CHAPTER 6

ADJOINT AND INVERSE OF A MATRIX

6.1 ADJOINT OF A SQUARE MATRIX

ADJOINT *Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.*

Thus, $\quad \text{adj } A = [C_{ij}]^T \Rightarrow (\text{adj } A)_{ij} = C_{ji} = \text{Cofactor of } a_{ji} \text{ in } A.$

If $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then, $\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$,

where C_{ij} denotes the cofactor of a_{ij} in A.

ILLUSTRATION 1 *Find the adjoint of matrix* $A = [a_{ij}] = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

SOLUTION We have,

Cofactor of $a_{11} = s$, Cofactor of $a_{12} = -r$, Cofactor of $a_{21} = -q$ and, Cofactor of $a_{22} = p$.

$$\therefore \quad \text{adj } A = \begin{bmatrix} s & -r \\ -q & p \end{bmatrix}^T = \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$$

RULE *It is evident from this example that the adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.*

If $A = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$, *then by the above rule, we obtain* $\text{adj } A = \begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$.

ILLUSTRATION 2 *Find the adjoint of matrix* $A = [a_{ij}] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$.

SOLUTION Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 3 \end{vmatrix} = 9, \quad C_{12} = -\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix} = -3, \quad C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$$

$$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1, \quad C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4, \quad C_{23} = -\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4, \quad C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5, \quad C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

THEOREM 1 *Let A be a square matrix of order n. Then,* $A\,(\text{adj } A) = |A|\, I_n = (\text{adj } A)\, A$.

PROOF Let $A = [a_{ij}]$, and let C_{ij} be cofactor of a_{ij} in A. Then,

$$(\text{adj } A)_{ij} = C_{ji} \quad \text{for all } i, j = 1, 2, \dots, n$$

Since A and adj A are both square matrices of the same order $n \times n$. Therefore, both A (adj A) and (adj A) A exist and are of the same order $n \times n$.

Now,

$$(A\,(\text{adj } A))_{ij} = \sum_{r=1}^{n} (A)_{ir}\,(\text{adj } A)_{rj} \quad \text{[By definition of multiplication of two matrices]}$$

$$= \sum_{r=1}^{n} a_{ir}\,C_{rj} = \begin{cases} |A| & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases} \quad \text{[By property 1 and 2 of section 5.4]}$$

Thus, each diagonal element of A (adj A) is equal to $|A|$ and all non-diagonal elements are equal to zero.

$$\therefore \quad A\,(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix} = |A|\,I_n$$

Similarly, we obtain

$$((\text{adj } A)\,A)_{ij} = \sum_{r=1}^{n} (\text{adj } A)_{ir}\,(A)_{rj}$$

$$\Rightarrow \quad ((\text{adj } A)\,A)_{ij} = \sum_{r=1}^{n} C_{ri}\,a_{rj}$$

$$\Rightarrow \quad ((\text{adj } A)\,A)_{ij} = \begin{cases} |A| & , \text{ if } i = j \\ 0 & , \text{ if } i \neq j \end{cases} \quad \text{[By property 1 and 2 of section 5.4]}$$

Hence, $A\,(\text{adj } A) = |A|\,I_n = (\text{adj } A)\,A$.

Q.E.D.

ILLUSTRATION 3 *Compute the adjoint of the matrix A given by* $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$ *and verify that* $A\,(\text{adj } A) = |A|\,I = (\text{adj } A)\,A$.

SOLUTION We have,

$$|A| = \begin{vmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{vmatrix} = 1\,(0 - 6) - 4\,(0 - 0) + 5\,(3 - 0) = 9$$

Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = -6, \quad C_{12} = -\begin{vmatrix} 3 & 6 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{13} = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$$

$$C_{21} = -\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} = 5, \quad C_{22} = \begin{vmatrix} 1 & 5 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{23} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{31} = \begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = 14, \quad C_{32} = -\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = 9, \quad C_{33} = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -6 & 0 & 3 \\ 5 & 0 & -1 \\ 14 & 9 & -10 \end{bmatrix}^T = \begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix}$$

Now, $A\,(\text{adj}\,A) = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I$

and, $(\text{adj}\,A)\,A = \begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix}\begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I$

Hence, $A\,(\text{adj}\,A) = |A|\,I = (\text{adj}\,A)\,A$.

6.2 INVERSE OF A MATRIX

INVERSE *A square matrix of order n is invertible if there exists a square matrix B of the same order such that* $AB = I_n = BA$.

In such a case, we say that the inverse of A is B and we write, $A^{-1} = B$.

THEOREM 1 *Every invertible matrix possesses a unique inverse.*

PROOF Let A be an invertible matrix of order $n \times n$. Let B and C be two inverses of A. Then,

$AB = BA = I_n$ ...(i) and $AC = CA = I_n$ ...(ii)

Now, $AB = I_n$

$\Rightarrow\ C\,(AB) = C\,I_n$ [Pre-multiplying both sides by C)

$\Rightarrow\ (CA)\,B = C\,I_n$ [By associativity of multiplication]

$\Rightarrow\ I_n\,B = C\,I_n$ [$\because\ CA = I_n$ from (ii)]

$\Rightarrow\ B = C$ [$\because\ I_n\,B = B$ and $C\,I_n = C$]

Hence, an invertible matrix possesses a unique inverse.

Q.E.D.

COROLLARY *If A is an invertible matrix, then* $(A^{-1})^{-1} = A$.

PROOF Since A^{-1} is inverse of A. Therefore,

$AA^{-1} = I = A^{-1}A \Rightarrow A$ is the inverse of A^{-1} i.e. $A = (A^{-1})^{-1}$.

THEOREM 2 *A square matrix is invertible iff it is non-singular.*

PROOF Let A be an invertible matrix. Then, there exists a matrix B such that

$AB = I_n = BA$

$\Rightarrow\ |AB| = |I_n|$

$\Rightarrow\ |A|\,|B| = 1$ [$\because\ |AB| = |A|\,|B|$]

$\Rightarrow\ |A| \neq 0 \Rightarrow A$ is a non-singular matrix.

Conversely, let A be a non-singular square matrix of order n. Then,

$A\,(\text{adj}\,A) = |A|\,I_n = (\text{adj}\,A)\,A$ [See Theorem 1 on page 6.1]

$\Rightarrow\ A\left(\dfrac{1}{|A|}\text{adj}\,A\right) = I_n = \left(\dfrac{1}{|A|}\text{adj}\,A\right)A$ $\left[\because\ |A| \neq 0\ \therefore\ \dfrac{1}{|A|}\text{ exists}\right]$

$\Rightarrow\ A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$ [By definition of inverse]

Hence, A is an invertible matrix such that $A^{-1} = \dfrac{1}{|A|}\text{adj}\,A$.

Q.E.D.

REMARK *This theorem provides us a formula for finding the inverse of a non-singular square matrix. The inverse of A is given by*

$$A^{-1} = \frac{1}{|A|}\cdot\text{adj}\,A$$

In order to find the inverse of a square matrix, we may use the following algorithm.

ALGORITHM

Step I *Find* $|A|$

Step II *If* $|A|=0$, *then write "A is a singular matrix and hence not invertible".*
Else write "A is a non-singular and hence invertible".

Step III *Calculate the cofactors of elements of A.*

Step IV *Write the matrix of cofactors of elements of A and then obtain its transpose to obtain adj A.*

Step V *Find the inverse of A by using the formula:* $A^{-1}=\dfrac{1}{|A|}\text{adj }A.$

ILLUSTRATION 1 *Find the inverse of the matrix* $\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$.

SOLUTION Let $A=\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Then, $|A|=\begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}=8+3=11\neq 0.$

So, A is a non-singular matrix and therefore it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then, the cofactors of elements of A are given by

$$C_{11}=4,\ C_{12}=-3,\ C_{21}=-(-1)=1 \text{ and } C_{22}=2.$$

$$\therefore \quad \text{adj } A=\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}^T=\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\text{Hence,}\quad A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}=\begin{bmatrix} 4/11 & 1/11 \\ -3/11 & 2/11 \end{bmatrix}$$

ILLUSTRATION 2 *Find the inverse of the matrix* $A=\begin{bmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{bmatrix}$.

SOLUTION We find that

$$|A|=\begin{vmatrix} 8 & 4 & 2 \\ 2 & 9 & 4 \\ 1 & 2 & 8 \end{vmatrix}=8(72-8)-4(16-4)+2(4-9)=454\neq 0$$

Thus, A is a non-singular matrix and hence it is invertible. Let C_{ij} be cofactor of a_{ij} in A. Then,

$$C_{11}=\begin{vmatrix} 9 & 4 \\ 2 & 8 \end{vmatrix}=64,\quad C_{12}=-\begin{vmatrix} 2 & 4 \\ 1 & 8 \end{vmatrix}=-12,\quad C_{13}=\begin{vmatrix} 2 & 9 \\ 1 & 2 \end{vmatrix}=-5$$

$$C_{21}=-\begin{vmatrix} 4 & 2 \\ 2 & 8 \end{vmatrix}=-28,\quad C_{22}=\begin{vmatrix} 8 & 2 \\ 1 & 8 \end{vmatrix}=62,\quad C_{23}=-\begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix}=-12$$

$$C_{31}=\begin{vmatrix} 4 & 2 \\ 9 & 4 \end{vmatrix}=-2,\quad C_{32}=-\begin{vmatrix} 8 & 2 \\ 2 & 4 \end{vmatrix}=-28,\quad C_{33}=\begin{vmatrix} 8 & 4 \\ 2 & 9 \end{vmatrix}=64$$

$$\therefore \quad \text{adj } A=\begin{bmatrix} 64 & -12 & -5 \\ -28 & 62 & -12 \\ -2 & -28 & 64 \end{bmatrix}^T=\begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$$

$$\text{Hence,}\quad A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{454}\begin{bmatrix} 64 & -28 & -2 \\ -12 & 62 & -28 \\ -5 & -12 & 64 \end{bmatrix}$$

6.3 SOME USEFUL RESULTS ON INVERTIBLE MATRICES

In this section, we shall discuss some useful results on inverse of a matrix. We shall state and prove these results as theorems given below.

THEOREM 1 *(Cancellation Laws) Let A, B, C be square matrices of the same order n. If A is a non-singular matrix, then*

(i) $AB = AC \Rightarrow B = C$ [*Left cancellation law*] (ii) $BA = CA \Rightarrow B = C$ [*Right Cancellation law*]

PROOF (i) Since A is a non-singular matrix i.e. $|A| \neq 0$. So, A^{-1} exists.

Now, $AB = AC$

$\Rightarrow \quad A^{-1}(AB) = A^{-1}(AC)$ [Pre-multiplying both sides by A^{-1}]

$\Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C$ [By associativity of multiplication]

$\Rightarrow \quad I_n B = I_n C$ [$\because A^{-1}A = I_n$]

$\Rightarrow \quad B = C$ [$\because I_n B = B$ and $I_n C = C$]

Similarly, we can prove that $BA = CA \Rightarrow B = C$.

Q.E.D.

REMARK *The result $AB = AC \Rightarrow B = C$ is true only when $|A| \neq 0$. Otherwise we can find matrices such that $AB = AC$ but $B \neq C$ as given below.*

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, B = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$. *Then*

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix} \text{ and } AC = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -6 & 0 \end{bmatrix}.$$

Clearly, $AB = AC$ but $B \neq C$.

THEOREM 2 *(Reversal Law) If A and B are invertible matrices of the same order, then show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

PROOF It is given that A and B are invertible matrices.

