CHAPTER 2

FUNCTIONS

2.1 INTRODUCTION

The concept of function is of paramount importance in Mathematics and among other disciplines as well. In earlier class we have introduced the notion of function and we have learnt about some special functions like identity function, constant function, polynomial function, rational function, modulus function, greatest integer function, signum function etc. along with their graphs. Addition, subtraction, multiplication and division of two real functions have also been studied in the earlier class. In this chapter, we would like to extend our study about functions from where we finished in earlier class. We will study about various kinds of functions, composition of functions and inverse of a function. Let us first recapitulate what we have learnt about functions in earlier class.

2.2 RECAPITULATION

FUNCTION AS A SET OF ORDERED PAIRS *Let A and B be two non-empty sets. A relation f from A to B i.e. a sub set of $A \times B$ is called a function (or a mapping or a map) from A to B, if*

(i) *for each $a \in A$ there exists $b \in B$ such that $(a, b) \in f$.*

(ii) $(a, b) \in f$ *and* $(a, c) \in f \Rightarrow b = c$.

Thus, a non-void subset of $A \times B$ is a function from A to B if each element of A appears in some ordered pair in f and no two ordered pairs in f have the same first element.

If $(a, b) \in f$, then b in called the image of a under f.

FUNCTION AS A CORRESPONDENCE *Let A and B be two non-empty sets. Then a function 'f' from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that:*

(i) *all elements of set A are associated to elements in set B.*

(ii) *an element of set A is associated to a unique element in set B.*

In other words, a function 'f' from a set A to a set B associates each element of set A to a unique element of set B.

Terms such as "map" (or "mapping"), "correspondence" are used as synonyms for "function".

If f is a function from a set A to a set B, then we write $f : A \to B$ or $A \xrightarrow{f} B$, which is read as *f is a function from A to B* or *f maps A to B.*

If an element $a \in A$ is associated to an element $b \in B$, then b is called "*the f-image of a*" or "*image of a under f*" or" *the value of the function f at a*". Also, a is called the pre-image of b under the function f. We write it as $b = f(a)$.

The set A is known as the domain of f and the set B is known as the co-domain of f. The set of all f-images of elements of A is known as the range of f or image set of A under f and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\}$ = Range of f.

A visual representation of a function is shown in Fig. 2.1.

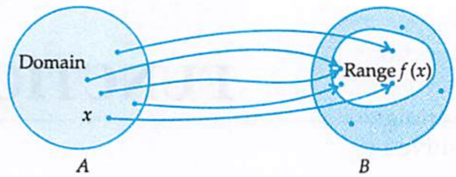


Fig. 2.1 Visual representation of a function.

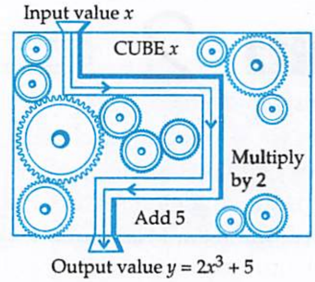


Fig. 2.2 Function as a machine.

FUNCTION AS A MACHINE *A function can also be regarded as a machine that gives unique output in set B corresponding to each input from the set A just as the function 'machine' shown in Fig. 2.2 which generate an output* $y = 2x^3 + 5$ *for each input x.*

Usually, real functions are described by using a mathematical formula. It is traditional to let x denote the input and y the corresponding output and to describe the function we write an equation relating x and y. In such an equation x and y are called variables. Because the value of the variable y is determined by that of the variable x, so we call y *the dependent variable* and x the *independent variable*.

If A and B are two sets having m and n elements respectively, then total number of functions from A to B is n^m.

A function $f : A \to B$ is called a real valued function if B is a subset of R (set of all real numbers).

If A and B both are subsets of R, then f is called a real function.

In order to represent a real function $y = f(x)$ geometrically as a graph, we use a cartesian coordinate system on which units for the independent variable x are marked on the horizontal axis i.e. x-axis and units for the dependent variable y on the vertical axis i.e. y-axis.

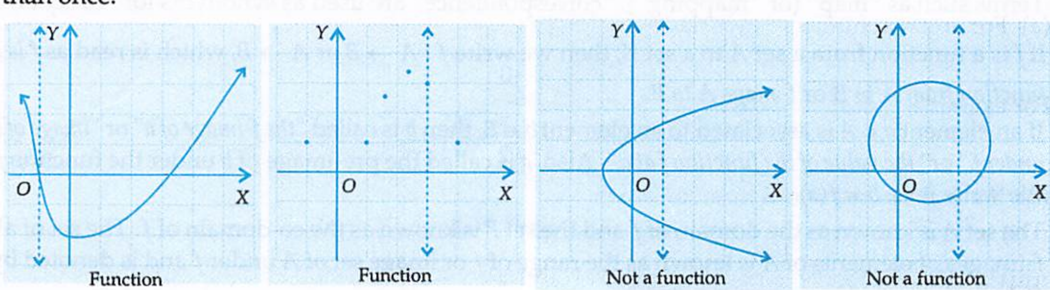
GRAPH OF A FUNCTION *The graph of a real function f consists of points whose coordinates* (x, y) *satisfy* $y = f(x)$, *for all* $x \in$ Domain (f).

In this section, we shall discuss graphs of some standard real functions.

By the definition of a real function f, for a given x in its domain there is only one number $y = f(x)$ in its range. Geometrically, this means that any vertical line $x = a$ crosses the graph of $f(x)$ at most once only. This observation leads to the following useful criterion for checking whether a curve in a plane is the graph of a function or not.

VERTICAL LINE TEST *A curve in a plane represents the graph of a real function if and only if no vertical line intersects it more than once.*

The curves shown in Fig. 2.3 (a) are the graphs of function whereas the curves shown in Fig. 2.3 (b) are not the graphs of functions as there exist vertical lines which intersect them more than once.



(a) (b)

Fig. 2.3 Vertical line test

Following are some standard real functions which will occur very frequently in the study of calculus.

CONSTANT FUNCTION *If k is a fixed real number, then a function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.*

Sometimes we also call it the constant function k.

We observe that the domain of the constant function $f(x) = k$ is the set R of all real numbers and range of f is the singleton set $\{k\}$.

The graph of a constant function $f(x) = k$ is a straight line parallel to x-axis (see Fig. 2.4) which is above or below x-axis according as k is positive or negative. If $k = 0$, then the straight line is coincident to x-axis.

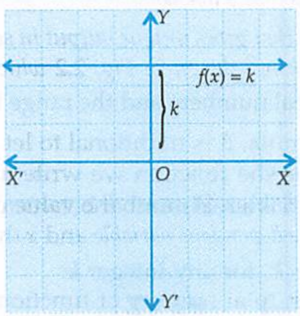


Fig. 2.4 Constant function

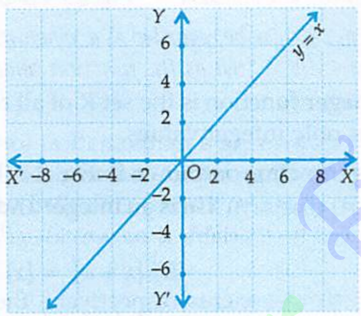


Fig. 2.5 Identity function

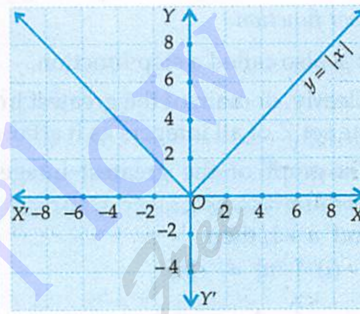


Fig. 2.6 Modulus function

IDENTITY FUNCTION *The function that associates each real number to itself is called the identity function and is usually denoted by I.*

Thus, the function $I : R \to R$ defined by $I(x) = x$ for all $x \in R$ is called the identity function.

Clearly, the domain and range of the identity function are both equal to R.

The graph of the identity function is a straight line passing through the origin and inclined at an angle of 45° with X-axis as shown in Fig. 2.5.

MODULUS FUNCTION *The function $f(x)$ defined by $f(x) = |x| = \begin{cases} x & , \text{ when } x \geq 0 \\ -x & , \text{ when } x < 0 \end{cases}$ is called the modulus function.*

It is also called the absolute value function.

We observe that the domain of the modulus function is the set R of all real numbers and the range is the set of all non-negative real numbers i.e. $R^+ = \{x \in R : x \geq 0\}$.

The graph of the modulus function is as shown in Fig. 2.6 for $x > 0$, the graph coincides with the graph of the identity function i.e. the line $y = x$ and for $x < 0$, it is coincident to the line $y = -x$.

The modulus function has the following properties:

(a) For any real number x, we have: $\sqrt{x^2} = |x|$

For example, $\sqrt{\cos^2 x} = |\cos x| = \begin{cases} \cos x & , \quad 0 \leq x \leq \dfrac{\pi}{2} \\ -\cos x & , \quad \dfrac{\pi}{2} < x \leq \pi \end{cases}$

(b) If a, b are positive real numbers, then

(i) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$ (ii) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or, $x \geq a$

(iii) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$ (iv) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a$ or, $x > a$

(v) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(vi) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow \in (-b, -a) \cup (a, b)$

(c) For real numbers x and y, we have

(i) $|x + y| = |x| + |y|$, if $(x \geq 0$ and $y \geq 0)$ or, $(x < 0$ and $y < 0)$

(ii) $|x - y| = |x| - |y|$, if $(x \geq 0$ and $|x| \geq |y|)$ or, $(x \leq 0, y \leq 0$ and $|x| \geq |y|)$

(iii) $|x \pm y| \leq |x| + |y|$ (iv) $|x \pm y| > ||x| - |y||$

GREATEST INTEGER FUNCTION (FLOOR FUNCTION) For any real number x, we use the symbol $[x]$ or, $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x. For example,

$$[2.75] = 2, [3] = 3, [0.74] = 0, [-7.45] = -8 \text{ etc.}$$

The function $f : R \to R$ defined by $f(x) = [x]$ for all $x \in R$ is called the greatest integer function or the floor function.

It is also called a step function.

Clearly, domain of the greatest integer function is the set R of all real numbers and the range is the set Z of all integers as it attains only integer values.

The graph of the greatest integer function is shown in Fig. 2.7.

PROPERTIES OF GREATEST INTEGER FUNCTION If n is an integer and x is a real number between n and $n + 1$, then

(i) $[-n] = -[n]$

(ii) $[x + k] = [x] + k$ for any integer k.

(iii) $[-x] = -[x] - 1$

(iv) $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $[x] - [-x] = \begin{cases} 2[x] + 1, & \text{if } x \notin Z \\ 2[x], & \text{if } x \in Z \end{cases}$

(vi) $[x] \geq k \Rightarrow x > k$, where $k \in Z$

(vii) $[x] \leq k \Rightarrow x < k + 1$, where $k \in Z$

(viii) $[x] > k \Rightarrow x \geq k + 1$, were $k \in Z$

(ix) $[x] < k \Rightarrow x < k$, where $k \in Z$

(x) $[x + y] = [x] + [y + x - [x]]$ for all $x, y \in R$

(xi) $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$.

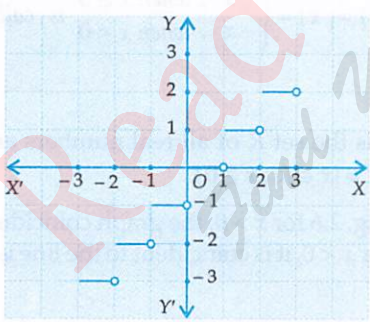


Fig. 2.7 Greatest integer function

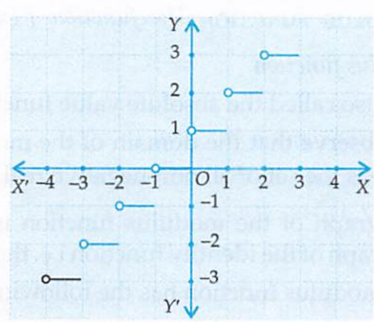


Fig. 2.8 Smallest integer function

SMALLEST INTEGER FUNCTION (CEILING FUNCTION) For any real number x, we use the symbol $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

For example, $\lceil 4.7 \rceil = 5, \lceil -7.2 \rceil = -7, \lceil 5 \rceil = 5, \lceil 0.75 \rceil = 1$ etc.

The function $f : R \to R$ defined by $f(x) = \lceil x \rceil$ for all $x \in R$ is called the smallest integer function or the ceiling function.

It is also a step function. We observe that the domain of the smallest integer function is the set R of all real numbers and its range is the set Z of all integers. The graph of the smallest integer function is as shown in Fig. 2.8.

PROPERTIES OF SMALLEST INTEGER FUNCTION Following are some properties of smallest integer function:

(i) $\lceil -n \rceil = -\lceil n \rceil$, where $n \in Z$ (ii) $\lceil -x \rceil = -\lceil x \rceil + 1$, where $x \in R - Z$

(iii) $\lceil x + n \rceil = \lceil x \rceil + n$, where $x \in R - Z$ and $n \in Z$ (iv) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 1, & \text{if } x \notin Z \\ 0, & \text{if } x \in Z \end{cases}$

(v) $\lceil x \rceil + \lceil -x \rceil = \begin{cases} 2\lceil x \rceil - 1, & \text{if } x \notin Z \\ 2\lceil x \rceil, & \text{if } x \in Z \end{cases}$

FRACTIONAL PART FUNCTION For any real number x, we use the symbol $\{x\}$ to denote the fractional part or decimal part of x. For example, $\{3.45\} = 0.45$, $\{-2.75\} = 0.25$, $\{-0.55\} = 0.45$, $\{3\} = 0$, $\{-7\} = 0$ etc.

The function $f : R \to R$ defined by $f(x) = \{x\}$ for all $x \in R$ is called the fractional part function.

We observe that the domain of the fractional part function is the set R of all real numbers and the range is the set $[0, 1)$. It is evident from the definition that

$$f(x) = \{x\} = x - [x] \quad \text{for all } x \in R$$

The graph of the fractional part function is as shown in Fig. 2.9.

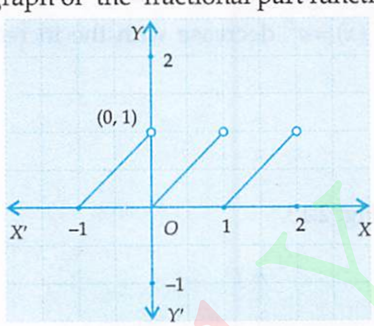


Fig. 2.9 Fractional part function

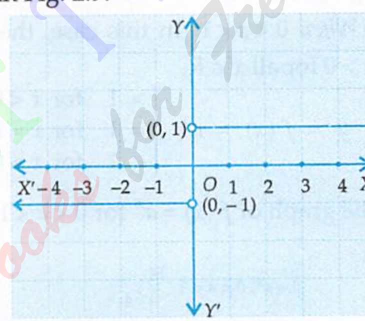


Fig. 2.10 Signum function

SIGNUM FUNCTION *The function f defined by* $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ or, $f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ *is called the signum function.*

The domain of the signum function is the set R of all real numbers and the range is the set $\{-1, 0, 1\}$ The graph of the signum function is as shown in Fig. 2.10.

EXPONENTIAL FUNCTION *If a is a positive real number other than unity, then a function that associates each $x \in R$ to a^x is called the exponential function.*

In other words, a function $f : R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$ is called the exponential function.

We observe that the domain of an exponential function is R the set of all real numbers and the range is the set $(0, \infty)$ as it attains only positive values. As $a > 0$ and $a \neq 1$. So, we have the following cases.

CASE I *When* $a > 1$: We observe that the values of $y = f(x) = a^x$ increase as the values of x increase.

Also, $$f(x) = a^x \begin{cases} < 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ > 1 & \text{for } x > 0. \end{cases}$$

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Thus, the graph of $f(x) = a^x$ for $a > 1$ as shown in Fig. 2.11.

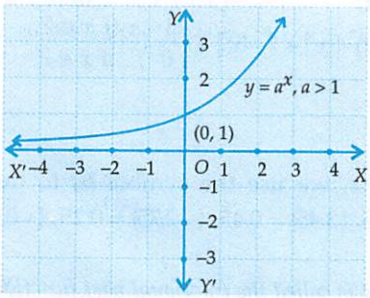

Fig. 2.11 Exponential function

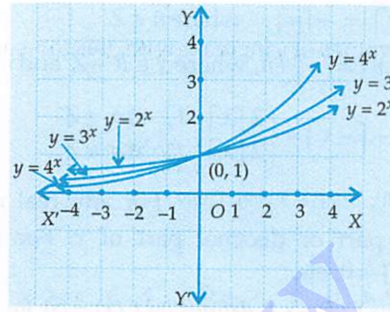

Fig. 2.12 Exponential functions on same scale

We also observe that:

$$2^x < 3^x < 4^x < \ldots \text{ for all } x > 0;\ 2^x = 3^x = 4^x = \ldots = 1 \text{ for } x = 0;\ 2^x > 3^x > 4^x > \ldots \text{ for } x < 0$$

So, the graphs of $f(x) = 2^x$, $f(x) = 3^x$, $f(x) = 4^x$ etc. are as shown in Fig. 2.12.

Case II *When* $0 < a < 1$: In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0$ for all $x \in R$.

Also, $$y = f(x) = a^x \begin{cases} > 1 & \text{for } x < 0 \\ = 1 & \text{for } x = 0 \\ < 1 & \text{for } x > 0 \end{cases}$$

Thus, the graph of $f(x) = a^x$ for $0 < a < 1$ is as shown in Fig. 2.13.

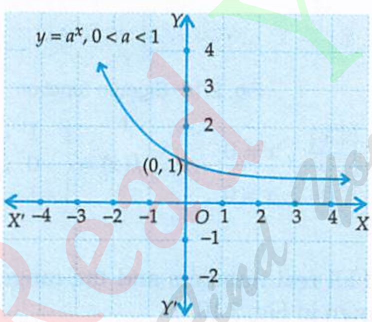

Fig. 2.13 Exponential Function

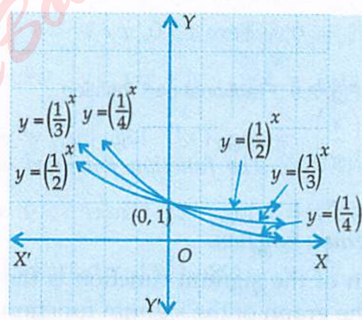

Fig. 2.14 Exponential functions on same scale

The graphs of $f(x) = a^x$, $0 < a < 1$ for different values of a are shown in Fig. 2.14.

REMARK *We have,* $2 < e < 3$. *Therefore, graph of* $f(x) = e^x$ *is identical to that of* $f(x) = a^x$ *for* $a > 1$ *and the graph of* $f(x) = e^{-x}$ *is identical to that of* $f(x) = a^x$ *for* $0 < a < 1$.

LOGARITHMIC FUNCTION *If* $a > 0$ *and* $a \neq 1$, *then the function defined by* $f(x) = \log_a x$, $x > 0$ *is called the logarithmic function.*

In earlier classes we have learnt that the logarithmic function and the exponential function are inverse functions.

i.e. $\log_a x = y \Leftrightarrow x = a^y$

We observe that the domain of the logarithmic function is the set of all positive real numbers i.e. $(0, \infty)$ and the range is the set R of all real numbers.

As $a > 0$ and $a \neq 1$. So, we have the following cases.

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Case I *When* $a>1$: In this case, we obtain: $y=\log_a x \begin{cases} <0 & \text{for } 0<x<1 \\ =0 & \text{for } x=1 \\ >0 & \text{for } x>1 \end{cases}$

Also, the values of y increase with the increase in x. So, the graph of $y=\log_a x$ is as shown in Fig. 2.15.

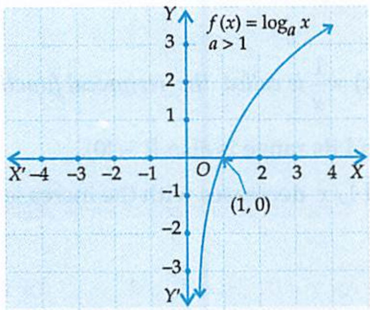

Fig. 2.15 Logarithmic function $f(x)=\log_a x, a>1$

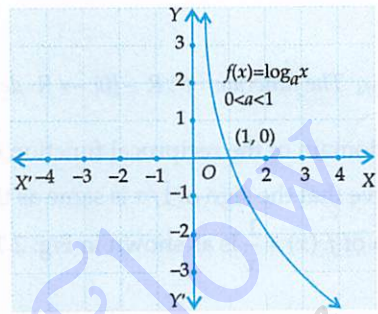

Fig. 2.16 Logarithmic function $f(x)=\log_a x, 0<a<1$

Case II When $0<a<1$: In this case, we obtain: $y=\log_a x \begin{cases} >0 & \text{for } 0<x<1 \\ =0 & \text{for } x=1 \\ <0 & \text{for } x>1 \end{cases}$

Also, the values of y decrease with the increase in x. So, the graph of $y=\log_a x$ is as shown in Fig. 2.16.

