CHAPTER 19

DEFINITE INTEGRALS

19.1 FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

STATEMENT *Let $\phi(x)$ be the primitive or antiderivative of a continuous function $f(x)$ defined on $[a, b]$ i.e. $\frac{d}{dx}\{\phi(x)\} = f(x)$. Then the definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x)\,dx$ and is equal to* $[\phi(b) - \phi(a)]$.

i.e. $$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

The numbers a and b are called the limits of integration, 'a' is called the lower limit and 'b' the upper limit. The interval $[a, b]$ is called the interval of integration.

If we use the notation $\left[\phi(x)\right]_a^b$ to denote $\phi(b) - \phi(a)$. Then,

$$\int_a^b f(x)\,dx = \left[\phi(x)\right]_a^b \qquad \text{[From (i)]}$$

$$\Rightarrow \int_a^b f(x)\,dx = \left(\text{Value of } \phi(x) \text{ at } x = b\right) - \left(\text{Value of } \phi(x) \text{ at } x = a\right)$$

$$\Rightarrow \int_a^b f(x)\,dx = (\text{Value of antiderivative at } x = b) - (\text{Value of antiderivative at } x = a)$$

REMARK 1 *In the above statement it does not matter which anti-derivative is used to evaluate the definite integral, because if $\int f(x)\,dx = \phi(x) + C$, then*

$$\int_a^b f(x)\,dx = \left[\phi(x) + C\right]_a^b = \left\{\phi(b) + C\right\} - \left\{\phi(a) + C\right\} = \phi(b) - \phi(a)$$

In other words, to evaluate the definite integral there is no need to keep the constant of integration.

REMARK 2 $\int_a^b f(x)\,dx$ *is read as "the integral of $f(x)$ from a to b" or integral of $f(x)$ over $[a, b]$.*

19.2 EVALUATION OF DEFINITE INTEGRALS

To evaluate the definite integral $\int_a^b f(x)\,dx$ of a continuous function $f(x)$ defined on $[a, b]$, we may use the following algorithm.

ALGORITHM

Step I *Find the indefinite integral* $\int f(x)\,dx$. *Let this be* $\phi(x)$. *There is no need to keep the constant of integration.*

STEP II *Evaluate* $\phi(b)$ *and* $\phi(a)$.

STEP III *Calculate* $\phi(b) - \phi(a)$.

The number obtained in Step III is the value of the definite integral $\int_a^b f(x)\,dx$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int_1^2 x^2\,dx$ (ii) $\int_{-4}^{-1} \frac{1}{x}\,dx$ (iii) $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx$ (iv) $\int_0^1 \frac{1}{2x-3}\,dx$

SOLUTION We have,

(i) $$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

(ii) $$\int_{-4}^{-1} \frac{1}{x}\,dx = \Big[\log|x|\Big]_{-4}^{-1} = \Big[\log|-1| - \log|-4|\Big] = \log 1 - \log 4 = 0 - \log 4 = -\log 4$$

(iii) $$\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\,dx = \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{\left(\sqrt{1+x}+\sqrt{x}\right)\left(\sqrt{1+x}-\sqrt{x}\right)}\,dx$$

$$= \int_0^1 \left(\sqrt{1+x}-\sqrt{x}\right)dx = \left[\frac{2}{3}(1+x)^{3/2} - \frac{2}{3}x^{3/2}\right]_0^1$$

$$= \left[\frac{2}{3}(1+1)^{3/2} - \frac{2}{3}(1)^{3/2}\right] - \left[\frac{2}{3}(1+0)^{3/2} - \frac{2}{3}(0)^{3/2}\right]$$

$$= \frac{2}{3}\left(2^{3/2}-1\right) - \frac{2}{3}(1-0) = \frac{2}{3}\left(2\sqrt{2}-2\right) = \frac{4}{3}\left(\sqrt{2}-1\right)$$

(iv) $$\int_0^1 \frac{1}{2x-3}\,dx = \frac{1}{2}\Big[\log\,(2x-3)\Big]_0^1$$

$$= \frac{1}{2}\Big[\log|-1| - \log|-3|\Big] = \frac{1}{2}\Big(\log 1 - \log 3\Big) = \frac{1}{2}(0 - \log 3) = -\frac{1}{2}\log 3$$

EXAMPLE 2 *If* $\int_0^1 (3x^2+2x+k)\,dx = 0$, *find* k.

SOLUTION We have,

$$\int_0^1 (3x^2+2x+k)\,dx = 0 \Rightarrow \Big[x^3 + x^2 + kx\Big]_0^1 = 0 \Rightarrow (1+1+k) - 0 = 0 \Rightarrow k = -2$$

EXAMPLE 3 *If* $\int_1^a (3x^2 + 2x + 1)\,dx = 11$, *find real values of a.*

SOLUTION We have,

$$\int_1^a (3x^2 + 2x + 1)\,dx = 11$$

$$\Rightarrow \left[x^3 + x^2 + x\right]_1^a = 11 \Rightarrow (a^3 + a^2 + a) - (1 + 1 + 1) = 11 \Rightarrow a^3 + a^2 + a - 3 = 11$$

$$\Rightarrow a^3 + a^2 + a - 14 = 0 \Rightarrow (a - 2)(a^2 + 3a + 7) = 0 \Rightarrow a = 2 \left[\because a^2 + 3a + 7 \neq 0 \text{ for any } a \in R\right]$$

EXAMPLE 4 *If* $\int_a^b x^3\,dx = 0$ *and if* $\int_a^b x^2\,dx = \frac{2}{3}$, *find a and b.*

SOLUTION We have,

$$\int_a^b x^3\,dx = 0 \Rightarrow \left[\frac{x^4}{4}\right]_a^b = 0 \Rightarrow \frac{1}{4}(b^4 - a^4) = 0 \Rightarrow b^4 - a^4 = 0$$

$$\Rightarrow (b^2 - a^2)(b^2 + a^2) = 0 \Rightarrow b^2 - a^2 = 0 \Rightarrow b = -a \qquad [\because b \neq a]$$

Now, $\int_a^b x^2\,dx = \frac{2}{3}$

$$\Rightarrow \left(\frac{x^3}{3}\right)_a^b = \frac{2}{3} \Rightarrow \frac{1}{3}(b^3 - a^3) = \frac{2}{3} \Rightarrow b^3 - a^3 = 2 \Rightarrow (-a)^3 - a^3 = 2 \qquad [\because b = -a]$$

$$\Rightarrow -2a^3 = 2 \Rightarrow a^3 = -1 \Rightarrow a = -1$$

$$\therefore \quad b = -a \Rightarrow b = 1$$

Hence, $a = -1$ and $b = 1$.

EXAMPLE 5 *Evaluate:* $\int_0^{\pi/2} \sqrt{1 - \cos 2x}\,dx$.

SOLUTION Let $I = \int_0^{\pi/2} \sqrt{1 - \cos 2x}\,dx$. Then,

$$I = \int_0^{\pi/2} \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int_0^{\pi/2} |\sin x|\,dx = \sqrt{2}\int_0^{\pi/2} \sin x\,dx \left[\because \sin x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{2}\right]$$

$$\Rightarrow I = \sqrt{2}\left[-\cos x\right]_0^{\pi/2} = \sqrt{2}\left[\left(-\cos\frac{\pi}{2}\right) - (-\cos 0)\right] = \sqrt{2}(0 + 1) = \sqrt{2}$$

EXAMPLE 6 *Evaluate:*

(i) $\int_0^{\pi/4} \tan^2 x\,dx$ (ii) $\int_0^{\pi/2} \sin^2 x\,dx$ (iii) $\int_0^{\pi/4} \sin 3x \sin 2x\,dx$

SOLUTION (i) Let $\int_0^{\pi/4} \tan^2 x\,dx$. Then,

$$I=\int_0^{\pi/4}(\sec^2 x-1)\,dx=\Big[\tan x-x\Big]_0^{\pi/4}=\left(\tan\frac{\pi}{4}-\frac{\pi}{4}\right)-\left(\tan\,0-0\right)=1-\frac{\pi}{4}$$

(ii) Let $I=\int_0^{\pi/2}\sin^2 x\,dx$. Then,

$$I=\frac{1}{2}\int_0^{\pi/2}(1-\cos 2x)\,dx=\frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\pi/2}=\frac{1}{2}\left\{\left(\frac{\pi}{2}-\frac{1}{2}\sin\pi\right)-\left(0-\frac{1}{2}\sin 0\right)\right\}=\frac{\pi}{4}$$

(iii) Let $I=\int_0^{\pi/4}\sin\,3x\sin\,2x\,dx$. Then,

$$I=\frac{1}{2}\int_0^{\pi/4}(2\sin\,3x\sin\,2x)\,dx=\frac{1}{2}\int_0^{\pi/4}(\cos x-\cos 5x)\,dx=\frac{1}{2}\left[\sin\,x-\frac{1}{5}\sin 5x\right]_0^{\pi/4}$$

$$\Rightarrow\quad I=\frac{1}{2}\left\{\left(\sin\frac{\pi}{4}-\frac{1}{5}\sin\frac{5\pi}{4}\right)-\left(\sin\,0-\frac{1}{5}\sin 0\right)\right\}=\frac{1}{2}\left\{\frac{1}{\sqrt{2}}+\frac{1}{(\sqrt{2})\,5}\right\}=\frac{6}{2\,(5\sqrt{2})}=\frac{3\sqrt{2}}{10}$$

EXAMPLE 7 *Evaluate:*

(i) $\int_0^{\pi}\sin^3 x\,dx$ (ii) $\int_0^{\pi/2}\cos^3 x\,dx$ (iii) $\int_0^{\pi/2}\sin^4 x\,dx$

SOLUTION (i) Let $I=\int_0^{\pi}\sin^3 x\,dx$. Using $\sin\,3x=3\sin x-4\sin^3 x$, we obtain

$$I=\int_0^{\pi}\frac{3\sin\,x-\sin\,3x}{4}\,dx=\frac{1}{4}\int_0^{\pi}(3\sin\,x-\sin\,3x)\,dx=\frac{1}{4}\left[-3\cos x+\frac{1}{3}\cos 3x\right]_0^{\pi}$$

$$\Rightarrow\quad I=\frac{1}{4}\left\{\left(-3\cos\pi+\frac{1}{3}\cos 3\pi\right)-\left(-3\cos 0+\frac{1}{3}\cos 0\right)\right\}=\frac{1}{4}\left\{\left(3-\frac{1}{3}\right)-\left(-3+\frac{1}{3}\right)\right\}=\frac{4}{3}$$

(ii) Let $I=\int_0^{\pi/2}\cos^3 x\,dx$. Using $\cos\,3x=4\cos^3 x-3\cos x$, we obtain

$$I=\int_0^{\pi/2}\frac{\cos\,3x+3\cos x}{4}\,dx=\frac{1}{4}\int_0^{\pi/2}(\cos\,3x+3\cos x)\,dx=\frac{1}{4}\left[\frac{1}{3}\sin 3x+3\sin x\right]_0^{\pi/2}$$

$$\Rightarrow\quad I=\frac{1}{4}\left\{\left(\frac{1}{3}\sin\frac{3\pi}{2}+3\sin\frac{\pi}{2}\right)-\left(\frac{1}{3}\sin 0+3\sin\,0\right)\right\}=\frac{1}{4}\left\{\left(-\frac{1}{3}+3\right)-(0+0)\right\}=\frac{2}{3}$$

(iii) Let $I=\int_0^{\pi/2}\sin^4 x\,dx$. Then,

$$I=\frac{1}{4}\int_0^{\pi/2}(2\sin^2 x)^2\,dx=\frac{1}{4}\int_0^{\pi/2}(1-\cos 2x)^2\,dx$$

$$\Rightarrow\quad I=\frac{1}{4}\int_0^{\pi/2}(1-2\cos 2x+\cos^2 2x)\,dx=\frac{1}{4}\int_0^{\pi/2}1-2\cos 2x+\frac{1+\cos 4x}{2}\,dx$$

$$\Rightarrow \quad I = \frac{1}{8}\int_0^{\pi/2} (3 - 4\cos 2x + \cos 4x)\, dx = \frac{1}{8}\left[3x - \frac{4}{2}\sin 2x + \frac{1}{4}\sin 4x\right]_0^{\pi/2}$$

$$\Rightarrow \quad I = \frac{1}{8}\left[\left\{\frac{3\pi}{2} - 2\sin\pi + \frac{1}{4}\sin 2\pi\right\} - \left\{0 - 0 + 0\right\}\right] = \frac{1}{8}\left[\frac{3\pi}{2} - 0 + 0\right] = \frac{3\pi}{16}$$

EXAMPLE 8 *If* $\int_0^a \sqrt{x}\, dx = 2a \int_0^{\pi/2} \sin^3 x\, dx$, *find the value of integral* $\int_a^{a+1} x\, dx$.

SOLUTION We have,

$$\int_0^a \sqrt{x}\, dx = \frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{2}{3}a^{3/2} \qquad \ldots(i)$$

Let $I = \int_0^{\pi/2} \sin^3 x\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{3\sin x - \sin 3x}{4}\, dx = \frac{1}{4}\int_0^{\pi/2} (3\sin x - \sin 3x)\, dx = \frac{1}{4}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\pi/2}$$

$$\Rightarrow \quad I = \frac{1}{4}\left[\left(-3\cos\frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2}\right) - \left(-3 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[0 - \left(-3 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[3 - \frac{1}{3}\right] = \frac{2}{3} \qquad \ldots(ii)$$

It is given that

$$\int_0^a \sqrt{x}\, dx = 2a \int_0^{\pi/2} \sin^3 x\, dx$$

$$\Rightarrow \quad \frac{2}{3}a^{3/2} = 2a\left(\frac{2}{3}\right) \qquad \text{[Using (i) and (ii)]}$$

$$\Rightarrow \quad a^{3/2} = 2a \Rightarrow a^3 = 4a^2 \Rightarrow a^2(a-4) = 0 \Rightarrow a = 0, 4.$$

When $a = 4$, we obtain

$$\int_a^{a+1} x\, dx = \int_4^5 x\, dx = \left[\frac{x^2}{2}\right]_4^5 = \frac{25}{2} - \frac{16}{2} = \frac{9}{2}.$$

When $a = 0$, we obtain

$$\int_a^{a+1} x\, dx = \int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}.$$

Hence, $\int_a^{a+1} x dx = \frac{9}{2}$ or, $\frac{1}{2}$

EXAMPLE 9 *Evaluate:*

(i) $\int_0^4 \frac{1}{\sqrt{x^2 + 2x + 3}}\, dx$ (ii) $\int_0^a \frac{1}{\sqrt{ax - x^2}}\, dx$

SOLUTION (i) Let $I = \int_0^4 \frac{1}{\sqrt{x^2 + 2x + 3}}\, dx$. Then,

$$I = \int_0^4 \frac{1}{\sqrt{x^2+2x+3}}\,dx = \int_0^4 \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\,dx = \left[\log\left|(x+1)+\sqrt{(x+1)^2+(\sqrt{2})^2}\right|\right]_0^4$$

$$\Rightarrow \quad I = \left[\log\left|x+1+\sqrt{x^2+2x+3}\right|\right]_0^4 = \log(5+\sqrt{16+8+3}) - \log(1+\sqrt{3})$$

$$\Rightarrow \quad I = \log(5+3\sqrt{3}) - \log(1+\sqrt{3}) = \log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$$

(ii) Let $I = \int_0^a \frac{1}{\sqrt{ax-x^2}}\,dx$. Then,

$$I = \int_0^a \frac{1}{\sqrt{-\left\{x^2-ax+\frac{a^2}{4}-\frac{a^2}{4}\right\}}}\,dx = \int_0^a \frac{1}{\sqrt{\left(\frac{a}{2}\right)^2-\left(x-\frac{a}{2}\right)^2}}\,dx$$

$$\Rightarrow \quad I = \left[\sin^{-1}\left(\frac{x-\frac{a}{2}}{\frac{a}{2}}\right)\right]_0^a = \left[\sin^{-1}\left(\frac{2x-a}{a}\right)\right]_0^a = \sin^{-1}1 - \sin^{-1}(-1) = 2\sin^{-1}(1) = 2\left(\frac{\pi}{2}\right) = \pi$$

EXAMPLE 10 *Evaluate:*

(i) $\int_{1/4}^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx$ (ii) $\int_2^4 \frac{x}{x^2+1}\,dx$ (iii) $\int_0^1 \frac{2x}{5x^2+1}\,dx$ (iv) $\int_0^2 \frac{5x+1}{x^2+4}\,dx$

SOLUTION (i) Let $I = \int_{1/4}^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx$. Then,

$$I = \int_{1/4}^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx = \int_{1/4}^{1/2} \frac{1}{\sqrt{-\left\{x^2-x+\frac{1}{4}-\frac{1}{4}\right\}}}\,dx = \int_{1/4}^{1/2} \frac{1}{\sqrt{-\left\{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right\}}}$$

$$\Rightarrow \quad I = \int_{1/4}^{1/2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}} = \left[\sin^{-1}\left(\frac{x-1/2}{1/2}\right)\right]_{1/4}^{1/2}$$

$$\Rightarrow \quad I = \left[\sin^{-1}(2x-1)\right]_{1/4}^{1/2} = \left[\sin^{-1}0 - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \left[0+\sin^{-1}\frac{1}{2}\right] = \frac{\pi}{6}$$

(ii) Let $I = \int_2^4 \frac{x}{x^2+1}\,dx$. Then,

$$I = \frac{1}{2}\int_2^4 \frac{2x}{x^2+1}\,dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\log(x^2+1)\right]_2^4 = \frac{1}{2}[\log(4^2+1) - \log(2^2+1)] = \frac{1}{2}(\log 17 - \log 5) = \frac{1}{2}\log\left(\frac{17}{5}\right)$$

(iii) Let $I = \int_0^1 \frac{2x}{5x^2+1}\,dx = \frac{1}{5}\int_0^1 \frac{10x}{5x^2+1}\,dx = \frac{1}{5}\left[\log(5x^2+1)\right]_0^1 = \frac{1}{5}(\log 6 - \log 1) = \frac{1}{5}\log 6$

(iv) Let $I = \int_0^2 \frac{5x+1}{x^2+4}\,dx$. Then,

$$I = \int_0^2 \frac{5x}{x^2+4}\,dx + \int_0^2 \frac{1}{x^2+4}\,dx = \frac{5}{2}\int_0^2 \frac{2x}{x^2+4}\,dx + \int_0^2 \frac{1}{x^2+2^2}\,dx$$

$$\Rightarrow \quad I = \frac{5}{2}\left[\log(x^2+4)\right]_0^2 + \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$\Rightarrow \quad I = \frac{5}{2}(\log 8 - \log 4) + \frac{1}{2}\left\{\tan^{-1}(1) - \tan^{-1} 0\right\} = \frac{5}{2}\log\left(\frac{8}{4}\right) + \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{5}{2}\log 2 + \frac{\pi}{8}$$

EXAMPLE 11 *Evaluate:*

(i) $\int_0^1 x e^x\,dx$ [NCERT] (ii) $\int_1^2 \frac{\log x}{x^2}\,dx$ (iii) $\int_0^{\pi/2} x \sin x\,dx$

(iv) $\int_0^1 \left\{xe^x + \sin\frac{\pi x}{4}\right\} dx$ [NCERT] (v) $\int_0^1 x \log(1+2x)\,dx$ [NCERT EXEMPLAR]

SOLUTION (i) Let $I = \int_0^1 x e^x\,dx$. Then,

$$I = \int_0^1 \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{e^x}\,dx = \left[x e^x\right]_0^1 - \int_0^1 1 \cdot e^x\,dx = \left[xe^x\right]_0^1 - \left[e^x\right]_0^1 = (e-0) - (e - e^0) = 1$$

(ii) Let $I = \int_1^2 \frac{\log x}{x^2}\,dx$. Then,

$$I = \int_1^2 \underset{\mathrm{I}}{\log x} \cdot \underset{\mathrm{II}}{\frac{1}{x^2}}\,dx = \left[(\log x)\left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x}\left(-\frac{1}{x}\right)dx \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad I = \left[-\frac{1}{x}\log x\right]_1^2 - \left[\frac{1}{x}\right]_1^2 = \left(-\frac{1}{2}\log 2\right) + (1 \times \log 1) - \left(\frac{1}{2} - \frac{1}{1}\right)$$

$$\Rightarrow \quad I = -\frac{1}{2}\log 2 + \frac{1}{2} = \frac{1}{2}(-\log 2 + 1) = \frac{1}{2}(-\log 2 + \log e) = \frac{1}{2}\log\left(\frac{e}{2}\right)$$

(iii) Let $I = \int_0^{\pi/2} \underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\sin x}\,dx$. Then,

$$I = \left[-x \cos x\right]_0^{\pi/2} - \int_0^{\pi/2} 1 \times (-\cos x)\,dx \qquad \text{[Integrating by parts]}$$

$$\Rightarrow \quad I = \left[-x \cos x\right]_0^{\pi/2} + \left[\sin x\right]_0^{\pi/2} = \left(-\frac{\pi}{2}\cos\frac{\pi}{2} + 0 \cos 0\right) + \left(\sin\frac{\pi}{2} - \sin 0\right) = 1$$

(iv) Let $I = \int_0^1 \left(xe^x + \sin \frac{\pi x}{4} \right) dx$. Then,

$$I = \int_0^1 xe^x\, dx + \int_0^1 \sin \frac{\pi x}{4}\, dx = \left[x\, e^x \right]_0^1 - \int_0^1 1 \cdot e^x\, dx - \frac{4}{\pi} \left[\cos \frac{\pi x}{4} \right]_0^1$$

$$\Rightarrow \quad I = \left[x\, e^x \right]_0^1 - \left[e^x \right]_0^1 - \frac{4}{\pi} \left[\cos \frac{\pi x}{4} \right]_0^1 = (1 \times e^1 - 0 \times e^0) - (e^1 - e^0) - \frac{4}{\pi} \left(\cos \frac{\pi}{4} - \cos 0 \right)$$

$$\Rightarrow \quad I = (e - 0) - (e - 1) - \frac{4}{\pi} \left(\frac{1}{\sqrt{2}} - 1 \right) = 1 + \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi}$$

(v) Let $I = \int_0^1 \underset{\text{II}}{x} \underset{\text{I}}{\log (1 + 2x)}\, dx$. Then,

$$I = \left[\frac{x^2}{2} \log (1 + 2x) \right]_0^1 - \int_0^1 \frac{2}{1 + 2x} \times \frac{x^2}{2}\, dx = \left(\frac{1}{2} \log 3 - 0 \right) - \int_0^1 \frac{x^2}{1 + 2x}\, dx$$

$$\Rightarrow \quad I = \frac{1}{2} \log 3 - \int_0^1 \left\{ \left(\frac{x}{2} - \frac{1}{4} \right) + \frac{1}{4\,(1 + 2x)} \right\} dx \qquad \left[\because \frac{x^2}{1 + 2x} = \frac{x}{2} - \frac{1}{4} + \frac{1}{4\,(1 + 2x)} \right]$$

$$\Rightarrow \quad I = \frac{1}{2} \log 3 - \left[\frac{x^2}{4} - \frac{x}{4} + \frac{1}{8} \log (1 + 2x) \right]_0^1 = \frac{1}{2} \log 3 - \left\{ \left(\frac{1}{4} - \frac{1}{4} + \frac{1}{8} \log 3 \right) - 0 \right\} = \frac{3}{8} \log 3$$

EXAMPLE 12 *Evaluate:*

(i) $\int_1^2 \frac{5x^2}{x^2 + 4x + 3}\, dx$ [NCERT, CBSE 2010] (ii) $\int_1^3 \frac{1}{x^2 (x + 1)}\, dx$ [NCERT]

SOLUTION (i) Let $I = \int_1^2 \frac{5x^2}{x^2 + 4x + 3}\, dx$. Then,

$$I = 5 \int_1^2 \frac{x^2}{x^2 + 4x + 3}\, dx = 5 \int_1^2 \left(1 - \frac{4x + 3}{x^2 + 4x + 3} \right) dx$$

$$\Rightarrow \quad I = 5 \int_1^2 1 \cdot dx - 5 \int_1^2 \frac{4x + 3}{x^2 + 4x + 3}\, dx = 5 \int_1^2 1 \cdot dx - 5 \int_1^2 \frac{2\,(2x + 4) - 5}{x^2 + 4x + 3}\, dx$$

$$\Rightarrow \quad I = 5 \int_1^2 1 \cdot dx - 5 \left[\int_1^2 \left\{ \frac{2\,(2x + 4)}{x^2 + 4x + 3} - \frac{5}{x^2 + 4x + 3} \right\} dx \right]$$

$$\Rightarrow \quad I = 5 \int_1^2 1 \cdot dx - 10 \int_1^2 \frac{2x + 4}{x^2 + 4x + 3}\, dx + 25 \int_1^2 \frac{1}{x^2 + 4x + 3}\, dx$$

$$\Rightarrow \quad I = 5 \int_1^2 1 \cdot dx - 10 \int_1^2 \frac{2x + 4}{x^2 + 4x + 3}\, dx + 25 \int_1^2 \frac{1}{(x + 2)^2 - 1^2}\, dx$$

$$\Rightarrow \quad I = 5 \left[x \right]_1^2 - 10 \left[\log (x^2 + 4x + 3) \right]_1^2 + 25 \times \frac{1}{2\,(1)} \left[\log \left| \frac{x + 2 - 1}{x + 2 + 1} \right| \right]_1^2$$

$$\Rightarrow \quad I = 5(2-1) - 10\left(\log 15 - \log 8\right) + \frac{25}{2}\left\{\log\left(\frac{3}{5}\right) - \log\left(\frac{2}{4}\right)\right\}$$

$$\Rightarrow \quad I = 5 - 10\log\left(\frac{15}{8}\right) + \frac{25}{2}\log\left(\frac{3}{5}\times\frac{4}{2}\right) = 5 - 10\log\frac{15}{8} + \frac{25}{2}\log\left(\frac{6}{5}\right)$$

(ii) Let $\dfrac{1}{x^2(x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2}$ ...(i)

Then, $1 = Ax^2 + (Bx+C)(x+1)$ ...(ii)

Putting $x=0, x=-1$ respectively in (ii), we get: $C=1$ and $A=1$

Equating coefficients of x^2 on both sides of (ii), we get: $0 = A + B \Rightarrow B = -A = -1$

Substituting the values of A, B and C in (i), we obtain

$$\frac{1}{x^2(x+1)} = \frac{1}{x+1} + \frac{-x+1}{x^2} = \frac{1}{x+1} - \frac{x}{x^2} + \frac{1}{x^2} = \frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}$$

$$\therefore \quad \int_1^3 \frac{1}{x^2(x+1)}\,dx = \int_1^3 \left(\frac{1}{x+1} - \frac{1}{x} + \frac{1}{x^2}\right)dx = \left[\log|x+1| - \log|x| - \frac{1}{x}\right]_1^3$$

$$= \left(\log 4 - \log 3 - \frac{1}{3}\right) - (\log 2 - \log 1 - 1) = \log 4 - \log 3 - \log 2 - \frac{1}{3} + 1$$

$$= \log\left(\frac{4}{2\times 3}\right) - \frac{1}{3} + 1 = \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

EXAMPLE 13 *Evaluate:*

(i) $\displaystyle\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx$ (ii) $\displaystyle\int_0^{\pi/6} (2+3x^2)\cos 3x\,dx$

SOLUTION (i) Let $I = \displaystyle\int_0^{\sqrt{2}} \sqrt{2-x^2}\,dx = \int_0^{\sqrt{2}} \sqrt{(\sqrt{2})^2 - x^2}\,dx$

$$\Rightarrow \quad I = \left[\frac{1}{2}x\sqrt{2-x^2} + \frac{1}{2}(\sqrt{2})^2 \sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}} = \left\{0 + \sin^{-1}(1)\right\} - \left\{0 + \sin^{-1} 0\right\} = \frac{\pi}{2}$$

(ii) Let $I = \displaystyle\int_0^{\pi/6} \underset{\text{I}}{(2+3x^2)}\ \underset{\text{II}}{\cos 3x}\,dx$. Then,

$$I = \left[(2+3x^2)\times\frac{1}{3}\sin 3x\right]_0^{\pi/6} - \int_0^{\pi/6} 6x \times \frac{1}{3}\sin 3x\,dx$$

$$\Rightarrow \quad I = \left[\frac{1}{3}(2+3x^2)\sin 3x\right]_0^{\pi/6} - 2\int_0^{\pi/6} \underset{\text{I}}{x}\ \underset{\text{II}}{\sin 3x}\,dx$$

$$\Rightarrow \quad I = \left[\frac{1}{3}(2+3x^2)\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\pi/6} - \int_0^{\pi/6} -\frac{\cos 3x}{3}\,dx\right]$$

$$\Rightarrow \quad I = \left[\frac{1}{3}(2+3x^2)\sin 3x\right]_0^{\pi/6} - 2\left[\left[\frac{-x\cos 3x}{3}\right]_0^{\pi/6} + \frac{1}{9}\left[\sin 3x\right]_0^{\pi/6}\right]$$

$$\Rightarrow \quad I = \left[\frac{1}{3}\left(2+\frac{\pi^2}{12}\right)\sin\frac{\pi}{2} - \frac{1}{3}(2+0)\,0\right] - 2\left[\left\{\left(-\frac{\pi}{18}\cos\frac{\pi}{2}\right) + \frac{0\cos 0}{3}\right\} + \frac{1}{9}\left\{\sin\frac{\pi}{2} - \sin 0\right\}\right]$$

$$\Rightarrow \quad I = \left[\frac{1}{3}\left(2+\frac{\pi^2}{12}\right) - \frac{2}{3}\times 0\right] - 2\left[(0-0) + \frac{1}{9}(1-0)\right] = \frac{2}{3} + \frac{\pi^2}{36} - \frac{2}{9} = \frac{\pi^2}{36} + \frac{4}{9} = \frac{1}{36}(\pi^2+16)$$

EXAMPLE 14 *Evaluate:*

(i) $\displaystyle\int_1^2 \frac{1}{(x+1)(x+2)}\,dx$ (ii) $\displaystyle\int_1^2 \frac{1}{x(1+x^2)}\,dx$

SOLUTION (i) Let $\dfrac{1}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. ...(i)

Then, $1 = A(x+2) + B(x+1)$ ...(ii)

Putting $x+2=0$ or, $x=-2$ in (ii), we get: $B=-1$. Putting $x+1=0$ or, $x=-1$ in (ii), we get: $A=1$

Substituting the values of A and B in (i), we get

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\therefore \quad I = \int_1^2 \frac{1}{(x+1)(x+2)}\,dx = \int_1^2 \frac{1}{x+1}\,dx - \int_1^2 \frac{1}{x+2}\,dx = \Big[\log(x+1)\Big]_1^2 - \Big[\log(x+2)\Big]_1^2$$

$$\Rightarrow \quad I = (\log 3 - \log 2) - (\log 4 - \log 3) = 2\log 3 - \log 2 - \log 4 = \log 9 - \log 8 = \log\left(\frac{9}{8}\right)$$

(ii) Let $\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2}$. ...(i)

Then, $1 = A(1+x^2) + (Bx+C)x$ ...(ii)

Putting $x=0$ in (ii), we get: $A=1$. Comparing the coefficients of x^2 and x in (ii), we get

$A+B=0$ and $C=0 \Rightarrow B=-1$ and $C=0$ $[\because A=1]$

Substituting the values of A, B and C in (i), we get

$$\frac{1}{x(1+x^2)} = \frac{1}{x} - \frac{x}{1+x^2}$$

$$\therefore \quad I = \int_1^2 \frac{1}{x(1+x^2)}\,dx = \int_1^2 \frac{1}{x}\,dx - \frac{1}{2}\int_1^2 \frac{2x}{1+x^2}\,dx = \Big[\log x\Big]_1^2 - \frac{1}{2}\Big[\log(1+x^2)\Big]_1^2$$

$$\Rightarrow \quad I = (\log 2 - \log 1) - \frac{1}{2}(\log 5 - \log 2) = \log 2 - \frac{1}{2}\log 5 + \frac{1}{2}\log 2 = \frac{3}{2}\log 2 - \frac{1}{2}\log 5$$

EXAMPLE 15 *Evaluate:* $\displaystyle\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx.$

SOLUTION Let $I = \displaystyle\int_0^{2\pi} \underset{\text{II}}{e^x}\ \underset{\text{I}}{\sin}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$. On integrating by parts, we get

$$I = \left[\sin\left(\frac{\pi}{4}+\frac{x}{2}\right) e^x\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} \underset{\text{II}}{e^x}\ \underset{\text{I}}{\cos}\left(\frac{\pi}{4}+\frac{x}{2}\right) dx$$

$$\Rightarrow \quad I = \left[\sin\frac{5\pi}{4}\, e^{2\pi} - \sin\frac{\pi}{4}\right] - \frac{1}{2}\left[\left\{e^x \cos\left(\frac{\pi}{4}+\frac{x}{2}\right)\right\}_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4}+\frac{x}{2}\right) dx\right]$$

$$\Rightarrow \quad I = \left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2} I\right] = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) + \left(\frac{e^{2\pi}+1}{2\sqrt{2}}\right) - \frac{1}{4} I$$

$$\Rightarrow \quad I + \frac{1}{4} I = \frac{e^{2\pi}+1}{2\sqrt{2}}(1-2) \Rightarrow \frac{5\,I}{4} = -\frac{e^{2\pi}+1}{2\sqrt{2}} \Rightarrow I = -\frac{\sqrt{2}}{5}\left(e^{2\pi}+1\right)$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 16 *Evaluate:*

(i) $\displaystyle\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$ (ii) $\displaystyle\int_0^{\pi/4} \sqrt{1-\sin 2x}\, dx$ [CBSE 2004]

SOLUTION (i) Let $I = \displaystyle\int_0^{\pi/4} \sqrt{1+\sin 2x}\, dx$. Then,

$$I = \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\; dx = \int_0^{\pi/4} \sqrt{(\sin x + \cos x)^2}\; dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} |\cos x + \sin x|\, dx = \int_0^{\pi/4} (\cos x + \sin x)\, dx = \Big[\sin x - \cos x\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - \left(\sin 0 - \cos 0\right) = \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (0-1) = 1$$

(ii) Let $I = \displaystyle\int_0^{\pi/4} \sqrt{1-\sin 2x}\, dx$. Then,

$$I = \int_0^{\pi/4} \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x\; dx} = \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2}$$

$$\Rightarrow \quad I = \int_0^{\pi/4} |\cos x - \sin x|\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} (\cos x - \sin x)\, dx \qquad \left[\begin{array}{l}\because\ 0 < x < \pi/4\ \therefore\ \cos x > \sin x \Rightarrow \cos x - \sin x > 0 \\ \Rightarrow\ |\cos x - \sin x| = \cos x - \sin x\end{array}\right]$$

$$\Rightarrow \quad I = \Big[\sin x + \cos x\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - \left(\sin 0 + \cos 0\right) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(0+1\right) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

EXAMPLE 17 *Evaluate:* $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

SOLUTION Let $I = \int_{\pi/4}^{\pi/2} \sqrt{1 - \sin 2x}\ dx$. Then,

$$I = \int_{\pi/4}^{\pi/2} \sqrt{\cos^2 x + \sin^2 x - 2 \sin x \cos x\ dx} = \int_{\pi/4}^{\pi/2} \sqrt{(\cos x - \sin x)^2}\ dx$$

$$\Rightarrow \quad I = \int_{\pi/4}^{\pi/2} |\cos x - \sin x|\ dx \qquad \left[\because \sqrt{x^2} = |x|\right]$$

$$\Rightarrow \quad I = \int_{\pi/4}^{\pi/2} -(\cos x - \sin x)\ dx \qquad \left[\begin{array}{l} \because \cos x < \sin x \text{ for } \frac{\pi}{4} < x < \frac{\pi}{2} \therefore \cos x - \sin x < 0 \\ \Rightarrow |\cos x - \sin x| = -(\cos x - \sin x) \end{array}\right]$$

$$\Rightarrow \quad I = \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\ dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{\pi/2}$$

$$\Rightarrow \quad I = \left\{-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right\} - \left\{-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right\} = \left(0 - 1\right) - \left(-\frac{2}{\sqrt{2}}\right) = \sqrt{2} - 1$$

EXAMPLE 18 *Evaluate:* $\int_{\pi/4}^{\pi/2} \cos 2x \log \sin x\ dx$ [CBSE 2003]

SOLUTION Let $I = \int_{\pi/4}^{\pi/2} \underset{\text{II}}{\cos 2x} \underset{\text{I}}{\log \sin x}\ dx$. Then,

$$I = \left[\frac{1}{2} (\log \sin x) \sin 2x\right]_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} \frac{1}{2} \cot x \sin 2x\ dx$$

$$\Rightarrow \quad I = \left[0 - \frac{1}{2} \log\left(\frac{1}{\sqrt{2}}\right)\right] - \int_{\pi/4}^{\pi/2} \cos^2 x\ dx = \frac{1}{4} \log 2 - \frac{1}{2} \int_{\pi/4}^{\pi/2} (1 + \cos 2x)\ dx$$

$$\Rightarrow \quad I = \frac{1}{4} \log 2 - \frac{1}{2}\left[x + \frac{1}{2} \sin 2x\right]_{\pi/4}^{\pi/2} = \frac{1}{4} \log 2 - \frac{1}{2}\left[\left(\frac{\pi}{2} + 0\right) - \left(\frac{\pi}{4} + \frac{1}{2}\right)\right] = \frac{1}{4} \log 2 - \frac{\pi}{8} + \frac{1}{4}$$

EXAMPLE 19 *Evaluate:* $\int_0^\infty \frac{1}{(x^2 + a^2)(x^2 + b^2)}\ dx$.

