CHAPTER 22

BRIEF REVIEW OF CARTESIAN SYSTEM OF RECTANGULAR CO-ORDINATES

22.1 CARTESIAN CO-ORDINATE SYSTEM

RECTANGULAR CO-ORDINATE AXES Let $X'OX$ and $Y'OY$ be two mutually perpendicular lines through any point O in the plane of the paper. We call the point O, the origin. Now choose a convenient unit of length and starting from the origin as zero, mark off a number scale on the horizontal line $X'OX$, positive to the right of the origin O and negative to the left of origin O. Also, mark off the same scale on the vertical line $Y'OY$, positive upwards and negative downwards of the origin O.

The line $X'OX$ is called the x-axis or axis of x, the line $Y'OY$ is known as the y-axis or axis of y, and the two lines taken together are called the co-ordinate axes or the axes of coordinates.

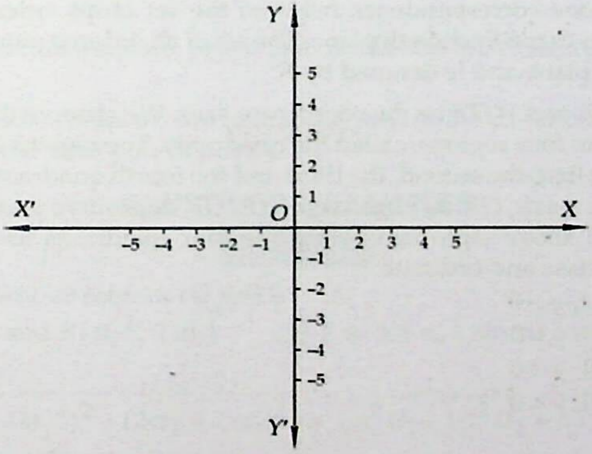

Fig. 22.1

CARTESIAN CO-ORDINATES OF A POINT Let $X'OX$ and $Y'OY$ be the co-ordinate axes, and let P be any point in the plane. Draw perpendiculars PM and PN from P on x *and* y-axis respectively. The length of the directed line segment OM in the units of scale chosen is called the *x-coordinate or abscissa of point P*. Similarly, the length of the directed line segment ON on the same scale is called the *y-coordinate or ordinate* of point P. Let $OM = x$ and $ON = y$. Then the position of the point P in the plane with respect to the coordinate axes is represented by the ordered (x, y). The ordered pair (x, y) is called the coordinates of point P.

Thus, for a given point, the abscissa and ordinate are the distances of the given point from y-axis and x-axis respectively.

The above system of coordinating an ordered pair (x, y) with every point in a plane is called the *Rectangular Cartesian coordinate system.*

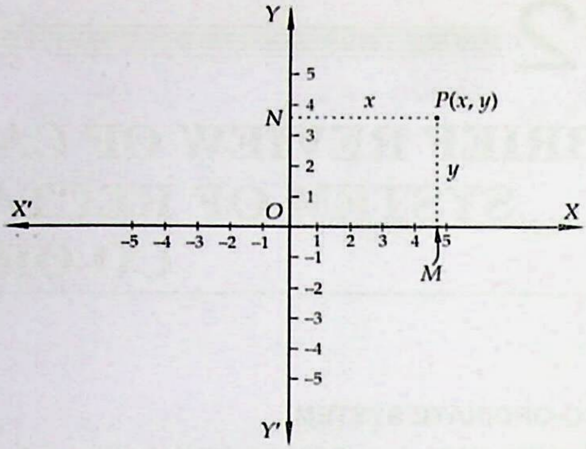

Fig. 22.2

It follows from the above discussion that corresponding to every point P in the Euclidean plane there is a unique ordered pair (x, y) of real numbers called its Cartesian coordinates. Conversely, when we are given an ordered pair (x, y) and a Cartesian coordinate system, we can determine a point in the Euclidean plane having its coordinates (x, y). For this we mark off a directed line segment $OM = x$ on the x-axis and another directed line segment $ON = y$ on y-*axis*. Now, draw perpendiculars at M and N to X and Y axes respectively. The point of intersection of these two perpendiculars determines point P in the Euclidean space having coordinates (x, y).

Thus, there is one-to-one correspondence between the set of all ordered pairs (x, y) of real numbers and the points in the Euclidean plane. The set of all ordered pairs (x, y) of real numbers is called the Cartesian plane and is denoted by R^2.

QUADRANTS Let $X'OX$ and $Y'OY$ be the coordinate axes. We observe that the two axes divide the Euclidean plane into four regions, called the quadrants. The regions $XOY, X'OY, X'OY'$ and $Y'OX$ are known as the first, the second, the third and the fourth quadrants respectively. The ray OX is taken as positive x-*axis*, OX' as negative x-axis, OY as positive y-axis and OY' as negative y-axis. In view of the above sign convention the four quadrants are characterised by the following signs of abscissa and ordinate.

I Quadrant : $x > 0, y > 0$
II Quadrant : $x < 0, y > 0$
III Quadrant : $x < 0, y < 0$
IV Quadrant : $x > 0, y < 0$

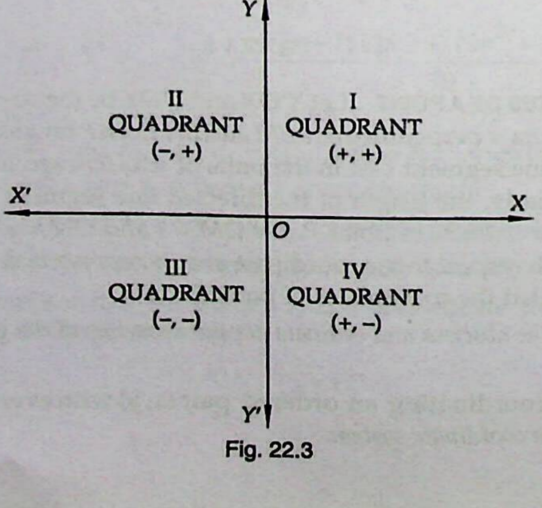

Fig. 22.3

The coordinates of the origin are taken as (0, 0). The coordinates of any point on x-axis are of the form (x, 0) and the coordinates of any point on y-axis are of the form (0, y). Thus, if the abscissa of a point is zero, it would lie somewhere on the y-axis and if its ordinate is zero it would lie on x-axis.

It follows from the above discussion that by simply looking at the coordinates of a point we can tell in which quadrant it would lie.

22.2 DISTANCE BETWEEN TWO POINTS

The distance between any two points in the plane is the length of the line segment joining them.

The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e. $$PQ = \sqrt{(\text{Difference of abscisse})^2 + (\text{Difference of ordinates})^2}$$

SOME USEFUL POINTS

(I) In order to prove that a given figure is a

(i) square, prove that the four sides are equal and the diagonals are also equal.

(ii) rhombus, prove that the four sides are equal.

(iii) rectangle, prove that opposite sides are equal and the diagonals are also equal.

(iv) a parallelogram, prove that the opposite sides are equal.

(v) parallelogram but not a rectangle, prove that its opposite sides are equal but the diagonals are not equal.

(vi) a rhombus but not a square, prove that its all sides are equal but the diagonals are not equal.

(II) For three points to be collinear, prove that the sum of the distances between two pairs of points is equal to the third pair of points.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the points*

(i) $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$ (ii) $L(a\cos\alpha, a\sin\alpha)$ and $M(a\cos\beta, a\sin\beta)$

SOLUTION (i) Clearly,

$$AB = \sqrt{(at_2^2 - at_1^2)^2 + (2at_2 - 2at_1)^2} = \sqrt{a^2(t_2 - t_1)^2(t_2 + t_1)^2 + 4a^2(t_2 - t_1)^2}$$

$$\Rightarrow \quad AB = a(t_2 - t_1)\sqrt{(t_2 + t_1)^2 + 4}$$

(ii)
$$LM = \sqrt{(a\cos\beta - a\cos\alpha)^2 + (a\sin\beta - a\sin\alpha)^2}$$
$$= \sqrt{a^2(\cos\beta - \cos\alpha)^2 + a^2(\sin\beta - \sin\alpha)^2}$$
$$= a\sqrt{(\cos\beta - \cos\alpha)^2 + (\sin\beta - \sin\alpha)^2}$$
$$= a\sqrt{\cos^2\beta + \cos^2\alpha + \sin^2\beta + \sin^2\alpha - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}$$
$$= a\sqrt{(\cos^2\beta + \sin^2\beta) + (\cos^2\alpha + \sin^2\alpha) - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)}$$
$$= a\sqrt{1 + 1 - 2\cos(\alpha - \beta)} = a\sqrt{2\{1 - \cos(\alpha - \beta)\}}$$
$$= a\sqrt{2 \times 2\sin^2\left(\frac{\alpha - \beta}{2}\right)} = 2a\sin\left(\frac{\alpha - \beta}{2}\right)$$

