\log\left|\frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}}\right| + C$$

$$\Rightarrow \quad I = x + \log|x^2+3x+2| - 2\log\left|\frac{x+1}{x+2}\right| + C$$

EXERCISE 18.20

BASIC

Evaluate the following integrals:

1. $\int \frac{x^2+x+1}{x^2-x}\,dx$
2. $\int \frac{x^2+x-1}{x^2+x-6}\,dx$
3. $\int \frac{(1-x^2)}{x(1-2x)}\,dx$ [CBSE 2010]
4. $\int \frac{x^2+1}{x^2-5x+6}\,dx$ [NCERT]
5. $\int \frac{x^2}{x^2+7x+10}\,dx$
6. $\int \frac{x^2+x+1}{x^2-x+1}\,dx$
7. $\int \frac{(x-1)^2}{x^2+2x+2}\,dx$
8. $\int \frac{x^3+x^2+2x+1}{x^2-x+1}\,dx$
9. $\int \frac{x^2(x^4+4)}{x^2+4}\,dx$
10. $\int \frac{x^2}{x^2+6x+12}\,dx$ [CBSE 2005]
11. $\int \frac{x^3+1}{x^3-x}\,dx$ [CBSE 2020]

ANSWERS

1. $x+\log|x^2-x|+2\log\left|\frac{x-1}{x}\right|+C$
2. $x+\log\left|\frac{x-2}{x+3}\right|+C$
3. $\frac{1}{2}x+\log|x|-\frac{3}{4}\log|1-2x|+C$
4. $x-5\log|x-2|+10\log|x-3|+C$
5. $x-\frac{7}{2}\log|x^2+7x+10|+\frac{29}{6}\log\left|\frac{x+2}{x+5}\right|+C$
6. $x+\log|x^2-x+1|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$
7. $x-2\log|x^2+2x+2|+3\tan^{-1}(x+1)+C$
8. $\frac{1}{2}x^2+2x+\frac{3}{2}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C$

9. $\frac{1}{5}x^5 - \frac{4}{3}x^3 + 20x - 40\tan^{-1}\left(\frac{x}{2}\right) + C$

10. $x - 3\log|x^2 + 6x + 12| + 2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C$

11. $x + \log\left|\frac{x-1}{x}\right| + C$

18.10.7 INTEGRALS OF THE FORM $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

In order to evaluate this type of integrals, we use the following algorithm:

ALGORITHM

Step I *Write the numerator $px + q$ in the following form:*

$$px + q = \lambda\left\{\frac{d}{dx}(ax^2 + bx + c)\right\} + \mu \quad i.e.\quad px + q = \lambda(2ax + b) + \mu$$

Step II *Obtain the values of* λ and μ *by equating the coefficients of like powers of x on both sides.*

Step III *Replace $px + q$ by $\lambda(2ax + b) + \mu$ in the given integral to get*

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx = \lambda\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}\,dx + \mu\int\frac{1}{\sqrt{ax^2+bx+c}}\,dx$$

Step IV *Integrate RHS in step III and put the values of λ and μ obtained in step II.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{2x+3}{\sqrt{x^2+4x+1}}\,dx$ (ii) $\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$ [CBSE 2010] (iii) $\int \sqrt{\frac{1+x}{x}}\,dx$

SOLUTION (i) Let $2x + 3 = \lambda\frac{d}{dx}(x^2 + 4x + 1) + \mu$. Then, $2x + 3 = \lambda(2x + 4) + \mu$.

Comparing the coefficients of like powers of x, we get: $2\lambda = 2$ and $4\lambda + \mu = 3 \Rightarrow \lambda = 1$ and $\mu = -1$

$$\therefore \quad I = \int\frac{2x+3}{\sqrt{x^2+4x+1}}\,dx = \int\frac{(2x+4)-1}{\sqrt{x^2+4x+1}}\,dx = \int\frac{2x+4}{\sqrt{x^2+4x+1}}\,dx - \int\frac{1}{\sqrt{x^2+4x+1}}\,dx$$

$$\Rightarrow \quad I = \int\frac{dt}{\sqrt{t}} - \int\frac{1}{\sqrt{(x+2)^2-(\sqrt{3})^2}}\,dx, \text{ where } t = x^2 + 4x + 1$$

$$\Rightarrow \quad I = 2\sqrt{t} - \log\left|(x+2) + \sqrt{x^2+4x+1}\right| + C = 2\sqrt{x^2+4x+1} - \log|x+2+\sqrt{x^2+4x+1}| + C$$

