CHAPTER 5

DETERMINANTS

5.1 DETERMINANTS

DEFINITION *Every square matrix can be associated to an expression or a number which is known as its determinant. If $A=[a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or, $|A|$ or,*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}$$

5.1.1 DETERMINANT OF A SQUARE MATRIX OF ORDER 1

If $A=[a_{11}]$ is a square matrix of order 1, then the determinant of A is defined as

$$|A| = a_{11} \quad or, \quad |a_{11}| = a_{11}$$

5.1.2 DETERMINANT OF A SQUARE MATRIX OF ORDER 2

If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then the expression $a_{11}\,a_{22}-a_{12}\,a_{21}$ is defined as the determinant of A.

i.e. $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{12}\,a_{21} \quad \text{...(i)}$$

Thus, the determinant of a square matrix of order 2 is equal to the product of the diagonal elements minus the product of off-diagonal elements.

ILLUSTRATION *Evaluate:*

(i) $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$ (iii) $\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix}$

(iv) $\begin{vmatrix} x^2+xy+y^2 & x+y \\ x^2-xy+y^2 & x-y \end{vmatrix}$ (v) $\begin{vmatrix} 1 & \log_b a \\ \log_a b & 1 \end{vmatrix}$

SOLUTION By definition, we obtain

(i) $$\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix} = 5\times 3 - 4\times -2 = 15+8 = 23$$

(ii) $$\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = \sin^2\theta - (-\cos^2\theta) = \sin^2\theta + \cos^2\theta = 1.$$

(iii) $$\begin{vmatrix} x-1 & 1 \\ x^3 & x^2+x+1 \end{vmatrix} = (x-1)(x^2+x+1) - x^3 = (x^3-1) - x^3 = -1.$$

(iv) $$\begin{vmatrix} x^2+xy+y^2 & x+y \\ x^2-xy+y^2 & x-y \end{vmatrix} = (x^2+xy+y^2)(x-y) - (x^2-xy+y^2)(x+y)$$

$$= (x^3 - y^3) - (x^3 + y^3) = -2y^3$$

(v) $\begin{vmatrix} 1 & \log_a b \\ \log_a b & 1 \end{vmatrix} = 1 - \log_b a \times \log_a b = 1 - 1 = 0$ $\left[\because \log_b a = \dfrac{1}{\log_a b}\right]$

5.1.3 DETERMINANT OF A SQUARE MATRIX OF ORDER 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then the expression

$$a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{32} a_{21} - a_{11} a_{23} a_{32} - a_{22} a_{13} a_{31} - a_{12} a_{21} a_{33}$$

is defined as the determinant of A

i.e. $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{32} a_{21} - a_{11} a_{23} a_{32} - a_{22} a_{31} a_{13} - a_{33} a_{12} a_{21} \quad \text{...(ii)}$$

or, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$\Rightarrow$ $|A| = a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{12}(a_{33} a_{21} - a_{23} a_{31}) + a_{13}(a_{32} a_{21} - a_{22} a_{31})$

$\Rightarrow$ $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ [Using notation given in 5.1.2]

$\Rightarrow$ $|A| = (-1)^{1+1} a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Thus the determinant of a square matrix of order 3 is the sum of the product of elements a_{1j} in first row with $(-1)^{1+j}$ times the determinant of a 2×2 sub-matrix obtained by leaving the first row and column passing through the element.

The above expansion of $|A|$ is known as the expansion along first row. For example, if $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ is a square matrix of order 3, then

$$|A| = \begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix} = (-1)^{1+1} \times 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + (-1)^{1+3} 4\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-2-1) + 2(-1-0) + 4(1-0) = -9 - 2 + 4 = -7$$

There are three rows and three columns in a square matrix of order 3. The expression (ii) for the determinant of a square matrix of order 3 can be arranged in various forms to obtain the expansion of $|A|$ along any of its rows or columns. Infact, to expand $|A|$ about a row or a column we multiply each element a_{ij} in i^{th} row with $(-1)^{i+j}$ times the determinant of the sub-matrix obtained by leaving the row and column passing through the element.

For example,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = (-1)^{2+1} a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+3} a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

is the expansion of $|A|$ about second row.

The expansion of $|A|$ about 2nd column is given as

$$|A| = (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{2+2} a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{3+2} a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

ILLUSTRATION 1 *Evaluate* $\Delta = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ *by expanding it along the second row.*

SOLUTION By using the definition, of expansion along second row, we obtain

$$\Delta = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = (-1)^{2+1}(1) \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + (-1)^{2+2}(2) \begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} + (-1)^{2+3}(3) \begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + 2 \begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} - 3 \begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix} = -(-9+2) + 2(-6-4) - 3(2+6) = -37.$$

ILLUSTRATION 2 *Evaluate the determinant* $D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ *by expanding it along first column.*

SOLUTION By using the definition, of expansion along first column, we obtain

$$D = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = (-1)^{1+1}(2) \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} + (-1)^{2+1}(1) \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + (-1)^{3+1}(-2) \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix} = 2(-6-3) - (-9+2) - 2(9+4) = -37.$$

NOTE 1 *Only square matrices have their determinants. The matrices which are not square do not have determinants.*

NOTE 2 *The determinant of a square matrix of order 3 can be expanded along any row or column.*

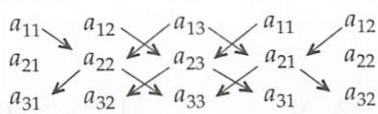
NOTE 3 *If a row or a column of a determinant consists of all zeros, then the value of the determinant is zero.*

5.1.4 DETERMINANT OF A SQUARE MATRIX OF ORDER 3 BY USING SARRUS DIAGRAM

The determinant of a square matrix of order 3 can be evaluated by the following procedure:

Consider the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ of the square matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

In order to find the value of the determinant, we first enlarge the determinant by adjoining the first two columns on the right and draw broken lines parallel and perpendicular to the diagonal as shown below.



The value of the determinant is the sum of the products of elements in lines parallel to the diagonal minus the sum of the product of elements in lines perpendicular to the diagonal.

i.e. $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$

NOTE *This method does not work for determinants of order more than 3.*

ILLUSTRATION 1 *Evaluate* $\Delta = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 2 \\ 1 & -1 & 5 \end{vmatrix}$ *by using Sarrus diagram.*

SOLUTION First we enlarge the determinant by adjoining the first two columns on the right and then draw the broken lines parallel and perpendicular to the diagonal as shown below.

$$\begin{array}{ccccc} 2 & -1 & 3 & 2 & -1 \\ 4 & 1 & 2 & 4 & 1 \\ 1 & -1 & 5 & 1 & -1 \end{array}$$

To find the value of Δ, we find the sum of the products of elements in lines parallel to the diagonal and subtract from it the sum of the products of elements in lines perpendicular to them as given below.

$$\Delta = [2 \times 1 \times 5 + (-1) \times 2 \times 1 + 3 \times 4 \times (-1)] - [3 \times 1 \times 1 + 2 \times 2 \times (-1) + (-1) \times 4 \times 5]$$
$$= [10 - 2 - 12] - [3 - 4 - 20] = (-4) - (-21) = 17.$$

ILLUSTRATION 2 *Evaluate* $\Delta = \begin{vmatrix} -1 & 6 & -2 \\ 2 & 1 & 1 \\ 4 & 1 & -3 \end{vmatrix}$ *by two methods.*

SOLUTION We have,

$$\Delta = -1 \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} - 6 \begin{vmatrix} 2 & 1 \\ 4 & -3 \end{vmatrix} + (-2) \begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} \qquad \text{[Expanding along first row]}$$
$$= -(-3 - 1) - 6(-6 - 4) - 2(2 - 4) = 4 + 60 + 4 = 68$$

To find Δ by a Sarrus diagram, first enlarge the determinant by adjoining the first two columns on the right and then draw the broken lines parallel and perpendicular to the diagonal as shown below.

$$\begin{array}{ccccc} -1 & 6 & -2 & -1 & 6 \\ 2 & 1 & 1 & 4 & 1 \\ 4 & 1 & -3 & 4 & 1 \end{array}$$

Now, we find the sum of the products of elements in lines parallel to the diagonal and subtract from it the sum of the products of elements in lines perpendicular to them as given below.

$$\Delta = [-1 \times 1 \times -3 + 6 \times 1 \times 4 + -2 \times 2 \times 1] - [-2 \times 1 \times 4 + -1 \times 1 \times 1 + 6 \times 2 \times -3]$$
$$= (3 + 24 - 4) - (-8 - 1 - 36) = 68.$$

5.1.5 DETERMINANT OF A SQUARE MATRIX OF ORDER 4 OR MORE

To evaluate the determinant of a square matrix of order 4 or more we follow the same procedure as discussed in evaluating the determinant of a square matrix of order 3.

For example,

$$\Delta = \begin{vmatrix} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ 3 & 1 & 2 & 1 \\ 1 & -1 & 0 & 2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (-1)^{1+1}(1) \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ -1 & 0 & 2 \end{vmatrix} + (-1)^{1+2}(2) \begin{vmatrix} 2 & -2 & 3 \\ 3 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} + (-1)^{1+3}(-1) \begin{vmatrix} 2 & 1 & 3 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$
$$+ (-1)^{1+4}(3) \begin{vmatrix} 2 & 1 & -2 \\ 3 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 1(16) - 2(12) + (-1)(-11) - 3(14) = -39.$$

REMARK *It is evident from the above discussion that every square matrix $A = [a_{ij}]$ of order n can be associated to a number (real or complex) or an expression which is called determinant of the square matrix A. Thus, determinant may be thought as a function from the set M of all square matrices to the set of all numbers (real or complex).*

5.2 SINGULAR MATRIX

DEFINITION *A square matrix is a singular matrix if its determinant is zero. Otherwise, it is a non-singular matrix.*

ILLUSTRATION 1 *For what value of x the matrix* $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ *is singular?*

SOLUTION The matrix A is singular, if

$$|A| = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow 1\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ x & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ x & 2 \end{vmatrix} = 0$$

$$\Rightarrow (-6-2) + 2(-3-x) + 3(2-2x) = 0 \Rightarrow -8-6-2x+6-6x = 0 \Rightarrow -8x-8 = 0 \Rightarrow x = -1.$$

ILLUSTRATION 2 *Determine the values of x for which the matrix* $A = \begin{bmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{bmatrix}$ *is singular.*

SOLUTION Given matrix A is a singular matrix, if

$$|A| = 0$$

$$\Rightarrow \begin{vmatrix} x+1 & -3 & 4 \\ -5 & x+2 & 2 \\ 4 & 1 & x-6 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)\begin{vmatrix} x+2 & 2 \\ 1 & x-6 \end{vmatrix} - (-3)\begin{vmatrix} -5 & 2 \\ 4 & x-6 \end{vmatrix} + 4\begin{vmatrix} -5 & x+2 \\ 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)\{(x+2)(x-6) - 2\} + 3\{-5x + 30 - 8\} + 4\{-5 - 4x - 8\} = 0$$

$$\Rightarrow (x+1)(x^2 - 4x - 14) + 3(-5x + 22) + 4(-4x - 13) = 0$$

$$\Rightarrow x(x^2 - 3x - 49) = 0 \Rightarrow x = 0,\ x = \frac{1}{2}(3 \pm \sqrt{205})$$

5.3 MINORS AND COFACTOR

MINOR *Let $A = [a_{ij}]$ be a square matrix of order n. The minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.*

For example, if $A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}$, then

M_{11} = Minor of $a_{11} = 2$, M_{12} = Minor of $a_{12} = -3$, M_{21} = Minor of $a_{21} = -7$, M_{22} = Minor of $a_{22} = 4$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{bmatrix}$, then

M_{11} = Minor of a_{11}
= Determinant of the 2×2 square sub-matrix obtained by leaving first row and first column of A

$$= \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 2.$$

Similarly, we obtain

$$M_{12} = \text{Minor of } a_{12} = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -7, \quad M_{13} = \text{Minor of } a_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 8$$

$$M_{21} = \text{Minor of } a_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = 18, \quad M_{22} = \text{Minor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3 \text{ etc.}$$

COFACTOR *Let $A = [a_{ij}]$ be a square matrix of order n. The cofactor C_{ij} of a_{ij} in A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order $(n-1)$ obtained by leaving i^{th} row and j^{th} column of A.*

It follows from this definition that

$$C_{ij} = \text{Cofactor of } a_{ij} \text{ in } A = (-1)^{i+j} M_{ij}, \quad \text{where } M_{ij} \text{ is minor of } a_{ij} \text{ in } A.$$

Thus, we have $C_{ij} = \begin{cases} M_{ij}, & \text{if } i+j \text{ is even} \\ -M_{ij}, & \text{if } i+j \text{ is odd} \end{cases}$

For example, if $A = \begin{bmatrix} 4 & -7 \\ -3 & 2 \end{bmatrix}$, then

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 2, \quad C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-3) = 3,$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = -(-7) = 7, \quad \text{and} \quad C_{22} = (-1)^{2+2} M_{22} = 4$$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{bmatrix}$, then

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 2,$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -(-9+2) = 7$$

$$C_{13} = (-1)^{1+3} M_{13} = M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 8, C_{23} = (-1)^{2+3} M_{23} = -M_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = 8 \text{ etc.}$$

REMARK *In some books the minors and cofactors are defined for the elements of a determinant. Infact, minors and cofactors are defined for the elements of a square matrix.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$, find the determinant of the matrix $A^2 - 2A$.*

SOLUTION We have, $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$$\therefore \quad A^2 - 2A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^2 - 2A = \begin{bmatrix} 1+6 & 3+3 \\ 2+2 & 6+1 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7-2 & 6-6 \\ 4-4 & 7-2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\therefore \quad |A^2 - 2A| = \begin{vmatrix} 5 & 0 \\ 0 & 5 \end{vmatrix} = 25 - 0 = 25.$$

EXAMPLE 2 *If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$.*

[NCERT]

SOLUTION We have,

$$A=\begin{bmatrix}1 & 2\\ 4 & 2\end{bmatrix} \Rightarrow 2A=\begin{bmatrix}2 & 4\\ 8 & 4\end{bmatrix}$$

$$\therefore \quad |A|=\begin{vmatrix}1 & 2\\ 4 & 2\end{vmatrix}=2-8=-6 \text{ and } |2A|=\begin{vmatrix}2 & 4\\ 8 & 4\end{vmatrix}=8-32=-24=4\times(-6)$$

Clearly, $|2A|=4|A|$.

EXAMPLE 3 *If* $\begin{vmatrix}x-2 & -3\\ 3x & 2x\end{vmatrix}=3$, *find the values of x.*

SOLUTION We have,

$$\begin{vmatrix}x-2 & -3\\ 3x & 2x\end{vmatrix}=3$$

$$\Rightarrow \quad (x-2)\times 2x-(-3)\times 3x=3 \Rightarrow 2x(x-2)+9x=3 \Rightarrow 2x^2-4x+9x=3$$

$$\Rightarrow \quad 2x^2+5x-3=0 \Rightarrow (2x-1)(x+3)=0 \Rightarrow 2x-1=0 \text{ or, } x+3=0 \Rightarrow x=\frac{1}{2},-3.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Let* $\begin{vmatrix}3 & y\\ x & 1\end{vmatrix}=\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}$. *Find possible values of x and y if x, y are natural numbers.*

SOLUTION We have,

$$\begin{vmatrix}3 & y\\ x & 1\end{vmatrix}=\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}$$

$$\Rightarrow \quad 3-xy=3-8 \Rightarrow xy=8 \Rightarrow x=1,\ y=8;\ x=2,\ y=4;\ x=4,\ y=2;\ x=8,\ y=1$$

EXAMPLE 5 *Evaluate the determinant* $\Delta=\begin{vmatrix}\log_3 512 & \log_4 3\\ \log_3 8 & \log_4 9\end{vmatrix}$.

SOLUTION We have,

$$\Delta=\begin{vmatrix}\log_3 512 & \log_4 3\\ \log_3 8 & \log_4 9\end{vmatrix}=\begin{vmatrix}\log_3 2^9 & \log_{2^2} 3\\ \log_3 2^3 & \log_{2^2} 3^2\end{vmatrix}$$

$$\Rightarrow \quad \Delta=\begin{vmatrix}9\log_3 2 & \frac{1}{2}\log_2 3\\ 3\log_3 2 & \frac{2}{2}\log_2 3\end{vmatrix} \qquad \left[\because \log_{a^p} m^n=\frac{n}{p}\log_a m\right]$$

$$\Rightarrow \quad \Delta=\begin{vmatrix}9\log_3 2 & \frac{1}{2}\log_2 3\\ 3\log_3 2 & \log_2 3\end{vmatrix}=(9\log_3 2)\times(\log_2 3)-\left(\frac{1}{2}\log_2 3\right)(3\log_3 2)$$

$$\Rightarrow \quad \Delta=9(\log_3 2\times\log_2 3)-\frac{3}{2}(\log_2 3\times\log_3 2)=9-\frac{3}{2}=\frac{15}{2} \qquad [\because \log_b a\times\log_a b=1]$$

EXAMPLE 6 *Find the minors and cofactors of elements of the matrix* $A=[a_{ij}]=\begin{bmatrix}1 & 3 & -2\\ 4 & -5 & 6\\ 3 & 5 & 2\end{bmatrix}$.

SOLUTION Let M_{ij} and C_{ij} denote respectively the minor and cofactor of element a_{ij} in A. Then,

$$M_{11}=\begin{vmatrix}-5 & 6\\ 5 & 2\end{vmatrix}=-10-30=-40 \Rightarrow C_{11}=M_{11}=-40$$

$$M_{12}=\begin{vmatrix}4 & 6\\ 3 & 2\end{vmatrix}=8-18=-10 \Rightarrow C_{12}=-M_{12}=10$$

$$M_{13} = \begin{vmatrix} 4 & -5 \\ 3 & 5 \end{vmatrix} = 20 + 15 = 35 \Rightarrow C_{13} = M_{13} = 35$$

$$M_{21} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 6 + 10 = 16 \Rightarrow C_{21} = -M_{21} = -16$$

$$M_{22} = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 2 + 6 = 8 \Rightarrow C_{22} = M_{22} = 8$$

$$M_{23} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 5 - 9 = -4 \Rightarrow C_{23} = -M_{23} = 4$$

$$M_{31} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 18 - 10 = 8 \Rightarrow C_{31} = M_{31} = 8$$

$$M_{32} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 6 + 8 = 14 \Rightarrow C_{32} = -M_{32} = -14$$

$$M_{33} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = -5 - 12 = -17 \Rightarrow C_{33} = M_{33} = -17$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Evaluate the determinant* $\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$. *Also, prove that* $2 \le \Delta \le 4$

SOLUTION We have,

$$\Delta = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

On expanding along first row, we obtain

$$\Delta = 1 \times \begin{vmatrix} 1 & \sin\theta \\ -\sin\theta & 1 \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & \sin\theta \\ -1 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} -\sin\theta & 1 \\ -1 & -\sin\theta \end{vmatrix}$$

$$= 1 \times (1 + \sin^2\theta) - \sin\theta(-\sin\theta + \sin\theta) + 1 \times (\sin^2\theta + 1)$$

$$= (1 + \sin^2\theta) - 0 + (\sin^2\theta + 1) = 2 + 2\sin^2\theta = 2(1 + \sin^2\theta)$$

We know that

$-1 \le \sin\theta \le 1$ for all θ

$\Rightarrow \quad 0 \le \sin^2\theta \le 1$ for all θ

$\Rightarrow \quad 1 + 0 \le 1 + \sin^2\theta \le 1 + 1$ for all θ

$\Rightarrow \quad 1 \le 1 + \sin^2\theta \le 2$ for all $\theta \Rightarrow 2 \le 2(1 + \sin^2\theta) \le 4$ for all $\theta \Rightarrow 2 \le \Delta \le 4$. for all θ

EXAMPLE 8 *If* $[\cdot]$ *denotes the greatest integer less than or equal to the real number under consideration, and* $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, *then find the value of the following determinant:*

$$\Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$$

SOLUTION We have, $-1 \le x < 0$, $0 \le y < 1$ and $1 \le z < 2 \Rightarrow [x] = -1$, $[y] = 0$ and $[z] = 1$.

$$\therefore \quad \Delta = \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 0 \times \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - 0 \times \begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} \qquad \text{[Expanding along first row]}$$

$$\Rightarrow \quad \Delta = 0\,(2-0) - 0\,(-2+1) + 1 \times (0+1) = 1.$$

EXAMPLE 9 *Prove that the determinant* $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ *is independent of* θ. **[NCERT]**

SOLUTION We have,

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = x \begin{vmatrix} -x & 1 \\ 1 & x \end{vmatrix} - \sin\theta \begin{vmatrix} -\sin\theta & 1 \\ \cos\theta & x \end{vmatrix} + \cos\theta \begin{vmatrix} -\sin\theta & -x \\ \cos\theta & 1 \end{vmatrix} \qquad \text{[Expanding along first row]}$$

$$\Rightarrow \quad \Delta = x(-x^2-1) - \sin\theta(-x\sin\theta - \cos\theta) + \cos\theta(-\sin\theta + x\cos\theta)$$

$$\Rightarrow \quad \Delta = -x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$\Rightarrow \quad \Delta = -x^3 - x + x(\sin^2\theta + \cos^2\theta) = -x^3 - x + x = -x^3,$$ which is independent of θ.

EXERCISE 5.1

BASIC

1. Write the minors and cofactors of each element of the first column of the following matrices and hence evaluate the determinant in each case:

   (i) $A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$ (iii) $A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$

   (iv) $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ (v) $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$

2. Evaluate the following determinants:

   (i) $\begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$ (ii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (iii) $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$ (iv) $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

   [CBSE 2008]

3. Evaluate: $\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}^2$.

4. Show that $\begin{vmatrix} \sin 10^\circ & -\cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1$

5. Evaluate $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ by two methods.

BASED ON LOTS

6. Evaluate: $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$ **[NCERT]**

7. Evaluate: $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$ **[NCERT]**

8. If $A=\begin{bmatrix}2 & 5\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & -3\\ 2 & 5\end{bmatrix}$, verify that $|AB|=|A||B|$.

