CHAPTER 31

MEAN AND VARIANCE OF A RANDOM VARIABLE

31.1 INTRODUCTION

Corresponding to every outcome of a random experiment, we can associate a real number. This correspondence between the elements of the sample space associated to a random experiment and the set of real numbers is defined as a random variable. If a random variable assumes countable number of values, it is called a discrete random variable. Otherwise, it is known as continuous random variable. We shall study these two types of random variables in the following sections.

31.2 DISCRETE RANDOM VARIABLE

DEFINITION *Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event $w \in S$ to a unique real number $X(w)$ is called a random variable.*

In other words, a random variable is a real valued function having domain as the sample space associated with a random experiment.

Thus, a random variable associated with a given random experiment associates every event to a unique real number as discussed below.

Consider a random experiment of tossing three coins. The sample space of eight possible outcomes of this experiment is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

Let X be a real valued function on S, defined by

$$X(w) = \text{Number of heads in } w \in S.$$

Then, X is a random variable such that:

$$X(HHH) = 3, X(HHT) = 2, X(HTH) = 2, X(THH) = 2$$

$$X(HTT) = 1, X(THT) = 1, X(TTH) = 1, \text{ and } X(TTT) = 0$$

Also, if w denotes the event "getting two heads", then $w = \{HTH, THH, HHT\}$ and, $X(w) = 2$.

Similarly, X associates every other compound event to a unique real number.

For the random variable X, we have range $(X) = \{0, 1, 2, 3\}$ and we say that X is a random variable such that it assumes values 0, 1, 2, 3. This random variable can also be described as the number of heads in a single throw of three coins.

Now, consider the random experiment of throwing an unbiased die. Let Y be a real valued function defined on the sample space $S = \{1, 2, 3, 4, 5, 6\}$ associated with the random experiment, defined by

$$Y(w) = \begin{cases} 1, \text{ if the outcome is an even number} \\ -1, \text{ if the outcome is an odd number} \end{cases}$$

Clearly, Y is a random variable such that:

$$Y(1) = -1, Y(2) = 1, Y(3) = -1, Y(4) = 1, Y(5) = -1 \text{ and } Y(6) = 1$$

Here, range $(Y) = \{-1, 1\}$. Therefore, we say that Y is a random variable such that it assumes values -1 and 1.

ILLUSTRATION 1 *Consider a random experiment of tossing three coins. Let X be a real valued function defined on the sample space $S = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}$ such that*

$$X(w) = \text{Number of tails in } w \in S.$$

Then, X is a random variable such that: $X(HHH) = 0, X(HHT) = 1, X(HTH) = 1, X(THH) = 1,$ $X(HTT)\ 2, X(THT) = 2, X(TTH) = 2$ and $X(TTT) = 3$.

Clearly, range of X is $\{0, 1, 2, 3\}$.

ILLUSTRATION 2 *Consider a random experiment of throwing a six faced die. Let X denote the number on the upper face of the die. Then, $X(1) = 1, X(2) = 2, X(3) = 3, X(4) = 4, X(5) = 5$ and $X(6) = 6$.*

Clearly, X is a random variable which assumes values 1, 2, 3, 4, 5, 6 i.e. range of $X = \{1, 2, 3, 4, 5, 6\}$.

ILLUSTRATION 3 *Let there be a bag containing 5 white, 4 red and 3 green balls. Three balls are drawn. If X denotes the number of green balls in the draw. Then, X can assume values 0, 1, 2, 3. Clearly, X is a random variable with its range $= \{0, 1, 2, 3\}$.*

ILLUSTRATION 4 *A pair of dice is thrown. If X denotes the sum of the numbers on two dice, then X assumes values 2, 3, 4,..., 12. Clearly, X is a random variable with its range $\{2, 3, 4, \ldots, 12\}$.*

31.3 PROBABILITY DISTRIBUTION

In the previous section, we have defined random variable. Now, consider a random experiment in which three coins are tossed simultaneously (or a coin is tossed three times). Let X be a random variable defined on the sample space

$$S = \{HHH, HTH, THH, HHT, THT, TTH, HTT, TTT\} \text{ such that}$$

$$X(w) = \text{Number of heads in } w \in S.$$

Clearly, X assumes value 0, 1, 2, 3.

Now, $P(X = 0)$ = Probability of getting no head $= P(TTT) = \frac{1}{8}$

$P(X = 1)$ = Probability of getting one head $= (HTT$ or THT or $TTH) = \frac{3}{8}$

$P(X = 2)$ = Probability of getting two heads $= P(HHT$ or THH or $HTH) = \frac{3}{8}$

and, $P(X = 3)$ = Probability of getting 3 heads $= P(HHH) = \frac{1}{8}$

These values of X and the corresponding probabilities can be exhibited as under:

X:	0	1	2	3
$P(X)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This tabular representation of the values of a random variable X and the corresponding probabilities is known as its probability distribution.

The formal definition of the probability distribution of a random variable is as given below.

PROBABILITY DISTRIBUTION *If a random variable X takes values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then*

X:	x_1	x_2	x_3	......	x_n
$P(X)$:	p_1	p_2	p_3	......	p_n

is known as the probability distribution of X.

Thus, a tabular description giving the values of the random variable along with the corresponding probabilities is called its probability distribution.

REMARK 1 *The probability distribution of a random variable X is defined only when we have the various values of the random variable e.g.* $x_1, x_2, \ldots, x_n$ *together with respective probabilities* $p_1, p_2, \ldots, p_n$ *satisfying* $\sum_{i=1}^{n} p_i = 1$.

REMARK 2 *If X is a random variable with the probability distribution*

$$\begin{array}{rcccc} X: & x_1 & x_2 & \ldots\ldots & x_n \\ P(X): & p_1 & p_2 & \ldots\ldots & p_n \end{array}$$

Then,

$$P(X \le x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_i) = p_1 + p_2 + \ldots + p_i$$
$$P(X < x_i) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_{i-1}) = p_1 + p_2 + \ldots\ldots p_{i-1}$$
$$P(X \ge x_i) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_n) = p_i + p_{i+1} + \ldots + p_n$$
$$P(X > x_i) = P(X = x_{i+1}) + P(X = x_{i+2}) + \ldots + P(X = x_n) = p_{i+1} + p_{i+2} + \ldots + p_n$$

Also, $P(X \ge x_i) = 1 - P(X < x_i)$, $P(X > x_i) = 1 - P(X \le x_i)$,

$$P(X \le x_i) = 1 - P(X > x_i) \text{ and, } P(X < x_i) = 1 - P(X \ge x_i)$$
$$P(x_i \le X \le x_j) = P(X = x_i) + P(X = x_{i+1}) + \ldots + P(X = x_j)$$
$$P(x_i < X < x_j) = P(X = x_{i+1}) + P(X = x_{i+2}) + \ldots + P(X = x_{j-1})$$

The graphical representation of a probability distribution is as follows:

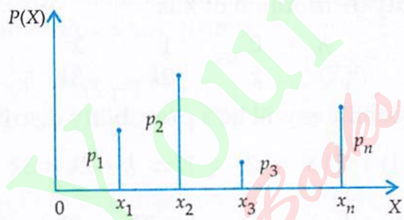

Fig. 31.1

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Determine which of the following can be probability distributions of a random variable X :*

(i)

X:	0	1	2
P(X):	0.4	0.4	0.2

(ii)

X :	0	1	2
P(X):	0.6	0.1	0.2

(iii)

X:	0	1	2	3	4
P(X):	0.1	0.5	0.2	−0.1	0.3

SOLUTION (i) Clearly, $P(X=0) + P(X=1) + P(X=2) = 0.4 + 0.4 + 0.2 = 1$.

Hence, the given distribution of probabilities is a probability distribution of random variable X.

(ii) We have, $P(X=0) + P(X=1) + P(X=2) = 0.6 + 0.1 + 0.2 = 0.9 \ne 1$.

Hence, the given distribution of probabilities is not a probability distribution.

(iii) We have, $P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) = 0.1 + 0.5 + 0.2 - 0.1 + 0.3 = 1$

But, $P(X=3) = -0.1 < 0$. So, the given distribution of probabilities is not a probability distribution.

EXAMPLE 2 *An unbiased die is rolled. If the random variable X is defined as*

$$X(w) = \begin{cases} 1, \text{ if the outcome } w \text{ is an even number} \\ 0, \text{ if the outcome } w \text{ is an odd number} \end{cases}$$

Find the probability distribution of X.

SOLUTION In a single throw of a die either we get an even number or we get an odd number. Thus, the possible values of the random variable X are 0 and 1.

Now,

$$P(X=0) = \text{Probability of getting an odd number} = \frac{3}{6} = \frac{1}{2}$$

$$P(X=1) = \text{Probability of getting an even number} = \frac{3}{6} = \frac{1}{2}$$

Thus, the probability distribution of the random variable X is given by

X:	0	1
$P(X)$:	$\frac{1}{2}$	$\frac{1}{2}$

EXAMPLE 3 *The random variable X has a probability distribution P (X) of the following form, where k is some number:*

$$P(X=x) = \begin{cases} k & , \text{if } x=0 \\ 2k & , \text{if } x=1 \\ 3k & , \text{if } x=2 \\ 0 & , \text{otherwise} \end{cases}$$

(i) *Determine the value of k* (ii) *Find $P(X<2)$, $P(X \le 2)$ and, $P(X \ge 2)$.* **[CBSE 2019]**

SOLUTION (i) The probability distribution of X is

X:	0	1	2
$P(X)$:	k	$2k$	$3k$

The given distribution of probabilities will be a probability distribution, if

$$P(X=0)+P(X=1)+P(X=2) = 1 \Rightarrow k+2k+3k = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$$

(ii) $P(X<2) = P(X=0)+P(X=1) = k+2k = 3k = \frac{3}{6} = \frac{1}{2}$

(iii) $P(X \le 2) = P(X=0)+P(X=1)+P(X=2) = k+2k+3k = 6k = 1$

(iii) $P(X \ge 2) = 1 - P(X<2) = 1 - \frac{1}{2} = \frac{1}{2}$

EXAMPLE 4 *Let X denote the number of hours you study during a randomly selected school day. The probability that X can take the value x has the following form, where k is some unknown constant.*

$$P(X=x) = \begin{cases} 0.1 & , \text{if } x=0 \\ kx & , \text{if } x=1 \text{ or } 2 \\ k(5-x) & , \text{if } x=3 \text{ or } 4 \\ 0 & , \text{otherwise} \end{cases}$$

(i) Find the value of k

What is the probability that you study (ii) At least two hours? (iii) Exactly two hours? (iv) At most two hours? **[NCERT]**

SOLUTION The probability distribution of X is

X:	0	1	2	3	4
$P(X)$:	0.1	k	$2k$	$2k$	k

(i) The given distribution is a probability distribution.

$\therefore \quad P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4) = 1$

$\Rightarrow \quad 0.1+k+2k+2k+k = 1 \Rightarrow 6k = 0.9 \Rightarrow k = 0.15$

(ii) Required probability $= P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4)$
$= 2k + 2k + k = 5k = 5 \times 0.15 = 0.75$

(iii) Required probability $= P(X = 2) = 2k = 2 \times 0.15 = 0.3$

(iv) Required probability $= P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$
$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3 \times 0.15 = 0.55$

EXAMPLE 5 *A random variable X has the following probability distribution:*

X:	0	1	2	3	4	5	6	7
$P(X)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Find each of the following:

(i) k (ii) $P(X < 6)$ (iii) $P(X \geq 6)$ (iv) $P(0 < X < 5)$ **[CBSE 2011]**

SOLUTION (i) Since the sum of all the probabilities in a probability distribution is always unity. Therefore,

$$P(X=0) + P(X=1) + ... + P(X=7) = 1$$

$$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0 \Rightarrow (10k-1)(k+1) = 0 \Rightarrow 10k - 1 = 0 \Rightarrow k = \frac{1}{10} \quad [\because k \geq 0 \therefore k+1 \neq 0]$$

(ii) $$P(X < 6) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$$
$$= 0 + k + 2k + 2k + 3k + k^2$$
$$= k^2 + 8k = \left(\frac{1}{10}\right)^2 + \frac{8}{10} = \frac{81}{100} \quad [\because k = 1/10]$$

(iii) $$P(X \geq 6) = P(X = 6) + P(X = 7)$$
$$= 2k^2 + 7k^2 + k = 9k^2 + k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100} \quad [\because k = 1/10]$$

ALITER $$P(X \geq 6) = 1 - P(X < 6) = 1 - \frac{81}{100} = \frac{19}{100}$$

(v) $$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= k + 2k + 2k + 3k = 8k = \frac{8}{10} = \frac{4}{5} \quad [\because k = 1/10]$$

EXAMPLE 6 *A random variable X can take all non-negative integral values and the probability that X takes the value r is proportional to α^r $(0 < \alpha < 1)$. Find $P(X = 0)$.*

SOLUTION We have, $P(X = r) \propto \alpha^r \Rightarrow P(X = r) = \lambda \alpha^r, r = 0, 1, 2, ...$

Since sum of all the probabilities in a probability distribution is 1.

$$\therefore \quad P(X = 0) + P(X = 1) + P(X = 2) + ... = 1$$

$$\Rightarrow \quad \lambda\alpha^0 + \lambda\alpha^1 + \lambda\alpha^2 + ... = 1 \quad [\because P(X = r) = \lambda \alpha^r \text{ (given)}]$$

$$\Rightarrow \quad \lambda(1 + \alpha + \alpha^2 + \alpha^3 + ...) = 1 \Rightarrow \lambda\left(\frac{1}{1-\alpha}\right) = 1 \Rightarrow \lambda = 1 - \alpha$$

$$\therefore \quad P(X = r) = (1-\alpha)\alpha^r, \quad r = 0, 1, 2, ... \text{ Hence, } P(X = 0) = (1-\alpha)\alpha^0 = (1-\alpha).$$

EXAMPLE 7 *Find the probability distribution of X, the number of heads in two tosses of a coin (or a simultaneous toss of two coins).*

SOLUTION When two coins are tossed, there may be 1 head, 2 heads or no head at all. Thus, the possible values of X are 0, 1, 2.

Now,

$$P(X=0) = P(\text{Getting no head}) = P(TT) = \frac{1}{4}$$

$$P(X=1) = P(\text{Getting one head}) = P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}$$

$$P(X=2) = P(\text{Getting two heads}) = P(HH) = \frac{1}{4}$$

Thus, the required probability distribution of X is given by

X:	0	1	2
$P(X)$:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

EXAMPLE 8 *Three cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.* **[CBSE 2001]**

SOLUTION Let X denote the number of aces in a sample of 3 cards drawn from a well shuffled pack of 52 playing cards. Since there are four aces in the pack, therefore in the sample of 3 cards drawn either there can be no ace or there can be one ace or two aces or three aces. Thus, X can take values 0, 1, 2, and 3.

