CHAPTER 23

SCALAR OR DOT PRODUCT

23.1 INTRODUCTION

In the previous chapter, we have introduced the notion of multiplication of a vector by a scalar. In this chapter, we will introduce the notion of product of two vectors. The product of two vectors results in two different ways, viz. a scalar and a vector. Correspondingly, there are two kinds of products, the one a scalar and the other a vector. But, before defining these products we define the angle between two vectors as under.

23.2 ANGLE BETWEEN TWO VECTORS

Let two non-zero vectors $\vec{a}$ and $\vec{b}$ be represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively. Then the angle between $\vec{a}$ and $\vec{b}$ is the angle between their directions when these directions both converge or both diverge from their point of intersection.

It is evident that if θ is the measure of the angle between two vectors, then $0 \leq \theta \leq \pi$.

For $\theta = \frac{\pi}{2}$, the vectors are said to be perpendicular or orthogonal and for $\theta = 0$ or π, the vectors are said to be parallel.

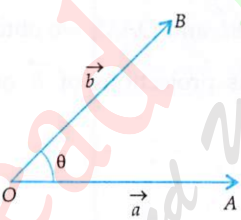

Fig. 23.1

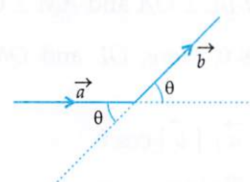

Fig. 23.2

23.3 THE SCALAR OR DOT PRODUCT

DEFINITION *Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors inclined at an angle θ. Then the scalar product of $\vec{a}$ with $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and is defined as the scalar* $|\vec{a}||\vec{b}|\cos\theta$.

Thus, $\quad \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta.$

Clearly, the scalar product of two vectors is a scalar quantity, due to which this product is called scalar product. Since we are putting a dot between $\vec{a}$ and $\vec{b}$. Therefore, it is also called dot product.

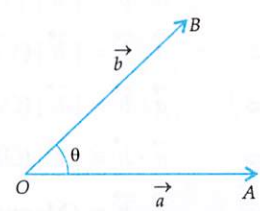

Fig. 23.3

REMARK 1 *If $\vec{a}$ or, $\vec{b}$ or, both is a zero vector, then θ is not defined as $\vec{0}$ has no direction. In this case their dot product $\vec{a} \cdot \vec{b}$ is defined as the scalar zero.*

REMARK 2 *Let $\vec{a} = \vec{AB}$ be a given vector and l be a given line. Let P and Q be the feet of perpendiculars drawn from A and B respectively on line l as shown in Fig. 23.4. Then, PQ is defined as the projection of $\vec{a}$ on line l. Let AM be perpendicular on BQ. Then, $AM = PQ$.*

Let θ be the angle between $\vec{a}$ and line l. Then, $\angle BAM = \theta$.

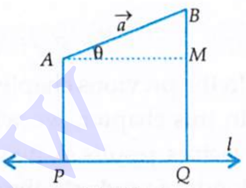

Fig. 23.4

In Δ AMB, we have

$$\cos\theta = \frac{AM}{AB}$$

$$\Rightarrow \quad \cos\theta = \frac{PQ}{AB} \qquad [\because AM = PQ]$$

$$\Rightarrow \quad PQ = AB\cos\theta \Rightarrow PQ = |\vec{a}|\cos\theta$$

Hence, projection of a vector $\vec{a}$ on a line l is $|\vec{a}|\cos\theta$, where θ is the angle between $\vec{a}$ and the line l.

If $\vec{OA} = \vec{a}$ and l is a line passing through O as shown in Fig. 23.5. Then, OM is the projection of $\vec{a}$ on line l.

Clearly, $OM = OA\cos\theta = |\vec{a}|\cos\theta$.

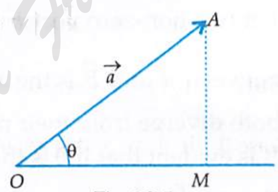

Fig. 23.5

23.3.1 GEOMETRICAL INTERPRETATION OF SCALAR PRODUCT

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ respectively. Let θ be the angle between $\vec{OA}$ and $\vec{OB}$. Draw $BL \perp OA$ and $AM \perp OB$. From Δ's OBL and OAM, we obtain: $OL = OB\cos\theta$ and $OM = OA\cos\theta$. Here, OL and OM are known as projections of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$ respectively.

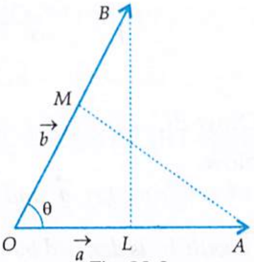

Fig. 23.6

Now, $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{a}|(OB\cos\theta)$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{a}|(OL)$

$\Rightarrow \quad \vec{a} \cdot \vec{b} =$ (Magnitude of $\vec{a}$) (Projection of $\vec{b}$ on $\vec{a}$) ...(i)

Again,

$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{b}|(|\vec{a}|\cos\theta)$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{b}|(OA\cos\theta)$

$\Rightarrow \quad \vec{a} \cdot \vec{b} = |\vec{b}|(OM)$

$\Rightarrow \quad \vec{a} \cdot \vec{b} =$ (Magnitude of $\vec{b}$) (Projection of $\vec{a}$ on $\vec{b}$) ...(ii)

Thus geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

REMARK 3 *From (i) and (ii), we obtain*

Projection of $\vec{b}$ *on* $\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \dfrac{\vec{a}}{|\vec{a}|}\cdot\vec{b} = \hat{a}\cdot\vec{b}$ *and, Projection of* $\vec{a}$ *on* $\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$

Thus, the projection of $\vec{a}$ *on* $\vec{b}$ *is the dot product of* $\vec{a}$ *with the unit vector along* $\vec{b}$ *and the projection of* $\vec{b}$ *on* $\vec{a}$ *is the dot product of* $\vec{b}$ *with the unit vector along* $\vec{a}$.

REMARK 4 *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *we have* : $\dfrac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}} = \dfrac{|\vec{a}|}{|\vec{b}|}$.

23.3.2 PROPERTIES OF SCALAR PRODUCT

PROPERTY I *(Commutativity) The scalar product of two vectors is commutative i.e.* $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

PROOF Case I Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors and let θ be the angle between them. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta, \text{ and } \vec{b}\cdot\vec{a} = |\vec{b}|\,|\vec{a}|\cos\theta$$

But, $|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{b}|\,|\vec{a}|\cos\theta \Rightarrow \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Case II If $\vec{a}$ or $\vec{b}$ is a zero vector, then $\vec{a}\cdot\vec{b} = 0$ and $\vec{b}\cdot\vec{a} = 0$. So, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

Hence, $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.

PROPERTY II *(Distributivity of scalar product over vector addition) The scalar product of vectors is distributive over vector addition. i.e.*

(i) $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ *(Left distributivity)*

(ii) $(\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a}$ *(Right distributivity)*

PROOF (i) Let $\vec{OA}, \vec{OB}$ and $\vec{BC}$ represent vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then,

$$\vec{OC} = \vec{OB} + \vec{BC} = \vec{b} + \vec{c}.$$

Draw $BL \perp OA$ and $CM \perp OA$.

Now,

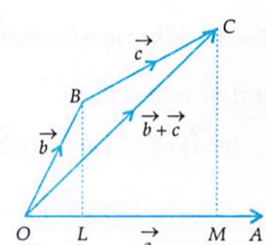

Fig. 23.7

$$\begin{aligned}
\vec{a}\cdot(\vec{b}+\vec{c}) &= |\vec{a}| \text{ (Projection of } \vec{b}+\vec{c} \text{ on } \vec{a})\\
&= |\vec{a}| \text{ (Projection of } \vec{OC} \text{ on } \vec{OA})\\
&= |\vec{a}|\,(OM)\\
&= |\vec{a}|\,(OL + LM)\\
&= |\vec{a}|\,(OL) + |\vec{a}|\,(LM)\\
&= |\vec{a}| \text{ (Projection of } \vec{b} \text{ on } \vec{a}) + |\vec{a}| \text{ (Projection of } \vec{c} \text{ on } \vec{a}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}.
\end{aligned}$$

Hence, $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

(ii) By commutativity of scalar product, we get

$$(\vec{b}+\vec{c})\cdot\vec{a} = \vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c} \qquad \text{[From (i)]}$$

$$= \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a} \qquad \text{[By commutativity of scalar product]}$$

Hence, $(\vec{b}+\vec{c})\cdot\vec{a} = \vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}$.

PROPERTY III *Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}$ is perpendicular to $\vec{b}$.*

PROOF We have,

$$\vec{a}\cdot\vec{b}=0$$

$$\Leftrightarrow \quad |\vec{a}|\,|\vec{b}|\cos\theta = 0$$

$$\Leftrightarrow \quad \cos\theta = 0 \qquad [\because |\vec{a}|\neq 0, |\vec{b}|\neq 0]$$

$$\Leftrightarrow \quad \theta = \frac{\pi}{2} \Leftrightarrow \vec{a} \text{ is perpendicular to } \vec{b}$$

REMARK 1 *Since $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular unit vectors along the coordinate axes.*

$$\therefore \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = 0; \quad \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = 0; \quad \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$$

PROPERTY IV *For any vector $\vec{a}$, we have: $\vec{a}\cdot\vec{a} = |\vec{a}|^2$*

PROOF For any vector $\vec{a}$, we find that $\vec{a}\cdot\vec{a} = |\vec{a}|\,|\vec{a}|\cos 0 = |\vec{a}|^2$

REMARK 2 *Students should keep in mind that $\vec{a}^2$ has no meaning but wherever it is used it denotes* $|\vec{a}|^2$.

REMARK 3 *Since $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along the coordinate axes. Therefore,*

$$\hat{i}\cdot\hat{i} = |\hat{i}|^2 = 1, \hat{j}\cdot\hat{j} = |\hat{j}|^2 = 1 \text{ and } \hat{k}\cdot\hat{k} = |\hat{k}|^2 = 1.$$

PROPERTY V *If m is a scalar and $\vec{a}, \vec{b}$ be any two vectors, then $(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$*

PROOF We have the following cases:

Case I *When $m>0$:* Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $m>0$, therefore the angle between $m\vec{a}$ and $\vec{b}$ is also θ.

$$\therefore \quad (m\vec{a})\cdot\vec{b} = |m\vec{a}|\,|\vec{b}|\cos\theta = |m|\,|\vec{a}|\,|\vec{b}|\cos\theta = m|\vec{a}|\,|\vec{b}|\cos\theta \qquad [\because m>0 \therefore |m|=m]$$

$$= m(\vec{a}\cdot\vec{b})$$

The angle between $\vec{a}$ and $m\vec{b}$ will also be θ.

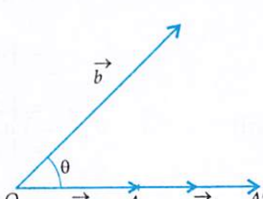

Fig. 23.8

$$\therefore \quad \vec{a}\cdot(m\vec{b}) = |\vec{a}|\,|m\vec{b}|\cos\theta = |\vec{a}|\,|m|\,|\vec{b}|\cos\theta$$

$$= m|\vec{a}|\,|\vec{b}|\cos\theta = m(\vec{a}\cdot\vec{b})$$

Hence, $(m\vec{a})\cdot\vec{b} = m(\vec{a}\cdot\vec{b}) = \vec{a}\cdot(m\vec{b})$

Case II *When* $m < 0$: Let θ be the angle between $\vec{a}$ and $\vec{b}$. As $m < 0$, therefore, the angle between $m\vec{a}$ and $\vec{b}$ is $(\pi - \theta)$.

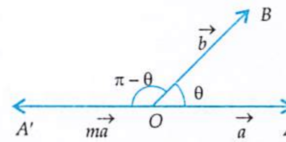

Fig. 23.9

$$\therefore \quad (m\vec{a}) \cdot \vec{b} = |m\vec{a}|\,|\vec{b}|\cos(\pi - \theta) = -|m|\,|\vec{a}|\,|\vec{b}|\cos\theta$$
$$= m(\vec{a} \cdot \vec{b})$$

In this case, the angle between $\vec{a}$ and $m\vec{b}$ is also $(\pi - \theta)$.

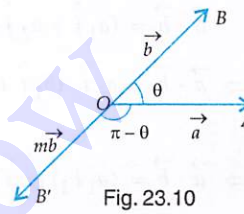

Fig. 23.10

$$\therefore \quad \vec{a} \cdot m\vec{b} = |\vec{a}|\,|m\vec{b}|\cos(\pi - \theta)$$
$$= -|\vec{a}|\,|m|\,|\vec{b}|\cos\theta = m|\vec{a}|\,|\vec{b}|\cos\theta)$$
$$= m(\vec{a} \cdot \vec{b})$$

Hence, $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$

Case III *If* $m = 0$, *then the result* $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (m\vec{b})$ *trivially holds good.*

PROPERTY VI *If m, n are scalars and* $\vec{a}$, $\vec{b}$ *be two vectors. Then,*

$$m\vec{a} \cdot n\vec{b} = mn(\vec{a} \cdot \vec{b}) = (mn\vec{a}) \cdot \vec{b} = \vec{a} \cdot (mn\vec{b}).$$

PROOF Proceed as in (V)

PROPERTY VII *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *we have*

(i) $\vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}) = (-\vec{a}) \cdot \vec{b}$ (ii) $(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$

PROOF Proceed as in (VI)

PROPERTY VIII *For any two vectors* $\vec{a}$ *and* $\vec{b}$, *we have*

(i) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a} \cdot \vec{b}$ (ii) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a} \cdot \vec{b}$

(iii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

PROOF (i) We have,

$$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \qquad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$$
$$= (\vec{a} + \vec{b}) \cdot \vec{a} + (\vec{a} + \vec{b}) \cdot \vec{b} \qquad \text{[By left distributivity]}$$
$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \qquad \text{[By right distributivity]}$$
$$= |\vec{a}|^2 + 2\,\vec{a} \cdot \vec{b} + |\vec{b}|^2 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$
$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a} \cdot \vec{b}$$

(ii)
$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \qquad [\because \vec{a} \cdot \vec{a} = |\vec{a}|^2]$$
$$= (\vec{a} - \vec{b}) \cdot \vec{a} - (\vec{a} - \vec{b}) \cdot \vec{b} \qquad \text{[By left distributivity]}$$
$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} \qquad \text{[By right distributivity]}$$
$$= |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a} \cdot \vec{b} \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$$

(iii) $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = (\vec{a}+\vec{b})\cdot\vec{a} - (\vec{a}+\vec{b})\cdot\vec{b}$ [By left distributivity]

$= \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{b}$ [By right distributivity]

$= |\vec{a}|^2 - |\vec{b}|^2$ $[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$

23.3.3 SCALAR PRODUCT IN TERMS OF COMPONENTS

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$$\vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$\Rightarrow \vec{a}\cdot\vec{b} = a_1\hat{i}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

[By right distributivity of scalar product]

$$\Rightarrow \vec{a}\cdot\vec{b} = (a_1 b_1)(\hat{i}\cdot\hat{i}) + (a_1 b_2)(\hat{i}\cdot\hat{j}) + (a_1 b_3)(\hat{i}\cdot\hat{k}) + a_2 b_1(\hat{j}\cdot\hat{i}) + a_2 b_2(\hat{j}\cdot\hat{j})$$
$$+ a_2 b_3(\hat{j}\cdot\hat{k}) + a_3 b_1(\hat{k}\cdot\hat{i}) + a_3 b_2(\hat{k}\cdot\hat{j}) + a_3 b_3(\hat{k}\cdot\hat{k})$$

[By left distributivity of scalar product]

$$\Rightarrow \vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \left[\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{i}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\right]$$

Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.

ILLUSTRATION *If* $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} + 3\hat{k}$, *find* $\vec{a}\cdot\vec{b}$.

SOLUTION Clearly, $\vec{a}\cdot\vec{b} = (2\hat{i} - \hat{j} + 2\hat{k})\cdot(3\hat{i} + 2\hat{j} + 3\hat{k}) = (2)(3) + (-1)(2) + (2)(3) = 10.$

23.3.4 ANGLE BETWEEN TWO VECTORS

Let $\vec{a}$, $\vec{b}$ be two vectors inclined at an angle θ. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \theta = \cos^{-1}\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right\} \quad ...(i)$$

This formula is used to find the angle between two given vectors.

If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3,\ |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^3},\ |\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Substituting the values of $\vec{a}\cdot\vec{b}$, $|\vec{a}|$ and $|\vec{b}|$ in (i), we have

$$\theta = \cos^{-1}\left\{\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right\}$$

ILLUSTRATION 1 *Find the angle between two vectors* $\vec{a}$ *and* $\vec{b}$ *with magnitude* 2 *and* 1 *respectively, and such that* $\vec{a}\cdot\vec{b} = \sqrt{3}$.

SOLUTION We have $\vec{a}\cdot\vec{b} = \sqrt{3}$, $|\vec{a}| = 2$ and $|\vec{b}| = 1$. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2\times 1} = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$

ILLUSTRATION 2 *Find the angle between the vectors* $\hat{i}-2\hat{j}+3\hat{k}$ *and* $3\hat{i}-2\hat{j}+\hat{k}$.

