CHAPTER 21

DIFFERENTIAL EQUATIONS

21.1 SOME DEFINITIONS

DIFFERENTIAL EQUATION *An equation containing an independent variable, dependent variable and differential coefficients of dependent variable with respect to independent variable is called a differential equation.*

For instance,

(i) $\dfrac{dy}{dx} = 2xy$ (ii) $\dfrac{d^2y}{dx^2} = 4x$ (iii) $\dfrac{dy}{dx} = \sin x + \cos x$

(iv) $\dfrac{dy}{dx} + 2xy = x^3$ (v) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x^2$ (vi) $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2} = k\dfrac{d^2y}{dx^2}$

(vii) $y = x\dfrac{dy}{dx} + \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ (viii) $(x^2 + y^2)\, dx - 2xy\, dy = 0$ (ix) $\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(1 + \dfrac{dy}{dx}\right)^3 = 0$

are examples of differential equations.

ORDER OF A DIFFERENTIAL EQUATION *The order of a differential equation is the order of the highest order derivative appearing in the equation.*

ILLUSTRATION 1 In the equation $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^x$, the order of highest order derivative is 2. So, it is a differential equation of order 2. The equation $\dfrac{d^3y}{dx^3} - 6\left(\dfrac{dy}{dx}\right)^2 - 4y = 0$ is of order 3, because the order of highest order derivative in it is 3.

NOTE *The order of a differential equation is a positive integer.*

DEGREE OF A DIFFERENTIAL EQUATION *The degree of a differential equation is the degree of the highest order derivative, when differential coefficients are made free from radicals and fractions.*

In other words, the degree of a differential equation is the power of the highest order derivative occurring in a differential equation when it is written as a polynomial in differential coefficients.

ILLUSTRATION 2 Consider the differential equation $\dfrac{d^3y}{dx^3} - 6\left(\dfrac{dy}{dx}\right)^2 - 4y = 0.$

In this equation the power of highest order derivative is 1. So, it is a differential equation of degree 1.

ILLUSTRATION 3 Consider the differential equation $x\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(\dfrac{dy}{dx}\right)^4 + y^2 = 0$

In this equation, the order of the highest order derivative is 3 and its power is 2. So, it is a differential equation of order 3 and degree 2.

ILLUSTRATION 4 The differential equation $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ when expressed as a polynomial in derivatives becomes $(x^2-1)\left(\frac{dy}{dx}\right)^2 - 2xy\frac{dy}{dx} + (y^2-1) = 0$. In this equation, the power of highest order derivative is 2. So, its degree is 2.

ILLUSTRATION 5 Consider the differential equation $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2} = k\left(\frac{d^2y}{dx^2}\right)$

The order of highest order differential coefficient is 2. So, its order is 2. To find its degree we express the differential equation as a polynomial in derivatives. When expressed as a polynomial in derivatives it becomes $k^2\left(\frac{d^2y}{dx^2}\right)^2 - \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 = 0$. Clearly, the power of the highest order differential coefficient is 2. So, its degree is 2.

ILLUSTRATION 6 The differential equation $(x^2+y^2)\,dx - 2xy\,dy = 0$ may be written as $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$. So, it is a differential equation of order 1 and degree 1.

ILLUSTRATION 7 Consider the differential equation $y = px + \sqrt{a^2p^2+b^2}$, where $p = \frac{dy}{dx}$. The order of the highest order derivative is 1. So, its order is 1. To determine its degree we express it as a polynomial in differential coefficients as follows:

$$y = px + \sqrt{a^2p^2+b^2}$$

$$\Rightarrow \quad (y-px)^2 = a^2p^2+b^2$$

$$\Rightarrow \quad p^2(x^2-a^2) - 2xyp + y^2 - b^2 = 0 \Rightarrow (x^2-a^2)\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + y^2 - b^2 = 0$$

Clearly, the power of highest order differential coefficient is 2. So, its degree is 2.

ILLUSTRATION 8 Consider the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \sin\left(\frac{dy}{dx}\right) = 0$. We observe that the highest order derivative present in the differential equation is $\frac{d^2y}{dx^2}$. So, its order is 2. Since the differential equation cannot be expressed as a polynomial in differential coefficients. So, its degree is not defined.

LINEAR AND NON-LINEAR DIFFERENTIAL EQUATIONS *A differential equation is a linear differential equation if it is expressible in the form*

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = Q$$

where $P_0, P_1, P_2, \ldots, P_{n-1}, P_n$ and Q are either constants or functions of independent variable x.

Thus, if a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these, and also the coefficient of the various terms are either constants or functions of the independent variable, then it is said to be linear differential equation. Otherwise, it is a non linear differential equation.

It follows from the above definition that a differential equation will be non-linear differential equation if

(i) its degree is more than one.
(ii) any of the differential coefficient has exponent more than one.
(iii) exponent of the dependent variable is more than one.
(iv) products containing dependent variable and its differential coefficients are present.

ILLUSTRATION 9 The differential equation $\left(\frac{d^3y}{dx^3}\right)^3 - 6\left(\frac{d^2y}{dx^2}\right)^2 - 4y = 0$, is a non-linear differential equation, because its degree is 3, more than one.

ILLUSTRATION 10 The differential equation $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 9y = x$, is a non-linear differential equation, because differential coefficient $\frac{dy}{dx}$ has exponent 2.

ILLUSTRATION 11 The differential equation $(x^2 + y^2)\, dx - 2xy\, dy = 0$ is a non-linear differential equation, because the exponent of dependent variable y is 2 and it involves the product of y and $\frac{dy}{dx}$.

ILLUSTRATION 12 Consider the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin x$. This is a linear differential equation of order 2 and degree 1.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Determine the order and degree of each of the following differential equations. State also if they are linear or non-linear.*

(i) $\dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = K$ (ii) $\frac{d^2y}{dx^2} = 1 + \sqrt{\frac{dy}{dx}}$ (iii) $y = \frac{dy}{dx} + \frac{c}{dy/dx}$ (iv) $y + \frac{dy}{dx} = \frac{1}{4}\int y\, dx$

SOLUTION (i) The given differential equation when written as a polynomial in derivatives becomes

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$$

The highest order differential coefficient in this equation is $\frac{d^2y}{dx^2}$ and its power is 2. Therefore, the given differential equation is a non-linear differential equation of second order and second degree.

(ii) The given differential equation when written as a polynomial in derivatives becomes

$$\left(\frac{d^2y}{dx^2}-1\right)^2=\frac{dy}{dx} \Rightarrow \left(\frac{d^2y}{dx^2}\right)^2-2\frac{d^2y}{dx^2}-\frac{dy}{dx}+1=0$$

Clearly, it is a non-linear differential equation of second order and second degree.

(iii) The given differential equation when written as a polynomial in $\frac{dy}{dx}$ is $\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}+c=0.$

Clearly, it is a non-linear differential equation of order 1 and degree 2.

(iv) We have,

$$y+\frac{dy}{dx}=\frac{1}{4}\int y\,dx \Rightarrow \frac{dy}{dx}+\frac{d^2y}{dx^2}=\frac{1}{4}y \qquad \text{[On differentiating with respect to } x \text{]}$$

Clearly, this is a differential equation of order 2 and degree 1. Also, it is a linear differential equation.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 2 *In each of the following differential equations indicate its degree, wherever possible. Also, give the order of each of them.*

(i) $\frac{dy}{dx}+\sin\left(\frac{dy}{dx}\right)=0$

(ii) $\frac{d^5y}{dx^5}+e^{dy/dx}+y^2=0$

(iii) $\frac{d^4y}{dx^4}+\sin\left(\frac{d^3y}{dx^3}\right)=0$ [NCERT]

(iv) $\left(\frac{d^2y}{dx^2}\right)^2+\cos\left(\frac{dy}{dx}\right)=0$

SOLUTION (i) The highest order derivative present in the differential equation is $\frac{dy}{dx}$. So, it is of order 1. Clearly, LHS of the differential equation cannot be expressed as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(ii) The highest order differential coefficient present in the differential equation is $\frac{d^5y}{dx^5}$. So, it is of order 5. We observe that the LHS of the differential equation is not expressible as a polynomial in $\frac{dy}{dx}$. So, its degree is not defined.

(iii) The highest order derivative present in the given differention equation is 4, so the order of the given differential equation is 4. As it is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

(iv) The order of the highest order derivative present in the given differential equation is 2. So, its order is 2. The given differential equation is not expressible as a polynomial in differential coefficients. So, its degree is not defined.

EXERCISE 21.1

BASIC

Determine the order and degree of each of the following differential equations. State also whether they are linear or non-linear (1-27):

1. $\frac{d^3x}{dt^3}+\frac{d^2x}{dt^2}+\left(\frac{dx}{dt}\right)^2=e^t$

2. $\frac{d^2y}{dx^2}+4y=0$

3. $\left(\frac{dy}{dx}\right)^2+\frac{1}{dy/dx}=2$

4. $\sqrt{1+\left(\frac{dy}{dx}\right)^2}=\left(c\frac{d^2y}{dx^2}\right)^{1/3}$

5. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+xy=0$

6. $\sqrt[3]{\frac{d^2y}{dx^2}}=\sqrt{\frac{dy}{dx}}$

7. $\dfrac{d^4y}{dx^4}=\left\{c+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}$

8. $x+\left(\dfrac{dy}{dx}\right)=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

9. $y\dfrac{d^2x}{dy^2}=y^2+1$

10. $s^2\dfrac{d^2t}{ds^2}+st\dfrac{dt}{ds}=s$

11. $x^2\left(\dfrac{d^2y}{dx^2}\right)^3+y\left(\dfrac{dy}{dx}\right)^4+y^4=0$

12. $\dfrac{d^3y}{dx^3}+\left(\dfrac{d^2y}{dx^2}\right)^3+\dfrac{dy}{dx}+4y=\sin x$

13. $(xy^2+x)\,dx+(y-x^2y)\,dy=0$

14. $\sqrt{1-y^2}\,dx+\sqrt{1-x^2}\,dy=0$

15. $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^{2/3}$

16. $2\dfrac{d^2y}{dx^2}+3\sqrt{1-\left(\dfrac{dy}{dx}\right)^2}-y=0$

17. $5\dfrac{d^2y}{dx^2}=\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{3/2}$

18. $y=x\dfrac{dy}{dx}+a\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

19. $y=px+\sqrt{a^2p^2+b^2}$, where $p=\dfrac{dy}{dx}$

20. $\dfrac{dy}{dx}+e^y=0$

21. $x^2\dfrac{d^2y}{dx^2}=\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^4$ [CBSE 2019]

BASED ON HOTS

22. $\left(\dfrac{d^2y}{dx^2}\right)^2+\left(\dfrac{dy}{dx}\right)^2=x\sin\left(\dfrac{d^2y}{dx^2}\right)$

23. $(y'')^2+(y')^3+\sin y=0$

24. $\dfrac{d^2y}{dx^2}+5x\left(\dfrac{dy}{dx}\right)-6y=\log x$ [NCERT]

25. $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+y\sin y=0$

26. $\dfrac{d^2y}{dx^2}+3\left(\dfrac{dy}{dx}\right)^2=x^2\log\left(\dfrac{d^2y}{dx^2}\right)$

27. $\left(\dfrac{dy}{dx}\right)^3-4\left(\dfrac{dy}{dx}\right)^2+7y=\sin x$ [NCERT]

28. Find the product of the order and the degree of the differential equation

$$\left\{\frac{d}{dx}(xy^2)\right\}\frac{dy}{dx}+y=0$$

[CBSE 2022]

29. Find the sum of the order and degree of the differential equation: $\left(x+\dfrac{dy}{dx}\right)^2=\left(\dfrac{dy}{dx}\right)^2+1$.

[CBSE 2022]

30. Find the value of $2a-3b$, if a and b represent respectively the order and the degree of the differential equation $x\left\{y\left(\dfrac{d^2y}{dx^2}\right)^3+x\left(\dfrac{dy}{dx}\right)^2-\dfrac{y}{x}\dfrac{dy}{dx}\right\}=0$ [CBSE 2022]

ANSWERS

	Order	Degree	Linear/Non-linear
1.	3	1	Non-linear
2.	2	1	Linear
3.	1	3	Non-linear
14.	1	1	Non-linear
15.	2	3	Non-linear
16.	2	2	Non-linear

4.	2	2	Non-linear
5.	2	1	Non-linear
6.	2	2	Non-linear
7.	4	2	Non-linear
8.	1	2	Linear
9.	2	1	Linear
10.	2	1	Non-linear
11.	2	3	Non-linear
12.	3	1	Non-linear
13.	1	1	Non-linear
27.	1	3	Non-linear
17.	2	2	Non-linear
18.	1	2	Non-linear
19.	1	2	Non-linear
20.	1	1	Non-linear
21.	2	1	Non-linear
22.	2	Undefined	Non-linear
23.	2	2	Non-linear
24.	2	1	Non-linear
25.	3	1	Linear
26.	2	Undefined	Non-linear

28. Order = 1, Degree = 2, Product = 2 29. Order = 1, Degree = 1, Sum = 2 30. – 5

HINTS TO SELECTED PROBLEMS

24. The differential equation $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x$ has highest order differential coefficient $\frac{d^2y}{dx^2}$ which is of order 2 and its exponent is 1. Hence, it is of order 2, degree 1 and it is non-linear.

27. The order of the highest order differential coefficient in the differential equation is one and its highest exponent is 3. So, the given differential equation is of order 1, degree 3 and it is non-linear.

21.2 FORMATION OF DIFFERENTIAL EQUATIONS

Consider the family of curves given by $y = A\,e^x$, where A is the parameter. For different values of A, we obtain different members of the family.

Differentiating $y = Ae^x$ with respect to x, we get $\frac{dy}{dx} = Ae^x$.

On eliminating the parameter A between $y = A\,e^x$ and $\frac{dy}{dx} = Ae^x$, we get $\frac{dy}{dx} = y$. This is the differential equation of the family of curves represented by the equation $y = Ae^x$.

Thus, by eliminating one arbitrary constant, a differential equation of first order is obtained. In other words, one parameter family of curves is represented by a first order differential equation.

Now, consider a two parameter family of curves given by

$$y = A\cos 2x + B\sin 2x \qquad \text{...(i)}$$

where A and B are arbitrary constants.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x \qquad \text{...(iii)}$$

Eliminating A and B from equations (i), (ii) and (iii), we get

$$\frac{d^2y}{dx^2} = -4y \Rightarrow \frac{d^2y}{dx^2} + 4y = 0.$$

Here, we note that by eliminating two arbitrary constants, a differential equation of second order is obtained. In other words, a two parameter family of curves is represented by a second order differential equation.

Similarly, one can see that by eliminating three arbitrary constants a differential equation of third order is obtained or, three parameter family of curves is represented by a third order differential equation.

Thus, from the examples cited above it can be concluded that if an equation involves n arbitrary constants, a differential equation of nth order can be obtained by eliminating these n arbitrary constants. In other words, an n-parameter family of curves is represented by an nth order differential equation.

Formulating a differential equation from a given equation representing a family of curves means finding a differential equation whose solution is the given equation. If an equation, representing a family of curves, contains n arbitrary constants, then we differentiate the given equation n times to obtain n more equations. Using all these equations, we eliminate the constants. The equation so obtained is the differential equation of order n for the family of given curves.

In order to formulate a differential equation from a given relation containing independent variable (x) dependent variable (y) and some arbitrary constants, we may follow the following algorithm:

ALGORITHM

Step I *Write the given equation involving independent variable* x (say), *dependent variable* y (say) *and the arbitrary constants.*

Step II *Obtain the number of arbitrary constants in Step I. Let there be n arbitrary constants.*

Step III *Differentiate the relation in step I n times with respect to x.*

Step IV *Eliminate arbitrary constants with the help of n equations involving differential coefficients obtained in step III and an equation in Step I. The equation so obtained is the desired differential equation.*

The following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Form the differential equation of the family of curves represented* $c\,(y+c)^2 = x^3$, *where c is a parameter.*

SOLUTION The equation of the family of curves is $c\,(y+c)^2 = x^3$ ...(i)

Clearly, it is one parameter family of curves, so we shall get a differential equation of first order. Differentiating (i) with respect to x, we get

$$2c\,(y+c)\,\frac{dy}{dx} = 3x^2 \qquad \text{...(ii)}$$

Dividing (i) by (ii), we get

$$\frac{c\,(y+c)^2}{2c\,(y+c)\left(\dfrac{dy}{dx}\right)} = \frac{x^3}{3x^2} \Rightarrow y + c = \frac{2x}{3}\frac{dy}{dx} \Rightarrow c = \frac{2x}{3}\frac{dy}{dx} - y$$

Substituting this value of c in (i), we get

$$\left(\frac{2x}{3}\frac{dy}{dx} - y\right)\left(\frac{2}{3}x\frac{dy}{dx}\right)^2 = x^3 \Rightarrow \frac{4}{9}\left(\frac{dy}{dx}\right)^2\left(\frac{2x}{3}\frac{dy}{dx} - y\right) = x$$

$$\Rightarrow \quad \frac{8}{27}x\left(\frac{dy}{dx}\right)^3 - \frac{4}{9}\left(\frac{dy}{dx}\right)^2 y = x \Rightarrow 8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right)^2 = 27x$$

This is the required differential equation of the family of curves represented by (i).

EXAMPLE 2 *Form the differential equation of the family of curves represented by $y = c(x-c)^2$, where c is a parameter.*

SOLUTION The equations of the family of curves is $y = c(x-c)^2$ ...(i)

This equation contains only one parameter. So, we differentiate it only once. Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = 2c(x-c) \quad ...(ii)$$

From (i) and (ii), we get

$$\frac{y}{\frac{dy}{dx}} = \frac{c(x-c)^2}{2c(x-c)} \Rightarrow \frac{y}{\frac{dy}{dx}} = \frac{x-c}{2} \Rightarrow x - c = \frac{2y}{\frac{dy}{dx}} \Rightarrow c = x - \frac{2y}{\frac{dy}{dx}}$$

Substituting this value of c in (i), we get

$$y = \left(x - \frac{2y}{\frac{dy}{dx}}\right)\left(\frac{2y}{\frac{dy}{dx}}\right)^2 \Rightarrow y\left(\frac{dy}{dx}\right)^3 = 4y^2\left(x\frac{dy}{dx} - 2y\right) \Rightarrow \left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right),$$

which is the required differential equation.

EXAMPLE 3 *Form the differential equation representing the family of curves $y = A\cos(x+B)$, where A and B are parameters.* **[CBSE 2007]**

SOLUTION The equation of the family of curves is $y = A\cos(x+B)$ ...(i)

This equation contains two arbitrary constants. So, let us differentiate it two times to obtain a differential equation of second order.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = -A\sin(x+B) \quad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = -A\cos(x+B) = -y \quad \text{[Using (i)]}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} + y = 0$, which is the required differential equation of the given family of curves.

EXAMPLE 4 *Form the differential equation of the family of curves $y = a\sin(bx+c)$, a and c being parameters.* **[NCERT]**

SOLUTION The equation of the family of curves is $y = a\sin(bx+c)$ ...(i)

Clearly, it contains two arbitrary constants. So, we differentiate it two times to get a differential equation of second order. Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = ab\cos(bx+c) \quad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = -ab^2\sin(bx+c) = -b^2y \quad \text{[Using (i)]}$$

$\Rightarrow \quad \frac{d^2y}{dx^2} + b^2y = 0$, which is the required differential equation.

EXAMPLE 5 *Form the differential equation corresponding to $y^2 = a(b-x)(b+x)$ by eliminating parameters a and b.* **[CBSE 2004]**

SOLUTION The equation of the family of curves is $y^2 = a(b^2 - x^2)$ ...(i)

Clearly, there are two arbitrary constants in this equation. So, we shall differentiate it two times to get a differential equation of second order.

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = -2ax \Rightarrow y\frac{dy}{dx} = -ax \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -a \Rightarrow a = -\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} \qquad \text{...(iii)}$$

Substituting the value of a obtained from (iii) in (ii), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$, which is the required differential equation.

EXAMPLE 6 *Form the differential equation corresponding to* $y^2 = m(a^2 - x^2)$ *by eliminating parameters m and a.*

SOLUTION The equation of the family of curves is

$$y^2 = m(a^2 - x^2) \qquad \text{...(i)}$$

This equation contains two parameters. So we shall differentiate it two times to get a differential equation of second order.

Differentiating both sides of (i) with respect to x, we get

$$2y\frac{dy}{dx} = m(-2x) \Rightarrow y\frac{dy}{dx} = -mx \qquad \text{...(ii)}$$

Differentiating both sides of (ii) with respect to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = -m \qquad \text{...(iii)}$$

Putting this value of $-m$ in (ii), we get

$$x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$$, which is the required differential equation.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 7 *Find the differential equation of all circles touching the*

(i) *x-axis at the origin* **[NCERT, CBSE 2005, 08, 10]** (ii) *y-axis at the origin* **[NCERT]**

SOLUTION (i) The equation of the family of circles touching x-axis at the origin is

$$(x-0)^2 + (y-a)^2 = a^2 \Rightarrow x^2 + y^2 - 2ay = 0 \qquad \text{...(i)}$$

where a is a parameter.

This equation contains only one arbitrary constant. So, we differentiate it once with respect to x, so that

$$2x + 2y\frac{dy}{dx} - 2a\frac{dy}{dx} = 0 \Rightarrow a = \frac{x + y\,(dy/dx)}{dy/dx} \qquad \text{...(ii)}$$

Substituting the value of a from (ii) in (i), we get

$$x^2 + y^2 = 2y\left(\frac{x + y\,(dy/dx)}{dy/dx}\right)$$

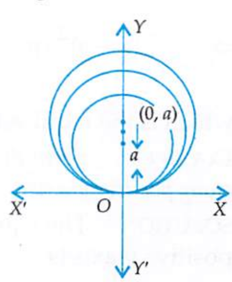

Fig. 21.1

$$\Rightarrow \quad (x^2 - y^2)\frac{dy}{dx} = 2xy,$$ which is the required differential equation.

(ii) The equation of the family of circles touching y-axis at the origin is

$$(x-a)^2 + (y-0)^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0 \qquad \text{...(i)}$$

where a is a parameter.

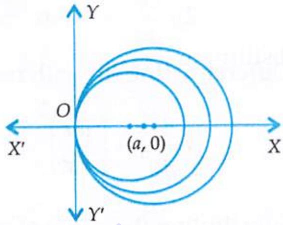

Fig. 21.2

This equation contains only one arbitrary constant. So, we differentiate it only once with respect to x, so that

$$2x + 2y\frac{dy}{dx} - 2a = 0 \Rightarrow a = x + y\frac{dy}{dx} \qquad \text{...(ii)}$$

Substituting the value of a from (ii) in (i), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0 \text{ or, } y^2 - x^2 = 2xy\frac{dy}{dx},$$ which is the required differential equation.

EXAMPLE 8 *Find the differential equation of all the circles in the first quadrant which touch the coordinate axes.* **[NCERT, CBSE 2010]**

SOLUTION The equation of the family of circles in the first quadrant which touch the coordinate axes is

$$(x-a)^2 + (y-a)^2 = a^2 \qquad \text{...(i)}$$

where a is a parameter.

This equation contains one arbitrary constant, so we shall differentiate it once only to get a differential equation of first order.

Differentiating (i) with respect to x, we get

$$2(x-a) + 2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x - a + (y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a = \frac{x + y\frac{dy}{dx}}{1 + \frac{dy}{dx}} \Rightarrow a = \frac{x+py}{1+p}, \text{ where } p = \frac{dy}{dx}$$

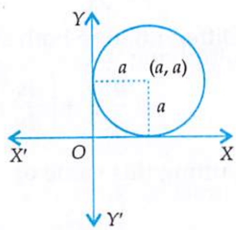

Fig. 21.3

Substituting the value of a in (i), we get

$$\left(x - \frac{x+py}{1+p}\right)^2 + \left(y - \frac{x+py}{1+p}\right)^2 = \left(\frac{x+py}{1+p}\right)^2$$

$$\Rightarrow \quad (xp - py)^2 + (y-x)^2 = (x+py)^2 \Rightarrow (x-y)^2 p^2 + (x-y)^2 = (x+py)^2$$

$$\Rightarrow \quad (x-y)^2(p^2+1) = (x+py)^2 \quad \Rightarrow (x-y)^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \left(x + y\frac{dy}{dx}\right)^2,$$

which is the required differential equation.

EXAMPLE 9 *Form the differential equation of family of parabolas having vertex at the origin and axis along positive y-axis.* **[NCERT, CBSE 2010, 11]**

SOLUTION The equation of the family of parabolas having vertex at the origin and axis along positive y-axis is

$$x^2 = 4ay, \text{ where } a \text{ is a parameter.} \qquad \text{...(i)}$$

This is a one parameter family of curves. So, we differentiate it once only.

Differentiating wtih respect to x, we get

$$2x = 4a\frac{dy}{dx} \Rightarrow a = \frac{x}{2\frac{dy}{dx}}$$

Substituting the value of a in (i), we get

$$x^2 = 4 \times \frac{x}{2\left(\frac{dy}{dx}\right)} \times y \Rightarrow x\frac{dy}{dx} = 2y,$$ which is the required differential equation.

EXAMPLE 10 *Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.* **[NCERT]**

SOLUTION The equation of the family of ellipses having centre at the origin and foci on y-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ where } b > a \qquad ...(i)$$

This is two parameter family of ellipses. So, we differentiate (i) twice with respect to x to obtain a differential equation of order 2.

Differentiating with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \Rightarrow \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \qquad ...(iii)$$

Multiplying throughout by x, we get

$$\frac{x}{a^2} + \frac{x}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{xy}{b^2}\frac{d^2y}{dx^2} = 0 \qquad ...(iv)$$

Subtracting (ii) from (iv), we get

$$\frac{1}{b^2}\left\{x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\left(\frac{dy}{dx}\right)\right\} = 0 \Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} = 0,$$

which is the required differential equation.

EXAMPLE 11 *Form the differential equation not containing the arbitrary constants and satisfied by the equation $y = ae^{bx}$, a and b are arbitrary constants.*

SOLUTION The given equation is

$$y = ae^{bx} \qquad ...(i)$$

Clearly, it contains two arbitrary constants. So, we shall differentiate it two times and get a differential equation of second order.

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = ae^{bx} \cdot b \Rightarrow \frac{dy}{dx} = by \qquad \text{[By using (i)]} \quad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = b\frac{dy}{dx} \qquad ...(iii)$$

From (ii), we get $b = \frac{1}{y}\frac{dy}{dx}$. Substituting this value of b in (iii), we get

$$\frac{d^2y}{dx^2} = \left(\frac{1}{y}\frac{dy}{dx}\right)\frac{dy}{dx} \Rightarrow y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$$, which is the required differential equation

EXAMPLE 12 *Show that the differential equation representing one parameter family of curves* $(x^2 - y^2) = c\,(x^2 + y^2)^2$ *is* $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$ [NCERT, CBSE 2017]

SOLUTION The given equation of one parameter family of curves is

$$x^2 - y^2 = c\,(x^2 + y^2)^2 \qquad \text{...(i)}$$

Differentiating (i) with respect to x, we get

$$2x - 2y\frac{dy}{dx} = 2c\,(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) \Rightarrow \left(x - y\frac{dy}{dx}\right) = 2c\,(x^2 + y^2)\left(x + y\frac{dy}{dx}\right) \qquad \text{...(ii)}$$

On substituting the value of c obtained from (i) in (ii), we get

$$\left(x - y\frac{dy}{dx}\right) = \frac{2\,(x^2 - y^2)\;(x^2 + y^2)}{(x^2 + y^2)^2}\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad (x^2 + y^2)\left(x - y\frac{dy}{dx}\right) = 2\,(x^2 - y^2)\left(x + y\frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left\{x\,(x^2 + y^2) - 2x\,(x^2 - y^2)\right\} = \frac{dy}{dx}\left\{2y\,(x^2 - y^2) + y\,(x^2 + y^2)\right\}$$

$$\Rightarrow \quad (3xy^2 - x^3) = \frac{dy}{dx}(3x^2y - y^3)$$

$\Rightarrow \quad (x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, which is the given differential equation.

EXAMPLE 13 *Represent the following family of curves by forming the corresponding differential equations (a, b are parameters):*

(i) $\frac{x}{a} + \frac{y}{b} = 1$ (ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ [NCERT, CBSE 2007] (iii) $(y - b)^2 = 4\,(x - a)$

SOLUTION The equation of the family of curves is $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

where a, b are parameters.

It is a two parameter family of curves. So, we will differentiate it twice with respect to x.

Differentiating (i) with respect to x, we get

$$\frac{1}{a} + \frac{1}{b}\frac{dy}{dx} = 0 \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{b}\frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$$, which is the required differential equation.

(ii) The equation of the family of curves is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

It is a two parameter family of curves. So, we will differentiate it twice to obtain the differential equation.

Differentiating with respect to x, we get

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$\frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0$$

Multiplying both sides by x, we get

$$\frac{x}{a^2} + \frac{x}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{xy}{b^2}\frac{d^2y}{dx^2} = 0 \qquad ...(iv)$$

Subtracting (ii) from (iv), we get

$$\frac{1}{b^2}\left\{x\left(\frac{dy}{dx}\right)^2 + xy\frac{d^2y}{dx^2} - y\frac{dy}{dx}\right\} = 0 \Rightarrow x\left(\frac{dy}{dx}\right)^2 + xy\frac{x^2y}{dx^2} - y\frac{dy}{dx} = 0,$$

which is the required differential equation.

(iii) The equation of the family of curves is $(y-b)^2 = 4(x-a)$ ...(i)

It is a two parameter family of curves. So, we will differentiate it twice with respect to x.

Differentiating (i) with respect to x, we get

$$2(y-b)\frac{dy}{dx} = 4 \Rightarrow (y-b)\frac{dy}{dx} = 2 \qquad ...(ii)$$

Differentiating with respect to x, we get

$$(y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad ...(iii)$$

From (ii), we get: $y - b = \dfrac{2}{\frac{dy}{dx}}$.

Substituting this value of $y-b$ in (iii), we get

$$\frac{2}{\frac{dy}{dx}} \times \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow 2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0,$$ which is the required differential equation.

EXAMPLE 14 *Obtain the differential equation of all circles of radius r.* **[CBSE 2010, 2015]**

SOLUTION The equation of the family of circles of radius r is

$$(x-a)^2 + (y-b)^2 = r^2 \qquad ...(i)$$

where a and b are a parameters.

Clearly equation (i) contains two arbitrary constants. So, let us differentiate it two times with respect to x.

Differentiating (i) with respect to x, we get

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0 \Rightarrow (x-a) + (y-b)\frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating (ii) with respect to x, we get

$$1 + (y-b)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad ...(iii)$$

$$\Rightarrow \qquad y - b = -\frac{1 + (dy/dx)^2}{d^2y/dx^2} \qquad \text{...(iv)}$$

Putting this value of $(y - b)$ in (ii), we obtain

$$x - a = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \frac{dy}{dx}}{d^2y/dx^2} \qquad \text{...(v)}$$

Substituting the values of $(x - a)$ and $(y - b)$ in (i), we get

$$\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^2 \left(\frac{dy}{dx}\right)^2}{(d^2y/dx^2)^2} + \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^2}{(d^2y/dx^2)^2} = r^2 \Rightarrow \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3 = r^2 \left(\frac{d^2y}{dx^2}\right)^2$$

This is the required differential equation.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *Show that the differential equation that represents the family of all parabolas having their axis of symmetry coincident with the axis of x is* $yy_2 + y_1^2 = 0$.

SOLUTION The equation that represents a family of parabolas having their axis of symmetry coincident with the axis of x is

$$y^2 = 4a(x - h) \qquad \text{...(i)}$$

where a and h are parameters.

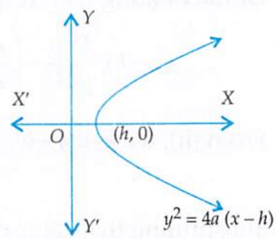

Fig. 21.4

This equation contains two parameters a and h, so we will differentiate it twice to obtain a second order differential equation.

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow y\frac{dy}{dx} = 2a \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \Rightarrow yy_2 + y_1^2 = 0,$$ which is the required differential equation.

EXAMPLE 16 *Find the differential equation of (i) all non-horizontal lines in a plane (ii) all non-vertical lines in a plane* **[NCERT EXEMPLAR]**

SOLUTION The general equation of a line in a plane is $ax + by = 1$. If it is parallel to x-axis, then it is of the form $y =$ constant. So, $b \neq 0$ and $a = 0$. Also, if $ax + by = 1$ represents a line parallel to y-axis, then it should be of the form $x =$ constant. So, $a \neq 0$ and $b = 0$. Thus, $ax + by = 1$ represents a family of non-horizontal lines in a plane, if $a \neq 0$ and a family of non-vertical lines, if $b \neq 0$.

(i) The equation of the family of non-horizontal lines in a plane is $ax + by = 1$, where $a \neq 0$ and b can take any real value. It is a two parameter family of curves with $a \neq 0$ and $b \in R$.

Now, $\quad ax + by = 1$

Differentiating both sides with respect to y, we obtain

$$a\frac{dx}{dy} + b = 0$$

Differentiating both sides with respect to y, we obtain

$$a\frac{d^2x}{dy^2}=0 \Rightarrow \frac{d^2x}{dy^2}=0 \qquad [\because\ a \neq 0]$$

This is the required differential equation.

(ii) The equation of the family of non-vertical lines in plane is $ax+by=1$, where $b \neq 0$ and $a \in R$. It is a two parameter family of curves with $b \neq 0$ and $a \in R$.

Now, $ax+by=1$

Differentiating both sides with respect to x, we obtain

$$a+b\frac{dy}{dx}=0$$

Differentiating both sides with respect to x, we obtain

$$b\frac{d^2y}{dx^2}=0 \Rightarrow \frac{d^2y}{dx^2}=0 \qquad [\because\ b \neq 0]$$

This is the required differential equation.

EXERCISE 21.2

BASIC

1. Form the differential equation of the family of curves represented by $y^2=(x-c)^3$.
2. Form the differential equation corresponding to $y=e^{mx}$ by eliminating m.
3. Form the differential equations from the following primitives where constants are arbitrary:
   (i) $y^2=4ax$ [CBSE 2017] (ii) $y=cx+2c^2+c^3$ (iii) $xy=a^2$ (iv) $y=ax^2+bx+c$
4. Find the differential equation of the family of curves $y=Ae^{2x}+Be^{-2x}$, where A and B are arbitrary constants.
5. Find the differential equation of the family of curves, $x=A\cos nt+B\sin nt$, where A and B are arbitrary constants. [CBSE 2007]
6. Form the differential equation corresponding to $y^2=a(b-x^2)$ by eliminating a and b.
7. Form the differential equation corresponding to $y^2-2ay+x^2=a^2$ by eliminating a. [CBSE 2005]
8. Form the differential equation corresponding to $(x-a)^2+(y-b)^2=r^2$ by eliminating a and b.
9. Show that the differential equation of which $y=2(x^2-1)+ce^{-x^2}$ is a solution, is $\frac{dy}{dx}+2xy=4x^3$.
10. Form the differential equation having $y=(\sin^{-1}x)^2+A\cos^{-1}x+B$, where A and B are arbitrary constants, as its general solution. [NCERT EXEMPLAR]
11. Form the differential equation of the family of curves represented by the equation (a being the parameter):
    (i) $(2x+a)^2+y^2=a^2$ (ii) $(2x-a)^2-y^2=a^2$ (iii) $(x-a)^2+2y^2=a^2$ [NCERT]
12. Represent the following families of curves by forming the corresponding differential equations (a, b being parameters):
    (i) $x^2+y^2=a^2$ (ii) $x^2-y^2=a^2$ (iii) $y^2=4ax$ (iv) $x^2+(y-b)^2=1$
    (v) $(x-a)^2-y^2=1$ (vi) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (vii) $y^2=4a(x-b)$ (viii) $y=ax^3$

(ix) $x^2 + y^2 = ax^3$ (x) $y = e^{ax}$

13. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants. [CBSE 2018]

14. Form the differential equation representing the family of curves $y = e^{2x}(a+bx)$, where a and b are arbitrary constants. [CBSE 2019]

BASED ON LOTS

15. Find the differential equation of all the circles which pass through the origin and whose centres lie on y-axis.

16. Find the differential equation of all the circles which pass through the origin and whose centres lie on x-axis.

17. Assume that a rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

18. Find the differential equation of all the parabolas with latus rectum '$4a$' and whose axes are parallel to x-axis.

19. Form the differential equation representing the family of ellipses having centre at the origin and foci on x-axis. [NCERT, CBSE 2007]

20. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin. [NCERT]

21. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. [NCERT, CBSE 2016]

ANSWERS

1. $8\left(\frac{dy}{dx}\right)^3 = 27y$ **2.** $x\frac{dy}{dx} = y \log y$

3. (i) $2x\frac{dy}{dx} = y$ (ii) $y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^3$ (iii) $y + x\frac{dy}{dx} = 0$ (iv) $\frac{d^3y}{dx^3} = 0$

4. $\frac{d^2y}{dx^2} = 4y$ **5.** $\frac{d^2x}{dt^2} + n^2x = 0$ **6.** $y\frac{dy}{dx} = x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\}$

7. $p^2(x^2 - 2y^2) - 4pxy - x^2 = 0,\ p = \frac{dy}{dx}$ **8.** $(1+p^2)^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2,\ p = \frac{dy}{dx}$

10. $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$

11. (i) $y^2 - 4x^2 - 2xy\frac{dy}{dx} = 0$ (ii) $2x\frac{dy}{dx} = 4x^2 + y^2$ (iii) $2y^2 - x^2 = 4xy\frac{dy}{dx}$

12. (i) $x + y\frac{dy}{dx} = 0$ (ii) $x - y\frac{dy}{dx} = 0$ (iii) $y - 2x\frac{dy}{dx} = 0$

(iv) $x^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = \left(\frac{dy}{dx}\right)^2$ (v) $y^2\left(\frac{dy}{dx}\right)^2 - y^2 = 1$

(vi) $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}$ (vii) $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ (viii) $x\frac{dy}{dx} = 3y$

(ix) $x^2 + 3y^2 = 2xy\frac{dy}{dx}$ (x) $x\frac{dy}{dx} = y\log y$

13. $y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ 14. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$ 15. $(x^2 - y^2)\frac{dy}{dx} = 2xy$

16. $(x^2 - y^2) + 2xy\frac{dy}{dx} = 0$ 17. $\frac{dr}{dt} = -k$ 18. $2a\, y_2 + y_1^3 = 0$

19. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ 20. $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

21. $\left(x + y\frac{dy}{dx}\right)^2 = (x+y)^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$

HINTS TO SELECTED PROBLEMS

1. The equation of the family of curves is $y^2 = (x-c)^3$. Differentiating with respect to x, we get

$$2y\,y_1 = 3(x-c)^2 \Rightarrow (2y\,y_1)^3 = 27(x-c)^6$$

$$\Rightarrow 8y^3\,y_1^3 = 27(y^2)^2 \qquad [\because (x-c)^3 = y^2]$$

$\Rightarrow 8y_1^3 = 27y$, which is the required differential equation.

2. The equation of the family of curves is $y = e^{mx}$. Differentiating with respect to x, we get

$$y_1 = me^{mx} \Rightarrow y_1 = my.$$

Now, $y = e^{mx} \Rightarrow \log y = mx \Rightarrow m = \frac{\log y}{x}$

Substituting this value of m in $y_1 = my$, we get $xy_1 = y\log y$, which is the required differential equation.

3. (i) The equation of the family of primitives is $y^2 = 4ax$. Differentiating with respect to x

$$2y\frac{dy}{dx} = 4a \Rightarrow a = \frac{1}{2}y\frac{dy}{dx}$$

Substituting this value of a in $y^2 = 4ax$, we get $y^2 = 4\left(\frac{1}{2}y\frac{dy}{dx}\right)$ or $y = 2\frac{dy}{dx}$ as the required differential equation.

(ii) The equation of the family of primitives is $y = cx + 2c^2 + c^3 \Rightarrow \frac{dy}{dx} = c$

Substitute this value of c in $y = cx + 2c^2 + c^3$, to get the required differential equation.

(iii) The equations of family of primitives is $xy = a^2$. Differentiating with respect to x, we get $x\frac{dy}{dx} + y = 0$, which is the required differential equation.

(iv) Differentiate y three times to get $\frac{d^3y}{dx^3} = 0$ as the required differential equation.

4. Differentiate y two times to get $\frac{d^2y}{dx^2} = 4y$ as the required differential equation.

5. Differentiate x two times with respect to t to obtain $\frac{d^2x}{dt^2} = -n^2\ x$ as the required differential equation.

7. The equation of one parameter family of curves is $y^2 - 2ay + x^2 = a^2$.

Differentiating with respect to x, we get: $2y\frac{dy}{dx} - 2a\frac{dy}{dx} + 2x = 0 \Rightarrow a = \frac{x + y\frac{dy}{dx}}{\frac{dy}{dx}}$

Substituting the value of a in $y^2 - 2ay + x^2 = a^2$, we obtain the required differential equation.

11. (iii) The equation of the one parameter family of curves is

$$(x-a)^2 + 2y^2 = a^2 \quad \text{...(i)}$$

Differentiating with respect to x, we get

$$2(x-a) + 4y\frac{dy}{dx} = 0 \Rightarrow x - a = -2y\frac{dy}{dx} \Rightarrow a = x + 2y\frac{dy}{dx}$$

Substituting the value of a in (i), we get

$$4y^2\left(\frac{dy}{dx}\right)^2 + 2y^2 = \left(x + 2y\frac{dy}{dx}\right)^2 \Rightarrow 2y^2 - x^2 = 4xy\frac{dy}{dx}$$

This is the required differential equation.

