CHAPTER 12

MATHEMATICAL INDUCTION

12.1 STATEMENTS

A sentence or description which can be judged to be true or false is called a statement.

Following are some examples of statements:

EXAMPLE 1 *2 divides 6.*

EXAMPLE 2 *Jaipur is the capital of Rajasthan.*

EXAMPLE 3 *There are 5 days in a week.*

EXAMPLE 4 $(x+1)$ *is a factor of* $x^2 - 3x + 2$.

EXAMPLE 5 $A \cup B = B \cup A$.

Clearly, statements in Examples 1, 2 and 5 are true statements whereas statements in Examples 3 and 4 are false.

MATHEMATICAL STATEMENTS *Statements involving mathematical relations are known as the mathematical statements.*

Clearly, statements in examples 1, 4 and 5 are mathematical statements. In this chapter, we shall be mainly discussing mathematical statements concerning natural numbers. We shall be using notations $P(n)$ or $P_1(n)$ or $P_2(n)$ etc. to denote such statements.

EXAMPLE 1 *Let* $P(n)$ *be the statement "*$10n + 3$ *is prime". Then,*

$P(2)$ is the statement "$10 \times 2 + 3$ is prime" i.e. "23 is prime".

Clearly, $P(2)$ is true.

$P(3)$ is the statement "$10 \times 3 + 3$ is prime" i.e. "33 is prime".

Clearly $P(3)$ is not true.

EXAMPLE 2 *If* $P(n)$ *is the statement "*$n^3 + n$ *is divisible by 3", is the statement* $P(3)$ *true ? Is the statement* $P(4)$ *true ?*

SOLUTION $P(3)$ is the statement "$3^3 + 3 = 30$ is divisible by 3".

Clearly, it is true.

$P(4)$ is the statement "$4^3 + 4 = 68$ is divisible by 3".

Clearly, it is not true.

EXAMPLE 3 *If* $P(n)$ *is the statement "*$n(n+1)(n+2)$ *is divisible by 12", prove that the statements* $P(3)$ *and* $P(4)$ *are true, but that* $P(5)$ *is not true.*

SOLUTION $P(3)$ is the statement "$3(3+1)(3+2) = 60$ is divisible by 12".

It is true.

$P(4)$ is the statement "$4(4+1)(4+2) = 120$ is divisible by 12".

It is also true.

$P(5)$ is the statement "$5(5+1)(5+2) = 210$ is divisible by 12".

Clearly it is not true.

EXAMPLE 4 *Let $P(n)$ be the statement "7 divides $(2^{3n}-1)$". What is $P(n+1)$?*

SOLUTION $P(n+1)$ is the statement "7 divides $(2^{3(n+1)}-1)$".

Clearly, $P(n+1)$ is obtained by replacing n by $(n+1)$ in $P(n)$.

EXAMPLE 5 *If $P(n)$ is the statement "$n^2>100$", prove that whenever $P(r)$ is true, $P(r+1)$ is also true.*

SOLUTION The statement $P(n)$ is "$n^2>100$". Let $P(r)$ be true. Then $r^2>100$.

We wish to prove that the statement $P(r+1)$ is true i.e. "$(r+1)^2>100$".

Now,

$P(r)$ is true

$\Rightarrow\quad r^2>100$

$\Rightarrow\quad r^2+2r+1>100+2r+1$ [Adding $(2r+1)$ on both sides]

$\Rightarrow\quad (r+1)^2>100+2r+1$

$\Rightarrow\quad (r+1)^2>100$

$\Rightarrow\quad P(r+1)$ is true [$\because$ $100+2r+1>100$ for every natural number r]

Thus, whenever $P(r)$ is true, $P(r+1)$ is also true.

EXAMPLE 6 *Let $P(n)$ be the statement "$3^n>n$". If $P(n)$ is true, prove that $P(n+1)$ is true.*

SOLUTION We are given that $P(n)$ is true i.e. $3^n>n$, and we wish to prove that $P(n+1)$ is true i.e. $3^{(n+1)}>(n+1)$.

Now,

$P(n)$ is true

$\Rightarrow\quad 3^n>n$

$\Rightarrow\quad 3.3^n>3n$ [Multiplying both sides by 3]

$\Rightarrow\quad 3^{n+1}>n+2n$

$\Rightarrow\quad 3^{n+1}>n+1$ [$\because$ $2n>1$ for every $n\in N \Rightarrow 2n+n>n+1$ for every $n\in N$]

$\Rightarrow\quad P(n+1)$ is true

EXAMPLE 7 *If $P(n)$ is the statement "$2^{3n}-1$ is an integral multiple of 7", and if $P(r)$ is true, prove that $P(r+1)$ is true.*

SOLUTION Let $P(r)$ be true. Then, $2^{3r}-1$ is an integral multiple of 7.

We wish to prove that $P(r+1)$ is true i.e. $2^{3(r+1)}-1$ is an integral multiple of 7.

Now,

$P(r)$ is true

$\Rightarrow\quad 2^{3r}-1$ is an integral multiple of 7

$\Rightarrow\quad 2^{3r}-1=7\lambda$, for some $\lambda\in N$.

$\Rightarrow\quad 2^{3r}=7\lambda+1$ ...(i)

Now, $2^{3(r+1)}-1=2^{3r}\times 2^3-1=(7\lambda+1)\times 8-1$ [Using (i)]

$\Rightarrow\quad 2^{3(r+1)}-1=56\lambda+8-1=56\lambda+7=7(8\lambda+1)$

$\Rightarrow\quad 2^{3(r+1)}-1=7\mu$, where $\mu=8\lambda+1\in N$

$\Rightarrow\quad 2^{3(r+1)}-1$ is an integral multiple of 7

$\Rightarrow\quad P(r+1)$ is true

EXERCISE 12.1

1. If $P(n)$ is the statement "$n(n+1)$ is even", then what is $P(3)$?
2. If $P(n)$ is the statement "$n^3 + n$ is divisible by 3", prove that $P(3)$ is true but $P(4)$ is not true.
3. If $P(n)$ is the statement "$2^n \geq 3n$", and if $P(r)$ is true, prove that $P(r+1)$ is true.
4. If $P(n)$ is the statement "$n^2 + n$ is even", and if $P(r)$ is true, then $P(r+1)$ is true.
5. Given an example of a statement $P(n)$ such that it is true for all $n \in N$.
6. If $P(n)$ is the statement "$n^2 - n + 41$ is prime", prove that $P(1)$, $P(2)$ and $P(3)$ are true. Prove also that $P(41)$ is not true.
7. Give an example of a statement $P(n)$ which is true for all $n \geq 4$ but $P(1)$, $P(2)$ and $P(3)$ are not true. Justify your answer.

ANSWERS

1. $P(3) : 3(3+1)$ is even
5. $P(n) : 1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$
7. $P(n) : 2n < n!$

HINTS TO SELECTED PROBLEMS

3. Let $P(r)$ be true. Then, $P(r)$ is true

$$\Rightarrow 2^r \geq 3r$$
$$\Rightarrow 2 \cdot 2^r \geq 6r$$
$$\Rightarrow 2^{r+1} \geq 3r + 3r$$
$$\Rightarrow 2^{r+1} \geq 3r + 3 \qquad [\because 3r \geq 3 \Rightarrow 3r + 3r \geq 3r + 3]$$
$$\Rightarrow 2^{r+1} \geq 3(r+1) \Rightarrow P(r+1) \text{ is true}$$

5. See the statement in Q. No. 4

12.2 THE PRINCIPLES OF MATHEMATICAL INDUCTION

FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n = 1$.

and, (II) $P(m+1)$ is true, whenever $P(m)$ is true.

i.e. $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Then, $P(n)$ is true for all natural numbers n.

SECOND PRINCIPLE OF MATHEMATICAL INDUCTION

Let $P(n)$ be a statement involving the natural number n such that

(I) $P(1)$ is true i.e. $P(n)$ is true for $n = 1$.

and, (II) $P(m+1)$ is true, whenever $P(n)$ is true for all n, where $1 \leq n \leq m$.

Then, $P(n)$ is true for all natural numbers.

ILLUSTRATIVE EXAMPLES

Type I PROBLEMS BASED UPON FIRST PRINCIPLE OF MATHEMATICAL INDUCTION

Recall that the first principle of mathematical induction consists of two parts. First we must show that the given statement $P(n)$ is true for $n = 1$. The second part has two steps. The first step is to assume that the statement $P(n)$ is true for some $m \in N$. The second step is to use this assumption to prove that the statement $P(n)$ is true for $n = m + 1$.

