CHAPTER 24

VECTOR OR CROSS PRODUCT

24.1 VECTOR (OR CROSS) PRODUCT

DEFINITION *Let $\vec{a}, \vec{b}$ be two non-zero non-parallel vectors. Then the vector product $\vec{a} \times \vec{b}$, in that order, is defined as a vector whose magnitude is $|\vec{a}|\,|\vec{b}| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$ and whose direction is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in such a way that $\vec{a}, \vec{b}$ and this direction constitute a right handed system.*

In other words,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta},$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

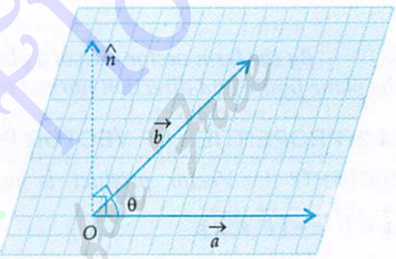

Fig. 24.1

When we say that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system it means that if we rotate vector $\vec{a}$ into the vector $\vec{b}$, then $\hat{\eta}$ will point in the direction perpendicular to the plane of $\vec{a}$ and $\vec{b}$ in which a right handed screw will move if it is turned in the same manner.

NOTE 1 *If one of $\vec{a}$ or $\vec{b}$ or both is $\vec{0}$, then θ is not defined as $\vec{0}$ has no direction and so $\hat{\eta}$ is not defined. In this case, we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 2 *If $\vec{a}$ and $\vec{b}$ are collinear i.e. if $\theta = 0$ or π, then the direction of $\hat{\eta}$ is not well defined. So in this case also we define $\vec{a} \times \vec{b} = \vec{0}$.*

NOTE 3 *$\vec{a} \times \vec{b}$ is read as $\vec{a}$ cross $\vec{b}$. Since we are putting cross between $\vec{a}$ and $\vec{b}$ that is why it is called cross-product. As the resulting quantity is a vector so it is also known as the vector product.*

24.1.1 GEOMETRICAL INTERPRETATION OF VECTOR PRODUCT

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors represented by $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively and let θ be the angle between them. Complete the parallelogram $OACB$. Draw $BL \perp OA$.

In ΔOBL, we have

$$\sin\theta = \frac{BL}{OB} \Rightarrow BL = OB \sin\theta = |\vec{b}| \sin\theta \quad \text{...(i)}$$

Now,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (OA)(BL)\, \hat{\eta} \quad \text{[Using (i)]}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (\text{Base} \times \text{height})\, \hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (\text{Area of parallelogram } OACB)\, \hat{\eta}$$

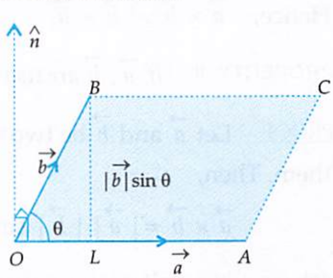

Fig. 24.2

$\Rightarrow \qquad \vec{a} \times \vec{b}$ = Vector area of the parallelogram $OACB$.

Thus, $\vec{a} \times \vec{b}$ *is a vector whose magnitude is equal to the area of the parallelogram having* $\vec{a}$ *and* $\vec{b}$ *as its adjacent sides and whose direction* $\hat{\eta}$ *is perpendicular to the plane of* $\vec{a}$ *and* $\vec{b}$ *such that* $\vec{a}, \vec{b}, \hat{\eta}$ *form a right handed system.*

In other words, $\vec{a} \times \vec{b}$ *represents the vector area of the parallelogram having adjacent sides along* $\vec{a}$ *and* $\vec{b}$.

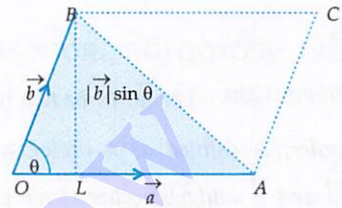

Fig. 24.3

Thus, area of parallelogram $OACB = |\vec{a} \times \vec{b}|$.

Clearly, Area of $\Delta OAB = \frac{1}{2}$ Area of parallelogram $OACB$

$$= \frac{1}{2} |\vec{a} \times \vec{b}| = \frac{1}{2} |\vec{OA} \times \vec{OB}|$$

NOTE *By the term vector area of a plane figure we mean that a vector of magnitude equal to the area of the plane figure and direction normal to the plane of the figure in the sense of right handed rotation.*

24.2 PROPERTIES OF VECTOR PRODUCT

PROPERTY I *Vector product is not commutative i.e. if* $\vec{a}$ *and* $\vec{b}$ *are any two vectors, then* $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

PROOF Let $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors and let θ be the angle between them. Then,

$$\vec{a} \times \vec{b} = |\vec{a}| \, |\vec{b}| \sin \theta \, \hat{\eta}_1,$$

where $\hat{\eta}_1$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}_1$ form a right-handed system.

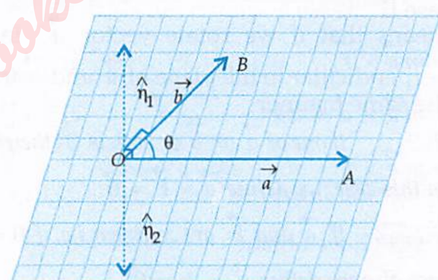

Fig. 24.4

and, $\quad \vec{b} \times \vec{a} = |\vec{b}| \, |\vec{a}| \sin \theta \, \hat{\eta}_2,$

where $\hat{\eta}_2$ is a unit vector perpendicular to the plane of $\vec{b}$ and $\vec{a}$ such that $\vec{b}, \vec{a}, \hat{\eta}_2$ form a right-handed system.

Obviously, $\hat{\eta}_1 = -\hat{\eta}_2 \Rightarrow |\vec{a}| \, |\vec{b}| \sin \theta \, \hat{\eta}_1 = -|\vec{a}| \, |\vec{b}| \sin \theta \, \hat{\eta}_2 \Rightarrow \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

Hence, $\quad \vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

PROPERTY II *If* $\vec{a}, \vec{b}$ *are two vectors and m is a scalar, then* $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

PROOF Let $\vec{a}$ and $\vec{b}$ be two vectors represented by $\vec{OA}$ and $\vec{OB}$ and let θ be the angle between them. Then,

$$\vec{a} \times \vec{b} = |\vec{a}| \, |\vec{b}| \sin \theta \, \hat{\eta},$$

where $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ form a right handed system.

We have the following cases:

Case I When $m > 0$: In this case, $m\vec{a} = \vec{OA_1}$ and $\vec{a} = \vec{OA}$ are in the same direction. Also, $\vec{b} = \vec{OB}$ and $m\vec{b} = \vec{OB_1}$ are in the same direction.

$$\therefore \quad m\vec{a} \times \vec{b} = |m\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = |m|\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = m|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta} \qquad [\because m > 0 \; \therefore |m| = m]$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = m\left(|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\right)$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b})$$

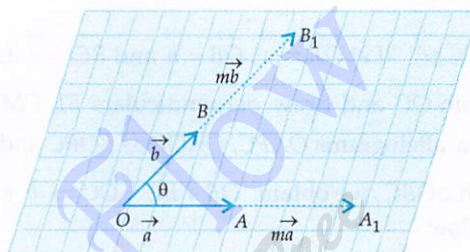

Fig. 24.5

and, $$\vec{a} \times m\vec{b} = |\vec{a}|\,|m\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = |\vec{a}|\,|m|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = m\left\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\right\}$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = m(\vec{a} \times \vec{b})$$

Thus, in this case, we obtain: $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

Case II *When* $m < 0$: In this case, the angle between $m\vec{a} = \vec{OA_1}$ and $\vec{b} = \vec{OB}$ is $(\pi + \theta)$.

$$\therefore \quad m\vec{a} \times \vec{b} = |m\vec{a}|\,|\vec{b}|\sin(\pi + \theta)\,\hat{\eta}$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = -|m|\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = m\left\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\right\} \quad \begin{bmatrix} \because m < 0 \\ \therefore |m| = -m \end{bmatrix}$$

$$\Rightarrow \quad m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b})$$

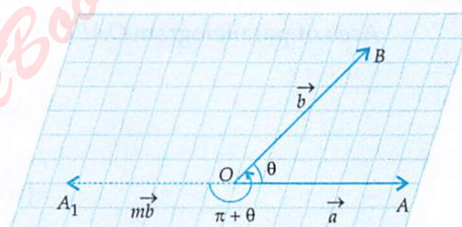

Fig. 24.6

The angle between $\vec{a}$ and $m\vec{b}$ is $(\pi + \theta)$.

$$\therefore \quad \vec{a} \times m\vec{b} = |\vec{a}|\,|m\vec{b}|\sin(\pi + \theta)\,\hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = -|\vec{a}|\,|m|\,|\vec{b}|\sin\theta\,\hat{\eta}$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = m\left\{|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}\right\} \qquad [\because |m| = -m]$$

$$\Rightarrow \quad \vec{a} \times m\vec{b} = m(\vec{a} \times \vec{b})$$

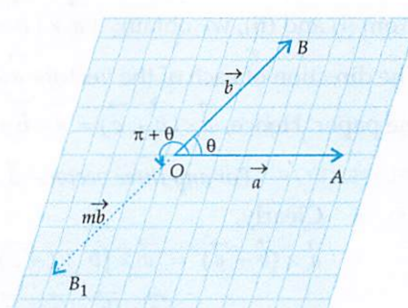

Fig. 24.7

Thus, in this case also, we obtain:

$$\vec{a} \times m\vec{b} = m(\vec{a} \times \vec{b}) = m\vec{a} \times \vec{b}$$

Case III *When* $m = 0$: In this case, using note 1 in section 24.1, we obtain

$$m\vec{a} \times \vec{b} = \vec{0} \times \vec{b} = \vec{0}, \; \vec{a} \times m\vec{b} = \vec{a} \times \vec{0} = \vec{0} \text{ and, } m(\vec{a} \times \vec{b}) = 0(\vec{a} \times \vec{b}) = \vec{0}$$

So, in this case also, we obtain: $m\vec{a} \times \vec{b} = m(\vec{a} \times \vec{b}) = \vec{a} \times m\vec{b}$.

Hence, $m\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times m\vec{b}$ for all values of m.

PROPERTY III *If $\vec{a}, \vec{b}$ are two vectors and m, n are scalars, then*

$$m\vec{a}\times n\vec{b} = mn(\vec{a}\times\vec{b}) = m(\vec{a}\times n\vec{b}) = n(m\vec{a}\times\vec{b})$$

PROPERTY IV *(Distributivity of vector product over vector addition). Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors. Then,*

(i) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ *[Left distributivity]*

(ii) $(\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a}+\vec{c}\times\vec{a}$ *[Right distributivity]*

PROOF Let $\vec{OA}=\vec{a}$, $\vec{OB}=\vec{b}$ and $\vec{BC}=\vec{c}$ as shown in Fig. 24.8.

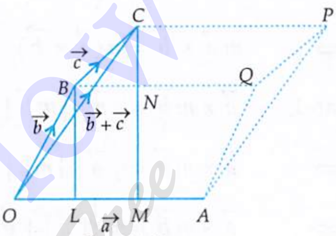

Fig. 24.8

Join OC and draw perpendiculars BL, CM and BN. Complete parallelograms $OAPC$, $OAQB$ as $BQPC$ and shown in Fig. 24.8.

In ΔOBC, we obtain: $\vec{OB}+\vec{BC}=\vec{OC} \Rightarrow \vec{b}+\vec{c}=\vec{OC}$.

Now,

$|\vec{a}\times(\vec{b}\times\vec{c})| = |\vec{OA}\times\vec{OC}|$ = Area of parallelogram $OAPC$ ...(i)

$|\vec{a}\times\vec{b}| = |\vec{OA}\times\vec{OB}|$ = Area of parallelogram $OAQB$ ...(ii)

and, $|\vec{a}\times\vec{c}| = |\vec{OA}\times\vec{BC}| = |\vec{BQ}\times\vec{BC}|$ = Area of parallelogram $BQPC$ ...(iii)

Now,

Area of parallelogram $OAPC = OA\times CM = OA\times(CN+NM) = OA\times(CN+BL)$

$= OA\times CN + OA\times BL$

= Area of parallelogram $BQPC$ + Area of parallelogram $OAQB$

$= |\vec{a}\times\vec{c}| + |\vec{a}\times\vec{b}|$ [Using (ii) and (iii)]

$= |\vec{a}\times\vec{b}| + |\vec{a}\times\vec{c}|$ ...(iv)

From (i) and (ii), we obtain: $|\vec{a}\times(\vec{b}+\vec{c})| = |\vec{a}\times\vec{b}| + |\vec{a}\times\vec{c}|$

The direction of each of the vectors $\vec{a}\times(\vec{b}+\vec{c})$, $\vec{a}\times\vec{b}$ and $\vec{a}\times\vec{c}$ is perpendicular to the plane of the paper. Hence, $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$.

