# CHAPTER 21

# SOME SPECIAL SERIES

## 21.1 SUM TO $n$ TERMS OF SOME SPECIAL SERIES

In this chapter, we intend to discuss the sum to $n$ terms of some other special series viz. series of natural numbers, series of square of natural numbers, series of cubes of natural numbers etc.

### 21.1.1 SUM OF FIRST n NATURAL NUMBERS

**THEOREM** *Prove that* $: 1 + 2 + 3 + \ldots + n = \dfrac{n(n+1)}{2}$.

PROOF Let $S_n = 1 + 2 + 3 + \ldots + n = \sum\limits_{k=1}^{n} k$

Clearly, it is an arithmetic series with first term $a = 1$, common difference $d = 1$ and last term $l = n$.

$$\therefore \quad S_n = \frac{n}{2}(1+n) = \frac{n(n+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n}{2}(a+l)\right]$$

$$\text{Hence, } \sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

### 21.1.2 SUM OF THE SQUARES OF FIRST $n$ NATURAL NUMBERS

**THEOREM** *Prove that* $: 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

PROOF Consider the identity $(x+1)^3 - x^3 = 3x^2 + 3x + 1$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and $n$ successively, we get

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$
$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$
$$4^3 - 3^3 = 3.3^2 + 3.3 + 1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$n^3 - (n-1)^3 = 3.(n-1)^2 + 3.(n-1) + 1$$
$$(n+1)^3 - n^3 = 3.n^2 + 3.n + 1$$

Adding column wise, we obtain

$$(n+1)^3 - 1^3 = 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + 3 + \ldots + n) + \underbrace{(1 + 1 + \ldots + 1)}_{n \text{ terms}}$$

$$\Rightarrow \quad (n+1)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + n$$

$$\Rightarrow \quad n^3 + 3n^2 + 3n = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n \qquad \left[\because \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\right]$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = n^3 + 3n^2 + 3n - \frac{3n(n+1)}{2} - n$$

$$\Rightarrow \quad 3\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$

### 21.1.3 SUM OF THE CUBES OF FIRST *n* NATURAL NUMBERS

**THEOREM** *Prove that* : $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2.$

PROOF Consider the identity

$$(x+1)^4 - x^4 = 4x^3 + 6x^2 + 4x + 1 \qquad \ldots(i)$$

Putting $x = 1, 2, 3, \ldots, (n-1)$ and $n$ successively, we get

$$2^4 - 1^4 = 4.1^3 + 6.1^2 + 4.1 + 1$$
$$3^4 - 2^4 = 4.2^3 + 6.2^2 + 4.2 + 1$$
$$4^4 - 3^4 = 4.3^3 + 6.3^2 + 4.3 + 1$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$n^4 - (n-1)^4 = 4(n-1)^3 + 6(n-1)^2 + 4(n-1) + 1$$
$$(n+1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$$

Adding column wise, we get

$$(n+1)^4 - 1^4 = 4(1^3 + 2^3 + \ldots + n^3) + 6(1^2 + 2^2 + 3^2 + \ldots + n^2) + 4(1 + 2 + 3 + \ldots + n) + \underbrace{(1 + 1 + \ldots + 1)}_{n \text{ terms}}$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left(\sum_{k=1}^{n} k^2\right) + 4\left(\sum_{k=1}^{n} k\right) + n$$

$$\Rightarrow \quad n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum_{k=1}^{n} k^3\right) + 6\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 4n^3 + 6n^2 + 4n - n(n+1)(2n+1) - 2n(n+1) - n$$

$$\Rightarrow \quad 4\left(\sum_{k=1}^{n} k^3\right) = n^4 + 2n^3 + n^2 = n^2(n+1)^2$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\Rightarrow \quad \sum_{k=1}^{n} k^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2$$

Hence, $$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = \left(\sum_{k=1}^{n} k\right)^2.$$

REMARK 1 *Sometimes for the sake of convenience the sum of a sequence is also denoted by putting the Greek letter Σ (Sigma) before its general term. For example,* $1+2+3+\ldots+n$ *can be written as* $\Sigma n$, $1^2+2^2+\ldots+n^2$ *is denoted by* $\Sigma n^2$ *and* $1^3+2^3+\ldots+n^3$ *by* $\Sigma n^3$.

*Thus, we have*

$$\Sigma n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\Sigma n^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Sigma n^3 = 1^3 + 2^3 + \ldots + n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

and, $$\Sigma a = \underbrace{a + a + \ldots + a}_{(n \text{ terms})} = na$$

REMARK 2 *Proceeding as above, we also have*

$$\Sigma n^4 = 1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

To find the sum to $n$ terms of a given series of natural numbers, we may follow the following algorithm:

**ALGORITHM**

STEP I *Write nth term of the given series.*

STEP II *Simplify nth term and express it as a polynomial in n i.e.* $T_n = an^3 + bn^2 + cn + d$

STEP III *Take the summation from 1 to n.*

$$\text{i.e.} \sum_{k=1}^{n} T_k = a\left(\sum_{k=1}^{n} k^3\right) + b\left(\sum_{k=1}^{n} k^2\right) + c\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} d.$$

STEP IV *Use the formulae for* $\sum_{k=1}^{n} k$, $\sum_{k=1}^{n} k^2$ *and* $\sum_{k=1}^{n} k^3$ *and obtain the sum.*

Following examples will illustrate the above algorithm.

## ILLUSTRATIVE EXAMPLES

### LEVEL-1

**EXAMPLE 1** *Find the sum to n terms of the series* $1^2 + 3^2 + 5^2 + \ldots$ *to n terms.*

SOLUTION Let $T_n$ be the $n$th term of this series and $S_n$ denote the sum of its $n$ terms. Then,

$$T_n = [1 + (n-1) \times 2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

and, $$S_n = \sum_{k=1}^{n} T_k$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n} (4k^2 - 4k + 1)$$

$$\Rightarrow \quad S_n = 4\left(\sum_{k=1}^{n} k^2\right) - 4\left(\sum_{k=1}^{n} k\right) + \sum_{k=1}^{n} 1$$

$$\Rightarrow \quad S_n = 4\,\frac{n(n+1)(2n+1)}{6} - 4\left\{\frac{n(n+1)}{2}\right\} + n$$

$$\Rightarrow \quad S_n = \frac{n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\} = \frac{n}{3}\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{n}{3}(4n^2 - 1)$$