Now, $P(X=0)$ = Probability of getting no ace

$$= \text{Probability of getting 3 other cards} = \frac{^{48}C_3}{^{52}C_3} = \frac{4324}{5525}$$

$$P(X=1) = \text{Probability of getting one ace and two other cards} = \frac{^4C_1 \times {}^{48}C_2}{^{52}C_3} = \frac{1128}{5525}$$

$$P(X=2) = \text{Probability of getting two aces and one other card} = \frac{^4C_2 \times {}^{48}C_1}{^{52}C_3} = \frac{72}{5525}$$

and, $$P(X=3) = \text{Probability of getting 3 aces} = \frac{^4C_3}{^{52}C_3} = \frac{1}{5525}$$

Thus, the probability distribution of random variable X is given by

X:	0	1	2	3
$P(X)$:	$\frac{4324}{5525}$	$\frac{1128}{5525}$	$\frac{72}{5525}$	$\frac{1}{5525}$

It is to note here that the sum of the probabilities is 1 which is the condition for a distribution to be a probability distribution.

EXAMPLE 9 *An urn contains 4 white and 6 red balls. Four balls are drawn at random from the urn. Find the probability distribution of the number of white balls.*

SOLUTION Let X denote the number of white balls drawn from the urn. Since there are 4 white balls, therefore X can take values 0, 1, 2, 3 and 4.

Now, $P(X=0)$ = Probability of getting no white ball

$$= \text{Probability that 4 balls drawn are red} = \frac{^6C_4}{^{10}C_4} = \frac{1}{14}$$

$$P(X=1) = \text{Probability of getting one white ball} = \frac{^4C_1 \times {}^6C_3}{^{10}C_4} = \frac{8}{21}$$

$$P(X=2) = \text{Probability of getting two white balls} = \frac{^4C_2 \times {}^6C_2}{^{10}C_4} = \frac{6}{14}$$

$$P(X=3) = \text{Probability of getting three white balls} = \frac{{}^4C_3 \times {}^6C_1}{{}^{10}C_4} = \frac{4}{35}$$

and, $$P(X=4) = \text{Probability of getting 4 white balls} = \frac{{}^4C_4}{{}^{10}C_4} = \frac{1}{210}.$$

Thus, the probability distribution of X is given by

X :	0	1	2	3	4
$P(X)$:	$\frac{1}{14}$	$\frac{8}{21}$	$\frac{6}{14}$	$\frac{4}{35}$	$\frac{1}{210}$

EXAMPLE 10 *Four bad oranges are mixed accidently with 16 good oranges. Find the probability distribution of the number of bad oranges in a draw of two oranges.* **[CBSE 2002C]**

SOLUTION Let X denote the number of bad oranges in a draw of 4 oranges drawn from group of 16 good oranges and 4 bad oranges. Since there are 4 bad oranges in the group, therefore X can take values 0, 1 and 2.

Now, $P(X=0)$ = Probability of getting no bad orange = Probability of getting 2 good oranges

$$= \frac{{}^{16}C_2}{{}^{20}C_2} = \frac{12}{19}$$

$$P(X=1) = \text{Probability of getting one bad orange} = \frac{{}^4C_1 \times {}^{16}C_1}{{}^{20}C_2} = \frac{32}{95}$$

and, $$P(X=2) = \text{Probability of getting two bad oranges} = \frac{{}^4C_2}{{}^{20}C_2} = \frac{3}{95}$$

Thus, the probability distribution of X is given by

X :	0	1	2
$P(X)$:	$\frac{12}{19}$	$\frac{32}{95}$	$\frac{3}{95}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 11 *An unbiased die is thrown twice. Find the probability distribution of the number of sixes.*

SOLUTION Let X denote the number of times six occurs i.e. the number of sixes. Since the die is thrown twice, X can take values 0, 1 and 2.

Let S_i denote the event that a six occurs on the die in ith throw and F_i denote the event that the six does not occur in the ith throw. Then,

$$P(X=0) = \text{Probability of not getting six in both the throws}$$
$$= P(F_1 \text{ and } F_2) = P(F_1 \cap F_2)$$
$$= P(F_1)\,P(F_2) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36} \qquad [\because F_1, F_2 \text{ are independent events}]$$

$$P(X=1) = \text{Probability of getting one six in two throws}$$
$$= P[(F_1 \text{ and } S_2) \text{ or } (S_1 \text{ and } F_2)]$$
$$= P[(F_1 \cap S_2) \cup (S_1 \cap F_2)]$$
$$= P(F_1 \cap S_2) + P(S_1 \cap F_2) \qquad [\text{By addition Theorem}]$$
$$= P(F_1)\,P(S_2) + P(S_1)\,P(F_2) \qquad [\text{By multiplication theorem for independent events}]$$
$$= \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} = \frac{10}{36} = \frac{5}{18}$$

and, $P(X=2)$ = Probability of getting sixes in both the throws

$$= P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \quad \text{[By Multiplication Theorem]}$$

Thus, the probability distribution of X is

X:	0	1	2
$P(X)$:	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

EXAMPLE 12 *Two cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability distribution of the number of kings.*

SOLUTION Let X denote the number of kings. Then, X can take values 0, 1 or 2.

Let S_i denote the event of getting a king in the i^{th} draw and F_i denote the event of not getting a king in the i^{th} draw. Then,

$P(X=0)$ = Probability of not getting a king in the two draws

= P (Not a king in 1st draw and not a king in second draw)

$$= P(F_1 \cap F_2) = P(F_1)\,P(F_2) = \frac{48}{52} \times \frac{48}{52} = \frac{144}{169} \quad [\because F_1 \text{ and } F_2 \text{ are independent}]$$

$P(X=1)$ = Probability of getting one king in the two draws

$$= P((S_1 \cap F_2) \text{ or } (F_1 \cap S_2)) = P(S_1 \cap F_2) + P(F_1 \cap S_2)$$

$$= P(S_1)\,P(F_2) + P(F_1)\,P(S_2) \quad \text{[By multiplication theorem]}$$

$$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52} = \frac{24}{169}$$

and, $P(X=2)$ = Probability of getting kings in both the draws

$$= P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169} \quad \text{[By multiplication theorem]}$$

Thus, the probability distribution of X is

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

EXAMPLE 13 *A coin is tossed until a head appears or the tail appears 4 times in succession. Find the probability distribution of the number of tosses.*

SOLUTION Let S be the sample space associated with the given random experiment. Then,

$S = \{H, TH, TTH, TTTH, TTTT\}$

Let X denote the number of tosses. Then, X can take values 1, 2, 3 and 4

Now, $P(X=1) = P(H) = \frac{1}{2}$, $P(X=2) = P(TH) = P(T)\,P(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$$P(X=3) = P(TTH) = P(T)\,P(T)\,P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

and, $P(X=4) = P(TTTH \text{ or } TTTT) = P(TTTH) + P(TTTT)$

$$= P(T)\,P(T)\,P(T)\,P(H) + P(T)\,P(T)\,P(T)\,P(T) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}.$$

Thus, the probability distribution of X is given by

X:	1	2	3	4
$P(X)$:	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$

EXAMPLE 14 *An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in a random draw of three balls.*

SOLUTION When three balls are drawn, there may be all red, 2 red, 1 red or no red ball at all. Thus, if X denotes the number of red balls in a random draw of three balls. Then, X can take values 0, 1, 2, 3.

Now,

$$P(X=0) = P(\text{Getting no red ball})$$

$$= P(\text{Getting three white balls}) = \frac{{}^4C_3}{{}^7C_3} = \frac{4\times3\times2}{7\times6\times5} = \frac{4}{35}$$

$$P(X=1) = P(\text{Getting one red and two white balls}) = \frac{{}^3C_1\times{}^4C_2}{{}^7C_3} = \frac{18}{35}$$

$$P(X=2) = P(\text{Getting two red and one white ball}) = \frac{{}^3C_2\times{}^4C_1}{{}^7C_3} = \frac{12}{35}$$

$$P(X=3) = P(\text{Getting three red balls}) = \frac{{}^3C_3}{{}^7C_3} = \frac{1}{35}$$

Thus, the probability distribution of the number of red balls is given by

X:	0	1	2	3
$P(X)$:	$\frac{4}{35}$	$\frac{18}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

EXAMPLE 15 *Two cards are drawn without replacement from a well-shuffled deck of 52 cards. Determine the probability distribution of the number of face cards (i.e. Jack, Queen, King and Ace).*

SOLUTION Let X denote the number of face cards in two draws. Then, X can take values 0, 1, 2.

Let F_i denote the event of getting a face card in i^{th} draw. Then,

$$P(X=0) = \text{Probability of getting no face card}$$

$$= P(\overline{F_1}\cap\overline{F_2}) = P(\overline{F_1})\,P(\overline{F_2}/\overline{F_1}) = \frac{36}{52}\times\frac{35}{51} = \frac{105}{221} \quad \text{[By multiplication theorem]}$$

$$P(X=1) = \text{Probability of getting one face card and one other card}$$

$$= P((F_1\cap\overline{F_2})\cup(\overline{F_1}\cap F_2))$$

$$= P(F_1\cap\overline{F_2}) + P(\overline{F_1}\cap F_2) \quad \text{[By addition theorem]}$$

$$= P(F_1)\,P(\overline{F_2}/F_1) + P(\overline{F_1})\,P(F_2/\overline{F_1}) = \frac{16}{52}\times\frac{36}{51} + \frac{36}{52}\times\frac{16}{51} = \frac{96}{221}$$

$$P(X=2) = \text{Probability of getting both face cards}$$

$$= P(F_1\cap F_2) = P(F_1)\,P(F_2/F_1) = \frac{16}{52}\times\frac{15}{51} = \frac{20}{221}$$

Hence, the required probability distribution is

X:	0	1	2
$P(X)$:	$\frac{105}{221}$	$\frac{96}{221}$	$\frac{20}{221}$

EXAMPLE 16 *Find the probability distribution of the number of green balls drawn when 3 balls are drawn, one by one, without replacement from a bag containing 3 green and 5 white balls.*

SOLUTION Let X denote the total number of green balls drawn in three draws without replacement. Clearly, there may be all green, 2 green, 1 green or no green at all. Thus, X can assume values 0, 1, 2, and 3. Let G_i denote the event of getting a green ball in i^{th} draw.

Now,

$$P(X=0) = \text{Probability of getting no green ball in three draws}$$
$$= P(\overline{G_1} \cap \overline{G_2} \cap \overline{G_3}) = P(\overline{G_1})\, P(\overline{G_2}/\overline{G_1})\, P(\overline{G_3}/\overline{G_1} \cap \overline{G_2}) = \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{5}{28}$$

$$P(X=1) = \text{Probability of getting one green ball in three draws}$$
$$= P\left((G_1 \cap \overline{G_2} \cap \overline{G_3}) \cup (\overline{G_1} \cap G_2 \cap \overline{G_3}) \cup (\overline{G_1} \cap \overline{G_2} \cap G_3)\right)$$
$$= P(G_1 \cap \overline{G_2} \cap \overline{G_3}) + P(\overline{G_1} \cap G_2 \cap \overline{G_3}) + P(\overline{G_1} \cap \overline{G_2} \cap G_3)$$
$$= P(G_1)\, P(\overline{G_2}/G_1)\, P(\overline{G_3}/G_1 \cap \overline{G_2}) + P(\overline{G_1})\, P(G_2/\overline{G_1})\, P(\overline{G_3}/\overline{G_1} \cap G_2) + P(\overline{G_1})\, P(\overline{G_2}/\overline{G_1})\, P(G_3/\overline{G_1} \cap \overline{G_2})$$
$$= \frac{3}{8} \times \frac{5}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{4}{6} + \frac{5}{8} \times \frac{4}{7} \times \frac{3}{6} = \frac{15}{28}$$

$$P(X=2) = P\left((G_1 \cap G_2 \cap \overline{G_3}) \cap (\overline{G_1} \cap G_2 \cap G_3) \cup (G_1 \cap \overline{G_2} \cap G_3)\right)$$
$$= P(G_1)\, P(G_2/G_1)\, P(\overline{G_3}/G_1 \cap G_2) + P(\overline{G_1})\, P(G_2/\overline{G_1})\, P(G_3/\overline{G_1} \cap G_2) + P(G_1)\, P(\overline{G_2}/G_1)\, P(G_3/G_1 \cap \overline{G_2})$$
$$= \frac{3}{8} \times \frac{2}{7} \times \frac{5}{6} + \frac{5}{8} \times \frac{3}{7} \times \frac{2}{6} + \frac{3}{8} \times \frac{5}{7} \times \frac{2}{6} = \frac{15}{56}$$

and,

$$P(X=3) = P(G_1 \cap G_2 \cap G_3) = P(G_1)\, P(G_2/G_1)\, P(G_3/G_1 \cap G_2) = \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} = \frac{1}{56}$$

Thus, the probability distribution of the number of green balls is given by

X:	0	1	2	3
$P(X)$:	$\frac{5}{28}$	$\frac{15}{28}$	$\frac{15}{56}$	$\frac{1}{56}$

EXAMPLE 17 *From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the sample is drawn randomly, find*

(i) *the probability distribution of X* **[CBSE 2010]** (ii) $P(X \le 1)$
(iii) $P(X < 1)$ (iv) $P(0 < X < 2)$

SOLUTION (i) Clearly, X can assume values 0, 1, 2, 3 such that

$$P(X = 0) = (\text{Probability of getting no defective item}) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{1}{6}$$

$$P(X = 1) = (\text{Probability of getting one defective item}) = \frac{{}^3C_1 \times {}^7C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = (\text{Probability of getting two defective items}) = \frac{{}^3C_2 \times {}^7C_2}{{}^{10}C_4} = \frac{3}{10}$$

and, $$P(X = 3) = (\text{Probability of getting three defective items}) = \frac{{}^3C_3 \times {}^7C_1}{{}^{10}C_4} = \frac{1}{30}$$

Hence, the probability distribution of X is

X:	0	1	2	3
$P(X)$:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$

(iii) $P(X < 1) = P(X = 0) = \frac{1}{6}$

(iv) $P(0 < X < 2) = P(X = 1) = \frac{1}{2}$

EXAMPLE 18 *We take 8 identical slips of paper, write the number 0 on one of them, the number 1 on three of the slips, the number 2 on three of the slips and the number 3 on one of the ships. These slips are folded, put in a box and thoroughly mixed. One slip is drawn at random from the box. If X is the random variable denoting the number written on the drawn slip, find the probability distribution of X*

SOLUTION Clearly, X takes values 0, 1, 2, 3 such that

$$P(X = 0) = (\text{Probability of getting a slip written 0 on it}) = \frac{1}{8}$$

$$P(X = 1) = (\text{Probability of getting a slip written 1 on it}) = \frac{3}{8}$$

$$P(X = 2) = (\text{Probability of getting a slip written 2 on it}) = \frac{3}{8}$$

and, $$P(X = 3) = (\text{Probability of getting a slip written 3 on it}) = \frac{1}{8}$$

Hence, the probability distribution of X is

X:	0	1	2	3
$P(x)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 19 *A coin is biased so that the head is 3 times as likely to occur as tail. If coin is tossed twice, find the probability distribution for the number of tails.* **[NCERT]**

SOLUTION Let p be the probability of getting a tail in a single toss of a coin. Then, probability of getting a head is $3p$.