PROPERTY V *For any three vectors $\vec{a}, \vec{b}, \vec{c}$, prove that $\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$.*

PROOF Clearly,

$\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\{\vec{b}+(-\vec{c})\} = \vec{a}\times\vec{b}+\vec{a}\times(-\vec{c})$ [By Property (IV)]

$= \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$ $[\because \vec{a}\times(-\vec{c}) = -(\vec{a}\times\vec{c})]$

PROPERTY VI *The vector product of two non-zero vectors is zero vector iff they are parallel (collinear).*

i.e. $\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$, *where $\vec{a}$ and $\vec{b}$ are non-zero vectors.*

PROOF Let $\vec{a}, \vec{b}$ be two non-zero vectors. Then,

$\vec{a}\times\vec{b}=\vec{0} \Leftrightarrow |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{\eta}=\vec{0} \Leftrightarrow \sin\theta=0 \Leftrightarrow \theta=0$ or, π $[\because |\vec{a}|\neq 0, |\vec{b}|\neq 0]$

$\Leftrightarrow \quad \vec{a},\ \vec{b}$ are parallel vectors

REMARK 1 *It follows from the above property that* $\vec{a} \times \vec{a} = \vec{0}$ *for every non-zero vector* $\vec{a}$ *which in turn implies that* $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$.

REMARK 2 *Vector product of orthonormal triad of unit vectors* $\hat{i}, \hat{j}, \hat{k}$ *is given by*

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$$

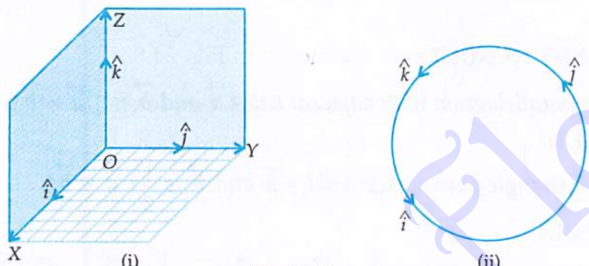

Fig. 24.9

24.3 VECTOR PRODUCT IN TERMS OF COMPONENTS

Let $\vec{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$ and $\vec{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ be two vectors. Then,

$$\vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times (b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k})$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times b_1 \hat{i} + (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times b_2 \hat{j} + (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}) \times b_3 \hat{k}$$

[By left distributivity]

$$\Rightarrow \quad \vec{a} \times \vec{b} = a_1 b_1 (\hat{i} \times \hat{i}) + (a_2 b_1)(\hat{j} \times \hat{i}) + (a_3 b_1)(\hat{k} \times \hat{i}) + a_1 b_2 (\hat{i} \times \hat{j}) + a_2 b_2 (\hat{j} \times \hat{j})$$
$$+ a_3 b_2 (\hat{k} \times \hat{j}) + a_1 b_3 (\hat{i} \times \hat{k}) + a_2 b_3 (\hat{j} \times \hat{k}) + a_3 b_3 (\hat{k} \times \hat{k})$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

[Using Remark 2 in section 24.2]

$$\Rightarrow \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

24.4 VECTORS NORMAL TO THE PLANE OF TWO GIVEN VECTORS

Let $\vec{a}, \vec{b}$ be two non-zero, non-parallel vectors and let θ be the angle between them. Then,

$$\vec{a} \times \vec{b} = |\vec{a}|\, |\vec{a}| \sin\theta\, \hat{\eta},$$

where $\hat{\eta}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ such that $\vec{a}, \vec{b}, \hat{\eta}$ from a right-handed system

$$\therefore \quad \vec{a} \times \vec{b} = |\vec{a} \times \vec{b}|\, \hat{\eta} \Rightarrow \hat{\eta} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$$

Thus, $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

Note that $-\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ is also a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$

Vectors of magnitude 'λ' normal to the plane of $\vec{a}$ and $\vec{b}$ are given by $\pm \dfrac{\lambda(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}$.

24.5 SOME IMPORTANT RESULTS

RESULT I *The area of a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$.*

PROOF See section 24.1.1

RESULT II *The area of a triangle with adjacent sides $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$.*

PROOF See section 24.1.1

RESULT III *The area of a triangle ABC is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$ or, $\frac{1}{2}|\vec{BC} \times \vec{BA}|$ or, $\frac{1}{2}|\vec{CB} \times \vec{CA}|$.*

PROOF See section 24.1.1

RESULT IV *The area of a parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}|\vec{a} \times \vec{b}|$.* [NCERT EXEMPALR]

PROOF Let $ABCD$ be a parallelogram. With A as the origin, let the position vectors of B and D be $\vec{p}$ and $\vec{q}$ respectively. Then, $\vec{AB} = \vec{p}$ and $\vec{AD} = \vec{q}$. But, $\vec{BC} = \vec{AD}$. Therefore, $\vec{BC} = \vec{q}$

Using triangle law of addition of vectors in ΔABC, we get

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{AB} + \vec{AD} = \vec{AC} \Rightarrow \vec{p} + \vec{q} = \vec{AC} \quad ...(i)$$

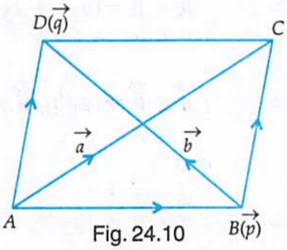

Fig. 24.10

Let $\vec{AC} = \vec{a}$ and $\vec{BD} = \vec{b}$ be the diagonals of the parallelogram $ABCD$. Then, from (i), we get

$$\vec{p} + \vec{q} = \vec{a} \quad ...(ii)$$

and, $\vec{BD}$ = Position vector of D – Position vector of B

$$\Rightarrow \quad \vec{b} = \vec{q} - \vec{p} \quad ...(iii)$$

Adding (ii) and (iii), we obtain: $2\vec{q} = \vec{a} + \vec{b} \Rightarrow \vec{q} = \frac{1}{2}(\vec{a} + \vec{b})$

Subtracting (iii) from (ii), we obtain: $2\vec{p} = \vec{a} - \vec{b} \Rightarrow \vec{p} = \frac{1}{2}(\vec{a} - \vec{b})$

Now,

$$\vec{p} \times \vec{q} = \frac{1}{2}(\vec{a} - \vec{b}) \times \frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{4}\left\{(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})\right\}$$

$$\Rightarrow \quad \vec{p} \times \vec{q} = \frac{1}{4}\left\{\vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}\right\} = \frac{1}{4}(\vec{a} \times \vec{b} + \vec{a} \times \vec{b}) = \frac{1}{2}(\vec{a} \times \vec{b})$$

Hence, area of parallelogram $ABCD = |\vec{p} \times \vec{q}| = |\frac{1}{2}(\vec{a} \times \vec{b})| = \frac{1}{2}|\vec{a} \times \vec{b}|$.

RESULT V *The area of a plane quadrilateral ABCD is $\frac{1}{2}|\vec{AC} \times \vec{BD}|$, where AC and BD are its diagonals.*

PROOF Let the diagonals AC and BD intersect at O. Then,

Vector area of quadrilateral $ABCD$

= (Vector area of triangle ABC) + (Vector area of triangle ADC)

$$= \frac{1}{2}(\vec{AB} \times \vec{AC}) + \frac{1}{2}(\vec{AC} \times \vec{AD}) = -\frac{1}{2}(\vec{AC} \times \vec{AB}) + \frac{1}{2}(\vec{AC} \times \vec{AD})$$

$$= \frac{1}{2}\left\{\vec{AC} \times (\vec{AD} - \vec{AB})\right\}$$

$$= \frac{1}{2}\left\{\vec{AC} \times (\vec{BA} + \vec{AD})\right\} \quad [\because -\vec{AB} = \vec{BA}]$$

$$= \frac{1}{2}(\vec{AC} \times \vec{BD}) \quad [\because \vec{BA} + \vec{AD} = \vec{BD}]$$

Hence, area of quadrilateral $ABCD = \frac{1}{2}|\vec{AC} \times \vec{BD}|$

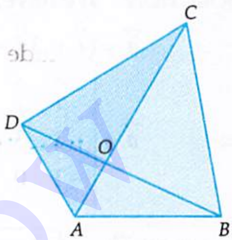

Fig. 24.11

24.6 LAGRANGE'S IDENTITY

THEOREM *If $\vec{a}, \vec{b}$ are any two vectors, then*

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}$$

or, $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$ **[CBSE 2004]**

PROOF We know that

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\therefore \quad |\vec{a} \times \vec{b}|^2 = \left(|\vec{a}||\vec{b}|\sin\theta\right)^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta = |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta = |\vec{a}|^2|\vec{b}|^2 - \left(|\vec{a}||\vec{b}|\cos\theta\right)^2$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}|^2 = \begin{vmatrix} |\vec{a}|^2 & \vec{a} \cdot \vec{b} \\ \vec{a} \cdot \vec{b} & |\vec{b}|^2 \end{vmatrix} = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}$$

Hence, $|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \Rightarrow |\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$ Q.E.D.

REMARK *Lagranges identity provides relation between the dot and cross product of two vectors.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE VECTOR PRODUCT OF GIVEN VECTORS**

EXAMPLE 1 *Find $\vec{a} \times \vec{b}$, if $\vec{a} = 2\hat{i} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.*

SOLUTION We have, $\vec{a} = 2\hat{i} + 0\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$.

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = (0-1)\hat{i} - (2-1)\hat{j} + (2-0)\hat{k} = -\hat{i} - \hat{j} + 2\hat{k}$$

EXAMPLE 2 *Find the magnitude of $\vec{a}$ given by $\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 3\hat{k})$*

SOLUTION We have,

$$\vec{a} = (\hat{i} + 3\hat{j} - 2\hat{k}) \times (-\hat{i} + 0\hat{j} + 3\hat{k})$$

$$\Rightarrow \quad \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 3 \end{vmatrix} = (9-0)\hat{i} - (3-2)\hat{j} + (0+3)\hat{k} = 9\hat{i} - \hat{j} + 3\hat{k}$$

$$\therefore \quad |\vec{a}| = \sqrt{9^2 + (-1)^2 + 3^2} = \sqrt{91}.$$

EXAMPLE 3 *Find λ and μ, if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.* **[NCERT]**

SOLUTION We have,

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}.$$

$$\Rightarrow \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

$$\Rightarrow \quad (6\mu - 27\lambda)\hat{i} - (2\mu - 27)\hat{j} + (2\lambda - 6)\hat{k} = \vec{0}$$

$$\Rightarrow \quad 6\mu - 27\lambda = 0,\ 2\mu - 27 = 0 \text{ and } 2\lambda - 6 = 0 \Rightarrow \lambda = 3 \text{ and } \mu = \frac{27}{2}$$

ALITER We have,

$$(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$$

$$\Rightarrow \quad 2\hat{i} + 6\hat{j} + 27\hat{k} \text{ is parallel to } \hat{i} + \lambda\hat{j} + \mu\hat{k}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{\lambda}{6} = \frac{\mu}{27} \quad \left[\because \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \text{ are parallel, iff } \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}\right]$$

$$\Rightarrow \quad \lambda = 3 \text{ and } \mu = \frac{27}{2}$$

EXAMPLE 4 *For any vector $\vec{a}$, prove that $|\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$*

SOLUTION Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$. Then,

$$\vec{a} \times \hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{i} = a_1(\hat{i} \times \hat{i}) + a_2(\hat{j} \times \hat{i}) + a_3(\hat{k} \times \hat{i}) = -a_2\hat{k} + a_3\hat{j}$$

$$\Rightarrow \quad |\vec{a} \times \hat{i}|^2 = a_2^2 + a_3^2$$

$$\vec{a} \times \hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{j} = a_1(\hat{i} \times \hat{j}) + a_2(\hat{j} \times \hat{j}) + a_3(\hat{k} \times \hat{j}) = a_1\hat{k} - a_3\hat{i}$$

$$\Rightarrow \quad |\vec{a} \times \hat{j}|^2 = a_1^2 + a_3^2$$

and, $\vec{a} \times \hat{k} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \hat{k} = a_1(\hat{i} \times \hat{k}) + a_2(\hat{j} \times \hat{k}) + a_3(\hat{k} \times \hat{k}) = -a_1\hat{j} + a_2\hat{i}$