In order to prove that a statement is true for all natural numbers using first principle of mathematical induction, we may use the following algorithm:

ALGORITHM

STEP I *Obtain P (n) and understand its meaning.*

STEP II *Prove that the statement P (1) is true i.e. P (n) is true for n = 1.*

STEP III *Assume that the statement P (n) is true for n = m (say) i.e. P (m) is true.*

STEP IV *Using assumption in step III prove that P (m + 1) is true.*

STEP V *Combining the results of step II and step IV, conclude by the first principle of mathematical induction that P (n) is true for all n ∈ N.*

The following examples illustrate the above algorithm.

LEVEL-1

EXAMPLE 1 *Prove by the principle of mathematical induction that for all $n \in N$, $n^2 + n$ is even natural number*

SOLUTION Let $P(n)$ be the statement "$n^2 + n$ is even".

STEP I We have, $P(n): n^2 + n$ is even

$\because \quad 1^2 + 1 = 2$, which is even

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$P(m)$ is ture $\Rightarrow m^2 + m$ is even $\Rightarrow m^2 + m = 2\lambda$ for some $\lambda \in N$ ...(i)

Now, we shall show that $P(m+1)$ is true. For this we have to show that $(m+1)^2 + (m+1)$ is an even natural number.

Now,

$$(m+1)^2 + (m+1) = (m^2 + 2m + 1) + (m+1) = (m^2 + m) + (2m + 2)$$

$$\Rightarrow \quad (m+1)^2 + (m+1) = m^2 + m + 2(m+1) = 2\lambda + 2(m+1) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad (m+1)^2 + (m+1) = 2(\lambda + m + 1) = 2\mu, \text{ where } \mu = \lambda + m + 1 \in N$$

$\Rightarrow \quad (m+1)^2 + (m+1)$ is an even natural number

$\Rightarrow \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $n^2 + n$ is even for all $n \in N$.

EXAMPLE 2 *Prove by the principle of mathematical induction that : $n(n+1)(2n+1)$ is divisible by 6 for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement "$n(n+1)(2n+1)$ is divisible by 6".

i.e. $P(n): n(n+1)(2n+1)$ is divisible by 6

STEP I We have, $P(1): 1(1+1)(2+1)$ is divisible by 6.

$\because \quad 1(1+1)(2+1) = 6$ which is divisible by 6

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$m(m+1)(2m+1)$ is divisible by 6

$\Rightarrow \quad m(m+1)(2m+1) = 6\lambda$, for some $\lambda \in N$ ...(i)

Now, we shall show that $P(m+1)$ is true. For this we have to show that

$(m+1)(m+1+1)\{2(m+1)+1\}$ is divisible by 6.

Now,

$$\begin{aligned}(m+1)(m+1+1)\{2(m+1)+1\} &= (m+1)(m+2)\{(2m+1)+2\}\\ &= (m+1)(m+2)(2m+1)+2(m+1)(m+2)\\ &= m(m+1)(2m+1)+2(m+1)(2m+1)+2(m+1)(m+2)\\ &= m(m+1)(2m+1)+2(m+1)(2m+1+m+2)\\ &= m(m+1)(2m+1)+2(m+1)(3m+3)\\ &= m(m+1)(2m+1)+6(m+1)^2 = 6\lambda+6(m+1)^2 \quad \text{[Using (i)]}\\ &= 6\{\lambda+(m+1)^2\}, \text{ which is divisible by 6}\end{aligned}$$

$\Rightarrow$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 3 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1)$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1+4+7+\ldots+(3n-2) = \frac{1}{2}n(3n-1)$$

STEP I We have,

$$P(1): 1 = \frac{1}{2}\times(1)\times(3\times1-1).$$

$$\because \quad 1 = \frac{1}{2}\times(1)\times(3\times1-1)$$

So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$1+4+7+\ldots+(3m-2) = \frac{1}{2}m(3m-1) \qquad \ldots\text{(i)}$$

We wish to show that $P(m+1)$ is true. For this we have to show that

$$1+4+7+\ldots+(3m-2)+\{3(m+1)-2\} = \frac{1}{2}(m+1)\{3(m+1)-1\}$$

Now, $1+4+7+\ldots+(3m-2)+\{3(m+1)-2\}$

$$\begin{aligned}&= \frac{1}{2}m(3m-1)+\{3(m+1)-2\} \qquad \text{[Using (i)]}\\ &= \frac{1}{2}m(3m-1)+(3m+1) = \frac{1}{2}\{3m^2-m+6m+2\}\\ &= \frac{1}{2}\{3m^2+5m+2\} = \frac{1}{2}(m+1)(3m+2) = \frac{1}{2}(m+1)\{3(m+1)-1\}\end{aligned}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 4 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^2+2^2+3^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)$$

STEP I We have,

$$P(1): 1^2 = \frac{1}{6}(1)(1+1)(2\times1+1)$$

$\because \quad 1^2 = 1 = \frac{1}{6}(1)(1+1)(2\times 1+1)$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$1^2 + 2^2 + 3^2 + \ldots + m^2 = \frac{1}{6} m(m+1)(2m+1) \quad \ldots(i)$$

We wish to show that $P(m+1)$ is true. For this we have to show that

$$1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 = \frac{1}{6}(m+1)\{(m+1)+1\}\ \{2(m+1)+1)\}$$

Now, $1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2$

$= \{1^2 + 2^2 + 3^2 + \ldots + m^2\} + (m+1)^2$

$= \frac{1}{6} m(m+1)(2m+1) + (m+1)^2$ [Using (i)]

$= \frac{1}{6}(m+1)\{m(2m+1) + 6(m+1)\} = \frac{1}{6}(m+1)\{2m^2 + 7m + 6\}$

$= \frac{1}{6}(m+1)(m+2)(2m+3) = \frac{1}{6}(m+1)\{(m+1)+1\}\{2(m+1)+1\}$

$\therefore \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 5 *Using the principle of mathematical induction prove that:*

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

STEP I We have,

$$P(1): 1^3 = \left\{\frac{1(1+1)}{2}\right\}^2$$

Clearly, $1^3 = 1 = \left\{\frac{1(1+1)}{2}\right\}^2$

$\therefore \quad P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1^3 + 2^3 + 3^3 + \ldots + m^3 = \left\{\frac{m(m+1)}{2}\right\}^2 \quad \ldots(i)$$

We shall now prove that $P(m+1)$ is true. For this we have to prove that

$$1^3 + 2^3 + 3^3 + \ldots + m^3 + (m+1)^3 = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

Now,

$1^3 + 2^3 + 3^3 + \ldots + m^3 + (m+1)^3$

$= \left\{1^3 + 2^3 + \ldots + m^3\right\} + (m+1)^3$

$= \left\{\frac{m(m+1)}{2}\right\}^2 + (m+1)^3$ [Using (i)]

$$= (m+1)^2 \left\{\frac{m^2}{4} + (m+1)\right\}$$

$$= (m+1)^2 \left\{\frac{m^2+4m+4}{4}\right\} = \frac{(m+1)^2 (m+2)^2}{4} = \left\{\frac{(m+1)\{(m+1)+1\}}{2}\right\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 6 *Using the principle of mathematical induction, prove that*

$$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

STEP I We have,

$$P(1): 1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}$$

$$\because \quad 1.2.3 = 6 \text{ and } \frac{1(1+1)(1+2)(1+3)}{4} = \frac{2 \times 3 \times 4}{4} = 6$$

$$\therefore \quad 1.2.3 = \frac{1(1+1)(1+2)(1+3)}{4}$$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) = \frac{m(m+1)(m+2)(m+3)}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+3) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

Now, $1.2.3 + 2.3.4 + \ldots + m(m+1)(m+2) + (m+1)(m+2)(m+2)$

$$= \frac{m(m+1)(m+2)(m+3)}{4} + (m+1)(m+2)(m+3) \quad \text{[Using (i)]}$$

$$= (m+1)(m+2)(m+3)\left(\frac{m}{4} + 1\right) = \frac{(m+1)(m+2)(m+3)(m+4)}{4}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 7 *Using the principle of mathematical induction prove that*

$$1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1} + 3}{4} \text{ for all } n \in N$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1.3 + 2.3^2 + 3.3^3 + \ldots + n.3^n = \frac{(2n-1)\,3^{n+1} + 3}{4}$$

STEP I $P(1): 1.3 = \frac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4}$

$$\because \quad 1.3 = 3 \text{ and } \frac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4} = \frac{9+3}{4} = 3$$

$$\therefore \quad 1.3 = \frac{(2 \times 1 - 1) \times 3^{1+1} + 3}{4}$$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m = \frac{(2m-1)\, 3^{m+1} + 3}{4} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1} = \frac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$$

Now,

$$1.3 + 2.3^2 + 3.3^3 + \ldots + m.3^m + (m+1).3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3}{4} + (m+1)\, 3^{m+1}$$

$$= \frac{(2m-1)\, 3^{m+1} + 3 + (4m+4)\, 3^{m+1}}{4}$$

$$= \frac{(2m-1) \times 3^{m+1} + (4m+4) \times 3^{m+1} + 3}{4}$$

$$= \frac{(2m-1+4m+4)\, 3^{m+1} + 3}{4}$$

$$= \frac{(6m+3)\, 3^{m+1} + 3}{4} = \frac{(2m+1)\, 3^{m+2} + 3}{4} = \frac{\{2(m+1)-1\}\, 3^{(m+1)+1} + 3}{4}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principal of mathematical induction $P(n)$ is true for all $n \in N$ i.e., the given result is true for all $n \in N$.