SOLUTION Let $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$. Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

We have $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} = 1\times 3+(-2)\times(-2)+3\times 1 = 3+4+3 = 10,\ |\vec{a}|=\sqrt{14} \text{ and } |\vec{b}|=\sqrt{14}$$

$$\therefore \quad \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{10}{14} = \frac{5}{7} \Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Hence, the angle between the given vectors is $\cos^{-1}\left(\frac{5}{7}\right)$.

23.3.5 COMPONENTS OF A VECTOR ALONG AND PERPENDICULAR TO VECTOR

Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$. Draw $BM \perp OA$. In ΔOBM, we obtain

$$\vec{OB} = \vec{OM} + \vec{MB} \Rightarrow \vec{b} = \vec{OM} + \vec{MB}$$

Thus, $\vec{OM}$ and $\vec{MB}$ are components of $\vec{b}$ along $\vec{a}$ and perpendicular to $\vec{a}$ respectively.

Now,

$$\vec{OM} = (OM)\,\hat{a} = (OB\cos\theta)\,\hat{a} = \left\{|\vec{b}|\cos\theta\right\}\hat{a} = \left\{|\vec{b}|\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|\,|\vec{b}|}\right\}\hat{a} \qquad \left[\because \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right]$$

$$\Rightarrow \vec{OM} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right\}\hat{a} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right\}\frac{\vec{a}}{|\vec{a}|} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$$

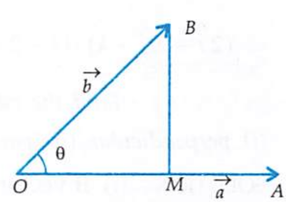

Fig. 23.11

Now,

$$\vec{b} = \vec{OM} + \vec{MB} \Rightarrow \vec{MB} = \vec{b} - \vec{OM} = \vec{b} - \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$$

Thus, the components of $\vec{b}$ along and perpendicular to $\vec{a}$ are $\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$ and $\vec{b}-\left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right\}\vec{a}$ respectively.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find $\vec{a}\cdot\vec{b}$ when*

(i) $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$ (ii) $\vec{a} = (1, 1, 2)$ and $\vec{b} = (3, 2, -1)$

SOLUTION (i) We have, $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and, $\vec{b} = 6\hat{i} - 3\hat{j} + 2\hat{k}$.

$\therefore \quad \vec{a}\cdot\vec{b} = (2\hat{i} + 2\hat{j} - \hat{k})\cdot(6\hat{i} - 3\hat{j} + 2\hat{k}) = (2)(6) + (2)(-3) + (-1)(2) = 12 - 6 - 2 = 4.$

(ii) We have, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$.

$\therefore \quad \vec{a}\cdot\vec{b} = (\hat{i} + \hat{j} + 2\hat{k})\cdot(3\hat{i} + 2\hat{j} - \hat{k}) = (1)(3) + (1)(2) + (2)(-1) = 3 + 2 - 2 = 3.$

EXAMPLE 2 *Find $(\vec{a} + 3\vec{b})\cdot(2\vec{a} - \vec{b})$, if $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$.* [CBSE 2002]

SOLUTION We have, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\therefore \quad \vec{a} + 3\vec{b} = (\hat{i} + \hat{j} + 2\hat{k}) + 3(3\hat{i} + 2\hat{j} - \hat{k}) = 10\hat{i} + 7\hat{j} - \hat{k}$

and, $\quad 2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k}) = -\hat{i} + 0\hat{j} + 5\hat{k}$

$$\therefore \quad (\vec{a} + 3\vec{b})\cdot(2\vec{a} - \vec{b}) = (10\hat{i} + 7\hat{j} - \hat{k})\cdot(-\hat{i} + 0\hat{j} + 5\hat{k})$$
$$= (10)(-1) + (7)(0) + (-1)(5) = -10 + 0 - 5 = -15$$

ALITER We have, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\Rightarrow \quad |\vec{a}| = \sqrt{1+1+4} = \sqrt{6}, |\vec{b}| = \sqrt{9+4+1} = \sqrt{14}$ and $\vec{a}\cdot\vec{b} = 3 + 2 - 2 = 3$.

$$\therefore \quad (\vec{a} + 3\vec{b})\cdot(2\vec{a} - \vec{b}) = \vec{a}\cdot 2\vec{a} - \vec{a}\cdot\vec{b} + 3\vec{b}\cdot 2\vec{a} - 3\vec{b}\cdot\vec{b}$$
$$= 2|\vec{a}|^2 - \vec{a}\cdot\vec{b} + 6(\vec{b}\cdot\vec{a}) - 3|\vec{b}|^2$$
$$= 2|\vec{a}|^2 + 5(\vec{a}\cdot\vec{b}) - 3|\vec{b}|^2 = 2(6) + 5(3) - 3(14) = -15.$$

EXAMPLE 3 *Find the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other.*

SOLUTION If the vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other, then

$$\vec{a}\cdot\vec{b} = 0$$

$\Rightarrow (2\hat{i} + \lambda\hat{j} + \hat{k})\cdot(\hat{i} - 2\hat{j} + 3\hat{k}) = 0 \Rightarrow (2)(1) + \lambda(-2) + (1)(3) = 0 \Rightarrow -2\lambda + 5 = 0 \Rightarrow \lambda = 5/2.$

EXAMPLE 4 *Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are (i) perpendicular (ii) parallel.*

SOLUTION (i) If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then

$$\vec{a}\cdot\vec{b} = 0 \Rightarrow (3\hat{i} + 2\hat{j} + 9\hat{k})\cdot(\hat{i} + p\hat{j} + 3\hat{k}) = 0 \Rightarrow 3 + 2p + 27 = 0 \Rightarrow p = -15$$

(ii) We know that the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are parallel iff

$$\vec{a} = \lambda\vec{b}$$

$\Leftrightarrow \quad (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) = \lambda (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}) \Leftrightarrow a_1 = \lambda b_1, a_2 = \lambda b_2, a_3 = \lambda b_3$

$\Leftrightarrow \quad \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} \; (= \lambda)$

So, given vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ will be parallel iff

$$\frac{3}{1} = \frac{2}{p} = \frac{9}{3} \Rightarrow 3 = \frac{2}{p} \Rightarrow p = \frac{2}{3}$$

EXAMPLE 5 *Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively, and such that $\vec{a} \cdot \vec{b} = \sqrt{6}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. It is given that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$.

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3} \times 2} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Hence, the measure of the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

EXAMPLE 6 *Find the angle between two vectors $\vec{a}$ and $\vec{b}$ having the same length $\sqrt{2}$ and their scalar product is -1.*

SOLUTION Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. It is given that $|\vec{a}| = |\vec{b}| = \sqrt{2}$ and $\vec{a} \cdot \vec{b} = -1$.

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{-1}{\sqrt{2} \times \sqrt{2}} = -\frac{1}{2} \Rightarrow \cos\theta = \cos\frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3} \quad [\because 0 \le \theta \le \pi]$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $2\pi/3$.

EXAMPLE 7 *Find the angle between the vectors $5\hat{i} + 3\hat{j} + 4\hat{k}$ and $6\hat{i} - 8\hat{j} - \hat{k}$.*

SOLUTION Let $\vec{a} = 5\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{b} = 6\hat{i} - 8\hat{j} - \hat{k}$ be the given vectors and let θ be the angle between them. Then,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$

Now, $\vec{a} \cdot \vec{b} = (5\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (6\hat{i} - 8\hat{j} - \hat{k}) = (5)(6) + 3(-8) + 4(-1) = 2$

$$|\vec{a}| = \sqrt{5^2 + 3^2 + 4^2} = \sqrt{50} \text{ and, } |\vec{b}| = \sqrt{6^2 + (-8)^2 + (-1)^2} = \sqrt{101}$$

$$\therefore \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} \Rightarrow \cos\theta = \frac{2}{\sqrt{50}\,\sqrt{101}} = \frac{2}{5\sqrt{2}\,\sqrt{101}} = \frac{\sqrt{2}}{5\sqrt{101}} \Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{2}}{5\sqrt{101}}\right).$$

EXAMPLE 8 *If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Deduce that $\vec{AB}$ and $\vec{CD}$ are collinear.* **[NCERT, CBSE 2019]**

SOLUTION Let θ be the angle between the lines AB and CD. Then, θ is also the angle between vectors $\vec{AB}$ and $\vec{CD}$.

Now, $\vec{AB} = (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}$

and, $\vec{CD} = (\hat{i} - 6\hat{j} - \hat{k}) - (3\hat{i} + 2\hat{j} - 3\hat{k}) = -2\hat{i} - 8\hat{j} + 2\hat{k}$

$\therefore \quad |\vec{AB}| = \sqrt{1+16+1} = 3\sqrt{2}, |\vec{CD}| = \sqrt{4+64+4} = 6\sqrt{2}$

and, $(\vec{AB} \cdot \vec{CD}) = (\hat{i} + 4\hat{j} - \hat{k}) \cdot (-2\hat{i} - 8\hat{j} + 2\hat{k}) = -2 - 32 - 2 = -36$

Now, $\cos\theta = \dfrac{\vec{AB} \cdot \vec{CD}}{|\vec{AB}|\,|\vec{CD}|}$

$\Rightarrow \quad \cos\theta = \dfrac{-36}{3\sqrt{2} \times 6\sqrt{2}} = -1 = \cos\pi$ $\quad [\because 0 \le \theta \le \pi]$

$\Rightarrow \quad \theta = \pi \Rightarrow \vec{AB}$ and $\vec{CD}$ are unlike parallel vectors.

Hence, lines AB and CD are parallel.

ALITER We observe that $\vec{CD} = -2\vec{AB}$. Therefore, $\vec{AB}$ and $\vec{CD}$ are unlike parallel vectors.

EXAMPLE 9 *Find the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on $2\hat{i} + 6\hat{j} + 3\hat{k}$.*

SOLUTION We know that: Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

Here, $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$.

$\therefore \quad \vec{a} \cdot \vec{b} = (7\hat{i} + \hat{j} - 4\hat{k}) \cdot (2\hat{i} + 6\hat{j} + 3\hat{k}) = 14 + 6 - 12 = 8$ and $|\vec{b}| = \sqrt{2^2 + 6^2 + 3^2} = 7$

Hence, Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{8}{7}$

EXAMPLE 10 *Let $\vec{a}$ and $\vec{b}$ be two vectors of the same magnitude such that the angle between them is $60°$ and $\vec{a} \cdot \vec{b} = 8$. Find $|\vec{a}|$ and $|\vec{b}|$.* [NCERT]

SOLUTION We have,

$$\vec{a} \cdot \vec{b} = 8$$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \cos 60° = 8$ $\quad [\because$ Angle between $\vec{a}$ and $\vec{b}$ is $60°]$

$\Rightarrow \quad |\vec{a}|^2 \times \dfrac{1}{2} = 8$ $\quad [\because |\vec{a}| = |\vec{b}|$ (given)$]$

$\Rightarrow \quad |\vec{a}|^2 = 16 \Rightarrow |\vec{a}| = 4$

Hence, $|\vec{a}| = |\vec{b}| = 4$.

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 11 *For any vector $\vec{r}$, prove that $\vec{r} = (\vec{r}\cdot\hat{i})\,\hat{i} + (\vec{r}\cdot\hat{j})\,\hat{j} + (\vec{r}\cdot\hat{k})\,\hat{k}$.*

SOLUTION Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be an arbitrary vector. Then,

$$\vec{r}\cdot\hat{i} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{i} = x(\hat{i}\cdot\hat{i}) + y(\hat{j}\cdot\hat{i}) + z(\hat{k}\cdot\hat{i}) = x(1) + y(0) + z(0) = x$$

$$\vec{r}\cdot\hat{j} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{j} = x(\hat{i}\cdot\hat{j}) + y(\hat{j}\cdot\hat{j}) + z(\hat{k}\cdot\hat{j}) = x(0) + y(1) + z(0) = y$$

and, $$\vec{r}\cdot\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k})\cdot\hat{k} = x(\hat{i}\cdot\hat{k}) + y(\hat{j}\cdot\hat{k}) + z(\hat{k}\cdot\hat{k}) = x(0) + y(0) + z(1) = z$$

Substituting the values of x, y, z in $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we obtain

$$\vec{r} = (\vec{r}\cdot\hat{i})\,\hat{i} + (\vec{r}\cdot\hat{j})\,\hat{j} + (\vec{r}\cdot\hat{k})\,\hat{k}.$$

REMARK 1 *In the above example, we have seen that for any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we have*

$$\vec{r}\cdot\hat{i} = x,\ \vec{r}\cdot\hat{j} = y \text{ and } \vec{r}\cdot\hat{k} = z$$

Also,

$\vec{r}\cdot\hat{i}$ = Projection of $\vec{r}$ on X-axis, $\vec{r}\cdot\hat{j}$ = Projection of $\vec{r}$ on Y-axis, $\vec{r}\cdot\hat{k}$ = Projection of $\vec{r}$ on Z-axis

$\therefore$ x = Projection of $\vec{r}$ on X-axis, y = Projection of $\vec{r}$ on Y-axis, z = Projection of $\vec{r}$ on Z-axis.

Thus, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then x, y and z respectively are the projections of $\vec{r}$ on X, Y and Z-axes. These projections are also equal to $\vec{r}\cdot\hat{i}$, $\vec{r}\cdot\hat{j}$ and $\vec{r}\cdot\hat{k}$ respectively.

REMARK 2 *For any vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. We call $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ as components of $\vec{r}$ along x, y and z-axes respectively.*

REMARK 3 *If a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ makes angles α, β and γ with x, y and z-axes respectively, then*

$$\cos\alpha = \frac{\vec{r}\,.\,\hat{i}}{|\vec{r}\,|\,|\hat{i}\,|} \Rightarrow \cos\alpha = \frac{x}{|\vec{r}|} \Rightarrow \alpha = \cos^{-1}\left(\frac{x}{|\vec{r}\,|}\right)$$

Similarly, we obtain

$$\beta = \cos^{-1}\left(\frac{y}{|\vec{r}\,|}\right) \text{ and } \gamma = \cos^{-1}\left(\frac{z}{|\vec{r}\,|}\right)$$

Again,

$$\cos\alpha = \frac{x}{|\vec{r}\,|},\ \cos\beta = \frac{y}{|\vec{r}\,|} \text{ and } \cos\gamma = \frac{z}{|\vec{r}\,|} \Rightarrow x = |\vec{r}\,|\cos\alpha,\ y = |\vec{r}\,|\cos\beta \text{ and } z = |\vec{r}\,|\cos\gamma$$

$$\therefore\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = |\vec{r}\,|\{(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{i} + (\cos\gamma)\,\hat{k}\}$$

Also, $$|\vec{r}\,|^2 = x^2 + y^2 + z^2$$

$$\Rightarrow\quad |\vec{r}\,|^2 = |\vec{r}\,|^2\cos^2\alpha + |\vec{r}\,|^2\cos^2\beta + |\vec{r}\,|^2\cos^2\gamma \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Thus, if a vector $\vec{r}$ makes angles α, β and γ with $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively, then

$$\vec{r} = |\vec{r}\,|\{(\cos\alpha)\,\hat{i} + (\cos\beta)\,\hat{j} + (\cos\gamma)\,\hat{k}\} \text{ and, } \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 12 *Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and satisfying $\vec{d}\cdot\vec{c} = 21$.*

SOLUTION Let $\vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$. Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$\therefore \quad \vec{d}\cdot\vec{a} = 0$ and $\vec{d}\cdot\vec{b} = 0$.

Now, $\vec{d}\cdot\vec{a} = 0 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(4\hat{i} + 5\hat{j} - \hat{k}) = 0 \Rightarrow 4x + 5y - z = 0$ ...(i)

$\vec{d}\cdot\vec{b} = 0 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(\hat{i} - 4\hat{j} + 5\hat{k}) = 0 \Rightarrow x - 4y + 5z = 0$ ...(ii)

$\vec{d}\cdot\vec{c} = 21 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} + \hat{j} - \hat{k}) = 21 \Rightarrow 3x + y - z = 21$ ...(iii)

Solving (i), (ii) and (iii), we get $x = 7$, $y = z = -7$. Hence, $\vec{d} = 7\hat{i} - 7\hat{j} - 7\hat{k}$.

EXAMPLE 13 *Dot products of a vector with vectors $3\hat{i} - 5\hat{k}$, $2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively -1, 6 and 5. Find the vector.*

SOLUTION Let $\vec{a} = 3\hat{i} + 0\hat{j} - 5\hat{k}$, $\vec{b} = 2\hat{i} + 7\hat{j} + 0\hat{k}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$ be three given vectors.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be a vector such that its dot products with $\vec{a}$, $\vec{b}$ and $\vec{c}$ are -1, 6 and 5 respectively. Then,

$\vec{r}\cdot\vec{a} = -1 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(3\hat{i} + 0\hat{j} - 5\hat{k}) = -1 \Rightarrow 3x + 0y - 5z = -1$ ...(i)

$\vec{r}\cdot\vec{b} = 6 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(2\hat{i} + 7\hat{j} + 0\hat{k}) = 6 \Rightarrow 2x + 7y + 0z = 6$ ...(ii)

and, $\vec{r}\cdot\vec{c} = 5 \Rightarrow (x\hat{i} + y\hat{j} + z\hat{k})\cdot(\hat{i} + \hat{j} + \hat{k}) = 5 \Rightarrow x + y + z = 5$ ...(iii)

Solving (i), (ii) and (iii), we get $x = 3$, $y = 0$, $z = 2$.

