CHAPTER 17

COMBINATIONS

17.1 INTRODUCTION

In the previous chapter, we have studied arrangements of a certain number of objects by taking some of them or all at a time. Most of the times we are not interested in arranging the objects, but we are more concerned in selecting a number of objects from given number of objects. In other words, we do not want to specify the ordering of selected objects. For example, a company may want to select 3 persons out of 10 applicants, a student may want to choose three books from his library at a time etc.

Suppose we want to select three persons out of 4 persons A, B, C and D. We may choose A, B, C or A, B, D or A, C, D or B, C, D. Note that we have not listed A, B, C ; B, C, A; C, A, B; B, A, C; C, B, A and A, C, B separately here, because they represent the same selection A, B, C. But, they give rise to different arrangements. It is evident from the above discussion that in a selection the order in which objects are arranged is immaterial.

17.2 COMBINATIONS

COMBINATIONS *Each of the different selections made by taking some or all of a number of objects, irrespective of their arrangements is called a combination.*

ILLUSTRATION 1 *List the different combinations formed of three letters A, B, C taken two at a time.*

SOLUTION The different combinations formed of three letters A, B, C are: AB, AC, BC.

ILLUSTRATION 2 *Write all combinations of four letters A, B, C, D taken two at a time.*

SOLUTION Various combinations of two letters out of four letters A, B, C, D are:

$$AB, \ AC, \ AD, \ BC, \ BD, \ CD.$$

DIFFERENCE BETWEEN A PERMUTATION AND COMBINATION

(i) In a combination only selection is made whereas in a permutation not only a selection is made but also an arrangement in a definite order is considered.

(ii) In a combination, the ordering of the selected objects is immaterial whereas in a permutation, the ordering is essential. For example, A, B and B, A are same as combinations but different as permutations.

(iii) Practically to find the permutations of n different items, taken r at a time, we first select r items from n items and then arrange them. So, usually the number of permutations exceeds the number of combinations.

(iv) Each combination corresponds to many permutations. For example, the six permutations ABC, ACB, BCA, BAC, CBA and CAB correspond to the same combination ABC.

REMARK *Generally we use the word 'arrangements' for permutations and the word 'selections' for combinations.*

NOTATION *The number of all combinations of n objects, taken r at a time is generally denoted by* $C(n, r)$ *or,* ${}^nC_r$ or, $\binom{n}{r}$.

Thus, ${}^nC_r$ or $C(n, r)$ = Number of ways of selecting r objects from n objects.

Clearly, ${}^nC_r$ is defined only when n and r are non-negative integers such that $0 \le r \le n$.

THEOREM *The number of all combinations of n distinct objects, taken r at a time is given by*

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

PROOF Let the number of combinations of n distinct objects taken r at a time be x. Consider one of these x ways. There are r objects in this selection which can be arranged in $r!$ ways. Thus, each of the x combinations gives rise to $r!$ permutations. So, x combinations will give rise to $x \times (r!)$ permutations. Consequently, the number of permutations of n things, taken r at a time is $x \times (r!)$. But, this number is also equal to ${}^nP_r$.

$$ \therefore \quad x\,(r!) = {}^nP_r $$

$$ \Rightarrow \quad x = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right] $$

$$ \Rightarrow \quad {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

Q.E.D.

REMARK 1 *We have,*

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} $$

$$ \Rightarrow \quad {}^nC_r = \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)\,(n-r)\,(n-r-1)\ldots 3.2.1}{\{(n-r)\,(n-r-1)\ldots 3.2.1\}\,\{1.2.3\ldots.r\}} $$

$$ \Rightarrow \quad {}^nC_r = \frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{1.2.3\ldots.r} $$

Sometimes this form of ${}^nC_r$ is also very convenient to use.

REMARK 2 *We have,* ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$

Putting $r = n$, we obtain

$$ {}^nC_n = \frac{n!}{(n-n)!\,n!} = \frac{n!}{n!\,0!} = 1 \qquad [\because\ 0! = 1] $$

Putting $r = 0$, we obtain

$$ {}^nC_0 = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1 $$

Thus, ${}^nC_n = {}^nC_0 = 1.$

REMARK 3 *We have,*

$$ {}^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{1}{r!}\left(\frac{n!}{(n-r)!}\right) = \frac{{}^nP_r}{r!}. $$

17.3 PROPERTIES OF ${}^nC_r$ OR, $C(n, r)$

In this section, we shall discuss some important properties of ${}^nC_r$.

PROPERTY 1 *For $0 \le r \le n$, we have* ${}^nC_r = {}^nC_{n-r}$.

PROOF We have,

$$ {}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r $$

REMARK 1 *The use of this property simplifies the calculation of ${}^nC_r$ when r is large.*

For example, if we want to calculate ${}^{20}C_{19}$, by using this property, we get

$$ {}^{20}C_{19} = {}^{20}C_{20-19} = {}^{20}C_1 = 20. $$

REMARK 2 *The above property can be restated as follows:*
If x and y are non-negative integers such that $x + y = n$, *then* ${}^nC_x = {}^nC_y$

This can also be stated as : ${}^nC_x = {}^nC_y \Rightarrow x = y$, *or* $x + y = n$

ILLUSTRATION 1 *If* ${}^nC_7 = {}^nC_4$, *find the value of n.*

SOLUTION We know that : ${}^nC_x = {}^nC_y \Leftrightarrow x + y = n$ or $x = y$.

$\therefore \quad {}^nC_7 = {}^nC_4 \Rightarrow n = 7 + 4 = 11$

PROPERTY 2 *Let n and r be non-negative integers such that* $r \le n$. *Then,* ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

PROOF We have,

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{\{n-(r-1)\}!\,(r-1)!}\\
&= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!}\\
&= \frac{n!}{(n-r)!\,r\,(r-1)!} + \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!}\\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\}\\
&= \frac{n!}{(n-r)!\,(r-1)!}\left\{\frac{n-r+1+r}{r\,(n-r+1)}\right\}\\
&= \frac{n!\,(n+1)}{(n-r)!\,(r-1)!\,r\,(n-r+1)}\\
&= \frac{(n+1)\,n!}{(n-r+1)\,(n-r)!\,r\,(r-1)!} = \frac{(n+1)!}{(n-r+1)!\,r!} = \frac{(n+1)!}{((n+1)-r)!\,r!} = {}^{n+1}C_r.
\end{aligned}
$$

REMARK 3 *This property is known as Pascal's rule and it can also be proved by giving combinatorial arguments.*

ILLUSTRATION 2 *Find the value of the expression* ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.

SOLUTION We have,

$$
\begin{aligned}
&{}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3\\
&= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3\\
&= {}^{47}C_3 + {}^{47}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3\\
&= ({}^{47}C_3 + {}^{47}C_4) + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3\\
&= {}^{48}C_4 + {}^{48}C_3 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]\\
&= ({}^{48}C_3 + {}^{48}C_4) + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]\\
&= {}^{49}C_4 + {}^{49}C_3 + {}^{50}C_3 + {}^{51}C_3\\
&= ({}^{49}C_3 + {}^{49}C_4) + {}^{50}C_3 + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]\\
&= {}^{50}C_4 + {}^{50}C_3 + {}^{51}C_4\\
&= ({}^{50}C_3 + {}^{50}C_4) + {}^{51}C_3 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]\\
&= {}^{51}C_4 + {}^{51}C_3 = {}^{52}C_4 && [\because {}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r]
\end{aligned}
$$

PROPERTY 3 *Let n and r be non-negative integers such that* $1 \le r \le n$. *Then,* ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$.

PROOF We have,

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$

$$\Rightarrow \quad {}^nC_r = \frac{n\,(n-1)!}{\{(n-1)-(r-1)\}!\,r\,(r-1)!} = \frac{n}{r}\,\frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!} = \frac{n}{r}\;{}^{n-1}C_{r-1}$$

REMARK 4 *This property is very useful to find the value of* ${}^nC_r$.

For example, ${}^{10}C_3 = \frac{10}{3} \times {}^9C_2 = \frac{10}{3} \times \frac{9}{2} \times {}^8C_1 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1}\,{}^7C_0 = \frac{10}{3} \times \frac{9}{2} \times \frac{8}{1} \times 1 = 120$

REMARK 5 *By using the above property, we obtain that*

$${}^nC_r = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot \frac{n-2}{r-2} \cdots \frac{n\,(r-1)}{2} \cdot \frac{n-(r-1)}{1}$$

For example, ${}^9C_4 = \frac{9}{4} \times \frac{8}{3} \times \frac{7}{2} \times \frac{6}{1} = 126.$

PROPERTY 4 If $1 \le r \le n$, then $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$.

PROOF We have,

$$n \cdot {}^{n-1}C_{r-1} = n \cdot \frac{(n-1)!}{\{(n-1)-(r-1)\}!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{n!}{(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = \frac{(n-r+1) \cdot n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\left\{\frac{n!}{(n-r+1)!\,(r-1)!}\right\}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\left\{\frac{n!}{\{n-(r-1)\}!\,(r-1)!}\right\}$$

$$\Rightarrow \quad n \cdot {}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$$

PROPERTY 5 ${}^nC_x = {}^nC_y \Rightarrow x = y$ *or,* $x + y = n$.

PROOF We have,

$${}^nC_x = {}^nC_y$$

$$\Rightarrow \quad {}^nC_x = {}^nC_y = {}^nC_{n-y} \qquad [\because {}^nC_y = {}^nC_{n-y}]$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x = n - y$$

$$\Rightarrow \quad x = y \quad \text{or} \quad x + y = n.$$

REMARK 6 *If* ${}^nC_x = {}^nC_y$ *and* $x \ne y$, *then* $x + y = n$.