EXAMPLE 8 *Prove by the principle of mathematical induction that for all $n \in N$:*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

STEP I We have, $P(1) : \frac{1}{1.2} = \frac{1}{1+1}$

$$\because \quad \frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$

So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} = \frac{m}{m+1} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+1+1)} = \frac{(m+1)}{(m+1)+1}$$

Now, $$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)} + \frac{1}{(m+1)((m+1)+1)}$$

$$= \left\{\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{m(m+1)}\right\} + \frac{1}{(m+1)((m+1)+1)}$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)((m+1)+1)} = \frac{m}{m+1} + \frac{1}{(m+1)(m+2)} \quad \text{[Using (i)]}$$

$$= \frac{1}{(m+1)}\left\{\frac{m}{1} + \frac{1}{m+2}\right\} = \frac{1}{(m+1)} \times \frac{(m^2+2m+1)}{(m+2)} = \frac{(m+1)^2}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2} = \frac{(m+1)}{(m+1)+1}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by the principle of mathematical induction, the given statement is true for all $n \in N$.

EXAMPLE 9 *Using the principle of mathematical induction prove that*

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+n} = \frac{2n}{n+1}$$

STEP I We have, $P(1): 1 = \frac{2 \times 1}{1+1}$

Clearly, $\frac{2\times 1}{1+1} = \frac{2}{2} = 1$

$\therefore$ $1 = \frac{2 \times 1}{1+1}$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} = \frac{2m}{m+1} \quad \ldots\text{(i)}$$

We shall now show that $P(m+1)$ is true. For this we will prove that

$$1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3+\ldots+m} + \frac{1}{1+2+3+\ldots+(m+1)} = \frac{2(m+1)}{(m+1)+1}$$

Now, $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots + \frac{1}{1+2+3\ldots+m} + \frac{1}{1+2+3+\ldots+(m+1)}$

$$= \frac{2m}{m+1} + \frac{1}{1+2+3+\ldots+(m+1)} \quad \text{[Using (i)]}$$

$$= \frac{2m}{m+1} + \frac{1}{\frac{(m+1)(m+2)}{2}} \quad \left[\because 1+2+\ldots+m+(m+1) = \frac{(m+1)(m+2)}{2}\right]$$

$$= \frac{2m}{m+1} + \frac{2}{(m+1)(m+2)} \quad \text{[Using (i)]}$$

$$= \frac{2}{m+1}\left\{m + \frac{1}{(m+2)}\right\} = \frac{2}{m+1}\left\{\frac{m^2+2m+1}{(m+2)}\right\} = \frac{2}{m+1} \times \frac{(m+1)^2}{m+2} = \frac{2(m+1)}{(m+1)+1}$$

$\therefore$ $P(m+1)$ is true.

Hence, by the principle of of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 10 *Prove by induction that the sum* $S_n = n^3 + 3n^2 + 5n + 3$ *is divisible by 3 for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : S_n = n^3 + 3n^2 + 5n + 3 \quad \text{is divisible by 3}$$

STEP I We have, $P(1) : S_1 = 1^3 + 3(1)^2 + 5(1) + 3$ is divisible by 3.

Since, $1^3 + 3(1)^2 + 5(1) + 3 = 12$, which is divisible by 3

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$S_m = m^3 + 3m^2 + 5m + 3$ is divisible by 3

$\Rightarrow$ $S_m = m^3 + 3m^2 + 5m + 3 = 3\lambda$, for some $\lambda \in N$ ...(i)

We now wish to show that $P(m+1)$ is true. For this we have to show that $(m+1)^3 + 3(m+1)^2 + 5(m+1) + 3$ is divisible by 3.

Now, $(m+1)^3 + 3(m+1)^2 + 5(m+1) + 3$

$= (m^3 + 3m^2 + 5m + 3) + 3m^2 + 9m + 9$

$= 3\lambda + 3(m^2 + 3m + 3)$

$= 3(\lambda + m^2 + 3m + 3)$ [Using (i)]

$= 3\mu$, where $\mu = \lambda + m^2 + 3m + 3 \in N$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by the principle of mathematical induction the statement is true for all $n \in N$.

EXAMPLE 11 *Prove by the principle of mathematical induction that for all* $n \in N$:

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n) : \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

STEP I We have, $P(1) : \frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$.

Clearly, $\frac{1}{1.3} = \frac{1}{(2 \times 1 + 1)}$.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2m-1)(2m+1)} = \frac{m}{2m+1} \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true. For this we shall show that

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

Now,

$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots + \frac{1}{(2m-1)(2m+1)} + \frac{1}{(2m+1)(2m+3)}$$

$$= \frac{m}{2m+1} + \frac{1}{(2m+1)(2m+3)} \quad \text{[Using (i)]}$$

$$= \frac{2m^2+3m+1}{(2m+1)(2m+3)} = \frac{(2m+1)(m+1)}{(2m+1)(2m+3)} = \frac{m+1}{2m+3}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction, the given result is true for all $n \in N$.

EXAMPLE 12 *Using the principle of mathematical induction, prove that*

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.45} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)} \text{ for all } n \in N.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.45} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

STEP I We have,

$$P(1) = \frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$$

$$\because \quad \frac{1}{1.2.3} = \frac{1}{6} \text{ and } \frac{1(1+3)}{4(1+1)(1+2)} = \frac{4}{4\times 2\times 3} = \frac{1}{6}$$

$$\therefore \quad \frac{1}{1.2.3} = \frac{1(1+3)}{4(1+1)(1+2)}$$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.45} + \ldots + \frac{1}{m(m+1)(m+2)} = \frac{m(m+3)}{4(m+1)(m+2)} \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true.

i.e., $$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \ldots + \frac{1}{m(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} = \frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

Now,

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.45} + \ldots + \frac{1}{m(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)}$$

$$= \frac{m(m+3)}{4(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} \quad \text{[Using (i)]}$$

$$= \frac{m(m+3)^2+4}{4(m+1)(m+2)(m+3)}$$

$$= \frac{m^3+6m^2+9m+4}{4(m+1)(m+2)(m+3)} = \frac{(m+1)^2(m+4)}{4(m+1)(m+2)(m+3)} = \frac{(m+1)(m+4)}{4(m+2)(m+3)}$$

$\therefore$ $P(m+1)$ is true.

Hence, $P(n)$ is true for all $n \in N$.

EXAMPLE 13 *If x and y are any two distinct integers, then prove by mathematical induction that $(x^n - y^n)$ is divisible by $(x-y)$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: $(x^n - y^n)$ is divisible by $(x-y)$

STEP I $P(1): (x^1 - y^1)$ is divisible by $(x-y)$.

$\because \quad x^1 - y^1 = (x - y)$ is divisible by $(x - y)$.

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$(x^m - y^m)$ is divisible by $(x - y)$

$\Rightarrow \quad (x^m - y^m) = \lambda(x - y)$, for some $\lambda \in Z$ ...(i)

We shall now show that $P(m + 1)$ is true. For this it is sufficient to show that $(x^{m+1} - y^{m+1})$ is divisible by $(x - y)$.

Now,

$$\begin{aligned} & x^{m+1} - y^{m+1} \\ &= x^{m+1} - x^m y + x^m y - y^{m+1} \\ &= x^m (x - y) + y(x^m - y^m) \\ &= x^m (x - y) + y\lambda(x - y) \qquad \text{[Using (i)]} \\ &= (x - y)(x^m + y\lambda), \text{ which is divisible by } (x - y) \end{aligned}$$

So, $P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m + 1)$ is true

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

i.e. $(x^n - y^n)$ is divisible by $(x - y)$ for all $n \in N$.