Hence, the required vector $\vec{r}$ is given by $\vec{r} = 3\hat{i} + 0\hat{j} + 2\hat{k}$.

EXAMPLE 14 *Show that the projection vector of $\vec{a}$ on $\vec{b}$ $(\neq \vec{0})$ (component of $\vec{a}$ along $\vec{b}$) is $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$.*

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. As shown in Fig. 23.12, length OL is the projection of $\vec{a}$ on $\vec{b}$ and $\vec{OL}$ is the projection vector of $\vec{a}$ on $\vec{b}$. In ΔOLA, we obtain

$$\cos\theta = \frac{OL}{OA}$$

$$\Rightarrow \quad OL = OA\cos\theta = |\vec{a}|\cos\theta = |\vec{a}|\left\{\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}||\vec{b}|}\right\} \qquad \left[\text{Using : } \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

$$\Rightarrow \quad OL = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$\therefore \quad \vec{OL} = (OL)\,\hat{b} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right\}\hat{b} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right\}\frac{\vec{b}}{|\vec{b}|} = \left\{\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right\}\vec{b}$$

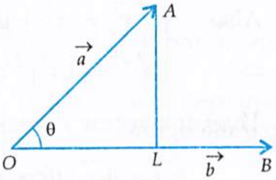

Fig. 23.12

NOTE *This may be used as a standard result.*

EXAMPLE 15 *Show that the projection vector of $\vec{b}$ on $\vec{a} \neq 0$ is* $\left(\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}$.

SOLUTION Proceed as in Example 14.

EXAMPLE 16 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that:*

(i) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b}$

(ii) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$

(iii) $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$. *Interpret the result geometrically.*

(iv) $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Leftrightarrow \vec{a} \perp \vec{b}$. *Interpret the result geometrically.*

(v) $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}| \Leftrightarrow \vec{a}$ *is parallel to* $\vec{b}$

(vi) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \Leftrightarrow \vec{a}, \vec{b}$ *are orthogonal.* **[NCERT]**

SOLUTION (i) We have,

$$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) \qquad [|\vec{x}|^2 = \vec{x}\cdot\vec{x}]$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot\vec{a} + (\vec{a}+\vec{b})\cdot\vec{b} \qquad \left[\begin{array}{r}\text{By distributivity of dot}\\ \text{product over vector addition}\end{array}\right]$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} \qquad \left[\begin{array}{r}\text{By distributivity of dot}\\ \text{product over vector addition}\end{array}\right]$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2\,\vec{a}\cdot\vec{b} + |\vec{b}|^2 \qquad [\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}]$$

(ii) Proceed as in (i).

(iii) Adding (i) and (ii), we obtain

$$|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$$

Geometrical Interpretation: Let $ABCD$ be a parallelogram having sides $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.

In ΔABC, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{a} + \vec{b} = \vec{AC} \qquad \text{...(i)}$$

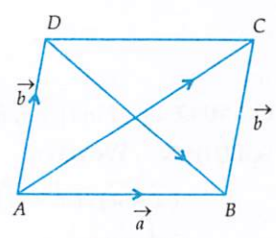

Fig. 23.13

In ΔABD, we obtain

$$\vec{AD} + \vec{DB} = \vec{AB}$$

$$\Rightarrow \quad \vec{AB} - \vec{AD} = \vec{DB} \Rightarrow \vec{a} - \vec{b} = \vec{DB} \qquad \text{...(ii)}$$

Thus, $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ represent two diagonals of a parallelogram whose two adjacent sides are $\vec{a}$ and $\vec{b}$. Therefore, $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left(|\vec{a}|^2 + |\vec{b}|^2\right)$ implies that the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals of the parallelogram.

(iv) $\quad |\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

$$\Leftrightarrow \quad |\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2$$

$\Leftrightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b}) \Leftrightarrow 4(\vec{a}\cdot\vec{b})=0 \Leftrightarrow \vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

As discussed above that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ represent diagonals of a parallelogram whose two adjacent sides are $\vec{a}$ and $\vec{b}$.

$\therefore \; |\vec{a}+\vec{b}|=|\vec{a}-\vec{b}| \Leftrightarrow \vec{a}\perp\vec{b} \Leftrightarrow$ Diagonals of a parallelogram are equal iff it is a rectangle.

(v) $\quad |\vec{a}+\vec{b}|=|\vec{a}|+|\vec{b}|$

$\Leftrightarrow \quad |\vec{a}+\vec{b}|^2=\left(|\vec{a}|+|\vec{b}|\right)^2$

$\Leftrightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2\,\vec{a}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2+2|\vec{a}|\,|\vec{b}|$

$\Leftrightarrow \quad \vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}| \Leftrightarrow |\vec{a}|\,|\vec{b}|\cos\theta=|\vec{a}|\,|\vec{b}| \Leftrightarrow \cos\theta=1 \Leftrightarrow \theta=0 \Leftrightarrow \vec{a}\parallel\vec{b}$

(vi) We have,

$\quad |\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$

$\Leftrightarrow \quad |\vec{a}|^2+|\vec{b}|^2+2(\vec{a}\cdot\vec{b})=|\vec{a}|^2+|\vec{b}|^2$

$\Leftrightarrow \quad 2(\vec{a}\cdot\vec{b})=0 \Leftrightarrow \vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}$ and $\vec{b}$ are orthogonal.

EXAMPLE 17 *If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2, |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$, find $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.*

SOLUTION Using distribution of dot product over vector addition, we get

$$\begin{aligned}(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b}) &= 6(\vec{a}\cdot\vec{a})+21(\vec{a}\cdot\vec{b})-10(\vec{b}\cdot\vec{a})-35(\vec{b}\cdot\vec{b})\\ &= 6|\vec{a}|^2+21(\vec{a}\cdot\vec{b})-10(\vec{a}\cdot\vec{b})-35|\vec{b}|^2\\ &= 6|\vec{a}|^2+11(\vec{a}\cdot\vec{b})-35|\vec{b}|^2\\ &= 6\times 2^2+11\times 1-35\times 1^2 \qquad [\because |\vec{a}|=2, |\vec{b}|=1, \vec{a}\cdot\vec{b}=1]\\ &= 24+11-35=0\end{aligned}$$

EXAMPLE 18 *Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=15$.*

SOLUTION We have,

$\quad (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=15$

$\Rightarrow \quad |\vec{x}|^2-|\vec{a}|^2=15 \qquad [\because (\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{a}|^2-|\vec{b}|^2]$

$\Rightarrow \quad |\vec{x}|^2-1=15 \qquad [\because \vec{a}$ is a unit vector $\therefore |\vec{a}|=1]$

$\Rightarrow \quad |\vec{x}|^2=16 \Rightarrow |\vec{x}|=4$

EXAMPLE 19 *Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})=27$ and $|\vec{a}|=2|\vec{b}|$.*

SOLUTION We have,

$\quad (\vec{a}-\vec{b})\cdot(\vec{a}+\vec{b})=27$

$\Rightarrow \quad |\vec{a}|^2-|\vec{b}|^2=27$

$\Rightarrow \quad 4|\vec{b}|^2 - |\vec{b}|^2 = 27 \qquad [\because |\vec{a}| = 2|\vec{b}| \text{ (Given)}]$

$\Rightarrow \quad 3|\vec{b}|^2 = 27 \Rightarrow |\vec{b}|^2 = 9 \Rightarrow |\vec{b}| = 3 \qquad [\because |\vec{b}| > 0]$

$\therefore \quad |\vec{a}| = 2|\vec{b}| \Rightarrow |\vec{a}| = 2 \times 3 = 6$. Thus, $|\vec{a}| = 6$ and $|\vec{b}| = 3$

EXAMPLE 20 *If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$, and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular.*

SOLUTION We have, $|\vec{a}| = \sqrt{25+1+9} = \sqrt{35}$ and $|\vec{b}| = \sqrt{1+9+25} = \sqrt{35}$

Now, $\quad (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 35 - 35 = 0 \Rightarrow (\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ are perpendicular to each other.

ALITER We have, $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$.

$\therefore \quad \vec{a} + \vec{b} = 6\hat{i} + 2\hat{j} - 8\hat{k}$ and $\vec{a} - \vec{b} = 4\hat{i} - 4\hat{j} + 2\hat{k}$

$\Rightarrow \quad (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = (6\hat{i} + 2\hat{j} - 8\hat{k})\cdot(4\hat{i} - 4\hat{j} + 2\hat{k}) = 24 - 8 - 16 = 0$

$\Rightarrow \quad \vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular to each other.

EXAMPLE 21 *If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = 6$, Find $|\vec{a}+\vec{b}|$ and $|\vec{a}-\vec{b}|$.*

SOLUTION We know that

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2 \text{ and } |\vec{a}-\vec{b}|^2 = |\vec{a}|^2 - 2(\vec{a}\cdot\vec{b}) + |\vec{b}|^2$$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 9 + 12 + 4$ and $|\vec{a}-\vec{b}|^2 = 9 - 12 + 4$

$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 25$ and $|\vec{a}-\vec{b}|^2 = 1 \Rightarrow |\vec{a}+\vec{b}| = 5$ and $|\vec{a}-\vec{b}| = 1$.

EXAMPLE 22 *Two vectors $\vec{a}$ and $\vec{b}$, prove that the vector $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is orthogonal to the vector $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.* **[NCERT, CBSE 2020]**

SOLUTION Let $\vec{\alpha} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $\vec{\beta} = |\vec{a}|\vec{b} - |\vec{b}|\vec{a}$. Then,

$$\vec{\alpha}\cdot\vec{\beta} = \left\{|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right\}\cdot\left\{|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right\}$$

$\Rightarrow \quad \vec{\alpha}\cdot\vec{\beta} = |\vec{a}|^2(\vec{b}\cdot\vec{b}) - |\vec{a}||\vec{b}|(\vec{b}\cdot\vec{a}) + |\vec{b}||\vec{a}|(\vec{a}\cdot\vec{b}) - |\vec{b}|^2(\vec{a}\cdot\vec{a})$

$\Rightarrow \quad \vec{\alpha}\cdot\vec{\beta} = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b}) + |\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b}) - |\vec{b}|^2|\vec{a}|^2 = 0$

$\therefore \quad \vec{\alpha}$ is perpendicular (or orthogonal) to $\vec{\beta}$.

EXAMPLE 23 *If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.* **[CBSE 2008, 2014]**

SOLUTION Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$.

We have,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow \quad |\vec{a} + \vec{b}| = |-\vec{c}|$

$\Rightarrow \quad |\vec{a} + \vec{b}| = |\vec{c}|$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b} = |\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta = |\vec{c}|^2$

$\Rightarrow \quad 9 + 25 + 2(3)(5)\cos\theta = 49 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$

EXAMPLE 24 *If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that* $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$.

SOLUTION We know that $\vec{x}\cdot\vec{x} = |\vec{x}|^2$.

$\therefore \quad |\hat{a} - \hat{b}|^2 = (\hat{a} - \hat{b})\cdot(\hat{a} - \hat{b})$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = \hat{a}\cdot\hat{a} - \hat{a}\cdot\hat{b} - \hat{b}\cdot\hat{a} + \hat{b}\cdot\hat{b}$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = |\hat{a}|^2 - 2\,\hat{a}\cdot\hat{b} + |\hat{b}|^2$ $\quad [\because \hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{a}]$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = |\hat{a}|^2 - 2|\hat{a}|\,|\hat{b}|\cos\theta + |\hat{b}|^2$ $\quad [\because \hat{a}\cdot\hat{b} = |\hat{a}|\,|\hat{b}|\cos\theta]$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = 2 - 2\cos\theta$ $\quad [\because |\hat{a}| = |\hat{b}| = 1]$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = 2(1 - \cos\theta) = 2\left(2\sin^2\frac{\theta}{2}\right) = 4\sin^2\frac{\theta}{2}$

$\Rightarrow \quad \sin^2\frac{\theta}{2} = \frac{1}{4}|\hat{a} - \hat{b}|^2 \Rightarrow \sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$

EXAMPLE 25 *For any two vectors $\vec{a}$ and $\vec{b}$, prove that* $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$ *(Triangle inequality)* [NCERT]

SOLUTION We have,

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\cdot\vec{b})$$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta \quad$ ...(i)

Now, $\cos\theta \le 1$ for all θ

$\Rightarrow \quad 2|\vec{a}|\,|\vec{b}|\cos\theta \le 2|\vec{a}|\,|\vec{b}|$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|\cos\theta \le |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}|\,|\vec{b}|$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 \le \left\{|\vec{a}| + |\vec{b}|\right\}^2 \Rightarrow |\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$

EXAMPLE 26 *(Cauchy-Schawarz inequality) For any two vectors $\vec{a}$ and $\vec{b}$, prove that* $(\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2$ *and hence show that* $(a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$. [NCERT]

SOLUTION Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \Rightarrow (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

Now, $\cos^2\theta \le 1 \Rightarrow |\vec{a}|^2|\vec{b}|^2\cos^2\theta \le |\vec{a}|^2|\vec{b}|^2 \Rightarrow (\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3,\ |\vec{a}|^2 = a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2} \text{ and, } |\vec{b}|^2 = b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2}$$

$\therefore$ $(\vec{a}\cdot\vec{b})^2 \le |\vec{a}|^2|\vec{b}|^2 \Rightarrow (a_1b_1 + a_2b_2 + a_3b_3)^2 \le (a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2})(b_1^{\ 2} + b_2^{\ 2} + b_3^{\ 2})$

EXAMPLE 27 *If $\vec{a}, \vec{b}$ are two vectors such that $|\vec{a}+\vec{b}| = |\vec{a}|$, then prove that $2\vec{a}+\vec{b}$ is perpendicular to $\vec{b}$.* **[CBSE 2013]**

SOLUTION We have,

$$|\vec{a}+\vec{b}| = |\vec{a}|$$

$\Rightarrow$ $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2$ [Squaring both sides]

$\Rightarrow$ $|\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a}\,.\,\vec{b}) = |\vec{a}|^2 \Rightarrow |\vec{b}|^2 + 2(\vec{a}\,.\,\vec{b}) = 0$ ...(i)

Now, $(2\vec{a}+\vec{b})\,.\,\vec{b} = 2(\vec{a}\,.\,\vec{b}) + (\vec{b}\,.\,\vec{b}) = 2(\vec{a}\,.\,\vec{b}) + |\vec{b}|^2 = 0$ [Using (i)]

Hence, $(2\vec{a}+\vec{b})$ is perpendicular to $\vec{b}$.

EXAMPLE 28 *Find the angle between unit vectors $\vec{a}$ and $\vec{b}$ so that $\sqrt{3}\,\vec{a}-\vec{b}$ is also a unit vector.* **[CBSE 2020]**

SOLUTION Let θ be the angle between $\vec{a}$ and $\vec{b}$. We have, $|\vec{a}| = 1$, $|\vec{b}| = 1$ and $|\sqrt{3}\,\vec{a}-\vec{b}| = 1$.

Now, $|\sqrt{3}\,\vec{a}-\vec{b}| = 1$

$\Rightarrow$ $|\sqrt{3}\,\vec{a}-\vec{b}|^2 = 1$

$\Rightarrow$ $(\sqrt{3}\,\vec{a}-\vec{b})\,.\,(\sqrt{3}\,\vec{a}-\vec{b}) = 1$

$\Rightarrow$ $\sqrt{3}\,\vec{a}\,.\,(\sqrt{3}\,\vec{a}-\vec{b}) - \vec{b}\,.\,(\sqrt{3}\,\vec{a}-\vec{b}) = 1$

$\Rightarrow$ $3|\vec{a}|^2 - \sqrt{3}(\vec{a}\,.\,\vec{b}) - \sqrt{3}(\vec{b}\,.\,\vec{a}) + |\vec{b}|^2 = 1$

$\Rightarrow$ $3 - 2\sqrt{3}(\vec{a}\,.\,\vec{b}) + 1 = 1 \Rightarrow 4 - 2\sqrt{3}\,|\vec{a}|\,|\vec{b}|\cos\theta = 1 \Rightarrow 2\sqrt{3}\cos\theta = 3 \Rightarrow \cos\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \theta = \dfrac{\pi}{6}$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\dfrac{\pi}{6}$.

EXAMPLE 29 *If a unit vector $\vec{a}$ makes angle $\pi/4$ with $\hat{i}$, $\pi/3$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find the components of $\vec{a}$ and the angle θ.*

SOLUTION We know that if a vector $\vec{a}$ makes angles α, β and γ with $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

It is given that: $\alpha = \dfrac{\pi}{4}$, $\beta = \dfrac{\pi}{3}$ and $\gamma = \theta$ an acute angle.