Since "Getting head" and "Getting tail" are mutually exclusive and exhaustive events in a single toss of a coin.

$\therefore \quad P(H) + P(T) = 1 \Rightarrow p + 3p = 1 \Rightarrow p = \frac{1}{4}$

$\therefore \quad P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$

Let X denote the number of tails in two tosses of a coin. Then, X can take values 0, 1, 2.

Now,

$$P(X = 0) = \text{Probability of getting no tail} = \text{Probability of getting both heads}$$
$$= P(HH)$$
$$= P(H) \times P(H) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \qquad [\because \text{ Two trials are independent}]$$

$$P(X = 1) = \text{Probability of getting one tail and one head.}$$
$$= P(HT) + P(TH) = P(H)\,P(T) + P(T)\,P(H) = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{3}{4} = \frac{3}{8}$$

$$P(X = 2) = \text{Probability of getting both tails} = P(TT) = P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

Hence, the probability distribution of X is

X:	0	1	2
$P(X)$:	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

EXAMPLE 20 *A die is loaded in such a way that an even number is twice likely to occur as an odd number. If the die is tossed twice, find the probability distribution of the random variable X representing the perfect squares in the two tosses.*

SOLUTION Let p be the probability of getting on odd number in a single throw of a die. Then, probability of getting on even number is $2p$.

We have,

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1 \quad [\because \text{ Sum of the probabilities} = 1]$$

$$\Rightarrow \quad p + 2p + p + 2p + p + 2p = 1 \Rightarrow 9p = 1 \Rightarrow p = \frac{1}{9}$$

Now,

Probability of getting a perfect square i.e. 1 or 4 in a single throw of a die

$$= P(1) + P(4) = p + 2p = 3p = \frac{3}{9} = \frac{1}{3}$$

Since X denotes the number of perfect squares in two tosses. Then, X can take values 0, 1, 2 such that

$P(X = 0)$ = Probability of not getting perfect squares in both the tosses $= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(X = 1)$ = Probability of getting perfect squares in one of the two tosses

$$= \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$$

$P(X = 2)$ = Probability of getting perfect squares in both two tosses $= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

Hence, the probability distribution of X is

X:	0	1	2
$P(X)$:	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

EXAMPLE 21 *Two biased dice are thrown together. For the first die $P(6) = \frac{1}{2}$, other scores being equally likely while for the second die, $P(1) = \frac{2}{5}$ and other scores are equally likely. Find the probability distribution of 'the number of ones seen'.* [NCERT EXEMPLAR]

SOLUTION For the first die, it is given that $P(6) = \frac{1}{2}$ and other scores are equally likely.

i.e. $\quad P(1) = P(2) = P(3) = P(4) = P(5) = p_1$ (say).

$$\therefore \quad P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1 \Rightarrow 5p_1 + \frac{1}{2} = 1 \Rightarrow p_1 = \frac{1}{10}$$

So, for the first die, we have

$$P(1) = P(2) = P(3) = P(4) = P(5) = \frac{1}{10} \text{ and } P(6) = \frac{1}{2}$$

For the second die, it is given that $P(1) = \frac{2}{5}$ and other scores are equally likely.

i.e. $P(2) = P(3) = P(4) = P(5) = P(6) = p_2$ (say).

$\therefore$ $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1 \Rightarrow \frac{2}{5} + 5p_2 = 1 \Rightarrow p_2 = \frac{3}{25}$

So, for the second die, we obtain

$$P(1) = \frac{2}{5} \text{ and } P(2) = P(3) = P(4) = P(5) = P(6) = \frac{3}{25}$$

When two dice are thrown, there may not be one on both the dice or one of the dice may show one or both of them show one. This, if X denotes 'the number of ones seen'. Then, X can take values 0, 1 and 2 such that

$P(X = 0) =$ Probability of not getting one on both dice

$=$ (Probability of not getting one on first die) $\times$ (Probability of not getting one on second die)

$$= \left(1 - \frac{1}{10}\right) \times \left(1 - \frac{2}{5}\right) = \frac{9}{10} \times \frac{3}{5} = \frac{27}{50}$$

$P(X = 1) =$ Probability of getting one on one die and other number on the other die

$$= \frac{1}{10} \times \left(1 - \frac{2}{5}\right) + \left(1 - \frac{1}{10}\right) \times \frac{2}{5} = \frac{21}{50}$$

$P(X = 2) =$ Probability of getting one on both dice $= \frac{1}{10} \times \frac{2}{5} = \frac{2}{50}$

Thus, the probability distribution of X is as given below.

X:	0	1	2
$P(X)$:	$\frac{27}{50}$	$\frac{21}{50}$	$\frac{2}{50}$

EXERCISE 31.1

BASIC

1. Which of the following distributions of probabilities of a random variable X are the probability distributions?

(i)

X:	3	2	1	0	−1
$P(X)$:	0.3	0.2	0.4	0.1	0.05

(ii)

X:	0	1	2
$P(X)$:	0.6	0.4	0.2

(iii)

X:	0	1	2	3	4
$P(X)$:	0.1	0.5	0.2	0.1	0.1

(iv)

X:	0	1	2	3
$P(X)$:	0.3	0.2	0.4	0.1

2. A random variable X has the following probability distribution:

Values of X:	−2	−1	0	1	2	3
$P(X)$:	0.1	k	0.2	$2k$	0.3	k

Find the value of k.

3. A random variable X has the following probability distribution:

Values of X:	0	1	2	3	4	5	6	7	8
$P(X)$:	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Determine:

(i) The value of a (ii) $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 5)$.

4. The probability distribution function of a random variable X is given by

x_i:	0	1	2
p_i:	$3c^3$	$4c - 10c^2$	$5c - 1$

where $c > 0$

Find: (i) c (ii) $P(X<2)$ (iii) $P(1<X\le 2)$

5. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X=x_1)=3P(X=x_2)=P(X=x_3)=5P(X=x_4)$. Find the probability distribution of X.
6. A random variable X takes the values 0, 1, 2 and 3 such that:
   $P(X=0)=P(X>0)=P(X<0)$; $P(X=-3)=P(X=-2)=P(X=-1)$;
   $P(X=1)=P(X=2)=P(X=3)$. Obtain the probability distribution of X.
7. Two cards are drawn from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces.
8. Find the probability distribution of the number of heads, when three coins are tossed.
9. Four cards are drawn simultaneously from a well shuffled pack of 52 playing cards. Find the probability distribution of the number of aces.
10. (i) A bag contains 4 red and 6 black balls. Three balls are drawn at random. Find the probability distribution of the number of red balls.
    (ii) Two balls are drawn at random from a bag containing 2 red balls and 3 blue balls, without replacement. Let the variable X denote the number of red balls. Find the probability distribution of X. **[CBSE 2022]**
11. Five defective mangoes are accidently mixed with 15 good ones. Four mangoes are drawn at random from this lot. Find the probability distribution of the number of defective mangoes.
12. Two dice are thrown together and the number appearing on them noted. X denotes the sum of the two numbers. Assuming that all the 36 outcomes are equally likely, what is the probability distribution of X?

BASED ON LOTS

13. A class has 15 students whose ages are 14, 17, 15, 14, 21, 19, 20, 16, 18, 17, 20, 17, 16, 19 and 20 years respectively. One student is selected in such a manner that each has the same chance of being selected and the age X of the selected student is recorded. What is the probability distribution of the random variable X? **[NCERT]**
14. Five defective bolts are accidently mixed with twenty good ones. If four bolts are drawn at random from this lot, find the probability distribution of the number of defective bolts.
15. Two cards are drawn successively with replacement from well shuffled pack of 52 cards. Find the probability distribution of the number of aces. **[CBSE 2011]**
16. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of kings. **[CBSE 2012]**
17. Two cards are drawn successively without replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of aces. **[CBSE 2001]**
18. Find the probability distribution of the number of white balls drawn in a random draw of 3 balls without replacement, from a bag containing 4 white and 6 red balls.
19. Find the probability distribution of Y in two throws of two dice, where Y represents the number of times a total of 9 appears.
20. From a lot containing 25 items, 5 of which are defective, 4 are chosen at random. Let X be the number of defectives found. Obtain the probability distribution of X if the items are chosen without replacement.
21. Three cards are drawn successively with replacement from a well-shuffled deck of 52 cards. A random variable X denotes the number of hearts in the three cards drawn. Determine the probability distribution of X.
22. An urn contains 4 red and 3 blue balls. Find the probability distribution of the number of blue balls in a random draw of 3 balls with replacement.
23. Two cards are drawn simultaneously from a well-shuffled deck of 52 cards. Find the probability distribution of the number of successes, when getting a spade is considered a success.
24. A fair die is tossed twice. If the number appearing on the top is less than 3, it is a success. Find the probability distribution of number of successes. **[CBSE 2004]**

25. An urn contains 5 red and 2 black balls. Two balls are randomly selected. Let X represent the number of black balls. What are the possible values of X. Is X a random variable?

26. Let X represent the difference between the number of heads and the number of tails when a coin is tossed 6 times. What are possible values of X?

27. From a lot of 10 bulbs, which includes 3 detectives, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs. **[NCERT, CBSE 2010]**

28. Four balls are to be drawn without replacement from a box containing 8 red and 4 white balls. If X denotes the number of red balls drawn, find the probability distribution of X. **[NCERT EXEMPLAR]**

29. The probability distribution of a random variable X is given below:

X:	0	1	2	3
$P(X)$:	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

(i) Determine the value of k (ii) Determine $P(X \le 2)$ and $P(X > 2)$

(iii) Find $P(X \le 2) + P(X > 2)$

30. Let X denote the number of colleges where you will apply after your results and $P(X = x)$ denotes your probability of getting admission in x number of colleges. It is given that

$$P(X = x) = \begin{cases} kx & \text{, if } x = 0, 1 \\ 2kx & \text{, if } x = 2 \\ k(5 - x), & \text{if } x = 3 \text{ or } 4 \\ 0 & \text{, if } x > 4 \end{cases}$$

where k is a positive constant. Find the value of k. Also find the probability that you will get admission in (i) exactly one college (ii) at most 2 colleges (iii) at least 2 colleges. **[CBSE 2016]**

31. A bag contains 19 tickets, numbered 1 to 19. A ticket is drawn at random and then another ticket is drawn without replacing the first one in the bag. Find the probability distribution of even numbers on the ticket. **[CBSE 2020]**

ANSWERS

1. (iii) and (iv) **2.** $k = 0.1$ **3.** (i) $a = \frac{1}{81}$ (ii) $\frac{1}{9}, \frac{8}{9}$

4. (i) $\frac{1}{3}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$

5.

X:	x_1	x_2	x_3	x_4
$P(X)$:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

6.

X:	-3	-2	-1	0	1	2	3
$P(X)$:	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

7.

X:	0	1	2
$P(X)$:	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

8.

X:	0	1	2	3
$P(X)$:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

9.

X:	0	1	2	3	4
$P(X)$:	$\frac{^{48}C_4}{^{52}C_4}$	$\frac{^{4}C_1 \times {^{48}C_3}}{^{52}C_4}$	$\frac{^{4}C_2 \times {^{48}C_2}}{^{52}C_4}$	$\frac{^{4}C_3 \times {^{48}C_1}}{^{52}C_4}$	$\frac{^{4}C_4}{^{52}C_4}$

10. (i)

X:	0	1	2	3
$P(X)$:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii)

X:	0	1	2
$P(X)$:	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

11.

X:	0	1	2	3	4
$P(X)$:	$\frac{91}{323}$	$\frac{455}{969}$	$\frac{70}{323}$	$\frac{10}{323}$	$\frac{1}{969}$

12.

X:	2	3	4	5	6	7	8	9	10	11	12
$P(X)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

13.

X:	14	15	16	17	18	19	20	21
$P(X)$:	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

14.

X:	0	1	2	3	4
$P(X)$:	$\frac{969}{2530}$	$\frac{114}{253}$	$\frac{38}{253}$	$\frac{4}{253}$	$\frac{1}{2530}$

15.

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

16.

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

17.

X:	0	1	2
$P(X)$:	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

18.

X:	0	1	2	3
$P(X)$:	$\frac{5}{30}$	$\frac{15}{30}$	$\frac{9}{30}$	$\frac{1}{30}$

19.

Y:	x_1	x_2	x_3
$P(X)$:	$\frac{64}{81}$	$\frac{16}{81}$	$\frac{1}{81}$

20. $P(X = r) = \frac{^{5}C_r \cdot {^{20}C_{4-r}}}{^{25}C_r}$, $r = 0, 1, 2, 3, 4.$

21.

X:	0	1	2	3
P (X):	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

22.

X:	0	1	2	3
P(X):	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

23.

X:	0	1	2
P(X):	$\frac{19}{34}$	$\frac{13}{34}$	$\frac{2}{34}$

24.

X:	0	1	2
P (X):	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

25. 0, 1, 2. Yes

26. $-6, -4, -2, 0, 2, 4, 6$

27.

X:	0	1	2
P (X):	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

28.

X:	0	1	2	3	4
P(X):	$\frac{1}{495}$	$\frac{32}{495}$	$\frac{168}{495}$	$\frac{224}{495}$	$\frac{70}{495}$

29. (i) $\frac{8}{15}$ (ii) $\frac{14}{15}, \frac{1}{15}$ (iii) 1

30. $k = \frac{1}{8}$ (i) $\frac{1}{8}$ (ii) $\frac{5}{8}$ (iii) $\frac{7}{8}$

31.