9. If $A=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 0 & 4\end{bmatrix}$, then show that $|3A| = 27|A|$.

10. Find the values of x, if

(i) $\begin{vmatrix}2 & 4\\ 5 & 1\end{vmatrix}=\begin{vmatrix}2x & 4\\ 6 & x\end{vmatrix}$ [NCERT] (ii) $\begin{vmatrix}3 & x\\ x & 1\end{vmatrix}=\begin{vmatrix}3 & 2\\ 4 & 1\end{vmatrix}$ [NCERT]

(iii) $\begin{vmatrix}x+1 & x-1\\ x-3 & x+2\end{vmatrix}=\begin{vmatrix}4 & -1\\ 1 & 3\end{vmatrix}$ [CBSE 2013] (iv) $\begin{vmatrix}2x & 5\\ 8 & x\end{vmatrix}=\begin{vmatrix}6 & 5\\ 8 & 3\end{vmatrix}$ [NCERT EXEMPLAR]

11. Find the integral value of x, if $\begin{vmatrix}x^2 & x & 1\\ 0 & 2 & 1\\ 3 & 1 & 4\end{vmatrix} = 28$.

12. For what value of x the matrix A is singular?

(i) $A=\begin{bmatrix}1+x & 7\\ 3-x & 8\end{bmatrix}$ [CBSE 2012] (ii) $A=\begin{bmatrix}x-1 & 1 & 1\\ 1 & x-1 & 1\\ 1 & 1 & x-1\end{bmatrix}$

ANSWERS

1. *Minors* / *Cofactors*

(i) $M_{11}=-1, M_{21}=20$; $C_{11}=-1, C_{21}=-20$

(ii) $M_{11}=-12, M_{21}=-16, M_{31}=-4,$; $C_{11}=-12, C_{21}=16, C_{31}=-4$

(iii) $M_{11}=a(b^2-c^2), M_{21}=b(a^2-c^2),$ $M_{31}=c(a^2-b^2)$; $C_{11}=a(b^2-c^2), C_{21}=-b(a^2-c^2),$ $C_{31}=c(a^2-b^2)$

(iv) $M_{11}=bc-f^2, M_{21}=hc-fg, M_{31}=hf-bg$; $C_{11}=bc-f^2, C_{21}=fg-ch, C_{31}=hf-bg$

(v) $M_{11}=-9, M_{21}=9, M_{31}=-9, M_{41}=0$; $C_{11}=-9, C_{21}=-9, C_{31}=-9, C_{41}=0$

2. (i) $5x^2+8x$ (ii) 1 (iii) 0 (iv) $a^2+b^2+c^2+d^2$ 3. 0 5. -140

6. 0 7. 1 10. (i) $\pm\sqrt{3}$ (ii) $\pm 2\sqrt{2}$ (iii) 2 (iv) ± 3 11. 2 12. (i) $\frac{13}{15}$ (ii) $-1, 2$

HINTS TO SELECTED PROBLEMS

6. We have, $\Delta = \begin{vmatrix}0 & \sin\alpha & -\cos\alpha\\ -\sin\alpha & 0 & \sin\beta\\ \cos\alpha & -\sin\beta & 0\end{vmatrix}$

On expanding along first row, we get

$$\Delta=0\begin{vmatrix}0 & \sin\beta\\ -\sin\beta & 0\end{vmatrix}-\sin\alpha\begin{vmatrix}-\sin\alpha & \sin\beta\\ \cos\alpha & 0\end{vmatrix}-\cos\alpha\begin{vmatrix}-\sin\alpha & 0\\ \cos\alpha & -\sin\beta\end{vmatrix}$$

$\Rightarrow \Delta=-\sin\alpha(0-\sin\beta\cos\alpha)-\cos\alpha(\sin\alpha\sin\beta)=\sin\alpha\cos\alpha\sin\beta-\sin\alpha\cos\alpha\sin\beta=0$

On expanding along first row, we get

$$\Delta=0\begin{vmatrix}0 & \sin\beta\\ -\sin\beta & 0\end{vmatrix}-\sin\alpha\begin{vmatrix}-\sin\alpha & \sin\beta\\ \cos\alpha & 0\end{vmatrix}-\cos\alpha\begin{vmatrix}-\sin\alpha & 0\\ \cos\alpha & -\sin\beta\end{vmatrix}$$

$\Rightarrow \Delta = -\sin\alpha\,(0 - \sin\beta\cos\alpha) - \cos\alpha\,(\sin\alpha\sin\beta) = \sin\alpha\cos\alpha\sin\beta - \sin\alpha\cos\alpha\sin\beta = 0$

7. We have, $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

On expanding along first row, we get

$$\Delta = \cos\alpha\cos\beta \begin{vmatrix} \cos\beta & 0 \\ \sin\alpha\sin\beta & \cos\alpha \end{vmatrix} - \cos\alpha\sin\beta \begin{vmatrix} -\sin\beta & 0 \\ \sin\alpha\cos\beta & \cos\alpha \end{vmatrix} - \sin\alpha \begin{vmatrix} -\sin\beta & \cos\beta \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \cos\alpha\cos\beta\,(\cos\alpha\cos\beta - 0) - \cos\alpha\sin\beta\,(-\cos\alpha\sin\beta - 0) - \sin\alpha\,(-\sin\alpha\sin^2\beta - \sin\alpha\cos^2\beta)$$

$$\Rightarrow \quad \Delta = \cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha$$

$$\Rightarrow \quad \Delta = \cos^2\alpha\,(\cos^2\beta + \sin^2\beta) + \sin^2\alpha = \cos^2\alpha + \sin^2\alpha = 1$$

10. (i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} \Rightarrow 2 - 20 = 2x^2 - 24 \Rightarrow 2x^2 = 6 \Rightarrow x = \pm\sqrt{3}$

(ii) $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \Rightarrow 3 - x^2 = 3 - 8 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$

5.4 PROPERTIES OF DETERMINANTS

In section 5.1, we have defined the determinant of a square matrix of order 4 or less. Infact, these definitions are consequences of the general definition of the determinant of a square matrix of any order which needs so many advanced concepts. These concepts are beyond the scope of this book. Using the said definition and some other advanced concepts we can prove the following properties. But, the concepts used in the definition itself are very advanced. Therefore we mention these properties and verify them for a determinant of a square matrix of order 3.

PROPERTY 1 *Let $A = [a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with their cofactors is always equal to $|A|$ or, det (A).*

i.e. $\sum_{j=1}^{n} a_{ij}\,C_{ij} = |A|$ and, $\sum_{i=1}^{n} a_{ij}\,C_{ij} = |A|$.

VERIFICATION Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3. Then, by definition, we obtain

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow \quad |A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

[Expanding along first row]

$\Rightarrow |A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ [By using the definition of cofactors]

Similarly, we obtain

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23},\ |A| = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33},$$

$$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \quad \text{etc.}$$

PROPERTY 2 *Let $A=[a_{ij}]$ be a square matrix of order n, then the sum of the product of elements of any row (column) with the cofactors of the corresponding elements of some other row (column) is zero.*

i.e. $\sum\limits_{j=1}^{n} a_{ij}\, C_{kj} = 0 \quad$ *and,* $\quad \sum\limits_{i=1}^{n} a_{ij}\, C_{ik} = 0.$

VERIFICATION Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3. Then, the sum of the product of elements of first row with the cofactors of elements in second row is given by

$$\begin{aligned}
& a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} \\
&= a_{11}(-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{2+2}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\
&= -a_{11}(a_{12}a_{33} - a_{13}a_{32}) + a_{12}(a_{11}a_{33} - a_{13}a_{31}) - a_{13}(a_{11}a_{32} - a_{12}a_{31}) \\
&= 0
\end{aligned}$$

Similarly, we obtain

$$a_{11} C_{31} + a_{12} C_{32} + a_{13} C_{33} = 0, \quad a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13} = 0 \text{ etc.}$$

PROPERTY 3 *Let $A=[a_{ij}]$ be a square matrix of order n, then $|A| = |A^T|$.*

By the abuse of language this property is also stated as follows:

The value of a determinant remains unchanged if its rows and columns are interchanged.

VERIFICATION Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a square matrix of order 3. Then, $A^T=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$.

Now, $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\Rightarrow \quad |A| = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

[Expanding along first row]

$\Rightarrow \quad |A| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ...(i)

and, $|A^T| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Rightarrow \quad |A^T| = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$

[Expanding along first column]

$\Rightarrow \quad |A^T| = a_1(b_2c_3 - b_3c_2) - b_1\cdot(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ...(ii)

From (i) and (ii), we obtain $|A| = |A^T|$.

PROPERTY 4 *Let $A=[a_{ij}]$ be a square matrix of order $n\ (\geq 2)$ and let B be a matrix obtained from A by interchanging any two rows (columns) of A, then $|B| = -|A|$.*

Conventionally this property is also stated as:

If any two rows (columns) of a determinant are interchanged, then the value of the determinant changes by minus sign only.

VERIFICATION Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a square matrix of order 3 and let B be the matrix obtained from A by interchanging first and third row i.e. $B = \begin{bmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{bmatrix}$.

Then, $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + (-1)^{1+2} b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (-1)^{1+3} c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

[Expanding along first row]

$\Rightarrow \quad |A| = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$ ...(i)

and, $|B| = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = (-1)^{3+1} a_1 \begin{vmatrix} b_3 & c_3 \\ b_2 & c_2 \end{vmatrix} + (-1)^{3+2} b_1 \begin{vmatrix} a_3 & c_3 \\ a_2 & c_2 \end{vmatrix} + (-1)^{3+3} c_1 \begin{vmatrix} a_3 & b_3 \\ a_2 & b_2 \end{vmatrix}$

[Expanding along first row]

$\Rightarrow \quad |B| = -[a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)]$ ...(ii)

From (i) and (ii), we obtain : $|B| = -|A|$.

PROPERTY 5 *If any two rows (columns) of a square matrix $A = [a_{ij}]$ of order n (> 2) are identical, then its determinant is zero i.e. $|A| = 0$.*

Conventionally this property is stated as:

If any two rows or columns of a determinant are identical, then its value is zero.

VERIFICATION Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{bmatrix}$ be a matrix having first and third rows identical and let B be the matrix obtained from A by interchanging the first and third rows. Then, by property 4, we obtain

$|B| = -|A|$ ...(i)

But, $B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{bmatrix} = A$. Therefore, $|B| = |A|$ ...(ii)

From (i) and (ii), we obtain

$|A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0.$

PROPERTY 6 *Let $A = [a_{ij}]$ be a square matrix of order n, and let B be the matrix obtained from A by multiplying each element of a row (column) of A by a scalar k, then $|B| = k\,|A|$.*

Conventionally this property is also stated as:

If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

VERIFICATION Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a square matrix order 3, and let B be a matrix obtained from A by multiplying each element of second row by the same constant k, then

$$B = \begin{bmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Now,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \qquad \text{...(i)}$$

and, $$|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_2 & kb_2 & kc_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad |B| = a_1(kb_2c_3 - kb_3c_2) - b_1(ka_2c_3 - ka_3c_2) + c_1(ka_2b_3 - kb_2a_3)$$

[On expanding along first row]

$$\Rightarrow \quad |B| = k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)] \qquad \text{...(ii)}$$

From (i) and (ii), we obtain $|B| = k|A|$.

REMARK 1 *It follows from the above property that we can take out any common factor from any one row or any one column of a given determinant.*

REMARK 2 *Let $A = [a_{ij}]$ be a square matrix of order n, then $|kA| = k^n|A|$, because k is common from each row of kA.*

PROPERTY 7 *Let A be a square matrix such that each element of a row (column) of A is expressed as the sum of two or more terms. Then, the determinant of A can be expressed as the sum of the determinants of two or more matrices of the same order.*

Conventionally this property is also stated as:

If each element of a row (column) of a determinant is expressed as a sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants.

VERIFICATION Let $A = \begin{bmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a square matrix such that each element in first row of A is the sum of two elements. Then,

$$|A| = \begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad |A| = (a_1+\alpha_1)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - (b_1+\beta_1)\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + (c_1+\gamma_1)\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \begin{bmatrix}\text{Expanding along} \\ \text{first row}\end{bmatrix}$$

$$\Rightarrow \quad |A| = \left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\} + \left\{\alpha_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \beta_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + \gamma_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}\right\}$$

$$\Rightarrow \quad |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad |A| = |B| + |C|, \text{ where } B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } C = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

PROPERTY 8 *Let A be a square matrix and B be a matrix obtained from A by adding to a row (column) of A a scalar multiple of another row (column) of A, then $|B| = |A|$.*

This property is conventionally stated as:

If each element of a row (column) of a determinant is multiplied by the same constant and then added to the corresponding elements of some other row (column), then the value of the determinant remains same.

VERIFICATION Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a matrix and let $B = \begin{bmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{bmatrix}$ be the matrix obtained from A by multiplying the elements of second column by k and then adding them to the corresponding elements of first column. Then,

$$|B| = \begin{vmatrix} a_1 + kb_1 & b_1 & c_1 \\ a_2 + kb_2 & b_2 & c_2 \\ a_3 + kb_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad |B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kb_1 & b_1 & c_1 \\ kb_2 & b_2 & c_2 \\ kb_3 & b_3 & c_3 \end{vmatrix} \qquad \text{[Using property 7]}$$

$$\Rightarrow \quad |B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \qquad \text{[Using property 6]}$$

$$\Rightarrow \quad |B| = |A| + k\,0 = |A| \qquad \text{[Using property 5]}$$

PROPERTY 9 *Let A be a square matrix order $n\,(\geq 2)$ such that each element in a row (column) of A is zero, then $|A| = 0$.*

Conventionally this property is also stated as:

If each element of a row (column) of a determinant is zero, then its value is zero.

VERIFICATION Let $A = \begin{bmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a square matrix. Then,

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - 0 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + 0 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0. \qquad \begin{bmatrix} \text{Expanding along} \\ \text{first row} \end{bmatrix}$$

PROPERTY 10 *If $A = [a_{ij}]$ is a diagonal matrix of order $n\,(\geq 2)$, then $|A| = a_{11} \times a_{22} \times a_{33} \times \ldots \times a_{nn}$.*

PROPERTY 11 *If A and B are square matrices of the same order, then $|AB| = |A|\,|B|$.*

PROPERTY 12 *Let $A = [a_{ij}]$ be a square matrix of order n and let c_{ij} = cofactor of a_{ij} in A for $i, j = 1, 2, \ldots, n$. If $C = [c_{ij}]$ is the matrix of cofactors of elements in A, then $|C| = |A|^{n-1}$.*

VERIFICATION Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$, where C_{ij} = cofactor of a_{ij} in A. Then,

$$CA^T = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\Rightarrow \quad CA^T = \begin{bmatrix} a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} & a_{21}c_{11} + a_{22}c_{12} + a_{23}c_{13} & a_{31}c_{11} + a_{32}c_{12} + a_{13}c_{33} \\ a_{11}c_{21} + a_{12}c_{22} + a_{13}c_{23} & a_{21}c_{21} + a_{22}c_{22} + a_{23}c_{23} & a_{31}c_{21} + a_{32}c_{22} + a_{33}c_{23} \\ a_{11}c_{31} + a_{12}c_{32} + a_{13}c_{33} & a_{21}c_{31} + a_{22}c_{32} + a_{23}c_{33} & a_{31}c_{31} + a_{32}c_{32} + a_{33}c_{33} \end{bmatrix}$$

$$\Rightarrow \quad CA^T = \begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \qquad \text{[Using Properties 1 and 2]}$$

$$\Rightarrow \quad |CA^T| = \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix} = |A|^3$$

$$\Rightarrow \quad |C|\,|A^T| = |A|^3 \Rightarrow |C|\,|A| = |A|^3 \Rightarrow |C| = |A|^2 \qquad [\because |A^T| = |A|]$$

5.5 EVALUATION OF DETERMINANTS

If A is a square matrix of order 2, then its determinant can be easily found. But, to evaluate determinants of square matrices of higher orders, we should always try to introduce zeros at maximum number of places in a particular row (column) by using the properties given in section 5.4 and then we should expand the determinant along that row (column).

We shall be using the following notations to evaluate a determinant:

(i) R_i to denote i^{th} row.

(ii) $R_i \leftrightarrow R_j$ to denote the interchange of i^{th} and j^{th} rows.

(iii) $R_i \to R_i + \lambda R_j$ to denote the addition of λ times the elements of j^{th} row to the corresponding elements of i^{th} row.

(iv) $R_i(\lambda)$ to denote the multiplication of all elements of i^{th} row by λ.

Similar notations are used to denote column operations if R is replaced by C.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **DETERMINANTS IN WHICH TWO ROWS (COLUMNS) BECOME IDENTICAL BY APPLYING THE PROPERTIES OF DETERMINANTS**

EXAMPLE 1 *Without expanding evaluate the determinant* $\begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$.

SOLUTION Let $\Delta = \begin{vmatrix} 41 & 1 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$. Applying $C_1 \to C_1 + (-8)C_3$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

EXAMPLE 2 *If w is a complex cube root of unity. Show that* $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix} = 0$.

SOLUTION Let $\Delta = \begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+w+w^2 & w & w^2 \\ w+w^2+1 & w^2 & 1 \\ w^2+1+w & 1 & w \end{vmatrix} = \begin{vmatrix} 0 & w & w^2 \\ 0 & w^2 & 1 \\ 0 & 1 & w \end{vmatrix} \qquad [\because 1+w+w^2=0]$$

$$\Rightarrow \quad \Delta = 0 \qquad [\because C_1 \text{ consists of all zeros}]$$

EXAMPLE 3 *Show that* $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

SOLUTION Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$. Applying $C_2 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix} \qquad \text{[Taking out } a+b+c \text{ common from } C_2]$$

$$\Rightarrow \quad \Delta = (a+b+c) \times 0 = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

EXAMPLE 4 *Show that* $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}.$ **[NCERT, CBSE 2009]**

SOLUTION Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix} = 0 \qquad [\because C_1 \text{ consists of all zeros}]$$

EXAMPLE 5 *Show that* $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0.$ **[NCERT]**

SOLUTION Let $\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$. Applying $C_3 \to C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix} \qquad \text{[Taking out } ab+bc+ca \text{ common from } C_3]$$

$$\Rightarrow \quad \Delta = (ab+bc+ca) \times 0 = 0. \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 6 *Without expanding prove that :* $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$ **[NCERT]**

SOLUTION Let $\Delta = \begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$. Applying $R_1 \to R_1 + R_2$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} \qquad [\text{Taking out } (x+y+z) \text{ common from } R_1]$$

$$\Rightarrow \quad \Delta = (x+y+z) \times 0 = 0 \qquad [\because R_1 \text{ and } R_3 \text{ are identical}]$$

EXAMPLE 7 *Without expanding show that:* $\Delta = \begin{vmatrix} \text{cosec}^2\theta & \cot^2\theta & 1 \\ \cot^2\theta & \text{cosec}^2\theta & -1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$

[NCERT EXEMPLAR]

SOLUTION Applying $C_1 \to C_1 - C_2$, we obtain

$$\Delta = \begin{vmatrix} \text{cosec}^2\theta - \cot^2\theta & \cot^2\theta & 1 \\ \cot^2\theta - \text{cosec}^2\theta & \text{cosec}^2\theta & -1 \\ 42-40 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 1 & \cot^2\theta & 1 \\ -1 & \text{cosec}^2\theta & -1 \\ 2 & 40 & 2 \end{vmatrix} = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 8 *Find the value of the determinant* $\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$. **[CBSE 2009]**

SOLUTION Taking $3x$ common from R_3, we get

$$\Delta = 3x\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 3x \times 0 = 0 \qquad [\because R_1 \text{ and } R_3 \text{ are identical}]$$

EXAMPLE 9 *Without expanding show that* $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$

[NCERT EXEMPLAR, CBSE 2001 C]

SOLUTION Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$. Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ and $R_3 \to R_3(c)$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ba \\ ca^2b^2 & abc & ac+bc \end{vmatrix} \qquad \left[\because R_1, R_2\, R_3 \text{ are multiplied by } a, b \text{ and } c \text{ respectively, therefore we divide by } abc\right]$$

$$\Rightarrow \quad \Delta = \frac{1}{abc}(abc)^2\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ba \\ ab & 1 & ac+bc \end{vmatrix} \qquad [\text{Taking out } abc \text{ common from } C_1 \text{ and } C_2]$$

$$\Rightarrow \quad \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix} \qquad [\text{Applying } C_3 \to C_3 + C_1]$$

$$\Rightarrow \quad \Delta = abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix} \qquad \text{[Taking out } ab+bc+ca \text{ common from } C_3]$$

$$\Rightarrow \quad \Delta = abc\,(ab+bc+ca) \times 0 = 0 \qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 10 *Without expanding evaluate the determinant* $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

SOLUTION Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix} \qquad [\because \sin(A+B) = \sin A\cos B + \cos A\sin B]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \qquad [\text{Applying } C_3 \to C_3 - (\cos\delta)\,C_1 - (\sin\delta)\,C_2]$$

$$\Rightarrow \quad \Delta = 0 \qquad [\because C_3 \text{ consists of all zeros}]$$

EXAMPLE 11 *Without expanding evaluate the determinant* $\begin{vmatrix} (a^x+a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (a^y+a^{-y})^2 & (a^y-a^{-y})^2 & 1 \\ (a^z+a^{-z})^2 & (a^z-a^{-z})^2 & 1 \end{vmatrix}$, *where* $a, > 0$ *and* $x, y, z \in R$.

SOLUTION Let Δ be the given determinant. Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} (a^x+a^{-x})^2-(a^x-a^{-x})^2 & (a^x-a^{-x})^2 & 1 \\ (a^y+a^{-y})^2-(a^y-a^{-y})^2 & (a^y-a^{-y})^2 & 1 \\ (a^z+a^{-z})^2-(a^z-a^{-z})^2 & (a^z-a^{-z})^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 4 & (a^x-a^{-x})^2 & 1 \\ 4 & (a^y-a^{-y})^2 & 1 \\ 4 & (a^z-a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Using}: (a+b)^2-(a-b)^2 = 4ab]$$

$$\Rightarrow \quad \Delta = 4\begin{vmatrix} 1 & (a^x-a^{-x})^2 & 1 \\ 1 & (a^y-a^{-y})^2 & 1 \\ 1 & (a^z-a^{-z})^2 & 1 \end{vmatrix} \qquad [\text{Taking out 4 common from } C_1]$$

$$\Rightarrow \quad \Delta = 4 \times 0 = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 12 *If a, b, c are in A.P., find the value of* $\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$. **[NCERT]**

SOLUTION Applying $R_2 \to R_2\,(2)$, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 6y+10 & 12y+16 & 18y+2b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 2b-(a+c) \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$ Applying $R_2 \to R_2 - (R_1 + R_3)$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 0 & 0 & 0 \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$ [$\because$ a, b, c are in A.P. $\therefore$ $2b = a + c$]

$$\Rightarrow \quad \Delta = 0$$ [$\because$ R_2 consists of zeros only]

REMARK *The transformation $R_1 \to R_1 + R_3 - 2R_2$ can also be used to get the value of Δ.*

EXAMPLE 13 *Without expanding evaluate the determinant* $\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$.

SOLUTION Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$ [Applying $C_1 \to C_1 - 2C_2$ and $C_3 \to C_3 - 10C_2$]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + 3R_3$]

$$\Rightarrow \quad \Delta = 2\,(39)\begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix}$$ [Taking 2 common from R_1 and 39 common from R_2]

$$\Rightarrow \quad \Delta = 78 \times 0$$ [$\because$ R_1 and R_2 are identical]

Type II EVALUATING DETERMINANTS BY USING THE PROPERTIES OF DETERMINANTS AND PROVING IDENTITIES

EXAMPLE 14 *If* $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$ *and* $\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, *without expanding prove that* $\Delta_1 = \Delta_2$.

SOLUTION Applying $C_1 \to C_1(x)$, $C_2 \to C_2(y)$ and $C_3 \to C_3(z)$, we get

$$\Delta_2 = \frac{1}{xyz}\begin{vmatrix} x & y & z \\ xyz & xyz & xyz \\ x^2 & y^2 & z^2 \end{vmatrix} = \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$ [Taking xyz common from R_2]

$$\Rightarrow \quad \Delta_2 = -\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$ [Applying $R_2 \leftrightarrow R_1$]

$$\Rightarrow \quad \Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} = \Delta_1$$ [Applying $R_2 \leftrightarrow R_3$]

EXAMPLE 15 *Let* $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ *and* $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ yz & zx & xy \end{vmatrix}$, *then show that* $\Delta_1 = \Delta$.

SOLUTION We have,

$$\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{xyz}\begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 x, C_2 \to C_2 y, C_3 \to C_3 z]$$

$$\Rightarrow \quad \Delta_1 = \frac{xyz}{xyz}\begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} \qquad [\text{Taking } xyz \text{ common from } R_3]$$

$$\Rightarrow \quad \Delta_1 = \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta_1 = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta. \qquad [\text{Interchanging rows and columns}]$$

EXAMPLE 16 *If* $\Delta_1 = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ *and* $\Delta_2 = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}$, *without expanding or evaluating* Δ_1 *and* Δ_2, *show that* $\Delta_1 + \Delta_2 = 0$.

SOLUTION Taking -1 common from second row, we obtain

$$\Delta_2 = -\begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix}$$

$$\Rightarrow \quad \Delta_2 = \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} \qquad [\text{Taking } (-1) \text{ common from } C_2]$$

$$\Rightarrow \quad \Delta_2 = \begin{vmatrix} q & p & r \\ b & a & c \\ y & x & z \end{vmatrix} \qquad [\text{Interchanging row and columns}]$$

$$\Rightarrow \quad \Delta_2 = -\begin{vmatrix} p & q & r \\ a & b & c \\ x & y & z \end{vmatrix} \qquad [\text{Applying } C_2 \leftrightarrow C_1]$$

$$\Rightarrow \quad \Delta_2 = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \qquad [\text{Applying } R_1 \leftrightarrow R_2]$$

$$\Rightarrow \quad \Delta_2 = -\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \qquad [\text{Applying } R_2 \leftrightarrow R_3]$$

$\Rightarrow \quad \Delta_2 = -\Delta_1 \Rightarrow \Delta_1 + \Delta_2 = 0$

EXAMPLE 17 *If A is a skew-symmetric matrix of odd order n, then $|A| = O$.*

SOLUTION Since A is a skew-symmetric matrix. Therefore,

$$A^T = -A$$

$\Rightarrow \quad |A^T| = |-A|$

$\Rightarrow \quad |A^T| = (-1)^n |A|$ $\quad [\because |kA| = k^n |A|]$

$\Rightarrow \quad |A| = (-1)^n |A|$ $\quad [\because |A^T| = |A|]$

$\Rightarrow \quad |A| = -|A| \Rightarrow 2|A| = 0 \Rightarrow |A| = 0$ $\quad [\because n \text{ is odd}]$

Hence, the determinant of a skew-symmetric matrix of odd order is zero.