18. The equation of the family of parabolas is $(y-k)^2 = 4a(x-h)$, where h and k are arbitrary constants.

Differentiate this relation twice and eliminate h and k to get the differential equation.

19. The equation of the family of ellipses having centre at the origin and foci on x-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

It is a two parameter family of curves. Differentiating (i) twice with respect to x, we get

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0 \text{ and } \frac{2}{a^2} + \frac{2}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{2y}{b^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0 \quad \text{...(ii)} \qquad \text{and,} \quad \frac{1}{a^2} + \frac{1}{b^2}\left(\frac{dy}{dx}\right)^2 + \frac{y}{b^2}\frac{d^2y}{dx^2} = 0 \quad \text{...(iii)}$$

Multiplying (iii) by x and subtracting from (ii), we get

$$\frac{1}{b^2}\left\{y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 - xy\frac{d^2y}{dx^2}\right\} = 0 \Rightarrow xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$$

This is the required differential equation.

20. The equation of the family of hyperbolas having foci on x-axis and centre at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(i)}$$

Differentiating this twice with respect to x, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \quad \text{and} \quad \frac{2}{a^2}-\frac{2}{b^2}\left(\frac{dy}{dx}\right)^2-\frac{2y}{b^2}\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \quad \frac{x}{a^2}-\frac{y}{b^2}\frac{dy}{dx}=0 \quad \text{...(ii)} \quad \frac{1}{a^2}-\frac{1}{b^2}\left(\frac{dy}{dx}\right)^2-\frac{y}{b^2}\frac{d^2y}{dx^2}=0 \quad \text{...(iii)}$$

Multiplying (iii) by x and subtracting from (ii), we get

$$\frac{1}{b^2}\left(-y\frac{dy}{dx}+x\left(\frac{dy}{dx}\right)^2+xy\frac{d^2y}{dx^2}\right)=0 \Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

This is the required differential equation.

21. The equation of the family of circles in second quadrant and touching the coordinates axes is $(x+a)^2+(y-a)^2=a^2$, where $a>0$ or, $x^2+y^2+2ax-2ay+a^2=0$ ...(i)

Differentiating (i) with respect to x, we get

$$2x+2y\frac{dy}{dx}+2a-2a\frac{dy}{dx}=0 \Rightarrow x+y\frac{dy}{dx}+a\left(1-\frac{dy}{dx}\right)=0 \Rightarrow a=\frac{x+py}{p-1}, \text{where } p=\frac{dy}{dx}$$

Substituting the value of a in $(x+a)^2+(y-a)^2=a^2$, we get

$$\left(x+\frac{x+py}{p-1}\right)^2+\left(y-\frac{x+py}{p-1}\right)^2=\left(\frac{x+py}{p-1}\right)^2$$

$$\Rightarrow \quad (x+y)^2p^2+(-y-x)^2=(x+py)^2 \Rightarrow (x+y)^2(1+p^2)=(x+py)^2, \text{ where } p=\frac{dy}{dx}$$

This is the required differential equation.

21.3 SOLUTION OF A DIFFERENTIAL EQUATION

SOLUTION *The solution of a differential equation is a relation between the variables involved which satisfies the differential equation.*

Such a relation and the derivatives obtained therefrom when substituted in the differential equation, makes left hand, and right hand sides identically equal.

For example, $y=e^x$ is a solution of the differential equation $\frac{dy}{dx}=y$.

Consider the differential equation $\frac{d^2y}{dx^2}+y=0$ ...(i)

Also, consider the relation $y=A\cos x+B\sin x$, where A and B are arbitrary constants. ...(ii)

Differentiating (ii), with respect to x, we get: $\frac{dy}{dx}=-A\sin x+B\cos x$

Differentiating this with respect to x, we get

$$\frac{d^2y}{dx^2}=-A\cos x-B\sin x \Rightarrow \frac{d^2y}{dx^2}=-y \Rightarrow \frac{d^2y}{dx^2}+y=0$$

This shows that $y=A\cos x+B\sin x$ satisfies the differential equation (i) and hence it is a solution of differential equation given in (i).

It can be easily verified that $y=3\cos x+2\sin x$, $y=A\cos x$, $y=B\sin x$ etc., are also solutions of differential equation given in (i).

We find that the solution $y = 3\cos x + 2\sin x$ does not contain any arbitrary constant whereas solutions $y = A\cos x$, $y = B\sin x$ contain only one arbitrary constant. The solution $y = A\cos x + B\sin x$ contains two arbitrary constants, so it is known as the general solution of (i) whereas all other solutions are particular solutions.

GENERAL SOLUTION *The solution which contains as many as arbitrary constants as the order of the differential equation is called the general solution of the differential equation.*

For example, $y = A\cos x + B\sin x$ is the general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0.$

But, $y = A\cos x$ is not the general solution as it contains one arbitrary constant.

PARTICULAR SOLUTION *Solution obtained by giving particular values to the arbitrary constants in the general solution of a differential equation is called a particular solution.*

For example, $y = 3\cos x + 2\sin x$ is a particular solution of the differential equation (i).

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that* $y = Ax + \frac{B}{x}$, $x \neq 0$ *is a solution of the differential equation* $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

SOLUTION We have, $y = Ax + \frac{B}{x}, x \neq 0$ ...(i)

Differentiating both sides of (i) with respect to x, we get

$$\frac{dy}{dx} = A - \frac{B}{x^2} \quad \text{... (ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{2B}{x^3} \quad \text{...(iii)}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$, we get

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2\left(\frac{2B}{x^3}\right) + x\left(A - \frac{B}{x^2}\right) - \left(Ax + \frac{B}{x}\right) = \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

Thus, the function $y = Ax + \frac{B}{x}$ satisfies the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$

Hence, $y = Ax + \frac{B}{x}$ is a solution of the given differential equation.

EXAMPLE 2 *Show that the function* $y = (A + Bx)e^{3x}$ *is a solution of the equation* $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

SOLUTION We have, $y = (A + Bx)e^{3x}$ ...(i)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = Be^{3x} + 3e^{3x}(A + Bx) \quad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = 6Be^{3x} + 9e^{3x}(A + Bx) \qquad \text{...(iii)}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{dy}{dx^2}$ on the LHS of the given differential equation, we obtain

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \{6Be^{3x} + 9e^{3x}(A + Bx)\} - 6\{Be^{3x} + 3e^{3x}(A + Bx)\} + \{9(A + Bx)e^{3x}\} = 0$$

Thus, $y = (A + Bx)e^{3x}$ satisfies the given differential equation. Hence, it is a solution of the given differential equation.

EXAMPLE 3 *Show that* $y = ae^{2x} + be^{-x}$ *is a solution of the differential equation* $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$.

SOLUTION We have, $y = ae^{2x} + be^{-x}$ ...(i)

Differentiating (i) with respect to x, we get

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \qquad \text{...(ii)}$$

Differentiating (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$

$$\therefore \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = (4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2(ae^{2x} + be^{-x}) = 0$$

So, $y = ae^{2x} + be^{-x}$ satisfies the given differential equation. Hence, it is a solution of the given differential equation.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Show that* $y = a\cos(\log x) + b\sin(\log x)$ *is a solution of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

SOLUTION We have, $y = a\cos(\log x) + b\sin(\log x)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x} \Rightarrow x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating both sides with respect to x, we obtain

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\{a\cos(\log x) + b\sin(\log x)\}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0,$$ which is same as the given differential equation.

Hence, $y = a\cos(\log x) + b\sin(\log x)$ is a solution of the given differential equation.

EXAMPLE 5 *Show that* $y = cx + \frac{a}{c}$ *is a solution of the differential equation* $y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}$.

SOLUTION We have, $y = cx + \frac{a}{c}$ ...(i)

Differentiating with respect to x, we get

$$\frac{dy}{dx} = c \quad \text{...(ii)}$$

$$\therefore \quad x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}} = xc + \frac{a}{c} \qquad \left[\text{Putting } \frac{dy}{dx} = c\right]$$

$$\Rightarrow \quad x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}} = y \qquad \text{[Using (i)]}$$

This shows that $y = cx + \frac{a}{c}$ is a solution of the given differential equation.

EXAMPLE 6 *Show that* $xy = ae^x + be^{-x} + x^2$ *is a solution of the differential equation*

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = 0.$$ **[NCERT]**

SOLUTION We have,

$$xy = ae^x + be^{-x} + x^2 \quad \text{... (i)}$$

Differentiating with respect to x, we get

$$x\frac{dy}{dx} + y = ae^x - be^{-x} + 2x$$

Differentiating this with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = ae^x + be^{-x} + 2$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = ae^x + be^{-x} + 2 \quad \text{...(ii)}$$

Using (i) and (ii) we obtain

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy + x^2 - 2 = \left(ae^x + be^{-x} + 2\right) - \left(ae^x + be^{-x} + x^2\right) + x^2 - 2 = 0$$

Thus, $xy = ae^x + be^{-x} + x^2$ is a solution of the given differential equation.

EXAMPLE 7 *Verify that the function* $y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx, C_1, C_2$ *are arbitrary constants is a solution of the differential equation* $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)y = 0.$ **[NCERT]**

SOLUTION We have,

$$y = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx \quad \text{...(i)}$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = C_1\left\{a\,e^{ax}\cos bx - b\,e^{ax}\sin bx\right\} + C_2\left\{a\,e^{ax}\sin bx + b\,e^{ax}\cos bx\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = a\left\{C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx\right\} + b\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = a\,y + b\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\} \quad \text{...(ii)}$$

Differentiating with respect to x, we get

$$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + b\left\{-a\,C_1 e^{ax} \sin bx - b\,C_1 e^{ax} \cos bx + a\,C_2 e^{ax} \cos bx - b\,C_2 e^{ax} \sin bx\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a\frac{dy}{dx} + ab\left\{-C_1 e^{ax} \sin bx + C_2 e^{ax} \cos bx\right\} - b^2\left\{C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx\right\}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = a\frac{dy}{dx} + a\left\{\frac{dy}{dx} - ay\right\} - b^2 y \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2 + b^2)\,y = 0$$

Hence, the given function is a solution of the given differential equation.

EXERCISE 21.3

BASIC

1. Show that $y = be^x + ce^{2x}$ is a solution of the differential equation, $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0.$

2. Verify that $y = 4 \sin 3x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 9y = 0.$

3. Show that $y = ae^{2x} + be^{-x}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$

4. Show that the function $y = A \cos x + B \sin x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0.$ **[NCERT]**

5. Show that the function $y = A \cos 2x - B \sin 2x$ is a solution of the differential equation $\frac{d^2y}{dx^2} + 4y = 0.$ **[CBSE 2007]**

6. Show that $y = Ae^{Bx}$ is a solution of the differential equation $\frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2.$

7. Verify that $y = \frac{a}{x} + b$ is a solution of the differential equation $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0.$

8. Verify that $y^2 = 4ax$ is a solution of the differential equation $y = x\frac{dy}{dx} + a\frac{dx}{dy}.$

9. Show that $Ax^2 + By^2 = 1$ is a solution of the differential equation $x\left\{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right\} = y\frac{dy}{dx}.$

10. Show that $y = ax^3 + bx^2 + c$ is a solution of the differential equation $\frac{d^3y}{dx^3} = 6a.$

11. Show that $y = \frac{c-x}{1+cx}$ is a solution of the differential equation $(1+x^2)\frac{dy}{dx} + (1+y^2) = 0$.

12. Show that $y = e^x (A\cos x + B\sin x)$ is the solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$ **[NCERT]**

BASED ON LOTS

13. Verify that $y = cx + 2c^2$ is a solution of the differential equation $2\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0$.

14. Verify that $y = -x - 1$ is a solution of the differential equation $(y-x)\,dy - (y^2 - x^2)\,dx = 0$.

15. Verify that $y^2 = 4a(x+a)$ is a solution of the differential equations $y\left\{1 - \left(\frac{dy}{dx}\right)^2\right\} = 2x\frac{dy}{dx}$.

16. Verify that $y = ce^{\tan^{-1}x}$ is a solution of the differential equation

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} = 0.$$

17. Verify that $y = e^{m\cos^{-1}x}$ satisfies the differential equation $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

18. Verify that $y = \log(x + \sqrt{x^2+a^2})^2$ satisfies the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

19. Show that the differential equation of which $y = 2(x^2-1) + ce^{-x^2}$ is a solution is

$$\frac{dy}{dx} + 2xy = 4x^3$$

20. Show that $y = e^{-x} + ax + b$ is solution of the differential equation $e^x\frac{d^2y}{dx^2} = 1$.

21. For each of the following differential equations verify that the accompanying function is a solution:

Differentials equation	*Function*	*Differentials equation*	*Function*
(i) $x\frac{dy}{dx} = y$	$y = ax$	(ii) $x + y\frac{dy}{dx} = 0$	$y = \pm\sqrt{a^2 - x^2}$
(ii) $x\frac{dy}{dx} + y = y^2$	$y = \frac{a}{x+a}$	(iv) $x^3\frac{d^2y}{dx^2} = 1$	$y = ax + b + \frac{1}{2x}$
(v) $y = \left(\frac{dy}{dx}\right)^2$	$y = \frac{1}{4}(x \pm a)^2$		

HINTS TO SELECTED PROBLEMS

4. We have,

$$y = A\cos x + B\sin x$$

$$\Rightarrow \frac{dy}{dx} = -A\sin x + B\cos x \text{ and } \frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = -y$ or, $\dfrac{d^2y}{dx^2} + y = 0$

This is the given differential equation. So, $y = A\cos x + B\sin x$ satisfies the given differential equation. Hence, $y = A\cos x + B\sin x$ is the solution of the given differential equation.

12. We have, $y = e^x (A\cos x + B\sin x)$ ...(i)

$$\Rightarrow \quad \frac{dy}{dx} = e^x (A\cos x + B\cos x) + e^x(-A\sin x + B\cos x)$$

$$\Rightarrow \quad \frac{dy}{dx} = y + e^x(-A\sin x + B\cos x) \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) - y \quad \text{...(iii)}$$

Subtracting (ii) from (iii), we get

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = \frac{dy}{dx} - 2y \Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

This is the given differential equation. So, $y = e^x(A\cos x + B\sin x)$ satisfies the differential equation. Hence, it is the solution of the given differential equation.

21.4 INITIAL VALUE PROBLEMS

In section 21.2, we have seen that a first order differential equation represents a one-parameter family of curves, a second order differential equation represents a two-parameter family of curves, and so on. Therefore, if we wish to specify a particular member of such a family of curves, then in addition to the differential equation we require some other conditions for the specification of the parameter(s). These conditions are generally prescribed by assigning values to the unknown function (dependent variable) and its various order derivatives at some point of the domain of definition of independent variable.

For example, the differential equation $\dfrac{dy}{dx} = 4x$ represents one parameter family of curves given by $y = 2x^2 + C$, where C is a parameter. In order to specify a particular member, say $y = 2x^2 + 3$, of this family, we require the differential equation $\dfrac{dy}{dx} = 4x$ and one more condition, namely $y(1) = 5$ i.e. $y = 5$ when $x = 1$.

Similarly, the differential equation $\dfrac{d^2y}{dx^2} - 6 = 0$ represents two parameter family of curves given by $y = 3x^2 + ax + b$ where a and b are parameters.

Now, if we want to specify a particular member, say $y = 3x^2 - 2x + 1$, of this family. Then, we require the differential equation $\dfrac{d^2y}{dx^2} - 6 = 0$ and two conditions, namely $y(0) = 1$ and $y'(0) = -2$.

It follows from the above discussion that to specify a particular member of a family of curves, we require the differential equation representing the given family of curves and the values of dependent variable and its various order derivatives at some point of the domain of definition. These values are generally prescribed at only one point of the domain of definition of independent variable and are generally referred to as *initial values* or *initial conditions*.

The differential equation with these initial values or initial conditions is generally known as an *initial value problem.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Verify that the function defined by* $y = \sin x - \cos x$, $x \in R$ *is a solution of the initial value problem* $\frac{dy}{dx} = \sin x + \cos x$, $y(0) = -1$.

SOLUTION We have,

$$y = \sin x - \cos x \Rightarrow \frac{dy}{dx} = \cos x + \sin x, \text{ which is the given differential equation.}$$

Thus, $y = \sin x - \cos x$ satisfies the differential given equation and hence it is a solution.
Also, when $x = 0$, $y = \sin 0 - \cos 0 = 0 - 1 = -1$ i.e. $y(0) = -1$.
Hence, $y = \sin x - \cos x$ is a solution of the given initial value problem.

EXAMPLE 2 *Show that* $y = x^2 + 2x + 1$ *is the solution of the initial value problem* $\frac{d^3y}{dx^3} = 0$, $y(0) = 1$, $y'(0) = 2$, $y''(0) = 2$.

SOLUTION We have,

$$y = x^2 + 2x + 1 \Rightarrow \frac{dy}{dx} = 2x + 2, \frac{d^2y}{dx^2} = 2 \text{ and } \frac{d^3y}{dx^3} = 0, \text{ which is the given differential equation.}$$

Thus, $y = x^2 + 2x + 1$ satisfies the differential equation $\frac{d^3y}{dx^3} = 0$.

Also, $y = x^2 + 2x + 1, \frac{dy}{dx} = 2x + 2$ and $\frac{d^2y}{dx^2} = 2$

$$\Rightarrow \quad y(0) = 0 + 0 + 1 = 1, \left(\frac{dy}{dx}\right)_{x=0} = 2 \text{ and } \left(\frac{d^2y}{dx^2}\right)_{x=0} = 2$$

$$\Rightarrow \quad y(0) = 1, y'(0) = 2 \text{ and } y''(0) = 2.$$

Hence, $y = x^2 + 2x + 1$ is the solution of the initial value problem.

EXAMPLE 3 *Show that the function* ϕ *defined by* $\phi(x) = \cos x$ $(x \in R)$; *satisfies the initial value problem* $\frac{d^2y}{dx^2} + y = 0$, $y(0) = 1$, $y'(0) = 0$.

SOLUTION We have,

$$\phi(x) = \cos x \Rightarrow \phi'(x) = -\sin x \text{ and } \phi''(x) = -\cos x \Rightarrow \phi'(x) = -\sin x \text{ and } \phi''(x) = -\phi(x)$$

Clearly, if we replace ϕ by y in $\phi''(x) = -\phi(x)$, we obtain

$$\frac{d^2y}{dx^2} = -y \text{ or, } \frac{d^2y}{dx^2} + y = 0, \text{ which is the given differential equation.}$$

Thus, $\phi(x)$ satisfies the given differential equation for all $x \in R$.

Also, $\phi(0) = \cos 0 = 1$ and $\phi'(0) = -\sin 0 = 0 \Rightarrow y(0) = 1$ and $y'(0) = 0$

So, ϕ satisfies the initial conditions. Hence, ϕ satisfies the initial value problem.

EXERCISE 21.4

BASIC

For each of the following initial value problems verify that the accompanying function is a solution:

Differential equation	Function
1. $x\dfrac{dy}{dx} = 1, y(1) = 0$	$y = \log x$
2. $\dfrac{dy}{dx} = y, y(0) = 1$	$y = e^x$
3. $\dfrac{d^2y}{dx^2} + y = 0, y(0) = 0, y'(0) = 1$	$y = \sin x$
4. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0, y(0) = 2, y'(0) = 1$	$y = e^x + 1$
5. $\dfrac{dy}{dx} + y = 2, y(0) = 3$	$y = e^{-x} + 2$
6. $\dfrac{d^2y}{dx^2} + y = 0, y(0) = 1, y'(0) = 1$	$y = \sin x + \cos x$
7. $\dfrac{d^2y}{dx^2} - y = 0, y(0) = 2, y'(0) = 0$	$y = e^x + e^{-x}$
8. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0, y(0) = 2, y'(0) = 3$	$y = e^x + e^{2x}$
9. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0, y(0) = 1, y'(0) = 2$	$y = xe^x + e^x$

21.5 GENERAL FORM OF A FIRST-ORDER FIRST-DEGREE DIFFERENTIAL EQUATION

A differential equation of first order and first degree involves the independent variable x, dependent variable y and $\dfrac{dy}{dx}$. So, it can be put in any one of the following forms:

$$\frac{dy}{dx} = f(x, y) \quad \text{...(i)}$$

or, $$\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)} \quad \text{...(ii)}$$

or, In general $f\left(x, y, \dfrac{dy}{dx}\right) = 0$, where $f(x,y)$ and $g(x, y)$ are obviously the functions of x and y.

In the chapter on differentials and approximations, we have proved that if dx and dy denote differentials of variables x and y, then $dy = \dfrac{dy}{dx}dx$. Therefore, equations (i) & (ii) can be written as

$$dy = f(x, y)\,dx \text{ and, } dy = \frac{\phi(x, y)}{\psi(x, y)}dx \text{ or, } \phi(x, y)\,dx = \psi(x, y)\,dy$$

Hence, a first-order first-degree differential equation is expressible in one of the following forms:

$$\frac{dy}{dx} = f(x, y) \text{ or, } \frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)} \text{ or, } dy = f(x, y)\,dx \text{ or, } f(x, y)\,dx + g(x, y)\,dy = 0$$

21.5.1 GEOMETRICAL INTERPRETATION OF THE DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

The general form of a first order and first degree differential equation is

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{...(i)}$$

We know that the tangent of the direction of a curve in cartesian rectangular coordinates at any point is given by $\frac{dy}{dx}$, so the equation in (i) can be known as an equation which establishes the relationship between the coordinates of a point and the slope of the tangent i.e., $\frac{dy}{dx}$ to the integral curve at that point. Solving the differential equation given by (i) means finding those curves for which the direction of tangent at each point coincides with the direction of the field. All the curves represented by the general solution when taken together will give the locus of the differential equation. Since there is one arbitrary constant in the general solution of the equation of first order, the locus of the equation can be said to be made up of single infinity of curves.

21.5.2 SOLUTION OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

As discussed earlier a first order and first degree differential equation can be written as

$$f(x, y)\, dx + g(x, y)\, dy = 0 \text{ or, } \frac{dy}{dx} = \frac{f(x, y)}{g(x, y)} \text{ or, } \frac{dy}{dx} = \phi(x, y)$$

where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y.

It is not always possible to solve this type of equations. The solution of this type of differential equations is possible only when it falls under the category of some standard forms. In the following section we will discuss some of the standard forms and methods of obtaining their solutions.

21.6 METHODS OF SOLVING A FIRST ORDER FIRST DEGREE

In this section, we shall discuss several techniques of obtaining solutions of following types of differential equations.

(i) Differential equations of the form $\frac{dy}{dx} = f(x)$.

(ii) Differential equations of the form $\frac{dy}{dx} = f(y)$.

(iii) Differential equations in variable separable form.

(iv) Differential equations reducible to variable separable form.

(v) Homogeneous differential equations.

(vi) Linear differential equations.

21.6.1 DIFFERENTIAL EQUATIONS OF THE TYPE $\frac{dy}{dx} = f(x)$

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed below.

We have, $\frac{dy}{dx} = f(x) \Leftrightarrow dy = f(x)\, dx$

Integrating both sides, we obtain

$\int dy = \int f(x)\, dx + C$ or, $y = \int f(x)\, dx + C$, which gives the general solution of the differential equation.

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:*

(i) $\frac{dy}{dx} = \frac{x}{x^2+1}$ (ii) $(e^x + e^{-x}) \frac{dy}{dx} = (e^x - e^{-x})$ [NCERT]

SOLUTION (i) We have,

$$\frac{dy}{dx} = \frac{x}{x^2+1} \Rightarrow dy = \frac{x}{x^2+1} dx$$

Integrating both sides, we get

$$\int dy = \int \frac{x}{x^2+1} dx \Rightarrow \int dy = \frac{1}{2} \int \frac{2x}{x^2+1} dx \Rightarrow y = \frac{1}{2} \log|x^2+1| + C$$

Clearly, $y = \frac{1}{2} \log|x^2+1| + C$ is defined for all $x \in R$.

Hence, $y = \frac{1}{2} \log|x^2+1| + C$, $x \in R$ is the solution of the given differential equation.

(ii) The given differential equations is

$$(e^x + e^{-x}) \frac{dy}{dx} = (e^x - e^{-x}) \Rightarrow \frac{dy}{dx} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

Integrating both sides, we get

$$\int dy = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\Rightarrow \int dy = \int \frac{dt}{t}, \text{ where } e^x + e^{-x} = t \Rightarrow y = \log|t| + C \Rightarrow y = \log|e^x + e^{-x}| + C.$$

Clearly, $y = \log|e^x + e^{-x}| + C$ is defined for all $x \in R$. Hence, $y = \log|e^x + e^{-x}| + C$, $x \in R$ is the solution of the given differential equation.

EXAMPLE 2 *Solve the following differential equations:*

(i) $(x+2)\frac{dy}{dx} = x^2 + 4x - 9$, $x \neq -2$ (ii) $\frac{dy}{dx} = \sin^3 x \cos^2 x + x e^x$

SOLUTION (i) We have,

$$(x+2)\frac{dy}{dx} = x^2 + 4x - 9 \Rightarrow \frac{dy}{dx} = \frac{x^2+4x-9}{x+2} \Rightarrow dy = \left(\frac{x^2+4x-9}{x+2}\right) dx \quad [\because x \neq -2]$$

Integrating both sides, we get

$$\int dy = \int \frac{x^2+4x-9}{x+2} dx \Rightarrow \int dy = \int \left(x + 2 - \frac{13}{x+2}\right) dx \Rightarrow y = \frac{x^2}{2} + 2x - 13 \log|x+2| + C$$

Clearly, it is defined for all $x \in R$, except $x = -2$. Hence, $y = \frac{x^2}{2} + 2x - 13 \log|x+2| + C$, $x \in R - \{2\}$ is the solution of the given differential equation.

(ii) We have, $\frac{dy}{dx} = \sin^3 x \cos^2 x + x e^x \Rightarrow dy = (\sin^3 x \cos^2 x + x e^x)\, dx$

Integrating both sides, we get

$$\int dy = \int (\sin^3 x \cos^2 x + x e^x)\, dx$$

$$\Rightarrow \int dy = \int \sin^3 x \cos^2 x\, dx + \int x e^x\, dx$$

$$\Rightarrow \quad \int dy = \int \cos^2 x\,(1-\cos^2 x)\sin x\,dx + \int \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^x}\,dx$$

$$\Rightarrow \quad y = -\int t^2(1-t^2)\,dt + \left\{x\,e^x - \int e^x\,dx\right\}, \text{ where } t=\cos x$$

$$\Rightarrow \quad y = -\left\{\frac{t^3}{3}-\frac{t^5}{5}\right\} + (xe^x - e^x) + C \Rightarrow y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x\,e^x - e^x + C$$

Clearly, $-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x\,e^x - e^x + C$ is defined for all $x \in R$.

Hence, $y = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + x\,e^x - e^x + C,\ x \in R$ is a solution of the given differential equation.

EXAMPLE 3 *Solve:*

(i) $\frac{dy}{dx} = \frac{1}{\sin^4 x + \cos^4 x}$ (ii) $\frac{dy}{dx} = \frac{3e^{2x}+3e^{4x}}{e^x + e^{-x}}$

SOLUTION (i) We have,

$$\frac{dy}{dx} = \frac{1}{\sin^4 x + \cos^4 x} \Rightarrow dy = \frac{1}{\sin^4 x + \cos^4 x}\,dx$$

Integrating both sides, we get

$$\int dy = \int \frac{1}{\sin^4 x + \cos^4 x}\,dx$$

$$\Rightarrow \quad \int dy = \int \frac{\sec^4 x}{\tan^4 x + 1}\,dx \quad \left[\text{Dividing numerator and denominator on RHS by } \cos^4 x\right]$$

$$\Rightarrow \quad \int dy = \int \frac{\sec^2 x \cdot \sec^2 x}{\tan^4 x + 1}\,dx$$

$$\Rightarrow \quad \int dy = \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^4 x + 1}\,dx \Rightarrow \int dy = \int \frac{1+t^2}{1+t^4}\,dt, \text{ where } t = \tan x$$

$$\Rightarrow \quad \int dy = \int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt \quad \left[\text{Dividing numerator and denominator by } t^2\right]$$

$$\Rightarrow \quad \int dy = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}\,dt$$

$$\Rightarrow \quad \int dy = \int \frac{du}{u^2+(\sqrt{2})^2}, \text{ where } t-\frac{1}{t}=u \Rightarrow y = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+C \Rightarrow y = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right)+C$$

This is the required primitive of the given differential equation.

(ii) We have, $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3e^{2x}(1+e^{2x})}{e^x + \frac{1}{e^x}} = \frac{3e^{2x}(1+e^{2x})}{\frac{e^{2x}+1}{e^x}} = \frac{3e^{3x}(1+e^{2x})}{(1+e^{2x})} = 3e^{3x}$$

$$\Rightarrow \quad dy = 3e^{3x}\,dx$$

$$\Rightarrow \quad \int dy = 3\int e^{3x}\,dx \qquad \text{[Integrating both sides]}$$

$\Rightarrow \quad y = 3\left(\frac{e^{3x}}{3}\right) + C \Rightarrow y = e^{3x} + C$, which is the required solution.

EXAMPLE 4 Solve the initial value problem $e^{(dy/dx)} = x + 1;\ y(0) = 5$.

SOLUTION We are given that

$$e^{dy/dx} = x + 1 \Rightarrow \frac{dy}{dx} = \log(x+1) \Rightarrow dy = \log(x+1)\,dx$$

Integrating both sides, we get

$$\int 1 \cdot dy = \int \underset{\mathrm{I}}{\log(x+1)} \cdot \underset{\mathrm{II}}{1}\,dx$$

$$\Rightarrow \quad y = x\log(x+1) - \int \frac{x}{x+1}\,dx \Rightarrow y = x\log(x+1) - \int \frac{x+1-1}{x+1}\,dx$$

$$\Rightarrow \quad y = x\log(x+1) - \int\left(1 - \frac{1}{x+1}\right)dx \Rightarrow y = x\log(x+1) - x + \log(x+1) + C \qquad \text{...(i)}$$

It is given that $y(0) = 5$ i.e., when $x = 0$, we have $y = 5$.

$\therefore \quad 5 = \log 1 - 0 + \log 1 + C \Rightarrow C = 5$ [Substituting $x = 0$, $y = 5$ in (i)]

Substituting the value of C in (i), we get: $y = x\log(x+1) - x + \log(x+1) + 5$

We observe that $x\log(x+1) - x + \log(x+1) + 5$ is defined for all $x \in (-1, \infty)$.

Hence, $y = x\log(x+1) - x + \log(x+1) + 5$, where $x \in (-1, \infty)$ is the solution of the given initial value problem.

EXERCISE 21.5

BASIC

Solve the following differential equations (1-15):

1. $\frac{dy}{dx} = x^2 + x - \frac{1}{x},\ x \neq 0$
2. $\frac{dy}{dx} = x^5 + x^2 - \frac{2}{x},\ x \neq 0$
3. $\frac{dy}{dx} + 2x = e^{3x}$
4. $(x^2+1)\frac{dy}{dx} = 1$
5. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$ [CBSE 2002]
6. $(x+2)\frac{dy}{dx} = x^2 + 3x + 7$
7. $\frac{dy}{dx} = \tan^{-1} x$
8. $\frac{dy}{dx} = \log x$
9. $\frac{1}{x}\frac{dy}{dx} = \tan^{-1} x,\ x \neq 0$
10. $\frac{dy}{dx} = \cos^3 x \sin^2 x + x\sqrt{2x+1}$
11. $(\sin x + \cos x)\,dy + (\cos x - \sin x)\,dx = 0$
12. $\sin^4 x \frac{dy}{dx} = \cos x$
13. $\sqrt{1-x^4}\,dy = x\,dx$

14. $\frac{dy}{dx} = x \log x$

15. $\frac{dy}{dx} = x e^x - \frac{5}{2} + \cos^2 x$

Solve the following initial value problems: (16-20)

16. $\sin\left(\frac{dy}{dx}\right) = k \,;\, y(0) = 1$

17. $e^{dy/dx} = x + 1 \,;\; y(0) = 3$

18. $C'(x) = 2 + 0.15\, x \,;\, C(0) = 100$

19. $x \frac{dy}{dx} + 1 = 0 \,;\, y(-1) = 0$

20. $x(x^2 - 1) \frac{dy}{dx} = 1;\, y(2) = 0$ [CBSE 2012]

BASED ON LOTS

21. $\frac{dy}{dx} - x \sin^2 x = \frac{1}{x \log x}$

22. $\frac{dy}{dx} = x^5 \tan^{-1}(x^3)$

23. $\cos x \frac{dy}{dx} - \cos 2x = \cos 3x$

24. $\sqrt{a + x}\, dy + x\, dx = 0$

25. $(1 + x^2) \frac{dy}{dx} - x = 2 \tan^{-1} x$ [NCERT, CBSE 2007]

26. $(x^3 + x^2 + x + 1) \frac{dy}{dx} = 2x^2 + x$ [NCERT, CBSE 2010]

ANSWERS

1. $y = \frac{x^3}{3} + \frac{x^2}{2} - \log|x| + C$
2. $y = \frac{x^6}{6} + \frac{x^3}{3} - 2\log|x| + C$
3. $y + x^2 = \frac{1}{3} e^{3x} + C$
4. $y = \tan^{-1} x + C$
5. $y = 2 \tan \frac{x}{2} - x + C$
6. $y = \frac{x^2}{2} + x + 5 \log|x + 2| + C$
7. $y = x \tan^{-1} x - \frac{1}{2} \log|1 + x^2| + C$
8. $y = x(\log x - 1) + C$
9. $y = \frac{1}{2}(x^2 + 1) \tan^{-1} x - \frac{1}{2} x + C$
10. $y = \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + \frac{1}{10}(2x + 1)^{5/2} - \frac{1}{6}(2x + 1)^{3/2} + C$
11. $y + \log|\sin x + \cos x| = C$
12. $y = -\frac{1}{3} \operatorname{cosec}^3 x + C$
13. $y = \frac{1}{2} \sin^{-1}(x^2) + C$
14. $y = \frac{1}{2} x^2 \log x - \frac{x^2}{4} + C$
15. $y = x e^x - e^x - 2x + \frac{1}{4} \sin 2x + C$
16. $y - 1 = x \sin^{-1}(k)$
17. $y = (x + 1) \log(x + 1) - x + 3$
18. $C(x) = 2x + (0{\cdot}15) \frac{x^2}{2} + 100$
19. $y = -\log|x|$
20. $y = \log \frac{4}{3} \left(\frac{x^2 - 1}{x^2}\right)$
21. $y = \frac{x^2}{4} - \frac{x \sin 2x}{4} - \frac{\cos 2x}{8} + \log|\log x| + C$
22. $y = \frac{1}{6}\left(x^6 \tan^{-1} x^3 - x^3 + \tan^{-1} x^3\right) + C$
23. $y = \sin 2x - x + 2 \sin x - \log|\sec x + \tan x| + C$
24. $y + \frac{2}{3}(a + x)^{3/2} - 2a\sqrt{a + x} = C$
25. $y = \frac{1}{2} \log|1 + x^2| + (\tan^{-1} x)^2 + C$
26. $y = \frac{1}{2} \log|x + 1| + \frac{3}{4} \log(x^2 + 1) - \frac{1}{2} \tan^{-1} x + C$

23. We have, $\frac{dy}{dx} = \frac{\cos 3x + \cos 2x}{\cos x} \Rightarrow \frac{dy}{dx} = \frac{4\cos^3 x - 3\cos x + 2\cos^2 x - 1}{\cos x}$

$$\Rightarrow \frac{dy}{dx} = 4\cos^2 x - 3 + 2\cos x - \sec x = 2 + 2\cos 2x - 3 + 2\cos x - \sec x$$

$$\Rightarrow y = \sin 2x - x + 2\sin x - \log(\sec x + \tan x) + C$$

24. We have,

$$dy = -\frac{x}{\sqrt{a+x}}\,dx \Rightarrow dy = -\frac{(x+a-a)}{\sqrt{a+x}}\,dx = -\left\{\sqrt{a+x} + \frac{a}{\sqrt{a+x}}\right\}dx$$

Integrate both sides to obtain the general solution.

25. We have,

$$(1+x^2)\frac{dy}{dx} - x = 2\tan^{-1} x \Rightarrow (1+x^2)\frac{dy}{dx} = x + 2\tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{1+x^2} + 2\,\frac{\tan^{-1}}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1} x \cdot \frac{1}{1+x^2}$$

Integrating both sides, we get

$$y = \frac{1}{2}\log(1+x^2) + 2\int \tan^{-1} x\, d(\tan^{-1} x)$$

or, $y = \frac{1}{2}\log(1+x^2) + 2\,\frac{(\tan^{-1} x)^2}{2} + C$ or, $y = \frac{1}{2}\log(1+x^2) + (\tan^{-1} x)^2 + C$

26. We have,

$$(x^3+x^2+x+1)\frac{dy}{dx} = 2x^2+x \Rightarrow (x+1)(x^2+1)\frac{dy}{dx} = 2x^2+x \Rightarrow \frac{dy}{dx} = \frac{2x^2+x}{(x+1)(x^2+1)}$$

Integrating both sides with respect to x, we get

$$y = \int \frac{2x^2+x}{(x+1)(x^2+1)}\,dx \Rightarrow y = \int \frac{1}{2(x+1)} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}\,dx \qquad \text{[Using partial fractions]}$$

$$\Rightarrow y = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{3x-1}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{3x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$\Rightarrow y = \frac{1}{2}\log|x+1| + \frac{3}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$\Rightarrow y = \frac{1}{2}\log|x+1| + \frac{3}{4}\log|x^2+1| - \frac{1}{2}\tan^{-1} x + C$$

21.6.2 DIFFERENTIAL EQUATIONS OF THE TYPE $\frac{dy}{dx} = f(y)$

To solve this type of differential equations we integrate both sides to obtain the general solution as discussed under:

$$\frac{dy}{dx} = f(y) \Rightarrow \frac{dx}{dy} = \frac{1}{f(y)}, \text{ provided that } f(y) \neq 0 \Rightarrow dx = \frac{1}{f(y)}\,dy$$

Integrating both sides, we obtain

$$\int dx = \int \frac{1}{f(y)}\,dy + C \text{ or, } x = \int \frac{1}{f(y)}\,dy + C$$

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:*

(i) $\frac{dy}{dx}=\frac{1}{y^2+\sin y}$ (ii) $\frac{dy}{dx}=\sec y$

SOLUTION (i) We have,

$$\frac{dy}{dx}=\frac{1}{y^2+\sin y} \Rightarrow \frac{dx}{dy}=y^2+\sin y \Rightarrow dx=(y^2+\sin y)\,dy$$

Integrating both sides, we obtain

$$\int dx=\int (y^2+\sin y)\,dy \Rightarrow x=\frac{y^3}{3}-\cos y+C, \text{ which is the required solution.}$$

(ii) We have,

$$\frac{dy}{dx}=\sec y \Rightarrow \frac{dx}{dy}=\frac{1}{\sec y}=\cos y \Rightarrow dx=\cos y\,dy$$

Integrating both sides, we obtain

$$\int dx=\int \cos y\,dy \Rightarrow x=\sin y+C, \text{ which is the required solution.}$$

EXAMPLE 2 *Solve:* $\frac{dy}{dx}+y=1$

SOLUTION We have,

$$\frac{dy}{dx}+y=1 \Rightarrow \frac{dy}{dx}=1-y \Rightarrow \frac{dx}{dy}=\frac{1}{1-y} \Rightarrow dx=\frac{1}{1-y}\,dy$$

Integrating both sides, we get

$$\int dx=\int \frac{1}{1-y}\,dy \Rightarrow x=-\log|1-y|+C, \text{ which is the required solution.}$$

EXAMPLE 3 *Find the particular solution of the differential equation* $\frac{dy}{dx}=2y^2$, *given* $y=1$ *when* $x=1$.

[CBSE 2022]

SOLUTION We have,

$$\frac{dy}{dx}=2y^2 \Rightarrow \frac{dx}{dy}=\frac{1}{2y^2}$$

Integrating both sides with respect to y, we get

$$\int dx=\int \frac{1}{2y^2}\,dy \Rightarrow x=-\frac{1}{2y}+C \quad \text{...(i)}$$

It is given that $y=1$ when $x=1$. Putting $x=1$ and $y=1$ in (i), we get

$$1=-\frac{1}{2}+C \Rightarrow C=\frac{3}{2}$$

Putting $C=3/2$ in (i), we get

$$x=-\frac{1}{2y}+\frac{3}{2} \Rightarrow y=\frac{1}{3-2x}, \text{ which is the required particular solution.}$$

EXERCISE 21.6

BASIC

Solve the following differential equations:

1. $\dfrac{dy}{dx}+\dfrac{1+y^2}{y}=0$
2. $\dfrac{dy}{dx}=\dfrac{1+y^2}{y^3}$
3. $\dfrac{dy}{dx}=\sin^2 y$
4. $\dfrac{dy}{dx}=\dfrac{1-\cos 2y}{1+\cos 2y}$

ANSWERS

1. $x+\dfrac{1}{2}\log|1+y^2|=C$ 2. $x=\dfrac{y^2}{2}-\dfrac{1}{2}\log|y^2+1|+C$ 3. $x+\cot y=C$ 4. $x+\cot y+y=C$

21.6.3 EQUATIONS IN VARIABLE SEPARABLE FORM

If the differential equation can be put in the form $f(x)\,dx=g(y)\,dy$, we say that the variables are separable and such equations can be solved by integrating on both sides. The solution is given by $\int f(x)\,dx=\int g(y)\,dy+C$, where C is an arbitrary constant.