**EXAMPLE 2** *Find the sum of the series* $2^2 + 4^2 + 6^2 + \ldots + (2n)^2$

**SOLUTION** Let $T_n$ be the $n$th term of this series and $S_n$ denote the sum of its $n$ terms. Then,

$$T_n = (2n)^2 = 4n^2$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} 4k^2 = 4\sum_{k=1}^{n} k^2 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} = \frac{2}{3}n(n+1)(2n+1)$$

**EXAMPLE 3** *Find the sum to n terms of the series* $1.2.3 + 2.3.4 + 3.45 + \ldots$ **[NCERT]**

**SOLUTION** Let $T_n$ the $n$th term of the given series. Then,

$T_n$ = ($n$th term of the sequence formed by first digits in each term)
× ($n$th term of the sequence of second digits in each term)
× ($n$th term of the sequence of third digits in each term)

$$\Rightarrow \quad T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2, 3, 4, \ldots) \times (n\text{th term of } 3, 4, 5, \ldots)$$

$$\Rightarrow \quad T_n = \{1 + (n-1)\times 1\} \times \{2 + (n-1)\times 1\} \times \{3 + (n-1)\times 1\}$$

$$\Rightarrow \quad T_n = n(n+1)(n+2)$$

Let $S_n$ denote the sum to $n$ terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$\Rightarrow \quad S_n = \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\} = \frac{n(n+1)}{4}\left\{n^2 + n + 4n + 2 + 4\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{4}(n^2 + 5n + 6) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

**EXAMPLE 4** *Find the sum of n terms of the series* $1.2^2 + 2.3^2 + 3.4^2 + \ldots$

**SOLUTION** Let $T_n$ be the $n$th term of the given series. Then,

$T_n$ = ($n$th term of the sequence formed by first digits in each term)
× ($n$th term of the sequence formed by second digits in each term)

$$\Rightarrow \quad T_n = (n\text{th term of } 1, 2, 3, \ldots) \times (n\text{th term of } 2^2, 3^2, 4^2 \ldots)$$

$$\Rightarrow \quad T_n = n(n+1)^2 = n^3 + 2n^2 + n$$

Let $S_n$ denote the sum to $n$ terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^3 + 2k^2 + k)$$

$$\Rightarrow \quad S_n = \sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right\} = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 3n + 8n + 4 + 6}{6}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2}\left\{\frac{3n^2 + 11n + 10}{6}\right\} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

**EXAMPLE 5** *Sum the series* $3.8 + 6.11 + 9.14 + \dots$ *to n terms.* **[NCERT]**

SOLUTION Let $T_n$ be the $n$th term of the given series. Then,

$$T_n = (n\text{th term of } 3, 6, 9, \dots) \times (n\text{th term of } 8, 11, 14, \dots)$$

$$\Rightarrow \quad T_n = [3 + (n-1)\times 3] \times [8 + (n-1)\times 3] = 3n(3n+5) = 9n^2 + 15n$$

Let $S_n$ denote the sum to $n$ terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (9k^2 + 15k) = 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 15\left\{\frac{n(n+1)}{2}\right\} = \frac{3}{2}n(n+1)[2n+1+5] = 3n(n+1)(n+3)$$

**EXAMPLE 6** *Find the sum of n terms of the series whose nth term is*

(i) $2n^2 - 3n + 5$ (ii) $n^2 + 2^n$ **[NCERT]**

SOLUTION (i) We have, $T_n = 2n^2 - 3n + 5$.

Let $S_n$ denote the sum of $n$ terms of the series whose $n$th term is $T_n$. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k^2 - 3k + 5) = 2\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 5$$

$$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} - 3\left\{\frac{n(n+1)}{2}\right\} + 5n$$

$$\Rightarrow \quad S_n = \frac{n}{6}\left\{2(n+1)(2n+1) - 9(n+1) + 30\right\} = \frac{n}{6}\left(4n^2 + 6n + 2 - 9n - 9 + 30\right)$$

$$\Rightarrow \quad S_n = \frac{n}{6}(4n^2 - 3n + 23)$$

(ii) We have, $T_n = n^2 + 2^n$

Let $S_n$ denote the sum of $n$ terms of the series having $T_n$ as its $n$th term. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (k^2 + 2^k) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + (2^1 + 2^2 + 2^3 + \dots + 2^n)$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + 2\left(\frac{2^n - 1}{2-1}\right) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

**EXAMPLE 7** *Find the sum of the following series to n terms:*

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\dots$$ **[NCERT]**

**SOLUTION** Let $T_n$ be the $n$th term of the given series. Then,

$$T_n = \frac{1^3+2^3+3^3+\dots+n^3}{1+3+5+\dots+(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\left\{1+(2n-1)\right\}} = \frac{(n+1)^2}{4} = \frac{1}{4}(n^2+2n+1)$$

Let $S_n$ denote the sum of $n$ terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{4}(k^2+2k+1) = \frac{1}{4}\left\{\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1\right\}$$

$$\Rightarrow \quad S_n = \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6} + 2\left(\frac{n(n+1)}{2}\right) + n\right\} = \frac{n}{24}(2n^2+9n+13)$$

**EXAMPLE 8** *Show that:* $\dfrac{1\times 2^2+2\times 3^2+\dots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\dots+n^2\times(n+1)} = \dfrac{3n+5}{3n+1}.$ **[NCERT]**

**SOLUTION** Let $T_n$ and $T_n'$ be the $n$th terms of the series in numerator and denominator of LHS. Then,

$T_n$ = $n$th term of the series $1\times 2^2+2\times 3^2+\dots+n\times(n+1)^2$

$T_n = n(n+1)^2 = n^3+2n^2+n$

and, $T_n'$ = $n$th term of the series $1^2\times 2+2^2\times 3+\dots+n^2\times(n+1)$

$T_n' = n^2(n+1) = n^3+n^2$

$$\therefore \quad \text{LHS} = \frac{\sum_{k=1}^{n} T_k}{\sum_{k=1}^{n} T_k'} = \frac{\sum_{k=1}^{n}(k^3+2k^2+k)}{\sum_{k=1}^{n}(k^3+k^2)} = \frac{\sum_{k=1}^{n} k^3 + 2\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k}{\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2}$$

$$\Rightarrow \quad \text{LHS} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$\Rightarrow \quad \text{LHS} = \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$

$$\Rightarrow \quad \text{LHS} = \frac{\frac{3n^2+3n+8n+4+6}{6}}{\frac{3n^2+3n+4n+2}{6}} = \frac{3n^2+11n+10}{3n^2+7n+2} = \frac{(3n+5)(n+2)}{(3n+1)(n+2)} = \frac{3n+5}{3n+1} = \text{RHS}$$