$\Rightarrow \quad |\vec{a} \times \hat{k}|^2 = a_1^2 + a_2^2$

$\therefore \quad |\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = a_2^2 + a_3^2 + a_1^2 + a_3^2 + a_1^2 + a_2^2$

$\Rightarrow \quad |\vec{a} \times \hat{i}|^2 + |\vec{a} \times \hat{j}|^2 + |\vec{a} \times \hat{k}|^2 = 2(a_1^2 + a_2^2 + a_3^2) = 2|\vec{a}|^2$

EXAMPLE 5 *If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{i}$, find the value of $(\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy$.* [CBSE 2015]

SOLUTION Proceeding as in Example 4, we obtain

$$\vec{r} \times \hat{i} = -y\hat{k} + z\hat{j} = 0\hat{i} + z\hat{j} - y\hat{k} \text{ and } \vec{r} \times \hat{j} = x\hat{k} - z\hat{i} = -z\hat{i} + 0\hat{j} + x\hat{k}$$

$\therefore \quad (\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) = (0\hat{i} \times z\hat{j} - y\hat{k}) \cdot (-z\hat{i} + 0\hat{j} + x\hat{k}) = 0 \times -z + z \times 0 + (-y)\,x = -yx$

$\Rightarrow \quad (\vec{r} \times \hat{i}) \cdot (\vec{r} \times \hat{j}) + xy = -xy + xy = 0$

Type II ON FINDING A UNIT VECTOR AND A VECTOR OF GIVEN MAGNITUDE PERPENDICULAR TO TWO GIVEN VECTORS

EXAMPLE 6 *Find a unit vector perpendicular to both the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$.*

SOLUTION Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$. Then, [CBSE 2020]

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 1 & 2 & -1 \end{vmatrix} = (2-6)\,\hat{i} - (-1-3)\,\hat{j} + (2+2)\,\hat{k} = -4\hat{i} + 4\hat{j} + 4\hat{k}.$$

$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{(-4)^2 + 4^2 + 4^2} = 4\sqrt{3}.$

Hence, a unit vector perpendicular to vectors $\vec{a}$ and $\vec{b}$ is given by

$$\hat{\eta} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{(-4\hat{i} + 4\hat{j} + 4\hat{k})}{4\sqrt{3}} = \frac{1}{\sqrt{3}}(-\hat{i} + \hat{j} + \hat{k})$$

EXAMPLE 7 *Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.* [NCERT]

SOLUTION We have, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$\therefore \quad \vec{a} + \vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{a} - \vec{b} = 0\hat{i} - \hat{j} - 2\hat{k}$

A vector $\vec{c}$ perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by their vector product.

$\therefore \quad \vec{c} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times (0\hat{i} - \hat{j} - 2\hat{k})$

$$\Rightarrow \quad \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i} + 4\hat{j} - 2\hat{k} \Rightarrow |\vec{c}| = \sqrt{4 + 16 + 4} = 2\sqrt{6}$$

$\therefore \quad$ Required unit vector $= \dfrac{1}{|\vec{c}|}\vec{c} = \dfrac{1}{2\sqrt{6}}(-2\hat{i} + 4\hat{j} - 2\hat{k}) = -\dfrac{1}{\sqrt{6}}\hat{i} + \dfrac{2}{\sqrt{6}}\hat{j} - \dfrac{1}{\sqrt{6}}\hat{k}$

EXAMPLE 8 *Find a vector of magnitude 9, which is perpendicular to both the vectors* $4\hat{i} - \hat{j} + 3\hat{k}$ *and* $-2\hat{i} + \hat{j} - 2\hat{k}$.

SOLUTION Let $\vec{a} = 4\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = -2\hat{i} + \hat{j} - 2\hat{k}$. Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = (2-3)\hat{i} - (-8+6)\hat{j} + (4-2)\hat{k} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$$

$$\therefore \quad \text{Required vector} = 9\left\{\frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}\right\} = \frac{9}{3}(-\hat{i} + 2\hat{j} + 2\hat{k}) = -3\hat{i} + 6\hat{j} + 6\hat{k}.$$

EXAMPLE 9 *Find a unit vector perpendicular to the plane ABC where A, B and C are the points* $(3, -1, 2)$ *and* $(1, -1, -3)$, $(4, -3, 1)$ *respectively.*

SOLUTION The vector $\vec{AB} \times \vec{AC}$ is perpendicular to the vectors $\vec{AB}$ and $\vec{AC}$ both.

$$\therefore \quad \text{Required unit vector} = \frac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|}$$

Now, $\vec{AB}$ = P.V. of B − P.V. of $A = (\hat{i} - \hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} + 0\hat{j} - 5\hat{k}$

and, $\vec{AC}$ = P.V. of C − P.V. of A $= (4\hat{i} - 3\hat{j} + \hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k}) = \hat{i} - 2\hat{j} - \hat{k}$

$$\therefore \quad \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 0 & -5 \\ 1 & -2 & -1 \end{vmatrix} = (0-10)\hat{i} - (2+5)\hat{j} + (4-0)\hat{k} = -10\hat{i} - 7\hat{j} + 4\hat{k}$$

$$\Rightarrow \quad |\vec{AB} \times \vec{AC}| = \sqrt{(-10)^2 + (-7)^2 + 4^2} = \sqrt{165}$$

Hence, required vector $= \dfrac{\vec{AB} \times \vec{AC}}{|\vec{AB} \times \vec{AC}|} = \dfrac{1}{\sqrt{165}}(-10\hat{i} - 7\hat{j} + 4\hat{k})$

EXAMPLE 10 *Let* $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = 3\hat{j} - \hat{k}$ *and* $\vec{c} = 7\hat{i} - \hat{k}$. *Find a vector* $\vec{d}$ *which is perpendicular to both* $\vec{a}$ *and* $\vec{b}$, *and* $\vec{c} \cdot \vec{d} = 1$. **[NCERT]**

SOLUTION Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore, it is parallel to $\vec{a} \times \vec{b}$. So, let

$$\vec{d} = \lambda(\vec{a} \times \vec{b}) \Rightarrow \vec{d} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 3 & -1 \end{vmatrix} = \lambda(\hat{i} + \hat{j} + 3\hat{k}) \qquad \text{...(i)}$$

It is given that

$$\vec{c} \cdot \vec{d} = 1 \Rightarrow (7\hat{i} - \hat{k}) \cdot \lambda(\hat{i} + \hat{j} + 3\hat{k}) = 1 \Rightarrow 7\lambda - 3\lambda = 1 \Rightarrow \lambda = \frac{1}{4}$$

Putting $\lambda = \frac{1}{4}$ in (i), we get: $\vec{d} = \frac{1}{4}(\hat{i} + \hat{j} + 3\hat{k})$.

Type III ON FINDING THE AREA OF THE PARALLELOGRAM

EXAMPLE 11 *Find the area of the parallelogram determined by the vectors $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.*

SOLUTION Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$. The vector area of the parallelogram whose adjacent sides are represented by the vectors $\vec{a}$ and $\vec{b}$ is $\vec{a} \times \vec{b}$.

Now,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = (2+6)\hat{i} - (1-9)\hat{j} + (-2-6)\hat{k} = 8\hat{i} + 8\hat{j} - 8\hat{k}$$

$\therefore$ Area of the parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{8^2 + 8^2 + (-8)^2} = 8\sqrt{3}$ square units

EXAMPLE 12 *Show that the area of a parallelogram having diagonals $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ is $5\sqrt{3}$ square units.* **[CBSE 2008]**

SOLUTION Let $\vec{a} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} - 3\hat{j} + 4\hat{k}$. Then,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = (4-6)\hat{i} - (12+2)\hat{j} + (-9-1)\hat{k} = -2\hat{i} - 14\hat{j} - 10\hat{k}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = \sqrt{(-2)^2 + (-14)^2 + (-10)^2} = \sqrt{300}$$

$\therefore$ Area of the parallelogram $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{300} = 5\sqrt{3}$ sq. units.

Type IV ON FINDING THE AREA OF A QUADRILATERAL

EXAMPLE 13 *If $A(0, 1, 1)$, $B(2, 3, -2)$, $C(22, 19, -5)$ and $D(1, -2, 1)$ are the vertices of a quadrilateral ABCD, find its area.*

SOLUTION We know that the area Δ of quadrilateral $ABCD$ is given by $\Delta = \frac{1}{2}|\vec{AC} \times \vec{BD}|$.

Now,

$\vec{AC}$ = Position vector of C − Position vector of $A = (22\hat{i} + 19\hat{j} - 5\hat{k}) - (0\hat{i} + \hat{j} + \hat{k}) = 22\hat{i} + 18\hat{j} - 6\hat{k}$

and $\vec{BD}$ = Position vector of D − Position vector of $B = (\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + 3\hat{j} - 2\hat{k}) = -\hat{i} - 5\hat{j} + 3\hat{k}$

$$\therefore \quad \vec{AC} \times \vec{BD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 22 & 18 & -6 \\ -1 & -5 & 3 \end{vmatrix} = (54-30)\hat{i} - (66-6)\hat{j} + (-110+18)\hat{k} = 24\hat{i} - 60\hat{j} - 92\hat{k}$$

$$\Rightarrow \quad |\vec{AC} \times \vec{BD}| = \sqrt{(24)^2 + (-60)^2 + (-92)^2} = \sqrt{576 + 3600 + 8464} = \sqrt{12640}$$

Hence, Area of quadrilateral $ABCD = \frac{1}{2}|\vec{AC} \times \vec{BD}| = \frac{1}{2}\sqrt{12640} = \sqrt{3160}$ square units.

Type V ON FIND $|\vec{a}\times\vec{b}|$ WHEN $\vec{a}\cdot\vec{b}$, $|\vec{a}|$ AND $|\vec{b}|$ ARE GIVEN

EXAMPLE 14 *Given* $|\vec{a}|=10, |\vec{b}|=2$ *and* $\vec{a}\cdot\vec{b}=12$, *find* $|\vec{a}\times\vec{b}|$.

SOLUTION We know that

$$(\vec{a}\cdot\vec{b})^2+|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2$$

$$\therefore \quad 12^2+|\vec{a}\times\vec{b}|^2=(10)^2\times(2)^2 \qquad \left[\because \vec{a}\cdot\vec{b}=12, |\vec{a}|=10 \text{ and } |\vec{b}|=2\right]$$

$$\Rightarrow \quad |\vec{a}\times\vec{b}|^2=400-144=256 \Rightarrow |\vec{a}\times\vec{b}|=16$$

EXAMPLE 15 *Show that* $(\vec{a}\times\vec{b})^2=|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$

SOLUTION We have,

$$(\vec{a}\times\vec{b})^2=|\vec{a}\times\vec{b}|^2=\left\{|\vec{a}|\,|\vec{b}|\sin\theta\right\}^2=|\vec{a}|^2\,|\vec{b}|^2\sin^2\theta$$

$$\Rightarrow \quad (\vec{a}\times\vec{b})^2=\left\{|\vec{a}|^2\,|\vec{b}|^2\right\}(1-\cos^2\theta)=|\vec{a}|^2\,|\vec{b}|^2-|\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$\Rightarrow \quad (\vec{a}\times\vec{b})^2=(\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})-(\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{b}) \qquad [\because \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta]$$

$$\Rightarrow \quad (\vec{a}\times\vec{b})^2=\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}$$

Type VI ON FINDING THE AREA OF A TRIANGLE

EXAMPLE 16 *Find the area of the triangle whose vertices are* $A(3,-1,2)$, $B(1,-1,-3)$ *and* $C(4,-3,1)$. **[CBSE 2010]**

SOLUTION Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the position vectors of points A, B and C respectively. Then,

$$\vec{a}=3\hat{i}-\hat{j}+2\hat{k},\ \vec{b}=\hat{i}-\hat{j}-3\hat{k} \text{ and } \vec{c}=4\hat{i}-3\hat{j}+\hat{k}$$

We know that: Area of $\Delta ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|$

Now, $\vec{AB}=\vec{b}-\vec{a}=(\hat{i}-\hat{j}-3\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=-2\hat{i}+0\hat{j}-5\hat{k}$

and, $\vec{AC}=\vec{c}-\vec{a}=(4\hat{i}-3\hat{j}+\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=\hat{i}-2\hat{j}-\hat{k}$

$$\therefore \quad \vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=(0-10)\hat{i}-(2+5)\hat{j}+(4-0)\hat{k}=-10\hat{i}-7\hat{j}+4\hat{k}$$

$$\Rightarrow \quad |\vec{AB}\times\vec{AC}|=\sqrt{(-10)^2+(-7)^2+4^2}=\sqrt{165}$$

Hence, area of $\Delta ABC=\frac{1}{2}|\vec{AB}\times\vec{AC}|=\frac{1}{2}\sqrt{165}$.