SOLUTION Let $x^2 = y$. Then,

$$\frac{1}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{(y + a^2)(y + b^2)}$$

Let $$\frac{1}{(y + a^2)(y + b^2)} = \frac{A}{y + a^2} + \frac{B}{y + b^2} \qquad \text{...(i)}$$

$$\Rightarrow \quad 1 = A(y + b^2) + B(y + a^2) \qquad \text{...(ii)}$$

Putting $y = -a^2$ and $y = -b^2$ successively in (ii), we get: $A = \frac{1}{b^2 - a^2}$ and $B = \frac{1}{a^2 - b^2}$. Substituting these values in (i), we obtain

$$\frac{1}{(y+a^2)\,(y+b^2)} = \frac{1}{a^2-b^2}\left\{\frac{1}{y+b^2} - \frac{1}{y+a^2}\right\}$$

$$\Rightarrow \quad \frac{1}{(x^2+a^2)\,(x^2+b^2)} = \frac{1}{a^2-b^2}\left\{\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right\} \qquad [\because\ y = x^2]$$

$$\therefore \quad I = \int_0^\infty \frac{1}{(x^2+a^2)\,(x^2+b^2)}\,dx = \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right) dx$$

$$\Rightarrow \quad I = \frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b} - \frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$\Rightarrow \quad I = \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\infty\right) - \left(\frac{1}{b}\tan^{-1}0 - \frac{1}{a}\tan^{-1}0\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{a^2-b^2}\left[\left(\frac{\pi}{2b} - \frac{\pi}{2a}\right) - (0-0)\right] = \frac{\pi}{2\,ab\,\,(a+b)}$$

EXAMPLE 20 *If $f(x)$ is of the form $f(x) = a + bx + cx^2$, show that $\int_0^1 f(x)\,dx = \frac{1}{6}\left\{f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right\}$.*

SOLUTION We have, $f(x) = a + bx + cx^2$. Therefore, $f(0) = a$, $f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{4}$ and $f(1) = a + b + c$.

$$\therefore \quad \frac{1}{6}\left\{f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right\} = \frac{1}{6}\left\{a + 4\left(a + \frac{b}{2} + \frac{c}{4}\right) + (a+b+c)\right\} = \frac{1}{6}(6a + 3b + 2c) \qquad \text{...(i)}$$

Now,

$$\int_0^1 f(x)\,dx = \int_0^1 (a + bx + cx^2)\,dx = \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]_0^1 = \left(a + \frac{b}{2} + \frac{c}{3}\right) - 0 = \frac{1}{6}(6a + 3b + 2c) \qquad \text{...(ii)}$$

From (i) and (ii), we get: $\int_0^1 f(x)\,dx = \frac{1}{6}\left\{f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right\}$

EXERCISE 19.1

BASIC

Evaluate the following definite integrals (1-68):

1. $\int_4^9 \frac{1}{\sqrt{x}}\,dx$ **2.** $\int_{-2}^3 \frac{1}{x+7}\,dx$ **3.** $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$ **4.** $\int_0^1 \frac{1}{1+x^2}\,dx$

5. $\int_2^3 \frac{x}{x^2+1}\,dx$ **6.** $\int_0^\infty \frac{1}{a^2+b^2x^2}\,dx$ **7.** $\int_{-1}^1 \frac{1}{1+x^2}\,dx$ **8.** $\int_0^\infty e^{-x}\,dx$

9. $\int_0^1 \frac{x}{x+1}\,dx$ **10.** $\int_0^{\pi/2} (\sin x + \cos x)\,dx$ **11.** $\int_{\pi/4}^{\pi/2} \cot x\,dx$

12. $\int_0^{\pi/4} \sec x \; dx$ 13. $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x \; dx$ [NCERT] 14. $\int_0^1 \frac{1-x}{1+x} \; dx$

15. $\int_0^{\pi} \frac{1}{1+\sin x} \; dx$ 16. $\int_{-\pi/4}^{\pi/4} \frac{1}{1+\sin x} dx$ 17. $\int_0^{\pi/2} \cos^2 x \; dx$ [NCERT, CBSE 2002]

18. $\int_0^{\pi/2} \cos^3 x \; dx$ 19. $\int_0^{\pi/6} \cos x \cos 2x \, dx$ 20. $\int_0^{\pi/2} \sin x \sin 2x \; dx$

21. $\int_{\pi/3}^{\pi/4} (\tan x + \cot x)^2 \; dx$ 22. $\int_0^{\pi/2} \cos^4 x \; dx$

23. $\int_0^{\pi/2} (a^2 \cos^2 x + b^2 \sin^2 x) \, dx$ 24. $\int_0^{\pi/2} \sqrt{1+\sin x} \; dx$ 25. $\int_0^{\pi/2} \sqrt{1+\cos x} \; dx$

26. $\int_0^{\pi/2} x^2 \sin x \, dx$ [CBSE 2014] 27. $\int_0^{\pi/2} x \cos x \; dx$ 28. $\int_0^{\pi/2} x^2 \cos x \, dx$

29. $\int_0^{\pi/4} x^2 \sin x \, dx$ 30. $\int_0^{\pi/2} x^2 \cos 2x \, dx$ 31. $\int_1^2 \log x \; dx$ 32. $\int_1^3 \frac{\log x}{(x+1)^2} \; dx$

33. $\int_1^e \frac{e^x}{x} \; (1 + x \log x) \; dx$ 34. $\int_1^e \frac{\log x}{x} dx$ 35. $\int_1^2 \frac{x+3}{x(x+2)} \, dx$

36. $\int_0^1 \frac{2x+3}{5x^2+1} \; dx$ [NCERT] 37. $\int_0^2 \frac{1}{4+x-x^2} \, dx$ [NCERT]

38. $\int_0^1 \frac{1}{2x^2+x+1} \; dx$ 39. $\int_0^1 \sqrt{x(1-x)} \, dx$ 40. $\int_0^2 \frac{1}{\sqrt{3+2x-x^2}} \, dx$ 41. $\int_0^4 \frac{1}{\sqrt{4x-x^2}} \; dx$

42. $\int_{-1}^1 \frac{1}{x^2+2x+5} \, dx$ [NCERT] 43. $\int_1^4 \frac{x^2+x}{\sqrt{2x+1}} \; dx$ 44. $\int_0^1 x(1-x)^5 \, dx$

45. $\int_0^1 \left(xe^{2x} + \sin \frac{\pi x}{2} \right) dx$ 46. $\int_0^1 \left(xe^x + \cos \frac{\pi x}{4} \right) dx$

47. $\int_0^1 \frac{1}{\sqrt{1+x} - \sqrt{x}} \; dx$ [NCERT] 48. $\int_1^2 \frac{x}{(x+1)(x+2)} \, dx$ [NCERT]

49. $\int_0^{\pi/2} \sin^3 x \, dx$ [NCERT] 50. $\int_0^{\pi} \left(\sin^2 \frac{x}{2} - \cos^2 \frac{x}{2} \right) \, dx$ [NCERT]

51. $\int_1^2 \frac{1}{\sqrt{(x-1)(2-x)}} \, dx$ [NCERT EXEMPLAR]

52. If $\int_0^k \frac{1}{2+8x^2} \; dx = \frac{\pi}{16}$, find the value of k. 53. If $\int_0^a 3x^2 dx = 8$, find the value of a.

BASED ON LOTS

54. $\int_0^{\pi/2} x^2 \cos^2 x \, dx$

55. $\int_e^{e^2} \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

56. $\int_1^2 \left(\frac{x-1}{x^2} \right) e^x dx$ [CBSE 2002]

57. $\int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x} \right) dx$ [NCERT]

58. $\int_0^{2\pi} e^{x/2} \sin\left(\frac{x}{2} + \frac{\pi}{4} \right) dx$

59. $\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4} + \frac{x}{2} \right) dx$

60. $\int_0^{\pi} e^{2x} \sin\left(\frac{\pi}{4} + x \right) dx$ [CBSE 2016]

61. $\int_1^2 e^{2x} \left(\frac{1}{x} - \frac{1}{2x^2} \right) dx$ [NCERT, CBSE 2020]

BASED ON HOTS

62. $\int_{\pi}^{3\pi/2} \sqrt{1 - \cos 2x} \, dx$

63. $\int_0^{2\pi} \sqrt{1 + \sin \frac{x}{2}} \, dx$

64. $\int_0^{\pi/4} (\tan x + \cot x)^{-2} dx$

65. $\int_0^1 x \log(1 + 2x) \, dx$

66. $\int_{\pi/6}^{\pi/3} (\tan x + \cot x)^2 dx$

67. $\int_0^{\pi/4} (a^2 \cos^2 x + b^2 \sin^2 x) \, dx$

68. $\int_0^1 \frac{1}{1 + 2x + 2x^2 + 2x^3 + x^4} dx$

ANSWERS

1. 2
2. $\log 2$
3. $\frac{\pi}{6}$
4. $\frac{\pi}{4}$
5. $\frac{1}{2} \log 2$
6. $\frac{\pi}{2\,ab}$
7. $\frac{\pi}{2}$
8. 1
9. $\log\left(\frac{e}{2} \right)$
10. 2
11. $\frac{1}{2} \log 2$
12. $\log(\sqrt{2} + 1)$
13. $\log(\sqrt{2} - 1) - \log(2 - \sqrt{3})$
14. $2 \log 2 - 1$
15. 2
16. 2
17. $\frac{\pi}{4}$
18. $\frac{2}{3}$
19. $\frac{5}{12}$
20. $\frac{2}{3}$
21. $-\frac{2}{\sqrt{3}}$
22. $\frac{3\pi}{16}$
23. $\frac{\pi}{4}(a^2 + b^2)$
24. 2
25. 2
26. $\pi - 2$
27. $\frac{\pi}{2} - 1$
28. $\frac{\pi^2}{4} - 2$
29. $\sqrt{2} + \frac{\pi}{2\sqrt{2}} - \frac{\pi^2}{16\sqrt{2}} - 2$
30. $-\frac{\pi}{4}$
31. $2 \log 2 - 1$
32. $\frac{3}{4} \log 3 - \log 2$
33. e^e
34. $\frac{1}{2}$
35. $\frac{1}{2} \log 6$
36. $\frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1} \sqrt{5}$
37. $\frac{1}{\sqrt{17}} \log\left(\frac{21 + 5\sqrt{17}}{4} \right)$
38. $\frac{2}{\sqrt{7}} \left\{ \tan^{-1} \frac{5}{\sqrt{7}} - \tan^{-1} \frac{1}{\sqrt{7}} \right\}$
39. $\frac{\pi}{8}$
40. $\frac{\pi}{3}$
41. π
42. $\frac{\pi}{8}$
43. $\frac{57 - \sqrt{3}}{5}$
44. $\frac{1}{42}$

45. $\frac{e^2}{4}+\frac{1}{4}+\frac{2}{\pi}$ 46. $1+\frac{2\sqrt{2}}{\pi}$ 47. $\log\left(\frac{32}{27}\right)$ 48. $\frac{2}{3}$ 49. 0

50. $\frac{e^4-2e^2}{4}$ 51. $\frac{1}{2}$ 52. 2 53. $\sqrt{2}$ 54. $\frac{\pi^3}{48}-\frac{\pi}{8}$

55. $\frac{e^2}{2}-e$ 56. $\frac{e^2}{2}-e$ 57. $e^{\pi/2}$ 58. 0 59. $-\frac{3\sqrt{2}}{5}\left(e^{2\pi}+1\right)$

60. $\frac{2^{5/2}}{3}$ 61. π 62. 8 63. $\frac{\pi}{32}$ 64. $\frac{3}{8}\log_e 3$

65. $\frac{4}{\sqrt{3}}$ 66. $(a^2+b^2)\frac{\pi}{8}+\frac{1}{4}(a^2-b^2)$ 67. $\frac{1}{4}\log(2e)$

19.3 EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

Consider the integral $\int_a^b f\left(g(x)\right)g'(x)\,dx$. Let $g(x)=t$, then $g'(x)\,dx=dt$. Also, when $x=a$, $t=g(a)$ and $t=g(b)$ for $x=b$.

$$\therefore \quad \int_a^b f\left(g(x)\right)g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt$$

Thus, if the variable in a definite integral is changed, then the substitution in terms of new variable is effected at three places :

(i) in the integrand (ii) in the differential, say, dx (iii) in the limits

Also, limits of the new variable t are simply the values of t corresponding to the values of the original variable x, and so they are obtained by putting the values of x in the substitutional relation between x and t.

We may use the following algorithm to evaluate definite integrals by making some substitution.

ALGORITHM

Step I *Obtain the definite integral and express it in the form* $I=\int_a^b f(g(x)\,g'(x)\,dx$.

Step II *Put* $t=g(x)$ *which gives* $g'(x)\,dx=dt$.

Step III *Put* $x=a$ *(the lower limit for x) in* $t=g(x)$ *to get* $t=g(a)$ *as the lower limit for t.*
Put $x=b$ *(the upper limit for x) in* $t=g(x)$ *to get* $t=g(b)$ *as the upper limit for t.*

Step IV *Substitute* $g'(x)\,dx=dt$ *and replace old limits of integration by the new limits to get*

$$I=\int_{g(a)}^{g(b)} f(t)\,dt$$

Evaluate this integral by using standard methods of integration discussed in section 19.1.

Following examples will illustrate the above procedure.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int_0^4 \frac{1}{x+\sqrt{x}}\,dx$ (ii) $\int_0^1 \frac{2x}{5x^2+1}\,dx$

SOLUTION (i) Let $x = t^2$. Then, $dx = d(t^2) \Rightarrow dx = 2t\,dt$.

When $x = 0, x = t^2 \Rightarrow t^2 = 0 \Rightarrow t = 0$. When $x = 4, t^2 = x \Rightarrow t^2 = 4 \Rightarrow t = 2$

$$\therefore \quad I = \int_0^4 \frac{1}{x + \sqrt{x}}\,dx = \int_0^2 \frac{2t\,dt}{t^2 + t} = 2\int_0^2 \frac{1}{t+1}\,dt = 2\Big[\log\,(t+1)\Big]_0^2 = 2\,[\log 3 - \log 1] = 2\log 3$$

(ii) Let $5x^2 + 1 = t$. Then, $d(5x^2 + 1) = dt \Rightarrow 10\,x dx = dt$.

When $x = 0, t = 5x^2 + 1 \Rightarrow t = 1$. When $x = 1, t = 5x^2 + 1 \Rightarrow t = 6$

$$\therefore \quad \int_0^1 \frac{2x}{5x^2+1}\,dx = \int_1^6 \frac{2x}{t} \times \frac{dt}{10x} = \frac{1}{5}\int_1^6 \frac{1}{t}\,dt = \frac{1}{5}\Big[\log t\Big]_1^6 = \frac{1}{5}(\log 6 - \log 1) = \frac{1}{5}\log 6$$

EXAMPLE 2 *Evaluate:*

(i) $\displaystyle\int_0^1 \sin^{-1} x\,dx$ **[NCERT]** (ii) $\displaystyle\int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta$ **[CBSE 2004, 2022]** (iv) $\displaystyle\int_0^{1/2} \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$ **[CBSE 2015]**

(v) $\displaystyle\int_0^{\pi/4} \frac{1}{\cos^3 x\sqrt{2\sin 2x}}\,dx$ **[CBSE 2015]**

SOLUTION (i) Let $t = \sin^{-1} x$ or, $x = \sin t$. Then, $dx = d(\sin t) = \cos t\,dt$

Also, $x = 0, \Rightarrow t = \sin^{-1} 0 = 0$ and $x = 1 \Rightarrow t = \sin^{-1} 1 = \frac{\pi}{2}$.

$$\therefore \quad I = \int_0^1 \sin^{-1} x\,dx = \int_0^{\pi/2} \underset{\text{I}}{t}\ \underset{\text{II}}{\cos t}\,dt = \Big[t\sin t\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin t\,dt$$

$$= \Big[t\sin t\Big]_0^{\pi/2} + \Big[\cos t\Big]_0^{\pi/2} = \frac{\pi}{2}\sin\frac{\pi}{2} - 0 + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1$$

(ii) Let $\cos\theta = t$. Then, $d(\cos\theta) = dt \Rightarrow -\sin\theta\,d\theta = dt$.

Also, $\theta = 0 \Rightarrow t = \cos 0 = 1$ and, $\theta = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{2} = 0$

$$\therefore \quad I = \int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta = \int_1^0 \sqrt{t}\,\sin^3\theta\left(\frac{-dt}{\sin\theta}\right) = -\int_1^0 \sqrt{t}\,\sin^2\theta\,dt$$

$$\Rightarrow \quad I = -\int_1^0 \sqrt{t}\,(1-t^2)\,dt = -\int_1^0 \left(\sqrt{t} - t^{5/2}\right)dt = -\left[\frac{2}{3}t^{3/2} - \frac{2}{7}t^{7/2}\right]_1^0 = -\left[0 - \left(\frac{2}{3} - \frac{2}{7}\right)\right] = \frac{8}{21}$$

(iii) Let $\sin\theta = t$. Then, $d(\sin\theta) = dt \Rightarrow \cos\theta\,d\theta = dt$.

Also, $\theta = 0 \Rightarrow t = \sin 0 = 0$ and, $\theta = \frac{\pi}{2} \Rightarrow t = \sin\frac{\pi}{2} = 1$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)}\,d\theta = \int_0^1 \frac{1}{(1+t)(2+t)}\,dt$$

$$\Rightarrow \quad I = \int_0^1 \left\{\frac{1}{1+t} - \frac{1}{2+t}\right\}dt = \Big[\log(1+t)\Big]_0^1 - \Big[\log(2+t)\Big]_0^1 \quad \text{[By using partial fractions]}$$

$$\Rightarrow \quad I = (\log 2 - \log 1) - (\log 3 - \log 2) = \log 2 - \log 3 + \log 2 = 2\log 2 - \log 3 = \log\left(\frac{4}{3}\right)$$

(iv) Let $I = \int_0^{1/2} \frac{1}{(1+x^2)\sqrt{1-x^2}}\,dx$. Let $x = \sin\theta$. Then, $dx = \cos\theta\, d\theta$.

Also, $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$ and $x = \frac{1}{2} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$

$$\therefore \quad I = \int_0^{\pi/6} \frac{1}{(1+\sin^2\theta)\sqrt{1-\sin^2\theta}}\cos\theta\, d\theta = \int_0^{\pi/6} \frac{1}{1+\sin^2\theta}\,d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/6} \frac{\sec^2\theta}{\sec^2\theta + \tan^2\theta}\,d\theta = \int_0^{\pi/6} \frac{\sec^2\theta}{1+2\tan^2\theta}\,d\theta \qquad \text{[Dividing } N^r \text{ and } D^r \text{ by } \cos^2\theta]$$

Let $\tan\theta = t$. Then, $\sec^2\theta\, d\theta = dt$. Also, $\theta = 0 \Rightarrow t = \tan 0 = 0$ and $\theta = \frac{\pi}{6} \Rightarrow t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.

$$\therefore \quad I = \int_0^{1/\sqrt{3}} \frac{1}{1+2t^2}\,dt = \frac{1}{2}\int_0^{1/\sqrt{3}} \frac{1}{(1/\sqrt{2})^2 + t^2}\,dt = \frac{1}{2}\times\frac{1}{(1/\sqrt{2})}\left[\tan^{-1}\sqrt{2}t\right]_0^{1/\sqrt{3}} = \frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$$

(v) Let $I = \int_0^{\pi/4} \frac{1}{\cos^3 x\sqrt{2\sin 2x}}\,dx$. Then,

$$I = \int_0^{\pi/4} \frac{1}{\cos^3 x\sqrt{4\sin x\cos x}}\,dx = \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2} x \cos^{-7/2} x\, dx$$

We observe that the sum of the exponents of $\sin x$ and $\cos x$ is $\left(-\frac{1}{2}\right)+\left(-\frac{7}{2}\right) = -4$, which is a negative even integer. So, we divide numerator and denominator by $\cos^4 x$ to get

$$I = \frac{1}{2}\int_0^{\pi/4} \frac{\frac{1}{\cos^4 x}}{\frac{\sin^{1/2} x\cos^{7/2} x}{\cos^4 x}}\,dx = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^4 x}{\sqrt{\tan x}}\,dx = \frac{1}{2}\int_0^{\pi/4} \frac{(1+\tan^2 x)}{\sqrt{\tan x}}\sec^2 x\, dx$$

Let $\tan x = t$. Then $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{4} \Rightarrow t = \tan\frac{\pi}{4} = 1$.

$$\therefore \quad I = \frac{1}{2}\int_0^1 \frac{1+t^2}{\sqrt{t}}\,dt = \frac{1}{2}\int_0^1 (t^{-1/2} + t^{3/2})\,dt = \frac{1}{2}\left[2t^{1/2} + \frac{2}{5}t^{5/2}\right]_0^1 = \frac{1}{2}\times\frac{12}{5} = \frac{6}{5}$$

EXAMPLE 3 *Evaluate:*

(i) $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx$ (ii) $\int_0^{1/\sqrt{2}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\,dx$ [CBSE 2007]

(iii) $\int_0^1 \frac{x\tan^{-1} x}{(1+x^2)^{3/2}}\,dx$ (iv) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$ [NCERT, CBSE 2002]

SOLUTION (i) Let $x = a\sin\theta$. Then, $dx = d(a\sin\theta) = a\cos\theta\, d\theta$.

Also, $x = 0 \Rightarrow a\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

And, $\quad x=a \Rightarrow a\sin\theta=a \Rightarrow \sin\theta=1 \Rightarrow \theta=\dfrac{\pi}{2}$

$$\therefore \quad I=\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\,dx=\int_0^{\pi/2}\frac{(a\sin\theta)^4}{\sqrt{a^2-a^2\sin^2\theta}}\,a\cos\theta\,d\theta=a^4\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$$\Rightarrow \quad I=a^4\times\frac{3\pi}{16}=\frac{3\pi a^4}{16}$$ [See Example 7 (iii) on page 19.4]

(ii) Let $\sin^{-1}x=\theta$ or, $x=\sin\theta$. Then, $dx=d(\sin\theta)=\cos\theta\,d\theta$

Now, $\quad x=0 \Rightarrow \sin\theta=0 \Rightarrow \theta=0$ and $x=\dfrac{1}{\sqrt{2}} \Rightarrow \sin\theta=\dfrac{1}{\sqrt{2}} \Rightarrow \theta=\dfrac{\pi}{4}$

$$\therefore \quad I=\int_0^{1/\sqrt{2}}\frac{\sin^{-1}x}{(1-x^2)^{32}}\,dx$$

$$\Rightarrow \quad I=\int_0^{\pi/4}\frac{\theta}{\cos^3\theta}\cos\theta\,d\theta=\int_0^{\pi/4}\underset{\mathrm{I}}{\theta}\,\underset{\mathrm{II}}{\sec^2\theta}\,d\theta=\Big[\theta\tan\theta\Big]_0^{\pi/4}-\int_0^{\pi/4}1\times\tan\theta\,d\theta$$

$$\Rightarrow \quad I=\Big[\theta\ \tan\theta\Big]_0^{\pi/4}+\Big[\log\ \cos\theta\Big]_0^{\pi/4}=\left(\frac{\pi}{4}-0\right)+\left\{\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right\}=\frac{\pi}{4}-\frac{1}{2}\log 2$$

(iii) Let $\tan^{-1}x=\theta$ or, $x=\tan\theta$. Then, $dx=\sec^2\theta\,d\theta$.

Also, $\quad x=0 \Rightarrow \tan\theta=0 \Rightarrow \theta=0$ and, $x=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\dfrac{\pi}{4}$

$$\therefore \quad I=\int_0^1\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx=\int_0^{\pi/4}\frac{\theta\tan\theta}{\sec^3\theta}\sec^2\theta\,d\theta=\int_0^{\pi/4}\underset{\mathrm{I}}{\theta}\,\underset{\mathrm{II}}{\sin\theta}\,d\theta$$

$$\Rightarrow \quad I=\Big[-\theta\ \cos\ \theta\Big]_0^{\pi/4}-\int_0^{\pi/4}(-\cos\theta)\ d\theta=\Big[-\theta\ \cos\theta\Big]_0^{\pi/4}+\Big[\sin\ \theta\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I=\left(-\frac{\pi}{4\sqrt{2}}-0\right)+\left(\frac{1}{\sqrt{2}}-0\right)=\frac{4-\pi}{4\sqrt{2}}$$

(iv) Let $I=\displaystyle\int_0^1\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$. Then,

$$I=\int_0^1 2\tan^{-1}x\,dx \qquad \left[\because\ \sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x\right]$$

$$\Rightarrow \quad I=2\int_0^1\underset{\mathrm{I}}{\tan^{-1}x}\cdot\underset{\mathrm{II}}{1}\ dx=2\left[\Big[x\tan^{-1}x\Big]_0^1-\int_0^1\frac{1}{1+x^2}\cdot x\ dx\right]$$

$$\Rightarrow \quad I=2\left[\Big[x\tan^{-1}x\Big]_0^1-\frac{1}{2}\int_0^1\frac{2x}{1+x^2}\ dx\right]=2\left[\Big[x\tan^{-1}x\Big]_0^1-\frac{1}{2}\Big[\log(1+x^2)\Big]_0^1\right]$$

$$\Rightarrow \quad I=2\left[(1\times\tan^{-1}1-0\tan^{-1}0)-\frac{1}{2}(\log 2-\log 1)\right]=2\left\{\left(\frac{\pi}{4}-0\right)-\frac{1}{2}\log 2\right\}=\frac{\pi}{2}-\log 2$$

ALITER Let $x = \tan\theta$. Then, $dx = d(\tan\theta) = \sec^2\theta\, d\theta$

Now, $x = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$ and $x = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore \quad I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx = \int_0^{\pi/4} \sin^{-1}(\sin 2\theta)\ \sec^2\theta\, d\theta$$

$$\Rightarrow \quad I = \int_0^{\pi/4} 2\theta \sec^2\theta\, d\theta = 2\int_0^{\pi/4} \underset{\mathrm{I}}{\theta}\ \underset{\mathrm{II}}{\sec^2\theta}\, d\theta = 2\left\{\Big[\theta\tan\theta\Big]_0^{\pi/4} - \int_0^{\pi/4} 1\times\tan\theta\, d\theta\right\}$$

$$\Rightarrow \quad I = 2\left\{\Big[\theta\tan\theta\Big]_0^{\pi/4} + \Big[\log\cos\theta\Big]_0^{\pi/4}\right\} = 2\left\{\left(\frac{\pi}{4}\tan\frac{\pi}{4} - 0\right) + \left(\log\cos\frac{\pi}{4} - \log 1\right)\right\}$$

$$\Rightarrow \quad I = 2\left\{\frac{\pi}{4} + \log\frac{1}{\sqrt{2}}\right\} = \frac{\pi}{2} + 2\log\frac{1}{\sqrt{2}} = \frac{\pi}{2} + \log\frac{1}{2} = \frac{\pi}{2} - \log 2$$

EXAMPLE 4 *Evaluate:* $\int_0^{\pi/4} \tan^3 x\, dx$ **[NCERT, CBSE 2004]**

SOLUTION Let $I = \int_0^{\pi/4} \tan^3 x\, dx$. Then,

$$I = \int_0^{\pi/4} \tan^2 x\ \tan x\, dx = \int_0^{\pi/4} (\sec^2 x - 1)\ \tan x\, dx = \int_0^{\pi/4} \sec^2 x \tan x\, dx - \int_0^{\pi/4} \tan x\, dx$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$ and, $x = \frac{\pi}{4} \Rightarrow t = 1$

$$\therefore \quad I = \int_0^1 t\, dt - \int_0^{\pi/4} \tan x\, dx = \left[\frac{t^2}{2}\right]_0^1 - \Big[\log\sec x\Big]_0^{\pi/4}$$

$$\Rightarrow \quad I = \left(\frac{1}{2} - 0\right) - \log\sec\frac{\pi}{4} + \log\sec 0 = \frac{1}{2} - \log\sqrt{2} + \log 1 = \frac{1}{2} - \frac{1}{2}\log 2 = \frac{1}{2}(1 - \log 2)$$

EXAMPLE 5 *Evaluate:*

(i) $\int_0^{\pi} \frac{1}{5 + 4\cos x}\, dx$ **[CBSE 2005]** (ii) $\int_0^{\pi/2} \frac{1}{3 + 2\cos x}\, dx$

SOLUTION (i) Let $I = \int_0^{\pi} \frac{1}{5 + 4\cos x}\, dx$. Then,

$$I = \int_0^{\pi} \frac{1}{5 + 4\left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)}\, dx = \int_0^{\pi} \frac{1 + \tan^2\frac{x}{2}}{5\left(1 + \tan^2\frac{x}{2}\right) + 4\left(1 - \tan^2\frac{x}{2}\right)}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{1+\tan^2 \frac{x}{2}}{9+\tan^2 \frac{x}{2}}\, dx = \int_0^{\pi} \frac{\sec^2 \frac{x}{2}}{9+\tan^2 \frac{x}{2}}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt \Rightarrow dx = \frac{2dt}{\sec^2 \frac{x}{2}}$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan \frac{\pi}{2} = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{\sec^2 \frac{x}{2}}{9+t^2} \times \frac{2dt}{\sec^2 \frac{x}{2}} = 2\int_0^{\infty} \frac{dt}{3^2+t^2} = \frac{2}{3}\left[\tan^{-1} \frac{t}{3}\right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{2}{3}\left(\tan^{-1}\infty - \tan^{-1} 0\right) = \frac{2}{3}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{3}$$

(ii) Let $I = \int_0^{\pi/2} \frac{1}{3+2\cos x}\, dx$. Then,

$$\Rightarrow I = \int_0^{\pi/2} \frac{1}{3+2\left(\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)}\, dx = \int_0^{\pi/2} \frac{1+\tan^2 \frac{x}{2}}{3\left(1+\tan^2 \frac{x}{2}\right)+2\left(1-\tan^2 \frac{x}{2}\right)}\, dx = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{5+\tan^2 \frac{x}{2}}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt \Rightarrow dx = \frac{2\,dt}{\sec^2 \frac{x}{2}}$