**EXAMPLE 9** *If $S_1$, $S_2$, $S_3$ are the sums of first n natural numbers, their squares, their cubes respectively, show that* $9S_2^2 = S_3(1 + 8S_1)$. **[NCERT]**

**SOLUTION** We have,

$$S_1 = \sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

and, $$S_3 = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\therefore \quad 9S_2^2 = 9\left\{\frac{n(n+1)(2n+1)}{6}\right\}^2 = \frac{9}{36}\left\{n(n+1)(2n+1)\right\}^2 = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots \text{(i)}$$

and, $$S_3(1 + 8S_1) = \left\{\frac{n(n+1)}{2}\right\}^2 \left\{1 + 8 \times \frac{n(n+1)}{2}\right\} = \left\{\frac{n(n+1)}{2}\right\}^2 \left(4n^2 + 4n + 1\right)$$

$$\Rightarrow \quad S_3(1 + 8S_1) = \frac{n^2(n+1)^2(2n+1)^2}{4} = \frac{1}{4}\left\{n(n+1)(2n+1)\right\}^2 \quad \ldots \text{(ii)}$$

From (i) and (ii), we obtain

$$9S_2^2 = S_3(1 + 8S_1).$$

**EXAMPLE 10** *Find the sum to n terms of the series:* $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$ **[NCERT]**

**SOLUTION** Let $T_n$ be the $n$th term of the given series. Then,

$$T_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

Let $S_n$ be the sum to $n$ terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{6}(2k^3 + 3k^2 + k)$$

$$\Rightarrow \quad S_n = \frac{2}{6}\left(\sum_{k=1}^{n} k^3\right) + \frac{3}{6}\sum_{k=1}^{n} k^2 + \frac{1}{6}\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \frac{1}{3}\left(\sum_{k=1}^{n} k^3\right) + \frac{1}{2}\left(\sum_{k=1}^{n} k^2\right) + \frac{1}{6}\left(\sum_{k=1}^{n} k\right)$$

$$\Rightarrow \quad S_n = \frac{1}{3}\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{1}{6}\left\{\frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{12}\left\{n(n+1) + (2n+1) + 1\right\} = \frac{n(n+1)}{12}(n^2 + 3n + 2) = \frac{n}{12}(n+1)^2(n+2).$$

## LEVEL-2

**EXAMPLE 11** *The sequence N of natural numbers is divided into classes as follows:*

$$\begin{array}{cccccc} & & 1 & 2 & & \\ & 3 & 4 & 5 & 6 & \\ 7 & 8 & 9 & 10 & 11 & 12 \\ \end{array}$$

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*Show that the sum of the numbers in nth row is* $n(2n^2 + 1)$.

SOLUTION Since the first row consists of 2 natural numbers, second row 4 natural numbers, third row 6 natural numbers and so on. So, the total number of natural numbers in $n$th row is $2n$. Now,

Total number of natural numbers upto the end of $n$th row

$$= 2 + 4 + 6 + \ldots + 2n = 2(1 + 2 + \ldots + n) = \frac{2n(n+1)}{2} = n(n+1).$$

$\therefore$ Total number of natural numbers upto the end of $(n-1)$th row $= (n-1)(n-1+1) = n(n-1)$.

Let $S_n$ denote the sum of first $n$ natural numbers. Then,

Sum of the natural numbers in $n$th row

= Sum of the natural numbers upto the end of $n$th row

– Sum of the natural numbers upto the end of $(n-1)$th row

$= S_{n(n+1)} - S_{n(n-1)}$

$= S_m - S_p$, where $m = n(n+1)$, $p = n(n-1)$.

$$= \frac{m(m+1)}{2} - \frac{p(p+1)}{2} \qquad \left[\text{Using: } S_n = \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)\{n(n+1)+1\}}{2} - \frac{n(n-1)\{n(n-1)+1\}}{2}$$

$$= \frac{n}{2}\left\{(n+1)(n^2+n+1) - (n-1)(n^2-n+1)\right\} = \frac{n}{2}(4n^2+2) = n(2n^2+1).$$

**EXAMPLE 12** *If* $S_k = \dfrac{1+2+\ldots+k}{k}$, *find the value of* $S_1^{\,2} + S_2^{\,2} + \ldots + S_n^{\,2}$.

SOLUTION We have,

$$S_k = \frac{1+2+\ldots+k}{k} = \frac{k(k+1)}{2k} = \frac{k+1}{2}$$

$$\therefore \quad S_1^{\,2} + S_2^{\,2} + \ldots + S_n^{\,2} = \sum_{k=1}^{n} S_k^{\,2} = \sum_{k=1}^{n}\left(\frac{k+1}{2}\right)^2 = \frac{1}{4}\sum_{k=1}^{n}(k+1)^2$$

$$= \frac{1}{4}\left\{2^2 + 3^2 + \ldots + (n+1)^2\right\} = \frac{1}{4}\left\{1^2 + 2^2 + \ldots + (n+1)^2 - 1^2\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+1+1)\{2(n+1)+1\}}{6} - 1\right\}$$

$$= \frac{1}{4}\left\{\frac{(n+1)(n+2)(2n+3)}{6} - 1\right\} = \frac{n}{24}(2n^2 + 9n + 13)$$

**EXAMPLE 13** *Sum to n terms the series:* $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots$

SOLUTION Clearly, $n$th term of the given series is negative or positive according as $n$ is even or odd respectively.