NOTE *There is no need of introducing arbitrary constants of integration on both sides as they can be combined together to give just one arbitrary constant.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:*

(i) $(x+1)\dfrac{dy}{dx}=2xy$ (ii) $\cos x\,(1+\cos y)\,dx-\sin y\,(1+\sin x)\,dy=0$

SOLUTION (i) Given differential equation is

$$(x+1)\frac{dy}{dx}=2xy \Rightarrow (x+1)\,dy=2xy\,dx \Rightarrow \frac{dy}{y}=\frac{2x}{x+1}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = 2\int \frac{x}{x+1}\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \int \frac{1}{y}\,dy = 2\int \frac{x+1-1}{x+1}\,dx \Rightarrow \int \frac{1}{y}\,dy = 2\int\left(1-\frac{1}{x+1}\right)dx$$

$\Rightarrow \log y=2\left\{x-\log|x+1|\right\}+C$, which is the solution of the given differential equation.

(ii) We have, $\cos x\,(1+\cos y)\,dx-\sin y\,(1+\sin x)\,dy=0$

$$\Rightarrow \frac{\cos x}{1+\sin x}\,dx-\frac{\sin y}{1+\cos y}\,dy=0$$

$$\Rightarrow \int\frac{\cos x}{1+\sin x}\,dx+\int\frac{-\sin y}{1+\cos y}\,dy=0 \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \log|1+\sin x|+\log|1+\cos y|=\log C$$

$$\Rightarrow \log\left\{|1+\sin x|\cdot|1+\cos y|\right\}=\log C$$

$$\Rightarrow |1+\sin x||1+\cos y|=C$$

$\Rightarrow (1+\sin x)(1+\cos y)=C$, which is the required solution.

EXAMPLE 2 *Solve:*

(i) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ **[NCERT, CBSE 2007]**

(ii) $e^x \sqrt{1-y^2}\, dx + \dfrac{y}{x}\, dy = 0$ **[CBSE 2012, 2014]**

SOLUTION (i) We have,

$$\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$$

$$\Rightarrow \quad \sec^2 x \tan y\, dx = -\sec^2 y \tan x\, dy$$

$$\Rightarrow \quad \frac{\sec^2 x}{\tan x}\, dx = -\frac{\sec^2 y}{\tan y}\, dy$$

$$\Rightarrow \quad \int \frac{\sec^2 x}{\tan x}\, dx = -\int \frac{\sec^2 y}{\tan y}\, dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|\tan x| = -\log|\tan y| + \log C$$

$$\Rightarrow \quad \log|(\tan x)(\tan y)| = \log C$$

$\Rightarrow \quad |\tan x \tan y| = C$, which is the solution of the given differential equation.

(ii) Given differential equation is

$$e^x \sqrt{1-y^2}\, dx + \frac{y}{x}\, dy = 0 \Rightarrow e^x \sqrt{1-y^2}\, dx = -\frac{y}{x}\, dy \Rightarrow x e^x\, dx = -\frac{y}{\sqrt{1-y^2}}\, dy$$

$$\Rightarrow \quad \int \underset{\text{I}}{x}\, \underset{\text{II}}{e^x}\, dx = -\int \frac{y}{\sqrt{1-y^2}}\, dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad xe^x - \int e^x\, dx = \frac{1}{2}\int \frac{dt}{\sqrt{t}}, \text{ where } t = 1-y^2$$

$$\Rightarrow \quad xe^x - e^x = \frac{1}{2}\left(\frac{t^{1/2}}{1/2}\right) + C$$

$\Rightarrow \quad xe^x - e^x = \sqrt{t} + C \Rightarrow xe^x - e^x = \sqrt{1-y^2} + C$ is the required solution.

EXAMPLE 3 *Solve the differential equation* $(1+e^{2x})\, dy + (1+y^2)\, e^x\, dx = 0$ *given that when* $x = 0$, $y = 1$. **[NCERT, CBSE 2004, 05]**

SOLUTION We are given that

$$(1+e^{2x})\, dy + (1+y^2)\, e^x dx = 0 \Rightarrow (1+e^{2x})\, dy = -(1+y^2)\, e^x dx \Rightarrow \frac{1}{1+y^2}\, dy = -\frac{e^x}{1+e^{2x}}\, dx$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\, dy = -\int \frac{e^x}{1+e^{2x}}\, dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{1+y^2}\, dy = -\int \frac{dt}{1+t^2}, \quad \text{where } t = e^x$$

$$\Rightarrow \quad \tan^{-1} y = -\tan^{-1}(t) + C \Rightarrow \tan^{-1} y = -\tan^{-1}(e^x) + C \qquad \text{...(i)}$$

It is given that $y = 1$, when $x = 0$. So, putting $x = 0$, $y = 1$ in (i), we get

$$\tan^{-1} 1 = -\tan^{-1}(e^0) + C \Rightarrow \frac{\pi}{4} = -\frac{\pi}{4} + C \Rightarrow C = \frac{\pi}{2}$$

Putting $C = \dfrac{\pi}{2}$ in (i), we obtain

$$\tan^{-1} y = -\tan^{-1}(e^x) + \frac{\pi}{2}$$

$$\Rightarrow \quad \tan^{-1} y + \tan^{-1}(e^x) = \frac{\pi}{2} \Rightarrow \tan^{-1} y = \frac{\pi}{2} - \tan^{-1}(e^x) \Rightarrow \tan^{-1} y = \cot^{-1}(e^x)$$

$$\Rightarrow \quad \tan^{-1} y = \tan^{-1}\left(\frac{1}{e^x}\right) \Rightarrow y = \frac{1}{e^x}$$, which is the required solution.

EXAMPLE 4 *Solve the differential equation:* $(1+y^2)(1+\log x)\,dx + x\,dy = 0$ *given that when* $x = 1$, $y = 1$. **[CBSE 2011]**

SOLUTION The given differential equation is

$$(1+y^2)(1+\log x)\,dx + x\,dy = 0$$

$$\Rightarrow \quad (1+\log x)(1+y^2)\,dx = -x\,dy \Rightarrow \frac{(1+\log x)}{x}\,dx = -\frac{1}{1+y^2}\,dy$$

$$\Rightarrow \quad \int \frac{1+\log x}{x}\,dx = -\int \frac{1}{1+y^2}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int t\,dt = -\int \frac{1}{1+y^2}\,dy, \quad \text{where } 1+\log x = t$$

$$\Rightarrow \quad \frac{t^2}{2} = -\tan^{-1} y + C \Rightarrow \frac{1}{2}(1+\log x)^2 = -\tan^{-1} y + C \qquad \text{...(i)}$$

It is given that when $x = 1$, $y = 1$. So, putting $x = 1$, $y = 1$ in (i), we obtain

$$\frac{1}{2}\left(1+\log 1\right)^2 = -\tan^{-1} 1 + C \Rightarrow \frac{1}{2} = -\frac{\pi}{4} + C \Rightarrow C = \frac{1}{2} + \frac{\pi}{4}$$

Putting $C = \frac{1}{2} + \frac{\pi}{4}$ in (i), we obtain

$$\frac{1}{2}(1+\log x)^2 = -\tan^{-1} y + \frac{1}{2} + \frac{\pi}{4} \Rightarrow \tan^{-1} y = \frac{\pi}{4} + \frac{1}{2} - \frac{1}{2}(1+\log x)^2$$

$$\Rightarrow \quad y = \tan\left\{\frac{\pi}{4} + \frac{1}{2} - \frac{1}{2}(1+\log x)\right\}$$, which is the solution of the given differential equation.

EXAMPLE 5 *Solve the differential equation* $x(1+y^2)\,dx - y(1+x^2)\,dy = 0$, *given that* $y = 0$, *when* $x = 1$. **[CBSE 2014]**

SOLUTION The given differential equation is

$$x(1+y^{2)}\,dx - y(1+x^2)\,dy = 0$$

$$\Rightarrow \quad x(1+y^2)\,dx = y(1+x^2)\,dy \Rightarrow \frac{x}{1+x^2}\,dx = \frac{y}{1+y^2}\,dy \Rightarrow \frac{2x}{1+x^2}\,dx = \frac{2y}{1+y^2}\,dy$$

Integrating both sides, we get

$$\int \frac{2x}{1+x^2}\,dx = \int \frac{2y}{1+y^2}\,dy$$

$$\Rightarrow \quad \log|1+x^2| = \log|1+y^2| + \log C$$

$$\Rightarrow \quad \log\left|\frac{1+x^2}{1+y^2}\right| = \log C \Rightarrow \frac{1+x^2}{1+y^2} = C \Rightarrow (1+x^2) = (1+y^2)\,C \qquad \text{...(i)}$$

It is given that when $x = 1$, $y = 0$. So, putting $x = 1$ and $y = 0$ in (i), we get

$$(1+1) = (1+0)\,C \Rightarrow C = 2$$

Putting $C = 2$ in (i), we get: $(1 + x^2) = 2(1 + y^2)$, which is the required solution.

EXAMPLE 6 *Solve the following differential equations:*

(i) $\frac{dy}{dx} = 1 + x + y + xy$ [CBSE 2014] (ii) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$ [CBSE 2002]

SOLUTION (i) We are given that

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1+x) + y(1+x) \Rightarrow \frac{dy}{dx} = (1+x)(1+y) \Rightarrow \frac{1}{1+y}\,dy = (1+x)\,dx$$

$$\Rightarrow \quad \int \frac{1}{1+y}\,dy = \int (1+x)\,dx \qquad \text{[Integrating both sides]}$$

$\Rightarrow \quad \log|1+y| = x + \frac{x^2}{2} + C$, which is the general solution of the given differential equation.

(ii) The given differential equation is

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad y - ay^2 = \frac{dy}{dx}(a+x) \Rightarrow (y - ay^2)\,dx = (a+x)\,dy \Rightarrow \frac{dx}{a+x} = \frac{dy}{y - ay^2}$$

$$\Rightarrow \quad \int \frac{1}{a+x}\,dx = \int \frac{1}{y - ay^2}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{a+x}\,dx = \int \left(\frac{1}{y} + \frac{a}{1 - ay}\right)dy \qquad \text{[By using partial fractions on RHS]}$$

$$\Rightarrow \quad \log|x+a| = \log|y| - \log|1 - ay| + \log C$$

$$\Rightarrow \quad \log\left|\frac{(x+a)(1-ay)}{y}\right| = \log C \Rightarrow \frac{(x+a)(1-ay)}{y} = C$$

$\Rightarrow \quad (x+a)(1-ay) = Cy$, which is the general solution of the given differential equation.

EXAMPLE 7 *Solve:*

(i) $(x^2 - yx^2)\,dy + (y^2 + x^2y^2)\,dx = 0$ [CBSE 2012, 2014]

(ii) $3e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$ [CBSE 2011, 2012, NCERT]

SOLUTION (i) The given differential equations is

$$x^2(1-y)\,dy + y^2(1+x^2)\,dx = 0$$

$$\Rightarrow \quad x^2(1-y)\,dy = -y^2(1+x^2)\,dx \Rightarrow \frac{1-y}{y^2}\,dy = -\left(\frac{1+x^2}{x^2}\right)dx, \text{ if } x, y \neq 0$$

$$\Rightarrow \quad \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \left(\frac{1}{x^2} + 1\right)dx$$

$$\Rightarrow \quad \int \left(\frac{1}{y} - \frac{1}{y^2}\right)dy = \int \left(\frac{1}{x^2} + 1\right)dx \qquad \text{[Integrating both sides]}$$

$\Rightarrow \quad \log|y| + \frac{1}{y} = -\frac{1}{x} + x + C$, which is the general solution of the differential equation.

(ii) We are given that

$$3e^x \tan y\,dx + (1-e^x)\sec^2 y\,dy = 0 \Rightarrow 3e^x \tan y\,dx = -(1-e^x)\sec^2 y\,dy$$

$$\Rightarrow \frac{3e^x}{1-e^x}\,dx = -\frac{\sec^2 y}{\tan y}\,dy$$

$$\Rightarrow 3\int \frac{e^x}{1-e^x}\,dx = -\int \frac{\sec^2 y}{\tan y}\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow 3\int \frac{e^x}{e^x-1}\,dx = \int \frac{\sec^2 y}{\tan y}\,dy$$

$$\Rightarrow 3\log|e^x-1| = \log|\tan y| + \log C \Rightarrow \log\left(\frac{|e^x-1|^3}{|\tan y|}\right) = \log C \Rightarrow \frac{(e^x-1)^3}{\tan y} = C$$

$\Rightarrow \quad (e^x-1)^3 = C\tan y$, which the general solution of the given differential equation.

EXAMPLE 8 *Solve:*

(i) $\sin^3 x \dfrac{dx}{dy} = \sin y$ (ii) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ [NCERT]

SOLUTION (i) The given differential equation is

$$\sin^3 x \frac{dx}{dy} = \sin y$$

$$\Rightarrow \sin^3 x\,dx = \sin y\,dy$$

$$\Rightarrow \int \sin^{3x} dx = \int \sin y\,dy \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \int \frac{3\sin x - \sin 3x}{4}\,dx = \int \sin y\,dy$$

$$\Rightarrow -\frac{3}{4}\cos x + \frac{1}{12}\cos 3x = -\cos y + C \Rightarrow \cos y - \frac{3}{4}\cos x + \frac{1}{12}\cos 3x = C$$

Hence, $\cos y - \frac{3}{4}\cos x + \frac{1}{12}\cos 3x = C$ gives the required solution.

(ii) The given differential equation is

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0 \Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,dy = -\sqrt{1-y^2}\,dx \Rightarrow \frac{1}{\sqrt{1-y^2}}\,dy = -\frac{1}{\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \int \frac{1}{\sqrt{1-y^2}}\,dy = -\int \frac{1}{\sqrt{1-x^2}}\,dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \sin^{-1} y = -\sin^{-1} x + \sin^{-1} C$$

$$\Rightarrow \sin^{-1} y + \sin^{-1} x = \sin^{-1} C \Rightarrow \sin^{-1}\left\{y\sqrt{1-x^2} + x\sqrt{1-y^2}\right\} = \sin^{-1} C$$

$\Rightarrow y\sqrt{1-x^2} + x\sqrt{1-y^2} = C$, which gives the general solution of the given differential equation.

EXAMPLE 9 *Solve:*

(i) $\dfrac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ (ii) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ [NCERT]

SOLUTION (i) We have,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} \Rightarrow dy = (e^{x-y} + x^2 e^{-y})\, dx \Rightarrow e^y\, dy = (e^x + x^2)\, dx$$

$$\Rightarrow \int e^y\, dy = \int (e^x + x^2)\, dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + C,$$ which is the required solution.

(ii) The given differential equation is

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow (1+x^2)\, dy = (1+y^2)\, dx \Rightarrow \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$$

$$\Rightarrow \int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \tan^{-1} y = \tan^{-1} x + \tan^{-1} C$$

$$\Rightarrow \tan^{-1} y - \tan^{-1} x = \tan^{-1} C \Rightarrow \tan^{-1}\left(\frac{y-x}{1+xy}\right) = \tan^{-1} C \Rightarrow \frac{y-x}{1+xy} = C$$

$$\Rightarrow y - x = C(1+xy),$$ which is the required solution.

EXAMPLE 10 *Solve:*

(i) $\frac{dy}{dx} = e^{x+y}$ (ii) $\log\left(\frac{dy}{dx}\right) = ax + by$

SOLUTION (i) The given differential equation is

$$\frac{dy}{dx} = e^{x+y} \Rightarrow \frac{dy}{dx} = e^x e^y \Rightarrow dy = e^x e^y\, dx \Rightarrow e^{-y}\, dy = e^x\, dx$$

$$\Rightarrow \int e^{-y}\, dx = \int e^x\, dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow -e^{-y} = e^x + C,$$ which is the required solution.

(ii) We are given that

$$\log\left(\frac{dy}{dx}\right) = ax + by$$

$$\Rightarrow \frac{dy}{dx} = e^{ax+by} \Rightarrow \frac{dy}{dx} = e^{ax} e^{by} \Rightarrow dy = e^{ax} e^{by}\, dx \Rightarrow e^{-by}\, dy = e^{ax}\, dx$$

$$\Rightarrow -\frac{1}{b} e^{-by} = \frac{1}{a} e^{ax} + C \qquad \text{[Integrating both sides]}$$

This is the required solution.

EXAMPLE 11 *Solve the initial value problem* $y' = y \cot 2x$, $y\left(\frac{\pi}{4}\right) = 2.$

SOLUTION Given differential equation is

$$y' = y \cot 2x \Rightarrow \frac{dy}{dx} = y \cot 2x \Rightarrow \frac{1}{y} dy = \cot 2x\, dx$$

$$\Rightarrow \log y = \frac{1}{2} \log \sin 2x + \log C \qquad \text{[Integrating both sides]}$$

$$\Rightarrow 2 \log y = \log \sin 2x + 2 \log C \Rightarrow \log y^2 = \log \sin 2x + \log C^2 \Rightarrow y^2 = C^2 \sin 2x \qquad ...(i)$$

It is given that $y = 2$ when $x = \pi/4$. Putting $x = \pi/4$ and $y = 2$ in (i), we get

$$4 = C^2 \sin\frac{\pi}{2} \Rightarrow C^2 = 4$$

Putting $C^2 = 4$ in (i), we get $y^2 = 4\sin 2x$ as the required solution.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 12 *Solve the initial value problem:* $x(x\,dy - y\,dx) = y\,dx$, $y(1) = 1$.

SOLUTION The given differential equation is

$$x(x\,dy - y\,dx) = y\,dx \Rightarrow x^2\,dy = y(x+1)\,dx \Rightarrow \frac{1}{y}\,dy = \frac{x+1}{x^2}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int\left(\frac{1}{x} + \frac{1}{x^2}\right)dx \Rightarrow \log|y| = \log|x| - \frac{1}{x} + C \Rightarrow \log|y| - \log|x| = -\frac{1}{x} + C$$

$$\Rightarrow \log\left\{\frac{|y|}{|x|}\right\} = -\frac{1}{x} + C \Rightarrow \log\left|\frac{y}{x}\right| = -\frac{1}{x} + C \Rightarrow \left|\frac{y}{x}\right| = e^{-1/x + C} \quad \text{...(i)}$$

It is given that $y = 1$ when $x = 1$. Putting $x = 1$ and $y = 1$ in (i), we get

$$1 = e^{-1+C} \Rightarrow e^0 = e^{-1+C} \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$\left|\frac{y}{x}\right| = e^{-1/x + 1} \Rightarrow \frac{y}{x} = \pm e^{1 - 1/x} \Rightarrow y = xe^{1-1/x} \text{ or } y = -xe^{1-1/x}$$

But, $y = -x\,e^{1-1/x}$ is not satisfied by $y(1) = 1$. Hence, $y = xe^{1-1/x}$, $x \neq 0$ is the required solution.

EXAMPLE 13 *Solve:* $\frac{dy}{dx} = y \sin 2x$, *it being given that* $y(0) = 1$. [CBSE 2004]

SOLUTION We have,

$$\frac{dy}{dx} = y \sin 2x \Rightarrow \frac{1}{y}\,dy = \sin 2x\,dx \Rightarrow \int \frac{1}{y}\,dy = \int \sin 2x\,dx \quad \text{[Integrating both sides]}$$

$$\Rightarrow \log|y| = -\frac{1}{2}\cos 2x + C \quad \text{...(i)}$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$. Putting $x = 0$ and $y = 1$ in (i), we get

$$0 = -\frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

Putting $C = \frac{1}{2}$ in (i), we get

$$\log|y| = -\frac{1}{2}\cos 2x + \frac{1}{2} \Rightarrow \log|y| = \frac{1}{2}(1 - \cos 2x) \Rightarrow \log|y| = \sin^2 x \Rightarrow |y| = e^{\sin^2 x}$$

$$\Rightarrow y = \pm e^{\sin^2 x} \Rightarrow y = e^{\sin^2 x} \text{ or, } y = -e^{\sin^2 x}$$

But, $y = -e^{\sin^2 x}$ is not satisfied by $y(0) = 1$. Hence, $y = e^{\sin^2 x}$ is the required solution.

EXAMPLE 14 *Solve the initial value problem:* $x\,dy + y\,dx = xy\,dx$, $y(1) = 1$

SOLUTION We have,

$$x\,dy + y\,dx = xy\,dx \Rightarrow x\,dy = (x-1)\,y\,dx \Rightarrow \frac{1}{y}\,dy = \left(1 - \frac{1}{x}\right)dx \Rightarrow \int \frac{1}{y}\,dy = \int\left(1 - \frac{1}{x}\right)dx$$

$$\Rightarrow \log|y| = x - \log|x| + C \Rightarrow \log|y| + \log|x| = x + C \Rightarrow \log|xy| = x + C$$

$$\Rightarrow \quad |xy| = e^{x+C} \qquad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. $y = 1$ when $x = 1$. Putting $x = 1$ and $y = 1$ in (i), we get

$$1 = e^{1+C} \Rightarrow e^0 = e^{1+C} \Rightarrow C = -1$$

Putting $C = -1$ in (i), we get

$$\therefore \quad |xy| = e^{x-1} \Rightarrow xy = \pm e^{x-1} \Rightarrow y = \pm \frac{1}{x} e^{x-1} \Rightarrow y = \frac{1}{x} e^{x-1} \text{ or, } y = -\frac{1}{x} e^{x-1}$$

But, $y = -\frac{1}{x} e^{x-1}$ is not satisfied by $y(1) = 1$. Also, $y = \frac{1}{x} e^{x-1}$ is defined for all $x \neq 0$.

Hence, $y = \frac{1}{x} e^{x-1}$, $x \in R - \{0\}$ is the required solution.

EXAMPLE 15 *Find the equation of the curve passing through the point* $(0, \pi/4)$ *whose differential equation is* $\sin x \cos y \, dx + \cos x \sin y \, dy = 0$. **[NCERT]**

SOLUTION We have,

$$\sin x \cos y \, dx + \cos x \sin y \, dy = 0$$

$$\Rightarrow \frac{\sin x}{\cos x} dx + \frac{\sin y}{\cos y} dy = 0 \qquad \text{[On separating the variables]}$$

$$\Rightarrow \tan x \, dx + \tan y \, dy = 0 \Rightarrow \int \tan x \, dx + \int \tan y \, dy = 0 \Rightarrow -\log|\cos x| - \log|\cos y| = \log C$$

$$\Rightarrow -\log\left(|\cos x|\,|\cos y|\right) = \log C \Rightarrow \log|\cos x \cos y| = \log\left(\frac{1}{C}\right) \Rightarrow \left|\cos x \cos y\right| = \frac{1}{C}$$

$$\Rightarrow \cos x \cos y = C_1 \text{ where } C_1 = \pm \frac{1}{C}. \qquad \text{...(i)}$$

It is given that the curve passes through $(0, \pi/4)$. Putting $x = 0$ and $y = \pi/4$ in (i), we get

$$\cos 0 \cos \frac{\pi}{4} = C_1 \Rightarrow C_1 = \frac{1}{\sqrt{2}}. \text{ Putting } C_1 = \frac{1}{\sqrt{2}} \text{ in (i), we get}$$

$$\cos x \cos y = \frac{1}{\sqrt{2}} \Rightarrow \cos y = \frac{1}{\sqrt{2}} \sec x \Rightarrow y = \cos^{-1}\left(\frac{1}{\sqrt{2}} \sec x\right)$$

Hence, $y = \cos^{-1}\left(\frac{1}{\sqrt{2}} \sec x\right)$ is the required curve.

EXAMPLE 16 *Solve the initial value problem:* $dy = e^{2x+y} dx$, $y(0) = 0$.

SOLUTION We have,

$$dy = e^{2x+y} dx \Rightarrow dy = e^{2x} \cdot e^y dx \Rightarrow e^{-y} dy = e^{2x} dx \qquad \text{[On separating the variables]}$$

$$\Rightarrow \int e^{-y} dy = \int e^{2x} dx \Rightarrow -e^{-y} = \frac{e^{2x}}{2} + C \qquad \text{...(i)}$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$. Putting $x = 0$ and $y = 0$ in (i), we get

$$-1 = \frac{1}{2} + C \Rightarrow C = -\frac{3}{2}$$

Putting $C = -\frac{3}{2}$ in (i), we get

$$-e^{-y} = \frac{e^{2x}}{2} - \frac{3}{2} \Rightarrow e^{-y} = \frac{3 - e^{2x}}{2} \Rightarrow e^y = \frac{2}{3 - e^{2x}} \Rightarrow y = \log\left(\frac{2}{3 - e^{2x}}\right)$$

This is the required solution.

EXAMPLE 17 *Solve the following initial value problems:*

(i) $(x+1)\dfrac{dy}{dx}=2e^{-y}-1,\ y(0)=0$ [CBSE 2012, 19, NCERT] (ii) $y-x\dfrac{dy}{dx}=2\left(1+x^2\dfrac{dy}{dx}\right),\ y(1)=1$

SOLUTION (i) We have,

$$(x+1)\frac{dy}{dx} = 2e^{-y}-1$$

$$\Rightarrow \quad (x+1)\,dy = (2e^{-y}-1)\,dx$$

$$\Rightarrow \quad \frac{1}{x+1}dx=\frac{1}{2e^{-y}-1}dy \qquad \text{[On separating the variables]}$$

$$\Rightarrow \quad \int\frac{1}{x+1}dx=\int\frac{1}{2e^{-y}-1}dy \Rightarrow \int\frac{1}{x+1}dx=\int\frac{e^y}{2-e^y}dy \Rightarrow \int\frac{1}{x+1}dx=-\int\frac{e^y}{e^y-2}dy$$

$$\Rightarrow \quad \log|x+1|=-\log|e^y-2|+\log C \Rightarrow \log|x+1|+\log|e^y-2|=\log C$$

$$\Rightarrow \quad \log|(x+1)(e^y-2)|=\log C \Rightarrow |(x+1)(e^y-2)| = C \qquad ...(i)$$

It is given that $y(0)=0$ i.e. $y=0$ when $x=0$. Putting $x=0$ and $y=0$ in (i), we get

$$|(0+1)(1-2)| = C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get

$$|(x+1)(e^y-2)|=1 \Rightarrow (x+1)(e^y-2)=-1 \quad \text{[In the neighbourhood of origin } e^y-2<0\text{]}$$

$$\Rightarrow \quad e^y-2=-\frac{1}{x+1} \Rightarrow e^y=\left(2-\frac{1}{x+1}\right) \Rightarrow y=\log\left(2-\frac{1}{x+1}\right),$$ which is the required solution.

(ii)
$$y-x\frac{dy}{dx} = 2\left(1+x^2\frac{dy}{dx}\right)$$

$$\Rightarrow \quad y-2 = 2x^2\frac{dy}{dx}+x\frac{dy}{dx} \Rightarrow y-2 = x(2x+1)\frac{dy}{dx} \Rightarrow (y-2)\,dx = x(2x+1)\,dy$$

$$\Rightarrow \quad \frac{1}{x(2x+1)}dx=\frac{1}{y-2}dy \Rightarrow \int\frac{1}{x(2x+1)}dx=\int\frac{1}{y-2}dy \Rightarrow \int\left(\frac{1}{x}-\frac{2}{2x+1}\right)dx=\int\frac{1}{y-2}dy$$

$$\Rightarrow \quad \log|x|-\log|2x+1| = \log|y-2|+\log C \Rightarrow \log\left|\frac{x}{2x+1}\right| = \log|y-2|+\log C$$

$$\Rightarrow \quad \log\left|\frac{x}{2x+1}\right|-\log|y-2|=\log C \Rightarrow \log\left|\frac{x}{2x+1}\times\frac{1}{y-2}\right|=\log C \Rightarrow \left|\frac{x}{(2x+1)(y-2)}\right|=C \qquad ...(i)$$

It is given that $y(1)=1$ i.e. $y=1$ when $x=1$. Putting $x=1$ and $y=1$ in (i), we get

$\left|-\dfrac{1}{3}\right|=C \Rightarrow C=\dfrac{1}{3}$. Putting $C = \dfrac{1}{3}$ in (i), we get

$$\left|\frac{x}{(2x+1)(y-2)}\right|=\frac{1}{3} \Rightarrow \frac{x}{(2x+1)(y-2)}=\pm\frac{1}{3} \Rightarrow y-2=\pm\frac{3x}{2x+1} \Rightarrow y=2\pm\frac{3x}{2x+1}$$

But, $y = 2 + \dfrac{3x}{2x+1}$ is not satisfied by $y(1) = 1$ Hence, $y = 2 - \dfrac{3x}{2x+1}$, where $x \neq -\dfrac{1}{2}$ is the required solution.

EXAMPLE 18 *Show that the general solution of the differential equation* $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$ *is given by* $x + y + 1 = A(1 - x - y - 2xy)$, *where A is a parameter.* **[NCERT]**

SOLUTION We have,

$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \quad \frac{1}{y^2+y+1}\,dy = -\frac{1}{x^2+x+1}\,dx \qquad \text{[On separating the variables]}$$

$$\Rightarrow \quad \frac{1}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dy = -\frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

Integrating both sides, we get

$$\int \frac{1}{\left(y+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dy = -\int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$\Rightarrow \quad \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = -\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + \tan^{-1}\left(\frac{2y+1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}C$$

$$\Rightarrow \quad \tan^{-1}\left\{\frac{2x+1+2y+1}{3-(2x+1)(2y+1)}\right\} = \frac{\sqrt{3}}{2}C \Rightarrow \frac{2x+2y+2}{2-2x-2y-4xy} = \tan\left(\frac{\sqrt{3}}{2}C\right)$$

$$\Rightarrow \quad x+y+1 = A(1-x-y-2xy), \text{ where } A = \tan\left(\frac{\sqrt{3}}{2}C\right), \text{ which is the required solution.}$$

EXAMPLE 19 *Find the particular solution of the differential equation:* $\log\left(\dfrac{dy}{dx}\right) = 3x + 4y$ *given that* $y = 0$ *when* $x = 0$. **[NCERT, CBSE 2014]**

SOLUTION We have,

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y \Rightarrow \frac{dy}{dx} = e^{3x+4y} \Rightarrow \frac{dy}{dx} = e^{3x}e^{4y} \Rightarrow e^{-4y}\,dy = e^{3x}\,dx$$

On integrating, we get

$$-\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} + C \Rightarrow 4e^{3x} + 3e^{-4y} + 12C = 0 \qquad \text{...(i)}$$

It is given that $y = 0$ when $x = 0$. Substituting $x = 0$ and $y = 0$ in (i), we get

$$4 + 3 + 12C = 0 \Rightarrow C = -\frac{7}{12}$$

Substituting the value of C in (i), we get $4e^{3x} + 3e^{-4y} - 7 = 0$ as a particular solution of the given differential equation.

EXAMPLE 20 *Find the equation of the curve passing through the point* (1, 1) *whose differential equation is:* $x\,dy = (2x^2 + 1)\,dx$.

SOLUTION We have,

$$x\,dy = (2x^2+1)\,dx \Rightarrow dy = \left(\frac{2x^2+1}{x}\right)dx \Rightarrow dy = \left(2x + \frac{1}{x}\right)dx$$

On integrating both sides, we get

$$\int dy = \int \left(2x + \frac{1}{x}\right)dx \Rightarrow y = x^2 + \log|x| + C \qquad \text{...(i)}$$

This equation represents the family of solution curves of the given differential equation. We have to find a particular member of this family which passes through the point (1, 1).

Substituting $x = 1$, $y = 1$ in (i), we get: $1 = 1 + 0 + C \Rightarrow C = 0$

Putting $C = 0$ in (i), we get $y = x^2 + \log|x|$ as the equation of the required curve.

EXAMPLE 21 *Find the equation of the curve passing through the point* (– 2, 3) *given that the slope of the tangent to the curve at any point* (x, y) *is* $\frac{2x}{y^2}$.

SOLUTION We know that the slope of the tangent to a curve at any points (x, y) is given by $\frac{dy}{dx}$.

It is given that

$$\frac{dy}{dx} = \frac{2x}{y^2} \Rightarrow y^2\,dy = 2x\,dx$$

On integrating both sides, we get

$$\int y^2\,dy = \int 2x\,dx \Rightarrow \frac{y^3}{3} = x^2 + C \qquad \text{...(i)}$$

This equation represents the family of solution curves of given differential equation. We have to find a particular member of this family which passes through the point (– 2, 3).

Substituting $x = -2$ and $y = 3$ in (i), we get: $9 = 4 + C \Rightarrow C = 5$.

Putting $C = 5$ in (i), we get $\frac{y^3}{3} = x^2 + 5$ as the equation of the required curve.

EXAMPLE 22 *In a bank principal increases at the rate of 5% per year. In how many years ₹ 1000 double itself.* **[NCERT]**

SOLUTION Let P be the principal at any time t. Then,

$$\frac{dP}{dt} = \frac{5P}{100} \Rightarrow \frac{dP}{dt} = \frac{P}{20} \Rightarrow \frac{1}{P}\,dP = \frac{1}{20}\,dt$$

Integrating both sides, we get

$$\int \frac{1}{P}\,dP = \int \frac{1}{20}\,dt \Rightarrow \log P = \frac{1}{20}t + \log C \Rightarrow \log\frac{P}{C} = \frac{1}{20}t \Rightarrow P = C\,e^{t/20} \qquad \text{...(i)}$$

It is given that $P = 1000$ when $t = 0$. Substituting these values in (i), we get: $1000 = C$

Substituting $C = 1000$ in (i), we get: $P = 1000\,e^{t/20}$ ...(ii)

Let t_1 years be the time required to double the principal i.e. at $t = t_1$, $P = 2000$.

Substituting these values in (ii), we get

$$2000 = 1000\, e^{t_1/20} \Rightarrow e^{t_1/20} = 2 \Rightarrow \frac{t_1}{20} = \log_e 2 \Rightarrow t_1 = 20 \log_e 2$$

Hence, the principal doubles in 20 $\log_e$ 2 years.

EXAMPLE 23 *Find the equation of the curve passing through the point $(0, -2)$ given that at any point (x, y) on the curve the product of the slope of its tangent and y coordinate of the point is equal to the x-coordinate of the point.* **[NCERT]**

SOLUTION We know that the slope of the tangent at any point (x, y) on the curve is given by $\frac{dy}{dx}$. According to the given statement, we have

$$y \frac{dy}{dx} = x \quad \text{...(i)}$$

$$\Rightarrow \quad y\, dy = x\, dx$$

On integrating both sides, we get

$$\int y\, dy = \int x\, dx \Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \quad \text{...(ii)}$$

This is the equation of the family of solution curves of differential equation (i). We have to find a particular member of the family which passes through $(0, -2)$.

Substituting $x = 0$ and $y = -2$ in (ii), we get: $\frac{4}{2} = 0 + C \Rightarrow C = 2$

Putting $C = 2$ in (ii), we get:

$$\frac{y^2}{2} = \frac{x^2}{2} + 2 \Rightarrow y^2 = x^2 + 4, \text{ which is the equation of the required curve.}$$

EXAMPLE 24 *At any point (x, y) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point $(-4, -3)$. Find the equation of the curve given that it passes through $(-2, 1)$.* **[NCERT]**

SOLUTION The slope of the tangent at any point $P(x, y)$ is given by $\frac{dy}{dx}$. The slope of the line segment joining $P(x, y)$ and $A(-4, -3)$ is $\frac{y+3}{x+4}$.

According to the given condition, we have

$$\frac{dy}{dx} = 2\left(\frac{y+3}{x+4}\right) \quad \text{...(i)}$$

$$\Rightarrow \quad \frac{1}{y+3}\, dy = \frac{2}{x+4}\, dx$$

On integrating both sides, we get

$$\int \frac{1}{y+3}\, dy = 2\int \frac{1}{x+4}\, dx \Rightarrow \log(y+3) = 2\log(x+4) + \log C \Rightarrow (y+3) = C(x+4)^2 \quad \text{...(ii)}$$

This represents the family of solutions of differential equation (i). We have to find a particular member of this family which passes through $(-2, 1)$.

Substituting $x = -2$, $y = 1$ in (ii), we get: $4 = C(-2+4)^2 \Rightarrow C = 1$.

Putting $C = 1$ in (ii), we get $y + 3 = (x+4)^2$ as the required equation of the curve.

EXAMPLE 25 *Find the equation of the curve through the point $(1, 0)$ if the slope of the tangent to the curve at any point (x, y) is $\frac{y-1}{x^2+x}$.* **[NCERT EXEMPLAR]**

SOLUTION Let $P(x, y)$ be an arbitrary point on the curve. Then, the slope of the tangent at P is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{y-1}{x^2+x} \Rightarrow \frac{1}{x^2+x}\,dx = \frac{1}{y-1}\,dy \Rightarrow \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \frac{1}{y-1}\,dy$$

Integrating, we obtain

$$\log x - \log(x+1) = \log(y-1) + \log C \Rightarrow \log\left(\frac{x}{x+1}\right) = \log C\,(y-1) \Rightarrow \frac{x}{x+1} = C(y-1) \quad \ldots(i)$$

It is given that the curve passes through the point $(1, 0)$. So, putting $x = 1$, $y = 0$ in (i), we obtain

$$\frac{1}{2} = C(0-1) \Rightarrow C = -\frac{1}{2}$$

Putting $C = -1/2$ in (i), we obtain $(x+1)(y-1) + 2x = 0$ as the equation of the curve.

EXERCISE 21.7

BASIC

Solve the following differential equations:

1. $(x-1)\frac{dy}{dx} = 2\,xy$
2. $(1+x^2)\,dy = xy\,dx$
3. $\frac{dy}{dx} = (e^x + 1)\,y$
4. $(x-1)\frac{dy}{dx} = 2x^3 y$
5. $xy\,(y+1)\,dy = (x^2+1)\,dx$
6. $e^{y-x}\frac{dy}{dx} = 1$ or, $\log\left(\frac{dy}{dx}\right) = x - y$ [CBSE 2020, 2022]
7. $x\cos y\,dy = (xe^x \log x + e^x)\,dx$ [CBSE 2007, 2022]
8. $\frac{dy}{dx} = e^{x+y} + x^2 e^y$
9. $x\frac{dy}{dx} + y = y^2$
10. $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$
11. $x\cos^2 y\,dx = y\cos^2 x\,dy$
12. $xy\,dy = (y-1)(x+1)\,dx$
13. $x\frac{dy}{dx} + \cot y = 0$
14. $\frac{dy}{dx} = \frac{x e^x \log x + e^x}{x\cos y}$
15. $\frac{dy}{dx} = e^{x+y} + e^y x^3$
16. $y\sqrt{1+x^2} + x\sqrt{1+y^2}\,\frac{dy}{dx} = 0$
17. $\sqrt{1+x^2}\,dy + \sqrt{1+y^2}\,dx = 0$
18. $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0$ [CBSE 2010]
19. $\frac{dy}{dx} = \frac{e^x(\sin^2 x + \sin 2x)}{y\,(2\log y + 1)}$
20. $\frac{dy}{dx} = \frac{x\,(2\log x + 1)}{\sin y + y\cos y}$ [CBSE 2014]
21. $(1-x^2)\,dy + xy\,dx = xy^2\,dx$
22. $\tan y\,dx + \sec^2 y\tan x\,dy = 0$
23. $(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$
24. $\tan y\frac{dy}{dx} = \sin(x+y) + \sin(x-y)$
25. $\cos x\cos y\frac{dy}{dx} = -\sin x\sin y$
26. $\frac{dy}{dx} + \frac{\cos x\sin y}{\cos y} = 0$
27. $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

28. $y(1+e^x)\,dy = (y+1)\,e^x\,dx$ [CBSE 2020]
29. $(y+xy)dx + (x-xy^2)dy = 0$ [CBSE 2002]

30. $\frac{dy}{dx} = 1 - x + y - xy$ [CBSE 2002C]

BASED ON LOTS

31. $(y^2+1)\,dx - (x^2+1)\,dy = 0$
32. $dy + (x+1)(y+1)\,dx = 0$

33. $\frac{dy}{dx} = (1+x^2)(1+y^2)$
34. $(x-1)\frac{dy}{dx} = 2x^3 y$

35. $\frac{dy}{dx} = e^{x+y} + e^{-x+y}$
36. $\frac{dy}{dx} = (\cos^2 x - \sin^2 x)\cos^2 y$

37. (i) $(xy^2+2x)\,dx + (x^2y+2y)\,dy = 0$ (ii) $\operatorname{cosec} x \log y \frac{dy}{dx} + x^2y^2 = 0$ [CBSE 2014]

38. (i) $xy\frac{dy}{dx} = 1 + x + y + xy$ (ii) $y(1-x^2)\frac{dy}{dx} = x(1+y^2)$ [CBSE 2007]

(iii) $ye^{x/y}dx = (xe^{x/y} + y^2)\,dy,\ y \neq 0$ [NCERT EXEMPLAR]

(iv) $(1+y^2)\tan^{-1}x\,dx + 2y(1+x^2)\,dy = 0$ [NCERT EXEMPLAR]

Solve the following initial value problems: (39-45)

39. $\frac{dy}{dx} = y\tan 2x,\ y(0) = 2$
40. $2x\frac{dy}{dx} = 3y,\ y(1) = 2$

41. $xy\frac{dy}{dx} = y + 2,\ y(2) = 0$
42. $\frac{dy}{dx} = 2e^x y^3,\ y(0) = \frac{1}{2}$

43. $\frac{dr}{dt} = -rt,\ r(0) = r_0$
44. $\frac{dy}{dx} = y\sin 2x,\ y(0) = 1$

45. (i) $\frac{dy}{dx} = y\tan x,\ y(0) = 1$ [CBSE 2010] (ii) $2x\frac{dy}{dx} = 5y,\ y(1) = 1$

(iii) $\frac{dy}{dx} = 2e^{2x}y^2,\ y(0) = -1$ (iv) $\cos y\frac{dy}{dx} = e^x,\ y(0) = \frac{\pi}{2}$

(v) $\frac{dy}{dx} = 2xy,\ y(0) = 1$

(vi) $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2,\ y(0) = 1$ [CBSE 2012, 2019]

(vii) $xy\frac{dy}{dx} = (x+2)(y+2),\ y(1) = -1$ [CBSE 2012]

(viii) $\frac{dy}{dx} = 1 + x + y^2 + xy^2$ when $y = 0,\ x = 0$ [NCERT EXEMPLAR]

(ix) $2(y+3) - xy\frac{dy}{dx} = 0,\ y(1) = -2$ [NCERT EXEMPLAR]

(x) $e^x\tan y\,dx + (2-e^x)\sec^2 y\,dy = 0,\ y(0) = \frac{\pi}{4}$ [CBSE 2018]

46. Solve the differential equation $x\frac{dy}{dx} + \cot y = 0$, given that $y = \frac{\pi}{4}$, when $x = \sqrt{2}$.

47. Solve the differential equation $(1+x^2)\frac{dy}{dx} + (1+y^2) = 0$, given that $y = 1$, when $x = 0$.

48. Solve the differential equation $\frac{dy}{dx} = \frac{2x(\log x + 1)}{\sin y + y\cos y}$, given that $y = 0$, when $x = 1$.