Now, $x = 0 \Rightarrow t = \tan 0 = 0$, and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{4} = 1$

$$\therefore \quad I = \int_0^1 \frac{\sec^2 \frac{x}{2}}{5+t^2} \times \frac{2dt}{\sec^2 \frac{x}{2}} = 2\int_0^1 \frac{dt}{(\sqrt{5})^2+t^2}$$

$$\Rightarrow I = 2 \times \frac{1}{\sqrt{5}}\left[\tan^{-1}\left[\frac{t}{\sqrt{5}}\right]\right]_0^1 = \frac{2}{\sqrt{5}}\left[\tan^{-1} \frac{1}{\sqrt{5}} - \tan^{-1} 0\right] = \frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

EXAMPLE 6 *Evaluate:*

(i) $\int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}\, dx$ (ii) $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\, dx$ [NCERT] (iii) $\int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$

SOLUTION (i) Let $I = \int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{1}{\frac{2\left(1-\tan^2 \frac{x}{2}\right)}{1+\tan^2 \frac{x}{2}} + \frac{4\left(2\tan \frac{x}{2}\right)}{1+\tan^2 \frac{x}{2}}}\, dx = \int_0^{\pi/2} \frac{1+\tan^2 \frac{x}{2}}{2-2\tan^2 \frac{x}{2}+8\tan \frac{x}{2}}\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{2 - 2\tan^2 \frac{x}{2} + 8\tan \frac{x}{2}}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\sec^2 \frac{x}{2}\, dx = dt \Rightarrow dx = 2\frac{dt}{\sec^2 \frac{x}{2}}$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$, and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{4} = 1$.

$$\therefore \quad I = \int_0^1 \frac{\sec^2 \frac{x}{2}}{2 - 2t^2 + 8t} \times \frac{2dt}{\sec^2 \frac{x}{2}} = \int_0^1 \frac{1}{1 - t^2 + 4t}\, dt = \int_0^1 \frac{1}{-(t^2 - 4t - 1)}\, dt$$

$$\Rightarrow \quad I = \int_0^1 \frac{1}{-\left(t^2 - 4t + 4 - 4 - 1\right)} 1 = \int_0^1 \frac{1}{-\left\{(t-2)^2 - 5\right\}}\, dt = \int_0^1 \frac{1}{(\sqrt{5})^2 - (t-2)^2}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\log \left|\frac{\sqrt{5} + t - 2}{\sqrt{5} - t + 2}\right|\right]_0^1 = \frac{1}{2\sqrt{5}}\left[\log\left(\frac{\sqrt{5} - 1}{\sqrt{5} + 1}\right) - \log\left(\frac{\sqrt{5} - 2}{\sqrt{5} + 2}\right)\right]$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\log\left\{\frac{(\sqrt{5} - 1)(\sqrt{5} + 2)}{(\sqrt{5} + 1)(\sqrt{5} - 2)}\right\}\right] = \frac{1}{2\sqrt{5}}\log\left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}}\right)$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\log\left(\frac{3 + \sqrt{5}}{3 - \sqrt{5}} \times \frac{3 + \sqrt{5}}{3 + \sqrt{5}}\right) = \frac{1}{2\sqrt{5}}\log\left(\frac{3 + \sqrt{5}}{2}\right)^2 = \frac{2}{2\sqrt{5}}\log\left(\frac{3 + \sqrt{5}}{2}\right)$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{5}}\log\left(\frac{3 + \sqrt{5}}{2}\right)$$

(ii) Let $I = \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx$. Let $\cos x = t$. Then, $-\sin x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \cos 0 = 1$ and $x = \frac{\pi}{2} \Rightarrow t = \cos \frac{\pi}{2} = 0$

$$\therefore \quad I = \int_1^0 \frac{\sin x}{1 + t^2}\left(\frac{-dt}{\sin x}\right)$$

$$\Rightarrow \quad I = -\int_1^0 \frac{dt}{1 + t^2} = -\left[\tan^{-1} t\right]_1^0 = -\left[\tan^{-1} 0 - \tan^{-1} 1\right] = -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4}$$

(iii) We have, $I = \int_0^{\pi/2} \frac{1}{4\sin^2 x + 5\cos^2 x}\, dx$. Dividing nominator and denominator by $\cos^2 x$, we obtain

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sec^2 x}{4\tan^2 x + 5}\, dx$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore \quad I = \int_0^\infty \frac{dt}{4t^2+5} = \frac{1}{4}\int_0^\infty \frac{1}{t^2+\left(\frac{\sqrt{5}}{2}\right)^2}\,dt = \frac{1}{4}\times\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\left[\tan^{-1}\left(\frac{t}{\frac{\sqrt{5}}{2}}\right)\right]_0^\infty$$

$$\Rightarrow \quad I = \frac{1}{2\sqrt{5}}\left[\tan^{-1}\left(\frac{2t}{\sqrt{5}}\right)\right]_0^\infty = \frac{1}{2\sqrt{5}}\left(\tan^{-1}\infty - \tan^{-1}0\right) = \frac{1}{2\sqrt{5}}\times\frac{\pi}{2} = \frac{\pi}{4\sqrt{5}}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Evaluate*:

(i) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{3\cos x + \sin x}\,dx$ (ii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos x+\sin x}\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\,dx$ (iv) $\displaystyle\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$ [CBSE 2014, 2018]

(v) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$ [CBSE 2010, 2020] (vi) $\displaystyle\int_0^{\pi/2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$ [CBSE 2002, 2003]

SOLUTION (i) Let $\cos x = K(3\cos x + \sin x) + L\,\frac{d}{dx}(3\cos x + \sin x)$. Then,

$$\cos x = K(3\cos x+\sin x) + L(-3\sin x + \cos x) \qquad \text{...(i)}$$

Comparing coefficients of $\cos x$ and $\sin x$, we get: $3K + L = 1$ and $K - 3L = 0$.

Solving these two equations, we get: $K = \frac{3}{10}$ and $L = \frac{1}{10}$.

Substituting the values of K and L in (i), we get

$$\cos x = \frac{3}{10}(3\cos x+\sin x) + \frac{1}{10}(-3\sin x+\cos x)$$

$$\therefore \quad I = \int_0^{\pi/2}\frac{\cos x}{(3\cos x+\sin x)}\,dx = \int_0^{\pi/2}\frac{\frac{3}{10}(3\cos x+\sin x)+\frac{1}{10}(-3\sin x+\cos x)}{3\cos x+\sin x}\,dx$$

$$\Rightarrow \quad I = \frac{3}{10}\int_0^{\pi/2}\frac{3\cos x+\sin x}{3\cos x+\sin x}\,dx + \frac{1}{10}\int_0^{\pi/2}\frac{-3\sin x+\cos x}{3\cos x+\sin x}\,dx$$

$$\Rightarrow \quad I = \frac{3}{10}\int_0^{\pi/2} 1\cdot dx + \frac{1}{10}\int_0^{\pi/2}\frac{-3\sin x+\cos x}{3\cos x+\sin x}\,dx$$

$$\Rightarrow \quad I = \frac{3}{10}\Big[x\Big]_0^{\pi/2} + \frac{1}{10}\Big[\log|3\cos x+\sin x|\Big]_0^{\pi/2}$$

$$\Rightarrow \quad I = \frac{3}{10}\left(\frac{\pi}{2}-0\right) + \frac{1}{10}(\log 1 - \log 3) = \frac{3\pi}{20} - \frac{1}{10}\log 3$$

(ii) We have,

$$I = \int_0^{\pi/2}\frac{\cos x}{1+\cos x+\sin x}\,dx = \int_0^{\pi/2}\frac{\cos x}{(1+\cos x)+\sin x}\,dx = \int_0^{\pi/2}\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{1-\tan^2 \frac{x}{2}}{2+2\tan\frac{x}{2}}\, dx \qquad \left[\text{Dividing numerator and denominator by } \cos^2\frac{x}{2}\right]$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^{\pi/2} \frac{\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)}{1+\tan\frac{x}{2}}\, dx = \frac{1}{2}\int_0^{\pi/2}\left(1-\tan\frac{x}{2}\right) dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[x+2\log\cos\frac{x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+2\log\cos\frac{\pi}{4}\right)-(0+2\log 1)\right]$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{2}+2\log\frac{1}{\sqrt{2}}\right] = \frac{1}{2}\left[\frac{\pi}{2}+\log\frac{1}{2}\right] = \frac{1}{2}\left(\frac{\pi}{2}-\log 2\right)$$

(iii) We have,

$$I = \int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\, dx = \int_0^{\pi/2} \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^3}\, dx = \int_0^{\pi/2} \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}\, dx$$

Let $\cos\frac{x}{2}+\sin\frac{x}{2} = t$. Then,

$$\Rightarrow \quad d\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right) = dt \Rightarrow \frac{1}{2}\left(-\sin\frac{x}{2}+\cos\frac{x}{2}\right)dx = dt \Rightarrow \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)dx = -2dt$$

Also, $x = 0 \Rightarrow t = \cos 0 + \sin 0 = 1$ and $x = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{4}+\sin\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$$\therefore \quad I = \int_1^{\sqrt{2}} \frac{2dt}{t^2} = 2\int_1^{\sqrt{2}} \frac{1}{t^2}\, dt = 2\left[-\frac{1}{t}\right]_1^{\sqrt{2}} = 2\left(-\frac{1}{\sqrt{2}}+1\right) = 2-\sqrt{2}$$

(iv) Let $I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}\, dx$. Here, we express the denominator in terms $\sin x - \cos x$ which is integration of numerator.

Clearly, $(\sin x-\cos x)^2 = \sin^2 x+\cos^2 x - 2\sin x\cos x = 1-\sin 2x$

$$\Rightarrow \quad \sin 2x = 1-(\sin x-\cos x)^2$$

$$\therefore \quad I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\{1-(\sin x-\cos x)^2\}}\, dx = \int_0^{\pi/4} \frac{\sin x+\cos x}{25-16(\sin x-\cos x)^2}\, dx$$

Let $\sin x - \cos x = t$. Then, $d(\sin x-\cos x) = dt \Rightarrow (\cos x+\sin x)\,dx = dt$.

Also, $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$ and $x = \frac{\pi}{4} \Rightarrow t = \sin\frac{\pi}{4}-\cos\frac{\pi}{4} = 0$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25-16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\frac{25}{16}-t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{4}\right)^2-t^2} = \frac{1}{16}\times\frac{1}{2(5/4)}\left[\log\left|\frac{5/4+t}{5/4-t}\right|\right]_{-1}^{0}$$

$$\Rightarrow \quad I = \frac{1}{40}\left[\log 1 - \log\left(\frac{1/4}{9/4}\right)\right] = \frac{1}{40}\left[\log 1 - \log\left(\frac{1}{9}\right)\right] = \frac{1}{40}\,[\log 1 + \log 9] = \frac{1}{40}\log 9$$

(v) We have,

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx = \int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{1-(\cos x - \sin x)^2}}\,dx$$

Let $t = (\cos x - \sin x)$. Then, $dt = d(\cos x - \sin x) \Rightarrow -(\sin x + \cos x)\,dx = dt$

Also, $x = \frac{\pi}{6} \Rightarrow t = \cos\frac{\pi}{6} - \sin\frac{\pi}{6} = \frac{\sqrt{3}-1}{2}$ and, $x = \frac{\pi}{3} \Rightarrow t = \cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{1-\sqrt{3}}{2}$

$$\therefore \quad I = -\int_{\frac{\sqrt{3}-1}{2}}^{\frac{1-\sqrt{3}}{2}} \frac{1}{\sqrt{1-t^2}}\,dt = -\left[\sin^{-1} t\right]_{\frac{\sqrt{3}-1}{2}}^{\frac{1-\sqrt{3}}{2}} = -\left[\sin^{-1}\frac{1-\sqrt{3}}{2} - \sin^{-1}\frac{\sqrt{3}-1}{2}\right]$$

$$\Rightarrow \quad I = -\sin^{-1}\frac{1-\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}-1}{2} = 2\sin^{-1}\frac{\sqrt{3}-1}{2} \qquad \left[\because \sin^{-1}(-x) = -\sin^{-1}x\right]$$

(vi) Let $I = \int_0^{\pi/2} \left\{\sqrt{\tan x} + \sqrt{\cot x}\right\} dx$. Then,

$$\Rightarrow \quad I = \int_0^{\pi/2} \left\{\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right\} dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}\,dx = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{2\sin x \cos x}}\,dx$$

$$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$$

Let $\sin x - \cos x = t$. Then, $d(\sin x - \cos x) = dt \Rightarrow (\cos x + \sin x)\,dx = dt$

Also, $x = 0 \Rightarrow t = -1$, and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \quad I = \sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\left[\sin^{-1} t\right]_{-1}^{1}$$

$$\Rightarrow \quad I = \sqrt{2}\left\{\sin^{-1} 1 - \sin^{-1}(-1)\right\} = \sqrt{2}\left\{2\sin^{-1}(1)\right\} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \sqrt{2}\pi$$

EXAMPLE 8 *Evaluate:*

(i) $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx$ **[CBSE 2003C]** (ii) $\int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\,dx$ **[CBSE 2013]**

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\,dx = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + (1-\sin^2 x)^2}\,dx$

Let $\sin^2 x = t$. Then, $d(\sin^2 x) = dt \Rightarrow 2\sin x \cos x\,dx = dt \Rightarrow \sin 2x\,dx = dt$

Also, $x = 0 \Rightarrow t = \sin^2 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \sin^2\frac{\pi}{2} = 1$

$$\therefore \quad I = \int_0^1 \frac{1}{t^2 + (1-t)^2}\, dt = \int_0^1 \frac{1}{2t^2 - 2t + 1}\, dt = \frac{1}{2}\int_0^1 \frac{1}{t^2 - t + \frac{1}{2}}\, dt$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}} = \frac{1}{2}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2} \times \frac{1}{\left(\frac{1}{2}\right)}\left[\tan^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow \quad I = \left[\tan^{-1}(2t-1)\right]_0^1 = \tan^{-1} 1 - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

ALITER Let $I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} = \int_0^{\pi/2} \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$. Dividing the numerator and denominator by $\cos^4 x$, we obtain

$$I = \int_0^{\pi/2} \frac{2 \tan x \sec^2 x}{\tan^4 x + 1}\, dx.$$

Let $t = \tan^2 x$. Then, $dt = d(\tan^2 x) = 2 \tan x \sec^2 x\, dx$.

Also, $x = 0 \Rightarrow t = \tan^2 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan^2 \frac{\pi}{2} = \infty$.

$$\therefore \quad I = \int_0^{\infty} \frac{1}{t^2 + 1^2}\, dt = \left[\tan^{-1} t\right]_0^{\infty} = \left(\tan^{-1} \infty - \tan^{-1} 0\right) = \frac{\pi}{2}$$

(ii) Let $I = \int_0^{\pi/4} \frac{\sin 2x}{\cos^4 x + \sin^4 x}\, dx$. Then, $I = \int_0^{\pi/4} \frac{2 \sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$. Dividing numerator and denominator by $\cos^4 x$, we get

$$I = \int_0^{\pi/4} \frac{2 \tan x \sec^2 x}{1 + \tan^4 x}\, dx$$

Let $\tan^2 x = t$. Then, $d(\tan^2 x) = dt \Rightarrow 2 \tan x \sec^2 x\, dx = dt$

Also, $x = 0 \Rightarrow t = \tan^2 0 = 0$ and $x = \frac{\pi}{4} \Rightarrow t = \tan^2 \frac{\pi}{4} = 1$

Substituting $t = \tan^2 x$ and $2 \tan x \sec^2 x\, dx = dt$, we get

$$I = \int_0^1 \frac{1}{1 + t^2}\, dt = \left[\tan^{-1} t\right]_0^1 = \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}$$

EXAMPLE 9 *Evaluate:* $\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x}\, dx$ **[CBSE 2012]**

SOLUTION Let $I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4 \sin^2 x}\, dx$. Then,

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x + 4(1 - \cos^2 x)}\, dx = \int_0^{\pi/2} \frac{\cos^2 x}{4 - 3\cos^2 x}\, dx = -\frac{1}{3}\int_0^{\pi/2} \frac{-3\cos^2 x}{4 - 3\cos^2 x}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} \frac{(4-3\cos^2 x)-4}{4-3\cos^2 x}\,dx = -\frac{1}{3}\int_0^{\pi/2}\left\{1-\frac{4}{4-3\cos^2 x}\right\}dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} 1\cdot dx + \frac{4}{3}\int_0^{\pi/2}\frac{1}{4-3\cos^2 x}\,dx$$

$$\Rightarrow \quad I = -\frac{1}{3}\int_0^{\pi/2} 1\cdot dx + \frac{4}{3}\int_0^{\pi/2}\frac{\sec^2 x}{4(1+\tan^2 x)-3}\,dx \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } \cos^2 x]$$

$$\Rightarrow \quad I = -\frac{1}{3}\Big[x\Big]_0^{\pi/2} + \frac{4}{3}\int_0^{\pi/2}\frac{\sec^2 x}{1+4\tan^2 x}\,dx = -\frac{1}{3}\left(\frac{\pi}{2}-0\right)+\frac{4}{3}\int_0^{\infty}\frac{1}{1+4t^2}\,dt, \text{ where } t=\tan x$$

$$\Rightarrow \quad I = -\frac{\pi}{6}+\frac{4}{3}\times\frac{1}{2}\Big[\tan^{-1}2t\Big]_0^{\infty} = -\frac{\pi}{6}+\frac{2}{3}\times\left(\frac{\pi}{2}-0\right)=\frac{\pi}{6}$$

EXAMPLE 10 *Evaluate:*

(i) $\displaystyle\int_0^a \frac{1}{(x^2+a^2)^2}\,dx$ (ii) $\displaystyle\int_0^{\infty}\frac{x^2}{(a^2+x^2)^{5/2}}\,dx$

SOLUTION (i) Let $I=\displaystyle\int_0^a \frac{1}{(x^2+a^2)^2}\,dx$. Let $x=a\tan\theta$. Then, $dx = d(a\tan\theta)=a\sec^2\theta\,d\theta$.

Also, $x=0\Rightarrow a\tan\theta=0\Rightarrow \tan\theta=0\Rightarrow\theta=0$ and, $x=a\Rightarrow a\tan\theta=a\Rightarrow\tan\theta=1\Rightarrow\theta=\dfrac{\pi}{4}$

$$\therefore \quad I=\int_0^{\pi/4}\frac{1}{(a^2\tan^2\theta+a^2)^2}\,a\sec^2\theta\,d\theta=\frac{1}{a^3}\int_0^{\pi/4}\cos^2\theta\,d\theta$$

$$\Rightarrow \quad I=\frac{1}{2a^3}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta=\frac{1}{2a^3}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/4}=\frac{1}{2a^3}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{1}{8a^3}(\pi+2)$$

(ii) Let $I=\displaystyle\int_0^{\infty}\frac{x^2}{(a^2+x^2)^{5/2}}\,dx$. Let $x=a\tan\theta$. Then, $dx=a\sec^2\theta\,d\theta$.

Also, $x=0\Rightarrow a\tan\theta=0\Rightarrow\tan\theta=0\Rightarrow\theta=0$ and, $x=\infty\Rightarrow a\tan\theta=\infty\Rightarrow\tan\theta=\infty\Rightarrow\theta=\dfrac{\pi}{2}$

$$\therefore \quad I=\int_0^{\pi/2}\frac{a^2\tan^2\theta}{(a^2+a^2\tan^2\theta)^{5/2}}\,a\sec^2\theta\,d\theta=\frac{1}{a^3}\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta=\frac{1}{a^2}\int_0^1 t^2\,dt, \text{ where } t=\sin\theta$$

$$\Rightarrow \quad I=\frac{1}{a^2}\left[\frac{t^3}{3}\right]_0^1=\frac{1}{3a^2}$$

EXAMPLE 11 *Evaluate:* $\displaystyle\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$

SOLUTION Let $I=\displaystyle\int_0^1 x\sqrt{\frac{1-x^2}{1+x^2}}\,dx$. Let $x^2=t$. Then, $d(x^2)=dt\Rightarrow 2x\,dx=dt$

Also, $x=0\Rightarrow t=0$ and $x=1\Rightarrow t=1$

$$\therefore\ I = \int_0^1 x\sqrt{\frac{1-t}{1+t}} \times \frac{dt}{2x} = \frac{1}{2}\int_0^1 \sqrt{\frac{1-t}{1+t}}\, dt = \frac{1}{2}\int_0^1 \sqrt{\frac{1-t}{1+t} \times \frac{1-t}{1-t}}\, dt = \frac{1}{2}\int_0^1 \frac{1-t}{\sqrt{1-t^2}}\, dt$$

$$\Rightarrow I = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt + \frac{1}{4}\int_0^1 \frac{-2t}{\sqrt{1-t^2}}\, dt = \frac{1}{2}\left[\sin^{-1} t\right]_0^1 + \frac{1}{4}\left[2\sqrt{1-t^2}\right]_0^1 \left[\because \int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}\right]$$

$$\Rightarrow I = \frac{1}{2}\left(\sin^{-1} 1 - \sin^{-1} 0\right) + \frac{1}{4}\left(2\times 0 - 2\times 1\right) = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) + \frac{1}{4}\left(0-2\right) = \frac{\pi}{4} - \frac{1}{2}$$

EXAMPLE 12 *Evaluate* $\displaystyle\int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2 \sin^2 x)^2}\, dx$ **[NCERT EXEMPLAR]**

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx$. Dividing numerator and denominator by $\cos^4 x$, we obtain

$$I = \int_0^{\pi/2} \frac{\sec^4 x}{(a^2 + b^2 \tan^2 x)^2}\, dx = \int_0^{\pi/2} \frac{1+\tan^2 x}{(a^2+b^2\tan^2 x)^2} \sec^2 x\, dx$$

Let $t = \tan x$. Then, $dt = \sec^2 x\, dx$. Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$.

$$\therefore\quad I = \int_0^\infty \frac{1+t^2}{(a^2+b^2t^2)^2}\, dt = \frac{1}{b^2}\int_0^\infty \frac{b^2+b^2t^2}{(a^2+b^2t^2)^2}\, dt = \frac{1}{b^2}\int_0^\infty \frac{(a^2+b^2t^2)+(b^2-a^2)}{(a^2+b^2t^2)^2}\, dt$$

$$\Rightarrow\quad I = \frac{1}{b^2}\int_0^\infty \frac{1}{a^2+b^2t^2}\, dt + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{1}{(a^2+b^2t^2)^2}\, dt$$

$$\Rightarrow\quad I = \frac{1}{b^2}\left[\frac{1}{ab}\tan^{-1}\frac{bt}{a}\right]_0^\infty + \frac{b^2-a^2}{b^2}\int_0^\infty \frac{1}{(a^2+b^2t^2)^2}\, dt$$

$$\Rightarrow\quad I = \frac{1}{b^2}\left\{\frac{1}{ab}\left(\frac{\pi}{2}-0\right)\right\} + \frac{b^2-a^2}{b^2} I_1, \text{ where } I_1 = \int_0^\infty \frac{1}{(a^2+b^2t^2)^2}\, dt$$

$$\Rightarrow\quad I = \frac{\pi}{2ab^3} + \frac{b^2-a^2}{b^2} I_1 \qquad \ldots\text{(i)}$$

Let $bt = a\tan\theta$. Then, $b\, dt = a\sec^2\theta\, d\theta$.

Also, $t = 0 \Rightarrow a\tan\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0$ and, $t = \infty \Rightarrow a\tan\theta = \infty \Rightarrow \tan\theta = \infty \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore\quad I_1 = \int_0^{\pi/2} \frac{1}{(a^2+a^2\tan^2\theta)^2}\frac{a}{b}\sec^2\theta\, d\theta$$

$$\Rightarrow\quad I_1 = \frac{1}{a^3b}\int_0^{\pi/2} \cos^2\theta\, d\theta = \frac{1}{2a^3b}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \frac{1}{2a^3b}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{4a^3b}$$

Substituting the value of I_1 in (i), we get

$$I = \frac{\pi}{2ab^3} + \frac{b^2 - a^2}{b^2} \times \frac{\pi}{4a^3 b} = \frac{\pi}{4a^3 b^3}(2a^2 + b^2 - a^2) = \frac{\pi}{4a^3 b^3}(a^2 + b^2)$$

ALITER Let $I = \int_0^{\pi/2} \frac{1}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} dx$. Then,

$$I = \int_0^{\pi/2} \frac{\sec^4 x}{(a^2 + b^2 \tan^2 x)^2} dx = \int_0^{\pi/2} \frac{(1 + \tan^2 x)}{(a^2 + b^2 \tan^2 x)^2} \sec^2 x \, dx$$

Let $b \tan x = a \tan t$. Then, $d(b \tan x) = d(a \tan t) \Rightarrow b \sec^2 x \, dx = a \sec^2 t \, dt \Rightarrow dx = \frac{a \sec^2 t}{b \sec^2 x} dt$

Also, $x = 0 \Rightarrow a \tan t = b \tan 0 \Rightarrow \tan t = 0 \Rightarrow t = 0$ and, $x = \frac{\pi}{2} \Rightarrow a \tan t = b \tan \frac{\pi}{2} \Rightarrow \tan t = \infty \Rightarrow t = \frac{\pi}{2}$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(1 + \frac{a^2}{b^2} \tan^2 t\right)}{(a^2 + a^2 \tan^2 t)^2} \times \frac{a \sec^2 t}{b} dt = \int_0^{\pi/2} \frac{\left(1 + \frac{a^2}{b^2} \tan^2 t\right)}{a^4 \sec^4 t} \times \frac{a}{b} \sec^2 t \, dt$$

$$\Rightarrow \quad I = \frac{1}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 t + a^2 \sin^2 t) \, dt = \frac{1}{2a^3 b^3} \int_0^{\pi/2} (2b^2 \cos^2 t + 2a^2 \sin^2 t) \, dt$$

$$\Rightarrow \quad I = \frac{1}{2a^3 b^3} \int_0^{\pi/2} \left\{ b^2 (1 + \cos 2t) + a^2 (1 - \cos 2t) \right\} dt = \frac{1}{2a^3 b^3} \int_0^{\pi/2} \left\{ a^2 + b^2 + (b^2 - a^2) \cos 2t \right\} dt$$

$$\Rightarrow \quad I = \frac{1}{2a^3 b^3} \left[(a^2 + b^2) t + \left(\frac{b^2 - a^2}{2} \right) \sin 2t \right]_0^{\pi/2}$$

$$\Rightarrow \quad I = \frac{1}{2a^3 b^3} (a^2 + b^2) \left(\frac{\pi}{2} - 0 \right) + \left(\frac{b^2 - a^2}{2} \right) (\sin \pi - \sin 0) = \frac{(a^2 + b^2) \pi}{4a^3 b^3}.$$

EXAMPLE 13 *Evaluate :* $\int_0^1 x (\tan^{-1} x)^2 \, dx$ **[NCERT EXEMPLAR]**

SOLUTION Let $I = \int_0^1 x (\tan^{-1} x)^2 \, dx$. Then,

$$I = \int_0^1 \underset{\text{I}}{(\tan^{-1} x)^2} \underset{\text{II}}{x} \, dx = \left[(\tan^{-1} x)^2 \frac{x^2}{2} \right]_0^1 - \int_0^1 2 \frac{\tan^{-1} x}{1 + x^2} \times \frac{x^2}{2} dx$$

$$\Rightarrow \quad I = \left\{ \frac{1}{2} (\tan^{-1} 1)^2 - 0 \right\} - \int_0^1 \frac{x^2}{1 + x^2} \tan^{-1} x \, dx = \frac{1}{2} \left(\frac{\pi}{4} \right)^2 - \int_0^1 \frac{(1 + x^2) - 1}{1 + x^2} \tan^{-1} x \, dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{32} - \int_0^1 \left\{ \tan^{-1} x - \frac{1}{1 + x^2} \tan^{-1} x \right\} dx = \frac{\pi^2}{32} - \int_0^1 \underset{\text{I}}{\tan^{-1} x} \, \underset{\text{II}}{1} \, dx + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{32} - \left\{ \left[x \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx \right\} + \int_0^1 \frac{\tan^{-1} x}{1 + x^2} dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{32} - \left\{ \left(\frac{\pi}{4} - 0 \right) - \frac{1}{2} \int_0^1 \frac{2x}{1+x^2}\, dx \right\} + \int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$$

$$\Rightarrow \quad I = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2} \left[\log(1+x^2) \right]_0^1 + \left[\frac{1}{2} (\tan^{-1} x)^2 \right]_0^1 = \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2} (\log 2 - 0) + \left(\frac{\pi^2}{32} - 0 \right)$$

$$\Rightarrow \quad I = \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2} \log 2$$

EXAMPLE 14 *If* $I_n = \int_0^{\pi/4} \tan^n x\, dx$, *prove that* $I_n + I_{n+2} = \frac{1}{n+1}$.

SOLUTION We have,

$$I_n = \int_0^{\pi/4} \tan^n x \Rightarrow I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x$$

$$\therefore \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\, dx + \int_0^{\pi/4} \tan^{n+2} x\, dx = \int_0^{\pi/4} \left(\tan^n x + \tan^{n+2} x \right) dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x\, (1 + \tan^2 x)\, dx = \int_0^{\pi/4} \tan^n x \sec^2 x\, dx$$

Let $t = \tan x$. Then, $dt = \sec^2 x\, dx$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \frac{\pi}{4} \Rightarrow t = \tan \frac{\pi}{4} = 1$

$$\therefore \quad I_n + I_{n+2} = \int_0^1 t^n\, dt = \left[\frac{t^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

EXAMPLE 15 *If* $I_n = \int_0^{\pi/4} \tan^n x\, dx$, *show that* $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \frac{1}{I_5 + I_7}, \ldots$ *from an A.P. Find the common difference of this progression.*

SOLUTION We have,

$$I_n = \int_0^{\pi/4} \tan^n x\, dx \Rightarrow I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x\, dx$$

$$\therefore \quad I_n + I_{n+2} = \int_0^{\pi} \tan^n x\, dx + \int_0^{\pi/4} \tan^{n+2} x\, dx = \int_0^{\pi/4} \tan^n x\, (1 + \tan^2 x)\, dx$$

$$\Rightarrow \quad I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x \sec^2 x\, dx = \int_0^1 t^n\, dt, \text{ where } t = \tan x.$$

$$\Rightarrow \quad I_n + I_{n+2} = \left[\frac{t^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}, \quad n = 2, 3, 4, 5, \ldots$$

$$\Rightarrow \quad \frac{1}{I_n + I_{n+2}} = n+1, \quad n = 2, 3, 4, 5, \ldots$$

$$\Rightarrow \quad \frac{1}{I_2+I_4}=3, \frac{1}{I_3+I_5}=4, \frac{1}{I_4+I_6}=5, \frac{1}{I_5+I_7}=6, \ldots$$

Clearly, 3, 4, 5, 6, ... is an AP with common difference 1.

Hence, $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$, ... is an AP with common difference 1.