EXAMPLE 14 *Using principle of mathematical induction, prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n) : (x^{2n} - y^{2n})$ is divisisble by $(x + y)$.

STEP I $P(1) : (x^2 - y^2)$ is divisible by $(x + y)$.

$\because \quad (x^2 - y^2) = (x - y)(x + y)$, which is divisble by $(x + y)$

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$x^{2m} - y^{2m}$ is divisible by $(x + y)$

$\Rightarrow \quad x^{2m} - y^{2m} = \lambda(x + y)$ ...(i)

We shall now show that $P(m + 1)$ is true i.e., $x^{2m+2} - y^{2m+2}$ is divisible by $(x + y)$.

Now,

$$\begin{aligned} & x^{2m+2} - y^{2m+2} = x^{2m+2} - x^{2m} y^2 + x^{2m} y^2 - y^{2m+2} \\ \Rightarrow \quad & x^{2m+2} - y^{2m+2} = x^{2m}(x^2 - y^2) + y^2(x^{2m} - y^{2m}) \\ \Rightarrow \quad & x^{2m+2} - y^{2m+2} = x^{2m}(x^2 - y^2) + y^2 \lambda(x + y) \qquad \text{[Using (i)]} \\ \Rightarrow \quad & x^{2m+2} - y^{2m+2} = (x + y)\left\{x^{2m}(x - y) + \lambda y^2\right\} \end{aligned}$$

Clearly, it is divisible by $(x + y)$.

$\therefore \quad P(m + 1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m + 1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $x^{2n} - y^{2n}$ is divisible by $(x + y)$ for all $n \in N$.

EXAMPLE 15 *Using principle of mathematical induction, prove that*
(i) $41^n - 14^n$ *is a multiple of* 27 (ii) $7^n - 3^n$ *is divisible by* 4.

SOLUTION (i) Let $P(n)$ be the statement given by
$P(n): 41^n - 14^n$ is a multiple of 27.

STEP I $P(1): 41^1 - 14^1$ is a multiple of 27.
$\because$ $41^1 - 14^1 = 41 - 14 = 27$, which is a multiple of 27.
So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,
$41^m - 14^m$ is a multiple of 27
$\Rightarrow$ $41^m - 14^m = 27\lambda$ for some $\lambda \in N$ ...(i)

Now, $41^{m+1} - 14^{m+1} = 41^{m+1} - 41 \times 14^m + 41 \times 14^m - 14^{m+1}$
$\Rightarrow$ $41^{m+1} - 14^{m+1} = 41(41^m - 14^m) + (41 - 14)14^m$
$\Rightarrow$ $41^{m+1} - 14^{m+1} = 41 \times 27\lambda + 27 \times 14^m$ [Using (i)]
$\Rightarrow$ $41^{m+1} - 14^{m+1} = 27(41\lambda + 14^m)$, which is a multiple of 27.
$\therefore$ $P(m+1)$ is true.
Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.
Hence, $P(n)$ is true for all $n \in N$.
(*ii*) Proceed as in (i).

EXAMPLE 16 *Using the principle of mathematical induction, prove that* $(2^{3n} - 1)$ *is divisible by 7 for all* $n \in N$. **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by
$P(n): 2^{3n} - 1$ is divisible by 7

STEP I $P(1): 2^{3 \times 1} - 1$ is divisible by 7.
Clearly, $2^{3 \times 1} - 1 = 8 - 1 = 7$, which is divisible by 7.
So, $P(1)$ is true

STEP II Let $P(m)$ be true. Then,
$2^{3m} - 1$ is divisible by 7.
$\Rightarrow$ $2^{3m} - 1 = 7\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that $2^{3(m+1)} - 1$ is divisible by 7.
Now,
$$2^{3(m+1)} - 1 = 2^{3m} \times 2^3 - 1 = (7\lambda + 1)2^3 - 1 \quad \text{[Using (i)]}$$
$$= 56\lambda + 8 - 1 = 7(8\lambda + 1), \text{ which is divisible by 7}$$
$\therefore$ $P(m+1)$ is true.
Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true
Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$ i.e. $2^{3n} - 1$ is divisible by 7 for all $n \in N$.

EXAMPLE 17 *Prove by the principle of induction that for all* $n \in N$, $(10^{2n-1} + 1)$ *is divisible by* 11.

SOLUTION Let $P(n)$ be the statement given by
$P(n): 10^{2n-1} + 1$ is divisible by 11

STEP I We have

$P(1): 10^{2\times 1-1}+1$ is divisible by 11.

Since $10^{2\times 1-1}+1=11$, which is divisible by 11.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$10^{2m-1}+1$ is divisible by 11

$\Rightarrow$ $10^{2m-1}+1 = 11\lambda$, for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true. For this we have to show that $10^{2(m+1)}+1$ is divisible by 11.

Now, $10^{2(m+1)-1}+1 = 10^{2m+1}+1 = 10^{2m-1}\times 10^2+1$

$\Rightarrow$ $10^{2(m+1)-1}+1 = (11\lambda-1)100+1$ [Using (i)]

$\Rightarrow$ $10^{2(m+1)-1}+1 = 1100\lambda-99 = 11(100\lambda-9) = 11\mu$, where $\mu = 100\lambda-9 \in N$

$\Rightarrow$ $10^{2(m+1)}-1$ is divisible by 11

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(m)$ is true for all $n \in N$ i.e. $10^{2n-1}+1$ is divisible by 11 for all $n \in N$.

EXAMPLE 18 *Prove that for $n \in N$, $10^n + 3.4^{n+2} + 5$ is divisible by 9.*

SOLUTION Let $P(n)$ be the statement given by

$P(n): 10^n + 3.4^{n+2}+5$ is divisible by 9

STEP I $P(1): 10^1+3(4^{1+2})+5$ is divisible by 9.

$\because$ $10^1+3(4^{1+2})+5 = 10+192+5 = 207$, which is divisible by 9

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$10^m+3(4^{m+2})+5$ is divisible by 9

$\Rightarrow$ $10^m+3(4^{m+2})+5 = 9\lambda, \lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that $10^{(m+1)}+3(4^{m+3})+5$ is divisible by 9.

Now,

$$\begin{aligned}10^{m+1}+3(4^{m+3})+5 &= 10^m\ (10)+3(4^{m+3})+5\\ &= \{9\lambda-3(4^{m+2})-5\}\times 10+3\times 4^{m+3}+5 \quad \text{[Using (i)]}\\ &= 90\lambda-30\times 4^{m+2}-50+3\times 4\times 4^{m+2}+5\\ &= 90\lambda-30\times 4^{m+2}+12\times 4^{m+2}-45\\ &= 90\lambda-18\times 4^{m+2}-45\\ &= 9(10\lambda-2\times 4^{m+2}-5) = 9\mu, \text{ where } \mu = 10\lambda-2\times 4^{m+2}-5\end{aligned}$$

$\Rightarrow$ $10^{m+1}+3.4^{m+3}+5$ is divisible by 9

$\Rightarrow$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 19 *Prove by induction that the sum of the cubes of three consecutive natural numbers is divisible by 9.*

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: Sum of the cubes of three consecutive natural numbers starting from n is divisible by 9.

STEP I $P(1)$: Sum of the cubes of first three consecutive natural numbers is divisible by 9.

Since $1^3 + 2^3 + 3^3 = 36$, which is divisible by 9.

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then, sum of the cubes of three consecutive natural numbers starting with m is divisible by 9.

i.e. $m^3 + (m+1)^3 + (m+2)^3$ is divisible by 9

$\Rightarrow$ $m^3 + (m+1)^3 + (m+2)^3 = 9\lambda,\ \lambda \in N$

We shall now show that $P(m+1)$ is true for which we have to show that

$(m+1)^3 + (m+2)^3 + (m+3)^3$ is divisible by 9.

Now, $(m+1)^3 + (m+2)^3 + (m+3)^3$

$= (m+1)^3 + (m+2)^3 + m^3 + 9m^2 + 27m + 27$

$= m^3 + (m+1)^3 + (m+2)^3 + 9(m^2 + 3m + 3)$

$= 9\lambda + 9(m^2 + 3m + 3)$ [Using (i)]

$= 9(\lambda + m^2 + 3m + 3)$, which is divisible by 9.

