CHAPTER 31

MATHEMATICAL REASONING

31.1 INTRODUCTION

In this chapter, we shall learn about some basics of mathematical reasoning. As all of us know that the main asset that makes humans far more superior than the other species is the ability to reasoning. The ability of reasoning varies from person to person. Also, it is the ability of reasoning which makes one person superior than the other. In this chapter, we shall discuss the process of reasoning especially in the context of mathematics. In mathematical language, there are two kinds of reasoning.

(i) Inductive reasoning. (ii) Deductive reasoning.

In the chapter on Mathematical induction, we have already discussed the inductive reasoning. In this, chapter, we shall discuss some basics of deductive reasoning.

31.2 STATEMENTS

In reasoning we communicate our ideas or thoughts with the help of sentences in a particular language. The following types of sentences are normally used in our every day communication.

ASSERTIVE SENTENCE *A sentence that makes an assertion is called an assertive sentence or a declarative sentence.*

For example, "Mars supports life" is an assertive or a declarative sentence. "Any two individuals are always related" is also a declarative sentence.

IMPERATIVE SENTENCE *A sentence that expresses a request or a command is called an imperative sentence.*

For example, "Please bring me a cup of tea" is an imperative sentence.

EXCLAMATORY SENTENCE *A sentence that expresses some strong feeling is called an exclamatory sentence.*

For example, "How big is the whale fish !" is an exclamatory sentence.

INTERROGATIVE SENTENCE *A sentence that asks some question is called an interrogative sentence.*

For example, "What is your age ?" is an interrogative sentence.

In this chapter, we shall be discussing about a specific type of sentences which will be called as statements or propositions.

STATEMENT *A statement or a proposition is an assertive (or a declarative) sentence which is either true or false but not both.*

A statement is assumed to be either true or false. A true statement is also known as a valid statement. If a statement is false, we say that it is an invalid statement. A statement cannot be both true and false at the same time. This fact is known as the law of the excluded middle.

A sentence which is both true and false simultaneously is not a statement, rather it is a paradox.

ILLUSTRATION 1 *Consider the following sentences:*

(i) *Washington D.C. is in America.* (ii) *Moon revolves around the Earth.*

(iii) *Two plus three is five.* (iv) *Every square is a rectangle.*

(v) *The sun is a star.*

Each of these sentences is a true declarative sentence and so, each of them is a statement.

ILLUSTRATION 2 *Consider the following sentences:*

(i) *Three plus four is 6.* (ii) *The earth is a star.*
(iii) *Every rectangle is a square.* (iv) *New Delhi is in Nepal.*
(v) *Every relation is a function.*

Each of these sentences is a false declarative sentence and hence each of them is a statement.

ILLUSTRATION 3 *Consider the following sentences:*

(i) *Give me a glass of water.* (ii) *Switch on the light.*
(iii) *Bring some fruits from the fruit shop.* (iv) *Do your home work.*
(v) *Please do me a favour.*

We observe that each of these sentences is an imperative sentence. In other words, each of them either expresses a request or a command. So, they are not statements.

ILLUSTRATION 4 *Consider the following sentences:*

(i) *How are you ?* (ii) *Where are you going ?*
(iii) *Is every set finite ?* (iv) *Have you ever seen Taj Mahal ?*
(v) *Where is your pen ?*

Clearly, each of these sentences is asking a question. So, they cannot be assigned, true or false. Hence, none of the them is a statement.

ILLUSTRATION 5 *Consider the following sentences:*

(i) *May God bless you !* (ii) *May you live long !*

Each of these sentences is an optive. So, we cannot assign true or false to them and hence none of them is a statement.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Which of the following sentences are statements or propositions? Justify your answer.*

(i) *The set of prime integers is infinite.* (ii) *Paris is in England.*
(iii) *The moon is made of green cheese.* (iv) *May god bless you !*
(v) *Who are you ?* (vi) *The number x is a positive integer.*

SOLUTION (i) The "set of prime integers is infinite" is a true declarative sentence. So, it is a true statement.

(ii) "Paris is in England" is a false declarative sentence. So, it is a false statement.

(iii) The sentence " The moon is made of green cheese" is a false declarative sentence. So, it is a false statement.

(iv) The sentence "May god bless you!" is an exclamatory sentence. So, it is not a statement.

(v) The sentence "who are you?" is an interrogative sentence. So, it is not a statement.

(vi) The sentence "The number x is a positive integer" is not a statement unless the variable x is assigned a specific value.

<u>REMARK</u> *The sentence "This sentence is false" cannot be assigned a truth value of either true or false, because either assignment contradicts the sense of the sentence. Although it is a declarative sentence, but it is not a proposition.*

EXAMPLE 2 *Which of the following is a statement (or proposition) ?*

(i) $x + 2 = 9.$ (ii) *6 has three prime factors.* (iii) $x^2 + 5x + 6 = 0.$

SOLUTION (i) The sentence : $x + 2 = 9$ is an open sentence. Its truth value cannot be confirmed unless we are given the value of x. So, it is not a statement.

(ii) The sentence "6 has three prime factors" is a false statement, because 6 has two prime factors, viz. 2 and 3.

(iii) $x^2 + 5x + 6 = 0$ is not a statement, because its truth or falsity cannot be confirmed without knowing the value of x.

EXAMPLE 3 *Check whether the following sentences are statements. Give reasons for your answer.*

(i) 18 *is less than* 16. (ii) *Every set is a finite set.*

(iii) *The sun is a star.* (iv) *Mathematics is fun.*

(v) There is no rain without clouds. (vi) *How far is Chennai from here?*

SOLUTION (i) This sentence is always false, because $18 > 16$. Hence, it is a statement.

(ii) This sentence is always false, because there are sets which are not finite. Hence, it is a statement.

(iii) Since the sun is a star (it is a scientific fact). So, the given sentence is a always true. Hence, it is a statement.

(iv) Mathematics is a fun is true for those who like mathematics. But, for others, it may not be true. So, the given sentence may or may not be true. Hence, it is not a statement.

(v) It is scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence, it is a statement.

(vi) It is an interrogative sentence. Hence, it is not a statement.

EXERCISE 31.1

LEVEL-1

1. Find out which of the following sentences are statements and which are not. Justify your answer.

(i) Listen to me, Ravi ! (ii) Every set is a finite set.

(iii) Two non-empty sets have always a non-empty intersection.

(iv) The cat pussy is black. (v) Are all circles round ?

(vi) All triangles have three sides. (vii) Every rhombus is a square.

(viii) $x^2 + 5|x| + 6 = 0$ has no real roots. (ix) This sentence is a statement.

(x) Is the earth round ? (xi) Go !

(xii) The real number x is less than 2. (xiii) There are 35 days in a month.

(xiv) Mathematics is difficult. (xv) All real numbers are complex numbers.

(xvi) The product of (-1) and 8 is 8.

2. Give three examples of sentences which are not statements. Give reasons for the answers.

ANSWERS

1. Statements : (ii) (iii) (vi) (vii) (viii) (xiii) (xv) (xvi)

31.3 NEGATION OF A STATEMENT

The denial of a statement p is called its negation and is written as ~ p, and read as 'not p'.

Negation of any statement p is formed by writing "It is not the case that ....." or "It is false that ...." before p or, if possible by inserting in p the word "not".

Let us consider the statement:

p: All integers are rational numbers.