<u>CASE I</u> *When n is even* : In this case the given series is

$$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \ldots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + (5^2 - 6^2) + \ldots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + \ldots + \left\{(n-1)-(n)\right\}(n-1+n)$$

$$= -(1+2+3+4+\ldots+(n-1)+n) = -\frac{n(n+1)}{2}.$$

<u>CASE II</u> *When n is odd :* In this case the given series is

$$(1^2-2^2)+(3^2-4^2)+\ldots+\{(n-2)^2-(n-1)^2\}+n^2$$

$$= (1-2)(1+2)+(3-4)(3+4)+\ldots+\left\{(n-2)-(n-1)\right\}\left\{(n-2)+(n-1)\right\}+n^2$$

$$= -\{1+2+3+4+\ldots+(n-2)+(n-1)\}+n^2 = -\frac{(n-1)(n-1+1)}{2}+n^2 = \frac{n(n+1)}{2}.$$

**EXAMPLE 14** *Find the sum of all possible products of the first n natural numbers taken two by two.*

SOLUTION We have,

$$(x_1+x_2+\ldots+x_n)^2 = (x_1{}^2+x_2{}^2+\ldots+x_n{}^2) + 2\text{ (Sum of all possible products taken two at a time)}$$

or,
$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i^2\right) + 2\left(\sum_{i=1,\, i<j}^{n} \sum_{j=1}^{n} x_i\, x_j\right)$$

$$\Rightarrow \quad \sum_{i=1,\, i<j}^{n} \sum_{j=1}^{n} x_i\, x_j = \frac{1}{2}\left\{\left(\sum_{i=1}^{n} x_i\right)^2 - \left(\sum_{j=1}^{n} x_i^2\right)\right\}$$

$$\therefore \text{ Required sum} = \frac{1}{2}\left\{\left(\sum_{k=1}^{n} k\right)^2 - \left(\sum_{k=1}^{n} k^2\right)\right\} = \frac{1}{2}\left[\left\{\frac{n(n+1)}{2}\right\}^2 - \frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{1}{2}\left[\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - \frac{2n+1}{3}\right\}\right] = \frac{n(n+1)}{4}\left\{\frac{3n^2+3n-4n-2}{6}\right\}$$

$$= \frac{n(n+1)(3n^2-n-2)}{24} = \frac{n(n+1)(n-1)(3n+2)}{24}$$

**EXAMPLE 15** *Find the sum of the series:* $1.n + 2.(n-1) + 3.(n-2) + \ldots + (n-1).2 + n.1.$

SOLUTION Let $T_r$ be the $r$th term of the given series. Then,

$$T_r = r\{n-(r-1)\} = r(n-r+1) = r\{(n+1)-r\} = (n+1)r - r^2$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r$$

$$= \sum_{r=1}^{n}\left\{(n+1)r - r^2\right\}$$

$$= (n+1)\left(\sum_{r=1}^{n} r\right) - \left(\sum_{r=1}^{n} r^2\right)$$

$$= (n+1)\,\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(n+2)}{6}$$

**EXAMPLE 16** *Find the sum of the series*

$$(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots \text{ to } n \text{ terms}$$ **[NCERT EXEMPLAR]**

**SOLUTION** Let $S$ be the sum of the given series. Then,

$$S=(3^3-2^3)+(5^3-4^3)+(7^3-6^3)+\ldots\ldots \text{ to } n \text{ terms.}$$

$$\Rightarrow \quad S=\sum_{r=1}^{n}\left\{(2r+1)^3-(2r)^3\right\}$$

$$\Rightarrow \quad S=\sum_{r=1}^{n}\left\{(2r+1)-(2r)\right\}\left\{(2r+1)^2+(2r+1)(2r)+(2r)^2\right\}$$

$$\Rightarrow \quad S=\sum_{r=1}^{n}(12r^2+6r+1)$$

$$\Rightarrow \quad S=12\sum_{r=1}^{n}r^2+6\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$$

$$\Rightarrow \quad S=12\times\frac{n(n+1)(2n+1)}{6}+6\times\frac{n(n+1)}{2}+n$$

$$\Rightarrow \quad S=2n(n+1)(2n+1)+3n(n+1)+n$$

$$\Rightarrow \quad S=n(4n^2+6n+2+3n+3+1)=n(4n^2+9n+6)$$

## EXERCISE 21.1

### LEVEL-1

*Find the sum of the following series to n terms: (1-7)*

1. $1^3+3^3+5^3+7^3+\ldots$
2. $2^3+4^3+6^3+8^3+\ldots$
3. $1.2.5+2.3.6+3.4.7+\ldots$
4. $1.2.4+2.3.7+3.4.10+\ldots$
5. $1+(1+2)+(1+2+3)+(1+2+3+4)+\ldots$
6. $1\times2+2\times3+3\times4+4\times5+\ldots$ **[NCERT]**
7. $3\times1^2+5\times2^2+7\times3^2+\ldots$ **[NCERT]**
8. Find the sum of the series whose $n$th term is:
   (i) $2n^3+3n^2-1$ (ii) $n^3-3^n$
   (iii) $n(n+1)(n+4)$ **[NCERT]** (iv) $(2n-1)^2$ **[NCERT]**
9. Find the $20^{th}$ term and the sum of 20 terms of the series:
   $2\times4+4\times6+6\times8+\ldots$ **[NCERT]**

### ANSWERS

1. $n^2(2n^2-1)$ 2. $2\{n(n+1)\}^2$ 3. $\frac{n}{12}(n+1)(3n^2+23n+34)$

4. $\frac{n}{12}(n+1)(9n^2+25n+14)$ 5. $\frac{n(n+1)(n+2)}{6}$

6. $\frac{n}{3}(n+1)(n+2)$ 7. $\frac{n}{6}(n+1)(3n^2+5n+1)$

8. (i) $\frac{n}{2}(n^3+4n^2+4n-1)$ (ii) $\left\{\frac{n(n+1)}{2}\right\}^2-\frac{3}{2}(3^n-1)$

(iii) $\frac{n(n+1)}{12}(3n^2+23n+34)$ (iv) $\frac{n}{3}(2n+1)(2n-1)$ 9. 1680, 12320

## HINTS TO NCERT & SELECTED PROBLEMS

1. Clearly, $T_n = (2n)^2 = 4n^2$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = 4 \sum_{k=1}^{n} k^2 = 4\,\frac{n(n+1)(2n+1)}{6} = \frac{2}{3}\,n(n+1)(2n+1)$$

2. Clearly, $T_n = (2n-1)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k-1)^3 = 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

3. Clearly, $T_n = (2n)^3$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (2k)^3 = 8\sum_{k=1}^{n} k^3 = 8\left\{\frac{n(n+1)}{2}\right\}^2 = 2n^2(n+1)^2$$

8. Let $T_r$ be the $r^{\text{th}}$ term of the given series and $S_n$ denote the sum of its $n$ terms. Then,

$$T_r = r(r+1),\ r = 1, 2, 3, \ldots$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} r(r+1)$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} (r^2 + r) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$\Rightarrow \quad S_n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+4)}{6} = \frac{n(n+1)(n+2)}{3}$$