ILLUSTRATION 3 *If* ${}^nC_{15} = {}^nC_8$, *find the value of* ${}^nC_{21}$.

SOLUTION We have,

$${}^nC_{15} = {}^nC_8 \Rightarrow n = (15+8) = 23 \qquad [{}^nC_x = {}^nC_y \Rightarrow x + y = n]$$

$$\therefore \quad {}^nC_{21} = {}^{23}C_{21} = {}^{23}C_{23-21} \qquad [\because {}^nC_r = {}^nC_{n-r}]$$

$= {}^{23}C_2 = \frac{23}{2} \times \frac{22}{1} \times {}^{21}C_0$ $\left[\because {}^nC_r = \frac{n}{r} \times \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}\right]$

$= \frac{23}{2} \times \frac{22}{1} \times 1 = 23 \times 11 = 253$ $[\because {}^nC_0 = 1]$

ILLUSTRATION 4 *If* ${}^{10}C_x = {}^{10}C_{x+4}$, *find the value of x.*

SOLUTION We have, ${}^{10}C_x = {}^{10}C_{x+4} \Rightarrow x + x + 4 = 10 \Rightarrow 2x = 6 \Rightarrow x = 3.$

PROPERTY 6 *If n is an even natural number, then the greatest of the values* ${}^nC_0, {}^nC_1, {}^nC_2, ..., {}^nC_n$ is ${}^nC_{n/2}$

If n is an odd natural number, then the greatest of the values ${}^nC_0, {}^nC_1, {}^nC_2, ... {}^nC_n$ is ${}^nC_{n-1/2} = {}^nC_{n+1/2}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following: (i)* ${}^{10}C_8$ *(ii)* ${}^{100}C_{98}$ *(iii)* ${}^{52}C_{52}$

SOLUTION (i) ${}^{10}C_8 = {}^{10}C_{10-8}$ $[\because {}^nC_r = {}^nC_{n-r}]$

$= {}^{10}C_2 = \frac{10}{2} \times \frac{9}{1} \times {}^8C_0$ $\left[\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}\right]$

$= \frac{10}{2} \times \frac{9}{1} \times 1$ $[\because {}^nC_0 = 1]$

$= 45$ $[\because {}^nC_r = {}^nC_{n-r}]$

(ii) ${}^{100}C_{98} = {}^{100}C_{100-98}$ $\left[\because {}^nC_r = {}^nC_{n-r}\right]$

$= {}^{100}C_2 = \frac{100}{2} \times \frac{99}{1} \times {}^{98}C_0$ $[\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}]$

$= \frac{100}{2} \times \frac{99}{1} \times 1$ $[\because {}^nC_0 = 1]$

$= 4950$

(iii) ${}^{52}C_{52} = 1$ $[\because {}^nC_n = 1]$

EXAMPLE 2 *If* ${}^nC_8 = {}^nC_6$, *find* ${}^nC_2$

SOLUTION If ${}^nC_x = {}^nC_y$ and $x \neq y$, then $x + y = n$.

$\therefore$ ${}^nC_8 = {}^nC_6 \Rightarrow n = (8 + 6) = 14$

Now, ${}^nC_2 = {}^{14}C_2 = \frac{14}{2} \times \frac{13}{1} \times {}^{12}C_0$ $[\because {}^nC_r = \frac{n}{r} . {}^{n-1}C_{r-1}]$

$= \frac{14}{2} \times \frac{13}{1} \times 1 = 91$ $[\because {}^nC_0 = 1]$

EXAMPLE 3 *If* ${}^nP_r = 720$ and ${}^nC_r = 120$, find r.

SOLUTION We know that

$${}^nC_r = \frac{{}^nP_r}{r!}$$

$\therefore$ $120 = \frac{720}{r!}$ $\left[\because {}^nC_r = 120 \text{ and } {}^nP_r = 720\right]$

$\Rightarrow$ $r! = 6 \Rightarrow r! = 3! \Rightarrow r = 3.$

EXAMPLE 4 *If the ratio* ${}^{2n}C_3 : {}^{n}C_3$ *is equal to 11 : 1, find n.*

SOLUTION We have,

$$ {}^{2n}C_3 : {}^{n}C_3 = 11:1 $$

$$ \Rightarrow \quad \frac{{}^{2n}C_3}{{}^{n}C_3} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{\dfrac{(2n)!}{(2n-3)!\,3!}}{\dfrac{n!}{(n-3)!\,(3!)}} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)!}{(2n-3)!} \times \frac{(n-3)!}{n!} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)\,(2n-1)\,(2n-2)\,(2n-3)!}{(2n-3)!} \times \frac{(n-3)!}{n\,(n-1)\,(n-2)\,(n-3)!} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{(2n)\,(2n-1)\,(2n-2)}{n\,(n-1)\,(n-2)} = \frac{11}{1} $$

$$ \Rightarrow \quad \frac{4\,(2n-1)}{n-2} = \frac{11}{1} \Rightarrow 8n-4 = 11n-22 \Rightarrow 3n = 18 \Rightarrow n = 6 $$

EXAMPLE 5 *Prove that:* ${}^{2n}C_n = \dfrac{2^n\,\{1\cdot 3\cdot 5\cdot \ldots (2n-1)\}}{n!}$.

SOLUTION We have,

$$ {}^{2n}C_n = \frac{2n!}{(2n-n)!\,n!} = \frac{(2n)!}{n!\,n!} $$

$$ = \frac{(2n)\,(2n-1)\,(2n-2)\ldots 5\cdot 4\cdot 3\cdot 2\cdot 1}{n!\,n!} $$

$$ = \frac{\{1\cdot 3\cdot 5\cdot \ldots (2n-1)\}\,\{2\cdot 4\cdot 6\cdot \ldots 2n\}}{n!\,n!} $$

$$ = \frac{\{1\cdot 3\cdot 5\cdot \ldots (2n-1)\} \times 2^n\,\{1\cdot 2\cdot 3\cdot \ldots n\}}{n!\,n!} $$

$$ = \frac{\{1\cdot 3\cdot 5\cdot \ldots (2n-1)\} \times 2^n \times n!}{n!\,n!} = 2^n\,\frac{\{1\cdot 3\cdot 5 \ldots (2n-1)\}}{n!} $$

EXAMPLE 6 *If* ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, *find n.*

SOLUTION We have,

$$ {}^{n+2}C_8 : {}^{n-2}P_4 = 57:16 $$

$$ \Rightarrow \quad \frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16} $$

$$ \Rightarrow \quad \frac{(n+2)!}{8!\,(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16} $$

$$ \Rightarrow \quad \frac{(n+2)\,(n+1)\,n\,(n-1)\,(n-2)!}{8!} \times \frac{1}{(n-2)!} = \frac{57}{16} $$

$$ \Rightarrow \quad (n+2)\,(n+1)\,n\,(n-1) = \frac{57}{16} \times 8! = \frac{19\times 3}{16} \times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1 $$

$$ \Rightarrow \quad (n+2)\,(n+1)\,(n-1)\,n = 143640 $$

$$ \Rightarrow \quad (n-1)\,n\,(n+1)\,(n+2) = 19\times 3\times 7\times 6\times 5\times 4\times 3 $$

$\Rightarrow \quad (n-1)\,n\,(n+1)\,(n+2) = 19 \times (3 \times 7) \times (6 \times 3) \times (4 \times 5)$

$\Rightarrow \quad (n-1)\,n\,(n+1)\,(n+2) = 18 \times 19 \times 20 \times 21 \Rightarrow n-1 = 18 \Rightarrow n = 19$

EXAMPLE 7 *If* ${}^nC_{r-1} = 36$, ${}^nC_r = 84$ *and* ${}^nC_{r+1} = 126$, *then find* ${}^rC_2$.

SOLUTION We know that

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$$

It is given that ${}^nC_r = 84$ and ${}^nC_{r+1} = 126$

$$\therefore \quad \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \quad \frac{r+1}{n-r} = \frac{2}{3} \Rightarrow 2n - 5r = 3 \quad \text{...(i)}$$

Replacing r by $(r-1)$ in $\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{r+1}{n-r}$, we get

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{r}{n-(r-1)}$$

$$\Rightarrow \quad \frac{36}{84} = \frac{r}{n-r+1} \quad \left[\because {}^nC_{r-1} = 36 \text{ and } {}^nC_r = 84\right]$$

$$\Rightarrow \quad \frac{r}{n-r+1} = \frac{3}{7} \Rightarrow 3n - 10r = -3 \quad \text{...(ii)}$$

Solving (i) and (ii), we get $r = 3$.

$$\therefore \quad {}^rC_2 = {}^3C_2 = \frac{3!}{(3-2)!\,2!} = 3.$$

NOTE *Students are advised to learn that* $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ *as it is a very useful result.*

EXAMPLE 8 *If* ${}^nP_r = {}^nP_{r+1}$ *and* ${}^nC_r = {}^nC_{r-1}$, *find the values of n and r.*

SOLUTION We have,

$${}^nP_r = {}^nP_{r+1}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!}$$

$$\Rightarrow \quad \frac{1}{(n-r)\,(n-r-1)!} = \frac{1}{(n-r-1)!}$$

$$\Rightarrow \quad n - r = 1 \quad \text{...(i)}$$

and, $\quad {}^nC_r = {}^nC_{r-1}$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r!} = \frac{n!}{(n-r+1)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{n!}{(n-r)!\,r\,(r-1)!} = \frac{n!}{(n-r+1)\,(n-r)!\,(r-1)!}$$

$$\Rightarrow \quad \frac{1}{r} = \frac{1}{n-r+1} \Rightarrow n-r+1 = r \Rightarrow n - 2r = -1 \quad \text{...(ii)}$$

Solving (i) and (ii), we obtain $n = 3$ and $r = 2$.