EXAMPLE 17 *If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the vertices A, B, C of a triangle ABC, show that the area of triangle ABC is $\frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$. Deduce the condition for points $\vec{a}, \vec{b}, \vec{c}$ to be collinear.*

SOLUTION We know that: Area of $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}|$

Now, $\vec{AB} = \vec{b}-\vec{a}$, and $\vec{AC} = \vec{c}-\vec{a}$

$\therefore \quad \vec{AB}\times\vec{AC} = (\vec{b}-\vec{a})\times(\vec{c}-\vec{a}) = \vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{a}\times\vec{c}+\vec{a}\times\vec{a}$ [By distributivity]

$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a}+\vec{0}$ $[\because \vec{a}\times\vec{a}=\vec{0}]$

$\Rightarrow \quad \vec{AB}\times\vec{AC} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$

$\therefore \quad$ Area of $\Delta ABC = \frac{1}{2}|\vec{AB}\times\vec{AC}| = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

If the points, A, B, C are collinear, then

Area of $\Delta ABC = 0$

$\Rightarrow \quad \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = 0$

$\Rightarrow \quad |\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}| = 0 \Rightarrow \vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a} = \vec{0}$

Thus, $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$ is the required condition of collinearity of three points having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 18 *Show that distance of the point $\vec{c}$ from the line joining $\vec{a}$ and $\vec{b}$ is*

$$\frac{|\vec{b}\times\vec{c}+\vec{c}\times\vec{a}+\vec{a}\times\vec{b}|}{|\vec{b}-\vec{a}|}$$

SOLUTION Let ABC be a triangle and let $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of its vertices A, B and C respectively. Let CM be the perpendicular from C on AB. Then,

Area of $\Delta ABC = \frac{1}{2}(AB)(CM) = \frac{1}{2}|\vec{AB}|(CM)$

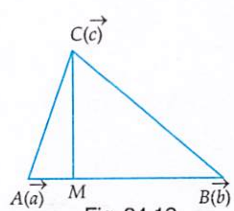

Fig. 24.12

Also, Area of $\Delta ABC = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$ [See Example 18]

$\therefore \quad \frac{1}{2}|\vec{AB}|(CM) = \frac{1}{2}|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|$

$$\Rightarrow \quad CM = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{AB}|} = \frac{|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|}{|\vec{b}-\vec{a}|}$$

Type VII MISCELLANEOUS EXAMPLES

EXAMPLE 19 *Prove that the points A, B and C with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively are collinear if and only if $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$.* **[CBSE 2020]**

SOLUTION The points A, B and C are collinear

$\Leftrightarrow$ Vectors $\vec{AB}$ and $\vec{BC}$ are parallel vectors.

$\Leftrightarrow$ $\vec{AB} \times \vec{BC} = \vec{0}$

$\Leftrightarrow$ $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{b}) = \vec{0}$ $[\because \vec{AB} = \vec{b} - \vec{a}$ and $\vec{BC} = \vec{c} - \vec{b}]$

$\Leftrightarrow$ $(\vec{b} - \vec{a}) \times \vec{c} - (\vec{b} - \vec{a}) \times \vec{b} = \vec{0}$

$\Leftrightarrow$ $(\vec{b} \times \vec{c} - \vec{a} \times \vec{c}) - (\vec{b} \times \vec{b} - \vec{a} \times \vec{b}) = \vec{0}$

$\Leftrightarrow$ $(\vec{b} \times \vec{c} + \vec{c} \times \vec{a}) - (\vec{0} - \vec{a} \times \vec{b}) = \vec{0}$ $[\because -(\vec{a} \times \vec{c}) = \vec{c} \times \vec{a}$ and $\vec{b} \times \vec{b} = \vec{0}]$

$\Leftrightarrow$ $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a} = \vec{0}$.

EXAMPLE 20 *For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. Show that $\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b}) = \vec{0}$*

SOLUTION We have,

$$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{a} + \vec{c} \times \vec{b} \quad \text{[Using distributive law]}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{b} \times \vec{c} = \vec{0}. \quad [\because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \text{ etc.}]$$

EXAMPLE 21 *If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.* **[CBSE 2001, 2009, 2016]**

SOLUTION Recall that two non-zero vectors are parallel iff their cross-product is zero vector. Therefore, to prove that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$, it is sufficient to show that $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{0}$.

Now, $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c}) = \vec{a} \times (\vec{b} - \vec{c}) - \vec{d} \times (\vec{b} - \vec{c})$

$$= \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c} \quad \text{[Using distributive law]}$$

$$= \vec{c} \times \vec{d} - \vec{b} \times \vec{d} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c} \quad \left[\because \vec{a} \times \vec{b} = \vec{c} \times \vec{d},\ \vec{a} \times \vec{c} = \vec{b} \times \vec{d}\right]$$

$$= \vec{c} \times \vec{d} - \vec{b} \times \vec{d} + \vec{b} \times \vec{d} - \vec{c} \times \vec{d} \quad \left[\because -(\vec{d} \times \vec{b}) = \vec{b} \times \vec{d} \,\&\, \vec{d} \times \vec{c} = -(\vec{c} \times \vec{d})\right]$$

$$= \vec{0}$$

Hence, $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$

EXAMPLE 22 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ are four non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d}$, then show that $\vec{a} - 2\vec{d}$ is parallel to $2\vec{b} - \vec{c}$, where $\vec{a} \neq 2\vec{d}$, $\vec{c} \neq 2\vec{b}$.* **[CBSE 2022]**

SOLUTION In order to prove that $\vec{a} - 2\vec{d}$ is parallel to $2\vec{b} - \vec{c}$, it is sufficient to show that $(\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) = 0$.

Now,

$$(\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) = \vec{a} \times (2\vec{b} - \vec{c}) - 2\vec{d} \times (2\vec{b} - \vec{c}) = \vec{a} \times 2\vec{b} - \vec{a} \times \vec{c} - 2\vec{d} \times 2\vec{b} + 2\vec{d} \times \vec{c}$$

$$= 2(\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c}) + 4(\vec{b} \times \vec{d}) - 2(\vec{c} \times \vec{d})$$

$$= 2(\vec{c} \times \vec{d}) - 4(\vec{b} \times \vec{d}) + 4(\vec{b} \times \vec{d}) - 2(\vec{c} \times \vec{d}) = \vec{0} \qquad [\because \vec{a} \times \vec{b} = \vec{c} \times \vec{d},\ \vec{a} \times \vec{c} = 4(\vec{b} \times \vec{d})]$$

$\therefore$ $\quad \vec{a} - 2\vec{d}$ is parallel to $2\vec{b} - \vec{c}$.

EXAMPLE 23 *If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that*

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$ **[CBSE 2001, 2004]**

SOLUTION We have,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$\Rightarrow \quad \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0}$ [Taking cross-product on left with $\vec{a}$]

$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$ [Using distributive law]

$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{c} \times \vec{a} = \vec{0}$ $[\because \vec{a} \times \vec{a} = \vec{0}$ and $\vec{a} \times \vec{c} = -\vec{c} \times \vec{a}]$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a}$ ...(i)

Again, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \quad \vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$ [Taking cross-product on left with $\vec{b}$]

$\Rightarrow \quad \vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$

$\Rightarrow \quad -\vec{a} \times \vec{b} + \vec{0} + \vec{b} \times \vec{c} = \vec{0}$ $[\because \vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}]$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{b} \times \vec{c}$ ...(ii)

Form (i) and (ii), we obtain: $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

EXAMPLE 24 *Prove that the normal to the plane containing three points whose position vectors are $\vec{a}, \vec{b}, \vec{c}$ lies in the direction $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$.* **[CBSE 2001C]**

SOLUTION Let A, B and C be the points having position vectors $\vec{a}$ $\vec{b}$ and $\vec{c}$ respectively. Then, $\vec{AB} \times \vec{AC}$ is a vector normal to the plane containing the points A, B and C.

Now,

$$\vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$$

$\Rightarrow \quad \vec{AB} \times \vec{AC} = \vec{b} \times (\vec{c} - \vec{a}) - \vec{a} \times (\vec{c} - \vec{a})$ [By distributivity]

$\Rightarrow \quad \vec{AB} \times \vec{AC} = \vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}$

$\Rightarrow \quad \vec{AB} \times \vec{AC} = \vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0} = \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$ $[\because \vec{a} \times \vec{a} = \vec{0}]$

Hence, $\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}$ is a vector normal to the plane containing points having position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

EXAMPLE 25 *If $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $\vec{a} \neq \vec{0}$ and $\vec{b} \neq \vec{c}$, show that $\vec{b} = \vec{c} + t\,\vec{a}$ for some scalar t.*

SOLUTION We have,

$$\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or, } \vec{b} - \vec{c} = \vec{0} \text{ or, } \vec{a} \parallel (\vec{b} - \vec{c})$$

$$\Rightarrow \quad \vec{a} = \vec{0} \text{ or, } \vec{b} = \vec{c} \text{ or, } \vec{a} \parallel (\vec{b} - \vec{c})$$

$$\Rightarrow \quad \vec{a} \parallel (\vec{b} - \vec{c}) \qquad [\because \vec{a} \neq \vec{0} \text{ and } \vec{b} \neq \vec{c}]$$

$$\Rightarrow \quad \vec{b} - \vec{c} = t\,\vec{a} \text{ for some scalar } t \Rightarrow \vec{b} = \vec{c} + t\,\vec{a}$$

EXAMPLE 26 *Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ and interpret it geometrically.* **[NCERT]**

SOLUTION We have,

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{b} \qquad [\because \vec{a} \times \vec{a} = \vec{0} = \vec{b} \times \vec{b} \text{ and } -\vec{b} \times \vec{a} = \vec{a} \times \vec{b}]$$

$$= 2(\vec{a} \times \vec{b})$$

Geometrical Interpretation: Let *ABCD* be a parallelogram. Taking *A* as the origin, let the position vectors of *B* and *D* be $\vec{a}$ and $\vec{b}$ respectively. Then, $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$.

Applying the triangle law of addition of vectors in ΔABC, we get

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \vec{AC} = \vec{a} + \vec{b}$$

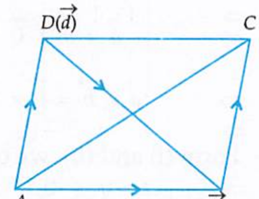

Fig. 24.13

Applying triangle law in ΔABD, we get

$$\vec{AD} + \vec{DB} = \vec{AB} \Rightarrow \vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}$$

$$\therefore \quad (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b}) \Leftrightarrow \vec{DB} \times \vec{AC} = 2(\vec{AB} \times \vec{AD})$$

The LHS of the above equality represents the vector area of parallelogram having $\vec{DB}$ and $\vec{AC}$ as adjacent sides and the RHS is twice the vector area of parallelogram having $\vec{AB}$ and $\vec{AD}$ as adjacent sides. So, it can be interpreted geometrically as follows:

The area of a parallelogram whose adjacent sides are the diagonals of a given parallelogram is twice the area of the given parallelogram.