(ii) Let $x + 2 = \lambda\frac{d}{dx}(x^2 + 5x + 6) + \mu$. Then, $x + 2 = \lambda(2x + 5) + \mu$. Comparing the coefficients of like powers of x, we get: $1 = 2\lambda$ and $5\lambda + \mu = 2 \Rightarrow \lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$

$$\therefore\ I = \int\frac{x+2}{\sqrt{x^2+5x+6}}\,dx = \int\frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}\,dx = \frac{1}{2}\int\frac{2x+5}{\sqrt{x^2+5x+6}}\,dx - \frac{1}{2}\int\frac{1}{\sqrt{x^2+5x+6}}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx, \text{ where } t = x^2+5x+6$$

$$\Rightarrow I = \sqrt{t} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + C = \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right| + C$$

(iii) Let $I = \int \sqrt{\frac{1+x}{x}}\,dx = \int \sqrt{\frac{1+x}{x}} \times \sqrt{\frac{1+x}{1+x}}\,dx = \int \frac{1+x}{\sqrt{x(1+x)}}\,dx = \int \frac{1+x}{\sqrt{x^2+x}}\,dx$

Let $x+1 = \lambda \frac{d}{dx}(x^2+x)+\mu$. Then, $x+1 = \lambda(2x+1)+\mu$.

Comparing the coefficients of like powers of x, we get: $1 = 2\lambda$ and $\lambda+\mu = 1 \Rightarrow \lambda = \frac{1}{2}, \mu = \frac{1}{2}$

$$\therefore\ I = \int \sqrt{\frac{1+x}{x}}\,dx = \int \frac{x+1}{\sqrt{x^2+x}}\,dx = \int \frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{\sqrt{x^2+x}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x}}\,dx + \frac{1}{2}\int \frac{1}{\sqrt{x^2+x}}\,dx$$

$$\Rightarrow I = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt + \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\,dx, \quad \text{where } t = x^2+x$$

$$\Rightarrow I = \sqrt{t} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + C = \sqrt{x^2+x} + \frac{1}{2}\log\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x}\right| + C$$

EXAMPLE 2 *Evaluate:*

(i) $\int \sqrt{\frac{a-x}{a+x}}\,dx$ (ii) $\int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

SOLUTION (i) Let $I = \int \sqrt{\frac{a-x}{a+x}}\,dx = \int \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\,dx = \int \frac{a-x}{\sqrt{a^2-x^2}}\,dx$. Then,

$$I = \int \frac{a}{\sqrt{a^2-x^2}}\,dx - \int \frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$\Rightarrow I = a\int \frac{1}{\sqrt{a^2-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}\,dx = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int \frac{-2x}{\sqrt{a^2-x^2}}\,dx$$

Putting $a^2-x^2 = t$ and $-2x\,dx = dt$, we get

$$I = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = a\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$$\Rightarrow I = a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{t} + C = a\sin^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2-x^2} + C$$

(ii) Let $I = \int x\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$. Putting $x^2 = t$, and $2x\,dx = dt$ or, $dx = \frac{dt}{2x}$, we get

$$I = \int x\sqrt{\frac{a^2-t}{a^2+t}}\,\frac{dt}{2x} = \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t}}\,dt = \frac{1}{2}\int \sqrt{\frac{a^2-t}{a^2+t} \times \frac{a^2-t}{a^2-t}}\,dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int \frac{a^2 - t}{\sqrt{a^4 - t^2}}\, dt = \frac{1}{2}\int \frac{a^2}{\sqrt{a^4 - t^2}}\, dt - \frac{1}{2}\int \frac{t\, dt}{\sqrt{a^4 - t^2}}$$

$$\Rightarrow \quad I = \frac{1}{2}a^2 \int \frac{1}{\sqrt{(a^2)^2 - t^2}}\, dt + \frac{1}{4}\int \frac{-2t}{\sqrt{a^4 - t^2}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}a^2 \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4}\int \frac{du}{\sqrt{u}}, \quad \text{where } a^4 - t^2 = u$$

$$\Rightarrow \quad I = \frac{1}{2}a^2 \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{4}\left(\frac{u^{1/2}}{1/2}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{2}a^2 \sin^{-1}\left(\frac{t}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - t^2} + C = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x^2}{a^2}\right) + \frac{1}{2}\sqrt{a^4 - x^4} + C$$