LEVEL-2

EXAMPLE 9 *Prove that the product of r consecutive positive integers is divisible by r!.*

SOLUTION Let the r consecutive positive integers be $(n+1), (n+2), (n+3), \ldots, (n+r)$. Then,

$$\begin{aligned} \text{Product} &= (n+1)(n+2)(n+3)\ldots(n+r) \\ &= \frac{n!(n+1)(n+2)(n+3)\ldots(n+r)}{n!} \\ &= \frac{1\cdot 2\cdot 3\cdot \ldots n\cdot(n+1)(n+2)\ldots(n+r)}{n!} \\ &= \frac{(n+r)!}{n!} = \frac{(n+r)!}{r!\{(n+r)-r\}!}(r!) \\ &= \left({}^{n+r}C_r\right) r!, \text{ which is divisible by } r! \qquad [\because {}^{n+r}C_r \text{ is an integer}] \end{aligned}$$

EXERCISE 17.1

LEVEL-1

1. Evaluate the following:
   (i) ${}^{14}C_3$ (ii) ${}^{12}C_{10}$ (iii) ${}^{35}C_{35}$ (iv) ${}^{n+1}C_n$ (v) $\sum_{r=1}^{5} {}^5C_r$
2. If ${}^nC_{12} = {}^nC_5$, find the value of n.
3. If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.
4. If ${}^nC_{10} = {}^nC_{12}$, find ${}^{23}C_n$.
5. If ${}^{24}C_x = {}^{24}C_{2x+3}$, find x.
6. If ${}^{18}C_x = {}^{18}C_{x+2}$, find x.
7. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
8. If ${}^8C_r - {}^7C_3 = {}^7C_2$, find r.
9. If ${}^{15}C_r : {}^{15}C_{r-1} = 11:5$, find r.
10. If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57:16$, find n.
11. If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225:11$, find r.
12. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then find n.
13. If ${}^{2n}C_3 : {}^nC_2 = 44:3$, find n.
14. If ${}^{16}C_r = {}^{16}C_{r+2}$, find ${}^rC_4$.
15. If $\alpha = {}^mC_2$, then find the value of ${}^{\alpha}C_2$.

LEVEL-2

16. Prove that the product of $2n$ consecutive negative integers is divisible by $(2n)!$
17. For all positive integers n, show that ${}^{2n}C_n + {}^{2n}C_{n-1} = \frac{1}{2}({}^{2n+2}C_{n+1})$.
18. Prove that: ${}^{4n}C_{2n} : {}^{2n}C_n = [1\cdot 3\cdot 5\ldots(4n-1)] : [1\cdot 3\cdot 5\ldots(2n-1)]^2$.
19. Evaluate ${}^{20}C_5 + \sum_{r=2}^{5} {}^{25-r}C_4$.
20. Let r and n be positive integers such that $1 \le r \le n$. Then prove the following:
    (i) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ (ii) $n\,{}^{n-1}C_{r-1} = (n-r+1)\,{}^nC_{r-1}$
    (iii) $\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$ (iv) ${}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2} = {}^{n+2}C_r$.

ANSWERS

1. (i) 364 (ii) 66 (iii) 1 (iv) $(n+1)$ (v) 31 2. 17 3. 66

4. 23 **5.** 7 **6.** 8 **7.** 3 **8.** 3, 5 **9.** 5 **10.** 19 **11.** 7

12. 14, 7 **13.** 6 **14.** 35 **15.** $\frac{(m+1)(m)(m-1)(m-2)}{8}$ **19.** 42504

HINTS TO NCERT & SELECTED PROBLEM

16. Let $(-r), (-r-1), (-r-2), \dots, (-r-2n+1)$ be $2n$ consecutive negative integers. Then, their product P is given by

$$P = (-1)^{2n}\, r\,(r+1)\,(r+2)\dots(r+2n-1)$$

$$\Rightarrow\ P = \frac{(r-1)!\,(r)\,(r+1)\dots(r+2n-1)}{(r-1)!}$$

$$\Rightarrow\ P = \frac{(r+2n-1)!}{(r-1)!} = \frac{(r+2n-1)!}{(r-1)!\,(2n)!}(2n)! = \left({}^{r+2n-1}C_{2n}\right)(2n)!$$

Clearly, P is divisible by $(2n)!$

17.4 PRACTICAL PROBLEMS ON COMBINATIONS

In this section, we intend to discuss some problems in real life where the formula for ${}^nC_r$ and its meaning can be applied.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *From a class of 32 students, 4 are to be chosen for a competition. In how many ways can this be done ?*

SOLUTION Out of 32 students, 4 students can be selected in ${}^{32}C_4$ ways.

$\therefore$ Required number of ways ${}^{32}C_4 = \frac{32!}{28!\,4!}$.

EXAMPLE 2 *Three gentlemen and three ladies are candidates for two vacancies. A voter has to vote for two candidates. In how many ways can one cast his vote ?*

SOLUTION Clearly, there are 6 candidates and a voter has to vote for any two of them. So, the required number of ways is the number of ways of selecting 2 out of 6 i.e. ${}^6C_2$.

Hence, the required number of ways $= {}^6C_2 = \frac{6!}{2!\,4!} = 15$.

EXAMPLE 3 *If there are 12 persons in a party, and if each two of them shake hands with each other, how many handshakes happen in the party?*

SOLUTION It is to note here that, when two persons shake hands, it is counted as one handshake, not two. So, this is a problem on combinations.

The total number of handshakes is same as the number of ways of selecting 2 persons among 12 persons i.e. ${}^{12}C_2 = \frac{12!}{10! \times 2!} = 66$.

EXAMPLE 4 *A question paper has two parts, Part A and Part B, each containing 10 questions. If a student has to choose 8 from Part A and 5 from Part B, in how many ways can he choose the questions ?*

SOLUTION There are 10 questions in Part A out of which 8 questions can be chosen in ${}^{10}C_8$ ways. Similarly, 5 questions can be chosen from part B containing 10 questions in ${}^{10}C_5$ ways.

Hence, the total number of ways of selecting 8 questions from part A and 5 from part B

$$= {}^{10}C_8 \times {}^{10}C_5 = \frac{10!}{8!\,2!} \times \frac{10!}{5! \times 5!} = 11340.$$

EXAMPLE 5 *In how many ways a committee of 5 members can be selected from 6 men and 5 women, consisting of 3 men and 2 women ?*

SOLUTION Three men out of 6 men can be selected in ${}^6C_3$ ways. Two women out of 5 women can be selected in ${}^5C_2$ ways. Therefore, by the fundamental principle of counting, 3 men out of 6 men and 2 women out of 5 women can be selected in

$${}^6C_3 \times {}^5C_2 = \left(\frac{6 \times 5 \times 4}{3 \times 2 \times 1} \times \frac{5 \times 4}{2 \times 1}\right) = 200 \text{ ways.}$$

EXAMPLE 6 *In how many ways can a cricket eleven be chosen out of a batch of 15 players if*

(i) there is no restriction on the selection? *(ii) a particular player is always chosen?*

(iii) a particular player is never chosen?

SOLUTION (i) The total number of ways of selecting 11 players out of 15 is

$${}^{15}C_{11} = {}^{15}C_{15-11} = {}^{15}C_4 = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 1365$$

(ii) If a particular player is always chosen. This means that 10 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{10} = {}^{14}C_{14-10} = {}^{14}C_4 = 1001$

(iii) If a particular player is never chosen. This means that 11 players are selected out of the remaining 14 players.

$\therefore$ Required number of ways $= {}^{14}C_{11} = {}^{14}C_{14-11} = {}^{14}C_3 = 364$

EXAMPLE 7 *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least five women have to be included in a committee ? In how many of these committees (i) the women are in majority (ii) the men are in majority ?*

SOLUTION There are 9 women and 8 men. A committee of 12, consisting of at least 5 women, can be formed by choosing :

(i) 5 women and 7 men (ii) 6 women and 6 men (iii) 7 women and 5 men
(iv) 8 women and 4 men (v) 9 women and 3 men

$\therefore$ Total number of ways of forming the committee

$= {}^9C_5 \times {}^8C_7 + {}^9C_6 \times {}^8C_6 + {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3$

$= 126 \times 8 + 84 \times 28 + 36 \times 56 + 9 \times 70 + 1 \times 56 = 6062$

Clearly, women are in majority in (iii), (iv) and (v) cases as discussed above.

So, total number of committees in which women are in majority

$= {}^9C_7 \times {}^8C_5 + {}^9C_8 \times {}^8C_4 + {}^9C_9 \times {}^8C_3 = 36 \times 56 + 9 \times 70 + 1 \times 56 = 2702$

Clearly, men are in majority in only (i) case as discussed above.

So, total number of committees in which men are in majority $= {}^9C_5 \times {}^8C_7 = 126 \times 8 = 1008$.

EXAMPLE 8 *A committee of three persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women?* **[NCERT]**

SOLUTION There are 5 persons (2 men and 3 women). In order consitute a committee of 3 persons we need to select three persons out of given 5 persons. This can be done in ${}^5C_3$ ways.

So, the committee can be formed in $^5C_3 = \frac{5!}{3!\,2!} = 10$ ways.

Now, 1 man can be selected from 2 men in $^2C_1$ ways and 2 women can be selected from 3 women in $^3C_2$ ways.

Therefore, required number of committees is $^2C_1 \times {}^3C_2 = 2 \times 3 = 6$

EXAMPLE 9 *What is the number of ways of choosing 4 cards from a pack of 52 playing cards? In how many of these*

(i) *four cards are of the same suit?* (ii) *four cards belong to four different suits?*
(iii) *four cards are face cards?* (iv) *two are red cards and two are black cards?*
(v) *cards are of the same colour?* **[NCERT]**

SOLUTION Four cards can be chosen from 52 playing cards in $^{52}C_4$ ways.