$\therefore$ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \quad \cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{3}+\cos^2\theta = 1$$

$$\Rightarrow \quad \frac{1}{2}+\frac{1}{4}+\cos^2\theta = 1$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2} \qquad [\because \theta \text{ is an acute angle} \therefore \cos\theta > 0]$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} \Rightarrow \gamma = \frac{\pi}{3}$$

Now, $\vec{a} = |\vec{a}|\{(\cos\alpha)\hat{i}+(\cos\beta)\hat{j}+(\cos\gamma)\hat{k}\}$

$$\Rightarrow \quad \vec{a} = \left(\cos\frac{\pi}{4}\right)\hat{i}+\left(\cos\frac{\pi}{3}\right)\hat{j}+\left(\cos\frac{\pi}{3}\right)\hat{k} \qquad \left[\because |\vec{a}|=1, \alpha=\frac{\pi}{4}, \beta=\frac{\pi}{3} \text{ and } \gamma=\frac{\pi}{3}\right]$$

$$\Rightarrow \quad \vec{a} = \frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}.$$

Thus, the components of $\vec{a}$ are $\frac{1}{\sqrt{2}}\hat{i}$, $\frac{1}{2}\hat{j}$, $\frac{1}{2}\hat{k}$.

EXAMPLE 30 *Find the components of a unit vector which is perpendicular to the vectors $\hat{i}+2\hat{j}-\hat{k}$ and $3\hat{i}-\hat{j}+2\hat{k}$.*

SOLUTION Let $\vec{a} = x\hat{i}+y\hat{j}+z\hat{k}$ be a unit vector perpendicular to the vectors $\vec{b} = \hat{i}+2\hat{j}-\hat{k}$ and $\vec{c} = 3\hat{i}-\hat{j}+2\hat{k}$. Then,

$$\vec{a}\cdot\vec{b} = 0 \text{ and } \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \quad (x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}+2\hat{j}-\hat{k}) = 0 \text{ and, } (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j}+2\hat{k}) = 0$$

$$\Rightarrow \quad x+2y-z = 0 \text{ and, } 3x-y+2z = 0$$

$$\Rightarrow \quad \frac{x}{4-1} = \frac{y}{-3-2} = \frac{z}{-1-6} \qquad \text{[Using cross–multiplication]}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{y}{-5} = \frac{z}{-7} = \lambda \text{ (say)} \Rightarrow x = 3\lambda, y = -5\lambda, z = -7\lambda$$

It is given that $\vec{a}$ is a unit vector. Therefore,

$$|\vec{a}| = 1$$

$$\Rightarrow \quad x^2+y^2+z^2 = 1$$

$$\Rightarrow \quad 9\lambda^2+25\lambda^2+49\lambda^2 = 1 \Rightarrow 83\lambda^2 = 1 \Rightarrow \lambda = \pm\frac{1}{\sqrt{83}} \qquad [\because x=3\lambda, y=-5\lambda, z=-7\lambda]$$

$$\therefore \quad x = \pm\frac{3}{\sqrt{83}}, y = \mp\frac{5}{\sqrt{83}} \text{ and } z = \mp\frac{7}{\sqrt{83}}$$

Hence, the components of $\vec{a}$ are $\frac{3}{\sqrt{83}}\hat{i}, \frac{-5}{\sqrt{83}}\hat{j}, \frac{-7}{\sqrt{83}}\hat{k}$ or, $\frac{-3}{\sqrt{83}}\hat{i}, \frac{5}{\sqrt{83}}\hat{j}, \frac{7}{\sqrt{83}}\hat{k}$.

ALITER Required components are the components of vectors $\pm(\vec{a}\times\vec{b})$.

EXAMPLE 31 *The scalar product of the vector $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of the vectors $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.* **[NCERT, CBSE 2014]**

SOLUTION Let $\vec{a} = \hat{i}+\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$. Then,

$$\vec{b}+\vec{c} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}.$$

Let $\hat{r}$ denote the unit vector along $\vec{b}+\vec{c}$. Then,

$$\hat{r} = \frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+36+4}} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}} \quad \text{...(i)}$$

Now, $(\hat{i}+\hat{j}+\hat{k})\cdot\hat{r} = 1$ [Given]

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}} = 1$$

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot\{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}\} = \sqrt{(2+\lambda)^2+40}$$

$$\Rightarrow \quad 2+\lambda+6-2 = \sqrt{(2+\lambda)^2+40} \Rightarrow (\lambda+6)^2 = (2+\lambda)^2+40 \Rightarrow 8\lambda = 8 \Rightarrow \lambda = 1.$$

Putting $\lambda = 1$ in (i), we obtain $\hat{r} = \frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k})$

EXAMPLE 32 *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, then show that $\vec{a} = 0$ or, $\vec{b} = \vec{c}$ or, $\vec{a}\perp(\vec{b}-\vec{c})$.*

SOLUTION We have,

$$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$$

$$\Rightarrow \quad \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c} = 0$$

$$\Rightarrow \quad \vec{a}\cdot(\vec{b}-\vec{c}) = 0$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or, } \vec{b}-\vec{c} = \vec{0} \text{ or, } \vec{a}\perp(\vec{b}-\vec{c}) \Rightarrow \vec{a} = \vec{0} \text{ or, } \vec{b} = \vec{c} \text{ or, } \vec{a}\perp(\vec{b}-\vec{c})$$

EXAMPLE 33 *Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$, $3\hat{i}-4\hat{j}-4\hat{k}$ form the sides of a right angled triangle.* **[NCERT]**

SOLUTION Let $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$, $\vec{b} = \hat{i}-3\hat{j}-5\hat{k}$ and $\vec{c} = 3\hat{i}-4\hat{j}-4\hat{k}$. Clearly, $\vec{a}+\vec{b} = \vec{c}$.

Thus, if $\vec{AC} = \vec{b}$, $\vec{CB} = \vec{a}$ and $\vec{AB} = \vec{c}$, then $\vec{AC} + \vec{CB} = \vec{AB}$.

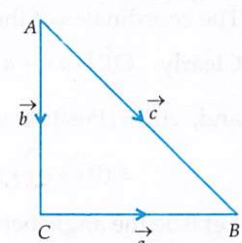

Fig. 23.14

Hence, vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ form the sides of a triangle.

Now, $\vec{a}\cdot\vec{b} = (2\hat{i}-\hat{j}+\hat{k})\cdot(\hat{i}-3\hat{j}-5\hat{k}) = 2+3-5 = 0.$

Therefore, $\vec{AC}$ is perpendicular to $\vec{CB}$. Hence, ΔABC is a right angled triangle.

EXAMPLE 34 *Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right angled triangle. Also, find the remaining angles of the triangle.* **[NCERT]**

SOLUTION We have,

$\vec{AB}$ = Position vector of B – Position vector of $A = (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\vec{BC}$ = Position vector of C – Position vector of $B = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}$

and, $\vec{CA}$ = Position vector of A – Position vector of $C = (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k}) = -\hat{i} + 3\hat{j} + 5\hat{k}$

Clearly, $\vec{AB} + \vec{BC} + \vec{CA} = (-\hat{i} - 2\hat{j} - 6\hat{k}) + (2\hat{i} - \hat{j} + \hat{k}) + (-\hat{i} + 3\hat{j} + 5\hat{k}) = \vec{0}$.

So, A, B and C are the vertices of a triangle.

Now, $\vec{BC} \cdot \vec{CA} = (2\hat{i} - \hat{j} + \hat{k}) \cdot (-\hat{i} + 3\hat{j} + 5\hat{k}) = -2 - 3 + 5 = 0 \Rightarrow \vec{BC} \perp \vec{CA} \Rightarrow \angle BCA = \dfrac{\pi}{2}$

Hence, ΔABC is a right angled triangle.

Clearly, $\angle BAC$ is the angle between the vectors $\vec{AB}$ and $\vec{AC}$.

$$\therefore \quad \cos A = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{(-\hat{i} - 2\hat{j} - 6\hat{k}) \cdot (\hat{i} - 3\hat{j} - 5\hat{k})}{\sqrt{(-1)^2 + (-2)^2 + (-6)^2}\,\sqrt{1^2 + (-3)^2 + (-5)^2}}$$

$$\Rightarrow \quad \cos A = \frac{-1 + 6 + 30}{\sqrt{1 + 4 + 36}\,\sqrt{1 + 9 + 25}} = \frac{35}{\sqrt{41}\,\sqrt{35}} = \sqrt{\frac{35}{41}}$$

$$\Rightarrow \quad A = \cos^{-1}\sqrt{\frac{35}{41}} \Rightarrow \angle BAC = \cos^{-1}\sqrt{\frac{35}{41}}$$

Since $\angle ABC$ is the angle between $\vec{BA}$ and $\vec{BC}$. Therefore,

$$\cos B = \frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}|\,|\vec{BC}|} = \frac{(\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k})}{\sqrt{1^2 + 2^2 + 6^2}\,\sqrt{2^2 + (-1)^2 + (1)^2}} = \frac{2 - 2 + 6}{\sqrt{41}\,\sqrt{6}} = \sqrt{\frac{6}{41}}$$

$$\Rightarrow \quad B = \cos^{-1}\sqrt{\frac{6}{41}} \Rightarrow \angle ABC = \cos^{-1}\left(\sqrt{\frac{6}{41}}\right)$$

Hence, the remaining angles of the triangle are $\cos^{-1}\sqrt{\dfrac{35}{41}}$ and $\cos^{-1}\sqrt{\dfrac{6}{41}}$

EXAMPLE 35 *Show that the angle between two diagonals of a cube is* $\cos^{-1}\left(\dfrac{1}{3}\right)$.

SOLUTION Let a be the length of an edge of the cube and let one corner be at the origin as shown in Fig. 23.15. Clearly, OP, AR, BS and CQ are the diagonals of the cube. Consider the diagonals OP and AR. The coordinates of the vertices are shown in Fig. 23.15.

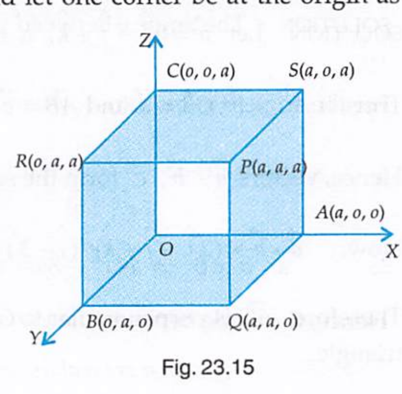

Fig. 23.15

Clearly, $\vec{OP} = a\hat{i} + a\hat{j} + a\hat{k}$.

and, $\vec{AR}$ = (Position vector of R) – (Position vector of A)

$$= (0\hat{i} + a\hat{j} + a\hat{k}) - (a\hat{i} + 0\hat{j} + 0\hat{k}) = -a\hat{i} + a\hat{j} + a\hat{k}.$$

Let θ be the angle between $\vec{OP}$ and $\vec{AR}$. Then,

$$\cos\theta = \frac{\vec{OP} \cdot \vec{AR}}{|\vec{OP}|\,|\vec{AR}|}$$

$$\Rightarrow \quad \cos\theta = \frac{(a\hat{i} + a\hat{j} + a\hat{k}) \cdot (-a\hat{i} + a\hat{j} + a\hat{k})}{\sqrt{a^2 + a^2 + a^2}\,\sqrt{(-a)^2 + a^2 + a^2}} = \frac{-a^2 + a^2 + a^2}{\sqrt{3}a\,\sqrt{3}a} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Similarly, angle θ between the other pairs of diagonals is $\cos^{-1}\left(\frac{1}{3}\right)$.

EXAMPLE 36 *If with reference to a right handed system of mutually perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$, we have $\vec{\alpha} = 3\hat{i} - \hat{j}$, and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$. Express $\vec{\beta}$ in the form $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$.* **[CBSE 2012, 2013 NCERT]**

SOLUTION It is given that $\vec{\beta_1}$ is parallel to $\vec{\alpha}$. Therefore, $\vec{\beta}_1 = \lambda\vec{\alpha}$ for some scalar λ ...(i)

It is also given that: $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2 \Rightarrow \vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 = \vec{\beta} - \lambda\vec{\alpha}$ ...(ii)

It is also given that $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$. Therefore,

$$\vec{\beta}_2 \cdot \vec{\alpha} = 0 \Rightarrow (\vec{\beta} - \lambda\vec{\alpha}) \cdot \vec{\alpha} = 0 \Rightarrow \vec{\beta} \cdot \vec{\alpha} - \lambda(\vec{\alpha} \cdot \vec{\alpha}) = 0 \Rightarrow \lambda = \frac{\vec{\beta} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} \quad \text{...(iii)}$$

Now, $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k} \Rightarrow \vec{\beta} \cdot \vec{\alpha} = 6 - 1 + 0 = 5$ and $\vec{\alpha} \cdot \vec{\alpha} = 9 + 1 = 10$

Substituting these values in (iii), we get: $\lambda = \dfrac{\vec{\beta} \cdot \vec{\alpha}}{\vec{\alpha} \cdot \vec{\alpha}} = \dfrac{5}{10} = \dfrac{1}{2}$

$$\therefore \quad \vec{\beta}_1 = \lambda\vec{\alpha} \Rightarrow \vec{\beta}_1 = \frac{1}{2}(3\hat{i} - \hat{j}) = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$$

and, $$\vec{\beta}_2 = \vec{\beta} - \vec{\beta}_1 \Rightarrow \vec{\beta}_2 = (2\hat{i} + \hat{j} - 3\hat{k}) - \left(\frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\right) = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$$

Hence, $\vec{\beta}_1 = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}$ and $\vec{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$.

EXAMPLE 37 *Find the values of x for which the angle between the vectors $\vec{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse.* **[CBSE 2013]**

SOLUTION The angle θ between vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$.

For the angle θ to be obtuse, we must have

$$\cos\theta < 0 \Rightarrow \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} < 0 \qquad [\because |\vec{a}|, |\vec{b}| > 0]$$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} < 0 \Rightarrow 14x^2 - 8x + x < 0 \Rightarrow 7x(2x-1) < 0 \Rightarrow x(2x-1) < 0 \Rightarrow 0 < x < \frac{1}{2}$$

Hence, the angle between the given vectors is obtuse if $x \in (0, 1/2)$.

EXAMPLE 38 *If l, m, n are scalars and $\vec{a}, \vec{b}, \vec{c}$ are vectors, prove that*

$$|l\vec{a} + m\vec{b} + n\vec{c}|^2 = l^2|\vec{a}|^2 + m^2|\,|\vec{b}|^2 + n^2|\vec{c}|^2 + 2\left\{lm(\vec{a} \cdot \vec{b}) + mn(\vec{b} \cdot \vec{c}) + nl(\vec{c} \cdot \vec{a})\right\}$$

Also, deduce that $|l\vec{a} + m\vec{b} + n\vec{c}|^2 = l^2|\vec{a}|^2 + m^2|\vec{b}|^2 + n^2|\vec{c}|^2$ if $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

SOLUTION We know that $|\vec{r}|^2 = \vec{r}\cdot\vec{r}$.

$$\therefore\ |l\vec{a}+m\vec{b}+n\vec{c}|^2=(l\vec{a}+m\vec{b}+n\vec{c})\cdot(l\vec{a}+m\vec{b}+n\vec{c})$$

$$=l\vec{a}\cdot(l\vec{a}+m\vec{b}+n\vec{c})+m\vec{b}\cdot(l\vec{a}+m\vec{b}+n\vec{c})+n\vec{c}\cdot(l\vec{a}+m\vec{b}+n\vec{c})$$

$$=(l\vec{a}\cdot l\vec{a})+(l\vec{a}\cdot m\vec{b})+(l\vec{a}\cdot n\vec{c})+(m\vec{b}\cdot l\vec{a})+(m\vec{b}\cdot m\vec{b})$$
$$+(m\vec{b}\cdot n\vec{c})+(n\vec{c}\cdot l\vec{a})+(n\vec{c}\cdot m\vec{b})+(n\vec{c}\cdot n\vec{c})$$

$$=l^2(\vec{a}\cdot\vec{a})+lm(\vec{a}\cdot\vec{b})+\ln(\vec{a}\cdot\vec{c})+ml(\vec{b}\cdot\vec{a})+m^2(\vec{b}\cdot\vec{b})$$
$$+mn(\vec{b}\cdot\vec{c})+nl(\vec{c}\cdot\vec{a})+nm(\vec{c}\cdot\vec{b})+n^2(\vec{c}\cdot\vec{c})$$

$$=l^2(\vec{a}\cdot\vec{a})+m^2(\vec{b}\cdot\vec{b})+n^2(\vec{c}\cdot\vec{c})+2lm(\vec{a}\cdot\vec{b})+2mn(\vec{b}\cdot\vec{c})+2nl(\vec{c}\cdot\vec{a})$$

$$=l^2|\vec{a}|^2+m^2|\vec{b}|^2+n^2|\vec{c}|^2+2\left\{lm(\vec{a}\cdot\vec{b})+mn(\vec{b}\cdot\vec{c})+nl(\vec{c}\cdot\vec{a})\right\}$$

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors, then $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$.

$$\therefore\quad |l\vec{a}+m\vec{b}+n\vec{c}|^2=l^2|\vec{a}|^2+m^2|\vec{b}|^2+n^2|\vec{c}|^2$$

EXAMPLE 39 *For any three vectors $\vec{a},\vec{b},\vec{c}$, prove that*

$$|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2|+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$

SOLUTION We know that

$$|\vec{r}|^2=\vec{r}\cdot\vec{r}$$

$$\therefore\quad |\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|^2=\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})+\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|^2=\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}$$

$$\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|^2=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2(\vec{a}\cdot\vec{b})+2(\vec{b}\cdot\vec{c})+2(\vec{c}\cdot\vec{a})$$

$$\Rightarrow\quad |\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$

EXAMPLE 40 *If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Also, find the angle.*

[NCERT, CBSE 2005, 2011, 2013, 2017]

SOLUTION Let $|\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda$ (say). Since $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors. Therefore,

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0 \quad \text{...(i)}$$

We know that $|\vec{x}|^2=\vec{x}\cdot\vec{x}$

$$\therefore\quad |\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{c}+2\vec{a}\cdot\vec{b}+2\vec{b}\cdot\vec{c}+2\vec{c}\cdot\vec{a}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2|+|\vec{b}|^2+|\vec{c}|^2 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 3\lambda^2 \qquad [\because |\vec{a}|=|\vec{b}|=|\vec{c}|=\lambda]$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}\,\lambda \qquad \text{...(ii)}$$

Suppose $\vec{a}+\vec{b}+\vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then,

$$\cos\theta_1 = \frac{\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|} = \frac{\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{|\vec{a}|^2}{|\vec{a}|\,|\vec{a}+\vec{b}+\vec{c}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|} = \frac{\lambda}{\sqrt{3}\,\lambda} = \frac{1}{\sqrt{3}} \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, we obtain: $\theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ and $\theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$. Therefore, $\theta_1 = \theta_2 = \theta_3$.