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 20 *Using principle of mathematical induction prove that $4^n + 15n - 1$ is divisible by 9 for all natural numbers n.*

SOLUTION Let $P(n)$ be the statement given by

$P(n) : 4^n + 15n - 1$ is divisible by 9

STEP I $P(1) : 4^1 + 15 \times 1 - 1$ is divisible by 9.

$\because$ $4^1 + 15 \times 1 - 1 = 18$, which is divisible by 9

$\therefore$ $P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$4^m + 15m - 1$ is divisible by 9

$\Rightarrow$ $4^m + 15m - 1 = 9\lambda$, for some $\lambda \in N$

We shall now show that $P(m+1)$ is true, for this we have to show that $4^{m+1} + 15(m+1) - 1$ is divisible by 9.

Now,

$$4^{m+1} + 15(m+1) - 1 = 4^m . 4 + 15(m+1) - 1$$
$$= (9\lambda - 15m + 1) \times 4 + 15(m+1) - 1$$
$$= 36\lambda - 45m + 18 = 9(4\lambda - 5m + 2), \text{ which is divisible by 9.}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e., $4^n + 15n - 1$ is divisible by 9.

EXAMPLE 21 *Prove that: $2.7^n + 3.5^n - 5$ is divisible by 24, for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by

$P(n): 2.7^n + 3.5^n - 5$ is divisible by 24.

STEP I We have,

$P(1): 2 \times 7^1 + 3 \times 5^1 - 5$ is divisible by 24

$\because\ 2 \times 7^1 + 3 \times 5^1 - 5 = 14 + 15 - 5 = 24$, which is divisible by 24.

$\therefore\ P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$2 \times 7^m + 3 \times 5^m - 5$ is divisible by 24

$\Rightarrow\ 2 \times 7^m + 3 \times 5^m - 5 = 24\lambda$ for some $\lambda \in N$

$$\Rightarrow\ 3 \times 5^m = 24\lambda + 5 - 2 \times 7^m \quad \text{...(i)}$$

Now, $2 \times 7^{m+1} + 3 \times 5^{m+1} - 5$

$= 2 \times 7^{m+1} + (3 \times 5^m) 5 - 5$

$= 2 \times 7^{m+1} + (24\lambda + 5 - 2 \times 7^m) 5 - 5$ [Using (i)]

$= 2 \times 7^{m+1} + 120\lambda + 25 - 10 \times 7^m - 5$

$= (2 \times 7^{m+1} - 10 \times 7^m) + 120\lambda + 20$

$= (2 \times 7 \times 7^m - 10 \times 7^m) + 120\lambda + 24 - 4$

$= (14 - 10) 7^m - 4 + 24(5\lambda + 1)$

$= 4(7^m - 1) + 24(5\lambda + 1)$

$= 4 \times 6\mu + 24(5\lambda + 1)$ [$\because\ 7^m - 1$ is a multiple of 6 for all $m \in N$ $\therefore 7^m - 1 = 6\mu, \mu \in N$]

$= 24(\mu + 5\lambda + 1)$, which is divisible by 24.

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 22 *Prove that :*

(i) $\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{n}\right) = (n+1)$ *for all* $n \in N$.

(ii) $\left(1 + \frac{1}{3}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\dots\left(1 + \frac{2n+1}{n^2}\right\} = (n+1)^2$ *for all* $n \in N$.

SOLUTION (i) Let $P(n)$ be the statement given by

$$P(n): \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{n}\right) = n + 1$$

STEP I We have,

$$P(1): \left(1 + \frac{1}{1}\right) = (1 + 1)$$

$$\because\ \left(1 + \frac{1}{1}\right) = 2 = (1 + 1)$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{m}\right)=m+1 \quad \text{....(i)}$$

Now,

$P(m)$ is true

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{m}\right)=(m+1) \quad \text{[From (i)]}$$

$$\Rightarrow \quad \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{m}\right)\left(1+\frac{1}{m+1}\right)$$

$$=(m+1)\left(1+\frac{1}{m+1}\right) \quad \left[\text{Multiplying both sides by } \left(1+\frac{1}{m+1}\right)\right]$$

$$=\frac{(m+1)(m+2)}{m+1}=m+2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

(ii) Let $P(n)$ be the statement given by

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2n+1}{n^2}\right)=(n+1)^2$$

STEP I We have,

$$P(1):\left(1+\frac{3}{1}\right)=(1+1)^2$$

$\because$ $1+\frac{3}{1}=1+3=4=(1+1)^2$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)=(m+1)^2 \quad \text{...(i)}$$

We shall now prove that $P(m+1)$ is true.

i.e. $$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots.\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2(m+1)+1}{(m+1)^2}\right)=\left\{(m+1)+1\right\}^2$$

Now,

$P(m)$ is true

$$\Rightarrow \quad \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)=(m+1)^2 \quad \text{[From (i)]}$$

$$\Rightarrow \quad \left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{2m+1}{m^2}\right)\left(1+\frac{2m+3}{(m+1)^2}\right)$$

$$=(m+1)^2\left(1+\frac{2m+3}{(m+1)^2}\right) \quad \left[\text{Multiplying both sides by } 1+\frac{2m+3}{(m+1)^2}\right]$$

$$=(m+1)^2\left\{\frac{(m+1)^2+2m+3)}{(m+1)^2}\right\}=(m^2+4m+4)=(m+2)^2=\{(m+1)+1\}^2$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 23 *Prove by induction that* $4 + 8 + 12 + \ldots + 4n = 2n(n+1)$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 4 + 8 + 12 + \ldots + 4n = 2n(n+1)$$

STEP I $P(1): 4 = 2 \times 1 \times (1+1)$, which is true.

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$4 + 8 + 12 + \ldots + 4m = 2m(m+1) \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true

i.e. $4 + 8 + \ldots + 4m + 4(m+1) = 2(m+1)\{(m+1)+1\}$.

Now,

$$4 + 8 + \ldots + 4m + 4(m+1)$$
$$= 2m(m+1) + 4(m+1) \quad \text{[Using (i)]}$$
$$= (m+1)(2m+4) = 2(m+1)(m+2) = 2(m+1)\{(m+1)+1\}$$

$\therefore$ $P(m+1)$ is true.

Thus $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true

Hence, by induction $P(n)$ is true for all $n \in N$.

LEVEL-2

EXAMPLE 24 *For all positive integer n, prove that* $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2}{3}n^3 - \frac{n}{105}$ *is an integer*

SOLUTION Let $P(n)$ be the statement given by

$$P(1): \frac{n^7}{7} + \frac{n^5}{5} + 2\frac{n^3}{3} - \frac{n}{105} \text{ is an integer}$$

STEP I $P(1): \frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105}$ is an integer.

Since $\frac{1}{7} + \frac{1}{5} + \frac{2}{3} - \frac{1}{105} = \frac{15 + 21 + 70 - 1}{105} = 1$, which is an integer.

So, $P(1)$ is true.

STEP II Let $P(m)$ be true. Then, $\frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105}$ is an integer

$$\text{Let } \frac{m^7}{7} + \frac{m^5}{5} + \frac{2m^3}{3} - \frac{m}{105} = \lambda, \lambda \in Z \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true for which we have to show that

$$\frac{(m+1)^7}{7} + \frac{(m+1)^5}{5} + \frac{2(m+1)^3}{3} - \frac{(m+1)}{105} \text{ is an integer.}$$

Now, $\frac{(m+1)^7}{7} + \frac{(m+1)^5}{5} + \frac{2(m+1)^3}{3} - \frac{(m+1)}{105}$

$$= \frac{1}{7}(m^7 + 7m^6 + 21m^5 + 35m^4 + 35m^3 + 21m^2 + 7m + 1)$$
$$+ \frac{1}{5}(m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1) + \frac{2}{3}(m^3 + 3m^2 + 3m + 1) - \frac{m}{105} - \frac{1}{105}$$

$$= \left\{\frac{m^7}{7} + \frac{m^5}{5} + 2\frac{m^3}{3} - \frac{m}{105}\right\} + m^6 + 3m^5 + 6m^4 + 7m^3 + 7m^2 + 4m + 1$$

$$= \lambda + m^6 + 3m^5 + 6m^4 + 7m^3 + 7m^2 + 4m + 1 \qquad \text{[Using (i)]}$$

$= $ an integer

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^7}{7} + \frac{n^5}{5} + 2\frac{n^3}{3} - \frac{n}{105}$ is an integer.