Now, $^{52}C_4 = \frac{52!}{48!\,4!} = \frac{49 \times 50 \times 51 \times 52}{2 \times 3 \times 4} = 270725$

Hence, required number of ways = 270725

(i) There are four suits (diamond, spade, club and heart) of 13 cards each. Therefore, there are $^{13}C_4$ ways of choosing 4 diamond cards, $^{13}C_4$ ways of choosing 4 club cards, $^{13}C_4$ ways of choosing 4 spade cards and $^{13}C_4$ ways of choosing heart cards.

$\therefore$ Required number of ways $= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 = 4 \times {}^{13}C_4 = 4 \times \frac{13!}{9!\,4!} = 2860$

(ii) There are 13 cards in each suit. Four cards drawn belong to four different suits means one card is drawn from each suit. Out of 13 diamond cards one card can be drawn in $^{13}C_1$ ways. Similarly, there are $^{13}C_1$ ways of choosing one club card, $^{13}C_1$ ways of choosing one spade card and $^{13}C_1$ ways of choosing one heart card.

$\therefore$ Number of ways of selecting one card from each suit $= {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13^4$

(iii) There are 12 face cards out of which 4 cards can be chosen in $^{12}C_4$ ways.

$\therefore$ Required number of ways $= {}^{12}C_4 = \frac{12!}{4!\,8!} = 495$

(iv) There are 26 red cards and 26 black cards. Therefore, 2 red cards can be chosen in $^{26}C_2$ ways and 2 black cards can be chosen in $^{26}C_2$ ways. Hence, 2 red and 2 black cards can be chosen in

$^{26}C_2 \times {}^{26}C_2 = \left(\frac{26!}{24!\,2!}\right)^2 = (325)^2 = 105625$ ways.

(v) Out of 26 red cards, 4 red cards can be chosen in $^{26}C_4$ ways. Similarly, 4 black cards can be chosen in 26C$_4$ ways.

Hence, 4 red or 4 black cards can be chosen in $^{26}C_4 + {}^{26}C_4 = 2 \times {}^{26}C_4 = 2 \times \frac{26!}{4!\,22!} = 29900$ ways.

EXAMPLE 10 *Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included ?*

SOLUTION The committee can be formed in the following ways:

(i) By selecting 2 men and 1 woman (ii) By selecting 1 man and 2 women

Now, 2 men out of 5 men and 1 woman out of 2 woman can be chosen in $^5C_2 \times {}^2C_1$ ways.

And, 1 man out of 5 men and 2 women out of 2 women can be chosen in $^5C_1 \times {}^2C_2$ ways.

$\therefore$ Total number of ways of forming the committee $= {}^5C_2 \times {}^2C_1 + {}^5C_1 \times {}^2C_2 = 20 + 5 = 25.$

EXAMPLE 11 *In how many ways can a cricket team be selected from a group of 25 players containing 10 batsmen, 8 bowlers, 5 all-rounders and 2 wicket keepers ? Assume that the team of 11 players requires 5 batsmen, 3 all-rounder, 2 bowlers and 1 wicket keeper.*

SOLUTION The selection of team is divided into four phases:

(i) Selection of 5 batsmen out of 10. This can be done in ${}^{10}C_5$ ways.

(ii) Selection of 3 all-rounders out of 5. This can be done in ${}^5C_3$ ways.

(iii) Selection of 2 bowlers out of 8. This can be done in ${}^8C_2$ ways.

(iv) Selection of one wicket keeper out of 2. This can be done in ${}^2C_1$ ways.

e selection of team is completed by completing all the four phases.

∴ The team can be selected in ${}^{10}C_5 \times {}^5C_3 \times {}^8C_2 \times {}^2C_1 = 141120$ ways.

MPLE 12 *A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be , when*

(i) at least two ladies are included? *(ii) at most two ladies are included ?*

LUTION (i) A committee of 5 persons, consisting of at least two ladies, can be formed in the ollowing ways:

I Selecting 2 ladies out of 4 and 3 gents out of 6. This can be done in ${}^4C_2 \times {}^6C_3$ ways.

II Selecting 3 ladies out of 4 and 2 gents out of 6. This can be done in ${}^4C_3 \times {}^6C_2$ ways.

III Selecting 4 ladies out of 4 and 1 gent out of 6. This can be done in ${}^4C_4 \times {}^6C_1$ ways.

Since the committee is formed in each case. Therefore, by the fundamental principle of addition,

The total number of ways of forming the committee $= {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$

$= 120 + 60 + 6 = 186$

(ii) A committee of 5 persons, consisting of at most two ladies, can be constituted in the following ways :

I Selecting 5 gents only out of 6. This can be done in ${}^6C_5$ ways.

II Selecting 4 gents only out of 6 and one lady out of 4. This can be done in ${}^6C_4 \times {}^4C_1$ ways.

III Selecting 3 gents only out of 6 and two ladies out of 4. This can be done is ${}^6C_3 \times {}^4C_2$ ways.

Since the committee is formed in each case. So, the total number of ways of forming the committee $= {}^6C_5 + {}^6C_4 \times {}^4C_1 + {}^6C_3 \times {}^4C_2 = 6 + 60 + 120 = 186$.

EXAMPLE 13 *A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be selected so that there are at least two balls of each colour ?*

SOLUTION The selection of 6 balls, consisting of at least two balls of each colour from 5 red and 6 white balls, can be made in the following ways :

(i) By selecting 2 red balls out of 5 and 4 white balls out of 6. This can be done in ${}^5C_2 \times {}^6C_4$ ways.

(ii) By selecting 3 red balls out of 5 and 3 white balls out of 6. This can be done in ${}^5C_3 \times {}^6C_3$ ways.

(iii) By selecting 4 red balls out of 5 and 2 white balls out of 6. This can be done in ${}^5C_4 \times {}^6C_2$ ways.

Since the selection of 6 balls can be completed in any one of the above ways.

Hence, by the fundamental principle of addition, the total number of ways to select the balls

$= {}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2 = 10 \times 15 + 10 \times 20 + 5 \times 15 = 425.$

EXAMPLE 14 *For the post of 5 teachers, there are 23 applicants, 2 posts are reserved for SC candidates and there are 7 SC candidates among the applicants. In how many ways can the selection be made ?*

SOLUTION Clearly, there are 7 SC candidates and 16 other candidates. We have to select 2 out of 7 SC candidates and 3 out of 16 other candidates. This can be done in ${}^7C_2 \times {}^{16}C_3$ ways.

$\therefore$ The number of ways of making the selection $= {}^7C_2 \times {}^{16}C_3 = 11760.$

EXAMPLE 15 *How many triangles can be formed by joining the vertices of a hexagon ?*

SOLUTION There are 6 vertices of a hexagon. One triangle is formed by selecting a group of 3 vertices from given 6 vertices. This can be done in ${}^6C_3$ ways.

$\therefore$ Number of triangles $= {}^6C_3 = \frac{6!}{3!\,3!} = 20.$

EXAMPLE 16 *How many diagonals are there in a polygon with n sides ?*

SOLUTION A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon. Number of line segments obtained by joining the vertices of an n sided polygon taken two at a time

$= $ Number of ways of selecting 2 out of $n = {}^nC_2 = \frac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon.

$\therefore$ Number of diagonals of the polygon $= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$

EXAMPLE 17 *A polygon has 44 diagonals. Find the number of its sides.*

SOLUTION Let there be n sides of the polygon. We know that the number of diagonals of n sided polygon is $\frac{n(n-3)}{2}$.

$\therefore$ $\frac{n(n-3)}{2} = 44 \Rightarrow n^2 - 3n - 88 = 0 \Rightarrow (n-11)(n+8) = 0 \Rightarrow n = 11$ $\quad (\because n > 0)$

Hence, there are 11 sides of the polygon.

EXAMPLE 18 *How many chords can be drawn through 21 points on a circle?*

SOLUTION A chord is obtained by joining any two points on a circle. Therefore, total number of chords drawn through 21 points is same as the number of ways of selecting 2 points out of 21 points. This can be done in ${}^{21}C_2$ ways.

Hence, total number of chords $= {}^{21}C_2 = \frac{21!}{19!\,2!} = 21 \times 10 = 210.$

LEVEL-2

EXAMPLE 19 *A person wishes to make up as many different parties as he can out of his 20 friends such that each party consists of the same number of persons. How many friends should he invite ?*

SOLUTION Suppose he invites r friends at a time. Then the total number of parties is ${}^{20}C_r$. We have to find the maximum value of ${}^{20}C_r$ which is for $r = 10$, because ${}^nC_r$ is maximum for $r = n/2$, when n is even.

Hence, he should invite 10 friends at a time in order to form the maximum number of parties.

EXAMPLE 20 *If m parallel lines in plane are intersected by a family of n parallel lines. Find the number of parallelograms formed.*

SOLUTION A parallelogram is formed by choosing two straight lines from the set of m parallel lines and two straight lines from the set of n parallel lines.

Two straight lines from the set of m parallel lines can be chosen in ${}^{m}C_2$ ways and two straight lines from the set of n parallel lines can be chosen in ${}^{n}C_2$ ways.

Hence, the number of parallelograms formed = ${}^{m}C_2 \times {}^{n}C_2$

$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2} = \frac{mn(m-1)(n-1)}{4}$$

EXAMPLE 21 *There are 10 points in a plane, no three of which are in the same straight line, excepting 4 points, which are collinear. Find the (i) number of straight lines obtained from the pairs of these points; (ii) number of triangles that can be formed with the vertices as these points.*

SOLUTION (i) Number of straight lines formed joining the 10 points, taking 2 at a time = ${}^{10}C_2$

$$= \frac{10!}{2!\,8!} = 45.$$

Number of straight lines formed by joining the four points, taking 2 at a time = ${}^{4}C_2 = \frac{4!}{2!\,2!} = 6$

But, 4 collinear points, when joined pairwise give only one line.