EXAMPLE 2 *Show that four points (0, – 1), (6, 7), (– 2, 3) and (8, 3) are the vertices of a rectangle.*

SOLUTION Let $A(0, -1)$, $B(6, 7)$, $C(-2, 3)$ and $D(8, 3)$ be the given points. Then,

$$AD = \sqrt{(8-0)^2 + (3+1)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$BC = \sqrt{(6+2)^2 + (7-3)^2} = \sqrt{64+16} = 4\sqrt{5}$$

$$AC = \sqrt{(-2-0)^2 + (3+1)^2} = \sqrt{4+16} = 2\sqrt{5}$$

and, $$BD = \sqrt{(8-6)^2 + (3-7)^2} = \sqrt{4+16} = 2\sqrt{5}$$

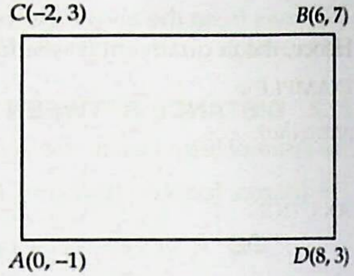

Fig. 22.4

$\therefore$ $AD = BC$ and $AC = BD$.

So, $ADBC$ is a parallelogram.

Now, $AB = \sqrt{(6-0)^2 + (7+1)^2} = \sqrt{36+64} = 10$ and, $CD = \sqrt{(8+2)^2 + (3-3)^2} = 10$.

Clearly, $AB^2 = AD^2 + DB^2$ and $CD^2 = CB^2 + BD^2$. Hence, $ADBC$ is a rectangle.

LEVEL-2

EXAMPLE 3 *If the two vertices of an equilateral triangle be (0, 0), $(3, \sqrt{3})$, find the third vertex.*

SOLUTION $O(0, 0)$ and $A(3, \sqrt{3})$ be the given points and let $B(x, y)$ be the third vertex of equilateral $\Delta\, OAB$. Then,

$$OA = OB = AB \Rightarrow OA^2 = OB^2 = AB^2 \quad \text{...(i)}$$

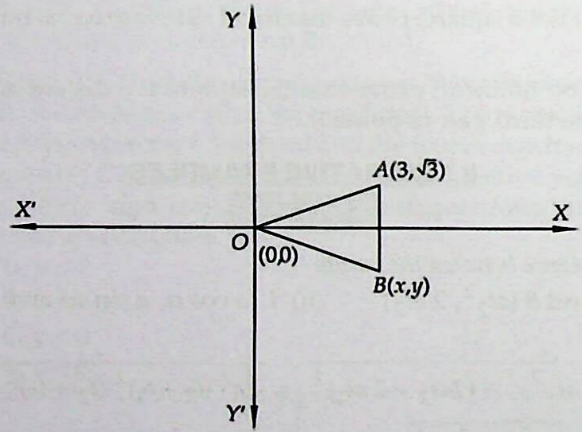

Fig. 22.5

Clearly, $OA^2 = (3-0)^2 + (\sqrt{3}-0)^2 = 12$, $OB^2 = x^2 + y^2$

and, $AB^2 = (x-3)^2 + (y-\sqrt{3})^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

Now, $OA^2 = OB^2 = AB^2$ [From (i)]

$\Rightarrow$ $OA^2 = OB^2$ and $OB^2 = AB^2$

$\Rightarrow$ $x^2 + y^2 = 12$ and $x^2 + y^2 = x^2 + y^2 - 6x - 2\sqrt{3}\,y + 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $6x + 2\sqrt{3}\,y = 12$

$\Rightarrow$ $x^2 + y^2 = 12$ and $3x + \sqrt{3}\,y = 6$

$\Rightarrow$ $x^2 + \left(\dfrac{6-3x}{\sqrt{3}}\right)^2 = 12$ $\left[\because 3x + \sqrt{3}\,y = 6 \therefore y = \dfrac{6-3x}{\sqrt{3}}\right]$

$\Rightarrow \quad 3x^2 + (6-3x)^2 = 36 \Rightarrow 12x^2 - 36x = 0 \Rightarrow x = 0, 3$

Putting $x = 0$ and 3 respectively in $y = \dfrac{6-3x}{\sqrt{3}}$, we get $y = 2\sqrt{3}$, and $y = -\sqrt{3}$ respectively.

Hence, the coordinates of the third vertex B are $(0, 2\sqrt{3},)$ or $(3, -\sqrt{3})$.

EXAMPLE 4 *If the segments joining the points A (a, b) and B (c, d) subtends an angle θ at the origin, prove that:* $\cos\theta = \dfrac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}$.

SOLUTION Let O be the origin. Then,

$$OA^2 = a^2 + b^2, OB^2 = c^2 + d^2 \text{ and, } AB^2 = (c-a)^2 + (d-b)^2$$

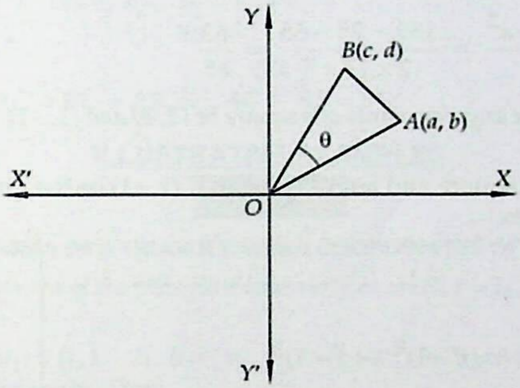

Fig. 22.6

Using cosine formula in ΔOAB, we have

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta$$

$$\Rightarrow \quad (c-a)^2 + (d-b)^2 = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

$$\Rightarrow \quad c^2 + a^2 - 2ac + d^2 + b^2 - 2bd = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

$$\Rightarrow \quad 2(ac+bd) = 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{ac+bd}{\sqrt{a^2+b^2}\sqrt{c^2+d^2}}.$$

EXAMPLE 5 *Find the coordinates of the circumcentre of the triangle whose vertices are (8, 6), (8, –2) and (2, –2). Also, find its circum-radius.*

SOLUTION Let $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$ be the vertices of the given triangle and let $P(x, y)$ be the circumcentre of this triangle. Then, $PA^2 = PB^2 = PC^2$

Now, $\quad PA^2 = PB^2$

$$\Rightarrow \quad (x-8)^2 + (y-6)^2 = (x-8)^2 + (y+2)^2$$

$$\Rightarrow \quad x^2 + y^2 - 16x - 12y + 100 = x^2 + y^2 - 16x + 4y + 68$$

$$\Rightarrow \quad 16y = 32 \Rightarrow y = 2$$

Again, $\quad PB^2 = PC^2$

$$\Rightarrow \quad (x-8)^2 + (y+2)^2 = (x-2)^2 + (y+2)^2$$

$$\Rightarrow \quad x^2 + y^2 - 16x + 4y + 68 = x^2 + y^2 - 4x + 4y + 8$$

$$\Rightarrow \quad 12x = 60 \Rightarrow x = 5.$$

So, the coordinates of the circumcentre P are $(5, 2)$.

Also, Circum-radius $= PA = PB = PC = \sqrt{(5-8)^2 + (2-6)^2} = 5.$

EXAMPLE 6 *The vertices of a triangle are A (1, 1), B (4, 5) and C (6, 13). Find cos A.*

SOLUTION We know that: $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where $a = BC, b = CA$ and $c = AB$ are the sides of the triangle ABC.

Here,

$$a = BC = \sqrt{(4-6)^2 + (5-13)^2} = \sqrt{68},\ b = CA = \sqrt{(6-1)^2 + (13-1)^2} = \sqrt{169} = 13,$$

and, $$c = AB = \sqrt{(4-1)^2 + (5-1)^2} = 5$$

$$\therefore \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{169 + 25 - 68}{2 \times 13 \times 5} = \frac{63}{65}$$

EXAMPLE 7 *Let the opposite angular points of a square be (3, 4) and (1, –1). Find the coordinates of the remaining angular points.*

SOLUTION Let $ABCD$ be a square and let $A\ (3, 4)$ and $C\ (1, -1)$ be the given angular points. Let $B\ (x, y)$ be the unknown vertex.