EXAMPLE 18 *Prove that:* $\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0.$

SOLUTION Let $\Delta = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix}$. Then,

$$\Delta = (-1)^3 \begin{vmatrix} 0 & -a & b \\ a & 0 & c \\ -b & -c & 0 \end{vmatrix} \quad \text{[Taking } (-1) \text{ common from each row]}$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} \quad \text{[Interchanging rows and columns]}$$

$\Rightarrow \quad \Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0.$

<u>ALITER</u> Clearly, $A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{bmatrix}$ is a skew-symmetric matrix of odd order. Therefore, $|A| = 0$.

EXAMPLE 19 *Without expanding or evaluating show that* $\begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix} = 0.$

[NCERT EXEMPLAR]

SOLUTION Let $\Delta = \begin{vmatrix} 0 & b-a & c-a \\ a-b & 0 & c-b \\ a-c & b-c & 0 \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} 0 & -(a-b) & -(a-c) \\ (a-b) & 0 & -(b-c) \\ (a-c) & (b-c) & 0 \end{vmatrix} = \begin{vmatrix} 0 & -(a-b) & -(a-c) \\ (a-b) & 0 & -(b-c) \\ (a-c) & (b-c) & 0 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (-1)^3 \begin{vmatrix} 0 & (a-b) & (a-c) \\ -(a-b) & 0 & (b-c) \\ -(a-c) & -(b-c) & 0 \end{vmatrix} \quad \text{[Taking } -1 \text{ common from each row]}$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} 0 & -(a-b) & -(a-c) \\ (a-b) & 0 & -(b-c) \\ (a-c) & (b-c) & 0 \end{vmatrix} \quad \text{[Interchanging rows and columns]}$$

$$\Rightarrow \quad \Delta = -\Delta \Rightarrow 2\Delta = 0 \Rightarrow \Delta = 0$$

ALITER Clearly, given determinant is the determinant of a skew-symmetric matrix of odd order. So, its value is zero.

EXAMPLE 20 *Without expanding, prove that* $\begin{vmatrix} a+bx & c+dx & p+qx \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix}$. [NCERT]

SOLUTION Applying $R_1 \to R_1 - x R_2$ to Δ, we get

$$\Delta = \begin{vmatrix} a(1-x^2) & c(1-x^2) & p(1-x^2) \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix} = (1-x^2)\begin{vmatrix} a & c & p \\ ax+b & cx+d & px+q \\ u & v & w \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1-x^2)\begin{vmatrix} a & c & p \\ b & d & q \\ u & v & w \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - xR_1]$$

EXAMPLE 21 *Prove that:* $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$. [NCERT, CBSE 2011]

SOLUTION Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$. Then,

$$\Delta = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad [\text{Taking } a, b \text{ and } c \text{ common from } R_1, R_2 \text{ and } R_3 \text{ respectively}]$$

$$\Rightarrow \quad \Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \left[\begin{array}{l}\text{Taking } a, b \text{ and } c \text{ common from } C_1, C_2 \text{ and } C_3 \\ \text{respectively}\end{array}\right]$$

$$\Rightarrow \quad \Delta = a^2b^2c^2\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 + C_1, C_3 \to C_3 + C_1]$$

$$\Rightarrow \quad \Delta = a^2b^2c^2 \times (-1) \times \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$\Rightarrow \quad \Delta = a^2b^2c^2(-1)(0-4) = 4a^2b^2c^2$$

EXAMPLE 22 *Prove that:* $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$. [NCERT]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$. Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = 1 \times \begin{vmatrix} x & 0 \\ 0 & y \end{vmatrix} - 0 \times \begin{vmatrix} 1 & 0 \\ 1 & y \end{vmatrix} + 0 \times \begin{vmatrix} 1 & x \\ 1 & 0 \end{vmatrix} = xy \qquad \text{[On expanding along } R_1\text{]}$$

EXAMPLE 23 *Evaluate:* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$. **[NCERT]**

SOLUTION Let Δ be the given determinant. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \quad \text{[Taking out } (b-a) \text{ common from } R_2 \text{ \& } (c-a) \text{ from } R_3\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix} \qquad \text{[Applying } R_3 \to R_3 - R_2\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & 1 \end{vmatrix} \qquad \text{[Taking out } (c-b) \text{ common from } R_3\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b) \times 1 \times \begin{vmatrix} 1 & b+a \\ 0 & 1 \end{vmatrix} \qquad \text{[Expanding along } C_1\text{]}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(c-b) \times 1 = (a-b)(b-c)(c-a)$$

REMARK *The reader is advised to remember the value of this determinant as a standard result.*

EXAMPLE 24 *Show that:* $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$. **[CBSE 2000, 2010 C, 2011]**

SOLUTION Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$. Taking x, y and z common from C_1, C_2 and C_3 respectively, we get

$$\Delta = xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = xyz\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \qquad \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1\text{]}$$

$$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x)\begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix} \quad \begin{bmatrix}\text{Taking } (y-x) \text{ and } (z-x) \text{ common from} \\ C_2 \text{ from } C_3 \text{ respectively.}\end{bmatrix}$$

$$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x) \times 1 \times \begin{vmatrix} 1 & 1 \\ y+x & z+x \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$\Rightarrow \quad \Delta = xyz\,(y-x)\,(z-x)\,(z+x-y-x) = xyz\,(x-y)\,(y-z)\,(z-x)$$

EXAMPLE 25 *Prove that:* $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma).$

[NCERT, CBSE 2007C, 2008, 2010 C]

SOLUTION Let $\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$. Applying $R_3 \to R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{Taking out } (\alpha+\beta+\gamma) \text{ common from } R_3]$$

$$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad [\text{Applying } C_1 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{bmatrix}\text{Taking } (\beta-\alpha) \text{ common from} \\ C_2 \text{ and } (\gamma-\alpha) \text{ from } C_3\end{bmatrix}$$

$$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha) \times 1 \times \begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \quad [\text{Expanding along } R_3]$$

$$\Rightarrow \quad \Delta = (\alpha+\beta+\gamma)\,(\beta-\alpha)\,(\gamma-\alpha)\,(\gamma+\alpha-\beta-\alpha) = (\alpha-\beta)\,(\beta-\gamma)\,(\gamma-\alpha)\,(\alpha+\beta+\gamma).$$

EXAMPLE 26 *In a* ΔABC, *if* $\begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} = 0$, *then prove that* ΔABC *is an isosceles triangle.*

[NCERT EXEMPLAR]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\sin A & 1+\sin B & 1+\sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin A+\sin^2 A & \sin B+\sin^2 B & \sin C+\sin^2 C \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} \qquad [\text{Applying } R_3 \to R_3 - R_2]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ \sin A & \sin B - \sin A & \sin C - \sin A \\ \sin^2 A & \sin^2 B - \sin^2 A & \sin^2 C - \sin^2 A \end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = (\sin B - \sin A)\,(\sin C - \sin A) \begin{vmatrix} 1 & 0 & 0 \\ \sin A & 1 & 1 \\ \sin^2 A & \sin B + \sin A & \sin C + \sin A \end{vmatrix}$$

[Taking $\sin B - \sin A$ common from C_2 and $\sin C - \sin A$ from C_3]

$\Rightarrow \quad \Delta = (\sin B - \sin A)\,(\sin C - \sin A)\,\{(\sin C + \sin A) - (\sin B + \sin A)\}$ [Expanding along R_1]

$\Rightarrow \quad \Delta = (\sin B - \sin A)\,(\sin C - \sin A)\,(\sin C - \sin B)$

Now, $\quad \Delta = 0$

$\Rightarrow \quad (\sin B - \sin A)\,(\sin C - \sin A)\,(\sin C - \sin B) = 0$

$\Rightarrow \quad$ either $\sin B - \sin A = 0$ or, $\sin C - \sin A = 0$ or, $\sin C - \sin B = 0$

$\Rightarrow \quad$ either $\sin A - \sin B = 0$ or, $\sin C = \sin A = 0$ or, $\sin C - \sin B = 0$

$\Rightarrow \quad A = B$ or $C = A$ or $B = C \Rightarrow BC = CA$ or, $AB = BC$ or $CA = AB \Rightarrow \Delta ABC$ is isosceles

EXAMPLE 27 *In a ΔABC, if* $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$, *show that ΔABC is an isosceles.* **[NCERT EXEMPLAR, CBSE 2016]**

SOLUTION Proceed as in Example 26.

EXAMPLE 28 *Prove that:* $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)\,(b-c)\,(c-a)\,(a+b+c)$

[NCERT, CBSE 2011, 2012, 2013]

SOLUTION Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we obtain

$$\Delta = \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3 - a^3 \\ 0 & c-a & c^3 - a^3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 1 & b^2 + a^2 + ab \\ 0 & 1 & c^2 + a^2 + ac \end{vmatrix} \qquad [\text{Taking out } (b-a) \text{ from } R_2 \text{ and } (c-a) \text{ from } R_3]$$

$$\Rightarrow \quad \Delta = (b-a)\,(c-a) \begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b^2 - c^2) + (ab - ac) \\ 0 & 1 & c^2 + a^2 + ac \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_3]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & (b-c)(b+c+a) \\ 0 & 1 & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & a & a^3 \\ 0 & 0 & a+b+c \\ 0 & 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{[Taking out } (b-c) \text{ common from } R_2]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\times 1\times\begin{vmatrix} 0 & a+b+c \\ 1 & c^2+a^2+ac \end{vmatrix} \qquad \text{[Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\{0-(a+b+c)\} = (a-b)(b-c)(c-a)(a+b+c).$$

EXAMPLE 29 *Show that* $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca).$

[NCERT, CBSE 2007, 2011, 2013, 2014]

SOLUTION Let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$. Multiplying C_1, C_2 and C_3 by a, b and c respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} \qquad \text{[Taking } abc \text{ common from } R_3]$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} a^2 & b^2 & c^2 \\ 1 & 1 & 1 \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad \text{[Applying } R_2 \leftrightarrow R_3]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad \text{[Applying } R_1 \leftrightarrow R_2]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2-a^2 & c^2-a^2 \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix} \qquad \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & (b-a)(b+a) & (c-a)(c+a) \\ a^3 & (b-a)(b^2+ba+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3 & b^2+a^2+ab & c^2+ac+a^2 \end{vmatrix} \qquad \left[\begin{array}{l}\text{Taking } (b-a) \text{ and } (c-a) \text{ common} \\ \text{from } C_2 \text{ and } C_3 \text{ respectively}\end{array}\right]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\times 1\times\begin{vmatrix} b+a & c+a \\ b^2+a^2+ab & c^2+a^2+ac \end{vmatrix} \qquad \text{[Expanding along } R_1]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ b^2-c^2+ab-ac & c^2+a^2+ac \end{vmatrix} \qquad \text{[Applying } C_1 \to C_1 - C_2]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)\begin{vmatrix} b-c & c+a \\ (b^2-c^2)+a(b-c) & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)\begin{vmatrix} 1 & c+a \\ b+c+a & c^2+a^2+ac \end{vmatrix} \qquad \text{[Taking } (b-c) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)(c^2+a^2+ac-bc-c^2-ac-ab-ac-a^2)$$

$$\Rightarrow \quad \Delta = (b-a)(c-a)(b-c)(-bc-ab-ac) = (a-b)(b-c)(c-a)(ab+bc+ca).$$

EXAMPLE 30 *If* $x \neq y \neq z$ *and* $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, *then prove that* $xyz = -1$.

[NCERT, CBSE 2011, 2020]

SOLUTION We have,

$$\Delta = \begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} \qquad \left[\begin{array}{l} \text{Since each element of third} \\ \text{column is sum of two elements} \end{array}\right]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \left[\begin{array}{l} \text{Taking } x, y \text{ and } z \text{ common from } C_1, \\ C_2, \text{ and } C_3 \text{ in second determinant} \end{array}\right]$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \text{[Interchanging } C_2 \text{ and } C_3 \text{ in first determinant]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad \text{[Interchanging } C_1 \text{ and } C_2 \text{ in first determinant]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}(1+xyz) \qquad \text{[Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (y-x)(z-x)\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix}(1+xyz) \qquad \left[\begin{array}{l} \text{Taking } (y-x) \text{ and } (z-x) \\ \text{common } R_2 \text{ and } R_3 \text{ resp.} \end{array}\right]$$

$$\Rightarrow \quad \Delta = (y-x)(z-x) \times 1 \times \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix}(1+xyz) \qquad \text{[Expanding along } C_1]$$

$\Rightarrow \quad \Delta = (y-x)(z-x)(z+x-y-x)(1+xyz)$

$\Rightarrow \quad \Delta = (y-x)(z-x)(z-y)(1+xyz) = (x-y)(y-z)(z-x)(1+xyz)$

$\therefore \quad \Delta = 0$

$\Rightarrow \quad (x-y)(y-z)(z-x)(1+xyz) = 0$

$\Rightarrow \quad 1+xyz = 0$ $\quad [\because\ x \neq y \neq z \Rightarrow x-y \neq 0,\ y-z \neq 0 \text{ and } z-x \neq 0]$

$\Rightarrow \quad xyz = -1.$

EXAMPLE 31 *For any scalar p prove that* $\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x).$

[NCERT, CBSE 2010]

SOLUTION We have,

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix} \quad [\because \text{ Each element in III column is sum of two elements}]$$

$$\Rightarrow \quad \Delta = -\begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad \left[\begin{array}{l} \text{Interchanging } C_1 \text{ and } C_3 \text{ in first det.} \\ \text{Taking } x, y, z \text{ common from } R_1, R_2, R_3 \\ \text{respectively and } p \text{ from } C_3 \text{ in 2nd det.} \end{array}\right]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + pxyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad [\text{Interchanging } C_2 \text{ and } C_3 \text{ in first determinant}]$$

$$\Rightarrow \quad \Delta = (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1+pxyz) \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (1+pxyz)(y-x)(z-x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y+x \\ 0 & 1 & z+x \end{vmatrix} \quad \left[\begin{array}{l} \text{Taking } (y-x) \text{ and } (z-x) \text{ common} \\ \text{from } R_2 \text{ and } R_3 \text{ respectively} \end{array}\right]$$

$$\Rightarrow \quad \Delta = (1+pxyz)(y-x)(z-x) \begin{vmatrix} 1 & y+x \\ 1 & z+x \end{vmatrix} \quad [\text{Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = (1+pxyz)(y-x)(z-x)(z+x-y-x) = (1+pxyz)(x-y)(y-z)(z-x)$$

EXAMPLE 32 *Using properties of determinants, show that* $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$ **[CBSE 2002]**

SOLUTION Let $\Delta = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & -bc \\ 1 & b & -ca \\ 1 & c & -ab \end{vmatrix}$$ [Each element of third column is sum of two elements]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$ [Taking (– 1) common from C_3 of second determinant]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$ [Multiplying R_1, R_2 and R_3 of second determinant by a, b and c respectively]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$ [Taking abc common from C_3 of second determinant]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$ [Applying $C_2 \leftrightarrow C_3$ in second determinant]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$ [Applying $C_1 \leftrightarrow C_2$ in second determinant]

$$\Rightarrow \quad \Delta = 0.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 33 *Prove that* : $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$ **[NCERT EXEMPLAR]**

SOLUTION Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} a^2+2a-3 & 2a-2 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$ [Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$]

$$\Rightarrow \quad \Delta = \begin{vmatrix} (a+3)(a-1) & 2(a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a-1)^2 \begin{vmatrix} a+3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} \qquad \text{[Taking } (a-1) \text{ common from } R_2 \text{ and } R_3]$$

$$\Rightarrow \quad \Delta = (a-1)^2 \begin{vmatrix} a+3 & 2 \\ 2 & 1 \end{vmatrix} = (a-1)^2 (a+3-4) = (a-1)^3 \qquad \text{[Expanding along } C_3]$$

EXAMPLE 34 *Let a, b and c denote the sides BC, CA and AB respectively of* ΔABC. *If* $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, *then find the value of* $\sin^2 A + \sin^2 B + \sin^2 C$.

SOLUTION We have,

$$\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1 & a & b \\ 0 & c-a & a-b \\ 0 & b-a & c-b \end{vmatrix} = 0 \qquad \text{[Applying } R_2 \to R_2 - R_1,\ R_3 - R_1]$$

$$\Rightarrow \quad \begin{vmatrix} c-a & a-b \\ b-a & c-b \end{vmatrix} \qquad \text{[Expanding along } C_1]$$

$$\Rightarrow \quad (c-a)(c-b) - (a-b)(b-a) = 0$$

$$\Rightarrow \quad a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0 \qquad \text{[Multiplying both sides by 2]}$$

$$\Rightarrow \quad (a-b)^2 + (b-a)^2 + (c-a)^2 = 0$$

$$\Rightarrow \quad a-b=0,\ b-c=0 \text{ and } c-a=0 \Rightarrow a=b=c \Rightarrow \Delta ABC \text{ is equilateral} \Rightarrow A=B=C=\frac{\pi}{3}$$

$$\therefore \quad \sin^2 A + \sin^2 B + \sin^2 C = 3\sin^2\frac{\pi}{3} = 3 \times \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4}$$

EXAMPLE 35 *If* $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, *using properties of determinants, find the value of* $f(2x) - f(x)$.

[CBSE 2015]

SOLUTION We have,

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix} = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} \qquad \text{[Applying } R_3 \to R_3 - xR_2,\ R_2 \to R_2 - xR_1]$$

$$\Rightarrow \quad f(x) = a \begin{vmatrix} a+x & -1 \\ 0 & a+x \end{vmatrix} \qquad \text{[Expanding along } C_1]$$

$\Rightarrow \quad f(x) = a(a+x)^2 \Rightarrow f(2x) = a(a+2x)^2$ [Replacing x by $2x$]

$\therefore \quad f(2x) - f(x) = a(a+2x)^2 - a(a+x)^2 = a\{(a+2x+a+x)(a+2x-a-x)\} = ax(2a+3x)$

EXAMPLE 36 *Show that:* $\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2+px-2q^2)$. **[NCERT EXEMPLAR]**

SOLUTION Let $\Delta = \begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} x-p & p & q \\ p-x & x & q \\ 0 & q & x \end{vmatrix} \qquad \text{[Applying } C_1 \to C_1 - C_2\text{]}$$

$$\Rightarrow \quad \Delta = (x-p)\begin{vmatrix} 1 & p & q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \qquad \text{[Taking } (x-p) \text{ common from } C_1\text{]}$$

$$\Rightarrow \quad \Delta = (x-p)\begin{vmatrix} 1 & p & q \\ 0 & x+p & 2q \\ 0 & q & x \end{vmatrix} \qquad \text{[Applying } R_2 \to R_2 + R_1\text{]}$$

$$\Rightarrow \quad \Delta = (x-p)\begin{vmatrix} x+p & 2q \\ q & x \end{vmatrix} = (x-p)(x^2+px-2q^2) \qquad \text{[Expanding along } C_1\text{]}$$

EXAMPLE 37 *If $m \in N$ and $m \geq 2$, prove that:* $\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix} = 1.$

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$. Then,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 + {}^{m+1}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 + {}^{m+1}C_2 \end{vmatrix} \qquad \left[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r\right]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 \end{vmatrix} \qquad \text{[Applying } C_3 \to C_3 - C_2\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^mC_0 + {}^mC_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 + {}^mC_2 & {}^{m+1}C_1 \end{vmatrix} \qquad \text{[Applying : } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \text{ in } C_2\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ {}^mC_1 & {}^mC_0 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 & {}^{m+1}C_1 \end{vmatrix} \qquad \text{[Applying } C_2 \to C_2 - C_1\text{]}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} {}^mC_0 & {}^{m+1}C_0 \\ {}^mC_1 & {}^{m+1}C_1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ m & m+1 \end{vmatrix} = (m+1-m) = 1 \qquad [\text{Expanding along } R_1]$$

EXAMPLE 38 *Evaluate:* $\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix}$.

SOLUTION We have,

$$\Delta = \begin{vmatrix} 10! & 11! & 12! \\ 11! & 12! & 13! \\ 12! & 13! & 14! \end{vmatrix} = \begin{vmatrix} 10! & 11\times 10! & 12\times 11\times 10! \\ 11! & 12\times 11! & 13\times 12\times 11! \\ 12! & 13\times 12! & 14\times 13\times 12! \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 10!\times 11!\times 12! \begin{vmatrix} 1 & 11 & 132 \\ 1 & 12 & 156 \\ 1 & 13 & 182 \end{vmatrix} \qquad \left[\begin{array}{l}\text{Taking } 10!, 11! \text{ and } 12! \text{ common} \\ \text{from } R_1, R_2 \text{ and } R_3 \text{ respectively}\end{array}\right]$$

$$\Rightarrow \quad \Delta = 10!\times 11!\times 12! \begin{vmatrix} 1 & 11 & 132 \\ 0 & 1 & 24 \\ 0 & 2 & 50 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = 10!\times 11!\times 12! \times \begin{vmatrix} 1 & 24 \\ 2 & 50 \end{vmatrix} = (10!\times 11!\times 12!)\times 2 \qquad [\text{Expanding along } C_1]$$

EXAMPLE 39 *Prove that:* $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$. **[CBSE 2002 C, 2009, 2014]**

SOLUTION Let $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$.

Since each element in the first column of Δ is the sum of two elements. Therefore, Δ can be expressed as the sum of two determinants given by

$$\Delta = \begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix} + \begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix} = x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + yx^2 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = x^3 \begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix} + yx^2 \times 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical in the second determinant}]$$

$$\Rightarrow \quad \Delta = x^3 \begin{vmatrix} 0 & 0 & 1 \\ 3 & 2 & 2 \\ 7 & 5 & 3 \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 - C_3, C_2 \to C_2 - C_3]$$

$$\Rightarrow \quad \Delta = x^3 \times 1 \times \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix} = x^3 (15-14) = x^3 \qquad [\text{Expanding along } R_1]$$

EXAMPLE 40 *Show that:* $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$. **[CBSE 2009]**

SOLUTION Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$. Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - pC_2]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_3, C_2 \to C_2 - C_3]$$

$$\Rightarrow \quad \Delta = 5 - 4 = 1 \quad [\text{Expanding along } R_1]$$

EXAMPLE 41 *Show that:* $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3$. **[NCERT, CBSE 2012]**

SOLUTION Let $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$.