49. Find the particular solution of $e^{dy/dx} = x + 1$, given that $y = 3$, when $x = 0$.

50. Find the solution of the differential equation $\cos y\, dy + \cos x \sin y\; dx = 0$ given that $y = \pi/2$, when $x = \pi/2$.

51. Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

52. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $\frac{dy}{dx} = e^x \sin x$. [NCERT]

53. For the differential equation $xy\frac{dy}{dx} = (x+2)(y+2)$. Find the solution curve passing through the point $(1, -1)$. [CBSE 2012, NCERT]

54. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t seconds. [NCERT]

55. In a bank principal increases at the rate of $r\%$ per year. Find the value of r if ₹ 100 double itself in 10 years ($\log_e 2 = 0.6931$). [NCERT]

56. In a bank principal increases at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$). [NCERT]

57. In a culture the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present. [NCERT]

58. If $y(x)$ is a solution of the different equation $\left(\frac{2+\sin x}{1+y}\right)\frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y(\pi/2)$. [CBSE 2014, NCERT EXEMPLAR]

59. Find the particular solution of the differential equation $(1-y^2)(1+\log x)\,dx + 2xy\,dy = 0$ given that $y = 0$ when $x = 1$. [CBSE 2016]

ANSWERS

1. $2x + 2\log|x-1| = \log y + C$
2. $y = C\sqrt{1+x^2}$
3. $\log|y| = e^x + x + C$
4. $\log|y| = \frac{2}{3}x^3 + x^2 + 2x + 2\log|x-1| + C$
5. $\frac{y^3}{3} + \frac{y^2}{2} = \frac{x^2}{2} + \log|x| + C$
6. $e^y = e^x + C$
7. $\sin y = e^x \log x + C$
8. $-e^{-y} = e^x + \frac{x^3}{3} + C$
9. $y - 1 = C\,xy$
10. $(e^y + 1)\sin x = C$
11. $x\tan x - y\tan y = \log|\sec x| - \log|\sec y| + C$
12. $y - x = \log|x| - \log|y-1| + C$
13. $x = C\cos y$
14. $\sin y = e^x \log x + C$
15. $e^x + e^{-y} + \frac{x^4}{4} = C$
16. $\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| + \frac{1}{2}\log\left|\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right| = C$
17. $\left(y+\sqrt{1+y^2}\right)\left(x+\sqrt{1+x^2}\right) = C$
18. $\sqrt{1+y^2} + \sqrt{1+x^2} + \frac{1}{2}\log\left|\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right| = C$

19. $y^2 \log y = e^x \sin^2 x + C$

20. $y \sin y = x^2 \log x + C$

21. $\log|y-1| - \log|y| = -\frac{1}{2}\log|1-x^2| + C$

22. $\sin x \tan y = C$

23. $\tan^{-1} x + \tan^{-1} y + \frac{1}{2}\log\{(1+x^2)(1+y^2)\} = C$

24. $2\cos x + \sec y = C$

25. $\sin y = C\cos x.$

26. $\log|\sin y| = -\sin x + C$

27. $\sqrt{1-x^2} + \sqrt{1-y^2} = C$

28. $y - \log|y+1| = \log|1+e^x| + C$

29. $\log x + x + \log y - \frac{1}{2}y^2 = C$

30. $\log(1+y) = x - \frac{x^2}{2} + C$

31. $\tan^{-1} x - \tan^{-1} y = C$

32. $\log|y+1| + \frac{x^2}{2} + x = C$

33. $\tan^{-1} y = x + \frac{x^3}{3} + C$

34. $y = C|x-1|^2 e^{(2/3)x^3 + x^2 + 2x}$

35. $e^{-x} - e^{-y} = e^x + C$

36. $\tan y = \frac{1}{2}\sin 2x + C$

37. (i) $y^2 + 2 = \frac{A}{x^2+2}$

(ii) $-\left(\frac{1+\log y}{y}\right) - x^2\cos x + 2(x\sin x + \cos x) = C$

38. (i) $y = x + \log x(1+y) + C$

(ii) $(1+y^2)(1-x^2) = C$

(iii) $e^{x/y} = y + C$

(iv) $\frac{1}{2}(\tan^{-1} x)^2 + \log(1+y^2) = C$

39. $y = \frac{2}{\sqrt{\cos 2x}}$

40. $y^2 = 4x^3$

41. $y - 2\log(y+2) = \log\left(\frac{x}{8}\right)$

42. $y^2(8-4e^x) = 1$

43. $r = r_0\, e^{-t^2/2}$

44. $y = e^{\sin^2 x}$

45. (i) $y = \sec x,\ x \in (-\pi/2, \pi/2)$

(ii) $y = |x|^{5/2}$

(iii) $y = -e^{-2x}$

(iv) $y = \sin^{-1}(e^x)$

(v) $y = e^{x^2}$

(vi) $\tan^{-1} y = x + x^3 + \frac{\pi}{4}$

(vii) $y - 2\log(y+2) = x + 2\log x - 2$

(viii) $y = \tan\left(x + \frac{x^2}{2}\right)$

(ix) $x^2(y+3)^3 = e^{y+2}$

(x) $\tan y = (2 - e^x)$

46. $x = 2\cos y$

47. $x + y = 1 - xy$

48. $2y\sin y = 2x^2\log x + x^2 - 1$

49. $y = (x+1)\log|x+1| - x + 3$

50. $\log\sin y + \sin x = 1$

51. $y = \frac{1}{2x^2+1}$

52. $2y = e^x(\sin x - \cos x) + 1$

53. $y - x + 2 = \log\left\{x^2(y+2)^2\right\}$

54. $r = (63t + 27)^{1/3}$ 55. 6.931% 56. ₹1648 57. $\dfrac{2 \log 2}{\log \dfrac{11}{10}}$

58. $(1 + y)(2 + \sin x) = 2,\ y(\pi/2) = -1/3$ 59. $-2 \log (1 - y^2) + 2 \log x + (\log x)^2 = 0$

HINTS TO SELECTED PROBLEMS

38. (iii) We have,

$$y e^{x/y} dx - x e^{x/y}\, dy = y^2\, dy$$

$$\Rightarrow \quad (y\, dx - x\, dy)\, e^{x/y} = y^2\, dy \Rightarrow \left(\frac{y\, dx - x\, dy}{y^2}\right) e^{x/y} = dy \Rightarrow e^{x/y}\, d\left(\frac{x}{y}\right) = dy$$

Integrating both sides, we obtain: $e^{x/y} = y + C$, which is the required solution.

52. The differential equation of the curve is $\dfrac{dy}{dx} = e^x \sin x$. Integrating both sides with respect to x, we get

$$y = \int e^x \sin x \ dx$$

$$\Rightarrow \quad y = \frac{e^x}{2} (\sin x - \cos x) + C \quad ...(i) \qquad \left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)\right]$$

It is given that the curve given by equation (i) passes through (0, 0).

$$\therefore \quad 0 = \frac{1}{2} (0 - 1) + C \Rightarrow C = \frac{1}{2} \qquad \text{[Putting } x = 0, y = 0 \text{ in (i)]}$$

Putting $C = \dfrac{1}{2}$ in (i), we get: $y = \dfrac{1}{2}\left\{e^x (\sin x - \cos x) + 1\right\}$ as the equation of the curve.

53. We have, $xy \dfrac{dy}{dx} = (x + 2)(y + 2) \Rightarrow \dfrac{y}{y + 2} dy = \dfrac{x + 2}{x}\, dx \Rightarrow \left(1 - \dfrac{2}{y + 2}\right) dy = \left(1 + \dfrac{2}{x}\right) dx$

$$\Rightarrow \int \left(1 - \frac{2}{y + 2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx \Rightarrow y - 2 \log (y + 2) = x + 2 \log x + C \quad ...(i)$$

It is given that the solution curve given by (i) passes through $(1, -1)$. Putting $x = 1$ and $y = -1$ in (i), we get: $-1 - 2 \log 1 = 1 + 2 \log 1 + C \Rightarrow C = -2$.

Putting $C = -2$ in (i), we get: $y - x + 2 = 2 \log \{x (y + 2)\}$ as the solution curve.

54. Let r be the radius and V be the volume of the balloon at any instant 't'. Then, $V = \dfrac{4}{3} \pi r^3$

It is given that

$$\frac{dV}{dt} = -\lambda, \text{ where } \lambda > 0 \Rightarrow \frac{d}{dt}\left(\frac{4}{3} \pi r^3\right) = -\lambda \Rightarrow 4\pi r^2 \frac{dr}{dt} = -\lambda \Rightarrow 4\pi r^2\, dr = -\lambda\, dt$$

On integrating, we get: $\dfrac{4}{3} \pi r^3 = -\lambda t + C$ ...(i)

It is given that at $t = 0$, $r = 3$. Putting $t = 0$ and $r = 3$ in (i), we get: $C = 36\pi$

Putting $C = 36\pi$ in (i), we get: $\dfrac{4}{3} \pi r^3 = -\lambda t + 36 \pi$ ...(ii)

It is also given that at $t = 3$, $r = 6$. Putting $t = 3$ and $r = 6$ in (i), we get (given)

$$288\pi = -3\lambda + 36\pi \Rightarrow \lambda = -84\pi$$

Putting $\lambda = -84\pi$ in (ii), we get

$$\frac{4}{3}\pi r^3 = 84\pi t + 36\pi \Rightarrow r^3 = 63t + 27 \Rightarrow r = (63t+27)^{1/3}$$

55. Let P be the principal at any instant t. It is given that

$$\frac{dP}{dt} = \frac{r}{100}P \Rightarrow \frac{dP}{P} = \frac{r}{100}dt \Rightarrow \log P = \frac{rt}{100} + C \quad \text{...(i)}$$

Initially i.e. at $t = 0$, let $P = P_0$. Putting $t = 0$ and $P = P_0$ in (i), we get: $\log P_0 = C$

Putting $C = \log P_0$ in (i), we obtain

$$\log P = \frac{rt}{100} + \log P_0 \Rightarrow \log \frac{P}{P_0} = \frac{rt}{100} \quad \text{...(ii)}$$

Substituting $P_0 = 100$, $P = 2P_0 = 200$ and $t = 10$ in (ii), we get

$$\log 2 = \frac{r}{10} \Rightarrow r = 10 \log 2 = 10 \times 0.6931 = 6.931$$

56. Let at any point of time t, the principal be P.

It is given that the principal increases at the rate of 5% per year.

$$\therefore \quad \frac{dP}{dt} = \frac{5P}{100} \Rightarrow \frac{dP}{P} = \frac{1}{20}dt \Rightarrow \log P = \frac{t}{20} + \log C \quad \text{...(i)}$$

Initially i.e. at $t = 0$, it is given that $P =$ ₹ 1000

$$\therefore \quad \log 1000 = \log C$$

Substituting the value of $\log C$ in (i), we get

$$\log P = \frac{t}{20} + \log 1000$$

Putting $t = 10$, we get: $\log \frac{P}{1000} = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} \Rightarrow P = 1000 \times 1.648 = 1648$

57. Let at any time the bacteria count be N. It is given that

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = \lambda N \Rightarrow \frac{1}{N}dN = \lambda\, dt$$

On integrating, we get

$$\log N = \lambda t + \log C \quad \text{...(i)}$$

It is given that at $t = 0$, $N = 100000$

$$\therefore \quad \log C = \log 100000$$

Putting the value of $\log C$ in (i), we get

$$\log N = \lambda t + \log 100000$$

It is also given that at $t = 2$, $N = 110000$. Putting $t = 2$, $N = 110000$ in (i), we get

$$\log 110000 = 2\lambda + \log 100000 \Rightarrow \frac{1}{2}\log\frac{11}{10} = \lambda$$

Substituting the values of $\log C$ and λ in (i), we get

$$\log N = \frac{1}{2}\log\left(\frac{11}{10}\right)t + \log 100000 \quad \text{...(ii)}$$

When $N = 200000$, let $t = T$. Substituting these values in (ii), we get

$$\log 200000 = \frac{T}{2}\log\left(\frac{11}{10}\right) + \log 100000 \Rightarrow \log 2 = \frac{T}{2}\log\frac{11}{10} \Rightarrow T = 2\,\frac{\log 2}{\log \frac{11}{10}}$$

21.6.4 EQUATIONS REDUCIBLE TO VARIABLE SEPARABLE FORM

Differential equations of the form $\frac{dy}{dx} = f(ax + by + c)$ can be reduced to variable separable form by the substitution $ax + by + c = v$ as discussed in the following examples.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:*

(i) $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$ (ii) $\frac{dy}{dx} = \cos(x+y)$ (iii) $\frac{dy}{dx} = (4x + y + 1)^2$

SOLUTION (i) We are given that

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x+y)$$

Let $x + y = v$. Then, $1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\therefore \quad \frac{dv}{dx} - 1 = \sin v \Rightarrow \frac{dv}{dx} = 1 + \sin v \Rightarrow \frac{1}{1 + \sin v}\, dv = dx$$

$$\Rightarrow \quad \int \frac{1}{1+\sin v}\, dv = \int dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int dx = \int \frac{1 - \sin v}{1 - \sin^2 v}\, dv \Rightarrow \int dx = \int \frac{1 - \sin v}{\cos^2 v}\, dv \Rightarrow \int dx = \int (\sec^2 v - \tan v \sec v)\, dv$$

$$\Rightarrow \quad x = \tan v - \sec v + C \Rightarrow x = \tan(x+y) - \sec(x+y) + C,$$ which is the required solution.

(ii) We are given that $\frac{dy}{dx} = \cos(x+y)$. Let $x + y = v$. Then, $1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Putting $x + y = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ in the given differential equation, we get

$$\frac{dv}{dx} - 1 = \cos v \Rightarrow \frac{dv}{dx} = 1 + \cos v \Rightarrow \frac{1}{1+\cos v}\, dv = dx$$

$$\Rightarrow \quad \frac{1}{2}\sec^2\frac{v}{2}\, dv = dx. \Rightarrow \int \frac{1}{2}\sec^2\frac{v}{2}\, dv = \int 1 \cdot dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \tan\frac{v}{2} = x + C \Rightarrow \tan\left(\frac{x+y}{2}\right) = x + C,$$ which is the required solution.

(iii) We are given that $\frac{dy}{dx} = (4x + y + 1)^2$.

Let $4x + y + 1 = v$, Then, $4 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 4$

Putting $4x + y + 1 = v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 4$ in the given differential equation, we get

$$\frac{dv}{dx} - 4 = v^2 \Rightarrow \frac{dv}{dx} = v^2 + 4 \Rightarrow dv = (v^2 + 4)\, dx$$

$$\Rightarrow \quad \frac{1}{v^2+4}\,dv = dx \Rightarrow \int \frac{1}{v^2+4}\,dv = \int 1 \cdot dx \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C \Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + C, \text{ which is the required solution.}$$

EXAMPLE 2 *Solve:* $(x+y)^2 \frac{dy}{dx} = a^2$

SOLUTION Let $x+y=v$. Then, $1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Putting $x+y=v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ the given differential equation, we get

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2 \Rightarrow v^2 \frac{dv}{dx} = a^2 + v^2$$

$$\Rightarrow v^2\,dv = (a^2+v^2)\,dx \Rightarrow \frac{v^2}{v^2+a^2}\,dv = dx \qquad \text{[By separating the variables]}$$

$$\Rightarrow \left(1 - \frac{a^2}{v^2+a^2}\right) dv = dx \Rightarrow \int 1 \cdot dv - a^2 \int \frac{1}{v^2+a^2}\,dv = \int dx + C \qquad \text{[On integration]}$$

$$\Rightarrow v - a\tan^{-1}\left(\frac{v}{a}\right) = x + C \Rightarrow (x+y) - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + C, \text{ which is the required solution.}$$

EXAMPLE 3 *Solve the initial value problem:* $\cos(x+y)\,dy = dx,\ y(0) = 0$.

SOLUTION The given differential equation can be written as $\frac{dy}{dx} = \frac{1}{\cos(x+y)}$

Let $x+y=v$. Then, $1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Putting $x+y=v$ and $\frac{dy}{dx} = \frac{dv}{dx} - 1$ the given differential equation, we get

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{1}{\cos v} \Rightarrow \frac{dv}{dx} = \frac{1+\cos v}{\cos v} \Rightarrow \frac{\cos v}{1+\cos v}\,dv = dx \Rightarrow \frac{\cos v\,(1-\cos v)}{1-\cos^2 v}\,dv = dx$$

$$\Rightarrow \quad (\cot v \operatorname{cosec} v - \cot^2 v)\,dv = dx \Rightarrow (\cot v \operatorname{cosec} v - \operatorname{cosec}^2 v + 1)\,dv = dx$$

$$\Rightarrow \quad -\operatorname{cosec} v + \cot v + v = x + C \qquad \text{[On integrating]}$$

$$\Rightarrow \quad -\operatorname{cosec}(x+y) + \cot(x+y) + x + y = x + C$$

$$\Rightarrow \quad -\operatorname{cosec}(x+y) + \cot(x+y) + y = C$$

$$\Rightarrow \quad -\frac{1-\cos(x+y)}{\sin(x+y)} + y = C \Rightarrow -\tan\left(\frac{x+y}{2}\right) + y = C \qquad \text{...(i)}$$

It is given that: $y(0)=0$ i.e. $y=0$ when $x=0$. Putting $x=0$ and $y=0$ in (i), we get $C=0$.
Putting $C=0$ in (i), we get

$$-\tan\left(\frac{x+y}{2}\right) + y = 0 \Rightarrow y = \tan\left(\frac{x+y}{2}\right), \text{ which is the required solution.}$$

EXAMPLE 4 *Solve:* $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$.

SOLUTION Let $x+y=v$. Then, $1+\frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$

Putting $x+y=v$ and $\frac{dy}{dx}=\frac{dv}{dx}-1$ in the given differential equation, we get

$$\frac{dv}{dx}-1 = \cos v+\sin v$$

$$\Rightarrow \quad \frac{dv}{dx} = 1+\cos v+\sin v \Rightarrow \frac{1}{1+\cos v+\sin v}\,dv = dx \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{1+\cos v+\sin v}\,dv = \int 1\cdot dx + C \qquad \text{[On integrating]}$$

$$\Rightarrow \quad \int \frac{1}{1+\frac{1-\tan^2(v/2)}{1+\tan^2(v/2)}+\frac{2\tan(v/2)}{1+\tan^2(v/2)}}\,dv = x+C$$

$$\Rightarrow \quad \int \frac{\sec^2(v/2)}{2\{(1+\tan(v/2)\}}\,dv = x+C \Rightarrow \log|1+\tan(v/2)| = x+C$$

$$\Rightarrow \quad \log\left|1+\tan\left(\frac{x+y}{2}\right)\right| = x+C, \text{ which is the required solution.}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Solve the following initial value problems:*

(i) $(x+y+1)^2\,dy = dx$, $y(-1) = 0$ (ii) $(x-y)(dx+dy) = dx-dy$, $y(0)=-1$ **[NCERT]**

SOLUTION (i) The given differential equation is

$$(x+y+1)^2\,dy = dx \Rightarrow \frac{dy}{dx} = \frac{1}{(x+y+1)^2} \qquad ...(i)$$

Let $x+y+1=v$. Then, $1+\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=\frac{dv}{dx}-1$.

Putting $x+y+1=v$ and $\frac{dy}{dx}=\frac{dv}{dx}-1$ in (i), we get

$$\frac{dv}{dx}-1 = \frac{1}{v^2} \Rightarrow \frac{dv}{dx} = \frac{1+v^2}{v^2} \Rightarrow \frac{v^2}{v^2+1}\,dv = dx$$

$$\Rightarrow \quad \int \frac{v^2}{v^2+1}\,dv = \int dx \Rightarrow \int \frac{v^2+1-1}{v^2+1}\,dv = \int dx \Rightarrow \int\left(1-\frac{1}{v^2+1}\right)dv = \int dx$$

$$\Rightarrow \quad v-\tan^{-1}v = x+C \Rightarrow (x+y+1)-\tan^{-1}(x+y+1) = x+C$$

$$\Rightarrow \quad y+1-\tan^{-1}(x+y+1) = C \qquad ...(ii)$$

It is given that $y(-1)=0$ i.e. $y=0$ when $x=-1$. Putting $x=1$ and $y=0$ in (ii), we get

$$1-\tan^{-1}0 = C \Rightarrow C = 1$$

Putting $C=1$ in (ii), we get: $y = \tan^{-1}(x+y+1) \Rightarrow x+y+1 = \tan y$, which is the required solution.

(ii) The given differential equations is

$$(x-y)(dx+dy) = dx-dy \Rightarrow (x-y-1)\,dx = -(x-y+1)\,dy \Rightarrow \frac{dy}{dx} = -\frac{x-y-1}{x-y+1} \qquad ...(i)$$

Let $x-y=v$. Then, $1-\frac{dy}{dx}=\frac{dv}{dx} \Rightarrow \frac{dy}{dx}=1-\frac{dv}{dx}$. Putting $x-y=v$ and $\frac{dy}{dx}=1-\frac{dv}{dx}$ in (i), we get

$$1-\frac{dv}{dx}=-\frac{v-1}{v+1} \Rightarrow \frac{dv}{dx}=\frac{v-1}{v+1}+1 \Rightarrow \frac{dv}{dx}=\frac{2v}{v+1}$$

$$\Rightarrow \quad \frac{v+1}{v}\,dv=2\,dx \Rightarrow \left(1+\frac{1}{v}\right)dv=2\,dx \Rightarrow \int\left(1+\frac{1}{v}\right)dv=2\int dx$$

$$\Rightarrow \quad v+\log|v|=2x+C \Rightarrow x-y+\log|x-y|=2x+C \Rightarrow \log|x-y|=x+y+C \quad \text{...(ii)}$$

It is given that $y(0)=-1$ i.e. when $x=0$, $y=-1$. Putting $x=0$ and $y=-1$ in (ii), we get

$$\log 1=-1+C \Rightarrow C=1$$

Putting $C=1$ in (ii), we get

$$\log|x-y|=x+y+1 \Rightarrow |x-y|=e^{x+y+1} \Rightarrow x-y=\pm e^{x+y+1}$$

But, $y(0)=-1$ does not satisfy $x-y=-e^{x+y+1}$. Hence, $x-y=e^{x+y+1}$ gives the required solution.

EXAMPLE 6 *Find the equation of the curve passing through origin if the slope of the tangent to the curve at any point (x, y) is equal to the square of the difference of the abscissa and ordinate of the pont.*

[NCERT EXEMPLAR]

SOLUTION The slope of the tangent at any point (x, y) on the curve $y=f(x)$ is given by $\frac{dy}{dx}$. It is given that the slope is equal to $(x-y)^2$.

$$\therefore \quad \frac{dy}{dx}=(x-y)^2 \quad \text{...(i)}$$

This is a differential equation in reducible to variable separable form. Let $x-y=v$. Then, $1-\frac{dy}{dx}=\frac{dv}{dx}$. Substituting these values in (i), we obtain

$$1-\frac{dv}{dx}=v^2 \Rightarrow \frac{dv}{dx}=1-v^2 \Rightarrow \frac{1}{1-v^2}\,dv=dx$$

Integrating, we obtain

$$\frac{1}{2}\log\left|\frac{1+v}{1-v}\right|=x+C \Rightarrow \log\left|\frac{1+v}{1-v}\right|=2x+2C \Rightarrow \log\left|\frac{1+x-y}{1-x+y}\right|=2x+2C \quad \text{...(ii)}$$

It is given that the curve passes through the origin. Therefore, $y=0$ when $x=0$. Putting $x=0$, $y=0$ in (ii), we obtain $C=0$. Putting $C=0$ in (ii), we obtain

$$\log\left|\frac{1+x-y}{1-x+y}\right|=2x \text{ or, } \log\left|\frac{1+x-y}{1-x+y}\right|=e^{2x} \text{ or, } (1+x-y)=\pm(1-x+y)\,e^{2x},$$

which is the equation of the curve.

EXERCISE 21.8

BASIC

Solve the following differential equations:

1. $\frac{dy}{dx}=(x+y+1)^2$
2. $\frac{dy}{dx}\cos(x-y)=1$
3. $\frac{dy}{dx}=\frac{(x-y)+3}{2(x-y)+5}$
4. $\frac{dy}{dx}=(x+y)^2$
5. $(x+y)^2\,\frac{dy}{dx}=1$

BASED ON LOTS

6. $\cos^2(x-2y) = 1 - 2\frac{dy}{dx}$ 7. $\frac{dy}{dx} = \sec(x+y)$ 8. $\frac{dy}{dx} = \tan(x+y)$

9. $(x+y)(dx-dy) = dx+dy$ 10. $(x+y+1)\frac{dy}{dx} = 1$

11. $\frac{dy}{dx} + 1 = e^{x+y}$ [NCERT EXEMPLAR]

ANSWERS

1. $\tan^{-1}(x+y+1) = x + C$ 2. $\cot\left(\frac{x-y}{2}\right) = y + C$

3. $2(x-y) + \log(x-y+2) = x + C$ 4. $x + y = \tan(x+C)$

5. $y - \tan^{-1}(x+y) = C$ 6. $x = \tan(x-2y) + C$

7. $y = \tan\left(\frac{x+y}{2}\right) + C$ 8. $y - x + \log|\sin(x+y) + \cos(x+y)| = C$

9. $\frac{1}{2}(y-x) + \frac{1}{2}\log|x+y| = C$ 10. $x = C\,e^{y} - y - 2$ 11. $-e^{-(x+y)} = x + C$

21.6.5 HOMOGENEOUS DIFFERENTIAL EQUATIONS

HOMOGENEOUS FUNCTION *A function $f(x, y)$ is called a homogeneous function of degree n, if*

$$f(\lambda x, \lambda y) = \lambda^n f(x, y)$$

For example, $f(x, y) = x^2 - y^2 + 3xy$ is a homogeneous function degree 2, because

$$f(\lambda x, \lambda y) = \lambda^2 x^2 - \lambda^2 y^2 + 3\,\lambda x \cdot \lambda y = \lambda^2 (x^2 - y^2 + 3xy) = \lambda^2 f(x, y)$$

Consider the following functions:

$$F(x, y) = 2x - 3y,\ G(x, y) = \sin\left(\frac{y}{x}\right) \text{ and, } H(x, y) = \sin x + \cos y$$

We find that

$$F(\lambda x, \lambda y) = 2\lambda x - 3\lambda y = \lambda^1 (2x - 3y) = \lambda^1 F(x, y)$$

Therefore, $F(x,y)$ is a homogeneous function of degree 1.

$$G(\lambda x, \lambda y) = \sin\left(\frac{\lambda y}{\lambda x}\right) = \lambda^0 \sin\left(\frac{y}{x}\right) = \lambda^0 G(x, y)$$

Therefore, $G(x,y)$ is a homogeneous function of zero degree.

$$H(\lambda x, \lambda y) = \sin \lambda x + \cos \lambda y \neq \lambda^n H(x, y) \text{ for any } n.$$

Therefore, $H(x, y)$ is not a homogeneous function.

Let $F_1(x, y) = x^2 + 2xy$ be a function of x, y. Then,

$$F_1(\lambda x, \lambda y) = \lambda^2 x^2 + 2\,\lambda x\ \lambda y = \lambda^2 (x^2 + 2xy) = \lambda^2 F_1(x, y)$$

$\therefore$ $F_1(x, y)$ is a homogeneous function of degree 2.

Also, $$F_1(x, y) = x^2 + 2xy = x^2\left\{1 + \left(\frac{2y}{x}\right)\right\} = x^2\ \phi_1\left(\frac{y}{x}\right)$$

and, $$F_1(x, y) = x^2 + 2xy = y^2\left\{\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right)\right\} = y^2\,\Psi_1\left(\frac{x}{y}\right)$$

Thus a homogeneous function $f(x, y)$ of degree n can always be written as

$$f(x, y) = x^n f\left(\frac{y}{x}\right) \text{ or, } f(x, y) = y^n f\left(\frac{x}{y}\right)$$

HOMOGENEOUS DIFFERENTIAL EQUATION *If a first-order first degree differential equation is expressible in the form* $\frac{dy}{dx} = \frac{f(x, y)}{g(x, y)}$, *where* $f(x, y)$ *and* $g(x, y)$ *are homogeneous functions of the same degree, then it is called a homogeneous differential equation.*

Such type of equations can be reduced to variable separable form by the substitution $y = vx$ as explained below:

The given differential equation can be written as

$$\frac{dy}{dx} = \frac{x^n f(y/x)}{x^n g(y/x)} = \frac{f(y/x)}{g(y/x)} = F\left(\frac{y}{x}\right) \text{ (say)}$$

Let $y = vx$. Then, $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Substituting these values in $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$, we get

$$v + x\frac{dv}{dx} = F(v) \Rightarrow \frac{1}{F(v) - v}\, dv = \frac{dx}{x}$$

On integration, we get

$$\int \frac{1}{F(v) - v}\, dv = \int \frac{dx}{x} + C$$, where C is an arbitrary constant of integration.

Replacing v by $\frac{y}{x}$ after integration, we obtain the complete solution of the given differential equation.

Following algorithm may be used to solve a homogeneous differential equation.

ALGORITHM

Step I *Put the differential equation in the form* $\frac{dy}{dx} = \frac{\phi(x, y)}{\psi(x, y)}$.

Step II *Put* $y = vx$ *and* $\frac{dy}{dx} = v + x\frac{dv}{dx}$ *in the equation in Step I and cancel out x from the right hand side.*

The equation reduces to the form $v + x\frac{dv}{dx} = F(v)$.

Step III *Shift v on RHS and separate the variables v and x.*

Step IV *Integrate both sides to obtain the solution in terms of v and x.*

Step V *Replace v by* $\frac{y}{x}$ *in the solution obtained in Step IV to obtain the solution in terms of x and y.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the differential equation* $x^2\, dy + y(x + y)\, dx = 0$, *given that* $y = 1$ *when* $x = 1$. **[CBSE 2013]**

SOLUTION The given differential equation is

$$x^2\, dy + y(x + y)\, dx = 0$$

$$\Rightarrow \quad x^2\,dy = -y(x+y)\,dx \Rightarrow \frac{dy}{dx} = -\frac{y(x+y)}{x^2} \Rightarrow \frac{dy}{dx} = -\left(\frac{xy+y^2}{x^2}\right) \quad \text{...(i)}$$

Since each of the functions $xy+y^2$ and x^2 is a homogeneous function of degree 2. Therefore, equation (i) is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -\left(\frac{vx^2+v^2x^2}{x^2}\right) \Rightarrow v + x\frac{dv}{dx} = -(v+v^2) \Rightarrow x\frac{dv}{dx} = -2v - v^2$$

$$\Rightarrow \quad x\,dv = -(v^2+2v)\,dx \Rightarrow \frac{1}{v^2+2v}\,dv = -\frac{dx}{x} \quad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \frac{1}{v^2+2v}\,dv = -\int \frac{1}{x}\,dx \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{1}{v^2+2v+1-1}\,dv = -\int \frac{1}{x}\,dx \Rightarrow \int \frac{1}{(v+1)^2-1^2}\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{2\times 1}\log\left|\frac{v+1-1}{v+1+1}\right| = -\log x + \log C \Rightarrow \frac{1}{2}\log\left|\frac{v}{v+2}\right| = -\log x + \log C$$

$$\Rightarrow \quad \log\left|\frac{v}{v+2}\right| + 2\log x = 2\log C \Rightarrow \log\left|\frac{v}{v+2}\right| + \log x^2 = \log k, \text{ where } k = C^2$$

$$\Rightarrow \quad \log\left|\frac{vx^2}{v+2}\right| = \log k \Rightarrow \left|\frac{vx^2}{v+2}\right| = k \Rightarrow \left|\frac{\frac{y}{x}\times x^2}{\frac{y}{x}+2}\right| = k \Rightarrow \left|\frac{x^2y}{y+2x}\right| = k \quad \left[\because \frac{y}{x} = v\right]$$

It is given that $y = 1$, when $x = 1$. Putting $x = 1, y = 1$ in (ii), we get $k = 1/3$. Putting $k = 1/3$ in (ii), we get

$$\left|\frac{x^2y}{y+2x}\right| = \frac{1}{3} \Rightarrow 3x^2y = \pm(y+2x). \text{ But, } 3x^2y = -(y+2x) \text{ is not satisfied by } y(1) = 1.$$

Therefore, $3x^2y = y + 2x \Rightarrow y = \frac{2x}{3x^2-1}$, which gives the required solution.

EXAMPLE 2 *Solve the differential equation* $(x+y)\,dy + (x-y)\,dx = 0$, *given that* $y = 1$ *when* $x = 1$.

SOLUTION The given differential equation is

$$(x+y)\,dy + (x-y)\,dx = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-y}{x+y} \Rightarrow \frac{dy}{dx} = \frac{y-x}{x+y} \quad \text{...(i)}$$

Since each of the functions $y - x$ and $x + y$ is a homogeneous function of degree 1. Therefore, equation (i) is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx-x}{x+vx} \Rightarrow v + x\frac{dv}{dx} = \frac{v-1}{v+1} \Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1} \Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2+1}{v+1}\right) \Rightarrow \frac{v+1}{v^2+1}\,dv = -\frac{dx}{x}$$

[By separating the variables]

$$\Rightarrow \quad \int \frac{v+1}{v^2+1}\,dv = -\int \frac{dx}{x}$$ [Integrating both sides]

$$\Rightarrow \quad \int \frac{v}{v^2+1}\,dv + \int \frac{1}{v^2+1}\,dv = -\int \frac{dx}{x} \Rightarrow \frac{1}{2}\int \frac{2v}{v^2+1}\,dv + \int \frac{1}{v^2+1^2}\,dv = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\log(v^2+1) + \tan^{-1}v = -\log|x| + C$$

$$\Rightarrow \quad \log(v^2+1) + 2\log|x| + 2\tan^{-1}v = 2C$$

$$\Rightarrow \quad \log(v^2+1) + \log x^2 + 2\tan^{-1}v = k, \text{ where } k = 2C$$

$$\Rightarrow \quad \log\{(v^2+1)\,x^2\} + 2\tan^{-1}v = k$$

$$\Rightarrow \quad \log\{(y^2/x^2)+1)\,x^2\} + 2\tan^{-1}(y/x) = k \qquad [\because v = yx]$$

$$\Rightarrow \quad \log(x^2+y^2) + 2\tan^{-1}(yx) = k \qquad \text{...(ii)}$$

It is given that $y = 1$, when $x = 1$. Putting $x = 1, y = 1$ in (ii), we get

$$\log 2 + 2\tan^{-1}(1) = k \Rightarrow k = \log 2 + 2(\pi/4) = (\pi/2) + \log 2$$

Substituting the value of k in (ii), we get

$$\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2$$

Hence, $\log(x^2+y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{2} + \log 2,\ x \neq 0$ is the required solution of the given differential equation.

EXAMPLE 3 *Solve the differential equation* $(x^2 - y^2)\,dx + 2xy\,dy = 0$; *given that* $y = 1$ *when* $x = 1$.

[NCERT, CBSE 2008]

SOLUTION We are given that

$$(x^2-y^2)\,dx + 2xy\,dy = 0 \Rightarrow (x^2-y^2)\,dx = -2xy\,dy \Rightarrow \frac{dy}{dx} = -\frac{x^2-y^2}{2xy} \Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy} \qquad \text{...(i)}$$

Since each of the functions $y^2 - x^2$ and $2xy$ is a homogeneous function of degree 2, the given differential equation is therefore homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x\cdot vx} \Rightarrow v + x\frac{dv}{dx} = \frac{v^2-1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v} \Rightarrow x\frac{dv}{dx} = -\left(\frac{v^2+1}{2v}\right) \Rightarrow \frac{2v}{v^2+1}\,dv = -\frac{dx}{x}$$

[By separating the variables]

$$\Rightarrow \quad \int \frac{2v}{v^2+1}\,dv = -\int \frac{dx}{x}$$ [Integrating both sides]

$\Rightarrow \quad \log(v^2+1) = -\log|x| + C \Rightarrow \log(v^2+1) + \log|x| = \log C$

$\Rightarrow \quad (v^2+1)|x| = C \Rightarrow \left\{(y^2/x^2)+1\right\}|x| = C$ $\quad [\because v = y/x]$

$\Rightarrow \quad (x^2+y^2) = C|x|$ ...(ii)

It is given that $y=1$ when $x=1$. So, putting $x=1, y=1$ in (ii), we get: $C=2$.

Substituting $C=2$ in (ii), we obtain $x^2+y^2 = 2|x| \Rightarrow x^2+y^2 = \pm 2x$

But, $x=1$, $y=1$ do not satisfy $x^2+y^2=-2x$. Hence, $x^2+y^2=2x$ is the required solution.

EXAMPLE 4 *Solve:* $x^2 y\,dx - (x^3+y^3)\,dy = 0$ **[CBSE 2002]**

SOLUTION The given differential equation is

$$x^2 y\,dx - (x^3+y^3)\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2 y}{x^3+y^3} \qquad \text{...(i)}$$

Since each of the functions $x^2 y$ and x^3+y^3 is a homogeneous function of degree 3, so the given differential equation is homogeneous. Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3} \Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3} \Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v-v-v^4}{1+v^3} \Rightarrow x\frac{dv}{dx} = -\frac{v^4}{1+v^3} \Rightarrow x(1+v^3)\,dv = -v^4\,dx$$

$$\Rightarrow \quad \frac{1+v^3}{v^4}\,dv = -\frac{dx}{x} \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \left(\frac{1}{v^4}+\frac{1}{v}\right)dv = -\frac{dx}{x} \Rightarrow \frac{v^{-3}}{-3} + \log|v| = -\log|x| + C \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad -\frac{1}{3v^3} + \log|v| + \log|x| = C \Rightarrow -\frac{1}{3}\frac{x^3}{y^3} + \log\left|\frac{y}{x}\cdot x\right| = C \qquad [\because v = y/x]$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \log|y| = C, \text{ which is the required solution.}$$

EXAMPLE 5 *Find the particular solution of the differential equation:* $(x^2+xy)\,dy = (x^2+y^2)\,dx$ *given that* $y=0$ *when* $x=1$. **[NCERT, CBSE 2005, 2013]**

SOLUTION The given differential equation is

$$(x^2+xy)\,dy = (x^2+y^2)\,dx \Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{x^2+xy} \qquad \text{...(i)}$$

Since each of the functions x^2+y^2 and x^2+xy is a homogeneous function of degree 2, so the given differential equation is homogeneous. Putting $y=vx$ and $\frac{dy}{dx}=v+x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{x^2+vx^2} \Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2-v-v^2}{1+v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1-v}{1+v} \Rightarrow x(1+v)\,dv = (1-v)\,dx \Rightarrow \frac{1+v}{1-v}\,dv = \frac{dx}{x} \qquad \text{[By separating variables]}$$

$$\Rightarrow \quad \frac{2-(1-v)}{1-v}\,dv = \frac{dx}{x} \Rightarrow \left(\frac{2}{1-v}-1\right)dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(\frac{2}{1-v}-1\right)dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{2}{1-v}\,dv - \int 1.dv = \int \frac{dx}{x} \Rightarrow -2\log|1-v| - v = \log|x| + \log C$$

$$\Rightarrow \quad \log|x| + \log C + 2\log|1-v| = -v \Rightarrow \log\left\{C|x|(1-v)^2\right\} = -v$$

$$\Rightarrow \quad C|x|(1-v)^2 = e^{-v} \Rightarrow C|x|(1-y/x)^2 = e^{-y/x} \Rightarrow C(x-y)^2 = |x|e^{-y/x} \quad \ldots\text{(ii)}$$

It is given that $y=0$ when $x=1$. Putting these values in (ii), we get: $C(1-0)^2 = e^0 \Rightarrow C=1$

Putting $C=1$ in (ii), we obtain $(x-y)^2 = |x|\ e^{-y/x}$, which is the particular solution of the given differential equation.