The negation of this statement is:

~ p: It is not the case that all integers are rational numbers.

or

$\sim p$: *It is false that all integers are rational numbers.*

or

$\sim p$: *At least one integer is not a rational numbers.*

Consider now the statement: $p : 7 > 9$

The negation of this statement is: $\sim p : 7 \ngtr 9$ or $\sim p : 7 \leq 9$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write the negation of the following statements:*

(i) *New Delhi is a city.*
(ii) *I went to my class yesterday.*
(iii) $\sqrt{7}$ *is rational.*
(iv) *The number 2 is greater than 7.*
(v) $\sqrt{2}$ *is not a complex number.*

SOLUTION (i) The given statement is:

p: *New Delhi is a city.*

The negation of this statement is:

$\sim p$: *It is not the case that New Delhi is a city.*

or

$\sim p$: *It is false that New Delhi is a city.*

or

$\sim p$: *New Delhi is not a city.*

(ii) Let q denote the given statement i.e. q: *I went to my class yesterday.*

The negation of this statement is:

$\sim q$: *I did not go to my class yesterday.*

or

$\sim q$: *It is not the case that I went to my class yesterday.*

or

$\sim q$: *It is false that I went to my class yesterday.*

or

$\sim q$: *I was absent from my class yesterday.*

(iii) Let r denote the given statement i.e. $r : \sqrt{7}$ is rational

The negation $\sim r$ of this statement is given by:

$\sim r$: *It is not the case that* $\sqrt{7}$ *is rational.*

or

$\sim r$: $\sqrt{7}$ *is not rational.*

or

$\sim r$: *It is false that* $\sqrt{7}$ *is rational.*

(iv) Let the given statement be denoted by s i.e. s: *The number 2 is greater than 7.*

The negative $\sim s$ of this statement is given by

$\sim s$: *The number 2 is not greater than 7.*

or

$\sim s$: *The number 2 is less than or equal to 7.*

or

$\sim s$: *It is false that the number 2 is greater than 7.*

(v) Let the given statement be denoted by u i.e. $u : \sqrt{2}$ *is not a complex number.*

The negation $\sim v$ of this statement is given by:

$\sim u$: *$\sqrt{2}$ is a complex number.*

or

$\sim u$: *It is false that $\sqrt{2}$ is not a complex number.*

EXAMPLE 2 *Write the negation of the following statements and check whether the resulting statements are true:*

(i) *The sum of 2 and 5 is 9.* (ii) *Every natural number is greater than 0.*

(iii) *Australia is a continent.*

(iv) *There does not exist a quadrilateral which has all its sides equal.*

SOLUTION (i) The negation of the given statement is

It is false that the sum of 2 and 5 is 9.

or

The sum of 2 and 5 is not equal to 9.

We know that $2 + 5 = 7 \neq 9$. So, this statement is true.

(ii) The negation of the given statement is:

It is false that every natural number is greater than 0.

or

There exists a natural number which is not greater than 0.

We know that all natural numbers are greater than 0. So, this statement is false.

(iii) The negation of the given statement is:

It is false that Australia is a continent.

or

Australia is not a continent.

We know that Australia is a continent. So, this statement is false.

(iv) The negation of the given statement is:

It is not the case that there does not exist a quadrilateral which has all its sides equal.

or

There exists a quadrilateral which has all its sides equal.

We know that square and rhombus are quadrilaterals having all sides equal. So, this statement is true.

NOTE *It should be noted that the negation of "Every or For all" is "there exists" and vice-versa.*

EXAMPLE 3 *Write the negation of the following statements:*

(i) *Everyone in Germany speaks German.* (ii) *All primes are odd.*

(iii) *All mathematicians are man.* (iv) *All triangles are not equilateral triangles.*

(v) *All complex numbers are real numbers.* (vi) *Every natural number is an integer.*

(vii) *All cats scratch.*

SOLUTION (i) The negation of the given statement is:

It is false that everyone in Germany speaks German.

or

There exists a person in Germany who does not speak German.

or

At least one person in Germany does not speak German.

(ii) The negation of the given statement is:

There exists a prime which is not odd.

or

Some primes are not odd.

or

At least one prime is not odd.

(iii) The negation of the given statement is:

Some mathematicians are not man

or

There exists a mathematician who is not man.

or

At least one mathematician is not man.

or

It is false that all mathematicians are man.

(iv) The negation of the given statement is:

Some triangles are equilateral.

or

There exists a triangle which is equilateral.

or

At least one triangle is equilateral.

(v) The negation of the given statement is:

Some complex numbers are not real numbers.

or

There exists a complex number which is not a real number.

or

At least one complex number is not a real number.

(vi) The negation of the given statement is:

There exists a natural number which is not an integer.

or

At least one natural number is not an integer.

or

Some natural numbers are not integers.

(vii) The negation of the given statement is:

Some cats do not scratch.

or

There exists a cat which does not scratch.

or

At least one cat does not scratch.

EXERCISE 31.2

LEVEL-1

1. Write the negation of the following statements:
   (i) Banglore is the capital of Karnataka.
   (ii) It rained on July 4, 2005.
   (iii) Ravish is honest.
   (iv) The earth is round.
   (v) The sun is cold.

2. (i) All birds sing. (ii) Some even integers are prime.
   (iii) There is a complex number which is not a real number.
   (iv) I will not go to school.
   (v) Both the diagonals of a rectangle have the same length.
   (vi) All policemen are thieves.
3. Are the following pairs of statements are negation of each other:
   (i) The number x is not a rational number.
   The number x is not an irrational number.
   (ii) The number x is not a rational number.
   The number x is an irrational number.
4. Write the negation of the following statements:
   (i) p : For every positive real number x, the number $(x-1)$ is also positive.
   (ii) q : For every real number x, either $x>1$ or $x<1$.
   (iii) r : There exists a number x such that $0<x<1$.
5. Check whether the following pair of statements are negation of each other. Give reasons for your answer.
   (i) $a+b = b+a$ is true for every real number a and b.
   (ii) There exist real numbers a and b for which $a+b = b+a$.

ANSWERS

1. (i) Banglore is not the capital of Karnataka.
   (ii) It did not rain on July 4, 2005. (iii) Ravish is not honest.
   (iv) The earth is not round. (v) The sun is not cold.
2. (i) Some birds do not sing.
   or
   At least one bird does not sing.
   or
   There exists a bird which does not sing.
   (ii) No even integer is prime. (iii) All complex numbers are real numbers.
   (iv) I will go to school.
   (v) There is at least one rectangle whose both diagonals do not have the same length.
   (vi) No policemen is a thief.
3. (i) Yes (ii) No
4. (i) $\sim p$: There exists a positive real number x such that the number $(x-1)$ is not positive.
   (ii) $\sim q$: There exists a real number such that neither $x>1$ nor $x<1$.
   (iii) $\sim r$: For every real number x, either $x \leq 0$ or $x \geq 1$.
5. (i) No. The negation of first statement is
   (ii) There exist real numbers a and b for which $a+b \neq b+a$.

31.4 COMPOUND STATEMENTS

In Mathematical reasoning, we generally come across two types of statements namely, simple statements and compound statements as defined below.

SIMPLE STATEMENTS *Any statement whose truth value does not explicity depend on another statement is said to be a simple statement.*

In other words, a statement is said to be simple if it cannot be broken down into simpler statements, that is, if it is not composed of simpler statements.

ILLUSTRATION 1 *Consider the following statements :*

(i) $\sqrt{2}$ *is an irrational number.* (ii) *The set of real numbers is an infinite set.* (iii) $2+5<4$.

All these statements are simple statements.

COMPOUND STATEMENTS *If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.*

ILLUSTRATION 2 *Each of the following statements is a compound statement:*

(i) "Roses are red and violets are blue" is a compound statement as it is composed of the statements ; "Roses are red" and "Violets are blue".

(ii) "The school works or a holiday is declared" is a compound statement as it is a combination of the statements : "The school works" and " A holiday is declared".

(iii) "John is intelligent or studies every night" is also a compound statement as it is composed of the statements : "John is intelligent" and "John studies every night".

(iv) "If it rains, then the school may be closed" is a compound statement as it is obtained by connecting two simple statements:
"It rains" and "The school may be closed" by using the phrase 'if .... then'.

(v) "A quadrilateral is a rhombus if and only if its diagonals are at right angles" is a compound statement obtained by connecting two simple statements : "A quadrilateral is a rhombus" and "Diagonals of a quadrilateral intersect at right angles" by using the phrase 'if and only if'.

The simple statements which form a compound statement are known as its sub-statements or component statements.

The fundamental property of a compound statement is that its truth value is completely determined by the truth values of the sub-statements together with the way in which they are connected to form the compound statement.

31.5 BASIC CONNECTIVES

In the previous section, we have learnt that the words 'or' & 'and' connect two or more simple statements to form a compound statement. These are called sentenial connectives or simply connectives. In this section, we shall learn how the truth and falsity of a compound statement depends upon the truth value of the component statements.