Since each element of the second column is sum of two elements. Therefore, Δ can be written as the sum of two determinants as follows:

$$\Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + \begin{vmatrix} a & b & a+b+c \\ 2a & 2b & 4a+3b+2c \\ 3a & 3b & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \begin{vmatrix} 1 & 1 & a+b+c \\ 2 & 2 & 4a+3b+2c \\ 3 & 3 & 10a+6b+3c \end{vmatrix} \quad \left[\begin{array}{l} \text{Taking } a \text{ and } b \text{ common} \\ \text{from } C_1 \text{ and } C_2 \text{ of second} \\ \text{determinant} \end{array}\right]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & a & a+b+c \\ 2a & 3a & 4a+3b+2c \\ 3a & 6a & 10a+6b+3c \end{vmatrix} + ab \times 0 \quad [\because C_2 \ \& \ C_3 \text{ are identical in second determinant}]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & a & a \\ 2a & 3a & 4a \\ 3a & 6a & 10a \end{vmatrix} + \begin{vmatrix} a & a & b \\ 2a & 3a & 3b \\ 3a & 6a & 6b \end{vmatrix} + \begin{vmatrix} a & a & c \\ 2a & 3a & 2c \\ 3a & 6a & 3c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 6 & 6 \end{vmatrix} + a^2 c \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} + a^2 b \times 0 + a^2 c \times 0 \quad \left[\begin{array}{l} \because C_2 \text{ and } C_3 \text{ are identical in second det.} \\ \text{and } C_1 \text{ and } C_3 \text{ are identical in third det.} \end{array}\right]$$

$$\Rightarrow \quad \Delta = a^3 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = a^3 \times 1 \times \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = a^3 (7-6) = a^3 \quad [\text{Expanding along } R_1]$$

ALITER Let $\Delta = \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$. Taking '$a$' common from C_1, we obtain

$$\Delta = a\begin{vmatrix} 1 & a+b & a+b+c \\ 2 & 3a+2b & 4a+3b+2c \\ 3 & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = a\begin{vmatrix} 1 & a & a+b \\ 2 & 3a & 4a+3b \\ 3 & 6a & 10a+6b \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - bC_1, C_3 \to C_3 - cC_1]$$

$$\Rightarrow \quad \Delta = a^2\begin{vmatrix} 1 & 1 & a+b \\ 2 & 3 & 4a+3b \\ 3 & 6 & 10a+6b \end{vmatrix} \quad [\text{Taking } a \text{ common from } C_2]$$

$$\Rightarrow \quad \Delta = a^2\begin{vmatrix} 1 & 1 & a \\ 2 & 3 & 4a \\ 3 & 6 & 10a \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - b\,C_2]$$

$$\Rightarrow \quad \Delta = a^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} \quad [\text{Taking } a \text{ common from } C_3]$$

$$\Rightarrow \quad \Delta = a^3\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = a^3 \times 1 = a^3 \quad [\text{Expanding along } R_1]$$

EXAMPLE 42 *Show that:* $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$. **[CBSE 2004, 2006, 2010, 2012, 2014]**

SOLUTION Let $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix} \quad [\text{Taking 2 common from } C_1]$$

$$\Rightarrow \quad \Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1, C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow \quad \Delta = 2(-1)^2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad [\text{Taking } (-1) \text{ common from both } C_2 \text{ and } C_3]$$

EXAMPLE 43 *Prove that* $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = abc + bc + ca + ab.$
[NCERT, CBSE 2004, 2009, 2012, 2014]

SOLUTION Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$. Taking a, b and c common from C_1, C_2 and C_3 respectively, we obtain

$$\Delta = abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & 1+\frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

$$\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow \quad \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix} \quad \left[\text{Taking } \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \text{common from } C_1\right]$$

$$\Rightarrow \quad \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\times 1 \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \qquad [\text{Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

EXAMPLE 44 *If a, b, c are the roots of the equation $x^3 + px + q = 0$, then find the value of the determinant*

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

SOLUTION From Example 43, we obtain : $\Delta = abc + ab + bc + ca$.

It is given that a, b, c are roots of the equation $x^3 + px + q = 0$.

$\therefore \quad a + b + c = 0 \quad ab + bc + ca = p$ and $abc = -q$

Hence, $\Delta = p - q$

EXAMPLE 45 *Prove that:* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$ **[CBSE 2006 C, 2010]**

SOLUTION Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$. Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$,

we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad [\text{Taking } (a+b+c) \text{ common from } C_1 \,\&\, C_2]$$

$$\Rightarrow \quad \Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \qquad [\text{Applying } R_3 \to R_3 - (R_1 + R_2)]$$

$$\Rightarrow \quad \Delta = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac-a^2 & 0 & a^2 \\ 0 & bc+ba-b^2 & b^2 \\ -2ab & -2ab & 2ab \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1\,(a),\, C_2 \to C_2\,(b)]$$

$$\Rightarrow \quad \Delta = \frac{(a+b+c)^2}{ab} \begin{vmatrix} ab+ac & a^2 & a^2 \\ b^2 & bc+ba & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_3,\, C_2 \to C_2 + C_3]$$

$$\Rightarrow \quad \Delta = \frac{(a+b+c)^2}{ab} \times ab \times 2ab \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ 0 & 0 & 1 \end{vmatrix} \qquad \begin{bmatrix} \text{Taking } a, b \text{ and } 2ab \text{ common} \\ \text{from } R_1, R_2 \text{ and } R_3 \text{ respectively} \end{bmatrix}$$

$$\Rightarrow \quad \Delta = 2ab\,(a+b+c)^2 \times 1 \times \begin{vmatrix} b+c & a \\ b & c+a \end{vmatrix} \qquad [\text{Expanding along } R_3]$$

$$\Rightarrow \quad \Delta = 2ab\,(a+b+c)^2\,\{(b+c)(c+a) - ab\} = 2abc\,(a+b+c)^3$$

EXAMPLE 46 *Show that:* $\begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$ **[NCERT, CBSE 2006, 10]**

SOLUTION Let $\Delta = \begin{vmatrix} (b+c)^2 & ba & ca \\ ab & (c+a)^2 & cb \\ ac & bc & (a+b)^2 \end{vmatrix}$. Multiplying R_1, R_2 and R_3 by a, b and c respectively, we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} a\,(b+c)^2 & ba^2 & ca^2 \\ ab^2 & (c+a)^2\,b & cb^2 \\ ac^2 & bc^2 & (a+b)^2\,c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{abc}\,abc \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} \qquad \begin{bmatrix} \text{Taking } a, b \text{ and } c \text{ common} \\ \text{from } C_1, C_2 \text{ and } C_3 \text{ respectively} \end{bmatrix}$$

$$\Rightarrow \quad \Delta = 2abc\,(a+b+c)^3 \qquad [\text{Proceed as in Example 45}]$$

EXAMPLE 47 *Show that* $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$

[NCERT, CBSE 2009, 2010 C, 2020]

SOLUTION Let $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

We shall try to introduce zeros at as many places as possible keeping in mind that we have to introduce the factor $1 + a^2 + b^2$.

Applying $C_1 \to C_1 - bC_3$ and $C_2 \to C_2 + aC_3$, we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix} \quad \text{[Taking } (1+a^2+b^2) \text{ common from both } C_1 \,\&\, C_2]$$

$$\Rightarrow \quad \Delta = (1+a^2+b^2)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - bR_1 + aR_2]$$

$$\Rightarrow \quad \Delta = (1+a^2+b^2)^2 \times 1 \times \begin{vmatrix} 1 & 2a \\ 0 & 1+a^2+b^2 \end{vmatrix} = (1+a^2+b^2)^3 \quad [\text{Expanding along } C_1]$$

EXAMPLE 48 *Show that:* $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2 b^2 c^2$.

SOLUTION Let $\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$. Multiplying R_1, R_2 and R_3 by a, b and c respectively, we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(b^2+c^2) & a^2 b & a^2 c \\ b^2 a & b(c^2+a^2) & b^2 c \\ c^2 a & c^2 b & c(a^2+b^2) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{abc}{abc} \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad \left[\begin{matrix} \text{Taking } a, b \text{ and } c \text{ common from} \\ C_1, C_2 \text{ and } C_3 \text{ respectively} \end{matrix}\right]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow \quad \Delta = 2 \begin{vmatrix} b^2+c^2 & a^2+c^2 & a^2+b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} \quad [\text{Taking 2 common from } R_1]$$

$$\Rightarrow \quad \Delta = 2\begin{vmatrix} b^2+c^2 & c^2+a^2 & a^2+b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \qquad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow \quad \Delta = 2\left\{ -c^2 \begin{vmatrix} -c^2 & -a^2 \\ -b^2 & 0 \end{vmatrix} + b^2 \begin{vmatrix} -c^2 & 0 \\ -b^2 & -a^2 \end{vmatrix} \right\} \qquad [\text{Expanding along } R_1]$$

$$\Rightarrow \quad \Delta = 2(a^2b^2c^2 + a^2b^2c^2) = 4a^2b^2c^2$$

EXAMPLE 49 *Prove that:* $\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$.

SOLUTION Let $\Delta = \begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix}$. Applying $C_3 \to C_3 - xC_1 - yC_2$, we get

$$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -x(ax+by)-y(bx+cy) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -(ax^2+2bxy+cy^2) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = -(ax^2+2bxy+cy^2)\begin{vmatrix} a & b \\ b & c \end{vmatrix} \qquad [\text{Expanding along } C_3]$$

$$\Rightarrow \quad \Delta = -(ax^2+2bxy+cy^2)(ac-b^2) = (b^2-ac)(ax^2+2bxy+cy^2).$$

EXAMPLE 50 *Without expanding the determinant, show that $(a+b+c)$ is a factor of the determinant* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

SOLUTION Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \qquad [\text{Taking } (a+b+c) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (a+b+c) \times 1 \times \begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} \qquad [\text{Expanding along } C_1]$$

$\Rightarrow \quad \Delta = (a+b+c)\{-(b-c)^2 - (a-c)(a-b)\}$

$\Rightarrow \quad \Delta = -(a+b+c)[(b-c)^2 + (a-c)(a-b)] = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Clearly $(a+b+c)$ is a factor of $-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Hence, $(a+b+c)$ is a factor of Δ.

EXAMPLE 51 *If a, b, c are roots of the equation $x^3+px+q=0$, prove that* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0.$

SOLUTION It is given that a, b, c are roots of the equation $x^3+px+q=0$.

$\therefore \quad a+b+c = 0,\ ab+bc+ca = p$ and $abc = -q$

From example 50, we have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \times (a^2+b^2+c^2-ab-bc-ca) = 0 \qquad [\because\ a+b+c=0]$$

EXAMPLE 52 *If a, b, c are positive and unequal, show that the value of the determinant* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ *is always negative.* [NCERT, CBSE 2010]

SOLUTION Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \qquad [\text{Taking } (a+b+c) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} \qquad [\text{Expanding along } C_1]$$

$\Rightarrow \quad \Delta = (a+b+c)\{-(c-b)^2 - (a-b)(a-c)\} = (a+b+c)(-a^2-b^2-c^2+ab+bc+ca)$

$\Rightarrow \quad \Delta = -\frac{1}{2}(a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca)$

$\Rightarrow \quad \Delta = -\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$\Rightarrow \quad \Delta < 0 \qquad [\because\ a+b+c>0, (a-b)^2>0, (b-c)^2>0, (c-a)^2>0]$

EXAMPLE 53 *If $a+b+c \neq 0$ and* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$*, then prove that $a=b=c$.* [NCERT EXEMPLAR]

SOLUTION We have,

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\} \quad \text{[See example 52]}$$

$$\therefore \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \Rightarrow -\frac{1}{2}(a+b+c)\left\{(a-b)^2+(b-c)^2+(c-a)^2\right\} = 0$$

$$\Rightarrow \quad (a-b)^2+(b-c)^2+(c-a)^2 = 0 \quad [\because a+b+c \neq 0]$$

$$\Rightarrow \quad a-b=0, b-c=0 \text{ and } c-a=0 \Rightarrow a=b=c.$$

EXAMPLE 54 *If a, b, c are real numbers, prove that*

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a+bw+cw^2)(a+bw^2+cw), \text{ where } w \text{ is a complex cube root of unity.}$$

SOLUTION Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ b+c+a & c & a \\ c+a+b & a & b \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix} \quad \text{[Taking } (a+b+c) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix} \quad \text{[Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\{-(b-c)^2-(a-c)(a-b)\}$$

$$\Rightarrow \quad \Delta = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$\Rightarrow \quad \Delta = -(a+b+c)(a+bw+cw^2)(a+bw^2+cw) \quad \left[\because a^2+b^2+c^2-ab-bc-ca = (a+bw+cw^2)(a+bw^2+cw)\right]$$

EXAMPLE 55 *Show that:* $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2).$

SOLUTION Let $\Delta = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$. Multiplying first column by a, we get

$$\Delta = \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + bC_2 + cC_3]$$

$$\Rightarrow \quad \Delta = \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix} \qquad [\text{Taking } a^2+b^2+c^2 \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{a}(a^2+b^2+c^2)\times 1 \times \begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix} \qquad [\text{Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{a}(a^2+b^2+c^2)\,(-bc+a^2+ac+ba+bc) = (a^2+b^2+c^2)\,(a+b+c)$$

EXAMPLE 56 *Show that:* $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3\,(a+b+c)\,(ab+bc+ca).$

[NCERT EXEMPLAR, CBSE 2006 C, 2013]

SOLUTION Let $\Delta \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix} \qquad [\text{Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\,\{(2b+a)\;(2c+a) - (-b+a)\,(-c+a)\}$$

$$\Rightarrow \quad \Delta = (a+b+c)\,\{(4bc+2ab+2ca+a^2) - (bc-ab-ac+a^2)\}$$

$$\Rightarrow \quad \Delta = (a+b+c)\,(3bc+3ab+3ca) = 3\,(a+b+c)\,(ab+bc+ca).$$

Type III SOLUTION OF DETERMINANT EQUATIONS

EXAMPLE 57 *Solve:* $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0.$ [CBSE 2004, 2005, 2011]

SOLUTION Let $\Delta = \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} \quad \text{[Taking } (3a-x) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (3a-x) \times 1 \times \begin{vmatrix} 2x & 0 \\ 0 & 2x \end{vmatrix} = (3a-x)\, 4x^2 \quad [\text{Expanding along } C_1]$$

$$\therefore \quad \Delta = 0 \Rightarrow (3a-x)\, 4x^2 = 0 \Rightarrow x = 0, 3a.$$

EXAMPLE 58 *Solve:* $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$ **[CBSE 2011]**

SOLUTION Let $\Delta = \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$. Applying $C_2 \to C_2 - 2C_1$ and $C_3 \to C_3 - 3C_1$, we get

$$\Delta = \begin{vmatrix} x-2 & 1 & 2 \\ x-4 & -1 & -4 \\ x-8 & -11 & -40 \end{vmatrix} = \begin{vmatrix} x-2 & 1 & 2 \\ -2 & -2 & -6 \\ -6 & -12 & -42 \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \Delta = (-2)(-6)\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix} \quad [\text{Taking } (-2) \,\&\, (-6) \text{ common from } R_2 \,\&\, R_3 \text{ respectively}]$$

$$\Rightarrow \Delta = 12\begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 0 & 1 & 4 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - R_2]$$

$$\Rightarrow \Delta = 12\left\{(x-2)\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}\right\} \quad [\text{Expanding along } C_1]$$

$$\Rightarrow \Delta = 12\,\{(x-2)(4-3) - (4-2)\} = 12\,(x-4)$$

$$\therefore \quad \Delta = 0 \Rightarrow 12\,(x-4) = 0 \Rightarrow x = 4.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I EVALUATING DETERMINANTS BY USING THE PROPERTIES OF DETERMINANTS AND PROVING IDENTITIES

EXAMPLE 59 *If a, b, c are all distinct and* $\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$. *Show that $abc\,(ab+bc+ca) = a+b+c$.*

SOLUTION Let $\Delta = \begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix}$. Then,

$$\Delta=\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix}+\begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix}=abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}-\begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta=abc\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix}-\begin{vmatrix} a & a^3 & 1 \\ b-a & b^3-a^3 & 0 \\ c-a & c^3-a^3 & 0 \end{vmatrix} \quad \left[\begin{array}{r}\text{Applying } R_2 \to R_2-R_1 \\ \text{and } R_3 \to R_3-R_1\end{array}\right]$$

$$\Rightarrow \quad \Delta=abc\,(b-a)\,(c-a)\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b+a & b^2+a^2+ab \\ 0 & c+a & c^2+a^2+ac \end{vmatrix}-(b-a)\,(c-a)\begin{vmatrix} a & a^3 & 1 \\ 1 & b^2+a^2+ab & 0 \\ 1 & c^2+a^2+ac & 0 \end{vmatrix}$$

$$\Rightarrow \quad \Delta=abc\,(b-a)\,(c-a)\begin{vmatrix} b+a & b^2+a^2+ab \\ c+a & c^2+a^2+ac \end{vmatrix}-(b-a)\,(c-a)\begin{vmatrix} 1 & b^2+a^2+ab \\ 1 & c^2+a^2+ac \end{vmatrix}$$

$$\Rightarrow \quad \Delta=abc\,(b-a)\,(c-a)\begin{vmatrix} b+a & b^2+a^2+ab \\ c-b & c^2-b^2+a\,(c-b) \end{vmatrix}-(b-a)\,(c-a)\begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & c^2-b^2+a\,(c-b) \end{vmatrix}$$

[Applying $R_2 \to R_2-R_1$]

$$\Rightarrow \quad \Delta=abc\,(b-a)\,(c-a)\,(c-b)\begin{vmatrix} b+a & b^2+a^2+ab \\ 1 & a+b+c \end{vmatrix}-(b-a)\,(c-a)\,(c-b)\begin{vmatrix} 1 & b^2+a^2+ab \\ 0 & a+b+c \end{vmatrix}$$

$$\Rightarrow \quad \Delta=abc\,(b-a)\,(c-a)\,(c-b)\,\{(b+a)\,(a+b+c)-(b^2+a^2+ab)\}$$
$$-(b-a)\,(c-a)\,(c-b)\,(a+b+c-0)$$

$\Rightarrow \quad \Delta=abc\,(a-b)\,(b-c)\,(c-a)\,(bc+ca+ab)-(a-b)\,(b-c)\,(c-a)\,(a+b+c)$

$\Rightarrow \quad \Delta=(a-b)\,(b-c)\,(c-a)\,\{abc\,(ab+bc+ca)-(a+b+c)\}$

Now, $\quad \Delta=0$

$\Rightarrow \quad (a-b)\,(b-c)\,(c-a)\,\{abc\,(ab+bc+ca)-(a+b+c)\}=0$

$\Rightarrow \quad abc\,(ab+bc+ca)-(a+b+c)=0 \qquad [\because\ a \neq b \neq c\ \therefore a-b \neq 0, b-c \neq 0, c-a \neq 0]$

$\Rightarrow \quad abc\,(ab+bc+ca)=a+b+c$

EXAMPLE 60 *If a, b, c are all positive and are pth, qth and rth terms of a G.P., then show that*

$$\Delta=\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix}=0$$

[CBSE 2020]

SOLUTION Let A be the first term and R be the common ratio of the G.P. Then, we have

$a=AR^{p-1} \Rightarrow \log a=\log A+(p-1)\log R$

$b=AR^{q-1} \Rightarrow \log b=\log A+(q-1)\log R$

$c=AR^{r-1} \Rightarrow \log c=\log A+(r-1)\log R$

$$\therefore \quad \Delta=\begin{vmatrix} \log A+(p-1)\log R & p & 1 \\ \log A+(q-1)\log R & q & 1 \\ \log A+(r-1)\log R & r & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta=\begin{vmatrix} \log A+(p-1)\log R & p-1 & 1 \\ \log A+(q-1)\log R & q-1 & 1 \\ \log A+(r-1)\log R & r-1 & 1 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2-C_3]$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 0 & p-1 & 1 \\ 0 & q-1 & 1 \\ 0 & r-1 & 1 \end{vmatrix} = 0. \quad [\text{Applying } C_1 \to C_1 - (\log A)\, C_3 - (\log R)\, C_2]$$

EXAMPLE 61 *If* $x+y+z=0$, *prove that* $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$\text{LHS} = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xa \begin{vmatrix} za & xb \\ xc & ya \end{vmatrix} - yb \begin{vmatrix} yc & xb \\ zb & ya \end{vmatrix} + zc \begin{vmatrix} yc & za \\ zb & xc \end{vmatrix}$$

$$= xa\,(yza^2 - x^2bc) - yb\,(y^2ac - zx\,b^2) + zc\,(xyc^2 - z^2ab)$$

$$= xyz\,(a^3 + b^3 + c^3) - abc\,(x^3 + y^3 + z^3)$$

$$= xyz\,(a^3 + b^3 + c^3) - 3abc\; xyz \quad [\because\ x+y+z=0\ \therefore\ x^3+y^3+z^3 = 3xyz]$$

$$= xyz\,(a^3 + b^3 + c^3 - 3abc) = xyz\,(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca) \quad \ldots\text{(i)}$$

$$\text{RHS} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = xyz \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ c & a & b \\ b & c & a \end{vmatrix} \quad [\text{Applying } R_1 \to R_1 + R_2 + R_3]$$

$$= xyz\,(a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ c & a & b \\ b & c & a \end{vmatrix} \quad [\text{Taking } a+b+c \text{ common from } R_1]$$

$$= xyz\,(a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ c & a-c & b-c \\ b & c-b & a-b \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1]$$

$$= xyz\,(a+b+c) \begin{vmatrix} a-c & b-c \\ c-b & a-b \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$= (xyz)\,(a+b+c)\,\{(a-c)\,(a-b) - (b-c)\,(c-b)\}$$

$$= (xyz)\,(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca) \quad \ldots\text{(ii)}$$

From (i) and (ii), we infer that $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

EXAMPLE 62 *Prove that:* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$. **[NCERT EXEMPLAR]**

SOLUTION Let $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ and let a C_{ij} = Cofactor of $(A)_{ij}$ in A. Then,

$$C_{11}=\begin{vmatrix} c & a \\ a & b \end{vmatrix}=bc-a^2,\ C_{12}=-\begin{vmatrix} b & a \\ c & b \end{vmatrix}=ac-b^2,\ C_{13}=\begin{vmatrix} b & c \\ c & a \end{vmatrix}=ab-c^2$$

$$C_{21}=-\begin{vmatrix} b & c \\ a & b \end{vmatrix}=ac-b^2,\ C_{22}=\begin{vmatrix} a & c \\ c & b \end{vmatrix}=ab-c^2,\ C_{23}=-\begin{vmatrix} a & b \\ c & a \end{vmatrix}=bc-a^2$$

$$C_{31}=\begin{vmatrix} b & c \\ c & a \end{vmatrix}=ab-c^2,\ C_{32}=-\begin{vmatrix} a & c \\ b & a \end{vmatrix}=bc-a^2,\ C_{33}=\begin{vmatrix} a & b \\ b & c \end{vmatrix}=ac-b^2$$

Let $C=[C_{ij}]$ be the matrix of cofactors of elements of A. Then,

$$|C|=|A|^{3-1} \qquad \text{[By Property 12]}$$

$$\Rightarrow \quad \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 \Rightarrow \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

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$$\text{LHS}=\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}. \text{ Applying } C_1\to C_1+C_2+C_3, \text{ we get}$$

$$=\begin{vmatrix} ab+bc+ca-(a^2+b^2+c^2) & ca-b^2 & ab-c^2 \\ ab+bc+ca-(a^2+b^2+c^2) & ab-c^2 & bc-a^2 \\ ab+bc+ca-(a^2+b^2+c^2) & bc-a^2 & ca-b^2 \end{vmatrix}$$

$$=\left\{ab+bc+ca-(a^2+b^2+c^2)\right\}\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 1 & ab-c^2 & bc-a^2 \\ 1 & bc-a^2 & ca-b^2 \end{vmatrix} \qquad \left[\begin{matrix}\text{Taking } ab+bc+ca-(a^2+b^2+c^2) \\ \text{common } C_1\end{matrix}\right]$$

$$=-(a^2+b^2+c^2-ab-bc-ca)\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 0 & a(b-c)+(b^2-c^2) & b(c-a)+(c^2-a^2) \\ 0 & c(b-a)+(b^2-a^2) & a(c-b)+(c^2-b^2) \end{vmatrix}$$

[Applying $R_2\to R_2-R_1$, $R_3\to R_3-R_1$]

$$=-(a^2+b^2+c^2-ab-bc-ca)\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 0 & (b-c)(a+b+c) & (c-a)(a+b+c) \\ 0 & (b-a)(a+b+c) & (c-b)(a+b+c) \end{vmatrix}$$

$$=-(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)\begin{vmatrix} 1 & ca-b^2 & ab-c^2 \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} \qquad \left[\begin{matrix}\text{Taking } (a+b+c) \\ \text{common from} \\ R_2 \text{ and } R_3\end{matrix}\right]$$

$$=-(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)\begin{vmatrix} b-c & c-a \\ b-a & c-b \end{vmatrix} \qquad \text{[Expanding along } C_1]$$

$$=-(a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)\ (2bc-b^2-c^2-bc+ac+ab-a^2)$$

$$=-(a+b+c)^2\ (a^2+b^2+c^2-ab-bc-ca)\ (-a^2-b^2-c^2+ab+bc+ca)$$

$$=(a+b+c)^2\ (a^2+b^2+c^2-ab-bc-ca)^2$$

$$=\{(a+b+c)\ (a^2+b^2+c^2-ab-bc-ca)\}^2 \qquad \text{...(i)}$$

Now, $\quad \text{RHS} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \{-(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)\}^2$ [See Example 50]

$$= (a+b+c)^2\,(a^2+b^2+c^2-ab-bc-ca)^2 \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: LHS = RHS i.e. $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$

EXAMPLE 63 *Prove that:* $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ *is divisible by* $a+b+c$ *and find the quotient.* **[NCERT EXEMPLAR, CBSE 2016]**

SOLUTION From Example 62, we obtain

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

From Example 50, we obtain

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)$$

$$\therefore \quad \begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix} = (a+b+c)^2\,(a^2+b^2+c^2-ab-bc-ca)^2$$

Clearly, RHS is divisible by $(a+b+c)$ and the quotient is $(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)^2$.