$\therefore \quad |A| \neq 0$ and $|B| \neq 0 \Rightarrow |A|\,|B| \neq 0 \Rightarrow |AB| \neq 0$ [$\because |AB| = |A|\,|B|$]

$\Rightarrow \quad AB$ is a invertible matrix.

Now,

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ [By associativity of multiplication]

$\Rightarrow \quad (AB)(B^{-1}A^{-1}) = (AI_n)A^{-1}$ [$\because BB^{-1} = I_n$]

$\Rightarrow \quad (AB)(B^{-1}A^{-1}) = AA^{-1}$ [$\because AI_n = A$]

$\Rightarrow \quad (AB)(B^{-1}A^{-1}) = I_n$ [$\because AA^{-1} = I_n$]

and, $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$ [By associativity of multiplication]

$\Rightarrow \quad (B^{-1}A^{-1})(AB) = B^{-1}(I_n B)$ [$\because A^{-1}A = I_n$]

$\Rightarrow \quad (B^{-1}A^{-1})(AB) = B^{-1}B$ [$\because I_n B = B$]

$\Rightarrow \quad (B^{-1}A^{-1})(AB) = I_n$ [$\because B^{-1}B = I_n$]

Thus, $(AB)(B^{-1}A^{-1}) = I_n = (B^{-1}A^{-1})(AB)$.

Hence, $(AB)^{-1} = B^{-1}A^{-1}$.

Q.E.D

REMARK *If A, B, C are invertible matrices, of the same order then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.*

THEOREM 3 *If A is an invertible square matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.*

PROOF Since A is invertible matrix.

$\therefore \quad |A| \neq 0 \Rightarrow |A^T| \neq 0 \Rightarrow A^T$ is also invertible [$\because |A^T| = |A|$]

Now,

$$AA^{-1} = I_n = A^{-1}A$$

$\Rightarrow \quad (AA^{-1})^T = (I_n)^T = (A^{-1}A)^T$

$\Rightarrow \quad (A^{-1})^T (A^T) = I_n = A^T (A^{-1})^T$ [By reversal law for transpose]

$\Rightarrow \quad (A^T)^{-1} = (A^{-1})^T$ [By definition of inverse]

Q.E.D

THEOREM 4 *The inverse of an invertible symmetric matrix is a symmetric matrix.*

PROOF Let A be an invertible symmetric matrix. Then, $|A| \neq 0$ and $A^T = A$.

Now, $(A^{-1})^T = (A^T)^{-1} = A^{-1}$ $[\because A^T = A]$

$\therefore \quad A^{-1}$ is a symmetric matrix.

ALITER Let A be a non-singular symmetric matrix. Then, A^{-1} exists.

Now, $AA^{-1} = I = A^{-1}A$

$\Rightarrow \quad (AA^{-1})^T = (I)^T = (A^{-1}A)^T$

$\Rightarrow \quad (A^{-1})^T A^T = I = A^T (A^{-1})^T$

$\Rightarrow \quad (A^{-1})^T A = I = A(A^{-1})^T$ $[\because A^T = A]$

$\Rightarrow \quad A^{-1} = (A^{-1})^T$ [By definition of inverse]

$\Rightarrow \quad A^{-1}$ is symmetric.

Q.E.D

THEOREM 5 *Let A be a non-singular square matrix of order n. Then, $|\text{adj } A| = |A|^{n-1}$.*

PROOF We have,

$$A(\text{adj } A) = |A| I_n$$

$$\Rightarrow \quad A(\text{adj } A) = \begin{bmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & |A| \end{bmatrix}_{n \times n}$$

$$\Rightarrow \quad |A(\text{adj } A)| = \begin{vmatrix} |A| & 0 & 0 & \dots & 0 \\ 0 & |A| & 0 & \dots & 0 \\ 0 & 0 & |A| & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & |A| \end{vmatrix} = |A|^n$$

$\Rightarrow \quad |A|\,|\text{adj } A| = |A|^n$ $[\because |AB| = |A|\,|B|]$

$\Rightarrow \quad |\text{adj } A| = |A|^{n-1}.$

Q.E.D

ILLUSTRATION 1 *If A is an invertible matrix of order 3 and $|A| = 5$, then find $|\text{adj } A|$.* [CBSE 2009]

SOLUTION Here A is an invertible matrix of order 3.

$\therefore \quad |\text{adj } A| = |A|^2$ [Using: $|\text{adj } A| = |A|^{n-1}$]

$\Rightarrow \quad |\text{adj } A| = 5^2 = 25$ $[\because |A| = 5 \text{ (given)}]$

THEOREM 6 *If A and B are non-singular square matrices of the same order, then*
$\text{adj } AB = (\text{adj } B)(\text{adj } A)$

PROOF Since A and B are non-singular square matrices of the same order. Therefore, AB exists such that $|AB| = |A|\,|B| \neq 0$ $[\because |A| \neq 0, |B| \neq 0]$

We know that $(AB)(\text{adj } AB) = |AB|\, I_n$ ...(i)

Also, $(AB)(\text{adj } B \text{ adj } A) = (A(B\cdot \text{adj } B) \text{ adj } A$ [By associativity of multiplication]

$= (A|B|I_n) \text{ adj } A$ [$\because$ $B \text{ adj } B = |B|\, I_n$]

$= |B|(A \text{ adj } A)$ [$\because$ $A I_n = A$]

$= |B|(|A|\, I_n)$ [$\because$ $A \text{ adj } A = |A|\, I_n$]

$= |A|\,|B|\, I_n$

$= |AB|\, I_n$ [$\because$ $|AB| = |A|\,|B|$]

Thus, $(AB)(\text{adj } B \text{ adj } A) = |AB|\, I_n$ ...(ii)

From (i) and (ii), we get

$(AB)(\text{adj } AB) = (AB)(\text{adj } B \cdot \text{adj } A)$

$\Rightarrow$ $(AB)^{-1}((AB)(\text{adj } AB)) = (AB)^{-1}((AB)(\text{adj } B \,.\, \text{adj } A))$

$\Rightarrow$ $((AB)^{-1}(AB))(\text{adj } AB) = ((AB)^{-1}(AB))(\text{adj } B \,.\, \text{adj } A)$

$\Rightarrow$ $I(\text{adj } AB) = I(\text{adj } B \,.\, \text{adj } A)$

$\Rightarrow$ $\text{adj } AB = \text{adj } B \,.\, \text{adj } A$ **Q.E.D**

THEOREM 7 *If A is an invertible square matrix, then adj $A^T = (\text{adj } A)^T$.*

PROOF Since A is an invertible matrix.

$\therefore$ $|A| \neq 0$

$\Rightarrow$ $|A^T| \neq 0$ [$\because$ $|A^T| = |A|$]

$\Rightarrow$ A^T is invertible.

We know that

$A \text{ adj } A = |A|\, I_n \Rightarrow (A \text{ adj } A)^T = (|A|\, I_n)^T \Rightarrow (\text{adj } A)^T (A^T) = |A|\, I_n$ ...(i)

Also,

$(\text{adj } A^T)(A^T) = |A^T|\, I_n \Rightarrow (\text{adj } A^T)(A^T) = |A|\, I_n$ ...(ii)

From (i) and (ii), we get

$(\text{adj } A^T)(A^T) = (\text{adj } A)^T (A^T) \Rightarrow \text{adj } A^T = (\text{adj } A)^T$ [By right cancellation law]

Q.E.D

THEOREM 8 *Prove that adjoint of a symmetric matrix is also a symmetric matrix.*

PROOF Let A be a symmetric matrix. Then, $A^T = A$

We know that

$(\text{adj } A)^T = (\text{adj } A^T)$

$\Rightarrow$ $(\text{adj } A)^T = \text{adj } A$ [$\because$ $A^T = A$]

$\Rightarrow$ $\text{adj } A$ is a symmetric matrix. **Q.E.D.**

THEOREM 9 *If A is a non-singular square matrix, then* $\text{adj}(\text{adj } A) = |A|^{n-2} A$.

PROOF We know that $B(\text{adj } B) = |B|\, I_n$ for every square matrix of order n.

Replacing B by adj A, we get

$(\text{adj } A)[\text{adj}(\text{adj } A)] = |\text{adj } A|\, I_n$

$\Rightarrow$ $(\text{adj } A)[\text{adj}(\text{adj } A)] = |A|^{n-1} I_n$ [$\because$ $|\text{adj } A| = |A|^{n-1}$]

$\Rightarrow$ $A\{(\text{adj } A)(\text{adj adj } A)\} = A\{|A|^{n-1} I_n\}$ [Pre-multiplying both sides by A]

$\Rightarrow$ $(A \text{ adj } A)(\text{adj adj } A) = |A|^{n-1}(AI_n)$ [By associativity of multiplication]

$\Rightarrow \quad |A| I_n (\text{adj adj } A) = |A|^{n-1} A$ $\qquad [\because AI_n = A \text{ and } A \text{ adj } A = |A| I_n]$

$\Rightarrow \quad |A| (I_n (\text{adj adj } A)) = |A|^{n-1} A$

$\Rightarrow \quad |A| (\text{adj adj } A) = |A|^{n-1} A$

$\Rightarrow \quad \text{adj adj } A = |A|^{n-2} A.$ $\qquad \left[\text{Multiplying both sides by } \frac{1}{|A|}\right]$

Q.E.D

COROLLARY *If A is a non-singular matrix of order n, then* $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

PROOF We know that

$$\text{adj}(\text{adj } A) = |A|^{n-2} A$$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = ||A|^{n-2} A|$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = |A|^{n(n-2)} |A|$ $\qquad [\because |kA| = k^n |A|]$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = |A|^{n^2-2n+1} = |A|^{(n-1)^2}$

ILLUSTRATION 2 *If A is an invertible matrix of order* 3×3 *such that* $|A| = 2$. *Then, find* adj (adj A).

SOLUTION Replacing n by 3 in the above theorem, we get

$$\text{adj}(\text{adj } A) = |A|^{3-2} A = |A| A = 2A$$

ILLUSTRATION 3 *If A is a square matrix of order 3 such that* $|A| = 2$, *then write the value of* $|adj\,(adj\,A)|$.

SOLUTION If A is a square matrix of order n, then $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$.

Here A is a square matrix of order 3 such that $|A| = 2$.

$\therefore \quad |\text{adj}(\text{adj } A)| = 2^{(3-1)^2} = 2^4 = 16$

ILLUSTRATION 4 *If* $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$, *then find* $|adj\,(adj\,A)|$.

SOLUTION Here, $|A| = \begin{vmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{vmatrix} = 3(3-0) - 0(2-0) - 1(8-0) = 1$

If A is a square matrix of order n, then $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

So, for the given matrix, we obtain : $|\text{adj}(\text{adj } A)| = |A|^4 = 1$

THEOREM 10 *If the product of two non-null square matrices is a null matrix, show that both of them must be singular matrices.*

PROOF Let A and B be two non-null square matrices of the same order $n \times n$. It is given that $AB = O$ (null matrix). If possible, let B be a non-singular matrix. Then, B^{-1} exists.

$\therefore \quad AB = O$

$\Rightarrow \quad (AB) B^{-1} = OB^{-1}$ $\qquad$ [Post-multiplying both sides by B^{-1}]

$\Rightarrow \quad A (BB^{-1}) = O$ $\qquad$ [By associativity of multiplication]

$\Rightarrow \quad AI_n = O$ $\qquad [\because BB^{-1} = I_n]$

$\Rightarrow \quad A = O.$

But, A is a non-null matrix. Therefore, our supposition is wrong. Hence, B is a singular matrix. Similarly it can be shown that A is a singular matrix. Q.E.D.