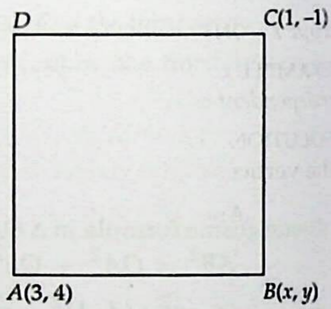

Fig. 22.7

Then, $AB = BC$

$\Rightarrow \quad AB^2 = BC^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-1)^2 + (y+1)^2$

$\Rightarrow \quad 4x + 10y - 23 = 0$

$\Rightarrow \quad x = \dfrac{23 - 10y}{4}$ ...(i)

Applying Pythagoras Theorem in triangle ABC, we obtain

$AB^2 + BC^2 = AC^2$

$\Rightarrow \quad (x-3)^2 + (y-4)^2 + (x-1)^2 + (y+1)^2 = (3-1)^2 + (4+1)^2$

$\Rightarrow \quad x^2 + y^2 - 4x - 3y - 1 = 0$ ...(ii)

Substituting the value of x from (i) into (ii), we get

$$\left(\frac{23 - 10y}{4}\right)^2 + y^2 - (23 - 10y) - 3y - 1 = 0$$

$\Rightarrow \quad 4y^2 - 12y + 5 = 0 \Rightarrow (2y-1)(2y-5) = 0 \Rightarrow y = \dfrac{1}{2}$ or, $\dfrac{5}{2}$

Putting $y = \dfrac{1}{2}$ and $y = \dfrac{5}{2}$ respectively in (i), we get $x = \dfrac{9}{2}$ and $x = -\dfrac{1}{2}$ respectively.

Hence, the required vertices of the square are $(9/2, 1/2)$ and $(-1/2, 5/2)$.

22.3 AREA OF A TRIANGLE

THEOREM *The area of a triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the absolute value of*

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \text{ or, } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

NOTE 1 *Sign of area: If the points A, B, C are plotted in the two dimensional plane and three points are taken in the anticlockwise sense then the area calculated of the triangle ABC will be positive while if the points are taken in clockwise sense then the area calculated will be negative. But, if the points are taken arbitrarily, then the area calculated may be positive or negative, the numerical value being the same in both cases. In case the area calculated is negative we will consider it positive.*

NOTE 2 *To find the area of a polygon we divide it into triangles and take numerical value of the area of each of the triangles.*

CONDITION OF COLLINEARITY OF THREE POINTS Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear iff

(i) Area of $\Delta\, ABC = 0$

i.e. $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ or, $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

OR

(ii) $AB + BC = AC$ or, $AC + BC = AB$ or, $AC + AB = BC$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE AREA OF A TRIANGLE WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 1 *Prove that the area of the triangle whose vertices are $(t, t-2)$, $(t+2, t+2)$ and $(t+3, t)$ is independent of t.*

SOLUTION Let $A = (x_1, y_1) = (t, t-2)$, $B = (x_2, y_2) = (t+2, t+2)$ and $C = (x_3, y_3) = (t+3, t)$ be the vertices of the given triangle. Then,

$$\begin{aligned} \text{Area of } \Delta\text{ ABC} &= \frac{1}{2}\left|\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}\right| \\ &= \frac{1}{2}\left|\{t(t+2-t) + (t+2)(t-t+2) + (t+3)(t-2-t-2)\}\right| \\ &= \frac{1}{2}\left|\{2t + 2t + 4 - 4t - 12\}\right| = |-4| = 4 \text{ sq. units.} \end{aligned}$$

Clearly, area of $\Delta\, ABC$ is independent of t.

Type II **ON FINDING THE AREA OF A QUADRILATERAL WHEN COORDINATES OF ITS VERTICES ARE GIVEN**

EXAMPLE 2 *Find the area of the quadrilateral ABCD whose vertices are respectively A (1, 1), B (7, –3), C (12, 2) and D (7, 21).*

SOLUTION Clearly, Area of quadrilateral $ABCD = |\text{Area of } \Delta\, ABC| + |\text{Area of } \Delta\, ACD|$

Now,

$$\begin{aligned} \text{Area of } \Delta\, ABC &= \frac{1}{2}\left|\{1\times(-3-2) + 7\times(2-1) + 12\times(1+3)\}\right| = \frac{1}{2}\left|(-5+7+48)\right| \\ &= 25 \text{ sq. units} \\ \text{Area of } \Delta\, ACD &= \frac{1}{2}\left|\{1\times(2-21) + 12\times(21-1) + 7\times(1-2)\}\right| = \frac{1}{2}\left|(-19+240-7)\right| \\ &= 107 \text{ sq. units} \end{aligned}$$

$\therefore$ Area of quadrilateral $ABCD = 25 + 107 = 132$ sq. units.

Type III **ON COLLINEARITY OF THREE POINTS**

EXAMPLE 3 *Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.*

SOLUTION Let $A = (x_1, y_1) = (a, b+c)$, $B = (x_2, y_2) = (b, c+a)$ and $C = (x_3, y_3) = (c, a+b)$ be three given points. Then,

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

$$= a\{(c+a)-(a+b)\} + b\{(a+b)-(b+c)\} + c\{(b+c)-(c+a)\}$$
$$= a(c-b) + b(a-c) + c(b-a) = 0$$

Hence, the given points are collinear.

Type IV ON FINDING THE DESIRED RESULT OR UNKNOWN WHEN THREE POINTS ARE COLLINEAR

EXAMPLE 4 *For what value of k are the points $(k, 2-2k)$ $(-k+1, 2k)$ and $(-4-k, 6-2k)$ are collinear?*

SOLUTION Let three given points be $A = (x_1, y_1) = (k, 2-2k)$, $B = (x_2, y_2) = (-k+1, 2k)$ and $C = (x_3, y_3) = (-4-k, 6-2k)$.

If the given points are collinear, then

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$
$$\Rightarrow \quad k(2k-6+2k) + (-k+1)(6-2k-2+2k) + (-4-k)(2-2k-2k) = 0$$
$$\Rightarrow \quad k(4k-6) - 4(k-1) + (4+k)(4k-2) = 0$$
$$\Rightarrow \quad 4k^2 - 6k - 4k + 4 + 4k^2 + 14k - 8 = 0$$
$$\Rightarrow \quad 8k^2 + 4k - 4 = 0 \Rightarrow 2k^2 + k - 1 = 0 \Rightarrow (2k-1)(k+1) = 0 \Rightarrow k = 1/2 \text{ or } k = -1.$$

Hence, the given points are collinear for $k = 1/2$ or $k = -1$.

LEVEL-2

Type V MIXED PROBLEMS BASED UPON THE CONCEPT OF AREA OF A TRIANGLE

EXAMPLE 5 *If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a triangle ABC, where $x_i, y_i, i = 1, 2, 3$ are integers. Then, the area of ΔABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$
$$\Rightarrow \quad \Delta = \text{A rational number} \qquad [\because x_i, y_i \text{ are integers}]$$

If possible, let the triangle ABC be an equilateral triangle, then its area is given by

$$\Delta = \frac{\sqrt{3}}{4}(\text{Side})^2 = \frac{\sqrt{3}}{4}(AB)^2 \qquad [\because AB = BC = CA]$$
$$\Rightarrow \quad \Delta = \frac{\sqrt{3}}{4}(\text{A positive integer}) \qquad [\because \text{Vertices are integers} \therefore AB^2 \text{ is a positive integer}]$$
$$\Rightarrow \quad \Delta = \text{An irrational number}$$

This is a contradiction to the fact that the area is a rational number. Hence, the triangle cannot be equilateral.

EXAMPLE 6 *If the coordinates of two points A and B are (3, 4) and (5, – 2) respectively. Find the coordinates of any point P, if $PA = PB$* and Area of $\Delta PAB = 10$.

SOLUTION Let the coordinates of P be (x, y). Then,

$$PA = PB$$
$$\Rightarrow \quad PA^2 = PB^2$$
$$\Rightarrow \quad (x-3)^2 + (y-4)^2 = (x-5)^2 + (y+2)^2 \Rightarrow x - 3y - 1 = 0 \qquad \text{...(i)}$$

Now, Area of $\Delta PAB = 10$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$
$$\Rightarrow \quad 6x + 2y - 26 = \pm 20$$

$\Rightarrow \quad 6x + 2y - 46 = 0 \quad \text{or,} \quad 6x + 2y - 6 = 0$

$\Rightarrow \quad 3x + y - 23 = 0 \quad \text{or,} \quad 3x + y - 3 = 0$ ...(ii)

Solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get: $x = 7, y = 2$.

Solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get: $x = 1, y = -0$.

Thus, the coordinates of P are $(7, 2)$ or $(1, 0)$.