Hence, LHS is divisible by $a+b+c$ and $(a+b+c)\,(a^2+b^2+c^2-ab-bc-ca)^2$ is the quotient.

EXAMPLE 64 *Find a quadratic polynomial $\phi(x)$ whose zeros are the maximum and minimum values of the function*

$$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$$

SOLUTION We have, $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 2 & 1+\cos^2 x & \sin 2x \\ 1 & \cos^2 x & 1+\sin 2x \end{vmatrix} \qquad [\text{Applying } C_1 \to C_1 + C_2]$$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 2 & \cos^2 x & \sin 2x \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$\Rightarrow \quad f(x) = 2 + \sin 2x$ [Expanding along C_1]

$\because \quad -1 \le \sin 2x \le 1$ for all $x \in R \Rightarrow 1 \le 2 + \sin 2x \le 3$ for all $x \in R \Rightarrow 1 \le f(x) \le 3$ for all $x \in R$

$\Rightarrow$ The maximum and minimum values of $f(x)$ are 3 and 1 respectively.

Thus, a quadratic polynomial having 1 and 3 as its roots is $\phi(x) = (x-1)(x-3)$ or, $\phi(x) = x^2 - 4x + 3$.

EXAMPLE 65 *Let* $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$. *Prove that:* $\int_0^{\pi/2} f(x)\, dx = -\frac{\pi}{4} - \frac{8}{15}$.

SOLUTION Applying $R_1 \to R_1 - \sec x\, R_3$, we obtain

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$$\Rightarrow \quad f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ 0 & \cos^2 x & \operatorname{cosec}^2 x \\ \sin^2 x & \cos^2 x & \cos^2 x \end{vmatrix} \qquad \text{Applying } C_1 \to C_1 - C_2$$

$$\Rightarrow \quad f(x) = (\sec^2 x + \cot x \operatorname{cosec} x - \cos x) \begin{vmatrix} 0 & \cos^2 x \\ \sin^2 x & \cos^2 x \end{vmatrix}$$

$$\Rightarrow \quad f(x) = -\sin^2 x \cos^2 x \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x \right)$$

$$\Rightarrow \quad f(x) = -\sin^2 x - \cos^3 x + \sin^2 x \cos^3 x = -\sin^2 x - \cos^3 x (1 - \sin^2 x) = -\sin^2 x - \cos^5 x$$

$$\therefore \quad \int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} (-\sin^2 x - \cos^5 x)\, dx = -\int_0^{\pi/2} \sin^2 dx - \int_0^{\pi/2} \cos^5 x\, dx$$

$$= -\frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x)\, dx - \int_0^{1} \cos^5 x\, dx$$

$$= -\frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi/2} - \int_0^1 (1 - 2t^2 + t^4)\, dt, \text{ where } t = \cos x$$

$$= -\frac{1}{2} \left[\left(\frac{\pi}{2} - 0 \right) - 0 \right] - \left[t - \frac{2}{3} t^3 + \frac{1}{5} t^5 \right]_0^1 = -\frac{\pi}{4} - \left(1 - \frac{2}{3} + \frac{1}{5} \right) = -\frac{\pi}{4} - \frac{8}{15}$$

Type II ON ADDITION OF DETERMINANTS

Two or more determinants can be added by using the following property:

$$\begin{vmatrix} a_1 & x & p \\ a_2 & y & q \\ a_3 & z & r \end{vmatrix} + \begin{vmatrix} b_1 & x & p \\ b_2 & y & q \\ b_3 & z & r \end{vmatrix} + \begin{vmatrix} c_1 & x & p \\ c_2 & y & q \\ c_3 & z & r \end{vmatrix} = \begin{vmatrix} a_1 + b_1 + c_1 & x & p \\ a_2 + b_2 + c_2 & y & q \\ a_3 + b_3 + c_3 & z & r \end{vmatrix}$$

i.e. the sum of two or more determinants having all columns (or rows) identical except a specific column (or row), say first, is a determinant whose first column (or row) is the sum of the corresponding elements of first columns (or rows) of various determinants and the remaining columns (or rows) remain same.

EXAMPLE 66 *Let* $\Delta_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$. *Show that* $\sum_{r=1}^{n} \Delta_r = 0$.

SOLUTION We have, $\Delta_r = \begin{vmatrix} r & x & \frac{n(n+1)}{2} \\ 2r-1 & y & n^2 \\ 3r-2 & z & \frac{n(3n-1)}{2} \end{vmatrix}$

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} r & x & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n} (2r-1) & y & n^2 \\ \sum_{r=1}^{n} (3r-2) & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

Now, $\sum_{r=1}^{n} r = 1+2+\ldots+n = \frac{n(n+1)}{2}$

$$\sum_{r=1}^{n} (2r-1) = 1+3+5+\ldots+(2n-1) = \frac{n}{2}\{1+(2n-1)\} = n^2$$

and, $\sum_{r=1}^{n} (3r-2) = 1+4+7+\ldots+(3n-2) = \frac{n}{2}\{1+(3n-2)\} = \frac{n(3n-1)}{2}$

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

EXAMPLE 67 *If* $\Delta_r = \begin{vmatrix} 2^{r-1} & 2.3^{r-1} & 4.5^{r-1} \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$. *Show that* $\sum_{r=1}^{n} \Delta_r = Constant$.

SOLUTION Using the properties of determinants, we have

$$\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & \sum_{r=1}^{n} 2 \times 3^{r-1} & \sum_{r=1}^{n} 4 \times 5^{r-1} \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$$

Now, $\sum_{r=1}^{n} 2^{r-1} = 1 + 2 + 2^2 + \ldots + 2^{n-1} = 1 \times \frac{(2^n - 1)}{(2-1)} = 2^n - 1$

$$\sum_{r=1}^{n} 2 \times 3^{r-1} = 2(1 + 3 + 3^2 + \ldots + 3^{n-1}) = 2 \times \left(\frac{3^n - 1}{3 - 1}\right) = 3^n - 1$$

and, $\sum_{r=1}^{n} 4 \times 5^{r-1} = 4\left(1 + 5 + 5^2 + \ldots + 5^{n-1}\right) = 4 \times \left(\frac{5^n - 1}{5 - 1}\right) = 5^n - 1.$

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ x & y & z \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix} = 0 \quad [\because R_1 \text{ and } R_3 \text{ are identical}]$$

EXAMPLE 68 *If m is a positive integer and* $D_r = \begin{vmatrix} 2r - 1 & {}^mC_r & 1 \\ m^2 - 1 & 2^m & m + 1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$. *Prove that* $\sum_{r=0}^{m} D_r = 0$.

SOLUTION Using properties of determinants, we have

$$\sum_{r=0}^{m} D_r = \begin{vmatrix} \sum_{r=0}^{m} (2r - 1) & \sum_{r=0}^{m} {}^mC_r & \sum_{r=0}^{m} 1 \\ m^2 - 1 & 2^m & m + 1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$$\because \quad \sum_{r=0}^{m} (2r - 1) = -1 + \{1 + 3 + 5 + \ldots + (2m - 1)\} = -1 + m^2 = m^2 - 1$$

$$\sum_{r=0}^{m} {}^mC_r = {}^mC_0 + {}^mC_1 + \ldots + {}^mC_m = 2^m \text{ and, } \sum_{r=0}^{m} 1 = (m + 1)$$

$$\therefore \quad \sum_{r=0}^{m} D_r = \begin{vmatrix} m^2 - 1 & 2^m & m + 1 \\ m^2 - 1 & 2^m & m + 1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0 \quad [\because R_1 \text{ and } R_2 \text{ are identical}]$$

Type III EVALUATION OF DETERMINANTS BY USING FACTOR HEOREM

If $f(x)$ is a polynomial such that $f(\alpha) = 0$, then $(x - \alpha)$ is a factor of $f(x)$.

For example, $x^3 - 6x^2 + 11x - 6$ vanishes for $x = 1$. Therefore, $(x - 1)$ is its factor.

Thus, if a determinant is a polynomial in x such that its value is zero for $x = a$, then $x - a$ is a factor of Δ.

EXAMPLE 69 *Without expanding evaluate the determinant* $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$.

SOLUTION If we put $a = b$ in Δ, we find that its two columns C_1 and C_2 become identical. Therefore, Δ becomes zero and thus $a - b$ is a factor of Δ. Similarly $b - c$ and $c - a$ are factors of Δ. The product of principal diagonal terms is bc^2 which is a third degree expression. Therefore, Δ is of third degree. Since $a - b$, $b - c$ and $c - a$ are factors of Δ. Therefore, $(a - b)(b - c)(c - a)$ is a third degree factor of Δ. Thus, there cannot be any other factor of Δ in terms of a, b and c. The only other factor of Δ can be a constant, say λ.

$$\therefore \quad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \lambda (a - b)(b - c)(c - a)$$

In order to find the value of λ, let us give values to a, b and c such that calculations are easy and the two sides do not vanish.

Putting $a = 0, b = 1, c = -1$, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = \lambda (-1)(1 - (-1))(-1 - 0) \Rightarrow 2 = 2\lambda \Rightarrow \lambda = 1.$$

$$\text{Hence,} \quad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a).$$

EXAMPLE 70 *Without expanding, show that*

$$\Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

SOLUTION If we put $a = b$, we observe that two rows R_1 and R_2 of Δ become identical, therefore $\Delta = 0$. Thus, $a - b$ is a factor of Δ. Similarly it can be easily shown that $b - c, c - a, x - y, y - z, z - x$ are factors of Δ. Therefore, $(a - b)(b - c)(c - a)(x - y)(y - z)(z - x)$ is a factor of Δ.

The product of diagonal elements of Δ is $(a - x)^2 (b - y)^2 (c - z)^2$ which is a sixth degree expression. Therefore, Δ can have six linear factors. Thus there cannot be any other factor of Δ except a constant λ (say).

$$\therefore \quad \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = \lambda (a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

In order to find the value of λ, we give some values to a, b, c, x, y, z such that two sides do not vanish together.

Putting $a = 0, b = -1, c = 1, x = 1, y = 0, z = -1$, we obtain

$$\begin{vmatrix} 1 & 0 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = \lambda (0 + 1)(-1 - 1)(1 - 0)(1 - 0)(0 + 1)(-1 - 1) \Rightarrow 8 = 4\lambda \Rightarrow \lambda = 2.$$

$$\therefore \quad \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

EXAMPLE 71 *Prove that:* $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a)$

SOLUTION Let $\Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$. Putting $b = -a$, we obtain

$$\Delta = \begin{vmatrix} -2a & 0 & a+c \\ 0 & 2a & c-a \\ c+a & c-a & -2c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = -2a \begin{vmatrix} 2a & c-a \\ c-a & -2c \end{vmatrix} + (a+c) \begin{vmatrix} 0 & 2a \\ c+a & c-a \end{vmatrix} \quad \text{[Expanding along } R_1\text{]}$$

$$\Rightarrow \quad \Delta = -2a\{-4ac - (c-a)^2\} - (a+c)\{2a(c+a)\}$$

$$\Rightarrow \quad \Delta = 2a\{(c-a)^2 + 4ac\} - 2a(c+a)^2 = 2a(c+a)^2 - 2a(c+a)^2 = 0$$

Therefore, by factor theorem $a+b$ is a factor of Δ. Similarly, we can show that $(b+c)$ and $(c+a)$ are factors of Δ. We find that Δ is a third degree homogeneous polynomial in a, b and c and $(b+c)(c+a)(a+b)$ is also a third degree homogeneous polynomial in a, b and c. Hence, we must have

$$\Delta = k(a+b)(b+c)(c+a), \text{ where } k \text{ is a constant.}$$

or, $$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(a+b)(b+c)(c+a) \quad \text{...(i)}$$

Putting $a = 0, b = 1$ and $c = 2$ in (i), we get

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 3 \\ 2 & 3 & -4 \end{vmatrix} = k(1)(3)(2) \Rightarrow 24 = 6k \Rightarrow k = 4$$

Hence, $\Delta = 4(a+b)(b+c)(c+a)$.

ALITER Let $a+b = 2C, b+c = 2A$ and $c+a = 2B$. Then,

$$a+b+b+c+c+a = 2C+2A+2B$$

$$\Rightarrow \quad a+b+c = A+B+C \Rightarrow a = (A+B+C)-(b+c) = (A+B+C)-2A = B+C-A.$$

Similarly, we obtain $b = C+A-B$ and $c = A+B-C$.

$$\therefore \quad \Delta = \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = \begin{vmatrix} 2A-2B-2C & 2C & 2B \\ 2C & 2B-2C-2A & 2A \\ 2B & 2A & 2C-2A-2B \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 8\begin{vmatrix} A-B-C & C & B \\ C & B-C-A & A \\ B & A & C-A-B \end{vmatrix} \quad \text{[Taking 2 common from } C_1, C_2 \text{ and } C_3\text{]}$$

$$\Rightarrow \quad \Delta = 8\begin{vmatrix} A-B & C+B & B \\ B-A & B-C & A \\ B+A & C-B & C-A-B \end{vmatrix} \quad \text{[Applying } C_1 \to C_1 + C_2 \text{ and } C_2 \to C_2 + C_3\text{]}$$

$$\Rightarrow \quad \Delta = 8\begin{vmatrix} A-B & C+B & B \\ 0 & 2B & A+B \\ 2B & 0 & C-B \end{vmatrix} \qquad [\text{Applying } R_2 \to R_2 + R_1 \text{ and } R_3 \to R_3 + R_2]$$

$$\Rightarrow \quad \Delta = 8\left\{(A-B)\begin{vmatrix} 2B & A+B \\ 0 & C-B \end{vmatrix} + 2B\begin{vmatrix} C+B & B \\ 2B & A+B \end{vmatrix}\right\} \qquad [\text{Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = 8\,[\{(A-B)\,2B\,(C-B) + 2B\,\{(C+B)(A+B) - 2B^2\}]$$

$$\Rightarrow \quad \Delta = 16B\,\{(A-B)(C-B) + (C+B)(A+B) - 2B^2\}$$

$$\Rightarrow \quad \Delta = 16B\,(2AC + 2B^2 - 2B^2) = 16B\,(2AC) = 32\,ABC$$

$$\Rightarrow \quad \Delta = 32\left(\frac{b+c}{2}\right)\left(\frac{c+a}{2}\right)\left(\frac{a+b}{2}\right) = 4\,(a+b)\,(b+c)\,(c+a).$$

EXERCISE 5.2

BASIC

1. *Evaluate the following determinant:*

(i) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$ (ii) $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$ (iii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(iv) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ (v) $\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$ (vi) $\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$

(vii) $\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$ (viii) $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$ [CBSE 2012]

2. Without expanding, show that the value of each of the following determinants is zero:

(i) $\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$ (iii) $\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$

(iv) $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix}$ (v) $\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$ (vi) $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$

(vii) $\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 26 & 2 & 3 \end{vmatrix}$ (viii) $\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$ (ix) $\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$

(x) $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ (xi) $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$ (xii) $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x}) & 1 \end{vmatrix}$

(xiii) $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

(xiv) $\begin{vmatrix} \sin^2 23° & \sin^2 67° & \cos 180° \\ -\sin^2 67° & -\sin^2 23° & \cos^2 180° \\ \cos 180° & \sin^2 23° & \sin^2 67° \end{vmatrix}$

(xv) $\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$

(xvi) $\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$

(xvii) $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$, where A, B, C are the angles of ΔABC.

Evaluate the following (3 – 9):

3. $\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$ [CBSE 2006]

4. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ [NCERT, CBSE 2006]

5. $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$ [NCERT]

6. $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$ [CBSE 2004]

7. $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

8. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$ [NCERT EXEMPLAR]

9. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$ [NCERT EXEMPLAR]

10. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$. [NCERT EXEMPLAR]

BASED ON LOTS

Prove the following identities (11 –45):

11. $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$ [CBSE 2009. 2019]

12. $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3$

13. $\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ [CBSE 2001, 2004, 2006 C, 2007]

14. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$ [NCERT, CBSE 2006C, 2008, 2014]

15. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$ [CBSE 2000C, 04, 07, NCERT EXEMPLAR]

16. $\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$ [CBSE 2002]

17. $\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$ [CBSE 2002, 2013, 2017]

18. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

19. $\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z).$

20. $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$ [CBSE 2020]

21. $\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$

22. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$ [CBSE 2012]

23. $\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$ [CBSE 2008]

24. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$ [NCERT, CBSE 2014, 2015]

25. $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$ [NCERT EXEMPLAR]

26. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$ [NCERT]

27. $\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a-b)$

28. $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = (a^3+b^3)^2$

29. $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$ [NCERT, CBSE 2014]

30. $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (a^3-1)^2$ [NCERT, CBSE 2013, 2014, 2015]

31. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$ [CBSE 2019]

32. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$ [NCERT, CBSE 2006 C]

33. $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$

34. $\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$ [CBSE 2003]

35. $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$

36. $\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$

37. $\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$ [CBSE 2014]

38. $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$ [CBSE 2007, 2011]

39. $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$ [NCERT EXEMPLAR]

40. $\begin{vmatrix} -a(b^2+c^2-a^2) & 2b^3 & 2c^3 \\ 2a^3 & -b(c^2+a^2-b^2) & 2c^3 \\ 2a^3 & 2b^3 & -c(a^2+b^2-c^2) \end{vmatrix} = abc(a^2+b^2+c^2)^3$

41. (i) $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^3 + 3a^2$ (ii) $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$ **[CBSE 2019]**

42. $\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$ **[CBSE 2014]**

43. $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)\,(x-z)^2.$ **[CBSE 2007]**

44. $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2\,(a+x+y+z)$ **[CBSE 2014]**

45. $\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2\,(a-b)\,(b-c)\,(c-a)\,(a+b+c)$ **[CBSE 2015]**

46. Without expanding, prove that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$.

47. Show that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ where a, b, c are in A.P. **[CBSE 2005]**

48. Show that $\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$, where α, β, γ are in AP. **[CBSE 2007]**

49. If a, b, c are real numbers such that $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$, then show that either $a+b+c=0$ or, $a=b=c$. **[NCERT]**

50. If $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, find the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$, $p \neq a, q = b, r \neq c$. **[CBSE 2014]**

51. Show that $x = 2$ is a root of the equation $\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0$ and solve it completely.

52. *Solve the following determinant equations:*

(i) $\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = 0$ **[CBSE 2003]** (ii) $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$ **[NCERT, CBSE 2011]**

(iii) $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ **[CBSE 2008]** (iv) $\begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix} = 0, a \neq b$

(v) $\begin{vmatrix} x+1 & 3 & 5 \\ 2 & x+2 & 5 \\ 2 & 3 & x+4 \end{vmatrix} = 0$ (vi) $\begin{vmatrix} 1 & x & x^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0, b \neq c$ (vii) $\begin{vmatrix} 15-2x & 11-3x & 7-x \\ 11 & 17 & 14 \\ 10 & 16 & 13 \end{vmatrix} = 0$

(viii) $\begin{vmatrix} 1 & 1 & x \\ p+1 & p+1 & p+x \\ 3 & x+1 & x+2 \end{vmatrix} = 0$ (ix) $\begin{vmatrix} 3 & -2 & \sin 3\theta \\ -7 & 8 & \cos 2\theta \\ -11 & 14 & 2 \end{vmatrix} = 0$ [NCERT EXEMPLAR]

(x) $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$ [CBSE 2019]

53. If a, b and c are all non-zero and $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$, then prove that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1=0$. [CBSE 2016]

54. If $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, then using properties of determinants, find the value of $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$, where $x, y, z \neq 0$. [CBSE 2017]

55. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9\,(3xyz+xy+yz+zx).$$ [CBSE 2018]

ANSWERS

1. (i) 0 (ii) -43 (iii) $abc+2fgh-af^2-bg^2-ch^2$ (iv) 40 (v) -8 (vi) 0 (vii) 512000 (viii) 0

3. $-(a+b+c)(a-b)(b-c)(c-a)$ 4. $(a-b)(b-c)(c-a)$

5. $\lambda^2(3x+\lambda)$ 6. $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$ 7. $(x-1)^2(x+2)$

8. $2x^3y^3z^3$ 9. $a^2(a+x+y+z)$ 50. 2 51. $-3, 1, 2$

52. (i) $0, -(a+b+c)$ (ii) $-a/3$ (iii) $\frac{2}{3}, \frac{11}{3}, \frac{11}{3}$ (iv) a, b (v) $1, 1, -9$

(vi) $b, c, -(b+c)$ (vii) 4 (viii) 1, 2 (ix) $\theta = n\pi$ or $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

(x) $x = -12, 0$ 54. 2

HINTS TO SELECTED PROBLEMS

2. (xvi) Take $\sqrt{5}$ common from C_2 and C_3 and apply $C_1 \to C_1 - \sqrt{3}\,C_2 - \sqrt{23}\,C_3$

4. Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$. Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-bc \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (a-b)(a-c)\begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix}$$ [Taking $(a-b)$ and $(a-c)$ common from R_1 and R_3 respectively.]