EXERCISE 19.2

BASIC

Evaluate the following integrals:

1. $\int_2^4 \frac{x}{x^2+1}\, dx$ [NCERT]
2. $\int_1^2 \frac{1}{x(1+\log x)^2}\, dx$
3. $\int_1^2 \frac{3x}{9x^2-1}\, dx$
4. $\int_0^{\pi/2} \frac{1}{5\cos x+3\sin x}\, dx$
5. $\int_0^a \frac{x}{\sqrt{a^2+x^2}}\, dx$
6. $\int_0^{\frac{1}{2}\ln 3} \frac{e^x}{1+e^{2x}}\, dx$ [CBSE 2022]
7. $\int_0^1 x e^{x^2}\, dx$
8. $\int_1^3 \frac{\cos(\log x)}{x}\, dx$
9. $\int_0^1 \frac{2x}{1+x^4}\, dx$
10. $\int_0^5 x\sqrt{5-x}\, dx$ [CBSE 2022]
11. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\, dx$
12. $\int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1+\cos\theta}}\, d\theta$
13. $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x}\, dx$
14. $\int_0^1 \frac{\sqrt{\tan^{-1} x}}{1+x^2}\, dx$
15. $\int_0^2 x\sqrt{x+2}\, dx$ [NCERT]
16. $\int_0^1 \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$
17. $\int_0^{\pi/2} \frac{\sin x\cos x}{1+\sin^4 x}\, dx$
18. $\int_0^{\pi/2} \frac{dx}{a\cos x+b\sin x}\, a, b>0$
19. $\int_0^{\pi/2} \frac{1}{5+4\sin x}\, dx$
20. $\int_0^{\pi} \frac{\sin x}{\sin x+\cos x}\, dx$
21. $\int_0^{\pi} \frac{1}{3+2\sin x+\cos x}\, dx$
22. $\int_0^1 \tan^{-1} x\, dx$ [CBSE2022]
23. $\int_0^{1/2} \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\, dx$
24. $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x}\, dx$
25. $\int_0^{\pi} \frac{1}{5+3\cos x}\, dx$
26. $\int_0^{\pi/2} \frac{1}{a^2\sin^2 x+b^2\cos^2 x}\, dx$

BASED ON LOTS

27. $\int_0^1 \frac{\tan^{-1} x}{1+x^2}\, dx$ [NCERT]
28. $\int_0^1 x\tan^{-1} x\, dx$
29. $\int_0^1 \frac{24x^3}{(1+x^2)^4}\, dx$
30. $\int_4^{12} x(x-4)^{1/3}\, dx$
31. $\int_0^{\pi/2} x^2\sin x\, dx$
32. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$ [CBSE 2004]
33. $\int_{-1}^1 5x^4\sqrt{x^5+1}\, dx$ [NCERT]
34. $\int_0^{\pi/2} \frac{\cos^2 x}{1+3\sin^2 x}\, dx$ [CBSE 2015]
35. $\int_0^{\pi/4} \sin^3 2t\cos 2t\, dt$ [NCERT]
36. $\int_0^{\pi} 5(5-4\cos\theta)^{1/4}\sin\theta\, d\theta$

37. $\int_0^{\pi/6} \cos^{-3} 2\theta \sin 2\theta \, d\theta$ 38. $\int_0^{(\pi)^{2/3}} \sqrt{x} \cos^2 x^{3/2} \, dx$ 39. $\int_1^2 \frac{1}{x(1+\log x)^2} dx$ [CBSE 2003]

40. $\int_4^9 \frac{\sqrt{x}}{(30-x^{3/2})^2} dx$ [NCERT] 41. $\int_0^{\pi} \sin^3 x (1+2\cos x)(1+\cos x)^2 \, dx$ [NCERT]

42. $\int_0^{\pi/2} 2\sin x \cos x \tan^{-1}(\sin x) \, dx$ [CBSE 2011] 43. $\int_0^{\pi/2} \sqrt{\sin\phi} \cos^5\phi \, d\phi$ [NCERT]

44. $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x}) \, dx$ [CBSE 2012] 45. $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x} dx$ [CBSE2011]

46. $\int_0^{\pi/4} \frac{\sin x + \cos x}{3+\sin 2x} dx$ [CBSE 2015] 47. $\int_0^1 \frac{1-x^2}{x^4+x^2+1} dx$

48. $\int_0^1 \frac{1-x^2}{(1+x^2)^2} dx$ 49. $\int_0^{\pi/2} \cos^5 x \, dx$

BASED ON HOTS

50. $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x) \, dx$ 51. $\int_0^1 (\cos^{-1} x)^2 \, dx$ 52. $\int_0^a \sin^{-1} \sqrt{\frac{x}{a+x}} \, dx$

53. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}} dx$ 54. $\int_0^a x \sqrt{\frac{a^2-x^2}{a^2+x^2}} \, dx$ 55. $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx$

56. $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2} dx$ 57. $\int_0^{\pi/2} \frac{\tan x}{1+m^2 \tan^2 x} dx$ 58. $\int_0^{1/2} \frac{1}{(1+x^2)\sqrt{1-x^2}} dx$

59. $\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4} dx$ 60. $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$

61. $\int_0^{\pi/2} \sqrt{\cos x - \cos^3 x} \, (\sec^2 x - 1) \cos^2 x \, dx$ 62. $\int_0^{\pi/2} \frac{\cos x}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^n} dx$

ANSWERS

1. $\frac{1}{2}\log\left(\frac{17}{5}\right)$ 2. $\frac{\log 2}{\log 2e}$ 3. $\frac{1}{6}(\log 35 - \log 8)$

4. $\frac{1}{\sqrt{34}}\log\left|\frac{8+\sqrt{34}}{8-\sqrt{34}}\right|$ 5. $a(\sqrt{2}-1)$ 6. $\frac{\pi}{3}$ 7. $\frac{e-1}{2}$

8. $\sin(\log 3)$ 9. $\frac{\pi}{4}$ 10. $\frac{20\sqrt{5}}{3}$ 11. $\frac{\pi}{4}$ 12. $2(\sqrt{2}-1)$

13. $\frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$ 14. $\frac{1}{12}\pi^{3/2}$ 15. $\frac{16}{15}(2+\sqrt{2})$ 16. $\frac{\pi}{2} - \log 2$

17. $\frac{\pi}{8}$ 18. $\frac{1}{\sqrt{a^2+b^2}} \log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)$ 19. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$

20. $\frac{\pi}{2}$ 21. $\frac{\pi}{4}$ 22. $\frac{\pi}{4}-\frac{1}{2}\log 2$ 23. $\frac{1}{2}-\frac{\pi}{4\sqrt{3}}$ 24. $\frac{1}{8}$

25. $\frac{\pi}{4}$ 26. $\frac{\pi}{2\,ab}$ 27. $\frac{\pi^2}{32}$ 28. $\frac{1}{2}\log 3$ 29. 1

30. $\frac{720}{7}$ 31. $\pi-2$ 32. $\frac{\pi}{2}-1$ 33. $\frac{4\sqrt{2}}{3}$ 34. $\frac{\pi}{6}$

35. $\frac{1}{8}$ 36. $9\sqrt{3}-1$ 37. $\frac{3}{4}$ 38. $\frac{\pi}{3}$ 39. $\frac{\log 2}{1+\log 2}$

40. $\frac{19}{99}$ 41. $\frac{8}{3}$ 42. $\frac{\pi}{2}-1$ 43. $\frac{64}{231}$ 44. $\frac{\pi}{\sqrt{2}}$

45. $\frac{\pi}{2}$ 46. $\frac{1}{4}\log_e 3$ 47. $\frac{\pi}{4}-\frac{1}{2}$ 48. $\frac{1}{2}$ 49. $\frac{8}{15}$

50. $\frac{\pi}{2}-1$ 51. $\pi-2$ 52. $a\left(\frac{\pi}{2}-1\right)$ 53. 1 54. $a^2\left(\frac{\pi}{4}-\frac{1}{2}\right)$

55. πa 56. $\log\left(\frac{9}{8}\right)$ 57. $\frac{\log|m|}{m^2-1}$ 58. $\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{\frac{2}{3}}$ 59. 6

60. $\frac{1}{6}$ 61. $\frac{8}{21}$ 62. $\frac{2}{2-n}\left(2^{1-\frac{n}{2}}-1\right)$

19.4 PROPERTIES OF DEFINITE INTEGRALS

In this section, we will study some fundamental properties of definite integrals which are very useful in evaluating integrals.

19.4.1 PROPERTY I

STATEMENT $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ *i.e., integration is independent of the change of variable.*

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then, $\frac{d}{dx}\{\phi(x)\} = f(x) \Rightarrow \frac{d}{dt}\{\phi(t)\} = f(t)$.

Hence, $\int_a^b f(x)\,dx = \left[\phi(x)\right]_a^b = \phi(b)-\phi(a)$ ...(i) and, $\int_a^b f(t)\,dt = \left[\phi(t)\right]_a^b = \phi(b)-\phi(a)$ ...(ii)

From (i) and (ii), we obtain: $\int_a^b f(x)\,dx = \int_b^a f(t)\,dt$ **Q.E.D.**

19.4.2 PROPERTY II

STATEMENT (*Order of integration*): $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

i.e., if the limits of a definite integral are interchanged then its value changes by minus sign only.

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \text{ and, } -\int_b^a f(x)\,dx = -[(\phi(a) - \phi(b)] = \phi(b) - \phi(a)$$

$$\therefore \quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Q.E.D.

19.4.3 PROPERTY III

STATEMENT *(Additivity):* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$.

PROOF Let $\phi(x)$ be a primitive of $f(x)$. Then,

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a) \qquad \text{...(i)}$$

and, $$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - \phi(a) \qquad \text{...(ii)}$$

From (i) and (ii), we obtain: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ Q.E.D.

GENERALIZATION *The above property can be generalized into the following form*

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \ldots + \int_{c_n}^b f(x)\,dx, \text{ where } a < c_1 < c_2 < c_3 \ldots < c_{n-1} < c_n < b$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int_{-1}^{1} f(x)\,dx$, *where* $f(x) = \begin{cases} 1-2x, & x \le 0 \\ 1+2x, & x \ge 0 \end{cases}$ (ii) $\int_1^4 f(x)\,dx$, *where* $f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$

SOLUTION (i) We have, $f(x) = \begin{cases} 1-2x, & x \le 0 \\ 1+2x, & x \ge 0 \end{cases}$

$$\therefore \quad \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx = \int_{-1}^{0} (1-2x)\,dx + \int_0^1 (1+2x)\,dx$$

$$\Rightarrow \quad \int_{-1}^{1} f(x)\,dx = \left[x - x^2\right]_{-1}^{0} + \left[x + x^2\right]_0^1 = [0 - (-1-1)] + [(1+1) - (0)] = 4$$

(ii) Using additivity of integration, we obtain

$$\int_1^4 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_1^2 (2x+8)\,dx + \int_2^4 6x\,dx = \left[x^2 + 8x\right]_1^2 + \left[3x^2\right]_2^4$$

$$= [(4+16) - (1+8)] + [48 - 12] = 47$$

EXAMPLE 2 *Evaluate:*

(i) $\int_0^1 |5x-3|\,dx$ (ii) $\int_0^\pi |\cos x|\,dx$ (iii) $\int_{-5}^5 |x-2|\,dx$ (iv) $\int_{-1}^1 e^{|x|}\,dx$

(v) $\int_0^2 |x^2+2x-3|\,dx$ (vi) $\int_1^4 (|x-1|+|x-2|+|x-3|)\,dx$ **[NCERT]**

(vii) $\int_{-1}^2 |x^3-x|\,dx$ **[NCERT, CBSE 2012, 13, 16]**

SOLUTION (i) Clearly, $|5x-3| = \begin{cases} -(5x-3), & \text{when } 5x-3<0 \text{ i.e., } x<\dfrac{3}{5} \\ 5x-3, & \text{when } 5x-3\ge 0 \text{ i.e., } x\ge\dfrac{3}{5} \end{cases}$

The graph of $y=|5x-3|$ is shown in Fig. 19.1.

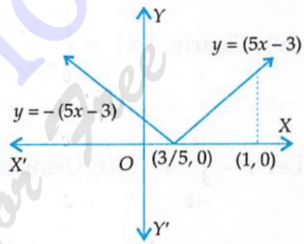

Fig. 19.1 Graph of $y = |5x-3|$

Let $I=\int_0^1 |5x-3|\,dx$. By using additive property, we obtain

$$I = \int_0^{3/5} |5x-3|\,dx + \int_{3/5}^1 |5x-3|\,dx$$

$$\Rightarrow I = \int_0^{3/5} -(5x-3)\,dx + \int_{3/5}^1 (5x-3)\,dx$$

$$\Rightarrow I = \left[3x-\frac{5x^2}{2}\right]_0^{3/5} + \left[\frac{5x^2}{2}-3x\right]_{3/5}^1 = \left(\frac{9}{5}-\frac{9}{10}\right)+\left(-\frac{1}{2}+\frac{9}{10}\right) = \frac{13}{10}$$

(ii) We know that: $|\cos x| = \begin{cases} \cos x & \text{when } 0\le x\le\dfrac{\pi}{2} \\ -\cos x & \text{when } \dfrac{\pi}{2}\le x\le\pi \end{cases}$

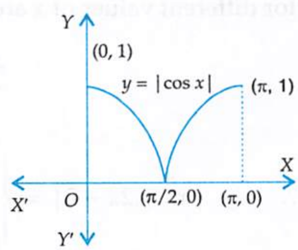

Fig. 19.2 Graph of $y = |\cos x|$

Let $I=\int_0^\pi |\cos x|\,dx$. Using additive property, we obtain

$$I = \int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx = \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx$$

$$\Rightarrow I = \Big[\sin x\Big]_0^{\pi/2} - \Big[\sin x\Big]_{\pi/2}^{0} = 1+1 = 2$$

(iii) Clearly, $|x-2| = \begin{cases} x-2 & \text{when } x-2\ge 0 \text{ i.e., } x\ge 2 \\ -(x-2) & \text{when } x-2<0 \text{ i.e., } x<2 \end{cases}$.

The graph of $y = |x-2|$ is shown in Fig. 19.3.

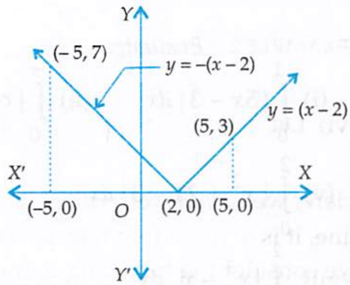

Fig. 19.3 Graph of $y = |x-2|$

Let $I = \int_{-5}^{5} |x-2|\, dx$. Using additive property, we obtain

$$\therefore \quad I = \int_{-5}^{2} |x-2|\, dx + \int_{2}^{5} |x-2|\, dx$$

$$\Rightarrow \quad I = \int_{-5}^{2} -(x-2)\, dx + \int_{2}^{5} (x-2)\, dx$$

$$\Rightarrow \quad I = \left[2x - \frac{x^2}{2}\right]_{-5}^{2} + \left[\frac{x^2}{2} - 2x\right]_{2}^{5} = \left(2 + \frac{45}{2}\right) + \left(\frac{5}{2} + 2\right) = 29$$

(iv) Clearly, $|x| = \begin{cases} x, & \text{when } x \geq 0 \\ -x, & \text{when } x < 0 \end{cases}$. The graph of $y = e^{|x|}$ is shown in Fig. 19.4.

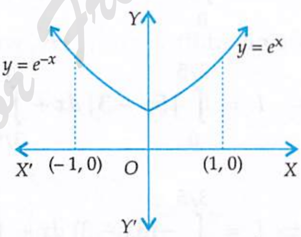

Fig. 19.4 Graph of $e^{|x|}$

Let $I = \int_{-1}^{1} e^{|x|}\, dx$. Using additive property, we obtain

$$I = \int_{-1}^{0} e^{|x|}\, dx + \int_{0}^{1} e^{|x|}\, dx = \int_{-1}^{0} e^{-x}\, dx + \int_{0}^{1} e^{x}\, dx$$

$$\Rightarrow \quad I = \left[-e^{-x}\right]_{-1}^{0} + \left[e^{x}\right]_{0}^{1} = (-1 + e^1) + (e^1 - 1) = 2e - 2$$

(v) Let $I = \int_{0}^{2} |x^2 + 2x - 3|\, dx$ Clearly, $x^2 + 2x - 3 = (x+3)(x-1)$. The signs of $x^2 + 2x - 3$ for different values of x are shown in Fig. 19.5.

Fig. 19.5 Signs of $x^2 + 2x - 3$

$$\therefore \quad |x^2 + 2x - 3| = \begin{cases} -(x^2 + 2x - 3) & , \text{ if } 0 < x < 1 \\ (x^2 + 2x - 3) & , \text{ if } 1 \leq x \leq 2 \end{cases}$$

Using additive property, we obtain

$$I = \int_{0}^{1} |x^2 + 2x - 3|\, dx + \int_{1}^{2} |x^2 + 2x - 3|\, dx = \int_{0}^{1} -(x^2 + 2x - 3)\, dx + \int_{1}^{2} (x^2 + 2x - 3)\, dx$$

$$\Rightarrow \quad I = -\left[\frac{x^3}{3} + x^2 - 3x\right]_{0}^{1} + \left[\frac{x^3}{3} + x^2 - 3x\right]_{1}^{2} = -\left\{\left(\frac{1}{3} + 1 - 3\right) - 0\right\} + \left\{\left(\frac{8}{3} + 4 - 6\right) - \left(\frac{1}{3} + 1 - 3\right)\right\}$$

$$\Rightarrow \quad I = \frac{5}{3} + \frac{2}{3} + \frac{5}{3} = 4$$

(vi) Let $I = \int_1^4 \left(|x-1| + |x-2| + |x-3| \right) dx$ and $f(x) = |x-1| + |x-2| + |x-3|$

Here, we have three critical points, namely $x = 1, 2, 3$. When these points are marked on real line, it is divided into four parts as shown in Fig. 19.6. Therefore, to remove the modulus sign, we consider the following four cases:

Fig. 19.6

(i) $x < 1$ (ii) $1 \le x < 2$ (iii) $2 \le x < 3$ and (iv) $x \ge 3$.

$$\therefore \quad f(x) = \begin{cases} -(x-1)-(x-2)-(x-3) = -3x+6 & , \text{ if } x < 1 \\ (x-1)-(x-2)-(x-3) = -x+4 & , \text{ if } 1 \le x < 2 \\ (x-1)+(x-2)-(x-3) = x & , \text{ if } 2 \le x < 3 \\ (x-1)+(x-2)+(x-3) = 3x-6 & , \text{ if } x \ge 3 \end{cases} \quad \ldots(i)$$

Using additive property, we obtain

$$\Rightarrow \quad I = \int_1^2 f(x)\, dx + \int_2^3 f(x)\, dx + \int_3^4 f(x)\, dx = \int_1^2 (-x+4)\, dx + \int_2^3 x\, dx + \int_3^4 (3x-6)\, dx$$

[Using (i)]

$$\Rightarrow \quad I = \left[-\frac{x^2}{2} + 4x \right]_1^2 + \left[\frac{x^2}{2} \right]_2^3 + \left[\frac{3x^2}{2} - 6x \right]_3^4$$

$$\Rightarrow \quad I = (-2+8) - \left(-\frac{1}{2} + 4 \right) + \left(\frac{9}{2} - \frac{4}{2} \right) + (24-24) - \left(\frac{27}{2} - 18 \right) = 6 - \frac{7}{2} + \frac{5}{2} + \frac{9}{2} = \frac{19}{2}$$

(vii) Let $I = \int_{-1}^2 |x^3 - x|\, dx$ and $f(x) = x^3 - x$. Clearly, $f(x) = x^3 - x = x(x-1)(x+1)$

The signs of $f(x)$ for different values of x are shown in Fig. 19.7.

Fig. 19.7 Signs of $f(x)$ for different values of x

We observe that: $f(x) > 0$ for all $x \in (-1, 0) \cup (1, 2)$ and, $f(x) < 0$ for all $x \in (0, 1)$

$$|f(x)| = \begin{cases} f(x), & x \in (-1, 0) \cup (1, 2) \\ -f(x), & x \in (0, 1) \end{cases} \Rightarrow |x^3 - x| = \begin{cases} x^3 - x, & x \in (-1, 0) \cup (1, 2) \\ -(x^3 - x), & x \in (0, 1) \end{cases}$$

$$\therefore \quad I = \int_{-1}^0 |x^3 - x|\, dx + \int_0^1 |x^3 - x|\, dx + \int_1^2 |x^3 - x|\, dx \quad \text{[Using additive property]}$$

$$\Rightarrow \quad I = \int_{-1}^0 (x^3 - x)\, dx - \int_0^1 (x^3 - x)\, dx + \int_1^2 (x^3 - x)\, dx$$

$$\Rightarrow I = \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{1}^{2} = -\left(\frac{1}{4}-\frac{1}{2}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)+\left(\frac{16}{4}-\frac{4}{4}\right)-\left(\frac{1}{4}-\frac{1}{2}\right)$$

$$\Rightarrow I = \frac{3}{4}+(4-2) = \frac{11}{4}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *If $a>0$, find* $\int_0^{3a} |x^2-a^2|\, dx$.

SOLUTION Let $f(x)=x^2-a^2$. Then, $f(x)=(x-a)(x+a)$. The sign of $f(x)$ for different values of x are shown in Fig. 19.8.

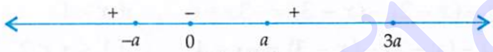

Fig. 19.8

$$\therefore \quad |f(x)| = \begin{cases} f(x) & \text{for } 0 \le x \le 3a \\ -f(x) & \text{for } 0 < x < a \end{cases} \Rightarrow |x^2-a^2| = \begin{cases} x^2-a^2 & \text{for } a \le x \le 3a \\ -(x^2-a^2) & \text{for } 0 < x < a \end{cases}$$

$$\therefore \quad \int_0^{3a} |x^2-a^2|\, dx = \int_0^{a} |x^2-a^2|\, dx + \int_a^{3a} |x^2-a^2|\, dx \qquad \text{[By additive property]}$$

$$= \int_0^{a} -(x^2-a^2)\, dx + \int_a^{3a} (x^2-a^2)\, dx = -\left[\frac{x^3}{3}-a^2x\right]_0^{a} + \left[\frac{x^3}{3}-a^2x\right]_a^{3a}$$

$$= -\left\{\left(\frac{a^3}{3}-a^3\right)-0\right\} + \left\{\left(9a^3-3a^3\right)-\left(\frac{a^3}{3}-a^3\right)\right\} = \frac{2a^3}{3}+6a^3+\frac{2a^3}{3} = \frac{22a^3}{3}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find* $\int e^{-x} \sin 2x\, dx$. *Hence show that* $\int_{-\pi/4}^{\pi/4} e^{-x} |\sin 2x|\, dx = \frac{1}{5}(4+e^{\pi/4}-e^{-\pi/4})$.

SOLUTION Let $I = \int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\sin 2x}\, dx$. Then,

$$I = -\frac{1}{2}e^{-x}\cos 2x - \int (-1)\, e^{-x} \times -\frac{1}{2}\cos 2x\, dx$$

$$\Rightarrow \quad I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int \underset{\text{I}}{e^{-x}} \underset{\text{II}}{\cos 2x}\, dx$$

$$\Rightarrow \quad I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left\{\frac{1}{2}e^{-x}\sin 2x - \int (-1)\, e^{-x} \times \frac{1}{2}\sin 2x\, dx\right\}$$

$$\Rightarrow \quad I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}\int e^{-x}\sin 2x\, dx$$

$$\Rightarrow \quad I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I$$

$$\Rightarrow \quad \frac{5}{4}I = -\frac{1}{4}e^{-x}(2\cos 2x+\sin 2x) \Rightarrow I = -\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x) + C \qquad \ldots\text{(i)}$$

$$\because \qquad |\sin 2x| = \begin{cases} -\sin 2x, \text{ if } -\frac{\pi}{4} \le x < 0 \\ \sin 2x, \text{ if } 0 \le x \le \frac{\pi}{4} \end{cases}$$

Using additive property, we obtain

$$I = \int_{-\pi/4}^{\pi/4} e^{-x} |\sin 2x| \, dx = \int_{-\pi/4}^{0} e^{-x} |\sin 2x| \, dx + \int_{0}^{\pi/4} e^{-x} |\sin 2x| \, dx$$

$$\Rightarrow \qquad I = -\int_{-\pi/4}^{0} e^{-x} \sin 2x \, dx + \int_{0}^{\pi/4} e^{-x} \sin 2x \, dx$$

$$\Rightarrow \qquad I = -\left[-\frac{1}{5} e^{-x} (\sin 2x + 2\cos 2x) \right]_{-\pi/4}^{0} + \left[-\frac{1}{5} e^{-x} (\sin 2x + 2\cos 2x \right]_{0}^{\pi/4}$$

$$\Rightarrow \qquad I = -\left[-\frac{2}{5} + \frac{1}{5} e^{\pi/4} (-1) \right] + \left[-\frac{1}{5} e^{-\pi/4} + \frac{2}{5} \right] = \frac{4}{5} + \frac{1}{5} (e^{\pi/4} - e^{-\pi/4}) = \frac{1}{5} (4 + e^{\pi/4} - e^{-\pi/4})$$

EXAMPLE 5 *Evaluate:* $\int_{1/e}^{e} |\log_e x| \, dx.$

SOLUTION We know that $\log_e x < 0$ for $x \in (0, 1)$ and $\log_e x \ge 0$ for $x \ge 1$.

$$\therefore \qquad |\log_e x| = \begin{cases} -\log_e x \quad , & \text{if } 1/e < x < 1 \\ \log_e x \quad , & \text{if } 1 < x < e \end{cases}$$

Let $I = \int_{1/e}^{e} |\log_e x| \, dx$. Then, by using additive property, we obtain

$$I = \int_{1/e}^{1} |\log_e x| \, dx + \int_{1}^{e} |\log_e x| \, dx = \int_{1/e}^{1} -\log_e x \, dx + \int_{1}^{e} \log_e x \, dx = -\int_{1/e}^{1} \log_e x \, dx + \int_{1}^{e} \log_e x \, dx$$

$$\Rightarrow I = -\Big[x (\log_e x - 1) \Big]_{1/e}^{1} + \Big[x (\log_e x - 1) \Big]_{1}^{e} \qquad \left[\because \int \log_e x \, dx = x (\log_e x - 1) \right]$$

$$\Rightarrow I = -\left[1 (0 - 1) - \frac{1}{e} (-1 - 1) \right] + \Big[e (1 - 1) - 1 (0 - 1) \Big] = -\left[-1 + \frac{2}{e} \right] + (0 + 1) = 2 - \frac{2}{e}$$

EXAMPLE 6 *Evaluate:*

(i) $\int_0^3 [x] \, dx$ (ii) $\int_0^2 [x^2] \, dx$ (iii) $\int_0^{1.5} [x^2] \, dx$

SOLUTION (i) Using additive property, we obtain

$$\int_0^3 [x] \, dx = \int_0^1 [x] \, dx + \int_1^2 [x] \, dx + \int_2^3 [x] \, dx = \int_0^1 0 \, dx + \int_1^2 1 \, . dx + \int_2^3 2 \, . dx$$

$$= 0 + \Big[x \Big]_1^2 + \Big[2x \Big]_2^3 = (2 - 1) + (6 - 4) = 3$$

(ii) Using additive property, we obtain

$$\int_0^2 [x^2] \, dx = \int_0^1 [x^2] \, dx + \int_1^{\sqrt{2}} [x^2] \, dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] \, dx + \int_{\sqrt{3}}^{2} [x^2] \, dx$$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1 \,.\, dx + \int_{\sqrt{2}}^{\sqrt{3}} 2\, dx + \int_{\sqrt{3}}^{2} 3\, dx = 0 + \Big[x\Big]_1^{\sqrt{2}} + 2\Big[x\Big]_{\sqrt{2}}^{\sqrt{3}} + 3\Big[x\Big]_{\sqrt{3}}^{2}$$

$$= 0 + (\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) = 5 - \sqrt{2} - \sqrt{3}$$

(iii) Using additive property, we obtain

$$\int_0^{1.5} [x^2]\, dx = \int_0^1 [x^2]\, dx + \int_1^{\sqrt{2}} [x^2]\, dx + \int_{\sqrt{2}}^{1.5} [x^2]\, dx = \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1 \,.\, dx + \int_{\sqrt{2}}^{1.5} 2\, dx$$

$$= 0 + \Big[x\Big]_1^{\sqrt{2}} + 2\Big[x\Big]_{\sqrt{2}}^{1.5} = 0 + (\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) = 2 - \sqrt{2}$$

EXAMPLE 7 *If* $[\cdot]$ *denotes the greatest integer function, then find the value of* $\int_1^2 [3x]\, dx$.

SOLUTION We observe that $3x \in [3, 6]$ when $x \in [1, 2]$.

$$\therefore \quad [3x] = \begin{cases} 3, & \text{if } 3 \le 3x < 4 \\ 4, & \text{if } 4 \le 3x < 5 \\ 5, & \text{if } 5 \le 3x < 6 \end{cases} \Rightarrow [3x] = \begin{cases} 3, & \text{if } 1 \le x < 4/3 \\ 4, & \text{if } 4/3 \le x < 5/3 \\ 5, & \text{if } 5/3 \le x < 2 \end{cases}$$

By using additive property, we obtain

$$\therefore \quad \int_1^2 [3x]\, dx = \int_1^{4/3} [3x]\, dx + \int_{4/3}^{5/3} [3x]\, dx + \int_{5/3}^{6/3} [3x]\, dx = \int_1^{4/3} 3\, dx + \int_{4/3}^{5/3} 4\, dx + \int_{5/3}^{6/3} 5\, dx$$

$$= 3\left(\frac{4}{3} - 1\right) + 4\left(\frac{5}{3} - \frac{4}{3}\right) + 5\left(\frac{6}{3} - \frac{5}{3}\right) = 1 + \frac{4}{3} + \frac{5}{3} = 4.$$

EXAMPLE 8 *Let* $f(x) = x - [x]$, *for every real number* x, *where* $[x]$ *is the greatest integer less than or equal to* x. *Then, evaluate* $\int_{-1}^{1} f(x)\, dx$.

SOLUTION We find that $x - [x] = \begin{cases} x - (-1) = x + 1, & \text{if } -1 \le x < 0 \\ x - 0 = x, & \text{if } 0 \le x < 1 \end{cases}$

By using the additive property, we obtain

$$\therefore \quad \int_{-1}^{1} \Big(x - [x]\Big) dx = \int_{-1}^{0} \Big(x - [x]\Big) dx + \int_0^1 \Big(x - [x]\Big) dx$$

$$= \int_{-1}^{0} (x + 1)\, dx + \int_0^1 (x - 0)\, dx$$

$$= \left[\frac{(x+1)^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

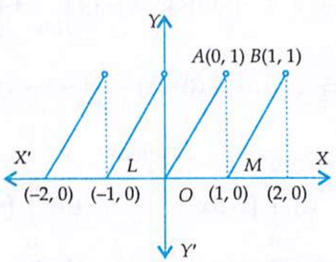

Fig. 19.9 Graph of $f(x) = x - [x]$

ALITER It is evident from the graph of curve $y = f(x)$ that

$$\int_{-1}^{1} f(x)\, dx = \text{Area of } \Delta OLA + \text{Area of } \Delta OMB = 2\,(\text{Area of } \Delta\, OMB) = 2 \times \frac{1}{2}(OM \times MB) = 1 \times 1 = 1.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Evaluate:*

(i) $\displaystyle\int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$ (ii) $\displaystyle\int_0^{1} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$

SOLUTION (i) We know that: $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1} x & , \text{ if } -1 < x < 1 \\ -\pi + 2\tan^{-1} x & , \text{ if } x > 1 \\ \pi + 2\tan^{-1} x & , \text{ if } x < -1 \end{cases}$

Using additive property, we obtain

$$\therefore \quad I = \int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx = \int_0^{1} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx + \int_1^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

$$\Rightarrow \quad I = \int_0^1 2\tan^{-1} x\, dx + \int_1^{\sqrt{3}} (-\pi + 2\tan^{-1} x)\, dx$$

$$\Rightarrow \quad I = \int_0^1 2\tan^{-1} x\, dx + \int_1^{\sqrt{3}} -\pi\, dx + \int_1^{\sqrt{3}} 2\tan^{-1} x\, dx$$

$$\Rightarrow \quad I = \left\{\int_0^1 2\tan^{-1} x\, dx + \int_1^{\sqrt{3}} 2\tan^{-1} x\, dx\right\} - \pi \int_1^{\sqrt{3}} 1 . dx = 2\int_0^{\sqrt{3}} \tan^{-1} x\, dx - \pi \int_1^{\sqrt{3}} 1 . dx$$

$$\Rightarrow \quad I = 2\left[\left\{x\tan^{-1} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\, dx\right] - \pi\left[x\right]_1^{\sqrt{3}}$$

$$\Rightarrow \quad I = 2\left[\left\{\sqrt{3}\tan^{-1}\sqrt{3} - 0\right\} - \frac{1}{2}\left[\log(1+x^2)\right]_0^{\sqrt{3}}\right] - \pi(\sqrt{3}-1)$$

$$\Rightarrow \quad I = 2\left[\frac{\pi}{3}\sqrt{3} - \frac{1}{2}(\log 4 - \log 1)\right] - \pi(\sqrt{3}-1) = \frac{2\pi}{3}\sqrt{3} - \log 4 - \pi(\sqrt{3}-1)$$

$$\Rightarrow \quad I = \pi\left(1 - \frac{1}{\sqrt{3}}\right) - \log 4$$

(ii) Clearly, $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) = \begin{cases} 3\tan^{-1} x & , \text{ if } -1/\sqrt{3} < x < 1\sqrt{3} \\ -\pi + 3\tan^{-1} x & , \text{ if } x > 1/\sqrt{3} \\ \pi + 3\tan^{-1} x & , \text{ if } x < -1/\sqrt{3} \end{cases}$

Using additive property, we obtain

$$I = \int_0^1 \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx = \int_0^{1/\sqrt{3}} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx + \int_{1/\sqrt{3}}^{1} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow \quad I = \int_0^{1/\sqrt{3}} 3\tan^{-1} x\, dx + \int_{1/\sqrt{3}}^{1} (-\pi + 3\tan^{-1} x)\, dx$$

$$\Rightarrow \quad I = \left\{ \int_0^{1/\sqrt{3}} 3\tan^{-1} x\, dx + \int_{1/\sqrt{3}}^{1} 3\tan^{-1} x\, dx \right\} + \int_{1/\sqrt{3}}^{1} -\pi\, dx$$

$$\Rightarrow \quad I = \int_0^1 3\tan^{-1} x - \pi \int_{1/\sqrt{3}}^{1} dx \qquad \text{[Using additive property]}$$

$$\Rightarrow \quad I = 3\left[x\tan^{-1} x - \frac{1}{2}\log(1+x^2) \right]_0^1 - \pi \Big[x \Big]_{1/\sqrt{3}}^{1} \qquad \left[\because \int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) \right]$$

$$\Rightarrow \quad I = 3\left[\left(\frac{\pi}{4} - \frac{1}{2}\log 2 \right) - 0 \right] - \pi\left[1 - \frac{1}{\sqrt{3}} \right] = \pi\left(\frac{1}{\sqrt{3}} - \frac{1}{4} \right) - \frac{3}{2}\log 2$$

EXAMPLE 10 *Prove that* $\displaystyle\int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt + \int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\, dt = 1$ *for all x for which* $\tan x$ *and* $\cot x$ *are defined.*