Following are some useful properties of logarithmic function:

(i) $\log_a 1 = 0$, where $a>0, a\neq 1$ (ii) $\log_a a = 1$, where $a>0, a\neq 1$

(iii) $\log_a (xy) = \log_a |x| + \log_a |y|$, where $a>0, a\neq 1$ and $xy>0$

(iv) $\log_a \left(\dfrac{x}{y}\right) = \log_a |x| - \log_a |y|$, where $a>0, a\neq 1$ and $\dfrac{x}{y}>0$

(v) $\log_a (x^n) = n \log_a |x|$, where $a>0, a\neq 1$ and $x^n>0$

(vi) $\log_{a^n} x^m = \dfrac{m}{n} \log_{|a|} |x|$, where $a>0, a\neq 1$ and $x^m>0, a^n>0$

(vii) $x^{\log_a y} = y^{\log_a x}$, where $x>0, y>0, a>0, a\neq 1$

(viii) If $a>1$, then the values of $f(x)=\log_a x$ increase with the increase in x.

i.e. $x<y \Leftrightarrow \log_a x < \log_a y$. Also, $\log_a x \begin{cases} <0 & \text{for } 0<x<1 \\ =0 & \text{for } x=1 \\ >0 & \text{for } x>1. \end{cases}$

(ix) If $0<a<1$, then the values of $f(x)=\log_a x$ decrease with the increase in x.

i.e. $x<y \Leftrightarrow \log_a x > \log_a y$. Also, $\log_a x \begin{cases} >0 & \text{for } 0<x<1 \\ =0 & \text{for } x=1 \\ <0 & \text{for } x>1 \end{cases}$

(x) $\log_a x = \dfrac{1}{\log_x a}$ for $a>0, a\neq 1$ and $x>0, x\neq 1$.

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REMARK *Functions $f(x) = \log_a x$ and $g(x) = a^x$ are inverse of each other. So, their graphs are mirror images of each other in the line mirror $y = x$.*

RECIPROCAL FUNCTION The function that associates a real number x to its reciprocal $\frac{1}{x}$ is called the reciprocal function. Since $\frac{1}{x}$ is not defined for $x = 0$. So, we define the reciprocal function as follows:

DEFINITION *The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$ is called the reciprocal function.*

Clearly, domain of the reciprocal function is $R - \{0\}$ and its range is also $R - \{0\}$.

We observe that the sign of $1/x$ is same as that of x and $1/x$ decreases with the increase in x. So, the graph of $f(x) = \frac{1}{x}$ is as shown in Fig. 2.17.

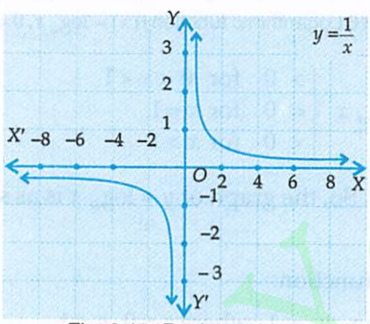

Fig. 2.17 Reciprocal function

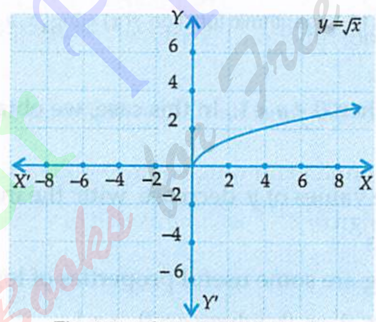

Fig. 2.18 Square root function

SQUARE ROOT FUNCTION The function that associates a real number x to $+\sqrt{x}$ is called the square root function. Since $\sqrt{x}$ is real for $x \geq 0$. So, we defined the square root function as follows:

DEFINITION *The function $f : R^+ \to R$ defined by $f(x) = +\sqrt{x}$ is called the square root function.*

Clearly, domain of the square root function is R^+ i.e. $[0, \infty)$ and its range is also $[0, \infty)$.

We observe that the values of $f(x) = +\sqrt{x}$ increase with the increase in x. So, the graph of $f(x) = +\sqrt{x}$ is as shown in Fig. 2.18.

SQUARE FUNCTION The function that associates a real number x to its square i.e. x^2 is called the square function. Since x^2 is defined for all $x \in R$. So, we define the square function as follows:

DEFINITION *The function $f : R \to R$ defined by $f(x) = x^2$ is called the square function.*

Clearly, domain of the square function is R and its range is the set of all non-negative real numbers i.e. $[0, \infty)$. The graph of $f(x) = x^2$ is parabola as shown in Fig. 2.19.

CUBE FUNCTION The function that associate a real number x to its cube is called the cube function. We observe that x^3 is meaningful for all $x \in R$. So, we define the cube function as follows:

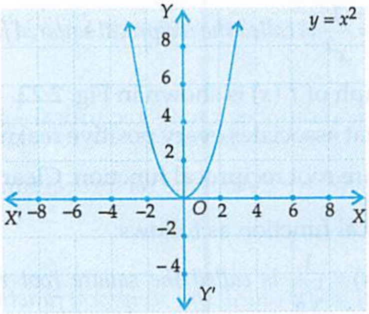

Fig. 2.19 Square function

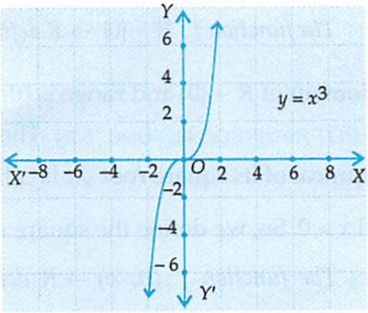

Fig. 2.20 Cube function

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^3$ is called the cube function.*

We observe that the sign of x^3 is same as that of x and the values of x^3 increase with the increase in x. So, the graph of $f(x) = x^3$ is as shown in Fig. 2.20. Clearly, the graph is symmetrical in opposite quadrants.

CUBE ROOT FUNCTION The function that associates a real number x to its cube root $x^{1/3}$ is called the cube root function. Clearly, $x^{1/3}$ is defined for all $x \in R$. So, we define the cube root function as follows:

DEFINITION *The function $f: R \to R$ defined by $f(x) = x^{1/3}$ is called the cube root function.*

Clearly, domain and range of the cube root function are both equal to R.

Also, the sign of $x^{1/3}$ is same as that of x and $x^{1/3}$ increase with the increase in x. So, the graph of $f(x) = x^{1/3}$ is as shown in Fig. 2.21.

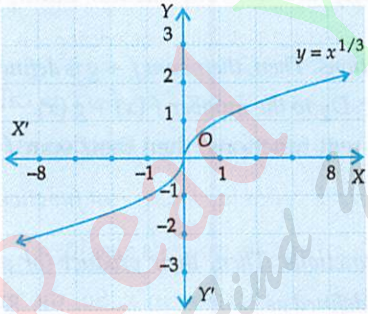

Fig. 2.21 Cube root function

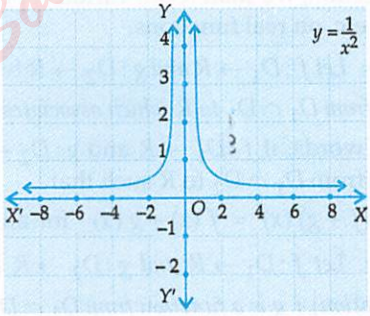

Fig. 2.22 Reciprocal squared function $f(x) = \dfrac{1}{x^2}$

REMARK 1 *A function $f: R \to R$ is said to be a polynomial function if $f(x)$ is a polynomial in x. For example, $f(x) = x^2 - x + 4$, $g(x) = x^3 + 3x^2 + \sqrt{2}\,x - 1$ etc are polynomial functions.*

REMARK 2 *A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$, is called a rational function. The domain of a rational function $f(x) = \dfrac{p(x)}{q(x)}$ is the set of all real numbers, except points where $q(x) = 0$.*

RECIPROCAL SQUARED FUNCTION The function that associates every non-zero real number x to the reciprocal of its square x^2 is called the reciprocal squared function. Clearly, $\dfrac{1}{x^2}$ is defined for non-zero x. So, we define the reciprocal squared function as follows:

DEFINITION *The function $f : R - \{0\} \to R$ defined by $f(x) = \dfrac{1}{x^2}$ is called the reciprocal squared function.*

Clearly, domain of $R - \{0\}$ and range is $(0, \infty)$. The graph of $f(x)$ is shown in Fig. 2.22.

SQUARE ROOT RECIPROCAL FUNCTION The function that associates every positive real number x to the reciprocal of its square root $\sqrt{x}$ is called the square root reciprocal function. Clearly, $\dfrac{1}{\sqrt{x}}$ is real for all $x > 0$. So, we define the square root reciprocal function as follows:

DEFINITION *The function $f : (0, \infty) \to R$ defined by $f(x) = \dfrac{1}{\sqrt{x}}$ is called the square root reciprocal function.*

Clearly, domain and range of f are both $(0, \infty)$. The graph of $f(x)$ is shown in Fig. 2.23.

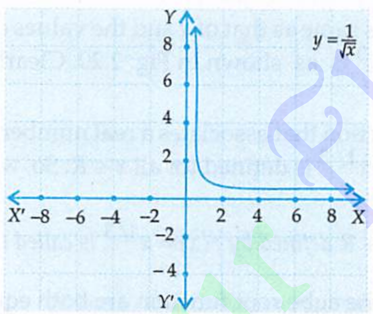

Fig. 2.23 Square root reciprocal function

2.2.1 OPERATIONS ON REAL FUNCTIONS

In this section, we shall recall various operations, namely addition, subtraction, multiplication, division etc. on real functions.

ADDITION *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their sum $f + g$ is defined as that function from $D_1 \cap D_2$ to R which associates each $x \in D_1 \cap D_2$ to the number $f(x) + g(x)$.*

In other words, if $f : D_1 \to R$ and $g : D_2 \to R$ are two real functions, then their sum $f + g$ is a function from $D_1 \cap D_2$ to R such that

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in D_1 \cap D_2$$

PRODUCT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then, their product (or pointwise multiplication) $f\,g$ is a function from $D_1 \cap D_2$ to R and is defined as*

$$(f\,g)(x) = f(x)\,g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

DIFFERENCE (SUBTRACTION) *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the difference of g from f is denoted by $f - g$ and is defined as*

$$(f - g)(x) = f(x) - g(x) \quad \textit{for all } x \in D_1 \cap D_2$$

QUOTIENT *Let $f : D_1 \to R$ and $g : D_2 \to R$ be two real functions. Then the quotient of f by g is denoted by $\dfrac{f}{g}$ and it is a function from $D_1 \cap D_2 - \{x : g(x) = 0\}$ to R defined by*

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \quad \textit{for all } x \in D_1 \cap D_2 - \{x : g(x) = 0\}$$

MULTIPLICATION OF A FUNCTION BY A SCALAR *Let $f : D \to R$ ba a real function and α be a scalar (real number). Then the product αf is a function from D to R and is defined as*

$$(\alpha f)(x) = \alpha f(x) \text{ for all } x \in D.$$

RECIPROCAL OF A FUNCTION *If $f : D \to R$ is a real function, then its reciprocal function $\dfrac{1}{f}$ is a function from $D - \{x : f(x) = 0\}$ to R and is defined as $\left(\dfrac{1}{f}\right)(x) = \dfrac{1}{f(x)}$.*

REMARK 1 *The sum, difference product and quotient are defined for real functions only on their common domain. These operations do not make any sense for general functions even if their domains are same, because the sum, difference, product and quotient may or may not be meaningful for the elements in their common domain.*

REMARK 2 *For any real function $f : D \to R$ and $n \in N$, we define*

$$(\underbrace{ff \dots f}_{n-\text{times}})(x) = \underbrace{f(x)f(x)\dots f(x)}_{n-\text{times}} = \{f(x)\}^n \text{ for all } x \in D$$

2.3 KINDS OF FUNCTIONS

If $f : A \to B$ is a function, then f associates all elements of set A to elements in set B such that an element of set A is associated to a unique element of set B. Following these two conditions we may associate different elements of set A to different elements of set B or more than one element of set A may be associated to the same element of set B. Similarly, there may be some elements in B which do not have their pre-images in A or all elements in B may have their pre-images in A. Corresponding to each of these possibilities we define a type of a function as given below.

2.3.1 ONE-ONE FUNCTION (INJECTION)

DEFINITION *A function $f : A \to B$ is said to be a one-one function or an injection if different elements of A have different images in B.*

Thus, $f : A \to B$ is one-one

$\Leftrightarrow$ $a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$ $\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all $a, b \in A$

ILLUSTRATION 1 *A function which associates to each country in the world, its capital, is one-one because different countries have their different capitals.*

ILLUSTRATION 2 *Let $f : A \to B$ and $g : X \to Y$ be two functions represented by the following diagrams:*

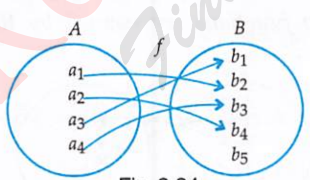

Fig. 2.24

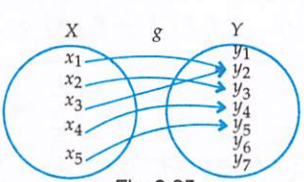

Fig. 2.25

Clearly, $f : A \to B$ is a one-one function. But, $g : X \to Y$ is not one-one because two distinct elements x_1 and x_3 have the same image under function g.

ILLUSTRATION 3 *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6\}$ and $f : A \to B$ be a function defined by $f(x) = x + 2$ for all $x \in A$.*

We observe that f as a set of ordered pairs can be written as $f = \{(1, 3), (2, 4), (3, 5), (4, 6)\}$

Clearly, different elements in A have different images under function f. So, $f : A \to B$ is an injection.

ILLUSTRATION 4 *Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Then, $f(1) = 4$, $f(2) = 5$ and $f(3) = 6$. Clearly, different elements of A have different images in B. So, f is a one-one function.*

Let $f: A \to B$ be a function such that A is an infinite set and we wish to check the injectivity of f. In such a case it is not possible to list the images of all elements of set A to see whether different elements of A have different images or not. The following algorithm provides a systematic procedure to check the injectivity of a function.

ALGORITHM

Step I *Take two arbitrary elements x, y (say) in the domain of f.*

Step II *Put $f(x) = f(y)$*

Step III *Solve $f(x) = f(y)$. If it gives $x = y$ only, then $f: A \to B$ is a one-one function (or an injection). Otherwise not.*

NOTE *Let $f: A \to B$ and let $x, y \in A$. Then, $x = y \Rightarrow f(x) = f(y)$ is always true from the definition. But, $f(x) = f(y) \Rightarrow x = y$ is true only when f is one-one.*

ILLUSTRATION 5 *Find whether the following functions are one-one or not:*

(i) *$f: R \to R$ given by $f(x) = x^3 + 2$ for all $x \in R$.*

(ii) *$f: Z \to Z$ given by $f(x) = x^2 + 1$ for all $x \in Z$*

SOLUTION (i) Let x, y be two arbitrary elements of R (domain of f) such that $f(x) = f(y)$. Then,

$$f(x) = f(y) \Rightarrow x^3 + 2 = y^3 + 2 \Rightarrow x^3 = y^3 \Rightarrow x = y$$

Hence, f is a one-one function from R to itself.

(ii) Let x, y be two arbitrary elements of Z such that $f(x) = f(y)$. Then,

$$f(x) = f(y) \Rightarrow x^2 + 1 = y^2 + 1 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

Here, $f(x) = f(y)$ does not provide the unique solution $x = y$ but it provides $x = \pm y$. So, f is not a one-one function. Infact, $f(2) = 2^2 + 1 = 5$ and $f(-2) = (-2)^2 + 1 = 5$. So, 2 and -2 are two distinct elements having the same image.

NOTE *If A and B are two sets having m and n elements respectively such that $m \le n$, then total numbers of one-one functions from A to B is ${}^nC_m \times m!$.*

2.3.2 MANY-ONE FUNCTION

DEFINITION *A function $f: A \to B$ is said to be a many-one function if two or more elements of set A have the same image in B.*

Thus, $f: A \to B$ is a many-one function if there exist $x, y \in A$ such that $x \ne y$ but $f(x) = f(y)$.

In other words, $f: A \to B$ is a many-one function if it is not a one-one function.

ILLUSTRATION 1 *Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams:*

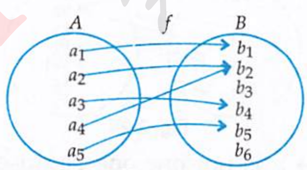

Fig. 2.26

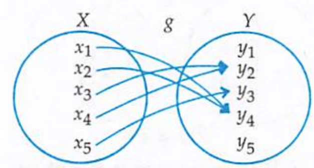

Fig. 2.27

Clearly, $a_2 \ne a_4$ but $f(a_2) = f(a_4)$ and $x_1 \ne x_2$ but $g(x_1) = g(x_2)$. So, f and g are many-one functions.

ILLUSTRATION 2 *Let $A = \{-1, 1, -2, 2\}$ and $B = \{1, 4, 9, 16\}$. Consider $f: A \to B$ given by $f(x) = x^2$. Then, $f(-1) = 1$, $f(1) = 1$, $f(-2) = 4$ and $f(2) = 4$. Thus, 1 and -1 have the same image. Similarly, 2 and -2 also have the same image. So, f is a many-one function.*

ILLUSTRATION 3 *Consider a function $f: Z \to Z$ given by $f(x) = |x|$ for all $x \in Z$. Then, f is a many-one function because for every $a \in Z$, $a \ne 0$, we find that*

$$a \ne -a, \text{ but } |a| = |-a| \Rightarrow f(a) = f(-a) \qquad [\because |a| = |-a|]$$

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ILLUSTRATION 4 *Show that the function $f: Z \to Z$ defined by $f(x) = x^2 + x$ for all $x \in Z$, is a many-one function.*

SOLUTION Let $x, y \in Z$. Then,

$$f(x) = f(y) \Rightarrow x^2 + x = y^2 + y \Rightarrow (x^2 - y^2) + (x - y) = 0$$

$$\Rightarrow \quad (x - y)(x + y + 1) = 0 \Rightarrow x = y \text{ or}, y = -x - 1.$$

Since $f(x) = f(y)$ does not provide the unique solution $x = y$ but it also provides $y = -x - 1$. This means that $x \neq y$ but $f(x) = f(y)$ when $y = -x - 1$. For example, if we put $x = 1$ in $y = -x - 1$ we obtain $y = -2$. This shows that 1 and –2 have the same image under f. Hence, f is a many-one function.

2.3.3 ONTO FUNCTION (SURJECTION)

DEFINITION *A function $f: A \to B$ is said to be an onto function or a surjection if every element of B is the f-image of some element of A i.e., if $f(A) = B$ or range of f is the co-domain of f.*

Thus, $f: A \to B$ is a surjection iff for each $b \in B$, there exists $a \in A$ such that $f(a) = b$.

INTO FUNCTION *A function $f: A \to B$ is an into function if there exists an element in B having no pre-image in A.*

In other words, $f: A \to B$ is an into function if it is not an onto function.

ILLUSTRATION 1 *Let $f: A \to B$ and $g: X \to Y$ be two functions represented by the following diagrams:*

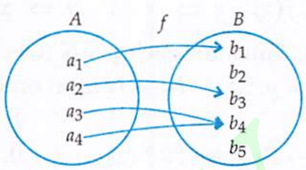

Fig. 2.28

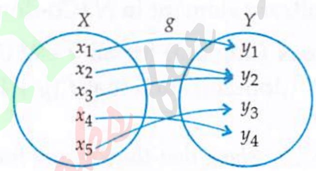

Fig. 2.29

Clearly, b_2 and b_5 are two elements in B which do not have their pre-images in A. So, $f: A \to B$ is an into function.

Under function g every element in Y has its pre-image X. So, $g: X \to Y$ is an onto function.

ILLUSTRATION 2 *Let $A = \{-1, 1, 2, -2\}$, $B = \{1, 4\}$ and $f: A \to B$ be a function defined by $f(x) = x^2$. Then, f is onto, because $f(A) = \{f(-1), f(1), f(2), f(-2)\} = \{1, 4\} = B$.*

ILLUSTRATION 3 *A function $f: N \to N$ defined by $f(x) = 2x$ is not an onto function, because $f(N) = \{2, 4, 6, \ldots\} \neq N$ (co-domain). In otherwords, range $(f) \neq$ co-domain of f.*

The following algorithm can be used to check the subjectivity of a real function.

ALGORITHM

Let $f: A \to B$ be the given function.

Step I *Choose an arbitrary element y in B.*

Step II *Put $f(x) = y$*

Step III *Solve the equation $f(x) = y$ for x and obtain x in terms of y. Let $x = g(y)$*

Step IV *If for all values of $y \in B$, the values of x obtained from $x = g(y)$ are in A, then f is onto. If there are some $y \in B$ for which x, given by $x = g(y)$, is not in A. Then, f is not onto.*

Following illustration will illustrate the above algorithm.

ILLUSTRATION 4 *Discuss the surjectivity of the following functions:*

(i) $f: R \to R$ *given by* $f(x) = x^3 + 2$ *for all* $x \in R$.

(ii) $f: R \to R$ *given by* $f(x) = x^2 + 2$ *for all* $x \in R$.

(iii) $f: Z \to Z$ *given by* $f(x) = 3x + 2$ *for all* $x \in Z$.

SOLUTION (i) Let y be an arbitrary element of R. Then, $f(x) = y \Rightarrow x^3 + 2 = y \Rightarrow x = (y-2)^{1/3}$.

Clearly, for all $y \in R$, $(y-2)^{1/3}$ is a real number. Thus, for all $y \in R$ (co-domain) there exists $x = (y-2)^{1/3}$ in R (domain) such that $f(x) = x^3 + 2 = y$. Hence, $f : R \to R$ is an onto function.

(ii) Clearly, $f(x) = x^2 + 2 \geq 2$ for all $x \in R$. So, negative real numbers in R(co-domain) do not have their pre-images in R(domain). Hence, f is not an onto function.

(iii) Let y be an arbitrary element of Z(co-domain). Then, $f(x) = y \Rightarrow 3x + 2 = y \Rightarrow x = \frac{y-2}{3}$.

Clearly, if $y = 0$, then $x = -2/3 \notin Z$. Thus, $y = 0 \in Z$ does not have its pre-image in Z(domain). Hence, f is not an onto function.