$\therefore$ Required number of straight lines = 45 − 6 + 1 = 40.

(ii) Number of triangles formed by joining the points, taking 3 at a time = ${}^{10}C_3 = \frac{10!}{3!7!} = 120.$

Number of triangles formed by joining the 4 points, taken 3 at a time = ${}^{4}C_3 = {}^{4}C_1 = 4.$

But, 4 collinear points cannot form a triangle when taken 3 at a time.

So, Required number of triangles = 120 − 4 = 116.

EXAMPLE 22 *In a plane there are 37 straight lines, of which 13 pass through the point A and 11 pass through the point B. Besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. Find the number of points of intersection of the straight lines.*

SOLUTION The number of points of intersection of 37 straight lines is ${}^{37}C_2$. But 13 straight lines out of the given 37 straight lines pass through the same point A. Therefore instead of getting ${}^{13}C_2$ points, we get merely one point A. Similarly, 11 straight lines out of the given 37 straight lines intersect at point B. Therefore instead of getting ${}^{11}C_2$ points, we get only one point B. Hence, the number of intersection points of the lines is ${}^{37}C_2 - {}^{13}C_2 - {}^{11}C_2 + 2 = 535.$

EXAMPLE 23 *From a class of 25 students, 10 are to be chosen for an excursion party. There are 3 students who decide that either all of them will join or none of them will join. In how many ways can they be chosen ?* **[NCERT]**

SOLUTION We have the following possibilities:

(i) *Three particular students join the excursion party:* In this case, we have to choose 7 students from the remaining 22 students. This can be done in ${}^{22}C_7$ ways.

(ii) *Three particular students do not join the excursion party:* In this case, we have to choose 10 students from the remaining 22 students. This can be done in ${}^{22}C_{10}$ ways.

Hence, the required number of ways = ${}^{22}C_7 + {}^{22}C_{10} = 817190.$

EXAMPLE 24 *A boy has 3 library tickets and 8 books of his interest in the library. Of these 8, he does not want to borrow Chemistry Part II, unless Chemistry Part I is also borrowed. In how many ways can he choose the three books to be borrowed ?*

SOLUTION We have the following two possibilities :

(i) *When Chemistry part I is borrowed*: In this case the boy may borrow Chemistry Part II. So, he has to select now two books out of the remaining 7 books of his interest. This can be done in ${}^{7}C_2$ ways.

(ii) *When Chemistry part I is not borrowed* : In this case the boy does not want to borrow Chemistry Part II. So, he has to select three books from the remaining 6 books. This can be done in ${}^6C_3$ ways.

Hence, the required number of ways $= {}^7C_2 + {}^6C_3 = 21 + 20 = 41$.

EXAMPLE 25 *In how many ways can 7 plus (+) signs and 5 minus (–) signs be arranged in a row so that no two minus signs are together ?*

SOLUTION The plus signs can be arranged in only one way, because all are identical, as shown below:

	+		+		+		+		+		+		+	

A blank box in the above arrangement shows available space for the minus signs. Since there are 7 plus signs, the number of blank boxes is therefore 8. The five minus signs are now to be arranged in the 8 boxes so that no two of them are together. Now, 5 boxes out of 8 can be chosen in ${}^8C_5$ ways. Since all minus signs are identical, so 5 minus signs can be arranged in 5 chosen boxes in only one way. Hence, the number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

EXAMPLE 26 *In how many ways can 21 identical books on English and 19 identical books on Hindi be placed in a row on a shelf so that two books on Hindi may not be together?*

SOLUTION In order that no two books on Hindi are together, we must first arrange all books in English in a row. Since all English books are identical, so they can be arranged in a row in only one way as shown below:

$$\times E \times E \times E \times E \times \ldots \times E \times E$$

Here E denotes the position of an English book and $\times$ that of a Hindi book.

Since there are 21 books on English, the number places mark $\times$ are therefore 22. Now, 19 books on Hindi are to be arranged in these 22 places so that no two of them are together. Out of 22 places 19 places for Hindi books can be chosen in ${}^{22}C_{19}$ ways. Since all books on Hindi are identical, so 19 books on Hindi can be arranged in 19 chosen places in only one way. Hence, the required number of ways $= 1 \times {}^{22}C_{19} \times 1 = 1540$.

EXERCISE 17.2

LEVEL-1

1. From a group of 15 cricket players, a team of 11 players is to be chosen. In how many ways can this be done?
2. How many different boat parties of 8, consisting of 5 boys and 3 girls, can be made from 25 boys and 10 girls?
3. In how many ways can a student choose 5 courses out of 9 courses if 2 courses are compulsory for every student?
4. In how many ways can a football team of 11 players be selected from 16 players? How many of these will (i) include 2 particular players? (ii) exclude 2 particular players?
5. There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed. Find the number of ways in which this can be done. Further find in how many of these committees:
   (i) a particular professor is included. (ii) a particular student is included.
   (iii) a particular student is excluded.
6. How many different products can be obtained by multiplying two or more of the numbers 3, 5, 7, 11 (without repetition)?

7. From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition; at least including 4 boys and 4 girls. The 2 girls who won the prizes last year should be included. In how many ways can the selection be made?
8. How many different selections of 4 books can be made from 10 different books, if
   (i) there is no restriction; (ii) two particular books are always selected;
   (iii) two particular books are never selected?
9. From 4 officers and 8 jawans in how many ways can 6 be chosen (i) to include exactly one officer (ii) to include at least one officer?
10. A sports team of 11 students is to be constituted, choosing at least 5 from class XI and at least 5 from class XII. If there are 20 students in each of these classes, in how many ways can the teams be constituted?
11. A student has to answer 10 questions, choosing at least 4 from each of part A and part B. If there are 6 questions in part A and 7 in part B, in how many ways can the student choose 10 questions?
12. In an examination, a student has to answer 4 questions out of 5 questions; questions 1 and 2 are however compulsory. Determine the number of ways in which the student can make the choice.
13. A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions. He is not permitted to attempt more than 5 questions from either group. In how many ways can he choose the 7 questions?
14. There are 10 points in a plane of which 4 are collinear. How many different straight lines can be drawn by joining these points.
15. Find the number of diagonals of (i) a hexagon (ii) a polygon of 16 sides.
16. How many triangles can be obtained by joining 12 points, five of which are collinear ?
17. In how many ways can a committee of 5 persons be formed out of 6 men and 4 women when at least one woman has to be necessarily selected ?
18. In a village, there are 87 families of which 52 families have at most 2 children. In a rural development programme, 20 families are to be helped chosen for assistance, of which at least 18 families must have at most 2 children. In how many ways can the choice be made ?
19. A group consists of 4 girls and 7 boys. In how many ways can a tem of 5 members be selected if the team has (i) no girl? (ii) at least one boy and one girl? (iii) at least 3 girls? **[NCERT]**
20. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committees would consist of 1 man and 2 women? **[NCERT]**
20. Find the number of (i) diagonals (ii) triangles formed in a decagon.
22. Determine the number of 5 cards combinations out of a deck of 52 cards if at least one of the 5 cards has to be a king ? **[NCERT]**
23. We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made ?
24. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls? **[NCERT]**
25. Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls and 5 blue balls if each selection consists of 3 balls of each colour. **[NCERT]**
26. Determine the number of 5 cards combinations out of a deck of 52 cards if there is exactly one ace in each combination. **[NCERT]**

27. In how many ways can one select a circket team of eleven from 17 players in which only 5 persons can bowl if each cricket team of 11 must include exactly 4 bowlers?

28. A bag contains 5 black and 6 red balls. Determine the number of ways in which 2 black and 3 red balls can be selected. **[NCERT]**

29. In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 specific courses are compulsory for every student? **[NCERT]**

30. A committee of 7 has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consits of :

   (i) exactly 3 girls? (ii) at least 3 girls? (iii) at most 3 girls? **[NCERT]**

31. In an examination, a question paper consists of 12 questions divided into two parts i.e., Part I and Part II, containing 5 and 7 questions, respectively. A student is required to attempt 8 questions in all, selecting at least 3 from each part. In how many ways can a student select the questions? **[NCERT]**

LEVEL-2

32. A parallelogram is cut by two sets of m lines parallel to its sides. Find the number of parallelograms thus formed.

33. Out of 18 points in a plane, no three are in the same straight line except five points which are collinear. How many (i) straight lines (ii) triangles can be formed by joining them ?

ANSWERS

1. 1365 2. 6375600 3. 35 4. 4368 (i) 2002 (ii) 364
5. 51300 (i) 10260 (ii) 7695 (iii) 43605 6. 11 7. 104874
8. (i) 210 (ii) 28 (iii) 70 9. (i) 224 (ii) 896 10. $2({}^{20}C_5 \times {}^{20}C_6)$
11. 266 12. 3 13. 780 14. 40 15. (i) 9 (ii) 104 16. 210 17. 246
18. ${}^{52}C_{18} \times {}^{35}C_2 + {}^{52}C_{19} \times {}^{35}C_1 + {}^{52}C_{20} \times {}^{35}C_0$ 19. (i) 21(ii) 441(iii) 91
20. (i) 35 (ii) 120 21. 10, 6 22. 886656 23. 22 24. 40 25. 2000
26. 778320 27. 3960 28. 200
29. 35 30. (i) 504 (ii) 588 (iii) 1630 31. 420 32. $({}^{m+2}C_2)^2$
33. (i) 144 (ii) 806

HINTS TO NCERT & SELECTED PROBLEMS

2. Required no. of boat parties = ${}^{25}C_5 \times {}^{10}C_3$.

3. Since 2 courses are compulsory. So, the student is to choose 3 courses out of the remaining 7 courses. This can be done in ${}^7C_3$ ways.