Hence, $\vec{a}+\vec{b}+\vec{c}$ is equally inclinded with $\vec{a}$, $\vec{b}$ and $\vec{c}$.

EXAMPLE 41 *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors of magnitudes 3, 4 and 5 respectively. If each one is perpendicular to the sum of the other two vectors, prove that $|\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$.*

[NCERT, CBSE 2010, 2013]

SOLUTION We have, $|\vec{a}|=3, |\vec{b}|=4$ and $|\vec{c}|=5$. It is given that

$$\vec{a}\perp(\vec{b}+\vec{c}),\ \vec{b}\perp(\vec{c}+\vec{a}) \text{ and } \vec{c}\perp(\vec{a}+\vec{b})$$

$$\Rightarrow \quad \vec{a}.(\vec{b}+\vec{c})=0,\ \vec{b}.(\vec{c}+\vec{a})=0 \text{ and } \vec{c}.(\vec{a}+\vec{b})=0$$

$$\Rightarrow \quad \vec{a}.\vec{b}+\vec{a}.\vec{c}=0,\ \vec{b}.\vec{c}+\vec{b}.\vec{a}=0 \text{ and } \vec{c}.\vec{a}+\vec{c}.\vec{b}=0$$

Adding all these, we obtain

$$2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0 \Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=0 \qquad \text{...(i)}$$

We know that

$$|\vec{a}+\vec{b}+\vec{c}|^2 = |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) \qquad \text{[See Example 39]}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2 = 3^2+4^2+5^2+0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}| = \sqrt{50} = 5\sqrt{2}.$$

EXAMPLE 42 *If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, find the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$.*

[NCERT, CBSE 2016, 2022]

SOLUTION We have, $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

Now, $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow |\vec{a}+\vec{b}+\vec{c}|=0 \Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2 = 0$

$$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \qquad \text{[See Example 39]}$$

$\Rightarrow \quad 3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \qquad [\because |\vec{a}|=|\vec{b}|=|\vec{c}|=1]$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=-\frac{3}{2}$

EXAMPLE 43 *Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=1, |\vec{b}|=4$ and $|\vec{c}|=2$.* **[NCERT]**

SOLUTION We have, $|\vec{a}|=1, |\vec{b}|=4$ and $|\vec{c}|=2$. It is given that

$\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow |\vec{a}+\vec{b}+\vec{c}|=0 \Rightarrow |\vec{a}+\vec{b}+\vec{c}|^2=0$

$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$ [See Example 39]

$\Rightarrow \quad 1+16+4+2\mu=0 \Rightarrow 2\mu=-21 \Rightarrow \mu=-\frac{21}{2} \qquad [\because |\vec{a}|=1, |\vec{b}|=4 \text{ and } |\vec{c}|=2]$

EXAMPLE 44 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are mutually perpendicular unit vectors, find $|2\vec{a}+\vec{b}+\vec{c}|$.*

SOLUTION We have, $\vec{a}\perp\vec{b}\perp\vec{c}$ and $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

$\therefore \quad \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0$ and $|\vec{a}|=|\vec{b}|=|\vec{c}|=1$

$\Rightarrow \quad |2\vec{a}+\vec{b}+\vec{c}|^2=4|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$ [See Example 38]

$\Rightarrow \quad |2\vec{a}+\vec{b}+\vec{c}|^2=4\times1+1+1=6 \Rightarrow |2\vec{a}+\vec{b}+\vec{c}|=\sqrt{6}.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 45 *Find the values of c for which the vectors $\vec{a}=(c\log_2 x)\hat{i}-6\hat{j}+3\hat{k}$ and $\vec{b}=(\log_2 x)\hat{i}+2\hat{j}+(2c\log_2 x)\hat{k}$ make an obtuse angle for any $x\in(0,\infty)$.*

SOLUTION Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$. Then, $\cos\theta=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$.

For θ to be an obtuse angle, we must have

$\Rightarrow \quad \cos\theta<0$ for all $x\in(0,\infty)$

$\Rightarrow \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}<0$ for all $x\in(0,\infty)$

$\Rightarrow \quad \vec{a}\cdot\vec{b}<0$ for all $x\in(0,\infty)$

$\Rightarrow \quad c(\log_2 x)^2-12+6c(\log_2 x)<0$ for all $x\in(0,\infty)$

$\Rightarrow \quad cy^2+6cy-12<0$ for all $y\in R$, where $y=\log_2 x \qquad [\because x>0 \Rightarrow y=\log_2 x\in R]$

$\Rightarrow \quad c<0$ and $36c^2+48c<0 \qquad [\because ax^2+bx+c<0$ for all $x \Rightarrow a<0$ and Discriminant $<0]$

$\Rightarrow \quad c<0$ and $c(3c+4)<0 \Rightarrow c<0$ and $-\frac{4}{3}<c<0 \Rightarrow c\in(-4/3, 0)$

EXAMPLE 46 *Find the values of 'a' for which the vector $\vec{r}=(a^2-4)\hat{i}+2\hat{j}-(a^2-9)\hat{k}$ makes acute angles with the coordinate axes.*

SOLUTION For vector $\vec{r}$ to be inclined with acute angles with the coordinate axes, we must have

$$\vec{r}\cdot\hat{i}>0,\ \vec{r}\cdot\hat{j}>0 \text{ and } \vec{r}\cdot\hat{k}>0$$

$$\Rightarrow \quad \vec{r}\cdot\hat{i}>0 \text{ and } \vec{r}\cdot\hat{k}>0 \qquad [\because \vec{r}\cdot\hat{j}=2>0]$$

$$\Rightarrow \quad (a^2-4)>0 \text{ and } -(a^2-9)>0 \qquad [\because \vec{r}\cdot\hat{i}=a^2-4 \text{ and } \vec{r}\cdot\hat{k}=-(a^2-9)]$$

$$\Rightarrow \quad (a-2)(a+2)>0 \text{ and } (a+3)(a-3)<0$$

$$\Rightarrow \quad a<-2 \text{ or, } a>2 \text{ and } -3<a<3 \Rightarrow a\in(-3,-2)\cup(2,3).$$

EXAMPLE 47 *If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors, prove that $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2 \le 9$.*

SOLUTION Let $\lambda=|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2$. Then,

$$\lambda=\left(|\vec{a}|^2+|\vec{b}|^2-2\,\vec{a}\cdot\vec{b}\right)+\left(|\vec{b}|^2+|\vec{c}|^2-2\,\vec{b}\cdot\vec{c}\right)+\left(|\vec{c}|^2+|\vec{a}|^2-2\,\vec{c}\cdot\vec{a}\right)$$

$$\Rightarrow \quad \lambda=\left\{2-(\vec{a}\cdot\vec{b})\right\}+\left\{2-2(\vec{b}\cdot\vec{c})\right\}+\left\{2-2(\vec{c}\cdot\vec{a})\right\} \qquad \left[\because |\vec{a}|=|\vec{b}|=|\vec{c}|=1\right]$$

$$\Rightarrow \quad \lambda=6-2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)$$

$$\Rightarrow \quad 2\left(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\right)=6-\lambda \qquad \ldots(i)$$

For any vector $\vec{r}$, we have $|\vec{r}|^2 \ge 0$.

$$\therefore \quad |\vec{a}+\vec{b}+\vec{c}|^2 \ge 0$$

$$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})\ge 0$$

$$\Rightarrow \quad 3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})\ge 0 \qquad \left[\because |\vec{a}|^2=|\vec{b}|^2=|\vec{c}|=1\right]$$

$$\Rightarrow \quad 3+6-\lambda\ge 0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 9-\lambda\ge 0 \Rightarrow \lambda\le 9 \Rightarrow |\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2\le 9$$

EXAMPLE 48 *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}|=1, |\vec{b}|=2$ and $|\vec{c}|=3$. If the projection of $\vec{b}$ along $\vec{a}$ is equal to the projection of $\vec{c}$ along $\vec{a}$ and $\vec{b}, \vec{c}$ are perpendicular to each other, find $|3\vec{a}-2\vec{b}+2\vec{c}|$.*

SOLUTION It is given that

Projection of $\vec{b}$ along $\vec{a}$ = Projection of $\vec{c}$ along $\vec{a}$

$$\Rightarrow \quad \frac{\vec{b}\cdot\vec{a}}{|\vec{a}|}=\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|} \Rightarrow \vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a} \Rightarrow \vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c} \qquad \ldots(i)$$

Also, $\vec{b}$ and $\vec{c}$ are perpendicular to each other $\Rightarrow \vec{b}\cdot\vec{c}=0$ ...(ii)

Now,

$$|3\vec{a}-2\vec{b}+2\vec{c}|^2=9|\vec{a}|^2+4|\vec{b}|^2+4|\vec{c}|^2+2\left\{-6(\vec{a}\cdot\vec{b})-4(\vec{b}\cdot\vec{c})+6(\vec{a}\cdot\vec{c})\right\}$$

[See Example 38]

$$\Rightarrow \quad |3\vec{a}-2\vec{b}+2\vec{c}|^2=9\times 1^2+4\times 2^2+4\times 3^2+2\left\{-6(\vec{a}\cdot\vec{b})-4\times 0+6(\vec{a}\cdot\vec{b})\right\}$$

[Using (i) and (ii)]

$$\Rightarrow \quad |3\vec{a}-2\vec{b}+2\vec{c}|^2=9+16+36+2\times 0=61 \Rightarrow |3\vec{a}-2\vec{b}+2\vec{c}|=\sqrt{61}$$

EXERCISE 23.1

BASIC

1. Find $\vec{a}\cdot\vec{b}$, when
   (i) $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=4\hat{i}-4\hat{j}+7\hat{k}$ (ii) $\vec{a}=\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{k}$
   (iii) $\vec{a}=\hat{j}-\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}-2\hat{k}$
2. For what value of λ are the vector $\vec{a}$ and $\vec{b}$ perpendicular to each other? where:
   (i) $\vec{a}=\lambda\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=4\hat{i}-9\hat{j}+2\hat{k}$ (ii) $\vec{a}=\lambda\hat{i}+2\hat{j}+\hat{k}$ and $\vec{b}=5\hat{i}-9\hat{j}+2\hat{k}$
   (iii) $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{b}=3\hat{i}+2\hat{j}-\lambda\hat{k}$ (iv) $\vec{a}=\lambda\hat{i}+3\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+3\hat{k}$
3. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}|=4$, $|\vec{b}|=3$ and $\vec{a}\cdot\vec{b}=6$. Find the angle between $\vec{a}$ and $\vec{b}$.
4. If $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=-\hat{j}+2\hat{k}$, find $(\vec{a}-2\vec{b})\cdot(\vec{a}+\vec{b})$.
5. Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where:
   (i) $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$ (ii) $\vec{a}=3\hat{i}-2\hat{j}-6\hat{k}$ and $\vec{b}=4\hat{i}-\hat{j}+8\hat{k}$
   (iii) $\vec{a}=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{b}=4\hat{i}+4\hat{j}-2\hat{k}$ (iv) $\vec{a}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$
   (v) $\vec{a}=\hat{i}+2\hat{j}-\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$
6. Find the angles which the vector $\vec{a}=\hat{i}-\hat{j}+\sqrt{2}\,\hat{k}$ makes with the coordinate axes.
7. (i) Dot product of a vector with $\hat{i}+\hat{j}-3\hat{k}$, $\hat{i}+3\hat{j}-2\hat{k}$ and $2\hat{i}+\hat{j}+4\hat{k}$ are 0, 5 and 8 respectively. Find the vector. [CBSE 2003]
   (ii) Dot products of a vector with vectors $\hat{i}-\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}-3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively 4, 0 and 2. Find the vector. [CBSE 2013]
8. If $|\vec{a}+\vec{b}|=60$, $|\vec{a}-\vec{b}|=40$ and $|\vec{b}|=46$, find $|\vec{a}|$
9. Show that the vector $\hat{i}+\hat{j}+\hat{k}$ is equally inclined with the coordinate axes.
10. Show that the vectors $\vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$ are mutually perpendicular unit vectors.
11. For any two vectors $\vec{a}$ and $\vec{b}$, show that: $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0 \Leftrightarrow |\vec{a}|=|\vec{b}|$.

12. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$, find λ such that $\vec{a}$ is perpendicular to $\lambda\vec{b} + \vec{c}$. **[NCERT EXEMPLAR]**

13. If $\vec{p} = 5\hat{i} + \lambda\hat{j} - 3\hat{k}$ and $\vec{q} = \hat{i} + 3\hat{j} - 5\hat{k}$, then find the value of λ, so that $\vec{p} + \vec{q}$ and $\vec{p} - \vec{q}$ are perpendicular vectors. **[CBSE 2013]**

14. Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$, $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angled triangle. **[CBSE 2005]**

15. If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ. **[NCERT]**

16. If A, B, C have position vectors (0, 1, 1,), (3, 1, 5), (0, 3, 3) respectively, show that $\Delta\, ABC$ is right angled at C.

17. Find the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. **[CBSE 2007]**

18. If $\vec{a} = 5\hat{i} - \hat{j} - 3\hat{k}$ and $\vec{b} = \hat{i} + 3\hat{j} - 5\hat{k}$, then show that the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ are orthogonal. **[CBSE 2004, 2019]**

19. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$. **[NCERT, CBSE 2011]**

20. If $\vec{a}$ is a unit vector, then find $|\vec{x}|$ in each of the following:

    (i) $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 8$ **[NCERT]** (ii) $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$ **[NCERT]**

21. Find $|\vec{a}|$ and $|\vec{b}|$, if

    (i) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 12$ and $|\vec{a}| = 2|\vec{b}|$ (ii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$ **[NCERT]**

    (iii) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 3$ and $|\vec{a}| = 2|\vec{b}|$

22. Find $|\vec{a} - \vec{b}|$, if

    (i) $|\vec{a}| = 2$, $|\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 8$ (ii) $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 1$

    (iii) $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$ **[NCERT]**

23. Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if

    (i) $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$ **[NCERT]** (ii) $|\vec{a}| = 3$, $|\vec{b}| = 3$ and $\vec{a} . \vec{b} = 1$

24. If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 30° such that $\vec{a} \cdot \vec{b} = 3$, find $|\vec{a}|$, $|\vec{b}|$.

25. Let $\vec{a} = 5\hat{i} - \hat{j} + 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \lambda\hat{k}$. Find λ such that $\vec{a} + \vec{b}$ is orthogonal to $\vec{a} - \vec{b}$.

26. If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, what can you conclude about the vector $\vec{b}$? **[NCERT, CBSE 2004]**

BASED ON LOTS

27. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ, then prove that

(i) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a}+\hat{b}|$ (ii) $\tan\frac{\theta}{2} = \frac{|\hat{a}-\hat{b}|}{|\hat{a}+\hat{b}|}$

28. If the sum of two unit vectors is a unit vector prove that the magnitude of their difference is $\sqrt{3}$. **[CBSE 2019]**

29. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular unit vectors, then prove that $|\vec{a}+\vec{b}+\vec{c}| = \sqrt{3}$.

30. If $\vec{\alpha} = 3\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{\beta} = 2\hat{i}+\hat{j}-4\hat{k}$, then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta_1}+\vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$. **[CBSE 2012]**

31. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a}\cdot\vec{b} = 0$. But, the converse need not be true. Justify your answer with an example. **[NCERT]**

32. Find the angles of a triangle whose vertices are $A(0, -1, -2)$, $B(3, 1, 4)$ and $C(5, 7, 1)$.

33. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is 1/2. **[NCERT]**

34. Show that the points whose position vectors are $\vec{a} = 4\hat{i}-3\hat{j}+\hat{k}$, $\vec{b} = 2\hat{i}-4\hat{j}+5\hat{k}$, $\vec{c} = \hat{i}-\hat{j}$ form a right triangle.

35. If the vertices A, B, C of ΔABC have position vectors (1, 2, 3), (–1, 0, 0), (0, 1, 2) respectively, what is the magnitude of $\angle ABC$?