SOLUTION Let $I_1 = \displaystyle\int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt$ and $I_2 = \displaystyle\int_{1/e}^{\cot x} \frac{1}{t(1+t^2)}\, dt$. Putting $t = \dfrac{1}{u}$ and $dt = -\dfrac{1}{u^2}\, du$ in I_2, we get

$$I_2 = \int_e^{\tan x} \frac{u^3}{1+u^2} \times -\frac{1}{u^2}\, du = -\int_e^{\tan x} \frac{u}{1+u^2}\, du$$

$$\Rightarrow \quad I_2 = -\int_e^{\tan x} \frac{t}{1+t^2}\, dt \qquad [\because \text{Integration is independent of change of variable}]$$

$$\Rightarrow \quad I_2 = -\left\{ \int_e^{1/e} \frac{t}{1+t^2}\, dt + \int_{1/e}^{\tan x} \frac{t}{1+t^2}\, dt \right\} = -\left\{ \int_e^{1/e} \frac{t}{1+t^2}\, dt + I_1 \right\}$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2} \int_e^{1/e} \frac{2t}{1+t^2}\, dt = -\frac{1}{2}\Big[\log(1+t^2) \Big]_e^{1/e}$$

$$\Rightarrow \quad I_1 + I_2 = -\frac{1}{2}\left[\log\left(\frac{1+e^2}{e^2} \right) - \log(1+e^2) \right] = -\frac{1}{2}\left\{ \log\frac{1}{e^2} \right\} = -\frac{1}{2} \times -2\log e = 1$$

EXERCISE 19.3

BASIC

Evaluate the following integrals:

1. (i) $\displaystyle\int_1^4 f(x)\, dx$, where $f(x) = \begin{cases} 4x+3, & \text{if } 1 \le x \le 2 \\ 3x+5, & \text{if } 2 \le x \le 4 \end{cases}$

(ii) $\displaystyle\int_0^9 f(x)\,dx$, where $f(x) \begin{cases} \sin x & , \quad 0 \le x \le \pi/2 \\ 1 & , \quad \pi/2 \le x \le 3 \\ e^{x-3} & , \quad 3 \le x \le 9 \end{cases}$

(iii) $\displaystyle\int_1^4 f(x)\,dx$, where $f(x) = \begin{cases} 7x+3 & , \quad \text{if } 1 \le x \le 3 \\ 8x & , \quad \text{if } 3 \le x \le 4 \end{cases}$ (iv) $\displaystyle\int_{-1}^{2} \frac{|x|}{x}\,dx$ [CBSE 2019]

2. $\displaystyle\int_{-4}^{4} |x+2|\,dx$ 3. $\displaystyle\int_{-3}^{3} |x+1|\,dx$ 4. $\displaystyle\int_{-1}^{1} |2x+1|\,dx$ 5. $\displaystyle\int_{-2}^{2} |2x+3|\,dx$

6. $\displaystyle\int_0^2 |x^2-3x+2|\,dx$ 7. $\displaystyle\int_1^3 |2x-1|\,dx$ [CBSE 2020] 8. $\displaystyle\int_{-6}^{6} |x+2|\,dx$

9. $\displaystyle\int_{-2}^{2} |x+1|\,dx$ 10. $\displaystyle\int_1^2 |x-3|\,dx$ 11. $\displaystyle\int_0^{\pi/2} |\cos 2x|\,dx$ 12. $\displaystyle\int_0^{2\pi} |\sin x|\,dx$

13. $\displaystyle\int_{-\pi/4}^{\pi/4} |\sin x|\,dx$ 14. $\displaystyle\int_1^4 |x-5|\,dx$ [CBSE 2020]

BASED ON LOTS

15. $\displaystyle\int_{-\pi/2}^{\pi/2} \left(\sin|x| + \cos|x|\right) dx$ [CBSE 2000] 16. $\displaystyle\int_0^4 |x-1|\,dx$ [NCERT]

17. $\displaystyle\int_1^4 \left\{|x-1| + |x-2| + |x-4|\right\} dx$ [CBSE 2011]

18. $\displaystyle\int_{-5}^{0} f(x)\,dx$, where $f(x) = |x| + |x+2| + |x+5|$ [CBSE 2005]

19. $\displaystyle\int_0^4 \left(|x| + |x-2| + |x-4|\right) dx$ [CBSE 2013]

20. $\displaystyle\int_{-1}^{2} \left(|x+1| + |x| + |x-1|\right) dx$ [NCERT EXEMPLAR]

21. $\displaystyle\int_{-2}^{2} x\,e^{|x|}\,dx$ 22. $\displaystyle\int_{-\pi/4}^{\pi/2} \sin x|\sin x|\,dx$ 23. $\displaystyle\int_0^{\pi} \cos x|\cos x|\,dx$

24. $\displaystyle\int_{-\pi/2}^{\pi/2} (2\sin|x| + \cos x|)\,dx$

BASED ON HOTS

25. $\displaystyle\int_{-\pi/2}^{\pi} \sin^{-1}(\sin x)\,dx$ 26. $\displaystyle\int_0^2 2x\,[x]\,dx$ 27. $\displaystyle\int_0^{2\pi} \cos^{-1}(\cos x)\,dx$

ANSWERS

1. (i) 37 (ii) $3-\frac{\pi}{2}+e^6$ (iii) 62 (iv) 1
2. 20
3. 10
4. $\frac{5}{2}$
5. $\frac{25}{2}$
6. 1
7. 6
8. 40
9. 5
10. $\frac{3}{2}$
11. 1
12. 4
13. $2-\sqrt{2}$
14. $\frac{15}{2}$
15. 4
16. 5
17. $\frac{23}{2}$
18. $\frac{63}{2}$
19. 20
20. $\frac{19}{2}$
21. 0
22. $\frac{\pi}{8}+\frac{1}{4}$
23. 0
24. 6
25. $\frac{\pi^2}{8}$
26. 3
27. π^2

19.4.4 PROPERTY IV

STATEMENT *If $f(x)$ is a continuous function defined on $[a, b]$, then* $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

PROOF Let $x = a+b-t$. Then, $dx = -\,dt$. Also, $x = a \Rightarrow t = b$ and $x = b \Rightarrow t = a$

$$\therefore \quad \int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt = \int_a^b f(a+b-t)\,dt \qquad \text{[By property II]}$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx \qquad \text{[By property I]}$$

Hence, $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:* $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx = \frac{b-a}{2}$.

SOLUTION Let $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\,dx$ ...(i)

Then, by using property IV, we obtain

$$I = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(a+b-(a+b-x))}\,dx = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}\,dx = \int_a^b 1\,dx = (b-a) = \frac{b-a}{2}$$

REMARK *The statement of the above example may be used as standard result in solving objective type questions.*

EXAMPLE 2 *Evaluate:*

(i) $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$ (ii) $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\,dx$ [NCERT, CBSE 2014]

SOLUTION (i) Let $I = \int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\, dx$ ...(i)

Then, by using property IV, we obtain

$$I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{3-(3-x)}+\sqrt{3-x}}\, dx \Rightarrow I = \int_1^2 \frac{\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\, dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_1^2 \frac{\sqrt{x}+\sqrt{3-x}}{\sqrt{x}+\sqrt{3-x}}\, dx = \int_1^2 1 \cdot dx = \Big[x\Big]_1^2 = 2-1 = 1 \Rightarrow I = \frac{1}{2}.$$

ALITER Here, $a=1, b=2$ and $f(x)=\sqrt{x}$. Clearly, $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx.$

Using Example 1, we obtain

$$I = \frac{b-a}{2} = \frac{2-1}{2} = \frac{1}{2}$$

(ii) Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\, dx = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\, dx$ ...(i)

Then, by using property IV, we obtain

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}\, dx \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\, dx = \int_{\pi/6}^{\pi/3} 1 \cdot dx = \Big[x\Big]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}.$$

ALITER Here $a=\frac{\pi}{6}, b=\frac{\pi}{3}$ and $f(x)=\sqrt{\sin x}$. Clearly, $I = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx.$

Using Example 1, we obtain: $I = \frac{b-a}{2} = \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{\pi}{12}$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Evaluate:* $\int_{-\pi/2}^{\pi/2} \frac{x \sin x}{e^x+1}\, dx$

SOLUTION Let $I = \int_{-\pi/2}^{\pi/2} \frac{x \sin x}{e^x+1}\, dx$ ...(i)

Using Property IV, we obtain

$$I = \int_{-\pi/2}^{\pi/2} \frac{(-\pi/2+\pi/2-x)\sin(-\pi/2+\pi/2-x)}{e^{\left(-\frac{\pi}{2}+\frac{\pi}{2}-x\right)}+1} = \int_{-\pi/2}^{\pi/2} \frac{-x\sin(-x)}{e^{-x}+1}\, dx = \int_{-\pi/2}^{\pi/2} \frac{x \sin x\, e^x}{e^x+1}\, dx \quad ...(ii)$$

Adding (i) and (ii), we obtain

$$2I=\int_{-\pi/2}^{\pi/2}\left(\frac{x\sin x}{e^x+1}+\frac{x\sin x\, e^x}{e^x+1}\right)dx=\int_{-\pi/2}^{\pi/2} x\sin x\left(\frac{e^x+1}{e^x+1}\right)dx=\int_{-\pi/2}^{\pi/2} \underset{\text{I}}{x}\ \underset{\text{II}}{\sin x}\, dx$$

$$\Rightarrow \quad 2I=\Big[-x\cos x\Big]_{-\pi/2}^{\pi/2}+\int_{-\pi/2}^{\pi/2}\cos x\, dx=\Big[-x\cos x\Big]_{-\pi/2}^{\pi/2}+\Big[\sin x\Big]_{-\pi/2}^{\pi/2}=(0-0)+1-(-1)$$

$$\Rightarrow \quad I=1$$

EXAMPLE 4 *Evaluate:* $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{1+e^x}dx$

SOLUTION Let $I=\displaystyle\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{1+e^x}dx$ ...(i)

Using Property IV, we obtain

$$I=\int_{-\pi/4}^{\pi/4}\frac{\sec^2\left(-\frac{\pi}{4}+\frac{\pi}{4}-x\right)}{1+e^{(-\pi/4+\pi/4-x)}}dx=\int_{-\pi/4}^{\pi/4}\frac{\sec^2(-x)}{1+e^{-x}}dx=\int_{-\pi/4}^{\pi/4}\frac{\sec^2 x}{1+e^{-x}}dx=\int_{-\pi/4}^{\pi/4}\frac{e^x\sec^2 x}{1+e^x}dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I=\int_{-\pi/4}^{\pi/4}\left(\frac{\sec^2 x}{1+e^x}+\frac{e^x\sec^2 x}{1+e^x}\right)dx=\int_{-\pi/4}^{\pi/4}\sec^2 x\, dx=\Big[\tan x\Big]_{-\pi/4}^{\pi/4}=1-(-1)=2.$$

$$\Rightarrow \quad I=1.$$

EXERCISE 19.4

BASIC

Evaluate of each of the following integrals (1-15):

1. $\displaystyle\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+e^{-\sin x}}dx$ **2.** $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$ **3.** $\displaystyle\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$

4. $\displaystyle\int_{-\pi/3}^{\pi/3}\frac{1}{1+e^{\tan x}}dx$ **5.** $\displaystyle\int_a^b\frac{x^{1/n}}{x^{1/n}+(a+b-x)^{1/n}}dx,\ n\in N, n\ge 2$

6. $\displaystyle\int_0^a\frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}dx$ [NCERT] **7.** $\displaystyle\int_0^5\frac{\sqrt[4]{x+4}}{\sqrt[4]{x+4}+\sqrt[4]{9-x}}dx$

8. $\displaystyle\int_0^7\frac{\sqrt[3]{x}}{\sqrt[3]{x}+\sqrt[3]{7-x}}dx$ **9.** $\displaystyle\int_{\pi/6}^{\pi/3}\frac{1}{1+\sqrt{\tan x}}dx$ [CBSE 2007, 2011]

BASED ON LOTS

10. $\displaystyle\int_0^{2\pi}\log(\sec x+\tan x)\,dx$ **11.** $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{\tan^2 x}{1+e^x}dx$ **12.** $\displaystyle\int_{-a}^{a}\frac{1}{1+a^x}dx,\ a>0$

13. $\displaystyle\int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{1+e^x}dx$ **14.** $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{x^{11}-3x^9+5x^7-x^5+1}{\cos^2 x}dx$

15. $\int_1^2 \log\left(\frac{3}{x}-1\right) dx$ [CBSE 2022]

16. $\int_0^{\pi/2} (2 \log \cos x - \log \sin 2x)\, dx$

17. If $f(a+b-x) = f(x)$, then prove that $\int_a^b x f(x)\, dx = \frac{a+b}{2} \int_a^b f(x)\, dx$.

ANSWERS

1. π	2. $\frac{\pi}{12}$	3. $\frac{\pi}{12}$	4. $\frac{\pi}{3}$	5. $\frac{b-a}{2}$	6. $\frac{a}{2}$
7. $\frac{5}{2}$	8. $\frac{7}{2}$	9. $\frac{\pi}{12}$	10. 0	11. $1-\frac{\pi}{4}$	12. a
13. $\frac{\pi}{4}$	14. 2	15. 0	16. $-\frac{\pi}{2}\log 2$		

19.4.5 PROPERTY V

STATEMENT *If $f(x)$ is a continuous function defined on $[0, a]$, then* $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$.

[CBSE 2019]

PROOF Let $x = a - t$. Then, $dx = d(a-t) \Rightarrow dx = -dt$. Also, $x = 0 \Rightarrow t = a$ and $x = a \Rightarrow t = 0$

$$\therefore \quad \int_0^a f(x)\, dx = -\int_a^0 f(a-t)\, dt$$

$$= \int_0^a f(a-t)\, dt \qquad \text{[By property II]}$$

$$= \int_0^a f(a-x)\, dx \qquad \text{[By property I]}$$

Hence, $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that:* $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$ [CBSE 2002C, 2007]

SOLUTION Let $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx.$ ...(i)

Using: $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\, dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\,dx + \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\,dx = \int_0^{\pi/2} 1 \cdot dx$$

$$\Rightarrow \quad 2I = \Big[x\Big]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

EXAMPLE 2 *Evaluate:* $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ **[NCERT]**

SOLUTION Let $\displaystyle I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ ..(i)

Using: $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} 1 \cdot dx = \Big[x\Big]_0^{\pi/2} = \frac{\pi}{2} - 0 \Rightarrow I = \frac{\pi}{4}$$

Hence $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$

EXAMPLE 3 *Evaluate:*

(i) $\displaystyle\int_0^{\pi/2} \log \tan x\,dx$ **[CBSE 2007]** (ii) $\displaystyle\int_0^{\pi/4} \log(1 + \tan x)\,dx$ **[NCERT, CBSE 2002C,03,04,11,13]**

SOLUTION (i) Let $\displaystyle I = \int_0^{\pi/2} \log \tan x\,dx$ ...(i)

Using : $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx \Rightarrow I = \int_0^{\pi/2} \log \cot x\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} (\log \tan x + \log \cot x)\,dx = \int_0^{\pi/2} \log(\tan x \times \cot x)\,dx = \int_0^{\pi/2} \log 1\,\,dx = \int_0^{\pi/2} 0\,\,dx = 0$$

$$\Rightarrow \quad I = 0$$

(ii) Let $\displaystyle I = \int_0^{\pi/4} \log(1 + \tan x)\,dx$ ...(i)

Using : $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I=\int_0^{\pi/4}\log\left\{1+\tan\left(\frac{\pi}{4}-x\right)\right\}dx=\int_0^{\pi/4}\log\left\{1+\frac{\tan\pi/4-\tan x}{1+\tan\pi/4\tan x}\right\}dx$$

$$\Rightarrow\quad I=\int_0^{\pi/4}\log\left\{1+\frac{1-\tan x}{1+\tan x}\right\}dx=\int_0^{\pi/4}\log\left\{\frac{1+\tan x+1-\tan x}{1+\tan x}\right\}dx$$

$$\Rightarrow\quad I=\int_0^{\pi/4}\log\left\{\frac{2}{1+\tan x}\right\}dx=\int_0^{\pi/4}\left\{\log 2-\log(1+\tan x)\right\}dx$$

$$\Rightarrow\quad I=\int_0^{\pi/4}\log 2\,dx-\int_0^{\pi/4}\log(1+\tan x)\,dx=(\log 2)\Big[x\Big]_0^{\pi/4}-I$$

$$\Rightarrow\quad 2I=\frac{\pi}{4}\log 2\Rightarrow I=\frac{\pi}{8}\log 2$$

EXAMPLE 4 *Evaluate:*

(i) $\displaystyle\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ [NCERT] (ii) $\displaystyle\int_0^{\pi/2}(2\log\sin x-\log\sin 2x)\,dx$ [NCERT, CBSE 2009]

SOLUTION (i) Let $I=\displaystyle\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$ ...(i)

Using: $\displaystyle\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$, we obtain

$$I=\int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx\Rightarrow I=\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\cos x\sin x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$2I=\int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx+\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\sin x\cos x}dx=\int_0^{\pi/2}\left(\frac{\sin x-\cos x}{1+\sin x\cos x}+\frac{\cos x-\sin x}{1+\sin x\cos x}\right)dx$$

$$\Rightarrow 2I=\int_0^{\pi/2}\frac{\sin x-\cos x+\cos x-\sin x}{1+\sin x\cos x}dx=0\Rightarrow I=0$$

(ii) We have,

$$I=\int_0^{\pi/2}(2\log\sin x-\log\sin 2x)\,dx=\int_0^{\pi/2}\left\{2\log\sin x-\log(2\sin x\cos x)\right\}dx$$

$$\Rightarrow I=\int_0^{\pi/2}\left\{2\log\sin x-\log 2-\log\sin x-\log\cos x\right\}dx$$

$$\Rightarrow I=\int_0^{\pi/2}\log\sin x\,dx-\int_0^{\pi/2}\log 2\,dx-\int_0^{\pi/2}\log\cos x\,dx$$

$$\Rightarrow I=\int_0^{\pi/2}\log\sin x\,dx-(\log 2)\int_0^{\pi/2}1\cdot dx-\int_0^{\pi/2}\log\cos\left(\frac{\pi}{2}-x\right)dx \quad \text{[Using property V]}$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log \sin x\, dx - (\log 2)\left[x\right]_0^{\pi/2} - \int_0^{\pi/2} \log \sin x\, dx = -(\log 2)\left(\frac{\pi}{2} - 0\right) = -\frac{\pi}{2}\log 2$$

EXAMPLE 5 *Evaluate:* $\displaystyle\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ **[CBSE 2009]**

SOLUTION Let $\displaystyle I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx$ ...(i)

Using : $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{e^{\cos(\pi - x)}}{e^{\cos(\pi - x)} + e^{-\cos(\pi - x)}}\, dx \Rightarrow I = \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx + \int_0^{\pi} \frac{e^{-\cos x}}{e^{-\cos x} + e^{\cos x}}\, dx = \int_0^{\pi} \frac{e^{\cos x} + e^{-\cos x}}{e^{\cos x} + e^{-\cos x}}\, dx = \int_0^{\pi} 1\, dx$$

$$\Rightarrow \quad 2I = \pi \Rightarrow I = \frac{\pi}{2}$$

EXAMPLE 6 *Prove that:* $\displaystyle\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a-x)\, dx.$ **[CBSE 2002C]**

SOLUTION Let $\displaystyle I = \int_0^{2a} f(x)\, dx$. Let $2a - x = t$. Then, $d(2a-x) = dt \Rightarrow -dx = dt \Rightarrow dx = -dt$

Also, $x = 0 \Rightarrow t = 2a - 0 = 2a$ and $x = a \Rightarrow t = 2a - a = a$

$$\therefore \quad I = \int_{2a}^{0} f(2a-t)(-dt) = -\int_{2a}^{0} f(2a-t)\, dt$$

$$\Rightarrow \quad I = \int_0^{2a} f(2a-t)\, dt \qquad \left[\because \int_a^b f(x)\, dx = -\int_b^a f(x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^{2a} f(2a-x)\, dx \qquad \left[\because \int_a^b f(x)\, dx = \int_a^b f(t)\, dt\right]$$

Hence, $\displaystyle\int_0^{2a} f(x)\, dx = \int_0^{2a} f(2a-x)\, dx$

EXAMPLE 7 *Evaluate:* $\displaystyle\int_0^1 x(1-x)^n\, dx$ **[NCERT, CBSE 2022]**

SOLUTION Let $\displaystyle I = \int_0^1 x(1-x)^n\, dx$. By using $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^1 (1-x)\{1-(1-x)\}^n\, dx = \int_0^1 (1-x)\, x^n\, dx = \int_0^1 (x^n - x^{n+1})\, dx$$

$$\Rightarrow \quad I = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \left\{\frac{1}{n+1} - \frac{1}{n+2}\right\} - (0-0) = \frac{1}{(n+1)(n+2)}$$

EXAMPLE 8 *Prove that:*

(i) $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$ (ii) $\int_0^1 \log\left(\frac{1}{x} - 1\right) dx = 0$

SOLUTION (i) Let $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$ ...(i)

Using: $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) dx \Rightarrow I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \sin 2x \left\{\log \tan x + \log \cot x\right\} dx = \int_0^{\pi/2} \sin 2x \log(\tan x \cot x)\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} (\sin 2x)\ (\log 1)\ dx = 0 \Rightarrow I = 0 \qquad [\because \log 1 = 0]$$

(ii) $$I = \int_0^1 \log\left(\frac{1}{x} - 1\right) dx = \int_0^1 \log\left(\frac{1-x}{x}\right) dx \quad \text{...(i)}$$

Using : $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^1 \log\left\{\frac{1-(1-x)}{1-x}\right\} dx \Rightarrow I = \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^1 \left\{\log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right)\right\} dx = \int_0^1 \log 1 \cdot dx = 0 \Rightarrow I = 0.$$

EXAMPLE 9 *Evaluate:*

(i) $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ [CBSE 2002, 03, 16, NCERT EXEMPLAR] (ii) $\int_0^{\pi/2} \frac{\sin^2 x}{1 + \sin x \cos x} dx$

SOLUTION (i) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$. Then, ...(i)

Using : $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} + \frac{\cos^2 x}{\sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{1}{\dfrac{2\tan x/2}{1+\tan^2 x/2} + \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}}\, dx = \int_0^{\pi/2} \frac{1+\tan^2 \frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{2\tan\frac{x}{2} + 1 - \tan^2\frac{x}{2}}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan\frac{x}{2}\right) = dt \Rightarrow \left(\sec^2\frac{x}{2}\right)\frac{1}{2}\, dx = dt \Rightarrow \sec^2\frac{x}{2}\, dx = 2\, dt$

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and, $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{4} = 1$

$$\therefore \quad 2I = \int_0^1 \frac{2dt}{2t+1-t^2} = 2\int_0^1 \frac{1}{(\sqrt{2})^2 - (t-1)^2}\, dt = 2 \times \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{1}{\sqrt{2}}\left\{\log\left(\frac{\sqrt{2}}{\sqrt{2}}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\} = \frac{1}{\sqrt{2}}\left\{0 - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right\}$$

$$\Rightarrow \quad 2I = -\frac{1}{\sqrt{2}}\log\left\{\frac{(\sqrt{2}-1)^2}{(\sqrt{2}+1)(\sqrt{2}-1)}\right\} = -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)^2 = -\frac{2}{\sqrt{2}}\log(\sqrt{2}-1)$$

$$\Rightarrow \quad I = -\frac{1}{\sqrt{2}}\log(\sqrt{2}-1)$$

(ii) Let $I = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cos x}\, dx$. Then, ...(i)

Using: $\int_0^a f(x)\, dx = \int_0^a f(a-x)$, we obtain

$$I = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}\, dx \Rightarrow I = \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos x\sin x}\, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{1+\sin x\cos x}\, dx = \int_0^{\pi/2} \frac{1}{1+\sin x\cos x}\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + \tan x}\, dx$$ [Dividing numerator and denominator by $\cos^2 x$]

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$.

$$\therefore \quad 2I = \int_0^\infty \frac{dt}{1+t^2+t} = \int_0^\infty \frac{dt}{(t+1/2)^2 + (\sqrt{3}/2)^2}\, dt = \frac{1}{\sqrt{3}/2}\left[\tan^{-1}\left(\frac{t+1/2}{\sqrt{3}/2}\right)\right]_0^\infty$$

$$\Rightarrow \quad 2I = \frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^\infty = \frac{2}{\sqrt{3}}\left[\tan^{-1}\infty - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{6}\right] = \frac{2\pi}{3\sqrt{3}}$$

$$\Rightarrow \quad I = \frac{\pi}{3\sqrt{3}}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *Evaluate:* $\int_0^1 \cot^{-1}(1 - x + x^2)\, dx$ **[CBSE 2008]**

SOLUTION Let $I = \int_0^1 \cot^{-1}(1 - x + x^2)\, dx$. Then,

$$I = \int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx \qquad \left[\because \cot^{-1} x = \tan^{-1}\frac{1}{x},\ x > 0\right]$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1}\left\{\frac{1}{1-x(1-x)}\right\} dx = \int_0^1 \tan^{-1}\left\{\frac{x+(1-x)}{1-x(1-x)}\right\} dx$$

$$\Rightarrow \quad I = \int_0^1 \{\tan^{-1} x + \tan^{-1}(1-x)\}\, dx = \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}\{1-(1-x)\}\, dx \qquad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow \quad I = \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx = 2\int_0^1 \tan^{-1} x\, dx = 2\int_0^1 \underset{\text{I}}{\tan^{-1} x} \cdot \underset{\text{II}}{1}\, dx$$

$$\Rightarrow \quad I = 2\left[x\tan^{-1} x\right]_0^1 - 2\int_0^1 \frac{x}{1+x^2}\, dx = 2\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{2x}{1+x^2}\, dx$$

$$\Rightarrow \quad I = 2\left[x\tan^{-1} x\right]_0^1 - \left[\log(1+x^2)\right]_0^1 = 2\left(\frac{\pi}{4} - 0\right) - (\log 2 - \log 1) = \frac{\pi}{2} - \log 2$$

EXAMPLE 11 *Evaluate:* $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx$

SOLUTION Let $I = \int_1^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x}\, dx$ ...(i)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)$, we obtain

$$I = \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{1 + \cos(\pi/2 - x) + \sin(\pi/2 - x)}\, dx \Rightarrow I = \int_0^{\pi/2} \frac{\sin x}{1 + \sin x + \cos x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos x + \sin x}{1 + \sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{(1 + \sin x + \cos x) - 1}{1 + \sin x + \cos x}\, dx = \int_0^{\pi/2} \left\{1 - \frac{1}{1 + \sin x + \cos x}\right\} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} 1 \,.\, dx - \int_0^{\pi/2} \frac{1}{1 + \sin x + \cos x}\, dx$$

$$\Rightarrow 2I = \Big[x\Big]_0^{\pi/2} - \int_0^{\pi/2} \frac{1 + \tan^2 x/2}{1 + \tan^2 x/2 + 2 \tan x/2 + 1 - \tan^2 x/2}\, dx$$

$$\Rightarrow 2I = \frac{\pi}{2} - \int_0^{\pi/2} \frac{\sec^2 x/2}{2 + 2 \tan x/2}\, dx = \frac{\pi}{2} - \int_0^{1} \frac{2\, dt}{2 + 2t}, \quad \text{where } t = \tan\frac{x}{2}$$

$$\Rightarrow 2I = \frac{\pi}{2} - \Big[\log\,(1 + t)\Big]_0^1 = \frac{\pi}{2} - \log 2 \;\Rightarrow\; I = \frac{\pi}{4} - \frac{1}{2}\log 2$$

ALITER
$$I = \int_0^{\pi/2} \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} = \int_0^{\pi/2} \frac{1 - \tan^2 x/2}{2\,(1 + \tan x/2)}\, dx \qquad \left[\begin{array}{l}\text{Dividing } N^r \text{ and } D^r \\ \text{by } \cos^2 x/2\end{array}\right]$$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^{\pi/2} \left(1 - \tan\frac{x}{2}\right) = \frac{1}{2}\left[x - 2 \log \sec\frac{x}{2}\right]_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} - \log 2\right)$$

EXAMPLE 12 *If f and g are continuous on [0, a] and satisfy $f(x) = f(a - x)$ and $g(x) + g(a - x) = 2$, show that*

$$\int_0^a f(x)\, g(x)\, dx = \int_0^a f(x)\, dx$$ [NCERT]

SOLUTION Using Property V, we obtain

$$\int_0^a f(x)\, g(x)\, dx = \int_0^a f(a - x)\, g(a - x)\, dx$$

$$\Rightarrow \int_0^a f(x)\, g(x)\, dx = \int_0^a f(x) \left\{2 - g(x)\right\} dx \qquad [\because\ g(a - x) = 2 - g(x)]$$

$$\Rightarrow \int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx - \int_0^a f(x)\, g(x)\, dx$$

$$\Rightarrow \quad 2\int_0^a f(x)\, g(x)\, dx = 2\int_0^a f(x)\, dx \Rightarrow \int_0^a f(x)\, g(x)\, dx = \int_0^a f(x)\, dx$$

19.4.6 PROPERTY VI

STATEMENT *If $f(x)$ is a continuous function defined on $[0, 2a]$, then*

$$\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx = \int_0^a \left\{f(x) + f(2a - x)\right\} dx$$

PROOF Let $I = \int_0^{2a} f(x)\,dx$. Then,

$$I = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \qquad \left[\because \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\right]$$

$$\Rightarrow \quad I = \int_0^a f(x)\,dx + I_1, \text{ where } I_1 = \int_a^{2a} f(x)\,dx \qquad \text{...(i)}$$

Let $2a - t = x$. Then, $dx = -dt$. Also, $x = a \Rightarrow t = a$ and $x = 2a \Rightarrow t = 0$

$$\therefore \quad I_1 = \int_a^{2a} f(x)\,dx = \int_a^0 f(2a-t)(-dt) = -\int_a^0 f(2a-t)\,dt = \int_0^a f(2a-t)\,dt = \int_0^a f(2a-x)\,dx$$

Substituting the value of I_1 in (i), we get

$$I = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a \left\{f(x) + f(2a-x)\right\} dx$$

i.e. $$\int_0^{2a} f(x)\,dx = \int_0^a \left\{f(x) + f(2a-x)\right\} dx.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$ **[CBSE 2013]**

SOLUTION Let $I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. Using $\int_0^{2a} f(x) = \int_0^a \{f(x) + f(2a-x)\}\,dx$, we obtain

$$I = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}\right\} dx = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \left\{\frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}}\right\} dx = \int_0^{\pi} 1\,dx = \Big[x\Big]_0^{\pi} = \pi$$

EXAMPLE 2 *Evaluate:* $\int_0^{\pi} \frac{1}{1+e^{\cos x}}\,dx.$

SOLUTION Let $\int_0^{\pi} \frac{1}{1+e^{\cos x}}\,dx$. Using $\int_0^{2a} f(x)\,dx = \int_0^a \{f(x) + f(2a-x)\}\,dx$, we obtain

$$I = \int_0^{\pi/2} \left\{\frac{1}{1+e^{\cos x}} + \frac{1}{1+e^{\cos(\pi - x)}}\right\} dx = \int_0^{\pi/2} \left\{\frac{1}{1+e^{\cos x}} + \frac{1}{1+e^{-\cos x}}\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left\{\frac{1}{1+e^{\cos x}} + \frac{e^{\cos x}}{1+e^{\cos x}}\right\} dx = \int_0^{\pi/2} \left\{\frac{1+e^{\cos x}}{1+e^{\cos x}}\right\} dx = \int_0^{\pi/2} 1 \cdot dx = \Big[x\Big]_0^{\pi/2} = \frac{\pi}{2}$$

19.4.7 PROPERTY VII

STATEMENT *Let $f(x)$ be a continuous function of x defined on $[0, a]$ such that $f(a-x) = f(x)$. Then,*

$$\int_0^a x\, f(x)\, dx = \frac{a}{2} \int_0^a f(x)\, dx.$$

PROOF Let $I = \int_0^a x\, f(x)\, dx$. Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^a (a-x)\, f(a-x)\, dx = \int_0^a (a-x)\, f(x)\, dx \qquad [\because f(a-x) = f(x)]$$

$$\Rightarrow \quad I = a \int_0^a f(x)\, dx - \int_0^a x\, f(x)\, dx = a \int_0^a f(x)\, dx - I$$

$$\Rightarrow \quad 2I = a \int_0^a f(x)\, dx \Rightarrow I = \frac{a}{2} \int_0^a f(x)\, dx.$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\displaystyle\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ [NCERT, CBSE 2008, 14]

(ii) $\displaystyle\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx$ [NCERT, CBSE 2003, 05, 08, 11, 12, 13, 14, 19]

SOLUTION (i) Let $I = \displaystyle\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$ ...(i)

Using: $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)}\, dx \Rightarrow I = \int_0^{\pi} \frac{\pi - x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx \Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$$

Using $\int_0^{2a} f(x)\, dx = \int_0^a \left\{ f(x) + f(2a-x) \right\} dx$, we obtain

$$2I = \pi \int_0^{\pi/2} \left\{ \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x} + \frac{1}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)} \right\} dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}\, dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx \qquad \text{[Dividing numerator and denominator by } \cos^2 x\text{]}$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx.$$

Let $\tan x = t$. Then, $d(\tan x) = dt \Rightarrow \sec^2 x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore \quad I = \pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} = \frac{\pi}{b^2} \int_0^{\infty} \frac{dt}{(a/b)^2 + t^2} = \frac{\pi}{b^2} \times \frac{1}{(a/b)} \left[\tan^{-1} \left(\frac{t}{a/b} \right) \right]_0^{\infty}$$

$$\Rightarrow \quad I = \frac{\pi}{ab} \left[\tan^{-1} \left(\frac{bt}{a} \right) \right]_0^{\infty} = \frac{\pi}{ab} \left(\tan^{-1} \infty - \tan^{-1} 0 \right) = \frac{\pi}{ab} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi^2}{2ab}.$$

(ii) $$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx \qquad \text{...(i)}$$

Using: $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{(\pi - x) \sin (\pi - x)}{1 + \cos^2 (\pi - x)}\, dx \Rightarrow I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x}\, dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{(x + \pi - x) \sin x}{1 + \cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx.$$

Let $\cos x = t$. Then, $d(\cos x) = dt \Rightarrow -\sin x\, dx = dt$.