X:	0	1	2
P (X):	$\frac{5}{19}$	$\frac{10}{19}$	$\frac{4}{19}$

HINTS TO SELECTED PROBLEMS

5. Let $P(X = x_3) = k$. Then,

$$P(X = x_1) = \frac{k}{2}, P(X = x_2) = \frac{k}{3}, P(X = x_4) = \frac{k}{5}$$

$$\therefore \quad P(X = x_1) + P(X = x_2) + P(X = x_3) + P(X = x_4) = 1$$

$$\Rightarrow \quad \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow k = \frac{30}{61}$$

6. Let $P(X = 0) = k$. Then, $P(X = 0) = P(X > 0) = P(X < 0) \Rightarrow P(X > 0) = k$ and $P(X < 0) = k$.

Now, $P(X = 0) + P(X < 0) + P(X > 0) = 1 \Rightarrow 3k = 1 \Rightarrow k = \frac{1}{3}$

$$\therefore \quad P(X < 0) = k$$

$$\Rightarrow \quad P(X = -1) + P(X = -2) + P(X = -3) = k$$

$$\Rightarrow \quad 3P(X = -1) = k \Rightarrow P(X = -1) = P(X = -2) = P(X = -3) = \frac{k}{3} = \frac{1}{9}$$

Similarly, $P(X > 0) = k \Rightarrow P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{9}$

13. We observe that X takes values 14, 15, 16, 17, 18, 19, 20 and 21.

$$P(X = 14) = \frac{2}{15}, P(X = 15) = \frac{1}{15}, P(X = 16) = \frac{2}{15}, P(X = 17) = \frac{3}{15},$$

$$P(X = 18) = \frac{1}{15}, P(X = 19) = \frac{2}{15}, P(X = 20) = \frac{3}{15}, P(X = 21) = \frac{1}{15}$$

Hence, the probability distribution of X is

X:	14	15	16	17	18	19	20	21
P (X):	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

27. Let X denote the number of defective bulbs in a sample of 2 bulbs drawn from a lot of 10 bulbs containing 3 defective and 7 non-defective bulbs. Then, X can take values 0, 1 and 2.

$$P(X = 0) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}, P(X = 1) = \frac{{}^3C_1 \times {}^7C_1}{{}^{10}C_2} = \frac{7}{15}, P(X = 2) = \frac{{}^3C_2}{{}^{10}C_2} = \frac{1}{15}$$

Hence, the probability distribution of X is as follows:

X:	0	1	2
$P(X)$:	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

31.4 MEAN OF A DISCRETE RANDOM VARIABLE

DEFINITION *If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with respective probabilities $p_1, p_2, p_3, \ldots, p_n$, then the mean $\overline{X}$ of X is defined as*

$$\overline{X} = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n \quad \text{or,} \quad \overline{X} = \sum_{i=1}^{n} p_i x_i$$

REMARK 1 *The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by $E(X)$.*

REMARK 2 *In case of a frequency distribution x_i / f_i; $i = 1, 2, \ldots, n$ the mean $\overline{X}$ is given by*

$$\overline{X} = \frac{1}{N}(f_1 x_1 + f_2 x_2 + \ldots + f_n x_n)$$

$$\Rightarrow \quad \overline{X} = \frac{f_1}{N} x_1 + \frac{f_2}{N} x_2 + \ldots + \frac{f_n}{N} x_n \Rightarrow \overline{X} = p_1 x_1 + p_2 x_2 + \ldots + p_n x_n, \text{ where } p_i = \frac{f_i}{N}$$

Thus, if we replace $\frac{f_i}{N}$ by p_i in the definition of mean, we obtain the mean of a discrete random variable. Consequently, the term 'mean' is appropriate for the sum $\sum_{i=1}^{n} p_i x_i$.

NOTE *The mean of a random variable means the mean of its probability distribution.*

ILLUSTRATION 1 *In a single throw of a die, if X denotes the number on its upper face. Find the mean of X.*

SOLUTION Clearly, X can take the values 1, 2, 3, 4, 5, 6 each with probabilities $\frac{1}{6}$.

So, the probability distribution of X is as given below:

X :	1	2	3	4	5	6
$P(X)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$\therefore \quad \overline{X} = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6$$

$$\Rightarrow \quad \overline{X} = \frac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6) = \frac{1}{6} \times \frac{6(6+1)}{2} = \frac{7}{2}$$

ILLUSTRATION 2 *If a pair of dice is thrown and X denotes the sum of the numbers on them. Find the probability distribution of X. Also, find the expectation of X.* **[NCERT]**

SOLUTION In a single throw of a pair of dice the sum of the numbers on them can be 2, 3, 4, ..., 12. So, X can assume values 2, 3, 4, ..., 12. The probability distribution of X is as given below:

X :	2	3	4	5	6	7	8	9	10	11	12
$P(X)$:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\therefore \quad E(X) = \frac{1}{36} \times 2 + \frac{2}{36} \times 3 + \frac{3}{36} \times 4 + \frac{4}{36} \times 5 + \frac{5}{36} \times 6 + \frac{6}{36} \times 7 + \frac{5}{36} \times 8 + \frac{4}{36} \times 9 + \frac{3}{36} \times 10 + \frac{2}{36} \times 11 + \frac{1}{36} \times 12$$

$$\Rightarrow \quad E(X) = \frac{1}{36}[2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12] = \frac{252}{36} = 7.$$

ILLUSTRATION 3 *A dealer in refrigerators estimates from his past experience the probabilities of his selling refrigerators in a day. These are as follows:*

No. of refrigerators sold in a day:	0	1	2	3	4	5	6
Probability:	0.03	0.20	0.23	0.25	0.12	0.10	0.07

Find the mean number of refrigerators sold in a day.

SOLUTION Let X denotes the number of refrigerators sold in a day. Then, the probability distribution of X is

x_i :	0	1	2	3	4	5	6
p_i :	0.03	0.20	0.23	0.25	0.12	0.10	0.07

$$\therefore \quad \overline{X} = 0.03 \times 0 + 0.20 \times 1 + 0.23 \times 2 + 0.25 \times 3 + 0.12 \times 4 + 0.10 \times 5 + 0.07 \times 6$$

$$\Rightarrow \quad \overline{X} = 0.20 + 0.46 + 0.75 + 0.48 + 0.50 + 0.42 = 2.81.$$

ILLUSTRATION 4 *A salesman wants to know the average number of units he sells per sales call. He checks his past sales records and comes up with the following probabilities:*

Sales (in units):	0	1	2	3	4	5
Probability:	0.15	0.20	0.10	0.05	0.30	0.20

What is the average number of units he sells per sale call ?

SOLUTION Let X denote the number of units. Then, X is a random variable with the following probability distribution

x_i:	0	1	2	3	4	5
p_i :	0.15	0.20	0.10	0.05	0.30	0.20

$$\therefore \quad \overline{X} = 0.15 \times 0 + 0.20 \times 1 + 0.10 \times 2 + 0.05 \times 3 + 0.30 \times 4 + 0.20 \times 5$$

$$\Rightarrow \quad \overline{X} = 0.20 + 0.20 + 0.15 + 1.20 + 1.00 = 2.75$$

Thus, the average number of units he would sell per sale call is 2.75.

31.5 VARIANCE OF A DISCRETE RANDOM VARIABLE

DEFINITION *If X is a discrete random variable which assumes values $x_1, x_2, x_3, \ldots, x_n$ with the respective probabilities $p_1, p_2, \ldots, p_n$, then variance of X is defined as*

$$\text{Var}(X) = p_1 (x_1 - \overline{X})^2 + p_2 (x_2 - \overline{X})^2 + \ldots + p_n (x_n - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2, \quad \text{where } \overline{X} = \sum_{i=1}^{n} p_i \, x_i \text{ is the mean of } X.$$

$$\text{Now,} \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i - \overline{X})^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i (x_i^{\,2} - 2 x_i \overline{X} + \overline{X}^2)$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^{\,2} - 2\overline{X}\left(\sum_{i=1}^{n} p_i x_i\right) + \overline{X}^{\,2}\left(\sum_{i=1}^{n} p_i\right)$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^{\,2} - 2\overline{X}\,.\,\overline{X} + \overline{X}^2 \qquad \left[\because \sum_{i=1}^{n} p_i = 1\right]$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^{\,2} - 2\overline{X}^2 + \overline{X}^2$$

$$\Rightarrow \quad \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \overline{X}^2 \Rightarrow \text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$$

Thus, $$\text{Var}(X) = \sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2$$

or, $$\text{Var}(X) = E(X^2) - \{E(X)\}^2$$

REMARK *If X is a random variable a, b are real numbers, then $aX+b$ is a random variable with mean $aX+b$ and variance a^2 Var (X).*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *A discrete random variable X has the following probability distribution:*

X:	1	2	3	4	5	6	7
$P(X)$:	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2+c$

Find the value of c. Also, find the mean of the distribution. [NCERT EXEMPLAR]

SOLUTION Since X is a random variable taking values 1, 2, …,7. Therefore,

$$P(X=1) + P(X=2) + \ldots + P(X=7) = 1$$

$$\Rightarrow \quad c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$$

$$\Rightarrow \quad 10c^2 + 9c - 1 = 0$$

$$\Rightarrow \quad (c+1)(10c-1) = 0 \Rightarrow 10c - 1 = 0 \Rightarrow c = \frac{1}{10} \qquad [\because P(X=1) = c > 0 \therefore c+1 \neq 0]$$

Now, $$\overline{X} = \sum_{i=1}^{7} x_i P(X = x_i)$$

$$\Rightarrow \quad \overline{X} = \sum_{i=1}^{7} i P(X=i) = 1 \times c + 2 \times 2c + 3 \times 2c + 4 \times 3c + 5 \times c^2 + 6 \times 2c^2 + 7 \times (7c^2 + c)$$

$$\Rightarrow \quad \overline{X} = 66c^2 + 30c = \frac{66}{100} + 3 = \frac{366}{100} = 3.66 \qquad \left[\because c = \frac{1}{10}\right]$$

EXAMPLE 2 *The probability distribution of a random variable X is given below:*

X:	0	1	2	3	4
$P(X)$:	0.1	0.25	0.3	0.2	0.15

Find (i) *Var* (X) (ii) *Var* $\left(\frac{X}{2}\right)$ [NCERT EXEMPLAR]

SOLUTION (i) *Computation of mean and variance*

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	0.1	0	0
1	0.25	0.25	0.25
2	0.3	0.6	1.2
3	0.2	0.6	1.8
4	0.15	0.6	2.40
		$\Sigma p_i x_i = 2.05$	$\Sigma p_i x_i^2 = 5.65$

Thus, we have $\Sigma p_i x_i = 2.05$ and $\Sigma p_i x_i^2 = 5.65$

$\therefore \quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 5.65 - (2.05)^2 = 1.4475$

(ii) We know that $\text{Var}(aX + b) = a^2 \text{ Var}(X)$.

$\therefore \quad \text{Var}\left(\frac{X}{2}\right) = \frac{1}{4} \text{Var}(X) = \frac{1}{4} \times 1.4475 = 0.361875$

EXAMPLE 3 *A random variable X has the following probability distribution:*

x_i :	-2	-1	0	1	2	3
p_i :	0.1	k	0.2	$2k$	0.3	k

(i) *Find the value of k.* (ii) *Calculate the mean of X.* (iii) *Calculate the variance of X.*

SOLUTION (i) The sum of the probabilities in a frequency distribution is always unity.

$\therefore \quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1 \Rightarrow 0.6 + 4k = 1 \Rightarrow 4k = 0.4 \Rightarrow k = 0.1$

Calculation of mean and variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.2	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.1	0.3	0.9
		$\Sigma p_i x_i = 0.8$	$\Sigma p_i x_i^2 = 2.8$

Thus, we have $\Sigma p_i x_i = 0.8$ and $\Sigma p_i x_i^2 = 2.8$

$\therefore \quad \text{Mean} = \Sigma p_i x_i = 0.8$ and, $\text{Variance} = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 2.8 - (0.8)^2 = 2.8 - 0.64 = 2.16$

EXAMPLE 4 *The probability distribution of a random variable X is given as under:*

$$P(X = x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}, \text{ where } k \text{ is a constant.}$$

Find (i) $P(X \geq 4)$ (ii) $E(X)$ (iii) $E(3X^2)$ **[NCERT EXEMPLAR]**

SOLUTION The probability distribution of X is as given below:

x_i:	1	2	3	4	5	6
$P(X = x_i)$:	k	$4k$	$9k$	$8k$	$10k$	$12k$

The sum of the probabilities in a probability distribution is 1.

$\therefore \quad P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1$

$\Rightarrow \quad k + 4k + 9k + 8k + 10k + 12k = 1 \Rightarrow 44k = 1 \Rightarrow k = \frac{1}{44}$

(i) $P(X \geq 4) = P(X = 4) + P(X = 5) + P(X = 6) = 8k + 10k + 12k = 30k = \frac{30}{44} = \frac{15}{22}$

(ii) $E(X) = 1\, P(X = 1) + 2\, P(X = 2) + 3\, P(X = 3) + 4\, P(X = 4) + 5P(X = 5) + 6\, P(X = 6)$

$\Rightarrow \quad E(X) = k + 8k + 27k + 32k + 50k + 72k = 190k = \frac{190}{44} = \frac{95}{22}$

(iii) The probability distribution of $3X^2$ is as given below:

$3x_i^2$:	3	12	27	48	75	108
p_i :	k	$4k$	$9k$	$8k$	$10k$	$12k$

$\therefore \quad E(3X^2) = 3 \times k + 12 \times 4k + 27 \times 9k + 48 \times 8k + 75 \times 10k + 108 \times 12k$

$\Rightarrow \quad E(3X^2) = 3k + 48k + 243k + 384k + 750k + 1296k = 2724k = \frac{2724}{44} = \frac{681}{11}$

EXAMPLE 5 *The probability distribution of the discrete random variables X and Y are given below:*

X:	0	1	2	3
$P(X)$:	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Y:	0	1	2	3
$P(Y)$:	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that: $E(Y^2) = 2\,E(X)$. **[NCERT EXEMPLAR]**

SOLUTION We find that:

$$E(X) = 0 \times \frac{1}{5} + 1 \times \frac{2}{5} + 2 \times \frac{1}{5} + 3 \times \frac{1}{5} = \frac{7}{5} \Rightarrow 2\,E(X) = \frac{14}{5} \quad \ldots(i)$$

The probability distribution of Y^2 is as given below.

y_i^2 :	0	1	4	9
p_i :	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

$$\therefore \quad E(Y^2) = 0 \times \frac{1}{5} + 1 \times \frac{3}{10} + 4 \times \frac{2}{5} + 9 \times \frac{1}{10} = \frac{28}{10} = \frac{14}{5} \quad \ldots(ii)$$

From (i) and (ii), we find that $E(Y^2) = 2\,E(X)$.

EXAMPLE 6 *Find the mean and variance of the number of heads in the two tosses of a coin.*

SOLUTION Let X denote the number of heads in the two tosses of a coin. Then, X can take values 0, 1 or 2 such that

$P(X = 0) =$ (Probability of getting no head) $= P(TT) = \frac{1}{4}$

$P(X = 1) =$ (Probability of getting one head) $= P(HT \text{ or } TH) = \frac{2}{4} = \frac{1}{2}$

and, $P(X = 2) =$ (Probability of getting both heads) $= P(HH) = \frac{1}{4}$

Thus, the probability distribution of X is as given below:

X :	0	1	2
$P(X)$:	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	1/4	0	0
1	1/2	1/2	1/2
2	1/4	1/2	1
		$\Sigma p_i x_i = 1$	$\Sigma p_i x_i^2 = 3/2$

Thus, we have

$$\Sigma p_i x_i = 1 \text{ and } \Sigma p_i x_i^2 = \frac{3}{2}$$

$\therefore$ $$\overline{X} = \text{Mean} = \Sigma p_i x_i = 1 \text{ and, } \mathrm{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

Hence, Mean = 1 and Variance = $\frac{1}{2}$

EXAMPLE 7 *Find the mean, variance and standard deviation of the number of heads in a simultaneous toss of three coins.* **[CBSE 2007, NCERT EXEMPLAR]**

SOLUTION Let X denote the number of heads in a simultaneous toss of three coins. Then, X can take values 0, 1, 2, 3.