31.5.1 THE WORD "AND"

Any two simple statements can be connected by the word "and" to form a compound statement. For example, consider the statement "The earth is round and the sun is cold". This statement can be broken into two component statements given by

p : *The earth is round.*

q : *The sun is cold.*

Let us now consider the statement "84 is divisible by 4, 7 and 12". The component statements of this statement are

p : *84 is divisible by 4.*

q : *84 is divisible by 7.*

r : *84 is divisible by 12.*

NOTE 1 *It should be noted that the word "and" is used as a connective as we use in the English language. But, 'and' is also used with other meanings. For example, in the statement "Ravish and Ravi are good friends" the word 'and' is not a connective. Similarly, in the statement "Mohan opened the door and ran away", the word 'and' is used in the sense of 'and then' because the action described in "Mohan ran away" occurs after action described in "Mohan opened the door".*

NOTE 2 *In our day-to-day life, the word 'and' is used between two statements which have some kind of relation. But, in reasoning it can be used even for the statements which are not related to each other. For example, "it is raining and 5 is a prime number" is a compound statement whose component statements are*

p : *It is raining.*

q: 5 *is a prime number.*

Clearly, these two statements are not related to each other.

We shall now see how the truth or falsity of a compound statement with "and" depends upon the truth or falsity of its component statements.

Consider the statement:

$p: 9 > 4$ *and* $2 < 7$

The component statements of this statement are:

$q: 9 > 4$

$r: 2 < 7$

Clearly, q and r are true statements. Also, p is a true statement.

Thus, if two statements are true, then their compound statement with "and" is also true.

Consider the statement:

p: *The earth is round and the sun is cold.*

Its component statements are:

q: *The earth is round.*

r: *The sun is cold.*

Clearly, statement q is true and r is false. Also, p is false.

Thus, if one of the two statements is true and the other is false, then the compound statement with "and" is false.

Let us now consider the statement:

$p: 5 < 12$ *and* $15 < 7$

Its component statements are :

$q: 5 < 12$

$r: 15 < 7$

Clearly, p, q and r are false statements.

Thus, the compound statement with "and" is a false, if the component statements are false.

The above discussion suggests us the following rules:

RULE 1 *The compound statement with "and" is true if all its component statements are true.*

RULE 2 *The compound statement with "and" is false if any or all of its compound statements is false.*

31.5.2 THE WORD "OR"

Any two statements can be connected by the word "OR" to form a compound statement. For example, consider the statement "The sun shines or it rains". This statement can be broken into two component statements given by:

p: *The sun shines.*

q: *It rains.*

Consider now the statement "Two lines in a plane either intersect at one point or they are parallel". The component statements of this statement are:

p: *Two lines in a plane intersect at a point.*

q: *Two lines in a plane are parallel.*

It should be noted that in addition to the connective the word "OR" is also used with other meanings in English language. For example, in the statement *"five or six children are playing in the playground"* the word "or" is used for indicating an approximate number of children. It is not used as a connective. As a connective also the word "OR" is used in two distinct ways in English language. Sometimes it is used in the sense of "p or q or both", i.e. at least one of the two alternatives occurs and sometimes it is used in the sense of "p or q but not both", i.e. exactly one of the two alternatives occurs. When it used for at least one of the two alternatives, we call it *inclusive* "OR". In case of exactly one of the two alternatives, it is called exclusive "OR".

Let us consider the statement given by:

p : *The school is closed if it is a holiday or Sunday.*

This means that the school remains closed on a holiday. It also remains closed on sunday. If a holiday falls on Sunday, then also the school remains closed. So, in this case, we are using the word "OR" as an inclusive "OR".

Consider now the following statement:

q : *An ice-cream or Coca-cola is available with a piza in piza-hut.*

This means that a person who does not want ice-cream can have a coca-cola with a piza or one does not want coca-cola can have an ice-cream along with a piza. That is who do not want an ice-cream can have coca-cola and vice-versa. A person cannot have both ice-cream and coca-cola with a piza. So, the "OR" used is an exclusive "OR".

NOTE *Throughout this chapter we will be using the word "OR" as an inclusive "OR" unless it is stated otherwise.*

We shall now see how the truth and falsity of the compound statement with an "OR" depends upon the truth and falsity of its component statements.

Consider the compound statement:

p : *Two lines intersect at a point or they are parallel.*

The component statements of this statement are :

q : *Two lines intersect at a point.*

r : *Two lines are parallel.*

We observe that when q is true r is false and when r is true q is false. Also, p is always true.

Thus, if one of the component statements is true, then the compound statement connected with "OR" is always true.

Consider another statement

p : 45 *is a multiple of* 4 *or* 6.

Its component statements are:

q : 45 *is multiple of* 4

r : 45 *is a multiple of* 6

We observe that both q and r are false. Also, p is a false.

Thus, if both the component statements are false, then the compound statement connected with "OR" is always false.

Again, consider the following statement:

p : *The earth is round or the sun is hot.*

Its component statements are:

q : *The earth is round.*

r : *The sun is hot.*

We observe that both q and r are true. Also, p is true.

Thus, if both the component statements are true, then the compound statement with "OR" is always true.

The above discussion suggests us the following rules for the compound statements with an "OR".

RULE 1 *A compound statement with an "OR" is true when one component statement is true or both the component statements are true.*

RULE 2 *A compound statement with an "OR" is false when both the component statements are false.*

NOTE *If p and q are two simple statements, then the negation of the compound statement*

(i) *p or q* is $\sim p$ *and* $\sim q$ i.e., $\sim (p \text{ or } q) \cong \sim p$ *and* $\sim q$.

(ii) *p and q is* $\sim p$ *or* $\sim q$ i.e., $\sim (p \text{ and } q) \cong \sim p$ *or* $\sim q$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the component statements of the following compound statements:*

(i) *There is somethings wrong with the bulb or with wiring.*

(ii) *It is raining and it is cold.*

(iii) *The roof is red and the wall is white.*

(iv) *The sun shines or it rains.*

(v) *0 is a positive number or a negative number.*

SOLUTION (i) The component statements are:

p : *There is some thing wrong with the bulb.*

q : *There is some thing wrong with the wiring.*

(ii) The component statements are:

p : *It is raining.*

q : *It is cold.*

(iii) The component statements are:

p : *The roof is red.*

q : *The wall is white..*

(iv) The component statements are:

p : *The sun shines.*

q : *It rains.*

(v) The component statements are:

p : *0 is a positive number.*

q : *0 is a negative number.*

EXAMPLE 2 *Find the component statement of the following and check whether they are true or not:*

(i) $\sqrt{2}$ *is a rational number or an irrational number.*

(ii) *All integers are positive or negative.*

(iii) *All primes are either even or odd.*

(iv) *24 is a multiple of 2, 4 and 8.*

SOLUTION (i) The component statements are:

p : $\sqrt{2}$ *is a rational number.*

q : $\sqrt{2}$ *is an irrational number.*

Clearly, p is false and q is true. The connecting word is 'or'

(ii) The component statements are:

p : *All integers are positive.*

q : *All integers are negative.*

Clearly, p and q both are false. The connecting word is 'or'.

(iii) The component statemetns are:

p : *All primes are even.*

q : *All primes are odd.*

Clearly, p and q both are false. Then correcting word is 'or'.

(iv) The component statements are:

p : *24 is a multiple of 2.*

q : *24 is a multiple of 4.*

r : *24 is a multiple of 8.*

Here, p, q and r are true statements. The connecting words are 'and'.

EXAMPLE 3 *For each of the following statements, determine whether an inclusive "OR" or exclusive "OR" is used. Give reasons for your answer.*

(i) *Sun rises or Moon sets.*

(ii) *All integers are positive or negative.*

(iii) *Two lines intersect at a point or are parallel.*

(iv) *The school is closed if it is a holiday or a Sunday.*

SOLUTION (i) Here "OR" is exclusive since sun rises and moon sets during day time.

(ii) Since all integers cannot be both positive as well as negative. Therefore, "OR" is an exclusive "OR".

(iii) Here "OR" is exclusive because it is not possible for two lines to intersect and parallel together.

(iv) Here "OR" is inclusive since school is closed on holiday as well as on Sunday.

EXAMPLE 4 *Write the component statements of the following compound statements and check whether the compound statement is true or false:*

(i) *50 is a multiple of both 2 and 5.*

(ii) *0 is less than every positive integer and every negative integer.*

(iii) *A line is straight and extends indefinitely in both directions.*

(iv) *All living things have two legs and two eyes.*

SOLUTION (i) The component statements of the given statement are:

p : *50 is multiple of 2.*

q : *50 is a multiple of 5.*

We observe that both p and q are true statements. Therefore, the compound statement is true.