9. Let $T_r$ be the $r^{\text{th}}$ term of the series and $S_n$ be the sum of its $n$ terms. Then,

$$T_r = (2r+1)\,r^2,\ r = 1, 2, 3, \ldots n$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r+1)\,r^2 = \sum_{r=1}^{n} (2r^3 + r^2) = 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2$$

$$\Rightarrow \quad S_n = 2\left\{\frac{n(n+1)}{2}\right\}^2 + \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)(3n^2+5n+1)}{6}$$

10. (iv) We have,

$$T_n = n(n+1)(n+4) = n^3 + 5n^2 + 4n$$

Let $S_n$ be the sum of $n$ terms. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^3 + 5r^2 + 4r)$$

$$\Rightarrow \quad S_n = \sum_{r=1}^{n} r^3 + 5\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r$$

$$\Rightarrow \quad S_n = \left\{\frac{n(n+1)}{2}\right\}^2 + \frac{5n(n+1)(2n+1)}{6} + 4\,\frac{n(n+1)}{2} = \frac{n(n+1)(3n^2+23n+34)}{12}$$

(v) We have, $T_n = (2n-1)^2$

Let $S_n$ be the sum of $n$ terms of the given series. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (2r-1)^2 = \sum_{r=1}^{n} (4r^2 - 4r + 1)$$

$$\Rightarrow \quad S_n = 4 \sum_{r=1}^{n} r^2 - 4 \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad S_n = 4 \frac{n(n+1)(2n+1)}{6} - 4 \frac{n(n+1)}{2} + n = \frac{n(2n-1)(2n+1)}{3}$$

11. Let $T_r$ be the $r^{th}$ term of the given series. Then,

$$T_r = 2r(2r+2), \ r = 1, 2, 3, \ldots$$

$$\Rightarrow \quad T_r = 4r^2 + 4r$$

$$\therefore \quad T_{20} = 4 \times 20^2 + 4 \times 20 = 1600 + 80 = 1680$$

Let $S_{20}$ be the sum of 20 terms. Then,

$$S_{20} = \sum_{r=1}^{20} T_r = \sum_{r=1}^{20} (4r^2 + 4r) = 4\left(\sum_{r=1}^{20} r^2\right) + \left(\sum_{r=1}^{20} r\right)$$

$$\Rightarrow \quad S_{20} = 4 \times \frac{20(20+1)(40+1)}{6} + 4 \times \frac{20(20+1)}{2} = 12320$$

## 21.2 METHOD OF DIFFERENCE

Sometimes the *nth* term of a series can not be determined by the methods discussed so far. If a series is such that the difference between successive terms are either in A.P. or in G.P., then we determine its *n*th term by the method of difference and then find the sum of the series by using the formulas for $\Sigma n$, $\Sigma n^2$ and $\Sigma n^3$. The method of difference is illustrated in the following examples.

### ILLUSTRATIVE EXAMPLES

**LEVEL-1**

**EXAMPLE 1** *Find the sum to n terms of the series:* $3 + 15 + 35 + 63 + \ldots$

**SOLUTION** The difference between the successive terms are $15 - 3 = 12$, $35 - 15 = 20$, $63 - 35 = 28, \ldots$ . Clearly, these differences are in A.P.

Let $T_n$ be the *nth* term and $S_n$ denote the sum to $n$ terms of the given series. Then,

$$S_n = 3 + 15 + 35 + 63 + \ldots + T_{n-1} + T_n \quad \text{...(i)}$$

Also, $$S_n = \quad 3 + 15 + 35 + \ldots \quad + T_{n-1} + T_n \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$0 = 3 + \left\{12 + 20 + 28 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 3 + \frac{(n-1)}{2}\left\{2 \times 12 + (n-1-1) \times 8\right\} = 3 + (n-1)(12 + 4n - 8)$$

$$\Rightarrow \quad T_n = 3 + (n-1)(4n+4) = 4n^2 - 1$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4k^2 - 1)$$

$$\Rightarrow \quad S_n = 4 \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 1 = 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} - n = \frac{n}{3}(4n^2 + 6n - 1)$$

<u>REMARK</u> *Instead of determining the nth term of a series by the method of difference as discussed in the above example, we can use the following steps to obtain the same.*

<u>STEP I</u> *Obtain the terms of the series and compute the differences between* $T_2 - T_1, T_3 - T_2, T_4 - T_3$, .. *etc. If these are in A.P., then take the nth term as* $T_n = an^2 + bn + c$, *where* $a, b, c$ *are constants. Determine constants* $a, b, c$ *by putting* $n = 1, 2, 3$ *and equating them with the values of corresponding terms of the given series.*

<u>STEP II</u> *If the differences* $T_2 - T_1, T_3 - T_2, T_4 - T_3$,.... *are in G.P. with common ratio r, then take* $T_n = ar^{n-1} + bn + c$ *and determine constants by putting* $n = 1, 2, 3$ *in* $T_n$.

<u>STEP III</u> *If the differences of the differences computed in step I are in A.P., then take* $T_n = an^3 + bn^2 + cn + d$ *and find the values of* $a, b, c, d$ *by putting* $n = 1, 2, 3, 4$.

<u>STEP IV</u> *If the differences of the differences computed in step I are in G.P. with common ratio r, then take* $T_n = ar^{n-1} + bn^2 + cn + d$ *and find the values of* $a, b, c, d$ *by putting* $n = 1, 2, 3, 4$.