Now, $P(X = 0) = P(TTT) = \frac{1}{8}$, $P(X = 1) = P(HTT \text{ or } TTH \text{ or } THT) = \frac{3}{8}$

$P(X = 2) = P(HHT \text{ or } THH \text{ or } HTH) = \frac{3}{8}$ and, $P(X = 3) = P(HHH) = \frac{1}{8}$

Thus, the probability distribution of X is given by

$X:$	0	1	2	3
$P(X):$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\Sigma p_i x_i = \frac{3}{2}$	$\Sigma p_i x_i^2 = 3$

Thus, we have

$$\Sigma p_i x_i = \frac{3}{2} \text{ and } \Sigma p_i x_i^2 = 3$$

$\therefore \quad \overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{3}{2}$ and, $\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4}$

$\therefore \quad \text{Standard deviation} = \sqrt{\text{Var}(X)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} = 0.87$

Hence, Mean $= \frac{3}{2}$, Variance $= \frac{3}{4}$ and, Standard deviation = 0.87

EXAMPLE 8 *Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation and variance of X.* **[NCERT]**

SOLUTION Clearly, X can take values 0, 1 and 2.

We have,

$$P(X=0) = (\text{Probability of not getting six on any dice}) = \frac{25}{36}$$

$$P(X=1) = (\text{Probability of getting one six}) = \frac{10}{36}$$

$$P(X=2) = (\text{Probability of getting two sixes}) = \frac{1}{36}$$

Thus, the probability distribution of X is given by

X :	0	1	2
$P(X)$:	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{25}{36}$	0	0
1	$\frac{10}{36}$	$\frac{10}{36}$	$\frac{10}{36}$
2	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
		$\Sigma p_i x_i = \frac{12}{36}$	$\Sigma p_i x_i^2 = \frac{14}{36}$

Thus, we have

$$\Sigma p_i x_i = \frac{12}{36} = \frac{1}{3} \text{ and } \Sigma p_i x_i^2 = \frac{7}{18}$$

$\therefore \quad E(X) = \Sigma p_i x_i = \frac{1}{3}$ and, $\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{7}{18} - \frac{1}{9} = \frac{5}{18}$

Hence, $E(X) = \frac{1}{3}$ and $\text{Var}(X) = \frac{5}{18}$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *The random variable X can take only the values 0, 1, 2. Given that $P(X=0) = P(X=1) = p$ and that $E(X^2) = E(X)$, find the value of p.*

SOLUTION Clearly,

$$P(X=0) + P(X=1) + P(X=2) = 1 \Rightarrow p + p + P(X=2) = 1 \Rightarrow P(X=2) = 1 - 2p.$$

So, the probability distribution of X is as given below:

x_i :	0	1	2
p_i :	p	p	$1-2p$

$\therefore \quad E(X) = 0 \times p + 1 \times p + 2(1-2p) = 2 - 3p$ and, $E(X^2) = 0^2 \times p + 1^2 \times p + 2^2(1-2p) = 4 - 7p$

It is given that: $E(X^2) = E(X) \Rightarrow 4 - 7p = 2 - 3p \Rightarrow p = \frac{1}{2}$

EXAMPLE 10 *The probability distribution of a discrete random variable X is given as under:*

X:	1	2	4	2A	3A	5A
P(X):	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate: (i) *The value of A, if* $E(X) = 2.94$ (ii) *Variance of X.*

SOLUTION (i) Using the definition of $E(X)$, we obtain

$$E(X) = \Sigma x_i P(X = x_i) = 1 \times \frac{1}{2} + 2 \times \frac{1}{5} + 4 \times \frac{3}{25} + 2A \times \frac{1}{10} + 3A \times \frac{1}{25} + 5A \times \frac{1}{25}$$

$$\Rightarrow \quad E(X) = \frac{1}{2} + \frac{2}{5} + \frac{12}{25} + \frac{A}{5} + \frac{3A}{25} + \frac{A}{5} = \frac{69}{50} + \frac{13A}{25}$$

It is given that $E(X) = 2.94$

$$\therefore \quad \frac{69}{50} + \frac{13}{25}A = 2.94 \Rightarrow 69 + 26A = 147 \Rightarrow 26A = 78 \Rightarrow A = 3.$$

(ii) The variance $V(X)$ of random variable X is given by $\text{Var}(X) = E(X^2) - (E(X))^2$.

Now,

$$E(X^2) = \Sigma x_i^2 P(X = x_i) = 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{5} + 4^2 \times \frac{3}{25} + (2A)^2 \times \frac{1}{10} + (3A)^2 \times \frac{1}{25} + (5A)^2 \times \frac{1}{25}$$

$$\Rightarrow \quad E(X^2) = \frac{1}{2} + \frac{4}{5} + \frac{48}{25} + \frac{2A^2}{5} + \frac{9A^2}{25} + A^2 = \frac{161}{50} + \frac{44}{25}A^2 = \frac{161}{50} + \frac{44}{25} \times 9 = \frac{953}{50} \qquad [\because A = 3]$$

$$\therefore \quad \text{Var}(X) = E(X^2) - (E(X))^2 = \frac{953}{50} - (2.94)^2 = 19.06 - 8.6436 = 10.4164.$$

EXAMPLE 11 *Let X be a discrete random variable whose probability distribution is defined as follows:*

$$P(X = x) = \begin{cases} k(x+1) & \text{for } x = 1, 2, 3, 4 \\ 2kx & \text{for } x = 5, 6, 7 \\ 0 & \text{otherwise} \end{cases}, \text{ where } k \text{ is a constant.}$$

Find : (i) k (ii) $E(X)$ (iii) *Standard deviation of X.*

SOLUTION (i) The probability distribution of X is as given below.

x_i :	1	2	3	4	5	6	7
p_i :	$2k$	$3k$	$4k$	$5k$	$10k$	$12k$	$14k$

The sum of the probabilities in a probability distribution is 1.

$$\therefore \quad P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) = 1$$

$$\Rightarrow \quad 2k + 3k + 4k + 5k + 10k + 12k + 14k = 1 \Rightarrow 50k = 1 \Rightarrow k = \frac{1}{50}$$

(ii) $E(X) = 1\ P(X=1) + 2P(X=2) + 3\ P(X=3) + 4\ P(X=4) + 5\ P(X=5) + 6\ P(X=6) + 7\ P(X=7)$

$= 1 \times 2k + 2 \times 3k + 3 \times 4k + 4 \times 5k + 5 \times 10k + 6 \times 12k + 7 \times 14k$

$= 2k + 6k + 12k + 20k + 50k + 72k + 98k = 260k = \frac{260}{50} = 5.2$

(iii) We have, $\text{Var}(X) = E(X^2) - (E(X))^2$ and $\sigma(X) = \sqrt{\text{Var}(X)}$

So, let us first compute $E(X^2)$.

$$E(X^2) = 1^2\ P(X=1) + 2^2 P(X=2) + 3^2 P(X=3) + 4^2 P(X=4) + 5^2 P(X=5) + 6^2 P(X=6) + 7^2 P(X=7)$$

$\Rightarrow \quad E(X^2) = 1^2 \times 2k + 2^2 \times 3k + 3^2 \times 4k + 4^2 \times 5k + 5^2 \times 10k + 6^2 \times 12k + 7^2 \times 14k$

$\Rightarrow \quad E(X^2) = 2k + 12k + 36k + 80k + 250k + 432k + 686k = 1498k = \frac{1498}{50} = 29.96$

$\therefore \quad \text{Var}(X) = E(X^2) - (E(X))^2 = 29.96 - (5.2)^2 = 2.92$

Hence, $\sigma(X) = \sqrt{2.92} = 1.708$

EXAMPLE 12 *A random variable X has the following probability distribution:*

x_i :	-2	-1	0	1	2	3
p_i :	0.1	k	0.2	$2k$	0.3	k

(i) *Find the value of k.* (ii) *Calculate the mean of X.* (iii) *Calculate the variance of X.*

SOLUTION (i) Since sum of the probabilities in a frequency distribution is always unity.

$\therefore \quad 0.1 + k + 0.2 + 2k + 0.3 + k = 1 \Rightarrow 0.6 + 4k = 1 \Rightarrow 4k = 0.4 \Rightarrow k = 0.1$

Calculation of mean and variance

x_i	p_i	$p_i\, x_i$	$p_i\, x_i^2$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.2	0	0
1	0.2	0.2	0.2
2	0.3	0.6	1.2
3	0.1	0.3	0.9
		$\Sigma\, p_i\, x_i = 0.8$	$\Sigma\, p_i\, x_i^2 = 2.8$

Thus, we have $\Sigma\, p_i\, x_i = 0.8$ and $\Sigma\, p_i\, x_i^2 = 2.8$

$\therefore \quad \text{Mean} = 0.8$ and, $\text{Variance} = \Sigma p_i x_i^2 - (\Sigma\, p_i\, x_i)^2 = 2.8 - (0.8)^2 = 2.8 - 0.64 = 2.16$

EXAMPLE 13 *The random variable X can take values 0, 1, 2, 3. Given that $P(X=0) = P(X=1) = p$ and $P(X=2) = P(X=3)$ such that $\sum p_i x_i^2 = 2\sum p_i x_i$, find the value of p.* **[CBSE 2017]**

SOLUTION Let $P(X=2) = P(X=3) = \lambda$.

It is given that X is a random variable taking values 0, 1, 2, and 3.

$\therefore \quad P(X=0) + P(X=1) + P(X=2) + P(X=3) = 1 \Rightarrow p + p + \lambda + \lambda = 1 \Rightarrow \lambda = \frac{1}{2} - p$

Thus, the probability distribution of X is

x_i :	0	1	2	3
p_i :	p	p	$\frac{1}{2} - p$	$\frac{1}{2} - p$

It is given that

$$\sum p_i x_i^2 = 2 \sum p_i x_i$$

$$\Rightarrow p \times 0^2 + p \times 1^2 + \left(\frac{1}{2} - p\right) \times 2^2 + \left(\frac{1}{2} - p\right) \times 3^2 = 2\left\{p \times 0 + p \times 1 + \left(\frac{1}{2} - p\right) \times 2 + \left(\frac{1}{2} - p\right) \times 3\right\}$$

$$\Rightarrow p + 2 - 4p + \frac{9}{2} - 9p = 2\left(p + 1 - 2p + \frac{3}{2} - 3p\right)$$

$$\Rightarrow \left(\frac{13}{2}-12p\right)=2\left(\frac{5}{2}-4p\right)\Rightarrow 13-24p=10-16p \Rightarrow 8p=3\Rightarrow p=\frac{3}{8}$$

EXAMPLE 14 *Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E (X) and Var (X).*

[NCERT, CBSE 2014, 2018]

SOLUTION We observe that X can take values 2, 3, 4, 5, 6 such that

$P(X=2)$ = Probability that the larger of two numbers is 2
= Probability of getting 1 in first selection and 2 in second selection or getting 2 in first selection and 1 in second selection

$$=\frac{1}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{1}{5}=\frac{2}{30}=\frac{1}{15}$$

$P(X=3)$ = Probability that the larger of two numbers is 3
= Probability of getting a number less than 3 in first selection and 3 in second selection or getting 3 in first selection and a number less than 3 in second selection.

$$=\frac{2}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{2}{5}=\frac{4}{30}=\frac{2}{15}$$

$P(X=4)$ = (Probability that the larger of two numbers is 4) $=\frac{3}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{3}{5}=\frac{6}{30}=\frac{1}{5}$

$P(X=5)=\frac{4}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{4}{5}=\frac{8}{30}=\frac{4}{15}$ and, $P(X=6)=\frac{5}{6}\times\frac{1}{5}+\frac{1}{6}\times\frac{5}{5}=\frac{10}{30}=\frac{1}{3}$

Thus, the probability distribution of X is

X:	2	3	4	5	6
$P(X)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{4}{15}$	$\frac{1}{3}$

$$\therefore \quad E(X)=\frac{1}{15}\times 2+\frac{2}{15}\times 3+\frac{1}{5}\times 4+\frac{4}{15}\times 5+\frac{1}{3}\times 6=\frac{70}{15}=\frac{14}{3}$$

EXAMPLE 15 *In a meeting 70% of the members favour a certain proposal, 30% being opposed. A member is selected at random and let X = 0 if he opposed, and X = 1 if he is in favour. Find E (X) and Var (X).*

SOLUTION It is given that **[NCERT]**

$P(X=0)$ = Probability that a member opposed a certain proposal $=\frac{30}{100}$

$P(X=1)$ = Probability that a member favoured a certain proposal $=\frac{70}{100}$

The probability distribution of X is

X:	0	1
$P(X)$:	$\frac{30}{100}$	$\frac{70}{100}$

$$\therefore \quad E(X)=\frac{30}{100}\times 0+\frac{70}{100}\times 1=\frac{7}{10} \text{ and, } E(X^2)=\frac{30}{100}\times 0^2+\frac{70}{100}\times 1^2=\frac{7}{10}$$

$$\therefore \quad \text{Var}(X)=E(X^2)-[E(X)]^2=\frac{7}{10}-\frac{49}{100}=\frac{21}{100}$$

Hence, $E(X)=\frac{7}{10}$ and $\text{Var}(X)=\frac{21}{100}$

EXAMPLE 16 *A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being choosen and the age X of the selected student is recorded. What is the probability distribution of random variable X? Find mean, variance and standard deviation of X.* **[NCERT]**

SOLUTION We observe that X takes values 14, 15, 16, 17, 18, 19, 20 and 21 such that

$$P(X=14)=\frac{2}{15},\ P(X=15)=\frac{1}{15},\ P(X=16)=\frac{2}{15},\ P(X=17)=\frac{3}{15}$$

$$P(X=18)=\frac{1}{15},\ P(X=19)=\frac{2}{15},\ P(X=20)=\frac{3}{15}\ P(X=21)=\frac{1}{15}$$

So, the probability distribution of X is as given below:

X :	14	15	16	17	18	19	20	21
P(X) :	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i\, x_i$	$p_i\, x_i^2$
14	$\frac{2}{15}$	$\frac{28}{15}$	$\frac{392}{15}$
15	$\frac{1}{15}$	$\frac{15}{15}$	$\frac{225}{15}$
16	$\frac{2}{15}$	$\frac{32}{15}$	$\frac{512}{15}$
17	$\frac{3}{15}$	$\frac{51}{15}$	$\frac{867}{15}$
18	$\frac{1}{15}$	$\frac{18}{15}$	$\frac{324}{15}$
19	$\frac{2}{15}$	$\frac{38}{15}$	$\frac{722}{15}$
20	$\frac{3}{15}$	$\frac{60}{15}$	$\frac{1200}{15}$
21	$\frac{1}{15}$	$\frac{21}{15}$	$\frac{441}{15}$
		$\Sigma\, p_i\, x_i = \frac{263}{15}$	$\Sigma\, p_i\, x_i^2 = \frac{4683}{15}$

We have,

$$\Sigma\, p_i\, x_i = \frac{263}{15} \text{ and } \Sigma\, p_i\, x_i^2 = \frac{4683}{15}$$

$$\therefore \quad \text{Mean} = \Sigma\, p_i\, x_i = \frac{263}{15} = 17.53$$

$$\text{Variance} = \Sigma\, p_i\, x_i^2 - (\Sigma\, p_i\, x_i)^2 = \frac{4683}{15} - \left(\frac{263}{15}\right)^2 = \frac{70245 - 69169}{225} = \frac{1076}{225}$$

$$\therefore \quad \text{Standard Deviation} = \sqrt{\text{Variance}} = \sqrt{\frac{1076}{225}} = \frac{\sqrt{1076}}{15} = \frac{32.80}{15} = 2.186$$

EXAMPLE 17 *Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as 'getting a number greater than 4'. Also, find the mean and variance of the distribution.* **[NCERT, CBSE 2020]**

SOLUTION Let X denote the number of successes in two tosses of a die. Then, X can take values 0, 1, 2.