(ii) The component statements of the given statements are:

p : *0 is less than every positive integer.*

q : *0 is less than every negative integer.*

We observe that p is true and q is false. Therefore, the compound statement is false.

(iii) The component statements of the given statement are:

p : *A line is straight.*

q : *A line extends indefinitely in both directions.*

We observe that both p and q are true. Therefore, the compound statement is true.

(iv) The component statements of the given statement are:

p : *All living things have two legs.*

q : *All living things have two eyes.*

We find that both p and q are false statements. Therefore, the compound statement is false.

EXAMPLE 5 *Write the component statements of the following compound statements and check whether the compound statement is true or false:*

(i) *125 is a multiple of 7 or 8.*

(ii) *Mumbai is the capital of Gujrat or Maharashtra.*

(iii) $\sqrt{2}$ *is a rational number or an irrational number.*

(iv) *The school is closed, if there is a holiday or Sunday.*

(v) *A rectangle is a quadrilateral or a 5-sided polygon.*

SOLUTION (i) The component statements of the given statement are:

p : *125 is a multiple of 7.*

q : *125 is a multiple of 8.*

We observe that both p and q are false statements. Therefore, the compound statement is also false.

(ii) The component statements of the given statement are:

p : *Mumbai is the Capital of Gujrat.*

q : *Mumbai is the Capital of Maharashtra.*

We find that p is false and q is true. Therefore, the compound statement is true.

(iii) The component statements are:

$p : \sqrt{2}$ *is a rational number.*

$q : \sqrt{2}$ *is an irrational number.*

Clearly, p is false and q is true. Therefore, the compound statement is true.

(iv) The component statements are:

p : *The school is closed if there is a holiday.*

q : *The school is closed if there is a Sunday.*

Both, p and q are true statements. Therefore, the compound statement is true.

(v) The component statements are :

p : *A rectangle is a quadrilateral.*

q : *A rectangle is a 5-sided polygon.*

We observe that p is true and q is false. Therefore, the compound statements is true.

EXAMPLE 6 *Write the negation of the following compound statements:*

(i) *It is daylight and all the people have arisen.*

(ii) *All the students completed their homework and the teacher is present.*

(iii) *All rational numbers are real and all real numbers are complex.*

(iv) *Square of an integer is positive or negative.*

(v) *The sand heats up quickly in the sun and does not cool down fast at night.*

SOLUTION (i) In writing down the negations of the above statements, we will be using the following results:

(a) $\sim (p \text{ or } q) \cong \sim p \text{ and } \sim q$.

(b) $\sim (p \text{ and } q) \cong \sim p \text{ or } \sim q$.

(c) ~ (*For all*) ≅ *There exists or some or atleast one.*

(d) ~ (*At least one or There exists or Some*) ≅ *For all or for every.*

(ii) The component statements of the given statement are:

p : *All the students completed their homework.*

q : *The teacher is present.*

The given statement is (p and q). So, its negation is :

$\sim p$ or $\sim q$ = *Some of the students did not complete their homework or the teacher is not present.*

(ii) The component statements of the given statement are :

p : *It is daylight.*

q : *All the people have risen.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$ ≡ *It is not daylight or it is false that all the people have arisen.*

≡ *It is night or someone has not arisen.*

(iii) The component statements of the given statement are:

p : *All rational numbers are real.*

q : *All real numbers are complex.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$: *Some rational are not real or some reals are not complex.*

(iv) The component statements of the given statements are:

p : *Square of an integer is positive.*

q : *Square of an integer is negative.*

The given statement is (p or q). So, its negation is:

$\sim p$ and $\sim q$ = *There exists an integer whose square is neither positive nor negative..*

(v) The component statements of the given statement are:

p : *The sand heats up quickly in the sun.*

q : *The sand does not cool down fast at night.*

The given statement is (p and q). So, its negation is:

$\sim p$ or $\sim q$ = *Either the sand does not heat up quickly in the sun or it cools down fast at night.*

EXERCISE 31.3

LEVEL-1

1. Find the component statements of the following compound statements:
   (i) The sky is blue and the grass is green.
   (ii) The earth is round or the sun is cold.
   (iii) All rational numbers are real and all real numbers are complex.
   (iv) 25 is a multiple of 5 and 8.
2. For each of the following statements, determine whether an inclusive "OR" or exclusive "OR" is used. Give reasons for your answer.
   (i) Students can take Hindi or Sanskrit as their third language.
   (ii) To entry a country, you need a passport or a voter registration card.
   (iii) A lady gives birth to a baby boy or a baby girl.
   (iv) To apply for a driving licence, you should have a ration card or a passport.
3. Write the component statements of the following compound statements and check whether the compound statement is true or false:
   (i) To enter into a public library children need an identity card from the school or a letter from the school authorities.
   (ii) All rational numbers are real and all real numbers are not complex.
   (iii) Square of an integer is positive or negative.
   (iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.
   (v) The sand heats up quickly in the sun and does not cool down fast at night.
4. Determine whether the following compound statements are true or false:
   (i) Delhi is in India and $2 + 2 = 4$.
   (ii) Delhi is in England and $2 + 2 = 4$.
   (iii) Delhi is in India and $2 + 2 = 5$.
   (iv) Delhi is in England and $2 + 2 = 5$.

ANSWERS

1. (i) p : The sky is blue
   q : The grass is green
   (ii) p : The earth is round
   q : The sun is cold
   (iii) p : All rational numbers are real
   q : All real numbers are complex
   (iv) p : 25 is a multiple of 5
   q : 25 is a multiple of 8.
2. (i) Exclusive "OR". A student cannot take both Hindi and Sanskrit.
   (ii) Inclusive "OR". Since a person can have both a passport and a voter registration card to enter a country.
   (iii) Exclusive "OR". A lady cannot give birth to a baby who is both a boy and a girl.
   (iv) Inclusive "OR". Since a person can have both a ration card and a passport to apply for a driving licence.
3. (i) p : To get into a public library children need an identity card.
   q : To get into a public library children need a letter from the school authorities.
   True.

(ii) p: All rational numbers are real.

q: All real numbers are not complex.

(iii) p: Square of an integer is positive.

q: Squre of an integer is negative. True.

(iv) $p: x = 2$ is a root of the equation $3x^2 - x - 10 = 0$.

$q: x = 3$ is a root of the equation $3x^2 - x - 10 = 0$. False.

(v) p: The sand heats up quickly in the sun.

q: The sand does not cool down fast at night. False.

4. (i) True (ii) False (iii) False (iv) False

31.6 QUANTIFIERS

In Mathematics we come across many mathematical statements containing phrases "There exists" and "For every". These two phrases are called quantifiers. Depending upon the context the phrase "There exists" can also be replaced by the equivalent phrases "There is" or, "There is at least one" or, "It is possible to find" or, "some".

Consider the following statements:

$p: x + 4 > 3$ *for all* $x \in N$.

q: *For every prime number* x, $\sqrt{x}$ *is an irrational number.*

r: *There exists a rectangle whose all sides are equal.*

s: *There exists* $x \in N$ *such that* $x + 4 < 7$ or, *For some* $x \in N$, $x + 4 < 7$.

The statement p means that for every natural number x, $x + 4 > 3$.

The statement q means that if S denotes the set of all prime numbers, then for all the members x of the set S, $\sqrt{x}$ is an irrational number.

The statement r means that there is at least one rectangle whose all sides are equal.

The statement s means that there is at least one natural number x such that $x + 4 < 7$.

Phrase "for every (or for all)" is called the *universal quantifier* and the phrase "There exists" is known as the *existential quantifier*.

Consider the statement: *For every* $x \in N$, $x + 5 > 4$

If $p(x)$ denotes $x + 5 > 4$, then the above statement can be written as

For every $x \in N$, $p(x)$

Consider the statement:

All Math Majors are male ...(i)

If M denotes the set of all Math majors, and $p(x)$ denotes 'x is male' then the above statement can be written as:

For every $x \in M$, *x is male* ...(ii)

or *For every* $x \in M$, $p(x)$ ...(iii)

The negation of statement (i) is

There exists at least one Math major who is female (not male).

or *There exists* $x \in M$ *such that x is not male*

or *There exists* $x \in M, \sim p(x)$.

Thus, we have

$\sim$ (For every $x \in M, p(x)$) $\cong$ (There exists $x \in M, \sim p(x)$)

This is true for any M and any $p(x)$.