THEOREM 11 *If A is a non-singular matrix, then prove that* $|A^{-1}| = |A|^{-1}$ i.e. $|A^{-1}| = \frac{1}{|A|}$.

PROOF Since $|A| \neq 0$, therefore A^{-1} exists such that

$$AA^{-1} = I = A^{-1}A$$

$\Rightarrow \quad |AA^{-1}| = |I|$ [Taking determinant of both sides]

$\Rightarrow \quad |A|\,|A^{-1}| = 1$ [$\because$ $|AB| = |A|\,|B|$ and $|I| = 1$]

$\Rightarrow \quad |A^{-1}| = \frac{1}{|A|}$. [$\because$ $|A| \neq 0$]

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I FINDING THE ADJOINT AND INVERSE OF A MATRIX

EXAMPLE 1 *If* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *find adj A.*

SOLUTION Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

C_{11} = Cofactor of $a_{11} = (-1)^{1+1} d = d$, C_{12} = Cofactor of $a_{12} = -(-1)^{1+2} c = -c$

C_{21} = Cofactor of $a_{21} = (-1)^{2+1} b = -b$ and, C_{22} = Cofactor of $a_{22} = (-1)^{2+2} a = a$

$$\therefore \quad \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

EXAMPLE 2 *If* $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$, *find adj A and verify that* $A\,(\text{adj } A) = (\text{adj } A)\,A = |A|\,I_3$.

[CBSE 2016]

SOLUTION Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} \sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = -\sin\alpha,$$

$$C_{13} = (-1)^{1+3}\begin{vmatrix} \sin\alpha & \cos\alpha \\ 0 & 0 \end{vmatrix} = 0, \quad C_{21} = (-1)^{2+1}\begin{vmatrix} -\sin\alpha & 0 \\ 0 & 1 \end{vmatrix} = \sin\alpha,$$

$$C_{22} = (-1)^{2+2}\begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} = \cos\alpha, \quad C_{23} = (-1)^{2+3}\begin{vmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{vmatrix} = 0,$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \end{vmatrix} = 0, \quad C_{32} = (-1)^{3+2}\begin{vmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} = 0,$$

and, $$C_{33} = (-1)^{3+3}\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

$\Rightarrow \quad |A| = (\cos\alpha)(\cos\alpha) + (-\sin\alpha)(-\sin\alpha) + 0 \times 0 = \cos^2\alpha + \sin^2\alpha = 1.$

Now,

$$A\,(\text{adj } A) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A\,(\text{adj } A) = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & 0 \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \sin^2 + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A\,(\text{adj } A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I \qquad [\because |A| = 1]$$

and,

$$(\text{adj } A)\,A = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad (\text{adj } A)\,A = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & 0 \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad (\text{adj } A)\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\,I \qquad [\because |A| = 1]$$

Hence, $A\,(\text{adj } A) = |A|\,I = (\text{adj } A)\,A$ is verified.

EXAMPLE 3 *If* $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, *show that* $A^{-1} = \frac{1}{19}A$.

SOLUTION We have, $|A| = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19 \neq 0$. Therefore, A is invertible.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = -2,\ C_{12} = -5,\ C_{21} = -3 \text{ and } C_{22} = 2.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -5 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$$

Now, $A^{-1} = \frac{1}{|A|}\text{adj } A \Rightarrow A^{-1} = \frac{1}{-19}\begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19}\begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} = \frac{1}{19}A.$

EXAMPLE 4 *Find the inverse of* $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ *and verify that* $A^{-1}A = I_3$.

SOLUTION We have, $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$.

$$\therefore \quad |A| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = (16-9) - 3(4-3) + 3(3-4) = 7 - 3 - 3 = 1 \neq 0.$$ So, A is invertible.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = -1, \quad C_{13} = (-1)^{1+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{21}=(-1)^{2+1}\begin{vmatrix}3&3\\3&4\end{vmatrix}=-3,\quad C_{22}=(-1)^{2+2}\begin{vmatrix}1&3\\1&4\end{vmatrix}=1,\quad C_{23}=(-1)^{2+3}\begin{vmatrix}1&3\\1&3\end{vmatrix}=0$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}3&3\\4&3\end{vmatrix}=-3,\quad C_{32}=(-1)^{3+2}\begin{vmatrix}1&3\\1&3\end{vmatrix}=0,\quad C_{33}=(-1)^{3+3}\begin{vmatrix}1&3\\1&4\end{vmatrix}=1$$

$$\therefore\quad \text{adj } A=\begin{bmatrix}C_{11}&C_{12}&C_{13}\\C_{21}&C_{22}&C_{23}\\C_{31}&C_{32}&C_{33}\end{bmatrix}^T=\begin{bmatrix}7&-1&-1\\-3&1&0\\-3&0&1\end{bmatrix}^T=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}$$

Now, $\quad A^{-1}=\dfrac{1}{|A|}\text{ adj } A$

$$\Rightarrow\quad A^{-1}=\frac{1}{1}\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}$$

$$\therefore\quad A^{-1}A=\begin{bmatrix}7&-3&-3\\-1&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&3&3\\1&4&3\\1&3&4\end{bmatrix}=\begin{bmatrix}7-3-3&21-12-9&21-9-12\\-1+1+0&-3+4+0&-3+3+0\\-1+0+1&-3+0+3&-3+0+4\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

EXAMPLE 5 *If* $A=\begin{bmatrix}1&\tan x\\-\tan x&1\end{bmatrix}$, *show that* $A^T A^{-1}=\begin{bmatrix}\cos 2x&-\sin 2x\\\sin 2x&\cos 2x\end{bmatrix}$.

SOLUTION Clearly, $|A|=\begin{vmatrix}1&\tan x\\-\tan x&1\end{vmatrix}=1+\tan^2 x\neq 0$

So, A is invertible. Let C_{ij} be the cofactor of a_{ij} in $A=[a_{ij}]$. Then,

$$C_{11}=(-1)^{1+1}1=1,\quad C_{12}=(-1)^{1+2}(-\tan x)=\tan x$$

$$C_{21}=(-1)^{2+1}\tan x=-\tan x\quad\text{and}\quad C_{22}=(-1)^{2+2}1=1$$

$$\therefore\quad \text{adj } A=\begin{bmatrix}C_{11}&C_{12}\\C_{21}&C_{22}\end{bmatrix}^T\begin{bmatrix}1&\tan x\\-\tan x&1\end{bmatrix}^T=\begin{bmatrix}1&-\tan x\\\tan x&1\end{bmatrix}.$$

Now,

$$A^{-1}=\frac{1}{|A|}\text{adj } A=\frac{1}{1+\tan^2 x}\begin{bmatrix}1&-\tan x\\\tan x&1\end{bmatrix}=\begin{bmatrix}\dfrac{1}{1+\tan^2 x}&-\dfrac{\tan x}{1+\tan^2 x}\\\dfrac{\tan x}{1+\tan^2 x}&\dfrac{1}{1+\tan^2 x}\end{bmatrix}$$

$$\therefore\quad A^T A^{-1}=\begin{bmatrix}1&-\tan x\\\tan x&1\end{bmatrix}\begin{bmatrix}\dfrac{1}{1+\tan^2 x}&\dfrac{-\tan x}{1+\tan^2 x}\\\dfrac{\tan x}{1+\tan^2 x}&\dfrac{1}{1+\tan^2 x}\end{bmatrix}$$

$$\Rightarrow\quad A^T A^{-1}=\begin{bmatrix}\dfrac{1}{1+\tan^2 x}-\dfrac{\tan^2 x}{1+\tan^2 x}&-\dfrac{\tan x}{1+\tan^2 x}-\dfrac{\tan x}{1+\tan^2 x}\\\dfrac{\tan x}{1+\tan^2 x}+\dfrac{\tan x}{1+\tan^2 x}&-\dfrac{\tan^2 x}{1+\tan^2 x}+\dfrac{1}{1+\tan^2 x}\end{bmatrix}$$

$$\Rightarrow\quad A^T A^{-1}=\begin{bmatrix}\dfrac{1-\tan^2 x}{1+\tan^2 x}&-\dfrac{2\tan x}{1+\tan^2 x}\\\dfrac{2\tan x}{1+\tan^2 x}&\dfrac{1-\tan^2 x}{1+\tan^2 x}\end{bmatrix}=\begin{bmatrix}\cos 2x&-\sin 2x\\\sin 2x&\cos 2x\end{bmatrix}$$

EXAMPLE 6 *If* $A=\begin{bmatrix}3 & 2\\ 7 & 5\end{bmatrix}$ *and* $B=\begin{bmatrix}6 & 7\\ 8 & 9\end{bmatrix}$, *verify that* $(AB)^{-1}=B^{-1}A^{-1}$. **[NCERT]**

SOLUTION Clearly, $|A|=\begin{vmatrix}3 & 2\\ 7 & 5\end{vmatrix}=15-14=1\neq 0$. So, A is invertible.

Let A_{ij} be the cofactor of elements a_{ij} in $A=[a_{ij}]$. Then,

$A_{11}=(-1)^{1+1}5=5,\quad A_{12}=(-1)^{1+2}7=-7, A_{21}=(-1)^{2+1}2=-2$ and $A_{22}=(-1)^{2+2}3=3.$

$$\therefore \quad \text{adj } A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}^T=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}^T=\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$$

Hence, $A^{-1}=\dfrac{1}{|A|}\text{adj } A=\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}$

Now, $B=\begin{bmatrix}6 & 7\\ 8 & 9\end{bmatrix}\Rightarrow |B|=\begin{vmatrix}6 & 7\\ 8 & 9\end{vmatrix}=54-56=-2\neq 0$. So, B is invertible.

Let B_{ij} be the cofactors of b_{ij} in $B=[b_{ij}]$. Then,

$B_{11}=(-1)^{1+1}9=9,\quad B_{12}=(-1)^{1+2}8=-8, B_{21}=(-1)^{2+1}7=-7$ and $B_{22}=(-1)^{2+2}6=6.$

$$\therefore \quad \text{adj } B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}^T=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}^T=\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}$$

Hence, $B^{-1}=\dfrac{1}{|B|}\text{adj } B=-\dfrac{1}{2}\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}$

We know that adj $AB=$ adj B . adj A.

$$\therefore \quad \text{adj } AB=\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}=\begin{bmatrix}94 & -39\\ -82 & 34\end{bmatrix}$$

We also know that $|AB|=|A|\,|B|$. Therefore, $|AB|=1\times -2=-2\neq 0$. So, AB is invertible.

$$\text{Hence,}\quad (AB)^{-1}=\frac{1}{|AB|}\text{adj}(AB)=\frac{1}{-2}\begin{bmatrix}94 & -39\\ -82 & 34\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}94 & -39\\ -82 & 34\end{bmatrix} \qquad \text{...(i)}$$

$$\text{Also,}\quad B^{-1}A^{-1}=-\frac{1}{2}\begin{bmatrix}9 & -7\\ -8 & 6\end{bmatrix}\begin{bmatrix}5 & -2\\ -7 & 3\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}94 & -39\\ -82 & 34\end{bmatrix} \qquad \text{...(ii)}$$

From (i) and (ii), we get : $(AB)^{-1}=B^{-1}A^{-1}$.