EXERCISE 18.21

BASIC

Evaluate the following integrals:

1. $\int \frac{x}{\sqrt{x^2 + 6x + 10}}\, dx$
2. $\int \frac{2x + 1}{\sqrt{x^2 + 2x - 1}}\, dx$
3. $\int \frac{x + 1}{\sqrt{4 + 5x - x^2}}\, dx$
4. $\int \frac{6x - 5}{\sqrt{3x^2 - 5x + 1}}\, dx$
5. $\int \frac{3x + 1}{\sqrt{5 - 2x - x^2}}\, dx$
6. $\int \frac{x}{\sqrt{8 + x - x^2}}\, dx$
7. $\int \frac{x + 2}{\sqrt{x^2 + 2x - 1}}\, dx$
8. $\int \frac{x + 2}{\sqrt{x^2 - 1}}\, dx$ [NCERT]
9. $\int \frac{x - 1}{\sqrt{x^2 + 1}}\, dx$ [NCERT]
10. $\int \frac{x}{\sqrt{x^2 + x + 1}}\, dx$
11. $\int \frac{x + 1}{\sqrt{x^2 + 1}}\, dx$
12. $\int \frac{2x + 5}{\sqrt{x^2 + 2x + 5}}\, dx$
13. $\int \frac{3x + 1}{\sqrt{5 - 2x - x^2}}\, dx$
14. $\int \sqrt{\frac{1 - x}{1 + x}}\, dx$
15. $\int \frac{2x + 1}{\sqrt{x^2 + 4x + 3}}\, dx$ [CBSE2000]
16. $\int \frac{2x + 3}{\sqrt{x^2 + 4x + 5}}\, dx$ [CBSE 2001]
17. $\int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx$ [CBSE 2011, 12]
18. $\int \frac{x + 2}{\sqrt{x^2 + 2x + 3}}$ [CBSE 2013]

ANSWERS

1. $\sqrt{x^2 + 6x + 10} - 3\log|(x + 3) + \sqrt{x^2 + 6x + 10}| + C$
2. $2\sqrt{x^2 + 2x - 1} - \log|x + 1 + \sqrt{x^2 + 2x - 1}| + C$
3. $-\sqrt{4 + 5x - x^2} + \frac{7}{2}\sin^{-1}\left(\frac{2x - 5}{\sqrt{41}}\right) + C$
4. $2\sqrt{3x^2 - 5x + 1} + C$
5. $-3\sqrt{5 - 2x - x^2} - 2\sin^{-1}\left(\frac{x + 1}{\sqrt{6}}\right) + C$
6. $-\sqrt{8 + x - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{2x - 1}{\sqrt{33}}\right) + C$
7. $\sqrt{x^2 + 2x - 1} + \log|(x + 1) + \sqrt{x^2 + 2x - 1}| + C$
8. $\sqrt{x^2 - 1} + 2\log|x + \sqrt{x^2 - 1}| + C$

9. $\sqrt{x^2+1}-\log|x+\sqrt{x^2+1}|+C$

10. $\sqrt{x^2+x+1}-\frac{1}{2}\left\{\log\left|\frac{2x+1}{2}+\sqrt{x^2+x+1}\right|\right\}+C$