36. A unit vector $\vec{a}$ makes angles $\frac{\pi}{4}$ and $\frac{\pi}{3}$ with $\hat{i}$ and $\hat{j}$ respectively and an acute angle θ with $\hat{k}$. Find the angle θ and components of $\vec{a}$. **[NCERT]**

37. Express the vector $\vec{a} = 5\hat{i}-2\hat{j}+5\hat{k}$ as the sum of two vectors such that one is parallel to the vector $\vec{b} = 3\hat{i}+\hat{k}$ and other is perpendicular to $\vec{b}$. **[CBSE 2005]**

38. Express $2\hat{i}-\hat{j}+3\hat{k}$ as the sum of a vector parallel and a vector perpendicular to $2\hat{i}+4\hat{j}-2\hat{k}$.

39. Decompose the vector $6\hat{i}-3\hat{j}-6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{i}+\hat{j}+\hat{k}$.

BASED ON HOTS

40. If $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, then prove that it is perpendicular to both $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

41. If $|\vec{a}| = a$ and $|\vec{b}| = b$, prove that $\left(\frac{\vec{a}}{a^2}-\frac{\vec{b}}{b^2}\right)^2 = \left(\frac{\vec{a}-\vec{b}}{ab}\right)^2$.

42. If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors such that $\vec{d}\cdot\vec{a} = \vec{d}\cdot\vec{b} = \vec{d}\cdot\vec{c} = 0$, then show that $\vec{d}$ is the null vector.

43. If a vector $\vec{a}$ is perpendicular to two non-collinear vectors $\vec{b}$ and $\vec{c}$, then $\vec{a}$ is perpendicular to every vector in the plane of $\vec{b}$ and $\vec{c}$.

44. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, show that the angle θ between the vectors $\vec{b}$ and $\vec{c}$ is given by

$$\cos\theta = \frac{|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{b}||\vec{c}|}.$$

45. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vector such $\vec{u} + \vec{v} + \vec{w} = \vec{0}$. If $|\vec{u}| = 3, |\vec{v}| = 4$ and $|\vec{w}| = 5$, then find $\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{u}$. **[CBSE 2012]**

46. Let $\vec{a} = x^2\hat{i} + 2\hat{j} - 2\hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = x^2\hat{i} + 5\hat{j} - 4\hat{k}$ be three vectors. Find the values of x for which the angle between $\vec{a}$ and $\vec{b}$ is acute and the angle between $\vec{b}$ and $\vec{c}$ is obtuse.

47. Find the values of x and y if the vectors $\vec{a} = 3\hat{i} + x\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + y\hat{k}$ are mutually perpendicular vectors of equal magnitude.

48. If $\vec{a}$ and $\vec{b}$ are two non-collinear unit vectors such that $|\vec{a} + \vec{b}| = \sqrt{3}$, find $(2\vec{a} - 5\vec{b})\cdot(3\vec{a} + \vec{b})$.

49. If $\vec{a}, \vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{b}|$, then prove that $\vec{a} + 2\vec{b}$ is perpendicular to $\vec{a}$.

50. Let $\hat{a}$ and $\hat{b}$ be unit vectors. If the vectors $\vec{c} = \hat{a} + 2\hat{b}$ and $\vec{d} = 5\hat{a} - 4\hat{b}$ are perpendicular to each other, then find the angle between the vectors $\hat{a}$ and $\hat{b}$. **[CBSE 2020]**

ANSWERS

1. (i) 19 (ii) 2 (iii) 5

2. (i) $\lambda = 4$ (ii) $\lambda = \frac{16}{5}$ (iii) 3 (iv) -3 **3.** $\frac{\pi}{3}$ **4.** -9

5. (i) $\frac{2\pi}{3}$ (ii) $\cos^{-1}\left(\frac{-34}{63}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\cos^{-1}\left(\frac{-3}{\sqrt{84}}\right)$ (v) $\cos^{-1}\left(-\frac{\sqrt{2}}{3}\right)$

6. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{\pi}{4}$ **7.** (i) $\hat{i} + 2\hat{j} + \hat{k}$ (ii) $2\hat{i} - \hat{j} + \hat{k}$ **8.** 22 **12.** $\lambda = -2$

13. $\lambda = \pm 1$ **15.** 8 **17.** 2 **19.** 0 **20.** (i) 3 (ii) $\sqrt{13}$

21. (i) $|\vec{a}| = 4, |\vec{b}| = 2$ (ii) $|\vec{a}| = \frac{8\sqrt{8}}{\sqrt{63}}, |\vec{b}| = \sqrt{\frac{8}{63}}$ (iii) $|\vec{a}| = 2, |\vec{b}| = 1$

22. (i) $\sqrt{13}$ (ii) $\sqrt{23}$ (iii) $\sqrt{5}$ **23.** (i) $\frac{\pi}{4}$ (ii) $\cos^{-1}\left(\frac{1}{9}\right)$

24. $|\vec{a}| = |\vec{b}| = \sqrt{2\sqrt{3}}$ **25.** $\sqrt{73}$ **26.** $\vec{b}$ is any vector

30. $\vec{\beta}_2 = \frac{1}{5}(13\hat{i} + 9\hat{j} - 15\hat{k}), \vec{\beta}_1 = -\frac{1}{5}(3\hat{i} + 4\hat{j} + 5\hat{k})$ **32.** $\angle A = \frac{\pi}{4}, \angle B = \frac{\pi}{2}, \angle C = \frac{\pi}{4}$

33. $|\vec{a}| = |\vec{b}| = 1$ **35.** $\cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$ **36.** $\frac{\pi}{3}, \frac{1}{\sqrt{2}}\hat{i}, \frac{1}{2}\hat{j}, \frac{1}{2}\hat{k}$

37. $6\hat{i}+2\hat{k}, -\hat{i}-2\hat{j}+3\hat{k}$
38. $\left(-\frac{1}{2}\hat{i}-\hat{j}+\frac{1}{2}\hat{k}\right)+\frac{5}{2}(\hat{i}+\hat{k})$
39. $-\hat{i}-\hat{j}-\hat{k}, 7\hat{i}-2\hat{j}-5\hat{k}$
45. -25
46. $(-3,-2)\cup(2,3)$
47. $x=-\frac{31}{12}, y=\frac{41}{12}$
48. $-\frac{11}{2}$
50. $\frac{\pi}{3}$

HINTS TO SELECTED PROBLEMS

15. It is given that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$.

$$\therefore (\vec{a}+\lambda\vec{b})\cdot\vec{c}=0 \Rightarrow \vec{a}\cdot\vec{c}+\lambda(\vec{b}\cdot\vec{c})=0 \Rightarrow \lambda=-\frac{\vec{a}\cdot\vec{c}}{\vec{b}\cdot\vec{c}}=-\frac{6+2+3\times 0}{-3+2+1\times 0}=-\frac{8}{-1}=8$$

19. We have, $|\vec{a}|=2, |\vec{b}|=1$ and $\vec{a}\cdot\vec{b}=1$

$$\therefore \quad (3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})=6(\vec{a}\cdot\vec{a})+21(\vec{a}\cdot\vec{b})-10(\vec{b}\cdot\vec{a})-35(\vec{b}\cdot\vec{b})$$

$$=6|\vec{a}|^2+11(\vec{a}\cdot\vec{b})-35|\vec{b}|^2=6\times 4+11\times 1-35\times 1=0$$

20. (i) It is given that $\vec{a}$ is a unit vector such that

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=8 \Rightarrow |\vec{x}|^2-|\vec{a}|^2=8 \Rightarrow |\vec{x}|^2-1=8 \Rightarrow |\vec{x}|^2=9 \Rightarrow |\vec{x}|=3$$

(ii) It is given that $|\vec{a}|$ is a unit vector such that

$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12 \Rightarrow |\vec{x}|^2-|\vec{a}|^2=12 \Rightarrow |\vec{x}|^2-1=12 \Rightarrow |\vec{x}|=\sqrt{13}$$

21. (i) We have,

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=12 \text{ and } |\vec{a}|=2|\vec{b}|$$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=12 \text{ and } |\vec{a}|=2|\vec{b}|$$

$$\Rightarrow 4|\vec{b}|^2-|\vec{b}|^2=12 \Rightarrow 3|\vec{b}|^2=12 \Rightarrow |\vec{b}|=2 \qquad \left[\because |\vec{a}|=2|\vec{b}|\right]$$

$$\therefore \quad |\vec{a}|=2|\vec{b}| \Rightarrow |\vec{a}|=4$$

(ii) We have, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=8 \text{ and } |\vec{a}|=8|\vec{b}|$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8 \Rightarrow 63|\vec{b}|^2=8 \Rightarrow |\vec{b}|=\frac{2\sqrt{2}}{\sqrt{63}} \qquad \left[\because |\vec{a}|=8|\vec{b}|\right]$$

$$\therefore \quad |\vec{a}|=8|\vec{b}| \Rightarrow |\vec{a}|=\frac{16\sqrt{2}}{\sqrt{63}}$$

(iii) We have, $|\vec{a}|=2|\vec{b}|$ and, $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=3$

$$\Rightarrow |\vec{a}|^2-|\vec{b}|^2=3 \text{ and, } |\vec{a}|=2|\vec{b}| \Rightarrow 4|\vec{b}|^2-|\vec{b}|^2=3 \Rightarrow 3|\vec{b}|^2=3 \Rightarrow |\vec{b}|=1$$

$$\therefore \quad |\vec{a}|=2|\vec{b}| \Rightarrow |\vec{a}|=2$$

22. (i) We have, $|\vec{a}|=2, |\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=8$

$\therefore \quad |\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b}) = 4+25-2\times 8 = 13 \Rightarrow |\vec{a}-\vec{b}| = \sqrt{13}$

(ii) We have, $|\vec{a}| = 3, |\vec{b}| = 4$ and $\vec{a}\cdot\vec{b} = 1$

$\therefore \quad |\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b}) = 9+16-2\times 1 = 23 \Rightarrow |\vec{a}-\vec{b}| = \sqrt{23}$

(iii) We have, $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a}\cdot\vec{b} = 4$

$\therefore \quad |\vec{a}-\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2-2(\vec{a}\cdot\vec{b}) = 4+9-2\times 4 = 5 \Rightarrow |\vec{a}-\vec{b}| = \sqrt{5}$

23. (i) We have, $|\vec{a}|=\sqrt{3}, |\vec{b}| = 2$ and $\vec{a}\cdot\vec{b} = \sqrt{6}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\sqrt{6}}{\sqrt{3}\times 2} = \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

26. We have, $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$

$\Rightarrow \quad |\vec{a}|^2 = 0$ and $(\vec{a}\perp\vec{b}$ or, $\vec{a} = \vec{0}$ or, $\vec{b}=\vec{0}) \Rightarrow \vec{a}=\vec{0}$ and $\vec{b}$ can be any vector.

28. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that $\hat{a}+\hat{b}$ is also a unit vector. Then, $|\hat{a}|=1, |\hat{b}|=1$, $|\hat{a}+\hat{b}|=1$. We know that:

$$|\hat{a}+\hat{b}|^2+|\hat{a}-\hat{b}|^2=2\left(|\hat{a}|^2+|\hat{b}|^2\right)$$

$\Rightarrow \quad 1+|\hat{a}-\hat{b}|^2=2(1+1) \Rightarrow |\hat{a}-\hat{b}|^2=3 \Rightarrow |\hat{a}-\hat{b}|=\sqrt{3}$

31. For vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-\hat{k}$, we obtain that $\vec{a}\cdot\vec{b}=2-1-1=0$. But neither $\vec{a}\neq\vec{0}$ nor $\vec{b}\neq\vec{0}$.

33. We have, $\vec{a}\cdot\vec{b} = \frac{1}{2}, |\vec{a}|=|\vec{b}|$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$.

Now, $\vec{a}\cdot\vec{b}=\frac{1}{2} \Rightarrow |\vec{a}|\,|\vec{b}|\cos\frac{\pi}{3}=\frac{1}{2} \Rightarrow |\vec{a}|^2=1 \Rightarrow |\vec{a}|=1$. Hence, $|\vec{a}| = |\vec{b}| = 1$

40. It is given that $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore, $\vec{c}\cdot\vec{a}=0$ and $\vec{c}\cdot\vec{b}=0$.

Now, $\vec{c}\cdot(\vec{a}+\vec{b})=\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}=0+0=0 \Rightarrow \vec{c}\perp(\vec{a}+\vec{b})$ $\quad [\because \vec{c}\cdot\vec{a}=0=\vec{c}\cdot\vec{b}]$

Similarly, $\vec{c}$ is perpendicular to $(\vec{a}-\vec{b})$.

43. Let $\vec{r}$ be an arbitrary vector in the plane of $\vec{b}$ and $\vec{c}$. Then,

$\vec{r}=x\vec{b}+y\vec{c}$ for some scalars x, y.

$\Rightarrow \quad \vec{r}\cdot\vec{a}=(x\vec{b}+y\vec{c})\cdot\vec{a}=x(\vec{b}\cdot\vec{a})+y(\vec{c}\cdot\vec{a})=x(0)+y(0)=0 \quad [\because \vec{a}\perp\vec{b}$ and $\vec{a}\perp\vec{c}]$

$\Rightarrow \quad \vec{r}\perp\vec{a}$

44. We have, $\vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{b}+\vec{c}=-\vec{a}$

$\Rightarrow \quad (\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c})=(-\vec{a})\cdot(-\vec{a})$

$$\Rightarrow \quad |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\theta = |\vec{a}|^2 \Rightarrow \cos\theta = \frac{|\vec{a}|^2 - |\vec{b}|^2 - |\vec{c}|^2}{2|\vec{b}||\vec{c}|}$$

23.4 SOME GEOMETRICAL PROBLEMS BASED UPON DOT PRODUCT

In this section, we will prove various geometrical problems by using the concept of dot product.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *(Cosine Formulae) If a, b, c are the lengths of the sides opposite respectively to the angles A, B, C of a triangle ABC, show that*

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ (ii) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ac}$ (iii) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then, $|\vec{a}| = |\vec{BC}| = a$, $|\vec{b}| = |\vec{CA}| = b$ and $|\vec{c}| = |\vec{AB}| = c$.

(i) Using triangle law of addition of vectors in ΔABC, we obtain

$$\Rightarrow \quad \vec{BC} + \vec{CA} = \vec{BA} \Rightarrow \vec{BC} + \vec{CA} = -\vec{AB} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{b} + \vec{c} = -\vec{a} \Rightarrow |\vec{b} + \vec{c}| = |-\vec{a}|$$

$$\Rightarrow \quad |\vec{b} + \vec{c}| = |\vec{a}| \qquad [\because |-\vec{a}| = |\vec{a}|]$$

$$\Rightarrow \quad |\vec{b} + \vec{c}|^2 = |\vec{a}|^2 \qquad [\because \vec{x}\cdot\vec{x} = |\vec{x}|^2]$$

$$\Rightarrow \quad |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b}\cdot\vec{c} = |\vec{a}|^2$$

$$\Rightarrow \quad |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos(\pi - A) = |\vec{a}|^2$$

$$\Rightarrow \quad b^2 + c^2 - 2bc\cos A = a^2 \Rightarrow 2bc\cos A = b^2 + c^2 - a^2 \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

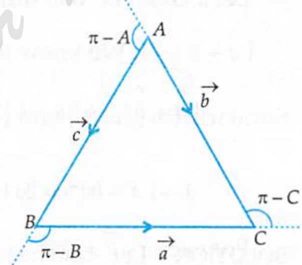

Fig. 23.16

(ii) Using triangle law of addition of vectors in ΔABC, we obtain

$$\vec{BC} + \vec{CA} = \vec{BA} \Rightarrow \vec{BC} + \vec{CA} = -\vec{AB} \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$$

$$\Rightarrow \quad \vec{a} + \vec{c} = -\vec{b} \Rightarrow |\vec{a} + \vec{c}| = |-\vec{b}| \Rightarrow |\vec{a} + \vec{c}| = |\vec{b}|$$

$$\Rightarrow \quad |\vec{a} + \vec{c}|^2 = |\vec{b}|^2 \Rightarrow |\vec{a}|^2 + |\vec{c}|^2 + 2\vec{a}\cdot\vec{c} = |\vec{b}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{c}|^2 + 2|\vec{a}||\vec{c}|\cos(\pi - B) = |\vec{b}|^2 \Rightarrow a^2 + c^2 + 2ac(-\cos B) = b^2$$

$$\Rightarrow \quad 2ac\cos B = a^2 + c^2 - b^2 \Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

(iii) Using triangle law of addition of vectors in ΔABC, we obtain

$$\vec{BC} + \vec{CA} = \vec{BA} \Rightarrow \vec{BC} + \vec{CA} = -\vec{AB} \Rightarrow \vec{a} + \vec{b} = -\vec{c}$$

$$\Rightarrow \quad |\vec{a} + \vec{b}| = |-\vec{c}| \Rightarrow |\vec{a} + \vec{b}| = |\vec{c}| \Rightarrow |\vec{a} + \vec{b}|^2 = |\vec{c}|^2$$

$$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b} = |\vec{c}|^2 \Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|\cos(\pi - C) = |\vec{c}|^2$$

$$\Rightarrow \quad a^2 + b^2 - 2\,ab\cos C = c^2 \Rightarrow 2ab\cos C = a^2 + b^2 - c^2 \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

EXAMPLE 2 *(Projection Formulae) If a, b, c are the lengths of the sides opposite respectively to the angles A, B, C of a triangle ABC, show that*

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then,

$$|\vec{a}| = |\vec{BC}| = a,\ |\vec{b}| = |\vec{CA}| = b \text{ and, } |\vec{c}| = |\vec{AB}| = c.$$

(i) Using triangle law of addition of vectors, we obtain

$$\vec{BC} + \vec{CA} = \vec{BA} \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{a} = -(\vec{b} + \vec{c})$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a} = -\vec{a}\cdot(\vec{b} + \vec{c}) \quad [\text{Taking producton both sides by } \vec{a}]$$

$$\Rightarrow \quad \vec{a}\cdot\vec{a} = -(\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c})$$

$$\Rightarrow \quad |\vec{a}|^2 = -\left\{|\vec{a}||\vec{b}|\cos(\pi - C) + |\vec{a}||\vec{c}|\cos(\pi - B)\right\}$$

$$\Rightarrow \quad a^2 = -(-ab\cos C - ac\cos B) \Rightarrow a = b\cos C + c\cos B$$

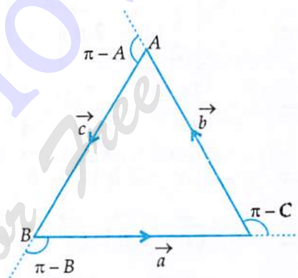

Fig. 23.17

Similarly results (ii) and (iii) can be proved.