NOTE *Students are advised not to find the product AB and $(AB)^{-1}$ by the usual technique.*

Type II FINDING THE INVERSE OF A MATRIX *A* WHEN IT SATISFIES SOME MATRIX EQUATION *f(A) = O*.

EXAMPLE 7 *Show that* $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$ *satisfies the equation* $x^2-6x+17=0$. *Hence, find* A^{-1}. **[CBSE 2007]**

SOLUTION We have, $A=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}$.

$$\therefore \quad A^2=AA=\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}=\begin{bmatrix}4-9 & -6-12\\ 6+12 & -9+16\end{bmatrix}=\begin{bmatrix}-5 & -18\\ 18 & 7\end{bmatrix}$$

$$-6A=(-6)\begin{bmatrix}2 & -3\\ 3 & 4\end{bmatrix}=\begin{bmatrix}-12 & 18\\ -18 & -24\end{bmatrix}\quad \text{and,}\quad 17I=17\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}17 & 0\\ 0 & 17\end{bmatrix}.$$

$$\therefore \quad A^2 - 6A + 17I_2 = \begin{bmatrix} -5 & -18 \\ 18 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 18 \\ -18 & -24 \end{bmatrix} + \begin{bmatrix} 17 & 0 \\ 0 & 17 \end{bmatrix}$$

$$\Rightarrow \quad A^2 - 6A + 17I_2 = \begin{bmatrix} -5-12+17 & -18+18+0 \\ 18-18+0 & 7-24+17 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

Hence, the matrix A satisfies the equation $x^2 - 6x + 17 = 0$.

Now,

$$A^2 - 6A + 17I_2 = O$$

$$\Rightarrow \quad A^2 - 6A = -17I_2$$

$$\Rightarrow \quad A^{-1}(A^2 - 6A) = A^{-1}(-17I_2) \quad \text{[Pre-multiplying both sides by } A^{-1}]$$

$$\Rightarrow \quad A^{-1}A^2 - 6A^{-1}A = -17(A^{-1}I_2)$$

$$\Rightarrow \quad A - 6I_2 = -17A^{-1}$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{17}(A - 6I_2) = \frac{1}{17}(6I_2 - A) = \frac{1}{17}\left\{\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}\right\} = \frac{1}{17}\begin{bmatrix} 4 & 3 \\ -3 & 2 \end{bmatrix}$$

EXAMPLE 8 *For the matrix* $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, *find* x and y *so that* $A^2 + xI = yA$.

Hence, find A^{-1}.

SOLUTION We have, $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$.

$$\therefore \quad A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now, $\quad A^2 + xI = yA$

$$\Rightarrow \quad \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$$\Rightarrow \quad 16 + x = 3y,\ y = 8,\ \ 7y = 56,\ 5y = 32 + x$$

Putting $y = 8$ in $16 + x = 3y$, we get: $x = 24 - 16 = 8$.

Clearly $x = 8$ and $y = 8$ also satisfy equations $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$.

Now, $|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$. So, A is invertible.

Putting $x = 8$, $y = 8$ in $A^2 + xI = yA$, we get

$$A^2 + 8I = 8A$$

$$\Rightarrow \quad A^{-1}(A^2 + 8I) = A^{-1}(8A) \quad \text{[Pre-multiplying throughout by } A^{-1}]$$

$$\Rightarrow \quad A^{-1}A^2 + 8A^{-1}I = 8(A^{-1}A)$$

$$\Rightarrow \quad A + 8A^{-1} = 8I \quad [\because A^{-1}A^2 = (A^{-1}A)A = IA = A,\ A^{-1}I = A^{-1} \text{ and, } A^{-1}A = I]$$

$$\Rightarrow \quad 8A^{-1} = 8I - A$$

$$\Rightarrow \quad A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\} = \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

EXAMPLE 9 *For the matrix* $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, *find the numbers* a *and* b *such that* $A^2 + aA + bI = O$.

Hence, find A^{-1}. **[NCERT]**

SOLUTION We have, $A=\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}$. Therefore, $A^2 = AA = \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} = \begin{bmatrix}11 & 8\\4 & 3\end{bmatrix}$.

Now, $A^2 + aA + bI = O$ ...(i)

$\Rightarrow \begin{bmatrix}11 & 8\\4 & 3\end{bmatrix} + a\begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} + b\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}11+3a+b & 8+2a\\4+a & 3+a+b\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$\Rightarrow 11+3a+b=0,\ 8+2a=0,\ 4+a=0$ and $3+a+b=0 \Rightarrow a=-4$ and $b=1$

Putting $a=-4$ and $b=1$ in (i), we get

$A^2 - 4A + I = O \Rightarrow 4A - A^2 = I \Rightarrow A(4I - A) = I$

$\Rightarrow A^{-1} = 4I - A = \begin{bmatrix}4 & 0\\0 & 4\end{bmatrix} - \begin{bmatrix}3 & 2\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & -2\\-1 & 3\end{bmatrix}$

EXAMPLE 10 *Show that the matrix* $A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$ *satisfies the equation* $A^2 - 4A - 5I_3 = O$ *and hence find* A^{-1}. [CBSE 2004]

SOLUTION We have, $A=\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}$

$\therefore A^2 = AA = \begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix}$, $4A = \begin{bmatrix}4 & 8 & 8\\8 & 4 & 8\\8 & 8 & 4\end{bmatrix}$ and, $5I_3 = \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$.

$\therefore A^2 - 4A - 5I_3 = \begin{bmatrix}9 & 8 & 8\\8 & 9 & 8\\8 & 8 & 9\end{bmatrix} - \begin{bmatrix}4 & 8 & 8\\8 & 4 & 8\\8 & 8 & 4\end{bmatrix} - \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$

$\Rightarrow A^2 - 4A - 5I_3 = \begin{bmatrix}9-4-5 & 8-8-0 & 8-8-0\\8-8-0 & 9-4-5 & 8-8-0\\8-8-0 & 8-8-0 & 9-4-5\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = O$

Now, $A^2 - 4A - 5I_3 = O$

$\Rightarrow A^2 - 4A = 5I_3$

$\Rightarrow A^{-1}A^2 - 4A^{-1}A = 5A^{-1}I_3$ [Pre-multiplying throughout by A^{-1}]

$\Rightarrow A - 4I = 5A^{-1}$ [$\because A^{-1}A^2 = (A^{-1}A)A = IA = A$]

$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$

$\Rightarrow A^{-1} = \frac{1}{5}\left\{\begin{bmatrix}1 & 2 & 2\\2 & 1 & 2\\2 & 2 & 1\end{bmatrix} - \begin{bmatrix}4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 4\end{bmatrix}\right\} = \frac{1}{5}\begin{bmatrix}-3 & 2 & 2\\2 & -3 & 2\\2 & 2 & -3\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-3/5 & 2/5 & 2/5\\2/5 & -2/5 & 2/5\\2/5 & 2/5 & -3/5\end{bmatrix}$

Type III FINDING THE INVERSE OF A MATRIX BY USING THE DEFINITION OF INVERSE

EXAMPLE 11 *If* $A=\begin{bmatrix}1 & -1 & 1\\2 & -1 & 0\\1 & 0 & 0\end{bmatrix}$, *show that* $A^{-1} = A^2$.

SOLUTION We know that a matrix B is the inverse of a matrix A if $AB = I = BA$. Here, we have to show that A^2 is the inverse of A. Therefore, it is sufficient to prove that $A^2 A = I$ or, $A^3 = I$.

Now, $A^2 = AA$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1-2+1 & -1+1+0 & 1+0+0 \\ 2-2+0 & -2+1+0 & 2+0+0 \\ 1+0+0 & -1+0+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

and, $\quad A^3 = A^2 A$

$$\Rightarrow \quad A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+1 & 0+0+0 & 0+0+0 \\ 0-2+2 & 0+1+0 & 0+0+0 \\ 1-2+1 & -1+1+0 & 1+0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, $A^2 = A^{-1}$.

Type IV ON SOLVING MATRIX EQUATIONS

EXAMPLE 12 *Find a* 2×2 *matrix B such that* $B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

SOLUTION Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$. Then, $|A| = \begin{vmatrix} 1 & -2 \\ 1 & 4 \end{vmatrix} = 6 \neq 0$. So, A is invertible.

The given matrix equation is

$$B\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\Rightarrow \quad BA = C$$

$$\Rightarrow \quad (BA)A^{-1} = CA^{-1} \qquad \text{[Post-multiplying throughout by } A^{-1}]$$

$$\Rightarrow \quad B(AA)^{-1} = CA^{-1} \Rightarrow BI = CA^{-1} \Rightarrow B = CA^{-1}.$$

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}4 = 4,\ C_{12} = (-1)^{1+2}1 = -1,\ C_{21} = (-1)^{2+1}(-2) = 2 \text{ and, } C_{22} = (-1)^{2+2}\,1 = 1.$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

Hence, $A^{-1} = \dfrac{1}{|A|}\text{ adj } A = \dfrac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$

Now,

$$B = CA^{-1}$$

$$\Rightarrow \quad B = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 24+0 & 12+0 \\ 0-6 & 0+6 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}$$

EXAMPLE 13 *Find the matrix A satisfying the matrix equation* $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

SOLUTION Let $B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$. Then,

$$|B| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4 - 3 = 1 \neq 0 \text{ and, } |C| = \begin{vmatrix} -3 & 2 \\ 5 & -3 \end{vmatrix} = 9 - 10 = -1 \neq 0.$$

So, B and C are invertible matrices. The given matrix equation is $BAC = I$.

Now, $\quad BAC = I$

$$\Rightarrow \quad B^{-1}(BAC)C^{-1} = B^{-1}IC^{-1}$$

$$\Rightarrow \quad (B^{-1}B)A(CC^{-1}) = B^{-1}C^{-1} \Rightarrow IAI = B^{-1}C^{-1} \Rightarrow A = B^{-1}C^{-1} \qquad \text{...(i)}$$

Let B_{ij} be the cofactor of elements b_{ij} in $B = [b_{ij}]$. Then,

$B_{11} = (-1)^{1+1} 2 = 2, B_{12} = (-1)^{1+2} 3 = -3, \quad B_{21} = (-1)^{2+1} 1 = -1$ and, $B_{22} = (-1)^{2+2} 2 = 2.$

$\therefore \quad \text{adj } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^T = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$

So, $\quad B^{-1} = \frac{1}{|B|} \text{adj } B = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}.$ $\quad [\because |B| = 1]$

Let C_{ij} be the cofactors of elements c_{ij} in $C = [c_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}(-3) = -3, C_{12} = (-1)^{1+2} 5 = -5, \quad C_{21} = (-1)^{2+1} 2 = -2$$

and, $\quad C_{22} = (-1)^{2+2}(-3) = -3.$

$\therefore \quad \text{adj } C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \begin{bmatrix} -3 & -5 \\ -2 & -3 \end{bmatrix}^T = \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

So, $\quad C^{-1} = \frac{1}{|C|} \text{adj } C = -\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ $\quad [\because |C| = -1]$

Substituting the values of B^{-1} and C^{-1} in (i), we get

$$A = B^{-1} C^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & 4-3 \\ -9+10 & -6+6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

EXAMPLE 14 *Find the matrix X for which* $\begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix} X = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}.$

SOLUTION Let $A = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix}$. Then the given matrix equation is $AX = B$.