Similarly, we have

$\sim$ (There exists $x \in M, p(x)$) $\cong$ (For every $x \in M, \sim p(x)$)

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Identify the quantifier in each of the following statements:*

(i) *For every real number x, $x + 4$ is greater than x.*

(ii) *There exists a real number which is twice of itself.*

(iii) *There exists a (living) person who is 200 years old.*

(iv) *For every $x \in N$, $x + 1 > x$.*

SOLUTION (i) For every (ii) There exists (iii) There exists (iv) For every.

EXAMPLE 2 *Write the negation of the following statements:*

(i) *For all positive integers x, we have $x + 2 > 8$.*

(ii) *Every living person is not 150 years old.*

(iii) *All students live in the dormitories.*

(iv) *Some students are 25 (years) or older.*

SOLUTION (i) There existsa positive integer $x$such that $x + 2 \not> 8$.

or There exists a positive integer x such that $x + 2 \leq 8$.

(ii) There exists a (living) person who is 150 years old.

(iii) Some students do not live in the dormitories.

or At least one student does not live in the dormitories.

or There exists a student who does not live in the dormitories.

(iv) None of the students is 25 or older

or All the students are under 25.

EXAMPLE 3 *Write the negation of each of the following statements:*

(i) *For every real number x, $x + 0 = x = 0 + x$.*

(ii) *For every real number x, x is less than $x + 1$*

(iii) *There exists a capital for every state in India.*

(iv) *There exists a number which is equal to its square.*

SOLUTION (i) There existsa real number $x$such that $x + 0 \neq x = 0 + x$

(ii) There exists a real number x such that x is not less than $x + 1$.

(iii) There exists a state in India which does not have its capital.

(iv) For every real number x, $x^2 \neq x$.

EXERCISE 31.4

LEVEL-1

1. Write the negation of each of the following statements:

   (i) For every $x \in N$, $x + 3 < 10$ (ii) There exists $x \in N$, $x + 3 = 10$

2. Negate each of the following statements:

   (i) All the students completed their homework.

   (ii) There exists a number which is equal to its square.

ANSWERS

1. (i) There exists $x \in N$ such that $x + 3 \geq 10$.

   (ii) For every $x \in N$, $x + 3 \neq 10$

2. (i) Some of the students did not complete their home work

   or There exists a student who did not complete his home work.

   (ii) For every real number x, $x^2 \neq x$.

31.7 IMPLICATIONS

In Mathematics we come across many statements of the form "if p then q", "p only if q" and "if and only if" such statements are called implications. In this section, we shall discuss about such statements.

IF-THEN IMPLICATION

Two statements connected by the connective phrase "if - then" give rise to a compound statement which is known as if-then implication.

For example,

If it rains, then the atmospheric humidity increases.

If $x = 4$, *then* $x^2 = 16$.

If ABCD is a parallelogram, then AB = CD

are implications.

If p and q are two statements forming the implication "if p then q", then we denote this implication by "$p \Rightarrow q$".

In the implication "$p \Rightarrow q$" p is called the antecedent and q the consequent.

We shall now see how the truth and falsity of an implication "$p \Rightarrow q$" depends upon the truth and falsity of its antecedent p and consequent q.

(i) *If both p and q are true, then* $p \Rightarrow q$ *is also true.*

Verification Let p denote the statement : "The number $N = 43221$ is divisible by 3"

and q denote the statement "The sum of the digits forming N is divisible by 3".

Clearly, p and q both are true.

Now, $p \Rightarrow q$: *If the number N is divisible by 3, then the sum of the digits forming N is divisible by 3.*

Clearly, $p \Rightarrow q$ is also true.

Thus, if p and q are true, then $p \Rightarrow q$ is also true.

(ii) *If p is true and q is false, then* $p \Rightarrow q$ *is false.*

Verification Consider the following statements:

p : *The number* $N = 43221$ *is divisible by* 3.

q : *The sum of the digits forming N is not divisible by* 3.

Clearly, p is true and q is false.

Now,

$p \Rightarrow q$: *If the number N is divisible by* 3, *then the sum of the digits forming N is not divisible by* 3.

Clearly, $p \Rightarrow q$ is false.

Thus, if p is true and q is false, then $p \Rightarrow q$ is false.

(iii) *If p is false and q is true, then* $p \Rightarrow q$ *is true.*

If p is false and q is true, then $p \Rightarrow q$ is assumed to be true. This assumption is made to be consistent with the other assumptions.

(iv) *If both p and q are false, then* $p \Rightarrow q$ *is true.*

Verification Consider the following statements:

p : *The number* $N = 43211$ *is divisible by* 3

q : *The sum of the digits forming N is divisible by* 3.

Clearly, p and q are false statements.

Now,

$p \Rightarrow q$: *If the number* $N = 43211$ *is divisible by* 3, *then the sum of the digits forming N is divisible by* 3.

Clearly, $p \Rightarrow q$ has truth value T.

Thus, if both p and q are false, then $p \Rightarrow q$ is true.

The above discussion suggests us the following rule for the implication "if-then".

RULE *The implication "if p, then q" is always true except when the antecedent p is true and the consequent q is false.*

The implication $p \Rightarrow q$ is same as each of the following:

(i) p is sufficient condition for q.

(ii) p only if q.

(iii) q is necessary condition for p.

(iv) $\sim q \Rightarrow \sim p$.

Consider the statement:

If a number is a multiple of 9, then it is a multiple of 3.

Clearly, it is an implication having antecedent (p) and consequent (q) as follows:

p : *a number is a multiple of 9.*

q : *a number is a multiple of 3.*

The above statement says that knowing that a number is a multiple of 9 is sufficient to conclude that it is a multiple of 3 i.e. $p \Rightarrow q$ is same as p is sufficient codition for q.

Also, the given statement says that a number is a multiple of 9 only if it is a multiple of 3 i.e. $p \Rightarrow q$ is same as p only if q.

The above statement also means that when a number is a multiple of 9, it is necessarily a multiple of 3 i.e. $p \Rightarrow q$ is same as q is necessary condition for p.

The above statement also says that if a number is not a multiple of 3, then it is not a multiple of 9 i.e. $p \Rightarrow q$ is same as $\sim q \Rightarrow \sim p$".

CONTRA POSITIVE *If p and q are two statements, then the contrapositive of the implication "if p then q" is "if ~ q, then ~ p."*

CONVERSE *If p and q are two statements, then the converse of the implication "if p, then q" is "if q, then p".*

INVERSE *If p and q are two statements, then the inverse of "If p, then q" is "If ~ p, then ~ q".*

Consider the statement:

If a number is divisible by 9, then it is divisible by 3

It is an implication with "if ... then" having antecedent (p) and consequent (q) as given below:

p : *a number is divisible by 9.*

q : *a number is divisible by 3.*

The given statement is "if p, then q." Its contrapositive is: If $\sim q$, then $\sim p$

i.e., *If a number is not divisible by 3, then it is not divisible by 9.*

The converse of the given statement is: If q, then p

i.e., *If a number is divisible by 3, then it is divisible by 9.*

IF AND ONLY IF IMPLICATION

If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.

Consider the statement:

A triangle is equilateral if and only if it is equiangular.

This is if and only if implication with the component statements:

p : *A triangle is equilateral.*

q : *A triangle is equiangular.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Rewrite the following statement with "if–then" in five different ways conveying the same meaning:*

If a natural number is odd, then its square is also odd.

SOLUTION The component statement of the given statement are:

p : *A natural number is odd.*

q : *Square of a natural number is odd.*

The given statement is: "If p, then q."

It is same as each of the following statements:

(i) $p \Rightarrow q$ i.e., x is an odd natural number $\Rightarrow x^2$ is an odd natural number

(ii) p is a sufficient condition for q.
i.e. Knowing that a natural number is odd is sufficient to conclude that its square is odd.

(iii) p only if q i.e., A natural number is odd only if its square is odd.

(iv) q is necessary condition for p
i.e. When a natural number is odd, its square is necessarily odd.

(v) $\sim q \Rightarrow \sim p$
i.e. If the square of a natural number is not odd, then the natural number is not odd.