EXAMPLE 25 *Prove by the principle of mathematical induction that* $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ *is a natural number for all* $n \in N$. **[NCERT EXEMPLAR]**

SOLUTION Let $P(n)$ be the statement given by

$P(n): \frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number

STEP I $P(1): \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$ is a natural number.

$\therefore$ $\frac{1}{5} + \frac{1}{3} + \frac{7}{15} = \frac{15}{15} = 1$, which is a natural number.

So, $P(1)$ is true.

STEP II *Let* $P(m)$ *be true.*

Then, $\frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15}$ is a natural number. Let $\frac{m^5}{5} + \frac{m^3}{3} + \frac{7m}{15} = \lambda$ ...(i)

We shall now show that $P(m+1)$ is true, for which it is sufficient to prove that $\frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{7(m+1)}{15}$ is a natural number.

Now, $$\frac{(m+1)^5}{5} + \frac{(m+1)^3}{3} + \frac{7(m+1)}{15}$$

$$= \frac{1}{5}(m^5 + 5m^4 + 10m^3 + 10m^2 + 5m + 1) + \frac{1}{3}(m^3 + 3m^2 + 3m + 1) + \frac{7}{15}m + \frac{7}{15}$$

$$= \left(\frac{m^5}{5} + \frac{m^3}{3} + \frac{7}{15}m\right) + (m^4 + 2m^3 + 3m^2 + 2m) + \frac{1}{5} + \frac{1}{3} + \frac{7}{15}$$

$$= \lambda + m^4 + 2m^3 + 3m^2 + 2m + 1 \qquad \text{[Using (i)]}$$

$= $ an integer

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

i.e. $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7}{15}n$ is a natural number for all $n \in N$.

EXAMPLE 26 ***Prove by the principle of mathematical induction that for all*** $n \in N$, 3^{2n} ***when divided by 8, the remainder is always 1.***

SOLUTION Let $P(n)$ be the statement given by

$P(n)$: 3^{2n} when divided by 8, the remainder is 1

or, $P(n) : 3^{2n} = 8\lambda + 1$ for some $\lambda \in N$

STEP I $P(1) : 3^2 = 8\lambda + 1$ for some $\lambda \in N$.

$\because \quad 3^2 = 8 \times 1 + 1 = 8\lambda + 1$, where $\lambda = 1$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then,

$3^{2m} = 8\lambda + 1$ for some $\lambda \in N$ ...(i)

We shall now show that $P(m+1)$ is true for which we have to show that $3^{2(m+1)}$ when divided by 8, the remainder is 1 i.e. $3^{2(m+1)} = 8\mu + 1$ for some $\mu \in N$.

Now, $3^{2(m+1)} = 3^{2m} \times 3^2 = (8\lambda + 1) \times 9$ [Using (i)]

$= 72\lambda + 9 = 72\lambda + 8 + 1 = 8(9\lambda + 1) + 1 = 8\mu + 1$, where $\mu = 9\lambda + 1 \in N$

$\Rightarrow \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. 3^{2n} when divided by 8 the remainder is always 1.

EXAMPLE 27 *Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in N$.*

SOLUTION Let $P(n)$ be the statement given by $P(n) : n < 2^n$.

STEP I $P(1) : 1 < 2^1$

$\because \quad 1 < 2^1$

$\therefore \quad P(1)$ is true

STEP II Let $P(m)$ be true. Then, $m < 2^m$

We shall now show that $P(m+1)$ is true for which we will have to prove that $(m+1) < 2^{m+1}$.

Now,

$P(m)$ is true

$\Rightarrow \quad m < 2^m$

$\Rightarrow \quad 2m < 2.2^m$

$\Rightarrow \quad 2m < 2^{m+1}$

$\Rightarrow \quad (m + m) < 2^{m+1}$

$\Rightarrow \quad m + 1 \le m + m < 2^{m+1}$ $[\because 1 \le m \;\therefore m + 1 \le m + m]$

$\Rightarrow \quad (m+1) < 2^{m+1}$

$\Rightarrow \quad P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for all $n \in N$ i.e. $n < 2^n$ for all $n \in N$.

EXAMPLE 28 *Prove by induction the inequality $(1 + x)^n \ge 1 + nx$ whenever x is positive and n is a positive integer.*

SOLUTION Let $P(n)$ be the statement given by $P(n) : (1 + x)^n \ge 1 + nx$

STEP I $P(1) : (1 + x)^1 \ge 1 + 1(x)$

$\because \quad (1+x)^1 \geq 1 + 1\,(x)$

$\therefore \quad P\,(1)$ is true

STEP II Let $P\,(m)$ be true. Then,

$(1+x)^m \geq 1 + m\,x$ ...(i)

We shall now prove that $P\,(m+1)$ is true whenever $P\,(m)$ is true. For this we have to show that $(1+x)^{m+1} \geq 1 + (m+1)\,x$.

Now, $P\,(m)$ is true

$\Rightarrow \quad (1+x)^m \geq 1 + mx$

$\Rightarrow \quad (1+x)\,(1+x)^m \geq (1+x)\,(1+m\,x)$ [Multiplying both sides by $(1+x)$]

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x + mx^2$

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x + mx^2 \geq 1 + (m+1)\,x$ [$\because m\,x^2 \geq 0$]

$\Rightarrow \quad (1+x)^{m+1} \geq 1 + (m+1)\,x$

$\Rightarrow \quad P\,(m+1)$ is true

Hence, by the principle of induction, $P\,(n)$ is true for all $n \in N$ i.e. $(1+x)^n \geq 1 + nx$ for all $n \in N$.

EXAMPLE 29 *Prove by induction that $(2n+7) < (n+3)^2$ for all natural numbers n. Using this, prove by induction that $(n+3)^2 \leq 2^{n+3}$ for all $n \in N$.*

SOLUTION Let $P\,(n)$ be the statement given by

$P\,(n) \;:\; (2n+7) < (n+3)^2$

STEP I $P\,(1) : (2 \times 1 + 7) < (1+3)^2$

$\because \quad (2 \times 1 + 7) = 9 < (1+3)^2$

$\therefore \quad P\,(1)$ is true

STEP II Let $P\,(m)$ be true. Then, $2m + 7 < (m+3)^2$. ... (i)

We shall now show that $P\,(m+1)$ is true whenever $P\,(m)$ is true. For this we have to show that $2\,(m+1) + 7 < (m+1+3)^2$.

Now,

$P\,(m)$ is true

$\Rightarrow \quad 2m + 7 < (m+3)^2$

$\Rightarrow \quad 2m + 7 + 2 < (m+3)^2 + 2$

$\Rightarrow \quad 2\,(m+1) + 7 < m^2 + 6m + 11$

$\Rightarrow \quad 2\,(m+1) + 7 < m^2 + 6m + 11 < m^2 + 8m + 16$

$\Rightarrow \quad 2\,(m+1) + 7 < (m+4)^2$

$\Rightarrow \quad \{2\,(m+1) + 7\} < \{(m+1) + 3\}^2$

$\Rightarrow \quad P\,(m+1)$ is true.

Hence, by the principle of mathematical induction, $P\,(n)$ is true for all $n \in N$.

Now, let $P'\,(n)$ be the statement given by $P'\,(n) \;:\; (n+3)^2 \leq 2^{n+3}$

STEP I $P'\,(1) \;:\; (1+3)^2 \leq 2^{1+3}$

$\because \quad (1+3)^2 = 16 \leq 2^{1+3}$

$\therefore \quad P'\,(1)$ is true

STEP II Let $P'(m)$ be true. Then, $(m+3)^2 \leq 2^{m+3}$.

We shall now show that $P'(m+1)$ is true whenever $P'(m)$ is true. For this we have to show that $\{(m+1)+3\}^2 \leq 2^{(m+1)+3}$.

Now, $P'(m)$ is true

$\Rightarrow (m+3)^2 \leq 2^{m+3}$

$\Rightarrow (m+3)^2 + (2m+7) \leq 2^{m+3} + (2m+7)$

$\Rightarrow (m+4)^2 \leq 2^{m+3} + (m+3)^2$ $\quad [\because 2m+7 < (m+3)^2 \therefore 2^{m+3} + (2m+7) < 2^{m+3} + (m+3)^2]$

$\Rightarrow (m+4)^2 \leq 2^{m+3} + 2^{m+3}$ $\quad [\because (m+3)^2 \leq 2^{m+3} \Rightarrow (m+3)^2 + 2^{m+3} \leq 2^{m+3} + 2^{m+3}]$

$\Rightarrow (m+4)^2 \leq 2.2^{m+3}$

$\Rightarrow (m+4)^2 \leq 2^{m+4}$

$\Rightarrow \{(m+1)+3\}^2 \leq 2^{(m+1)+3}$

$\Rightarrow P'(m+1)$ is true

Hence, by the principle of mathematical induction, $P'(n)$ is true for all $n \in N$ i.e. $(n+3)^2 \leq 2^{n+3}$ for all $n \in N$.