$\therefore \quad |A| = \begin{vmatrix} 1 & -4 \\ 3 & -2 \end{vmatrix} = -2 + 12 = 10 \neq 0.$

So, A is an invertible matrix. Let C_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$C_{11} = (-1)^{1+1}(-2) = -2, C_{12} = (-1)^{1+2} 3 = -3, C_{21} = (-1)^{2+1}(-4) = 4$$

and, $\quad C_{22} = (-1)^{2+2} 1 = 1.$

$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -3 \\ 4 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$ and, $A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{10} \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}.$

Now, $\quad AX = B$

$\Rightarrow \quad A^{-1}(AX) = A^{-1} B \Rightarrow (A^{-1} A) X = A^{-1} B \Rightarrow IX = A^{-1} B \quad \Rightarrow X = A^{-1} B.$

$\Rightarrow \quad X = \frac{1}{10} \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -16 & -6 \\ 7 & 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 32+28 & 12+8 \\ 48+7 & 18+2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 11/2 & 2 \end{bmatrix}.$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 15 *If* $A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix}$ *and* $A^{-1} = \begin{bmatrix} 1/2 & -4 & 5/2 \\ -1/2 & 3 & -3/2 \\ 1/2 & y & 1/2 \end{bmatrix}$, *find x, y.*

SOLUTION We know that

$$AA^{-1} = I_3$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -4 & 5/2 \\ -1/2 & 3 & -3/2 \\ 1/2 & y & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 3+3y & 0 \\ -\frac{1}{2}+\frac{x}{2} & 2+xy & -\frac{1}{2}+\frac{x}{2} \\ 0 & 1+y & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 3+3y=0 \quad -\frac{1}{2}+\frac{x}{2}=0, \quad 2+xy=1, \quad 1+y=0 \Rightarrow x=1, y=-1$$

EXAMPLE 16 *If* $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ *is such that* $A^T = A^{-1}$, *find* α.

SOLUTION It is given that

$$A^T = A^{-1}$$

$$\Rightarrow \quad AA^T = AA^{-1} \qquad \text{[Premultiplying by } A\text{]}$$

$$\Rightarrow \quad AA^T = I \qquad [\because AA^{-1} = 1]$$

Now,

$$AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$\Rightarrow \quad AA^T = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\sin\alpha\cos\alpha + \sin\alpha\cos\alpha \\ -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, $AA^T = I$ is true for all α. Hence, α can take any real value.

EXAMPLE 17 *If matrix* $A = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ *satisfies* $A^T = A^{-1}$, *find* x, y, z. **[NCERT EXEMPLAR]**

SOLUTION It is given that the matrix A satisfies the relation

$$A^T = A^{-1}$$

$$\Rightarrow \quad AA^T = I \qquad [\text{Putting } A^{-1} = A^T \text{ in } AA^{-1} = I]$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix} \begin{bmatrix} 0 & x & x \\ 2y & y & -y \\ z & -z & z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 4y^2+z^2 & 2y^2-z^2 & -2y^2+z^2 \\ 2y^2-z^2 & x^2+y^2+z^2 & x^2-y^2-z^2 \\ -2y^2+z^2 & x^2-y^2-z^2 & x^2+y^2+z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 4y^2+z^2=1, \quad 2y^2-z^2=0, \quad x^2+y^2+z^2=1, \quad x^2-y^2-z^2=0$$

Now, $4y^2+z^2=1$ and $2y^2-z^2=0 \Rightarrow (4y^2+z^2)+(2y^2-z^2)=1 \Rightarrow 6y^2=1 \Rightarrow y=\pm\frac{1}{\sqrt{6}}$

Putting $y=\pm\frac{1}{\sqrt{6}}$ in $2y^2-z^2=0$, we obtain $z=\pm\frac{1}{\sqrt{3}}$.

Substituting $y=\pm\frac{1}{\sqrt{6}}, z=\pm\frac{1}{\sqrt{3}}$ in $x^2-y^2-z^2=0$, we obtain:

$$x^2-\frac{1}{6}-\frac{1}{3}=0 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{1}{\sqrt{2}}$$

Hence, $x=\pm\frac{1}{\sqrt{2}}, y=\pm\frac{1}{\sqrt{6}}, z=\pm\frac{1}{\sqrt{3}}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type V ON FINDING *A* NON-SINGULAR MATRIX A WHEN | *A* | AND adj *A* ARE GIVEN

EXAMPLE 18 *Find the matrix A such that* $|A|=2$ *and* $\text{adj}\, A=\begin{bmatrix}2&2&0\\2&5&1\\0&1&1\end{bmatrix}$.

SOLUTION We know that

$$A(\text{adj}\, A)=|A|\, I$$

$$\Rightarrow A=|A|\, I\, (\text{adj}\, A)^{-1}$$

$$\Rightarrow A=|A|\, (\text{adj}\, A)^{-1}$$

$$\Rightarrow A=2\,(\text{adj}\, A)^{-1} \qquad [\because |A|=2]$$

$$\Rightarrow A=2B^{-1}, \text{ where } B=\text{adj}\, A$$

Now, $B=\text{adj}\, A=\begin{bmatrix}2&2&0\\2&5&1\\0&1&1\end{bmatrix}$

$$\therefore |B|=|\text{adj}\, A|=|A|^{3-1}=|A|^2=4 \qquad [\because |\text{adj}\, A|=|A|^{n-1}]$$

Let C_{ij} be the cofactor of $(B)_{ij}$ in matrix B. Then,

$$C_{11}=4, C_{12}=-2, C_{13}=2, C_{21}=-2, C_{22}=2, C_{23}=-2, C_{31}=2, C_{32}=-2 \text{ and } C_{33}=6$$

$$\therefore \text{adj}\, B=\begin{bmatrix}4&-2&2\\-2&2&-2\\2&-2&6\end{bmatrix}^T=\begin{bmatrix}4&-2&2\\-2&2&-2\\2&-2&6\end{bmatrix} \text{ and, } B^{-1}=\frac{1}{|B|}\text{adj}\, B=\frac{1}{4}\begin{bmatrix}4&-2&2\\-2&2&-2\\2&-2&6\end{bmatrix}$$

$$\Rightarrow (\text{adj}\, A)^{-1}=\frac{1}{4}\begin{bmatrix}4&-2&2\\-2&2&-2\\2&-2&6\end{bmatrix}$$

Hence, $A=2(\text{adj}\, A)^{-1}=\frac{1}{2}\begin{bmatrix}4&-2&2\\-2&2&-2\\2&-2&6\end{bmatrix}=\begin{bmatrix}2&-1&1\\-1&1&-1\\1&-1&3\end{bmatrix}$

ALITER We have,

$$A^{-1}=\frac{1}{|A|}\text{adj}\, A=\frac{1}{2}\begin{bmatrix}2&2&0\\2&5&1\\0&1&1\end{bmatrix}=\begin{bmatrix}1&1&0\\1&5/2&1/2\\0&1/2&1/2\end{bmatrix}$$

Since $A=(A^{-1})^{-1}$. So, let us find the inverse of A^{-1}.

$$|A^{-1}|=\frac{1}{|A|} \Rightarrow |A|^{-1}=\frac{1}{2}$$

Let C_{ij} be the cofact of $(i, j)^{\text{th}}$ element of A^{-1}. Then,

$$C_{11}=1, C_{12}=-\frac{1}{2}, C_{13}=\frac{1}{2}, C_{21}=-\frac{1}{2}, C_{22}=\frac{1}{2}, C_{23}=-\frac{1}{2}\, C_{31}=\frac{1}{2}, C_{32}=-\frac{1}{2} \text{ and } C_{33}=\frac{3}{2}$$

$$\therefore \quad \text{adj}(A^{-1}) = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 3/2 \end{bmatrix}$$

Hence, $A = (A^{-1})^{-1} = \dfrac{1}{|A^{-1}|} \text{adj}(A^{-1}) = 2 \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Type VI ON FINDING $(\text{adj } A)^{-1}$ WHEN MATRIX A IS GIVEN

EXAMPLE 19 *If A is a non-singular matrix, prove that :*

(i) *adj (A) is also non-singular.* (ii) $(\text{adj } A)^{-1} = \dfrac{1}{|A|} A$.

SOLUTION (i) We know that

$$A(\text{adj } A) = |A| I_n = (\text{adj } A) A$$

$\Rightarrow \quad |A(\text{adj } A)| = ||A| I_n|$

$\Rightarrow \quad |A| |\text{adj } A| = |A|^n |I_n|$

$\Rightarrow \quad |\text{adj } A| = |A|^{n-1}$ $\quad [\because |A| \neq 0]$

$\Rightarrow \quad |\text{adj } A| \neq 0 \Rightarrow$ adj A is non-singular. $\quad [\because |A| \neq 0]$

(ii) We know that

$$A(\text{adj } A) = |A| I_n = (\text{adj } A) A$$

$\Rightarrow \quad \left(\dfrac{1}{|A|} A\right)(\text{adj } A) = I_n = (\text{adj } A)\left(\dfrac{1}{|A|} A\right)$ $\quad [\because |A| \neq 0]$

$\Rightarrow \quad (\text{adj } A)^{-1} = \dfrac{1}{|A|} A.$

EXAMPLE 20 *If A is a non-singular matrix, prove that* $(\text{adj } A)^{-1} = (\text{adj } A^{-1})$.

SOLUTION We have,

$$AA^{-1} = I$$

$\Rightarrow \quad \text{adj}(AA^{-1}) = \text{adj}(I)$

$\Rightarrow \quad (\text{adj } A^{-1})(\text{adj } A) = I$ $\quad [\because \text{adj}(AB) = (\text{adj } B)(\text{adj } A)$ and $\text{adj}(I) = I]$

$\Rightarrow \quad (\text{adj } A)^{-1} = \text{adj } A^{-1}$

EXAMPLE 21 *Find the non-singular matrices A, if its is given that* $\text{adj}(A) = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 0 & -3 \\ 1 & -4 & 1 \end{bmatrix}$.

SOLUTION We know that $(\text{adj } A)^{-1} = \dfrac{A}{|A|}$. Therefore,

$$A = |A| (\text{adj } A)^{-1} = |A| \frac{1}{|\text{adj } A|} \text{adj}(\text{adj } A) \quad \ldots(i)$$

Now, $\text{adj}(A) = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 0 & -3 \\ 1 & -4 & 1 \end{bmatrix}$

$$\Rightarrow \quad |\text{adj } A| = \begin{vmatrix} -1 & -2 & 1 \\ 3 & 0 & -3 \\ 1 & -4 & 1 \end{vmatrix} = -1(0-12) + 2(3+3) + 1(-12-0) = 12$$

$$\Rightarrow \quad |A|^2 = 12 \Rightarrow |A| = \pm 2\sqrt{3} \qquad [\because \ |\text{adj } A| = |A|^{n-1}]$$

Let C_{ij} be the cofactor of $(\text{adj } A)_{ij}$ in $(\text{adj } A)$. Then,

$$C_{11} = -12, C_{12} = -6, C_{13} = -12, C_{21} = -2, C_{22} = -2, C_{23} = -6, C_{31} = 6, C_{32} = 0, C_{33} = 6$$

$$\therefore \quad \text{adj}(\text{adj } A) = \begin{bmatrix} -12 & -6 & -12 \\ -2 & -2 & -6 \\ 6 & 0 & 6 \end{bmatrix}^T = \begin{bmatrix} -12 & -2 & 6 \\ -6 & -2 & 0 \\ -12 & -6 & 6 \end{bmatrix}$$

Substituting the values of $|A|$, $|\text{adj } A|$ and $\text{adj}(\text{adj } A)$ in (i), we get

$$A = \pm \frac{2\sqrt{3}}{12} \begin{bmatrix} -12 & -2 & 6 \\ -6 & -2 & 0 \\ -12 & -6 & 6 \end{bmatrix} = \pm \frac{1}{\sqrt{3}} \begin{bmatrix} -6 & -1 & 3 \\ -3 & -1 & 0 \\ -6 & -3 & 3 \end{bmatrix}$$

EXAMPLE 22 *If* $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, *find* $(\text{adj } A)^{-1}$ *and* $(\text{adj } A^{-1})$.