EXAMPLE 7 *The coordinates of A, B, C are (6, 3), (– 3, 5) and (4, – 2) respectively and P is any point (x, y). Show that the ratio of the areas of triangles PBC and ABC is* $\left|\frac{x+y-2}{7}\right|$.

SOLUTION We have,

$$\frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2}\left| x(5+2) + (-3)(-2-y) + 4(y-5) \right|}{\frac{1}{2}\left| 6(5+2) + (-3)(-2-3) + 4(3-5) \right|}$$

$$\Rightarrow \quad \frac{\text{Area of } \Delta PBC}{\text{Area of } \Delta ABC} = \frac{|7x + 7y - 14|}{|42 + 15 - 8|} = \frac{7|x+y-2|}{49} = \left|\frac{x+y-2}{7}\right|$$

22.4 SECTION FORMULAE

(i) The coordinates of the point P which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

NOTE 1 *If P is the mid-point of AB, then it divides AB in the ratio 1 : 1, so its coordinates are* $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

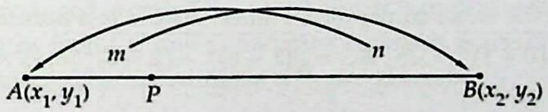

Fig. 22.8

NOTE 2 *The following diagram will help to remember the section formula.*

(ii) The coordinates of the point which divides the line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$ are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SECTION POINT WHEN THE SECTION RATIO IS GIVEN**

EXAMPLE 1 *Find the coordinates of points on the line joining the point P (3, – 4) and Q (– 2, 5) that is twice as far from P as from Q.*

SOLUTION Let $R(x, y)$ be the required point. Then, $PR = 2 \cdot RQ$ (given) i.e. $PR : RQ = 2 : 1$. Thus, R divides PQ internally or externally in the ratio $2 : 1$.

If R divides PQ internally in the ratio $2 : 1$, then the coordinates of R are given by

$$x = \frac{2 \times -2 + 1 \times 3}{2+1} \text{ and } y = \frac{2 \times 5 + 1 \times -4}{2+1} \Rightarrow x = -\frac{1}{3} \text{ and } y = 2.$$

So, the coordinates of R are $(-1/3, 2)$

If R divides PQ externally in the ratio $2:1$, then the coordinates of R are given by

$$x = \frac{2 \times -2 - 1 \times 3}{2-1} \text{ and } y = \frac{2 \times 5 - 1 \times -4}{2-1} \Rightarrow x = -7 \text{ and } y = 14.$$

Hence, the coordinates of R are $(-7, 14)$.

Type II **ON FINDING THE SECTION RATIO OR AN END POINT OF THE SEGMENT WHEN THE SECTION POINT IS GIVEN**

EXAMPLE 2 *Determine the ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points (1, 3) and (2, 7).*

SOLUTION Suppose the line $3x + y - 9 = 0$ divides the line segment joining A (1, 3) and B (2, 7) in the ratio $k:1$ at point C. Then, the coordinates of C are $\left(\frac{2k+1}{k+1}, \frac{7k+3}{k+1}\right)$. But, C lies on the line $3x + y - 9 = 0$.

$$\therefore \quad 3\left(\frac{2k+1}{k+1}\right) + \frac{7k+3}{k+1} - 9 = 0 \Rightarrow 6k + 3 + 7k + 3 - 9k - 9 = 0 \Rightarrow k = \frac{3}{4}$$

So, the required ratio is $3:4$ internally.

Type III **ON DETERMINATION OF THE TYPE OF A GIVEN QUADRILATERAL**

EXAMPLE 3 *Prove that:* $(4, -1), (6, 0), (7, 2)$ *and* $(5, 1)$ *are the vertices of a rhombus. Is it a square?*

SOLUTION Let the given points be A, B, C and D respectively. Then,

Coordinates of the mid-point of AC are $\left(\frac{4+7}{2}, \frac{-1+2}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Coordinates of the mid-point of BD are $\left(\frac{6+5}{2}, \frac{0+1}{2}\right) = \left(\frac{11}{2}, \frac{1}{2}\right)$

Thus, AC and BD have the same mid-point. Hence, $ABCD$ is a parallelogram.

Now, $AB = \sqrt{(6-4)^2 + (0+1)^2} = \sqrt{5}$, $BC = \sqrt{(7-6)^2 + (2-0)^2} = \sqrt{5}$

$\therefore \quad AB = BC$

So, $ABCD$ is a parallelogram whose adjacent sides are equal. Hence, $ABCD$ is a rhombus.

Now, $AC = \sqrt{(7-4)^2 + (2+1)^2} = 3\sqrt{2}$, and $BD = \sqrt{(6-5)^2 + (0-1)^2} = \sqrt{2}$.

Clearly, $AC \neq BD$. So, $ABCD$ is not a square.

Type IV **ON FINDING THE UNKNOWN VERTEX FROM GIVEN POINTS**

EXAMPLE 4 *If the coordinates of the mid-points of the sides of a triangle are (1, 2) (0, –1) and (2, –1). Find the coordinates of its vertices.*

SOLUTION Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC. Let $D(1, 2)$, $E(0, -1)$, and $F(2, -1)$ be the mid-points of sides BC, CA and AB respectively.

Since D is the mid-point of BC.

$$\therefore \quad \frac{x_2 + x_3}{2} = 1 \text{ and } \frac{y_2 + y_3}{2} = 2 \Rightarrow x_2 + x_3 = 2 \text{ and } y_2 + y_3 = 4 \quad \text{...(i)}$$

Similarly, E and F are the mid-points of CA and AB respectively.

$$\therefore \quad \frac{x_1 + x_3}{2} = 0 \text{ and } \frac{y_1 + y_3}{2} = -1 \Rightarrow x_1 + x_3 = 0 \text{ and } y_1 + y_3 = -2 \quad \text{...(ii)}$$

$$\text{and,} \quad \frac{x_1 + x_2}{2} = 2 \text{ and } \frac{y_1 + y_2}{2} = -1 \Rightarrow x_1 + x_2 = 4 \text{ and } y_1 + y_2 = -2 \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$(x_2 + x_3) + (x_1 + x_3) + (x_1 + x_2) = 2 + 0 + 4 \text{ and, } (y_2 + y_3) + (y_1 + y_3) + (y_1 + y_2) = 4 - 2 - 2$$

$\Rightarrow x_1 + x_2 + x_3 = 3$ and $y_1 + y_2 + y_3 = 0$ ...(iv)

From (i) and (iv), we get

$$x_1 + 2 = 3 \text{ and } y_1 + 4 = 0 \Rightarrow x_1 = 1 \text{ and } y_1 = -4$$

So, the coordinates of A are $(1, -4)$.

From (ii) and (iv), we get

$$x_2 + 0 = 3 \text{ and } y_2 - 2 = 0 \Rightarrow x_2 = 3 \text{ and } y_2 = 2$$

So, coordinates of B are (3, 2).

From (iii) and (iv), we get

$$x_3 + 4 = 3 \text{ and } y_3 - 2 = 0 \Rightarrow x_3 = -1 \text{ and } y_3 = 2$$

So, coordinates of C are (–1, 2).

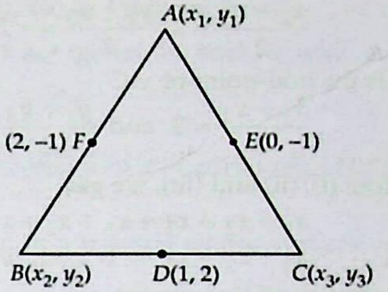

Fig. 22.9

Hence, the vertices of the triangle ABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

22.5 CENTROID, IN-CENTRE AND EX-CENTRES OF A TRIANGLE

(i) The coordinates of the centroid of the triangle whose vertices are (x_1, y_1) and (x_2, y_2) are (x_3, y_3) are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

(ii) The coordinates of the in-centre of a triangle whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ $C(x_3, y_3)$ are $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$, where $a = BC$, $b = CA$ and $C = AB$

(iii) Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ be the vertices of the triangle ABC, and let a, b, c be the lengths of the sides BC, CA, AB respectively. The circle which touches the side BC and two sides AB and AC produced is called the escribed circle opposite to the angle A. The bisectors of the external angles B and C meet at a point I_1 which is the centre of the escribed circle opposite to the angle A. The coordinates of I_1 are given by

$$\left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c}\right)$$

The coordinates of I_2 and I_3 (centres of escribed circles opposite to the angles B and C respectively) are given by

$$I_2\left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c}\right) \text{ and } I_3\left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c}\right)$$

respectively.