$\Rightarrow \quad \Delta = (a-b)(a-c)(-b+c) = (a-b)(b-c)(c-a)$ [Expanding along first column]

5. Let $\Delta = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 3x+\lambda & x & x \\ 3x+\lambda & x+\lambda & x \\ 3x+\lambda & x & x+\lambda \end{vmatrix} = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix} \quad \begin{bmatrix}\text{Taking } (3x+\lambda) \\ \text{common from } C_1\end{bmatrix}$$

$$\Rightarrow \quad \Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (3x+\lambda)\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} = \lambda^2(3x+\lambda) \quad [\text{Expanding along } C_1]$$

14. Let $\Delta = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \quad [\text{Taking } 2(a+b+c) \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \,\&\, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = 2(a+b+c)^3\begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2(a+b+c)^3 \times 1 = 2(a+b+c)^3 \quad [\text{Expanding along } C_1]$$

24. Let $\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$. Then,

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix} \quad [\text{Taking } a, b, c \text{ common from } C_1, C_2 \text{ and } C_3 \text{ respectively}]$$

$$\Rightarrow \quad \Delta = abc\begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow \quad \Delta = 2abc\begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 2abc\begin{vmatrix} a+c & -a & 0 \\ a+b & -a & -b \\ b+c & 0 & -b \end{vmatrix} \quad [\text{Applying } C_1 \to C_1 - C_2, C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad \Delta = 2abc \begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$$\Rightarrow \quad \Delta = 2abc \times abc \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$ [Taking c, a & b common from C_1, C_2 & C_3 respectively]

$$\Rightarrow \quad \Delta = 2a^2 b^2 c^2 \begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$ [Applying $R_3 \to R_3 - R_1$]

$$\Rightarrow \quad \Delta = 4\, a^2 b^2 c^2$$ [Expanding along C_1]

26. Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$. Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - q\,C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$ [Applying $C_3 \to C_3 - p\,C_2$]

$$\Rightarrow \quad \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$]

$$\Rightarrow \quad \Delta = 1\begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 7 - 6 = 1$$ [On expanding along R_1]

29. Let $\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$. Applying $R_1 \to R_1\,(a)$, $R_2 \to R_2\,(b)$ & $R_3 \to R_3\,(c)$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a\,(a^2+1) & a^2 b & a^2 c \\ ab^2 & b\,(b^2+1) & b^2 c \\ c^2 a & c^2 b & c\,(c^2+1) \end{vmatrix}$$

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ [Taking a, b, c common from C_1, C_2 & C_3 respectively]

$$\Rightarrow \quad \Delta = \begin{vmatrix} a^2+b^2+c^2+1 & a^2+b^2+c^2+1 & a^2+b^2+c^2+1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ [Applying $R_1 \to R_1 + R_2 + R_3$]

$$\Rightarrow \quad \Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$ [Taking $(a^2+b^2+c^2+1)$ common from R_1]

$$\Rightarrow \quad \Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$ [Applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$]

$$\Rightarrow \quad \Delta = (a^2 + b^2 + c^2 + 1) \qquad \text{[Expanding along } R_1]$$

30. Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+a+a^2 & a & a^2 \\ 1+a+a^2 & 1 & a \\ 1+a+a^2 & a^2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1+a+a^2) \begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & a \\ 1 & a^2 & 1 \end{vmatrix} \qquad \text{[Taking } 1+a+a^2 \text{ common from } C_1]$$

$$\Rightarrow \quad \Delta = (1+a+a^2) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a-a^2 \\ 0 & a^2-a & 1-a^2 \end{vmatrix} \qquad \text{[Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = (1+a+a^2) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a(1-a) \\ 0 & -a(1-a) & (1-a)(1+a) \end{vmatrix}$$

$$\Rightarrow \quad \Delta = (1+a+a^2)(1-a)^2 \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & -a & 1+a \end{vmatrix} \qquad \left[\begin{matrix}\text{Taking } (1-a) \text{ common from} \\ R_2 \text{ and } R_3 \text{ respectively}\end{matrix}\right]$$

$$\Rightarrow \quad \Delta = (1+a+a^2)(1-a)^2(1+a+a^2) \qquad \text{[Expanding along } C_1]$$

$$\Rightarrow \quad \Delta = \{(1-a)(1+a+a^2)\}^2 = (a^3-1)^2$$

32. Let $\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$. Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 2 \begin{vmatrix} b+c & c+a & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \qquad \text{[Taking 2 common from } R_1]$$

$$\Rightarrow \quad \Delta = 2 \begin{vmatrix} b+c & c+a & a+b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix} \qquad \text{[Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = 2 \begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix} \qquad \text{[Applying } R_1 \to R_1 + R_2 + R_3]$$

$$\Rightarrow \quad \Delta = \left\{ 0 \begin{vmatrix} 0 & -a \\ -a & 0 \end{vmatrix} - c \begin{vmatrix} -c & -a \\ -b & 0 \end{vmatrix} + b \begin{vmatrix} -c & 0 \\ -b & -a \end{vmatrix} \right\} = 2(0 + abc + abc) = 4abc$$

49. We have, $\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \\ 2(a+b+c) & b+c & c+a \end{vmatrix} = 0 \qquad \text{[Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 1 & a+b & b+c \\ 1 & b+c & c+a \end{vmatrix} = 0$$

$$\Rightarrow 2(a+b+c) \begin{vmatrix} 1 & c+a & a+b \\ 0 & b-c & c-a \\ 0 & b-a & c-b \end{vmatrix} = 0 \qquad \text{Applying } R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$$

$\Rightarrow \quad 2(a+b+c)\{(b-c)(c-b)-(c-a)(b-a)\} = 0$ [Expanding along C_1]

$\Rightarrow \quad 2(a+b+c)\{-(b^2-2bc+c^2)-(bc-ca-ab+a^2)\} = 0$

$\Rightarrow \quad 2(a+b+c)(-a^2-b^2-c^2+bc+ca+ab) = 0$

$\Rightarrow \quad 2(a+b+c)(a^2+b^2+c^2-ab-bc-ca) = 0$

$\Rightarrow \quad (a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ca) = 0$

$\Rightarrow \quad (a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\} = 0$

$\Rightarrow \quad a+b+c = 0$ or, $(a-b)^2+(b-c)^2+(c-a)^2 = 0 \Rightarrow a+b+c=0$ or, $a=b=c$

50. We have, $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$. Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$, we obtain

$$\begin{vmatrix} p-a & 0 & c-r \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 \Rightarrow (p-a)\begin{vmatrix} q-b & c-r \\ b & r \end{vmatrix} - 0\begin{vmatrix} 0 & c-r \\ a & r \end{vmatrix} + (c-r)\begin{vmatrix} 0 & q-b \\ a & b \end{vmatrix} = 0$$

$\Rightarrow \quad (p-a)\{r(q-b)-b(c-r)\}-a(c-r)(q-b) = 0$

$\Rightarrow \quad (p-a)(q-r)r+(p-q)(r-c)b+a(q-b)(r-c) = 0$

$\Rightarrow \quad \dfrac{r}{r-c}+\dfrac{b}{q-b}+\dfrac{a}{p-a} = 0$ [Dividing by $(p-a)(q-b)(r-c)$]

$\Rightarrow \quad \dfrac{r}{r-c}+\left(\dfrac{b}{q-b}+1\right)+\left(\dfrac{a}{p-a}+1\right) = 1+1$ [Adding 2 on both sides]

$\Rightarrow \quad \dfrac{r}{r-c}+\dfrac{q}{q-b}+\dfrac{p}{p-a} = 2$

52. (ii) We have, $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, \ a \neq 0$

$$\Rightarrow \begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0 \qquad \text{[Applying } C_1 \to C_1 + C_2 + C_3]$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0 \qquad \text{[Taking } (3x+a) \text{ common from } C_1]$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0 \qquad \text{[Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$\Rightarrow \quad (3x+a)a^2 = 0 \Rightarrow 3x+a = 0 \Rightarrow x = -a/3$ [$\because a \neq 0$]

5.6 APPLICATIONS OF DETERMINANTS TO COORDINATE GEOMETRY

5.6.1 AREA OF A TRIANGLE

We know that the area of triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by the expression:

$$\Delta = \frac{1}{2}\left\{x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)\right\} \qquad ...(i)$$

Also, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ [Expanding along C_1]

$$= x_1 (y_2 - y_3) - x_2 (y_1 - y_3) + x_3 (y_1 - y_2)$$
$$= x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) \qquad ...(ii)$$

From (i) and (ii), we obtain : $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Thus, the area of a triangle having vertices at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is the absolute value of Δ given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

NOTE *Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.*

5.6.2 CONDITION OF COLLINEARITY OF THREE POINTS

Let $A (x_1, y_1)$, $B (x_2, y_2)$ and $C (x_3, y_3)$ be three points. Then,

A, B, C are collinear $\Leftrightarrow$ Area of triangle $ABC = 0 \Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

5.6.3 EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Let the two point be $A (x_1, y_1)$ and $B (x_2, y_2)$. Let $P (x, y)$ be any point on the line joining A and B. Then, points P, A and B are collinear.

$$\therefore \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Thus, the equation of the line joining points (x_1, y_1) and (x_2, y_2) is given by $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the area of the triangle with vertices A* (5, 4), *B* (– 2, 4) *and C* (2, – 6).

SOLUTION The area Δ of triangle ABC is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ -7 & 0 & 0 \\ -3 & -10 & 0 \end{vmatrix} \quad \text{[Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\times 1\times\begin{vmatrix}-7 & 0\\ -3 & -10\end{vmatrix} = \frac{1}{2}(70-0) = 35 \text{ sq. units.} \qquad [\text{Expanding along } C_3]$$

EXAMPLE 2 *Show that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.*

SOLUTION We have,

$$\Delta = \begin{vmatrix}a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix} = \begin{vmatrix}a & a+b+c & 1\\ b & b+c+a & 1\\ c & c+a+b & 1\end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 + C_1]$$

$$\Rightarrow \quad \Delta = (a+b+c)\begin{vmatrix}a & 1 & 1\\ b & 1 & 1\\ c & 1 & 1\end{vmatrix} \qquad [\text{Taking } (a+b+c) \text{ common from } C_2]$$

$$\Rightarrow \quad \Delta = (a+b+c)\times 0 = 0. \qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$$

Hence, the given points are collinear.

EXAMPLE 3 *If the points (a_1, b_1), (a_2, b_2) and (a_1+a_2, b_1+b_2) are collinear, show that $a_1 b_2 = a_2 b_1$.*

SOLUTION If given points are collinear, then

$$\begin{vmatrix}a_1 & b_1 & 1\\ a_2 & b_2 & 1\\ a_1+a_2 & b_1+b_2 & 1\end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix}a_1 & b_1 & 1\\ a_2-a_1 & b_2-b_1 & 0\\ a_2 & b_2 & 0\end{vmatrix} = 0 \qquad [\text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \begin{vmatrix}a_2-a_1 & b_2-b_1\\ a_2 & b_2\end{vmatrix} = 0 \qquad [\text{Expanding along } C_3]$$

$$\Rightarrow \quad \begin{vmatrix}-a_1 & -b_1\\ a_2 & b_2\end{vmatrix} = 0 \qquad [\text{Applying } R_1 \to R_1 - R_2]$$

$$\Rightarrow \quad -a_1 b_2 + a_2 b_1 = 0 \Rightarrow a_1 b_2 = a_2 b_1$$

EXAMPLE 4 *If the points $(2, -3)$, $(\lambda, -1)$ and $(0, 4)$ are collinear, find the value of λ.*

SOLUTION If given points are collinear, then

$$\begin{vmatrix}2 & -3 & 1\\ \lambda & -1 & 1\\ 0 & 4 & 1\end{vmatrix} = 0 \Rightarrow \begin{vmatrix}2 & -3 & 1\\ \lambda-2 & 2 & 0\\ -2 & 7 & 0\end{vmatrix} = 0 \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \begin{vmatrix}\lambda-2 & 2\\ -2 & 7\end{vmatrix} = 0 \qquad [\text{Expanding along } C_3]$$

$$\Rightarrow \quad 7\lambda - 14 + 4 = 0 \Rightarrow \lambda = 10/7.$$

EXAMPLE 5 *Using determinants, find the area of the triangle whose vertices are $(-2, 4)$, $(2, -6)$ and $(5, 4)$. Are the given points collinear?*

SOLUTION Let Δ be the area of the triangle. Then,

$$\Delta = \frac{1}{2}\begin{vmatrix}-2 & 4 & 1\\ 2 & -6 & 1\\ 5 & 4 & 1\end{vmatrix} = \frac{1}{2}\begin{vmatrix}-2 & 4 & 1\\ 4 & -10 & 0\\ 7 & 0 & 0\end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{2}\begin{vmatrix}4 & -10\\ 7 & 0\end{vmatrix} \qquad [\text{By expanding along } C_1]$$

$$\Rightarrow \quad \Delta = \frac{1}{2}(70) = 35 \text{ sq. units.}$$

Clearly, $\Delta \neq 0$, therefore given points are not collinear.

EXAMPLE 6 *Find the equation of the line joining A (1, 3) and B (0, 0) using determinants and find k if D (k, 0) is a point such that area of Δ ABD is 3 sq. units.* **[CBSE 2013, 2020]**

SOLUTION Let $P(x, y)$ be any point on line AB. Then,

Area of $\Delta ABP = 0$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix} = 0 \Rightarrow \frac{1}{2}\{1(0-y)-3(0-x)+1(0-0)\} = 0$$

$\Rightarrow \quad 3x - y = 0$, which is the required equation of AB.

Now, Area of $\Delta ABD = 3$ sq. units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

$$\Rightarrow \quad \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6 \Rightarrow 1(0-0)-3(0-k)+1(0-0) = \pm 6 \Rightarrow 3k = \pm 6 \Rightarrow k = \pm 2$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are vertices of an equilateral triangle whose each side is equal to a, then prove that* $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$. **[NCERT EXEMPLAR]**

SOLUTION Let Δ be the area of triangle ABC. Then,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4\Delta = 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}$$

$$\Rightarrow \quad 16\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 \qquad \text{...(i)}$$

But, the area of an equilateral triangle with each side equal to a is $\frac{\sqrt{3}}{4}a^2$.

$$\therefore \quad \Delta = \frac{\sqrt{3}}{4}a^2 \Rightarrow 16\Delta^2 = 3a^4 \qquad \text{...(ii)}$$

From (i) and (ii), we obtain $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = 3a^4$.

EXAMPLE 8 *A triangle has its three sides equal to a, b and c. If the coordinates of its vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, show that*

$$\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

SOLUTION Let Δ be the area of triangle ABC. Then,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \Rightarrow 2\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad 4\Delta = 2\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix} \Rightarrow 16\Delta^2 = \begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 \qquad \text{...(i)}$$

We also know that the area of triangle ABC is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{1}{2}(a+b+c)$$

But, $s = \frac{1}{2}(a+b+c) \Rightarrow s-a = \frac{1}{2}(a+b+c) - a = \frac{1}{2}(b+c-a)$.

Similarly, $s-b = \frac{1}{2}(c+a-b)$ and $s-c = \frac{1}{2}(a+b-c)$.

$$\therefore \quad \Delta^2 = \frac{1}{2}(a+b+c) \times \frac{1}{2}(b+c-a) \times \frac{1}{2}(c+a-b) \times \frac{1}{2}(a+b-c)$$

$$\Rightarrow \quad 16\Delta^2 = (a+b+c)(b+c-a)(c+a-b)(a+b+c) \qquad \text{...(ii)}$$

From (i) and (ii), we get : $\begin{vmatrix} x_1 & y_1 & 2 \\ x_2 & y_2 & 2 \\ x_3 & y_3 & 2 \end{vmatrix}^2 = (a+b+c)(b+c-a)(c+a-b)(a+b-c)$

EXERCISE 5.3

BASIC

1. Find the area of the triangle with vertices at the points:
   (i) (3, 8), (– 4, 2) and (5, – 1)
   (ii) (2, 7), (1, 1) and (10, 8)
   (iii) (– 1, – 8), (– 2, – 3) and (3, 2)
   (iv) (0, 0), (6, 0) and (4, 3).
2. Using determinants show that the following points are collinear:
   (i) (5, 5), (– 5, 1) and (10, 7)
   (ii) (1, – 1), (2, 1) and (4, 5)
   (iii) (3, – 2), (8, 8) and (5, 2)
   (iv) (2, 3), (–1, – 2) and (5, 8)
3. If the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, prove that $a + b = ab$.
4. Using determinants prove that the points (a, b), (a', b') and $(a - a', b - b')$ are collinear if $ab' = a'b$.
5. Find the value of λ so that the points $(1, -5)$, $(-4, 5)$ and $(\lambda, 7)$ are collinear.
6. Find the value of x if the area of Δ is 35 square cms with vertices $(x, 4)$, $(2, -6)$ and $(5, 4)$.
7. Using determinants, find the area of the triangle whose vertices are (1, 4), (2, 3) and (–5, – 3). Are the given points collinear?
8. Using determinants, find the area of the triangle with vertices (– 3, 5), (3, – 6) and (7, 2).
9. Using determinants, find the value of k so that the points $(k, 2-2k)$, $(-k+1, 2k)$ and $(-4-k, 6-2k)$ may be collinear.
10. If the points $(x, -2)$, $(5, 2)$ and $(8, 8)$ are collinear, find x using determinants.
11. If the points $(3, -2)$, $(x, 2)$ and $(8, 8)$ are collinear, find x using determinant.

12. Using determinants, find the equation of the line joining the points
(i) $(1, 2)$ and $(3, 6)$ (ii) $(3, 1)$ and $(9, 3)$

13. Find values of k, if area of triangle is 4 square units whose vertices are
(i) $(k, 0), (4, 0)$ and $(0, 2)$ (ii) $(-2, 0), (0, 4)$ and $(0, k)$

ANSWERS

1. (i) $\frac{75}{2}$ sq. units (ii) $\frac{47}{2}$ sq. units (iii) 15 sq. units (iv) 9 sq. units
5. $\lambda = -5$
6. $x = -2, 12$
7. $\frac{13}{2}$ sq. units, No
8. 46 sq. units
9. $k = -1, 1/2$
10. $x = 3$
11. $x = 5$
12. (i) $y = 2x$ (ii) $x = 3y$
13. (i) $k = 0, 8$ (ii) $0, 8$

5.7 APPLICATIONS OF DETERMINANTS IN SOLVING A SYSTEM OF LINEAR EQUATIONS

Consider a system of simultaneous linear equations given by

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \quad \text{...(i)}$$

A set of values of the variables x, y, z which simultaneously satisfy these three equations is called a solution set.

For example, $x = 3$, $y = 4$ and $z = 6$ is the solution of the system of equations

$$\begin{aligned} 5x - 6y + 4z &= 15 \\ 7x + 4y - 3z &= 19 \\ 2x + y + 6z &= 46 \end{aligned}$$

A system of linear equations may have a unique solution, or many solutions, or no solution at all. If it has a solution (whether unique or not) the system is said to be ***consistent***. If it has no solution, it is called an ***inconsistent*** system.

If $d_1 = d_2 = d_3 = 0$ in (i), then the system of equations is said to be a ***homogeneous system***. Otherwise it is called a ***non-homogeneous system*** of equations.

5.7.1 SOLUTION OF A NON-HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

We now intend to solve a system of simultaneous linear equations by Cramer's rule named after the Swiss mathematician Gabriel Cramer.

THEOREM 1 *(Cramer's rule) The solution of the system of simultaneous linear equations*

$$a_1 x + b_1 y = c_1 \quad \text{...(i)}$$

$$a_2 x + b_2 y = c_2 \quad \text{...(ii)}$$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, *where* $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ *and* $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$ *provided that* $D \neq 0$.

PROOF We have, $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

$$\therefore \quad xD = x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 \\ a_2 x & b_2 \end{vmatrix}$$

$$\Rightarrow \quad xD = \begin{vmatrix} a_1 x + b_1 y & b_1 \\ a_2 x + b_2 y & b_2 \end{vmatrix} \quad \text{[Applying } C_1 \to C_1 + y\,C_2]$$

$$\Rightarrow \quad xD = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = D_1 \quad \text{[Using (i) and (ii)]}$$

Similarly, we obtain : $yD = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = D_2$

$\therefore \quad x = \dfrac{D_1}{D}$ and $y = \dfrac{D_2}{D}$, provided that $D \neq 0$. **Q.E.D.**

REMARK *Here* $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ *is the determinant of the coefficient matrix* $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$.

The determinant D_1 is obtained by replacing first column in D by the column on the right hand side of the given equations.

The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

THEOREM 2 *(Cramer's Rule) The solution of the system of linear equations*

$$a_1 x + b_1 y + c_1 z = d_1 \quad \text{...(i)}$$
$$a_2 x + b_2 y + c_2 z = d_2 \quad \text{...(ii)}$$
$$a_3 x + b_3 y + c_3 z = d_3 \quad \text{...(iii)}$$

is given by $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$ *and* $z = \dfrac{D_3}{D}$, *where*

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

provided that $D \neq 0$.

PROOF We have, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

$$\therefore \quad xD = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad xD = \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} \quad \text{[Applying } C_1 \to C_1 + y\,C_2 + z\,C_3]$$

$$\Rightarrow \quad xD = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1 \quad \text{[Using (i), (ii) and (iii)]}$$

Similarly, we obtain : $yD = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = D_2$ and $zD = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = D_3$.

$\therefore \quad x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$ and $z = \dfrac{D_3}{D}$, provided that $D \neq 0$. **Q.E.D.**

REMARK *Here D is the determinant of the coefficient matrix. The determinant D_1 is obtained by replacing the elements in first column of D by d_1, d_2, d_3. D_2 is obtained by replacing the elements in the second column of D by d_1, d_2, d_3 and to obtain D_3, replace elements in the third column of D by d_1, d_2, d_3.*

The above method of solving a system of three linear equations in three unknowns can be used exactly the same way to solve a system of n equations in n unknowns as stated below.

THEOREM 3 *(Cramer's Rule) Let there be a system of n simultaneous linear equations n unknowns given by*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad \vdots \quad &\ \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Let $D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ *and let D_j be the determinant obtained from D after replacing the j^{th} column by* $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$ *Then,* $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}, \ldots, x_n = \frac{D_n}{D}$, *provided that $D \neq 0$.*

5.7.2 CONDITIONS FOR CONSISTENCY

Case I *For a system of 2 simultaneous linear equations with 2 unknowns*

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$.

(ii) If $D = 0$ and $D_1 = D_2 = 0$, then the system is consistent and has infinitely many solutions.

(iii) If $D = 0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

Case II *For a system of 3 simultaneous linear equations in three unknowns*

(i) If $D \neq 0$, then the given system of equations is consistent and has a unique solution given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

(ii) If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the given system of equations may or may not be consistent. However, if it is consistent, then it has infinitely many solutions.

(iii) If $D = 0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then the given system of equations is inconsistent.

In order to solve a non-homogeneous system of simultaneous linear equations by Cramer's rule, we may use the following algorithm.

ALGORITHM

Step I *Obtain D, D_1, D_2 and D_3.*

Step II *Find the value of D.*

If $D \neq 0$, then the system of equations is consistent and has a unique solution. To find the solution, obtain the values of D_1, D_2 and D_3. The solution is given by

$x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$.

If $D = 0$, go to step III

Step III *Find the values of D_1, D_2, D_3.*

If at least one of these determinants is non-zero, then the system is inconsistent.

If $D_1 = D_2 = D_3 = 0$, then go to step IV.

Step IV *Take any two equations out of three given equations and shift one of the variables, say z, on the right hand side to obtain two equations in x, y. Solve these two equations by Cramer's rule to obtain x, y in terms of z. If these values of x and y satisfy the third equation, then the system is consistent and the values of x, y and z constitute a solution.*

If the values of x and y do not satisfy the third equation, then the system is inconsistent.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the following system of equations by Cramer's rule* $\begin{aligned} 2x - y &= 17 \\ 3x + 5y &= 6 \end{aligned}$

SOLUTION For the given system, we have

$$D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 2 \times 5 - (-1) \times 3 = 13 \neq 0$$

$$D_1 = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix} = 85 + 6 = 91 \text{ and } D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix} = 12 - 51 = -39.$$

So, by Cramer's rule, we obtain

$$x = \frac{D_1}{D} = \frac{91}{13} = 7 \text{ and } y = \frac{D_2}{D} = \frac{-39}{13} = -3.$$

Hence, $x = 7$ and $y = -3$ is the required solution.

EXAMPLE 2 *Solve the following system of equations using Cramer's rule:*

$5x - 7y + z = 11,\quad 6x - 8y - z = 15\quad$ and $\quad 3x + 2y - 6z = 7.$

SOLUTION The given system of equations is

$$\begin{aligned} 5x - 7y + z &= 11 \\ 6x - 8y - z &= 15 \\ 3x + 2y - 6z &= 7 \end{aligned}$$

$$\therefore \quad D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 5(48 + 2) + 7(-36 + 3) + 1(12 + 24) = 250 - 231 + 36 = 55 \neq 0$$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 11(48 + 2) + 7(-90 + 7) + 1(30 + 56) = 550 - 581 + 86 = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = 5(-90 + 7) - 11(-36 + 3) + 1(42 - 45) = -415 + 363 - 3 = -55$$

and $$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = 5(-56 - 30) + 7(42 - 45) + 11(12 + 24) = -430 - 21 + 396 = -55$$

So, by Cramer's rule, we obtain

$$x = \frac{D_1}{D} = \frac{55}{55} = 1, \quad y = \frac{D_2}{D} = -\frac{55}{55} = -1 \quad \text{and} \quad z = \frac{D_3}{D} = -\frac{55}{55} = -1.$$

Hence, $x = 1$, $y = -1$ and $z = -1$ is the solution of the given system of equations.

EXAMPLE 3 *Solve the system of equations $x + 2y = 3$ and $4x + 8y = 12$ by using determinants.*

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix} = 0, \quad D_1 = \begin{vmatrix} 3 & 2 \\ 12 & 8 \end{vmatrix} = 0 \text{ and } D_2 = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix} = 0.$$

Thus, $D = D_1 = D_2 = 0$

So, the given system has infinite number of solutions. Let $y = k$. Then,

$$x + 2y = 3 \Rightarrow x = 3 - 2k.$$

Hence, $x = 3 - 2k$, $y = k$ is the solution of the given system of equations, where k is an arbitrary real number.