ILLUSTRATION 5 *Show that the function $f : N \to N$ given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x \geq 2$, is onto but not one-one.* **[NCERT]**

SOLUTION It is given that $f(x) = \begin{cases} 1 & \text{, when } x = 1, 2 \\ x - 1, & \text{when } x \geq 2 \end{cases}$

Clearly, $f(1) = f(2) = 1$ i.e. 1 and 2 have the same image. So, $f : N \to N$ is a many-one function. Let y be an arbitrary element in N (Co-domain). Then, $f(x) = y \Rightarrow x - 1 = y \Rightarrow x = y + 1$. Clearly, $y + 1 \in N$ (domain) for all $y \in N$ (Co-domain). Thus, for each $y \in N$ (co-domain) there exists $y + 1 \in N$ (domain) such that $f(y+1) = y + 1 - 1 = y$. So, $f : N \to N$ is an onto function.

ILLUSTRATION 6 *Show that the Signum function $f : R \to R$, given by $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$ is neither one-one nor onto.* **[NCERT]**

SOLUTION Clearly, all positive real numbers have the same image equal to 1. So, f is a many-one function. We observe that the range of f is $\{-1, 0, 1\}$ which is not equal to the co-domain of f. So, f is not onto. Hence, f is neither one-one nor onto.

2.3.4 BIJECTION (ONE-ONE ONTO FUNCTION)

DEFINITION *A function $f : A \to B$ is a bijection if it is one-one as well as onto.*

In other words, a function $f : A \to B$ is a bijection, if it is

(i) *one-one i.e. $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.*

(ii) *onto i.e. for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.*

ILLUSTRATION 1 *Let $f : A \to B$ be a function represented by the diagram shown in Fig. 2.30. Clearly, f is a bijection since it is both injective as well as surjective.*

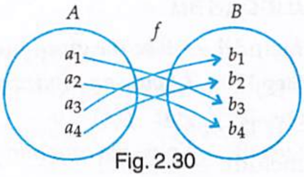

Fig. 2.30

ILLUSTRATION 2 *Prove that the function $f : Q \to Q$ given by $f(x) = 2x - 3$ for all $x \in Q$ is a bijection.*

SOLUTION We observe the following properties of f.

Injectivity: Let x, y be two arbitrary elements in Q. Then,

$$f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow 2x = 2y \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in Q$. So, f is an injective map.

Surjectivity : Let y be an arbitrary element of Q. Then, $f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y+3}{2}$.

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Clearly, for all $y \in Q$ the value of x given by $x = \dfrac{y+3}{2} \in Q$. Thus, for all $y \in Q$ (co-domain) there exists $x \in Q$ (domain) given by $x = \dfrac{y+3}{2}$ such that $f(x) = f\left(\dfrac{y+3}{2}\right) = 2\left(\dfrac{y+3}{2}\right) - 3 = y$. That is every element in the co-domain has its pre-image in x. So, f is a surjection.

Hence, $f: Q \to Q$ is a bijection.

ILLUSTRATION 3 *Show that the function $f: R \to R$ defined by $f(x) = 3x^3 + 5$ for all $x \in R$ is a bijection.*

SOLUTION We observe the following properties of f.

Injectivity: Let x, y be any two elements of R(domain). Then,

$$f(x) = f(y) \Rightarrow 3x^3 + 5 = 3y^3 + 5 \Rightarrow x^3 = y^3 \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$. So, f is an injective map.

Surjectivity : Let y be an arbitrary element of R(co-domain). Then,

$$f(x) = y \Rightarrow 3x^3 + 5 = y \Rightarrow x^3 = \frac{y-5}{3} \Rightarrow x = \left(\frac{y-5}{3}\right)^{1/3}$$

Thus, we find that for all $y \in R$ (co-domain) there exists $x = \left(\dfrac{y-5}{3}\right)^{1/3} \in R$ (domain) such that

$$f(x) = f\left(\left(\frac{y-5}{3}\right)^{1/3}\right) = 3\left[\left(\frac{y-5}{3}\right)^{1/3}\right]^3 + 5 = y - 5 + 5 = y$$

This shows that every element in the co-domain has its pre-image in the domain. So, f is a surjection. Hence, f is a bijection.

ILLUSTRATION 4 *Let $A = \{x \in R : -1 \le x \le 1\} = B$. Show that $f: A \to B$ given by $f(x) = x|x|$ is a bijection.*

SOLUTION We observe the following properties of f.

Injectivity: Let x, y be any two elements in A. Then, $x \ne y \Rightarrow x|x| \ne y|y| \Rightarrow f(x) \ne f(y)$.

So, $f: A \to B$ is an injective map.

Surjectivity: We have, $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \ge 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

If $0 \le x \le 1$, then $f(x) = x^2$ takes all values between 0 and 1 including these two points. Also, if $-1 \le x < 0$, then $f(x) = -x^2$ takes all values between -1 and 0 including -1. Therefore, $f(x)$ takes every value between -1 and 1 including -1 and 1. So, range of f is same as its co-domain. Hence, $f: A \to B$ is an onto function. Thus, $f: A \to B$ is both one-one and onto. Hence, it is a bijection.

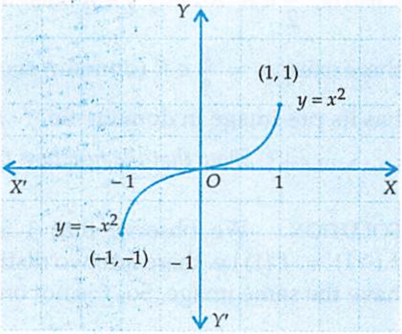

Fig. 2.31 Graph of $f(x) = x|x|$

ALITER We have, $f(x) = x|x| = \begin{cases} x^2, & \text{if } x \ge 0 \\ -x^2, & \text{if } x < 0 \end{cases}$

For $x \ge 0$, $f(x) = x^2$ represents a parabola opening upward and for $x < 0$, $f(x) = -x^2$ represents a parabola opening downward. So, the graph of $f(x)$ is as shown in Fig. 2.31.

It is evident from the graph of $f(x)$ that f is one-one and onto.

REMARK *It follows from the above discussion that if A and B are two finite sets and $f: A \to B$ is a function, then*

(i) *f is an injection* $\Rightarrow$ $n(A) \le n(B)$ (ii) *f is a surjection* $\Rightarrow$ $n(B) \le n(A)$

(iii) *f is a bijection* $\Rightarrow$ $n(A) = n(B)$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Let A be the set of all 50 students of class XII in a central school. Let $f: A \to N$ be a function defined by $f(x) =$ Roll number of student x. Show that f is one-one but not onto.*

SOLUTION Here, f associates each students to his (her) roll number. Since no two different students of the class can have the same roll number. Therefore, f is one-one.

We observe that $f(A) =$ Range of $f = \{1, 2, 3, \dots, 50\} \ne N$ i.e. range of f is not same as its co-domain. So, f is not onto.

EXAMPLE 2 *Show that the function $f: N \to N$, given by $f(x) = 2x$, is one-one but not onto.* **[NCERT]**

SOLUTION We observe the following properties of f.

Injectivity: Let $x_1, x_2 \in N$ such that $f(x_1) = f(x_2)$. Then, $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

Surjectivity: Clearly, f takes even values. Therefore, no odd natural number in N (co-domain) has its pre-image in domain. So, f is not onto.

EXAMPLE 3 *Prove that $f: R \to R$, given by $f(x) = 2x$, is one-one and onto.* **[NCERT]**

SOLUTION We observe the following properties of f.

Injectivity: Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$. Then, $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So, $f: R \to R$ is one-one.

Surjectivity: Let y be any real number in R (co-domain). Then, $f(x) = y \Rightarrow 2x = y \Rightarrow x = \frac{y}{2}$.

Clearly, $\frac{y}{2} \in R$ for any $y \in R$ such that $f\left(\frac{y}{2}\right) = 2\left(\frac{y}{2}\right) = y$. Thus, for each $y \in R$ (co-domain) there exists $x = \frac{y}{2} \in R$ (domain) such that $f(x) = y$. This means that each element in co-domain has its pre-image in domain. So, $f: R \to R$ is onto. Hence, $f: R \to R$ is a bijection.

EXAMPLE 4 *Show that the function $f: R \to R$, defined as $f(x) = x^2$, is neither one-one nor onto.* **[NCERT]**

SOLUTION We observe that 1 and $-1 \in R$ such that $f(-1) = f(1)$ i.e. there are two distinct elements in R which have the same image. So, f is not one-one.

Since $f(x)$ assumes only non-negative values. So, no negative real number in R (co-domain) has its pre-image in domain of f i.e. R. Consequently f is not onto.

These facts are evident from the graph of $f(x)$ as shown in Fig. 2.32.

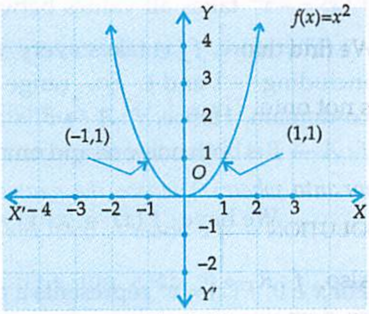

Fig. 2.32

EXAMPLE 5 *Show that $f : R \to R$, defined as $f(x) = x^3$, is a bijection.* **[NCERT]**

SOLUTION We observe the following properties of f.

Injectivity: Let $x, y \in R$ such that $f(x) = f(y)$. Then, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

So, $f : R \to R$ is one-one.

Surjectivity: Let $y \in R$ (co-domain). Then, $f(x) = y \Rightarrow x^3 = y \Rightarrow x = y^{1/3}$.

Clearly, $y^{1/3} \in R$ (domain) for all $y \in R$ (co-domain).

Thus, for each $y \in R$ (co-domain) there exists $x = y^{1/3} \in R$ (domain) such that $f(x) = x^3 = y$.

So, $f : R \to R$ is onto. Hence, $f : R \to R$ is a bijection.

EXAMPLE 6 *Show that the function $f : R_0 \to R_0$, defined as $f(x) = \frac{1}{x}$, is one-one onto, where R_0 is the set of all non-zero real numbers. Is the result true, if the domain R_0 is replaced by N with co-domain being same as R_0?* **[NCERT]**

SOLUTION We observe the following properties of f.

Injectivity: Let $x, y \in R_0$ such that $f(x) = f(y)$. Then, $f(x) = f(y) \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y$.

So, $f : R_0 \to R_0$ is one-one.

Surjectivity: Let y be an arbitrary element of R_0 (co-domain) such that $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{1}{x} = y \Rightarrow x = \frac{1}{y}$$

Clearly, $x = \frac{1}{y} \in R_0$ (domain) for all $y \in R_0$ (co-domain).

Thus, for each $y \in R_0$ (co-domain) there exits $x = \frac{1}{y} \in R_0$ (domain) such that $f(x) = \frac{1}{x} = y$. So, $f : R_0 \to R_0$ is onto.

Hence, $f : R_0 \to R_0$ is one-one onto. This is also evident from the graph of $f(x)$ as shown in Fig. 2.33.

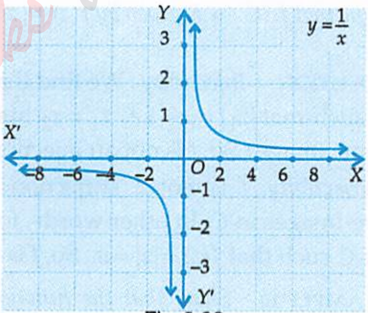

Fig. 2.33

Let us now consider $f : N \to R_0$ given by $f(x) = \frac{1}{x}$. For any $x, y \in N$, we find that

$$f(x) = f(y) \Rightarrow \frac{1}{x} = \frac{1}{y} \Rightarrow x = y. \text{ So, } f : N \to R_0 \text{ is one-one.}$$

We find that $\frac{2}{3}, \frac{3}{5}$ etc. in co-domain R_0 do not have their pre-image in domain N. So, $f : N \to R_0$ is not onto. Thus, $f : N \to R_0$ is one-one but not onto.

EXAMPLE 7 *Prove that the greatest integer function $f : R \to R$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.* **[NCERT]**

SOLUTION We observe that $f(x) = 0$ for all $x \in [0, 1)$. So, $f : R \to R$ is not one-one.

Also, $f : R \to R$ does not attain non-integral values. Therefore, non-integer points in R (co-domain) do not have their pre-images in the domain. So, $f : R \to R$ is not onto. Hence, $f : R \to R$ is neither one-one nor onto.

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This is also evident from the graph of the greatest integer function shown in Fig. 2.34.

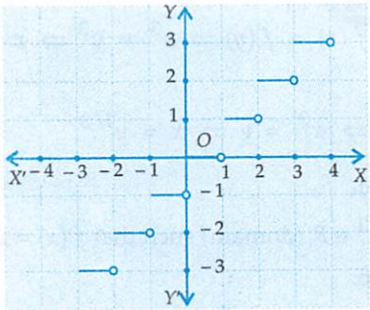

Fig. 2.34 Greatest integer function

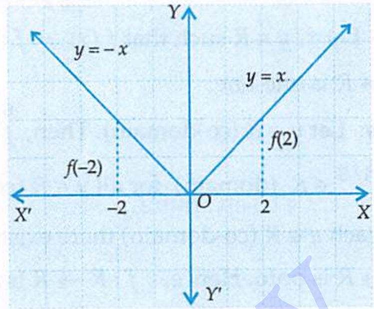

Fig. 2.35 Modulus function

EXAMPLE 8 *Show that the modulus function $f : R \to R$, given by $f(x) = |x|$ is neither one-one nor onto.* **[NCERT]**

SOLUTION We observe that $f(-2) = f(2)$. So, f is not one-one.

Also, $f(x) = |x|$ assumes only non-negative values. So, negative real numbers in R (co-domain) do not have their pre-images in R (domain). Hence, f is neither one-one nor onto.

This is also evident from the graph of $f(x) = |x|$ shown in Fig. 2.35.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 9 *Let C and R denote the set of all complex numbers and all real numbers respectively. Then show that $f : C \to R$ given by $f(z) = |z|$ for all $z \in C$ is neither one-one nor onto.*

[NCERT EXEMPLAR]

SOLUTION *Injectivity*: We find that $z_1 = 1 - i$ and $z_2 = 1 + i$ are two distinct complex numbers in C such that $|z_1| = |z_2|$ *i.e.* $z_1 \neq z_2$ but $f(z_1) = f(z_2)$. Thus, different elements in C may have the same image. So, f is not an injection.

Surjectivity: f is not a surjection, because negative real numbers in R do not have their pre-images in C. In other words, for every negative real number a there is no complex number $z \in C$ such that $f(z) = |z| = a$. So, f is not a surjection.

EXAMPLE 10 *Show that the function $f : R \to R$ given by $f(x) = ax + b$, where $a, b \in R, a \neq 0$ is a bijection.* **[CBSE 2010]**

SOLUTION *Injectivity*: Let x, y be any two real numbers. Then,

$$f(x) = f(y) \Rightarrow ax + b = ay + b \Rightarrow ax = ay \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$(domain). So, f is an injection.

Surjectivity: Let y be an arbitrary element of R(co-domain). Then,

$$f(x) = y \Rightarrow ax + b = y \Rightarrow x = \frac{y-b}{a}.$$

Clearly, $x = \frac{y-b}{a} \in R$ (domain) for all $y \in R$ (co-domain). Thus, for all $y \in R$ (co-domain) there exists $x = \frac{y-b}{a} \in R$ (domain) such that $f(x) = f\left(\frac{y-b}{a}\right) = a\left(\frac{y-b}{a}\right) + b = y$.

This means that every element in co-domain has its pre-image in domain. So, f is a surjection. Hence, f is a bijection.

EXAMPLE 11 *Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a mapping defined by $f(x) = \frac{x-1}{x-2}$, show that f is bijective.*

SOLUTION *Injectivity:* Let x, y be any two elements of A. Then,

$$f(x) = f(y)$$
$$\Rightarrow \quad \frac{x-1}{x-2} = \frac{y-1}{y-2}$$
$$\Rightarrow \quad (x-1)(y-2) = (x-2)(y-1) \Rightarrow xy - y - 2x + 2 = xy - x - 2y + 2 \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$. So, f is an injective map.

Surjectivity: Let y be an arbitrary element of B. Then,

$$f(x) = y \Rightarrow \frac{x-1}{x-2} = y \Rightarrow (x-1) = y(x-2) \Rightarrow x = \frac{1-2y}{1-y}$$

Clearly, $x = \frac{1-2y}{1-y}$ is a real number for all $y \neq 1$. Also, $\frac{1-2y}{1-y} \neq 2$ for any y, for, if we take $\frac{1-2y}{1-y} = 2$, then we get $1 = 2$, which is wrong. Thus, every element y in B has its pre-image x in A given by $x = \frac{1-2y}{1-y}$. So, f is a surjective map. Hence, f is a bijective map.

EXAMPLE 12 *Let A and B be two sets. Show that $f : A \times B \to B \times A$ defined by $f(a, b) = (b, a)$ is a bijection.* **[NCERT]**

SOLUTION *Injectivity:* Let (a_1, b_1) and $(a_2, b_2) \in A \times B$ such that

$$f(a_1, b_1) = f(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2 \text{ and } a_1 = a_2 \Rightarrow (a_1, b_1) = (a_2, b_2)$$

Thus, $f(a_1, b_1) = f(a_2, b_2) \Rightarrow (a_1, b_1) = (a_2, b_2)$ for all $(a_1, b_1), (a_2, b_2) \in A \times B$.

So, f is an injective map.

Surjectivity: Let (b, a) be an arbitrary element of $B \times A$. Then, $b \in B$ and $a \in A \Rightarrow (a, b) \in A \times B$. Thus, for all $(b, a) \in B \times A$ there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$. So, $f : A \times B \to B \times A$ is an onto function. Hence, f is a bijection.

EXAMPLE 13 *Let A be any non-empty set. Then, prove that the identity function on set A is a bijection.*

SOLUTION The identity function $I_A : A \to A$ is defined as $I_A(x) = x$ for all $x \in A$.

Injectivity: Let x, y be any two elements of A. Then,

$$I_A(x) = I_A(y) \Rightarrow x = y \qquad \text{[By definition of } I_A\text{]}$$

So, I_A is an injective map.

Surjectivity: Let $y \in A$. Then, there exists $x = y \in A$ such that $I_A(x) = x = y$.

So, I_A is a surjective map. Hence, $I_A : A \to A$ is a bijection.

EXAMPLE 14 *Consider the identity function $I_N : N \to N$ defined as, $I_N(x) = x$ for all $x \in N$. Show that although I_N is onto but $I_N + I_N : N \to N$ defined as $(I_N + I_N)(x) = I_N(x) + I_N(x) = x + x = 2x$ is not onto.* **[NCERT]**

SOLUTION We know that the identity function on a given set is always a bijection. Therefore, $I_N : N \to N$ is onto.

We find that, $(I_N + I_N)(x) = 2x$ for all $x \in N$. This means that under $I_N + I_N$, images of natural numbers are even natural numbers. So, odd natural numbers in N (co-domain) do not have their pre-images in domain N. For example, 1, 3, 5 etc. do not have their pre-images. So, $I_N + I_N : N \to N$ is not onto.

EXAMPLE 15 *Consider the function $f : [0, \pi/2] \to R$ given by $f(x) = \sin x$ and $g : [0, \pi/2] \to R$ given by $g(x) = \cos x$. Show that f and g are one-one, but $f + g$ is not one-one.* **[NCERT]**

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SOLUTION We observe that for any two distinct elements x_1 and x_2 in $[0, \pi/2]$

$\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$ [See graphs of $f(x) = \sin x$ & $f(x) = \cos x$]

$\Rightarrow \quad f(x_1) \neq f(x_2)$ and $g(x_1) \neq g(x_2) \Rightarrow f$ and g are one-one.

We find that

$$(f+g)(x) = f(x) + g(x) = \sin x + \cos x$$

$$\Rightarrow \quad (f+g)(0) = \sin 0 + \cos 0 = 1 \text{ and } (f+g)\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1$$

Thus, $0 \neq \dfrac{\pi}{2}$ but, $(f+g)(0) = (f+g)\left(\dfrac{\pi}{2}\right)$. So, $f+g$ is not one-one.

EXAMPLE 16 *Let $f: X \to Y$ be a function. Define a relation R on X given by $R = \{(a, b) : f(a) = f(b)\}$. Show that R is an equivalence relation on X.* **[NCERT, CBSE 2010]**

SOLUTION We observe the following properties of relation R.

Reflexivity: For any $a \in X$, we find that

$$f(a) = f(a) \Rightarrow (a, a) \in R \Rightarrow R \text{ is reflexive.}$$

Symmetry: Let $a, b \in X$ be such that $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$. So, R is symmetric.

Transitivity: Let $a, b, c \in X$ be such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R$ and $(b, c) \in R \Rightarrow f(a) = f(b)$ and $f(b) = f(c) \Rightarrow f(a) = f(c) \Rightarrow (a, c) \in R$.

So, R is transitive. Hence, R is an equivalence relation.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 17 *Show that the function $f: R \to R$ given by $f(x) = \cos x$ for all $x \in R$, is neither one-one nor onto.* **[NCERT EXEMPLAR]**

SOLUTION *Injectivity*: We know that $f(0) = \cos 0 = 1$ and $f(2\pi) = \cos 2\pi = 1$.

So, different elements in R may have the same image. Hence, f is not an injection.

Surjectivity: We know that $\cos x$ takes all values between -1 and 1. Therefore, its range is $[-1, 1]$. Thus, the range of $f(x)$ is not equal to its co-domain. So, f is not a surjection.

EXAMPLE 18 *Let $f: N - \{1\} \to N$ be defined by, $f(n)$ = the highest prime factor of n. Show that f is neither one-one nor onto. Find the range of f.*

SOLUTION We find that

$f(6)$ = (the highest prime factor of 6) = 3, $f(9)$ = (the highest prime factor of 9) = 3

and, $f(12)$ = (the highest prime factor of 12) = 3. Therefore, $f(6) = f(9) = f(12)$.