LEVEL-1

EXAMPLE 1 *If the coordinates of the mid-points of the sides of a triangle are (1, 1), (2, – 3) and (3, 4). Find its (i) centroid (ii) in-centre.*

SOLUTION Let $P(1, 1)$, $Q(2, -3)$, $R(3, 4)$ be the mid-points of sides AB, BC and CA respectively of triangle ABC. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of triangle ABC. Then, P is the mid-point of AB.

$$\therefore \quad \frac{x_1 + x_2}{2} = 1, \frac{y_1 + y_2}{2} = 1 \Rightarrow x_1 + x_2 = 2 \text{ and } y_1 + y_2 = 2 \quad \text{...(i)}$$

Q is the mid-point of BC.

$\therefore \quad \frac{x_2 + x_3}{2} = 2, \frac{y_2 + y_3}{2} = -3 \Rightarrow x_2 + x_3 = 4$ and $y_2 + y_3 = -6$ ...(ii)

R is the mid-point of AC

$\therefore \quad \frac{x_1 + x_3}{2} = 3$ and $\frac{y_1 + y_3}{2} = 4 \Rightarrow x_1 + x_3 = 6$ and $y_1 + y_3 = 8$ ...(iii)

From (i), (ii) and (iii), we get

$$x_1 + x_2 + x_2 + x_3 + x_1 + x_3 = 2 + 4 + 6 \text{ and, } y_1 + y_2 + y_2 + y_3 + y_1 + y_3 = 2 - 6 + 8$$

$\Rightarrow \quad x_1 + x_2 + x_3 = 6$ and, $y_1 + y_2 + y_3 = 2$ ...(iv)

From (i) and (iv), we get

$$x_3 + 2 = 6 \text{ and } 2 + y_3 = 2 \Rightarrow x_3 = 4, \ y_3 = 0$$

So, the coordinates of C are (4, 0).

From (ii) and (iv), we get

$$x_1 + 4 = 6 \text{ and } y_1 - 6 = 2 \Rightarrow x_1 = 2, \ y_1 = 8$$

So, coordinates of A are (2, 8).

From (iii) and (iv), we get

$$x_2 + 6 = 6 \text{ and } y_2 + 8 = 2 \Rightarrow x_2 = 0 \text{ and } y_2 = -6$$

So, coordinates of B are (0, – 6).

Now, $a = BC = \sqrt{(4-0)^2 + (0+6)^2} = \sqrt{52} = 2\sqrt{13}$

$b = AC = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{68} = 2\sqrt{17}$

and, $c = AB = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{200} = 10\sqrt{2}$

The coordinates of the in-centre of the triangle ABC are

$$\left(\frac{a x_1 + b x_2 + c x_3}{a+b+c}, \frac{a y_1 + b y_2 + c y_3}{a+b+c}\right)$$

or, $$\left(\frac{(2\sqrt{13})\,2 + (2\sqrt{17})\,0 + (10\sqrt{2})\,4}{2\sqrt{13} + 2\sqrt{17} + 10\sqrt{2}}, \frac{(2\sqrt{13})\,8 + (2\sqrt{17})(-6) + (10\sqrt{2})\,0}{2\sqrt{13} + 2\sqrt{17} + 10\sqrt{2}}\right)$$

or, $$\left(\frac{4\sqrt{13} + 40\sqrt{2}}{2\sqrt{13} + 2\sqrt{17} + 10\sqrt{2}}, \frac{16\sqrt{13} - 12\sqrt{17}}{2\sqrt{13} + 2\sqrt{17} + 10\sqrt{2}}\right)$$

or, $$\left(\frac{2\sqrt{13} + 20\sqrt{2}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}, \frac{8\sqrt{13} - 6\sqrt{17}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}\right)$$

EXAMPLE 2 *Two vertices of a triangle are (3, – 5) and (– 7, 4). If its centroid is (2, – 1), find the third vertex.*

SOLUTION Let the coordinates of the third vertex be (x, y). Then,

$$\frac{x + 3 - 7}{3} = 2 \text{ and } \frac{y - 5 + 4}{3} = -1$$

$\Rightarrow \quad x - 4 = 6$ and $y - 1 = -3 \Rightarrow x = 10$ and $y = -2$

Thus, the coordinates of the third vertex are (10, – 2).

EXERCISE 22.1

LEVEL-1

1. If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that: $OP \cdot OQ \cos\alpha = x_1 x_2 + y_1 y_2$.

2. The vertices of a triangle ABC are $A(0, 0)$, $B(2, -1)$ and $C(9, 2)$. Find $\cos B$.
3. Four points $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $D(x, 3x)$ are given in such a way that $\dfrac{\Delta DBC}{\Delta ABC} = \dfrac{1}{2}$, find x.
4. The points $A(2, 0)$, $B(9, 1)$, $C(11, 6)$ and $D(4, 4)$ ar the vertices of a quadrilateral $ABCD$. Determine whether $ABCD$ is a rhombus or not.
5. Find the coordinates of the centre of the circle inscribed in a triangle whose vertices are $(-36, 7)$, $(20, 7)$ and $(0, -8)$.
6. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle. **[NCERT]**
7. Find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ when (i) PQ is parallel to the y-axis (ii) PQ is parallel to the x-axis. **[NCERT]**
8. Find a point on the x-axis, which is equidistant from the point $(7, 6)$ and $(3, 4)$. **[NCERT]**

ANSWERS

2. $\dfrac{-11}{\sqrt{290}}$ 3. $\dfrac{11}{8}$ 4. No 5. $(-1, 0)$

6. $(0, a)$, $(0, -a)$ and $(-\sqrt{3}a, 0)$ or $(0, a)$, $(0, -a)$ and $(\sqrt{3}a, 0)$

7. (i) $|y_2 - y_1|$ (ii) $|x_2 - x_1|$ 8. $(15/2, 0)$

HINTS TO NCERT & SELECTED PROBLEM

6. Let BC be the base of equilateral triangle ABC. It is given that $BC = 2a$ and mid-point of BC is at the origin. Therefore, coordinates of B and C are $(0, a)$ and $(0, -a)$ respectively. As the triangle ABC is equilateral. So, vertex A lies on x-axis. Let its coordinates be $(x, 0)$.

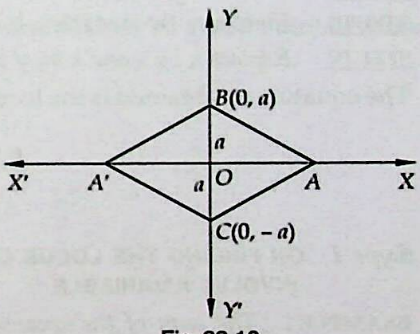

Fig. 22.10

Also, $AB = BC = AC \Rightarrow AB = AC = 2a$

Using Pythagoras theorem in ΔAOB, we get

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad (2a)^2 = OA^2 + a^2$$

$\Rightarrow \quad OA = \sqrt{3}a$ So, coordinates of A are $(-\sqrt{3}a, 0)$. Similarly, coordinates of A' are $(-\sqrt{3}a, 0)$.

Hence, the coordinates of the vertices of triangle are $A(\sqrt{3}a, 0)$, $B(0, a)$ and $C(0, -a)$ or, $A'(-\sqrt{3}a, 0)$, $B(0, a)$ and $C(0, -a)$.

7. (i) If PQ is parallel to y-axis, then $x_1 = x_2$

and, PQ = Absolute value of the difference of y-coordinates of P and $Q = |y_2 - y_1|$.

(ii) If PQ is parallel to x-axis, then $y_1 = y_2$

and, PQ = Absolute value of the difference of x-coordinates of P and $= |x_2 - x_1|$.

8. Let $P(x, 0)$ be a point on x-axis which is equidistant from the points $A(7, 6)$ and $B(3, 4)$. Then,

$$PA = PB$$

$$\Rightarrow \quad PA^2 = PB^2$$

$$\Rightarrow \quad (x-7)^2+(0-6)^2=(x-3)^2+(0-4)^2 \Rightarrow -14x+85 = -6x+25 \Rightarrow 8x=60 \Rightarrow x=\frac{15}{2}$$

Hence, $P\left(\frac{15}{2}, 0\right)$ is the required point.

22.6 LOCUS AND EQUATION TO A LOCUS

LOCUS *The curve described by a point which moves under given condition or conditions is called its locus.*

For example, suppose C is a point in the plane of the paper and P is a variable point in the plane of the paper such that its distance from C is always equal to a (say). It is clear that all the positions of the moving point P lie on the circumference of a circle whose centre is C and whose radius is a. The circumference of this circle is therefore the "Locus" of point P when it moves under the condition that its distance from the point C is always equal to constant a.