Also, $x = 0 \Rightarrow t = \cos 0 = 1$ and $x = \pi \Rightarrow t = \cos \pi = -1$.

$$\therefore \quad I = \frac{\pi}{2} \int_1^{-1} \frac{-1}{1 + t^2}\, dt = -\frac{\pi}{2} \int_1^{-1} \frac{1}{1 + t^2}\, dt = -\frac{\pi}{2} \left[\tan^{-1} t \right]_1^{-1}$$

$$\Rightarrow \quad I = -\frac{\pi}{2} \left\{ \tan^{-1} (-1) - \tan^{-1} (1) \right\} = -\frac{\pi}{2} \left(-\frac{\pi}{4} - \frac{\pi}{4} \right) = \frac{\pi^2}{4}.$$

EXAMPLE 2 *Evaluate:*

(i) $\int_0^{\pi} \frac{x}{1 + \sin x}\, dx$ **[NCERT, CBSE 2001, 04, 10, 12, 22]**

(ii) $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx$ **[NCERT, CBSE 2008, 10, 14]**

(iii) $\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx$ **[CBSE 2022]**

SOLUTION (i) Let $I = \int_0^{\pi} \frac{x}{1+\sin x}\, dx$ ...(i)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{\pi - x}{1+\sin(\pi - x)}\, dx \Rightarrow I = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\, dx = \pi \int_0^{\pi} \frac{1}{1+\sin x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi/2} \left\{ \frac{1}{1+\sin x} + \frac{1}{1+\sin(\pi - x)} \right\} dx \quad \left[\because \int_0^{2a} f(x)\, dx = \int_0^a \left\{ f(x) + f(2a-x) \right\} dx \right]$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{1}{1+\sin x}\, dx = 2\pi \int_0^{\pi/2} \frac{1-\sin x}{1-\sin^2 x}\, dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} (\sec^2 x - \tan x \sec x)\, dx = 2\pi \Big[\tan x - \sec x \Big]_0^{\pi/2} = 2\pi \left[\frac{\sin x - 1}{\cos x} \right]_0^{\pi/2}$$

$$\Rightarrow \quad 2I = 2\pi \left[\frac{\sin^2 x - 1}{\cos x(\sin x + 1)} \right]_0^{\pi/2} = 2\pi \left[\frac{-\cos x}{1+\sin x} \right]_0^{\pi/2} = 2\pi\,(0+1) = 2\pi$$

$$\Rightarrow \quad I = \pi$$

(ii) Let $I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx$ ...(i)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}\, dx \Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}\, dx \Rightarrow I = \int_0^{\pi} \frac{\pi \sin x - x \sin x}{1+\sin x}\, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x}\, dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi/2} \left\{ \frac{\sin x}{1+\sin x} + \frac{\sin(\pi - x)}{1+\sin(\pi - x)} \right\} dx \quad \left[\because \int_0^{2a} f(x)\, dx = \int_0^a \left\{ f(x) + f(2a-x) \right\} dx \right]$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi/2} \left(\frac{\sin x}{1+\sin x} + \frac{\sin x}{1+\sin x} \right) dx = 2\pi \int_0^{\pi/2} \frac{\sin x}{1+\sin x}\, dx = 2\pi \int_0^{\pi/2} \frac{\sin x(1-\sin x)}{1-\sin^2 x}\, dx$$

$$\Rightarrow \quad 2I = 2\pi \int_0^{\pi/2} \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx = 2\pi \int_0^{\pi/2} (\tan x \sec x - \tan^2 x)\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} (\tan x \sec x - \tan^2 x)\, dx = \pi \int_0^{\pi/2} \{\tan x \sec x - (\sec^2 x - 1)\}\, dx$$

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} (\sec x \tan x - \sec^2 x + 1)\, dx = \pi \left[\sec x - \tan x + x \right]_0^{\pi/2}$$

$$\Rightarrow \quad I = \pi \left[\frac{1 - \sin x}{\cos x} + x \right]_0^{\pi/2} = \pi \left[\frac{\cos x}{1 + \sin x} + x \right]_0^{\pi/2} = \pi \left[\left(0 + \frac{\pi}{2} \right) - (1 + 0) \right] = \frac{\pi}{2}(\pi - 2)$$

(iii) Let $I = \displaystyle\int_0^{\pi/2} \frac{x}{\sin x + \cos x}\, dx$ ...(i)

Using $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx \Rightarrow I = \int_0^{\pi/2} \frac{\frac{\pi}{2} - x}{\cos x + \sin x}\, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x + \frac{\pi}{2} - x}{\sin x + \cos x}\, dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\sin x + \cos x}\, dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{1}{\frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}\, dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sec^2 \frac{x}{2}}{-\tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 1}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2}\, dx = dt \Rightarrow dx = \dfrac{2\, dt}{\sec^2 \frac{x}{2}}$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \frac{\pi}{2} \Rightarrow t = \tan \frac{\pi}{4} = 1$.

$$\therefore \quad 2I = \frac{\pi}{2} \int_0^1 \frac{2\, dt}{-t^2 + 2t + 1} = \pi \int_0^1 \frac{dt}{-(t^2 - 2t - 1)} = \pi \int_0^1 \frac{dt}{-[(t-1)^2 - 2]}$$

$$\Rightarrow \quad 2I = \pi \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2} = \pi \times \frac{1}{2\sqrt{2}} \left[\log \left| \frac{\sqrt{2} + t - 1}{\sqrt{2} - t + 1} \right| \right]_0^1$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}} \left[\log 1 - \log \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right] = -\frac{\pi}{2\sqrt{2}} \log \left(\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) = \frac{\pi}{2\sqrt{2}} \log \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right)$$

$$\Rightarrow \quad 2I = \frac{\pi}{2\sqrt{2}} \log \left\{ \frac{(\sqrt{2} + 1)^2}{(\sqrt{2} - 1)(\sqrt{2} + 1)} \right\} = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1)^2 = \frac{\pi}{\sqrt{2}} \log (\sqrt{2} + 1)$$

$$\Rightarrow \quad I = \frac{\pi}{2\sqrt{2}} \log (\sqrt{2} + 1).$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Evaluate:* $\displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ **[CBSE 2010, 11, 14]**

SOLUTION Let $I = \displaystyle\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$. Using $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi/2} \frac{(\pi/2 - x)\sin(\pi/2 - x)\cos(\pi/2 - x)}{\sin^4(\pi/2 - x) + \cos^4(\pi/2 - x)}\,dx = \int_0^{\pi/2} \frac{(\pi/2 - x)\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x \sec^2 x}{1 + \tan^4 x}\,dx \qquad \text{[Dividing numerator and denominator by } \cos^4 x]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^{\pi/2} \frac{2\tan x \sec^2 x\,dx}{1 + (\tan^2 x)^2}$$

Let $t = \tan^2 x$. Then, $dt = d(\tan^2 x) = 2\tan x \sec^2 x\,dx$.

Also, $x = 0 \Rightarrow t = \tan^2 0$ and, $x = \frac{\pi}{2} \Rightarrow t = \tan^2\frac{\pi}{2} = \infty$.

$$\therefore \quad 2I = \frac{\pi}{4}\int_0^{\infty} \frac{1}{1+t^2}\,dt = \frac{\pi}{4}\left[\tan^{-1} t\right]_0^{\infty} = \frac{\pi}{4}\left\{\tan^{-1}\infty - \tan^{-1} 0\right\} = \frac{\pi}{4}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{8}$$

$$\Rightarrow \quad I = \frac{\pi^2}{16}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Prove that:* $\displaystyle\int_0^{\pi} \frac{x}{1 - \cos\alpha \sin x}\,dx = \frac{\pi(\pi - \alpha)}{\sin\alpha}$

SOLUTION Let $I = \displaystyle\int_0^{\pi} \frac{x}{1 - \cos\alpha\sin x}\,dx$. Using $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi} \frac{(\pi - x)}{1 - \cos\alpha\sin(\pi - x)}\,dx = \int_0^{\pi} \frac{\pi}{1 - \cos\alpha\sin x}\,dx - \int_0^{\pi} \frac{x}{1 - \cos\alpha\sin x}\,dx$$

$$\Rightarrow \quad I = \pi\int_0^{\pi} \frac{1}{1 - \cos\alpha\sin x}\,dx - I$$

$$\Rightarrow \quad 2I = \pi\int_0^{\pi} \frac{1}{1 - \cos\alpha\sin x}\,dx = \pi\int_0^{\pi} \frac{1 + \tan^2 x/2}{(1 + \tan^2 x/2) - 2\cos\alpha\tan x/2}\,dx$$

$$\Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

$$\Rightarrow \quad I = \frac{\pi}{2} \int_0^{\pi} \frac{\sec^2 x/2}{\tan^2 x/2 - 2\cos\alpha \tan x/2 + 1}\, dx$$

Let $\tan \frac{x}{2} = t$. Then, $d\left(\tan \frac{x}{2}\right) = dt \Rightarrow \sec^2 \frac{x}{2}\, dx = 2\, dt$.

Also, $x = 0 \Rightarrow t = \tan 0 = 0$ and $x = \pi \Rightarrow t = \tan \frac{\pi}{2} = \infty$.

$$\therefore \quad I = \frac{\pi}{2} \int_0^{\infty} \frac{2\, dt}{t^2 - 2t\cos\alpha + 1} = \pi \int_0^{\infty} \frac{1}{(t-\cos\alpha)^2 + (1-\cos^2\alpha)}\, dt = \pi \int_0^{\infty} \frac{1}{\sin^2\alpha + (t-\cos\alpha)^2}\, dt$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t-\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty} = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\infty - \tan^{-1}(-\cot\alpha)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \tan^{-1}(\cot\alpha)\right]$$

$$\Rightarrow \quad I = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \tan^{-1}\left\{\tan\left(\frac{\pi}{2} - \alpha\right)\right\}\right] = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right] = \frac{\pi(\pi-\alpha)}{\sin\alpha}$$

EXAMPLE 5 *Prove that:* $\displaystyle\int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx = \frac{\pi^2}{4a^3b^3}(a^2+b^2)$

SOLUTION Let $\displaystyle\int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx$. Using $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi} \frac{\pi - x}{\{a^2\cos^2(\pi-x) + b^2\sin^2(\pi-x)\}^2}\, dx = \int_0^{\pi} \frac{\pi - x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx$$

$$\Rightarrow I = \int_0^{\pi} \frac{\pi}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx - \int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx$$

$$\Rightarrow I = \pi \int_0^{\pi} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx - I$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\, dx \qquad \left[\text{Using } \int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx, \text{ if } f(2a-x) = f(x)\right]$$

$$\Rightarrow I = \pi \times \frac{\pi(a^2+b^2)}{4a^3b^3} \Rightarrow I = \frac{\pi^2(a^2+b^2)}{4a^3b^3}$$

[See example 12 on page 19.28]

19.4.8 PROPERTY VIII

STATEMENT *If $f(x)$ is a continuous function defined on $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\, dx = \int_0^{a} \{f(x) + f(-x)\}\, dx$$

PROOF Using additive property III, we obtain

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \quad \ldots(i)$$

Let $x=-t$. Then, $dx=-dt$. Also, $x=-a \Rightarrow t=a$ and $x=0 \Rightarrow t=0$.

$$\therefore \quad \int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-t)(-dt) = -\int_{a}^{0} f(-t)\,dt = \int_{0}^{a} f(-t)\,dt \quad \text{[By property II]}$$

$$\Rightarrow \quad \int_{-a}^{0} f(x) = \int_{0}^{a} f(-x)\,dx \quad \text{[By property I]} \quad \ldots(ii)$$

From (i) and (ii), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left\{ f(-x) + f(x) \right\} dx$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:* $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}}\,dx$

SOLUTION Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}}\,dx$. Using $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x)+f(-x)\}\,dx$, we obtain

$$I = \int_{0}^{\pi/2} \left\{ \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}} \right\} dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi/2} \left\{ \frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{1+e^{\sin x}} \right\} dx = \int_{0}^{\pi/2} \frac{1+e^{\sin x}}{1+e^{\sin x}}\,dx = \int_{0}^{\pi/2} 1\,dx = \Big[x\Big]_{0}^{\pi/2} = \frac{\pi}{2}$$

EXAMPLE 2 *Evaluate:* $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{x}}\,dx$ [CBSE 2015]

SOLUTION Let $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^{x}}\,dx$. Using $\displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \{f(x)+f(-x)\}\,dx$, we obtain

$$I = \int_{0}^{\pi/2} \left\{ \frac{\cos x}{1+e^{x}} + \frac{\cos(-x)}{1+e^{-x}} \right\} dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi/2} \left\{ \frac{\cos x}{1+e^{x}} + \frac{\cos x}{1+e^{-x}} \right\} dx = \int_{0}^{\pi/2} \left\{ \frac{1}{1+e^{x}} + \frac{e^{x}}{1+e^{x}} \right\} \cos x\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \left(\frac{1+e^x}{1+e^x} \right) \cos x \, dx = \int_0^{\pi/2} \cos x \, dx = \Big[\sin x \Big]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$$

19.4.9 PROPERTY IX

STATEMENT *If $f(x)$ is a continuous function defined on $[-a, a]$, then*

$$\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

PROOF Using property VIII, we obtain

$$\int_{-a}^{a} f(x)\, dx = \int_0^a \left\{ f(x) + f(-x) \right\} dx$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\, dx = \begin{cases} \int_0^a \{f(x) + f(x)\}\, dx, & \text{if } f(-x) = f(x) \\ \int_0^a \{f(x) - f(x)\}\, dx, & \text{if } f(-x) = -f(x) \end{cases}$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(-x) = f(x) \\ 0, & \text{if } f(-x) = -f(x) \end{cases}$$

$$\Rightarrow \quad \int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function.} \end{cases}$$

Q.E.D

REMARK *The graph of an even function is symmetric about y-axis that is the curve on the left side of y-axis is exactly identical to curve on its right side.*

So, $$\int_0^a f(x)\, dx = \int_{-a}^0 f(x)\, dx$$ (See Fig. 19.10)

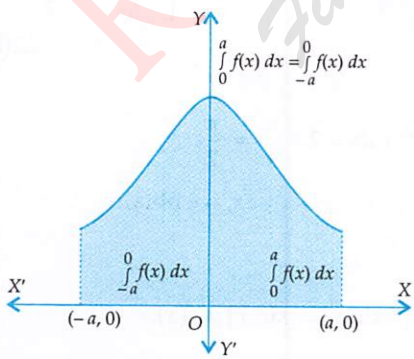

Fig. 19.10 Graph of an even function

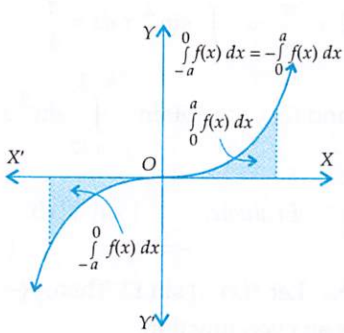

Fig. 19.11 Graph of an odd function

In case of an odd function the curve is symmetric in opposite quadrants. Consequently

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx \qquad \text{(See Fig. 19.11)}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate:*

(i) $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$ [NCERT] (ii) $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$ [NCERT]

SOLUTION (i) Let $f(x) = \sin^7 x$. We find that

$$f(-x) = \sin^7(-x) = \{\sin(-x)\}^7 = (-\sin x)^7 = -\sin^7 x = -f(x)$$

So, $f(x)$ is an odd function.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} f(x)\,dx = 0 \text{ i.e. } \int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0$$

(ii) Let $f(x) = \sin^2 x$. We find that

$$f(-x) = \sin^2(-x) = \{\sin(-x)\}^2 = (-\sin x)^2 = \sin^2 x = f(x)$$

So, $f(x)$ is an even function.

$$\therefore \quad \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \sin^2 x\,dx \qquad \text{...(i)}$$

$$\text{Let} \quad I = \int_{0}^{\pi/2} \sin^2 x\,dx \qquad \text{...(ii)}$$

$$\text{Then,} \quad I = \int_{0}^{\pi/2} \sin^2\left(\frac{\pi}{2} - x\right)dx = \int_{0}^{\pi/2} \cos^2 x\,dx \qquad \text{...(iii)}$$

Adding (ii) and (iii), we obtain

$$2I = \int_{0}^{\pi/2} \sin^2 x\,dx + \int_{0}^{\pi/2} \cos^2 x\,dx = \int_{0}^{\pi/2} (\sin^2 x + \cos^2 x)\,dx = \int_{0}^{\pi/2} 1\,.\,dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4} \Rightarrow \int_{0}^{\pi/2} \sin^2 x\,dx = \frac{\pi}{4} \qquad \text{...(iv)}$$

From (i) and (iv), we obtain $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_{0}^{\pi/2} \sin^2 x\,dx = 2 \times \frac{\pi}{4} = \frac{\pi}{2}$.

EXAMPLE 2 *Evaluate:* $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$

SOLUTION Let $f(x) = |\sin x|$. Then, $f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$.

So, $f(x)$ is an even function.

$$\therefore \quad I = \int_{-\pi/2}^{\pi/2} |\sin x|\,dx = 2\int_{0}^{\pi/2} |\sin x|\,dx = 2\int_{0}^{\pi/2} \sin x\,dx \qquad \left[\because \sin x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{2}\right]$$

$$\Rightarrow \quad I = 2\left[-\cos x\right]_0^{\pi/2} = 2\left\{-\cos\frac{\pi}{2} + \cos 0\right\} = 2$$

EXAMPLE 3 *Evaluate:*

(i) $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx$ (ii) $\displaystyle\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$ [CBSE 2002, 2008]

SOLUTION (i) Let $f(x) = x^3 \sin^4 x$. We find that

$$f(-x) = (-x)^3 \sin^4(-x) = -x^3\{\sin(-x)\}^4 = -x^3(-\sin x)^4 = -x^3 \sin^4 x = -f(x).$$

So, $f(x)$ is an odd function.

Hence, $\displaystyle\int_{-\pi/4}^{\pi/4} f(x)\, dx = 0$ i.e. $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \sin^4 x\, dx = 0$.

(ii) Let $I = \displaystyle\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\, dx$. Then,

$$I = \int_{-a}^{a} \sqrt{\frac{a-x}{a+x} \times \frac{a-x}{a-x}}\, dx = \int_{-a}^{a} \frac{a-x}{\sqrt{a^2-x^2}}\, dx = \int_{-a}^{a} \frac{a}{\sqrt{a^2-x^2}}\, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\, dx$$

$$\Rightarrow \quad I = a\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\, dx - \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\, dx = a I_1 - I_2 \quad \ldots(i)$$

where $I_1 = \displaystyle\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\, dx$ and $I_2 = \displaystyle\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}}\, dx$

Let $f(x) = \dfrac{1}{\sqrt{a^2-x^2}}$ and, $g(x) = \dfrac{x}{\sqrt{a^2-x^2}}$. Then,

$$f(-x) = \frac{1}{\sqrt{a^2-(-x)^2}} = \frac{1}{\sqrt{a^2-x^2}} = f(x) \text{ and, } g(-x) = \frac{-x}{\sqrt{a^2-(-x)^2}} = \frac{-x}{\sqrt{a^2-x^2}} = -g(x)$$

$\Rightarrow$ $f(x)$ is an even function and $g(x)$ is an odd function.

$$\therefore \quad I_1 = 2\int_0^a \frac{1}{\sqrt{a^2-x^2}}\, dx \text{ and } I_2 = 0 \Rightarrow I_1 = 2\left[\sin^{-1}\frac{x}{a}\right]_0^a \text{ and } I_2 = 0$$

$$I_1 = 2\left[\sin^{-1} 1 - \sin^{-1} 0\right] = 2\left[\frac{\pi}{2} - 0\right] = \pi \text{ and } I_2 = 0$$

Substituting the values of I_1 and I_2 in (i), we get: $I = I_1 - I_2 = a\pi - 0 = a\pi$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Evaluate:* $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x + \pi/4}{2 - \cos 2x}\, dx$

SOLUTION Let $I = \displaystyle\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x}\, dx$. Then,

$$I = \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx + \frac{\pi}{4}\int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx \qquad \ldots\text{(i)}$$

We observe that $\frac{x}{2-\cos 2x}$ is an odd function and $\frac{1}{2-\cos 2x}$ is an even function.

$$\therefore \quad \int_{-\pi/4}^{\pi/4} \frac{x}{2-\cos 2x}\,dx = 0 \text{ and, } \int_{-\pi/4}^{\pi/4} \frac{1}{2-\cos 2x}\,dx = 2\int_0^{\pi/4} \frac{1}{2-\cos 2x}\,dx.$$

Substituting these values in (i), we obtain

$$I = 0 + 2\left(\frac{\pi}{4}\right)\int_0^{\pi/4} \frac{1}{2-\cos 2x}\,dx = \frac{\pi}{2}\int_0^{\pi/4} \frac{1+\tan^2 x}{1+3\tan^2 x}\,dx = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{1^2+(\sqrt{3}\tan x)^2}\,dx$$

Let $\sqrt{3}\tan x = t$. Then, $dt = d(\sqrt{3}\tan x) = \sqrt{3}\sec^2 x\,dx$.

Also, $x = 0 \Rightarrow t = \sqrt{3}\tan 0 = 0$ and, $x = \frac{\pi}{4} \Rightarrow t = \sqrt{3}\tan\frac{\pi}{4} = \sqrt{3}$.

$$\therefore \quad I = \frac{\pi}{2\sqrt{3}}\int_0^{\sqrt{3}} \frac{1}{1^2+t^2}\,dt = \frac{\pi}{2\sqrt{3}}\left[\tan^{-1} t\right]_0^{\sqrt{3}} = \frac{\pi}{2\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1} 0\right) = \frac{\pi}{2\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi^2}{6\sqrt{3}}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Evaluate:* $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

SOLUTION Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$. Then,

$$I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx = I_1 + I_2, \text{ where}$$

$$I_1 = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}\,dx \text{ and } I_2 = \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx$$

We observe that $f(x) = \frac{2x}{1+\cos^2 x}$ is an odd function and $g(x) = \frac{2x\sin x}{1+\cos^2 x}$ is an even function.

$$\therefore \quad I_1 = \int_{-\pi}^{\pi} f(x)\,dx = 0 \text{ and, } I_2 = \int_{-\pi}^{\pi} g(x)\,dx = 2\int_0^{\pi} g(x)\,dx = 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}\,dx.$$

Now, $$I_2 = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \qquad \text{[Using Property IV]}$$

$$\Rightarrow \quad I_2 = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\Rightarrow \quad I_2 = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I_2$$

$$\Rightarrow \quad 2I_2 = 4\pi\int_0^{\pi} \frac{1}{1+\cos^2 x}\sin x\,dx = -4\pi\int_1^{-1} \frac{1}{1+t^2}\,dt, \text{ where } t = \cos x$$

$$\Rightarrow \quad 2I_2 = -4\pi\left[\tan^{-1} t\right]_1^{-1} = -4\pi\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = 2\pi^2$$

$$\Rightarrow \quad I_2 = \pi^2$$

Hence, $I = I_1 + I_2 = 0 + \pi^2 = \pi^2$.

EXAMPLE 6 *Evaluate* : $\displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$ [NCERT EXEMPLAR]

SOLUTION Let $I = \displaystyle\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}\,dx$. Then,

$$I = \int_{-1}^{1} \left\{\frac{x^3}{x^2+2|x|+1} + \frac{|x|+1}{x^2+2|x|+1}\right\} dx = \int_{-1}^{1} \frac{x^3}{x^2+2|x|+1}\,dx + \int_{-1}^{1} \frac{|x|+1}{x^2+2|x|+1}\,dx$$

$$\Rightarrow \quad I = \int_{-1}^{1} \frac{x^3}{|x|^2+2|x|+1}\,dx + \int_{-1}^{1} \frac{|x|+1}{|x|^2+2|x|+1}\,dx = \int_{-1}^{1} \frac{x^3}{\left(|x|+1\right)^2}\,dx + \int_{-1}^{1} \frac{|x|+1}{\left(|x|+1\right)^2}\,dx$$

$$\Rightarrow \quad I = 0 + 2\int_0^1 \frac{|x|+1}{\left(|x|+1\right)^2}\,dx \quad \left[\because \frac{x^3}{\left(|x|+1\right)^2} \text{ and } \frac{|x|+1}{\left(|x|+1\right)^2} \text{ are odd and even functions respectively}\right]$$

$$\Rightarrow \quad I = 2\int_0^1 \frac{1}{|x|+1}\,dx = 2\int_0^1 \frac{1}{x+1}\,dx = 2\left[\log(x+1)\right]_0^1 = 2(\log 2 - \log 1) = 2\log 2$$

EXAMPLE 7 *Evaluate* : $\displaystyle\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\,dx$. [CBSE 2015]

SOLUTION Let $I = \displaystyle\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\,dx$. Then,

$$I = \int_{-\pi}^{\pi} (\cos^2 ax + \sin^2 bx - 2\cos ax \sin bx)\,dx$$

$$\Rightarrow \quad I = \int_{-\pi}^{\pi} \cos^2 ax\,dx + \int_{-\pi}^{\pi} \sin^2 bx\,dx - 2\int_{-\pi}^{\pi} \cos ax \sin bx\,dx$$

$$\Rightarrow \quad I = 2\int_0^{\pi} \cos^2 ax\,dx + 2\int_0^{\pi} \sin^2 bx\,dx - 2\times 0 \quad \left[\begin{array}{l}\because \cos^2 ax \text{ and } \sin^2 bx \text{ are even functions} \\ \text{and } \cos ax \sin bx \text{ is an odd function}\end{array}\right]$$

$$\Rightarrow \quad I = \int_0^{\pi} (1+\cos 2ax)\,dx + \int_0^{\pi} (1-\cos 2bx)\,dx = \left[x + \frac{1}{2a}\sin 2ax\right]_0^{\pi} + \left[x - \frac{1}{2b}\sin 2bx\right]_0^{\pi}$$

$$\Rightarrow \quad I = \left(\pi + \frac{1}{2a}\sin 2a\pi - 0\right) + \left(\pi - \frac{1}{2b}\sin 2b\pi - 0\right) = 2\pi + \frac{1}{2a}\sin 2a\pi - \frac{1}{2b}\sin 2b\pi$$

$$\Rightarrow \quad I = \begin{cases} 2\pi + \dfrac{1}{2a\pi}\times 0 - \dfrac{1}{2b\pi}\times 0 = 2\pi, & \text{if } a \text{ and } b \text{ are integers} \\ 2\pi + \dfrac{1}{2a}\sin 2a\pi - \dfrac{1}{2b}\sin 2b\pi, & \text{if } a, b \text{ are not integers} \end{cases}$$

EXAMPLE 8 *Evaluate* $\displaystyle\int_{-1}^{3/2} |x \sin \pi x|\, dx.$ **[NCERT]**

SOLUTION We have, $-1 < x < \dfrac{3}{2} \Rightarrow -\pi < \pi x < \dfrac{3\pi}{2}$

Now,

$$-1 < x < 0 \Rightarrow -\pi < \pi x < 0 \Rightarrow \sin \pi x < 0 \Rightarrow x \sin \pi x > 0 \Rightarrow |x \sin \pi x| = x \sin \pi x$$

$$0 < x < 1 \Rightarrow 0 < \pi x < \pi \Rightarrow \sin \pi x > 0 \Rightarrow x \sin \pi x > 0 \Rightarrow |x \sin \pi x| = x \sin \pi x$$

and, $1 < x < \dfrac{3}{2} \Rightarrow \pi < \pi x < \dfrac{3\pi}{2} \Rightarrow \sin \pi x < 0 \Rightarrow x \sin \pi x < 0 \Rightarrow |x \sin \pi x| = -x \sin \pi x$

Thus, $|x \sin \pi x| = \begin{cases} x \sin \pi x, & \text{if } -1 < x \le 1 \\ -x \sin \pi x, & \text{if } 1 < x < \dfrac{3}{2} \end{cases}$

Using additive property, we obtain

$$\therefore \quad I = \int_{-1}^{3/2} |x \sin \pi x|\, dx = \int_{-1}^{1} |x \sin \pi x|\, dx + \int_{1}^{3/2} |x \sin \pi x|\, dx$$

$$\Rightarrow \quad I = \int_{-1}^{1} x \sin \pi x\, dx + \int_{1}^{3/2} (-x \sin \pi x)\, dx = \int_{-1}^{1} x \sin \pi x\, dx - \int_{1}^{3/2} x \sin \pi x\, dx$$

$$\Rightarrow \quad I = 2\int_{0}^{1} x \sin \pi x\, dx - \int_{1}^{3/2} x \sin \pi x\, dx \qquad [\because\ x \sin \pi x \text{ is an even function}]$$

Now, $\displaystyle\int \underset{\text{I}}{x} \underset{\text{II}}{\sin \pi x}\, dx = -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi}\int \cos \pi x\, dx = -\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x$

$$\therefore \quad I = 2\left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_0^1 - \left[-\frac{x}{\pi}\cos \pi x + \frac{1}{\pi^2}\sin \pi x\right]_1^{3/2}$$

$$\Rightarrow \quad I = 2\left\{\left(-\frac{1}{\pi}\cos \pi + \frac{1}{\pi^2}\sin \pi\right) - (0)\right\} - \left\{\left(-\frac{3}{2\pi}\cos\frac{3\pi}{2} + \frac{1}{\pi^2}\sin\frac{3\pi}{2}\right) - \left(-\frac{1}{\pi}\cos \pi + \frac{1}{\pi^2}\sin \pi\right)\right\}$$

$$\Rightarrow \quad I = 2\left\{\left(\frac{1}{\pi} + 0\right)\right\} - \left\{\left(0 + \frac{1}{\pi^2}(-1)\right) - \left(\frac{1}{\pi} + 0\right)\right\} = \frac{2}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi} = \frac{3\pi + 1}{\pi^2}$$

EXAMPLE 9 *Evaluate* : $\displaystyle\int_{-2}^{2} |x \cos \pi x|\, dx.$

SOLUTION Let $I = \displaystyle\int_{-2}^{2} |x \cos \pi x|\, dx$ and $f(x) = |x \cos \pi x|$. Then,

$f(-x) = |-x \cos(-\pi x)| = |-x \cos \pi x| = |x \cos \pi x| = f(x)$. So, $f(x)$ is an even function.

$$\therefore \quad I = \int_{-2}^{2} |x \cos \pi x|\, dx = 2\int_{0}^{2} |x \cos \pi x|\, dx \quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ if } f(-x) = f(x)\right]$$

$$\text{Now, } \quad f(x) = |x \cos \pi x| = \begin{cases} x \cos \pi x, & \text{if } 0 \le x \le \frac{1}{2} \\ -x \cos \pi x, & \text{if } \frac{1}{2} < x < \frac{3}{2} \\ x \cos \pi x, & \text{if } \frac{3}{2} \le x \le 2 \end{cases}$$

$$\therefore \quad I = 2\int_{0}^{2} |x \cos \pi x|\, dx$$

$$\Rightarrow \quad I = 2\left\{\int_{0}^{1/2} x \cos \pi x\, dx + \int_{1/2}^{3/2} -x \cos \pi x\, dx + \int_{3/2}^{2} x \cos \pi x\, dx\right\} \text{ [Using additive property]}$$

$$\Rightarrow \quad I = 2\left\{\left[\frac{x}{\pi}\sin \pi x + \frac{1}{\pi^2}\cos \pi x\right]_0^{1/2} - \left[\frac{x}{\pi}\sin \pi x + \frac{1}{\pi^2}\cos \pi x\right]_{1/2}^{3/2} + \left[\frac{x}{\pi}\sin \pi x + \frac{1}{\pi^2}\cos \pi x\right]_{3/2}^{2}\right\}$$

$$\Rightarrow \quad I = 2\left[\left\{\left(\frac{1}{2\pi} + 0\right) - \left(0 + \frac{1}{\pi^2}\right)\right\} - \left\{\left(-\frac{3}{2\pi} + 0\right) - \left(\frac{1}{2\pi} + 0\right) + \left(0 + \frac{1}{\pi^2}\right) - \left(-\frac{3}{2\pi} + 0\right)\right\}\right]$$

$$\Rightarrow \quad I = 2\left\{\left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) + \left(\frac{3}{2\pi} + \frac{1}{2\pi}\right) + \left(\frac{1}{\pi^2} + \frac{3}{2\pi}\right)\right\} = \frac{8}{\pi}$$

19.4.10 PROPERTY X

STATEMENT *If $f(x)$ is a continuous function defined on $[0, 2a]$, then*

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

PROOF Using property VI, we obtain

$$\int_0^{2a} f(x)\,dx = \int_0^a \left\{f(x) + f(2a - x)\right\} dx = \begin{cases} \int_0^a \left\{f(x) + f(x)\right\} dx, & \text{if } f(2a - x) = f(x) \\ \int_0^a \left\{f(x) - f(x)\right\} dx, & \text{if } f(2a - x) = -f(x) \end{cases}$$

$$\Rightarrow \quad \int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a - x) = f(x) \\ 0, & \text{if } f(2a - x) = -f(x) \end{cases}$$

Q.E.D.