4. We have to select 11 players out of 16. So, required number of ways = ${}^{16}C_{11}$.

   (i) Since 2 particular players are always included, so, we have to select 9 players out of the remaining 14 players. This can be done in ${}^{14}C_9$ ways.

   (ii) Since 2 particular players are excluded from every selection, so, we have to select 11 players from the remaining 14 players. This can be done in ${}^{14}C_{11}$ ways.

6. Total number of products = Number of ways of selecting 2 or 3 or all out of 4 numbers 3,5,7,11

$$= {}^4C_2 + {}^4C_3 + {}^4C_4 = 6 + 4 + 1 = 11.$$

7. Since two girls who won the prizes last year are to be included in every selection. So, we have to select 8 students out of 12 boys and 8 girls, choosing at least 4 boys and at least two girls. This can be done in ${}^{12}C_6 \times {}^8C_2 + {}^{12}C_5 \times {}^8C_3 + {}^{12}C_4 \times {}^8C_4 = 104874$ ways.

9. (i) Required number of ways = ${}^4C_1 \times {}^8C_5$

(ii) Required number of ways = Total no. of ways – No. of ways of selecting no officer
$= {}^{12}C_6 - {}^8C_6$.

10. Required number of ways = ${}^{20}C_5 \times {}^{20}C_6 + {}^{20}C_6 \times {}^{20}C_5$.

11. The various possibilities are : (i) 4 from part A and 6 from part B (ii) 5 from part A and 5 from part B (iii) 6 from part A and 4 from part B.
So, the required number of ways = ${}^6C_4 \times {}^7C_6 + {}^6C_5 \times {}^7C_5 + {}^6C_6 \times {}^7C_4 = 266$.

12. Required number of ways = ${}^3C_2$.

13. Required number of ways = ${}^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5 = 780$.

14. Number of straight lines = ${}^{10}C_2 - {}^4C_2 + 1$.

16. Number of triangles = ${}^{12}C_3 - {}^5C_3$.

18. 52 families have at most 2 children, while 35 families have more than 2 children. The selection of 20 families of which at least 18 families must have at most 2 children can be made as under:

(i) 18 families out of 52 and 2 families out of 35

or, (ii) 19 families out of 52 and 1 family out of 35

or, (iii) 20 families out of 52.

19. (i) From a group of 4 girls and 7 boys, a team of 5 consisting of no girls can be chosen in ${}^7C_5 = 21$ ways.

(ii) A team of 5 consisting of at least one boy and one girl can be chosen in
${}^7C_1 \times {}^4C_4 + {}^7C_2 \times {}^4C_3 + {}^7C_3 \times {}^4C_2 + {}^7C_4 \times {}^4C_1 = 441$ ways.

(iii) A team of 5 consisting of at least 3 girls can be chosen in
${}^4C_3 \times {}^7C_2 + {}^4C_4 \times {}^7C_1 = 91$ ways.

21. A committee of 3 persons out of 2 men and 3 women can be constituted in ${}^5C_3 = 10$ ways.
A committee of 1 man and 2 women can be constituted in ${}^2C_1 \times {}^3C_2 = 6$ ways.

22. Required number of combinations = Total number of 5 card combinations
– Number of 5 car combinations having no king.
$= {}^{52}C_5 - {}^{48}C_5 = 886656$.

24. Number of ways of selecting team = ${}^5C_3 \times {}^4C_3 = 40$.

25. Number of ways of selecting 9 balls = ${}^6C_3 \times {}^5C_3 \times {}^5C_3 = 2000$.

26. Out of 4 aces one ace can be selected in ${}^4C_1$ ways and from the remaining 48 cards, four cards can be selected in ${}^{48}C_4$ ways. So, number of 5 cards combinations consisting of exactly one ace = ${}^4C_1 \times {}^{48}C_4 = 778320$.

27. Required number of ways = ${}^5C_4 \times {}^{12}C_7$.

28. Out of 5 black and 6 red balls, 2 black and 3 red balls can be chosen in ${}^5C_2 \times {}^6C_3 = 200$ ways.

29. Required number of ways = Number of ways of selecting 3 courses out of 7 courses.
$= {}^7C_3$ ways = 35.

30. (i) A committee consisting of 3 girls and 4 boys can be formed in ${}^9C_4 \times {}^4C_3 = 504$ ways.

(ii) A committee consisting of at least 3 girls can be formed in ${}^9C_4 \times {}^4C_3 + {}^9C_3 \times {}^4C_4 = 588$ ways.

(iii) A committee of at most 3 girls can be formed in

$$ {}^9C_7 \times {}^4C_0 + {}^9C_6 \times {}^4C_1 + {}^9C_5 \times {}^4C_2 + {}^9C_4 \times {}^4C_3 = 1632 \text{ ways.} $$

31. At least 3 questions can be selected in the following ways:

Part I	Part II
3	5
4	4
5	3

So, required number of ways $= {}^5C_3 \times {}^7C_5 + {}^5C_4 \times {}^7C_4 + {}^5C_5 \times {}^7C_3 = 420.$

32. Each set of parallel lines consists of $(m + 2)$ lines and each parallelogram is formed by choosing two lines from the first set and two straight lines from the second set.

Hence, the total number of parallelograms $= {}^{m+2}C_2 \times {}^{m+2}C_2$.

17.5 MIXED PROBLEMS ON PERMUTATIONS AND COMBINATIONS

In this section, we intend to discuss some practical problems where both permutations and combinations are used as is illustrated in the following examples.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed ?*

SOLUTION Three consonants out of 7 and 2 vowels out of 4 can be chosen in ${}^7C_3 \times {}^4C_2$ ways. Thus, there are ${}^7C_3 \times {}^4C_2$ groups each containing 3 consonants and 2 vowels. Since each group contains 5 letters, which can be arranged among themselves in 5! ways.

Hence, the required number of words $= ({}^7C_3 \times {}^4C_2) \times 5! = 25200$.

EXAMPLE 2 *How many four-letter words can be formed using the letters of the word 'FAILURE', so that*

(i) *F is included in each word ?* (ii) *F is not included in any word ?*

SOLUTION There are 7 letters in the word 'FAILURE'.

(i) To include F in every 4 letter word, we first select four letters from the 7 letters of the word 'FAILURE' such that F is included in every selection. This can be done by selecting three letters from the remaining 6 letters i.e. A, I, L, U, R, E in ${}^6C_3$ ways. Now, there are 4 letters in each of ${}^6C_3$ selections. Consider one of these ${}^6C_3$ selections. This selection contains 4 letters which can be arranged in 4! ways. Thus, each of ${}^6C_3$ selections provides 4! words.

Hence, the total number of words $= {}^6C_3 \times 4! = 480$.

(ii) If F is not to be included in any word, then we first select 4 letters from the remaining 6 letters. This can be done in ${}^6C_4$ ways. Now, every selection has 4 letters which can be arranged in a row in 4! ways.

Hence, the total number of words $= {}^6C_4 \times 4! = 360$.

EXAMPLE 3 *How many words with or without meaning, can be formed using all the letters of the word EQUATION at a time so that vowels and consonants occur together?* **[NCERT]**

SOLUTION There are 5 vowels and 3 consonants in the word EQUATION. All vowels can be put together in 5! ways and all consonants can be put together in 3! ways. Considering all vowels as one letter and all consonants as at letter, vowels and consonants can be arranged in 2! ways. Therefore, vowels and consonants can be put together in $5! \times 3! \times 2!$ ways i.e. 1440 ways.

EXAMPLE 4 *How many five-letter words containing 3 vowels and 2 consonants can be formed using the letters of the word 'EQUATION' so that the two consonants occur together ?*

SOLUTION There are 5 vowels and 3 consonants in the word 'EQUATION'. Three vowels out of 5 and 2 consonants out of 3 can be chosen in ${}^5C_3 \times {}^3C_2$ ways. So, there are ${}^5C_3 \times {}^3C_2$ groups each containing 3 consonants and two vowels. Now, each group contains 5 letters which are to be arranged in such a way that 2 consonants occur together. Considering 2 consonants as one letter, we have 4 letters which can be arranged in 4! ways. But two consonants can be put together in 2! ways. Therefore, 5 letters in each group can be arranged in $4! \times 2!$ ways.

Hence, the required number of words $= ({}^5C_3 \times {}^3C_2) \times 4! \times 2! = 1440$.

EXAMPLE 5 *How many words with or without meaning, each 2 of vowels and 3 consonants can be formed from the letters of the word DAUGHTER?* **[NCERT]**

SOLUTION There are 3 vowels and 5 consonants in the word DAUGHTER out of which 2 vowels and 3 consonants can be chosen in ${}^3C_2 \times {}^5C_3$ ways. These selected five letters can now be arranged in 5! ways.

Hence, required number of words $= {}^3C_2 \times {}^5C_3 \times 5! = 3 \times 10 \times 120 = 3600$

EXAMPLE 6 *The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and 2 different consonants can be formed from the alphabet ?* **[NCERT]**

SOLUTION Out of 5 vowels and 21 consonants, 2 vowels and 2 consonants can be chosen in ${}^5C_2 \times {}^{21}C_2$ ways. These selected 4 letters can now be arranged in 4! ways. Therefore, by the fundamental principle of counting, required number of words is

$$ {}^5C_2 \times {}^{21}C_2 \times 4! = 10 \times 210 \times 24 = 50400. $$

EXAMPLE 7 *In how many ways can 5 girls and 3 boys be seated in a row so that no two boys are together?* **[NCERT]**

SOLUTION Since boys are to be separated. Therefore, let us first seat 5 girls. This can be done in 5! ways. For each such arrangement, three boys can be seated only at the cross marked places.

$$ \times G \times G \times G \times G \times G \times $$

There are 6 crossed marked places and three boys can be seated in ${}^6C_3 \times 3!$ ways. Hence, by the fundamental principle of counting, the total number of ways is $5! \times {}^6C_3 \times 3! = 14400$.