EXAMPLE 6 *Find the particular solution of the differential equation* $(3xy+y^2)\,dx + (x^2+xy)\,dy = 0$; *for* $x=1$, $y=1$. **[CBSE 2008, 2013]**

SOLUTION We are given that

$$(3xy+y^2)\,dx + (x^2+xy)\,dy = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{3xy+y^2}{x^2+xy}\right) \quad \ldots\text{(i)}$$

Since each of the functions $(3xy+y^2)$ and (x^2+xy) is a homogeneous function of degree 2, the given equation is, therefore, a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = -\left\{\frac{3v\,x^2 + x^2 v^2}{x^2 + vx^2}\right\} \Rightarrow v + x\frac{dv}{dx} = -\left\{\frac{3v+v^2}{1+v}\right\}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\left\{\frac{3v+v^2}{1+v} + v\right\} \Rightarrow x\frac{dv}{dx} = -\left\{\frac{2v^2+4v}{v+1}\right\}$$

$$\Rightarrow \quad (v+1)\,x\,dv = -(2v^2+4v)\,dx \Rightarrow \frac{v+1}{2v^2+4v}\,dv = -\frac{dx}{x} \quad \text{[By separating variables]}$$

$$\Rightarrow \quad \frac{(2v+2)\,dv}{v^2+2v} = -4\frac{dx}{x} \Rightarrow \int \frac{2v+2}{v^2+2v}\,dv = -4\int\frac{dx}{x} \quad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|v^2+2v| = -4\log|x| + \log C \Rightarrow \log|v^2+2v| = \log\left(\frac{C}{x^4}\right)$$

$$\Rightarrow \quad |v^2+2v| = \frac{C}{x^4} \Rightarrow \left|\frac{y^2}{x^2}+\frac{2y}{x}\right| = \frac{C}{x^4} \Rightarrow |y^2+2xy| = \frac{C}{x^2} \quad \ldots\text{(ii)}$$

It is given that $y=1$ when $x=1$. Putting these values in (ii), we get: $C=3$.

Putting $C=3$ in (ii), we obtain $\left|y^2+2xy\right| = \frac{3}{x^2}$ as the required solution.

EXAMPLE 7 *Solve:* $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$.

SOLUTION We are given that

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy \Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}$$

Clearly, the given equation is a homogeneous equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3} \Rightarrow v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} \Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} \Rightarrow x(v^3 - 3v)\,dv = (1 - v^4)\,dx \Rightarrow \frac{v^3 - 3v}{1 - v^4}\,dv = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{v^3 - 3v}{1 - v^4}\,dv = \int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \int \frac{v^3}{1 - v^4}\,dv - 3\int \frac{v}{1 - v^4}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\int \frac{-4v^3}{1 - v^4}\,dv - \frac{3}{2}\int \frac{2v}{1 - (v^2)^2}\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad -\frac{1}{4}\int \frac{-4v^3}{1 - v^4}\,dv - \frac{3}{2}\int \frac{dt}{1 - t^2} = \int \frac{dx}{x}, \text{ where } t = v^2.$$

$$\Rightarrow \quad -\frac{1}{4}\log|1 - v^4| - \frac{3}{2} \times \frac{1}{2 \times 1}\log\left|\frac{1 + t}{1 - t}\right| = \log|x| + \log C$$

$$\Rightarrow \quad -\frac{1}{4}\log|1 - v^4| - \frac{3}{4}\log\left|\frac{1 + v^2}{1 - v^2}\right| = \log|Cx|$$

$$\Rightarrow \quad -\log|(1 - v^4)| - 3\log\left|\frac{1 + v^2}{1 - v^2}\right| = 4\log|Cx| \Rightarrow \log\left|(1 - v^4)^{-1}\left(\frac{1 + v^2}{1 - v^2}\right)^{-3}\right| = \log|(Cx)^4|$$

$$\Rightarrow \quad \frac{1}{1 - v^4} \times \left(\frac{1 - v^2}{1 + v^2}\right)^3 = (Cx)^4 \Rightarrow (1 - v^2)^2 = (1 + v^2)^4\,(Cx)^4$$

$$\Rightarrow \quad 1 - v^2 = (1 + v^2)^2\,(Cx)^2 \Rightarrow 1 - y^2/x^2 = (1 + y^2/x^2)^2\,C^2x^2 \qquad [\because\ v = y/x]$$

$$\Rightarrow \quad x^2 - y^2 = (x^2 + y^2)^2\,C^2, \text{ which is the required solution.}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *Solve:* $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$ **[NCERT, CBSE 2005, 11, 19]**

SOLUTION The given differential equation can be written as $\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x}$, $x \neq 0$

Clearly, it is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in it, we get

$$v + x\frac{dv}{dx} = \frac{\sqrt{x^2 + v^2x^2} + vx}{x} \Rightarrow v + x\frac{dv}{dx} = \sqrt{1+v^2} + v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \sqrt{1+v^2} \Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x} \qquad \text{[By separating the variables]}$$

Integrating both sides, we get

$$\int \frac{1}{\sqrt{1+v^2}}\,dv = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log\left|v + \sqrt{1+v^2}\right| = \log|x| + \log C \Rightarrow \left|v + \sqrt{1+v^2}\right| = |Cx|$$

$$\Rightarrow \quad \left|\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}}\right| = |Cx| \qquad \left[\because v = \frac{y}{x}\right]$$

$$\Rightarrow \quad \left\{y + \sqrt{x^2+y^2}\right\}^2 = C^2x^4 \text{, which gives the required solution}$$

EXAMPLE 9 *Solve:*

$$y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}dx - x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}dy = 0 \qquad \text{[NCERT, CBSE 2010]}$$

SOLUTION The given differential equation can be written as

$$\frac{dy}{dx} = \frac{y\left\{x\cos\left(\frac{y}{x}\right) + y\sin\left(\frac{y}{x}\right)\right\}}{x\left\{y\sin\left(\frac{y}{x}\right) - x\cos\left(\frac{y}{x}\right)\right\}} \qquad \text{...(i)}$$

It can be checked that RHS does not change when x is replaced by λx and y by λy. So, the given differential equation is homogeneous. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = \frac{vx\,\{x\cos v + v\,x\sin v\}}{x\,\{vx\sin v - x\cos v\}} = \frac{v\,\{\cos v + v\sin v\}}{\{v\sin v - \cos v\}}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} = \frac{2v\cos v}{v\sin v - \cos v}$$

$$\Rightarrow \quad \frac{v\sin v - \cos v}{v\cos v}\,dv = 2\frac{dx}{x} \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad -\int \frac{\cos v - v\sin v}{v\cos v}\,dv = 2\int\frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad -\log|v\cos v| = 2\log|x| + \log C \Rightarrow \log\frac{1}{|v\cos v|} = \log|x^2| + \log C$$

$$\Rightarrow \quad \left|\frac{1}{v\cos v}\right| = |C|\,x^2 \Rightarrow \left|\frac{x}{y}\sec\left(\frac{y}{x}\right)\right| = |C|\,x^2$$

$$\Rightarrow \quad \left|xy\cos\left(\frac{y}{x}\right)\right| = \frac{1}{|C|} \Rightarrow \left|xy\cos\left(\frac{y}{x}\right)\right| = k, \text{ where } k = \frac{1}{|C|}$$

Hence, $\left|xy\cos\left(\frac{y}{x}\right)\right| = k,\ x \neq 0,\ k > 0$ gives the required solution.

EXAMPLE 10 *Solve:* $x\dfrac{dy}{dx} = y - x\tan\left(\dfrac{y}{x}\right)$. **[CBSE 2002]**

SOLUTION We are given that

$$x\frac{dy}{dx} = y - x\tan\frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right) \qquad \text{...(i)}$$

Clearly, the given differential equation is homogeneous. Putting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \tan v$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\tan v \Rightarrow \cot v\,dv = \frac{-dx}{x}, \text{ if } x \neq 0 \qquad \text{[By separating the variables]}$$

$$\Rightarrow \quad \int \cot v\,dv = -\int \frac{dx}{x} \qquad \text{[Integrating both sides]}$$

$$\Rightarrow \quad \log|\sin v| = -\log|x| + \log C \Rightarrow |\sin v| = \left|\frac{C}{x}\right| \Rightarrow |\sin(y/x)| = |C/x|$$

Hence, $\left|\sin\dfrac{y}{x}\right| = \left|\dfrac{C}{x}\right|$ gives the required solution.

REMARK *Sometimes a homogeneous differential equation is expressible in the form* $\dfrac{dx}{dy} = \dfrac{f(x, y)}{g(x, y)}$ *where* $f(x, y)$ *and* $g(x, y)$ *are homogeneous functions of same degree. In such a case, we substitute* $x = vy$ *and* $\dfrac{dx}{dy} = v + y\dfrac{dv}{xy}$ *to solve the differential equation.*

EXAMPLE 11 *Solve:* $2y\,e^{x/y}\,dx + (y - 2x\,e^{x/y})\,dy = 0$ **[CBSE 2012, 13, NCERT]**

SOLUTION We have,

$$2y\,e^{x/y}\,dx + \left(y - 2x\,e^{x/y}\right)dy = 0 \Rightarrow \frac{dx}{dy} = \frac{2x\,e^{x/y} - y}{2y\,e^{x/y}} \qquad \text{...(i)}$$

Clearly, the given differential equation is a homogeneous differential equation. As the right hand side of (i) is expressible as a function of $\dfrac{x}{y}$. So, we put $x = vy$ and $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ to obtain

$$v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} \Rightarrow y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v \Rightarrow y\frac{dv}{dy} = -\frac{1}{2e^v}$$

$$\Rightarrow \quad 2ye^v\,dv = -dy \Rightarrow 2e^v\,dv = -\frac{1}{y}dy \Rightarrow 2\int e^v\,dv = -\int \frac{1}{y}dy \qquad \text{[On integrating]}$$

$$\Rightarrow \quad 2e^v = -\log|y| + \log C \Rightarrow 2e^v = \log\left|\frac{C}{y}\right| \Rightarrow 2e^{x/y} = \log\left|\frac{C}{y}\right|$$

Hence, $2e^{x/y} = \log\left|\frac{C}{y}\right|$ gives the general solution of the given differential equation.

EXAMPLE 12 *Solve the following initial value problems:*

(i) $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0,\ y(1) = \frac{\pi}{2}$ **[CBSE 2012, 2020]**

(ii) $xe^{y/x} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0,\ y(1) = 0.$

SOLUTION (i) We have,

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) + x - y\sin\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{x - y\sin\left(\frac{y}{x}\right)}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = -\frac{1 - v\sin v}{\sin v} \Rightarrow x\frac{dv}{dx} = -\frac{1 - v\sin v}{\sin v} - v \Rightarrow x\frac{dv}{dx} = -\frac{1}{\sin v}$$

$$\Rightarrow \quad \sin v\, dv = -\frac{1}{x}dx, \text{ if } x \neq 0 \Rightarrow \int \sin v\, dv = -\int \frac{1}{x}dx \qquad \text{[On integrating]}$$

$$\Rightarrow \quad -\cos v = -\log|x| + C \Rightarrow -\cos\left(\frac{y}{x}\right) + \log|x| = C \qquad \text{...(i)}$$

It is given that $y(1) = \frac{\pi}{2}$ i.e. when $x = 1$, $y = \frac{\pi}{2}$. Putting $x = 1$ and $y = \frac{\pi}{2}$ in (i), we get

$$-\cos\frac{\pi}{2} + \log 1 = C \Rightarrow C = 0.$$

Putting $C = 0$ in (i), we get: $-\cos\left(\frac{y}{x}\right) + \log|x| = 0 \Rightarrow \log|x| = \cos\left(\frac{y}{x}\right)$

Hence, $\log|x| = \cos\left(\frac{y}{x}\right)$, is the required solution.

(ii) Given differential equation is

$$xe^{y/x} - y\sin\left(\frac{y}{x}\right) + x\frac{dy}{dx}\sin\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{dy}{dx} = \frac{y\sin\left(\frac{y}{x}\right) - xe^{y/x}}{x\sin\left(\frac{y}{x}\right)}$$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{v\sin v - e^v}{\sin v} \Rightarrow x\frac{dv}{dx} = \frac{v\sin v - e^v}{\sin v} - v \Rightarrow x\frac{dv}{dx} = -\frac{e^v}{\sin v}$$

$$\Rightarrow \quad e^{-v}\sin v\, dv = -\frac{dx}{x} \Rightarrow \int e^{-v}\sin v\, dv = -\int \frac{1}{x}dx$$

$$\Rightarrow \frac{e^{-v}}{2}(-\sin v - \cos v) = -\log|x| + \log C \qquad \left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx)\right]$$

$$\Rightarrow -\frac{1}{2}e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = -\log|x| + \log C$$

$$\Rightarrow e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = 2\log|x| - 2\log C$$

$$\Rightarrow e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = \log|x|^2 - 2\log C \qquad \text{...(ii)}$$

It is given that $y(1) = 0$ i.e. $y = 0$ when $x = 1$. Putting these values in (ii), we get

$$1 = 0 - 2\log C \Rightarrow \log C = -\frac{1}{2}$$

Putting $\log C = -\frac{1}{2}$ in (ii), we get: $e^{-y/x}\left\{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\} = \log|x|^2 + 1$ as the required solution.

EXAMPLE 13 *Solve each of the following initial value problems:*

(i) $2x^2\frac{dy}{dx} - 2xy + y^2 = 0,\ y(e) = e$ [CBSE 2012] (ii) $2xy + y^2 - 2x^2\frac{dy}{dx} = 0,\ y(1) = 2$ [NCERT]

SOLUTION (i) We have,

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2x^2},$$ which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v - v^2}{2} \Rightarrow 2x\frac{dv}{dx} = -v^2 \Rightarrow -\frac{2}{v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{-2}{v^2}dv = \int \frac{1}{x}dx \qquad \text{[On integrating]}$$

$$\Rightarrow \frac{2}{v} = \log|x| + C \Rightarrow \frac{2x}{y} = \log|x| + C \qquad \text{...(i)}$$

It is given that $y(e) = e$ i.e. $y = e$ when $x = e$. Putting $x = e$ and $y = e$ in (i), we get: $2 = 1 + C \Rightarrow C = 1$.
Putting $C = 1$ in (i), we get

$$\frac{2x}{y} = \log|x| + 1 \Rightarrow y = \frac{2x}{1 + \log|x|}$$

Hence, $y = \frac{2x}{1+\log|x|}$ gives the required solution.

(ii) We have,

$$2xy + y^2 - 2x^2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2},$$ which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v+v^2}{2} \Rightarrow 2v + 2x\frac{dv}{dx} = 2v + v^2 \Rightarrow 2x\frac{dv}{dx} = v^2$$

$$\Rightarrow \quad \frac{2}{v^2}\,dv = \frac{1}{x}\,dx \Rightarrow 2\int \frac{1}{v^2}\,dv = \int \frac{1}{x}dx \Rightarrow -\frac{2}{v} = \log|x| + C \Rightarrow -\frac{2x}{y} = \log|x| + C \quad \text{...(i)}$$

It is given that $y(1) = 2$ i.e. $y = 2$ when $x = 1$. Putting $x = 1$ and $y = 2$ in (i), we get

$$-1 = 0 + C \Rightarrow C = -1$$

Putting $C = -1$ in (i), we get: $-\frac{2x}{y} = \log|x| - 1 \Rightarrow y = \frac{2x}{1-\log|x|}$

Hence, $y = \frac{2x}{1-\log|x|}$ gives the solution of the given differential equation.

EXAMPLE 14 *Solve each of the following initial value problems:*

(i) $(x^2+y^2)\,dx + xy\,dy = 0,\ y(1) = 1$ (ii) $(xe^{y/x}+y)\,dx = x\,dy,\ y(1) = 1$ [CBSE 2012]

(iii) $(x^2-2y^2)\,dx + 2xy\,dy = 0,\ y(1) = 1$

SOLUTION (i) Given differential equation is

$(x^2+y^2)\,dx + xy\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2+y^2}{xy}$, which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = -\frac{1+v^2}{v} \Rightarrow x\frac{dv}{dx} = -\left(\frac{1+2v^2}{v}\right) \Rightarrow \frac{v\,dv}{2v^2+1} = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{4v}{2v^2+1}\,dv = -\frac{4}{x}\,dx \Rightarrow \int \frac{4v}{2v^2+1}\,dv = -\int \frac{4}{x}\,dx \quad \text{[On integrating both sides]}$$

$$\Rightarrow \quad \log(2v^2+1) = -4\log|x| + \log C \Rightarrow 2v^2 + 1 = \frac{|C|}{x^4} \Rightarrow (2y^2+x^2)\,x^2 = |C| \quad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. $y = 1$ when $x = 1$. Putting $x = 1$, $y = 1$ in (i), we get: $|C| = 3$

Putting $|C| = 3$ in (i), we get $(2y^2+x^2)\,x^2 = 3$ as the required solution.

(ii) Given differential equation is

$(xe^{y/x}+y)\,dx = x\,dy \Rightarrow \frac{dy}{dx} = e^{y/x} + \frac{y}{x}$, which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ it reduces to

$$v + x\frac{dv}{dx} = e^v + v \Rightarrow x\frac{dv}{dx} = e^v \Rightarrow e^{-v}\,dv = \frac{dx}{x}, \text{ if } x \neq 0$$

$$\Rightarrow \quad \int e^{-v}\,dv = \int \frac{1}{x}\,dx \Rightarrow -e^{-v} = \log|x| + C \Rightarrow -e^{-y/x} = \log|x| + C \quad \text{...(i)}$$

It is given that $y(1) = 1$ i.e. when $x = 1$, $y = 1$. Putting $x = 1$, $y = 1$ in (i), we get: $-e^{-1} = C$.

Putting $C = -\frac{1}{e}$ in (i), we get

$$-e^{-y/x} = \log|x| - \frac{1}{e} \Rightarrow e^{-y/x} = \frac{1}{e} - \log|x|$$

$$\Rightarrow \quad -\frac{y}{x} = \log\left(1 - e\log|x|\right) - 1 \Rightarrow y = x - x\log(1 - e\log|x|)$$

Hence, $y = x - x\log(1 - e\log|x|)$, is the solution of the given equation.

(iii) Given differential equation is

$(x^2 - 2y^2)\,dx + 2xy\,dy = 0 \Rightarrow \frac{dy}{dx} = \frac{2y^2 - x^2}{2xy}$, which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, it reduces to

$$v + x\frac{dv}{dx} = \frac{2v^2 - 1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{2v^2 - 1}{2v} - v \Rightarrow x\frac{dv}{dx} = -\frac{1}{2v}$$

$$\Rightarrow \quad 2v\,dv = -\frac{dx}{x} \Rightarrow \int 2v\,dv = -\int \frac{1}{x}\,dx \Rightarrow v^2 = -\log|x| + C \Rightarrow y^2 = -x^2\log|x| + Cx^2 \quad ...(i)$$

It is given that $y(1) = 1$ i.e. when $x = 1$, $y = 1$. Putting $x = 1$, $y = 1$ in (i), we get:

$$1 = 0 + C \Rightarrow C = 1$$

Putting $C = 1$ in (i), we get: $y^2 = -x^2\log|x| + x^2$ as the required solution.

EXERCISE 21.9

BASIC

Solve the following differential equations:

1. $x^2\,dy + y(x + y)\,dx = 0$ [CBSE 2010]
2. $\frac{dy}{dx} = \frac{y - x}{y + x}$ [CBSE 2004]
3. $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$
4. $x\frac{dy}{dx} = x + y$
5. $(x^2 - y^2)\,dx - 2xy\,dy = 0$
6. $\frac{dy}{dx} = \frac{x + y}{x - y}$ [NCERT]
7. $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$ [CBSE 2022]
8. $x^2\frac{dy}{dx} = x^2 - 2y^2 + xy$ [NCERT]
9. $xy\frac{dy}{dx} = x^2 - y^2$
10. $y\,e^{x/y}\,dx = (xe^{x/y} + y)\,dy$
11. $x^2\frac{dy}{dx} = x^2 + xy + y^2$
12. $(y^2 - 2xy)\,dx = (x^2 - 2xy)\,dy$
13. $2xy\,dx + (x^2 + 2y^2)\,dy = 0$
14. $3x^2\,dy = (3xy + y^2)\,dx$
15. $\frac{dy}{dx} = \frac{x}{2y + x}$
16. $(x + 2y)\,dx - (2x - y)\,dy = 0$
17. $\frac{dy}{dx} = \frac{y}{x} - \sqrt{\frac{y^2}{x^2} - 1}$
18. $\frac{dy}{dx} = \frac{y}{x}\left\{\log y - \log x + 1\right\}$
19. $\frac{dy}{dx} = \frac{y}{x} + \sin\left(\frac{y}{x}\right)$ [NCERT, CBSE 2022]
20. $y^2\,dx + (x^2 - xy + y^2)\,dy = 0$

BASED ON LOTS

21. $\left\{x\sqrt{x^2 + y^2} - y^2\right\}dx + xy\,dy = 0$
22. $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$ [CBSE 2022]
23. $\frac{y}{x}\cos\left(\frac{y}{x}\right)dx - \left\{\frac{x}{y}\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right)\right\}dy = 0$

24. $xy \log\left(\frac{x}{y}\right) dx + \left\{y^2 - x^2 \log\left(\frac{x}{y}\right)\right\} dy = 0$ [CBSE 201]

25. $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$ [NCERT]

26. $(x^2 + y^2)\frac{dy}{dx} = 8x^2 - 3xy + 2y^2$

27. $(x^2 - 2xy)\, dy + (x^2 - 3xy + 2y^2)\, dx = 0$

28. $(1 + e^{y/x})\, dy + e^{y/x}\left(1 - \frac{y}{x}\right) dx = 0$ [CBSE 2022]

29. $x\frac{dy}{dx} - y = 2\sqrt{y^2 - x^2}$

30. $x \cos\left(\frac{y}{x}\right) \cdot (y\, dx + x\, dy) = y \sin\left(\frac{y}{x}\right) \cdot (x\, dy - y\, dx)$ [CBSE 2013]

31. $(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

32. $(x - y)\frac{dy}{dx} = x + 2y$ [CBSE 2010, 2015]

33. $(2x^2 y + y^3)\, dx + (xy^2 - 3x^3)\, dy = 0$

34. $x\frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$ [CBSE 2022]

35. $y\, dx + \left\{x \log\left(\frac{y}{x}\right)\right\} dy - 2x\, dy = 0$

36. Solve each of the following initial value problems:

(i) $(x^2 + y^2)\, dx = 2xy\, dy$, $y(1) = 0$

(ii) $xe^{y/x} - y + x\frac{dy}{dx} = 0$, $y(e) = 0$

(iii) $\frac{dy}{dx} - \frac{y}{x} + \operatorname{cosec}\frac{y}{x} = 0$, $y(1) = 0$ [NCERT, CBSE 2009]

(iv) $(xy - y^2)\, dx - x^2\, dy = 0$, $y(1) = 1$

(v) $\frac{dy}{dx} = \frac{y(x + 2y)}{x(2x + y)}$, $y(1) = 2$

(vi) $(y^4 - 2x^3 y)\, dx + (x^4 - 2xy^3)\, dy = 0$, $y(1) = 1$

(vii) $x(x^2 + 3y^2)\, dx + y(y^2 + 3x^2)\, dy = 0$, $y(1) = 1$

(viii) $\left\{x \sin^2\left(\frac{y}{x}\right) - y\right\} dx + x\, dy = 0$, $y(1) = \frac{\pi}{4}$ [CBSE 2011, 2013]

(ix) $x\frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$, $y(2) = x$ [CBSE 2012]

37. Find the particular solution of the differential equation $x \cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$, given that when $x = 1$, $y = \frac{\pi}{4}$. [CBSE 2013, 2017]

38. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = x + 2y$, given that when $x = 1, y = 0$. [CBSE 2013, 2014, 2015]

39. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$ given that $y = 1$ when $x = 0$. [CBSE 2015, 2019]

40. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = Cx$. [CBSE 2017]

ANSWERS

1. $x^2 y = C(y + 2x)$
2. $\log(x^2 + y^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = k$
3. $x^2 + y^2 = Cx$
4. $y = x \log |x| + Cx$
5. $x(x^2 - 3y^2) = C$
6. $\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\log(x^2 + y^2) + C$
7. $\frac{y^2}{2x^2} = \log Cx$
8. $\frac{x + \sqrt{2}\,y}{x - \sqrt{2}\,y} = (Cx^2)^{\sqrt{2}}$
9. $x^2(x^2 - 2y^2) = Cx$
10. $e^{x/y} = \log y + C$
11. $\tan^{-1}\left(\frac{y}{x}\right) = C + \log |x|$
12. $x^2 y - xy^2 = C$
13. $3x^2 y + 2y^3 = C$
14. $-\frac{3x}{y} = \log |x| + C$
15. $(x + y)(2y - x)^2 = C$
16. $\sqrt{x^2 + y^2} = C\, e^{2\tan^{-1} y/x}$
17. $y + \sqrt{y^2 - x^2} = C$
18. $\log\left(\frac{y}{x}\right) = Cx$
19. $\tan\left(\frac{y}{2x}\right) = Cx$
20. $y = C\, e^{\tan^{-1}(y/x)}$
21. $\sqrt{x^2 + y^2} = x \log\left|\frac{C}{x}\right|$
22. $\tan\left(\frac{y}{x}\right) = \log\left|\frac{C}{x}\right|$
23. $\left|y \sin\left(\frac{y}{x}\right)\right| = C$
24. $\frac{x^2}{y^2}\left\{\log\left(\frac{x}{y}\right) - \frac{1}{2}\right\} + \log y^2 = C$
25. $x + y\, e^{x/y} = C$
26. $C|2x - y|^{5/8}(4x^2 + y^2)^{3/16} = e^{-3/8 \tan^{-1}(y/2x)}$
27. $\frac{y}{x} + \log x = C$
28. $y + x\, e^{y/x} = C$
29. $y + \sqrt{y^2 - x^2} = C\, x^3$
30. $\sec\left(\frac{y}{x}\right) = Cxy$
31. $\frac{x}{x + y} + \log x = C$
32. $\log|x^2 + xy + y^2| = 2\sqrt{3}\tan^{-1}\left(\frac{x + 2y}{\sqrt{3}\,x}\right) + C$
33. $x^2 y^{12} = C^4 |2y^2 - x^2|^5$
34. $x \sin\left(\frac{y}{x}\right) = C\left(1 + \cos\frac{y}{x}\right)$
35. $Cy = \log\left|\frac{y}{x}\right| - 1$

36. (i) $(x^2 - y^2) = x$ (ii) $y = x \log(\log|x|)$ (iii) $\log|x| = \cos\left(\dfrac{y}{x}\right) - 1$

(iv) $y = \dfrac{x}{1 + \log|x|}$ (v) $xy = 2|y - x|^{3/2}$ (vi) $x^3 + y^3 = 2xy$

(vii) $x^4 + 6x^2y^2 + y^4 = 8$ (viii) $\cot\left(\dfrac{y}{x}\right) = \log(ex)$ **37.** $\sin\dfrac{y}{x} = \log x + \dfrac{1}{\sqrt{2}}$

38. $\log\left(x^2 + xy + y^2\right) = 2\sqrt{3}\tan^{-1}\left(\dfrac{x + 2y}{\sqrt{3}x}\right) - \dfrac{\pi}{\sqrt{3}}$ **39.** $\log y = -\dfrac{x^2}{2y^2}$

21.6.6 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is linear if the dependent variable (y) and its derivative appear only in first degree. The general form of a linear differential equation of first order is

$$\frac{dy}{dx} + Py = Q \qquad \text{...(i)}$$

where P and Q are functions of x (or constants)

For example,

(i) $\dfrac{dy}{dx} + xy = x^3$ (ii) $x\dfrac{dy}{dx} + 2y = x^3$ (iii) $\dfrac{dy}{dx} + 2y = \sin x$ etc. are linear differential equations.

This type of differential equations are solved when they are multiplied by a factor, which is called integrating factor, because by multiplication of this factor the left hand side of the differential equation (i) becomes an exact differential of some function. The integrating factor for $\dfrac{dy}{dx} + Py = Q$ is $e^{\int P dx}$.

Multiplying both sides of (i) by $e^{\int P dx}$, we get

$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = Q\, e^{\int P dx} \Rightarrow \frac{d}{dx}\left\{y\, e^{\int P dx}\right\} = Q\, e^{\int P dx}$$

On integrating both sides with respect to x, we obtain

$$y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + C \qquad \text{...(ii)}$$

This is the required solution, where C is the constant of integration.

The solution (ii) in short may also be written as $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$

Following algorithm may be used to solve a linear differential equation.

ALGORITHM

Step I *Write the differential equation in the form* $\dfrac{dy}{dx} + Py = Q$ *and obtain P and Q*

Step II *Find integrating factor (I.F.) given by I.F.* $= e^{\int P dx}$

Step III *Multiply both sides of equation in Step I by I.F.*

Step IV *Integrate both sides of the equation obtained in step III with respect to x to obtain* $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C$, *which gives the required solution.*

Following examples will illustrate the procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the differential equation* $\frac{dy}{dx} - \frac{y}{x} = 2x^2$ **[CBSE 2004, 2007, 2011]**

SOLUTION We are given that $\frac{dy}{dx} + \left(-\frac{1}{x}\right) y = 2x^2$ ...(i)

Clearly, it is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = -\frac{1}{x}$ and $Q = 2x^2$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Multiplying both sides of (i) by I.F. $= \frac{1}{x}$, we obtain

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2} y = 2x$$

Integrating both sides with respect to x, we obtain

$$y\left(\frac{1}{x}\right) = \int 2x \, dx + C \qquad [\text{Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\,]$$

$\Rightarrow \quad \frac{y}{x} = x^2 + C \Rightarrow y = x^3 + Cx$, which is the required solution.

EXAMPLE 2 *Solve the differential equation:* $\frac{dy}{dx} + \frac{y}{2x} = 3x^2$.

SOLUTION The given differential equation is $\frac{dy}{dx} + \frac{1}{2x} \cdot y = 3x^2$ ...(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{2x}$ and $Q = 3x^2$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int (1/2x)\, dx} = e^{(1/2)\log x} = e^{\log x^{1/2}} = x^{1/2}$$

Multiplying both sides of (i) by I.F. $= \sqrt{x}$, we get

$$\sqrt{x}\frac{dy}{dx} + \frac{1}{2\sqrt{x}} y = 3x^{5/2}$$

Integrating both sides with respect to x, we get

$$y\sqrt{x} = \int 3x^{5/2}\, dx + C \qquad [\text{Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C]$$

$\Rightarrow \quad y\sqrt{x} = 3\left(\frac{x^{7/2}}{7/2}\right) + C \Rightarrow y\sqrt{x} = \frac{6}{7}x^{7/2} + C \Rightarrow y = \frac{6}{7}x^3 + Cx^{-1/2}$, which is the required solution.

EXAMPLE 3 *Solve the differential equation:* $x \log x \frac{dy}{dx} + y = \frac{2}{x}\log x$ **[CBSE 2010, 2014]**

SOLUTION The given differential equation is

$$x \log x \frac{dy}{dx} + y = \frac{2}{x}\log x \Rightarrow \frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x^2} \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{1}{x \log x} \text{ and } Q = \frac{2}{x^2}$$

$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{t} dt} = e^{\log t} = t = \log x, \text{ where } t = \log x$

Multiplying both sides of (i) by I.F. = $\log x$, we get

$$\log x \frac{dy}{dx} + \frac{1}{x} y = \frac{2}{x^2} \log x$$

Integrating both sides with respect to x, we get

$$y \log x = \int \frac{2}{x^2} \log x \, dx + C \qquad \left[\text{Using : } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y \log x = 2 \int \underset{\text{I}}{\log x}\, \underset{\text{II}}{x^{-2}}\, dx + C = 2 \left\{ \log x \left(\frac{x^{-1}}{-1} \right) - \int \frac{1}{x} \left(\frac{x^{-1}}{-1} \right) dx \right\} + C$$

$$\Rightarrow \quad y \log x = 2 \left\{ -\frac{\log x}{x} + \int x^{-2}\, dx \right\} + C = 2 \left\{ -\frac{\log x}{x} - \frac{1}{x} \right\} + C$$

$$\Rightarrow \quad y \log x = -\frac{2}{x}(1 + \log x) + C, \text{ which gives the required solution.}$$

EXAMPLE 4 *Solve the differential equation:* $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$ [CBSE 2010, 2014]

SOLUTION The given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1} \Rightarrow \frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{1}{(x^2 - 1)^2} \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{2x}{x^2 - 1} \text{ and } Q = \frac{1}{(x^2 - 1)^2}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int 2x/(x^2 - 1)\, dx} = e^{\log (x^2 - 1)} = (x^2 - 1)$$

Multiplying both sides of (i) by I.F. $= (x^2 - 1)$, we get

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$$

Integrating both sides with respect to x, we get

$$y (x^2 - 1) = \int \frac{1}{x^2 - 1} dx + C \qquad \left[\text{Using : } y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y (x^2 - 1) = \frac{1}{2} \log \left| \frac{x - 1}{x + 1} \right| + C, \text{ which is the required solution.}$$

EXAMPLE 5 *Solve:* $\frac{dy}{dx} + y \sec x = \tan x$ [CBSE 2008, 2012]

SOLUTION The given differential equation is

$$\frac{dy}{dx} + (\sec x)\, y = \tan x \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \sec x \text{ and } Q = \tan x$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \sec x\, dx} = e^{\log (\sec x + \tan x)} = (\sec x + \tan x)$$

Multiplying both sides of (i) by I.F. = (sec x + tan x), we get

$$(\sec x + \tan x)\frac{dy}{dx} + y \sec x (\sec x + \tan x) = \tan x (\sec x + \tan x)$$

Integrating both sides with respect to x, we get

$$y(\sec x + \tan x) = \int \tan x (\sec x + \tan x)\, dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \sec x + \tan^2 x)\, dx + C$$

$$\Rightarrow \quad y(\sec x + \tan x) = \int (\tan x \sec x + \sec^2 x - 1)\, dx + C$$

$\Rightarrow \quad y(\sec x + \tan x) = \sec x + \tan x - x + C$, which is the required solution.

EXAMPLE 6 *Solve:* $\cos^2 x \dfrac{dy}{dx} + y = \tan x$ **[NCERT, CBSE 2008, 2011]**

SOLUTION Given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x \Rightarrow \frac{dy}{dx} + (\sec^2 x)\, y = \tan x \sec^2 x \qquad \text{...(i)}$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where $P = \sec^2 x$ and $Q = \tan x \sec^2 x$

$$\therefore \quad I = \text{I.F.} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Multiplying both sides of (i) by I.F. = $e^{\tan x}$, we get

$$e^{\tan x}\frac{dy}{dx} + \sec^2 x\, e^{\tan x}\, y = e^{\tan x} \tan x \sec^2 x$$

Integrating both sides with respect to x, we get

$$y\, e^{\tan x} = \int e^{\tan x} \cdot \tan x \sec^2 x\, dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y\, e^{\tan x} = \int \underset{\text{I}}{t}\, \underset{\text{II}}{e^t}\, dt + C, \text{ where } t = \tan x$$

$$\Rightarrow \quad y\, e^{\tan x} = t\, e^t - \int e^t\, dt + C \qquad \text{[Integrating by parts]}$$

$\Rightarrow \quad y e^{\tan x} = t e^t - e^t + C \Rightarrow y e^{\tan x} = e^{\tan x}(\tan x - 1) + C$, which is the required solution.

EXAMPLE 7 *Solve* : $(x^2+1)\dfrac{dy}{dx} + 2xy = \sqrt{x^2+4}$. **[CBSE 2010]**

SOLUTION Given differential equation is

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4} \Rightarrow \frac{dy}{dx} + \frac{2x}{x^2+1}\, y = \frac{\sqrt{x^2+4}}{x^2+1} \qquad \text{...(i)}$$

This is a linear differential equation of the form $\dfrac{dy}{dx} + Py = Q$, where

$$P = \frac{2x}{x^2+1} \text{ and } Q = \frac{\sqrt{x^2+4}}{x^2+1}$$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int 2x/(x^2+1)\, dx} = e^{\log(x^2+1)} = (x^2+1)$$

Multiplying both sides of (i) by I.F. = (x^2+1), we get

$$(x^2+1)\frac{dy}{dx} + 2xy = \sqrt{x^2+4}$$

Integrating both sides with respect to x, we obtain

$$y(x^2+1) = \int \sqrt{x^2+4}\, dx + C \qquad \left[\text{Using} : y\,(I.\,F.) = \int Q\,(I.\,F.)\, dx + C\right]$$

$$\Rightarrow \quad y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4}+\frac{1}{2}(2)^2\log|x+\sqrt{x^2+4}|+C$$

$$\Rightarrow \quad y(x^2+1) = \frac{1}{2}x\sqrt{x^2+4}+2\log|x+\sqrt{x^2+4}|+C,$$ which is the required solution.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 8 *Solve:* $\frac{dy}{dx}-2y=\cos 3x$

SOLUTION We are given the differential equation

$$\frac{dy}{dx}+(-2)\,y = \cos 3x \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q, \text{ where } P = -2 \text{ and } Q = \cos 3x$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int -2\,dx} = e^{-2x}$$

Multiplying both sides of (i) by I.F. $= e^{-2x}$, we get

$$e^{-2x}\frac{dy}{dx}-2ye^{-2x} = e^{-2x}\cos 3x$$

Integrating both sides with respect to x, we get

$$ye^{-2x} = \int e^{-2x}\cos 3x\,dx + C \qquad \left[\text{Using: } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y\,e^{-2x} = I + C, \qquad ...(ii)$$

where $I = \int e^{-2x}\cos 3x\,dx$

Now, $\quad I = \int \underset{\text{I}}{e^{-2x}}\,\underset{\text{II}}{\cos 3x}\,dx$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x - \int \frac{(-2)}{3}e^{-2x}\sin 3x\,dx = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\int e^{-2x}\sin 3x\,dx$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\left\{-\frac{1}{3}e^{-2x}\cos 3x - \int (-2)\,e^{-2x}\times -\frac{1}{3}\cos 3x\,dx\right\}$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x + \frac{2}{3}\left\{-\frac{1}{3}e^{-2x}\cos 3x - \frac{2}{3}\int e^{-2x}\cos 3x\,dx\right\}$$

$$\Rightarrow \quad I = \frac{1}{3}e^{-2x}\sin 3x - \frac{2}{9}e^{-2x}\cos 3x - \frac{4}{9}I$$

$$\Rightarrow \quad \left(I+\frac{4}{9}I\right) = \frac{e^{-2x}}{9}(3\sin 3x - 2\cos 3x) \Rightarrow I = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)$$

Substituting the value of I in (ii), we get: $ye^{-2x} = \frac{e^{-2x}}{13}(3\sin 3x - 2\cos 3x)+C,$

which is the required solution.

EXAMPLE 9 *Solve:* $(1+x^2)\frac{dy}{dx}+2xy-4x^2 = 0$ *subject to the initial condition* $y(0)=0$.

SOLUTION The given differential equation can be written as **[CBSE 2019]**

$$\frac{dy}{dx}+\frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2} \qquad ...(i)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{4x^2}{1+x^2}$$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int 2x/(1+x^2)\,dx} = e^{\log(1+x^2)} = 1 + x^2$$

Multiplying both sides of (i) by I.F. $= (1 + x^2)$, we get

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

Integrating both sides with respect to x, we get

$$y(1+x^2) = \int 4x^2\,dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y(1+x^2) = \frac{4x^3}{3} + C \qquad \text{...(ii)}$$

It is given that $y = 0$, when $x = 0$. Putting $x = 0$ and $y = 0$ in (i), we get: $0 = 0 + C \Rightarrow C = 0$

Substituting $C = 0$ in (ii), we get $y = \frac{4x^3}{3(1+x^2)}$, which is the required solution.

EXAMPLE 10 *Solve:* $\frac{dy}{dx} + y = \cos x - \sin x$ **[CBSE 2019]**

SOLUTION The given differential equation is $\frac{dy}{dx} + y = \cos x - \sin x$ ...(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = 1$ and $Q = \cos x - \sin x$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int 1\,dx} = e^x$$

Multiplying both sides of (i) by I.F. $= e^x$, we get

$$e^x \frac{dy}{dx} + y = e^x(\cos x - \sin x)$$

Integrating both sides with respect to x, we get

$$ye^x = \int e^x(\cos x - \sin x)\,dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad ye^x = \int \underset{\text{II}}{e^x}\,\underset{\text{I}}{\cos x}\,dx - \int e^x \sin x\,dx + C$$

$$\Rightarrow \quad ye^x = e^x \cos x - \int -\sin x\, e^x\,dx - \int e^x \sin x + C \qquad [\text{Integrating 1st integral by parts}]$$

$$\Rightarrow \quad ye^x = e^x \cos x + \int e^x \sin x\,dx - \int e^x \sin x\,dx + C$$

$$\Rightarrow \quad ye^x = e^x \cos x + C, \text{ which is the required solution.}$$

EXAMPLE 11 *Solve:* $\frac{dy}{dx} + y\tan x = 2x + x^2 \tan x.$

SOLUTION The given differential equation is $\frac{dy}{dx} + y\tan x = 2x + x^2\tan x$ ...(i)

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \tan x$ and $Q = 2x + x^2 \tan x$.