REMARK *If $f(2a - x) = f(x)$, then the graph of $f(x)$ is symmetrical about $x = a$ as shown in Fig.* 19.12.

$$\therefore \quad \int_a^{2a} f(x)\,dx = \int_0^a f(x)\,dx$$

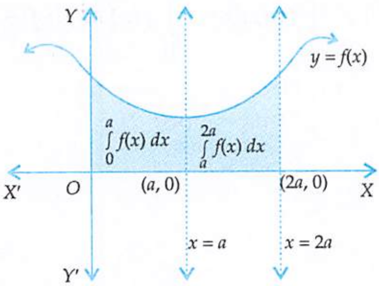

Fig. 19.12

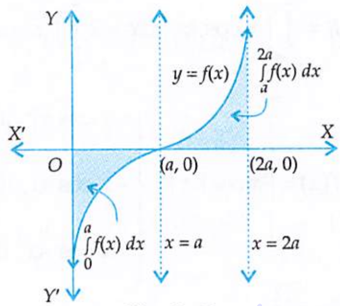

Fig. 19.13

If $f(2a-x)=-f(x)$, then the graph of $f(x)$ is as shown in Fig. 19.13.

$$\therefore \quad \int_0^a f(x)\,dx = -\int_a^{2a} f(x)\,dx.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate* : $\int_0^{2\pi} \cos^5 x\,dx$. [NCERT]

SOLUTION Let $f(x)=\cos^5 x$. Then, $f(2\pi-x)=\left\{\cos(2\pi-x)\right\}^5=\cos^5 x$

$$\therefore \quad \int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx \qquad \text{[Using Property X]}$$

Now, $f(\pi-x)=\left\{\cos(\pi-x)\right\}^5=-\cos^5 x=-f(x)$

$$\therefore \quad \int_0^{\pi} \cos^5 x\,dx = 0. \qquad \text{[Using Property X]}$$

Hence, $\int_0^{2\pi} \cos^5 x\,dx = 2\int_0^{\pi} \cos^5 x\,dx = 2\times 0 = 0.$

EXAMPLE 2 *Prove that* : $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -\frac{\pi}{2}\log 2.$ [NCERT, CBSE 2008]

SOLUTION Let $I=\int_0^{\pi/2} \log\sin x\,dx.$ ...(i)

Using $\int_0^a f(x)\,dx=\int_0^a f(a-x)$, we obtain

$$I=\int_0^{\pi/2} \log\sin\left(\frac{\pi}{2}-x\right)dx=\int_0^{\pi/2} \log\cos x\,dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I=\int_0^{\pi/2} \log\sin x\,dx+\int_0^{\pi/2} \log\cos x\,dx=\int_0^{\pi/2} (\log\sin x+\log\cos x)\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin x \cos x)\,dx = \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx = \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right)dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x\,dx - \int_0^{\pi/2} \log 2\,dx = \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}(\log 2)$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log \sin 2x\,dx - \frac{\pi}{2}\log 2. \qquad \text{...(iii)}$$

Let $I_1 = \int_0^{\pi/2} \log \sin 2x\,dx$. Putting $2x = t$, we get

$$I_1 = \int_0^{\pi} \log \sin t\,\frac{dt}{2} = \frac{1}{2}\int_0^{\pi} \log \sin t\,dt = \frac{1}{2}\times 2\int_0^{\pi/2} \log \sin t\,dt \qquad \text{[Using property X]}$$

$$\Rightarrow \quad I_1 = \int_0^{\pi/2} \log \sin x\,dx = I \qquad \text{[Using Property I]}$$

Putting $I_1 = I$ in (iii), we get: $2I = I - \frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{2}\log 2$

Hence, $\int_0^{\pi/2} \log \sin x\,dx = \int_0^{\pi/2} \log \cos x\,dx = -\frac{\pi}{2}\log 2$

EXAMPLE 3 *Prove that:* $\int_0^{\pi/2} \log|\tan x + \cot x|\,dx = \pi \log_e 2$

SOLUTION Let $I = \int_0^{\pi/2} \log|\tan x + \cot x|\,dx$. Then,

$$I = \int_0^{\pi/2} \log\left|\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right|dx = \int_0^{\pi/2} \log\left|\frac{1}{\sin x\cos x}\right|dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \log\left(\frac{1}{\sin x\cos x}\right)dx \qquad [\because \sin x > 0, \cos x > 0 \text{ for all } x \in (0, \pi/2)]$$

$$\Rightarrow \quad I = -\int_0^{\pi/2} \log(\sin x\cos x)\,dx = -\int_0^{\pi/2} \log \sin x\,dx - \int_0^{\pi/2} \log \cos x\,dx$$

$$\Rightarrow \quad I = -\left(-\frac{\pi}{2}\log_e 2\right) - \left(-\frac{\pi}{2}\log_e 2\right) = \pi \log_e 2 \qquad \text{[See Example 2]}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Evaluate:* $\int_0^{\pi} \log(1+\cos x)\,dx$. **[NCERT]**

SOLUTION Let $I = \int_0^{\pi} \log(1+\cos x)\,dx$. ...(i)

Using $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi} \log\{1 + \cos(\pi - x)\}\, dx = \int_0^{\pi} \log(1 - \cos x)\, dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} \{\log(1 + \cos x) + \log(1 - \cos x)\}\, dx = \int_0^{\pi} \log(1 - \cos^2 x)\, dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx$$

$$\Rightarrow \quad I = 2\int_0^{\pi/2} \log \sin x\, dx \qquad \text{[Using property X]}$$

$$\Rightarrow \quad I = 2 \times -\frac{\pi}{2} \log_e 2 = -\pi \log_e 2 \qquad \left[\because \int_0^{\pi/2} \log \sin x\, dx = -\frac{\pi}{2} \log_e 2\right]$$

EXAMPLE 5 *Evaluate:* $\int_0^{\pi} x \log \sin x\, dx$

SOLUTION Let $I = \int_0^{\pi} x \log \sin x\, dx$. Using $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$, we obtain

$$I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x)\, dx$$

$$\Rightarrow \quad I = \int_0^{\pi} (\pi - x) \log \sin x\, dx = \pi \int_0^{\pi} \log \sin x\, dx - \int_0^{\pi} x \log \sin x\, dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi/2} \log \sin x\, dx - I \qquad \text{[Using Property X]}$$

$$\Rightarrow \quad I = 2\pi \times -\frac{\pi}{2} \log 2 - I \qquad \text{[See Example 2]}$$

$$\Rightarrow \quad 2I = -\pi^2 \log 2 \Rightarrow I = -\frac{\pi^2}{2} \log 2$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *Evaluate:* $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$ **[NCERT EXEMPLAR]**

SOLUTION Let $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$. Then,

$$I = \int_{-\pi/4}^{\pi/4} \log\left\{\sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right)\right\} dx = \int_{-\pi/4}^{\pi/4} \log\left\{\sqrt{2}\left(\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right)\right\} dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log\left\{\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)\right\} dx = \int_{-\pi/4}^{\pi/4} \left\{\log \sqrt{2} + \log \sin\left(x + \frac{\pi}{4}\right)\right\} dx$$

$$\Rightarrow \quad I = \int_{-\pi/4}^{\pi/4} \log \sqrt{2}\, dx + \int_{-\pi/4}^{\pi/4} \log \sin\left(x + \frac{\pi}{4}\right) dx$$

$$\Rightarrow \quad I = \left(\frac{1}{2}\log 2\right)\left[x\right]_{-\pi/4}^{\pi/4} + \int_0^{\pi/2} \log \sin t\, dt, \text{ where } t = x + \frac{\pi}{4}$$

$$\Rightarrow \quad I = \left(\frac{1}{2}\log 2\right)\left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \frac{\pi}{2}\log 2 = -\frac{\pi}{4}\log 2 \qquad \left[\because \int_0^{\pi/2} \log \sin t\, dt = -\frac{\pi}{2}\log 2\right]$$

ALITER Let $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$ Then,

$$I = \int_0^{\pi/4} \left\{\log(\sin x + \cos x) + \log(\sin(-x) + \cos(-x)\right\} dx \qquad \text{[Using Prop. VIII]}$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \left\{\log(\sin x + \cos x) + \log(\cos - \sin x)\right\} dx$$

$$\Rightarrow \quad I = \int_0^{\pi/4} \log(\cos^2 x - \sin^2 x)\, dx = \int_0^{\pi/4} \log \cos 2x\, dx = \frac{1}{2}\int_0^{\pi/2} \log \cos t\, dt, \text{ where } t = 2x$$

$$\Rightarrow \quad I = \frac{1}{2} \times -\frac{\pi}{2}\log 2 \Rightarrow I = -\frac{\pi}{4}\log 2$$

EXAMPLE 7 *For $x > 0$, let $f(x) = \int_1^x \frac{\log_e t}{1+t}\, dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.*

SOLUTION We have,

$$f(x) = \int_1^x \frac{\log_e t}{1+t}\, dt \qquad \text{...(i)}$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log_e t}{1+t}\, dt$$

Let $t = \frac{1}{u}$. Then, $dt = -\frac{1}{u^2}\, du$. Also, $t = 1 \Rightarrow u = 1$ and, $t = \frac{1}{x} \Rightarrow \frac{1}{u} = \frac{1}{x} \Rightarrow u = x$

$$\therefore \quad f\left(\frac{1}{x}\right) = \int_1^x \frac{\log_e(1/u)}{1 + \frac{1}{u}} \times \frac{-1}{u^2}\, du = \int_1^x \frac{\log_e u}{(1+u)\,u}\, du = \int_1^x \frac{\log_e t}{(1+t)\,t}\, dt \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$f(x) + f\left(\frac{1}{x}\right) = \int_1^x \left\{\frac{\log_e t}{1+t} + \frac{\log_e t}{(1+t)\,t}\right\} dt = \int_1^x \frac{\log_e t}{1+t}\left(\frac{1+t}{t}\right) dt = \int_1^x \frac{\log_e t}{t}\, dt$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \int_0^{\log_e x} v\, dv, \text{ where } v = \log_e t \text{ and } \frac{1}{t}\, dt = dv$$

$$\Rightarrow \quad f(x) + f\left(\frac{1}{x}\right) = \left[\frac{v^2}{2}\right]_0^{\log_e x} = \frac{1}{2}(\log_e x)^2 - 0 = \frac{1}{2}\left(\log_e x\right)^2$$

Putting $x = e$, we get

$$f(e) + f\left(\frac{1}{e}\right) = \frac{(\log_e e)^2}{2} = \frac{1}{2}$$

EXAMPLE 8 *Show that:* $\int_0^{\pi/2} f(\sin 2x) \sin x \, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx.$

SOLUTION Let $I = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$ ...(i)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2} - x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} f\left\{\sin(\pi - 2x)\right\} \cos x \, dx = \int_0^{\pi/2} f(\sin 2x) \cos x \, dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x)\, dx = 2\int_0^{\pi/4} f(\sin 2x)(\sin x + \cos x)\, dx \quad \text{[Using Property X]}$$

$$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f(\sin 2x)\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right) dx = 2\sqrt{2} \int_0^{\pi/4} f(\sin 2x) \sin\left(x + \frac{\pi}{4}\right) dx$$

$$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f\left\{\sin 2\left(\frac{\pi}{4} - x\right)\right\} \sin\left(\frac{\pi}{4} - x + \frac{\pi}{4}\right) dx \quad \left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$\Rightarrow 2I = 2\sqrt{2} \int_0^{\pi/4} f\left\{\sin\left(\frac{\pi}{2} - 2x\right)\right\} \sin\left(\frac{\pi}{2} - x\right) dx = 2\sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$$

$$\Rightarrow I = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$$

Hence, $\int_0^{\pi/2} f(\sin 2x) \sin x \, dx = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx.$

EXAMPLE 9 *Prove that:* $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \pi^2$

SOLUTION Let $I = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$ ...(i)

Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we obtain

$$I = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n}(2\pi - x)}{\sin^{2n}(2\pi - x) + \cos^{2n}(2\pi - x)}\,dx = \int_0^{2\pi} \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} + \frac{(2\pi - x)\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx$$

$$\Rightarrow \quad I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad \text{[Using Property X]}$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx \quad \text{[Using Property X]} \quad \text{... (iii)}$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n}(\pi/2 - x)}{\sin^{2n}(\pi/2 - x) + \cos^{2n}(\pi/2 - x)}\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\Rightarrow \quad I = 4\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x}\,dx \quad \text{...(iv)}$$

Adding (iii) and (iv), we get

$$2I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} + \frac{\cos^{2n} x}{\sin^{2n} x + \cos^{2n} x}\,dx = 4\pi \int_0^{\pi/2} 1\,.\,dx = 4\pi \times \frac{\pi}{2}$$

$$\Rightarrow \quad I = \pi^2$$

EXAMPLE 10 *Prove that:* $\displaystyle\int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx$

SOLUTION Let $\displaystyle I = \int_0^{\pi} \frac{x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx$ ...(i)

Using $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we obtain

$$I = \int_0^{\pi} \frac{\pi - x}{\{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)\}^2}\,dx = \int_0^{\pi} \frac{\pi - x}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx \quad \text{...(ii)}$$

Adding (i) and (ii), we obtain

$$2I = \int_0^{\pi} \frac{\pi}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx \Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx$$

[Using Property X]

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{1}{(a^2\cos^2 x + b^2\sin^2 x)^2}\,dx$$

$$\Rightarrow \quad I = \pi \times \frac{\pi(a^2 + b^2)}{4a^3b^3} = \frac{\pi^2(a^2 + b^2)}{4a^3b^3}$$ [See Example 12 on page 19.28]

EXERCISE 19.4

BASIC

Evaluate the following integrals: (1-35)

1. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\tan x}$

2. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\cot x}\,dx$

3. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$

4. $\displaystyle\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x+\cos^{3/2} x}\,dx$ [NCERT]

5. $\displaystyle\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x+\cos^n x}\,dx$ [CBSE 2004]

6. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$ [CBSE 2015]

7. $\displaystyle\int_0^{a} \frac{1}{x+\sqrt{a^2-x^2}}\,dx$

8. $\displaystyle\int_0^{\infty} \frac{\log x}{1+x^2}\,dx$

9. $\displaystyle\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx$ [CBSE 2008]

10. $\displaystyle\int_0^{\infty} \frac{x}{(1+x)(1+x^2)}\,dx$

11. $\displaystyle\int_0^{\pi} \frac{x\tan x}{\sec x\,\operatorname{cosec} x}\,dx$ [CBSE 01, 07, 11, 2014]

12. $\displaystyle\int_0^{\pi} x\sin x\cos^4 x\,dx$

13. $\displaystyle\int_0^{\pi} x\sin^3 x\,dx$

14. $\displaystyle\int_0^{\pi} x\log\sin x\,dx$

15. $\displaystyle\int_0^{\pi} x\cos^2 x\,dx$

16. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\cot^{5/2} x}\,dx$ [CBSE 2022]

17. $\displaystyle\int_0^{\pi/2} \frac{\tan^7 x}{\tan^7 x+\cot^7 x}\,dx$

18. $\displaystyle\int_2^{8} \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx$

19. $\displaystyle\int_0^{\pi} x\sin x\cos^2 x\,dx$ [NCERT EXEMPLAR]

20. $\displaystyle\int_{-\pi/2}^{\pi/2} x\cos^2 x$ [CBSE 2020]

21. (i) $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^4 x\,dx$ (ii) $\displaystyle\int_{-\pi}^{\pi} (3\sin x-2)^2\,dx$ [CBSE 2022]

22. (i) $\displaystyle\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)dx$ (ii) $\displaystyle\int_{-\pi}^{\pi} (1-x^2)\sin x\cos^2 x\,dx$ [CBSE 2019] (iii) $\displaystyle\int_{-a}^{a} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right)d\theta,\ a>0$

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 x\,dx$

24. $\displaystyle\int_{-\pi/2}^{\pi/2} \log\left(\frac{2-\sin x}{2+\sin x}\right)dx$

25. $\displaystyle\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

26. $\displaystyle\int_0^{5} x\sqrt{5-x}\,dx$ [CBSE 2022]

27. $\displaystyle\int_0^{2} x\sqrt{2-x}\,dx$ [NCERT, CBSE 2007]

28. $\displaystyle\int_0^{1} \log\left(\frac{1}{x}-1\right)dx$ [CBSE 2011]

29. $\displaystyle\int_0^{2\pi} \sin^{100} x\cos^{101} x\,dx$

30. $\displaystyle\int_0^{\pi/2} \frac{a\sin x+b\cos x}{\sin x+\cos x}\,dx$

31. If $f(2a-x)=-f(x)$, prove that $\displaystyle\int_0^{2a} f(x)\,dx=0$.

32. If f is an integrable function, show that

    (i) $\displaystyle\int_{-a}^{a} f(x^2)\,dx=2\int_0^{a} f(x^2)\,dx$

    (ii) $\displaystyle\int_{-a}^{a} x f(x^2)\,dx=0$

33. $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx$ [NCERT]

34. $\displaystyle\int_0^{\pi} \frac{x}{1+\cos\alpha\sin x}\,dx,\ 0<\alpha<\pi$

35. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\, dx$ 36. $\int_0^{\pi} \log(1 - \cos x)\, dx$ 37. $\int_{-2}^{2} \frac{3x^3 + 2|x| + 1}{x^2 + |x| + 1}\, dx$

38. (i) $\int_{-3\pi/2}^{-\pi/2} \left\{\sin^2(3\pi + x) + (\pi + x)^3\right\} dx$ (ii) $\int_0^1 \tan^{-1}\left(\frac{1-2x}{1+x-x^2}\right) dx$ [CBSE 2020]

39. $\int_{-1}^{1} |x \cos \pi x|\, dx$ [NCERT EXEMPLAR] 40. $\int_0^{\pi} \left(\frac{x}{1+\sin^2 x} + \cos^7 x\right) dx$

41. $\int_0^{\pi} \frac{x}{1 + \sin \alpha \sin x}\, dx$ [CBSE 2016] 42. $\int_0^{3/2} |x \cos \pi x|\, dx$ [CBSE 2016]

43. $\int_0^1 |x \sin \pi x|\, dx$ [CBSE 2017] 44. $\int_0^{3/2} |x \sin \pi x|\, dx$ [CBSE 2017]

45. If f is an integrable function such that $f(2a - x) = f(x)$, then prove that

$$\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$$

46. If $f(x)$ is a continuous function defined on $[0, 2a]$. Then, prove that

$$\int_0^{2a} f(x)\, dx = \int_0^{a} \{f(x) + f(2a - x)\}\, dx$$

47. If $f(a + b - x) = f(x)$, then prove that $\int_a^b x\, f(x)\, dx = \left(\frac{a+b}{2}\right)\int_a^b f(x)\, dx.$

48. If $f(x)$ is a continuous function defined on $[-a, a]$, then prove that

$$\int_{-a}^{a} f(x)\, dx = \int_0^{a} \{f(x) + f(-x)\}\, dx$$

49. Prove that: $\int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\, dx$

50. Prove that $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$, and hence evaluate $\int_0^1 x^2 (1 - x)^n\, dx$ [CBSE 2019]

ANSWERS

1. $\frac{\pi}{4}$	2. $\frac{\pi}{4}$	3. $\frac{\pi}{4}$	4. $\frac{\pi}{4}$	5. $\frac{\pi}{4}$	6. $\frac{\pi}{4}$
7. $\frac{\pi}{4}$	8. 0	9. $\frac{\pi}{8} \log 2$	10. $\frac{\pi}{4}$	11. $\frac{\pi^2}{4}$	12. $\frac{\pi}{5}$
13. $\frac{2\pi}{3}$	14. $-\frac{\pi^2}{2} \log 2$	15. $\frac{\pi^2}{4}$	16. $\frac{\pi}{12}$	17. $\frac{\pi}{4}$	18. 3
19. $\frac{\pi}{3}$	20. 0	21. (i) $\frac{3\pi}{8}$ (ii) 12π		22. (i) 0 (ii) 0 (iii) 0	
23. $\frac{\pi}{4} - \frac{1}{2}$	24. 0	25. π^2	26. $\frac{20\sqrt{5}}{3}$	27. $\frac{16\sqrt{2}}{15}$	28. 0
29. 0	30. $\frac{\pi}{4}(a+b)$	33. $\pi\left(\frac{\pi}{2} - 1\right)$	34. $\frac{\pi \alpha}{\sin \alpha}$	35. $\frac{\pi^2}{16}$	36. $-\pi \log 2$

37. $2\log_e 7$ 38. (i) $\frac{\pi}{2}$ (ii) 0 39. $\frac{2}{\pi}$ 40. $\frac{\pi^2}{2\sqrt{2}}$ 41. $\frac{\pi(\pi/2-\alpha)}{\cos\alpha}$ 42. $\frac{1}{\pi}\left(\frac{5}{2}-\frac{1}{\pi}\right)$

43. $\frac{1}{\pi}$ 44. $\frac{2}{\pi}+\frac{1}{\pi^2}$ 50. $\frac{2}{(n+1)(n+2)(n+3)}$

19.5 INTEGRATION AS THE LIMIT OF A SUM

In this section, we shall consider integration as the limit of the sum of certain number of terms when the number of terms tends to infinity and each term tends to zero. As a matter of fact the summation aspect of definite integral is more fundamental and it was invented far before the differentiation was known.

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n equal parts each of width h by inserting $(n-1)$ points $a+h, a+2h, a+3h, \ldots, a+(n-1)h$ between a and b as shown in Fig. 19.14. Then,

$$nh = b-a \text{ or, } h = \frac{b-a}{n}$$

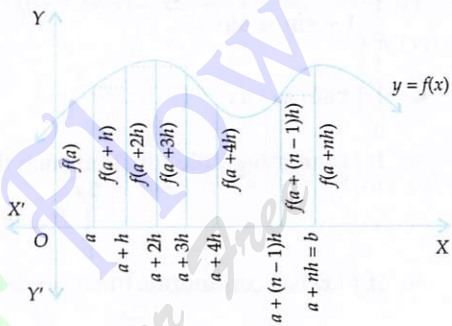

Fig. 19.14

Let S_n denote the sum of the areas of n rectangles shown in Fig. 19.14. Then,

$$S_n = h\cdot f(a) + h\cdot f(a+h) + h\cdot f(a+2h) + \ldots\ldots\ldots + h\cdot f(a+(n-1)h)$$

$$\Rightarrow \quad S_n = h\,[f(a) + f(a+h) + f(a+2h) + \ldots\ldots\ldots + f(a+(n-1)h)]$$

Clearly, S_n denotes the area which is close to the area of the region bounded the curve $y=f(x)$, x-axis and the ordinates $x=a, x=b$. It is evident that if n increases, the number of rectangles will increase and the width of rectangles will decrease. Consequently, S_n gives closer approximation to the area enclosed by the curve $y=f(x)$, x-axis and the ordinates $x=a, x=b$.

Thus, $\lim\limits_{n\to\infty} S_n$ gives the area of the region bounded by the curve $y=f(x)$, $y=0$ (x-axis), and ordinates $x=a$ and $x=b$. It can be proved that this limit exists for all continuous functions defined on closed integral $[a, b]$ and is defined as the definite integral of $f(x)$ over $[a, b]$.

$$\therefore \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} S_n$$

$$\text{or,} \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right], \text{ where } h=\frac{b-a}{n}$$

$$\text{or,} \quad \int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)h)\right] \quad \ldots\text{(i)}$$

$$[\because n\to\infty \Leftrightarrow h\to 0]$$

The process of evaluating a definite integral by using the above definition is called *integration from first principles or integration by ab-initio method or integration as the limit of a sum.*

REMARK *While finding S_n in the above discussion, we have taken the left end points of the sub-intervals. We can also take the right end-points of the subintervals throughout to obtain:*

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a+h) + f(a+2h) + f(a+3h) + \ldots + f(a+nh)\right], \quad h=\frac{b-a}{n}$$

It can be proved that this formula and the formula (i) give the same limit.

Following results will be helpful in evaluating definite integrals as limit of sums:

(i) $1+2+3+\ldots+(n-1) = \dfrac{n(n-1)}{2}$

(ii) $1^2+2^2+3^2+\ldots(n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

(iii) $1^3+2^3+3^3+\ldots+(n-1)^3 = \left\{\dfrac{n(n-1)}{2}\right\}^2$

(iv) $a+ar+ar^2+\ldots+ar^{n-1} = a\left(\dfrac{r^n-1}{r-1}\right),\ r\neq 1$

(v) $\sin a+\sin(a+h)+\sin(a+2h)+\ldots+\sin(a+(n-1)h) = \dfrac{\sin\left\{a+\left(\dfrac{n-1}{2}\right)h\right\}\ \sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}$

(vi) $\cos a+\cos(a+h)+\cos(a+2h)+\ldots+\cos(a+(n-1)h) = \dfrac{\cos\left\{a+\left(\dfrac{n-1}{2}\right)h\right\}\ \sin\left(\dfrac{nh}{2}\right)}{\sin\left(\dfrac{h}{2}\right)}$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 (x+4)\,dx$ (ii) $\int_0^2 (2x+1)\,dx$

SOLUTION (i) We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\Big[f(a)+f(a+h)+f(a+2h)+\ldots+f(a+(n-1)h)\Big],\ \text{where } h=\frac{b-a}{n}$$

Here, $a=0, b=2, f(x)=x+4$ and $h=\dfrac{2-0}{n}=\dfrac{2}{n}$

$$\therefore\quad I = \int_0^2 (x+4)\,dx$$

$$\Rightarrow\quad I = \lim_{h\to 0} h\Big[f(0)+f(0+h)+f(0+2h)+\ldots\ldots+f(0+(n-1)h)\Big]$$

$$\Rightarrow\quad I = \lim_{h\to 0} h\Big[(0+4)+(h+4)+(2h+4)+\ldots\ldots+((n-1)h+4)\Big]$$

$$\Rightarrow\quad I = \lim_{h\to 0} h\Big[4n+h\ (1+2+3+\ldots\ldots+(n-1))\Big]$$

$$\Rightarrow\quad I = \lim_{h\to 0} h\left\{4n+h\ \frac{n(n-1)}{2}\right\}$$

$$\Rightarrow \quad I = \lim_{n\to\infty} \frac{2}{n}\left\{4n + \frac{2}{n} \times \frac{n(n-1)}{2}\right\} \qquad \left[\because h = \frac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$$

$$\Rightarrow \quad I = \lim_{n\to\infty} \left\{8 + 2\left(1 - \frac{1}{n}\right)\right\} = 8 + 2(1-0) = 10$$

(ii) We have,

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\ h)\right], \text{ where } h = \frac{b-a}{n}$$

Here $\quad a = 0, b = 2, f(x) = 2x+1$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \quad I = \int_0^2 (2x+1)\,dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(0) + f(0+h) + f(0+2h) + f(0+3h) + \ldots\ldots + f(0+(n-1)\,h)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(0) + f(h) + f(2h) + f(3h) + \ldots\ldots + f((n-1)\,h)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[(0+1) + (2h+1) + (2\times 2h+1) + (2\times 3h+1) + \ldots\ldots + (2(n-1)\ h+1)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[n + 2h(1+2+3+\ldots\ldots+(n-1))\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left\{n + 2h \times \frac{n(n-1)}{2}\right\} = \lim_{n\to 0} h\left\{n + nh(n-1)\right\}$$

$$\Rightarrow \quad I = \lim_{n\to\infty} \frac{2}{n}\left\{n + n \times \frac{2}{n}\ (n-1)\right\} \qquad \left[\because h = \frac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$$

$$\Rightarrow \quad I = \lim_{n\to 0} \left\{2 + 4\left(\frac{n-1}{n}\right)\right\} = \lim_{n\to\infty} \left\{2 + 4\left(1 - \frac{1}{n}\right)\right\} = 2 + 4 = 6$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Evaluate the following integrals as limit of sums:*

(i) $\displaystyle\int_1^3 (2x+1)\,dx$ (ii) $\displaystyle\int_2^4 (2x-1)\,dx$

SOLUTION (i) We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a) + (n-h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $\quad a = 1,\ b = 3,\ f(x) = 2x+1$ and $h = \dfrac{3-1}{n} = \dfrac{2}{n}$

$$\therefore \quad I = \int_1^3 (2x+1)\,dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots\ldots + f(1+(n-1)\ h)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[(2\times 1+1) + \{2(1+h)+1\} + \{2(1+2h)+1\} + \ldots\ldots + \{2(1+(n-1)\ h+1\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[3 + (3+2h) + (3+2\times 2h) + (3+2\times 3h) + \ldots\ldots + (3+2(n-1)\ h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{3n + 2h\ (1+2+3+\ldots\ldots+(n-1)\right\}$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{3n + 2h\frac{n(n-1)}{2}\right\} = \lim_{h \to 0} h\left\{3n + hn\ (n-1)\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left\{3n + \frac{2}{n}\times n(n-1)\right\} \qquad \left[\because\ h = \frac{2}{n} \text{ and } h \to 0 \Rightarrow n \to \infty\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{6 + 4\left(\frac{n-1}{n}\right)\right\} = \lim_{n \to \infty}\left\{6 + 4\left(1 - \frac{1}{n}\right)\right\} = 6 + 4\ (1-0) = 10$$

(ii) We know that

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\ h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 2, b = 4, f(x) = 2x - 1$ and $h = \frac{4-2}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_2^4 f(x)\,dx = \lim_{h \to 0} h\left[f(2) + f(2+h) + f(2+2h) + \ldots + f(2+(n-1)\ h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{2(2)-1)\} + \{2(2+h)-1\} + \{2(2+2h)-1\} + \ldots + \{2(2+(n-1)\ h)-1\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[3 + (3+2h) + (3+4h) + (3+6h) + \ldots + \{3+2(n-1)\ h\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[3n + 2h(1+2+3+\ldots+(n-1)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{3n + 2h\times\frac{n(n-1)}{2}\right\} = \lim_{n \to \infty}\frac{2}{n}\left\{3n + 2\times\frac{2}{n}\times\frac{n(n-1)}{2}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left\{6 + 4\left(\frac{n-1}{n}\right)\right\} = \lim_{n \to \infty}\left\{6 + 4\left(1 - \frac{1}{n}\right)\right\} = 6 + 4(1-0) = 10$$

EXAMPLE 3 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 (x^2+3)\,dx$ **[CBSE 2001C]** (ii) $\int_1^3 (2x^2+5)\,dx$ **[CBSE 2010]**

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a+(n-1)\ h\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0,\ b = 2, f(x) = x^2 + 3$ and $h = \frac{2-0}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_0^2 (x^2+3)\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(0) + f(0+h) + f(0+2h) + \ldots + f\{0+(n-1)\ h\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f\{(n-1)\ h\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[(0+3) + (h^2+3) + (2^2h^2+3) + (3^2h^2+3) + \ldots + \{(n-1)^2h^2+3\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[3n + h^2(1^2+2^2+\ldots+(n-1)^2\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left\{3n + h^2 \frac{n(n-1)(2n-1)}{6}\right\} = \lim_{n \to \infty} \frac{2}{n}\left\{3n + \frac{4}{n^2} \times \frac{n(n-1)(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{6 + \frac{8}{6}\frac{(n-1)\ (2n-1)}{n^2}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{6 + \frac{8}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\} = 6 + \frac{8}{6}(1-0)(2-0) = 6 + \frac{8}{3} = \frac{26}{3}$$

(ii) We know that

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a+(n-1)h\}\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 1, b = 3, f(x) = 2x^2 + 5$ and $h = \dfrac{3-1}{n} = \dfrac{2}{n}$

$$\therefore \quad I = \int_1^3 (2x^2+5)\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots + f\{1+(n-1)h\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\{2(1)^2+5\} + \{2(1+h)^2+5\} + \{2(1+2h)^2+5\} + \ldots + \{2\{1+(n-1)h\}^2+5\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\left\{1^2 + (1+h)^2 + (1+2h)^2 + (1+3h)^2 + \ldots + \left(1+(n-1)\ h\right)^2\right\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\left\{n + 2h\left(1+2+3+\ldots+(n-1)\right) + h^2\left(1^2+2^2+\ldots+(n-1)^2\right)\right\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2\left\{n + 2h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right\} + 5n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{2n + 2h\,n(n-1) + 2h^2\frac{n(n-1)(2n-1)}{6} + 5n\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left\{7n + 2 \times \frac{2}{n}n(n-1) + 2 \times \frac{4}{n^2} \times \frac{n(n-1)(2n-1)}{6}\right\} \qquad \left[\because h = \frac{2}{n}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{14 + 8\left(\frac{n-1}{n}\right) + \frac{8}{3}\frac{(n-1)(2n-1)}{n^2}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \left\{14 + 8\left(1-\frac{1}{n}\right) + \frac{8}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}$$

$$\Rightarrow \quad I = 14 + 8(1-0) + \frac{8}{3}(1-0)(2-0) = 14 + 8 + \frac{16}{3} = \frac{82}{3}$$

EXAMPLE 4 *Evaluate the following integrals as limit of sums:*

(i) $\int_1^3 (x^2 + x)\, dx$ [CBSE 2000C] (ii) $\int_1^3 (x^2 + 5x)\, dx$ [CBSE 2010] (iii) $\int_a^b x^2\, dx$

SOLUTION (i) We know that

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \dots + f(a+(n-1)h)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 1, b = 3, f(x) = x^2 + x$ and $h = \frac{3-1}{n} = \frac{2}{n}$

$$\therefore \quad I = \int_1^3 (x^2 + x)\, dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + f(1+3h) + \dots + f\{1+(n-1)h\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{1^2 + 1\right\} + \left\{(1+h)^2 + (1+h)\right\} + \left\{(1+2h)^2 + (1+2h)\right\} + \dots\dots\dots + \left\{\left(1+(n-1)\,h\right)^2 + \left(1+(n-1)\,h\right)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{1^2 + (1+h)^2 + (1+2h)^2 + \dots + \left(1+(n-1)\,h\right)^2\right\} + \left\{1 + (1+h) + (1+2h) + \dots + (1+(n-1)\,h)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{n + 2h\left(1+2+3+\dots+(n-1)\right) + h^2\left(1^2 + 2^2 + \dots + (n-1)^2\right)\right\} + \left\{n + h\left(1+2+3+\dots+(n-1)\right)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[n + 2h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6} + n + h \times \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[2n + 3h \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left[2n + \frac{6}{n} \times \frac{n(n-1)}{2} + \frac{4}{n^2} \times \frac{n(n-1)(2n-1)}{6}\right] \qquad \left[\because h = \frac{2}{n}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[4 + 6\left(\frac{n-1}{n}\right) + \frac{4}{3}\frac{(n-1)(2n-1)}{n^2}\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[4 + 6\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left[4 + 6\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = 4 + 6(1-0) + \frac{4}{3}(1-0)(2-0) = 4 + 6 + \frac{8}{3} = \frac{38}{3}$$