EXAMPLE 2 *Write each of the following statements in the form "if–then".*

(i) *You get job implies that your credentials are good.*

(ii) *You can access the website only if you pay a subscription fee.*

(iii) *The Banana trees will bloom if it stays warm for a month.*

(iv) *A quadrilateral is a parallelogram if its diagonals bisect each other.*

(v) *To get A^+ in the class, it is necessary that you do all the exercises of the book.*

SOLUTION (i) We know that "If p, then q" is equivalent to "$p \Rightarrow q$".
Therefore, the given statement can be written as
"If you get a job, then your credentials are good"

(ii) We know that "p only if q" is equivalent to "If p, then q".
Therefore, the given statement can be written as:
"If you can access the website, then you pay a subscription fee".

(iii) The given statement can be written as:
"If it stays warm for a month, then the Banana trees will bloom".

(iv) The given statement can be written as:
"If the diagonals of aquadrilateral bisect each other, then it is a parallelogram".

(v) The given statement can be written as:
"If you get A^+ in the class, then you do all the exercise of the book."

EXAMPLE 3 *Write the contrapositive of the following statements:*

(i) *If a number is divisible by 9, then it is divisible by 3.*

(ii) *If you are born in India, then you are a citizen of India.*

(iii) *If a triangle is equilateral, it is isosceles.*

(iv) *If x is prime number, then x is odd.*

(v) *If two lines are parallel, then they do not intersect in the same plane.*

(vi) *x is an even number implies that x is divisible by 4.*

(vii) *Something is cold implies that it has low temperature.*

(viii) *You cannot comprehend geometry if you do not know how to reason deductively.*

SOLUTION We know that the contrapositive of the statement "If p, then q" is "if ~ q, then ~ p". Therefore contrapositive of the given statements are:

(i) If a number is not divisible by 3, it is not divisible by 9.
(ii) If you are not a citizen of India, then you were not born in India.
(iii) If a triangle is not isosceles, then it is not equilateral.
(iv) If a number x is not odd, then x is not prime.
(v) If two lines do not intersect in the same plane, then they are not parallel.
(vi) If x is not divisible by 4, then x is not an even number.
(vii) If something does not have low temperature, then it is not cold.
(viii) If you can comprehend geometry, then you know how to reason deductively.

EXAMPLE 4 *Write the converse of the following statements:*

(i) *If a number n is even, then n^2 is even.*
(ii) *If you do all the exercises in the book, you get an A grade in the class.*
(iii) *If two integers a and b are such that $a > b$, then $a - b$ is always a positive integer.*
(iv) *If x is prime number, then x is odd.*
(v) *If two lines are parallel, then they do not intersect in the same plane.*

SOLUTION (i) If a number n^2 is even, then is even.
(ii) If you get an A grade in the class, then you have done all the exercises of the book.
(iii) If two integers a and b are such that $a - b$ is always a positive integer, then $a > b$.
(iv) If x is an odd number, then x is a prime number.
(v) If two lines do not intersect in the same plane, then they are parallel.

EXAMPLE 5 *Write the component statements of each of the following statements. Also, check whether the statements are true or not.*

(i) *If a triangle ABC is equilateral, then it is isosceles.*
(ii) *If a and b are integers, then ab is a rational number.*

SOLUTION (i) The component statements of the given statement are:

p : *The triangle ABC is equilateral.*
q : *The triangle ABC is isosceles.*

Since an equilateral triangle is isosceles, so the given statement is true.

(ii) The component statements are:

p : *a and b are intergers.*
q : *ab is a rational number.*

Since the product of two integers is an integer and therefore a rational number. So, the compound statement is true.

EXAMPLE 6 *Given below are two pairs of statements. Combine these two statements using "if and only if".*

(i) p : *If a rectangle is a square, then all its four sides are equal.*
q : *If all the four sides of a rectangle are equal, then the rectangle is a square.*
(ii) p : *If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.*
q : *If a number is divisible by 3, then the sum of its divisible by 3.*

SOLUTION (i) A rectangle is a square if and only if all its four sides are equal.
(ii) A number is divisible by 3 if and only if the sum of its digits is divisible by 3.

EXAMPLE 7 *For the given statement identify the necessary and sufficient conditions:*

p : *"If you drive over 80 km per hour, then you will get a fine".*

SOLUTION Let p and q denote the statements:

p : *You drive over 80 km per hour.*

q: *You will get a fine.*

We know that the implications of "if p, then q" indicates that p is sufficient for q. It also indicates that q is necessary for p. Therefore, sufficient condition is "Driving over 80 km per hour" and the necessary conditions is "getting a fine".

EXERCISE 31.5

LEVEL-1

1. Write each of the following statements in the form "if p, then q".
   (i) You can access the website only if you pay a subscription fee.
   (ii) There is traffic jam whenever it rains.
   (iii) It is necessary to have a passport to log on to the server.
   (iv) It is necessary to be rich in order to be happy.
   (v) The game is cancelled only if it is raining.
   (vi) It rains only if it is cold.
   (vii) Whenever it rains it is cold.
   (viii) It never rains when it is cold.
2. State the converse and contrapositive of each of the following statements:
   (i) If it is hot outside, then you feel thirsty.
   (ii) I go to a beach whenever it is a sunny day.
   (iii) A positive integer is prime only if it has no divisors other than 1 and itself.
   (iv) If you live in Delhi, then you have winter clothes.
   (v) If a quadrilateral is a parallelogram, then its diagonals bisect each other.
3. Rewrite each of the following statements in the form "p if and only if q".
   (i) p: If you watch television, then your mind is free and if your mind is free, then you watch television.
   (ii) q: If a quadrilateral is equiangular, then it is a rectangle and if a quadrilateral is a you rectangle, then it is equiangular.
   (iii) r: For you to get an A grade, it is necessary and sufficient that you do all the homework you regularly.
   (iv) s: If a tumbler is half empty, then it is half full and if a tumbler is half full, then it is half empty.
4. Determine the contrapositive of each of the following statements:
   (i) If Mohan is a poet, then he is poor. (ii) Only if Max studies will he pass the test.
   (iii) If she works, she will earn money. (iv) If it snows, then they do not drive the car.
   (v) It never rains when it is cold. (vi) If Ravish skis, then it snowed.
   (vii) If x is less than zero, then x is not positive.
   (viii) If he has courage he will win.
   (ix) It is necessary to be strong in order to be a sailor.
   (x) Only if he does not tire will he win. (xi) If x is an integer and x^2 is odd, then x is odd.

ANSWERS

1. (i) If you access the website, then you pay a subscription fee.
   (ii) If it rains, then there is traffic jam.
   (iii) If you log on to the server, then you must have a passport.
   (iv) If he is happy, then he is rich.
   (v) If it is raining, then the game is cancelled.
   (vi) If it rains, then it is cold.
   (vii) If it rains, then it is cold.

(viii) If it is cold, then it never rains.

2. (i) Converse: If you feel thirsty, then it is hot outside.
   Contrapositive: If you do not feel thirsty, then it is not hot outside.
   (ii) Converse: If I go to a beach, then it is a sunny day.
   Contrapositive: If I do not go to a beach, then it is not a sunny day.
   (iii) Converse: If an integer has no divisors other than 1 and itself, then it is prime.
   Contrapositive: If an integer has some divisors other than 1 and itself, then it is not prime.
   (iv) Converse: If you have winter clothes, then you live in Delhi.
   Contrapositive: If you do not have winter clothes, then you do not live in Delhi.
   (v) Converse: If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
   Contrapositive: If the diagonals of a quadrilateral do not bisect each other, then the quadrilateral is not a parallelogram.

3. (i) You watch television if and only if your mind is free.
   (ii) A quadrilateral is a rectangle if and only if it is equiangular.
   (iii) You get an A grade if and only if you do all the homework regularly.
   (iv) A tumbler is half empty if and only if it is half full.

4. (i) If Mohan is not poor, then he is not a poet.
   (ii) If Max does not study, then he will not pass the test.
   (iii) If she does not earn money, then she does not work.
   (iv) If they do not drive the car, then there is no snow.
   (v) If it rains, then it is not cold.
   (vi) If it did not snow, then Ravish will not ski.
   (vii) If x is positive, then x is not less than zero.
   (viii) If he does not win, then he does not have courage.
   (ix) If he is not strong, then he is not a sailor.
   (x) If he tires, then he will not win.
   (xi) If x is even, then x^2 is even.

31.8 VALIDITY OF STATEMENTS

In this section, we will study validity of statements. Checking the validity of a statement means checking when it is true and when it is not true. This depends upon which of the special words "and", "or", and which of the implications "if-then", "if and only if", and which of the quantifiers "for every", "there exists", appear in the given statement.