LEVEL-2

EXAMPLE 8 *How many words can be formed by taking 4 letters at a time out of the letters of the word 'MATHEMATICS'.*

SOLUTION There are 11 letters viz. MM, AA, TT, H, E, I, C, S. All these letters are not distinct, so we cannot use ${}^nP_r$. We can choose 4 letters from the following ways:

(i) All the four distinct letters: There are 8 distinct letters viz. M, A, T, H, E, I, C, S out of which 4 can be chosen in ${}^8C_4$ ways. So, the total number of groups of 4 letters $= {}^8C_4$. Each such group has 4 letters which can be arranged in 4! ways.

Hence, the total number of words $= {}^8C_4 \times 4! = {}^8P_4 = 1680$.

(ii) Two distinct and two alike letters: There are 3 pairs of alike letters viz MM, AA, TT, out of which one pair can be chosen in ${}^3C_1$ ways. Now we have to choose two letters out of the remaining 7 different types of letters which can be done in ${}^7C_2$ ways. So, the total number of groups of 4 letters in which two are different and 2 are alike is ${}^3C_1 \times {}^7C_2$. Each such group has 4 letters of which 2 are alike and remaining two distinct and they can be arranged in $\frac{4!}{2!}$ ways.

Hence, the total number of words in which two letters are alike $= {}^3C_1 \times {}^7C_2 \times \frac{4!}{2!} = 756$.

(iii) Two alike of one kind and two alike of other kind: There are 3 pairs of 2 alike letters out of which 2 pairs can be chosen in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups of 4 letters each. In each group there are 4 letters of which 2 are alike of one kind and two alike of other kind. These 4 letters can be arranged in $\frac{4!}{2!\,2!}$ ways. Hence, the total number of words in which two letters are alike of one kind and two alike of other kind $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18$.

From (i), (ii) and (iii) the total number of 4 letter words $= 1680 + 756 + 18 = 2454$.

EXAMPLE 9 *Eighteen guests have to be seated, half on each side of a long table. Four particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the seating arrangement can be made.*

SOLUTION Since four particular guests want to sit on a particular side A (say) and three others on the other side B (say). So, we are left with 11 guests out of which we choose 5 for side A in ${}^{11}C_5$ ways and the remaining 6 for side B in ${}^6C_6$ ways. Hence, the number of selections for the two sides is ${}^{11}C_5 \times {}^6C_6$.

Now 9 persons on each side of the table can be arranged among themselves in 9! ways.

Hence, the total number of arrangements $= {}^{11}C_5 \times {}^6C_6 \times 9! \times 9! = \frac{11!}{6!\,5!} \times 9! \times 9!$

EXAMPLE 10 *How many four-letter words can be formed using the letter of the word 'INEFFECTIVE'?*

SOLUTION There are 11 letters in the word 'INEFFECTIVE'. viz. EEE, FF, II, C, T, N, V.

The four-letter words may consist of:

(i) 3 alike letters and 1 distinct letter
(ii) 2 alike letters of one kind and 2 alike letters of the second kind
(iii) 2 alike letters and 2 distinct letters
(iv) all different letters

Now we, shall discuss these four cases one by one:

(i) *3 alike letters and 1 distinct letter*: There is one set of three alike letters viz. EEE. So, three alike letters can be selected in one way. Out of the 6 different letters F, I, T, N, V, C one letter can be selected in ${}^6C_1$ ways. Thus, three alike and one different letter can be selected in $1 \times {}^6C_1 = {}^6C_1$ ways. So, there are ${}^6C_1$ groups each of which contains 3 alike letters and one different letter. These 4 letters can be arranged in $\frac{4!}{3!\,1!}$ ways.

Hence, the total number of words consisting of three alike and one distinct letters

$$= {}^6C_1 \times \frac{4!}{3!\,1!} = {}^6C_1 \times 4 = 24.$$

(ii) *2 alike letters of one kind and 2 alike letters of second kind:* There are three sets of two alike letters viz EE, FF, II. Out of these three sets two can be selected in ${}^3C_2$ ways. So, there are ${}^3C_2$ groups each of which contains 4 letters out of which 2 are alike of one type and two are alike of second type. Now, 4 letters in each group can be arranged in $\frac{4!}{2!\,2!}$ ways.

Hence, the total number of words consisting of two alike letters of one type and 2 alike letters of second type $= {}^3C_2 \times \frac{4!}{2!\,2!} = 18.$

(iii) *2 alike and 2 different letters:* Out of 3 sets of two alike letters one set can be chosen in ${}^3C_1$ ways. Now, from the remaining 6 distinct letters, 2 letters can be chosen in ${}^6C_2$ ways. Thus, 2 alike letters and 2 distinct letters can be selected in $({}^3C_1 \times {}^6C_2)$ ways. So, there are $({}^3C_1 \times {}^6C_2)$ groups of 4 letters each. Now, letters of each group can be arranged among themselves in $\frac{4!}{2!}$ ways.

Hence, the total number of words consisting of two alike letters and 2 distinct

$$= {}^3C_1 \times {}^6C_2 \times \frac{4!}{2!} = 540.$$

(iv) *All different letters:* There are 7 distinct letters E, F, I, T, N, V, C out of which 4 can be selected in ${}^7C_4$ ways. So, there are ${}^7C_4$ groups of 4 letters each. The letters in each of ${}^7C_4$ groups can be arranged in 4! ways.

So, the total number of 4 letter words in which all letters are distinct $= {}^7C_4 \times 4! = 840.$

Hence, the total number of 4-letter words $= 24 + 18 + 540 + 840 = 1422.$

EXAMPLE 11 *In how many ways can the letters of the word PERMUTATIONS be arranged if there are always 4 letters between P and S?* **[NCERT]**

SOLUTION There 12 letters in the given word of which 2 are T's. There can be 4 letters between P and S in one of the following ways:

(i) There are 2T's and 2 other letters from the remaining 8 letters (excluding 2T's and *P* and *S*).
(ii) One *T* and 3 other letters from the remaining 8 letters.
(iii) There is no *T* and 4 other letters.

Let us now find the number of words in each case.

(i) In the first case, 2 letters can be chosen from remaining 8 letters in ${}^8C_2$ ways. Now, 2T's and 2 other letters can be arranged between *P* and *S* in $\frac{4!}{2!}$ ways. Also, *P* and *S* can interchange their positions. So, 2T's and 2 other letters can be arranged between *P* and *S* in ${}^8C_2 \times \frac{4!}{2!} \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words, in this case $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7!$

(ii) In this case, 3 letters can be chosen from the remaining 8 letters in ${}^8C_3$ ways. Now, one T and 3 other letters from the remaining 8 letters can be arranged between *P* and *S* in 4! ways. Also, *P* and *S* can interchange their positions. So, one *T* and 3 other letters can be arranged between *P* and *S* in ${}^8C_3 \times 4! \times 2!$ ways. Considering these six letters as one letter and the remaining 6 letters can be arranged in 7! ways.

$\therefore$ Total number of words formed $= {}^8C_3 \times 4! \times 2! \times 7!$

(iii) In this case, 4 letters other than 2T's can be chosen from the remaining 8 letters in ${}^8C_4$ ways. These 4 letters can be arranged between P and S in 4! ways. Also, P and S can interchange their positions in 2! ways. Thus, 4 letters between P and S can be arranged in ${}^8C_4 \times 4! \times 2!$ ways. Taking these 6 letters as one letter with the remaining 6 letters (including 2T's), we have 7 letters which can be arranged in $\frac{7!}{2!}$ ways.

$\therefore$ Number of words formed $= {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

Hence, total number of words $= {}^8C_2 \times \frac{4!}{2!} \times 2! \times 7! + {}^8C_3 \times 4! \times 2! \times 7! + {}^8C_4 \times 4! \times 2! \times \frac{7!}{2!}$

$= 25401600$

EXERCISE 17.3

LEVEL-1

1. How many different words, each containing 2 vowels and 3 consonants can be formed with 5 vowels and 17 consonants?
2. There are 10 persons named $P_1, P_2, P_3 \ldots, P_{10}$. Out of 10 persons, 5 persons are to be arranged in a line such that is each arrangement P_1 must occur whereas P_4 and P_5 do not occur. Find the number of such possible arrangements.
3. How many words, with or without meaning can be formed from the letters of the word 'MONDAY', assuming that no letter is repeated, if (i) 4 letters are used at a time (ii) all letters are used at a time (iii) all letters are used but first letter is a vowel? [NCERT]
4. Find the number of permutations of n distinct things taken r together, in which 3 particular things must occur together.
5. How many words each of 3 vowels and 2 consonants canbe formed from the letters of the word INVOLUTE? [NCERT]
6. Find the number of permutations of n different things taken r at a time such that two specified things occur together? [NCERT]

LEVEL-2

7. Find the number of ways in which : (a) a selection (b) an arrangement, of four letters can be made from the letters of the word 'PROPORTION'.
8. How many words can be formed by taking 4 letters at a time from the letters of the word 'MORADABAD'?
9. A business man hosts a dinner to 21 guests. He is having 2 round tables which can accommodate 15 and 6 persons each. In how many ways can he arrange the guests?
10. Find the number of combinations and permutations of 4 letters taken from the word 'EXAMINATION'.
11. A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular side and two on the other side. In how many ways can they be seated?