So, f is a many-one function.

Clearly, image of any $n \in N - \{1\}$ is the largest prime number that divides n. So, the range of f consists of prime numbers only. Consequently, range of $f \neq N$ (co-domain). So, f is not onto function. Hence, f is neither one-one nor onto. The range of f is the set of all prime numbers.

EXAMPLE 19 *Let $A = \{1, 2\}$. Find all one-to-one functions from A to A.*

SOLUTION Let $f: A \to A$ be a one-one function. Then, $f(1)$ has two choices, namely, 1 or 2. So, $f(1) = 1$ or, $f(1) = 2$.

CASE I *When $f(1) = 1$*: As $f: A \to A$ is one-one. Therefore, $f(2) = 2$. Thus, we obtain: $f(1) = 1$ and $f(2) = 2$.

CASE II *When $f(1) = 2$*: Since $f: A \to A$ is one-one. Therefore, $f(2) = 1$. Thus, in this case, we obtain: $f(1) = 2$ and $f(2) = 1$.

So, there are two one-one functions say f and g from A to A given by $f(1)=1$, $f(2)=2$ and, $g(1)=2$, $g(2)=1$.

ALITER All one-to-one functions from A to itself can be expressed in the following two row notation as follows:

$$f=\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}, g=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$$

First row contains elements of the domain and second row contains the corresponding images. Clearly, each arrangement of second row provides a one-to-one function from A to itself.

EXAMPLE 20 *Show that the function* $f: R \to \{x \in R: -1 < x < 1\}$ *defined by* $f(x)=\dfrac{x}{1+|x|}$, $x \in R$ *is one-one onto function.* **[NCERT]**

SOLUTION We have, $f(x)=\dfrac{x}{1+|x|}=\begin{cases}\dfrac{x}{1+x}, \text{ if } x \geq 0\\[2mm] \dfrac{x}{1-x}, \text{ if } x<0\end{cases}$

So, following cases arise:

Case I *When* $x \geq 0$: In this case, we have $f(x)=\dfrac{x}{1+x}$.

Injectivity: Let $x, y \in R$ such that $x \geq 0$, $y \geq 0$. Then,

$$f(x)=f(y) \Rightarrow \frac{x}{1+x}=\frac{y}{1+y} \Rightarrow x+xy=y+xy \Rightarrow x=y. \text{ So, } f \text{ is an injective map.}$$

Surjectivity: When $x \geq 0$, we find that: $f(x)=\dfrac{x}{1+x} \geq 0$ and $f(x)<1$. Let $y \in [0, 1)$ be any real number. Then, $f(x)=y \Rightarrow \dfrac{x}{1+x}=y \Rightarrow x=\dfrac{y}{1-y}$.

Clearly, $x \geq 0$ for all $y \in [0, 1)$. Thus, for each $y \in [0, 1)$ there exists $x=\dfrac{y}{1-y} \geq 0$ such that $f(x)=y$.

So, f is an onto function from $[0, 1)$ to $[0, 1)$

Case II *When* $x<0$: In this case, we have $f(x)=\dfrac{x}{1-x}$.

Injectivity: Let $x, y \in R$ such that $x<0$, $y<0$. Then,

$$f(x)=f(y) \Rightarrow \frac{x}{1-x}=\frac{y}{1-y} \Rightarrow x-xy=y-xy \Rightarrow x=y. \text{ So, } f \text{ is an injective map.}$$

Surjectivity: When $x<0$, we find that: $f(x)=\dfrac{x}{1-x}<0$.

Also, $f(x)=\dfrac{x}{1-x}=-1+\dfrac{1}{1-x}>-1$. Therefore, $-1<f(x)<0$.

Let $y \in (-1, 0)$ be an arbitrary real number such that $f(x)=y$. Then,

$$f(x)=y \Rightarrow \frac{x}{1-x}=y \Rightarrow x=\frac{y}{1+y}$$

Clearly, $x<0$ for $y \in (-1, 0)$. Thus, for each $y \in (-1, 0)$ there exists $x=\dfrac{y}{1+y}<0$ such that $f(x)=y$.

So, f is an onto function from $(-1, 0)$ to $(-1, 0)$.

Hence, $f: R \to \{x \in R: -1<x<1\}$ is a one-one onto function.

EXAMPLE 21 *Show that the function $f : R \to R$ given by $f(x) = x^3 + x$ is a bijection.*

SOLUTION *Injectivity*: Let $x, y \in R$ such that

$$f(x) = f(y)$$
$$\Rightarrow \quad x^3 + x = y^3 + y$$
$$\Rightarrow \quad x^3 - y^3 + (x - y) = 0$$
$$\Rightarrow \quad (x - y)(x^2 + xy + y^2 + 1) = 0$$
$$\Rightarrow \quad x - y = 0 \quad [\because x^2 + xy + y^2 \geq 0 \text{ for all } x, y \in R \therefore x^2 + xy + y^2 + 1 \geq 1 \text{ for all } x, y \in R]$$
$$\Rightarrow \quad x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$. So, f is an injective map.

Surjectivity: Let y be an arbitrary element of R. Then,

$$f(x) = y \Rightarrow x^3 + x = y \Rightarrow x^3 + x - y = 0$$

We know that an odd degree equation has at least one real root. Therefore, for every real value of y, the equation $x^3 + x - y = 0$ has a real root α (say) such that

$$\alpha^3 + \alpha - y = 0 \Rightarrow \alpha^3 + \alpha = y \Rightarrow f(\alpha) = y$$

Thus, for every $y \in R$ there exists $\alpha \in R$ such that $f(\alpha) = y$. So, f is a surjective map.

Hence, $f : R \to R$ is a bijection.

EXAMPLE 22 *Show that $f : N \to N$ defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ is many-one onto function.*

[NCERT, CBSE 2009]

SOLUTION We observe that $f(1) = \dfrac{1+1}{2} = 1$ and $f(2) = \dfrac{2}{2} = 1$. Thus, $1, 2 \in N$ such that $1 \neq 2$ but $f(1) = f(2)$. So, f is a many-one function.

Surjectivity Let n be an arbitrary element of N.

If n is an odd natural number, then $2n - 1$ is also an odd natural number such that

$$f(2n-1) = \frac{2n - 1 + 1}{2} = n$$

If n is an even natural number, then $2n$ is also an even natural number such that

$$f(2n) = \frac{2n}{2} = n.$$

Thus, for every $n \in N$ (whether even or odd) there exists its pre-image in N. So, f is a surjection.

Hence, f is a many-one onto function.

EXAMPLE 23 *Show that the function $f : N \to N$ given by, $f(n) = n - (-1)^n$ for all $n \in N$ is a bijection.*

SOLUTION We have,

$$f(n) = n - (-1)^n \text{ for all } n \in N \Rightarrow f(n) = \begin{cases} n - 1, & \text{if } n \text{ is even} \\ n + 1, & \text{if } n \text{ is odd} \end{cases}$$

Injectivity: Let n, m be any two even natural numbers. Then, $f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$.

If n, m are any two odd natural numbers. Then, $f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$.

Thus in both the cases, $f(n) = f(m) \Rightarrow n = m$.

If n is even and m is odd, then $n \neq m$. Also $f(n)$ is odd and $f(m)$ is even. So, $f(n) \neq f(m)$.

Thus, $n \neq m \Rightarrow f(n) \neq f(m)$. So, f is an injective map.

Surjectivity: Let n be an arbitrary natural number. If n is an odd natural number, then there exists an even natural number $n+1$ such that $f(n+1) = n+1-1 = n$.

If n is an even natural number, then there exists an odd natural number $(n-1)$ such that $f(n-1) = n-1+1 = n$. Thus, every $n \in N$ has its pre-image in N.

So, $f: N \to N$ is a surjection. Hence, $f: N \to N$ is a bijection.

EXAMPLE 24 *Let $f: N \cup \{0\} \to N \cup \{0\}$ be defined by* $f(n) = \begin{cases} n+1, \text{ if } n \text{ is even} \\ n-1, \text{ if } n \text{ is odd} \end{cases}$. *Show that f is a bijection.*

SOLUTION *f is an injection :* Let $n, m \in N \cup \{0\}$.

If n and m are even, then $f(n) = f(m) \Rightarrow n+1 = m+1 \Rightarrow n = m$.

If n and m are odd, then $f(n) = f(m) \Rightarrow n-1 = m-1 \Rightarrow n = m$.

Thus, in both case, we obtain: $f(n) = f(m) \Rightarrow n = m$.

If n is odd and m is even, then $f(n) = n-1$ is even and $f(m) = m+1$ is odd. Therefore, $f(n) \neq f(m)$. Thus, $n \neq m \Rightarrow f(n) \neq f(m)$.

Similarly, if n is even and m is odd, then $n \neq m \Rightarrow f(n) \neq f(m)$.

Hence, f is an injection.

f is a surjection : Let n be an arbitrary element of $N \cup \{0\}$.

If n is an odd natural number, there exist an even natural number $n-1 \in N \cup \{0\}$ (domain) such that $f(n-1) = n-1+1 = n$.

If n is an even natural number, then there exists an odd natural number $n+1 \in N \cup \{0\}$ (domain) such that $f(n+1) = n+1-1 = n$. Also, $f(1) = 0$.

Thus, every element of $N \cup \{0\}$ (co-domain) has its pre-image in $N \cup \{0\}$ (domain). So, f is an onto function.

EXAMPLE 25 *Let A be a finite set. If $f: A \to A$ is a one-one function, show that f is onto also.*

SOLUTION Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$. In order to prove that f is onto function, we will have to show that every element in A (co-domain) has its pre-image in the domain A. In other words, range of $f = A$.

Since $f: A \to A$ is a one-one function. Therefore, $f(a_1), f(a_2), \ldots, f(a_n)$ are distinct elements of set A. But, A has only n elements. Therefore, $A = \{f(a_1), f(a_2), \ldots, f(a_n)\}$ i.e. Co-domain = Range.

Hence, $f: A \to A$ is onto.

EXAMPLE 26 *Let A be a finite set. If $f: A \to A$ is an onto function, show that f is one-one also.*

SOLUTION Let $A = \{a_1, a_2, \ldots, a_n\}$. In order to prove that f is a one-one function, we will have to show that $f(a_1), f(a_2), \ldots, f(a_n)$ are distinct elements of A.

Clearly, Range of $f = \{f(a_1), f(a_2), \ldots, f(a_n)\}$. Since $f: A \to A$ is an onto function. Therefore,

$$\text{Range of } f = A \Rightarrow \{f(a_1), f(a_2), \ldots, f(a_n)\} = A$$

But, A is a finite set consisting of n elements. Therefore, $f(a_1), f(a_2), f(a_3), \ldots, f(a_n)$ are distinct elements of A. Hence, $f: A \to A$ is one-one.

EXERCISE 2.1

BASIC

1. Give an example of a function
   (i) which is one-one but not onto.
   (ii) which is not one-one but onto.
   (iii) which is neither one-one nor onto. **[NCERT EXEMPLAR]**
2. Which of the following functions from A to B are one-one and onto?
   (i) $f_1 = \{(1, 3), (2, 5), (3, 7)\}$; $A = \{1, 2, 3\}$, $B = \{3, 5, 7\}$
   (ii) $f_2 = \{(2, a), (3, b), (4, c)\}$; $A = \{2, 3, 4\}$, $B = \{a, b, c\}$

(iii) $f_3 = \{(a, x), (b, x), (c, z), (d, z)\}$; $A = \{a, b, c, d\}$, $B = \{x, y, z\}$

3. Let $A = \{-1, 0, 1\}$ and $f = \{(x, x^2) : x \in A\}$. Show that $f : A \to A$ is neither one-one nor onto.

4. Classify the following functions as injection, surjection or bijection:

(i) $f : N \to N$ given by $f(x) = x^2$ (ii) $f : Z \to Z$ given by $f(x) = x^2$

(iii) $f : N \to N$ given by $f(x) = x^3$ (iv) $f : Z \to Z$ given by $f(x) = x^3$

(v) $f : R \to R$, defined by $f(x) = |x|$ (vi) $f : Z \to Z$, defined by $f(x) = x^2 + x$

(vii) $f : Z \to Z$, defined by $f(x) = x - 5$ (viii) $f : R \to R$, defined by $f(x) = \sin x$

(ix) $f : R \to R$, defined by $f(x) = x^3 + 1$ (x) $f : R \to R$, defined by $f(x) = x^3 - x$

(xi) $f : R \to R$, defined by $f(x) = \sin^2 x + \cos^2 x$

(xii) $f : Q - \{3\} \to Q$, defined by $f(x) = \dfrac{2x + 3}{x - 3}$

(xiii) $f : Q \to Q$, defined by $f(x) = x^3 + 1$ (xiv) $f : R \to R$, defined by $f(x) = 5x^3 + 4$

(xv) $f : R \to R$, defined by $f(x) = 3 - 4x$ (xvi) $f : R \to R$, defined by $f(x) = 1 + x^2$

(xvii) $f : R \to R$, defined by $f(x) = \dfrac{x}{x^2 + 1}$ **[CBSE 2018, NCERT EXEMPLAR]**

5. Let $A = [-1, 1]$. Then, discuss whether the following functions from A to itself are one-one, onto or bijective:

(i) $f(x) = \dfrac{x}{2}$ (ii) $g(x) = |x|$ (iii) $h(x) = x^2$ **[NCERT EXEMPLAR]**

6. Are the following set of ordered pairs functions? If so, examine whether the mapping is injective or surjective:

(i) $\{(x, y) : x$ is a person, y is the mother of $x\}$ (ii) $\{(a, b) : a$ is a person, b is an ancestor of $a\}$ **[NCERT EXEMPLAR]**

BASED ON LOTS

7. Prove that the function $f : N \to N$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto.

8. If $f : A \to B$ is an injection such that range of $f = \{a\}$. Determine the number of elements in A.

9. Show that the function $f : R - \{3\} \to R - \{1\}$ given by $f(x) = \dfrac{x - 2}{x - 3}$ is a bijection. **[CBSE 2012, NCERT EXEMPLAR]**

10. Let $A = \{1, 2, 3\}$. Write all one-one functions from A to itself.

11. If $f : R \to R$ be the function defined by $f(x) = 4x^3 + 7$, show that f is a bijection. **[CBSE 2011]**

12. Show that the exponential function $f : R \to R$, given by $f(x) = e^x$, is one-one but not onto. What happens if the co-domain is replaced by R_0^+ (set of all positive real numebrs).

13. Show that the logarithmic function $f : R_0^+ \to R$ given by $f(x) = \log_a x$, $a > 0$ is a bijection.

14. If $A = \{1, 2, 3\}$, show that a one-one function $f : A \to A$ must be onto. **[NCERT]**

15. If $A = \{1, 2, 3\}$, show that an onto function $f : A \to A$ must be one-one. **[NCERT]**

BASED ON HOTS

16. Find the number of all onto functions from the set $A = \{1, 2, 3, \ldots, n\}$ to itself. **[NCERT]**

17. Give examples of two one-one functions f_1 and f_2 from R to R such that $f_1 + f_2 : R \to R$, defined by $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ is not one-one.

18. Give examples of two surjective function f_1 and f_2 from Z to Z such that $f_1 + f_2$ is not surjective.

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19. Show that if f_1 and f_2 are one-one maps from R to R, then the product $f_1 \times f_2 : R \to R$ defined by $(f_1 \times f_2)(x) = f_1(x) f_2(x)$ need not be one-one.

20. Suppose f_1 and f_2 are non-zero one-one functions from R to R. Is $\frac{f_1}{f_2}$ necessarily one-one? Justify your answer. Here, $\frac{f_1}{f_2} : R \to R$ is given by $\left(\frac{f_1}{f_2}\right)(x) = \frac{f_1(x)}{f_2(x)}$ for all $x \in R$.

21. Given $A = \{2, 3, 4\}$, $B = \{2, 5, 6, 7\}$. Construct an example of each of the following:
   (i) an injective map from A to B
   (ii) a mapping from A to B which is not injective
   (iii) a mapping from A to B.

22. Show that $f : R \to R$, given by $f(x) = x - [x]$, is neither one-one nor onto.

23. Let $f : N \to N$ be defined by $f(n) = \begin{cases} n+1, & \text{if } n \text{ is odd} \\ n-1, & \text{if } n \text{ is even} \end{cases}$. Show that f is a bijection. [CBSE 2012, NCERT]

ANSWERS

2. f_1, f_2
4. (i) one-one but not onto (ii) Neither one-one not onto
   (iii) Injective but not surjective (iv) Injective but not surjective.
   (v) Neither an injection nor a surjection (vi) Neither Injective nor Surjective
   (vii) Bijective (viii) Neither injective nor surjective
   (ix) Bijective (x) Surjective but not injective
   (xi) Neither injective nor surjective (xii) Injective but not surjective
   (xiii) Injective (xiv) Bijective
   (xv) Bijective (xvi) Neither injective nor surjective.
   (xvii) Neither one-one nor onto
5. (i) one-one but not onto (ii) neither one-one nor onto
   (iii) neither one-one nor onto
6. (i) represents a function which is surjective but not injective
   (ii) does not represent a function
8. 1
10. (i) $f(1) = 1, f(2) = 2, f(3) = 3$ (ii) $f(1) = 1, f(2) = 3, f(3) = 2$
   (iii) $f(1) = 2, f(2) = 3, f(3) = 1$ (iv) $f(1) = 2, f(2) = 1, f(3) = 3$
   (v) $f(1) = 3, f(2) = 2, f(3) = 1$ (vi) $f(1) = 3, f(2) = 1, f(3) = 2$

HINTS TO SELECTED PROBLEMS

1. (i) $f : Z \to Z$ given by $f(x) = 3x + 2$ (ii) $f : Z \to \{0\}$ given by $f(x) = |x|$
   (iii) $f : Z \to Z$ given by $f(x) = 2x^2 + 1$

3. We have, $f(x) = x^2$, $x \in \{-1, 0, 1\}$. Clearly, $f(-1) = f(1)$. So, f is not one-one. Range $(f) = \{0, 1\} \neq A$. So, $f : A \to A$ is not onto.

7. We have, $f(x) = x^2 + x + 1$.

Injectivity: Let $x, y \in N$ be such that

$$f(x) = f(y) \Rightarrow x^2 + x + 1 = y^2 + y + 1 \Rightarrow x^2 - y^2 + x - y = 0$$

$$\Rightarrow (x - y)(x + y + 1) = 0 \Rightarrow x - y = 0 \Rightarrow x = y \qquad [\because x + y + 1 \neq 0 \text{ for any } x, y \in N]$$

So, f is a one-one function. Clearly, $f(x) = x^2 + x + 1 \geq 3$ for all $x \in N$.

So, $f(x)$ does not assume values 1 and 2. Therefore, $f : N \to N$ is not an onto function.

8. It is given that $f : A \to B$ is an injective map such that range of f is $\{a\}$. As f is an injective map, therefore different elements of A have different images in B. So, A has just one element.

9. $f : R - \{3\} \to R - \{1\}$ is given by $f(x) = \dfrac{x-2}{x-3}$

Injectivity: Let $x, y \in R - \{3\}$ be such that

$$f(x) = f(y) \Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3} \Rightarrow 1 + \frac{1}{x-3} = 1 + \frac{1}{y-3} \Rightarrow \frac{1}{x-3} = \frac{1}{y-3} \Rightarrow x - 3 = y - 3 \Rightarrow x = y$$

So, f is a one-one function.

Surjectivity: Let y be an arbitrary element of $R - \{1\}$. Then,

$$f(x) = y \Rightarrow \frac{x-2}{x-3} = y \Rightarrow x = \frac{2-3y}{1-y}$$

We find that, $x = 3 \Rightarrow 1 = 0$ which is an absurd result. Therefore, $x \neq 3$. Hence, $x \in R - \{3\}$ for all $y \in R - \{1\}$. Thus, for each $y \in R - \{1\}$ there exists $x = \dfrac{2-3y}{1-y} \in R - \{3\}$ such that $f(x) = y$. So, f is an onto function.

10. All one-one functions from $A = \{1, 2, 3\}$ to itself are obtained by arranging elements of the second row in the two row notation $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

12. For any $x, y \in R$, we find that : $f(x) = f(y) \Rightarrow e^x = e^y \Rightarrow x = y$. So, $f : R \to R$ is one-one. Clearly, range $(f) = (0, \infty) \neq R$. So, f is not onto.

13. $f : R_0^+ \to R$ is given by $f(x) = \log_a x$, $a > 0$. For any $x, y \in R_0^+$, we find that

$$f(x) = f(y) \Rightarrow \log_a x = \log_a y \Rightarrow x = y.$$ So, f is one-one.

For each $y \in R$, there exists $x = a^y \in R_0^+$ such that $f(x) = \log_a a^y = y$. So, f is onto. Hence, f is a bijection.

14. We have, $A = \{1, 2, 3\}$ and $f : A \to A$ is a one-one function. Therefore, $f(1), f(2), f(3)$ are distinct elements of A. But, A has three elements only. Therefore, $A = \{f(1), f(2), f(3)\}$ i.e., range $(f) = A$. So, f is onto.

15. We have, $A = \{1, 2, 3\}$. It is given that $f : A \to A$ is an onto function. Therefore,

$\{f(1), f(2), f(3)\} = A$

$\Rightarrow f(1), f(2), f(3)$ are distinct elements of $A \Rightarrow f : A \to A$ is one-one.

16. Since every onto function from A to itself is one-one (See example 22). Therefore, total number of onto functions from A to itself is same as the number of bijections from A to itself, which is equal to n!.