**EXAMPLE 2** *Find the sum to n terms of the series:* $1 + 5 + 12 + 22 + 35 + \dots$

SOLUTION The sequence of differences between successive terms is 4, 7, 10, 13, ..., which is clearly an A.P. Let $T_n$ be the $n$th term of the sequence and $S_n$ be the sum of its $n$ terms. Then,

$$S_n = 1 + 5 + 12 + 22 + 35 + \dots + T_{n-1} + T_n \qquad \text{...(i)}$$

Also, $$S_n = \quad 1 + 5 + 12 + 22 + \dots \quad + T_{n-1} + T_n \qquad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$0 = 1 + \left\{4 + 7 + 10 + 13 + \dots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad T_n = 1 + \frac{(n-1)}{2}\left\{2 \times 4 + (n-1-1) \times 3\right\} = 1 + \left(\frac{n-1}{2}\right)(3n+2)$$

$$\Rightarrow \quad T_n = \frac{1}{2}(3n^2 - n)$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2}(3k^2 - k) = \frac{3}{2}\sum_{k=1}^{n} k^2 - \frac{1}{2}\sum_{k=1}^{n} k$$

$$\Rightarrow \quad S_n = \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

<u>ALITER</u> The given series is : $1 + 5 + 12 + 22 + 35 + \dots$

The sequence of differences between successive terms is : $4, 7, 10, 13, \dots$

Clearly, it is an A.P. So, let the $n$th term $T_n$ of the given series be

$$T_n = an^2 + bn + c \qquad \text{...(i)}$$

Putting $n = 1, 2, 3,$ we get

$$T_1 = a + b + c \quad \Rightarrow a + b + c = 1 \qquad [\because T_1 = 1]$$

$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 5 \qquad [\because T_2 = 5]$$

$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 12 \qquad [\because T_3 = 12]$$

Solving these equations, we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \text{ and } c = 0$$

Substituting the values of $a, b, c$ in (i), we get

$$T_n = \frac{3}{2}n^2 - \frac{1}{2}n = \frac{1}{2}(3n^2 - n)$$

$$\therefore \text{ Sum of the given series} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{2}(3r^2 - r) = \frac{3}{2}\sum_{r=1}^{n} r^2 - \frac{1}{2}\sum_{r=1}^{n} r$$

$$= \frac{3}{2}\left\{\frac{n(n+1)(2n+1)}{6}\right\} - \frac{1}{2}\left\{\frac{n(n+1)}{2}\right\} = \frac{n^2(n+1)}{2}$$

**EXAMPLE 3** *Find the sum of first n terms of the following series:*

(i) $3 + 7 + 13 + 21 + 31 + \ldots$ [NCERT]

(ii) $5 + 11 + 19 + 29 + 41 + \ldots$ [NCERT]

SOLUTION (i) The given series is: $3 + 7 + 13 + 21 + 31 + \ldots$

The sequence of the differences between the successive terms of this series is

$4, 6, 8, 10, \ldots$

Clearly, it is an A.P. with common difference 2. So, let the $n$th term of the given series be

$$T_n = an^2 + bn + c \qquad \ldots(\text{i})$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 3 \qquad [\because T_1 = 3]$$

$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 7 \qquad [\because T_2 = 7]$$

$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13 \qquad [\because T_3 = 13]$$

Solving these equations, we get: $a = b = c = 1$

$$\therefore \quad T_n = n^2 + n + 1$$

$$\text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + r + 1) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n = \frac{n}{3}(n^2 + 3n + 5)$$

(ii) The given series is: $5 + 11 + 19 + 29 + 41 + \ldots$

The sequence of the differences between the successive terms is: $6, 8, 10, 12, \ldots$

Clearly, it is an A.P. So, $n$th term of the given series is given by

$$T_n = an^2 + bn + c \qquad \ldots(\text{i})$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 5 \qquad [\because T_1 = 5]$$

$$T_2 = 4a + 2b + c \Rightarrow 4a + 2b + c = 11 \qquad [\because T_2 = 11]$$

$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 19 \qquad [\because T_3 = 19]$$

Solving these three equations, we get : $a = 1, b = 3$ and $c = 1$.

$$\therefore \quad T_n = n^2 + 3n + 1$$

$$\text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (r^2 + 3r + 1) = \sum_{r=1}^{n} r^2 + 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + n = \frac{n(n+2)(n+4)}{3}$$

**LEVEL-2**

**EXAMPLE 4** *Sum the following series to n terms:* $5 + 7 + 13 + 31 + 85 + \ldots$

SOLUTION The sequence of differences between successive terms is 2, 6, 18, 54, ...

Clearly, it is a G.P. Let $T_n$ be the *nth* term of the given series and $S_n$ be the sum of its $n$ terms. Then,

$$S_n = 5 + 7 + 13 + 31 + 85 + \ldots + T_{n-1} + T_n \quad \ldots(i)$$

Also, $$S_n = \quad 5 + 7 + 13 + 31 + \ldots \quad + T_{n-1} + T_n \quad \ldots(ii)$$

Subtracting (ii) from (i), we get

$$0 = 5 + \left\{2 + 6 + 18 + 54 + \ldots + (T_n - T_{n-1})\right\} - T_n$$

$$\Rightarrow \quad 0 = 5 + \frac{2(3^{n-1} - 1)}{(3-1)} - T_n$$

$$\Rightarrow \quad T_n = 5 + (3^{n-1} - 1) = 4 + 3^{n-1}$$

$$\therefore \quad S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} (4 + 3^{k-1}) = \sum_{k=1}^{n} 4 + \sum_{k=1}^{n} 3^{k-1}$$

$$\Rightarrow \quad S_n = 4n + (1 + 3 + 3^2 + \ldots + 3^{n-1})$$

$$\Rightarrow \quad S_n = 4n + 1 \times \left(\frac{3^n - 1}{3-1}\right) = 4n + \left(\frac{3^n - 1}{2}\right) = \frac{1}{2}\left(3^n + 8n - 1\right)$$

ALITER The given series is: $5 + 7 + 13 + 31 + 85 + \ldots$

The sequence of the differences between successive terms is $2, 6, 18, 54, \ldots$

Clearly, it is a G.P. with common ratio 3. So, let the $n$th term of the given series be

$$T_n = a \cdot 3^{n-1} + bn + c \quad \ldots(i)$$

Putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c \Rightarrow a + b + c = 5 \quad [\because T_1 = 5]$$
$$T_2 = 3a + 2b + c \Rightarrow 3a + 2b + c = 7 \quad [\because T_2 = 7]$$
$$T_3 = 9a + 3b + c \Rightarrow 9a + 3b + c = 13 \quad [\because T_3 = 13]$$

Solving these equations, we get: $a = 1, b = 0$ and $c = 4$

Substituting the values of $a, b, c$ in (i), we get

$$T_n = 3^{n-1} + 4$$

$$\therefore \quad \text{Sum of the series} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} (3^{r-1} + 4) = \sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n} 4$$

$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$

$$= \frac{3^n - 1}{3-1} + 4n = \frac{3^n - 1}{2} + 4n = \frac{1}{2}(3^n - 1 + 8n)$$