Let us now discuss some techniques or rules to find when a statement is valid or true.

31.8.1 VALIDITY OF STATEMENTS WITH "AND"

If p and q are mathematical statements, then in order to show that the statement "p and q" is true, we follow the following steps:

STEP I *Show that the statement p is true.*

STEP II *Show that the statement q is true.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Given below are two statements:*

p : *80 is a multiple of* 5.

q : *80 is a multiple of* 4.

Write the compound statement connecting these two statements with "and" and checks its validity.

SOLUTION The compound statement is: "80 *is multiple of* 5 *and* 4."

We know that 80 is a multiple of 5 as well as 4. So, p and q are true statements.

Hence, the compound statement is also true i.e. the compound statement "p and q" is a valid statement.

EXAMPLE 2 *If p and q are two statements given by:*

p : 25 *is multiple of* 5.

q : 25 *is a multiple of* 8.

Write the compound statement connecting these two statements with "and" and check its validity.

SOLUTION The compound statement is : "25 *is a multiple of* 5 *and* 8"

Since 25 is a multiple of 5 but it is not a multiple of 8. Therefore, p is true but q is not true.

Hence, the compound statements is not true i.e., the statement "p and q" is not a valid statement.

31.8.2 VALIDITY OF STATEMENTS WITH "OR"

If p and q are mathematical statements, then in order to show that the compound statement "p or q" is true, we proceed as follows:

Assuming that p is false, show that q must be true.

or Assuming that q is false, show that p must be true.

Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Given below are two statements:*

p : 25 *is a multiple of* 5.

q : 25 *is a multiple of* 8.

Write the compound statement connecting these two statements with "OR" and check its validity.

SOLUTION The compound statement is: "25 *is a multiple of* 5 *or* 8".

Let us assume that the statement q is false i.e. 25 is not a multiple of 8. Clearly, p is true.

Thus, if we assume that q is false, then p is true. Hence, the compound statement is true i.e. valid.

EXAMPLE 2 *Check the validity of the following statement:*

"Square of an integer is positive or negative "

SOLUTION The given statement is a compound statement with "OR" whose component statements are:

p : *Square of an integer is positive.*

q : *Square of an integer is negative.*

Let us assume that p is false i.e. square of an integer is not positive. Then, for any integer x, we have

$$x^2 \ngtr 0 \Rightarrow x^2 < 0 \Rightarrow q \text{ is true.}$$

Thus, if we assume that p is false, then q is true.

Hence, "p or q" is a valid statement. In other words, the given statement is true.

31.8.3 VALIDITY OF STATEMENTS WITH "IF-THEN"

If p and q are two mathematical statements, then to prove the validity of the statement "if p, then q", we may use any one of the following methods.

(i) DIRECT METHOD

<u>STEP I</u> *Assume that p is true.*

<u>STEP II</u> *Prove that q is true.*

(ii) CONTRAPOSITIVE METHOD

STEP I *Assume that q is false.*

STEP II *Prove that p is false.*

(iii) CONTRADICTION METHOD

STEP I *Assume that p is true and q is false.*

STEP II *Obtain a contradication from step I.*

Following examples will illustrate the above methods.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Check whether the following statement is true or not:*
"If x and y are odd integers, then xy is an odd integer"

SOLUTION Let p and q be the statements given by

p : *x and y are odd integers.*

q : *xy is an odd integer.*

Then, the given statement is: If p, then q.

Direct Method: Let p be true. Then,

p is true

$\Rightarrow$ x and y are odd integers

$\Rightarrow$ $x = 2m+1,\ y = 2n+1$ for some integers m, n.

$\Rightarrow$ $xy = (2m+1)(2n+1)$

$\Rightarrow$ $xy = 2(2mn+m+n)+1$

$\Rightarrow$ xy is an odd integer

$\Rightarrow$ q is true.

Thus, p is true $\Rightarrow$ q is true.

Hence, "If p, then q" is a true statement.

Contrapositive Method: Let q be not true. Then,

q is not true

$\Rightarrow$ xy is an even integer

$\Rightarrow$ either x is even or y is even or both x and y are even

$\Rightarrow$ p is not true.

Thus, q is false $\Rightarrow$ p is false

Hence, "If p, then q" is a true statement.

EXAMPLE 2 *Check whether the following statement is true or false by proving its contrapositive.*
"If x, y are integers such that xy is odd, then both x and y are odd integers"

SOLUTION Let p and q be two statements given by

p : *xy is an odd integer*

q : *Both x and y are odd integers.*

Let q be not true. Then,

q is not true

$\Rightarrow$ It is false that both x and y are odd integers

$\Rightarrow$ At least one of x and y is an even integer.

Let $x = 2n$ for some integer n. Then,

$xy = 2ny$ for some integer n.

$\Rightarrow$ xy is an even integer

$\Rightarrow$ xy is not an odd integer

$\Rightarrow$ $\sim p$ is true

Thus, $\sim q$ is true $\Rightarrow \sim p$ is true

Hence, the given statement is true.

EXAMPLE 3 *Show that the statement*

p : If x is a real number such that $x^3 + 4x = 0$, then x is 0" is true by

(i) *direct method* (ii) *method of contradiction* (iii) *method of contrapositive*

SOLUTION Let q and r be the statements given by

$q : x$ is a real number such that $x^3 + 4x = 0$.

$r : x$ is 0.

Then, p : If q, then r.

(i) *Direct method*: Let q be true. Then,

q is true

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$

$\Rightarrow$ x is a real number such that $x(x^2 + 4) = 0$

$\Rightarrow$ $x = 0$ $[\because x \in R \therefore x^2 + 4 \neq 0]$

$\Rightarrow$ r is true.

Thus, q is true $\Rightarrow r$ is true.

Hence, p is true.

(ii) *Method of contradiction*: If possible, let p be not true. Then,

p is not true

$\Rightarrow$ $\sim p$ is true

$\Rightarrow$ $\sim (q \Rightarrow r)$ is true $[\because p : q \Rightarrow r]$

$\Rightarrow$ q and $\sim r$ is true $[\because \sim (q \Rightarrow r) \cong q \text{ and } \sim r]$

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$ and $x \neq 0$

$\Rightarrow$ $x = 0$ and $x \neq 0$

This a contradiction.

Hence, p is true.

(iii) *Method of contrapositive:* Let r be not true. Then,

r is not true

$\Rightarrow$ $x \neq 0, x \in R$

$\Rightarrow$ $x(x^2 + 4) \neq 0, x \in R$

$\Rightarrow$ q is not true

Thus, $\sim r \Rightarrow \sim q$.

Hence, $p : q \Rightarrow r$ is true.

EXAMPLE 4 *Show that the following statement is true by the method of contrapositive.*

p : If x is an integer and x^2 is even, then x is also even.

SOLUTION Let q and r be the statements given by

q : If x is an integer and x^2 is even

r : x is an even integer.

Then, p : "If q, then r".

If possible, let r be false. Then,

r is false

$\Rightarrow$ x is not an even integer

$\Rightarrow$ x is an odd integer

$\Rightarrow$ $x = (2n+1)$ for some integer n

$\Rightarrow$ $x^2 = 4n^2 + 4n + 1$

$\Rightarrow$ $x^2 = 4n(n+1) + 1$

$\Rightarrow$ x^2 is an odd integer [$\because$ $4n(n+1)$ is even]

$\Rightarrow$ q is false.

Thus, r is false $\Rightarrow$ q is false.

Hence, p: "If q, then r" is a true statement.

31.8.4 VALIDITY OF STATEMENTS WITH "IF AND ONLY IF"

In order to prove the validity of the statement "p if and only if q", we proceed as follows:

STEP I Show that: If p is true, then q is true.

STEP II Show that: If q is true, then p is true.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Using the words "necessary and sufficient" rewrite the statement*

"The integer n is odd if and only if n^2 is odd"

Also check whether the statement is true.

SOLUTION The given statement can be re-written as

"The necessary and sufficient condition that the integer n is odd is n^2 must be odd"

Let p and q be the statements given by

p: *the integer n is odd.*

q: *n^2 is odd.*

The given statement is

"p if and only if q"

In order to check its validity, we have to check the validity of the following statements.