Let A and B be two fixed points in the plane of the paper and P be a variable point in the plane of the paper which moves in such a way that its distances from A and B are always equal. Obviously all the positions of the moving point P lie on the perpendicular bisector of AB. Thus, the "locus" of P is the perpendicular bisector of AB when it moves under the condition that its distances from A and B are always equal.

EQUATION TO THE LOCUS OF A POINT *The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.*

In order to find the locus of a point, we may use the following algorithm.

ALGORITHM

STEP I *Assume the coordinates of the point say, (h, k) whose locus is to be found.*

STEP II *Write the given condition in mathematical form involving h, k.*

STEP III *Eliminate the variable (s), if any.*

STEP IV *Replace h by x and k by y in the result obtained in step III.*

The equation so obtained is the locus of the point which moves under some stated condition (s).

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE LOCUS OF A POINT WHEN GIVEN GEOMETRICAL CONDITIONS DO NOT INVOLVE A VARIABLE**

EXAMPLE 1 *The sum of the squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant quantity $2c^2$. Find the equation to its locus.*

SOLUTION Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$, $B(-a, 0)$ be the given points. Then,

$$PA^2+PB^2=2c^2 \qquad \text{[Given]}$$

$$\Rightarrow \quad (h-a)^2+(k-0)^2+(h+a)^2+(k-0)^2=2c^2$$

$$\Rightarrow \quad h^2-2ah+a^2+k^2+h^2+2ah+a^2+k^2=2c^2$$

$$\Rightarrow \quad 2h^2+2k^2+2a^2=2c^2$$

$$\Rightarrow \quad h^2+k^2=c^2-a^2$$

Hence, locus of (h, k) is $x^2+y^2=c^2-a^2$.

EXAMPLE 2 *Find the equation to the locus of a point equidistant from the points $A(1, 3)$ and $B(-2, 1)$.*

SOLUTION Let $P(h, k)$ be any point on the locus. Then,

$$PA = PB \qquad \text{[Given]}$$

$$\Rightarrow \quad PA^2 = PB^2 \Rightarrow (h-1)^2 + (k-3)^2 = (h+2)^2 + (k-1)^2 \Rightarrow 6h + 4k = 5$$

Hence, locus of (h, k) is $6x + 4y = 5$.

EXAMPLE 3 *Find the locus of a point such that the sum of its distances from the points (0, 2) and (0, – 2) is 6.*

SOLUTION Let $P(h, k)$ be any point on the locus and let $A(0, 2)$ and $B(0, -2)$ be the given points. By the given condition

$$PA + PB = 6$$

$$\Rightarrow \sqrt{(h-0)^2 + (k-2)^2} + \sqrt{(h-0)^2 + (k+2)^2} = 6$$

$$\Rightarrow \sqrt{h^2 + (k-2)^2} = 6 - \sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow h^2 + (k-2)^2 = 36 - 12\sqrt{h^2 + (k+2)^2} + h^2 + (k+2)^2$$

$$\Rightarrow -8k - 36 = -12\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow (2k+9) = 3\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow (2k+9)^2 = 9(h^2 + (k+2)^2)$$

$$\Rightarrow 4k^2 + 36k + 81 = 9h^2 + 9k^2 + 36k + 36$$

$$\Rightarrow 9h^2 + 5k^2 = 45$$

Hence, locus of (h, k) is $9x^2 + 5y^2 = 45$.

EXAMPLE 4 *Find the locus of a point, so that the join of (– 5, 1) and (3, 2) subtends a right angle at the moving point.*

SOLUTION Let $P(h, k)$ be a moving point and let $A(-5, 1)$ and $B(3, 2)$ be given points. By the given condition

$$\angle APB = 90°$$

$\therefore$ ΔAPB is a right angle triangle

$$\Rightarrow AB^2 = AP^2 + PB^2$$

$$\Rightarrow (3+5)^2 + (2-1)^2 = (h+5)^2 + (k-1)^2 + (h-3)^2 + (k-2)^2$$

$$\Rightarrow 65 = 2(h^2 + k^2 + 2h - 3k) + 39$$

$$\Rightarrow h^2 + k^2 + 2h - 3k - 13 = 0$$

Hence, locus of (h, k) is $x^2 + y^2 + 2x - 3y - 13 = 0$.

EXAMPLE 5 *A point moves so that the sum of its distances from* $(ae, 0)$ and $(-ae, 0)$ *is 2a, prove that the equation to its locus is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*, where* $b^2 = a^2(1-e^2)$.

SOLUTION Let $P(h, k)$ be the moving point such that the sum of its distances from $A(ae, 0)$ and $B(-ae, 0)$ is $2a$. Then,

$$PA + PB = 2a$$

$$\Rightarrow \sqrt{h-ae)^2 + (k-0)^2} + \sqrt{(h+ae)^2 + (k-0)^2} = 2a$$

$$\Rightarrow \sqrt{(h-ae)^2 + k^2} = 2a - \sqrt{(h+ae)^2 + k^2}$$

$$\Rightarrow (h-ae)^2 + k^2 = 4a^2 + (h+ae)^2 + k^2 - 4a\sqrt{(h+ae)^2 + k^2} \quad \text{[Squaring both sides]}$$

$$\Rightarrow -4aeh - 4a^2 = -4a\sqrt{(h+ae)^2 + k^2}$$

$$\Rightarrow \quad (eh + a) = \sqrt{(h + ae)^2 + k^2}$$

$$\Rightarrow \quad (eh + a)^2 = (h + ae)^2 + k^2$$

$$\Rightarrow \quad e^2 h^2 + 2aeh + a^2 = h^2 + a^2 e^2 + 2aeh + k^2$$

$$\Rightarrow \quad h^2 (1 - e^2) + k^2 = a^2 (1 - e^2)$$

$$\Rightarrow \quad \frac{h^2}{a^2} + \frac{k^2}{a^2 (1 - e^2)} = 1$$

Hence, locus of (h, k) is $\frac{x^2}{a^2} + \frac{y^2}{a^2 (1 - e^2)} = 1$ or, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $b^2 = a^2 (1 - e^2)$

EXAMPLE 6 *Find the equation to the locus of a point which moves so that the sum of its distances from (3, 0) and (–3, 0) is less than 9.*

SOLUTION Let $P(h, k)$ be the moving point such that the sum of its distances from A (3, 0) and B (– 3, 0) is less than 9. Then,

$$PA + PB < 9$$

$$\Rightarrow \quad \sqrt{(h - 3)^2 + (k - 0)^2} + \sqrt{(h + 3)^2 + (k - 0)^2} < 9$$

$$\Rightarrow \quad \sqrt{(h - 3)^2 + k^2} < 9 - \sqrt{(h + 3)^2 + k^2}$$

$$\Rightarrow \quad (h - 3)^2 + k^2 < \{9 - \sqrt{(h + 3)^2 + k^2}\}^2 \qquad [\because\ x < y \Rightarrow x^2 < y^2 \text{ for } x, y > 0]$$

$$\Rightarrow \quad (h - 3)^2 + k^2 < 81 + (h + 3)^2 + k^2 - 18\sqrt{(h + 3)^2 + k^2}$$

$$\Rightarrow \quad -12h - 81 < -18\sqrt{(h + 3)^2 + k^2}$$

$$\Rightarrow \quad 4h + 27 > 6\sqrt{(h + 3)^2 + k^2}$$

$$\Rightarrow \quad (4h + 27)^2 > 36\,[(h + 3)^2 + k^2]$$

$$\Rightarrow \quad 20\,h^2 + 36\,k^2 < 405$$

Hence, the locus of (h, k) is $20\,x^2 + 36y^2 < 405$.

LEVEL-2

Type II ON FINDING THE LOCUS OF A POINT WHEN GIVEN GEOMETRICAL CONDITIONS INVOLVE SOME VARIABLE(S)

EXAMPLE 7 *Find the locus of the point of intersection of the lines* $x \cos \alpha + y \sin \alpha = a$ and $x \sin \alpha - y \cos \alpha = b$, where α is a variable.

SOLUTION Let P (h, k) be the point of intersection of the given lines. Then,

$$h \cos \alpha + k \sin \alpha = a \qquad \text{...(i)}$$

and, $$h \sin \alpha - k \cos \alpha = b \qquad \text{...(ii)}$$

Here α is a variable. So, we have to eliminate α. Squaring and adding (i) and (ii), we get

$$(h \cos \alpha + k \sin \alpha)^2 + (h \sin \alpha - k \cos \alpha)^2 = a^2 + b^2 \Rightarrow h^2 + k^2 = a^2 + b^2$$

Hence, the locus of (h, k) is $x^2 + y^2 = a^2 + b^2$.