(ii) $\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\Big(a + (n-1)\,h\Big)\right]$, where $h = \frac{b-a}{n}$

Here $a = 1, b = 3, f(x) = x^2 + 5x$ and $h = \frac{3-1}{n} = \frac{2}{n}$.

$$\therefore \quad I = \int_1^3 (x^2 + 5x)\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots + f\Big(1 + (n-1)\;h\Big)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{(1^2 + 5 \times 1)\right\} + \left\{(1+h)^2 + 5\,(1+h)\right\} + \left\{(1+2h)^2 + 5\,(1+2h)\right\} + \ldots\ldots + \left\{\Big(1 + (n-1)\;h\Big)^2 + 5\Big(1 + (n-1)\;h\Big)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0}\left[\left[\left\{(1^2 + (1+h)^2 + (1+2h)^2 + \ldots + \Big(1 + (n-1)\;h\Big)^2\right\} + 5\left\{1 + (1+h) + (1+2h) + \ldots + \Big(1 + (n-1)\;h\Big)\right\}\right]\right.$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{n + 2h\,(1 + 2 + 3 + \ldots + (n-1) + h^2\,(1^2 + 2^2 + \ldots + (n-1)^2)\right\} + 5\left\{n + h\Big(1 + 2\;+ \ldots + (n-1)\Big)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[6n + 7h\Big(1 + 2 + 3 + \ldots + (n-1)\Big) + h^2\Big(1^2 + 2^2 + \ldots + \ldots + (n-1)^2\Big)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{6n + 7h \times \frac{n\,(n-1)}{2} + h^2 \times \frac{n\,(n-1)\,(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty} \frac{2}{n}\left\{6n + \frac{14}{n} \times \frac{n\,(n-1)}{2} + \frac{4}{n^2} \times \frac{n\,(n-1)\,(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left\{12 + 14\left(\frac{n-1}{n}\right) + \frac{8}{6}\,\frac{(n-1)\,(2n-1)}{n^2}\right\}$$

$$\Rightarrow \quad I = \lim_{n \to \infty}\left\{12 + 14\left(1 - \frac{1}{n}\right) + \frac{4}{3}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)\right\} = 12 + 14 + \frac{4}{3} \times 2 = 12 + 14 + \frac{8}{3} = \frac{86}{3}$$

(iii) $\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\Big(a + (n-1)\;h\Big)\right]$, where $h = \frac{b-a}{n}$

Here, $f(x) = x^2$

$$\therefore \quad I = \int_a^b x^2\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[a^2 + (a+h)^2 + (a+2h)^2 + (a+3h)^2 + \ldots + \Big(a + (n-1)\,h\Big)^2\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[na^2 + 2ah\left\{1+2+3+\ldots+(n-1)\right\} + h^2\left\{1^2+2^2+\ldots+(n-1)^2\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{na^2 + 2ah \times \frac{n(n-1)}{2} + h^2 \times \frac{n(n-1)(2n-1)}{6}\right\}$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left\{(nh)\, a^2 + a\,(nh)(nh-h) + \frac{1}{6}(nh)(nh-h)(2nh-h)\right\}$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left[(b-a)\, a^2 + a\,(b-a)(b-a-h) + \frac{1}{6}(b-a)(b-a-h)\left\{2(b-a)-h\right\}\right]$$

$$\left[\because\ h = \frac{b-a}{n}\ \therefore\ nh = b-a\right]$$

$$\Rightarrow \quad I = \left\{(b-a)\, a^2 + a\,(b-a)^2 + \frac{2}{6}(b-a)^3\right\} = \frac{(b-a)}{3}\left[3a^2 + 3a\,(b-a) + (b-a)^2\right]$$

$$\Rightarrow \quad I = \frac{(b-a)}{3}\left(3a^2 + 3ab - 3a^2 + b^2 - 2ab + a^2\right) = \frac{1}{3}(b-a)(a^2+ab+b^2) = \frac{1}{3}(b^3-a^3)$$

EXAMPLE 5 *Evaluate the following integrals as limit of sums:*

(i) $\int_0^2 e^x\,dx$ (ii) $\int_{-1}^1 e^x\,dx$ [NCERT]

SOLUTION (i) We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\left(a+(n-1)h\right)\right], \text{ where } h = \frac{b-a}{n}$$

Here, $a = 0,\ b = 2,\ f(x) = e^x$ and $h = \dfrac{2-0}{n} = \dfrac{2}{n}$

$$\therefore \quad I = \int_0^2 e^x\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[f(0) + f(h) + f(2h) + \ldots + f\left((n-1)h\right)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^0 + e^h + e^{2h} + \ldots + e^{(n-1)h}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^0\left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}\right] \qquad \left[\text{Using}: a + ar + \ldots + ar^{n-1} = a\left(\frac{r^n-1}{r-1}\right)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left\{\frac{e^{nh}-1}{e^h-1}\right\} = \lim_{h \to 0} \frac{h}{h}\left\{\frac{e^2-1}{\left(\dfrac{e^h-1}{h}\right)}\right\} \qquad \left[\because\ h = \frac{2}{n} \Rightarrow nh = 2\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \frac{e^2-1}{\left(\dfrac{e^h-1}{h}\right)} = \frac{e^2-1}{1} = e^2 - 1 \qquad \left[\lim_{h \to 0} \frac{e^h-1}{h} = 1\right]$$

(ii) $\displaystyle\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f\Big(a+(n-1)\,h\Big)\right]$, where $h=\dfrac{b-a}{n}$.

Here, $a=-1, b=1, f(x)=e^x$ and $h=\dfrac{1-(-1)}{n}=\dfrac{2}{n}$

$$\therefore \quad I = \int_{-1}^{1} e^x\,dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(-1)+f(-1+h)+f(-1+2h)+\ldots+f\Big(-1+(n-1)\,h\Big)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[e^{-1}+e^{-1+h}+e^{-1+2h}+\ldots+e^{-1+(n-1)\,h}\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\,e^{-1}\left[1+e^h+e^{2h}+\ldots+e^{(n-1)\,h}\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\,e^{-1}\left[\left\{\frac{(e^h)^n-1}{e^h-1}\right\}\right] \qquad \left[\text{Using: } a+ar+\ldots+ar^{n-1}=a\left(\frac{r^n-1}{r-1}\right)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} e^{-1}\left\{\frac{e^2-1}{\left(\dfrac{e^h-1}{h}\right)}\right\} \qquad \left[\because\ h=\frac{2}{n}\Rightarrow nh=2\right]$$

$$\Rightarrow \quad I = e^{-1}\left(\frac{e^2-1}{1}\right) = e-e^{-1} \qquad \left[\because\ \lim_{h\to 0}\frac{e^h-1}{h}=1\right]$$

EXAMPLE 6 *Evaluate the following integrals as a limit of sums.*

(i) $\displaystyle\int_0^1 e^{2-3x}\,dx$ [NCERT] (ii) $\displaystyle\int_2^4 2^x\,dx$ (iii) $\displaystyle\int_1^3 (e^{2-3x}+x^2+1)\,dx$ [CBSE 2015]

SOLUTION We know that

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+\ldots+f\Big(a+(n-1)\,h\Big)\right], \text{ where } h=\frac{b-a}{n}$$

(i) Here, $a=0, b=1$ and $f(x)=e^{2-3x}$. Therefore, $h=\dfrac{1}{n}\Rightarrow nh=1$.

$$\therefore \quad I = \int_0^1 e^{2-3x}\,dx$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[f(0)+f(h)+f(2h)+\ldots+f\Big((n-1)\,h\Big)\right]$$

$$\Rightarrow \quad I = \lim_{h\to 0} h\left[e^2+e^{2-3h}+e^{2-3(2h)}+\ldots+e^{2-3(n-1)\,h}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} he^2 \left[1 + e^{-3h} + e^{-3(2h)} + e^{-3(3h)} + \ldots + e^{-3(n-1)h}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} he^2 \left\{\frac{(e^{-3h})^n - 1}{e^{-3h} - 1}\right\}$$

$$\Rightarrow \quad I = \lim_{h \to 0} he^2 \left[\frac{e^{-3nh} - 1}{e^{-3h} - 1}\right] = e^2 \lim_{h \to 0} \frac{e^{-3} - 1}{\left(\frac{e^{-3h} - 1}{-3h}\right)} \times -\frac{1}{3} \qquad [\because nh = 1]$$

$$\Rightarrow \quad I = e^2 (e^{-3} - 1) \times -\frac{1}{3} = -\frac{1}{3}(e^{-1} - e^2) = \frac{1}{3}(e^2 - e^{-1})$$

(ii) Here, $a = 2, b = 4$ and $f(x) = 2^x$. Therefore, $h = \frac{4-2}{n} \Rightarrow nh = 2$

Substituting these values in

$$\Rightarrow \quad \int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\left(a + (n-1)h\right)\right], \text{ we get}$$

$$\Rightarrow \quad \int_2^4 2^x\,dx = \lim_{h \to 0} h\left[f(2) + f(2+h) + f(2+2h) + \ldots + f\left(2 + (n-1)h\right)\right]$$

$$\Rightarrow \quad \int_2^4 2^x\,dx = \lim_{h \to 0} h\left\{2^2 + 2^{2+h} + 2^{2+2h} + \ldots + 2^{2+(n-1)h}\right\}$$

$$\Rightarrow \quad \int_2^4 2^x\,dx = \lim_{h \to 0} 4h\left\{1 + 2^h + 2^{2h} + \ldots + 2^{(n-1)h}\right\}$$

$$\Rightarrow \quad \int_2^4 2^x\,dx = \lim_{h \to 0} 4h\left\{\frac{(2^h)^n - 1}{2^h - 1}\right\} = 4 \lim_{h \to 0} \left\{\frac{2^{nh} - 1}{\left(\frac{2^h - 1}{h}\right)}\right\}$$

$$\Rightarrow \quad \int_2^4 2^x\,dx = 4 \times \left(\frac{2^2 - 1}{\log 2}\right) = \frac{12}{\log 2} \qquad \left[\because nh = 2 \text{ and } \lim_{h \to 0} \frac{2^h - 1}{h} = \log 2\right]$$

(iii) Here, $a = 1, b = 3$ and $f(x) = e^{2-3x} + x^2 + 1$. Therefore, $h = \frac{3-1}{n} = \frac{2}{n}$ and $nh = 2$.

Substituting these values in

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f(a + (n-1)h\right], \text{ we obtain}$$

$$I = \int_1^3 (e^{2-3x} + x^2 + 1)\,dx = \lim_{h \to 0} h\left[f(1) + f(1+h) + f(1+2h) + \ldots + f\left(1 + (n-1)h\right)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[(e^{2-3\times 1} + 1^2 + 1) + \{e^{2-3(1+h)} + (1+h)^2 + 1\} + \{e^{2-3(1+2h)} + (1+2h)^2 + 1\} + \ldots + \{e^{2-3\{1+(n-1)h\}} + \{1+(n-1)h\}^2 + 1\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\left\{e^{-1} + e^{-1-3h} + e^{-1-3(2h)} + \ldots + e^{-1-3(n-1)h}\right\} + \left\{1^2 + (1+h)^2 + (1+2h)^2 + (1+3h)^2 + \ldots + \{1+(n-1)h\}^2\right\} + n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^{-1}\left\{1 + e^{-3h} + e^{-3(2h)} + e^{-3(3h)} + \ldots e^{-3(n-1)h}\right\} + \left\{n + 2h\left(1+2+3+\ldots+(n-1)\right)\right\} + h^2\left\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right\} + n\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[e^{-1}\left\{\frac{(e^{-3h})^n - 1}{e^{-3h} - 1}\right\} + \left\{2n + 2h\,\frac{n(n-1)}{2} + \frac{h^2}{6}\,n(n-1)(2n-1)\right\}\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0}\left[he^{-1}\left(\frac{e^{-3nh} - 1}{e^{-3h} - 1}\right) + 2nh + nh(nh-h) + \frac{1}{6}nh(nh-h)(2nh-h)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0}\left[e^{-1} \times \frac{(e^{-6} - 1)}{-3\left(\dfrac{e^{-3h} - 1}{-3h}\right)} + 4 + 2(2-h) + \frac{2}{6}(2-h)(4-h)\right] \qquad [\because\ nh = 2]$$

$$\Rightarrow \quad I = -\frac{e^{-1}}{3}(e^{-6} - 1) + 4 + 2(2-0) + \frac{1}{3}(2-0)(4-0) = -\frac{1}{3}(e^{-7} - e^{-1}) + \frac{32}{3}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 7 *Evaluate* $\int_a^b \sin x \, dx$ *as limit of sums.*

SOLUTION We have,

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h\left[f(a) + f(a+h) + f(a+2h) + \ldots + f\left(a + (n-1)h\right)\right], \text{ where } h = \frac{b-a}{n}.$$

Here, $f(x) = \sin x$

$$\therefore \quad I = \int_a^b \sin x\,dx$$

$$\Rightarrow \quad I = \lim_{h \to 0} h\left[\sin a + \sin(a+h) + \sin(a+2h) + \ldots + \sin\left(a + (n-1)\,h\right)\right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h \left[\frac{\sin\left(a + (n-1)\frac{h}{2}\right) \sin \frac{nh}{2}}{\sin \frac{h}{2}} \right] = \lim_{h \to 0} h \left[\frac{\sin\left(a + \frac{nh}{2} - \frac{h}{2}\right) \sin \frac{nh}{2}}{\sin \frac{h}{2}} \right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} h \left[\frac{\sin\left(a + \frac{b-a}{2} - \frac{h}{2}\right) \sin\left(\frac{b-a}{2}\right)}{\sin \frac{h}{2}} \right] \qquad [\because nh = b - a]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left[\frac{h/2}{\sin h/2} \times 2 \sin\left(\frac{a+b}{2} - \frac{h}{2}\right) \sin\left(\frac{b-a}{2}\right) \right]$$

$$\Rightarrow \quad I = \lim_{h \to 0} \left(\frac{h/2}{\sin \frac{h}{2}} \right) \times \lim_{h \to 0} 2 \sin\left(\frac{a+b}{2} - \frac{h}{2}\right) \sin\left(\frac{b-a}{2}\right) = 2 \sin\left(\frac{a+b}{2}\right) \sin\left(\frac{b-a}{2}\right)$$

$$\Rightarrow \quad I = \cos a - \cos b \qquad [\because 2 \sin A \sin B = \cos(A - B) - \cos(A + B)]$$

EXERCISE 19.5

BASIC

Evaluate the following integrals as limit of sums:

1. $\int_0^3 (x+4)\,dx$
2. $\int_0^2 (x+3)\,dx$
3. $\int_1^3 (3x-2)\,dx$
4. $\int_{-1}^1 (x+3)\,dx$
5. $\int_0^5 (x+1)\,dx$
6. $\int_1^3 (2x+3)\,dx$
7. $\int_3^5 (2-x)\,dx$

BASED ON LOTS

8. $\int_0^2 (x^2+1)\,dx$
9. $\int_1^2 x^2\,dx$
10. $\int_2^3 (2x^2+1)\,dx$
11. $\int_1^2 (x^2-1)\,dx$
12. $\int_0^2 (x^2+4)\,dx$
13. $\int_1^4 (x^2-x)\,dx$ [NCERT, CBSE 2010, 12]
14. $\int_0^1 (3x^2+5x)\,dx$
15. $\int_0^2 e^x\,dx$ [NCERT]
16. $\int_a^b e^x\,dx$
17. $\int_a^b \cos x\,dx$
18. $\int_0^{\pi/2} \sin x\,dx$
19. $\int_0^{\pi/2} \cos x\,dx$
20. $\int_1^4 (3x^2+2x)\,dx$
21. $\int_0^2 (3x^2-2)\,dx$
22. $\int_0^2 (x^2+2)\,dx$ [CBSE 2000]
23. $\int_0^4 (x+e^{2x})\,dx$ [NCERT]
24. $\int_0^2 (x^2+x)\,dx$ [CBSE 2005]
25. $\int_0^2 (x^2+2x+1)\,dx$ [CBSE 2007]

26. $\int_0^3 (2x^2 + 3x + 5)\, dx$ [CBSE 2007]

27. $\int_a^b x\, dx$ [NCERT]

28. $\int_0^5 (x+1)\, dx$ [NCERT]

29. $\int_2^3 x^2\, dx$ [NCERT]

30. $\int_1^4 (x^2 - x)\, dx$ [NCERT, CBSE 2020]

31. $\int_0^2 (x^2 - x)\, dx$ [CBSE 2012]

32. $\int_1^3 (2x^2 + 5x)\, dx$ [CBSE 2012]

33. $\int_1^3 (3x^2 + 1)\, dx$ [CBSE 2014]

34. $\int_1^3 (x^2 + 3x + e^x)\, dx$ [CBSE 2018]

ANSWERS

1. $\frac{33}{2}$ 2. 8 3. 8 4. 6 5. $\frac{35}{2}$ 6. 14 7. -4 8. $\frac{14}{3}$ 9. $\frac{7}{3}$ 10. $\frac{41}{3}$ 11. $\frac{4}{3}$ 12. $\frac{32}{3}$ 13. $\frac{27}{2}$ 14. $\frac{7}{2}$ 15. $e^2 - 1$ 16. $e^b - e^a$ 17. $\sin b - \sin a$ 18. 1 19. 1 20. 78 21. 4 22. $\frac{20}{3}$ 23. $\frac{15 + e^8}{2}$ 24. $\frac{14}{3}$ 25. $\frac{26}{3}$ 26. $\frac{93}{2}$ 27. $\frac{b^2 - a^2}{2}$ 28. $\frac{35}{2}$ 29. $\frac{19}{3}$ 30. $\frac{38}{3}$ 31. $\frac{2}{3}$ 32. $\frac{112}{3}$ 33. 28 34. $\frac{62}{3} + e^3 - e$

REVISION EXERCISE

BASIC

Evaluate the following integrals:

1. $\int_0^4 x\sqrt{4-x}\, dx$

2. $\int_1^2 x\sqrt{3x-2}\, dx$

3. $\int_0^5 x\sqrt{4-x}\, dx$

4. $\int_0^1 \cos^{-1} x\, dx$

5. $\int_0^1 \tan^{-1} x\, dx$

6. $\int_0^1 \frac{1-x}{1+x}\, dx$

7. $\int_0^{\pi/3} \frac{\cos x}{3 + 4\sin x}\, dx$

8. $\int_0^{\pi/2} \frac{\sin^2 x}{(1+\cos x)^2}\, dx$

9. $\int_0^{\pi/2} \frac{\sin x}{\sqrt{1+\cos x}}\, dx$

10. $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\, dx$

11. $\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)\,(1+\cos x)^2\, dx$

12. $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)}\, dx$

13. $\int_0^{\pi/4} \sin 2x \sin 3x\, dx$

14. $\int_0^1 \sqrt{\frac{1-x}{1+x}}\, dx$

15. $\int_1^2 \frac{1}{x^2} e^{-1/x}\, dx$

16. $\int_0^{\pi/4} \cos^4 x \sin^3 x\, dx$

17. $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\, dx$

18. $\int_0^{\pi/2} x^2 \cos 2x \, dx$

19. $\int_0^1 \log(1+x) \, dx$

20. $\int_2^4 \frac{x^2+x}{\sqrt{2x+1}} \, dx$

21. $\int_1^2 \frac{x+3}{x(x+2)} \, dx$

22. $\int_0^{\pi/4} e^x \sin x \, dx$

23. $\int_0^1 |2x-1| \, dx$

24. $\int_{-\pi/2}^{\pi/2} \sin^9 x \, dx$

25. $\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx$

26. $\int_{-a}^{a} \frac{x e^{x^2}}{1+x^2} \, dx$

27. $\int_0^{\pi/2} \frac{1}{1+\cot^7 x} \, dx$

28. $\int_0^{2\pi} \cos^7 x \, dx$

29. $\int_0^a \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}} \, dx$

30. $\int_0^{\pi/2} \frac{1}{1+\tan^3 x} \, dx$

31. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \, dx$

32. $\int_0^{\pi} x \sin x \cos^4 x \, dx$

33. $\int_{-\pi/4}^{\pi/4} |\tan x| \, dx$

34. $\int_0^{\pi} \frac{x}{a^2-\cos^2 x} \, dx, \ a>1$

35. $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} \, dx$

36. $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}} \, dx$

37. $\int_0^{\pi/2} \frac{x}{\sin^2 x + \cos^2 x} \, dx$

38. $\int_{-\pi}^{\pi} x^{10} \sin^7 x \, dx$

39. $\int_0^{\pi} \frac{dx}{6-\cos x}$

40. $\int_0^{\pi/2} \frac{1}{2\cos x + 4\sin x} \, dx$

41. $\int_{\pi/6}^{\pi/2} \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x} \, dx$

42. $\int_0^{\pi/2} \frac{dx}{4\cos x + 2\sin x}$

Evaluate the following definite integrals as limits of sums: (43-51)

43. $\int_0^4 x \, dx$

44. $\int_0^2 (2x^2+3) \, dx$

45. $\int_1^4 (x^2+x) \, dx$

46. $\int_{-1}^{1} e^{2x} \, dx$

47. $\int_2^3 e^{-x} \, dx$

48. $\int_1^3 (2x^2+5x) \, dx$

49. $\int_1^3 (x^2+3x) \, dx$

50. $\int_0^2 (x^2+2) \, dx$

51. $\int_0^3 (x^2+1) \, dx$

BASED ON LOTS

52. $\int_0^1 x(\tan^{-1} x)^2 \, dx$

53. $\int_0^1 (\cos^{-1} x)^2 \, dx$

54. $\int_0^{\pi/4} \tan^4 x \, dx$

55. $\int_1^3 |x^2-2x| \, dx$

56. $\int_0^{\pi/2} |\sin x - \cos x| \, dx$

57. $\int_0^1 |\sin 2\pi x| \, dx$

58. $\int_1^3 |x^2-4| \, dx$

59. $\int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2 \sin^2 x} \, dx$

60. $\int_0^{1.5} [x^2] \, dx$

61. $\int_0^{\pi} \frac{x}{1+\cos\alpha \sin x}\,dx$

62. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$

63. $\int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x}\,dx$

64. $\int_0^{\pi} \cos 2x \log \sin x\,dx$

65. $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx$

66. $\int_0^{1} \cot^{-1}(1 - x + x^2)\,dx$

67. $\int_0^{1} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$

68. $\int_0^{1} \tan^{-1}\left(\frac{2x}{1-x^2}\right)dx$

69. $\int_0^{1/\sqrt{3}} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

ANSWERS

1. $\frac{128}{15}$ 2. $\frac{326}{135}$ 3. $\frac{16}{3}$ 4. 1 5. $\frac{\pi}{4} - \frac{1}{2}\log 2$

6. $2\log 2 - 1$ 7. $\frac{1}{4}\log\left(\frac{3+2\sqrt{3}}{3}\right)$ 8. $2 - \frac{\pi}{2}$ 9. $2(\sqrt{2}-1)$

10. $\frac{\pi}{4}$ 11. $\frac{8}{3}$ 12. $\frac{\pi}{4}$ 13. $\frac{3}{5\sqrt{2}}$ 14. $\frac{\pi}{2} - 1$

15. $\frac{\sqrt{e}-1}{e}$ 16. $\frac{2}{35}$ 17. $\frac{3}{2}$ 18. $-\frac{\pi}{4}$ 19. $\log\left(\frac{4}{e}\right)$

20. $\frac{57}{5} - \sqrt{5}$ 21. $\frac{1}{2}\log 6$ 22. $\frac{1}{2}$ 23. $\frac{1}{2}$ 24. 0

25. 0 26. 0 27. $\frac{\pi}{4}$ 28. 0 29. $\frac{a}{2}$

30. $\frac{\pi}{4}$ 31. $\frac{\pi^2}{2}$ 32. $\frac{\pi}{5}$ 33. $\log 2$ 34. $\frac{\pi^2}{2a\sqrt{a^2-1}}$

35. $\frac{\pi}{2}(\pi - 2)$ 36. $\frac{1}{2}$ 37. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ 38. $\frac{\pi^2}{8}$ 39. $\frac{\pi}{\sqrt{35}}$

40. $\frac{1}{2\sqrt{5}}\log\left\{\frac{\sqrt{5}+1}{2(\sqrt{5}-2)}\right\}$ 41. $\tan^{-1}\left(\frac{1}{3}\right)$ 42. $\frac{1}{\sqrt{5}}\log\left\{\frac{\sqrt{5}+1}{\sqrt{5}-1}\right\}$

43. 8 44. $\frac{34}{3}$ 45. $\frac{27}{2}$ 46. $\frac{1}{2}(e^2 - e^{-2})$ 47. $e^{-2} - e^{-3}$

48. $\frac{112}{3}$ 49. $\frac{62}{3}$ 50. $\frac{20}{3}$ 51. 12

52. $\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2$ 53. $\pi - 2$ 54. $\frac{\pi}{4} - \frac{2}{3}$ 55. 2

56. $2(\sqrt{2}-1)$ 57. $\frac{2}{\pi}$ 58. 4 59. $\frac{\pi^2}{2ab}$ 60. $2 - \sqrt{2}$

61. $\frac{\pi\alpha}{\sin\alpha}$ 62. $\frac{\pi^2}{16}$ 63. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ 64. $-\frac{\pi}{2}$ 65. 0

66. $\frac{\pi}{2} - \log 2$ 67. $\frac{\pi}{2} - \log 2$ 68. $\frac{\pi}{2} - \log 2$ 69. $\frac{\pi}{2\sqrt{3}} - \frac{3}{2}\log\frac{4}{3}$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then $a =$ ....

2. The value of $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x \, dx$ is ....

3. The value of $\int_0^{\pi/2} e^{\sin x} \cos x \, dx$ is ....

4. $\int_0^a \frac{x}{\sqrt{a^2+x^2}} dx =$ ....

5. The value of the integral $\int_{1/\pi}^{2/\pi} \frac{\sin\left(\frac{1}{x}\right)}{x^2} dx$ is ....

6. The value of the integral $\int_0^1 \frac{\tan^{-1} x}{1+x^2} dx$ is ....

7. The value of the integral $\int_1^2 e^x \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$ is ....

8. $\int_1^e \frac{e^x}{x}(1+\log x)\, dx =$ ....

9. The value of the integral $\int_{-2}^{2} (ax^5 + bx^3 + cx + d)\, dx$, where a, b, c, d are constants, depends only on ....

10. $\int_0^{\pi} \frac{1}{1+\sin x} dx =$ ....

11. $\int_0^{\pi/2} \frac{\sin x \cos x}{1+\sin^4 x} dx =$ ....

12. $\int_0^{\pi/4} \tan^6 x \sec^2 x \, dx =$ ....

13. The value of $\int_0^{\pi/4} \frac{1+\tan x}{1-\tan x} dx$ is ....

14. The value of $\int_0^2 \frac{3^{\sqrt{x}}}{\sqrt{x}} dx$ is ....

15. The value of $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$ is ....

16. If $f(a-x) = x$ and $\int_0^a f(x)\, dx = k \int_0^{a/2} f(x)\, dx$, then $k =$ ....

17. The value of the integral $\int_0^{2\pi} \cos^7 x \sin^4 x \, dx$ is ....

18. The value of the integral $\int_{-1}^{1} x|x| \, dx$ is ....

19. The value of the integral $\int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right) dx$ is ....

20. The value of the integral $\int_{-1}^{1} |1-x| \, dx$ is ....

21. The value of the integral $\int_0^{\pi/2} \frac{1}{1+\tan^3 x} dx$ is ....

22. If $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}} \, dx = k\pi$, then $k =$ ....

23. If $f(x) = f(a-x)$ and $\int_0^a x f(x) \, dx = k \int_0^a f(x) \, dx$, then $k =$ ....

24. The value of the integral $\int_0^{10} \frac{x^{10}}{(10-x)^{10} + x^{10}} dx$ is ....

25. $\int_{-1}^{1} e^{|x|} \, dx =$ ....

ANSWERS

1. $a = \frac{1}{2}$	2. 0	3. $e-1$	4. $a(\sqrt{2}-1)$	5. 1
6. $\frac{\pi^2}{32}$	7. $\frac{e^2}{2} - e$	8. e^e	9. d	10. 2
11. $\frac{\pi}{8}$	12. $\frac{1}{7}$	13. $-\frac{1}{2}\log_e 2$	14. $\frac{2}{\log_e 3}(3^{\sqrt{2}}-1)$	15. $\frac{\pi}{4}$
16. 2	17. 0	18. 0	19. 0	20. 2
21. $\frac{\pi}{4}$	22. a	23. $\frac{a}{2}$	24. 5	25. $2(e-1)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question: Evaluate each of the following integrals: (1-30)

1. $\int_0^{\pi/2} \sin^2 x \, dx.$

2. $\int_0^{\pi/2} \cos^2 x \, dx.$

3. $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$

4. $\int_{-\pi/2}^{\pi/2} \cos^2 x \, dx.$

5. $\int_{-\pi/2}^{\pi/2} \sin^3 x \, dx.$ [CBSE 2010]

6. $\int_{-\pi/2}^{\pi/2} x \cos^2 x \, dx.$

7. $\int_0^{\pi/4} \tan^2 x \, dx.$

8. $\int_0^1 \frac{1}{x^2+1} dx.$

9. $\int_{-2}^{1} \frac{|x|}{x} dx.$

10. $\int_0^{\infty} e^{-x}\, dx.$ 11. $\int_0^4 \frac{1}{\sqrt{16-x^2}}\, dx.$ 12. $\int_0^3 \frac{1}{x^2+9}\, dx.$

13. $\int_0^{\pi/2} \sqrt{1-\cos 2x}\, dx.$ 14. $\int_0^{\pi/2} \log \tan x\, dx.$ 15. $\int_0^{\pi/2} \log\left(\frac{3+5\cos x}{3+5\sin x}\right) dx.$

16. $\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx,\ n \in N.$ 17. $\int_0^{\pi} \cos^5 x\, dx$

18. $\int_{-\pi/2}^{\pi/2} \log\left(\frac{a-\sin\theta}{a+\sin\theta}\right) d\theta$ 19. $\int_{-1}^{1} x|x|\, dx$

20. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\, dx$ 21. $\int_0^1 \frac{1}{1+x^2}\, dx$ [CBSE 2008] 22. $\int_0^{\pi/4} \tan x\, dx$ [CBSE 2014]

23. $\int_2^3 \frac{1}{x} dx$ [CBSE 2012] 24. $\int_0^2 \sqrt{4-x^2}\, dx$ [CBSE 2012]

25. $\int_0^1 \frac{2x}{1+x^2}\, dx$ [CBSE 2012] 26. $\int_0^1 x e^{x^2}\, dx$ [CBSE 2014]

27. $\int_0^{\pi/4} \sin 2x\, dx$ [CBSE 2014] 28. $\int_e^{e^2} \frac{1}{x \log x}\, dx$ [CBSE 2014]

29. $\int_e^{\pi/2} e^x(\sin x - \cos x)\, dx$ [CBSE 2014] 30. $\int_2^4 \frac{x}{x^2+1} dx$ [CBSE 2014]

31. If $\int_0^1 (3x^2+2x+k)\, dx = 0$, find the value of k. [CBSE 2009]

32. If $\int_0^a 3x^2\, dx = 8$, write the value of a. [CBSE 2012]

33. If $f(x) = \int_0^x t \sin t\, dt$, then write the value of $f'(x)$. [CBSE 2014]

34. If $\int_0^a \frac{1}{4+x^2} dx = \frac{\pi}{8}$, find the value of a. [CBSE 2014]

35. Write the coefficient a, b, c of which the value of the integral $\int_{-3}^{3} (ax^2+bx+c)\, dx$ is independent.

36. Evaluate $\int_2^3 3^x\, dx.$ [CBSE 2016]

If $[\cdot]$ and $\{\cdot\}$ denote respectively the greatest integer and fractional part functions respectively, evaluate the following integrals:

37. $\int_0^2 [x]\, dx$ 38. $\int_0^{1.5} [x]\, dx$ 39. $\int_0^1 \{x\}\, dx$ 40. $\int_0^1 e^{\{x\}}\, dx$

41. $\int_0^2 x[x]\ dx$

42. $\int_0^1 2^{x-[x]}\ dx$

43. $\int_1^2 \log_e [x]\ dx$

44. $\int_0^{\sqrt{2}} [x^2]\ dx$

45. $\int_0^{\pi/4} \sin\{x\}\ dx$

ANSWERS

1. $\frac{\pi}{4}$
2. $\frac{\pi}{4}$
3. $\frac{\pi}{2}$
4. $\frac{\pi}{2}$
5. 0
6. 0
7. $1-\frac{\pi}{4}$
8. $\frac{\pi}{4}$
9. -1
10. 1
11. $\frac{\pi}{2}$
12. $\frac{\pi}{12}$
13. $\sqrt{2}$
14. 0
15. 0
16. $\frac{\pi}{4}$
17. 0
18. 0
19. 0
20. $\frac{b-a}{2}$
21. $\frac{\pi}{4}$
22. $\frac{1}{2}\log 2$
23. $\log_e\left(\frac{2}{3}\right)$
24. π
25. $\log_e 2$
26. $\frac{1}{2}(e-1)$
27. $\frac{1}{2}$
28. $\log_e 2$
29. 1
30. $\frac{1}{2}\log\left(\frac{17}{5}\right)$
31. -2
32. $a=2$
33. $f'(x) = x\sin x$
34. $a=2$
35. b
36. $\frac{18}{\log_e 3}$
37. 1
38. $\frac{1}{2}$
39. $\frac{1}{2}$
40. $e-1$
41. $\frac{3}{2}$
42. $\frac{1}{\log_e 2}$
43. 0
44. 2
45. $\frac{\sqrt{2}-1}{\sqrt{2}}$