EXAMPLE 27 *For any two vectors $\vec{a}$ and $\vec{b}$, show that:*

$$(1 + |\vec{a}|^2)(1 + |\vec{b}|^2) = \{(1 - \vec{a}.\vec{b}\}^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2 \qquad \textbf{[CBSE 2002]}$$

SOLUTION We have,

$$(1 - \vec{a}.\vec{b})^2 + |\vec{a} + \vec{b} + (\vec{a} \times \vec{b})|^2$$

$$= \left\{1 - 2(\vec{a}.\vec{b}) + (\vec{a}.\vec{b})^2\right\} + \left\{(\vec{a} + \vec{b} + \vec{a} \times \vec{b}).(\vec{a} + \vec{b} + \vec{a} \times \vec{b})\right\}$$

$$= \left\{1 - 2(\vec{a}.\vec{b}) + (\vec{a}.\vec{b})^2\right\} + \left\{(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{b}) \cdot (\vec{a} + \vec{b}) + (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})\right\}$$

$$=\left\{1-2(\vec{a}.\vec{b})+(\vec{a}.\vec{b})^2\right\}+\left\{|\vec{a}+\vec{b}|^2+\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{b}\cdot(\vec{a}\times\vec{b})+(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{a}\times\vec{b})\cdot\vec{b}+|\vec{a}\times\vec{b}|^2\right\}$$

$$=\left\{1-2(\vec{a}.\vec{b})+(\vec{a}.\vec{b})^2\right\}+\left\{|\vec{a}+\vec{b}|^2+|\vec{a}\times\vec{b}|^2\right\} \quad \left[\begin{array}{l}\because \vec{a}\perp(\vec{a}\times\vec{b}),\ \vec{b}\perp(\vec{a}\times\vec{b}) \\ \therefore \vec{a}.(\vec{a}\times\vec{b})=\vec{b}.(\vec{a}\times\vec{b})=0\end{array}\right]$$

$$=1-2(\vec{a}.\vec{b})+(\vec{a}.\vec{b})^2+|\vec{a}|^2+|\vec{b}|^2+2(\vec{a}.\vec{b})+|\vec{a}\times\vec{b}|^2$$

$$=1+|\vec{a}|^2+|\vec{b}|^2+(\vec{a}.\vec{b})^2+|\vec{a}\times\vec{b}|^2$$

$$=1+|\vec{a}|^2+|\vec{b}|^2+|\vec{a}|^2|\vec{b}|^2=\left(1+|\vec{a}|^2\right)\left(1+|\vec{b}|^2\right) \quad \left[\because (\vec{a}.\vec{b})^2+|\vec{a}\times\vec{b}|^2=|\vec{a}|^2|\vec{b}|^2\right]$$

Hence, $\left(1+|\vec{a}|^2\right)\left(1+|\vec{b}|^2\right)=1-\left(\vec{a}.\vec{b}\right)^2+|\vec{a}+\vec{b}+\vec{a}\times\vec{b}|^2$

EXAMPLE 28 *If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=\hat{j}-\hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{a}.\vec{b}=3$.* **[CBSE 2008, 2013, NCERT EXEAMPLAR]**

SOLUTION Let $\vec{b}=x\hat{i}+y\hat{j}+z\hat{k}$. Then,

$$\vec{a}\times\vec{b}=\vec{c}$$

$$\Rightarrow \quad \begin{vmatrix}\hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z\end{vmatrix}=(\hat{j}-\hat{k})$$

$$\Rightarrow \quad (z-y)\hat{i}-(z-x)\hat{j}+(y-x)\hat{k}=0\hat{i}+\hat{j}-\hat{k}$$

$\Rightarrow \quad z-y=0, -(z-x)=1, y-x=-1$ [On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$]

$\Rightarrow \quad y=z, x-z=1, x-y=1$

$\Rightarrow \quad x-z=1, x-y=1$ [$\because$ These two equations are equivalent to $y=z$]

$\Rightarrow \quad z=x-1$ and $y=x-1$. ...(i)

It is given that

$$\vec{a}.\vec{b}=3$$

$$\Rightarrow \quad (\hat{i}+\hat{j}+\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})=3$$

$\Rightarrow \quad x+y+z=3 \Rightarrow x+x-1+x-1=3 \Rightarrow 3x=5 \Rightarrow x=\frac{5}{3}$ [Using (i)]

$\therefore \quad y=x-1 \Rightarrow y=\frac{5}{3}-1=\frac{2}{3}$ and $y=z \Rightarrow z=\frac{2}{3}$.

Hence, $\vec{b}=\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{2}{3}\hat{k}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 29 *Let $\overrightarrow{OA}=\vec{a}$, $\overrightarrow{OB}=10\vec{a}+2\vec{b}$, and $\overrightarrow{OC}=\vec{b}$ where O is origin. Let p denote the area of the quadrilateral OABC and q denote the area of the parallelogram with OA and OC as adjacent sides. Prove that $p=6q$.*

SOLUTION We have,

$$p = \text{Area of the quadrilateral } OABC$$

$$\Rightarrow \quad p = \frac{1}{2}|\vec{OB} \times \vec{AC}|$$

$$\Rightarrow \quad p = \frac{1}{2}|\vec{OB} \times (\vec{OC} - \vec{OA})| \qquad [\because \vec{AC} = \vec{OC} - \vec{OA}]$$

$$\Rightarrow \quad p = \frac{1}{2}|(10\vec{a} + 2\vec{b}) \times (\vec{b} - \vec{a})| \qquad [\because \vec{OB} = 10\vec{a} + 2\vec{b}, \vec{OC} = \vec{b} \text{ and } \vec{OA} = \vec{a}]$$

$$\Rightarrow \quad p = \frac{1}{2}|10\vec{a} \times (\vec{b} - \vec{a}) + 2\vec{b} \times (\vec{b} - \vec{a})|$$

$$\Rightarrow \quad p = \frac{1}{2}|10(\vec{a} \times \vec{b}) - 10(\vec{a} \times \vec{a}) + 2(\vec{b} \times \vec{b}) - 2(\vec{b} \times \vec{a})|$$

$$\Rightarrow \quad p = \frac{1}{2}|10(\vec{a} \times \vec{b}) - 0 + 0 + 2(\vec{a} \times \vec{b})| = \frac{1}{2}|12(\vec{a} \times \vec{b})| = 6|\vec{a} \times \vec{b}| \qquad \text{...(i)}$$

and, q = Area of the parallelogram with OA and OC as adjacent sides

$$\Rightarrow \quad q = |\vec{OA} \times \vec{OC}| = |\vec{a} \times \vec{b}| \qquad \text{...(ii)}$$

From (i) and (ii), we get: $p = 6q$

EXAMPLE 30 *ABCD is a quadrilateral such that* $\vec{AB} = \vec{b}$, $\vec{AD} = \vec{d}$, $\vec{AC} = m\vec{b} + p\vec{d}$. *Show that the area of the quadrilateral ABCD is* $\frac{1}{2}|m+p|\,|\vec{b} \times \vec{d}|$.

SOLUTION We have, $\vec{AB} = \vec{b}$, $\vec{AD} = \vec{d}$ and $\vec{AC} = m\vec{b} + p\vec{d}$.

Now, $\vec{AB} + \vec{BD} = \vec{AD} \Rightarrow \vec{BD} = \vec{AD} - \vec{AB} \Rightarrow \vec{BD} = \vec{d} - \vec{b}$

Let Δ be the area of quadrilateral $ABCD$. Then,

$$\Delta = \frac{1}{2}|\vec{AC} \times \vec{BD}|$$

$$\Rightarrow \quad \Delta = \frac{1}{2}|(m\vec{b} + p\vec{d}) \times (\vec{d} - \vec{b})| = \frac{1}{2}|m\vec{b} \times \vec{d} + p\vec{d} \times \vec{d} - m\vec{b} \times \vec{b} - p\vec{d} \times \vec{b}|$$

$$\Rightarrow \quad \Delta = \frac{1}{2}|m(\vec{b} \times \vec{d}) - p(\vec{d} \times \vec{b})| = \frac{1}{2}|m(\vec{b} \times \vec{d}) + p(\vec{b} \times \vec{d})|$$

$$\Rightarrow \quad \Delta = \frac{1}{2}|(m+p)(\vec{b} \times \vec{d})| = \frac{1}{2}|m+p|\,|\vec{b} \times \vec{d}|.$$

EXAMPLE 31 *If* A, B, C, D *be any four points in space, prove that*

$$|\vec{AB} \times \vec{CD} + \vec{BC} \times \vec{AD} + \vec{CA} \times \vec{BD}| = 4\,(\textit{Area of triangle } ABC).$$

SOLUTION Taking A as the origin, let the position vectors of B, C and D be $\vec{b}$, $\vec{c}$ and $\vec{d}$ respectively. Then, $\vec{AB} = \vec{b}$, $\vec{CD} = \vec{d} - \vec{c}$, $\vec{BC} = \vec{c} - \vec{b}$, $\vec{AD} = \vec{d}$, $\vec{CA} = -\vec{c}$ and, $\vec{BD} = \vec{d} - \vec{b}$

$$\therefore \quad |\vec{AB} \times \vec{CD} + \vec{BC} \times \vec{AD} + \vec{CA} \times \vec{BD}| = |\vec{b} \times (\vec{d} - \vec{c}) + (\vec{c} - \vec{b}) \times \vec{d} + (-\vec{c}) \times (\vec{d} - \vec{b})|$$

$$= |\vec{b} \times \vec{d} - \vec{b} \times \vec{c} + \vec{c} \times \vec{d} - \vec{b} \times \vec{d} - \vec{c} \times \vec{d} + \vec{c} \times \vec{b}|$$

$$= |-\vec{b} \times \vec{c} + \vec{c} \times \vec{b}| = |-2(\vec{b} \times \vec{c})| = 2|\vec{b} \times \vec{c}|$$

$$= 2|\vec{AB} \times \vec{AC}|$$

$$= 4\,(\text{Area of } \Delta\, ABC) \qquad \left[\because \text{ Area of } \Delta\, ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|\right]$$

EXAMPLE 32 *A triangle OAB is determined by the vectors $\vec{a}$ and $\vec{b}$ as shown in Fig. 24.14. Show that the triangle has the area Δ given by* $\Delta = \frac{1}{2}\sqrt{|\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$.

SOLUTION Clearly, $\Delta = \frac{1}{2}\,|\vec{OA} \times \vec{OB}|$

$\Rightarrow \quad \Delta^2 = \frac{1}{4}\,|\vec{a}\times\vec{b}|^2 = \frac{1}{4}\left\{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2\right\}$

$\Rightarrow \quad \Delta = \frac{1}{2}\sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$

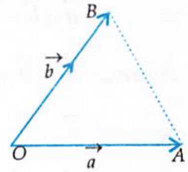

Fig. 24.14

EXAMPLE 33 *Let $\vec{a}, \vec{b}, \vec{c}$ be unit vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c} = 0$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{6}$, prove that $\vec{a} = \pm 2(\vec{b}\times\vec{c})$.*

SOLUTION We have,

$\vec{a}\cdot\vec{b} = 0$ and $\vec{a}\cdot\vec{c} = 0$

$\Rightarrow \quad \vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a}$ is perpendicular to both $\vec{b}$ and $\vec{c}$

$\Rightarrow \quad \vec{a}$ is perpendicular to the plane containing vectors $\vec{b}$ and $\vec{c}$.

$\Rightarrow \quad \vec{a}$ is parallel to $\vec{b}\times\vec{c}$ $\quad \left[\because \vec{b}\times\vec{c}\text{ is perpendicular to the plane containing } \vec{b} \text{ and } \vec{c}\right]$

$\Rightarrow \quad \vec{a} = \lambda(\vec{b}\times\vec{c})$ for some scalar λ ...(i)

$\Rightarrow \quad |\vec{a}| = |\lambda|\,|\vec{b}\times\vec{c}|$

$\Rightarrow \quad |\vec{a}| = |\lambda|\,|\vec{b}|\,|\vec{c}|\sin\frac{\pi}{6} \Rightarrow 1 = \frac{|\lambda|}{2} \Rightarrow |\lambda| = 2 \Rightarrow \lambda = \pm 2 \quad [\because |\vec{a}| = |\vec{b}| = |\vec{c}| = 1]$

Putting this value of λ in (i), we get: $\vec{a} = \pm 2(\vec{b}\times\vec{c})$.

EXAMPLE 34 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ and $\vec{b}$ and $\vec{c}$ are not parallel vectors, prove that $\vec{a} = \lambda\vec{b} + \mu\vec{c}$, where λ and μ are scalars.*

SOLUTION We have,

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \Rightarrow \vec{a} = \vec{0}$ or, $\vec{b}\times\vec{c} = \vec{0}$ or, $\vec{a}\perp(\vec{b}\times\vec{c}) \Rightarrow \vec{a} = \vec{0}$ or, $\vec{b}\,||\,\vec{c}$ or, $\vec{a}\perp(\vec{b}\times\vec{c})$

But, $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane containing $\vec{b}$ and $\vec{c}$.