EXAMPLE 8 *A rod of length l slides with its ends on two perpendicular lines. Find the locus of its mid-point.*

SOLUTION Let the two perpendicular lines be the coordinate axes. Let AB be a rod of length l. Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ respectively. As the rod slides the values of a and b change. So a and b are two variables. Let $P(h, k)$ be the mid-point of the rod AB in one of the infinite positions it attains. Then,

$$h = \frac{a+0}{2} \text{ and } k = \frac{0+b}{2}$$

$$\Rightarrow \quad h = \frac{a}{2} \text{ and } k = \frac{b}{2} \quad \text{...(i)}$$

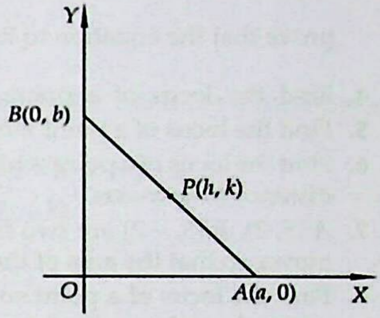

Fig. 22.11

From $\Delta\, OAB$, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow \quad a^2 + b^2 = l^2$$

$$\Rightarrow \quad (2h)^2 + (2k)^2 = l^2 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad 4h^2 + 4k^2 = l^2$$

Hence, the locus of (h, k) is $4x^2 + 4y^2 = l^2$.

EXAMPLE 9 *AB is a variable line sliding between the coordinate axes in such a way that A lies on x-axis and B lies on y-axis. If P is a variable point on AB such that $PA = b$, $PB = a$ and $AB = a + b$, find the equation of the locus of P.*

SOLUTION Let $P(h, k)$ be a variable point on AB such that $\angle OAB = \theta$, where θ is a variable. From triangles ALP and PMB, we have

$$\sin\theta = \frac{k}{b} \quad \text{...(i)} \qquad \cos\theta = \frac{h}{a} \quad \text{...(ii)}$$

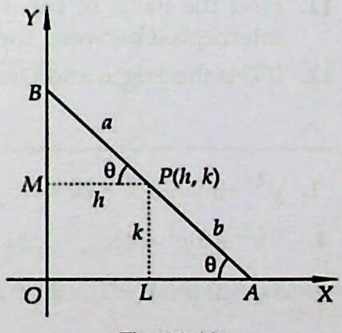

Fig. 22.12

Here, θ is a variable. So, we have to eliminate θ.

Squaring (i) and (ii) and adding, we get: $\dfrac{k^2}{b^2} + \dfrac{h^2}{a^2} = 1$

Hence, the locus of (h, k) is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

EXAMPLE 10 *If O is the origin and Q is a variable point on $x^2 = 4y$. Find the locus of the mid-point of OQ.*

SOLUTION Let the coordinates of Q be (a, b) and let $P(h, k)$ be the mid-point of OQ. Then,

$$h = \frac{a+0}{2} = \frac{a}{2} \text{ and } k = \frac{0+b}{2} = \frac{b}{2} \Rightarrow a = 2h \text{ and } b = 2k \quad \text{...(i)}$$

Here, a and b are two variables which are to be eliminated. Since, (a, b) lies on $x^2 = 4y$.

$$\therefore \quad a^2 = 4b$$

$$\Rightarrow \quad (2h)^2 = 4(2k) \quad \text{[Using (i)]}$$

$$\Rightarrow \quad h^2 = 2k$$

Hence, the locus of (h, k) is $x^2 = 2y$.

EXERCISE 22.2

LEVEL-1

1. Find the locus of a point equidistant from the point (2, 4) and the y-axis.
2. Find the equation of the locus of a point which moves such that the ratio of its distances from (2, 0) and (1, 3) is 5 : 4.
3. A point moves as so that the difference of its distances from $(ae, 0)$ and $(-ae, 0)$ is $2a$, prove that the equation to its locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.
4. Find the locus of a point such that the sum of its distances from (0, 2) and (0, –2) is 6.
5. Find the locus of a point which is equidistant from (1, 3) and x-axis.
6. Find the locus of a point which moves such that its distance from the origin is three times its distance from x-axis.
7. A (5, 3), B (3, – 2) are two fixed points; find the equation to the locus of a point P which moves so that the area of the triangle PAB is 9 units.
8. Find the locus of a point such that the line segments having end points (2, 0) and (–2, 0) subtend a right angle at that point.
9. If A (–1, 1) and B (2, 3) are two fixed points, find the locus of a point P so that the area of $\Delta PAB = 8$ sq. units.

LEVEL-2

10. A rod of length l slides between the two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.
11. Find the locus of the mid-point of the portion of the line $x \cos \alpha + y \sin \alpha = p$ which is intercepted between the axes.
12. If O is the origin and Q is a variable point on $y^2 = x$. Find the locus of the mid-point of OQ.

ANSWERS

1. $y^2 - 8y - 4x + 20 = 0$
2. $9x^2 + 9y^2 + 14x - 150y + 186 = 0$
4. $9x^2 + 5y^2 = 45$
5. $x^2 - 2x - 6y + 10 = 0$
6. $x^2 = 8y^2$
7. $5x - 2y - 1 = 0$ or $5x - 2y - 37 = 0$
8. $x^2 + y^2 = 4$
9. $2x - 3y - 11 = 0$, $2x - 3y + 21 = 0$
10. $\frac{x^2}{4} + y^2 = \frac{l^2}{9}$
11. $p^2(x^2 + y^2) = 4x^2y^2$
12. $2y^2 = x$

HINTS TO SELECTED PROBLEM

8. Let $P(h, k)$ be the variable point and let $A(2, 0)$ and $B(-2, 0)$ be the given points. Then, $\angle APB = \pi/2 \Rightarrow AB^2 = PA^2 + PB^2$.

22.7 SHIFTING OF ORIGIN

Let O be the origin and let $x'Ox$ and $y'Oy$ be the axis of x and y respectively. Let O' and P be two points in the plane having coordinates (h, k) and (x, y) respectively referred to $X'OX$ and $Y'OY$ as the coordinate axes. Let the origin be transferred to O' and let $X'O'X$ and $Y'O'Y$ be new rectangular axes. Let the coordinates of P referred to new axes as the coordinate axes be (X, Y). Then,

$$O'N = X,\ PN = Y,\ OM = x,\ PM = y,\ OL = h \text{ and, } O'L = k.$$

Now, $x = OM = OL + LM = OL + O'N = h + X$

and, $y = PM = PN + NM = PN + O'L = Y + k$

$\therefore$ $x = X + h$ and $y = Y + k$.

Thus, if (x, y) are coordinates of a point referred to old axes and (X, Y) are the coordinates of the same point referred to new axes, then

$$x = X + h \text{ and } y = Y + k$$

i.e., (Old x-coordinate) = (New x-coordinate) + h

and, (Old y-coordinate) = (New y-coordinate) + k.

If therefore the origin is shifted at a point (h, k) we must substitute $X + h$ and $Y + k$ for x and y respectively.

The transformation formula from new axes to old axes is: $X = x - h, Y = y - k$

The coordinates of the old origin referred to the new axes are $(-h, -k)$.

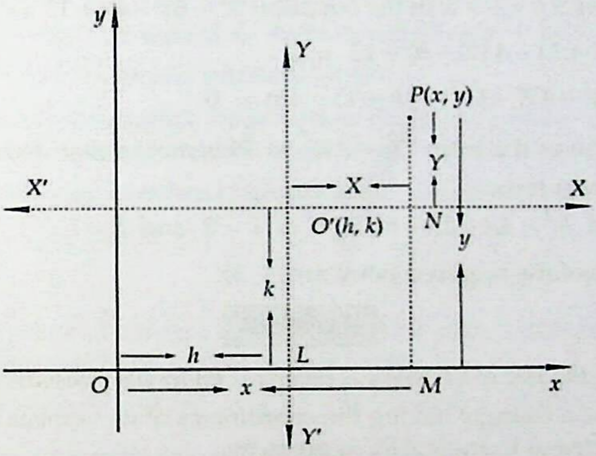

Fig. 22.13

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If the axes are shifted to the point* $(1, -2)$ *without rotation, what do the following equations become?*

(i) $2x^2 + y^2 - 4x + 4y = 0$ (ii) $y^2 - 4x + 4y + 8 = 0$

SOLUTION (i) Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0 \Rightarrow 2X^2 + Y^2 = 6.$$

(ii) Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get

$$(Y-2)^2 - 4(X+1) + 4(Y-2) + 8 = 0 \qquad \Rightarrow Y^2 = 4X$$

EXAMPLE 2 *At what point the origin be shifted, if the coordinates of a point (4, 5) become (–3, 9)?*

SOLUTION Let (h, k) be the point to which the origin is shifted. Then,

$x = 4,\ y = 5,\ X = -3,\ Y = 9$

$\therefore$ $x = X + h$ and $y = Y + k \Rightarrow 4 = -3 + h$ and $5 = 9 + k \Rightarrow h = 7$ and $k = -4$

Hence, the origin must be shifted to $(7, -4)$

EXAMPLE 3 *Shift the origin to a suitable point so that the equation* $y^2 + 4y + 8x - 2 = 0$ *will not contain term in* y *and the constant term.*

SOLUTION Let the origin be shifted to (h, k). Then, $x = X + h$ and $y = Y + k$.
Substituting $x = X + h,\ y = Y + k$ in the equation $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$$\Rightarrow \quad Y^2 + (4 + 2k)Y + 8X + (k^2 + 4k + 8h - 2) = 0$$

For this equation to be free from the term containing Y and the constant term, we must have

$$4 + 2k = 0 \text{ and } k^2 + 4k + 8h - 2 = 0 \Rightarrow k = -2 \text{ and } h = \frac{3}{4}.$$

Hence, the origin is shifted at the point $(3/4, -2)$.