17. Let $f_1 : R \to R$ and $f_2 : R \to R$ be given by $f_1(x) = x$ and $f_2(x) = -x$.

Clearly, f_1 and f_2 are one-one. But, $(f_1 + f_2)(x) = x - x = 0$ for all $x \in R$ is not one-one.

18. Let $f_1 : Z \to Z$ and $f_2 : Z \to Z$ be given by $f_1(x) = x$ and $f_2(x) = -x$. Then, f_1 and f_2 are surjections, but $f_1 + f_2 : Z \to Z$ is not surjection. Because, $(f_1 + f_2)(x) = x - x = 0$ for all $x \in Z$.

19. Take $f_1(x) = x$ and $f_2(x) = x$.

20. Take $f_1 : R \to R$ given by $f_1(x) = x^3$ and $f_2 : R \to R$ given $f_2(x) = x$.

22. We have, $f(x) = x - [x]$. Clearly, $f(x) = 0$ for all $x \in Z$. So, $f : R \to R$ is a many-one function. We find that, range $(f) = [0, 1) \neq R$. So, f is an into function.

2.4 COMPOSITION OF FUNCTIONS

Let A, B and C be three non-void sets and let $f : A \to B$, $g : B \to C$ be two functions. Since f is a function from A to B, therefore for each $x \in A$ there exists a unique element $f(x) \in B$. Again, since g is a function from B to C, therefore corresponding to $f(x) \in B$ there exists a unique element $g(f(x)) \in C$. Thus, for each $x \in A$ there exists a unique element $g(f(x)) \in C$.

It follows from the above discussion that f and g when considered together define a new function from A to C. This function is called the composition of f and g and is denoted by gof. We define it formally as follows:

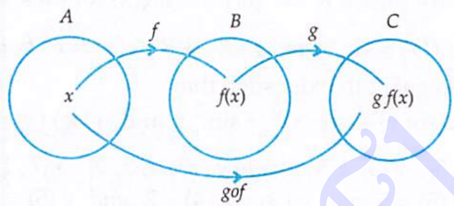

Fig. 2.36

DEFINITION *Let* $f : A \to B$ *and* $g : B \to C$ *be two functions. Then a function* $gof : A \to C$ *defined by*

$$(gof)(x) = g(f(x)), \text{ for all } x \in A \text{ is called the composition of } f \text{ and } g.$$

NOTE 1 *It is evident from the definition that gof is defined only if for each* $x \in A$, $f(x)$ *is an element of domain of g so that we can take its g-image. Hence, for the composition gof to exist, the range of f must be a subset of the domain of g as shown in Fig. 2.37.*

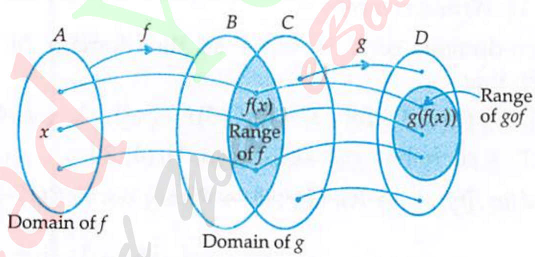

Fig. 2.37

NOTE 2 *It should be noted that gof exists iff the range of f is a subset of domain of g. Similarly, fog exists if range of g is a subset of domain of f.*

NOTE 3 *In order to visualize how functional composition works, let us think gof in terms of an "assembly line" in which f and g are arranged in series with output f (x) becoming the input of g as shown in Fig. 2.38.*

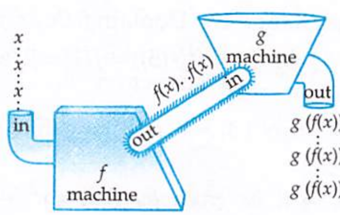

Fig. 2.38

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ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Let R be the set of real numbers. If* $f : R \to R$; $f(x) = x^2$ *and* $g : R \to R; g(x) = 2x + 1$. *Then, find fog and gof. Also, show that* $fog \neq gof$.

SOLUTION Clearly, range of f is a subset of domain of g and range of g is a subset of domain of f. So, fog and gof both exist.

Now, $(gof)(x) = g(f(x)) = g(x^2) = 2(x^2) + 1 = 2x^2 + 1$

And, $(fog)(x) = f(g(x)) = f(2x+1) = (2x+1)^2$

Clearly, $2x^2 + 1 \neq (2x+1)^2$ for all $x \in R \Rightarrow gof(x) = fog(x)$ for all $x \in R \Rightarrow gof \neq fog$.

EXAMPLE 2 *Let* $f : R \to R$; $f(x) = \sin x$ *and* $g : R \to R$; $g(x) = x^2$ *find fog and gof.*

SOLUTION Clearly, fog and gof both exist such that

$(gof)(x) = g(f(x)) = g(\sin x) = (\sin x)^2 = \sin^2 x$ and, $(fog)(x) = f(g(x)) = f(x^2) = \sin x^2$.

EXAMPLE 3 *Let* $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ *and* $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ *be functions defined as* $f(2) = 3, f(3) = 4, f(4) = f(5) = 5$ *and,* $g(3) = g(4) = 7$ *and* $g(5) = g(9) = 11$. *Find gof.* **[NCERT]**

SOLUTION We find that Range of $f = \{3, 4, 5\}$. Clearly, it is a subset of domain of g. So, gof exists and $gof : \{2, 3, 4, 5\} \to \{7, 11, 15\}$ such that

$gof(2) = g(f(2)) = g(3) = 7;$ $gof(3) = g(f(3)) = g(4) = 7$

$gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = g(f(5)) = g(5) = 11$

Hence, $gof : \{2, 3, 4, 5\} \to \{7, 11, 15\}$ such that $gof = \{(2, 7), (3, 7), (4, 11), (5, 11)\}$

EXAMPLE 4 *Let* $f : \{1, 3, 4\} \to \{1, 2, 5\}$ *and* $g : \{1, 2, 5\} \to \{1, 3\}$ *be given by* $f = \{(1, 2), (3, 5), (4, 1)\}$ *and* $g = \{(1, 3), (2, 3), (5, 1)\}$. *Write down gof.* **[NCERT]**

SOLUTION Clearly, co-domain of f is same as the domain of g. So, gof exists and $gof : \{1, 3, 4\} \to \{1, 3\}$ such that

$gof(1) = g(f(1)) = g(2) = 3;$ $gof(3) = g(f(3)) = g(5) = 1;$ $gof(4) = g(f(4)) = g(1) = 3$

Hence, $gof : \{1, 3, 4\} \to \{1, 3\}$ such that $gof = \{(1, 3), (3, 1), (4, 3)\}$.

EXAMPLE 5 *Find gof and fog, if* $f : R \to R$ *and* $g : R \to R$ *are given by* $f(x) = |x|$ *and* $g(x) = |5x - 2|$. **[NCERT]**

SOLUTION $gof(x) = g(f(x)) = g(|x|) = |5|x| - 2| = \begin{cases} |5x - 2|, \text{ if } x \geq 0 \\ |-5x - 2|, \text{ if } x < 0 \end{cases}$

and, $fog(x) = f(g(x)) = f(|5x - 2|) = ||5x - 2|| = |5x - 2|$.

EXAMPLE 6 *If the functions f and g are given by* $f = \{(1, 2), (3, 5), (4, 1)\}$ *and* $g = \{(2, 3), (5, 1), (1, 3)\}$, *find range of f and g. Also, write down fog and gof as sets of ordered pairs.*

SOLUTION We find that: Range of f = Set of second components of ordered pairs in $f = \{2, 5, 1\}$

Similarly, Range of $g = \{3, 1\}$. We also find that: Domain $f = \{1, 3, 4\}$ and, Domain $g = \{2, 5, 1\}$

Clearly, Range $f \subset$ Domain g and, Range $g \subset$ Domain f. So, fog and gof both exist.

Now, $fog(2) = f(g(2)) = f(3) = 5$; $fog(5) = f(g(5)) = f(1) = 2$ and, $fog(1) = f(g(1)) = f(3) = 5$.

$\therefore$ $fog = \{(2, 5), (5, 2), (1, 5)\}$

And, $gof(1) = g(f(1)) = g(2) = 3$; $gof(3) = g(f(3)) = g(5) = 1$ and, $gof(4) = g(f(4)) = g(1) = 3$

$\therefore$ $gof = \{(1, 3), (3, 1), (4, 3)\}$

EXAMPLE 7 *If the function* $f : R \to R$ *be given by* $f(x) = x^2 + 2$ *and* $g : R - \{1\} \to R$ *be given by* $g(x) = \dfrac{x}{x-1}$. *Find fog and gof.* **[CBSE 2014]**

SOLUTION Clearly, range $f \subset$ domain g and, range $g \subset$ domain f. So, fog and gof both exist.

Now, $(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2}{(x-1)^2} + 2$

and, $(gof)(x) = g(f(x)) = g(x^2+2) = \frac{x^2+2}{(x^2+2)-1} = \frac{x^2+2}{x^2+1}$

Hence, $gof: R \to R$ and $fog: R - \{1\} \to R$ are given by $(gof)(x) = \frac{x^2+2}{x^2+1}$ and $(fog)(x) = \frac{x^2}{(x-1)^2} + 2$.

EXAMPLE 8 *If $f, g: R \to R$ are defined respectively by $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$, find*

(i) fog (ii) gof (iii) fof (iv) gog. **[NCERT EXEMPLAR]**

SOLUTION Clearly, Range f = Domain g and, Range g = Domain f. Therefore, fog, gof, fof and gog all exist.

(i) For any $x \in R$, we find that

$$(fog)(x) = f(g(x)) = f(2x-3) = (2x-3)^2 + 3(2x-3) + 1 = 4x^2 - 6x + 1$$

So, $fog: R \to R$ is defined by $(fog)(x) = 4x^2 - 6x + 1$ for all $x \in R$.

(ii) For any $x \in R$, we find that

$$(gof)(x) = g(f(x)) = g(x^2 + 3x + 1) = 2(x^2 + 3x + 1) - 3 = 2x^2 + 6x - 1$$

So, $gof: R \to R$ is defined by $(gof)(x) = 2x^2 + 6x - 1$ for all $x \in R$

(iii) For any $x \in R$, we find that

$$(fof)(x) = f(f(x)) = f(x^2 + 3x + 1) = (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$
$$= x^4 + 6x^3 + 14x^2 + 15x + 5$$

So, $fof: R \to R$ is defined by $(fof)(x) = x^4 + 6x^3 + 14x^2 + 15x + 5$

(iv) For any $x \in R$, we find that

$$(gog)(x) = g(g(x)) = g(2x-3) = 2(2x-3) - 3 = 4x - 9$$

So, $gog: R \to R$ is defined by $(gog)(x) = 4x - 9$.

EXAMPLE 9 *If $f: R \to R$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.* **[NCERT]**

SOLUTION We have, $f(x) = x^2 - 3x + 2$.

$\therefore \quad f(f(x)) = f(x^2 - 3x + 2) = f(y)$, where $y = x^2 - 3x + 2$.

$$= y^2 - 3y + 2 \qquad [\because f(x) = x^2 - 3x + 2 \therefore f(y) = y^2 - 3y + 2]$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 = x^4 - 6x^3 + 10x^2 - 3x.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 10 *If $f: R - \left\{\frac{7}{5}\right\} \to R - \left\{\frac{3}{5}\right\}$ be defined as $f(x) = \frac{3x+4}{5x-7}$ and $g: R - \left\{\frac{3}{5}\right\} \to R - \left\{\frac{7}{5}\right\}$ be defined as $g(x) = \frac{7x+4}{5x-3}$. Show that $gof = I_A$ and $fog = I_B$, where $B = R - \left\{\frac{3}{5}\right\}$ and $A = R - \left\{\frac{7}{5}\right\}$.*

[NCERT]

SOLUTION It is given that $f : A \to B$ and $g : B \to A$. Therefore, $gof : A \to A$ and $fog : B \to B$.

$$gof\,(x) = g\,(f\,(x)) = g\left(\frac{3x+4}{5x-7}\right) = \frac{7\left(\frac{3x+4}{5x-7}\right)+4}{5\left(\frac{3x+4}{5x-7}\right)-3} = \frac{21x+28+20x-28}{15x+20-15x+21} = \frac{41x}{41} = x$$

Therefore, $gof : A \to A$ is such that $gof\,(x) = x$ for all $x \in A$. Hence, $gof = I_A$.

Now, $$fog\,(x) = f\,(g(x)) = f\left(\frac{7x+4}{5x-3}\right) = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} = \frac{21x+12+20x-12}{35x+20-35x+21} = \frac{41x}{41} = x$$

$\therefore$ $fog : B \to B$ such that $fog\,(x) = x$ for all $x \in B$. Hence, $fog = I_B$.

EXAMPLE 11 *Let $f : R \to R$ be the signum function defined as $f\,(x) = \begin{cases} 1\,, & x > 0 \\ 0\,, & x = 0 \\ -1\,, & x < 0 \end{cases}$ and $g : R \to R$ be the greatest integer function given by $g\,(x) = [x]$. Then, prove that fog and gof coincide in* $[-1, 0)$. **[NCERT]**

SOLUTION For any $x \in [-1, 0)$, we find that

$$fog\,(x) = f\,(g\,(x)) = f\,([x]) = f\,(-1) = -1 \text{ and, } gof\,(x) = g\,(f\,(x)) = g\,(-1) = [-1] = -1$$

$\therefore$ $gof\,(x) = fog\,(x)$ for all $x \in [-1, 0)$

Hence, gof and fog coincide on $[-1, 0)$.

EXAMPLE 12 *Let f, g and h be functions from R to R. Show that:*

(i) $(f+g)\,oh = foh + goh$ (ii) $(fg)\,oh = (foh)\,(goh)$ **[NCERT]**

SOLUTION (i) Since f, g and h are functions from R to R. Therefore,

$$(f+g)\,oh : R \to R \text{ and } foh + goh : R \to R$$

For any $x \in R$, we find that

$$((f+g)\,oh)\,(x) = (f+g)\,(h\,(x)) = f\,(h\,(x)) + g\,(h\,(x)) = foh\,(x) + goh\,(x) = (foh + goh)\,(x)$$

$\therefore$ $(f+g)\,oh = foh + goh$

(ii) Clearly, $(fg)\,oh : R \to R$ and $(foh)\,(goh) : R \to R$ such that

$$\{(fg)\,oh\}\,(x) = (fg)\,(h\,(x)) = f\,(h\,(x))\,g\,(h\,(x)) = (foh)\,(x)\,(goh)\,(x) = \{(foh).(goh)\}\,(x) \text{ for all } x \in R$$

$\therefore$ $(fg)\,oh = (foh).(goh)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 13 *If $f : R \to R$ be defined by $f\,(x) = 2x$ for all $x \in R$. Find $g : R \to R$ such that $gof = I_R$.*

SOLUTION It is given that

$gof = I_R$

$\Rightarrow$ $gof\,(x) = I_R\,(x)$ for all $x \in R$

$\Rightarrow$ $g\,(f\,(x)) = x$ for all $x \in R$

$\Rightarrow$ $g\,(2x) = x$ for all $x \in R$

$\Rightarrow$ $g\,(y) = \frac{y}{2}$ for all $y \in R$, where $2x = y$

$\Rightarrow$ $g\,(x) = \frac{x}{2}$ for all $x \in R$.

Hence, $g : R \to R$ given by, $g\,(x) = \frac{x}{2}$ for all $x \in R$, is the required function.

EXAMPLE 14 *Let $A = \{x \in R : 0 \le x \le 1\}$. If $f : A \to A$ is defined by* $f(x) = \begin{cases} x, & \text{if } x \in Q \\ 1-x, & \text{if } x \notin Q \end{cases}$ *then prove that $fof\,(x) = x$ for all $x \in A$.* **[NCERT EXEMPLAR]**

SOLUTION Let $x \in A$. Then, either x is rational or x is irrational. So following cases arise:

Case I *When $x \in Q$:* In this case, we have $f(x) = x$.

$$\therefore \quad fof\,(x) = f(f(x)) = f(x) = x \qquad [\because f(x) = x]$$

Case II *When $x \notin Q$:* In this case, we have $f(x) = 1 - x$.

$$\therefore \quad fof\,(x) = f(f(x)) = f(1-x) \qquad [\because x \notin Q \therefore f(x) = 1-x]$$

$$= 1 - (1-x) = x \qquad [\because x \notin Q \Rightarrow 1 - x \notin Q \Rightarrow f(1-x) = 1 - (1-x)]$$

Thus, $fof\,(x) = x$ whether $x \in Q$ or, $x \notin Q$. Hence, $fof\,(x) = x$ for all $x \in A$.

EXAMPLE 15 *Let $f : R \to R$ and $g : R \to R$ be two functions such that $fog\,(x) = \sin x^2$ and $gof\,(x) = \sin^2 x$. Then, find $f(x)$ and $g(x)$.*

SOLUTION We have,

$$fog\,(x) = \sin x^2 \text{ and, } gof\,(x) = \sin^2 x$$

$$\Rightarrow \quad f(g(x)) = \sin(x^2) \text{ and, } g(f(x)) = (\sin x)^2 \Rightarrow f(x) = \sin x \text{ and, } g(x) = x^2.$$

EXAMPLE 16 *If $f : R \to R$ be given by $f(x) = \sin^2 x + \sin^2 (x + \pi/3) + \cos x \cos (x + \pi/3)$ for all $x \in R$, and $g : R \to R$ be such that $g(5/4) = 1$, then prove that $gof : R \to R$ is a constant function.*

SOLUTION We have,

$$f(x) = \sin^2 x + \sin^2 (x + \pi/3) + \cos x \cos(x + \pi/3)$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left\{2\sin^2 x + 2\sin^2\left(x + \frac{\pi}{3}\right) + 2\cos x \cos\left(x + \frac{\pi}{3}\right)\right\}$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left[1 - \cos 2x + 1 - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right) + \cos\frac{\pi}{3}\right]$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left[\frac{5}{2} - \cos 2x - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right)\right]$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left[\frac{5}{2} - \left\{\cos(2x) + \cos\left(2x + \frac{2\pi}{3}\right)\right\} + \cos\left(2x + \frac{\pi}{3}\right)\right]$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left[\frac{5}{2} - 2\cos\left(2x + \frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right]$$

$$\Rightarrow \quad f(x) = \frac{1}{2}\left[\frac{5}{2} - \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right)\right] = \frac{5}{4} \text{ for all } x \in R.$$

Therefore, for any $x \in R$, we obtain: $gof\,(x) = g(f(x)) = g\left(\frac{5}{4}\right) = 1$

Thus, $gof\,(x) = 1$ for all $x \in R$. Hence, $gof : R \to R$ is a constant function.

EXAMPLE 17 *Let $f : Z \to Z$ be defined by $f(n) = 3n$ for all $n \in Z$ and $g : Z \to Z$ be defined by*

$$g(n) = \begin{cases} \dfrac{n}{3}, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{if } n \text{ is not a multiple of } 3 \end{cases} \quad \text{for all } n \in Z. \text{ Show that } gof = I_Z \text{ and } fog \ne I_Z.$$

SOLUTION Since $f : Z \to Z$ and $g : Z \to Z$. Therefore, $gof : Z \to Z$ and $fog : Z \to Z$.

For any $n \in Z$, we obtain

$$gof\,(n) = g(f(n)) = g(3n) = \frac{3n}{3} = n \qquad \left[\because 3n \text{ is a multiple of } 3 \therefore f(3n) = \frac{3n}{3}\right]$$

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$\therefore \quad gof\,(n) = n$ for all $n \in Z \Rightarrow gof = I_Z$.

For any $n \in Z$, we have

$$fog\,(n) = f\,(g\,(n)) = \begin{cases} f\left(\dfrac{n}{3}\right) & \text{, if } n \text{ is a multiple of } 3 \\ f\,(0) & \text{, if } n \text{ is not a multiple of } 3 \end{cases}$$

$$\Rightarrow \quad fog\,(n) = \begin{cases} 3\left(\dfrac{n}{3}\right) & \text{, if } n \text{ is a multiple of } 3 \\ 3 \times 0 & \text{, if } n \text{ is not a multiple of } 3 \end{cases} = \begin{cases} n\text{, if } n \text{ is a multiple of } 3 \\ 0\text{, if } n \text{ is not a multiple of } 3 \end{cases}$$

Clearly, $fog\,(n) \neq n$ for all $n \in Z$. In fact, $fog\,(n) = n$ only for multiple of 3. So, $fog \neq I_Z$.

EXAMPLE 18 *Let $f : R \to R$ be a function given by $f\,(x) = ax + b$ for all $x \in R$. Find the constants a and b such that $fof = I_R$.*

SOLUTION We have,

$$fof = I_R$$

$$\Rightarrow \quad fof\,(x) = I_R\,(x) \text{ for all } x \in R$$

$$\Rightarrow \quad f\,(f\,(x)) = x \text{ for all } x \in R \qquad [\because I_R\,(x) = x \text{ for all } x \in R]$$

$$\Rightarrow \quad f\,(ax + b) = x \text{ for all } x \in R$$

$$\Rightarrow \quad a\,(ax + b) + b = x \text{ for all } x \in R$$

$$\Rightarrow \quad (a^2 - 1)\,x + ab + b = 0 \text{ for all } x \in R$$

$$\Rightarrow \quad a^2 - 1 = 0 \text{ and } ab + b = 0 \qquad [\because (a^2 - 1)\,x + (ab + b) = 0 \text{ is an identity in } x]$$

$$\Rightarrow \quad a = \pm 1 \text{ and } b\,(a + 1) = 0$$

When $a = 1$: $b\,(a + 1) = 0 \Rightarrow 2b = 0 \Rightarrow b = 0$. Therefore, $a = 1$ and $b = 0$.

When $a = -1$: $b\,(a + 1) = 0$ for all $b \in R$. Therefore, $a = -1$ and b can take any real value.