EXAMPLE 4 *Show that the following system of equations is inconsistent:*

$$2x + y = 3, \quad 4x + 2y = 5.$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0 \quad \text{and} \quad D_1 = \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = 1 \neq 0.$$

Thus, we have $D = 0$ and $D_1 \neq 0$. So, the given system is inconsistent.

EXAMPLE 5 *By using determinants, solve the following system of equations:*

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 3z &= 4 \\ x + 3y + 5z &= 7 \end{aligned}$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix} = 1 \times (10 - 9) - 1 \times (5 - 3) + 1 \times (3 - 2) = 0,$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 3 \\ 7 & 3 & 5 \end{vmatrix} = 1 \times (10 - 9) - 1 \times (20 - 21) + 1 \times (12 - 14) = 0,$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 7 & 5 \end{vmatrix} = 1 \times (20 - 21) - 1 \times (5 - 3) + 1 \times (7 - 4) = 0,$$

and, $$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{vmatrix} = 1 \times (14 - 12) - 1 \times (7 - 4) + 1 \times (3 - 2) = 0.$$

Thus, we have $D = D_1 = D_2 = D_3 = 0$.

So, either the system is consistent with infinitely many solutions or it is inconsistent.

Consider the first two equations, these equations can be written as $\begin{aligned} x + y &= 1 - z \\ x + 2y &= 4 - 3z \end{aligned}$

In order to solve these equations let us use Cramer's rule.

Here, $D = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1, \quad D_1 = \begin{vmatrix} 1 - z & 1 \\ 4 - 3z & 2 \end{vmatrix} = 2 - 2z - 4 + 3z = z - 2$

and, $D_2 = \begin{vmatrix} 1 & 1 - z \\ 1 & 4 - 3z \end{vmatrix} = 4 - 3z - 1 + z = 3 - 2z.$

$\therefore \quad x = \dfrac{D_1}{D} \text{ and } y = \dfrac{D_2}{D}$

$\Rightarrow \quad x = z - 2, \quad y = 3 - 2z.$

Let $z = k$, where k is any real number. Then, we get $x = k - 2$, $y = 3 - 2k$ and $z = k$.
These values satisfy the third equation.

Hence, $x = k - 2$, $y = 3 - 2k$, $z = k$ is a solution of the given system of equation for every value of k.

EXAMPLE 6 *Using determinants, show that the following system of linear equation is inconsistent:*

$$x - 3y + 5z = 4$$
$$2x - 6y + 10z = 11$$
$$3x - 9y + 15z = 12$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 1 & -3 & 5 \\ 2 & -6 & 10 \\ 3 & -9 & 15 \end{vmatrix} = 0 \qquad [\because C_2 \text{ is proportional to } C_1]$$

$$D_1 = \begin{vmatrix} 4 & -3 & 5 \\ 11 & -6 & 10 \\ 12 & -9 & 15 \end{vmatrix} = -15 \begin{vmatrix} 4 & 1 & 1 \\ 11 & 2 & 2 \\ 12 & 3 & 3 \end{vmatrix} = 0 \qquad [\because C_2 \text{ and } C_3 \text{ are identical}]$$

$$D_2 = \begin{vmatrix} 1 & 4 & 5 \\ 2 & 11 & 10 \\ 3 & 12 & 15 \end{vmatrix} = 5 \begin{vmatrix} 1 & 4 & 1 \\ 2 & 11 & 2 \\ 3 & 12 & 3 \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

and, $$D_3 = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -6 & 11 \\ 3 & -9 & 12 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 11 \\ 3 & 3 & 12 \end{vmatrix} = 0 \qquad [\because C_1 \text{ and } C_2 \text{ are identical}]$$

$\therefore \quad D = D_1 = D_2 = D_3 = 0.$

So, the given system of equations may or may not be consistent.

If we now put $z = k$ in any two of three equations, we find that the two equations obtained are inconsistent as they represent a pair of parallel lines. Hence, the given system of equations is inconsistent.

REMARK *If we examine the given system of equations closely, we find that the three equations represent parallel planes. So, they have no point in common. Consequently the given system has no solution.*

EXAMPLE 7 *Using Cramer's rule, solve the following system of linear equations:*

$$(a+b)\,x - (a-b)\,y = 4ab$$
$$(a-b)\,x + (a+b)\,y = 2\,(a^2 - b^2)$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} a+b & -(a-b) \\ a-b & a+b \end{vmatrix} = (a+b)^2 + (a-b)^2 = 2\,(a^2 + b^2) \neq 0$$

So, the given system of equations has a unique solution.

Now, $$D_1 = \begin{vmatrix} 4ab & -(a-b) \\ 2\,(a^2 - b^2) & (a+b) \end{vmatrix}$$

$$\Rightarrow \quad D_1 = 2\,(a+b) \begin{vmatrix} 2ab & -(a-b) \\ a-b & 1 \end{vmatrix} \qquad [\text{Taking 2 common from } C_1 \text{ and } (a+b) \text{ from } R_2]$$

$$\Rightarrow \quad D_1 = 2\,(a+b)\,\{2\,ab + (a-b)^2\} = 2\,(a+b)\,(a^2 + b^2)$$

and, $$D_2 = \begin{vmatrix} a+b & 4ab \\ a-b & 2\,(a^2 - b^2) \end{vmatrix}$$

$$\Rightarrow \quad D_2 = 2(a-b)\begin{vmatrix} a+b & 2ab \\ 1 & (a+b) \end{vmatrix} \quad \text{[Taking } (a-b) \text{ common from } R_2 \text{ and 2 from } C_2]$$

$$\Rightarrow \quad D_2 = 2(a-b)\{(a+b)^2 - 2ab\} = 2(a-b)(a^2+b^2)$$

By Cramer's rule, we obtain

$$x = \frac{D_1}{D} = \frac{2(a+b)\ (a^2+b^2)}{2(a^2+b^2)} = a+b \text{ and, } y = \frac{D_2}{D} = \frac{2(a-b)\ (a^2+b^2)}{2(a^2+b^2)} = a-b.$$

Hence, $x = a+b, y = a-b$ is the solution of the given system of equations.

EXAMPLE 8 *Using determinants, show that the following system of equations is inconsistent:*

$$2x - y + z = 4, \quad x + 3y + 2z = 12, \quad 3x + 2y + 3z = 10.$$

SOLUTION The given system of equations is

$$2x - y + z = 4$$
$$x + 3y + 2z = 12$$
$$3x + 2y + 3z = 10.$$

Here, $D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 2(9-4) + 1(3-6) + 1(2-9) = 0$

and, $D_1 = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 10 & 2 & 3 \end{vmatrix} = 4(9-4) + 1(36-20) + (24-30) = 30 \neq 0.$

Hence, the given system of equations is inconsistent.

EXAMPLE 9 *Solve the following system of equations by using determinants:*

$$x + y + z = 1$$
$$ax + by + cz = k$$
$$a^2 x + b^2 y + c^2 z = k^2$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$\Rightarrow \quad D = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$\Rightarrow \quad D = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \quad \left[\begin{matrix}\text{Taking } (b-a) \text{ and } (c-a) \text{ common} \\ \text{from } C_2 \text{ and } C_3 \text{ respectively}\end{matrix}\right]$$

$$\Rightarrow \quad D = (b-a)(c-a) \times 1 \times \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \quad \text{[Expanding along } R_1]$$

$$\Rightarrow \quad D = (b-a)(c-a)(c+a-b-a) = (b-c)(c-a)(a-b) \quad \text{...(i)}$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-k)(k-b) \quad \text{[Replacing } a \text{ by } k \text{ in (i)]}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (k-c)(c-a)(a-k) \qquad \text{[Replacing } b \text{ by } k \text{ in (i)]}$$

and, $$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)(b-k)(k-a) \qquad \text{[Replacing } c \text{ by } k \text{ in (i)]}$$

$\therefore \quad x = \dfrac{D_1}{D}, \; y = \dfrac{D_2}{D}$ and $z = \dfrac{D_3}{D}$

$\Rightarrow \quad x = \dfrac{(b-c)(c-k)(k-b)}{(b-c)(c-a)(a-b)}, \; y = \dfrac{(k-c)(c-a)(a-k)}{(b-c)(c-a)(a-b)}$ and $z = \dfrac{(a-b)(b-k)(k-a)}{(a-b)(b-c)(c-a)}$

Hence, $x = \dfrac{(c-k)(k-b)}{(c-a)(a-b)}, \; y = \dfrac{(k-c)(a-k)}{(b-c)(a-b)}$ and $z = \dfrac{(b-k)(k-a)}{(b-c)(c-a)}$

is the solution of given system of equations.

EXAMPLE 10 *The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number we get 12. Use determinants to find the numbers.*

SOLUTION Let the three numbers be x, y and z. Then, from the given conditions, we obtain

$$\begin{aligned} x + y + z &= 6 \\ x + 2z &= 7 \\ 3x + y + z &= 12 \end{aligned} \qquad \text{or,} \qquad \begin{aligned} x + y + z &= 6 \\ x + 0y + 2z &= 7 \\ 3x + y + z &= 12 \end{aligned}$$

Here, $$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1(0-2) - 1(1-6) + 1(1-0) = -2 + 5 + 1 = 4$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{vmatrix} = 6(0-2) - 1(7-24) + 1(7-0) = -12 + 17 + 7 = 12$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 7 & 2 \\ 3 & 12 & 1 \end{vmatrix} = 1(7-24) - 6(1-6) + 1(12-21) = -17 + 30 - 9 = 4$$

and, $$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 0 & 7 \\ 3 & 1 & 12 \end{vmatrix} = 1(0-7) - 1(12-21) + 6(1-0) = -7 + 9 + 6 = 8$$

$$\therefore \quad x = \frac{D_1}{D} = \frac{12}{4} = 3, \; y = \frac{D_2}{D} = \frac{4}{4} = 1 \text{ and } z = \frac{D_3}{D} = \frac{8}{4} = 2.$$

Thus, the three numbers are 3, 1 and 2.

EXAMPLE 11 *Solve the following system of equations by Cramer's rule:*

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4, \; \frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1 \text{ and } \frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2.$$

SOLUTION Let $\dfrac{1}{x} = u, \; \dfrac{1}{y} = v$ and $\dfrac{1}{z} = w$. Then, the above system of equations can be written as

$$\begin{aligned} 2u + 3v + 10w &= 4 \\ 4u - 6v + 5w &= 1 \\ 6u + 9v - 20w &= 2 \end{aligned}$$

Here, $D = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36) = 150 + 330 + 720 = 1200$

$$D_1 = \begin{vmatrix} 4 & 3 & 10 \\ 1 & -6 & 5 \\ 2 & 9 & -20 \end{vmatrix} = 4(120-45) - 3(-20-10) + 10(9+12) = 300 + 90 + 210 = 600$$

$$D_2 = \begin{vmatrix} 2 & 4 & 10 \\ 4 & 1 & 5 \\ 6 & 2 & -20 \end{vmatrix} = 2(-20-10) - 4(-80-30) + 10(8-6) = -60 + 440 + 20 = 400$$

and, $D_3 = \begin{vmatrix} 2 & 3 & 4 \\ 4 & -6 & 1 \\ 6 & 9 & 2 \end{vmatrix} = 2(-12-9) - 3(8-6) + 4(36+36) = -42 - 6 + 288 = 240$

$\therefore \quad u = \frac{D_1}{D} = \frac{600}{1200} = \frac{1}{2} \Rightarrow \frac{1}{x} = \frac{1}{2} \Rightarrow x = 2, \; v = \frac{D_2}{D} = \frac{400}{1200} = \frac{1}{3} \Rightarrow \frac{1}{y} = \frac{1}{3} \Rightarrow y = 3,$

and, $w = \frac{D_3}{D} = \frac{240}{1200} = \frac{1}{5} \Rightarrow \frac{1}{z} = \frac{1}{5} \Rightarrow z = 5$

Hence, $x = 2, y = 3$ and $z = 5$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *If $f(x) = ax^2 + bx + c$ is a quadratic function such that $f(1) = 8$, $f(2) = 11$ and $f(-3) = 6$, find $f(x)$ by using determinants. Also, find $f(0)$.*

SOLUTION We have, $f(x) = ax^2 + bx + c$

$\therefore \quad f(1) = 8 \Rightarrow a + b + c = 8$

$f(2) = 11 \Rightarrow 4a + 2b + c = 11$ and, $f(-3) = 6 \Rightarrow 9a - 3b + c = 6$

Thus, we obtain the following system of equations

$$a + b + c = 8$$
$$4a + 2b + c = 11$$
$$9a - 3b + c = 6$$

For this system of equations, we have

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & -3 & 1 \end{vmatrix} = 1(2+3) - 1(4-9) + 1(-12-18) = 5 + 5 - 30 = -20$$

$$D_1 = \begin{vmatrix} 8 & 1 & 1 \\ 11 & 2 & 1 \\ 6 & -3 & 1 \end{vmatrix} = 8(2+3) - 1(11-6) + 1(-33-12) = 40 - 5 - 45 = -10$$

$$D_2 = \begin{vmatrix} 1 & 8 & 1 \\ 4 & 11 & 1 \\ 9 & 6 & 1 \end{vmatrix} = 1(11-6) - 8(4-9) + 1(24-99) = 5 + 40 - 75 = -30$$

and, $D_3 = \begin{vmatrix} 1 & 1 & 8 \\ 4 & 2 & 11 \\ 9 & -3 & 6 \end{vmatrix} = 1(12+33) - 1(24-99) + 8(-12-18) = 45 + 75 - 240 = -120$

$\therefore \quad a = \frac{D_1}{D} = \frac{-10}{-20} = \frac{1}{2}, \; b = \frac{D_2}{D} = \frac{-30}{-20} = \frac{3}{2}$ and $C = \frac{D_3}{D} = \frac{-120}{-20} = 6$

Hence, $f(x) = \frac{1}{2}x^2 + \frac{3}{2}x + 6$. Consequently, $f(0) = 6$.

EXAMPLE 13 *Determine the values of λ for which the following system of equations fail to have a unique solution:*

$$\lambda x + 3y - z = 1$$
$$x + 2y + z = 2$$
$$-\lambda x + y + 2z = -1$$

Does it have any solution for this value of λ?

SOLUTION The given system of equations will fail to have unique solution, if

$$D = 0$$

i.e. $$\begin{vmatrix} \lambda & 3 & -1 \\ 1 & 2 & 1 \\ -\lambda & 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow$ $$\lambda(4-1) - 3(2+\lambda) - (1+2\lambda) = 0$$

$\Rightarrow$ $$3\lambda - 6 - 3\lambda - 1 - 2\lambda = 0 \Rightarrow -2\lambda - 7 = 0 \Rightarrow \lambda = -\frac{7}{2}$$

For $\lambda = -\frac{7}{2}$, we obtain : $D_1 = \begin{vmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix} = -16 \neq 0.$

Thus, for $\lambda = -\frac{7}{2}$, we have $D = 0$ and $D_1 \neq 0$.

Hence, the given system of equations has no solution for $\lambda = -\frac{7}{2}$.

EXAMPLE 14 *For what values of a and b, the following system of equations is consistent?*

$$x + y + z = 6$$
$$2x + 5y + az = b$$
$$x + 2y + 3z = 14$$

SOLUTION The given system of equations is consistent, if $D \neq 0$ or, if $D = 0$, then $D_1 = D_2 = D_3 = 0$.

We have,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & a \\ 1 & 2 & 3 \end{vmatrix} = 15 - 2a - 6 + a + 4 - 5 = 8 - a$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ b & 5 & a \\ 14 & 2 & 3 \end{vmatrix} = 6(15-2a) - (3b - 14a) + (2b - 70) = 2a - b + 20$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 2 & b & a \\ 1 & 14 & 3 \end{vmatrix} = (3b - 14a) - 6(6-a) + (28-b) = -8a + 2b - 8$$

and, $$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 5 & b \\ 1 & 2 & 14 \end{vmatrix} = (70 - 2b) - (28 - b) + 6(4-5) = 36 - b$$

Now, $D \neq 0 \Rightarrow a - 8 \neq 0$ $a \neq 8$.

Thus, the given system of equations will be consistent and will have unique solution for $a \neq 8$.

For $a = 8$, we have

$$D = 0 \text{ and } D_1 = 36 - b, D_2 = 2b - 72, D_3 = 36 - b$$

Clearly, $D_1 = D_2 = D_3 = 0$ for $b = 36$.

Thus, for $a = 8$ and $b = 36$, we obtain : $D = D_1 = D_2 = D_3 = 0$.

Putting $a = 8$ and $b = 36$ the given system of equations reduces to

$$x + y + z = 6$$
$$2x + 5y + 8z = 36$$
$$x + 2y + 3z = 14$$

Taking $z = k$, first and third equations become

$$x + y = 6 - k$$
$$x + 2y = 14 - 3k$$

Solving these equations by Cramer's rule, we get

$$x = \frac{\begin{vmatrix} 6-k & 1 \\ 14-3k & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 12 - 2k - 14 + 3k = k - 2, \quad y = \frac{\begin{vmatrix} 1 & 6-k \\ 1 & 14-3k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} = 14 - 3k - 6 + k = 8 - 2k$$

Thus, we obtain : $x = k - 2, y = 8 - 2k, z = k$. Clearly, these values satisfy the second equation. Thus, the given system of equations will be consistent and will have infinitely many solutions for $a = 8$ and $b = 36$.

Hence, the given system of equation will be consistent if $a \neq 8$ and $b \in R$ or, if $a = 8$ and $b = 36$.

EXAMPLE 15 *For what values of a and b, the system of equations*

$$2x + ay + 6z = 8$$
$$x + 2y + bz = 5$$
$$x + y + 3z = 4$$

has: (i) *a unique solution* (ii) *infinitely many solutions* (iii) *no solution.*

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 2 & a & 6 \\ 1 & 2 & b \\ 1 & 1 & 3 \end{vmatrix}$$

$\Rightarrow$ $D = 2(6 - b) - a(3 - b) + 6(1 - 2)$

$\Rightarrow$ $D = 12 - 2b - 3a + ab - 6 = 6 - 3a - 2b + ab = (b - 3)(a - 2)$

$$D_1 = \begin{vmatrix} 8 & a & 6 \\ 5 & 2 & b \\ 4 & 1 & 3 \end{vmatrix}$$

$\Rightarrow$ $D_1 = 8(6 - b) - a(15 - 4b) + 6(5 - 8)$

$\Rightarrow$ $D_1 = 48 - 8b - 15a + 4ab - 18 = 30 - 15a - 8b + 4ab = (a - 2)(4b - 15)$

$$D_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & b \\ 1 & 4 & 3 \end{vmatrix}$$

$\Rightarrow$ $D_2 = 2(15 - 4b) - 8(3 - b) + 6(4 - 5) = 30 - 8b - 24 + 8b - 6 = 0$

and, $D_3 = \begin{vmatrix} 2 & a & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 2(8-5) - a(4-5) + 8(1-2) = 6 + a - 8 = a - 2$

(i) For unique solution, we must have

$D \neq 0 \Rightarrow (a-2)(b-3) \neq 0 \Rightarrow$ Neither $a \neq 2$, nor $b \neq 3$.

(ii) For infinitely many solutions, we must have

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow \quad (a-2)(b-3) = 0, \ (a-2)(4b-15) = 0$ and $a - 2 = 0$

$\Rightarrow \quad a = 2.$

Putting $a = 2$ in the given system of equations, we obtain

$$2x + 2y + 6z = 8$$
$$x + 2y + bz = 5$$
$$x + y + 3z = 4$$

This system is equivalent to the system

$$x + y + 3z = 4$$
$$x + 2y + bz = 5$$

Putting $z = k$, we get

$$x + y = 4 - k$$
$$x + 2y = 5 - bk$$

Solving these two equations, we get : $x = 3 - 2k + bk, \ y = 1 - bk + k$

Thus, the given system has infinitely many solutions given by

$$x = 3 - 2k + bk, y = 1 - bk + k, z = k, \text{ where } k \in R.$$

Hence, the system has infinitely many solutions for $a = 2$.

(iii) For no solution, we must have

$D = 0$ and at least one of D_1, D_2 and D_3 is non-zero.

Clearly, for $b = 3$, we have

$D = 0$ and $D_3 \neq 0$.

Hence, the system has no solution for $b = 3$.

EXERCISE 5.4

BASIC

Solve the following systems of linear equations by Cramer's rule:

1. $x - 2y = 4$, $-3x + 5y = -7$
2. $2x - y = 1$, $7x - 2y = -7$
3. $2x - y = 17$, $3x + 5y = 6$
4. $3x + y = 19$, $3x - y = 23$
5. $2x - y = -2$, $3x + 4y = 3$
6. $3x + ay = 4$, $2x + ay = 2, a \neq 0$
7. $2x + 3y = 10$, $x + 6y = 4$
8. $5x + 7y = -2$, $4x + 6y = -3$
9. $9x + 5y = 10$, $3y - 2x = 8$
10. $x + 2y = 1$, $3x + y = 4$

Solve the following system of the linear equations by Cramer's rule:

11. $3x + y + z = 2$, $2x - 4y + 3z = -1$, $4x + y - 3z = -11$
12. $x - 4y - z = 11$, $2x - 5y + 2z = 39$, $-3x + 2y + z = 1$
13. $6x + y - 3z = 5$, $x + 3y - 2z = 5$, $2x + y + 4z = 8$

14. $x + y = 5$
$y + z = 3$
$x + z = 4$

15. $2y - 3z = 0$
$x + 3y = -4$
$3x + 4y = 3$

16. $5x - 7y + z = 11$
$6x - 8y - z = 15$
$3x + 2y - 6z = 7$

17. $2x - 3y - 4z = 29$
$-2x + 5y - z = -15$
$3x - y + 5z = -11$

18. $x + y = 1$
$x + z = -6$
$x - y - 2z = 3$

19. $x + y + z + 1 = 0$
$ax + by + cz + d = 0$
$a^2x + b^2y + c^2z + d^2 = 0$

20. $x + y + z + w = 2$
$x - 2y + 2z + 2w = -6$
$2x + y - 2z + 2w = -5$
$3x - y + 3z - 3w = -3$

21. $2x - 3z + w = 1$
$x - y + 2w = 1$
$-3y + z + w = 1$
$x + y + z = 1$

Show that each of the following systems of linear equations is inconsistent:

22. $2x - y = 5$
$4x - 2y = 7$

23. $3x + y = 5$
$-6x - 2y = 9$

24. $3x - y + 2z = 3$
$2x + y + 3z = 5$
$x - 2y - z = 1$

25. $3x - y + 2z = 6$
$2x - y + z = 2$
$3x + 6y + 5z = 20.$

Show that each of the following systems of linear equations has infinite number of solutions and solve (26 - 30)

26. $x - y + z = 3$
$2x + y - z = 2$
$-x - 2y + 2z = 1$

27. $x + 2y = 5$
$3x + 6y = 15$

28. $x + y - z = 0$
$x - 2y + z = 0$
$3x + 6y - 5z = 0$

29. $2x + y - 2z = 4$
$x - 2y + z = -2$
$5x - 5y + z = -2$

30. $x - y + 3z = 6$
$x + 3y - 3z = -4$
$5x + 3y + 3z = 10$

31. A salesman has the following record of sales during three months for three items A, B and C which have different rates of commission.

Month	*Sale of units* A	B	C	*Total commission drawn (in ₹)*
Jan	90	100	20	800
Feb	130	50	40	900
March	60	100	30	850

Find out the rates of commission on items A, B and C by using determinant method.

32. An automobile company uses three types of steel S_1, S_2 and S_3 for producing three types of cars C_1, C_2 and C_3. Steel requirements (in tons) for each type of cars are given below:

Cars / *Steel*	C_1	C_2	C_3
S_1	2	3	4
S_2	1	1	2
S_3	3	2	1

Using Cramer's rule, find the number of cars of each type which can be produced using 29, 13 and 16 tonnes of steel of three types respectively.