Let S_i = Getting a success in i^{th} toss and, F_i = Getting a failure in i^{th} toss. Then,

$$P(S_1) = \text{Probability of getting a number greater than 4 in first toss} = \frac{2}{6} = \frac{1}{3}$$

Also, $P(S_2) = \dfrac{1}{3}$

$\therefore \quad P(F_1) = P(F_2) = \dfrac{2}{3}$

Now, $P(X=0)$ = Probability of getting no success in two tosses of a die

$$= P(F_1 \cap F_2)$$

$$= P(F_1) \times P(F_2) \qquad \text{[By multiplication theorem]}$$

$$= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

$P(X=1)$ = Probability of getting one success in two tosses of a die.

$$= P\{(S_1 \cap F_2) \cup (F_1 \cap S_2)\}$$

$$= P(S_1 \cap F_2) + P(F_1 \cap S_2) = P(S_1)\,P(F_2) + P(F_1)\,P(S_2) = \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}.$$

and, $P(X=2)$ = Probability of getting two successes in two tosses of a die

$$= P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Thus, the probability distribution of X is given by

X:	0	1	2
$P(X)$:	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{4}{9}$	0	0
1	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{4}{9}$
2	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{4}{9}$
		$\Sigma p_i x_i = \frac{6}{9}$	$\Sigma p_i x_i^2 = \frac{8}{9}$

Thus, we have

$$\Sigma p_i x_i = \frac{6}{9} = \frac{2}{3} \text{ and } \Sigma p_i x_i^2 = \frac{8}{9}$$

$$\therefore \quad \overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{2}{3} \text{ and, } \operatorname{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{8}{9} - \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Hence, Mean $= \dfrac{2}{3}$ and, Variance $= \dfrac{4}{9}$.

EXAMPLE 18 *Find the probability distribution of the number of sixes in three tosses of a die. Find also the mean and variance of the distribution.*

SOLUTION Let X denote the number of sixes in three tosses of a die. Then, X can take values 0, 1, 2, 3. Let S_i denote the event of getting a six in i^{th} toss, $i = 1, 2, 3$. Then,

$$P(S_i) = \frac{1}{6} \text{ and } P(\overline{S_i}) = \frac{5}{6},\ i = 1, 2, 3.$$

Now,
$$P(X=0) = P(\overline{S_1} \cap \overline{S_2} \cap \overline{S_3})$$
$$= P(\overline{S_1})\, P(\overline{S_2})\, P(\overline{S_3}) \qquad [\because \overline{S_1}, \overline{S_2}, \overline{S_3} \text{ are independent events}]$$
$$= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$$

$$P(X=1) = P\left((S_1 \cap \overline{S_2} \cap \overline{S_3}) \cup (\overline{S_1} \cap S_2 \cap \overline{S_3}) \cup (\overline{S_1} \cap \overline{S_2} \cap S_3)\right)$$
$$= P(S_1 \cap \overline{S_2} \cap \overline{S_3}) + P(\overline{S_1} \cap S_2 \cap \overline{S_3}) + P(\overline{S_1} \cap \overline{S_2} \cap S_3)$$
$$= P(S_1)\, P(\overline{S_2})\, P(\overline{S_3}) + P(\overline{S_1})\, P(S_2)\, P(\overline{S_3}) + P(\overline{S_1})\, P(\overline{S_2})\, P(S_3)$$
$$= \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{75}{216}$$

$$P(X=2) = P\left((S_1 \cap S_2 \cap \overline{S_3}) \cap (\overline{S_1} \cap S_2 \cap S_3) \cap (S_1 \cap \overline{S_2} \cap S_3)\right)$$
$$= P(S_1 \cap S_2 \cap \overline{S_3}) + P(\overline{S_1} \cap S_2 \cap S_3) + P(S_1 \cap \overline{S_2} \cap S_3)$$
$$= P(S_1)\, P(S_2)\, P(\overline{S_3}) + P(\overline{S_1})\, P(S_2)\, P(S_3) + P(S_1)\, P(\overline{S_2})\, P(S_3)$$
$$= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{15}{216}$$

and,
$$P(X=3) = P(S_1 \cap S_2 \cap S_3) = P(S_1)\, P(S_2)\, P(S_3) = \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216}$$

Thus, the probability distribution of X is given by

X:	0	1	2	3
$P(X)$:	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

Computation of mean and variance

x_i	$P(X = x_i) = p_i$	$p_i x_i$	$p_i x_i^2$
0	$\frac{125}{216}$	0	0
1	$\frac{75}{216}$	$\frac{75}{216}$	$\frac{75}{216}$
2	$\frac{15}{216}$	$\frac{30}{216}$	$\frac{60}{216}$
3	$\frac{1}{216}$	$\frac{3}{216}$	$\frac{9}{216}$
		$\Sigma p_i x_i = \frac{108}{216} = \frac{1}{2}$	$\Sigma p_i x_i^2 = \frac{144}{216}$

Thus, we have $\Sigma p_i x_i = \frac{108}{216} = \frac{1}{2}$ and $\Sigma p_i x_i^2 = \frac{144}{216}$

$$\therefore \quad \text{Mean} = \overline{X} = \Sigma p_i x_i = \frac{1}{2}$$

and, $\quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \dfrac{144}{216} - \left(\dfrac{1}{2}\right)^2 = \dfrac{90}{216} = \dfrac{5}{12}$

Hence, Mean $= \dfrac{1}{2}$ and Variance $= \dfrac{5}{12}$

EXAMPLE 19 *A die is tossed twice. A "success" is "getting an odd number" on a random toss. Find the variance of the number of successes.*

SOLUTION Let X denote the number of successes in two tosses of a die. Then, X can take values 0, 1, 2.

Let S_i and F_i denote the success and failure respectively in i^{th} toss. Then,

$$P(S_i) = \text{Probability of getting an odd number in } i^{th} \text{ toss} = \frac{3}{6} = \frac{1}{2}$$

and, $\quad P(F_i) = \text{Probability of not getting an odd number in } i^{th} \text{ toss} = \left(1 - \dfrac{1}{2}\right) = \dfrac{1}{2}$

Now, $\quad P(X=0) = $ Probability of getting no success in two tosses of a die

$$= P(F_1 \cap F_2)$$

$$= P(F_1)\,P(F_2) \qquad \text{[By Multiplication Theorem]}$$

$$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \qquad \left[\because P(F_1) = P(F_2) = \frac{1}{2}\right]$$

$P(X=1) = $ Probability of getting one success in two tosses of a die

$$= P\Big((S_1 \cap F_2) \cup (F_1 \cap S_2)\Big)$$

$$= P(S_1 \cap F_2) + P(F_1 \cap S_2) = P(S_1)\,P(F_2) + P(F_1)\,P(S_2) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}.$$

and, $\quad P(X=2) = $ Probability of getting two successes in two tosses of a die

$$= P(S_1 \cap S_2) = P(S_1)\,P(S_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Thus, the probability distribution of X i.e. the number of successes in two tosses of a die, is given by

X :	0	1	2
P(X) :	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Computation of variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
0	$\frac{1}{4}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{4}$	$\frac{1}{2}$	1
		$\Sigma p_i x_i = 1$	$\Sigma p_i x_i^2 = \frac{3}{2}$

Thus, we have $\Sigma p_i x_i = 1$ and $\Sigma p_i x_i^2 = \dfrac{3}{2}$

$$\therefore \quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{3}{2} - 1 = \frac{1}{2}$$

EXAMPLE 20 *Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Compute the variance of the number of aces.* **[CBSE 2010, 2019]**

SOLUTION Let A_i denote the event of getting an ace in ith draw, where $i = 1, 2$. Further, let X denote the number of aces in two draws. Then, X can take values 0, 1, 2.

Now, $P(X=0)$ = Probability of getting no ace in two successive draws

$$\Rightarrow \quad P(X=0) = P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1})\, P(\overline{A_2}/\overline{A_1}) = \frac{48}{52} \times \frac{47}{51} = \frac{564}{663}$$

$P(X=1)$ = Probability of getting an ace in one of the two draws

$$= P\left((A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2)\right)$$

$$= P(A_1 \cap \overline{A_2}) + P(\overline{A_1} \cap A_2)$$

$$= P(A_1)\, P(\overline{A_2}/A_1) + P(\overline{A_1})\, P(A_2/\overline{A_1}) = \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} = \frac{96}{663}$$

$P(X=2)$ = Probability of getting an ace in each draw

$$= P(A_1 \cap A_2) = P(A_1)\, P(A_2/A_1) = \frac{4}{52} \times \frac{3}{51} = \frac{3}{663}$$

Thus, the probability distribution of X is given by

X:	0	1	2
$P(X)$:	$\frac{564}{663}$	$\frac{96}{663}$	$\frac{3}{663}$

$$\therefore \quad \Sigma p_i x_i = 0 \times \frac{564}{663} + 1 \times \frac{96}{663} + 2 \times \frac{3}{663} = \frac{102}{663}$$

$$\text{and,} \quad \Sigma p_i x_i^2 = \frac{564}{663} \times 0 + \frac{96}{663} \times 1 + \frac{3}{663} \times 4 = \frac{108}{663}$$

$$\text{Hence,} \quad \text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{108}{663} - \left(\frac{102}{663}\right)^2 = \frac{108 \times 663 - (102)^2}{(663)^2} = \frac{61200}{663 \times 663} = \frac{400}{2873}$$

EXAMPLE 21 *From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. If the items in the sample are drawn one by one without replacement, find*

(i) *The probability distribution of X* (ii) *Mean of X* (iii) *Variance of X*

SOLUTION (i) Clearly, X can assume values 0, 1, 2, 3 such that

$$P(X = 0) = \frac{{}^7C_4}{{}^{10}C_4} = \frac{1}{6}, \qquad P(X = 1) = \frac{{}^3C_1 \times {}^7C_3}{{}^{10}C_4} = \frac{1}{2}$$

$$P(X = 2) = \frac{{}^3C_2 \times {}^7C_2}{{}^{10}C_4} = \frac{3}{10}, \text{ and } P(X = 3) = \frac{{}^3C_3 \times {}^7C_1}{{}^{10}C_4} = \frac{1}{30}.$$

So, the probability distribution of X is as given below.

X:	0	1	2	3
$P(X)$:	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Computation of mean and variance

x_i	$P(X = x_i) = p_i$	$p_i\, x_i$	$p_i\, x_i^{\,2}$
0	$\frac{1}{6}$	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{6}{5}$
3	$\frac{1}{30}$	$\frac{1}{10}$	$\frac{3}{10}$
		$\Sigma\, p_i\, x_i = \frac{12}{10}$	$\Sigma\, p_i\, x_i^{\,2} = 2$

Thus, we have $\Sigma\, p_i\, x_i = \frac{12}{10} = \frac{6}{5}$ and $\Sigma\, p_i\, x_i^{\,2} = 2$

$\therefore \quad \overline{X} = \text{Mean} = \Sigma\, p_i\, x_i = \frac{6}{5}$

and, $\quad \text{Var}(X) = \Sigma\, p_i\, x_i^{\,2} - (\Sigma\, p_i\, x_i)^2 = 2 - \left(\frac{6}{5}\right)^2 = 2 - \frac{36}{25} = \frac{14}{25}$

Hence, Mean $= \frac{6}{5}$ and Variance $= \frac{14}{25}$.

EXAMPLE 22 *In a game, a person is paid ₹ 5 if he gets all heads or all tails when three coins are tossed, and he will pay ₹ 3 if either one or two heads show. What can he expect to win on the average per game?*

SOLUTION Let X be the amount received by the person. Then, X can take values 5 and -3 such that

$P(X = 5)$ = Probability of getting all heads or all tails when three coins are tossed $= \frac{2}{8} = \frac{1}{4}$

$P(X = -3)$ = Probability of getting one or two heads $= \frac{6}{8} = \frac{3}{4}$

$\therefore$ Expected amount to win, on the average, per game $= \overline{X} = \Sigma\, p_i\, x_i = 5 \times \frac{1}{4} + -3 \times \frac{3}{4} = -1$

Thus, the person will, on the average, lose ₹ 1 per toss of the coins.

EXAMPLE 23 *Let X denote the number of vowels in word selected at random from this sentence. Find the expected value and standard deviation of the random variable X. (Consider X as a word with one letter).*

SOLUTION There are 12 words in the following sentence.

"Find the expected value and standard deviation of the random variable X".

Clearly, X can take values 0, 1, 2, 3, 4, 5 such that

$P(X = 0) = P$ (Selecting a word containing no vowel) $= P$ (Selecting X) $= \frac{1}{12}$

$P(X = 1) = P$ (Selecting a word containing one vowel)

$\Rightarrow \quad P(X = 1) = P$ (Selecting a word from the words 'The', 'Find', 'and', of 'The') $= \frac{5}{12}$

$P(X = 2) = P$ (Selecting a word containing two vowels)

$= P$ (Selecting a word from the words 'standard', 'random') $= \frac{2}{12}$

$P(X=3) = P$ (Selecting a word containing three vowels)

$= P$ (Selecting a word from the word 'expected', 'value') $= \dfrac{2}{12}$

$P(X=4) = P$ (Selecting a word containing four vowels)

$= P$ (Selecting the word 'variable') $= \dfrac{1}{12}$

$P(X=5) = P$ (Selecting a word containing five vowels)

$= P$ (Selecting the word 'deviation') $= \dfrac{1}{12}$

So, the probability distribution of X is as given below:

X:	0	1	2	3	4	5
$P(X)$:	$\dfrac{1}{12}$	$\dfrac{5}{12}$	$\dfrac{2}{12}$	$\dfrac{2}{12}$	$\dfrac{1}{12}$	$\dfrac{1}{12}$

$$\therefore \quad E(X) = 0 \times \frac{1}{12} + 1 \times \frac{5}{12} + 2 \times \frac{2}{12} + 3 \times \frac{2}{12} + 4 \times \frac{1}{12} + 5 \times \frac{1}{12} = \frac{0+5+4+6+4+5}{12} = 2.$$

$$E(X^2) = 0^2 \times \frac{1}{12} + 1^2 \times \frac{5}{12} + 2^2 \times \frac{2}{12} + 3^2 \times \frac{2}{12} + 4^2 \times \frac{1}{12} + 5^2 \times \frac{1}{12}$$

$$\Rightarrow \quad E(X^2) = \frac{0+5+8+18+16+25}{12} = \frac{72}{12} = 6$$

$$\therefore \quad \text{Var}(X) = E(X^2) - [E(X)]^2 = 6 - 2^2 = 2.$$

$$\Rightarrow \quad \text{Standard deviation of } X = \sqrt{\text{Var}(X)} = \sqrt{2}$$

Hence, $E(X) = 2$ and Standard deviation $= \sqrt{2}$.