11. $\sqrt{x^2+1}+\log|x+\sqrt{x^2+1}|+C$

12. $2\sqrt{x^2+2x+5}+3\log|x+1+\sqrt{x^2+2x+5}|+C$

13. $-3\sqrt{5-2x-x^2}-2\sin^{-1}\left(\frac{x+1}{\sqrt{6}}\right)+C$

14. $\sin^{-1}x+\sqrt{1-x^2}+C$

15. $2\sqrt{x^2+4x+3}-3\log|x+2+\sqrt{x^2+4x+3}|+C$

16. $2\sqrt{x^2+4x+5}-\log\left|x+2+\sqrt{x^2+4x+5}\right|+C$

17. $5\sqrt{x^2+4x+10}-7\log\left|x+2+\sqrt{x^2+4x+10}\right|+C$

18. $\sqrt{x^2+2x+3}+\log\left|(x+1)+\sqrt{x^2+2x+3}\right|+C$

18.10.8 INTEGRALS OF THE FORM $\int \frac{1}{a\sin^2 x+b\cos^2 x}\,dx$, $\int \frac{1}{a+b\sin^2 x}\,dx$, $\int \frac{1}{a+b\cos^2 x}\,dx$, $\int \frac{1}{(a\sin x+b\cos x)^2}\,dx$, $\int \frac{1}{a+b\sin^2 x+c\cos^2 x}\,dx$

To evaluate this type of integrals we use the following algorithm.

ALGORITHM

Step I *Divide numerator and denominator both by* $\cos^2 x$.

Step II *Replace* $\sec^2 x$, *if any, in denominator by* $1+\tan^2 x$.

Step III *Put* $\tan x=t$ *so that* $\sec^2 x\,dx=dt$. *This substitution reduces the integral in the form* $\int \frac{1}{at^2+bt+c}\,dt$.

Step IV *Evaluate the integral obtained in step III by using the methods discussed earlier.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}\,dx$ (ii) $\int \frac{1}{1+3\sin^2 x+8\cos^2 x}\,dx$

(iii) $\int \frac{\sin x}{\sin 3x}\,dx$ (iv) $\int \frac{1}{(2\sin x+3\cos x)^2}\,dx$

SOLUTION (i) Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$I=\int \frac{1}{a^2\sin^2 x+b^2\cos^2 x}\,dx=\int \frac{\sec^2 x}{a^2\tan^2 x+b^2}\,dx$$

Putting $\tan x=t$ and $\sec^2 x\,dx=dt$, we get

$$I=\int \frac{dt}{a^2t^2+b^2}=\frac{1}{a^2}\int \frac{dt}{t^2+(b/a)^2}=\frac{1}{a^2}\times\frac{1}{b/a}\tan^{-1}\left(\frac{t}{b/a}\right)+C$$

$$\Rightarrow \quad I = \frac{1}{ab}\tan^{-1}\left(\frac{at}{b}\right) + C = \frac{1}{ab}\tan^{-1}\left(\frac{a\tan x}{b}\right) + C$$

(ii) Dividing the numerator and denominator of the given integrand by $\cos^2 x$, we get

$$I = \int \frac{1}{1 + 3\sin^2 x + 8\cos^2 x}\,dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{\sec^2 x + 3\tan^2 x + 8}\,dx = \int \frac{\sec^2 x}{1 + \tan^2 x + 3\tan^2 x + 8}\,dx = \int \frac{\sec^2 x\,dx}{4\tan^2 x + 9}$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\Rightarrow \quad I = \int \frac{dt}{4t^2 + 9} = \frac{1}{4}\int \frac{dt}{t^2 + (3/2)^2} = \frac{1}{4} \times \frac{1}{3/2}\tan^{-1}\left(\frac{t}{3/2}\right) + C$$

$$\Rightarrow \quad I = \frac{1}{6}\tan^{-1}\left(\frac{2t}{3}\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{2\tan x}{3}\right) + C$$

(iii) Let $I = \int \frac{\sin x}{\sin 3x}\,dx = \int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx = \int \frac{1}{3 - 4\sin^2 x}\,dx$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx$$ [Dividing numerator and denominator by $\cos^2 x$]

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$I = \int \frac{dt}{3(1 + t^2) - 4t^2} = \int \frac{dt}{3 - t^2} = \int \frac{1}{(\sqrt{3})^2 - t^2}\,dt$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3} + t}{\sqrt{3} - t}\right| + C = \frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right| + C$$

(iv) Dividing the numerator and denominator by $\cos^2 x$, we get

$$I = \int \frac{1}{(2\sin x + 3\cos x)^2}\,dx = \int \frac{\sec^2 x}{(2\tan x + 3)^2}\,dx$$

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\therefore \quad I = \int \frac{dt}{(2t + 3)^2} = -\frac{1}{2(2t + 3)} + C = -\frac{1}{2(2\tan x + 3)} + C$$