EXAMPLE 3 *Prove using vectors: The median to the base of an isosceles triangle is perpendicular to the base.*

SOLUTION Let ABC be an isosceles triangle with $AB = AC$ and let D be the mid-point of BC. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, Position vector of $D = \dfrac{\vec{b} + \vec{c}}{2}$, $\vec{AB} = \vec{b}$ and, $\vec{AC} = \vec{c}$.

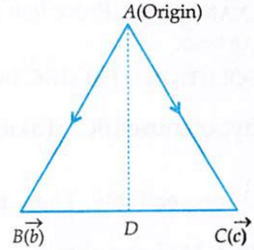

Fig. 23.18

Now,

$$\vec{AD} = \text{Position vector of } D - \text{Position vector of } A = \frac{\vec{b} + \vec{c}}{2} - \vec{0}$$

$$= \frac{1}{2}(\vec{b} + \vec{c})$$

and, $\vec{BC}$ = Position vector of C – Position vector of $B = \vec{c} - \vec{b}$

$$\therefore \quad \vec{AD}\cdot\vec{BC} = \frac{1}{2}(\vec{b} + \vec{c})\cdot(\vec{c} - \vec{b}) = \frac{1}{2}\left\{(\vec{c} + \vec{b})\cdot(\vec{c} - \vec{b})\right\} = \frac{1}{2}\left(|\vec{c}|^2 - |\vec{b}|^2\right)$$

$$\Rightarrow \quad \vec{AD}\cdot\vec{BC} = \frac{1}{2}(AC^2 - AB^2) = \frac{1}{2}(AC^2 - AC^2) = 0 \qquad [\because AB = AC]$$

$$\Rightarrow \quad \vec{AD} \perp \vec{BC} \Rightarrow AD \perp BC$$

Hence, the median AD is perpendicular to the base BC of ΔABC.

EXAMPLE 4 *Prove using vectors: If two medians of a triangle are equal, then it is isosceles.*

SOLUTION Let ABC be a triangle and let BE and CF be its equal medians. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of E and F are $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

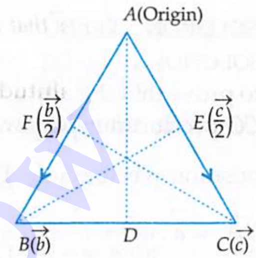

Fig. 23.19

$$\therefore \quad \vec{BE} = \text{P.V. of } E - \text{P.V. of } B = \frac{1}{2}\vec{c} - \vec{b} = \frac{1}{2}(\vec{c} - 2\vec{b}),$$

$$\vec{CF} = \text{P.V. of F} - \text{P.V. of C} = \frac{1}{2}\vec{b} - \vec{c} = \frac{1}{2}(\vec{b} - 2\vec{c})$$

It is given that $BE = CF$

$$\Rightarrow \quad |\vec{BE}| = |\vec{CF}| \Rightarrow |\vec{BE}|^2 = |\vec{CF}|^2$$

$$\Rightarrow \quad \left|\frac{1}{2}(\vec{c} - 2\vec{b})\right|^2 = \left|\frac{1}{2}(\vec{b} - 2\vec{c})\right|^2$$

$$\Rightarrow \quad \frac{1}{4}|\vec{c} - 2\vec{b}|^2 = \frac{1}{4}|\vec{b} - 2\vec{c}|^2$$

$$\Rightarrow \quad |\vec{c} - 2\vec{b}|^2 = |\vec{b} - 2\vec{c}|^2$$

$$\Rightarrow \quad (\vec{c} - 2\vec{b})\cdot(\vec{c} - 2\vec{b}) = (\vec{b} - 2\vec{c})\cdot(\vec{b} - 2\vec{c})$$

$$\Rightarrow \quad \vec{c}\cdot\vec{c} - 4(\vec{b}\cdot\vec{c}) + 4(\vec{b}\cdot\vec{b}) = (\vec{b}\cdot\vec{b}) - 4(\vec{b}\cdot\vec{c}) + 4(\vec{c}\cdot\vec{c})$$

$$\Rightarrow \quad |\vec{c}|^2 - 4(\vec{b}\cdot\vec{c}) + 4|\vec{b}|^2 = |\vec{b}|^2 - 4(\vec{b}\cdot\vec{c}) + 4|\vec{c}|^2$$

$$\Rightarrow \quad 3|\vec{b}|^2 = 3|\vec{c}|^2 \Rightarrow |\vec{b}|^2 = |\vec{c}|^2 \Rightarrow |\vec{b}| = |\vec{c}| \Rightarrow AB = AC$$

Hence, triangle ABC is an isosceles triangle.

EXAMPLE 5 *Prove that the mid-point of the hypotenuse of a right angled triangle is equidistant from its vertices.*

SOLUTION Let ABC be a right angled triangle, right angled at A and let D be the mid-point of hypotenuse BC. Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vector of D is $\frac{\vec{b} + \vec{c}}{2}$.

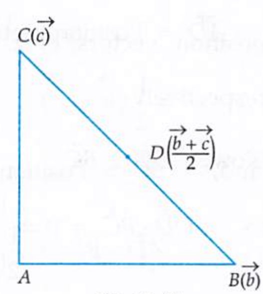

Fig. 23.20

In ΔABC, we obtain

$$AB \perp AC \Rightarrow \vec{AB} \perp \vec{AC} \Rightarrow \vec{b} \perp \vec{c} \Rightarrow \vec{b}\cdot\vec{c} = 0 \quad \text{...(i)}$$

Now,

$$AD^2 = |\vec{AD}|^2$$

$$\Rightarrow \quad AD^2 = \vec{AD}\cdot\vec{AD} = \left(\frac{\vec{b} + \vec{c}}{2}\right)\cdot\left(\frac{\vec{b} + \vec{c}}{2}\right)$$

$$\Rightarrow \quad AD^2 = \frac{1}{4}\left\{|\vec{b}|^2 + |\vec{c}|^2 + 2\,\vec{b}\cdot\vec{c}\right\} = \frac{1}{4}\left\{|\vec{b}|^2 + |\vec{c}|^2\right\} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad AD^2 = \frac{1}{4}(AB^2 + AC^2) = \frac{1}{4}BC^2 \quad [\because \Delta ABC \text{ is a right triangle}]$$

$\therefore \quad AD = \frac{1}{2} BC$ ...(ii)

Also, D is the mid-point of BC. Therefore, $BD = DC = \frac{1}{2} BC$. ...(iii)

From (ii) and (iii), we obtain $AD = BD = DC$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Prove that the altitudes of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let AD, BE be its two altitudes intersecting at O. In order to prove that the altitudes are concurrent, it is sufficient to prove that CO produced is perpendicular to AB. Taking O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then, $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{OC} = \vec{c}$.

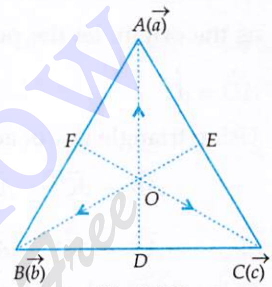

Fig. 23.21

Now,

$$AD \perp BC \Rightarrow \vec{OA} \perp \vec{BC} \Rightarrow \vec{OA} \cdot \vec{BC} = \vec{0}$$

$$\Rightarrow \vec{a} \cdot (\vec{c} - \vec{b}) = 0 \Rightarrow \vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b} = 0 \quad \text{...(i)}$$

and,

$$BE \perp CA \Rightarrow \vec{OB} \perp \vec{CA} \Rightarrow \vec{OB} \cdot \vec{CA} = 0 \Rightarrow \vec{b} \cdot (\vec{a} - \vec{c}) = 0 \Rightarrow \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{c} = 0 \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c} = 0 \Rightarrow (\vec{a} - \vec{b}) \cdot \vec{c} = 0 \Rightarrow \vec{BA} \cdot \vec{OC} = 0 \Rightarrow \vec{OC} \perp \vec{AB} \Rightarrow CF \perp AB.$$

Hence, the three altitudes AD, BE and CF are concurrent.

EXAMPLE 7 *Prove that the perpendicular bisectors of the sides of a triangle are concurrent.*

SOLUTION Let ABC be a triangle and let DO and EO be perpendicular bisectors of BC and CA intersecting at O. Join O to the mid-point F of AB. In order to prove that the perpendicular bisectors are concurrent, it is sufficient to prove that OF is perpendicular to AB. Taking O as the origin, let the position vectors of A, B, C be $\vec{a}, \vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of D, E and F are $\frac{\vec{b} + \vec{c}}{2}$, $\frac{\vec{c} + \vec{a}}{2}$ and $\frac{\vec{a} + \vec{b}}{2}$ respectively.

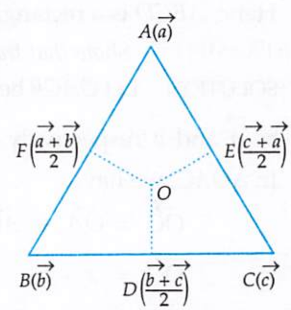

Fig. 23.22

Now, $\vec{OD} \perp \vec{BC}$

$$\Rightarrow \quad \vec{OD} \cdot \vec{BC} = 0 \Rightarrow \left(\frac{\vec{b} + \vec{c}}{2}\right) \cdot (\vec{c} - \vec{b}) = 0 \Rightarrow (\vec{b} + \vec{c}) \cdot (\vec{c} - \vec{b}) = 0 \Rightarrow |\vec{b}|^2 - |\vec{c}|^2 = 0 \text{ ...(i)}$$

and, $\vec{OE} \perp \vec{CA}$

$$\Rightarrow \quad \vec{OE} \cdot \vec{CA} = 0 \Rightarrow \left(\frac{\vec{c} + \vec{a}}{2}\right) \cdot (\vec{a} - \vec{c}) = 0 \Rightarrow (\vec{c} + \vec{a}) \cdot (\vec{a} - \vec{c}) = 0 \Rightarrow |\vec{c}|^2 - |\vec{a}|^2 = 0 \text{ ...(ii)}$$

Adding (i) and (ii), we get

$$|\vec{a}|^2 - |\vec{b}|^2 = 0 \Rightarrow (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0 \Rightarrow \frac{1}{2}(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0 \Rightarrow \vec{OF}\cdot\vec{BA} = 0$$

$$\Rightarrow \quad \vec{OF} \perp \vec{BA} \Rightarrow OF \perp AB.$$

Hence, the perpendicular bisectors of the sides of a triangle are concurrent.

EXAMPLE 8 *Prove using vectors : If the diagonals of a parallelogram are equal in length, then it is a rectangle.*

SOLUTION Let $ABCD$ be a parallelogram such that its diagonals AC and BD are equal. Taking A as the origin, let the position vectors of B and D be $\vec{b}$ and $\vec{d}$ respectively. Then, $\vec{AB} = \vec{b}$ and $\vec{AD} = \vec{d}$.

Using triangle law of addition of vectors in ΔABC, we obtain

$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\Rightarrow \quad \vec{AB} + \vec{AD} = \vec{AC} \Rightarrow \vec{b} + \vec{d} = \vec{AC} \quad [\because \vec{BC} = \vec{AD}]$$

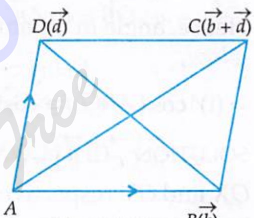

Fig. 23.23

Using triangle law of addition of vectors in ΔABD, we obtain

$$\vec{AB} + \vec{BD} = \vec{AD} \Rightarrow \vec{b} + \vec{BD} = \vec{d} \Rightarrow \vec{BD} = \vec{d} - \vec{b}$$

In parallelogram $ABCD$, it is given that

$$AC = BD \Rightarrow |\vec{AC}| = |\vec{BD}| \Rightarrow |\vec{AC}|^2 = |\vec{BD}|^2 \Rightarrow |\vec{b}+\vec{d}|^2 = |\vec{d}-\vec{b}|^2$$

$$\Rightarrow \quad |\vec{b}|^2 + |\vec{d}|^2 + 2(\vec{b}\cdot\vec{d}) = |\vec{d}|^2 + |\vec{b}|^2 - 2(\vec{b}\cdot\vec{d})$$

$$\Rightarrow \quad 4(\vec{b}\cdot\vec{d}) = 0 \Rightarrow \vec{b}\cdot\vec{d} = 0 \Rightarrow \vec{b} \perp \vec{d} \Rightarrow \vec{AB} \perp \vec{AD}$$

Henc, $ABCD$ is a rectangle.

EXAMPLE 9 *Show that the diagonals of a rhombus bisect each other at right angles.*

SOLUTION Let $OACB$ be a rhombus. Taking O as the origin, let the position vectors of A and B be $\vec{a}$ and $\vec{b}$ respectively. Then, $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

In ΔOAC, we have

$$\vec{OC} = \vec{OA} + \vec{AC}$$

$$\Rightarrow \quad \vec{OC} = \vec{a} + \vec{b} \qquad [\because \vec{AC} = \vec{OB} = \vec{b}]$$

So, the position vector of C is $\vec{a} + \vec{b}$.

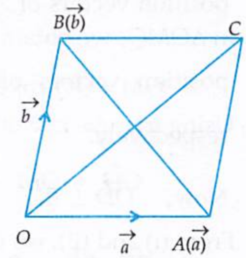

Fig. 23.24

$\therefore$ Position vector of the mid-point of OC is $\dfrac{\vec{a}+\vec{b}}{2}$.

Similarly, the position vector of mid-point of AB is $\dfrac{\vec{a}+\vec{b}}{2}$. Hence, the mid-point of OC coincides with the mid-point of AB.

Now, $\vec{OC}\cdot\vec{AB} = (\vec{a}+\vec{b})\cdot(\vec{b}-\vec{a}) = |\vec{b}|^2 - |\vec{a}|^2 = OB^2 - OA^2 = 0 \qquad [\because OB = OA]$

$$\Rightarrow \quad \vec{OC} \perp \vec{AB}.$$

Hence, the diagonals of a rhombus bisect each other at right angles.

EXAMPLE 10 *Using vector method, prove that the angle in a semi-circle is a right angle.*

SOLUTION Let O be the centre of the semi-circle and AA' be the diameter. Let P be any point on the circumference of the semi-circle.

Taking O as the origin, let the position vectors of A and P be $\vec{a}$ and $\vec{r}$ respectively. Then, the position vector of A' is $-\vec{a}$.

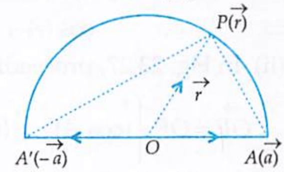

Fig. 23.25

Now,

$$\vec{AP} = (\text{Position vector of } P) - (\text{Position vector of } A) = \vec{r} - \vec{a}$$

and, $$\vec{A'P} = (\text{Position vector of } P) - (\text{Position vector of } A') = \vec{r} + \vec{a}$$

$$\therefore \quad \vec{AP} \cdot \vec{A'P} = (\vec{r} - \vec{a}) \cdot (\vec{r} + \vec{a}) = |\vec{r}|^2 - |\vec{a}|^2 = OP^2 - OA^2 = (\text{Radius})^2 - (\text{Radius})^2 = 0$$

$$\Rightarrow \quad \vec{AP} \perp \vec{A'P} \Rightarrow \angle APA' = \frac{\pi}{2}$$

Hence, angle in a semi-circle is a right angle.

EXAMPLE 11 *Using vectors: Prove that*

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$ (ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$

SOLUTION (i) Let OX and OY be the coordinate axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively.

Let $\angle XOP = A$ and $\angle XOQ = B$. Draw $PL \perp OX$ and $QM \perp OX$.

Clearly, angle between $\vec{OP}$ and $\vec{OQ}$ is $(A+B)$.