EXAMPLE 4 *Find the point to which the origin should be shifted so that the equation $y^2 - 6y - 4x + 13 = 0$ is transformed to the form $y^2 + Ax = 0$.*

SOLUTION Let the origin be shifted to the point (h, k). Then, $x = X + h$ and $y = Y + k$.
Substituting $x = X + h$ and $y = Y + k$ in the equation $y^2 - 6y - 4x + 13 = 0$, we get

$$(Y + k)^2 - 6(Y + k) - 4(X + h) + 13 = 0$$

$$\Rightarrow \quad Y^2 + (2k - 6)Y - 4X + (k^2 - 6k + 13 - 4h) = 0$$

This equation should be of the form $Y^2 + AX = 0$. This means that it should not contain term containing Y and constant term.

$$\therefore \quad 2k - 6 = 0 \text{ and } k^2 - 4h + 13 - 6k = 0 \Rightarrow k = 3 \text{ and } h = 1$$

Hence, the coordinates of the required point are $(1, 3)$.

LEVEL-2

EXAMPLE 5 *Prove that the area of a triangle is invariant under the translation of the axes.*

SOLUTION Let ABC be a triangle having the coordinates of its vertices as $A(x_1, y_1)$, $B(x_2, y_1)$ and $C(x_3, y_3)$. Then, area of triangle ABC is given by

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \quad \text{...(i)}$$

Let the origin be shifted at (h, k). Then, the new coordinates of the vertices are

$$A(x_1 + h, y_1 + k),\ B(x_2 + h, y_2 + k) \text{ and } C(x_3 + h, y_3 + k).$$

Therefore, the area of the triangle in the new-coordinate system is given by

$$\Delta_1 = \frac{1}{2}\Big[(x_1 + h)\{(y_2 + k) - (y_3 + k)\} + (x_2 + h)\{(y_3 + k) - (y_1 + k)\} + (x_3 + h)\{(y_1 + k) - (y_2 + k)\}\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[(x_1 + h)(y_2 - y_3) + (x_2 + h)(y_3 - y_1) + (x_3 + h)(y_1 - y_2)\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) + h(y_2 - y_3 + y_3 - y_1 + y_1 - y_2)\Big]$$

$$\Rightarrow \quad \Delta_1 = \frac{1}{2}\Big[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\Big] \quad \text{...(ii)}$$

From (i) and (ii), we get $\Delta = \Delta_1$.
Hence, the area of a triangle is invariant under the translation of the axes.

EXERCISE 22.3

LEVEL-1

1. What does the equation $(x-a)^2+(y-b)^2=r^2$ become when the axes are transferred to parallel axes through the point $(a-c, b)$?
2. What does the equation $(a-b)(x^2+y^2)-2abx=0$ become if the origin is shifted to the point $\left(\frac{ab}{a-b}, 0\right)$ without rotation?
3. Find what the following equations become when the origin is shifted to the point (1, 1)?
   (i) $x^2+xy-3x-y+2=0$ (ii) $x^2-y^2-2x+2y=0$
   (iii) $xy-x-y+1=0$ (iv) $xy-y^2-x+y=0$
4. At what point the origin be shifted so that the equation $x^2+xy-3x-y+2=0$ does not contain any first degree term and constant term?
5. Verify that the area of the triangle with vertices (2, 3), (5, 7) and (– 3 – 1) remains invariant under the translation of axes when the origin is shifted to the point (– 1, 3).
6. Find, what the following equations become when the origin is shifted to the ponit (1, 1).
   (i) $x^2+xy-3y^2-y+2=0$ (ii) $xy-y^2-x+y=0$
   (iii) $xy-x-y+1=0$ (iv) $x^2-y^2-2x+2y=0$
7. Find the point to which the origin should be shifted after a translation of axes so that the following equations will have no first degree terms:
   (i) $y^2+x^2-4x-8y+3=0$ (ii) $x^2+y^2-5x+2y-5=0$ (iii) $x^2-12x+4=0$
8. Verify that the area of the triangle with vertices (4, 6), (7, 10) and (1, – 2) remains invariant under the translation of axes when the origin is shifted to the point (– 2, 1). [NCERT]

ANSWERS

1. $X^2+Y^2-2CX=r^2-c^2$ 2. $(a-b)^2(X^2+Y^2)=a^2b^2$
3. (i) $x^2+xy=0$ (ii) $x^2-y^2=0$ (iii) $xy=0$ (iv) $xy-y^2=0$
4. (1, 1) 6. (i) $x^2-3y^2+xy+3x-6y=0$ (ii) $xy-y^2=0$
(iii) $xy=0$ (iv) $x^2-y^2=0$ 7. (i) (2, 4) (ii) (5/2, –1) (iii) $(6, k), k \in R$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. The vertices of a triangle are $O(0, 0)$, $A(a, 0)$ and $B(0, b)$. Write the coordinates of its circumcentre.
2. In Q.No. 1, write the distance between the circumcentre and orthocentre of ΔOAB.
3. Write the coordinates of the orthocentre of the triangle formed by points (8, 0), (4, 6) and (0, 0).
4. Three vertices of a parallelogram, taken in order, are (–1, – 6), (2, –5) and (7, 2). Write the coordinates of its fourth vertex.

5. If the points $(a, 0)$, $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ are collinear, write the value of $t_1 t_2$.
6. If the coordinates of t (ides AB and AC of a ΔABC are $(3, 5)$ and $(-3, -3)$ respectively, then write the length of side BC.
7. Write the coordinates of the circumcentre of a triangle whose centroid and orthocentre are at $(3, 3)$ and $(-3, 5)$ respectively.
8. Write the coordinates of the incentre of the triangle having its vertices at $(0, 0)$, $(5, 0)$ and $(0, 12)$.
9. If the points $(1, -1)$, $(2, -1)$ and $(4, -3)$ are the mid-points of the sides of a triangle, then write the coordinates of its centroid.
10. Write the area of the triangle having vertices at $(a, b + c)$, $(b, c + a)$, $(c, a + b)$.

ANSWERS

1. $\left(\frac{a}{2}, \frac{b}{2}\right)$	2. $\frac{1}{2}\sqrt{a^2 + b^2}$	3. $\left(\frac{8}{3}, 4\right)$	4. $(4, 1)$	5. -1
6. 20 units	7. $(6, 2)$	8. $(2, 2)$	9. $\left(\frac{7}{3}, \frac{-5}{3}\right)$	10. 0

SUMMARY

1. The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e. $$PQ = \sqrt{(\text{Difference of abscissae})^2 + (\text{Difference of ordinates})^2}$$

2. The distance of a point $P(x, y)$ from the origin $O(0, 0)$ is given by $OP = \sqrt{x^2 + y^2}$.
3. The area of the triangle, the coordinates of whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is the absolute value of

$$\frac{1}{2}\left[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right] \text{ or, } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

4. If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear, then $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$
5. The coordinates of the point dividing the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $m : n$ are

$$\left(\frac{m x_2 + n x_1}{m + n}, \frac{m y_2 + n y_1}{m + n}\right), \text{ internally; } \left(\frac{m x_2 - n x_1}{m - n}, \frac{m y_2 - n y_1}{m - n}\right), \text{ externally}$$

6. The coordinates of the mid-point of the line segment joining (x_1, y_1) and (x_2, y_2) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
7. The coordinates of the centroid of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_2, y_3) are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.