EXAMPLE 24 *In a game a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.* **[NCERT]**

SOLUTION The man may get six in the first throw and then he quits the game. He may get a number other than six in the first throw and in the second throw he may get six and quits the game. In the first two throws he gets a number other than six and in third throw he may get a six. He may not get six in any one of three throws.

Let X be the amount he wins/looses. Then, X can take values 1, 0, −1, − 3 such that

$P(X=1) = P$ (Getting six in first throw) $= \dfrac{1}{6}$

$P(X=0) = P$ (Getting an other number in first throw and six in second throw)

$$= \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$$

$P(X=-1) = P$ (Getting numbers other than 6 in first two throws and a six in third throw)

$$= \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$$

$P(X=-3) = P$ (Getting a number other than six in first three throw) $= \dfrac{5}{6} \times \dfrac{5}{6} \times \dfrac{5}{6} = \dfrac{125}{216}$

Thus, the probability distribution of X is as given below:

X:	1	0	−1	−3
$P(X)$:	$\dfrac{1}{6}$	$\dfrac{5}{36}$	$\dfrac{25}{216}$	$\dfrac{125}{216}$

$$\therefore \quad E(X) = 1 \times \frac{1}{6} + 0 \times \frac{5}{36} + (-1) \times \frac{25}{216} + (-3) \times \frac{125}{216} = \frac{36 + 0 - 25 - 375}{216} = -\frac{364}{216} = -\frac{91}{54}$$

EXAMPLE 25 *In a group of 30 scientists working on an experiment, 20 never commit error in their work and are reporting results elaborately. Two scientists are selected at random from the group. Find the probability distribution of the number of selected scientists who never commit error in the work and reporting. Also, find the mean of the distribution. What values are described in the question?*

[CBSE 2013]

SOLUTION Let X denote the number of selected scientists who never commit error in the work and reporting. Clearly, X can take values 0, 1, 2.

Now,

$P(X=0)$ = Probability that two scientists selected commit error either in the work or in reporting

$$= \frac{^{10}C_2}{^{30}C_2} = \frac{3}{29}$$

$P(X=1)$ = Probability that one out of two scientists selected does not commit error in the work and reporting while the other is not so

$$= \frac{^{20}C_1 \times {^{10}C_1}}{^{30}C_2} = \frac{40}{87}$$

$P(X=2)$ = Probability that two scientists selected do not commit error in the work and reporting

$$= \frac{^{20}C_2}{^{30}C_2} = \frac{38}{87}$$

The probability distribution of X is as given below:

X:	0	1	2
$P(X)$:	$\frac{3}{29}$	$\frac{40}{87}$	$\frac{38}{87}$

Let $\overline{X}$ be the mean of the distribution. Then,

$$\overline{X} = 0 \times \frac{3}{29} + 1 \times \frac{40}{87} + 2 \times \frac{38}{87} = \frac{116}{87} = 1.33$$

This means that on an average out of two selected scientists one scientists will not commit error in the work and reporting.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 26 *Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the mean and standard deviation of the number of aces.* **[CBSE 2012, NCERT EXEMPLAR]**

SOLUTION Let A_i $(i = 1, 2)$ denote the event of getting an ace in ith draw. Since the cards are drawn with replacement. Therefore,

$$P(A_i) = \text{Probability of getting an ace in } i^{\text{th}} \text{ draw} = \frac{^4C_1}{^{52}C_1} = \frac{4}{52} = \frac{1}{13}$$

$$\text{and,} \quad P(\overline{A_i}) = 1 - P(A_i) = 1 - \frac{1}{13} = \frac{12}{13}, \quad i = 1, 2.$$

Let X denote the number of aces in two draws. Then, X can take values 0, 1, 2.

Now, $P(X=0)$ = Probability of getting no ace in two draws

$$\Rightarrow \quad P(X=0) = P(\overline{A_1} \cap \overline{A_2}) = P(\overline{A_1})\, P(\overline{A_2}) = \frac{12}{13} \times \frac{12}{13} = \frac{144}{169}$$

$P(X=1)$ = Probability of getting an ace in either of the two draws

$$\Rightarrow \quad P(X=1) = P\left((A_1 \cap \overline{A_2}) \cup (\overline{A_1} \cap A_2)\right)$$

$$\Rightarrow \quad P(X=1) = P(A_1 \cap \overline{A_2}) + P(\overline{A_1} \cap A_2)$$

$$\Rightarrow \quad P(X=1) = P(A_1)\,P(\overline{A_2}) + P(\overline{A_1})\,P(A_2) = \frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13} = \frac{24}{169}$$

and, $P(X=2)$ = Probability of getting ace in each draw

$$\Rightarrow \quad P(X=2) = P(A_1 \cap A_2) = P(A_1)\,P(A_2) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

Thus, the probability distribution of X is given by

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$$\therefore \quad \Sigma p_i x_i = 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 2 \times \frac{1}{169} = \frac{26}{169}$$

and, $$\Sigma p_i x_i^2 = 0 \times \frac{144}{169} + 1 \times \frac{24}{169} + 4 \times \frac{1}{169} = \frac{28}{169}$$

Hence, $$\overline{X} = \text{Mean} = \Sigma p_i x_i = \frac{26}{169} = \frac{2}{13}$$

and, $$\text{Var}(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = \frac{28}{169} - \left(\frac{2}{13}\right)^2 = \frac{24}{169}$$

$$\therefore \quad \text{S.D.} = \sqrt{\text{Var}(X)} = \sqrt{\frac{24}{169}} = \frac{2\sqrt{6}}{13}$$

Hence, Mean $= \frac{2}{13}$ and S.D. $= \frac{2\sqrt{6}}{13}$.

EXAMPLE 27 *A coin weighted so that $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$ is tossed three times. Let X be the random variable which denotes the longer string of heads which occurs. Find the probability distribution, mean and variance of X.*

SOLUTION The random variable X is defined on the sample space S given by

$$S = \{TTT, HTT, THT, TTH, THH, HTH, HHT, HHH\}$$

Note that the string of heads means the sequence of consecutive heads.

Since X denotes the longest string of heads. Therefore,

$$X(TTT) = 0, X(THT) = 1, X(HTT) = 1, X(TTH) = 1, X(HTH) = 1,$$
$$X(HHT) = 2, X(THH) = 2 \text{ and } X(HHH) = 3.$$

Now, $$P(X=0) = P(TTT) = P(T)\,P(T)\,P(T) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$$

$$P(X=1) = P(THT \cup HTT \cup TTH \cup HTH)$$

$$\Rightarrow \quad P(X=1) = P(THT) + P(HTT) + P(TTH) + P(HTH)$$

$$\Rightarrow \quad P(X=1) = P(T)\,P(H)\,P(T) + P(H)\,P(T)\,P(T) + P(T)\,P(T)\,P(H) + P(H)\,P(T)\,P(H)$$

$$\Rightarrow \quad P(X=1) = 3\left(\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4}\right) + \frac{3}{4} \times \frac{1}{4} + \frac{3}{4} = \frac{18}{64}$$

$$P(X=2) = P(THH \cup HHT)$$

$$\Rightarrow \quad P(X=2) = P(THH) + P(HHT)$$

$$\Rightarrow \quad P(X=2) = P(T)\,P(H)\,P(H) + P(H)\,P(H)\,P(T) = 2\left(\frac{1}{4} \times \frac{3}{4} \times \frac{3}{4}\right) = \frac{18}{64}$$

and, $P(X=3) = P(HHH) = P(H)\,P(H)\,P(H) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

So, the probability distribution is as follows:

x_i :	0	1	2	3
p_i :	$\frac{1}{64}$	$\frac{18}{64}$	$\frac{18}{64}$	$\frac{27}{64}$

Calculation of mean and variance

x_i	p_i	$p_i\,x_i$	$p_i\,x_i^2$
0	$\frac{1}{64}$	0	0
1	$\frac{18}{64}$	$\frac{18}{64}$	$\frac{18}{64}$
2	$\frac{18}{64}$	$\frac{36}{64}$	$\frac{72}{64}$
3	$\frac{27}{64}$	$\frac{81}{64}$	$\frac{243}{64}$
		$\Sigma\,p_i\,x_i = \frac{135}{64}$	$\Sigma\,p_i\,x_i^2 = \frac{333}{64}$

Thus, we have $\Sigma\,p_i\,x_i = \frac{135}{64}$ and $\Sigma\,p_i\,x_i^2 = \frac{333}{64}$

$\therefore$ Mean $= \Sigma\,p_i\,x_i = \frac{135}{64} = 2.1$

and, Variance $= \Sigma\,p_i\,x_i^2 - (\text{Mean})^2 = \frac{333}{64} - (2.1)^2 = 5.2 - 4.41 = 0.79$

Hence, Mean = 21 and Variance = 0.79.

EXAMPLE 28 *A fair coin is tossed until a head or five tails occur. If X denotes the number of tosses of the coin, find the mean of X.*

SOLUTION The sample space related to the given random experiment is given by

$$S = \{H, TH, TTH, TTTH, TTTTH, TTTTT\}$$

Clearly, X assumes values 1, 2, 3, 4, 5 such that

$$P(X=1) = P(H) = \frac{1}{2}$$

$$P(X=2) = P(TH) = P(T)\,P(H) = \frac{1}{4}$$

$$P(X=3) = P(TTH) = P(T)\,P(T)\,P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$P(X=4) = P(TTTH) = P(T)\,P(T)\,P(T)\,P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$$

and, $P(X=5) = P(TTTTH \cup TTTTT)$

$= P(TTTTH) + P(TTTTT)$

$= P(T)\,P(T)\,P(T)\,P(T)\,P(H) + P(T)\,P(T)\,P(T)\,P(T)\,P(T) = \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5 = \frac{1}{16}$

So, the probability distribution of X is given by

x_i :	1	2	3	4	5
p_i :	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

$\therefore$ Mean $= \Sigma p_i x_i = \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{8} \times 3 + \frac{1}{16} \times 4 + \frac{1}{16} \times 5 = \frac{31}{16} = 1.9.$

EXAMPLE 29 *There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance.*

SOLUTION Clearly, X can take values from 3 to 9 such that

$P(X = 3) =$ Probability of getting 3 as the sum

$\Rightarrow$ $P(X = 3) = P$ {(Getting 1 in first draw and 2 in second draw) or (Getting 2 in first draw and 1 in second draw)}

$$\Rightarrow \quad P(X = 3) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}$$

$P(X = 4) = P$ {(Getting 1 in first draw and 3 in second draw) or (Getting 3 in first draw and 1 in second draw)}

$$\Rightarrow \quad P(X = 4) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}$$

$$P(X = 5) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{5}$$

$$P(X = 6) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{5}$$

$$P(X = 7) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{5}$$

$$P(X = 8) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{10} \text{ and } P(X = 9) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{1}{10}$$

Thus, the probability distribution of X is as given below:

X:	3	4	5	6	7	8	9
$P(X)$:	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

Computation of mean and variance

x_i	$p_i = P(X = x_i)$	$p_i x_i$	$p_i x_i^2$
3	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{9}{10}$
4	$\frac{1}{10}$	$\frac{4}{10}$	$\frac{16}{10}$
5	$\frac{1}{5}$	$\frac{5}{5}$	$\frac{25}{5}$
6	$\frac{1}{5}$	$\frac{6}{5}$	$\frac{36}{5}$
7	$\frac{1}{5}$	$\frac{7}{5}$	$\frac{49}{5}$
8	$\frac{1}{10}$	$\frac{8}{10}$	$\frac{64}{10}$
9	$\frac{1}{10}$	$\frac{9}{10}$	$\frac{81}{10}$
Total	$\Sigma p_i = 1$	$\Sigma p_i x_i = \frac{60}{10} = 6$	$\Sigma p_i x_i^2 = \frac{390}{10} = 39$

Thus, we have, $\Sigma\, p_i\, x_i = 6$ and $\Sigma p_i\, x_i^{\,2} = 39$

$\therefore \quad \overline{X} = \text{Mean} = \Sigma\, p_i\, x_i = 6$ and, $\text{Var}(X) = \Sigma\, p_i\, x_i^{\,2} - (\Sigma\, p_i\, x_i)^2 = 39 - 6^2 = 3$

EXAMPLE 30 *A biased die is such that $P(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seens', find the variance of the random variable X.*

[NCERT EXEMPLAR]

SOLUTION It is given that $P(4) = \frac{1}{10}$ and other scores being equally likely. So, let $P(1) = P(2) = P(3) = P(5) = P(6) = p$.

$$\therefore \quad P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1 \Rightarrow 5p + \frac{1}{10} = 1 \Rightarrow p = \frac{9}{50}$$

Thus, $P(1) = P(2) = P(3) = P(5) = P(6) = \frac{9}{50}$ and $P(4) = \frac{1}{10}$

When die is tossed twice, there may be no four or one of the two throws may result in a four or both the throws produce 4. So, X can take values 0, 1 and 2.