## 21.3 SUMMATION OF SOME SPECIAL SERIES

In this section, we shall discuss some problems for finding the sum of some series of the form

$$\frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \ldots + \ldots + \frac{1}{(a+(n-2)d)(a+(n-1)d)}$$

In order to find the sum of a finite number of terms of such series, we write its each term as the difference of two terms as given below

$$\frac{1}{a(a+d)} = \frac{1}{d}\left(\frac{1}{a} - \frac{1}{a+d}\right),$$

$$\frac{1}{(a+d)(a+2d)} = \frac{1}{d}\left(\frac{1}{a+d} - \frac{1}{a+2d}\right),$$

$$\frac{1}{(a+2d)(a+3d)} = \frac{1}{d}\left(\frac{1}{a+2d} - \frac{1}{a+3d}\right) \text{ and so on.}$$

$$\therefore \quad \frac{1}{a(a+d)} + \frac{1}{(a+d)(a+2d)} + \frac{1}{(a+2d)(a+3d)} + \dots + \frac{1}{\{a+(n-2)d\}\{a+(n-1)d\}}$$

$$= \frac{1}{d}\left\{\left(\frac{1}{a} - \frac{1}{a+d}\right) + \left(\frac{1}{a+d} - \frac{1}{a+2d}\right) + \left(\frac{1}{a+2d} - \frac{1}{a+3d}\right) + \dots + \left(\frac{1}{a+(n-2)d} - \frac{1}{a+(n-1)d}\right)\right\}$$

$$= \frac{1}{d}\left\{\frac{1}{a} - \frac{1}{a+(n-1)d}\right\} = \frac{n-1}{a\{a+(n-1)d\}}$$

Following examples will illustrate the above procedure.

## ILLUSTRATIVE EXAMPLES

### LEVEL-1

**EXAMPLE 1** *Find the sum to n terms of the series:*

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots + \frac{1}{n.(n+1)}$$ [NCERT]

**SOLUTION** Let $T_r$ be the $r$th term of the given series. Then,

$$T_r = \frac{1}{r(r+1)}, r = 1, 2, \dots, n$$

$$\Rightarrow \quad T_r = \frac{1}{r} - \frac{1}{r+1}, \; r = 1, 2, \dots, n$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

**EXAMPLE 2** *Find the sum to n terms of the series :* $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \dots$

**SOLUTION** Let $T_r$ be the $r$th term of the given series. Then,

$$T_r = \frac{1}{(2r-1)(2r+1)}, \; r = 1, 2, 3 \dots n$$

$$\Rightarrow \quad T_r = \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right), \; r = 1, 2, 3, \dots n$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\} = \frac{n}{2n+1}$$

**EXAMPLE 3** *Find the sum :* $\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$

SOLUTION We have,

$$\sum_{r=1}^{n} \frac{1}{(ar+b)(ar+a+b)}$$

$$= \sum_{r=1}^{n} \frac{1}{a}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{a}\sum_{r=1}^{n}\left(\frac{1}{ar+b} - \frac{1}{ar+a+b}\right)$$

$$= \frac{1}{b}\left[\left(\frac{1}{a+b} - \frac{1}{2a+b}\right) + \left(\frac{1}{2a+b} - \frac{1}{3a+b}\right) + \ldots + \left(\frac{1}{na+b} - \frac{1}{(n+1)a+b}\right)\right]$$

$$= \frac{1}{a}\left\{\frac{1}{a+b} - \frac{1}{(n+1)a+b}\right\} = \frac{n}{(a+b)\left\{(n+1)a+b\right\}}$$

## LEVEL-2

**EXAMPLE 4** *Find the sum to n terms of the series:* $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + \ldots$

SOLUTION Let $T_r$ be the $r$th term of the given series. Then,

$$T_r = \frac{(2r+1)}{r^2(r+1)^2}, r = 1, 2, 3, \ldots.$$

$$\Rightarrow \quad T_r = \frac{(r+1)^2 - r^2}{r^2(r+1)^2}, r = 1, 2, 3, \ldots$$

$$\Rightarrow \quad T_r = \left\{\frac{1}{r^2} - \frac{1}{(r+1)^2}\right\}, \ r = 1, 2, 3, \ldots$$

Let $S_n$ be the sum to $n$ terms of the given series. Then,

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \sum_{r=1}^{n}\left\{\frac{1}{r^2} - \frac{1}{(r+1)^2}\right\} = 1 - \frac{1}{(n+1)^2} = \frac{2n+n^2}{(n+1)^2}$$

**EXAMPLE 5** *Find the sum to n terms of the series:*

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \ldots$$

SOLUTION Let $T_r$ be the $r$th term of the given series. Then,

$$T_r = \frac{r}{1+r^2+r^4}, r = 1, 2, 3, \ldots, n$$

$$\Rightarrow \quad T_r = \frac{r}{(r^2+r+1)(r^2-r+1)} = \frac{1}{2}\left\{\frac{2r}{(r^2+r+1)(r^2-r+1)}\right\} = \frac{1}{2}\left\{\frac{(r^2+r+1)-(r^2-r+1)}{(r^2+r+1)(r^2-r+1)}\right\}$$

$$\Rightarrow \quad T_r = \frac{1}{2}\left\{\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right\}, r=1, 2, \dots, n$$

Let $S_n$ be the sum to $n$ terms of the given series. Then,

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \frac{1}{2}\left\{\sum_{r=1}^{n}\left(\frac{1}{r^2-r+1}-\frac{1}{r^2+r+1}\right)\right\}$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{13}\right)+\dots+\left(\frac{1}{n^2-n+1}-\frac{1}{n^2+n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1-\frac{1}{n^2+n+1}\right\} = \frac{n^2+n}{2(n^2+n+1)}$$

## EXERCISE 21.2

### LEVEL-1

*Sum the following series to n terms:*

1. $3+5+9+15+23+\dots$
2. $2+5+10+17+26+\dots$
3. $1+3+7+13+21+\dots$
4. $3+7+14+24+37+\dots$
5. $1+3+6+10+15+\dots$
6. $1+4+13+40+121+\dots$
7. $4+6+9+13+18+\dots$
8. $2+4+7+11+16+\dots$
9. $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\dots$
10. $\frac{1}{1.6}+\frac{1}{6.11}+\frac{1}{11.16}+\frac{1}{16.21}+\dots+\frac{1}{(5n-4)(5n+1)}$