Hence, either $a = 1$ and $b = 0$, or $a = -1$ and b can take any real value.

EXAMPLE 19 *Let $f, g : R \to R$ be a two function defined as $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Then, find fog and gof.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$f(x) = |x| + x = \begin{cases} x + x = 2x, & \text{if } x \geq 0 \\ -x + x = 0, & \text{if } x < 0 \end{cases} \text{ and, } g(x) = |x| - x = \begin{cases} x - x = 0, & \text{if } x \geq 0 \\ -x - x = -2x, & \text{if } x < 0 \end{cases}$$

The graphs of $f(x)$ and $g(x)$ are shown in Fig. 2.39 (i) and 2.39 (ii) respectively. It is event from these graphs that range $(f) = [0, \infty)$ and range $(g) = (-\infty, 0]$. Thus, range $(f) \subset$ domain (g) and range $(g) \subset$ domain (f). So, fog and gof both exist.

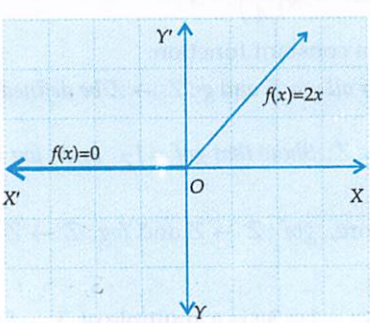

Fig. 2.39 (i) Graph of $f(x)$

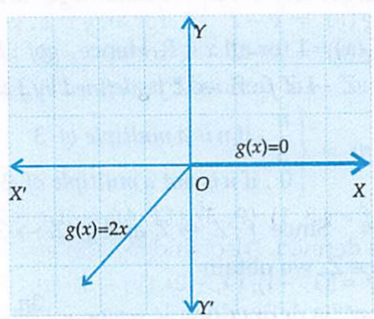

Fig. 2.39 (ii) Graph of $f(x)$

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Now,

$$fog(x) = f(g(x)) = \begin{cases} f(0) & , \text{ if } x \geq 0 \\ f(-2x) & , \text{ if } x < 0 \end{cases}$$

$$\Rightarrow \quad fog(x) = \begin{cases} 2 \times 0 = 0, & \text{if } x \geq 0 \\ 2(-2x) = -4x, & \text{if } x < 0 \end{cases} \qquad \left[\begin{array}{l} \because -2x > 0 \text{ if } x < 0 \\ \therefore f(-2x) = 2(-2x) = -4x, \text{ if } x < 0 \end{array}\right]$$

and, $$gof(x) = g(f(x)) = \begin{cases} g(2x) & , \text{ if } x \geq 0 \\ g(0) & , \text{ if } x < 0 \end{cases} = \begin{cases} 0, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases} = 0 \text{ for all } x \in R.$$

EXAMPLE 20 *Let $f : A \to A$ be a function such that $fof = f$. Show that f is onto if and only if f is one-one. Describe f in this case.*

SOLUTION We have, $fof = f$.

Let $f : A \to A$ be onto. Then, we have to prove that f is one-one.

Let $x, y, \in A$. Then, as $f : A \to A$ is onto there exist $\alpha, \beta \in A$ such that

$$f(\alpha) = x \text{ and } f(\beta) = y \qquad \text{...(i)}$$

Now, $f(x) = f(y)$

$\Rightarrow \quad f(f(\alpha)) = f(f(\beta))$ [Using (i)]

$\Rightarrow \quad fof(\alpha) = fof(\beta)$

$\Rightarrow \quad f(\alpha) = f(\beta)$ $[\because fof = f]$

$\Rightarrow \quad x = y$ [Using (i)]

So, f is one-one.

Thus, $f : A \to A$ is onto $\Rightarrow f : A \to A$ is one-one.

Conversely, let $f : A \to A$ be one-one. Then, we have to prove that f is onto.

Let y be an arbitrary element in A. Then,

$$fof = f \Rightarrow fof(y) = f(y) \Rightarrow f(f(y)) = f(y) \Rightarrow f(y) = y \quad [\because f : A \to A \text{ is one-one}]$$

Thus, for all $y \in A$, there exists $y \in A$ such that $f(y) = y$. Hence, f is onto.

Now, $fof = f$

$\Rightarrow \quad fof(x) = f(x)$ for all $x \in A$

$\Rightarrow \quad f(f(x)) = f(x)$ for all $x \in A$

$\Rightarrow \quad f(\alpha) = \alpha$ for all $\alpha = f(x) \in A$

Thus, $f(x) = x$ for all $x \in A$

EXAMPLE 21 *Let $f : Z \to Z$ be defined by $f(x) = x + 2$. Find $g : Z \to Z$ such that $gof = I_Z$.*

SOLUTION It is given that

$gof = I_Z$

$\Rightarrow \quad gof(x) = I_Z(x)$ for all $x \in Z$

$\Rightarrow \quad g(f(x)) = x$ for all $x \in Z$

$\Rightarrow \quad g(x+2) = x$ for all $x \in Z$

$\Rightarrow \quad g(y) = y - 2$ for all $y \in Z$, where $x + 2 = y$

$\Rightarrow \quad g(x) = x - 2$ for all $x \in Z$.

Hence, $g : Z \to Z$ defined by $g(x) = x - 2$ for all $x \in Z$, is the required function.

EXERCISE 2.2

BASIC

1. Let $f = \{(3, 1), (9, 3), (12, 4)\}$ and $g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}$. Show that gof and fog are both defined. Also, find fog and gof.
2. Let $f = \{(1, -1), (4, -2), (9, -3), (16, 4)\}$ and $g = \{(-1, -2), (-2, -4), (-3, -6), (4, 8)\}$. Show that gof is defined while fog is not defined. Also, find gof.

3. Let $A = \{a, b, c\}$, $B = \{u\, v, w\}$ and let f and g be two functions from A to B and from B to A respectively defined as: $f = \{(a, v), (b, u), (c, w)\}$, $g = \{(u, b), (v, a), (w, c)\}$.
Show that f and g both are bijections and find fog and gof.
4. Find fog (2) and gof (1) when: $f : R \to R$; $f(x) = x^2 + 8$ and $g : R \to R$; $g(x) = 3x^3 + 1$.
5. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = x^2$ and $g(x) = x + 1$. Show that $fog \neq gof$.
6. Let R^+ be the set of all non-negative real numbers. If $f : R^+ \to R^+$ and $g : R^+ \to R^+$ are defined as $f(x) = x^2$ and $g(x) = +\sqrt{x}$. Find fog and gof. Are they equal functions.

BASED ON LOTS

7. Find gof and fog when $f : R \to R$ and $g : R \to R$ are defined by
(i) $f(x) = 2x + 3$ and $g(x) = x^2 + 5$ (ii) $f(x) = 2x + x^2$ and $g(x) = x^3$
(iii) $f(x) = x^2 + 8$ and $g(x) = 3x^3 + 1$ (iv) $f(x) = x$ and $g(x) = |x|$
(v) $f(x) = x^2 + 2x - 3$ and $g(x) = 3x - 4$ (vi) $f(x) = 8x^3$ and $g(x) = x^{1/3}$
8. Let $f : R \to R$ and $g : R \to R$ be defined by $f(x) = x + 1$ and $g(x) = x - 1$. Show that $fog = gof = I_R$.
9. Verify associativity for the following three mappings : $f : N \to Z_0$ (the set of non-zero integers), $g : Z_0 \to Q$ and $h : Q \to R$ given by $f(x) = 2x$, $g(x) = 1/x$ and $h(x) = e^x$.
10. If $f(x) = \dfrac{1 - x}{1 + x}$, then find $fof(x)$. [CBSE 2020]
11. Consider $f : N \to N$, $g : N \to N$ and $h : N \to R$ defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$ for all $x, y, z \in N$. Show that $ho\,(gof) = (hog)\,of$. [NCERT]

BASED ON HOTS

12. Give examples of two functions $f : N \to N$ and $g : N \to N$ such that gof is onto but f is not onto. [NCERT]
13. Give examples of two functions $f : N \to Z$ and $g : Z \to Z$ such that gof is injective but g is not injective. [NCERT]
14. If $f : A \to B$ and $g : B \to C$ are one-one functions, show that gof is a one-one function.
15. If $f : A \to B$ and $g : B \to C$ are onto functions show that gof is an onto function.

ANSWERS

1. $gof = \{(3, 3), (9, 3), (12, 9)\}$ $fog = \{(1, 1), (3, 1), (4, 3)\ (5, 3)\}$
2. $gof = \{(1, -2), (4, -4), (9, -6), (16, 8)\}$
3. $fog = \{(u, u), (v, v), (w, w)\}$ $gof = \{(a, a), (b, b), (c, c)\}$ 4. $fog\,(2) = 633$, $gof\,(1) = 2188$
6. $fog\,(x) = x$, $gof(x) = x$, Yes
7. (i) $gof\,(x) = 4x^2 + 12x + 14$, $fog\,(x) = 2x^2 + 13$
(ii) $gof\,(x) = (x^2 + 2x)^3$, $fog\,(x) = 2x^3 + x^6$
(iii) $gof\,(x) = 3\,(x^2 + 8)^3 + 1$, $fog\,(x) = 9x^6 + 6x^3 + 9$
(iv) $gof\,(x) = |\,x\,|$, $fog\,(x) = |\,x\,|$ (v) $gof\,(x) = 3x^2 + 6x - 13$, $fog\,(x) = 9x^2 - 18x + 5$
(vi) $gof\,(x) = 2x$, $fog\,(x) = 8x$ 10. $fof(x) = x$

HINTS TO SELECTED PROBLEMS

2. We have, Range $g = \{-2, -4, -6, 8\}$, Domain $f = \{1, 4, 9, 16\}$,
Range $f = \{-1, -2, -3, 4\}$, Domain $g = \{-1, -2, -3, 4\}$.
Clearly, Range f = Domain g but, Range $g \not\subset$ Domain f. So, fog is not defined but gof is defined.

11. We have, $f(x) = 2x, g(y) = 3y + 4$ and $h(z) = \sin z$ for all $x, y, z \in N$

$\therefore \quad gof(x) = g(f(x)) = g(2x) = 3(2x) + 4 = 6x + 4$

$\Rightarrow \quad \{ho(gof)\}(x) = h\{(gof)(x)\} = h(6x+4) = \sin(6x+4)$ ...(i)

$(hog)(x) = h(g(x)) = h(3x+4) = \sin(3x+4)$

$\therefore \quad \{(hog)of\}(x) = (hog)(f(x)) = (hog)(2x) = \sin 2(3x+4) = \sin(6x+4)$ ...(ii)

From (i) and (ii), we get: $ho(gof) = (hog)of$

12. If $f(x) = x + 1$ and $g(x) = \begin{cases} x-1, \text{ if } x > 1 \\ 1 \ \ , \text{ if } x = 1 \end{cases}$, then $f : N \to N$ is not onto because

Range $(f) = N - \{1\} \neq$ Co–domain of f.

Now, $gof(x) = g(f(x)) = g(x+1) = x + 1 - 1 = x$ $[\because x + 1 > 1]$

Clearly, gof, being identity function, is onto.

13. Let $f : N \to N$ and $g : Z \to Z$ be given by $f(x) = x$ and $g(x) = |x|$. Then, g is not injective as $g(-2) = g(2) = 2$. We observe that $gof : N \to Z$ is given by $gof(x) = g(f(x)) = g(x) = |x| = x$

Clearly, gof is injective but g is not injective.

*2.4.1 PROPERTIES OF COMPOSITION OF FUNCTIONS

THEOREM 1 *The composition of functions is not commutative i.e. $fog \neq gof$.*

PROOF Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the function gof exists because the range of f is a subset of the domain of g. But, fog cannot exist unless the range of g is a subset of domain of f i.e. unless $C \subset A$. As such we find that fog does not exist if $C \not\subset A$ but fog will be a function from B to itself if $A = C$. Thus, if $A = C$.

$$f : A \to B \text{ and } g : B \to A \Rightarrow gof : A \to A \text{ and } fog : B \to B$$

Now, we find that both fog and gof exist but they cannot be equal if A and B are two distinct sets, which are their domains. However if $A = B = C$, then both gof and fog exist and both are from A to itself, even then they may not be equal as shown in Example 1 on page 2.35.

Hence, in general the composition of functions is not necessarily commutative.

THEOREM 2 *The composition of functions is associative i.e. if f, g, h are three functions such that $(fog)oh$ and $fo(goh)$ exist, then $(fog)\,oh = fo(goh)$.* **[NCERT]**

PROOF Let A, B, C, D be four non-void sets. Let $h : A \to B, g : B \to C$ and $f : C \to D$ be three functions. Then,

$$h : A \to B, g : B \to C, f : C \to D \Rightarrow fog : B \to D \text{ and } h : A \to B \Rightarrow (fog)\,oh : A \to D$$

Again, $h : A \to B, g : B \to C, f : C \to D \Rightarrow f : C \to D$ and $goh : A \to C \Rightarrow fo(goh) : A \to D$

Thus, $(fog)oh$ and $fo(goh)$ are functions from set A to set D.

Now, we shall show that $\{(fog)oh\}(x) = \{fo(goh)\}(x)$ for all $x \in A$.

Let x be an arbitrary element of A and let $y \in B, z \in C$ such that $h(x) = y$ and $g(y) = z$. Then,

$\{(fog)\,oh\}(x) = (fog)\{h(x)\}$

$= (fog)(y)$ $[\because h(x) = y]$

$= f(g(y))$

$= f(z)$ $[\because g(y) = z]$ ...(i)

And, $\{fo(goh)\}(x) = f\{(goh)(x)\}$

$= f\{g(h(x))\}$

$= f\{g(y)\}$ $[\because h(x) = y]$

$= f(z)$ $[\because g(y) = z]$ ...(ii)

May be skipped. Not from examination point of view.

From (i) and (ii), we obtain: $\{(fog)\ oh\}(x) = \{fo\,(goh)\}(x)$ for all $x \in A$.
Hence, $(fog)\ oh = fo\,(goh)$.

THEOREM 3 *The composition of two bijections is a bijection i.e. if f and g are two bijections, then gof is also a bijection.* **[NCERT]**

PROOF Let $f : A \to B$ and $g : B \to C$ be two bijections. Then, gof exists such that $gof : A \to C$.
We have to prove that gof is injective as well as surjective map.
Injectivity: Let x, y be two arbitrary elements of A. Then,

$\Rightarrow \quad (gof)(x) = (gof)(y)$
$\Rightarrow \quad g(f(x)) = g(f(y))$
$\Rightarrow \quad f(x) = f(y)$ [$\because g$ is an injective map]
$\Rightarrow \quad x = y$ [$\because f$ is an injective map]

Thus, $(gof)(x) = (gof)(y)$ for all $x, y \in A$. So, gof is an injective map.
Surjectivity: In order to prove the surjectivity of gof, we have to show that every element in C has its pre-image in A i.e. for all $z \in C$, there exists $x \in A$ such that $(gof)(x) = z$.

Let z be an arbitrary element of C. Then,
$z \in C \Rightarrow$ there exists $y \in B$ such that $g(y) = z$ [$\because$ g is a surjective map]
and, $y \in B \Rightarrow$ there exists $x \in A$ such that $f(x) = y$ [$\because$ f is a surjective map]
Thus, we find that for every $z \in C$, there exists $x \in A$ such that
$(gof)(x) = g(f(x)) = g(y) = z.$
i.e. every element of C is the gof-image of some element of A.
So, gof is a surjective map.
Hence, gof being both injective as well as surjective, is a bijective map.

THEOREM 4 *Let $f : A \to B$. Then, $f\,o\,I_A = I_B\,o\,f = f$ i.e. the composition of any function with the identity function is the function itself.*

PROOF Since $I_A : A \to A$ and $f : A \to B$, therefore $fo\,I_A : A \to B$. Now let x be an arbitrary element of A. Then,
$(fo\,I_A)(x) = f(I_A(x)) = f(x)$ [$\because$ $I_A(x) = x$ for all $x \in A$]
$\therefore \quad fo\,I_A = f$
Again, $f : A \to B$ and $I_B : B \to B \Rightarrow I_B\,o\,f : A \to B$.
Now, let x be an arbitrary element of B. Let $f(x) = y$. Then, $y \in B$.

$\therefore \quad (I_B\,o\,f)(x) = I_B(f(x))$
$\Rightarrow \quad (I_B\,o\,f)(x) = I_B(y)$ [$\because$ $f(x) = y$]
$\Rightarrow \quad (I_B\,o\,f)(x) = y$
$\Rightarrow \quad (I_B\,o\,f)(x) = f(x)$ [$\because$ $I_B(y) = y$ for all $y \in B$]
$\therefore \quad I_B\,o\,f = f$

Hence, $f\,o\,I_A = I_B\,o\,f = f$

THEOREM 5 *Let $f : A \to B, g : B \to A$ be two functions such that $gof = I_A$. Then, f is an injection and g is a surjection.*

PROOF *f is an injection:* Let $x, y \in A$ be such that $f(x) = f(y)$. Then,

$f(x) = f(y)$
$\Rightarrow \quad g(f(x)) = g(f(y))$
$\Rightarrow \quad gof(x) = gof(y)$
$\Rightarrow \quad I_A(x) = I_A(y)$ [$\because gof = I_A$ (Given)]
$\Rightarrow \quad x = y$ [By definition of I_A]

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in A$.

So, f is an injective map.

g is a surjection: As $g: B \to A$ therefore to prove that g is a surjection. It is sufficient to prove that every element in A has its pre-image in B.

Let x be an arbitrary element of A. Then, as $f: A \to B$ is a function therefore $f(x) \in B$. Let $f(x) = y$. Then,

$$
\begin{aligned}
& g(y) = g(f(x)) \\
\Rightarrow\ & g(y) = gof(x) \\
\Rightarrow\ & g(y) = I_A(x) && [\because gof = I_A] \\
\Rightarrow\ & g(y) = x
\end{aligned}
$$

Thus, for every $x \in A$ there exists $y = f(x) \in B$ such that $g(y) = x$. So, g is a surjection.

THEOREM 6 *Let $f: A \to B$ and $g: B \to A$ be two function such that $fog = I_B$. Then, f is a surjection and g is an injection.*

PROOF *f is a surjection*: In order to prove that $f: A \to B$ is a surjection, it is sufficient to prove that every element in B has its pre-image in A. Let b be an arbitrary element of B. Since $g: B \to A$. Therefore, $g(b) \in A$.

$$
\begin{aligned}
\text{Let}\quad & g(b) = a. \\
\therefore\quad & f(a) = f(g(b)) && [\because a = g(b)] \\
\Rightarrow\quad & f(a) = fog(b) \\
\Rightarrow\quad & f(a) = I_B(b) && [\because fog = I_B] \\
\Rightarrow\quad & f(a) = b
\end{aligned}
$$

Thus, for every $b \in B$ there exists $a \in A$ such that $f(a) = b$. So, f is a surjection.

g is an injection: Let x, y be any two elements of B such that $g(x) = g(y)$. Then,

$$
\begin{aligned}
& g(x) = g(y) \\
\Rightarrow\ & f(g(x)) = f(g(y)) \\
\Rightarrow\ & fog(x) = fog(y) \\
\Rightarrow\ & I_B(x) = I_B(y) && [\because fog = I_B] \\
\Rightarrow\ & x = y
\end{aligned}
$$

Thus, $g(x) = g(y) \Rightarrow x = y$ for all $x, y, \in B$.

So, g is an injection.

THEOREM 7 *Let $f: A \to B$ and $g: B \to C$ be two functions. Then,*

(i) *$gof: A \to C$ is onto $\Rightarrow$ $g: B \to C$ is onto*

(ii) *$gof: A \to C$ is one-one $\Rightarrow$ $f: A \to B$ is one-one*

(iii) *$gof: A \to C$ is onto and $g: B \to C$ is one-one $\Rightarrow$ $f: A \to B$ is onto*

(iv) *$gof: A \to C$ is one-one and $f: A \to B$ is onto $\Rightarrow$ $g: B \to C$ is one-one.*

PROOF (i) In order to prove that $g: B \to C$ is onto whenever $gof: A \to C$ is onto, it is sufficient to prove that for all $z \in C$ there exists $y \in B$ such that $g(y) = z$.

Let z be an arbitrary element of C. Since $gof: A \to C$ is onto. Therefore, there exists $x \in A$ such that

$$
\begin{aligned}
& gof(x) = z \\
\Rightarrow\ & g(f(x)) = z \\
\Rightarrow\ & g(y) = z, \text{ where } y = f(x) \in B.
\end{aligned}
$$

Thus, for all $z \in C$, there exists $y = f(x) \in B$ such that $g(y) = z$.

Hence, $g: B \to C$ is onto.

(ii) In order to prove that $f: A \to B$ is one-one, it is sufficient to prove that

$$f(x) = f(y) \Rightarrow x = y \text{ for all } x, y \in A.$$

Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$$f(x) = f(y)$$

$\Rightarrow \quad g(f(x)) = g(f(y))$ [$\because g: B \to C$ is a function]

$\Rightarrow \quad gof(x) = gof(y)$

$\Rightarrow \quad x = y$ [$\because gof: A \to C$ is one-one]

Hence, $f: A \to B$ is one-one.

(iii) In order to prove that $f: A \to B$ is onto, it is sufficient to prove that for all $y \in B$ there exists $x \in A$ such that $f(x) = y$. Let y be an arbitrary element of B. Then,

$g(y) \in C$ [$\because g: B \to C$]

Since $gof: A \to C$ is an onto function. Therefore, for any $g(y) \in C$ there exists $x \in A$ such that

$$go f(x) = g(y)$$

$\Rightarrow \quad g(f(x)) = g(y)$

$\Rightarrow \quad f(x) = y$ [$\because g$ is one-one]

Thus, for all $y \in B$ there exists $x \in A$ such that $f(x) = y$.