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

Multiplying both sides of (i) by I.F. = $\sec x$, we get

$$\sec x \frac{dy}{dx} + y \sec x \tan x = 2x \sec x + x^2 \sec x \tan x$$

Integrating both sides with respect to x, we get

$$y \sec x = \int (2x \sec x + x^2 \sec x \tan x)\, dx + C \qquad \left[\text{Using: } y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x \, dx + \int \underset{\text{I}}{x^2} \underset{\text{II}}{\sec x \tan x} \, dx + C$$

$$\Rightarrow \quad y \sec x = \int 2x \sec x \, dx + x^2 \sec x - \int 2x \sec x \, dx + C$$

$\Rightarrow \quad y \sec x = x^2 \sec x + C$, which is the required solution

EXAMPLE 12 *Solve:* $\dfrac{dy}{dx} + \dfrac{y}{x} = \cos x + \dfrac{\sin x}{x}$. **[CBSE 2017]**

SOLUTION The given differential equation is $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right) y = \cos x + \dfrac{\sin x}{x}$ ...(i)

This is a linear differential equation of the form

$\dfrac{dy}{dx} + Py = Q$, where $P = \dfrac{1}{x}$ and $Q = \cos x + \dfrac{\sin x}{x}$

$\therefore \quad$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

Multiplying both sides of (i) by I.F. = x, we get

$$x \frac{dy}{dx} + y = x \cos x + \sin x$$

Integrating both sides with respect to x, we get

$$yx = \int (x \cos x + \sin x)\, dx + C \qquad \left[\text{Using}: y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad xy = \int \underset{\text{I}}{x} \underset{\text{II}}{\cos x} \, dx + \int \sin x \, dx + C$$

$$\Rightarrow \quad xy = x \sin x - \int \sin x \, dx + \int \sin x \, dx + C \qquad [\text{Integrating 1st integral by parts}]$$

$\Rightarrow \quad xy = x \sin x + C \Rightarrow y = \sin x + \dfrac{C}{x}$, which gives the required solution.

EXAMPLE 13 *Solve:* $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

SOLUTION The given differential equation can be written as

$$\sec^2 y \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3 \Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \qquad \text{...(i)}$$

Let $\tan y = v$. Then, $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$. Putting $\tan y = v$ and $\sec^2 y \dfrac{dy}{dx} = \dfrac{dv}{dx}$ in (i), we get

$$\frac{dv}{dx} + (2x)\, v = x^3 \qquad \text{...(ii)}$$

This is a linear differential equation of the form $\dfrac{dv}{dx} + Pv = Q$, where $P = 2x$ and $Q = x^3$.

$\therefore \quad$ I.F. $= e^{\int 2x\, dx} = e^{x^2}$

Multiplying both sides of (ii) by e^{x^2}, we obtain

$$e^{x^2} \frac{dy}{dx} + e^{x^2} (2x)\, v = x^3 e^{x^2}$$

Integrating both sides with respect to x, we get

$$ve^{x^2} = \int x^3 e^{x^2}\, dx + C \qquad \left[\text{Using} : v\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}\int t\, e^t\, dt + C, \text{ where } t = x^2$$

$$\Rightarrow \quad ve^{x^2} = \frac{1}{2}(t-1)\, e^t + C \Rightarrow e^{x^2} \tan y = \frac{1}{2}(x^2-1)\, e^{x^2} + C,$$ which gives the required solution.

EXAMPLE 14 *Solve:* $\dfrac{dy}{dx} = -\dfrac{x + y\cos x}{1+\sin x}$ **[CBSE 2022]**

SOLUTION We have,

$$\frac{dy}{dx} = -\frac{x+y\cos x}{1+\sin x} \Rightarrow \frac{dy}{dx} + \frac{\cos x}{1+\sin x}\, y = -\frac{x}{1+\sin x} \qquad \text{...(i)}$$

This is a linear differential equation with $P = \dfrac{\cos x}{1+\sin x}$ and $Q = \dfrac{-x}{1+\sin x}$

$$\therefore \quad \text{I. F.} = e^{\int \frac{\cos x}{1+\sin x}\, dx} = e^{\log(1+\sin x)} = (1+\sin x)$$

Multiplying both sides of (i) by I.F. $= 1 + \sin x$, we get

$$(1+\sin x)\frac{dy}{dx} + y\cos x = -x$$

Integrating with respect to x, we get

$$y\,(1+\sin x) = \int -x\, dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad y\,(1+\sin x) = -\frac{x^2}{2} + C \Rightarrow y = \frac{2C - x^2}{2\,(1+\sin x)},$$ which gives the required solution.

EXAMPLE 15 *Solve each of the following initial value problems:*

(i) $\dfrac{dy}{dx} - y = e^x,\ y(0) = 1$ (ii) $x\dfrac{dy}{dx} + y = x\log x,\ y(1) = \dfrac{1}{4}$

SOLUTION (i) Given differential equation is

$$\frac{dy}{dx} - y = e^x \qquad \text{...(i)}$$

This is a linear differential equation with $P = -1$ and $Q = e^x$.

$$\therefore \quad \text{I.F.} = e^{\int -1\, dx} = e^{-x}$$

Multiplying both sides of (i) by e^{-x}, we get

$$\frac{dy}{dx}e^{-x} - ye^{-x} = e^x \cdot e^{-x}$$

Integrating both sides with respect to x, we get

$$ye^{-x} = \int e^x . e^{-x}\, dx + C \qquad \left[\text{Using} : y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + C\right]$$

$$\Rightarrow \quad ye^{-x} = x + C \qquad \text{...(ii)}$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$. Putting $x = 0$, $y = 1$ in (ii), we get

$$1 = 0 + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get: $ye^{-x} = x + 1 \Rightarrow y = (x+1)\, e^x$

Hence, $y = (x+1)\, e^x$ is the required solution.

(ii) Given differential equation is

$$x\frac{dy}{dx}+y = x\log x \Rightarrow \frac{dy}{dx}+\frac{1}{x}y = \log x \qquad ...(i)$$

This is linear differential equation of the form $\frac{dy}{dx}+Py=Q$ with $P=\frac{1}{x}$ and $Q=\log x$.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\log x} = x \qquad [\because x>0]$$

Multiplying both sides of (i) by I.F. $= x$, we get

$$x\frac{dy}{dx}+y = x\log x$$

Integrating with respect to x, we get

$$y\,x = \int \underset{\text{II}}{x}\,\underset{\text{I}}{\log x}\,dx \qquad [\text{Using}: y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C]$$

$$\Rightarrow \qquad yx = \frac{x^2}{2}(\log x)-\frac{1}{2}\int x\,dx \Rightarrow xy = \frac{x^2}{2}(\log x)-\frac{x^2}{4}+C \qquad ...(ii)$$

It is given that $y(1)=\frac{1}{4}$ *i.e.* $y=\frac{1}{4}$ when $x=1$. Putting $x=1$ and $y=\frac{1}{4}$ in (ii), we get

$$\frac{1}{4} = 0-\frac{1}{4}+C \Rightarrow C = \frac{1}{2}$$

Putting $C=\frac{1}{2}$ in (ii), we get: $xy = \frac{x^2}{2}(\log x)-\frac{x^2}{4}+\frac{1}{2} \Rightarrow y=\frac{1}{2}x\log x-\frac{x}{4}+\frac{1}{2x}$

Hence, $y=\frac{1}{2}x\log x-\frac{x}{4}+\frac{1}{2x}$ is the solution of the given differential equation.

EXAMPLE 16 *Solve each of the following initial value problems:*

(i) $\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{1}{(x^2+1)^2}$, $y(0)=0$ (ii) $(x^2+1)\,y'-2xy=(x^4+2x^2+1)\cos x$, $y(0)=0$

SOLUTION (i) Given differential equation is

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y = \frac{1}{(x^2+1)^2} \qquad ...(i)$$

This is a linear differential equation with $P=\frac{2x}{x^2+1}$ and $Q=\frac{1}{(x^2+1)^2}$.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{2x}{x^2+1}dx} = e^{\log(x^2+1)} = x^2+1$$

Multiplying both sides of (i) by I.F. $= x^2+1$, we get

$$(x^2+1)\frac{dy}{dx}+2xy = \frac{1}{x^2+1}$$

Integrating both sides with respect to x, we get

$$y(x^2+1) = \int \frac{1}{x^2+1}dx + C \qquad [\text{Using}: y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C]$$

$$\Rightarrow \qquad y(x^2+1) = \tan^{-1}x + C \qquad ...(ii)$$

It is given that $y(0)=0$ i.e. $y=0$ when $x=0$. Putting $x=0$, $y=0$ in (ii), we get

$$0 = 0 + C \Rightarrow C = 0$$

Putting $C=0$ in (ii), we get: $y(x^2+1) = \tan^{-1} x \Rightarrow y = \dfrac{\tan^{-1} x}{x^2+1}$, which gives the required solution.

(ii) Given differential equation is

$$(x^2+1)\, y' - 2xy = (x^4+2x^2+1)\cos x \Rightarrow \frac{dy}{dx} - \frac{2x}{x^2+1}\, y = (x^2+1)\cos x \qquad ...(i)$$

This is a linear differential equation with $P = \dfrac{-2x}{x^2+1}$ and $Q = (x^2+1)\cos x$.

$$\therefore \qquad \text{I.F.} = e^{\int \frac{-2x}{x^2+1}\, dx} = e^{-\log(x^2+1)} = (x^2+1)^{-1}$$

Multiplying (i) by $\dfrac{1}{x^2+1}$, we obtain

$$\frac{1}{x^2+1}\frac{dy}{dx} - \frac{2x}{(x^2+1)^2}\, y = \cos x$$

Integrating both sides with respect to x, we get

$$y \times \frac{1}{x^2+1} = \int \cos x\, dx + C \Rightarrow \frac{y}{x^2+1} = \sin x + C \qquad ...(ii)$$

It is given that $y(0)=0$ i.e. $y=0$ when $x=0$. Putting $x=0, y=0$ in (ii), we get: $C=0$

Putting $C=0$ in (ii), we get

$$\frac{y}{x^2+1} = \sin x \Rightarrow y = (x^2+1)\sin x, \text{ which gives the required solution.}$$

EXAMPLE 17 *Solve:*

(i) $x\dfrac{dy}{dx} + y - x + xy\cot x = 0$ **[CBSE 2011, 2012, NCERT]**

(ii) $(1+x^2)\, dy + 2\, xy\, dx = \cot x\, dx$ **[CBSE 2012]**

(iii) $y + \dfrac{d}{dx}(xy) = x(\sin x + \log x)$ **[NCERT EXEMPLAR]**

SOLUTION (i) Given differential equation is

$$x\frac{dy}{dx} + y - x + xy\cot x = 0 \Rightarrow x\frac{dy}{dx} + (1 + x\cot x)\, y = x \Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right) y = 1 \qquad ...(i)$$

This is a linear differential equation with $P = \dfrac{1}{x} + \cot x$ and $Q = 1$.

$$\therefore \qquad \text{I.F.} = e^{\int P\, dx} = e^{\int \left(\frac{1}{x} + \cot x\right) dx} = e^{\log x + \log \sin x} = e^{\log(x \sin x)} = x\sin x$$

Multiplying both sides of (i) by I.F. $= x\sin x$, we get

$$x\sin x\frac{dy}{dx} + (\sin x + x\cos x)\, y = x\sin x$$

Integrating with respect to x, we get

$$y(x\sin x) = \int \underset{\text{I}}{x}\, \underset{\text{II}}{\sin x}\, dx + C \Rightarrow xy\sin x = -x\cos x + \sin x + C, \text{ which is the required solution.}$$

(ii) Given differential equation is

$$(1+x^2)\,dy + 2xy\,dx = \cot x\,dx \Rightarrow \frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2} \qquad \ldots(i)$$

This is a linear differential equation with $P = \frac{2x}{1+x^2}$ and $Q = \frac{\cot x}{1+x^2}$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$$

Multiplying both sides of (i) by I.F. $=1+x^2$, we get

$$(1+x^2)\frac{dy}{dx} + 2\,xy = \cot x$$

Integrating both sides with respect to x, we get

$$y(1+x^2) = \int \cot x\,dx + C \Rightarrow y(1+x^2) = \log|\sin x| + C, \text{ which is the required solution.}$$

(iii) Given differential equation is

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x) \Rightarrow x\frac{dy}{dx} + 2y = x(\sin x + \log x) \Rightarrow \frac{dy}{dx} + \frac{2}{x}y = (\sin x + \log x) \quad \ldots(i)$$

This a linear differential equation with $P = \frac{2}{x}$ and $Q = (\sin x + \log x)$

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x^2} = e^{2\log x^2} = x^2$$

Multiplying both sides of (i) by I.F. $= x^2$, we obtain

$$x^2\frac{dy}{dx} + 2xy = x^2(\sin x + \log x)$$

Integrating both sides with respect to x, we obtain

$$yx^2 = \int x^2(\sin x + \log x)\,dx + C$$

$$\Rightarrow \quad yx^2 = \int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sin x}\,dx + \int \underset{\text{II}}{x^2}\,\underset{\text{I}}{\log x}\,dx + C$$

$$\Rightarrow \quad yx^2 = -x^2\cos x + 2\int x\cos x\,dx + \frac{x^3}{3}\log x - \frac{1}{3}\int x^2\,dx + C$$

$$\Rightarrow \quad yx^2 = -x^2\cos x + 2(x\sin x + \cos x) + \frac{x^3}{3}\log x - \frac{1}{9}x^3 + C, \text{ which is the required solution.}$$

EXAMPLE 18 *If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then show that $y(1) = -\frac{1}{2}$.*

[NCERT EXEAMPLAR]

SOLUTION We have,

$$(1+t)\frac{dy}{dt} - ty = 1 \Rightarrow \frac{dy}{dt} + \left(\frac{-t}{t+1}\right)y = \frac{1}{t+1} \qquad \ldots(i)$$

This is a linear differential equation with $P = \frac{-t}{t+1}$ and $Q = \frac{1}{t+1}$.

$$\therefore \quad \text{I.F.} = e^{\int \frac{-t}{t+1}dt} = e^{-\int \frac{(t+1)-1}{t+1}dt} = e^{-\int \left(1-\frac{1}{t+1}\right)dt} = e^{-(t-\log(t+1))} = e^{-t}e^{\log(t+1)} = (t+1)\,e^{-t}$$

Multiplying both sides of (i) by I.F. $=(t+1)\,e^{-t}$, we get

$$(t+1)\, e^{-t} \frac{dy}{dt} - t\, e^{-t} y = e^{-t}$$

Integrating both sides with respect to t we get

$$y\,(t+1)\, e^{-t} = \int e^{-t}\, dt + C \Rightarrow y\,(t+1)\, e^{-t} = -e^{-t} + C \Rightarrow y = -\frac{1}{t+1} + \frac{C}{t+1} e^{t} \qquad \text{...(ii)}$$

It is given that $y(0) = -1$ i.e. $y = -1$ when $t = 0$. Putting $t = 0$ and $y = -1$ in (ii), we get

$$-1 = -1 + C \Rightarrow C = 0$$

Putting $C = 0$ in (ii), we obtain $y = -\frac{1}{t+1}$. Hence, $y(1) = -\frac{1}{2}$.

21.6.7 LINEAR DIFFERENTIAL EQUATIONS OF THE FORM $\frac{dx}{dy} + Rx = S$

Sometimes a linear differential equation can be put in the form $\frac{dx}{dy} + Rx = S$, where R and S are functions of y or constants.

Note that here y is independent variable and x is a dependent variable.

The following algorithm is used to solve these types of equations.

ALGORITHM

Step I *Write the differential equation in the form* $\frac{dx}{dy} + Rx = S$ *and obtain* R *and* S.

Step II *Find I.F. by using I.F.* $= e^{\int R\, dy}$

Step III *Multiply both sides of the differential equation in Step I by I.F.*

Step IV *Integrate both sides of the equation obtained is Step III with respect to* y *to obtain the solution given by* $x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C$, *where* C *is the constant of integration.*

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve:* $y\, dx - (x + 2y^2)\, dy = 0$ **[NCERT]**

SOLUTION The given differential equation is

$$y\, dx - (x + 2y^2)\, dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x + 2y^2} \Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y} \Rightarrow \frac{dx}{dy} + \left(-\frac{1}{y}\right) x = 2y \qquad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dx}{dy} + Rx = S$, where $R = -\frac{1}{y}$ and $S = 2y$

$$\therefore \quad \text{I.F.} = e^{\int R dy} = e^{\int \frac{-1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$$

Multiplying both sides of (i) by I.F. $= y^{-1}$, we obtain

$$\frac{1}{y} \frac{dx}{dy} - \frac{1}{y^2} x = 2$$

Integrating both sides with respect to y, we get

$$x \times \frac{1}{y} = \int 2\,dy + C \qquad \left[\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad \frac{x}{y} = 2y + C,$$ which is the required solution.

EXAMPLE 2 *Solve:* $y\,dx + (x - y^3)\,dy = 0$ **[CBSE 2011]**

SOLUTION The given differential equation is

$$y\,dx + (x - y^3)\,dy = 0 \Rightarrow \frac{dx}{dy} + \frac{x}{y} = y^2 \qquad \text{...(i)}$$

This is a linear differential equation of the form $\frac{dx}{dy} + Rx = S$, where $R = \frac{1}{y}$ and $S = y^2$.

$$\therefore \quad \text{I.F.} = e^{\int R\,dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

Multiplying both sides of (i) by I.F. $= y$, we obtain

$$y\frac{dx}{dy} + x = y^3$$

Integrating both sides with respect to y, we get

$$xy = \int y^3\,dy + C \qquad \left[\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad xy = \frac{y^4}{4} + C,$$ which is the required solution.

EXAMPLE 3 *Solve:* $(x + 2y^3)\,dy = y\,dx$.

SOLUTION The given differential equation can be written as

$$\frac{dx}{dy} = \frac{x + 2y^3}{y} \Rightarrow \frac{dx}{dy} + \left(-\frac{1}{y}\right)x = 2y^2 \qquad \text{...(i)}$$

This is a linear differential equation of the form

$$\frac{dx}{dy} + Rx = S, \text{ where } R = -\frac{1}{y} \text{ and } S = 2y^2$$

We have, $\text{I.F.} = e^{\int R\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$

Multiplying both sides of (i) by I.F. $= \frac{1}{y}$, we get

$$\frac{1}{y}\frac{dx}{dy} - \frac{1}{y^2}x = 2y$$

Integrating both sides with respect to y, we get

$$x\left(\frac{1}{y}\right) = \int 2y\,dy + C \qquad \left[\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad \frac{x}{y} = y^2 + C \Rightarrow x = y^3 + Cy,$$ which is the required solution.

EXAMPLE 4 *Solve:* $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1.$ **[CBSE 2012, NCERT, CBSE 2015]**

SOLUTION We have,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \Rightarrow \frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \Rightarrow \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \qquad ...(i)$$

This is a linear differential equation with $P = \frac{1}{\sqrt{x}}$ and $Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$.

$$\therefore \quad \text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$$

Multiplying both sides of (i) by I.F. $= e^{2\sqrt{x}}$, we get

$$\frac{dy}{dx}e^{2\sqrt{x}} + \frac{y\,e^{2\sqrt{x}}}{\sqrt{x}} = \frac{1}{\sqrt{x}}$$

Integrating both sides with respect to x, we get

$$ye^{2\sqrt{x}} = \int e^{2\sqrt{x}} \cdot \frac{e^{-2\sqrt{x}}}{\sqrt{x}}\,dx + C \qquad \left[\text{Using}: y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + C\right]$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\,dx + C$$

$$\Rightarrow \quad y\,e^{2\sqrt{x}} = 2\sqrt{x} + C \Rightarrow y = (2\sqrt{x} + C)e^{-2\sqrt{x}},$$ which gives the required solution.

EXAMPLE 5 *Solve each of the following initial value problems:*

(i) $(x - \sin y)\,dy + (\tan y)\,dx = 0,\ y(0) = 0$

(ii) $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy,\ y(0) = 0$ **[NCERT]**

SOLUTION (i) Given differential equation is

$$(x - \sin y)\,dy + (\tan y)\,dx = 0 \Rightarrow \frac{dx}{dy} = -\left(\frac{x - \sin y}{\tan y}\right) \Rightarrow \frac{dx}{dy} + (\cot y)\,x = \cos y \qquad ...(i)$$

This is a linear differential equation with $R = \cot y$ and $S = \cos y$.

$$\therefore \quad \text{I.F.} = e^{\int \cot y} = e^{\log \sin y} = \sin y$$

Multiplying both sides of (i) by $\sin y$, we get

$$\frac{dx}{dy}\sin y + x\cos y = \cos y \sin y$$

Integrating both sides with respect to y, we get

$$x \sin y = \int \cos y \sin y\,dy + C \qquad \left[\text{Using}: x\,(\text{I.F.}) = \int S\,(\text{I.F.})\,dy + C\right]$$

$$\Rightarrow \quad x \sin y = \frac{1}{2}\int (\sin 2y)\,dy + C \Rightarrow x \sin y = -\frac{1}{4}\cos 2y + C \qquad ...(ii)$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$. Putting $x = 0$, $y = 0$ in (ii), we get

$$0 = -\frac{1}{4} + C \Rightarrow C = \frac{1}{4}$$

Putting $C = \frac{1}{4}$ in (ii), we get

$$x \sin y = -\frac{1}{4}\cos 2y + \frac{1}{4} \Rightarrow x \sin y = \frac{1}{2}\sin^2 y \Rightarrow 2x = \sin y \Rightarrow y = \sin^{-1} 2x$$

Hence, $y = \sin^{-1} 2x$ gives the required solution.

(ii) Given differential equation is

$$(1+y^2)\,dx = (\tan^{-1} y - x)\,dy \Rightarrow \frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2} \Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2} \quad ...(i)$$

This is a linear differential equation with $R = \dfrac{1}{1+y^2}$ and $S = \dfrac{\tan^{-1} y}{1+y^2}$.

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Multiplying both sides of (i) by I.F. $= e^{\tan^{-1} y}$, we get

$$\frac{dx}{dy} e^{\tan^{-1} y} + \frac{x}{1+y^2} e^{\tan^{-1} y} = \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}$$

Integrating both sides with respect to y, we get

$$x\,e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y}\,dy + C \qquad [\text{Using}: x(\text{I.F.}) = \int S(\text{I.F.})\,dy + C]$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = \int t\,e^t\,dt + C, \text{ where } t = \tan^{-1} y$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = e^t(t-1) + C \Rightarrow xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + C \quad ...(ii)$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$. Putting $x = 0$, $y = 0$ in (ii), we get

$$0 = e^0(0-1) + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get

$$xe^{\tan^{-1} y} = e^{\tan^{-1} y}(\tan^{-1} y - 1) + 1 \Rightarrow (x - \tan^{-1} y + 1)\,e^{\tan^{-1} y} = 1,$$

which gives the required solution.

EXAMPLE 6 *Solve each of the following initial value problems:*

(i) $ye^y\,dx = (y^3 + 2x\,e^y)\,dy$, $y(0) = 1$ (ii) $\sqrt{1-y^2}\,dx = (\sin^{-1} y - x)dy$, $y(0) = 0$

SOLUTION (i) We have,

$$ye^y\,dx = (y^3 + 2x\,e^y)\,dy \Rightarrow \frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y} \Rightarrow \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y} \quad ...(i)$$

This is a linear differential equation with $R = -2/y$ and $S = y^2 e^{-y}$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{2}{y} dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$$

Multiplying both sides of (i) by $\dfrac{1}{y^2}$, we get:

$$\frac{1}{y^2}\frac{dx}{dy} - \frac{2}{y^3}x = e^{-y} \quad ...(ii)$$

Integrating both sides of (ii) with respect to y, we get

$$x\left(\frac{1}{y^2}\right) = \int e^{-y}\,dy + C \qquad [\text{Using}: x(\text{I.F.}) = \int S(\text{I.F.})\,dy + C]$$

$$\Rightarrow \quad \frac{x}{y^2} = -e^{-y} + C \qquad ...(iii)$$

It is given that $y(0) = 1$ i.e. $y = 1$ when $x = 0$. Putting $x = 0$, $y = 1$ in (iii), we get

$$0 = -e^{-1} + C \Rightarrow C = \frac{1}{e}$$

Putting $C = \frac{1}{e}$ in (ii), we get: $\frac{x}{y^2} = -e^{-y} + \frac{1}{e} \Rightarrow x = y^2 (e^{-1} - e^{-y})$

Hence, $x = y^2 (e^{-1} - e^{-y})$ gives the required solution.

(ii) We are given that

$$\sqrt{1-y^2}\, dx = (\sin^{-1} y - x)\, dy \Rightarrow \frac{dx}{dy} = \frac{\sin^{-1} y}{\sqrt{1-y^2}} - \frac{x}{\sqrt{1-y^2}} \Rightarrow \frac{dx}{dy} + \frac{1}{\sqrt{1-y^2}} x = \frac{\sin^{-1} y}{\sqrt{1-y^2}} \qquad ...(i)$$

This is a linear differential equation with $R = \frac{1}{\sqrt{1-y^2}}$ and $S = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$.

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{\sqrt{1-y^2}} dy} = e^{\sin^{-1} y}$$

Multiplying both sides of (i) by I.F. $= e^{\sin^{-1} y}$, we get

$$e^{\sin^{-1} y} \frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} e^{\sin^{-1} y} = e^{\sin^{-1} y} \cdot \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

Integrating both sides with respect to y, we get

$$x\, e^{\sin^{-1} y} = \int e^{\sin^{-1} y} \frac{\sin^{-1} y}{\sqrt{1-y^2}} dy + C \qquad [\text{Using} : x\,(\text{I.F.}) = \int S\,(\text{I.F.})\, dy + C]$$

$$\Rightarrow \quad x e^{\sin^{-1} y} = \int t e^t\, dt + C, \text{ where } t = \sin^{-1} y$$

$$\Rightarrow \quad x e^{\sin^{-1} y} = e^t (t-1) + C \Rightarrow x\, e^{\sin^{-1} y} = e^{\sin^{-1} y} (\sin^{-1} y - 1) + C \qquad ...(ii)$$

It is given that $y(0) = 0$ i.e. $y = 0$ when $x = 0$. Putting $x = 0$, $y = 0$ in (ii), we get

$$0 = e^0 (0 - 1) + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we get:

$$x e^{\sin^{-1} y} = e^{\sin^{-1} y} (\sin^{-1} y - 1) + 1$$

$\Rightarrow e^{\sin^{-1} y} (x - \sin^{-1} y + 1) = 1 \Rightarrow x - \sin^{-1} y + 1 = e^{-\sin^{-1} y}$, which gives the required solution.

EXERCISE 21.10

BASIC

Solve the following differential equations:

1. $\frac{dy}{dx} + 2y = e^{3x}$ [CBSE 2017]
2. $4\frac{dy}{dx} + 8y = 5\, e^{-3x}$ [CBSE 2007]
3. $\frac{dy}{dx} + 2y = 6\, e^{x}$ [CBSE 2007C]
4. $\frac{dy}{dx} + y = e^{-2x}$
5. $x\frac{dy}{dx} = x + y$ [CBSE 2020]
6. $\frac{dy}{dx} + 2y = 4x$

7. $x\frac{dy}{dx}+y=x\,e^x$

8. $\frac{dy}{dx}+\frac{4x}{x^2+1}y+\frac{1}{(x^2+1)^2}=0$ [CBSE 2005]

9. $x\frac{dy}{dx}+y=x\log x$

10. $x\frac{dy}{dx}-y=(x-1)\,e^x$

11. $\frac{dy}{dx}+\frac{y}{x}=x^3$

12. $\frac{dy}{dx}+y=\sin x$ [NCERT]

13. $\frac{dy}{dx}+y=\cos x$

14. $\frac{dy}{dx}+2y=\sin x$ [NCERT]

15. $\frac{dy}{dx}=y\tan x-2\sin x$

16. $(1+x^2)\frac{dy}{dx}+y=\tan^{-1}x$ [CBSE 2009]

17. $\frac{dy}{dx}+y\tan x=\cos x$

18. $\frac{dy}{dx}+y\cot x=x^2\cot x+2x$ [NCERT, CBSE 2005]

19. $\frac{dy}{dx}+y\tan x=x^2\cos^2 x$

20. $(1+x^2)\frac{dy}{dx}+y=e^{\tan^{-1}x}$ [CBSE 2002, 2014]

BASED ON LOTS

21. $x\,dy=(2y+2x^4+x^2)\,dx$

22. $(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$ [CBSE 2016]

23. $y^2\frac{dx}{dy}+x-\frac{1}{y}=0$

24. $(2x-10y^3)\frac{dy}{dx}+y=0$

25. $(x+\tan y)\,dy=\sin 2y\,dx$

26. $dx+xdy=e^{-y}\sec^2 y\,dy$

27. $\frac{dy}{dx}=y\tan x-2\sin x$

28. $\frac{dy}{dx}+y\cos x=\sin x\cos x$

29. $(1+x^2)\frac{dy}{dx}-2xy=(x^2+2)(x^2+1)$ [CBSE 2005]

30. $(\sin x)\frac{dy}{dx}+y\cos x=2\sin^2 x\cos x$

31. $(x^2-1)\frac{dy}{dx}+2(x+2)y=2(x+1)$

32. $x\frac{dy}{dx}+2y = x\cos x$

33. $\frac{dy}{dx}-y = xe^x$ [CBSE 2002]

34. $\frac{dy}{dx}+2y = xe^{4x}$ [CBSE 2002C]

35. Solve the differential equation $(x+2y^2)\frac{dy}{dx}=y$, given that when $x=2$, $y=1$ [CBSE 2000]

36. Find one-parameter families of solution curves of the following differential equations: (or Solve the following differential equations)

(i) $\frac{dy}{dx}+3y = e^{mx}$, m is a given real number

(ii) $\frac{dy}{dx}-y = \cos 2x$

(iii) $x\frac{dy}{dx}-y = (x+1)\,e^{-x}$

(iv) $x\frac{dy}{dx}+y = x^4$

(v) $(x\log x)\frac{dy}{dx}+y = \log x$

(vi) $\frac{dy}{dx}-\frac{2xy}{1+x^2} = x^2+2$ [CBSE 2019]

(vii) $\frac{dy}{dx}+y\cos x = e^{\sin x}\cos x$

(viii) $(x+y)\frac{dy}{dx} = 1$

(ix) $\frac{dy}{dx}\cos^2 x = \tan x-y$

(x) $e^{-y}\sec^2 y\,dy = dx+x\,dy$

(xi) $x \log x \dfrac{dy}{dx} + y = 2 \log x$ [CBSE 2009] (xii) $x \dfrac{dy}{dx} + 2y = x^2 \log x$ [CBSE 2022]

37. Solve each of the following initial value problems:

(i) $y' + y = e^x,\ y(0) = \dfrac{1}{2}$ (ii) $x \dfrac{dy}{dx} - y = \log x,\ y(1) = 0$

(iii) $\dfrac{dy}{dx} + 2y = e^{-2x} \sin x,\ y(0) = 0$ (iv) $x \dfrac{dy}{dx} - y = (x+1)\, e^{-x},\ y(1) = 0$

(v) $(1 + y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0,\ y(0) = 0$

(vi) $\dfrac{dy}{dx} + y \tan x = 2x + x^2 \tan x,\ y(0) = 1$ (xi) $\dfrac{dy}{dx} + y \cot x = 2 \cos x,\ y\left(\dfrac{\pi}{2}\right) = 0$

(vii) $x \dfrac{dy}{dx} + y = x \cos x + \sin x,\ y\left(\dfrac{\pi}{2}\right) = 1$ [CBSE 2017]

(viii) $\dfrac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x,\ y\left(\dfrac{\pi}{2}\right) = 0$ [CBSE 2012, NCERT]

(ix) $\dfrac{dy}{dx} + 2y \tan x = \sin x;\ y = 0$ when $x = \dfrac{\pi}{3}$ [NCERT, CBSE 2014, 2018]

(x) $\dfrac{dy}{dx} - 3y \cot x = \sin 2x;\ y = 2$ when $x = \dfrac{\pi}{2}$ [NCERT]

(xi) $\dfrac{dy}{dx} + y \cot x = 2 \cos x,\ y\left(\dfrac{\pi}{2}\right) = 0$ [CBSE 2014]

(xii) $dy = \cos x\,(2 - y \operatorname{cosec} x)\, dx$ [NCERT EXEMPLAR]

(xiii) $\tan x \dfrac{dy}{dx} = 2x \tan x + x^2 - y;\ \tan x \neq 0$ given that $y = 0$ when $x = \dfrac{\pi}{2}$ [CBSE 2017]

38. Find the general solution of the differential equation $x \dfrac{dy}{dx} + 2y = x^2$. [NCERT]

39. Find the general solution of the differential equation $\dfrac{dy}{dx} - y = \cos x$. [NCERT]

40. Solve the differential equation $(y + 3x^2) \dfrac{dx}{dy} = x$ given that $y = 1$ when $x = 1$. [CBSE 2011, 22]

41. Find the particular solution of the differential equation $\dfrac{dx}{dy} + x \cot y = 2y + y^2 \cot y,\ y \neq 0$ given that $x = 0$ when $y = \dfrac{\pi}{2}$. [CBSE 2013]

42. Solve the following differential equation $(\cot^{-1} y + x)\, dy = (1 + y^2)\, dx$. [CBSE 2016]

ANSWERS

1. $y = \dfrac{1}{5} e^{3x} + C e^{-2x}$
2. $y = -\dfrac{5}{4} e^{-3x} + C e^{-2x}$
3. $y e^{2x} = 2 e^{3x} + C$
4. $y = -e^{-2x} + C e^{-x}$
5. $\dfrac{y}{x} = \log |x| + C$
6. $y = (2x - 1) + C e^{-2x}$
7. $y = \left(\dfrac{x-1}{x}\right) e^x + \dfrac{C}{x}$
8. $y (x^2 + 1)^2 = -x + C$
9. $4xy = 2x^2 \log |x| - x^2 + C$
10. $y = e^x + Cx$
11. $5xy = x^5 + C$
12. $y = C e^{-x} + \dfrac{1}{2} (\sin x - \cos x)$

13. $y = Ce^{-x} + \frac{1}{2}(\cos x + \sin x)$ 14. $y = Ce^{-2x} + \frac{1}{5}(2\sin x - \cos x)$

15. $2y \cos x = \cos 2x + C$ 16. $y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x}$ 17. $y \sec x = x + C$

18. $y \sin x = x^2 \sin x + C$ 19. $y \sec x = x^2 \sin x + 2x \cos x - 2 \sin x + C$

20. $2y\, e^{\tan^{-1} x} = e^{2\tan^{-1} x} + C$ 21. $y = x^4 + x^2 \log x + Cx^2$ 22. $2x\, e^{\tan^{-1} y} = e^{2\tan^{-1} y} + C$

23. $y = \left(\frac{y+1}{y}\right) + Ce^{1/y}$ 24. $x = 2y^3 + Cy^{-2}$ 25. $x = \tan y + C\sqrt{\tan y}$

26. $xe^y = \tan y + C$ 27. $y = \sec x(-\sin^2 x + C)$ 28. $y = \sin x - 1 + C\, e^{-\sin x}$

29. $y = (x + \tan^{-1} x + C)(x^2 + 1)$ 30. $y \sin x = \frac{2}{3}\sin^3 x + C$

31. $y = \frac{2(x+1)}{(x-1)^3}\left\{x^2 - 6x + 8\log(x+1)\right\} + C$ 32. $x^2 y = x^2 \sin x + 2x \cos x - 2 \sin x + C$

33. $y = \left(\frac{x^2}{2} + C\right)e^x$ 34. $y = \frac{x}{6}e^{4x} - \frac{1}{36}e^{4x} + Ce^{-2x}$ 35. $x = 2y^2$

36. (i) $y = \frac{e^{mx}}{m+3} + Ce^{-3x}$, if $m + 3 \neq 0$; $y = (x + C)\, e^{-3x}$, if $m + 3 = 0$

(ii) $y = \frac{1}{5}(-\cos 2x + 2\sin 2x) + C\, e^x$ (iii) $y = -e^{-x} + Cx$

(iv) $y = \frac{1}{5}x^4 + \frac{C}{x}$ (v) $y = \frac{1}{2}\log x + \frac{C}{\log x}$

(vi) $y = (x^2 + 1)\left\{(x + \tan^{-1} x) + C\right\}$ (vii) $y = \frac{1}{2}e^{\sin x} + C\, e^{-\sin x}$

(viii) $x + y + 1 = C\, e^y$ (ix) $y = \tan x - 1 + Ce^{-\tan x}$

(x) $x = e^{-y}(\tan y + C)$ (xi) $y = \log x + \frac{C}{\log x}$

(xii) $y = \frac{x^2}{16}(4\log|x| - 1) + \frac{C}{x^2}$

37. (i) $y = \frac{1}{2}e^x$ (ii) $y = x - 1 - \log x$

(iii) $y\, e^{2x} = 1 - \cos x$ (iv) $y = xe^{-1} - e^{-x}$

(v) $xe^{\tan^{-1} y} = \tan^{-1} y$ (vi) $y = x^2 + \cos x$

(vii) $y = \sin x$ (viii) $y \sin x = 2x^2 - \frac{\pi^2}{2}$

(ix) $y = \cos x - 2\cos^2 x$ (x) $y = 4\sin^3 x - 2\sin^2 x$

(xi) $y = -\cot x \cos x$ (xii) $y \sin x = -\frac{1}{2}\cos 2x + \frac{3}{2}$

38. $y = \frac{x^2}{4} + Cx^{-2}$ 39. $y = \left(\frac{\sin x - \cos x}{2}\right) + Ce^x$ 40. $\frac{y}{x} = 3x - 2$

41. $x \sin y = y^2 \sin y - \frac{\pi^2}{4}$ 42. $x = 1 - \cot^{-1} y + C\, e^{\tan^{-1} y}$

*21.7 APPLICATIONS OF DIFFERENTIAL EQUATIONS

In this section, we shall discuss some problems on the applications of differential equations in Science and Engineering. We shall also discuss problems on applications to other disciplines.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I APPLICATIONS ON GROWTH AND DECAY

EXAMPLE 1 *The surface area of a balloon being inflated changes at a constant rate. If initially, its radius is 3 units and after 2 seconds, it is 5 units, find the radius after t seconds.*

SOLUTION Let r be the radius and S be the surface area of the balloon at any time t. Then,

$$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \qquad \text{...(i)}$$

It is given that $\frac{dS}{dt} = \text{Const} = k$ (say). Putting $\frac{dS}{dt} = k$ in (i), we get

$$k = 8\pi r \frac{dr}{dt} \Rightarrow 8\pi r\, dr = k\, dt \qquad \text{[By separating the variables]}$$

Integrating both sides, we get: $4\pi r^2 = kt + C$ ...(ii)

We are given that at $t = 0$, $r = 3$. Putting $t = 0$ and $r = 3$ in (ii), we get: $36\pi = k(0) + C$ ...(iii)

Putting $t = 2$ and $r = 5$ in (ii), we get: $100\pi = 2k + C$ ...(iv)

Solving (iii) and (iv), we get, $C = 36\pi$ and $k = 32\pi$

Substituting the values of C and k in (ii), we obtain

$$4\pi r^2 = 32\pi t + 36\pi \Rightarrow r^2 = 8t + 9 \Rightarrow r = \sqrt{8t + 9}$$

EXAMPLE 2 *A population grows at the rate of 8% per year. How long does it take for the population to double? Use differential equation for it.*

SOLUTION Let P_0 be the initial population and let the population after t years be P. It is given that

$$\frac{dP}{dt} = \frac{8P}{100} \Rightarrow \frac{dP}{dt} = \frac{2P}{25} \Rightarrow \frac{dP}{P} = \frac{2}{25}\, dt \qquad \text{[By separating the variables]}$$

Integrating both sides

$$\int \frac{1}{P}\, dP = \frac{2}{25} \int 1.dt \Rightarrow \log P = \frac{2}{25} t + C \qquad \text{...(i)}$$

Initially i.e. at $t = 0$, we have $P = P_0$. Putting $t = 0$ and $P = P_0$ in (i), we get

$$\log P_0 = \frac{2 \times 0}{25} + C \Rightarrow C = \log P_0$$

Substituting $C = \log P_0$ in (i), we get

$$\log P = \frac{2}{25} t + \log P_0 \Rightarrow \log \frac{P}{P_0} = \frac{2}{25} t \Rightarrow t = \frac{25}{2} \log\left(\frac{P}{P_0}\right)$$

When $P = 2P_0$, we get: $t = \frac{25}{2} \log\left(\frac{2P_0}{P_0}\right) = \frac{25}{2} \log 2$

May be skipped in the first reading. Not from examination point of view.

Thus, the population is doubled in $\frac{25}{2}\log 2$ years.

EXAMPLE 3 *Suppose the growth of a population is proportional to the number present. If the population of a colony doubles in 25 days, in how many days will the population become triple?*

SOLUTION Let P_0 be the initial population and P be the population at any time t. It is given that

$$\frac{dP}{dt} \propto P$$

$$\Rightarrow \quad \frac{dP}{dt} = \lambda P, \quad \lambda \text{ is a constant}$$

$$\Rightarrow \quad \frac{dP}{P} = \lambda\, dt \Rightarrow \int \frac{1}{P}\, dP = \lambda \int dt \Rightarrow \log P = \lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $P = P_0$. Putting $t = 0$ and $P = P_0$ in (i), we get

$$\log P_0 = 0 + C \Rightarrow C = \log P_0$$

Putting $C = \log P_0$ in (i), we get

$$\log P = \lambda t + \log P_0 \Rightarrow \log\left(\frac{P}{P_0}\right) = \lambda t \qquad \text{...(ii)}$$

It is given that $P = 2P_0$ when $t = 25$ days. Putting $t = 25$ and $P = 2P_0$ in (ii), we get

$$\log 2 = 25\lambda \Rightarrow \lambda = \frac{1}{25}\log 2$$

Putting $\lambda = \frac{1}{25}\log 2$ in (ii), we get: $\log\left(\frac{P}{P_0}\right) = \left(\frac{1}{25}\log 2\right)t$ ...(iii)

Suppose the population is tripled in t_1 days. i.e. $P = 3P_0$ when $t = t_1$. Putting $P = 3P_0$ and $t = t_1$ in (iii), we get

$$\log 3 = \left(\frac{1}{25}\log 2\right)t_1 \Rightarrow t_1 = 25\left(\frac{\log 3}{\log 2}\right) \text{days}$$

Hence, the population is tripled in $25\left(\frac{\log 3}{\log 2}\right)$ days.