(i) *"If p, then q"* (ii) *"If q, then p"*

Checking the validity of "If p, then q":

The statement "if p, then q" is given by:

"If the integer n is odd, then n^2 is odd"

Let us assume that n is odd. Then,

$n = 2m + 1$, where m is an integer

$\Rightarrow$ $n^2 = (2m+1)^2$

$\Rightarrow$ $n^2 = 4m(m+1) + 1$

$\Rightarrow$ n^2 is an odd integer [$\because$ $4m(m+1)$ is even]

$\Rightarrow$ n^2 is odd.

Thus, n is odd $\Rightarrow$ n^2 is odd

$\therefore$ *"If p, then q" is true.*

Checking the validity of "If q, then p":

The statement "If q, then p" is given by

"*If n is an integer and n^2 is odd, then n is odd*"

To check the validity of this statements, we will use contrapositive method. So, let n be an even integer. Then,

n is even

$\Rightarrow$ $n = 2k$ for some integer k

$\Rightarrow$ $n^2 = 4k^2$

$\Rightarrow$ n^2 is an even integer

$\Rightarrow$ n^2 is not an odd integer.

Thus, n is not odd $\Rightarrow$ n^2 is not odd

$\therefore$ "If q, then p" is true.

Hence, "p if and only if q" is true.

31.8.5 VALIDITY OF STATEMENTS BY CONTRADICTION

Sometimes to check whether a statement p is true or not, we assume that p is not true i.e. $\sim p$ is true. Then, we arrive at some result which contradicts our supposition. Therefore, we conclude that p is true. This method is known as contradiction method.

Following examples will illustrate this method.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Verify by the method of contradiction that $\sqrt{7}$ is irrational.*

SOLUTION Let p be the statement given by $p : \sqrt{7}$ *is irrational.*

If possible, let p be not true i.e. let p be false. Then,

p is false

$\Rightarrow$ $\sqrt{7}$ is rational

$\Rightarrow$ $\sqrt{7} = \frac{a}{b}$, where a and b are integers having no common factor.

$\Rightarrow$ $7 = \frac{a^2}{b^2}$

$\Rightarrow$ $a^2 = 7b^2$

$\Rightarrow$ 7 divides a^2

$\Rightarrow$ 7 divides a

$\Rightarrow$ $a = 7c$ for some integer c

$\Rightarrow$ $a^2 = 49c^2$

$\Rightarrow$ $7b^2 = 49c^2$ $\quad [\because a^2 = 7b^2]$

$\Rightarrow$ $b^2 = 7c^2$

$\Rightarrow$ 7 divides b^2

$\Rightarrow$ 7 divides b

Thus, 7 is a common factor of both a and b. This contradicts that a and b have no common factor. So, the supposition $\sqrt{7}$ is rational is wrong. Hence, the statement "$\sqrt{7}$ is irrational" is true.

EXAMPLE 2 *Check the validity of the statement given below by contradiction method.*

"*p : The sum of an irrational number and a rational number is irrational*"

SOLUTION If possible, let p be not true. Then,

p is false

$\Rightarrow$ The sum of an irrational number and a rational number is not irrational

$\Rightarrow$ There exists an irrational number x (say) and a rational number y (say) such that $x+y$ is not irrational.

$\Rightarrow$ $x+y$ is rational say, z

$\Rightarrow$ $x+y=z$

$\Rightarrow$ $x=z-y$

$\Rightarrow$ x is rational [$\because$ $z-y$ is rational]

But, x is irrational. So, we arrive at a contradiction.

Thus, our supposition is wrong.

Hence, p is true.

31.8.6 INVALIDITY OF STATEMENTS BY COUNTER EXAMPLES

In order to show that a statement is false, we may give an example of a situation where the statement is not valid. Such an example is called a counter example. The name itself suggests that this is an example to counter the statement. Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *By giving an example, show that the following statement is false.*

"If n is an odd integer, then n is prime"

SOLUTION We observe that 9 is an odd integer which is not prime. Similarly, 21, 25 etc are odd integers which are not primes.

Hence, the given statement is false.

EXAMPLE 2 *Show that the statement*

"For any real numbers a and b, $a^2=b^2$ implies that $a=b$"

is not true by giving a counter example.

SOLUTION We observe that $(-2)^2 = 2^2$ but $-2 \neq 2$.

So, the given statement is not true.

EXAMPLE 3 *By giving a counter example, show that the following statement is not true:*

p: "The equation $x^2-1=0$ does not have a root lying between 0 and 2"

SOLUTION We observe that $x = 1$ is a root of $x^2-1=0$ and $x=1$ lies between 0 and 2.

So, the given statement is not true.

EXERCISE 31.6

LEVEL-1

1. Check the validity of the following statements:
   (i) p: 100 is a multiple of 4 and 5. (ii) q: 125 is a multiple of 5 and 7.
   (iii) r: 60 is a multiple of 3 or 5.
2. Check whether the following statement are true or not:
   (i) p: If x and y are odd integers, then $x+y$ is an even integer.
   (ii) q: If x, y are integers such that xy is even, then at least one of x and y is an even integer.

3. Show that the statement

   p: *"If x is a real number such that $x^3 + x = 0$, then x is 0"*

   is true by

   (i) direct method (ii) method of contrapositive (iii) method of contradition.

4. Show that the following statement is true by the method of contrapositive

   p: *"If x is an integer and x^2 is odd, then x is also odd"*

5. Show that the following statement is true

   "The integer n is even if and only if n^2 is even"

6. By giving a counter example, show that the following statement is not true.

   p: *"If all the angles of a triangle are equal, then the triangle is an obtuse angled triangle".*

7. Which of the following statements are true and which are false? In each case give a valid reason for saying so

   (i) p: Each radius of a circle is a chord of the circle.

   (ii) q: The centre of a circle bisects each chord of the circle.

   (iii) r: Circle is a particular case of an ellipse.

   (iv) s: If x and y are integers such that $x > y$, then $-x < -y$.

   (v) t: $\sqrt{11}$ is a rational number.

8. Determine whether the argument used to check the validity of the following statement is correct:

   p: "If x^2 is irrational, then x is rational"

   The statement is true because the number $x^2 = \pi^2$ is irrational, therefore $x = \pi$ is irrational.

ANSWERS

1. (i) True	(ii) False	(iii) True	2. (i) True	(ii) True
7. (i) False	(ii) False	(iii) True	(iv) True	(v) False

SUMMARY

1. A sentence is called a mathematically acceptable statement or simply a statement if it is either true or false but not both.
2. The denial of a statement is called negation of the statement. The negation of a statement p is denoted by $\sim p$ and is read as "not p".
3. A statement is called a compound statement if it is made up of two or more simple statements. The simple statements are called component statements of the compound statement.
4. Compound statements are obtained by using connecting words like "and", "or" etc and phrases "If-then", "Only if", "if and only if", "There exists", "For all" etc.
5. The compound statement with "AND" is

   (i) true if all its component statements are true.

   (ii) false if any of its component statements is false.

6. The compound statement with "OR" is

   (i) true when one component statement is true or both the component statements are true.

   (ii) false when both the component statements are false.

7. A sentence with "If p, then q" can be written in the following ways:
   (i) p implies q (denoted by $p \Rightarrow q$)
   (ii) p is sufficient condition for q
   (iii) q is necessary condition for p
   (iv) p only if q
   (v) $\sim q$ implies $\sim p$

8. (i) The contrapositive of the statement $p \Rightarrow q$ is the statement $\sim q \Rightarrow \sim p$.
   (ii) The converse of the statement $p \Rightarrow q$ is the statement $q \Rightarrow p$.
   (iii) The inverse of the statement $p \Rightarrow q$ is the statement $\sim p \Rightarrow \sim q$.

9. For all or for every is called universal quantifier. There exists is called existential quantifier.

10. A statement is said to valid or invalid according as it is true or false.

11. If p and q are two mathematical statements, then the statement
   (i) "p and q" is true if both p and q are true.
   (ii) "p or q" is true if p is false $\Rightarrow q$ is true or, q is false $\Rightarrow p$ is true.
   (iii) "If p, then q" is true if
   (a) p is true $\Rightarrow q$ is true
   or (b) q is false $\Rightarrow p$ is false
   or (c) p is true and q is false leads us to a contradiction.
   (iv) "p if and only if q" is true, if
   (a) p is true $\Rightarrow q$ is true
   and (b) q is true $\Rightarrow p$ is true.