EXAMPLE 30 *Prove that:* $1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 > \frac{n^3}{3}$$

STEP I $P(1): 1^2 > \frac{1^3}{3}$

$\because \quad 1^2 = 1 > \frac{1}{3} = \frac{1^3}{3}$

$\therefore \quad P(1)$ is true.

STEP II Let $P(n)$ be true for $n = m$. Then,

$$1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3} \qquad \ldots(i)$$

We shall now prove that $P(m+1)$ is true.

i.e. $\quad 1^2 + 2^2 + 3^2 + \ldots. + m^2 + (m+1)^2 > \frac{(m+1)^3}{3}$

Now, $P(m)$ is true

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 > \frac{m^3}{3}$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{m^3}{3} + (m+1)^2$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}(m^3 + 3m^2 + 6m + 3)$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m^3 + 3m^2 + 3m + 1) + (3m + 2)\right\}$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \ldots + m^2 + (m+1)^2 > \frac{1}{3}\left\{(m+1)^3 + (3m+2)\right\} > \frac{(m+1)^3}{3}$

$\Rightarrow \quad P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

EXAMPLE 31 *Prove that:* $1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$ *for all* $n \in N$.

SOLUTION Let $P(n)$ be the statement given by

$$P(n): 1 + 2 + 3 + \ldots + n < \frac{(2n+1)^2}{8}$$

STEP I We have,

$$P(1): 1 < \frac{(2 \times 1 + 1)^2}{8}$$

$$\because \quad 1 < \frac{(2 \times 1 + 1)^2}{8} = \frac{9}{8}$$

$\therefore \quad P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8} \quad \ldots \text{(i)}$$

We shall now show that $P(m+1)$ is true.

i.e., $\quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{\{2(m+1)+1\}^2}{8}$

Now,

$P(m)$ is true

$\Rightarrow \quad 1 + 2 + 3 + \ldots + m < \frac{(2m+1)^2}{8}$

$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2}{8} + (m+1)$

$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+1)^2 + 8(m+1)}{8}$

$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(4m^2 + 12m + 9)}{8}$

$\Rightarrow \quad 1 + 2 + 3 + \ldots + m + (m+1) < \frac{(2m+3)^2}{8} = \frac{\{2(m+1)+1\}^2}{8}$

$\therefore \quad P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow$ $P(m+1)$ is true.

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 32 *Prove by the principle of mathematical induction that for all* $n \in N$,

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \, \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} \quad \text{[NCERT EXEMPLAR]}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\left(\frac{n+1}{2}\right)\theta \sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$$

STEP I We have, $P(1): \sin\theta = \dfrac{\sin\left(\frac{1+1}{2}\right)\theta \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$

$$\because \quad \sin\theta = \frac{\sin\left(\frac{1+1}{2}\right)\theta . \sin\left(\frac{1\times\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta = \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} \quad \ldots(i)$$

We shall now show that $P(m+1)$ is true.

i.e. $$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

Now,

$$\sin\theta + \sin 2\theta + \ldots + \sin m\theta + \sin(m+1)\theta$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + \sin(m+1)\theta \quad \text{[Using (i)]}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}} + 2\sin\left(\frac{m+1}{2}\right)\theta \cos\left(\frac{m+1}{2}\right)\theta$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right)}{\sin\frac{\theta}{2}} + 2\cos\left(\frac{m+1}{2}\right)\theta\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + 2\sin\frac{\theta}{2}\cos\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}\right\}$$

$$= \sin\left(\frac{m+1}{2}\right)\theta \left\{\frac{\sin\left(\frac{m\theta}{2}\right) + \sin\left(\frac{m+2}{2}\right)\theta - \sin\frac{m\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \frac{\sin\left(\frac{m+1}{2}\right)\theta \ \sin\left(\frac{m+2}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{\sin\left\{\frac{(m+1)+1}{2}\right\}\theta \ \sin\left(\frac{m+1}{2}\right)\theta}{\sin\frac{\theta}{2}}$$

$\therefore$ $P(m+1)$ is true

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true for all $n \in N$.

EXAMPLE 33 *Using principle of mathematical induction, prove that*

$$\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin 2^n \alpha}{2^n \sin\alpha} \text{ for all } n \in N. \quad \textbf{[NCERT EXEMPLAR]}$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): \cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{n-1}\alpha) = \frac{\sin(2^n\alpha)}{2^n \sin\alpha}$$

STEP I $P(1): \cos\alpha = \dfrac{\sin(2^1\alpha)}{2^1\sin\alpha}$

$$\because \frac{\sin(2^1\alpha)}{2^1\sin\alpha} = \frac{\sin 2\alpha}{2\sin\alpha} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$$

$\therefore$ $P(1)$ is true.

STEP II Let $P(m)$ be true. Then,

$$\cos\alpha \cos 2\alpha \cos 4\alpha \ldots \cos(2^{m-1}\alpha) = \frac{\sin(2^m\alpha)}{2^m\sin\alpha} \quad \text{...(i)}$$

We shall now show that $P(m+1)$ is true. For this we have to show that

$$\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\cos(2^m\alpha) = \frac{\sin(2^{(m+1)}\alpha)}{2^{m+1}\sin\alpha}$$

We have,

$$\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\cos(2^m\alpha)$$

$$= \{\cos\alpha \cos 2\alpha \cos 2^2\alpha \ldots \cos(2^{m-1}\alpha)\}\cos(2^m\alpha)$$

$$= \frac{\sin(2^m\alpha)}{2^m\sin\alpha} \times \cos(2^m\alpha) \quad \text{[Using (i)]}$$

$$= \frac{2\sin(2^m\alpha)\cos(2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2 \cdot 2^m\alpha)}{2^{m+1}\sin\alpha} = \frac{\sin(2^{m+1}\alpha)}{2^{m+1}\sin\alpha}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(m)$ is true $\Rightarrow P(m+1)$ is true

Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type II **PROBLEMS BASED UPON SECOND PRINCIPLE OF MATHEMATICAL INDUCTION**

EXAMPLE 34 *Let* $U_1 = 1$, $U_2 = 1$ and $U_{n+2} = U_{n+1} + U_n$ for $n \geq 1$. *Use mathematical induction to show that:*

$$U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ for all } n \geq 1.$$

SOLUTION Let $P(n)$ be the statement given by

$$P(n): U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

We have,

$$U_1 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right\} = 1$$

and,

$$U_2 = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right\} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+5+2\sqrt{5}}{4}\right) - \left(\frac{1+5-2\sqrt{5}}{4}\right)\right\} = 1$$

$\therefore$ $P(1)$ and $P(2)$ are true.

Let $P(n)$ be true for all $n \leq m$.

i.e. $$U_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\} \text{ for all } n \leq m \quad \text{...(i)}$$

We shall now show that $P(n)$ is true for $n = m+1$.

i.e. $$U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right\}$$

We have,

$$U_{n+2} = U_{n+1} + U_n \quad \text{for } n \geq 1$$

$$\Rightarrow \quad U_{m+1} = U_m + U_{m-1} \quad \text{for } m \geq 2 \quad \text{[On replacing } n \text{ by } (m-1)]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\}$$

[Using (i)]

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left[\left\{\left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\right\} - \left\{\left(\frac{1-\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right\}\right]$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{3-\sqrt{5}}{2}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right\}$$

$$\Rightarrow \quad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right\}$$

$$\Rightarrow \qquad U_{m+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right\}$$

$\therefore$ $P(m+1)$ is true.

Thus, $P(n)$ is true for all $n \le m \Rightarrow P(n)$ is true for all $n \le m+1$.

Hence, $P(n)$ is true for all $n \in N$.