SOLUTION We have, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$\therefore \quad |A| = 2(4-1) + 1(-2+1) + 1(1-2) = 4$$

We know that $(\text{adj } A)^{-1} = \dfrac{1}{|A|} A$.

$$\therefore \quad (\text{adj } A)^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

We also know that $(\text{adj } A^{-1}) = (\text{adj } A)^{-1}$.

$$\therefore \quad \text{adj } A^{-1} = \frac{1}{4} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Type VII MISCELLANEOUS PROBLEMS

EXAMPLE 23 *Let A be a non-singular matrix. Show that $A^T A^{-1}$ is symmetric iff $A^2 = (A^T)^2$.*

SOLUTION First, let $A^T A^{-1}$ be symmetric. Then,

$$(A^T A^{-1})^T = A^T A^{-1}$$

$$\Rightarrow \quad (A^{-1})^T (A^T)^T = A^T A^{-1}$$

$$\Rightarrow \quad (A^T)^{-1} A = A^T A^{-1} \qquad \left[\because \ (A^{-1})^T = (A^T)^{-1}\right]$$

$$\Rightarrow \quad A^T\left((A^T)^{-1} A\right) A = A^T (A^T A^{-1}) A \Rightarrow \left(A^T (A^T)^{-1}\right) AA = (A^T A^T)(A^{-1} A)$$

$$\Rightarrow \quad I A^2 = (A^T)^2 I \Rightarrow A^2 = (A^T)^2 \text{ or, } (A^T)^2 = A^2$$

Conversely, let A be a non-singular matrix such that $A^2 = (A^T)^2$. Then,

$$A^2 = (A^T)^2$$

$\Rightarrow \quad AA = A^T A^T$

$\Rightarrow \quad (A^T)^{-1}(AA)\,A^{-1} = (A^T)^{-1}(A^T A^T)\,A^{-1}$ $\left[\begin{array}{l}\text{Pre and post multiplying by } (A^T)^{-1} \\ \text{and } A^{-1} \text{ respectively}\end{array}\right]$

$\Rightarrow \quad \left((A^{-1})^T A\right)(AA^{-1}) = \left((A^T)^{-1} A^T\right)(A^T A^{-1}) \Rightarrow \left((A^{-1})^T A\right) I = I\,(A^T A^{-1})$

$\Rightarrow \quad (A^{-1})^T A = A^T A^{-1} \Rightarrow (A^{-1})^T (A^T)^T = A^T A^{-1} \Rightarrow (A^T A^{-1})^T = A^T A^{-1}$

$\Rightarrow \quad A^T A^{-1}$ is a symmetric matrix.

EXERCISE 6.1

BASIC

1. Find the adjoint of each of the following matrices:

(i) $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$ **[CBSE 2020]** (ii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & \tan\alpha/2 \\ -\tan\alpha/2 & 1 \end{bmatrix}$

Verify that $(\text{adj } A)\,A = |A|\,I = A\,(\text{adj } A)$ for the above matrices.

2. Compute the adjoint of each of the following matrices:

(i) $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

Verify that $(\text{adj } A)\,A = |A|\,I = A\,(\text{adj } A)$ for the above matrices.

3. For the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, show that $A\,(\text{adj } A) = O$.

4. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj $A = A$.

5. If $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$, show that adj $A = 3A^T$.

6. Find $A\,(\text{adj } A)$ for the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$.

7. Find the inverse of each of the following matrices:

(i) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$

8. Find the inverse of each of the following matrices.

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ (vi) $\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

9. Find the inverse of each of the following matrices and verify that $A^{-1}A = I_3$.

(i) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$

10. For the following pairs of matrices verify that $(AB)^{-1} = B^{-1}A^{-1}$:

(i) $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$. Find $(AB)^{-1}$

12. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. [CBSE 2018]

13. If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, then show that $A - 3I = 2(I + 3A^{-1})$.

14. Find the inverse of the matrix $A = \begin{bmatrix} a & b \\ c & \dfrac{1+bc}{a} \end{bmatrix}$ and show that $aA^{-1} = (a^2 + bc + 1)I - aA$.

15. Given $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. Compute $(AB)^{-1}$. [NCERT]

16. Let $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $G(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$. Show that

(i) $[F(\alpha)]^{-1} = F(-\alpha)$ (ii) $[G(\beta)]^{-1} = G(-\beta)$ (iii) $[F(\alpha)\,G(\beta)]^{-1} = G(-\beta)\,F(-\alpha)$.

17. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, verify that $A^2 - 4A + I = O$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, find A^{-1}. [NCERT]

18. Show that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $A^2 + 4A - 42I = O$. Hence, find A^{-1}.

19. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence, find A^{-1}. [NCERT, CBSE 2007]

20. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, find x and y such that $A^2 - xA + yI = O$. Hence, evaluate A^{-1}.

21. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1}. [CBSE 2007]

22. Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $x^2 - 3x - 7 = 0$. Thus, find A^{-1}.

23. Show that $A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$ satisfies the equation $x^2 - 12x + 1 = 0$. Thus, find A^{-1}

24. For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11 I_3 = O$. Hence, find A^{-1}. [NCERT]

25. Show that the matrix, $A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation, $A^3 - A^2 - 3A - I_3 = O$.

Hence, find A^{-1}.

26. If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}. [NCERT]

27. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, prove that $A^{-1} = A^T$.

28. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, show that $A^{-1} = A^3$.

29. If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^2 = A^{-1}$.

30. Solve the matrix equation $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, where X is a 2×2 matrix.

31. Find the matrix X satisfying the matrix equation: $X \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

32. Find the matrix X for which: $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$.

33. Find the matrix X satisfying the equation: $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} X \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

34. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1} and prove that $A^2 - 4A - 5I = O$.

BASED ON LOTS

35. If A is a square matrix of order n, prove that $|A \text{ adj } A| = |A|^n$.

36. If $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$, find $(AB)^{-1}$. [CBSE 2012]

37. If $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$, find $(A^T)^{-1}$. [CBSE 2015]

38. Find the adjoint of the matrix $A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ and hence show that $A\,(\text{adj}\,A) = |A|\,I_3$.

[CBSE 2015]

39. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, find A^{-1} and show that $A^{-1} = \frac{1}{2}(A^2 - 3I)$. [NCERT EXEMPLAR]

ANSWERS

1. (i) $\begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (iii) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix}$

2. (i) $\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$

6. $\begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$

7. (i) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$ (iv) $\frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$

8. (i) $\frac{1}{18}\begin{bmatrix} -5 & 1 & 7 \\ 1 & 7 & -5 \\ 7 & -5 & 1 \end{bmatrix}$ (ii) $\frac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$ (iii) $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ (v) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ (vi) $\frac{1}{4}\begin{bmatrix} -8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$

9. (i) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ (ii) $\frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$

11. $\begin{bmatrix} -47 & \frac{39}{2} \\ 41 & -17 \end{bmatrix}$

14. $\begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

15. $\begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

18. $\frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

19. $\frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

20. $x = 9,\ y = 14,\ \frac{1}{14}\begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$

21. $\lambda = 1,\ A^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$

22. $\frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$

23. $\begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$

24. $\frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

25. $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$ **26.** $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ **30.** $\begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$ **31.** $\begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$

32. $\begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}$ **33.** $\begin{bmatrix} 9 & -14 \\ -16 & 25 \end{bmatrix}$ **34.** $\frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$ **36.** $\begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

37. $\begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$

HINTS TO SELECTED PROBLEMS

15. We have to find $(AB)^{-1}$ and we are given the values of A and B^{-1}. But, $(AB)^{-1} = B^{-1}A^{-1}$.

So, we need to find A^{-1}.

Now, $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \Rightarrow |A| = 5(3-4) - 0(2-2) + 4(4-3) = -5+4 = -1 \neq 0$

So, A^{-1} exists.

Let C_{ij} be cofactor of a_{ij} in $A = [a_{ij}]$. Then,

$C_{11} = 3-4 = -1, C_{12} = -(2-2) = 0, \quad C_{13} = 4-3 = 1, C_{21} = -(0-8) = 8$

$C_{22} = 5-4 = 1, \quad C_{23} = -(10-0) = -10, C_{31} = (0-12) = -12,$

$C_{32} = -(10-8) = -2, C_{33} = 15$

$$\therefore \quad \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}^T = \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|}\text{adj } A = \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix} = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$$

17. We have, $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$\therefore \quad A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$\Rightarrow \quad A^{-1}(A^2 - 4A + I) = A^{-1}O$ [Multiplying both sides by A^{-1}]

$\Rightarrow \quad A^{-1}A^2 - 4A^{-1}A + A^{-1}I = O \Rightarrow A - 4I + A^{-1} = O$

$$\Rightarrow \quad A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

19. We have, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$\therefore \quad A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

So, $A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Now, $A^2 - 5A + 7I = O$

$\Rightarrow \quad A^{-1}(A^2 - 5A + 7I) = A^{-1}O$ [Multiplying throughout by A^{-1}]

$\Rightarrow \quad A^{-1}A^2 - 5A^{-1}A + 7A^{-1}I = O$

$\Rightarrow \quad A - 5I + 7A^{-1} = O \Rightarrow 7A^{-1} = 5I - A$

$\Rightarrow \quad 7A^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

24. We have, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$\therefore \quad A^2 = AA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$

and, $A^3 = A^2A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$

$\therefore \quad A^3 - 6A^2 + 5A + 11I_3$

$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$

Now, $A^3 - 6A^2 + 5A + 11I_3 = O$

$\Rightarrow \quad A^{-1}(A^3 - 6A^2 + 5A + 11I_3) = A^{-1}O$ [Multiplying both sides by A^{-1}]

$\Rightarrow \quad A^2 - 6A + 5I + 11A^{-1} = O$

$\Rightarrow \quad 11A^{-1} = -A^2 + 6A - 5I$

$\Rightarrow \quad 11A^{-1} = -\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad 11A^{-1} = \begin{bmatrix} -4+6-5 & -2+6+0 & -1+6+0 \\ 3+6+0 & -8+12-5 & 14-18+0 \\ -7+12+0 & 3-6+0 & -14+18-5 \end{bmatrix} = \begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

$\Rightarrow \quad A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$

26. We have, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$\therefore \quad A^2 = AA = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$\text{and,} \quad A^3 = A^2 A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$\therefore \quad A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} + \begin{bmatrix} -36 & 30 & -30 \\ 30 & -36 & 30 \\ -30 & 30 & -36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9+0 & 21-30+9+0 \\ -21+30-9+0 & 22-36+18-4 & -21+30-9+0 \\ 21-30+9+0 & -21+30-9+0 & 22-36+18-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Now, $A^3 - 6A^2 + 9A - 4I = O$

$$\Rightarrow \quad A^{-1}(A^3 - 6A^2 + 9A - 4I) = A^{-1}O \qquad \text{[Multiplying both sides by } A^{-1}]$$

$$\Rightarrow \quad A^2 - 6A + 9I - 4A^{-1} = O$$

$$\Rightarrow \quad 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow \quad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -6 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$\Rightarrow \quad 4A^{-1} = \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5+6+0 & 6-12+9 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

6.4 ELEMENTARY TRANSFORMATIONS OR ELEMENTARY OPERATIONS OF A MATRIX

The following three operations applied on the rows (columns) of a matrix are called elementary row (column) transformations.