EXAMPLE 2 *Evaluate:* (i) $\int \frac{1}{3 + \sin 2x}\,dx$ (ii) $\int \frac{1}{2 - 3\cos 2x}\,dx$

SOLUTION (i) Let

$$I = \int \frac{1}{3 + \sin 2x}\,dx = \int \frac{1}{3(\sin^2 x + \cos^2 x) + 2\sin x\cos x}\,dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{3\tan^2 x + 2\tan x + 3}\,dx$$ [Dividing numerator and denominator by $\cos^2 x$]

Putting $\tan x = t$ and $\sec^2 x\,dx = dt$, we get

$$\therefore \quad I = \int \frac{dt}{3t^2 + 2t + 3} = \frac{1}{3}\int \frac{dt}{t^2 + \frac{2}{3}t + 1} = \frac{1}{3}\int \frac{dt}{\left(t + \frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}$$

$$\Rightarrow I = \frac{1}{3} \times \frac{1}{\left(\frac{2\sqrt{2}}{3}\right)} \tan^{-1}\left(\frac{t + \frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right) + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{3t+1}{2\sqrt{2}}\right) + C = \frac{1}{2\sqrt{2}} \tan^{-1}\left(\frac{3\tan x + 1}{2\sqrt{2}}\right) + C$$

(ii) Let $I = \int \frac{1}{2 - 3\cos 2x}\, dx$. Then,

$$\Rightarrow \quad I = \int \frac{1}{2 - 3(\cos^2 x - \sin^2 x)}\, dx$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{2\sec^2 x - 3 + 3\tan^2 x}\, dx \quad \text{[Dividing numerator and denominator by } \cos^2 x]$$

$$\Rightarrow \quad I = \int \frac{\sec^2 x}{2(1 + \tan^2 x) - 3 + 3\tan^2 x}\, dx = \int \frac{\sec^2 x}{5\tan^2 x - 1}\, dx$$

Putting $\tan x = t$ and $\sec^2 x\, dx = dt$, we get

$$\therefore \quad I = \int \frac{dt}{5t^2 - 1} = \frac{1}{5}\int \frac{dt}{t^2 - \left(\frac{1}{\sqrt{5}}\right)^2} = \frac{1}{5} \times \frac{1}{2\left(\frac{1}{\sqrt{5}}\right)} \log\left|\frac{t - \frac{1}{\sqrt{5}}}{t + \frac{1}{\sqrt{5}}}\right| + C$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}t - 1}{\sqrt{5}t + 1}\right| + C = \frac{1}{2\sqrt{5}} \log\left|\frac{\sqrt{5}\tan x - 1}{\sqrt{5}\tan x + 1}\right| + C$$

EXERCISE 18.22

BASIC

Evaluate the following integrals:

1. $\int \frac{1}{4\cos^2 x + 9\sin^2 x}\, dx$
2. $\int \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$
3. $\int \frac{2}{2 + \sin 2x}\, dx$
4. $\int \frac{\cos x}{\cos 3x}\, dx$
5. $\int \frac{1}{1 + 3\sin^2 x}\, dx$
6. $\int \frac{1}{3 + 2\cos^2 x}\, dx$
7. $\int \frac{1}{\cos 2x + 3\sin^2 x}\, dx$
8. $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$
9. $\int \frac{1}{\cos x(\sin x + 2\cos x)}\, dx$
10. $\int \frac{1}{\sin^2 x + \sin 2x}\, dx$
11. $\int \frac{1}{(\sin x - 2\cos x)(2\sin x + \cos x)}\, dx$

ANSWERS

1. $\frac{1}{6}\tan^{-1}\left(\frac{3\tan x}{2}\right) + C$
2. $\frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{5}}\right) + C$
3. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x + 1}{\sqrt{3}}\right) + C$
4. $\frac{1}{2\sqrt{3}}\log\left|\frac{1 + \sqrt{3}\tan x}{1 - \sqrt{3}\tan x}\right| + C$
5. $\frac{1}{2}\tan^{-1}(2\tan x) + C$
6. $\frac{1}{\sqrt{15}}\tan^{-1}\left(\frac{\sqrt{3}\tan x}{\sqrt{5}}\right) + C$