ANSWERS

1. $x=-6, y=-5$
2. $x=-3, y=-7$
3. $x=7, y=-3$
4. $x=7, y=-2$
5. $x=-\frac{5}{11}, y=\frac{12}{11}$
6. $x=2, y=-\frac{2}{a}$
7. $x=\frac{16}{3}, y=-\frac{2}{9}$
8. $x=\frac{9}{2}, y=-\frac{7}{2}$
9. $x=-\frac{10}{37}, y=\frac{92}{37}$
10. $x=\frac{7}{5}, y=-\frac{1}{5}$
11. $x=-1, y=2, z=3$
12. $x=-1, y=-5, z=8$
13. $x=1, y=2, z=1$
14. $x=3, y=2, z=1$
15. $x=5, y=-3, z=-2$
16. $x=1, y=-1, z=-1$
17. $x=2, y=-3, z=-4$
18. $x=-2, y=3, z=-4$
19. $x=-\frac{(b-c)(c-d)(d-b)}{(a-b)(b-c)(c-a)}, y=-\frac{(a-d)(d-c)(c-a)}{(a-b)(b-c)(c-a)}, z=-\frac{(a-b)(b-d)(d-a)}{(a-b)(b-c)(c-a)}$
20. $x=-2, y=3, z=\frac{3}{2}, w=-\frac{1}{2}$
21. $x=1, y=-\frac{2}{7}, z=\frac{2}{7}, w=-\frac{1}{7}$
25. $x=-3, y=-1, z=7$
26. $x=\frac{5}{3}, y=k-\frac{4}{3}, z=k$
27. $x=5-2k, y=k$
28. $x=k, y=2k, z=3k$
29. $x=\frac{6+3k}{5}, y=\frac{8+4k}{5}, z=k$
30. $x=\frac{7-3k}{2}, y=\frac{3k-5}{2}, z=k$

HINTS TO SELECTED PROBLEMS

31. Let x, y and z be the rates of commission in ₹ of items A, B and C respectively. Then, we have

$$90x+100y+20z=800, 130x+50y+40z=900 \text{ and, } 60x+100y+30z=850$$

5.7.3 SOLUTION OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

In the previous sub-section, we have learnt about the solution of a non- homogeneous system of linear equations and its consistency and inconsistency.

Let us now consider a homogeneous system of equations given by

$$\begin{aligned} a_1 x+b_1 y+c_1 z &= 0 \\ a_2 x+b_2 y+c_2 z &= 0 \\ a_3 x+b_3 y+c_3 z &= 0 \end{aligned}$$

For this system of equations, we have

$$D_1=\begin{vmatrix} 0 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}=0,\ D_2=\begin{vmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & 0 & c_3 \end{vmatrix}=0 \text{ and, } D_3=\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}=0.$$

If $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then $x=\frac{D_1}{D}=0, y=\frac{D_2}{D}=0$ and $z=\frac{D_3}{D}=0$.

Thus, if $D \neq 0$, then the homogeneous system of equations has unique solution $x=0, y=0, z=0$. This solution is called the trivial solution.

If $D=0$, then a homogeneous system of equations has infinitely many solutions. Solutions other than the trivial solution are called *non-trivial* or non-zero solutions.

In order to solve a homogeneous system of equations by Cramer's rule, we may use the following algorithm.

ALGORITHM

Step I *Obtain the system of equations and compute D i.e. the determinant of the coefficient matrix.*

Step II *If $D \neq 0$, then the system has only the trivial solution i.e. $x = y = z = 0$. So, $x = 0 = y = z$ is the only solution of the given system.*

Step III *If $D = 0$, then take any two out of three equations and replace one of the variables z (say) by k. Solve the system so obtained by Cramer's rule. The solution so obtained with $z = k$ gives a solution of the given system.*

REMARK *It is evident from the above discussion that a homogeneous system of equations will have non-trivial solution iff $|D| = 0$.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Solve the following system of homogeneous equations:*

$$3x - 4y + 5z = 0$$
$$x + y - 2z = 0$$
$$2x + 3y + z = 0$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix} = 46 \neq 0.$$

So, the given system of equations has only the trivial solution i.e. $x = y = z = 0$.

EXAMPLE 2 *Solve the following system of homogeneous equations:*

$$x + y - z = 0$$
$$x - 2y + z = 0$$
$$3x + 6y - 5z = 0$$

SOLUTION For the given system of equations, we have

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 1 \\ 3 & 6 & -5 \end{vmatrix} = 1 \times (10 - 6) - 1 \times (-5 - 3) - 1 \times (6 + 6) = 4 + 8 - 12 = 0$$

So, the system has infinitely many solutions. Consider the first two equations. Putting $z = k$ in first two equations, we get

$$x + y = k$$
$$x - 2y = -k$$

Solving these equations by Cramer's rule, we obtain

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} k & 1 \\ -k & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-k}{-3} = \frac{k}{3} \quad \text{and,} \quad y = \frac{D_2}{D} = \frac{\begin{vmatrix} 1 & k \\ 1 & -k \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}} = \frac{-2k}{-3} = \frac{2k}{3}$$

Thus, we have $x = \dfrac{k}{3}, y = \dfrac{2k}{3}, z = k$. Clearly, these values satisfy the third equation.

Hence, $x = \frac{k}{3}, y = \frac{2k}{3}, z = k$ gives the solution for each value of k.

EXAMPLE 3 *Find the value of λ for which the homogeneous system of equations:*

$$2x + 3y - 2z = 0$$
$$2x - y + 3z = 0$$
$$7x + \lambda y - z = 0$$

has non-trivial solutions. Find the solution.

SOLUTION The given system of equations will have non-trivial solution, if

$D = 0$

i.e. $$\begin{vmatrix} 2 & 3 & -2 \\ 2 & -1 & 3 \\ 7 & \lambda & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad 2(1 - 3\lambda) - 3(-2 - 21) - 2(2\lambda + 7) = 0$$

$$\Rightarrow \quad 2 - 6\lambda + 69 - 4\lambda - 14 = 0 \Rightarrow -10\lambda + 57 = 0 \Rightarrow \lambda = \frac{57}{10}.$$

Hence, the given system of equations will have non-trivial solutions, if $\lambda = \frac{57}{10}$.

Let us now find solutions for this value of λ.

Taking first two equations and replacing z by k, we get
$$2x + 3y = 2k$$
$$2x - y = -3k$$

Solving these two equations by Cramer's rule, we get

$$x = \frac{\begin{vmatrix} 2k & 3 \\ -3k & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}} = \frac{-2k + 9k}{-2 - 6} = \frac{-7k}{8}, \; y = \frac{\begin{vmatrix} 2 & 2k \\ 2 & -3k \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}} = \frac{-10k}{-8} = \frac{5k}{4}.$$

Substituting these values of x and y in the third equation i.e. $7x + \lambda y - z = 0$, we obtain

$$\text{LHS} = 7 \times \frac{-7k}{8} + \lambda \times \frac{5k}{4} - k = \frac{-49}{8}k + \frac{57}{10} \times \frac{5k}{4} - k = 0 = \text{RHS} \qquad \left[\because \lambda = \frac{57}{10}\right]$$

Hence, $x = -\frac{7k}{8}$, $y = \frac{5k}{4}$, $z = k$ gives the solution of the given system of equations for each value of k.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *If the system of equations*

$$x = cy + bz$$
$$y = az + cx$$
$$z = bx + ay$$

has a non-trivial solution, show that $a^2 + b^2 + c^2 + 2abc = 1$

SOLUTION The given system of equations can be written as

$$x - cy - bz = 0$$
$$cx - y + az = 0$$
$$bx + ay - z = 0$$

If it has a non-trivial solution, then

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1 \times (1 - a^2) + c(-c - ab) - b(ca + b) = 0$

$\Rightarrow \quad 1 - a^2 - c^2 - abc - abc - b^2 = 0 \Rightarrow a^2 + b^2 + c^2 + 2abc = 1.$

EXAMPLE 5 *If a, b, c are distinct real numbers and the system of equations*

$$ax + a^2 y + (a^3 + 1) z = 0$$
$$bx + b^2 y + (b^3 + 1) z = 0$$
$$cx + c^2 y + (c^3 + 1) z = 0$$

has a non-trivial solution, show that $abc = -1$.

SOLUTION It is given that the given system of homogeneous linear equations has a non-trivial solution.

$$\therefore \quad \begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = 0$$

$\Rightarrow \quad (a-b)(b-c)(c-a)(1+abc) = 0$ [See Example 30 on page 5.28]

$\Rightarrow \quad 1 + abc = 0$ $[\because\ a \neq b \neq c \therefore\ a - b \neq 0, b - c \neq 0, c - a \neq 0]$

$\Rightarrow \quad abc = -1.$

EXAMPLE 6 *If x, y, z are not all zero such that*

$$ax + y + z = 0$$
$$x + by + z = 0$$
$$x + y + cz = 0,$$

then prove that $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = 1.$

SOLUTION It is given that x, y, z are not all zero. This means that there are non-trivial solutions of the given system of equations.

$$\therefore \quad \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow abc - a - c - b + 2 = 0 \Rightarrow abc = a + b + c - 2 \quad \text{...(i)}$$

Now, $\dfrac{1}{1-a} + \dfrac{1}{1-b} + \dfrac{1}{1-c} = \dfrac{(1-b)(1-c) + (1-c)(1-a) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$

$$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{1 - (a+b+c) + (ab+bc+ca) - abc}$$

$$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{1 - (a+b+c) + (ab+bc+ca) - (a+b+c) + 2} \quad \text{[Using (i)]}$$

$$= \frac{3 - 2(a+b+c) + (ab+bc+ca)}{3 - 2(a+b+c) + (ab+bc+ca)} = 1.$$

EXERCISE 5.5

BASIC

Solve each of the following system of homogeneous linear equations:

1. $x + y - 2z = 0$
   $2x + y - 3z = 0$
   $5x + 4y - 9z = 0$

2. $2x + 3y + 4z = 0$
   $x + y + z = 0$
   $2x + 5y - 2z = 0$

3. $3x + y + z = 0$
   $x - 4y + 3z = 0$
   $2x + 5y - 2z = 0$

BASED ON HOTS

4. Find the real values of λ for which the following system of linear equations has non-trivial solutions. Also, find the non-trivial solutions
$$2\lambda x - 2y + 3z = 0$$
$$x + \lambda y + 2z = 0$$
$$2x + \lambda z = 0$$
5. If a, b, c are non-zero real numbers and if the system of equations
$$(a-1)x = y + z$$
$$(b-1)y = z + x$$
$$(c-1)z = x + y$$
has a non-trivial solution, then prove that $ab + bc + ca = abc$.

ANSWERS

1. $x = k, y = k, z = k$, where $k \in R$
2. $x = 0, y = 0, z = 0$
3. $x = -7k, y = 8k, z = 13k$, where, $k \in R$
4. $\lambda = 2, x = k,\ y = \frac{k}{2}, z = -k$ where $k \in R$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $A = \text{diag}(1, 2, 3)$, then $|A| =$ ....
2. If A is a skew-symmetric matrix of order 3×3, then $|A| =$ .... [CBSE 2020]
3. If the matrix $A = \begin{bmatrix} 1 & 3 & x+2 \\ 2 & 4 & 8 \\ 3 & 5 & 10 \end{bmatrix}$ is singular, then $x =$ ....
4. If A and B are non-singular square matrices of order n such that $A = kB$, then $\frac{|A|}{|B|} =$ ....
5. The set of real values of a for which the matrix $A = \begin{bmatrix} a & 2 \\ 2 & 4 \end{bmatrix}$ is non-singular is ....
6. If A is a 2×2 matrix such that $|A| = 5$, then $|5A| =$ ....
7. If A and B are square matrices of order 3 such that $|A| = -1, |B| = 3$, then $|3AB| =$ .... [CBSE 2020]
8. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, then $\alpha =$ ....
9. If $A = \begin{bmatrix} \ln x & -1 \\ -\ln x & 2 \end{bmatrix}$ and if $\det(A) = 2$, then $x =$............................ .
10. If I is the identity matrix of order 10, then determinant of I is ....
11. If $|A|$ denotes the value of the determinant of a square matrix of order 3, then $|-2A| =$ ....
12. Let $A = [a_{ij}]$ be a 3×3 matrix such that $|A| = 5$. If $C_{ij} =$ Cofactor of a_{ij} in A. Then $a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} =$ ....
13. In the above question, $a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} =$ ....
14. If $A = \text{diag}(2, 3, 4)$, then $|A^2| =$ ....
15. Let $A = [a_{ij}]$ be a square matrix of order 3 with $|A| = 2$ and let $C = [c_{ij}]$, where $c_{ij} =$ cofactor of a_{ij} in A. Then, $|C| =$ ....

16. The value of the determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3x & 6x & 9x \end{vmatrix}$ is ....

17. The value of the determinant $\Delta = \begin{vmatrix} \sec^2\theta & \tan^2\theta & 1 \\ \tan^2\theta & \sec^2\theta & -1 \\ 22 & 20 & 2 \end{vmatrix}$ is ....

18. The value of the determinant $\Delta = \begin{vmatrix} 0 & x-y & y-z \\ y-x & 0 & z-x \\ z-y & x-z & 0 \end{vmatrix}$ is ....

19. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be a square matrices of order 3 such that
$b_{i1} = 2\,a_{i1},\quad b_{i2} = 3\,a_{i2}$ and $b_{i3} = 4\,a_{i3}, i = 1, 2, 3$
If $|A| = 5$, then $|B| =$ ....

20. If A and B are square matrices of order 3 such that $|A| = -2, |B| = 4$, then $|2AB| =$ ....

21. The value of determinant $\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & \sin y \end{vmatrix}$ depends on..................... only.

[NCERT EXEMPLAR]

22. If $x, y, z \in R$, then the value of the determinant $\begin{vmatrix} (2^x + 2^{-x})^2 & (2^x - 2^{-x})^2 & 1 \\ (3^x + 3^{-x})^2 & (3^x - 3^{-x})^2 & 1 \\ (4^x + 4^{-x})^2 & (4^x - 4^{-x})^2 & 1 \end{vmatrix}$ is equal to............................ . [NCERT EXEMPLAR]

23. If $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 +$ ...., then $A =$........... .

[NCERT EXEMPLAR]

24. If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 =$ .... [NCERT EXEMPLAR]

25. If A is a matrix of order 3×3, then the number of minors in A is .... [NCERT EXEMPLAR]

26. If $x = -9$ is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are .... [NCERT EXEMPLAR]

27. $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} =$............................ . [NCERT EXEMPLAR]

28. If A and B are square matrices of order 3 and $|A| = 5, |B| = 5$, then $|3AB| =$ ....

29. If $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, then a, b, c are in............................ .

30. The value of the determinant $\begin{vmatrix} \sin A & \cos A & \sin A+\cos B \\ \sin B & \cos A & \sin B+\cos B \\ \sin C & \cos A & \sin C+\cos B \end{vmatrix}$ is.............................. .

31. If the determinant $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ splits into exactly k determinants of order 3, each element of which contains only one term, then $k=$............................. .

32. The maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ is ....

33. If A, B, C are the angles of a triangle, then $\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}=$........ .[NCERT EXEMPLAR]

34. The determinant $A=\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ is equal to is equal to............. [NCERT EXEMPLAR]

35. The value of the determinant $\Delta=\begin{vmatrix} \sin^2 33° & \sin^2 57° & \cos 180° \\ -\sin^2 57° & -\sin^2 33° & \cos^2 180° \\ \cos 180° & \sin^2 33° & \sin^2 57° \end{vmatrix}$ is equal to .... [NCERT EXEMPLAR]

36. If $\begin{vmatrix} 2-x & 2+x & 2+x \\ 2+x & 2-x & 2+x \\ 2+x & 2+x & 2-x \end{vmatrix}=0$, then $x=$............................ . [NCERT EXEMPLAR]

37. If A is a square matrix of order 3 and $|A|=2$, then $|-AA^T|=$.................... [CBSE 2020]

38. If $A=\begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 17 \\ 0 & -10 \end{bmatrix}$, then $|AB|=$.................... [CBSE 2020]

ANSWERS

1. 6	**2.** 0	**3.** 4	**4.** k^n	**5.** $R-\{1\}$
6. 125	**7.** -81	**8.** ± 3	**9.** e^2	**10.** 1
11. $-8\|A\|$	**12.** 5	**13.** 0	**14.** 576	**15.** 4
16. 0	**17.** 0	**18.** 0	**19.** 120	**20.** -64
21. y	**22.** zero	**23.** zero	**24.** $\frac{1}{2}$	**25.** 9
26. 2, 7	**27.** $(y-z)(z-x)(y-x+xyz)$	**28.** 405	**29.** AP	
30. zero	**31.** 8	**32.** $\frac{1}{2}$	**33.** zero	**34.** zero
35. zero	**36.** $0, -6$	**37.** -4	**38.** -100	

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If A is a singular matrix, then write the value of $|A|$.
2. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular? **[CBSE 2011]**
3. Write the value of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 2x & 3x & 4x \\ 5 & 6 & 8 \end{vmatrix}$.
4. State whether the matrix $\begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix}$ is singular or nonsingular.
5. Find the value of the determinant $\begin{vmatrix} 4200 & 4201 \\ 4202 & 4203 \end{vmatrix}$.
6. Find the value of the determinant $\begin{vmatrix} 101 & 102 & 103 \\ 104 & 105 & 106 \\ 107 & 108 & 109 \end{vmatrix}$
7. Write the value of the determinant $\begin{vmatrix} a & 1 & b+c \\ b & 1 & c+a \\ c & 1 & a+b \end{vmatrix}$.
8. If $A = \begin{bmatrix} 0 & i \\ i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find the value of $|A| + |B|$.
9. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$, find $|AB|$.
10. Evaluate: $\begin{vmatrix} 4785 & 4787 \\ 4789 & 4791 \end{vmatrix}$.
11. If w is an imaginary cube root of unity, find the value of $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$.
12. If $A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 3 & -2 \end{bmatrix}$, find $|AB|$.
13. If $A = [a_{ij}]$ is a 3×3 diagonal matrix such that $a_{11} = 1$, $a_{22} = 2$ and $a_{33} = 3$, then find $|A|$.
14. If $A = [a_{ij}]$ is a 3×3 scalar matrix such that $a_{11} = 2$, then write the value of $|A|$.
15. If I_3 denotes identity matrix of order 3×3, write the value of its determinant.
16. A matrix A of order 3×3 has determinant 5. What is the value of $|3A|$? **[CBSE 2012]**
17. On expanding by first row, the value of the determinant of 3×3 square matrix $A = [a_{ij}]$ is $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$, where C_{ij} is the cofactor of a_{ij} in A. Write the expression for its value on expanding by second column.
18. Let $A = [a_{ij}]$ be a square matrix of order 3×3 and C_{ij} denote cofactor of a_{ij} in A. If $|A| = 5$, write the value of $a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$.
19. In question 18, write the value of $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}$. **[CBSE 2020]**
20. Write the value of $\begin{vmatrix} \sin 20° & -\cos 20° \\ \sin 70° & \cos 70° \end{vmatrix}$.

21. If A is a square matrix satisfying $A^T A = I$, write the value of $|A|$.

22. If A and B are square matrices of the same order such that $|A| = 3$ and $AB = I$, then write the value of $|B|$.

23. A is a skew-symmetric of order 3, write the value of $|A|$.

24. If A is a square matrix of order 3 with determinant 4, then write the value of $|-A|$.

25. If A is a square matrix such that $|A| = 2$, write the value of $|AA^T|$.

26. Find the value of the determinant $\begin{vmatrix} 243 & 156 & 300 \\ 81 & 52 & 100 \\ -3 & 0 & 4 \end{vmatrix}$.

27. Write the value of the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ 6 & -9 & 15 \end{vmatrix}$.

28. If the matrix $\begin{bmatrix} 5x & 2 \\ -10 & 1 \end{bmatrix}$ is singular, find the value of x.

29. If A is a square matrix of order $n \times n$ such that $|A| = \lambda$, then write the value of $|-A|$.

30. Find the value of the determinant $\begin{vmatrix} 2^2 & 2^3 & 2^4 \\ 2^3 & 2^4 & 2^5 \\ 2^4 & 2^5 & 2^6 \end{vmatrix}$.

31. If A and B are non-singular matrices of the same order, write whether AB is singular or non-singular.

32. A matrix of order 3×3 has determinant 2. What is the value of $|A(3I)|$, where I is the identy matrix of order 3×3.

33. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then find the value of $|3AB|$.

34. Write the value of $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$. **[CBSE 2008]**

35. Write the cofactor of a_{12} in the matrix $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$. **[CBSE 2008]**

36. If $\begin{vmatrix} 2x+5 & 3 \\ 5x+2 & 9 \end{vmatrix} = 0$, find x. **[CBSE 2008]**

37. Find the value of x from the following: $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$ **[CBSE 2009]**

38. Write the value of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$ **[CBSE 2009]**

39. If $|A| = 2$, where A is 2×2 matrix, find $|\text{adj } A|$. **[CBSE 2010]**

40. What is the value of the determinant $\begin{vmatrix} 0 & 2 & 0 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$? **[CBSE 2010]**

41. For what value of x is the matrix $\begin{bmatrix} 6-x & 4 \\ 3-x & 1 \end{bmatrix}$ singular? **[CBSE 2011]**

42. A matrix A of order 3×3 is such that $|A| = 4$. Find the value of $|2A|$. **[CBSE 2011]**

43. Evaluate: $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$. **[CBSE 2011]**

44. If $A = \begin{bmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$. Write the cofactor of the element a_{32}. **[CBSE 2012]**

45. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of x. **[CBSE 2013]**

46. If $\begin{vmatrix} 2x & x+3 \\ 2(x+1) & x+1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 3 & 3 \end{vmatrix}$, then write the value of x. **[CBSE 2013]**

47. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x. **[CBSE 2014]**

48. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, write the value of x. **[CBSE 2014]**

49. If A is a 3×3 matrix, $|A| \neq 0$ and $|3A| = k|A|$ then write the value of k. **[CBSE 2014]**

50. Write the value of the determinant $\begin{vmatrix} p & p+1 \\ p-1 & p \end{vmatrix}$. **[CBSE 2014]**

51. Write the value of the determinant $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ -3 & -3 & -3 \end{vmatrix}$. **[CBSE 2015]**

52. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then for any natural number, find the value of Det (A^n). **[CBSE 2015]**

53. Find the maximum value of $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ **[CBSE 2016]**

54. If $x \in N$ and $\begin{vmatrix} x+3 & -2 \\ -3x & 2x \end{vmatrix} = 8$, then find the value of x. **[CBSE 2016]**

55. If $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix} = 8$, write the value of x. **[CBSE 2016]**

56. If A is a 3×3 matrix, then what will be the value of k if Det $(A^{-1}) = (\text{Det } A)^k$? **[CBSE 2017]**

57. A and B are square matrices of the same order 3, such that $AB = 2I$ and $|A| = 2$, write the value of $|B|$. **[CBSE 2019]**

58. A square matrix A is said to be singular, if .... **[CBSE 2020]**

ANSWERS

1. 0 **2.** 3 **3.** 0 **4.** Non-singular **5.** -2 **6.** 0 **7.** 0

8. 0 **9.** 0 **10.** -8 **11.** 0 **12.** -70 **13.** 6 **14.** 8 **15.** 1

16. 135 **17.** $a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$ **18.** 5 **19.** 0 **20.** 1 **21.** ± 1

22. 1/3 **23.** 0 **24.** -4 **25.** 4 **26.** 0 **27.** 0 **28.** -4 **29.** $(-1)^n \lambda$

30. 0 **31.** Non-singular **32.** 54 **33.** -81 **34.** $a^2 + b^2 + c^2 + d^2$ **35.** 46

36. -13 **37.** ± 2 **38.** 0 **39.** 8 **40.** 8 **41.** 2 **42.** 32 **43.** 0

44. 11 **45.** 2 **46.** 1 **47.** -2 **48.** ± 6 **49.** 27 **50.** 1 **51.** 0

52. 1 **53.** $\frac{1}{2}$ **54.** ± 2 **55.** $-2, -2w, -2w^2$ **56.** $k = -1$ **57.** 4 **58.** 0