$\therefore \quad \vec{a}\perp(\vec{b}\times\vec{c})$

$\Rightarrow \quad \vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow \quad \vec{a}, \vec{b}, \vec{c}$ are coplanar vectors

$\Rightarrow \quad \vec{a} = \lambda\vec{b} + \mu\vec{c}$ for some scalars λ and μ. (see Fig. 24.15)

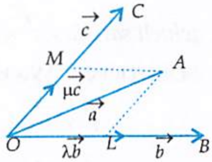

Fig. 24.15

EXAMPLE 35 *If $\vec{a}, \vec{b}, \vec{c}$ are vectors such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$, then show that $\vec{b} = \vec{c}$.*

SOLUTION We have,

$\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ and $\vec{a}\neq\vec{0}$

$\Rightarrow \quad \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a} \cdot (\vec{b} - \vec{c}) = 0$ and $\vec{a} \neq \vec{0} \Rightarrow \vec{b} - \vec{c} = \vec{0}$ or, $\vec{a} \perp (\vec{b} - \vec{c}) \Rightarrow \vec{b} = \vec{c}$ or, $\vec{a} \perp (\vec{b} - \vec{c})$ ...(i)

Again, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0}$ and $\vec{a} \neq \vec{0}$

$\Rightarrow \quad \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$ and $\vec{a} \neq \vec{0} \Rightarrow \vec{b} - \vec{c} = \vec{0}$ or, $\vec{a} \parallel (\vec{b} - \vec{c}) \Rightarrow \vec{b} = \vec{c}$ or, $\vec{a} \parallel (\vec{b} - \vec{c})$ ...(ii)

From (i) and (ii), it follows that $\vec{b} = \vec{c}$, because $\vec{a}$ cannot be both parallel and perpendicular to vectors $(\vec{b} - \vec{c})$.

EXAMPLE 36 *If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$; prove that $\vec{a}, \vec{b}, \vec{c}$ are mutually at right angles such that $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$.*

SOLUTION We have,

$$\vec{a} \times \vec{b} = \vec{c} \text{ and } \vec{a} = \vec{b} \times \vec{c}$$

$\Rightarrow \quad \vec{c} \perp \vec{a}, \ \vec{c} \perp \vec{b}$ and $\vec{a} \perp \vec{b}, \ \vec{a} \perp \vec{c}$

$\Rightarrow \quad \vec{a} \perp \vec{b}, \ \vec{b} \perp \vec{c}$ and $\vec{c} \perp \vec{a} \Rightarrow \vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors.

Again, $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{c}|$ and $|\vec{b} \times \vec{c}| = |\vec{a}|$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin\frac{\pi}{2} = |\vec{c}|$ and $|\vec{b}|\,|\vec{c}| \sin\frac{\pi}{2} = |\vec{a}|$ $\quad [\because \vec{a} \perp \vec{b}$ and $\vec{b} \perp \vec{c}]$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| = |\vec{c}|$ and $|\vec{b}|\,|\vec{c}| = |\vec{a}|$

$\Rightarrow \quad |b|^2\, |\vec{c}| = |\vec{c}|$ $\quad$ [Putting $|\vec{a}| = |\vec{b}|\,|\vec{c}|$ in $|\vec{a}|\,|\vec{b}| = |\vec{c}|$]

$\Rightarrow \quad |\vec{b}|^2 = 1 \Rightarrow |\vec{b}| = 1$ $\quad [\because |\vec{c}| \neq 0]$

Putting $|\vec{b}| = 1$ in $|\vec{a}|\,|\vec{b}| = |\vec{c}|$, we obtain $|\vec{a}| = |\vec{c}|$.

EXAMPLE 37 *Let $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ be three vectors. Find a vector $\vec{r}$ which satisfies $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$.*

SOLUTION We have,

$$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$$

$\Rightarrow \quad \vec{r} \times \vec{b} - \vec{c} \times \vec{b} = \vec{0}$

$\Rightarrow \quad (\vec{r} - \vec{c}) \times \vec{b} = \vec{0}$

$\Rightarrow \quad \vec{r} - \vec{c}$ is parallel to $\vec{b} \Rightarrow \vec{r} - \vec{c} = \lambda \vec{b}$ for some scalar $\lambda \Rightarrow \vec{r} = \vec{c} + \lambda \vec{b}$. ...(i)

It is also given that

$$\vec{r}\,.\,\vec{a}=0$$

$$\Rightarrow \quad (\vec{c}+\lambda\,\vec{b})\,.\,\vec{a}=0 \qquad [\because\ \vec{r}=\vec{c}+\lambda\,\vec{b}\ \text{(from (i)]}$$

$$\Rightarrow \quad (\vec{c}\,.\,\vec{a})+\lambda(\vec{b}\,.\,\vec{a})=0 \Rightarrow \lambda=-\frac{\vec{c}\,.\,\vec{a}}{\vec{b}\,.\,\vec{a}} \Rightarrow \lambda=-\left(\frac{8+7}{2+0+1}\right)=-\frac{15}{3}=-5$$

Putting $\lambda=-5$ in (i), we obtain

$$\vec{r}=\vec{c}-5\vec{b}=(4\hat{i}-3\hat{j}+7\hat{k})-5(\hat{i}+\hat{j}+\hat{k})=-\hat{i}-8\hat{j}+2\hat{k}$$

EXAMPLE 38 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors such that $|\vec{a}+\vec{b}+\vec{c}|=1$, $\vec{c}=\lambda(\vec{a}\times\vec{b})$ and $|\vec{a}|=\frac{1}{\sqrt{2}}$, $|\vec{b}|=\frac{1}{\sqrt{3}}$, $|\vec{c}|=\frac{1}{\sqrt{6}}$, find the angle between $\vec{a}$ and $\vec{b}$.*

SOLUTION We have,

$$\vec{c}=\lambda(\vec{a}\times\vec{b}) \Rightarrow \vec{c} \text{ is perpendicular to both } \vec{a} \text{ and } \vec{b} \Rightarrow \vec{c}\cdot\vec{a}=\vec{c}\cdot\vec{b}=0$$

Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then,

$$|\vec{a}+\vec{b}+\vec{c}|=1$$

$$\Rightarrow \quad |\vec{a}+\vec{b}+\vec{c}|^2=1$$

$$\Rightarrow \quad |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=1$$

$$\Rightarrow \quad \frac{1}{2}+\frac{1}{3}+\frac{1}{6}+2\left(\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}\cos\theta+0+0\right)=1 \qquad \left[\because\ \vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta=\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}\cos\theta\right]$$

$$\Rightarrow \quad 1+2\left(\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}\cos\theta\right)=1 \Rightarrow \sqrt{\frac{2}{3}}\cos\theta=0 \Rightarrow \cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$$

Hence, the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{2}$.

24.3 APPLICATIONS OF CROSS PRODUCT IN GEOMETRICAL PROBLEMS

In this section, we shall use cross-product in proving some geometrical results. Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *In a triangle ABC, prove by vector method that* $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$.

SOLUTION Let $\vec{BC}=\vec{a}$, $\vec{CA}=\vec{b}$ and $\vec{AB}=\vec{c}$. Then, $|\vec{a}|=a$, $|\vec{b}|=b$, $|\vec{c}|=c$.

Using triangle law of addition of vectors in ΔABC, we obtain

$$\vec{BC}+\vec{CA}=\vec{BA}$$

$$\Rightarrow \quad \vec{a}+\vec{b}=-\vec{c}$$

$$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c}=\vec{0}$$

$$\Rightarrow \quad \vec{a} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{a} \times \vec{0} \qquad \text{[Taking cross-product on the left of both sides]}$$

$$\Rightarrow \quad \vec{a} \times \vec{a} + \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = -\vec{a} \times \vec{c} \qquad [\because \vec{a} \times \vec{a} = \vec{0}]$$

$$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{c} \times \vec{a} \qquad \text{...(i)}$$

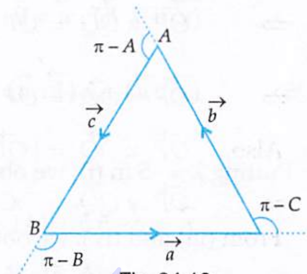

Fig. 24.16

Again,

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{b} \times (\vec{a} + \vec{b} + \vec{c}) = \vec{b} \times \vec{0}$$

$$\Rightarrow \quad \vec{b} \times \vec{a} + \vec{b} \times \vec{b} + \vec{b} \times \vec{c} = \vec{0}$$

$$\Rightarrow \quad \vec{b} \times \vec{a} + \vec{b} \times \vec{c} = 0 \qquad [\because \vec{b} \times \vec{b} = \vec{0}]$$

$$\Rightarrow \quad \vec{b} \times \vec{c} = -\vec{b} \times \vec{a} \Rightarrow \vec{b} \times \vec{c} = \vec{a} \times \vec{b} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$$

$$\Rightarrow \quad |\vec{a} \times \vec{b}| = |\vec{b} \times \vec{c}| = |\vec{c} \times \vec{a}|$$

$$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin(\pi - C) = |\vec{b}|\,|\vec{c}| \sin(\pi - A) = |\vec{c}|\,|\vec{a}| \sin(\pi - B)$$

$$\Rightarrow \quad ab \sin(\pi - C) = bc \sin(\pi - A) = ca \sin(\pi - B)$$

$$\Rightarrow \quad ab \sin C = bc \sin A = ca \sin B$$

$$\Rightarrow \quad \frac{\sin C}{c} = \frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \qquad \text{[Dividing throughout by } abc\text{]}$$

EXAMPLE 2 *Prove by vector method that*

(i) $\sin(A - B) = \sin A \cos B - \cos A \sin B$ (ii) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

SOLUTION Let OX and OY be two mutually perpendicular lines taken as axes and let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY respectively.

(i) Let $\vec{OP}$ and $\vec{OQ}$ be two vectors such that $\angle XOP = A$ and $\angle XOQ = B$. Then, $\angle POQ = A - B$.

Draw $PL \perp OX$ and $QM \perp OX$. Let $\hat{k}$ be the unit vector along Z-axis.

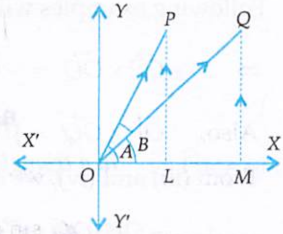

Fig. 24.17

In ΔOPL, we obtain: $OL = OP \cos A$, $PL = OP \sin A$

$$\therefore \quad \vec{OP} = \vec{OL} + \vec{LP} = (OP \cos A)\hat{i} + (OP \sin A)\hat{j} \qquad \text{...(i)}$$

In ΔOQM, we obtain: $OM = OQ \cos B$, $QM = OQ \sin B$

$$\therefore \quad \vec{OQ} = \vec{OM} + \vec{MQ} = (OQ \cos B)\hat{i} + (OQ \sin B)\hat{j} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\vec{OP} \times \vec{OQ} = \left\{(OP \cos A)\hat{i} + (OP \sin A)\hat{j}\right\} \times \left\{(OQ \cos B)\hat{i} + (OQ \sin B)\hat{j}\right\}$$

$$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{(\cos A)\hat{i} + (\sin A)\hat{j}\right\} \times \left\{(\cos B)\hat{i} + (\sin B)\hat{j}\right\}$$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{ \cos A \sin B\,(\hat{i} \times \hat{j}) + \sin A \cos B\,(\hat{j} \times \hat{i}) \right\}$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{ \cos A \sin B\,\hat{k} - \sin A \cos B\,\hat{k} \right\}$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \{\cos A \sin B - \sin A \cos B\}\,\hat{k}$ ...(iii)

Also, $\vec{OP} \times \vec{OQ} = |\vec{OP}|\,|\vec{OQ}| \sin (A - B)\,(-\hat{k})$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = -OP \cdot OQ \sin (A - B)\,\hat{k}$ ...(iv)

From (iii) and (iv), we obtain

$$-OP \cdot OQ \sin (A - B)\,\hat{k} = -OP \cdot OQ\ (\sin A \cos B - \cos A \sin B)\,\hat{k}$$

$\Rightarrow \quad \sin (A - B) = \sin A \cos B - \cos A \sin B.$

(ii) Let $\vec{OP}$ and $\vec{OQ}$ be two vectors such that $\angle XOP = A$ and $\angle XOQ = B$. Then, $\angle QOP = A + B$. Draw $PL \perp OX$ and $QM \perp OX$.