Hence, $f: A \to B$ is onto.

(iv) Let $y_1, y_2, \in B$ such that $g(y_1) = g(y_2)$. In order to prove that g is one-one, it is sufficient to prove that $y_1 = y_2$.

Since $f: A \to B$ is onto and $y_1, y_2 \in B$. So, there exist $x_1, x_2 \in A$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Now, $\quad g(y_1) = g(y_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad gof(x_1) = gof(x_2)$

$\Rightarrow \quad x_1 = x_2$ [$\because gof: A \to C$ is one-one]

$\Rightarrow \quad f(x_1) = f(x_2)$ [$\because f: A \to B$ is a function]

$\Rightarrow \quad y_1 = y_2$

Hence, $g: B \to C$ is one-one.

*2.5 COMPOSITION OF REAL FUNCTIONS

In the previous section, we have learnt about the composition of general functions. We have learnt that if $f: A \to B$ and $g: C \to D$, then

$gof: A \to D$ is defined as $gof(x) = g(f(x))$, provided that Range $(f) \subseteq$ Domain (g)

and,

$fog: C \to B$ is defined as $fog(x) = f(g(x))$, provided that Range $(g) \subseteq$ Domain (f)

In case of real functions f and g, even if range of f is not contained in domain of g, then gof is defined for those elements in domain of f which have their images in domain of g. Similarly, if range of g is not a subset of domain of f, then fog is defined for those elements in domain of g which have their images in the domain of f.

Thus, we may define the composition of two real functions as follows:

DEFINITION *Let $f: D_1 \to R$ and $g: D_2 \to R$ be two real functions. Then,*

$$gof: X = \{x \in D_1 : f(x) \in D_2\} \to R \text{ and, } fog: Y = \{x \in D_2 : g(x) \in D_1\} \to R \text{ are defined as}$$
$$gof(x) = g(f(x)) \text{ for all } x \in X \text{ and } fog(x) = f(g(x)) \text{ for all } x \in Y.$$

REMARK 1 *If Range $(f) \subseteq$ Domain (g), then $gof: D_1 \to R$ and if Range $(g) \subseteq$ Domain (f), then $fog: D_2 \to R$.*

May be skipped. Not from examination point of view.

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REMARK 2 *For any two real functions f and g, it may be possible that gof exists but fog does not. In some cases, even if both exist, they may not be equal.*

REMARK 3 *If Range $(f) \cap$ Domain $(g) = \phi$, then gof does not exist. In other words, gof exists if Range $(f) \cap$ Domain $(g) \neq \phi$. Similarly, fog exists if Range $(g) \cap$ Domain $(f) \neq \phi$*

REMARK 4 *If f and g are bijections, then fog and gof both are bijections.*

REMARK 5 *If $f : R \to R$ and $g : R \to R$ are real functions, then fog and gof both exist.*

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *If $f : R \to R$ and $g : R \to R$ be functions defined by $f(x) = x^2 + 1$ and $g(x) = \sin x$, then find fog and gof.*

SOLUTION We have, $f(x) = x^2 + 1$ and $g(x) = \sin x$.

Now, $x^2 \geq 0$ for all $x \in R$

$\Rightarrow$ $x^2 + 1 \geq 1$ for all $x \in R \Rightarrow f(x) \geq 1$ for all $x \in R \Rightarrow$ Range $(f) = [1, \infty)$

Also, $-1 \leq \sin x \leq 1$ for all $x \in R \Rightarrow$ Range $(g) = [-1, 1]$

Clearly, Range $(f) = [1, \infty) \subseteq$ Domain (g) and, Range $(g) = [-1, 1] \subseteq$ Domain (f).

So, $gof : R \to R$ and $fog : R \to R$ are given by $gof(x) = g(f(x)) = g(x^2 + 1) = \sin(x^2 + 1)$ and, $fog(x) = f(g(x)) = f(\sin x) = \sin^2 x + 1$ respectively.

EXAMPLE 2 *If $f : [0, \infty) \to R$ and $g : R \to R$ be defined as $f(x) = \sqrt{x}$ and $g(x) = -x^2 - 1$, then find gof and fog.*

SOLUTION Clearly, Domain $(f) = [0, \infty)$, Range $(f) = [0, \infty)$, Domain $(g) = R$

and, Range $(g) = (-\infty, -1]$ $\quad [\because -x^2 \leq 0$ for all $x \therefore -x^2 - 1 \leq -1$ for all $x \in R]$

Computation of gof : We observe that: Range $(f) = [0, \infty) \subseteq$ Domain (g)

$\therefore$ gof exists and Domain (gof) = Domain $(f) = [0, \infty)$

Also, $(gof)(x) = g(f(x)) = g(\sqrt{x}) = -(\sqrt{x})^2 - 1 = -x - 1$.

Thus, $gof : [0, \infty) \to R$ is given by $gof(x) = -x - 1$.

Computation of fog : We have, Range $(g) = [-\infty, -1)$. Clearly, it is not a subset of domain of f. So, fog does not exist.

EXAMPLE 3 *If $f(x) = e^x$ and $g(x) = \log_e x \ (x > 0)$, find fog and gof. Is fog = gof?* [CBSE 2002]

SOLUTION We observe that

Domain $(f) = R$, Range $(f) = (0, \infty)$, Domain $(g) = (0, \infty)$ and, Range $(g) = R$.

Computation of fog : We observe that: Range (g) = Domain (f)

$\therefore$ fog exists and fog : Domain $(g) \to R$ i.e. $fog : (0, \infty) \to R$ such that

$$fog(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$$

Thus, $fog : (0, \infty) \to R$ is defined as $fog(x) = x$.

Computation of gof : We have, Range $(f) = (0, \infty)$ = Domain (g)

$\therefore$ gof exists and gof : Domain $(f) \to R$ i.e. $gof : R \to R$ such that

$$gof(x) = g(f(x)) = g(e^x) = \log_e e^x = x \log_e e = x.$$

Thus, $gof : R \to R$ is defined as $gof(x) = x$.

We observe that Domain $(gof) \neq$ Domain (fog). Therefore, $gof \neq fog$.

EXAMPLE 4 *If $f(x)=\sqrt{x}\ (x>0)$ and $g(x)=x^2-1$ are two real functions, find fog and gof. Is $fog = gof$?* **[CBSE 2002]**

SOLUTION We observe that: Domain $(f)=[0,\infty)$, Range $(f)=[0,\infty)$, Domain $(g)=R$

and, Range $(g)=[-1,\infty)$ $\quad [\because\ x^2 \geq 0$ for all $x \in R \therefore x^2-1 \geq -1$ for all $x \in R]$

Computation of gof: We observe that: Range $(f)=[0,\infty) \subseteq$ Domain (g). Therefore, gof exists and $gof:[0,\infty)\to R$ such that $gof(x)=g(f(x))=g(\sqrt{x})=(\sqrt{x})^2-1=x-1$

Thus, $gof:[0,\infty)\to R$ is defined as $gof(x)=x-1$.

Computation of fog: We observe that: Range $(g)=[-1,\infty)\not\subseteq$ Domain (f)

$\therefore$ Domain $(fog)=\{x: x\in$ Domain (g) and $g(x)\in$ Domain $(f)\}=\{x: x\in R$ and $g(x)\in[0,\infty)\}$

$=\{x: x\in R$ and $x^2-1\in[0,\infty)\}=\{x: x\in R$ and $x^2-1\geq 0\}$

$=\{x: x\in R$ and $x\leq -1$ or, $x\geq 1\}=\{x: x\leq -1$ or $x\geq 1\}=(-\infty,-1]\cup[1,\infty)$

Also, $fog(x)=f(g(x))=f(x^2-1)=\sqrt{x^2-1}$

Thus, $fog:(-\infty,-1]\cup[1,\infty)\to R$ is defined as $fog(x)=\sqrt{x^2-1}$.

We find that fog and gof have distinct domains. Also, their formulas are not same. Hence, $fog \neq gof$.

EXAMPLE 5 *If $f(x)=\frac{1}{x}$ and $g(x)=0$ are two real functions, show that fog is not defined.*

SOLUTION We find that:

Domain $(f)=R-\{0\}$, Range $(f)=R-\{0\}$, Domain $(g)=R$ and, Range $(g)=\{0\}$.

Clearly, Range $(g)\cap$ Domain $(f)=\phi$ Hence, fog is not defined.

EXAMPLE 6 *Let $f(x)=[x]$ and $g(x)=|x|$. Find*

(i) $(gof)\left(\frac{-5}{3}\right)-(fog)\left(\frac{-5}{3}\right)$ (ii) $(gof)\left(\frac{5}{3}\right)-(fog)\left(\frac{5}{3}\right)$ (iii) $(f+2g)(-1)$

SOLUTION We have, $f(x)=[x]$ and $g(x)=|x|$. We find that Domain $(f)=R$ and, Domain $(g)=R$. Therefore, each of fog, gof and $f+2g$ has domain R.

(i) $$(gof)\left(\frac{-5}{3}\right)-(fog)\left(\frac{-5}{3}\right)=g\left\{f\left(\frac{-5}{3}\right)\right\}-f\left\{g\left(\frac{-5}{3}\right)\right\}=g\left\{\left[\frac{-5}{3}\right]\right\}-f\left\{\left|-\frac{5}{3}\right|\right\}$$

$$=g(-2)-f\left(\frac{5}{3}\right)=|-2|-\left[\frac{5}{3}\right]=2-1=1$$

(ii) $$(gof)\left(\frac{5}{3}\right)-(fog)\left(\frac{5}{3}\right)=g\left\{f\left(\frac{5}{3}\right)\right\}-f\left\{g\left(\frac{5}{3}\right)\right\}$$

$$=g\left\{\left[\frac{5}{3}\right]\right\}-f\left\{\left|\frac{5}{3}\right|\right\}=g(1)-f\left(\frac{5}{3}\right)=|1|-\left[\frac{5}{3}\right]=1-1=0$$

(iii) $(f+2g)(-1)=f(-1)+(2g)(-1)=f(-1)+2g(-1)=[-1]+2|-1|=-1+2\times 1=1.$

EXAMPLE 7 *Let f and g be real functions defined by $f(x)=\frac{x}{x+1}$ and $g(x)=\frac{1}{x+3}$. Describe the functions gof and fog (if they exist).*

SOLUTION We have, $f(x)=\frac{x}{x+1}$ and $g(x)=\frac{1}{x+3}$. We find that Domain $(f)=R-\{-1\}$ and, Range $(f)=R-\{1\}$, Domain $(g)=R-\{-3\}$ and, Range $(g)=R-\{0\}$.

Computation of gof : We observe that: Range $(f) \not\subset$ Domain (g)

$\therefore$ Domain $(gof) = \{x : x \in \text{Domain}(f) \text{ and } f(x) \in \text{Domain}(g)\}$

$$= \left\{x : x \in R - \{-1\} \text{ and } \frac{x}{x+1} \in R - \{-3\}\right\} = \left\{x \in R : x \neq -1 \text{ and } \frac{x}{x+1} \neq -3\right\}$$

$$= \left\{x \in R : x \neq -1 \text{ and } x \neq -\frac{3}{4}\right\} = R - \left\{-\frac{3}{4}, -1\right\} \quad \left[\because \frac{x}{x+1} = -3 \Rightarrow x = -\frac{3}{4}\right]$$

And, $\quad gof(x) = g(f(x)) = g\left(\dfrac{x}{x+1}\right) = \dfrac{1}{\dfrac{x}{x+1} + 3} = \dfrac{x+1}{4x+3}$

Hence, $gof : R - \left\{-\frac{3}{4}, -1\right\} \to R$ is defined as $gof(x) = \dfrac{x+1}{4x+3}$.

Computation of fog : We observe that: Range $(g) \not\subset$ Domain (f).

$\therefore$ Domain $(fog) = \{x : x \in \text{Domain}(g) \text{ and } g(x) \in \text{Domain}(f)\}$

$$= \left\{x : x \in R - \{-3\} \text{ and } \frac{1}{x+3} \in R - \{-1\}\right\} = \left\{x : x \neq -3 \text{ and } \frac{1}{x+3} \neq -1\right\}$$

$$= \{x : x \neq -3 \text{ and } x \neq -4\} \quad \left[\because \frac{1}{x+3} = -1 \Rightarrow x = -4\right]$$

$$= \{x \in R : x \neq -3, -4\} = R - \{-3, -4\}$$

And, $\quad fog(x) = f(g(x)) = f\left(\dfrac{1}{x+3}\right) = \dfrac{\dfrac{1}{x+3}}{\dfrac{1}{x+3} + 1} = \dfrac{1}{x+4}$

Hence, $fog : R - \{-3, -4\} \to R$ is defined as $fog(x) = \dfrac{1}{x+4}$.

EXAMPLE 8 *Let* $f(x) = \dfrac{x}{\sqrt{1+x^2}}$. *Then, show that* $(fofof)(x) = \dfrac{x}{\sqrt{1+3x^2}}$.

SOLUTION We have, $f(x) = \dfrac{x}{\sqrt{1+x^2}}$. Clearly, domain $(f) = R$. In order to find the range of f, we proceed as follows: Let $f(x) = y$. Then,

$$y = f(x) \Rightarrow \frac{x}{\sqrt{1+x^2}} = y \Rightarrow \frac{x^2}{1+x^2} = y^2 \Rightarrow x = \pm \frac{y}{\sqrt{1-y^2}}$$

Since x takes real values. Therefore, $1 - y^2 > 0 \Rightarrow y^2 - 1 < 0 \Rightarrow y \in (-1, 1)$.

Hence, Range $(f) = (-1, 1)$

Clearly, Range $(f) \subset$ Domain f. Therefore, $fof : R \to R$ and $fofof : R \to R$.

Now,

$$(fofof)(x) = ((fof)of)(x) = (fof)(f(x)) = (fof)\left(\frac{x}{\sqrt{1+x^2}}\right) = f\left(f\left(\frac{x}{\sqrt{1+x^2}}\right)\right)$$

$$= f\left(\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}}\right) = f\left(\frac{x}{\sqrt{1+2x^2}}\right) = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}} = \frac{x}{\sqrt{1+3x^2}}$$

EXAMPLE 9 *Let f be a real function defined by $f(x) = \sqrt{x-1}$. Find $(fofof)(x)$. Also, show that $fof \neq f^2$.*

SOLUTION We have, $f(x) = \sqrt{x-1}$. Clearly, Domain $(f) = [1, \infty)$ and Range $(f) = [0, \infty)$. We observe that range (f) is not a subset of domain of f.

$\therefore$ Domain $(fof) = \{x : x \in \text{Domain}(f) \text{ and } f(x) \in \text{Domain}(f)\}$

$= \{x : x \in [1, \infty) \text{ and } \sqrt{x-1} \in [1, \infty)\}$

$= \{x : x \in [1, \infty) \text{ and } \sqrt{x-1} \geq 1\} = \{x : x \in [1, \infty) \text{ and } x \geq 2\} = [2, \infty)$

Clearly, Range $(f) = [0, \infty) \not\subset$ Domain (fof).

$\therefore$ Domain $((fof)of) = \{x : x \in \text{Domain}(f) \text{ and } f(x) \in \text{Domain}(fof)\}$

$= \{x : x \in [1, \infty) \text{ and } f(x) \in [2, \infty)\} = \{x : x \in [1, \infty) \text{ and } \sqrt{x-1} \in [2, \infty)\}$

$= \{x : x \geq 1 \text{ and } \sqrt{x-1} \geq 2\} = \{x : x \geq 1 \text{ and } x - 1 \geq 4\}$

$= \{x : \geq 1 \text{ and } x \geq 5\} = [5, \infty)$

Now, $(fof)(x) = f(f(x)) = f(\sqrt{x-1}) = \sqrt{\sqrt{x-1}-1}$

$\therefore$ $(fofof)(x) = ((fof)of)(x) = (fof)(f(x)) = (fof)(\sqrt{x-1}) = \sqrt{\sqrt{\sqrt{x-1}-1}-1}$

Thus, $fof : [2, \infty) \to R$ and $fofof : [5, \infty) \to R$ are defined as

$fof(x) = \sqrt{\sqrt{x-1}-1}$ and $(fofof)(x) = \sqrt{\sqrt{\sqrt{x-1}-1}-1}$

Now, $f^2(x) = [f(x)]^2 = \left(\sqrt{x-1}\right)^2 = x - 1.$

$\therefore$ $f^2 : [1, \infty) \to R$ is given by $f^2(x) = x - 1$. Clearly, $fof \neq f^2$.

EXAMPLE 10 *If $f(x) = \frac{x-1}{x+1}$, $x \neq -1$, then show that $f(f(x)) = -\frac{1}{x}$ provided that $x \neq 0, -1$.*

SOLUTION We have, $f(x) = \frac{x-1}{x+1}$. Clearly, $f(x)$ is defined for all $x \in R$ except $x + 1 = 0\, x = -1$.

Therefore, Domain $(f) = R - \{-1\}$. Let us now find the range of f. Let $y = f(x)$. Then,

$$y = \frac{x-1}{x+1} \Rightarrow x = \frac{y+1}{1-y}$$

As x takes all real values other than -1. Therefore, y also takes all real values other than 1.

$\therefore$ Range $(f) = R - \{1\}$

We observe that Range $(f) \not\subset$ Domain (f).

$\therefore$ Domain $(fof) = \{x : x \in \text{Domain}(f) \text{ and } f(x) \in \text{Domain}(f)\}$

$$= \left\{x : x \in R - \{-1\} \text{ and } \frac{x-1}{x+1} \in R - \{-1\}\right\}$$

$$= \left\{x : x \neq -1 \text{ and } \frac{x-1}{x+1} \neq -1\right\} = \{x : x \neq -1 \text{ and } x \neq 0\} = R - \{-1, 0\}$$

And, $fof\,(x) = f\,(f\,(x)) = f\left(\dfrac{x-1}{x+1}\right) = \dfrac{\dfrac{x-1}{x+1}-1}{\dfrac{x-1}{x+1}+1} = \dfrac{-2}{2x} = -\dfrac{1}{x}$

Thus, $fof : R - \{-1, 0\} \to R$ is defined as $fof\,(x) = -\dfrac{1}{x}$ or, $f\,(f\,(x)) = -\dfrac{1}{x}$.

Hence, $f\,(f\,(x)) = -\dfrac{1}{x}$ for all $x \neq 0, -1$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 11 *If* $f\,(x) = \dfrac{3x-2}{2x-3}$, *prove that* $f\,(f\,(x)) = x$ *for all* $x \in R - \left\{\dfrac{3}{2}\right\}$.

SOLUTION We have, $f\,(x) = \dfrac{3x-2}{2x-3}$. Clearly, domain $(f) = R - \left\{\dfrac{3}{2}\right\}$. Let $y = f\,(x)$. Then,

$$y = \frac{3x-2}{2x-3} \Rightarrow 2xy - 3y = 3x - 2 \Rightarrow x = \frac{3y-2}{2y-3}$$

Clearly, $x \in R$ for all $y \in R$, $y \neq \dfrac{3}{2}$. Therefore, Range $(f) = R - \left\{\dfrac{3}{2}\right\}$.

Since, Range $(f) =$ Domain (f). Therefore, Domain (fof) = Domain (f). Thus, for any $x \in$ Domain $(fof) = R - \left\{\dfrac{3}{2}\right\}$, we obtain

$$(fof)\,(x) = f\,(f\,(x)) = f\left(\frac{3x-2}{2x-3}\right) = \frac{3\left(\dfrac{3x-2}{2x-3}\right)-2}{2\left(\dfrac{3x-2}{2x-3}\right)-3} = \frac{9x-6-4x+6}{6x-4-6x+9} = x$$

Hence, $(fof)\,(x) = f\,(f\,(x)) = x$ for all $x \in R - \left\{\dfrac{3}{2}\right\}$.

EXAMPLE 12 *If* $f(x) = \dfrac{1}{2x+1}$, $x \neq -\dfrac{1}{2}$, *then show that* $f(f(x)) = \dfrac{2x+1}{2x+3}$, *provided that* $x \neq -\dfrac{1}{2}, -\dfrac{3}{2}$.

SOLUTION We have, $f(x) = \dfrac{1}{2x+1}$. Clearly, domain $(f) = R - \left\{-\dfrac{1}{2}\right\}$. Let $y = \dfrac{1}{2x+1}$. Then,

$$y = \frac{1}{2x+1} \Rightarrow 2x + 1 = \frac{1}{y} \Rightarrow x = \frac{1-y}{2y}$$

Since x is a real number distinct from $-\dfrac{1}{2}$. Therefore, y can take any non-zero real value.

So, Range $(f) = R - \{0\}$. We observe that Range $(f) = R - \{0\} \not\subset$ Domain $(f) = R - \left\{-\dfrac{1}{2}\right\}$

$\therefore$ Domain $(fof) = \{x : x \in \text{Domain } (f) \text{ and } f\,(x) \in \text{Domain } (f)\}$

$$= \left\{x : x \in R - \left\{-\frac{1}{2}\right\} \text{ and } f\,(x) \in R - \left\{-\frac{1}{2}\right\}\right\} = \left\{x : x \neq -\frac{1}{2} \text{ and } f\,(x) \neq -\frac{1}{2}\right\}$$

$$= \left\{x : x \neq -\frac{1}{2} \text{ and } \frac{1}{2x+1} \neq -\frac{1}{2}\right\} = \left\{x : x \neq -\frac{1}{2} \text{ and } x \neq -\frac{3}{2}\right\} = R - \left\{-\frac{1}{2}, -\frac{3}{2}\right\}$$

And, $fof(x) = f(f(x)) = f\left(\dfrac{1}{2x+1}\right) = \dfrac{1}{2\left(\dfrac{1}{2x+1}\right)+1} = \dfrac{2x+1}{2x+3}$

Thus, $fof : R - \left\{-\dfrac{1}{2}, -\dfrac{3}{2}\right\} \to R$ is defined by $fof(x) = \dfrac{2x+1}{2x+3}$.