EXAMPLE 4 *It is known that, if the interest is compounded continuously, the principal changes at the rate equal to the product of the rate of bank interest per annum and the principal.*

(i) *If the interest is compounded continuously at 5% per annum, in how many years will ₹100 double itself?*

(ii) *At what interest rate will ₹ 100 double itself in 10 years?* ($\log_e 2 = 0.6931$).

(iii) *How much will ₹ 1000 be worth at 5% interest after 10 years?* ($e^{0.5} = 1.648$).

SOLUTION If P denotes the principal at any time t and the rate of interest be r % per annum compounded continuously, then according to the law given in the problem, we get

$$\frac{dP}{dt} = \frac{Pr}{100} \Rightarrow \frac{dP}{P} = \frac{r}{100}dt \Rightarrow \int \frac{1}{P}\, dP = \frac{r}{100}\int dt \Rightarrow \log P = \frac{rt}{100} + C \qquad \text{...(i)}$$

Let P_0 be the initial principal i.e. at $t = 0$, $P = P_0$. Putting $P = P_0$ in (i), we get: $\log P_0 = C$

Putting $C = \log P_0$ in (i), we get

$$\log P = \frac{rt}{100} + \log P_0 \Rightarrow \log\left(\frac{P}{P_0}\right) = \frac{rt}{100} \qquad \text{...(ii)}$$

(i) In this case, we have: $r = 5$, $P_0 = ₹100$ and $P = ₹200 = 2P_0$

Substituting these values in (ii), we obtain

$$\log 2 = \frac{5}{100} t \Rightarrow t = 20 \log_e 2 = 20 \times 0.6931 \text{ years} = 13.862 \text{ years.}$$

(ii) In this case, we have: $P_0 = ₹100$, $P = ₹200 = 2P_0$ and $t = 10$ years.

Substituting these values in (ii), we get

$$\log 2 = \frac{10r}{100} \Rightarrow r = 10 \log 2 = 10 \times 0.6931 = 6.931$$

Hence, $r = 6.931\%$ per annum.

(iii) In this case, we have: $P_0 = ₹1000$, $r = 5$ and $t = 10$

Substituting these values in (ii), we get

$$\log\left(\frac{P}{1000}\right) = \frac{5 \times 10}{100} = \frac{1}{2} = 0.5 \Rightarrow \frac{P}{1000} = e^{0.5} \Rightarrow P = 1000 \times 1.648 = 1648$$

Hence, $P = ₹1648$.

EXAMPLE 5 *It is given that the rate at which some bacteria multiply is proportional to the instantaneous number present. If the original number of bacteria doubles in two hours, in how many hours will it be five times?*

SOLUTION Let the original count of bacteria be N_0 and at any time t the count of bacteria be N. It is given that

$$\frac{dN}{dt} \propto N$$

$$\Rightarrow \quad \frac{dN}{dt} = \lambda N, \text{ where } \lambda \text{ is a constant}$$

$$\Rightarrow \quad \frac{dN}{N} = \lambda\, dt \Rightarrow \int \frac{1}{N} dN = \lambda \int dt \Rightarrow \log N = \lambda t + C \qquad \text{...(i)}$$

We have, $N = N_0$ at $t = 0$. Putting $t = 0$ and $N = N_0$ in (i), we obtain

$$\therefore \quad \log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = \lambda t + \log N_0 \Rightarrow \log\left(\frac{N}{N_0}\right) = \lambda t \qquad \text{...(ii)}$$

It is given that the original number of bacteria doubles in 2 hrs.

That is when $t = 2$ hours, $N = 2N_0$. Putting $t = 2$ and $N = 2N_0$ in (ii), we get

$$\log\left(\frac{2N_0}{N_0}\right) = 2\lambda \Rightarrow \lambda = \frac{1}{2}\log 2$$

Putting $\lambda = \frac{1}{2}\log 2$ in (ii), we get

$$\log\left(\frac{N}{N_0}\right) = \left(\frac{1}{2}\log 2\right) t \Rightarrow t = \frac{2}{\log 2}\log\left(\frac{N}{N_0}\right) \qquad \text{...(iii)}$$

Suppose the count of bacteria becomes 5 times i.e. $5\,N_0$ in t_1 hours. Putting $t = t_1$ and $N = 5N_0$ in (iii), we get

$$t_1 = \frac{2}{\log 2} \log\left(\frac{5\,N_0}{N_0}\right) = \frac{2}{\log 2}(\log 5) = \frac{2\log 5}{\log 2} \text{ hours.}$$

EXAMPLE *It is given that radium decomposes at a rate proportional to the amount present. If p % of the original amount of radium disappears in l years, what percentage of it will remain after 2l years?*

SOLUTION Let A_0 be the original amount of radium and A be the amount of radium at any time t. Then, the rate of decompose of radium is $\frac{dA}{dt}$. It is given that

$$\frac{dA}{dt} \propto A$$

$$\Rightarrow \quad \frac{dA}{dt} = -\lambda A, \text{ where } \lambda \text{ is a positive constant}$$

$$\Rightarrow \quad \frac{dA}{A} = -\lambda\, dt \Rightarrow \log A = -\lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $A = A_0$. Putting $t = 0$ and $A = A_0$ in (i), we get

$$\log A_0 = 0 + C \Rightarrow C = \log A_0$$

Putting $C = \log A_0$ in (i), we get

$$\log A = -\lambda t + \log A_0 \Rightarrow \log\left(\frac{A}{A_0}\right) = -\lambda t \qquad \text{...(ii)}$$

It is given that p % of the original amount of radium disintegrates in l years. This means that the amount of radium present at $t = l$ is $\left(A_0 - \frac{p}{100} \times A_0\right) = \left(1 - \frac{p}{100}\right) A_0$. Putting $A = A_0\left(1 - \frac{p}{100}\right)$ and $t = l$ in (ii), we get

$$\log\left(1 - \frac{p}{100}\right) = -\lambda l \Rightarrow \lambda = -\frac{1}{l}\log\left(1 - \frac{p}{100}\right)$$

Substituting the value of λ in (ii), we get

$$\log\left(\frac{A}{A_0}\right) = \frac{t}{l}\log\left(1 - \frac{p}{100}\right) \qquad \text{...(iii)}$$

Let A be the amount of radium available after $2l$ years.

Putting $t = 2l$ in (iii), we get

$$\log\left(\frac{A}{A_0}\right) = 2\log\left(1 - \frac{p}{100}\right)$$

$$\Rightarrow \quad \frac{A}{A_0} = \left(1 - \frac{p}{100}\right)^2$$

$$\Rightarrow \quad \frac{A}{A_0} \times 100 = \left(1 - \frac{p}{100}\right)^2 \times 100 \qquad \text{[Multiplying both sides by 100]}$$

$$\Rightarrow \quad \frac{A}{A_0} \times 100 = \left(10 - \frac{p}{10}\right)^2$$

Hence, required percent $= \left(10 - \dfrac{p}{10}\right)^2$.

EXAMPLE 7 *A radioactive substance disintegrates at a rate proportional to the amount of substance present. If 50% of the given amount disintegrates in 1600 years. What percentage of the substance disintegrates in 10 years?* $\left(\textit{Take } e^{-\frac{\log 2}{160}} = 0.9957\right)$

SOLUTION Let A denote the amount of the radioactive substance present at any instant t and let A_0 be the initial amount of the substance.

It is given that

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -\lambda A \qquad \text{...(i)}$$

where λ is the constant of proportionality such that $\lambda > 0$. Here, negative sign indicates that A decreases with the increase in t.

Now,

$$\frac{dA}{dt} = -\lambda A \Rightarrow \frac{1}{A}\, dA = -\lambda\, dt \Rightarrow \int \frac{1}{A}\, dA = -\lambda \int 1 . dt \Rightarrow \log A = -\lambda t + C \qquad \text{...(ii)}$$

Initially i.e. at $t = 0$, we have $A = A_0$. Putting $t = 0$ and $A = A_0$ in (ii), we get

$$\log A_0 = 0 + C \Rightarrow C = \log A_0$$

Putting $C = \log A_0$ in (ii), we get

$$\log A = -\lambda t + \log A_0 \Rightarrow \log\left(\frac{A}{A_0}\right) = -\lambda t \qquad \text{...(iii)}$$

It is given that $A = \dfrac{A_0}{2}$ at $t = 1600$ years. Putting $A = \dfrac{A_0}{2}$ and $t = 1600$ in (iii), we get

$$\log\left(\frac{1}{2}\right) = -1600\,\lambda \Rightarrow \lambda = \frac{1}{1600}\log 2$$

Substituting the value of λ in (iii), we get

$$\log\left(\frac{A}{A_0}\right) = -\left(\frac{1}{1600}\log 2\right)t \Rightarrow \frac{A}{A_0} = e^{-\frac{\log 2}{1600}t} \Rightarrow A = A_0\, e^{-\frac{\log 2}{1600}t}$$

Putting $t = 10$, we obtain the amount of the radioactive substance present after 10 years and is given by

$$A = A_0\,(0.9957) \qquad \left[\because e^{-\frac{\log 2}{160}} = 0.9957\right]$$

$\therefore$ Amount that disintegrates in 10 years $= A_0 - A = A_0 - 0.9957\, A_0 = 0.0043$

Hence, percentage of the amount disintegrated in 10 years $= \dfrac{0.0043 A_0}{A_0} \times 100 = 0.43$

Hence, 0.43% of the original amount disintegrates in 10 years.

EXAMPLE 8 *The rate at which radioactive substances decay is known to be proportional to the number of such nuclei that are present at the time in a given sample.*

(i) *In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. Find what percentage of the original radioactive nuclei will remain after 1000 years.*

(ii) *If 100 grams of a radioactive substance is present 1 year after the substance was produced and 75 grams is present 2 years after the substance was produced, how much radioactive substance was produced?*

SOLUTION (i) Let there be N radioactive nuclei in a sample at any time t and let N_0 be the initial number of radioactive nuclei. Then, the rate of decay of the radioactive substance is given by $\frac{dN}{dt}$.

It is given that $\frac{dN}{dt} \propto N$

$$\Rightarrow \quad \frac{dN}{dt} = -\lambda N, \text{ where } \lambda > 0 \text{ is a constant}$$

$$\Rightarrow \quad \frac{dN}{N} = -\lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{N}\, dN = -\lambda \int dt \qquad \text{[On integrating]}$$

$$\Rightarrow \quad \log N = -\lambda t + C \qquad \text{...(i)}$$

At $t = 0$, we have $N = N_0$. Putting $t = 0$ and $N = N_0$ in (i), we get

$$\log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = -\lambda t + \log N_0 \Rightarrow \log \frac{N}{N_0} = -\lambda t \qquad \text{...(ii)}$$

It is given that 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. Therefore, number of radioactive nuclei available at $t = 100$ is given by

$$N = N_0 - \frac{10}{100} \times N_0 = \frac{9N_0}{10}$$

Putting $N = \frac{9N_0}{10}$ and $t = 100$ in (ii), we get

$$\log \frac{9}{10} = -100\lambda \Rightarrow \lambda = -\frac{1}{100} \log \frac{9}{10}$$

Putting the value of λ in (ii), we get

$$\log \frac{N}{N_0} = \left(\frac{1}{100} \log \frac{9}{10}\right) t \qquad \text{...(iii)}$$

We have to find the value of N at $t = 1000$ years. Putting $t = 1000$ years in (iii), we get

$$\log \frac{N}{N_0} = 10 \log\left(\frac{9}{10}\right)$$

$$\Rightarrow \quad \log \frac{N}{N_0} = \log\left(\frac{9}{10}\right)^{10} \Rightarrow \frac{N}{N_0} = \left(\frac{9}{10}\right)^{10} \Rightarrow \frac{N}{N_0} \times 100 = \left(\frac{9}{10}\right)^{10} \times 100 = \frac{9^{10}}{10^8}$$

$\therefore$ Percentage of radioactive nuclei that remain after 1000 years $= \frac{N}{N_0} \times 100 = \frac{9^{10}}{10^8}$.

Hence, $\frac{9^{10}}{10^8}$ % of radioactive nuclei will remain after 1000 years.

(ii) Suppose N_0 grams of radioactive substance was produced and at any time t, N grams of substance is present. Then, the rate of decay of the radioactive substance is given by $\frac{dN}{dt}$.

It is given that $\frac{dN}{dt} \propto N$

$\Rightarrow \quad \frac{dN}{dt} = -\lambda N$, where $\lambda > 0$ is a constant

$$\Rightarrow \quad \frac{dN}{N} = -\lambda\, dt \Rightarrow \int \frac{1}{N}\, dN = -\lambda \int dt \Rightarrow \log N = -\lambda t + C \quad \text{...(i)}$$

At $t = 0$, we have $N = N_0$. Putting $t = 0$ and $N = N_0$ in (i), we get

$$\log N_0 = 0 + C \Rightarrow C = \log N_0$$

Putting $C = \log N_0$ in (i), we get

$$\log N = -\lambda t + \log N_0 \Rightarrow \log \frac{N}{N_0} = -\lambda t \quad \text{...(ii)}$$

It is given that after 1 years 100 grams of radioactive substance was present. i.e.
$N = 100$ grams at $t = 1$. Putting $t = 1$ and $N = 100$ in (ii), we get

$$\log\left(\frac{100}{N_0}\right) = -\lambda \quad \text{...(iii)}$$

After 2 years, 75 grams of radioactive substances was present i.e. at $t = 2$, $N = 75$ grams. Putting $N = 75$ grams and $t = 2$ in (ii), we get

$$\log\left(\frac{75}{N_0}\right) = -2\lambda \quad \text{...(iv)}$$

Eliminating λ from (iii) and (iv), we get

$$\log\left(\frac{75}{N_0}\right) = 2\log\left(\frac{100}{N_0}\right)$$

$$\log\left(\frac{75}{N_0}\right) = \log\left(\frac{100}{N_0}\right)^2 \Rightarrow \frac{75}{N_0} = \left(\frac{100}{N_0}\right)^2 \Rightarrow N_0 = \frac{100^2}{75} \Rightarrow N_0 = \frac{400}{3} \text{ grams.}$$

Hence, $\frac{400}{3}$ grams of radioactive substance was produced.

EXAMPLE 9 *In a college hostel accommodating 1000 students, one of them came in carrying a flu virus, then the hostel was isolated. If the rate at which the virus spreads is assumed to be proportional to the product of the number N of infected students and the number of non-infected students, and if the number of infected students is 50 after 4 years, then show that more than 95% of the students will be infected after 10 days.*

SOLUTION At any time t, we have

Number of infected students $= N$, Number of non-infected students $= 1000 - N$

It is given that at any time t, the rate at which the virus spreads i.e. $\frac{dN}{dt}$ is proportional to the product of number of infected and the number of non-infected students.

$\therefore \quad \frac{dN}{dt} \propto N(1000 - N)$

$\Rightarrow \quad \frac{dN}{dt} = \lambda N(1000 - N)$, where λ is the constant of proportionality.

$$\Rightarrow \quad \frac{1}{N(1000 - N)}\, dN = \lambda\, dt$$

$$\Rightarrow \quad \int \frac{1}{N(1000 - N)}\, dN = \lambda \int dt \quad \text{[On integrating]}$$

$$\Rightarrow \quad \frac{1}{1000}\int\left(\frac{1}{1000-N}+\frac{1}{N}\right)dN = \int dt$$

$$\Rightarrow \quad \frac{1}{1000}\{\log N - \log(1000-N)\} = \lambda t + C \Rightarrow \frac{1}{1000}\log\left(\frac{N}{1000-N}\right) = \lambda t + C \qquad \text{...(i)}$$

It is given that initially one student was carrying a flu virus. That is at $t=0$, we have $N=1$. Putting $t=0$ and $N=1$ in (i), we get

$$\frac{1}{1000}\log\frac{1}{999} = 0 + C \Rightarrow C = \frac{1}{1000}\log\left(\frac{1}{999}\right) = -\frac{\log 999}{1000}$$

Substituting the value of C in (i), we get

$$\frac{1}{1000}\log\left(\frac{N}{1000-N}\right) = \lambda t - \frac{\log 999}{1000}$$

$$\Rightarrow \quad \frac{1}{1000}\log\left(\frac{N}{1000-N}\right) + \frac{1}{1000}\log 999 = \lambda t \Rightarrow \frac{1}{1000}\log\left(\frac{999\,N}{1000-N}\right) = \lambda t \qquad \text{...(ii)}$$

After 4 years the number of infected students is 50. That is at $t=4$, $N=50$.

Substituting these values in (ii), we get

$$\frac{1}{1000}\log\left(\frac{49950}{950}\right) = 4\lambda \Rightarrow \lambda = \frac{1}{4000}\log\left(\frac{4995}{95}\right) = \frac{1}{4000}\log\left(\frac{999}{19}\right)$$

Putting the value of λ in (ii), we get

$$\frac{1}{1000}\log\left(\frac{999N}{1000-N}\right) = \left\{\frac{1}{4000}\log\left(\frac{999}{19}\right)\right\}t \Rightarrow 4\log\left(\frac{999N}{1000-N}\right) = \left(\log\frac{999}{19}\right)t \ \text{...(iii)}$$

We have to find the number of infected students after 10 years. So, putting $t=10$ in (iii), we get

$$4\log\left(\frac{999N}{1000-N}\right) = 10\times\log\left(\frac{999}{19}\right)$$

$$\Rightarrow \quad \log\left(\frac{999N}{1000-N}\right) = \frac{5}{2}\log\left(\frac{999}{19}\right)$$

$$\Rightarrow \quad \log\left(\frac{1000-N}{999N}\right) = -\frac{5}{2}\log\left(\frac{999}{19}\right) = \log\left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000-N}{999N} = \left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000}{999N} - \frac{1}{999} = \left(\frac{999}{19}\right)^{-5/2} \Rightarrow \frac{1000}{999N} = \frac{1}{999} + \left(\frac{999}{19}\right)^{-5/2}$$

$$\Rightarrow \quad \frac{1000}{N} = 1 + (999)^{-3/2}\times 19^{5/2} \Rightarrow N = \frac{1000}{1+(999)^{-3/2}\times 19^{5/2}} = 952 \text{ approximately}$$

Thus, percentage of infected students after 10 years is given by $\frac{N}{1000}\times 100 = \frac{952}{1000}\times 100 = 95.2.$

Hence, more than 95% students will be infected after 10 days.

EXAMPLE 10 *Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm and 1 hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.*

SOLUTION Let r be the radius, V be the volume and S be the surface area of the rain drop at any time t. Then,

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2$$

We are given that the rate of change of volume of the rain drop is proportional to surface area.

i.e. $\frac{dV}{dt} \propto S$

$\Rightarrow$ $\frac{dV}{dt} = kS$, k is the constant of proportionality

$$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k(4\pi r^2) \Rightarrow 4\pi r^2 \frac{dr}{dt} = k(4\pi r^2) \Rightarrow \frac{dr}{dt} = k \Rightarrow dr = k\,dt$$

Integrating both sides, we get

$$\int dr = k\int dt \Rightarrow r = kt + C \qquad \text{...(i)}$$

We are given that $r = 3$ at $t = 0$ and $r = 2$ at $t = 1$

$\therefore$ $3 = k(0) + C$ and $2 = k + C \Rightarrow C = 3$ and $k = -1$

Putting the values of C and k in (i), we get $r = 3 - t$, where $0 \le t \le 3$.

Type II ON NEWTON'S LAW OF COOLING

The Newton's law of cooling states that the temperature of a body changes at a rate which is proportional to the difference in temperature between that of the surrounding medium and that of the body itself.

EXAMPLE 11 *The temperature T of a cooling object drops at a rate proportional to the difference $T - S$, where S is constant temperature of surrounding medium. If initially $T = 150°C$, find the temperature of the cooling object at any time t.*

SOLUTION Let T be the temperature of the cooling object at any time t. Then,

$\frac{dT}{dt} = -k(T - S)$, where $k > 0$ is a constant.

$$\Rightarrow \frac{1}{T - S}\,dT = -k\,dt$$

Integrating both sides, we get

$$\int \frac{1}{T-S}\,dT = -k\int dt \Rightarrow \log|T - S| = -kt + \log C \qquad \text{...(i)}$$

It is given that at $t = 0$, $T = 150°C$. Putting $t = 0$ and $T = 150$ in (i), we get

$$\log|150 - S| = 0 + \log C$$

Putting the value of log C in (i), we get

$$\log|T - S| = -kt + \log|150 - S| \Rightarrow \log\left|\frac{T-S}{150-S}\right| = -kt \Rightarrow \frac{T-S}{150-S} = e^{-kt}$$

$\Rightarrow$ $T - S = (150 - S)\,e^{-kt}$, this gives the temperature T at any time t.

EXAMPLE 12 *Water at temperature* 100° C *cools in* 10 *minutes to* 80° C *in a room of temperature* 25° C. *Find*

(i) *the temperature of water after* 20 *minutes*

(ii) *the time when the temperature is* 40° C. $\left[\text{Given} : \log_e \frac{11}{15} = -0.3101,\ \log 5 = 1.6094\right]$

SOLUTION Let T be the temperature of water at any time t. Then, by Newton's law of cooling

$$\frac{dT}{dt} \propto (T-25)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda (T-25), \text{ where } \lambda > 0 \text{ is a constant.}$$

$$\Rightarrow \quad \frac{1}{T-25}\, dT = -\lambda\, dt \Rightarrow \int \frac{1}{T-25}\, dT = -\lambda \int dt \Rightarrow \log|T-25| = -\lambda t + C \quad \text{...(i)}$$

At $t=0$, we have $T=100°$ C. Substituting these values in (i), we get

$$\log 75 = 0 + C \Rightarrow C = \log 75$$

Putting $C = \log 75$ in (i), we get

$$\log|T-25| = -\lambda t + \log 75 \Rightarrow \log\left|\frac{T-25}{75}\right| = -\lambda t \quad \text{...(ii)}$$

It is also given that $T = 80°$ C at $t=10$.

$$\therefore \quad \log\left|\frac{80-25}{75}\right| = -10\lambda \Rightarrow \log\frac{11}{15} = -10\lambda \Rightarrow \lambda = -\frac{1}{10}\log\frac{11}{15}$$

Putting the value of λ in (ii), we get

$$\log\left|\frac{T-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)t \quad \text{...(iii)}$$

(i) We have to find the temperature of water after 20 minutes. Putting $t=20$ in (iii), we get

$$\log\left|\frac{T-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)\times 20$$

$$\Rightarrow \quad \log\left|\frac{T-25}{75}\right| = \log\left(\frac{11}{15}\right)^2 \Rightarrow \frac{T-25}{75} = \left(\frac{11}{15}\right)^2 \Rightarrow T-25 = \frac{121}{225}\times 75 \Rightarrow T = 65.33°\text{ C}$$

So, the temperature of water after 20 minutes is 65.33° C.

(ii) We have to find the time taken so that the temperature becomes 40°C. Putting $T = 40°$ C in (iii), we get

$$\log\left|\frac{40-25}{75}\right| = \left(\frac{1}{10}\log\frac{11}{15}\right)t$$

$$\Rightarrow \quad \log\frac{1}{5} = \frac{1}{10}\left(\log\frac{11}{15}\right)t \Rightarrow t = \frac{10\log\left(\frac{1}{5}\right)}{\log\left(\frac{11}{15}\right)} = \frac{-10\log 5}{\log\frac{11}{15}} = \frac{-10\times 1.6094}{-0.3101} = 53.46$$

Hence, the temperature becomes 40°C after 53.46 minutes.

EXAMPLE 13 *A thermometer reading 80° F is taken outside. Five minutes later the thermometer reads 60° F. After another 5 minutes the thermometer reads 50° F. What is the temperature outside?*

SOLUTION Let at any time t the thermometer reading be $T°$ F and the outside temperature be $S°$F. Then, by Newton's law of cooling

$$\frac{dT}{dt} \propto (T-S)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda(T-S) \Rightarrow \frac{1}{T-S}\, dT = -\lambda\, dt \Rightarrow \int \frac{1}{T-S}\, dT = -\lambda\int dt \Rightarrow \log(T-S) = -\lambda t + C \quad \text{...(i)}$$

It is given that $T = 80°$ F at $t = 0$. Substituting these values in (i), we get

$$\log(80-S) = 0 + C \Rightarrow C = \log(80-S)$$

Putting the value of C in (i), we get

$$\log(T-S) = -\lambda t + \log(80-S) \Rightarrow \log\left(\frac{T-S}{80-S}\right) = -\lambda t \qquad ...(ii)$$

It is given that: $T = 60°$ F at $t = 5$ and, $T = 50°$ F at $t = 10$

Substituting these values in (ii), we get

$$\log\left(\frac{60-S}{80-S}\right) = -5\lambda \text{ and } \log\left(\frac{50-S}{80-S}\right) = -10\lambda$$

$$\Rightarrow \quad 2\log\left(\frac{60-S}{80-S}\right) = \log\left(\frac{50-S}{80-S}\right) \qquad \text{[On eliminating } \lambda]$$

$$\Rightarrow \quad \left(\frac{60-S}{80-S}\right)^2 = \left(\frac{50-S}{80-S}\right) \Rightarrow (60-S)^2 = (50-S)(80-S)$$

$$\Rightarrow \quad 3600 - 120S + S^2 = 4000 - 130S + S^2 \Rightarrow 10S = 400 \Rightarrow S = 40° \text{ F.}$$

Hence, the out side temperature is 40° F.

EXAMPLE 14 *The doctor took the temperature of a dead body at 11.30 PM which was 94.6° F. He took the temperature of the body again after one hour, which was 93.4° F. If the temperature of the room was 70° F, estimate the time of death. Taking normal temperature of human body as 98.6° F.*

$$\left[\text{Given}: \log\frac{143}{123} = 0.15066, \log\frac{123}{117} = 0.05\right]$$

SOLUTION Let T be the temperature of the body at time t. Then, by Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T-70)$$

$$\Rightarrow \quad \frac{dT}{dt} = -\lambda(T-70), \text{ where } \lambda > 0 \text{ is a constant}$$

$$\Rightarrow \quad \frac{1}{T-70}\,dT = -\lambda\,dt \Rightarrow \int \frac{1}{T-70}\,dT = -\lambda\int dt \Rightarrow \log|T-70| = -\lambda t + C \qquad ...(i)$$

At $t = 0$, we have $T = 94.6°$ F and at $t = 1$, $T = 93.4°$ F

$$\therefore \quad \log(94.6-70) = 0 + C \text{ and } \log(93.4-70) = -\lambda + C$$

$$\Rightarrow \quad \log 24.6 = C \text{ and } \log 23.4 = -\lambda + C$$

$$\Rightarrow \quad C = \log 24.6 \text{ and } \lambda = \log 24.6 - \log 23.4 = \log\left(\frac{24.6}{23.4}\right) = \log\left(\frac{123}{117}\right)$$

Substituting these values in (i), we get

$$\log|T-70| = -\left(\log\frac{123}{117}\right)t + \log 24.6 \qquad ...(ii)$$

Let t_1 be the time that has elapsed after the death. Then, at $t = t_1$ we have $T = 98.6$.

Substituting these values in (ii), we have

$$\log(98.6-70) = -\left(\log\frac{123}{117}\right)t_1 + \log 24.6$$

$$\Rightarrow \quad \log 28.6 = -\left(\log \frac{123}{117}\right) t_1 + \log 24.6$$

$$\Rightarrow \quad \log\left(\frac{28.6}{24.6}\right) = -\left(\log \frac{123}{117}\right) t_1$$

$$\Rightarrow \quad \log \frac{143}{123} = -\left(\log \frac{123}{117}\right) t_1 \Rightarrow t_1 = -\frac{\log\left(\frac{143}{123}\right)}{\log \frac{123}{117}} \Rightarrow t_1 = -\frac{0.15066}{0.05} = -3.0132.$$

Hence, estimated time of death is 11.30 – 3.01 = 8.30 pm. (approx.)

Type III APPLICATIONS ON CO-ORDINATE GEOMETRY

EXAMPLE 15 *The slope of the tangent to the curve at any point is twice the ordinate at that point. The curve passes through the point (4, 3). Determine its equation.*

SOLUTION Let $P(x, y)$ be any point on the curve. Then, slope of the tangent at P is $\frac{dy}{dx}$.

It is given that the slope of the tangent at $P(x, y)$ is twice the ordinate i.e. $2y$.

$$\therefore \quad \frac{dy}{dx} = 2y \Rightarrow \frac{1}{y}\, dy = 2\, dx \Rightarrow \log y = 2x + \log C \Rightarrow y = C\, e^{2x} \quad \text{...(i)}$$

Since the curve passes through (4, 3). Therefore, $y = 3$ for $x = 4$. Putting $x = 4$ and $y = 3$ in (i), we get

$$3 = C\, e^{8} \Rightarrow C = 3\, e^{-8}$$

Putting the value of C in (i), we get: $y = 3\, e^{2x-8}$ as the required equation of the curve.

EXAMPLE 16 *The normal lines to a given curve at each point pass through (2, 0). The curve passes through (2, 3). Formulate the differential equation and hence find out the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the curve. The equation of the normal at $P(x, y)$ to the given curve is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \quad \text{...(i)}$$

It is given that the normal at each point passes through (2, 0). Therefore, (i) also passes through (2, 0). Putting $Y = 0$ and $X = 2$ in (i), we get

$$0 - y = -\frac{1}{\frac{dy}{dx}}(2 - x)$$

$$\Rightarrow \quad y\,\frac{dy}{dx} = 2 - x \Rightarrow y\, dy = (2 - x)\, dx$$

$$\Rightarrow \quad \frac{y^2}{2} = -\frac{(2-x)^2}{2} + C \Rightarrow y^2 = -(2-x)^2 + 2C \quad \text{...(ii)}$$

This passes through (2, 3). Therefore, $9 = 0 + 2C \Rightarrow C = \frac{9}{2}$.

Putting $C = \frac{9}{2}$ in (ii), we get: $y^2 = -(2-x)^2 + 9$, which is the required equation of the curve.

EXAMPLE 17 *The slope of the tangent to a curve at any point is reciprocal of twice the ordinate of that point. The curve passes through (4, 3). Formulate the differential equation and hence find the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the curve. Then the slope of the tangent at $P(x, y)$ is $\frac{dy}{dx}$.

It is given that the slope of the tangent at P is reciprocal of twice the ordinate of the point P.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2y} \Rightarrow 2y\, dy = dx$$

On integrating both sides, we obtain

$$y^2 = x + C \qquad \text{...(i)}$$

Since the curve passes through (4, 3). Therefore, putting $x = 4$ and $y = 3$ in (i), we get

$$9 = 4 + C \Rightarrow C = 5$$

Putting $C = 5$ in (i), we get: $y^2 = x + 5$, which is the required equation of the curve.

EXAMPLE 18 *The slope of the tangent at any point on a curve is λ times the slope of the line joining the point of contact to the origin. Formulate the differential equation and hence find the equation of the curve.*

SOLUTION Let $P(x, y)$ be any point on the given curve. Then,

Slope of the tangent at $P = \lambda$ (Slope of the line OP)

$$\Rightarrow \quad \frac{dy}{dx} = \lambda\left(\frac{y-0}{x-0}\right) \Rightarrow \frac{dy}{dx} = \frac{\lambda y}{x}$$

This is the required differential equation.

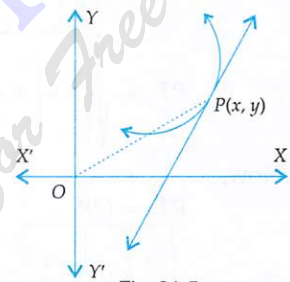

Fig. 21.5

Now, $\frac{dy}{dx} = \frac{\lambda y}{x}$

$$\Rightarrow \quad \frac{dy}{y} = \lambda \frac{dx}{x}$$

$$\Rightarrow \quad \log y = \lambda \log x + \log C \quad \text{[On integrating both sides]}$$

$$\Rightarrow \quad y = Cx^{\lambda}, \text{ which is the required equation of the curve.}$$

EXAMPLE 19 *The slope of the tangent to a curve at any point (x, y) on it is given by $\frac{y}{x} - \cot\frac{y}{x} \cdot \cos\frac{y}{x}$, $(x > 0, y > 0)$ and the curve passes through the point $(1, \pi/4)$. Find the equation of the curve.*

SOLUTION Let $y = f(x)$ be the given curve. Then, the slope of the tangent at $P(x, y)$ is $\frac{dy}{dx}$. But, the slope of the tangent at P is given as $\frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x}$. Therefore,

$$\frac{dy}{dx} = \frac{y}{x} - \cot\frac{y}{x}\cos\frac{y}{x} \qquad \text{...(i)}$$

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \cot v \cos v \Rightarrow x\frac{dv}{dx} = -\cot v \cos v$$

$$\Rightarrow \quad \sec v \tan v\, dv = -\frac{dx}{x} \Rightarrow \sec v = -\log|x| + \log C \qquad \text{[On integrating]}$$

$$\Rightarrow \quad \sec\left(\frac{y}{x}\right) = -\log|x| + C \qquad \text{...(ii)}$$

Since the curve passes through $(1, \pi/4)$. Therefore, $\sec \pi/4 = -\log 1 + C \Rightarrow \sqrt{2} = C$

Putting $C = \sqrt{2}$ in (ii), we get: $\sec\left(\frac{y}{x}\right) = -\log|x| + \sqrt{2}$, which is the required equation of the curve.

EXAMPLE 20 *If the tangent at any point P of a curve meets the axis of X in T. Find the curve for which $OP = PT$, O being the origin.*

SOLUTION Let $y = f(x)$ be the given curve and let $P(x, y)$ be a point on it. The equation of the tangent at P is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{...(i)}$$

This meets X-axis at T. So, the x-coordinate of T is obtained by putting $Y = 0$ in (i).

Putting $Y = 0$ in (i), we get

$$0 - y = \frac{dy}{dx}(X - x) \Rightarrow X = x - y\frac{dx}{dy}$$

Thus, the coordinates of T are $\left(x - y\frac{dx}{dy}, 0\right)$

$$\therefore \quad PT = \sqrt{\left\{x - \left(x - y\frac{dx}{dy}\right)\right\}^2 + (y - 0)^2} = \sqrt{y^2\left(\frac{dx}{dy}\right)^2 + y^2} \quad \text{[Given]}$$

Now,

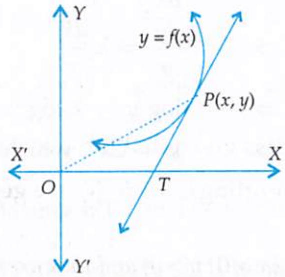

Fig. 21.6

$$PT = OP$$

$$\Rightarrow \sqrt{y^2 + y^2\left(\frac{dx}{dy}\right)^2} = \sqrt{x^2 + y^2}$$

$$\Rightarrow y^2 + y^2\left(\frac{dx}{dy}\right)^2 = x^2 + y^2$$

$$\Rightarrow y^2\left(\frac{dx}{dy}\right)^2 = x^2$$

$$\Rightarrow y\frac{dx}{dy} = \pm x \Rightarrow \frac{dx}{x} = \pm\frac{dy}{y} \Rightarrow \log x = \pm \log y + \log C \quad \text{[On integrating]}$$

$$\Rightarrow \log x = \log(C y) \quad \text{or,} \quad \log x = \log\left(\frac{C}{y}\right) \Rightarrow x = C y \text{ or, } x = \frac{C}{y}$$

Hence, the equations of the curve are $x = Cy$ or, $xy = C$.

EXAMPLE 21 *Show that the curve for which the normal at every point passes through a fixed point is a circle.*

SOLUTION Let $P(x, y)$ be any point on the given curve. The equation of the normal to the given curve at (x, y) is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \quad \text{...(i)}$$

It is given that the normal at every point passes through a fixed point (a, b) (say). Therefore,

$$b - y = -\frac{dx}{dy}(a - x) \quad \text{[Putting } X = a \text{ and } Y = b \text{ in (i)]}$$

$$\Rightarrow \quad -(y-b)\,dy = (x-a)\,dx \Rightarrow (x-a)\,dx + (y-b)\,dy = 0 \Rightarrow \frac{(x-a)^2}{2} + \frac{(y-b)^2}{2} = C$$

[On integrating]

$$\Rightarrow \quad (x-a)^2 + (y-b)^2 = 2C \Rightarrow (x-a)^2 + (y-b)^2 = r^2, \text{ where } r^2 = 2C$$

Clearly, this equation represents a circle.

EXAMPLE 22 *Find the curve in the xy-plane, passing through the point* $(1, 1)$, *so that the segment of any tangent drawn to the curve between its point of tangency and the y-axis is bisected at the x-axis.*

SOLUTION Let $P(x, y)$ be the point of tangency. The equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

It cuts X-axis and Y-axis at $A\left(x - y\dfrac{dx}{dy}, 0\right)$ and $B\left(0, y - x\dfrac{dy}{dx}\right)$ respectively.

It is given that A is the mid-point of BP.

$$\therefore \quad \frac{x}{2} = x - y\frac{dx}{dy} \text{ and } \frac{2y - x\dfrac{dy}{dx}}{2} = 0$$

$$\Rightarrow \quad x - 2y\frac{dx}{dy} = 0 \text{ and } 2y - x\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y - x\frac{dy}{dx} = 0 \Rightarrow x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad \frac{1}{y}\,dy = \frac{2}{x}\,dx \Rightarrow \int \frac{1}{y}\,dy = 2\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log y = 2\log x + \log C \Rightarrow y = Cx^2 \quad \text{...(i)}$$

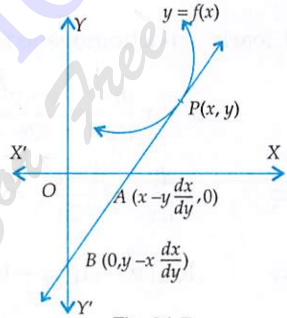

Fig. 21.7

It is given that the curve passes through $(1, 1)$. Putting $x = 1$, $y = 1$ in (i), we get $C = 1$.

Putting $C = 1$ in (i), we get, $y = x^2$ as the required equation of the curve.

EXAMPLE 23 *Find the equation of a curve passing through the point* $(1, 1)$. *If the tangent drawn at any point* $P(x, y)$ *on the curve meets the coordinate axes at A and B such that P is the mid-point of AB.*

[NCERT EXEMPLAR]

SOLUTION Let the equation of the curve be $y = f(x)$. then the equation of the tangent to the curve at $P(x, y)$ is $Y - y = \dfrac{dy}{dx}(X - x)$. It is given that the tangent at P cuts the coordinate axes at A and B. The coordinate of A and B are $\left(x - y\dfrac{dx}{dy}, 0\right)$ and $\left(0, y - x\dfrac{dy}{dx}\right)$ respectively.

It is given that $P(x, y)$ is the mid-point of AB.

$$\therefore \quad \frac{x - y\dfrac{dx}{dy} + 0}{2} = x \text{ and } \frac{0 + y - x\dfrac{dy}{dx}}{2} = y$$

$$\Rightarrow \quad x - y\frac{dx}{dy} = 2x \text{ and } y - x\frac{dy}{dx} = 2y$$

$$\Rightarrow \quad x = -y\frac{dx}{dy} \text{ and } y = -x\frac{dy}{dx}$$

$$\Rightarrow \quad y = -x\frac{dy}{dx}$$

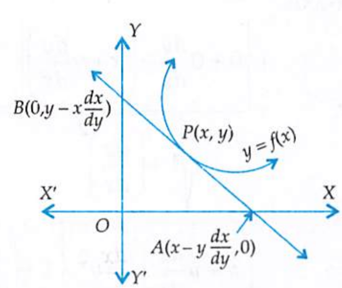

Fig. 21.8

$$\Rightarrow \quad \frac{1}{y}\,dy = -\frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int -\frac{1}{x}\,dx \qquad \text{[On integrating]}$$

$$\Rightarrow \quad \log|y| = -\log|x| + \log|C| \Rightarrow \log|xy| = \log|C| \Rightarrow |xy| = |C| \Rightarrow xy = C \quad \ldots(i)$$

This passes through (1, 1). Putting $x = 1$, $y = 1$ in (i), we obtain $C = 1$.

Putting $C = 1$ in $xy = C$, we obtain $xy = 1$ as the equation of the required curve.