In ΔOPL, we have

$OL = OP \cos A$ and $PL = OP \sin A$

$\therefore \quad \vec{OP} = \vec{OL} + \vec{LP} = (OP \cos A)\,\hat{i} + (OP \sin A)\,\hat{j}$ ...(i)

In ΔOQM, we have

$OM = OQ \cos B$, $MQ = OQ \sin B$

$\therefore \quad \vec{OQ} = \vec{OM} + \vec{MQ} = (OQ \cos B)\,\hat{i} - (OQ \sin B)\,\hat{j}$ ...(ii)

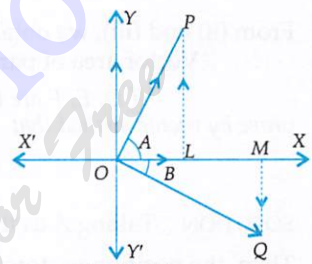

Fig. 24.18

From (i) and (ii), we obtain

$$\vec{OP} \times \vec{OQ} = \left\{ (OP \cos A)\,\hat{i} + (OP \sin A)\,\hat{j} \right\} \times \left\{ (OQ \cos B)\,\hat{i} - (OQ \sin B)\,\hat{j} \right\}$$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{ (\cos A)\,\hat{i} + (\sin A)\,\hat{j} \right\} \times \left\{ (\cos B)\,\hat{i} - (\sin B)\,\hat{j} \right\}$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{ -\cos A \sin B\,(\hat{i} \times \hat{j}) + \sin A \cos B\,(\hat{j} \times \hat{i}) \right\}$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = OP \cdot OQ \left\{ -(\cos A \sin B)\,\hat{k} - (\sin A \cos B)\,\hat{k} \right\}$

$\Rightarrow \quad \vec{OP} \times \vec{OQ} = -OP \cdot OQ \{\cos A \sin B + \sin A \cos B\}\,\hat{k}$ ...(iii)

Also, $\vec{OP} \times \vec{OQ} = |\vec{OP}|\,|\vec{OQ}| \sin (A + B)\,(-\hat{k}) = -OP \cdot OQ \sin (A + B)\,\hat{k}$ ...(iv)

From (iii) and (iv), we obtain

$$-OP \cdot OQ \sin (A + B)\,\hat{k} = -OP \cdot OQ\,(\cos A \sin B + \sin A \cos B)\,\hat{k}$$

$\Rightarrow \quad \sin (A + B) = \sin A \cos B + \cos A \sin B.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Prove by vector method that the parallelogram on the same base and between the same parallels are equal in area.*

SOLUTION Let $OACB$ and $OAED$ be parallelograms on the same base OA and between the same parallels OA and BE. Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Since $\vec{BD}$ is parallel to $\vec{OA}$. Therefore, $\vec{BD} = x\vec{OA} = x\,\vec{a}$ for some scalar x.

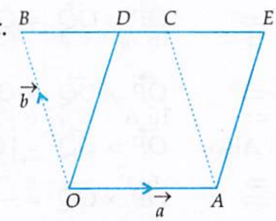

Fig. 24.19

Using triangle law of addition of vectors, in ΔOBD, we obtain

$$\vec{OB} + \vec{BD} = \vec{OD} \Rightarrow \vec{OD} = \vec{b} + x\,\vec{a} \qquad \ldots\text{(i)}$$

Now, Vector area of parallelogram $OACB = \vec{OA} \times \vec{OB} = \vec{a} \times \vec{b}$ ...(ii)

and,

Vector area of parallelogram $OAED = \vec{OA} \times \vec{OD}$

$$= \vec{a} \times (\vec{b} + x\,\vec{a}) \qquad \text{[Using (i)]}$$

$$= \vec{a} \times \vec{b} + x(\vec{a} \times \vec{a})$$

$$= \vec{a} \times \vec{b} \qquad [\because\ \vec{a} \times \vec{a} = \vec{0}] \qquad \ldots\text{(iii)}$$

From (ii) and (iii), we obtain

Vector area of parallelogram $OAED$ = Vector area of parallelogram $OACB$.

EXAMPLE 4 *If D, E, F are the mid-points of the sides BC, CA and AB respectively of a triangle ABC, prove by vector method that*

$$\textit{Area of } \Delta DEF = \frac{1}{4}\,(\textit{Area of } \Delta ABC).$$

SOLUTION Taking A as the origin, let the position vectors of B and C be $\vec{b}$ and $\vec{c}$ respectively. Then, the position vectors of D, E and F are $\frac{1}{2}(\vec{b} + \vec{c})$, $\frac{1}{2}\vec{c}$ and $\frac{1}{2}\vec{b}$ respectively.

$$\therefore \quad \vec{DE} = \text{P.V. of } E - \text{P.V. of } D = \frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b} + \vec{c}) = -\frac{\vec{b}}{2}$$

$$\text{and,} \quad \vec{DF} = \text{P.V of } F - \text{P.V. of } D = \frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b} + \vec{c}) = \frac{-\vec{c}}{2}$$

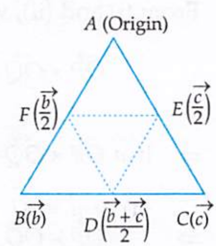

Fig. 24.20

$$\therefore \quad \text{Vector area of } \Delta DEF = \frac{1}{2}(\vec{DE} \times \vec{DF}) = \frac{1}{2}\left(-\frac{\vec{b}}{2} \times -\frac{\vec{c}}{2}\right) = \frac{1}{8}(\vec{b} \times \vec{c})$$

$$= \frac{1}{4}\left\{\frac{1}{2}(\vec{AB} \times \vec{AC})\right\} = \frac{1}{4}(\text{Vector area of } \Delta ABC)$$

Hence, Area of $\Delta DEF = \frac{1}{4}$ Area of ΔABC.

EXAMPLE 5 *Using vectors : Prove that if a, b, c are the lengths of three sides of a triangle, then its area Δ is given by $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$.*

SOLUTION Let $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$. Then, $|\vec{BC}| = |\vec{a}| = a$, $|\vec{CA}| = |\vec{b}| = b$ and $|\vec{AB}| = |\vec{c}| = c$. Using triangle law of addition of vectors in ΔABC, we obtain

$$\vec{BC} + \vec{CA} = \vec{BA} \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

Now,

$$\Delta = \frac{1}{2}|\vec{BC} \times \vec{BA}| = \frac{1}{2}|\vec{b} \times -\vec{c}| = \frac{1}{2}|\vec{b} \times \vec{c}|$$

$$\Rightarrow \quad 2\Delta = |\vec{b} \times \vec{c}|$$

$$\Rightarrow \quad 4\Delta^2 = |\vec{b} \times \vec{c}|^2$$

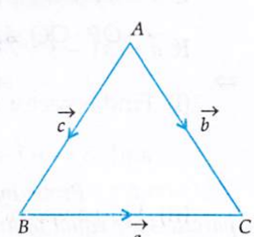

Fig. 24.21

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b} \times \vec{c}|^2$

$\Rightarrow \quad 16\,\Delta^2 = 4\left\{-(\vec{b}\cdot\vec{c})^2 + |\vec{b}|^2|\vec{c}|^2\right\}$ [Using Lagrange's identity]

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b}|^2|\vec{c}|^2 - 4(\vec{b}\cdot\vec{c})^2$

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b}|^2|\vec{c}|^2 - \left\{-2(\vec{b}\cdot\vec{c})\right\}^2$

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b}|^2|\vec{c}|^2 - \left\{|\vec{b}|^2 + |\vec{c}|^2 - |\vec{b}+\vec{c}|^2\right\}^2$

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b}|^2|\vec{c}|^2 - \left\{|\vec{b}|^2 + |\vec{c}|^2 - |-\vec{a}|^2\right\}^2$ $[\because \vec{a}+\vec{b}+\vec{c}=\vec{0} \Rightarrow \vec{b}+\vec{c}=-\vec{a}]$

$\Rightarrow \quad 16\,\Delta^2 = 4|\vec{b}|^2|\vec{c}|^2 - \left\{|\vec{b}|^2 + |\vec{c}|^2 - |\vec{a}|^2\right\}^2$

$\Rightarrow \quad 16\,\Delta^2 = (2bc)^2 - \left(b^2+c^2-a^2\right)^2 = (2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)$

$\Rightarrow \quad 16\,\Delta^2 = \left\{(b+c)^2 - a^2\right\}\left\{a^2-(b-c)^2\right\} = (b+c+a)(b+c-a)(a+b-c)(a-b+c)$

$\Rightarrow \quad 16\,\Delta^2 = (2s)(2s-2a)(2s-2c)(2s-2b) = 16\,s(s-a)(s-b)(s-c)$

$\Rightarrow \quad \Delta^2 = s(s-a)(s-b)(s-c) \Rightarrow \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

EXERCISE 24.1

BASIC

1. If $\vec{a} = \hat{i} + 3\hat{j} - 2\hat{k}$ and $\vec{b} = -\hat{i} + 3\hat{k}$, find $|\vec{a} \times \vec{b}|$.
2. (i) If $\vec{a} = 3\hat{i} + 4\hat{j}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the value of $|\vec{a} \times \vec{b}|$.

   (ii) If $\vec{a} = 2\hat{i} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, find the magnitude of $\vec{a} \times \vec{b}$.
3. (i) Find a unit vector perpendicular to both the vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} + \hat{j} - 2\hat{k}$.

   (ii) Find a unit vector perpendicular to the plane containing the vectors $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.
4. Find the magnitude of vector $\vec{a} = (3\hat{k} + 4\hat{j}) \times (\hat{i} + \hat{j} - \hat{k})$.
5. If $\vec{a} = 4\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{k}$, then find $|2\vec{b} \times \vec{a}|$.
6. If $\vec{a} = 3\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, find $(\vec{a} + 2\vec{b}) \times (2\vec{a} - \vec{b})$.
7. (i) Find a vector of magnitude 49, which is perpendicular to both the vectors $2\hat{i} + 3\hat{j} + 6\hat{k}$ and $3\hat{i} - 6\hat{j} + 2\hat{k}$.

   (ii) Find a vector whose length is 3 and which is perpendicular to the vector $\vec{a} = 3\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 6\hat{i} + 5\hat{j} - 2\hat{k}$.

8. Find the area of the parallelogram determined by the vectors :

(i) $2\hat{i}$ and $3\hat{j}$ (ii) $2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{i}-\hat{j}$

(iii) $3\hat{i}+\hat{j}-2\hat{k}$ and $\hat{i}-3\hat{j}+4\hat{k}$ (iv) $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$. **[CBSE 2002C]**

9. Find the area of the parallelogram whose diagonals are :

(i) $4\hat{i}-\hat{j}-3\hat{k}$ and $-2\hat{i}+\hat{j}-2\hat{k}$ (ii) $2\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$

(iii) $3\hat{i}+4\hat{j}$ and $\hat{i}+\hat{j}+\hat{k}$ (iv) $2\hat{i}+3\hat{j}+6\hat{k}$ and $3\hat{i}-6\hat{j}+2\hat{k}$

10. If $\vec{a}=2\hat{i}+5\hat{j}-7\hat{k}$, $\vec{b}=-3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}-3\hat{k}$, compute $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ and verify that these are not equal.

11. (i) If $|\vec{a}|=2$, $|\vec{b}|=5$ and $|\vec{a}\times\vec{b}|=8$, find $\vec{a}\cdot\vec{b}$.

(ii) If $|\vec{a}|=13$, $|\vec{b}|=5$ and $\vec{a}\cdot\vec{b}=60$, then find $|\vec{a}\times\vec{b}|$.

12. Given $\vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})$, $\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})$, $\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$, $\hat{i}$, $\hat{j}$, $\hat{k}$ being a right handed orthogonal system of unit vectors in space, show that $\vec{a}$, $\vec{b}$, $\vec{c}$ is also another system.

13. If $\vec{a}=\hat{i}+\hat{j}-2\hat{k}$, $\vec{b}=-\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{c}=-\hat{i}+2\hat{j}-\hat{k}$ are three vectors, then find a vector perpendicular to both the vectors $(\vec{a}+\vec{b})$ and $(\vec{b}-\vec{c})$. **[CBSE 2022]**

14. Find the angle between two vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a}\times\vec{b}|=\vec{a}\cdot\vec{b}$.

15. If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$. **[CBSE 2018]**

16. If $|\vec{a}|=2$, $|\vec{b}|=7$ and $\vec{a}\times\vec{b}=3\hat{i}+2\hat{j}+6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

17. If $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=400$ and $|\vec{a}|=5$, then write the value of $|\vec{b}|$. **[CBSE 2016]**

18. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}-5\hat{k}$ and $2\hat{i}+2\hat{j}+3\hat{k}$. Find the two unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram. **[CBSE 2016, 2022]**

19. Find a unit vector perpendicular to the plane ABC, where the coordinates of A, B and C are $A(3,-1,2)$, $B(1,-1,-3)$ and $C(4,-3,1)$.

20. Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i}+2\hat{j}+\hat{k}$ and $-\hat{i}+3\hat{j}+4\hat{k}$. **[NCERT EXEMPLAR]**

21. If $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$, and $\vec{b}=2\hat{i}+3\hat{j}-5\hat{k}$, then find $\vec{a}\times\vec{b}$. Verify that $\vec{a}$ and $\vec{a}\times\vec{b}$ are perpendicular to each other.