ANSWERS

1. 816000 2. ${}^7C_4 \times 5!$ 3. (i) 360 (ii) 720 (iii) 240 4. ${}^{n-3}C_{r-3}\,(r-2)!\,3!$
5. 2880 6. $2(r-1)\,{}^{n-2}P_{r-2}$ 7. (a) 53 (b) 758 8. 626
9. ${}^{21}C_{15} \times 14! \times 5!$ 10. 2454 11. ${}^{10}C_4 \times (8!)^2$

HINTS TO NCERT & SELECTED PROBLEMS

1. 2 vowels out of 5 and 3 consonants out of 17 can be chosen in ${}^5C_2 \times {}^{17}C_3$ ways.
Now, 5 letters in each selection can be arranged in 5! ways.
So, total number of words = ${}^5C_2 \times {}^{17}C_3 \times 5! = 816000$
3. (i) Total number of 4 letter words formed from the letters of the word 'MONDAY"
$= {}^6C_4 \times 4! = 360.$
(ii) Total number of words formed by using all letters of the word 'MONDAY'
$= 6! = 720$
(iii) There are two vowels A and O. So, first place can be filled in 2 ways and the remaining 5 places can be filled in 5! ways.
So, total number of words beginning with a vowel $= 2 \times 5! = 240.$
5. Required number of words = ${}^4C_3 \times {}^4C_2 \times 5!$
6. Out of $(n-2)$ remaining things select $(r-2)$ things in ${}^{n-2}C_{r-2}$ ways. Consider two specified things as one and mix it with $(r-2)$ selected things. Now we have $(r-1)$ things which can be arranged in $(r-1)!$ ways, but two specified things can be put together in 2! ways. Hence, required number of ways = ${}^{n-2}C_{r-2} \times (r-1)! \times 2!$.
9. Total number of ways = ${}^{21}C_{15} \times {}^6C_6 \times 14! \times 5!$
11. 4 persons wish to sit on side A(say) and two on the other side B(say). So, 10 persons are left, out of which 4 persons for side A can be selected in ${}^{10}C_4$ ways and 6 persons for side B from the remaining 6 persons in ${}^6C_6$ ways. Hence, the number of selections for two sides $= {}^{10}C_4 \times {}^6C_6$. Now, 8 persons on each side can be arranged amongst themselves in 8! ways.
Hence, the total number of seating arrangements = ${}^{10}C_4 \times {}^6C_6 \times 8! \times 8!$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write $\sum_{r=0}^{m} {}^{n+r}C_r$ in the simplified form.
2. If ${}^{35}C_{n+7} = {}^{35}C_{4n-2}$, then write the values of n.
3. Write the number of diagonals of an n-sided polygon.
4. Write the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ in the simplest form.
5. Write the value of $\sum_{r=1}^{6} {}^{56-r}C_3 + {}^{50}C_4$.
6. There are 3 letters and 3 directed envelopes. Write the number of ways in which no letter is put in the correct envelope.
7. Write the maximum number of points of intersection of 8 straight lines in a plane.
8. Write the number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines.
9. Write the number of ways in which 5 red and 4 white balls can be drawn from a bag containing 10 red and 8 white balls.
10. Write the number of ways in which 12 boys may be divided into three groups of 4 boys each.
11. Write the total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants.

ANSWERS

1. ${}^{n+m+1}C_{n+1}$ 2. 3, 6 3. $\frac{n(n-3)}{2}$ 4. ${}^{n+2}C_{r+1}$ 5. ${}^{56}C_4$

6. 2 7. 28 8. 18 9. ${}^{10}C_5 \times {}^8C_4$ 10. $\frac{12!}{(4!)^3\, 3!}$

11. ${}^4C_2 \times 5C_3 \times 5!$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If ${}^{20}C_r = {}^{20}C_{r-10}$, then ${}^{18}C_r$ is equal to
   (a) 4896 (b) 816 (c) 1632 (d) none of these
2. If ${}^{20}C_r = {}^{20}C_{r+4}$, then ${}^rC_3$ is equal to
   (a) 54 (b) 56 (c) 58 (d) none of these
3. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, then r is equal to
   (a) 5 (b) 4 (c) 3 (d) 2
4. If ${}^{20}C_{r+1} = {}^{20}C_{r-1}$, then r is equal to
   (a) 10 (b) 11 (c) 19 (d) 12
5. If $C(n, 12) = C(n, 8)$, then $C(22, n)$ is equal to
   (a) 231 (b) 210 (c) 252 (d) 303
6. If ${}^mC_1 = {}^nC_2$, then
   (a) $2m = n$ (b) $2m = n(n+1)$ (c) $2m = n(n-1)$ (d) $2n = m(m-1)$
7. If ${}^nC_{12} = {}^nC_8$, then $n =$
   (a) 20 (b) 12 (c) 6 (d) 30
8. If ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_x$, then $x =$
   (a) r (b) $r-1$ (c) n (d) $r+1$
9. If ${}^{(a^2-a)}C_2 = {}^{(a^2-a)}C_4$, then $a =$
   (a) 2 (b) 3 (c) 4 (d) none of these
10. ${}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$ is equal to
    (a) 30 (b) 31 (c) 32 (d) 33
11. Total number of words formed by 2 vowels and 3 consonants taken from 4 vowels and 5 consonants is equal to
    (a) 60 (b) 120 (c) 7200 (d) none of these
12. There are 12 points in a plane. The number of the straight lines joining any two of them when 3 of them are collinear, is
    (a) 62 (b) 63 (c) 64 (d) 65
13. Three persons enter a railway compartment. If there are 5 seats vacant, in how many ways can they take these seats ?
    (a) 60 (b) 20 (c) 15 (d) 125
14. In how many ways can a committee of 5 be made out of 6 men and 4 women containing at least one women ?

(a) 246 (b) 222 (c) 186 (d) none of these

15. There are 10 points in a plane and 4 of them are collinear. The number of straight lines joining any two of them is
(a) 45 (b) 40 (c) 39 (d) 38

16. There are 13 players of cricket, out of which 4 are bowlers. In how many ways a team of eleven be selected from them so as to include at least two bowlers ?
(a) 72 (b) 78 (c) 42 (d) none of these

17. If $C_0 + C_1 + C_2 + \ldots + C_n = 256$, then ${}^{2n}C_2$ is equal to
(a) 56 (b) 120 (c) 28 (d) 91

18. The number of ways in which a host lady can invite for a party of 8 out of 12 people of whom two do not want to attend the party together is
(a) $2 \times {}^{11}C_7 + {}^{10}C_8$ (b) ${}^{10}C_8 + {}^{11}C_7$
(c) ${}^{12}C_8 - {}^{10}C_6$ (d) none of these

19. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the number of circles that can be drawn so that each contains at least 3 of the given points is
(a) 216 (b) 156 (c) 172 (d) none of these

20. How many different committees of 5 can be formed from 6 men and 4 women on which exact 3 men and 2 women serve ?
(a) 6 (b) 20 (c) 60 (d) 120

21. If ${}^{43}C_{r-6} = {}^{43}C_{3r+1}$, then the value of r is
(a) 12 (b) 8 (c) 6 (d) 10 (e) 14

22. The number of diagonals that can be drawn by joining the vertices of an octagon is
(a) 20 (b) 28 (c) 8 (d) 16

23. The value of $\left({}^{7}C_0 + {}^{7}C_1\right) + \left({}^{7}C_1 + {}^{7}C_2\right) + \ldots + \left({}^{7}C_6 + {}^{7}C_7\right)$ is
(a) $2^7 - 1$ (b) $2^8 - 2$ (c) $2^8 - 1$ (d) 2^8

24. Among 14 players, 5 are bowlers. In how many ways a team of 11 may be formed with at least 4 bowlers?
(a) 265 (b) 263 (c) 264 (d) 275

25. A lady gives a dinner party for six guests. The number of ways in which they may be selected from among ten friends if two of the friends will not attend the party together is
(a) 112 (b) 140 (c) 164 (d) none of these

26. If ${}^{n+1}C_3 = 2 \cdot {}^{n}C_2$, then $n =$
(a) 3 (b) 4 (c) 5 (d) 6

27. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 6 (b) 9 (c) 12 (d) 18

ANSWERS

1. (b)	2. (b)	3. (c)	4. (a)	5. (a)	6. (c)	7. (a)	8. (d)
9. (b)	10. (b)	11. (c)	12. (c)	13. (a)	14. (a)	15. (b)	16. (b)
17. (b)	18. (c)	19. (b)	20. (d)	21. (a)	22. (a)	23. (b)	24. (c)
25. (b)	26. (c)	27. (d)					

SUMMARY

1. If n is a natural number and r is a non-negative integer such that $0 \leq r \leq n$, then

   (i) ${}^nC_r = \dfrac{n!}{(n-r)!\,r!}$ (ii) ${}^nC_r \times r! = {}^nP_r$

   (iii) ${}^nC_r = {}^nC_{n-r}$ (iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

   (v) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r} \times \dfrac{n-1}{r-1} \cdot {}^{n-2}C_{r-2} = \ldots = \dfrac{n}{r} \times \dfrac{n-1}{r-1} \times \dfrac{n-2}{r-2} \times \ldots \times \dfrac{n-(r-1)}{1}$

   (vi) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or, $x + y = n$

   (vii) If n is an even natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ is ${}^nC_{\frac{n}{2}}$.

   If n is an odd natural number, then the greatest among ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ is ${}^nC_{\frac{n-1}{2}}$ or, ${}^nC_{\frac{n+1}{2}}$.

2. The number of ways of selecting r items or objects from a group of n distinct items or objects is $\dfrac{n!}{(n-r)!\,r!} = {}^nC_r$.