EXAMPLE 24 *Find the equation of the curve passing through* $(2, 1)$, *if the slope of the tangent to the curve at any point* (x, y) *is* $\dfrac{x^2 + y^2}{2xy}$. **[NCERT EXEMPLAR]**

SOLUTION We know that the slope of the tangent at any point on a curve is $\dfrac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \qquad \text{[Given]}$$

Clearly, it is a homogeneous differential equation. Substituting $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, we get

$$v + x\frac{dv}{dx} = \frac{1 + v^2}{2v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1 - v^2}{2v} \Rightarrow \frac{2v}{1 - v^2}\,dv = \frac{1}{x}\,dx \Rightarrow \int \frac{2v}{v^2 - 1}\,dv = -\int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log|v^2 - 1| = -\log|x| + \log|C|$$

$$\Rightarrow \quad \log|v^2 - 1| = \log\left|\frac{C}{x}\right| \Rightarrow v^2 - 1 = \frac{C}{x} \Rightarrow y^2 - x^2 = Cx$$

It passes through $(2, 1)$. Therefore, $1 - 4 = 2C \Rightarrow C = -\dfrac{3}{2}$

Putting $C = -\dfrac{3}{2}$ in (i) we obtain $2x^2 - 2y^2 = 3x$ as the equation of the required curve.

EXAMPLE 25 *Find the equation of the curve passing through the point* $(1, 1)$, *if the perpendicular distance of the normal at* $P(x, y)$ *to the curve from the origin is equal to the distance of* P *from the x-axis.* **[NCERT EXEMPLAR]**

SOLUTION The equation of normal to the curve $y = f(x)$ at any point $P(x, y)$ is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x) \text{ or, } X + Y\frac{dy}{dx} - \left(x + y\frac{dy}{dx}\right) = 0 \qquad \ldots(i)$$

It is given that the distance of (i) from the origin $O\,(0, 0)$ is equal to the distance of P from the x-axis.

$$\therefore \quad \frac{\left|0 + 0\dfrac{dy}{dx} - \left(x + y\dfrac{dy}{dx}\right)\right|}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}} = |y|$$

$$\Rightarrow \quad \left(x + y\frac{dy}{dx}\right)^2 = y^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$$

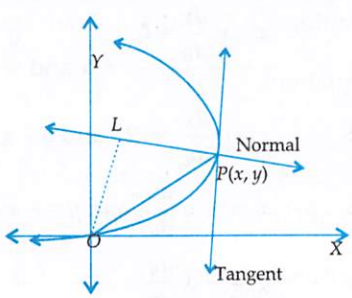

Fig. 21.9

$$\Rightarrow \qquad x^2 + 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 = y^2 + y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \qquad x^2 + 2xy\frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$, which is a homogeneous differential equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$, we obtain

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \Rightarrow x\frac{dv}{dx} = -\frac{v^2+1}{2v} \Rightarrow \frac{2v}{v^2+1}dv = -\frac{dx}{x}$$

Integrating, we obtain

$$\int \frac{2v}{v^2+1}\,dv = -\int \frac{dx}{x} + \log C$$

$$\Rightarrow \qquad \log(v^2+1) = -\log x + \log C \Rightarrow x(v^2+1) = C \Rightarrow x\left(\frac{y^2}{x^2}+1\right) = C \Rightarrow x^2 + y^2 = Cx \qquad \ldots\text{(ii)}$$

It is given that (ii) passes through (1, 1). Therefore, $1 + 1 = C \Rightarrow C = 2$

Putting $C = 2$ in (ii), we obtain $x^2 + y^2 = 2x$ as the equation of the required curve.

EXAMPLE 26 *Find the equation of a curve passing through* $(1, \pi/4)$, *if the slope of the tangent to the curve at any point* $P(x, y)$ *is* $\left(\frac{y}{x}\right) - \cos^2\left(\frac{y}{x}\right)$. **[NCERT EXEMPAR]**

SOLUTION The slope of the tangent to the curve at any point $P(x, y)$ is $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$.

$$\therefore \qquad \frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x} \qquad \ldots\text{(i)}$$

This is a homogeneous differential equation. Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (i), we get

$$v + x\frac{dv}{dx} = v - \cos^2 v \Rightarrow x\frac{dv}{dx} = -\cos^2 v \Rightarrow \sec^2 v\,dv = -\frac{dx}{x}$$

Integrating, we get

$$\int \sec^2 v\,dv = -\int \frac{1}{x}\,dx + C \Rightarrow \tan v = -\log x + C \Rightarrow \tan\left(\frac{y}{x}\right) = -\log x + C \qquad \ldots\text{(ii)}$$

It is given that the curve (ii) passes through $(1, \pi/4)$. Putting $x = 1$, $y = \frac{\pi}{4}$ in (iii), we get

$$\therefore \qquad \tan\frac{\pi}{4} = -\log 1 + C \Rightarrow C = 1$$

Putting $C = 1$ in (ii), we obtain $\tan\left(\frac{y}{x}\right) = -\log x + 1$ or, $\tan\left(\frac{y}{x}\right) + \log x = 1$, which is the required equation.

Type IV MISCELLANEOUS APPLICATIONS

EXAMPLE 27 *Experiments show that the rate of inversion of cane-sugar in dilute solution is proportional to the concentration* $y(t)$ *of the unaltered solution. Suppose that the concentration is* $\frac{1}{100}$ *at* $t = 0$ *and* $\frac{1}{300}$ *at* $t = 10$ *hours. Find* $y(t)$.

SOLUTION Let y be the concentration at time t. Then, the rate of inversion of the cane sugar is $\frac{dy}{dt}$. It is given that

$$\frac{dy}{dt} \propto y$$

$$\Rightarrow \quad \frac{dy}{dt} = -ky \qquad \left[\because \frac{dy}{dt} < 0\right]$$

$$\Rightarrow \quad \frac{dy}{y} = -k\,dt \Rightarrow \log y = -kt + C \qquad ...(i)$$

When $t = 0$, it is given that $y = \frac{1}{300}$. So, putting $t = 0$ and $y = \frac{1}{100}$ in (i), we get

$$\log \frac{1}{100} = C \Rightarrow C = \log 10^{-2} = -2 \log 10$$

Putting $C = -2$ in (i), we get: $\log y = -kt - 2 \log 10$ ...(ii)

When $t = 10$, it is given that $y = \frac{1}{300}$. Putting $y = \frac{1}{300}$ and $t = 10$ in (ii), we get

$$\log \frac{1}{300} = -10k - 2 \log 10$$

$$\Rightarrow \quad -\log 300 = -10\,k - 2 \log 10 \Rightarrow -(\log 3 + \log 100) = -10k - 2 \log 10$$

$$\Rightarrow \quad -(\log 3 + 2 \log 10) = -10\,k - 2 \log 10 \Rightarrow -\log 3 = -10k \Rightarrow k = \frac{1}{10} \log 3$$

Putting the value of k in (ii), we get

$$\log y = \left(-\frac{1}{10} \log 3\right) t - 2 \log 10 \Rightarrow y = \frac{1}{100}\, e^{\left(-\frac{1}{10} \log 3\right) t - 2}$$

EXAMPLE 28 *The equation of electromotive forces for an electric circuit containing resistance and self inductance is* $E = R\,i + L\,\frac{di}{dt}$, *where E is the electromotive force given to the circuit, R, the resistance and L, the coefficient of induction. Find the current i at time t when* (i) $E = 0$ *and* (ii) $E =$ *a non-zero constant.*

SOLUTION We are given that

$$L \frac{di}{dt} + Ri = E \Rightarrow \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L} \qquad ...(i)$$

This is a linear differential equation in i and t with I.F. $= e^{\int \frac{R}{L} dt} = e^{(R/L)t}$

Multiplying both sides of (i) by I.F. $= e^{(R/L)t}$, we get

$$e^{(R/L)t} \frac{di}{dt} + e^{(R/L)t} \frac{R}{L} i = \frac{E}{L} e^{(R/L)t}$$

Integrating both sides with respect to t, we get

$$i\, e^{(R/L)t} = \int \frac{E}{L} e^{(R/L)t}\, dt + C$$

$$\Rightarrow \quad i\, e^{(R/L)t} = \frac{E}{L} \left\{\frac{e^{(R/L)t}}{R/L}\right\} + C \Rightarrow i\, e^{(R/L)t} = \frac{E}{R} e^{(R/L)t} + C \Rightarrow i = \frac{E}{R} + C\, e^{(-R/L)t} \qquad ...(ii)$$

(i) When $E = 0$, from (ii), we get: $i = Ce^{-Rt/L}$

(ii) When $E =$ a non-zero constant, from (ii), we get: $i = \frac{E}{R} + C\, e^{(-R/L)t}$.

EXERCISE 21.11

BASED ON HOTS

1. The surface area of a balloon being inflated, changes at a rate proportional to time t. If initially its radius is 1 unit and after 3 seconds it is 2 units, find the radius after time t.
2. A population grows at the rate of 5% per year. How long does it take for the population to double?
3. The rate of growth of a population is proportional to the number present. If the population of a city doubled in the past 25 years, and the present population is 100000, when will the city have a population of 500000? [Given $\log_e 5 = 1.609$, $\log_e 2 = 0.6931$.]
4. In a culture, the bacteria count is 100000. The number is increased by 10% in 2 hours. In how many hours will the count reach 200000, if the rate of growth of bacteria is proportional to the number present?
5. If the interest is compounded continuously at 6% per annum, how much worth ₹ 1000 will be after 10 years? How long will it take to double ₹ 1000? [Given $e^{0.6} = 1.822$]
6. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given the number triples in 5 hrs, find how many bacteria will be present after 10 hours. Also find the time necessary for the number of bacteria to be 10 times the number of initial present. [Given $\log_e 3 = 1.0986$, $e^{2.1972} = 9$]
7. The population of a city increases at a rate proportional to the number of inhabitants present at any time t. If the population of the city was 200000 in 1990 and 250000 in 2000, what will be the population in 2010? **[NCERT]**
8. If the marginal cost of manufacturing a certain item is given by $C'(x) = \frac{dC}{dx} = 2 + 0.15\,x$.

   Find the total cost function $C(x)$, given that $C(0) = 100$.
9. A bank pays interest by continuous compounding, that is, by treating the interest rate as the instantaneous rate of change of principal. Suppose in an account interest accrues at 8% per year, compounded continuously. Calculate the percentage increase in such an account over one year. [Take $e^{0.08} \approx 1.0833$]
10. In a simple circuit of resistance R, self inductance L and voltage E, the current i at any time t is given by $L\frac{di}{dt} + R\,i = E$. If E is constant and initially no current passes through the circuit, prove that $i = \frac{E}{R}\left\{1 - e^{-(R/L)t}\right\}$.
11. The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.
12. Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year? [Use: $e^{-\frac{\log 2}{1590}} = 0.9996$]
13. The slope of the tangent at a point $P(x, y)$ on a curve is $\frac{-x}{y}$. If the curve passes through the point $(3, -4)$, find the equation of the curve.
14. Find the equation of the curve which passes through the point (2, 2) and satisfies the differential equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$.

15. Find the equation of the curve passing through the point $\left(1, \frac{\pi}{4}\right)$ and tangent at any point of which makes an angle $\tan^{-1}\left(\frac{y}{x} - \cos^2\frac{y}{x}\right)$ with x-axis.

16. Find the curve for which the intercept cut-off by a tangent on x-axis is equal to four times the ordinate of the point of contact.

17. Show that the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is $y + 2(x+1) = 2e^{2x}$.

18. The tangent at any point (x, y) of a curve makes an angle $\tan^{-1}(2x + 3y)$ with x-axis. Find the equation of the curve if it passes through $(1, 2)$.

19. Find the equation of the curve such that the portion of the x-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point $(1, 2)$.

20. Find the equation to the curve satisfying $x(x+1)\frac{dy}{dx} - y = x(x+1)$ and passing through $(1, 0)$.

21. Find the equation of the curve which passes through the point $(3, -4)$ and has the slope $\frac{2y}{x}$ at any point (x, y) on it.

22. Find the equation of the curve which passes through the origin and has the slope $x + 3y - 1$ at any point (x, y) on it.

23. At every point on a curve the slope is the sum of the abscissa and the product of the ordinate and the abscissa, and the curve passes through $(0, 1)$. Find the equation of the curve.

24. A curve is such that the length of the perpendicular from the origin on the tangent at any point P of the curve is equal to the abscissa of P. Prove that the differential equation of the curve is $y^2 - 2xy\frac{dy}{dx} - x^2 = 0$, and hence find the curve.

25. Find the equation of the curve which passes through the point $(1, 2)$ and the distance between the foot of the ordinate of the point of contact and the point of intersection of the tangent with x-axis is twice the abscissa of the point of contact.

26. The normal to a given curve at each point (x, y) on the curve passes through the point $(3, 0)$. If the curve contains the point $(3, 4)$, find its equation.

27. The rate of increase of bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 6 hours. Prove that the bacteria becomes 8 times at the end of 18 hours.

28. Radium decomposes at a rate proportional to the quantity of radium present. It is found that in 25 years, approximately 1.1% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one-half of the original amount of radium to decompose? [Given $\log_e 0.989 = 0.01106$ and $\log_e 2 = 0.6931$]

29. Show that all curves for which the slope at any point (x, y) on it is $\frac{x^2 + y^2}{2xy}$ are rectangular hyperbola.

30. The slope of the tangent at each point of a curve is equal to the sum of the coordinates of the point. Find the curve that passes through the origin.

31. Find the equation of the curve passing through the point $(0, 1)$ if the slope of the tangent to the curve at each of its point is equal to the sum of the abscissa and the product of the abscissa and the ordinate of the point. **[NCERT]**

32. The slope of a curve at each of its points is equal to the square of the abscissae of the point. Find the particular curve through the point $(-1, 1)$.

33. Find the equation of the curve that passes through the point $(0, a)$ and is such that at any point (x, y) on it, the product of its slope and the ordinate is equal to the abscissa.

34. The x-intercept of the tangent line to a curve is equal to the ordinate of the point of contact. Find the particular curve through the point $(1, 1)$.

ANSWERS

1. $r = \sqrt{1 + \frac{1}{3}t^2}$
2. $20 \log 2$ years
3. 58 years
4. $\dfrac{2 \log 2}{\log\left(\dfrac{11}{10}\right)}$ hours
5. ₹ 1822, 12 years
6. 9 times, $\dfrac{5 \log 10}{\log 3}$
7. 312500
8. $C(x) = 0.075x^2 + 2x + 100$
9. 8.33%
11. $\frac{1}{k} \log 2$, k is the constant of proportionality
12. 0.04%
13. $x^2 + y^2 = 25$
14. $2xy - 2x - y - 2 = 0$
15. $\tan\left(\dfrac{y}{x}\right) = \log\left(\dfrac{e}{x}\right)$
16. $e^{-xy} = Cy^4$
18. $y e^{-3x} = \left(-\dfrac{2}{3}x - \dfrac{2}{9}\right)e^{-3x} + \dfrac{26}{9}e^{-3}$
19. $xy = 2$
20. $y = \dfrac{x}{x+1}(x - 1 + \log x)$
21. $9y + 4x^2 = 0$
22. $3(x + 3y) = 2(1 - e^{3x})$
23. $y + 1 = 2e^{x^2/2}$
24. $x^2 + y^2 = Cx$
25. $y^2 = 4x$
26. $x^2 + y^2 - 6x - 7 = 0$
28. 1567 years.
30. $x + y = e^x - 1$
31. $1 + y = 2e^{x^2/2}$
32. $3y = x^3 + 4$
33. $x^2 - y^2 = -a^2$
34. $x + y \log y = y$

REVISION EXERCISE

1. Determine the order and degree (if defined) of the following differential equations:

   (i) $\left(\dfrac{ds}{dt}\right)^4 + 3s\dfrac{d^2s}{dt^2} = 0$ [NCERT]

   (ii) $y''' + 2y'' + y' = 0$ [NCERT]

   (iii) $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$ [NCERT]

   (iv) $y''' + 2y'' + y' = 0$ [NCERT]

   (v) $y'' + (y')^2 + 2y = 0$ [NCERT]

   (vi) $y'' + 2y' + \sin y = 0$ [NCERT]

   (vii) $y''' + y^2 + e^{y'} = 0$ [NCERT]

2. Verify that the function $y = e^{-3x}$ is a solution of the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0$.

3. In each of the following verify that the given functions (explicit or implicit) is a solution of the corresponding differential equation:

   (i) $y = e^x + 1$ $\quad y'' - y' = 0$

   (ii) $y = x^2 + 2x + C$ $\quad y' - 2x - 2 = 0$

   (iii) $y = \cos x + C$ $\quad y' + \sin x = 0$

   (iv) $y = \sqrt{1 + x^2}$ $\quad y' = \dfrac{xy}{1 + x^2}$

   (v) $y = x \sin x$ $\quad xy' = y + x\sqrt{x^2 - y^2}$

   (vi) $y = \sqrt{a^2 - x^2}$ $\quad x + y\dfrac{dy}{dx} = 0$

4. Form the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant.

5. Form the differential equation representing the family of curves $y = a \sin(x + b)$, where a, b are arbitrary constant.
6. Form the differential equation representing the family of parabolas having vertex at origin and axis along positive direction of x-axis.
7. Form the differential equation of the family of circles having centre on y-axis and radius 3 unit.
8. Form the differential equation of the family of parabolas having vertex at origin and axis along positive y-axis.
9. Form the differential equation of the family of ellipses having foci on y-axis and centre at the origin.
10. Form the differential equation of the family of hyperbolas having foci on x-axis and centre at the origin.
11. Verify that $xy = a e^{x} + b e^{-x} + x^2$ is a solution of the differential equation $x \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0$.
12. Show that $y = cx + 2c^2$ is a solution of the differential equation $2\left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} - y = 0$.
13. Show that $y^2 - x^2 - xy = a$ is a solution of the differential equation $(x - 2y)\dfrac{dy}{dx} + 2x + y = 0$.
14. Verify that $y = A\cos x + \sin x$ satisfies the differential equation $\cos x \dfrac{dy}{dx} + (\sin x)\, y = 1$.
15. Find the differential equation corresponding to $y = ae^{2x} + be^{-3x} + ce^{x}$ where a, b, c are arbitrary constants.
16. Show that the differential equation of all parabolas which have their axes parallel to y-axis is $\dfrac{d^3y}{dx^3} = 0$.
17. From $x^2 + y^2 + 2ax + 2by + c = 0$, derive a differential equation not containing a, b and c.

Solve the following differential equations:

18. $\dfrac{dy}{dx} = \sin^3 x \cos^4 x + x\sqrt{x+1}$
19. $\dfrac{dy}{dx} = \dfrac{1}{x^2 + 4x + 5}$
20. $\dfrac{dy}{dx} = y^2 + 2y + 2$
21. $\dfrac{dy}{dx} + 4x = e^{x}$
22. $\dfrac{dy}{dx} = x^2 e^{x}$
23. $\dfrac{dy}{dx} - x\sin^2 x = \dfrac{1}{x\log x}$
24. $(\tan^2 x + 2\tan x + 5)\dfrac{dy}{dx} = 2(1 + \tan x)\sec^2 x$
25. $\dfrac{dy}{dx} = \sin^3 x \cos^2 x + xe^{x}$
26. $\tan y\, dx + \tan x\, dy = 0$
27. $(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0$
28. $x\cos^2 y\, dx = y\cos^2 x\, dy$
29. $\cos y \log(\sec x + \tan x)\, dx = \cos x \log(\sec y + \tan y)\, dy$
30. $\operatorname{cosec} x\, (\log y)\, dy + x^2 y\, dx = 0$
31. $(1 - x^2)\, dy + xy\, dx = xy^2\, dx$
32. $\dfrac{dy}{dx} = \dfrac{\sin x + x\cos x}{y(2\log y + 1)}$
33. $x(e^{2y} - 1)\, dy + (x^2 - 1)\, e^{y}\, dx = 0$
34. $\dfrac{dy}{dx} + 1 = e^{x+y}$
35. $\dfrac{dy}{dx} = (x + y)^2$
36. $\cos(x + y)\, dy = dx$
37. $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$
38. $\dfrac{dy}{dx} = \dfrac{y(x - y)}{x(x + y)}$
39. $(x + y - 1)\, dy = (x + y)\, dx$
40. $\dfrac{dy}{dx} - y\cot x = \operatorname{cosec} x$
41. $\dfrac{dy}{dx} - y\tan x = -2\sin x$
42. $\dfrac{dy}{dx} - y\tan x = e^{x}\sec x$

43. $\frac{dy}{dx} - y \tan x = e^x$ **44.** $(1 + y + x^2 y)\, dx + (x + x^3)\, dy = 0$

45. $(x^2 + 1)\, dy + (2y - 1)\, dx = 0$ **46.** $y \sec^2 x + (y + 7) \tan x \frac{dy}{dx} = 0$

47. $(2ax + x^2) \frac{dy}{dx} = a^2 + 2ax$ **48.** $(x^3 - 2y^3)\, dx + 3x^2 y\, dy = 0$

49. $x^2\, dy + (x^2 - xy + y^2)\, dx = 0$ **50.** $y - x \frac{dy}{dx} = b\left(1 + x^2 \frac{dy}{dx}\right)$ **51.** $\frac{dy}{dx} + 2y = \sin 3x$

52. $\frac{dy}{dx} + y = 4x$ **53.** $\frac{dy}{dx} + 5y = \cos 4x$ **54.** $x \frac{dy}{dx} + x \cos^2\left(\frac{y}{x}\right) = y$

55. $\cos^2 x \frac{dy}{dx} + y = \tan x$ **56.** $x \cos x \frac{dy}{dx} + y\,(x \sin x + \cos x) = 1$

57. $(1 + y^2) + (x - e^{-\tan^{-1} y}) \frac{dy}{dx} = 0$ **58.** $y^2 + \left(x + \frac{1}{y}\right) \frac{dy}{dx} = 0$

59. $2 \cos x \frac{dy}{dx} + 4y \sin x = \sin 2x$, given that $y = 0$ when $x = \frac{\pi}{3}$.

60. $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$ **61.** $\frac{dy}{dx} + y \tan x = x^n \cos x, n \neq -1$

62. Find the general solution of the differential equation $\frac{dy}{dx} = \frac{x+1}{2-y}, y \neq 2$.

63. Find the particular solution of the differential equation $\frac{dy}{dx} = -4xy^2$ given that $y = 1$, when $x = 0$.

64. For each of the following differential equations, find the general solution:

(i) $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$ (ii) $\frac{dy}{dx} = \sqrt{4 - y^2}, -2 < y < 2$ **[NCERT]** (iii) $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

(iv) $y \log y\, dx - x\, dy = 0$ (v) $\frac{dy}{dx} = \sin^{-1} x$ (vi) $\frac{dy}{dx} + y = 1$

65. For each of the following differential equations, find a particular solution satisfying the given condition: **[NCERT]**

(i) $x(x^2 - 1) \frac{dy}{dx} = 1$, $y = 0$ when $x = 2$ (ii) $\cos\left(\frac{dy}{dx}\right) = a$, $y = 1$ when $x = 0$

(iii) $\frac{dy}{dx} = y \tan x$, $y = 1$ when $x = 0$

66. Solve the each of the following differential equations: **[NCERT]**

(i) $(x - y) \frac{dy}{dx} = x + 2y$ (ii) $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$

(iii) $y\, dx + x \log\left(\frac{y}{x}\right) dy - 2x\, dy = 0$ (iv) $\frac{dy}{dx} - y = \cos x$ (v) $x \frac{dy}{dx} + 2y = x^2, x \neq 0$

(vi) $\frac{dy}{dx} + 2y = \sin x$ (vii) $\frac{dy}{dx} + 3y = e^{-2x}$ (viii) $\frac{dy}{dx} + \frac{y}{x} = x^2$

(ix) $\frac{dy}{dx} + (\sec x)\, y = \tan x$ (x) $x \frac{dy}{dx} + 2y = x^2 \log x$

(xi) $x \log x \dfrac{dy}{dx} + y = \dfrac{2}{x} \log x$ (xii) $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx$

(xiii) $(x + y) \dfrac{dy}{dx} = 1$ (xiv) $y\,dx + (x - y^2)\, dy = 0$ (xv) $(x + 3y^2) \dfrac{dy}{dx} = y$

67. Find a particular solution of each of the following differential equations: [NCERT]

(i) $(1 + x^2) \dfrac{dy}{dx} + 2xy = \dfrac{1}{1 + x^2}$; $y = 0$, when $x = 1$

(ii) $(x + y)\, dy + (x - y)\, dx = 0$; $y = 1$ when $x = 1$

(iii) $x^2\, dy + (xy + y^2)\, dx = 0$; $y = 1$ when $x = 1$

68. Find the equation of the curve passing through the point (1, 1) whose differential equation is $x\, dy = (2x^2 + 1)\, dx, x \neq 0$. [NCERT]

69. Find the equation of a curve passing through the point (– 2, 3), given that the slope of the tangent to the curve at any point (x, y) is $\dfrac{2x}{y^2}$. [NCERT]

70. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $\dfrac{dy}{dx} = e^x \sin x$. [NCERT]

71. At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). Find the equation of the curve given that it passes through (– 2, 1). [NCERT]

72. Show that the family of curves for which the slope of the tangent at any point (x, y) on it is $\dfrac{x^2 + y^2}{2xy}$ is given by $x^2 - y^2 = Cx$. [NCERT]

73. Find the equation of a curve passing thought the point (0, 1). If the slope of the tangent to the curve at any point (x, y) is equal to the sum of the x-coordinate and the product of the x-coordinate and y-coordinate of that point. [NCERT]

74. Find the equation of the curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point. [NCERT]

75. Find the equation of the curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5. [NCERT]

76. The slope of the tangent to the curve at any point is the reciprocal of twice the ordinate at that point. The curve passes through the point (4,3). Determine its equation. [NCERT]

77. The decay rate of radium at any time t is proportional to its mass at that time. Find the time when the mass will be halved of its initial mass.

78. Experiments show that radium disintegrates at a rate proportional to the amount of radium present at the moment. Its half-life is 1590 years. What percentage will disappear in one year? [Use $e^{-\frac{\log 2}{1590}} = 0.9996$]

79. A wet porous substance in the open air loeses its moisture at a rate proportional to the moisture content. If a sheet hung in the wind loses half of its moisture during the first hour, when will it have lost 95% moisture, wheather conditions remaining the same.

ANSWERS

1.

	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
Order	2	3	3	3	2	2	3
Degree	1	1	2	1	1	1	Not specified

4. $x\dfrac{dy}{dx}-y=0$ 5. $\dfrac{d^2y}{dx^2}+y=0$ 6. $y^2-2xy\dfrac{dy}{dx}=0$

7. $(x^2-9)(y')^2+x^2=0$ 8. $xy'-2y=0$

9. $xyy''+x(y')^2-yy'=0$ 10. $xyy''+x(y')^2-yy'=0$

15. $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dx}+6y=0$ 17. $\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}\dfrac{d^3y}{dx^3}-3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2=0$

18. $y=\dfrac{1}{7}\cos^7 x-\dfrac{1}{5}\cos^5 x+\dfrac{2}{5}(x+1)^{5/2}-\dfrac{2}{3}(x+1)^{3/2}+C$

19. $y=\tan^{-1}(x+2)+C$ 20. $x=\tan^{-1}(y+1)+C$

21. $y+2x^2=e^x+C$ 22. $y=(x^2-2x+2)e^x+C$

23. $y=\dfrac{x^2}{4}-\dfrac{x}{4}\sin 2x-\dfrac{1}{8}\cos 2x+\log|\log x|+C$

24. $y=\log|\tan^2 x+2\tan x+5|+C$ 25. $y=\dfrac{\cos^5 x}{5}-\dfrac{\cos^3 x}{3}+(x-1)e^x+C$

26. $\sin x\sin y=C$ 27. $x+y+\log(xy)=C$

28. $x\tan x-y\tan y=\log|\sec x|-\log|\sec y|+C$

29. $[\log(\sec x+\tan x)]^2=[\log(\sec y+\tan y)]^2+C$

30. $\dfrac{1}{2}(\log|y|)^2+(2-x^2)\cos x+2x\sin x=C$ 31. $(y-1)^2|1-x^2|=C^2y^2$

32. $y^2\log y=x\sin x+C$ 33. $e^y+e^{-y}-\dfrac{1}{2}x^2+\log|x|=C$

34. $-1=(x+C)e^{x+y}$ 35. $x+y=\tan(x+C)$ 36. $y-C=\tan\left(\dfrac{x+y}{2}\right)$

37. $y-2x=Cx^2y$ 38. $C\,xy=e^{x/y}$ 39. $2(y-x)-\log(2x+2y-1)=C$

40. $y\operatorname{cosec} x=-\cot x+C$ 41. $y\sin x=C-\dfrac{1}{2}\cos 2x$ 42. $y\cos x=e^x+C$

43. $y\cos x=\dfrac{e^x}{2}(\cos x+\sin x)+C$ 44. $xy=-\tan^{-1}x+C$

45. $y=C\,e^{-2\tan^{-1}x}+\dfrac{1}{2}$ 46. $y^7\tan x=k\,e^{-y}$ 47. $y+C=\dfrac{a}{2}\left\{\log x+3\log(x+2a)\right\}$

48. $x^3+y^3=kx^2$ 49. $C=x\,e^{\tan^{-1}(y/x)}$ 50. $y=k(y-b)(1+bx)$

51. $y=\dfrac{3}{13}\left(\dfrac{2}{3}\sin 3x-\cos 3x\right)+C\,e^{-2x}$ 52. $y=4(x-1)+C\,e^{-x}$

53. $y=\dfrac{4}{41}\left(\sin 4x+\dfrac{5}{4}\cos 4x\right)+C\,e^{-5x}$ 54. $\tan\left(\dfrac{y}{x}\right)=\log x+C$

55. $y\,e^{\tan x}=C+e^{\tan x}(\tan x-1)$ 56. $xy\sec x=C+\tan x$

57. $xe^{\tan^{-1}y}=C+\tan^{-1}y$ 58. $x=C\,e^{1/y}+\left(\dfrac{1}{y}+1\right)$ 59. $y=\cos x-2\cos^2 x$

60. $xe^{\tan^{-1}y} = C + e^{\tan^{-1}y}(\tan^{-1}y - 1)$

61. $y \sec x = C + \dfrac{x^{n+1}}{n+1}$

62. $x^2 + y^2 + 2x - 4y + C = 0$

63. $y = \dfrac{1}{2x^2 + 1}$

64. (i) $y = 2\tan\dfrac{x}{2} - x + C$ (ii) $y = 2\sin(x + C)$ (iii) $\tan^{-1}y = x + \dfrac{x^3}{3} + C$

(iv) $y = e^{Cx}$ (v) $y = x\sin^{-1}x + \sqrt{1 - x^2} + C$ (vi) $y = 1 + Ce^{-x}$

65. (i) $y = \dfrac{1}{2}\log\left(\dfrac{x^2 - 1}{x^2}\right) - \dfrac{1}{2}\log\dfrac{3}{4}$ (ii) $\cos\left(\dfrac{y-2}{x}\right) = a$ (iii) $y = \sec x$

66. (i) $\log|x^2 + xy + y^2| = 2\sqrt{3}\tan^{-1}\left(\dfrac{x + 2y}{\sqrt{3}\,x}\right) + C$ (ii) $\sin\left(\dfrac{y}{x}\right) = \log|Cx|$

(iii) $Cy = \log\left|\dfrac{y}{x}\right| - 1$ (iv) $y = \dfrac{1}{2}(\sin x - \cos x) + Ce^x$ (v) $y = \dfrac{x^2}{4} + Cx^{-2}$

(vi) $y = \dfrac{1}{5}(2\sin x - \cos x) + Ce^{-2x}$ (vii) $y = e^{-2x} + Ce^{-3x}$

(viii) $xy = \dfrac{x^4}{4} + C$ (ix) $y(\sec x + \tan x) = \sec x + \tan x - x + C$

(x) $y = \dfrac{x^2}{16}(4\log x - 1) + Cx^{-2}$ (xi) $y\log x = -\dfrac{2}{x}(1 + \log x) + C$

(xii) $y = (1 + x)^{-2}\log\sin x + C(1 + x^2)^{-1}$ (xiii) $x + y + 1 = Ce^y$

(xiv) $x = \dfrac{y^2}{3} + \dfrac{C}{y}$ (xv) $x = 3y^2 + Cy$

67. (i) $y(1 + x^2) = \tan^{-1}x - \dfrac{\pi}{4}$ (ii) $\log(x^2 + y^2) + 2\tan^{-1}\dfrac{y}{x} = \dfrac{\pi}{2} + \log 2$

(iii) $y + 2x = 3x^2 y$

68. $y = x^2 + \log|x|$

69. $y = (3x^2 + 15)^{1/3}$

70. $2y - 1 = e^x(\sin x - \cos x)$

71. $(x + 4)^2 = y + 3$

73. $y = -1 + 2e^{x^2/2}$

74. $x + y + 1 = e^x$

75. $y = 4 - x - 2e^x$

76. $y^2 = x + 5$

77. $\dfrac{1}{k}\log 2$, k is the constant of proportionality

78. 0.04%

79. $\dfrac{\log 20}{\log 2}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The order of the differential equation representing the family of parabolas $y^2 = 4ax$ is................... .

2. The degree of the differential equation $\left(\dfrac{dy}{dx}\right)^3 + \left(\dfrac{d^2y}{dx^2}\right)^2 = 0$ is ....

3. The number of arbitrary constants in a particular solution of the differential equation $\tan x\, dx + \tan y\, dy = 0$ is ....

4. An appropriate substitution to solve the differential equation $\frac{dx}{dy} = \frac{x^2 \log\left(\frac{x}{y}\right) - x^2}{xy \log\left(\frac{x}{y}\right)}$ is........................ . [NCERT EXEMPLAR]

5. The integrating factor of the differential equation $x\frac{dy}{dx} - y = \sin x$ is .... [NCERT EXEMPLAR]

6. The general solution of the differential equation $\frac{dy}{dx} = e^{x-y}$ is .... [NCERT EXEMPLAR]

7. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = 1$ is ....

8. The differential equation representing the family of curves $y = A\sin x + B\cos x$ is ....

9. The linear differential equation $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dy}{dx} = 1, x \neq 0$ when written in the form $\frac{dy}{dx} + Py = Q$, then P =.................................. . [NCERT EXEMPLAR]

10. The order of the differential equation representing the family of ellipses having centre at origin and foci on x-axis is ....

11. The degree of the differential equation $\sqrt{1 + \frac{d^2y}{dx^2}} = \frac{dy}{dx} + x$ is ....

12. The integration factor of the differential equation $x\frac{dy}{dx} - y = x\cos x$ is ....

13. The degree of the differential equation $\frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} = 0$ is ....[NCERT EXEMPLAR]

14. The degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x$ is .... [CBSE 2020, NCERT EXEMPLAR]

15. The number of arbitrary constants in the general solution of the differential equation of order three is .... [NCERT EXEMPLAR]

16. The general solution of the differential equation of the type $\frac{dx}{dy} + Rx = S$, where R and S are functions of y, is ....

17. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is .... [NCERT EXEMPLAR]

18. The solution of the differential equation $\cot y\, dx = x\, dy$ is.......... . [NCERT EXEMPLAR]

19. The general solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is .... [CBSE 2020, NCERT EXEMPLAR]

20. The solution of the differential equation $y\, dx + (x + xy)\, dy = 0$ is .... [NCERT EXEMPLAR]

21. The order of the differential equation representing the family of circles $x^2+(y-a)^2=a^2$ is.............................
22. The number of arbitrary constants in the particular solution of a differential equation of order two is ....
23. The differential equation of all non-horizontal lines in a plane is ....
24. The differential equation of all non-vertical lines in a plane is ....
25. The integrating factor of all differential equation $(x^2+1)\dfrac{dy}{dx}+2xy=x^2-1$ is ....
26. The degree of the differential equation $y=x\left(\dfrac{dy}{dx}\right)^2+\left(\dfrac{dx}{dy}\right)^2$ is..................................... .
27. The order of the differential equation representing all circles of radius r is ....
28. The degree of the differential equation representing the family of curves $y=Ax+A^3$, where A is arbitrary constant, is
29. The general solution of the differential equation $\dfrac{dx}{x}+\dfrac{dy}{y}=0$ is... .
30. The order and degree of the differential equation $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)^4=y^4$ are .... and .... respectively. [NCERT EXEMPLAR]
31. The differential equation for which $y=a\cos x+b\sin x$ is a solution, is .... [NCERT EXEMPLAR]
32. The curve for which the slope of the tangent at any point is equal to the ratio of the abscissa and ordinate of the point, is .... [NCERT EXEMPLAR]
33. Family $y=Ax+A^3$ of curves will correspond to a differential equation of order .... and degree .... [NCERT EXEMPLAR]
34. The differential $x\,dy+y\,dy=0$ equation represents a family of ....
35. The differential equation of the family of curves $x^2+y^2-2ay=0$, where a is arbitrary constant, is ....
36. The order and degree of the differential equation $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)^4$ are .... and ....respectively. [CBSE 2020]
37. The integrating factor of the differential equation $d\dfrac{dy}{dx}-y=\log x$ is .... [CBSE 2020]

ANSWERS

1. 1	2. 2	3. zero	4. $x=vy$	5. $\dfrac{1}{x}$
6. $e^y=e^x+C$	7. $xy=\dfrac{x^2}{2}+C$	8. $\dfrac{d^2y}{dx^2}+y=0$	9. $\dfrac{1}{\sqrt{x}}$	10. 2
11. 1	12. $\dfrac{1}{x}$	13. not defined	14. 2	15. 3
16. $x\,e^{\int R\,dy}=\int(S\,e^{\int R dy})\,dy+C$		17. $\dfrac{e^x}{x}$	18. $x=C\sec y$	19. $y=\dfrac{1}{4}x^2+Cx^{-2}$
20. $xy=Ce^{-y}$	21. one	22. zero	23. $\dfrac{d^2x}{dy^2}=0$	24. $\dfrac{d^2y}{dx^2}=0$
25. x^2+1	26. 4	27. 2	28. 3	29. $xy=C$

30. 3, 2 31. $\frac{d^2y}{dx^2}+y=0$ 32. a rectangular hyperbola 33. One, Three

34. circles 35. $(x^2-y^2)\,dy=2xy\,dx$ 36. 2, 1 37. $\frac{1}{x}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define a differential equation.
2. Define order of a differential equation.
3. Define degree of a differential equation.
4. Write the differential equation representing the family of straight lines $y=Cx+5$, where C is an arbitrary constant.
5. Write the differential equation obtained by eliminating the arbitrary constant C in the equation $x^2-y^2=C^2$.
6. Write the differential equation obtained eliminating the arbitrary constant C in the equation $xy=C^2$.
7. Write the degree of the differential equation $a^2\frac{d^2y}{dx^2}=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/4}$.
8. Write the order of the differential equation $1+\left(\frac{dy}{dx}\right)^2=7\left(\frac{d^2y}{dx^2}\right)^3$.
9. Write the order and degree of the differential equation $y=x\frac{dy}{dx}+a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$.
10. Write the degree of the differential equation $\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=2x^2\log\left(\frac{d^2y}{dx^2}\right)$.
11. Write the order of the differential equation of the family of circles touching X-axis at the origin.
12. Write the order of the differential equation of all non-horizontal lines in a plane.
13. If $\sin x$ is an integrating factor of the differential equation $\frac{dy}{dx}+Py=Q$, then write the value of P.
14. Write the order of the differential equation of the family of circles of radius r.
15. Write the order of the differential equation whose solution is $y=a\cos x+b\sin x+c\,e^{-x}$.
16. Write the order of the differential equation associated with the primitive $y=C_1+C_2e^x+C_3e^{-2x+C_4}$, where C_1, C_2, C_3, C_4 are arbitrary constants.
17. What is the degree of the following differential equation?

$$5x\left(\frac{dy}{dx}\right)^2-\frac{d^2y}{dx^2}-6y=\log x$$ **[CBSE 2010]**

18. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^4+3x\frac{d^2y}{dx^2}=0$. **[CBSE 2013]**

19. Write the degree of the differential equation $x\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + x^3 = 0.$ **[CBSE 2013]**

20. Write the differential equation representing family of curves $y = mx$, where m is arbitrary constant. **[CBSE 2013]**

21. Write the degree of the differential equation $x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$ **[CBSE 2013]**

22. Write the degree of the differential equation $\left(1 + \frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2.$

23. Write degree of the differential equation

$$\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$$

24. Write the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\left(\frac{dy}{dx}\right)$

25. Write the order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0.$

26. The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0.$

27. How many arbitrary constants are there in the general solution of the differential equation of order 3.

28. Write the order of the differential equation representing the family of curves $y = ax + a^3$.

29. Find the sum of the order and degree of the differential equation $y = x\left(\frac{dy}{dx}\right)^3 + \frac{d^2y}{dx^2}.$ **[CBSE 2015]**

30. Find the solution of the differential equation $x\sqrt{1+y^2}\ dx + y\sqrt{1+x^2}\ dy = 0$ **[CBSE 2015]**

31. Form the differential equation representing the family of curves $y = A\sin x$, by eliminating the arbitrary constant A. **[CBSE 2019]**

ANSWERS

4. $x\frac{dy}{dx} - y + 5 = 0$ **5.** $x\,dx - y\,dy = 0$ **6.** $x\,dy + y\,dx = 0$ **7.** 4 **8.** 2
9. 1, 2 **10.** Not defined **11.** 1 **12.** 2 **13.** $\cot x$
14. 2 **15.** 3 **16.** 3 **17.** 1 **18.** 1
19. 3 **20.** $y = x\frac{dy}{dx}$ **21.** 2 **22.** 2
23. Not defined **24.** Not defined **25.** order 2 degree not defined
26. Not defined **27.** 3 **28.** one **29.** 3
30. $\sqrt{1+x^2} + \sqrt{1+y^2} = C$ **31.** $\frac{d^2y}{dx^2} + y = 0$