## ANSWERS

1. $\frac{n}{3}(n^2+8)$
2. $\frac{n}{6}(2n^2+3n+7)$
3. $\frac{n(n^2+2)}{3}$
4. $\frac{n}{2}(n^2+n+4)$
5. $\frac{n}{6}(n+1)(n+2)$
6. $\frac{1}{4}(3^{n+1}-2n-3)$
7. $\frac{n}{6}(n^2+3n+20)$
8. $\frac{n}{6}(n^2+3n+8)$
9. $\frac{n}{3n+1}$
10. $\frac{n}{5n+1}$

## VERY SHORT ANSWER QUESTIONS (VSAQs)

*Answer each of the following questions in one word or one sentence or as per exact requirement of the question:*

1. Write the sum of the series: $2+4+6+8+\dots+2n$.
2. Write the sum of the series: $1^2-2^2+3^2-4^2+5^2-6^2+\dots+(2n-1)^2-(2n)^2$.

3. Write the sum to $n$ terms of a series whose $r^{\text{th}}$ term is: $r + 2^r$.

4. If $\sum_{r=1}^{n} r = 55$, find $\sum_{r=1}^{n} r^3$.

5. If the sum of first $n$ even natural numbers is equal to $k$ times the sum of first $n$ odd natural numbers, then write the value of $k$.

6. Write the sum of 20 terms of the series: $1 + \frac{1}{2}(1+2) + \frac{1}{3}(1+2+3) + \ldots$

7. Write the 50th term of the series $2+3+6+11+18+\ldots$

8. Let $S_n$ denote the sum of the cubes of first $n$ natural numbers and $s_n$ denote the sum of first $n$ natural numbers. Then, write the value of $\sum_{r=1}^{n} \frac{S_r}{s_r}$.

## ANSWERS

1. $n(n+1)$
2. $-n(2n+1)$
3. $\frac{n(n+1)}{2} + 2^{n+1} - 2$
4. 3025
5. $\frac{n+1}{n}$
6. 115
7. $49^2 + 2$
8. $\frac{n(n+1)(n+2)}{6}$

## MULTIPLE CHOICE QUESTIONS (MCQs)

*Mark the correct alternative in each of the following:*

1. The sum to $n$ terms of the series $\frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{7}} + \ldots\ldots$ is

(a) $\sqrt{2n+1}$
(b) $\frac{1}{2}\sqrt{2n+1}$
(c) $\sqrt{2n+1}-1$
(d) $\frac{1}{2}\left\{\sqrt{2n+1}-1\right\}$

2. The sum of the series : $\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + \ldots + \frac{1}{\log_{2^n} 4}$ is

(a) $\frac{n(n+1)}{2}$
(b) $\frac{n(n+1)(2n+1)}{12}$
(c) $\frac{n(n+1)}{4}$
(d) none of these

3. The value of $\sum_{r=1}^{n} \left\{(2r-1)a + \frac{1}{b^r}\right\}$ is equal to

(a) $an^2 + \frac{b^{n-1}-1}{b^{n-1}(b-1)}$
(b) $an^2 + \frac{b^n - 1}{b^n(b-1)}$
(c) $an^3 + \frac{b^{n-1}-1}{b^n(b-1)}$
(d) none of these

4. If $\Sigma n = 210$, then $\Sigma n^2 =$

(a) 2870
(b) 2160
(c) 2970
(d) none of these

5. If $S_n = \sum_{r=1}^{n} \frac{1+2+2^2+\ldots \text{ Sum to } r \text{ terms}}{2^r}$, then $S_n$ is equal to

(a) $2^n - n - 1$
(b) $1 - \frac{1}{2^n}$
(c) $n - 1 + \frac{1}{2^n}$
(d) $2^n - 1$

6. If $1+\frac{1+2}{2}+\frac{1+2+3}{3}+\ldots$ to $n$ terms is $S$. Then, $S$ is equal to

(a) $\frac{n(n+3)}{4}$ (b) $\frac{n(n+2)}{4}$ (c) $\frac{n(n+1)(n+2)}{6}$ (d) $n^2$

7. Sum of $n$ terms of the series $\sqrt{2}+\sqrt{8}+\sqrt{18}+\sqrt{32}+\ldots$ is

(a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$ (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1

8. The sum of 10 terms of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+\ldots$ is

(a) $121(\sqrt{6}+\sqrt{2})$ (b) $243(\sqrt{3}+1)$ (c) $\frac{121}{\sqrt{3}-1}$ (d) $242(\sqrt{3}-1)$

9. The sum of the series $1^2+3^2+5^2+\ldots$ to $n$ terms is

(a) $\frac{n(n+1)(2n+1)}{2}$ (b) $\frac{n(2n-1)(2n+1)}{3}$ (c) $\frac{(n-1)^2(2n+1)}{6}$ (d) $\frac{(2n+1)^3}{3}$

10. The sum of the series $\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+\ldots$ to $n$ terms is

(a) $n-\frac{1}{2}(3^{-n}-1)$ (b) $n-\frac{1}{2}(1-3^{-n})$ (c) $n+\frac{1}{2}(3^n-1)$ (d) $n-\frac{1}{2}(3^n-1)$

### ANSWERS

1. (d) 2. (c) 3. (b) 4. (a) 5. (c) 6. (a) 7. (c) 8. (a)
9. (b) 10. (b)

## SUMMARY

1. For any $n \in N$, we have

(i) $\sum_{k=1}^{n} k = 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

(ii) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\ldots+n^3 = \left\{\frac{n(n+1)}{2}\right\}^2$

(iv) $\sum_{k=1}^{n} k^4 = 1^4+2^4+3^4+\ldots+n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

2. In a series $a_1+a_2+a_3+a_4+\ldots+a_n+\ldots$,

(i) if the differences $a_2-a_1, a_3-a_2, a_4-a_3, \ldots$ are in A.P., then the $n$th term is given by
$$a_n = an^2+bn+c, \text{ where } a, b, c \text{ are constants.}$$

(ii) if the differences $a_2-a_1, a_3-a_2, a_4-a_3, \ldots$ are in G.P. with common ratio $r$, then
$$a_n = ar^{n-1}+bn+c, \text{ where } a, b, c \text{ are constants.}$$

To determine constants $a, b, c$ we put $n=1, 2, 3$ and equate them with the values of corresponding terms of the given series.