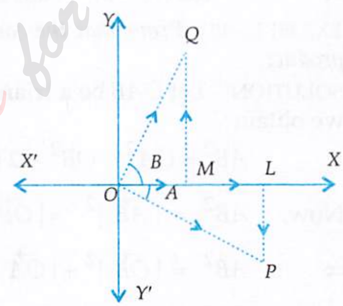

Fig. 23.26

In ΔOLP, we obtain

$$OL = OP \cos A \text{ and, } LP = OP \sin A.$$

$$\Rightarrow \quad \vec{OL} = (OP \cos A)\,\hat{i} \text{ and, } \vec{LP} = (OP \sin A)(-\hat{j}).$$

Using triangle law of addition of vectors in ΔOLP, we obtain

$$\vec{OL} + \vec{LP} = \vec{OP}$$

$$\Rightarrow \quad \vec{OP} = (OP \cos A)\,\hat{i} + (OP \sin A)(-\hat{j})$$

$$\Rightarrow \quad \vec{OP} = OP\left\{(\cos A)\,\hat{i} - (\sin A)\,\hat{j}\right\} \quad \text{...(i)}$$

In ΔOMQ, we obtain

$$OM = OQ \cos B \text{ and, } MQ = OQ \sin B. \Rightarrow \vec{OM} = (OQ \cos B)\,\hat{i} \text{ and, } \vec{MQ} = (OQ \sin B)\,\hat{j}$$

Using triangle law of addition of vectors in ΔOMQ, we obtain

$$\vec{OQ} = \vec{OM} + \vec{MQ} = (OQ \cos B)\,\hat{i} + (OQ \sin B)\,\hat{j} = OQ\left\{(\cos B)\,\hat{i} + (\sin B)\,\hat{j}\right\} \quad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\vec{OP} \cdot \vec{OQ} = OP\left\{(\cos A)\,\hat{i} - (\sin A)\,\hat{j}\right\} \cdot OQ\left\{(\cos B)\,\hat{i} + (\sin B)\,\hat{j}\right\}$$

$$\Rightarrow \quad \vec{OP} \cdot \vec{OQ} = OP \cdot OQ\,\{\cos A \cos B - \sin A \sin B\} \quad \text{...(iii)}$$

Using the definition of dot product, we obtain

$$\vec{OP} \cdot \vec{OQ} = |\vec{OP}|\,|\vec{OQ}| \cos(A+B) = OP \cdot OQ \cos(A+B) \quad \text{...(iv)}$$

From (iii) and (iv), we obtain

$\therefore \quad OP \cdot OQ \cos(A+B) = OP \cdot OQ \{\cos A \cos B - \sin A \sin B\}$

$\Rightarrow \quad \cos(A+B) = \cos A \cos B - \sin A \sin B$

(ii) In Fig. 23.27, proceeding as in (i), we obtain

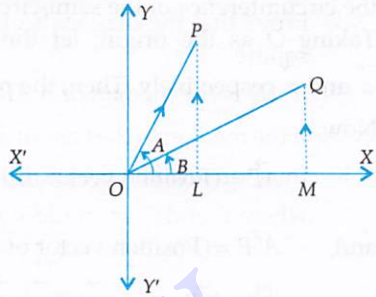

Fig. 23.27

$$\vec{OP} = OP\left\{(\cos A)\hat{i} + (\sin A)\hat{j}\right\}$$

and, $\vec{OQ} = OQ\left\{(\cos B)\hat{i} + (\sin B)\hat{j}\right\}$

$$\therefore \vec{OP} \cdot \vec{OQ} = OP\left\{(\cos A)\hat{i} + (\sin A)\hat{j}\right\} \cdot OQ\left\{(\cos B)\hat{i} + (\sin B)\hat{j}\right\}$$

$$\Rightarrow \vec{OP} \cdot \vec{OQ} = OP \cdot OQ\{\cos A \cos B + \sin A \sin B\} \qquad \text{...(i)}$$

Using the definition of dot product, we obtain

$$\vec{OP} \cdot \vec{OQ} = |\vec{OP}||\vec{OQ}|\cos(A-B) = OP \cdot OQ \cos(A-B) \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$OP \cdot OQ \cos(A-B) = OP \cdot OQ\{\cos A \cos B + \sin A \sin B\}$

$\Rightarrow \quad \cos(A-B) = \cos A \cos B + \sin A \sin B.$

EXAMPLE 12 *Prove that the cosine formula for triangles is equivalent to the definition of the scalar product.*

SOLUTION Let OAB be a triangle such that $\angle AOB = \theta$. Applying cosine formula in ΔOAB, we obtain

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta \qquad \text{...(i)}$$

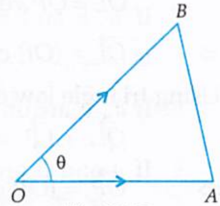

Fig. 23.28

Now, $\quad AB^2 = |\vec{AB}|^2 = |\vec{OB} - \vec{OA}|^2$

$\Rightarrow \quad AB^2 = |\vec{OB}|^2 + |\vec{OA}|^2 - 2\vec{OB} \cdot \vec{OA}$

$\Rightarrow \quad AB^2 = OB^2 + OA^2 - 2\vec{OB} \cdot \vec{OA} \qquad \text{...(ii)}$

From (i) and (ii), we obtain

$$OA^2 + OB^2 - 2(OA)\cdot(OB)\cos\theta = OB^2 + OA^2 - 2\vec{OB} \cdot \vec{OA}$$

$\Rightarrow \quad \vec{OB} \cdot \vec{OA} = (OB)(OA)\cos\theta,$ which is the definition of scalar product.

EXERCISE 23.2

BASED ON LOTS

1. In a triangle OAB, $\angle AOB = 90°$. If P and Q are points of trisection of AB, prove that $OP^2 + OQ^2 = \frac{5}{9}AB^2$.
2. Prove that: If the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.
3. (Pythagoras's Theorem) Prove by vector method that in a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
4. Prove by vector method that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.
5. Prove using vectors: The quadrilateral obtained by joining mid-points of adjacent sides of a rectangle is a rhombus.

BASED ON HOTS

6. Prove that the diagonals of a rhombus are perpendicular bisectors of each other.
7. Prove that the diagonals of a rectangle are perpendicular if and only if the rectangle is a square.
8. If AD is the median of $\Delta\, ABC$, using vectors, prove that $AB^2 + AC^2 = 2\,(AD^2 + CD^2)$.
9. If the median to the base of a triangle is perpendicular to the base, then triangle is isosceles.
10. In a quadrilateral $ABCD$, prove that $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4\, PQ^2$, where P and Q are middle points of diagonals AC and BD.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|\vec{a}+\vec{b}|$ is also a unit vector, then $|\vec{a}-\vec{b}|$ ....
2. If $|\vec{a}| = 3, |\vec{b}| = 4$ and $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{a}-\lambda\vec{b}$, then $\lambda =$ ....
3. If $\vec{a} = 2\hat{i}-7\hat{j}+\hat{k}$, $\vec{b} = \hat{i}+3\hat{j}-5\hat{k}$ and $\vec{a}\,.\, m\vec{b} = 120$, then $m =$ ....
4. The non-zero vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are related by $\vec{a} = 8\vec{b}$ and $\vec{c} = -7\vec{b}$, then the angle between $\vec{a}$ and $\vec{c}$ is ....
5. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, then $|\vec{a}+\vec{b}| =$ ....
6. If the angle between the vectors $\hat{i}+\hat{k}$ and $\hat{i}-\hat{j}+\alpha\hat{k}$ is $\frac{\pi}{3}$, then $\alpha =$ ....
7. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}-\vec{c} = \vec{0}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....
8. If $\vec{a}$, $\vec{b}$ are unit vectors such that $|\vec{a}+\vec{b}| = \sqrt{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....
9. If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then the projection of $\vec{a}$ on $\vec{b}$ is ....
10. Let $\vec{a}$, $\vec{b}$ be unit vectors such that $\vec{a}-\sqrt{2}\,\vec{b}$ is also a unit vector. Then the angle between $\vec{a}$ and $\vec{b}$ is ....
11. If $|\vec{a}| = 1, |\vec{b}| = 3$ and $|\vec{a}-\vec{b}| = \sqrt{7}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....
12. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three mutually perpendicular vectors then $|\vec{a}+\vec{b}+\vec{c}| =$ ....
13. If $\vec{a}$, $\vec{b}$ are non-zero vectors such that $\vec{a}\,.\,\vec{b} = -|\vec{a}|\ \ |\vec{b}|$, then the angle between $\vec{a}$ and $\vec{b}$ is ....
14. For any vector $\vec{r}$, $(\vec{r}\,.\hat{i})^2 + (\vec{r}\,.\hat{j})^2 + (\vec{r}\,.\hat{k})^2 =$ ....
15. If $\vec{a}$, $\vec{b}$ are non-zero vectors of same magnitude such that the angle between $\vec{a}$ and $\vec{b}$ is $\frac{2\pi}{3}$ and $\vec{a}\,.\,\vec{b} = -8$, then $|\vec{a}| =$ ....
16. For any non-zero vectors $\vec{r}$, the expression $(\vec{r}\,.\,\hat{i})\,\hat{i} + (\vec{r}\,.\,\hat{j})\,\hat{j} + (\vec{r}\,.\,\hat{k})\,\hat{k}$ equals ....

17. The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The acture angle between its diagonals is .... **[NCERT EXEMPLAR]**

18. If $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$, then $\dfrac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}}$ =......... **[CBSE 2020]**

ANSWERS

1. $\sqrt{3}$ 2. $\pm\dfrac{3}{4}$ 3. -5 4. π 5. $\sqrt{2}$ 6. 0

7. $\dfrac{2\pi}{3}$ 8. $\dfrac{\pi}{3}$ 9. $\vec{a}\cdot\vec{b}$ 10. $\dfrac{\pi}{4}$ 11. $\dfrac{\pi}{3}$

12. $\sqrt{a^2+b^2+c^2}$ 13. π 14. $|\vec{r}|^2$ 15. 4

16. $\vec{r}$ 17. $\dfrac{\pi}{4}$ 18. $\dfrac{3}{5\sqrt{2}}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. What is the angle between vectors $\vec{a}$ and $\vec{b}$ with magnitudes 2 and $\sqrt{3}$ respectively? Given $\vec{a}\cdot\vec{b} = \sqrt{3}$. **[CBSE 2008]**

2. If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}\cdot\vec{b} = 6, |\vec{a}| = 3$ and $|\vec{b}| = 4$. Write the projection of $\vec{a}$ on $\vec{b}$.

3. Find the cosine of the angle between the vectors $4\hat{i} - 3\hat{j} + 3\hat{k}$ and $2\hat{i} - \hat{j} - \hat{k}$.

4. If the vectors $3\hat{i} + m\hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} - 8\hat{k}$ are orthogonal, find m.

5. If the vectors $3\hat{i} - 2\hat{j} - 4\hat{k}$ and $18\hat{i} - 12\hat{j} - m\hat{k}$ are parallel, find the value of m.

6. If $\vec{a}$ and $\vec{b}$ are vectors of equal magnitude, write the value of $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})$.

7. If $\vec{a}$ and $\vec{b}$ are two vectors such that $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 0$, find the relation between the magnitudes of $\vec{a}$ and $\vec{b}$.

8. For any two vector $\vec{a}$ and $\vec{b}$ write when $|\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$ holds.

9. For any two vectors $\vec{a}$ and $\vec{b}$ write when $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$ holds.

10. If $\vec{a}$ and $\vec{b}$ are two vectors of the same magnitude inclined at an angle of 60° such that $\vec{a}\cdot\vec{b} = 8$, write the value of their magnitude.

11. If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, what can you conclude about the vector $\vec{b}$?

12. If $\vec{b}$ is a unit vector such that $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = 8$, find $|\vec{a}|$.

13. If $\hat{a}, \hat{b}$ are unit vectors such that $\hat{a}+\hat{b}$ is a unit vector, write the value of $|\hat{a}-\hat{b}|$.

14. If $|\vec{a}| = 2, |\vec{b}| = 5$ and $\vec{a} \cdot \vec{b} = 2$, find $|\vec{a} - \vec{b}|$.

15. If $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = -\hat{j} + \hat{k}$, find the projection of $\vec{a}$ on $\vec{b}$.

16. For any two non-zero vectors, write the value of $\dfrac{|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2}{|\vec{a}|^2 + |\vec{b}|^2}$.

17. Write the projections of $\vec{r} = 3\hat{i} - 4\hat{j} + 12\hat{k}$ on the coordinate axes.

18. Write the component of $\vec{b}$ along $\vec{a}$.

19. Write the value of $(\vec{a} \cdot \hat{i})\,\hat{i} + (\vec{a} \cdot \hat{j})\,\hat{j} + (\vec{a} \cdot \hat{k})\,\hat{k}$, where $\vec{a}$ is any vector.

20. Find the value of $\theta \in (0, \pi/2)$ for which vectors $\vec{a} = (\sin\theta)\,\hat{i} + (\cos\theta)\,\hat{j}$ and $\vec{b} = \hat{i} - \sqrt{3}\,\hat{j} + 2\hat{k}$ are perpendicular.

21. Write the projection of $\hat{i} + \hat{j} + \hat{k}$ along the vector $\hat{j}$.

22. Write a vector satisfying $\vec{a} \cdot \hat{i} = \vec{a} \cdot (\hat{i} + \hat{j}) = \vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$.

23. If $\vec{a}$ and $\vec{b}$ are unit vectors, find the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

24. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, write the value of $|\vec{a} + \vec{b}|$.

25. If $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors, write the value of $|\vec{a} + \vec{b} + \vec{c}|$.

26. Find the angle between the vectors $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$. [CBSE 2008]

27. For what value of λ are the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other? [CBSE 2008]

28. Find the projection of $\vec{a}$ on $\vec{b}$, if $\vec{a} \cdot \vec{b} = 8$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$. [CBSE 2009]

29. Write the value of p for which $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are parallel vectors. [CBSE 2009]

30. Find the value of λ if the vectors $2\hat{i} + \lambda\hat{j} + 3\hat{k}$ and $3\hat{i} + 2\hat{j} - 4\hat{k}$ are perpendicular to each other. [CBSE 2010]

31. If $|\vec{a}| = 2, |\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 3$, find the projection of $\vec{b}$ on $\vec{a}$. [CBSE 2010]

32. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2 respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$. [CBSE 2011]

33. Write the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $2\hat{i} - 3\hat{j} + 6\hat{k}$.

34. Find λ, when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units. [CBSE 2012]

35. For what value of λ are the vectores $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ perpendicular to each other? [CBSE 2012]

36. Write the projection of the vector $7\hat{i} + \hat{j} - 4\hat{k}$ on the vector $2\hat{i} + 6\hat{j} + 3\hat{k}$. [CBSE 2013, 14, 15]

37. Write the value of λ so that the vectors $\vec{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular to each other. **[CBSE 2013]**

38. Write the projection of $\vec{b} + \vec{c}$ on $\vec{a}$, when $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. **[CBSE 2013]**

39. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a} + \vec{b}| = 3$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. **[CBSE 2014]**

40. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2014]**

41. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + \vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2014]**

42. If $\vec{a}$ and $\vec{b}$ are unit vectors, then find the angle between $\vec{a}$ and $\vec{b}$, given that $(\sqrt{3}\,\vec{a} - \vec{b})$ is a unit vector. **[CBSE 2014]**

43. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is $60°$ and their scalar product is $\frac{9}{2}$. **[CBSE 2018]**

44. If $\vec{a}$ and $\vec{b}$ are two units vectors such that $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2022]**

45. If $\vec{a} = 2\hat{i} + y\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are two vectors for which the vector $\vec{a} + \vec{b}$ is perpendicular to the vector $\vec{a} - \vec{b}$, then find all possible value of y. **[CBSE 2022]**

ANSWERS

1. $\frac{\pi}{3}$
2. 3
3. $\frac{4}{\sqrt{51}}$
4. -2
5. -24
6. 0
7. $|\vec{a}| = |\vec{b}|$
8. $\vec{a}$ and $\vec{b}$ are parallel
9. $\vec{a}$ and $\vec{b}$ are perpendicular
10. 4
11. $\vec{b}$ is any non-zero vector
12. 3
13. $\sqrt{3}$
14. 5
15. $\frac{1}{\sqrt{2}}$
16. 2
17. $3, -4, 12$
18. $\frac{(\vec{a}\cdot\vec{b})}{|\vec{a}|^2}\vec{a}$
19. $\vec{a}$
20. $\theta = \frac{\pi}{3}$
21. 1
22. $\hat{i}$
23. $\frac{\pi}{2}$
24. $\sqrt{2}$
25. $\sqrt{3}$
26. $\cos^{-1}\left(\frac{-1}{3}\right)$
27. $\frac{5}{2}$
28. $\frac{8}{7}$
29. $\frac{2}{3}$
30. 3
31. $\frac{3}{2}$
32. $\frac{\pi}{4}$
33. 5
34. $\lambda = 5$
35. $\lambda = \frac{5}{2}$
36. $\frac{8}{7}$
37. $\lambda = \frac{5}{2}$
38. 2
39. 12
40. $\frac{\pi}{6}$
41. $\frac{2\pi}{3}$
42. $\frac{\pi}{6}$
43. $|\vec{a}| = |\vec{b}| = 3$
44. $\frac{\pi}{2}$
45. ± 3