(i) Interchange of any two rows (columns)

If i^{th} row (column) of a matrix is interchanged with the j^{th} row (column), it will be denoted by $R_i \leftrightarrow R_j$ $(C_i \leftrightarrow C_j)$.

For example, $A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 2 & 4 \end{bmatrix}$, then by applying $R_2 \leftrightarrow R_3$ we get: $B = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 4 \\ -1 & 2 & 1 \end{bmatrix}$.

(ii) Multiplying all elements of a row (column) of a matrix by a non-zero scalar

If the elements of i^{th} row (column) are multiplied by a non-zero scalar k, it will be denoted by $R_i \to R_i\ (k)\ [C_i \to C_i\ (k)]$ or $R_i \to kR_i\ [C_i \to kC_i]$

If $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 2 \\ -1 & 2 & -3 \end{bmatrix}$, then by applying $R_2 \to 3R_2$, we obtain: $B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 6 \\ -1 & 2 & -3 \end{bmatrix}$.

(iii) Adding to the elements of a row (column), the corresponding elements of any other row (column) multiplied by any scalar k.

If k times the elements of j^{th} row (column) are added to the corresponding elements of the i^{th} row (column), it will be denoted by $R_i \to R_i + k\,R_j\ (C_i \to C_i + k\,C_j)$.

If $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$, then the application of elementary operation $R_3 \to R_3 + 2\,R_1$ gives the

matrix $B = \begin{bmatrix} 2 & 1 & 3 & 1 \\ -1 & -1 & 0 & 2 \\ 4 & 3 & 9 & 3 \end{bmatrix}$.

If a matrix B is obtained from a matrix A by one or more elementary transformations, then A and B are equivalent matrices and we write $A \sim B$.

Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 1 & 2 & 4 \end{bmatrix}$. Then,

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \\ 3 & 1 & 2 & 4 \end{bmatrix} \qquad \text{Applying } R_2 \to R_2 + (-1)\,R_1$$

$$\Rightarrow \quad A \sim \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & -1 & 1 & -2 \\ 1 & 1 & 2 & 2 \end{bmatrix} \qquad \text{Applying } C_4 \to C_4 + (-1)\,C_3$$

An elementary transformation is called a row transformation or a column transformation according as it is applied to rows or columns.

ELEMENTARY MATRIX *A matrix obtained from an identity matrix by a single elementary operation (transformation) is called an elementary matrix.*

For example, $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

are elementary matrices obtained from I_3 by subjecting it to the elementary transformations $R_1 \to R_1 + 3\,R_2, C_1 \leftrightarrow C_3$ and $R_2 \leftrightarrow R_3$ respectively.

Consider a matrix $A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -3 & 5 & 1 & 2 \\ 2 & 1 & 5 & 3 \end{bmatrix}$. Let B be a matrix obtained from A by applying elementary transformation $R_2 \to R_2 + 2\,R_1$ and let E be the elementary matrix obtained from I_3 (as there are three rows in A) by subjecting it to the same transformation. Then,

$$B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ -1 & 3 & 5 & 2 \\ 2 & 1 & 5 & 3 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now,}\quad EA=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&-1&2&0\\-3&5&1&2\\2&1&5&3\end{bmatrix}=\begin{bmatrix}1&-1&2&0\\-1&3&5&2\\2&1&5&3\end{bmatrix}=B$$

Thus, we find that B can be obtained from A by pre-multiplying with an elementary matrix obtained from I_3 by subjecting it to the same elementary row transformation.

Let C be a matrix obtained from A by the application of transformation $C_3 \to C_3 + 2C_2$, and let E be the elementary matrix obtained from I_4 (as there are four columns in A) by subjecting it to the same column transformation. Then,

$$C=\begin{bmatrix}1&-1&0&0\\-3&5&11&2\\2&1&7&3\end{bmatrix}\text{ and } E=\begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\text{Now,}\quad AE=\begin{bmatrix}1&-1&2&0\\-3&5&1&2\\2&1&5&3\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0&0\\-3&5&11&2\\2&1&7&3\end{bmatrix}=C$$

Thus, C can also be obtained from A by post-multiplying with an elementary matrix obtained from I_4 by subjecting it to the same elementary column transformations.

We now state the results obtained in the above discussion as two theorems, the proofs of which are beyond the scope of this book.

THEOREM 1 *Every elementary row (column) transformation of an $m \times n$ matrix (not identity matrix) can be obtained by pre-multiplication (post-multiplication) with the corresponding elementary matrix obtained from the identity matrix I_m (I_n) by subjecting it to the same elementary row (column) transformation.*

THEOREM 2 *Let $C = AB$ be a product of two matrices. Any elementary row (column) transformation of AB can be obtained by subjecting the pre-factor A (post-factor B) to the same elementary row (column) transformation.*

ILLUSTRATION 1 *Verify Theorem 2, if* $A=\begin{bmatrix}2&1&-3\\0&1&-1\\1&2&-1\end{bmatrix}$, $B=\begin{bmatrix}4&0\\1&2\\2&3\end{bmatrix}$ *and the elementary row-operation is* $R_2 \to R_2 + (-2)R_1$.

SOLUTION We have,

$$AB=\begin{bmatrix}2&1&-3\\0&1&-1\\1&2&-1\end{bmatrix}\begin{bmatrix}4&0\\1&2\\2&3\end{bmatrix}=\begin{bmatrix}3&-7\\-1&-1\\4&1\end{bmatrix}$$

Applying $R_2 \to R_2 + (-2)R_1$ on AB, we get

$$AB\sim\begin{bmatrix}3&-7\\-7&13\\4&1\end{bmatrix}=P\text{ (say)}\qquad\text{...(i)}$$

Applying $R_2 \to R_2 + (-2)R_1$ on A, we get

$$A\sim\begin{bmatrix}2&1&-3\\-4&-1&5\\1&2&-1\end{bmatrix}=Q\text{ (say)}\qquad\text{...(ii)}$$

Now, $QB = \begin{bmatrix} 2 & 1 & -3 \\ -4 & -1 & 5 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -7 \\ -7 & 13 \\ 4 & 1 \end{bmatrix} = R$ (say) ...(iii)

From (i) and (iii), we get $P = R$.

Hence, the theorem is verified.

ILLUSTRATION 2 *Use elementary column operation $C_2 \to C_2 - 2C_1$ in the matrix equation*

$$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

[CBSE 2014]

SOLUTION If A, B, C are three matrices such that $C = AB$, then any elementary row (column) transformation of C can be obtained by subjecting the pre-factor (post-factor B) to the same elementary row (column) transformation. Therefore, given matrix equation after applying $C_2 \to C_2 - 2C_1$, becomes

$$\begin{bmatrix} 4 & 2-2\times 4 \\ 3 & 3-2\times 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0-2\times 2 \\ 1 & 1-2\times 1 \end{bmatrix} \quad \text{or,} \quad \begin{bmatrix} 4 & -6 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$$

ILLUSTRATION 3 *Apply elementary transformation $R_2 \to R_2 - 3R_1$ in the matrix equation*

$$\begin{bmatrix} 11 & -6 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix}$$

SOLUTION Any elementary row transformation on the LHS of the given equation is obtained by subjecting the pre-factor on the RHS of the same transformation. Therefore, given matrix equation, by applying $R_2 \to R_2 - 3R_1$, becomes

$$\begin{bmatrix} 11 & -6 \\ -27 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix}$$

6.4.1 METHOD OF FINDING THE INVERSE OF A MATRIX BY ELEMENTARY TRANSFORMATIONS

Let A be a non-singular matrix of order n. Then A can be reduced to the identity matrix by a finite sequence of elementary transformations only. As we have discussed every elementary row transformation of a matrix is equivalent to pre-multiplication by the corresponding elementary matrix. Therefore, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$(E_k E_{k-1} \dots E_2 E_1) A = I_n$$

$\Rightarrow \quad (E_k E_{k-1} \dots E_2 E_1) AA^{-1} = I_n A^{-1}$ [Post-multiplying by A^{-1}]

$\Rightarrow \quad (E_k E_{k-1} \dots E_2 E_1) I_n = A^{-1}$ [$\because I_n A^{-1} = A^{-1}$ and $AA^{-1} = I_n$]

$\Rightarrow \quad A^{-1} = (E_k E_{k-1} \dots E_2 E_1) I_n.$

Following algorithm may be used for finding the inverse of a non-singular matrix by elementary row transformations.

ALGORITHM

Step I *Obtain the square matrix, say A.*

Step II *Write $A = I_n A$*

Step III *Perform a sequence of elementary row operations successively on A on the LHS and the pre-factor I_n on the RHS till we obtain the result $I_n = BA$.*

Step IV *Write $A^{-1} = B$.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, *using elementary row transformations.*

SOLUTION We know that

$$A = IA$$

or, $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$ [Applying $R_2 \to R_2 + (-2) R_1$]

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A$ [Applying $R_1 \to R_1 + (-3) R_2$]

$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$.

EXAMPLE 2 *By using elementary row transformations find the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

SOLUTION We know that

$$A = IA$$

or, $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ [Applying $R_2 \to R_2 + (-3) R_1$]

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} A$ [Applying $R_1 \to R_1 + (-2) R_2$]

$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$.

We may use the following algorithm to find the inverse of a square matrix of order 3 by using elementary row transformations.

ALGORITHM

Step I *Introduce unity at the intersection of first row and first column either by interchanging two rows or by adding a constant multiple of elements of some other row to the first row.*

Step II *After introducing unity at* $(1, 1)^{th}$ *place introduce zeros at all other places in first column.*

Step III *Introduce unity at the intersection of 2nd row and 2nd column with the help of 2nd and 3rd rows.*

Step IV *Introduce zeros at all other places in the second column except at the intersection of 2nd row and 2nd column.*

Step V *Introduce unity at the intersection of 3rd row and third column.*

Step VI *Finally introduce zeros at all other places in the third column except at the intersection of third row and third column.*

EXAMPLE 3 *Using elementary row transformation find the inverse of the matrix* $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$.