22. Using vectors, find the area of the triangle with vertices: (i) $A(1,1,2)$, $B(2,3,5)$ and $C(1,5,5)$ (ii) $A(1,2,3)$, $B(2,-1,4)$ and $C(4,5,-1)$. **[CBSE 2011, 2020, NCERT EXEMPLAR]**

23. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find its area. **[CBSE 2012]**

24. If $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + \hat{k}$, $\vec{c} = 2\hat{j} - \hat{k}$ are three vectors, find the area of the parallelogram having diagonals $(\vec{a} + \vec{b})$ and $(\vec{b} + \vec{c})$. [CBSE 2014]

25. If $|\vec{a}| = \sqrt{26}$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 35$, find $\vec{a} \cdot \vec{b}$. [CBSE 2002]

26. Find the area of the triangle formed by O, A, B when $\vec{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$.

27. Using vectors find the area of the triangle with vertices, $A(2, 3, 5)$, $B(3, 5, 8)$ and $C(2, 7, 8)$. [CBSE 2010]

BASED ON LOTS

28. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$. [CBSE 2011]

29. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$, then show that $\vec{a} + \vec{c} = m\vec{b}$, where m is any scalar.

30. What inference can you draw if $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$. [NCERT]

31. If $\vec{a}, \vec{b}, \vec{c}$ are three unit vectors such that $\vec{a} \times \vec{b} = \vec{c}$, $\vec{b} \times \vec{c} = \vec{a}$, $\vec{c} \times \vec{a} = \vec{b}$. Show that $\vec{a}, \vec{b}, \vec{c}$ form an orthonormal right handed triad of unit vectors.

32. If a, b, c are the lengths of sides, BC, CA and AB of a triangle ABC, prove that $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$ and deduce that $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

33. If $\vec{p}$ and $\vec{q}$ are unit vectors forming an angle of 30°; find the area of the parallelogram having $\vec{a} = \vec{p} + 2\vec{q}$ and $\vec{b} = 2\vec{p} + \vec{q}$ as its diagonals.

34. For any two vectors $\vec{a}$ and $\vec{b}$, prove that $|\vec{a} \times \vec{b}|^2 = \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix}$.

35. Define $\vec{a} \times \vec{b}$ and prove that $|\vec{a} \times \vec{b}| = (\vec{a} \cdot \vec{b}) \tan\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

36. (i) Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c} \cdot \vec{d} = 15$. [CBSE 2010, 2012, 2015, NCERT]

(ii) Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$. [CBSE 2018]

37. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example. [NCERT]

38. If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then verify that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$. [NCERT]

39. If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$, then find the vector $\vec{c}$, given that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 4$. [CBSE 2022]

40. $ABCD$ is a parallelogram such that $\vec{AC} = \hat{i} + \hat{j}$ and $\vec{BD} = 2\hat{i} + \hat{j} + \hat{k}$. Find $\vec{AB}$ and $\vec{AD}$. Also, find the area of the parallelogram $ABCD$. [CBSE 2022]

ANSWERS

1. $\sqrt{91}$
2. (i) $\sqrt{26}$ (ii) $\sqrt{6}$
3. (i) $\frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$ (ii) $\pm\frac{1}{\sqrt{11}}(\hat{i}+\hat{j}-3\hat{k})$
4. $\sqrt{74}$
5. $\sqrt{504}$
6. $-25\hat{i}+35\hat{j}-55\hat{k}$
7. (i) $42\hat{i}+14\hat{j}-21\hat{k}$ (ii) $2\hat{i}-2\hat{j}+\hat{k}$
8. (i) 6 sq. units (ii) $3\sqrt{3}$ sq. units (iii) $10\sqrt{3}$ sq. units (iv) $4\sqrt{2}$ sq. units
9. (i) $\frac{15}{2}$ sq. units (ii) $\frac{\sqrt{6}}{2}$ sq. units (iii) $\frac{\sqrt{26}}{2}$ sq. units (iv) $\frac{49}{2}$ sq. units
11. (i) 6 (ii) 25
13. $9\hat{i}$
14. $\frac{\pi}{4}$
15. $\frac{2\sqrt{6}}{7}$
16. $\frac{\pi}{6}$
17. 4
18. $\frac{2}{\sqrt{6}}\hat{i}-\frac{1}{\sqrt{6}}\hat{j}-\frac{1}{\sqrt{6}}\hat{k}, -\frac{3}{5}\hat{j}-\frac{4}{5}\hat{k}, \sqrt{404}$ sq. units
19. $\frac{1}{\sqrt{165}}(10\hat{i}+7\hat{j}-4\hat{k})$
20. $\frac{1}{\sqrt{165}}(10\hat{i}+7\hat{j}-4\hat{k})$
21. $\hat{i}+11\hat{j}+7\hat{k}$
22. (i) $\sqrt{61}$ sq. units (ii) $\frac{\sqrt{274}}{2}$ sq. units
23. $\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}, 11\sqrt{5}$ sq. units
24. $\frac{1}{2}\sqrt{21}$ sq. units
25. 7
26. $3\sqrt{5}$ sq. units
27. $\frac{\sqrt{21}}{2}$ sq. units
28. $\frac{1}{3}(2\hat{i}-2\hat{j}-\hat{k})$
30. either $\vec{a}=\vec{0}$ or, $\vec{b}=\vec{0}$
33. $\frac{3}{4}$ sq. units
36. (i) $\frac{5}{3}(32\hat{i}-\hat{j}+14\hat{k})$ (ii) $-\frac{1}{3}(\hat{i}-16\hat{j}-13\hat{k})$
37. No, Take any two collinear vectors
39. $-3\hat{j}+\hat{k}$
40. $\vec{AB}=\frac{1}{2}(-\hat{i}-\hat{k}), \vec{AD}=\frac{1}{2}(3\hat{i}+2\hat{j}+\hat{k})$, Area $=\frac{\sqrt{3}}{2}$ sq. units

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The value of the expression $|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2$ is ....
2. If $|\vec{a}\times\vec{b}|^2+|\vec{a}.\vec{b}|^2=144$ and $|\vec{a}|=4$, then $|\vec{b}|$ is equal to ....
3. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a}\times\vec{b}$ is also a unit vector, then the angle between $\vec{a}$ and $\vec{b}$ is ....
4. For any two vectors $\vec{a}$ and $\vec{b}$, $(2\vec{a}+3\vec{b})\times(5\vec{a}+7\vec{b})=$.......................... .
5. The number of vectors of unit length perpendicular to vectors $\vec{a}=\hat{i}+\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$ is ....
6. If $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{j}+\hat{k}$, then $|\vec{a}\times\vec{b}|^2+|\vec{a}.\vec{b}|^2=$.......................... .
7. For any non-zero vector $\vec{r}$, $|\vec{r}\times\hat{i}|^2+|\vec{r}\times\hat{j}|^2+|\vec{r}\times\hat{k}|^2=\lambda|\vec{r}|^2$, then $\lambda=$........................ .

8. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of the vertices A, B and C respectively of a ΔABC such that $|\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}|=20$, then area of ΔABC is ....

9. If $\vec{a}$, $\vec{b}$ are two vectors such that $|\vec{a}|=1, |\vec{b}|=2$ and $|\vec{a}\times\vec{b}|=\sqrt{2}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....

10. For any two non-collinear vectors $\vec{a}$ and $\vec{b}$, the value of $\vec{a}\cdot(\vec{a}\times\vec{b})$ is ....

11. If $\vec{a}$ and $\vec{b}$ are two non-zero non-collinear vectors such that $|\vec{a}\times\vec{b}|=1$ and $|\vec{a}\cdot\vec{b}|=\sqrt{3}$, then the angle between $\vec{a}$ and $\vec{b}$ is ....

12. If three points with position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are collinear, then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=$...

ANSWERS

1. $|\vec{a}|^2|\vec{b}|^2$ 2. 3 3. $\frac{\pi}{2}$ 4. $\vec{b}\times\vec{a}$ 5. Two 6. 4
7. 2 8. 10 sq.units 9. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$ 10. 0 11. $\frac{\pi}{6}$ 12. $\vec{0}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Define vector product of two vectors.
2. Write the value $(\hat{i}\times\hat{j})\cdot\hat{k}+\hat{i}\cdot\hat{j}$.
3. Write the value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{j}\times\hat{i})$.
4. Write the value of $\hat{i}\cdot(\hat{j}\times\hat{k})+\hat{j}\cdot(\hat{k}\times\hat{i})+\hat{k}\cdot(\hat{i}\times\hat{j})$.
5. Write the value of $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$. [CBSE 2014]
6. Write the expression for the area of the parallelogram having $\vec{a}$ and $\vec{b}$ as its diagonals.
7. For any two vectors $\vec{a}$ and $\vec{b}$ write the value of $(\vec{a}\cdot\vec{b})^2+|\vec{a}\times\vec{b}|^2$ in terms of their magnitudes.
8. If $\vec{a}$ and $\vec{b}$ are two vectors of magnitudes 3 and $\frac{\sqrt{2}}{3}$ respectively such that $\vec{a}\times\vec{b}$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$.
9. If $|\vec{a}|=10, |\vec{b}|=2$ and $|\vec{a}\times\vec{b}|=16$, find $\vec{a}\cdot\vec{b}$.
10. For any two vectors $\vec{a}$ and $\vec{b}$, find $\vec{a}\cdot(\vec{b}\times\vec{a})$.
11. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}\times\vec{b}|=\sqrt{3}$ and $\vec{a}\cdot\vec{b}=1$, find the angle between.
12. For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ write the value of $\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$.
13. For any two vectors $\vec{a}$ and $\vec{b}$, find $(\vec{a}\times\vec{b})\cdot\vec{b}$.
14. Write the value of $\hat{i}\times(\hat{j}\times\hat{k})$.

15. If $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, then find $(\vec{a} \times \vec{b})\ \vec{a}$.

16. Write a unit vector perpendicular to $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$.

17. If $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = 144$ and $|\vec{a}| = 4$, find $|\vec{b}|$.

18. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then write the value of $|\vec{r} \times \hat{i}|^2$.

19. If $\vec{a}$ and $\vec{b}$ are unit vectors such that $\vec{a} \times \vec{b}$ is also a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

20. If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$, write the angle between $\vec{a}$ and $\vec{b}$.

21. If $\vec{a}$ and $\vec{b}$ are unit vectors, then write the value of $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2$.

22. If $\vec{a}$ is a unit vector such that $\vec{a} \times \hat{i} = \hat{j}$, find $\vec{a}.\hat{i}$.

23. If $\vec{c}$ is a unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b}$, write another unit vector perpendicular to $\vec{a}$ and $\vec{b}$.

24. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2 respectively and when $|\vec{a} \times \vec{b}| = \sqrt{3}$. [CBSE 2009]

25. Vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = \frac{2}{3}$ and $(\vec{a} \times \vec{b})$ is a unit vector. Write the angle between $\vec{a}$ and $\vec{b}$. [CBSE 2010]

26. Find λ, if $(2\hat{i} + 6\hat{j} + 14\hat{k}) \times (\hat{i} - \lambda\hat{j} + 7\hat{k}) = \vec{0}$. [CBSE 2010]

27. Write the value of the area of the parallelogram determined by the vectors $2\hat{i}$ and $3\hat{j}$ [CBSE 2012]

28. Write the value of $(\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{j} + \hat{k}) \cdot \hat{j}$ [CBSE 2012]

29. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$. [CBSE 2015]

30. Write the number of vectors of unit length perpendicular to both the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$. [CBSE 2016]

31. Write the angle between the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. [CBSE 2017]

ANSWERS

2. 1
3. 1
4. 3
5. 0
6. $\frac{1}{2}|\vec{a} \times \vec{b}|$
7. $|\vec{a}|^2 |\vec{b}|^2$
8. $45°, 135°$
9. ± 12
10. 0
11. $60°$
12. $\vec{0}$
13. $\vec{0}$
14. $\vec{0}$
15. Not meaningful
16. $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} + \hat{k})$
17. 3
18. $y^2 + z^2$
19. $\frac{\pi}{2}$
20. $\frac{\pi}{4}$
21. 1
22. 0
23. $-\vec{c}$
24. $\pi/3$
25. $\pi/3$
26. -3
27. 6 sq. units
28. 2
29. $\pm(\hat{i} - 11\hat{j} - 7\hat{k})$
30. 2
31. π