Now,

$$P(X=0) = \text{Probability of not getting a four in both the throws}$$
$$= \left(1 - \frac{1}{10}\right)\left(1 - \frac{1}{10}\right) = \frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$$

$$P(X=1) = \text{Probability of getting a four in one of the two throws}$$
$$= \frac{1}{10} \times \left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10}\right) \times \frac{1}{10} = \frac{1}{10} \times \frac{9}{10} + \frac{9}{10} \times \frac{1}{10} = \frac{18}{100}$$

$$P(X=2) = \text{Probability of getting a four in both the throws} = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100}$$

Thus, the probability distribution of X is as given below.

$$\begin{array}{llll} X: & 0 & 1 & 2 \\ P(X): & \frac{81}{100} & \frac{18}{100} & \frac{1}{100} \end{array}$$

Calculation of variance

x_i	p_i	$p_i\, x_i$	$p_i\, x_i^{\,2}$
0	$\frac{81}{100}$	0	0
1	$\frac{18}{100}$	$\frac{18}{100}$	$\frac{18}{100}$
2	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{4}{100}$
		$\Sigma\, p_i\, x_i = \frac{20}{100}$	$\Sigma\, p_i\, x_i^{\,2} = \frac{22}{100}$

Thus, we have

$$\Sigma\, p_i\, x_i = \frac{20}{100} = \frac{1}{5} \text{ and } \Sigma\, p_i\, x_i^{\,2} = \frac{22}{100} = \frac{11}{50}$$

$$\therefore \quad \text{Var}(X) = \Sigma\, p_i\, x_i^2 - (\Sigma p_i\, x_i)^2 = \frac{11}{50} - \left(\frac{1}{5}\right)^2 = \frac{9}{50} = 0.18$$

EXAMPLE 31 *Find the probability distribution of the maximum of two scores obtained when a die is thrown twice. Determine also the mean of the distribution.* **[NCERT EXEMPLAR]**

SOLUTION The sample space associated to the random experiment of throwing a die twice consists of 36 elementary events and is given by

$$S = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), (2, 2), (2, 6), \ldots, (6, 1), (6, 2), \ldots, (6, 6)\}$$

Let X denote the maximum of the two numbers obtained in two throws of a die. Then, X can take values 1, 2, 3, 4, 5 and 6.

$$P(X=1) = P(1, 1) = \frac{1}{36}$$

$$P(X=2) = P((1, 2)\,(2, 1), (2, 2)) = \frac{3}{36}$$

$$P(X=3) = P((1, 3)\,(3, 1), (2, 3), (3, 2), (3, 3)) = \frac{5}{36}$$

$$P(X=4) = P((1, 4)\,(4, 1), (2, 4), (4, 2), (3, 4), (4, 3), (4, 4)) = \frac{7}{36}$$

$$P(X=5) = P((1, 5)\,(5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4), (5, 5)) = \frac{9}{36}$$

$$P(X=6) = P((1, 6)\,(2, 6), (3, 6), (4, 6), (5, 6), (6, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)) = \frac{11}{36}$$

So, the probability distribution of X is as given below:

X :	1	2	3	4	5	6
$P(X)$:	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\therefore \quad \overline{X} = 1 \times \frac{1}{36} + 2 \times \frac{3}{36} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{9}{36} + 6 \times \frac{11}{36} = \frac{161}{36}$$

EXAMPLE 32 *There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.*

SOLUTION The sum X of the numbers on two cards drawn without replacement can take values 3, 4, 5, 6, 7, 8, 9.

The sum can be three if one of the cards drawn bears number 1 and other bears number 2.

$$P(X=3) = P\big((1, 2), (2, 1)\big) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{2}{20}$$

Similarly,

$$P(X=4) = P((1, 3), (3, 1)) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{2}{20}$$

$$P(X=5) = P((1, 4), (4, 1), (2, 3), (3, 2)) = \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{4} = \frac{4}{20}$$

$$P(X=6) = P((1, 5), (5, 1), (2, 4), (4, 2)) = \frac{4}{20}$$

$$P(X=7) = P((2, 5), (5, 2), (3, 4), (4, 3)) = \frac{4}{20}$$

$$P(X=8) = P\big((3, 5), (5, 3)\big) = \frac{2}{20}$$

$$P(X=9) = P\big((4, 5), (5, 4)\big) = \frac{2}{20}$$

Thus, the probability distribution of X is as given below.

X :	3	4	5	6	7	8	9
$P(X)$:	$\frac{2}{20}$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{4}{20}$	$\frac{2}{20}$	$\frac{2}{20}$

Calculation of variance

x_i	p_i	$p_i x_i$	$p_i x_i^2$
3	$\frac{2}{20}$	$\frac{6}{20}$	$\frac{18}{20}$
4	$\frac{2}{20}$	$\frac{8}{20}$	$\frac{32}{20}$
5	$\frac{4}{20}$	$\frac{20}{20}$	$\frac{100}{20}$
6	$\frac{4}{20}$	$\frac{24}{20}$	$\frac{144}{20}$
7	$\frac{4}{20}$	$\frac{28}{20}$	$\frac{196}{20}$
8	$\frac{2}{20}$	$\frac{16}{20}$	$\frac{128}{20}$
9	$\frac{2}{20}$	$\frac{18}{20}$	$\frac{162}{20}$
		$\Sigma p_i x_i = \frac{120}{20} = 6$	$\Sigma p_i x_i^2 = \frac{780}{20} = 39$

Thus, we have

$$\Sigma p_i x_i = 6 \text{ and } \Sigma p_i x_i^2 = 39$$

$\therefore$ Mean $= \Sigma p_i x_i = 6$ and, Variance $= \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2 = 39 - 36 = 3$

EXERCISE 31.2

BASIC

1. Find the mean and standard deviation of each of the following probability distributions:

(i) x_i : 2 3 4 **[NCERT EXEMPLAR]**
p_i : 0.2 0.5 0.3

(ii) x_i : 1 3 4 5
p_i : 0.4 0.1 0.2 0.3

(iii) x_i : -5 -4 1 2
p_i : 1/4 1/8 1/2 1/8

(iv) x_i : -1 0 1 2 3
p_i : 0.3 0.1 0.1 0.3 0.2

(v) x_i : 1 2 3 4
p_i : 0.4 0.3 0.2 0.1

(vi) x_i : 0 1 3 5
p_i : 0.2 0.5 0.2 0.1

(vii) x_i : -2 -1 0 1 2
p_i : 0.1 0.2 0.4 0.2 0.1

(viii) x_i : -3 -1 0 1 3
p_i : 0.05 0.45 0.20 0.25 0.05

(ix) x_i : 0 1 2 3 4 5 **[NCERT EXEMPLAR]**
p_i : $\frac{1}{6}$ $\frac{5}{18}$ $\frac{2}{9}$ $\frac{1}{6}$ $\frac{1}{9}$ $\frac{1}{18}$

2. A discrete random variable X has the probability distribution given below:

X : 0.5 1 1.5 2
$P(X)$: k k^2 $2k^2$ k

(i) Find the value of k. (ii) Determine the mean of the distribution. **[NCERT EXEMPLAR]**

3. Find the mean variance and standard deviation of the following probability distribution

x_i : a b
p_i : p q

where $p + q = 1$.

4. Find the mean and variance of the number of tails in three tosses of a coin. **[NCERT EXEMPLAR]**

5. Two cards are drawn simultaneously from a pack of 52 cards. Compute the mean and standard deviation of the number of kings. **[CBSE 2008, 2019]**
6. Find the mean, variance and standard deviation of the number of tails in three tosses of a coin.
7. Two bad eggs are accidently mixed up with ten good ones. Three eggs are drawn at random with replacement from this lot. Compute the mean for the number of bad eggs drawn.
8. A pair of fair dice is thrown. Let X be the random variable which denotes the minimum of the two numbers which appear. Find the probability distribution, mean and variance of X.
9. A fair coin is tossed four times. Let X denote the number of heads occurring. Find the probability distribution, mean and variance of X.
10. A fair die is tossed. Let X denote twice the number appearing. Find the probability distribution, mean and variance of X.
11. A fair die is tossed. Let X denote 1 or 3 according as an odd or an even number appears. Find the probability distribution, mean and variance of X.
12. A fair coin is tossed four times. Let X denote the longest string of heads occurring. Find the probability distribution, mean and variance of X.
13. Two cards are selected at random from a box which contains five cards numbered 1, 1, 2, 2, and 3. Let X denote the sum and Y the maximum of the two numbers drawn. Find the probability distribution, mean and variance of X and Y.
14. A die is tossed twice. A 'success' is getting an odd number on a toss. Find the variance of the number of successes.

BASED ON LOTS

15. A box contains 13 bulbs, out of which 5 are defective. 3 bulbs are randomly drawn, one by one without replacement, from the box. Find the probability distribution of the number of defective bulbs. **[CBSE 2005]**
16. In roulette, Fig. 31.2, the wheel has 13 numbers 0, 1, 2, ..., 12 marked on equally spaced slots. A player sets ₹ 10 on a given number. He receives ₹ 100 from the organiser of the game if the ball comes to rest in this slot; otherwise he gets nothing. If X denotes the player's net gain/loss, find $E(X)$.

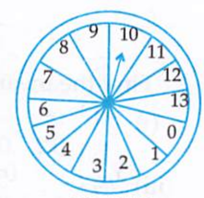

Fig. 31.2

17. Three cards are drawn at random (without replacement) from a well shuffled pack of 52 cards. Find the probability distribution of number of red cards. Hence find the mean of the distribution. **[CBSE 2014]**
18. An urn contains 5 red 2 black balls. Two balls are randomly drawn, without replacement. Let X represent the number of black balls drawn. What are the possible values of X? Is X a random variable? If yes, find the mean and variance of X. **[CBSE 2015]**
19. Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6 and 7. Let X denote the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X. **[CBSE 2015]**
20. In a game, a man wins ₹ 5 for getting a number greater than 4 and loses ₹ 1 otherwise, when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a number greater than 4. Find the expected value of the amount he wins/lose. **[CBSE 2016]**
21. Find the probability distribution of the random variable X, which denotes the number of doublets in four throws of a pair of dice. Hence, find the mean of the number of doublets (X). **[CBSE 2020]**
22. A discrete random variable has the following probability distribution

X :	0	1	2	3	4	5
$P(X)$:	$4c^2$	$3c^2$	$2c^2$	c^2	c	$2c$

(i) Find the value of c (ii) Find the mean of the distribution

(iii) Given $\Sigma p_i X_i^2 = 14$, find the variance of the distribution. **[CBSE 2020]**

ANSWERS

1. (i) 3.1, 0.7 (ii) 3, 1.7 (iii) – 1, 2.9 (iv) 1, 1.5

(v) Mean = 2, S.D = 1 (vi) Mean = 1.6, S.D = 1.497

(vii) Mean = 0, S.D = 1.095 (viii) Mean = – 0.2, S.D = 1.249

(viii) Mean $= \dfrac{35}{18}$ S.D. $= \dfrac{\sqrt{665}}{18}$ (ix) Mean $= \dfrac{35}{18}$, S.D. $= \dfrac{\sqrt{665}}{18}$

2. (i) $k = \dfrac{1}{3}$ (ii) $\overline{X} = \dfrac{23}{18}$ **3.** Mean $= ap + bq$ Var $= pq(a-b)^2$ $\sigma = |a-b|\sqrt{pq}$

4. Mean = 1.5, Var = 3/4 **5.** Mean $= \dfrac{34}{221}$, Var $= \dfrac{400}{2873}$

6. Mean = 1.5, Var $= \dfrac{3}{4}$, S.D. = 0.87 **7.** $\dfrac{1}{2}$

8.

x_i:	1	2	3	4	5	6	Mean = 2.5
p_i:	11/36	9/36	7/36	5/36	3/36	1/36	Var = 2.1

9.

x_i :	0	1	2	3	4	Mean = 2
p_i :	1/16	4/16	6/16	4/16	1/16	Var = 1

10.

x_i :	2	4	6	8	10	12	Mean = 7
p_i :	1/6	1/6	1/6	1/6	1/6	1/6	Var = 11.7

11.

x_i :	1	3	Mean = 2
p_i :	1/2	1/2	Var = 1

12.

x_i :	0	1	2	3	4	Mean = 1.7
p_i :	1/16	7/16	5/16	2/16	1/16	Var = 0.9

13.

x_i :	2	3	4	5	Mean = 3.6
p_i :	0.1	0.4	0.3	0.2	Var = 0.84
y_i :	1	2	3		Mean = 2.3
p_i :	0.1	0.5	0.4		Var = 0.41

14. 1/2

15.

X :	0	1	2	3
P (X) :	$\dfrac{28}{143}$	$\dfrac{70}{143}$	$\dfrac{40}{143}$	$\dfrac{5}{143}$

16. $E(X) = -\dfrac{30}{13}$

17.

X :	0	1	2	3
P (X) :	$\dfrac{703}{1700}$	$\dfrac{741}{1700}$	$\dfrac{117}{850}$	$\dfrac{11}{850}$;

Mean $= \dfrac{997}{1700}$

18. 0, 1, 2, ; Yes ; Mean $= \dfrac{4}{7}$, Var (X) $= \dfrac{50}{147}$ **19.** Mean $= \dfrac{17}{3}$, Var (X) $= \dfrac{14}{9}$ **20.** ₹ $\dfrac{19}{9}$

21.

X:	0	1	2	3	4
$P(X)$:	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{1296}$

; Mean $=\frac{2}{3}$

22. (i) $\frac{1}{5}$ (ii) $\frac{16}{5}$ (iii) $\frac{94}{125}$

HINTS TO SELECTED PROBLEM

16. The probability distribution of X is

X:	-10	90
$P(X)$:	$\frac{12}{13}$	$\frac{1}{13}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If X is a random variable with the following probability distribution :

X:	x_1	x_2	....	x_n
$P(X)$:	p_1	p_2	....	p_n

Then, Mean(X) =......................... . [NCERT EXEMPLAR]

2. In Q. No. 1, Var (X) = ...

3. A discrete random variable X has the following probability distribution :

X:	1	2	3	4	5	6	7
$P(X)$:	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2+c$

Then, $P(X \leq 2)$ = ...

4. If a discrete random variable X has the following probability distribution:

X:	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$P(X)$:	c	c^2	$2c^2$	c

The, c = ...

5. If a random variable X has the following probability distribution :

X:	0	1	2	3
$P(X)$:	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Then, $E(X^2)$ = ...

ANSWERS

1. $\sum_{i=1}^{n} p_i x_i$ 2. $\sum_{i=1}^{n} p_i x_i^2 - \left(\sum p_i x_i\right)^2$ 3. $\frac{3}{10}$ 4. $\frac{1}{3}$ 5. $\frac{14}{5}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the values of *'a'* for which the following distribution of probabilities becomes a probability distribution:

$X = x_i$:	-2	-1	0	1
$P(X = x_i)$:	$\frac{1-a}{4}$	$\frac{1+2a}{4}$	$\frac{1-2a}{4}$	$\frac{1+a}{4}$

2. For what value of k the following distribution is a probability distribution?

$X = x_i$:	0	1	2	3
$P(X = x_i)$:	$2k^4$	$3k^2 - 5k^3$	$2k - 3k^2$	$3k - 1$

3. If X denotes the number on the upper face of a cubical die when it is thrown, find the mean of X.
4. If the probability distribution of a random variable X is given by

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	$2k$	$4k$	$3k$	k

Write the value of k.

5. Find the mean of the following probability distribution:

$X = x_i$:	1	2	3
$P(X = x_i)$:	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{5}{8}$

6. If the probability distribution of a random variable X is as given below:

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	c	$2c$	$3c$	$4c$

Write the value of $P(X \le 2)$.

7. A random variable has the following probability distribution:

$X = x_i$:	1	2	3	4
$P(X = x_i)$:	k	$2k$	$3k$	$4k$

Write the value of $P(X \ge 3)$.

ANSWERS

1. $-\frac{1}{2} \le a \le \frac{1}{2}$ 2. $k = \frac{1}{2}$ 3. 3.5 4. $k = 0.1$ 5. $\frac{19}{8}$ 6. 0.3 7. $\frac{7}{10}$