SOLUTION We know that

$$A = IA$$

or, $$\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{[Applying } R_1 \to R_1 - R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{[Applying } R_2 \to R_2 + (-2)\, R_1 \,\&\, R_3 \to R_3 + (-3)\, R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A \quad \text{[Applying } R_2 \to R_2(1/2)]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A \quad \text{[Applying } R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 + 2R_2]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A \quad \text{[Applying } R_3 \to 1/4\, R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A \quad \begin{bmatrix} \text{Applying } R_2 \to R_1 + (1/2)\, R_3 \\ \&\, R_2 \to R_2 - (1/2)\, R_3 \end{bmatrix}$$

Hence, $$A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}.$$

EXAMPLE 4 *Find the inverse of the matrix* $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ *by using elementary row transformations.* **[CBSE 2010, 2019]**

SOLUTION We know that

$$A = I\,A$$

or, $$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \quad \text{[Applying } R_2 \to R_2 + R_1]$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} A \quad \text{[Applying } R_2 \to R_2 + 2R_3]$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A \quad \text{[Applying } R_1 \to R_1 + (-2)\, R_2,\ R_3 \to R_3 + 2R_2]$$

$$\Rightarrow \quad \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix} A \qquad \text{[Applying } R_1 \to R_1 + 2R_3\text{]}$$

Hence, $A^{-1} = \begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix}$

EXERCISE 6.2

BASIC

Find the inverse of each of the following matrices by using elementary row transformations:

1. $\begin{bmatrix}7 & 1\\4 & -3\end{bmatrix}$

2. $\begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}$

3. $\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$ [CBSE 2017]

4. $\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}$ [CBSE2010]

5. $\begin{bmatrix}3 & 10\\2 & 7\end{bmatrix}$

6. $\begin{bmatrix}0 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$

7. $\begin{bmatrix}2 & 0 & -1\\5 & 1 & 0\\0 & 1 & 3\end{bmatrix}$

8. $\begin{bmatrix}2 & 3 & 1\\2 & 4 & 1\\3 & 7 & 2\end{bmatrix}$

9. $\begin{bmatrix}3 & -3 & 4\\2 & -3 & 4\\0 & -1 & 1\end{bmatrix}$

10. $\begin{bmatrix}1 & 2 & 0\\2 & 3 & -1\\1 & -1 & 3\end{bmatrix}$

11. $\begin{bmatrix}2 & -1 & 3\\1 & 2 & 4\\3 & 1 & 1\end{bmatrix}$

12. $\begin{bmatrix}1 & 1 & 2\\3 & 1 & 1\\2 & 3 & 1\end{bmatrix}$

13. $\begin{bmatrix}2 & -1 & 4\\4 & 0 & 2\\3 & -2 & 7\end{bmatrix}$ [CBSE 2008]

14. $\begin{bmatrix}3 & 0 & -1\\2 & 3 & 0\\0 & 4 & 1\end{bmatrix}$ [CBSE 2009]

15. $\begin{bmatrix}1 & 3 & -2\\-3 & 0 & 1\\2 & 1 & 0\end{bmatrix}$ [CBSE 2011]

16. $\begin{bmatrix}-1 & 1 & 2\\1 & 2 & 3\\3 & 1 & 1\end{bmatrix}$ [CBSE 2012]

17. $\begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\-2 & -4 & -5\end{bmatrix}$ [CBSE 2018]

18. $\begin{bmatrix}2 & -3 & 5\\3 & 2 & -4\\1 & 1 & -2\end{bmatrix}$ [CBSE 2019]

ANSWERS

1. $\frac{1}{25}\begin{bmatrix}3 & 1\\4 & -7\end{bmatrix}$

2. $\begin{bmatrix}1 & -2\\-2 & 5\end{bmatrix}$

3. $\frac{1}{5}\begin{bmatrix}1 & 2\\2 & -1\end{bmatrix}$

4. $\begin{bmatrix}3 & -5\\-1 & 2\end{bmatrix}$

5. $\begin{bmatrix}7 & -10\\-2 & 3\end{bmatrix}$

6. $\begin{bmatrix}1/2 & -1/2 & 1/2\\-4 & 3 & -1\\5/2 & -3/2 & 1/2\end{bmatrix}$

7. $\begin{bmatrix}3 & -1 & 1\\-15 & 6 & -5\\5 & -2 & 2\end{bmatrix}$

8. $\begin{bmatrix}1 & 1 & -1\\-1 & 1 & 0\\2 & -5 & 2\end{bmatrix}$

9. $\begin{bmatrix}1 & -1 & 0\\-2 & 3 & -4\\-2 & 3 & -3\end{bmatrix}$

10. $\begin{bmatrix}-4/3 & 1 & 1/3\\7/6 & -1/2 & -1/6\\5/6 & -1/2 & 1/6\end{bmatrix}$

11. $\frac{-1}{30}\begin{bmatrix}-2 & 4 & -10\\11 & -7 & -5\\-5 & -5 & 5\end{bmatrix}$

12. $\frac{1}{11}\begin{bmatrix}-2 & 5 & -1\\-1 & -3 & 5\\7 & -1 & -2\end{bmatrix}$

13. $\begin{bmatrix} -2 & 1/2 & 1 \\ 11 & -1 & -6 \\ 4 & -1/2 & -2 \end{bmatrix}$ **14.** $\begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$ **15.** $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ 3 & 5 & 9 \end{bmatrix}$ **16.** $\begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

17. $\begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ **18.** $\begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If A is a unit matrix of order n, then A (adj A) = ...
2. If A is a non-singular square matrix such that $A^3 = I$, then A^{-1} =....................... .
3. If A and B are square matrices of the same order and $AB = 3I$, then A^{-1} = ...
4. If the matrix $A = \begin{bmatrix} 1 & a & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ is not invertible, then a = ...
5. If A is a singular matrix, then A (adj A) = ...
6. Let A be a square matrix of order 3 such that $|A| = 11$ and B be the matrix of cofactors of elements of A. Then, $|B|^2$ = ...
7. If A is a square matrix of order 2 such that $A(\text{adj}A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ = ...
8. If A is an invertible matrix of order 3 and $|A| = 3$, then $|\text{adj } A|$ = ...
9. If A is an invertible matrix of order 3 and $|A| = 5$, then adj (adj A) = ...
10. If A is an invertible matrix of order 3 and $|A| = 4$, then $|\text{adj (adj } A)|$ = ...
11. If A = diag (1, 2, 3), then $|\text{adj (adj } A)|$ = ...
12. If A is a square matrix of order 3 such that $|A| = \frac{5}{2}$, then $|A^{-1}|$ = ...
13. If A is a square matrix such that A (adj A) $= 10I$, then $|A|$ = ...
14. Let A be a square matrix of order 3 and $B = |A| A^{-1}$. If $|A| = -5$, then $|B|$ = ...
15. If k is a scalar and I is a unit matrix of order 3, then adj (kI) = ...
16. If $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ and A (adj A) $= \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then k = ...
17. If A is a non-singular matrix of order 3, then adj (adj A) is equal to ....
18. If $A = [a_{ij}]_{2\times 2}$, where $a_{ij} = \begin{cases} i+j, \text{ if } i \neq j \\ i^2 - 2j, \text{ if } i = j \end{cases}$, then A^{-1} = ...
19. If $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$ and $A^{-1} = \lambda$ (adj A), then λ = ...
20. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1}A^T$, then BB^T = ...
21. If A and B are two square matrices of the same order such that $B = -A^{-1}BA$, then $(A+B)^2$ = ...
22. If A is a non-singular matrix of order 3×3, then $(A^3)^{-1}$ = ...
23. If A be a square matrix such that $|\text{adj } A| = |A|^2$, then the order of A is ....

24. If $A = \begin{bmatrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{bmatrix}$, $xyz = 80$, $3x + 2y + 10z = 20$ and A adj $A = kI$, then $k =$...... . [NCERT EXEMPLAR]

25. For $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, write A^{-1}. [CBSE 2020]

ANSWERS

1. A 2. A^2 3. $\frac{1}{3}B$ 4. 1 5. Null matrix 6. 11^4 7. 10
8. 9 9. $5A$ 10. 4^4 11. 6^4 12. $\frac{2}{5}$ 13. 10 14. 25 15. $k^2 I$
16. 1 17. $|A|A$ 18. $\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ 19. $-\frac{1}{6}$ 20. I 21. $A^2 + B^2$
22. $(A^{-1})^3$ 23. 3×3 24. 81 25. $\begin{bmatrix} -1 & 4 \\ -1 & 3 \end{bmatrix}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the adjoint of the matrix $A = \begin{bmatrix} -3 & 4 \\ 7 & -2 \end{bmatrix}$.
2. If A is a square matrix such that $A\,(\text{adj}\,A) = 5I$, where I denotes the identity matrix of the same order. Then, find the value of $|A|$.
3. If A is a square matrix of order 3 such that $|A| = 5$, write the value of $|\text{adj}\,A|$. [CBSE 2009]
4. If A is a square matrix of order 3 such that $|\text{adj}\,A| = 64$, find $|A|$.
5. If A is a non-singular square matrix such that $|A| = 10$, find $|A^{-1}|$.
6. If A, B, C are three non-null square matrices of the same order, write the condition on A such that $AB = AC \Rightarrow B = C$.
7. If A is a non-singular square matrix such that $A^{-1} = \begin{bmatrix} 5 & 3 \\ -2 & -1 \end{bmatrix}$, then find $(A^T)^{-1}$.
8. If adj $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ and adj $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$, find adj AB.
9. If A is a symmetric matrix, write whether A^T is symmetric or skew-symmetric.
10. If A is a square matrix of order 3 such that $|A| = 2$, then write the value of adj (adj A).
11. If A is a square matrix of order 3 such that $|A| = 3$, then find the value of $|\text{adj}\,(\text{adj}\,A)|$.
12. If A is a square matrix of order 3 such that adj $(2A) = k$ adj (A), then write the value of k.
13. If A is a square matrix, then write the matrix adj $(A^T) - (\text{adj}\,A)^T$.
14. Let A be a 3×3 square matrix such that $A\,(\text{adj}\,A) = 2\,I$, where I is the identity matrix. Write the value of $|\text{adj}\,A|$.
15. If A is a non-singular symmetric matrix, write whether A^{-1} is symmetric or skew-symmetric.
16. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A\,(\text{adj}\,A) = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, then find the value of k.

17. If A is an invertible matrix such that $|A^{-1}| = 2$, find the value of $|A|$.

18. If A is a square matrix such that $A\,(\text{adj}\,A) = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then write the value of $|\text{adj}\,A|$.

19. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ be such that $A^{-1} = k\,A$, then find the value of k.

20. Let A be a square matrix such that $A^2 - A + I = O$, then write A^{-1} in terms of A.

21. If C_{ij} is the cofactor of the element a_{ij} of the matrix $A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$, then write the value of $a_{32}\,C_{32}$. **[CBSE 2013]**

22. Find the inverse of the matrix $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$

23. Find the inverse of the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

24. If $A = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}$, write adj A.

25. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find adj (AB).

26. If $A = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$, then find $|\text{adj}\,A|$. **[CBSE 2010]**

27. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, write A^{-1} in terms of A. **[CBSE 2011]**

28. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

29. Use elementary column operation $C_2 \to C_2 + 2C_1$ in the following matrix equation :

$$\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$ **[CBSE 2016]**

30. In the following matrix equation use elementary operation $R_2 \to R_2 + R_1$ and the equation thus obtained:

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 9 & -4 \end{bmatrix}$$ **[CBSE 2017]**

31. If A is a square matrix with $|A| = 4$ then find the value of $|A\,.\,(\text{adj}\,A)|$. **[CBSE 2019]**

ANSWERS

1. $\begin{bmatrix} -2 & -4 \\ -7 & -3 \end{bmatrix}$
2. 5
3. 25
4. ± 8
5. $\frac{1}{10}$
6. A must be invertible or $|A| \neq 0$
7. $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$
8. $\begin{bmatrix} -6 & 5 \\ -2 & -10 \end{bmatrix}$
9. symmetric
10. $2A$
11. 81
12. 4
13. Null matrix
14. 4
15. symmetric
16. 1
17. $\frac{1}{2}$
18. 25
19. $\frac{1}{19}$
20. $A^{-1} = (I - A)$
21. 110
22. $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$
23. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
24. $\begin{bmatrix} 0 & 3 \\ -2 & 1 \end{bmatrix}$
25. $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
26. -11
27. $A^{-1} = \frac{1}{19}A$
28. $A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$
29. $\begin{bmatrix} 2 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$
30. $\begin{bmatrix} 2 & 3 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 17 & -7 \end{bmatrix}$
31. $4^{O(A)}$