EXERCISE 12.2

LEVEL-1

Prove the following by the principle of mathematical induction: (1-38)

1. $1+2+3+\ldots+n = \frac{n(n+1)}{2}$ i.e, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$.
2. $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1+3+3^2+\ldots+3^{n-1} = \frac{3^n-1}{2}$
4. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)} = \frac{n}{n+1}$
5. $1+3+5+\ldots+(2n-1) = n^2$ i.e., the sum of first n odd natural numbers is n^2.
6. $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$
7. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)} = \frac{n}{3n+1}$
8. $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$
9. $\frac{1}{3.7}+\frac{1}{7.11}+\frac{1}{11.15}+\ldots+\frac{1}{(4n-1)(4n+3)} = \frac{n}{3(4n+3)}$
10. $1.2+2.2^2+3.2^3+\ldots+n.2^n = (n-1)\,2^{n+1}+2$
11. $2+5+8+11+\ldots+(3n-1) = \frac{1}{2}n(3n+1)$
12. $1.3+2.4+3.5+\ldots+n.(n+2) = \frac{1}{6}n(n+1)(2n+7)$
13. $1.3+3.5+5.7+\ldots+(2n-1)(2n+1) = \frac{n(4n^2+6n-1)}{3}$
14. $1.2+2.3+3.4+\ldots+n(n+1) = \frac{n(n+1)(n+2)}{3}$
15. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n} = 1-\frac{1}{2^n}$
16. $1^2+3^2+5^2+\ldots+(2n-1)^2 = \frac{1}{3}n(4n^2-1)$
17. $a+ar+ar^2+\ldots+ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right), r \ne 1$

18. $a+(a+d)+(a+2d)+\ldots+(a+(n-1)\,d) = \frac{n}{2}[2a+(n-1)\,d]$

19. $5^{2n}-1$ is divisible by 24 for all $n \in N$

20. $3^{2n}+7$ is divisible by 8 for all $n \in N$

21. $5^{2n+2}-24n-25$ is divisible by 576 for all $n \in N$

22. $3^{2n+2}-8n-9$ is divisible by 8 for all $n \in N$

23. $(ab)^n = a^n b^n$ for all $n \in N$

24. $n(n+1)(n+5)$ is a multiple of 3 for all $n \in N$

25. $7^{2n}+2^{3n-3}\,.\,3^{n-1}$ is divisible by 25 for all $n \in N$

26. $2.7^n+3.5^n-5$ is divisible by 24 for all $n \in N$

27. $11^{n+2}+12^{2n+1}$ is divisible by 133 for all $n \in N$

28. $1\times 1!+2\times 2!+3\times 3!+\ldots+n\times n! = (n+1)!-1$ for all $N \in N$. **[NCERT EXEMPLAR]**

29. n^3-7n+3 is divisible by 3 for all $n \in N$.

30. $1+2+2^2+\ldots+2^n = 2^{n+1}-1$ for all $n \in N$.

LEVEL-2

31. Prove that $7+77+777+\ldots+\underbrace{777\ldots\ldots7}_{n\text{-digits}} = \frac{7}{81}(10^{n+1}-9n-10)$ for all $n \in N$

32. Prove that $\frac{n^7}{7}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{n^2}{2}-\frac{37}{210}n$ is a positive integer for all $n \in N$

33. Prove that $\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer for all $n \in N$.

34. Prove that $\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$ for all $n \in N$ and $0 < x < \frac{\pi}{2}$

35. Prove that $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all natural numbers, $n \geq 2$.

36. Prove that $\frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$ for all $n \in N$.

37. Prove that $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2} < 2-\frac{1}{n}$ for all $n > 2, n \in N$.

38. Prove that $x^{2n-1}+y^{2n-1}$ is divisible by $x+y$ for all $n \in N$.

39. Prove that $\sin x+\sin 3x+\ldots+\sin(2n-1)\,x = \frac{\sin^2 nx}{\sin x}$ for all $n \in N$.

40. Prove that $\cos \alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots + \cos(\alpha+(n-1)\beta)$

$$= \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\} \sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

for all $n \in N$. **[NCERT EXEMPLAR]**

41. Prove that $\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$, for all natural numbers $n > 1$.

[NCERT EXEMPLAR]

42. Given $a_1 = \frac{1}{2}\left(a_0 + \frac{A}{a_0}\right), a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$ for $n \geq 2$,

where $a > 0, A > 0$.

Prove that $\frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}}$.

43. Let $P(n)$ be the statement: $2^n \geq 3n$. If $P(r)$ is true, show that $P(r+1)$ is true. Do you conclude that $P(n)$ is true for all $n \in N$?

44. Show by the Principle of Mathematical induction that the sum S_n of the n terms of the series $1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + 7^2 + \ldots$ is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

[NCERT EXEMPLAR]

45. Prove that the number of subsets of a set containing n distinct elements is 2^n for all $n \in N$. **[NCERT EXEMPLAR]**

46. A sequence $a_1, a_2, a_3, \ldots\ldots$ is defined by letting $a_1 = 3$ and $a_k = 7\, a_{k-1}$ for all natural numbers $k \geq 2$. Show that $a_n = 3.7^{n-1}$ for all $n \in N$.

[NCERT EXEMPLAR]

47. A sequence $x_1, x_2, x_3, \ldots$ is defined by letting $x_1 = 2$ and $x_k = \frac{x_{k-1}}{n}$ for all natural numbers $k, k \geq 2$. Show that $x_n = \frac{2}{n!}$ for all $n \in N$. **[NCERT EXEMPLAR]**

48. A sequence $x_0, x_1, x_2, x_3, \ldots$ is defined by letting $x_0 = 5$ and $x_k = 4 + x_{k-1}$ for all natural number k. Show that $x_n = 5 + 4n$ for all $n \in N$ using mathematical induction. **[NCERT EXEMPLAR]**

49. Using principle of mathematical induction prove that $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$ for all natural numbers $n \geq 2$. **[NCERT EXEMPLAR]**

50. The distributive law from algebra states that for all real numbers c, a_1 and a_2, we have
$c(a_1 + a_2) = ca_1 + ca_2$
Use this law and mathematical induction to prove that, for al natural numbers, $n \geq 2$, if $c, a_1, a_2, \ldots, a_n$ are any real numbers, then
$c(a_1 + a_2 + \ldots + a_n) = ca_1 + ca_2 + \ldots + ca_n$.

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per the requirement of the question.

1. State the first principle of mathematical induction.
2. Write the set of value of n for which the statement $P(n)$: $2n < n!$ is true.
3. State the second principle of mathematical induction.
4. If $P(n): 2 \times 4^{2n+1} + 3^{3n+1}$ is divisible by λ for all $n \in N$ is true, then find the value of λ.

ANSWERS

2. $\{n \in N : n \geq 4\}$ 4. 11

MULTIPLE CHOICES QUESTIONS (MCQS)

Make the correct alternative in each of the following.

1. If $x^n - 1$ is divisible by $x - \lambda$, then the least positive integral value of λ is
   (a) 1 (b) 2 (c) 3 (d) 4
2. For all $n \in N$, $3 \times 5^{2n+1} + 2^{3n+1}$ is divisible by
   (a) 19 (b) 17 (c) 23 (d) 25
3. If $10^n + 3 \times 4^{n+2} + \lambda$ is divisible by 9 for all $n \in N$, then the least positive integral value of λ is
   (a) 5 (b) 3 (c) 7 (d) 1
4. Let $P(n)$: $2^n < (1 \times 2 \times 3 \times \ldots \times n)$. Then the smallest positive integer for which $P(n)$ is true is
   (a) 1 (b) 2 (c) 3 (d) 4
5. A student was asked to prove a statement $P(n)$ by induction. He proved $P(k+1)$ is true whenever $P(k)$ is true for all $k > 5 \in N$ and also $P(5)$ is true. On the basis of this he could conclude that $P(n)$ is true.
   (a) for all $n \in N$ (b) for all $n > 5$ (c) for all $n \geq 5$ (d) for all $n < 5$
6. If $P(n) : 49^n + 16^n + \lambda$ is divisible by 64 for $n \in N$ is true, then the least negative integral value of λ is
   (a) -3 (b) -2 (c) -1 (d) -4

ANSWERS

1. (a) 2. (b) 3. (a) 4. (d) 5. (c) 6. (c)

SUMMARY

1. A sentence or description which can be judged to be true or false is called a statement. Statements involving mathematical relations are called mathematical statements.
2. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true.
   and, (ii) $P(m+1)$ is true, whenever $P(m)$ is true.
   Then, $P(n)$ is true for all $n \in N$.
   This is called first principle of mathematical induction.
3. Let $P(n)$ be a statement involving the natural number n such that
   (i) $P(1)$ is true
   and, (ii) $P(m+1)$ is true, whenever $P(n)$ is true for all $n \leq m$.
   Then, $P(n)$ is true for all $n \in N$.
   This is called second principle of mathematical induction.