Hence, $f(f(x)) = \dfrac{2x+1}{2x+3}$ for all $x \in R, x \neq -\dfrac{1}{2}, -\dfrac{3}{2}$.

EXERCISE 2.3

BASIC

1. Find fog and gof, if
   (i) $f(x) = e^x$, $g(x) = \log_e x$ (ii) $f(x) = x^2$, $g(x) = \cos x$
   (iii) $f(x) = |x|$, $g(x) = \sin x$ (iv) $f(x) = x+1$, $g(x) = e^x$
   (v) $f(x) = \sin^{-1} x$, $g(x) = x^2$ (vi) $f(x) = x+1$, $g(x) = \sin x$
   (vii) $f(x) = x+1$, $g(x) = 2x+3$ (viii) $f(x) = c, c \in R$, $g(x) = \sin x^2$
   (ix) $f(x) = x^2 + 2$, $g(x) = 1 - \dfrac{1}{1-x}$
2. Let $f(x) = x^2 + x + 1$ and $g(x) = \sin x$. Show that $fog \neq gof$.
3. If $f(x) = |x|$, prove that $fof = f$.
4. If $f(x) = 2x + 5$ and $g(x) = x^2 + 1$ be two real functions, then describe each of the following functions:
   (i) fog (ii) gof (iii) fof (iv) f^2
   Also, show that $fof \neq f^2$.

BASED ON LOTS

5. If $f(x) = \sin x$ and $g(x) = 2x$ be two real functions, then describe gof and fog. Are these equal functions?
6. Let f, g, h be real functions given by $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. Prove that $fog = go(fh)$.
7. Let f be any real function and let g be a function given by $g(x) = 2x$. Prove that $gof = f + f$.
8. If $f(x) = \sqrt{1-x}$ and $g(x) = \log_e x$ are two real functions, then describe functions fog and gof.
9. If $f : (-\pi/2, \pi/2) \to R$ and $g : [-1, 1] \to R$ be defined as $f(x) = \tan x$ and $g(x) = \sqrt{1-x^2}$ respectively. Describe fog and gof.
10. If $f(x) = \sqrt{x+3}$ and $g(x) = x^2 + 1$ be two real functions, then find fog and gof.
11. Let f be a real function given by $f(x) = \sqrt{x-2}$. Find each of the following:
    (i) fof (ii) $fofof$ (iii) $(fofof)(38)$ (iv) f^2
    Also, show that $fof \neq f^2$.
12. If $f, g : R \to R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Then, find fog and gof. Hence, find $fog(-3)$, $fog(5)$ and $gof(-2)$. **[CBSE 2016]**

BASED ON HOTS

13. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$. Find fof.

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ANSWERS

1. (i) $fog:(0,\infty)\to R$ given by $fog(x)=x$, $gof:R\to R$ given by $gof(x)=x$

(ii) $fog:R\to R$ given by $fog(x)=\cos^2 x$, $gof:R\to R$ given by $gof(x)=\cos x^2$

(iii) $fog:R\to R$ given by $fog(x)=|\sin x|$, $gof:R\to R$ given by $gof(x)=\sin|x|$

(iv) $fog:R\to R$ given by $fog(x)=e^x+1$, $gof:R\to R$ given by $gof(x)=e^{x+1}$

(v) $fog:[-1,1]\to R$ given by $fog(x)=\sin^{-1}(x^2)$,

$gof:[-1,1]\to R$ given by $gof(x)=(\sin^{-1}x)^2$

(vi) $fog:R\to R$ given by $fog(x)=\sin x+1$, $gof:R\to R$ given by $gof(x)=\sin(x+1)$

(vii) $fog:R\to R$ given by $fog(x)=2x+4$, $gof:R\to R$ given by $gof(x)=2x+5$

(viii) $fog:R\to R$ given by $fog(x)=c$, $gof:R\to R$ given by $gof(x)=\sin c^2$

(ix) $fog:R-\{1\}\to R$ given by $fog(x)=\dfrac{3x^2-4x+2}{(1-x)^2}$,

$gof:R\to R$ given by $gof(x)=\dfrac{x^2+2}{x^2+1}$

4. (i) $fog:R\to R$ is given by $fog(x)=2x^2+7$

(ii) $gof:R\to R$ is given by $gof(x)=4x^2+20x+26$

(iii) $fof:R\to R$ is given by $fof(x)=4x+15$

(iv) $f^2:R\to R$ is given by $f^2(x)=4x^2+20x+25$

5. (i) $gof:R\to R$ is defined as $gof(x)=2\sin x$

(ii) $fog:R\to R$ is defined as $fog(x)=\sin 2x$. No.

8. $fog:(0,e]\to R$ is given by $(fog)(x)=\sqrt{1-\log_e x}$

$gof:(-\infty,1)\to R$ is given by $(gof)(x)=\dfrac{1}{2}\log(1-x)$

9. $fog:[-1,1]\to R$ is defined as $fog(x)=\tan\sqrt{1-x^2}$

$gof:[-\pi/4,\pi/4]\to R$ is defined as $gof(x)=\sqrt{1-\tan^2 x}$

10. $fog:R\to R$ is defined as $fog(x)=\sqrt{x^2+4}$

$gof:[-3,\infty)\to R$ is defined as $gof(x)=x+4$

11. (i) $fof:[6,\infty)\to R$ is given by $fof(x)=\sqrt{\sqrt{x-2}-2}$

(ii) $fofof:[38,\infty)\to R$ is given by $(fofof)(x)=\sqrt{\sqrt{\sqrt{x-2}-2}-2}$

(iii) 0 (iv) $f^2\,[2,\infty)\to R$ is given by $f^2(x)=x-2$

12. $fog(x)=\begin{cases}0, & x\ge 0\\ -4x, & x<0\end{cases}$, $gof(x)=0$, for all x and, $fog(-3)=12$, $fog(5)=0$, $gof(-2)=0$

13. $fof(x)=\begin{cases}2+x, & \text{if } 0\le x\le 1\\ 2-x, & \text{if } 1<x\le 2\\ 4-x, & \text{if } 2<x\le 3\end{cases}$

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2.6 INVERSE OF AN ELEMENT

Let A and B be two sets and let $f: A \to B$ be a mapping. As we have discussed earlier that if $a \in A$ is associated to $b \in B$ under the function f, then 'b' is called the f image of 'a' and we write it as $b = f(a)$. We also say that 'a' is the pre-image or inverse element of 'b' under f and we write $a = f^{-1}(b)$.

It should be noted that the inverse of an element under a function may consist of a single element, two or more elements or no element depending on whether function is injective or many-one; onto or into. If $f: A \to B$ is a many-one and into function, then the inverse of some elements of B may or may not exist or the inverse of some element of B may consists of more than one element. If f is a bijection, then for each $b \in B$, $f^{-1}(b)$ exists and it consists of a single element only.

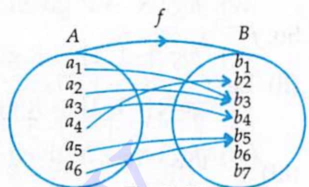

Fig. 2.40

If f is represented by Fig. 2.40 , then we find that: $f^{-1}(b_1) = \phi$, $f^{-1}(b_2) = a_4$, $f^{-1}(b_3) = \{a_1, a_2\}$, $f^{-1}(b_4) = a_3$, $f^{-1}(b_5) = \{a_5, a_6\}$, $f^{-1}(b_6) = \phi$ and, $f^{-1}(b_7) = \phi$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $f: Q \to Q$ is given by $f(x) = x^2$, then find:* (i) $f^{-1}(9)$ (ii) $f^{-1}(-5)$ (iii) $f^{-1}(0)$

SOLUTION (i) Let $f^{-1}(9) = x$. Then, $f(x) = 9 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$.

$\therefore \quad f^{-1}(9) = \{-3, 3\}$.

(ii) Let $f^{-1}(-5) = x$. Then, $f(x) = -5 \Rightarrow x^2 = -5$, which is not possible for any $x \in Q$.

$\therefore \quad f^{-1}(-5) = \phi$

(iii) Let $f^{-1}(0) = x$. Then, $f(x) = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$. So, $f^{-1}(0) = \{0\}$.

EXAMPLE 2 *If the function $f: R \to R$ be defined by $f(x) = x^2 + 5x + 9$, find $f^{-1}(8)$ and $f^{-1}(9)$.*

SOLUTION Let $f^{-1}(8) = x$. Then, $f(x) = 8 \Rightarrow x^2 + 5x + 9 = 8 \Rightarrow x = \dfrac{-5 \pm \sqrt{21}}{2}$ which are in R.

$$\therefore \quad f^{-1}(8) = \left\{\frac{-5+\sqrt{21}}{2}, \frac{-5-\sqrt{21}}{2}\right\}$$

Now, let $f^{-1}(9) = x$. Then, $f(x) = 9$.

$\Rightarrow \quad x^2 + 5x + 9 = 9 \Rightarrow x^2 + 5x = 0 \Rightarrow x(x+5) = 0 \Rightarrow x = 0, -5$, which are in R

$\therefore \quad f^{-1}(9) = \{0, -5\}$

EXAMPLE 3 *If the function $f: C \to C$ be defined by $f(x) = x^2 - 1$, find $f^{-1}(-5)$ and $f^{-1}(8)$.*

SOLUTION Let $f^{-1}(-5) = x$. Then,

$$f(x) = -5 \Rightarrow x^2 - 1 = -5 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4} \Rightarrow x = \pm 2i, \text{ which are in } C.$$

$\therefore \quad f^{-1}(-5) = \{2i, -2i\}$

Again, let $f^{-1}(8) = x$. Then, $f(x) = 8 \Rightarrow x^2 - 1 = 8 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$, which are in C.

$\therefore \quad f^{-1}(8) = \{-3, 3\}$

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EXAMPLE 4 *Let $f : R \to R$ be defined as $f(x) = x^2 + 1$. Find:*

(i) $f^{-1}(-5)$ (ii) $f^{-1}(26)$ (iii) $f^{-1}\{10, 37\}$

SOLUTION (i) Let $f^{-1}(-5) = x$. Then,

$f(x) = -5 \Rightarrow x^2 + 1 = -5 \Rightarrow x^2 = -6 \Rightarrow x = \pm\sqrt{-6}$, which is not in R.

So, $f^{-1}(-5) = \phi$.

(ii) Let $f^{-1}(26) = x$. Then, $f(x) = 26 \Rightarrow x^2 + 1 = 26 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$, which are real numbers

$\therefore \quad f^{-1}(26) = \{-5, 5\}$

(iii) $f^{-1}\{10, 37\} = \{x \in R : f(x) = 10 \text{ or } f(x) = 37\}, = \{x \in R : x^2 + 1 = 10 \text{ or } x^2 + 1 = 37\}$

$= \{x \in R : x^2 = 9 \text{ or } x^2 = 36\} = \{3, -3, 6, -6\}$

2.7 INVERSE OF A FUNCTION

Let A and B be two sets and let $f : A \to B$ be a function. If we follow a rule in which elements of B are associated to their pre-images, then we find that under such a rule there may be some elements in B which are not associated to elements in A. This happens when f is not an onto map. Therefore all elements in B will be associated to some elements in A if f is an onto map. Also, if it is a many-one function then under the said rule an element in B may be associated to more than one element in A. Therefore an element in B will be associated to a unique element in A if f is an injective map.

It follows from the above discussion that if $f : A \to B$ is a bijection, we can define a new function from B to A which associates each element $y \in B$ to its pre-image $f^{-1}(y) \in A$. Such a function is known as the inverse of function f and is denoted by f^{-1}.

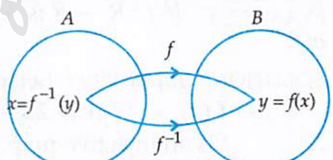

Fig. 2.41

DEFINITION *Let $f : A \to B$ be a bijection. Then a function $g : B \to A$ which associates each element $y \in B$ to a unique element $x \in A$ such that $f(x) = y$ is called the inverse of f.*

i.e., $f(x) = y \Leftrightarrow g(y) = x$

The inverse of f is generally denoted by f^{-1}.

Thus, if $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

In order to find the inverse of a bijection, we may follow the following algorithm.

ALGORITHM

Let $f : A \to B$ be a bijection. To find the inverse of f we follow the following steps:

Step I *Put $f(x) = y$, where $y \in B$ and $x \in A$.*

Step II *Solve $f(x) = y$ to obtain x in terms of y.*

Step III *In the relation obtained in step II replace x by $f^{-1}(y)$ to obtain the required inverse of f.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $f : A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.*

SOLUTION Clearly, $f(1) = 2$, $f(2) = 4$, $f(3) = 6$ and $f(4) = 8$.

$\therefore \quad f = \{(1, 2), (2, 4), (3, 6), (4, 8)\}$ which is clearly a bijection.

On interchanging the components of ordered pairs in f, we obtain

$$f^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4)\}.$$

EXAMPLE 2 *Let $S = \{1, 2, 3\}$. Determine whether the function $f : S \to S$ defined as below have inverse. Find f^{-1}, if it exists.*

(i) $f = \{(1, 1), (2, 2), (3, 3)\}$ (ii) $f = \{(1, 2), (2, 1), (3, 1)\}$ (iii) $f = \{(1, 3), (3, 2), (2, 1)\}$ **[NCERT]**

SOLUTION (i) Clearly, $f : S \to S$ is a bijection. So, f is invertible and its inverse is given by $f^{-1} = \{(1, 1), (2, 2), (3, 3)\}$.

(ii) Clearly, $f(2) = f(3) = 1$. Therefore, f is may-one and hence it is not invertible.

(iii) Clearly, $f : S \to S$ is a bijection and hence invertible. The inverse of f is given by

$$f^{-1} = \{(3, 1), (2, 3), (1, 2)\}.$$

EXAMPLE 3 *Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find the inverse $(f^{-1})^{-1}$ of f^{-1}. Show that $(f^{-1})^{-1} = f$.* **[NCERT]**

SOLUTION We have, $f = \{(1, a), (2, b), (3, c)\}$ ...(i)

Clearly, f is a bijection and hence invertible. The inverse of f is given by

$$f^{-1} = \{(a, 1), (b, 2), (c, 3)\} \Rightarrow (f^{-1})^{-1} = \{(1, a), (2, b), (3, c)\} \quad ...(ii)$$

From (i) and (ii), we find that: $(f^{-1})^{-1} = f$.

EXAMPLE 4 *If $f : R \to R$ is defined by $f(x) = 2x + 7$. Prove that f is a bijection. Also, find the inverse of f.*

SOLUTION *Injectivity* : Let x, y be any two elements of R. Then,

$$f(x) = f(y) \Rightarrow 2x + 7 = 2y + 7 \Rightarrow x = y$$

$\therefore$ f is an injective map.

Surjectivity : Let y be an arbitrary element of R(co-domain). Then,

$$y = f(x) \Rightarrow y = 2x + 7 \Rightarrow x = \frac{y - 7}{2}.$$

Clearly, $x = \dfrac{y-7}{2} \in R$ for all $y \in R$. Thus, for all $y \in R$ (co-domain) there exists $x = \dfrac{y-7}{2} \in R$ such that $f(x) = y$. In other words every element in R (co-domain) has its pre-image in R (domain). Therefore, f is a surjective map. Hence, f is a bijection. Consequently f^{-1} exists.

Inverse of f: Let $x \in R$ (domain) and $y \in R$ (co-domain) such that $f(x) = y$. Then,

$$f(x) = y \Rightarrow 2x + 7 = y \Rightarrow x = \frac{y-7}{2} \Rightarrow f^{-1}(y) = \frac{y-7}{2}.$$

Thus, $f^{-1} : R \to R$ is defined as $f^{-1}(x) = \dfrac{x-7}{2}$ for all $x \in R$.

EXAMPLE 5 *If $f : R \to R$ is a bijection given by $f(x) = x^3 + 3$, find $f^{-1}(x)$.*

SOLUTION Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow x^3 + 3 = y \Rightarrow x = (y - 3)^{1/3} \Rightarrow f^{-1}(y) = (y - 3)^{1/3}$$

Thus, $f^{-1} : R \to R$ is defined as $f^{-1}(x) = (x - 3)^{1/3}$ for all $x \in R$.

EXAMPLE 6 *Let $f : R \to R$ be defined by $f(x) = 3x - 7$. Show that f is invertible and hence find f^{-1}.*

SOLUTION In order to prove that f is invertible, it is sufficient to prove that f is a bijection.

Injectivity: Let $x, y \in R$. Then, $f(x) = f(y) \Rightarrow 3x - 7 = 3y - 7 \Rightarrow x = y$.

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Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$. So, f is an injection.

Surjectivity: Let y be an arbitrary element of R. Then,

$$f(x) = y \Rightarrow 3x - 7 = y \Rightarrow x = \frac{y+7}{3}$$

Clearly, $\frac{y+7}{3} \in R$ for all $y \in R$. Thus, for all $y \in R$ there exists $x = \frac{y+7}{3} \in R$ such that

$$f(x) = f\left(\frac{y+7}{3}\right) = 3\left(\frac{y+7}{3}\right) - 7 = y$$

So, f is a surjection. Hence, $f : R \rightarrow R$ is a bijection. Consequently, it is invertible.

Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow 3x - 7 = y \Rightarrow x = \frac{y+7}{3} \Rightarrow f^{-1}(y) = \frac{y+7}{3}$$

Therefore, $f^{-1} : R \rightarrow R$ is given by $f^{-1}(x) = \frac{x+7}{3}$.

EXAMPLE 7 *Show that $f : R - \{0\} \rightarrow R - \{0\}$ given by $f(x) = \frac{3}{x}$ is invertible and it is inverse of itself.*

SOLUTION In order to prove that f is invertible, it is sufficient to show that it is a bijection.

f is an injection: Let $x, y \in R - \{0\}$ such that $f(x) = f(y)$. Then,

$$f(x) = f(y) \Rightarrow \frac{3}{x} = \frac{3}{y} \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R - \{0\}$. So, f is an injection.

f is a surjection: Let y be an arbitrary element of $R - \{0\}$. Then, $f(x) = y \Rightarrow \frac{3}{x} = y \Rightarrow x = \frac{3}{y}$.

Thus, for each $y \in R - \{0\}$, there exists $x = \frac{3}{y} \in R - \{0\}$ such that $f(x) = f\left(\frac{3}{y}\right) = \frac{3}{3/y} = y$. So, f is a surjection. Hence, f is a bijection. Consequently, it is invertible.

Let $f(x) = y$. Then,

$$f(x) = y \Rightarrow \frac{3}{x} = y \Rightarrow x = \frac{3}{y} \Rightarrow f^{-1}(y) = \frac{3}{y}$$

Thus, f^{-1} is given by $f^{-1}(x) = \frac{3}{x}$ for all $x \in R - \{0\}$. Clearly, $f(x) = f^{-1}(x)$ for all $x \in R - \{0\}$.

Hence, f is inverse of itself.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *Let $f : N \cup \{0\} \rightarrow N \cup \{0\}$ be defined by $f(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$. Show that f is invertible and $f = f^{-1}$.* **[CBSE 2014, NCERT]**

SOLUTION In Example 24 on page 2.23, we have proved that f is a bijection. So, it is invertible. In order to find f^{-1}, let $n, m \in N \cup \{0\}$ such that

$$f(n) = m$$

$$\Rightarrow \quad n + 1 = m, \text{ if } n \text{ is even}$$

$$n - 1 = m, \text{ if } n \text{ is odd}$$

$$\Rightarrow \quad n = \begin{cases} m-1, & \text{if } m \text{ is odd} \\ m+1, & \text{if } m \text{ is even} \end{cases} \qquad \left[\begin{array}{l} \text{If } n \text{ is even, then } n+1 = m \text{ is odd} \\ \text{If } n \text{ is odd, then } n-1 = m \text{ is even} \end{array}\right]$$

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$$\Rightarrow \quad f^{-1}(m) = \begin{cases} m-1, & \text{if } m \text{ is odd} \\ m+1, & \text{if } m \text{ is even} \end{cases}$$

Hence, $f^{-1}(n) = \begin{cases} n+1, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases}$. Clearly, $f = f^{-1}$.

*2.7.1 PROPERTIES OF INVERSE OF A FUNCTION

THEOREM 1 *The inverse of a bijection is unique.* [NCERT]

PROOF Let $f : A \to B$ be a bijection. If possible, let $g : B \to A$ and $h : B \to A$ be two inverses of f. We have to prove that $g = h$. In order to prove this it is sufficient to show that $g(y) = h(y)$ for all $y \in B$. Let y be an arbitrary element of B.

Let $g(y) = x_1$ and $h(y) = x_2$. Then,

$g(y) = x_1 \Rightarrow f(x_1) = y$ [$\because$ g is inverse of f]

and $h(y) = x_2 \Rightarrow f(x_2) = y$ [$\because$ h is inverse of f]

$\therefore \quad f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 = x_2$ [$\because$ f is one-one]

$\Rightarrow \quad g(y) = h(y)$

Thus, $g(y) = h(y)$ for all $y \in B$. Hence, $g = h$

THEOREM 2 *The inverse of a bijection is also a bijection.*

PROOF Let $f : A \to B$ be a bijection and let $g : B \to A$ be its inverse. We have to show that g is a bijection.

Injectivity of g: Let $y_1, y_2, \in B$ such that $g(y_1) = x_1$ and $g(y_2) = x_2$.

Since g is the inverse of f.

$\therefore \quad g(y_1) = x_1 \Rightarrow f(x_1) = y_1$ and $g(y_2) = x_2 \Rightarrow f(x_2) = y_2$.

Now, $g(y_1) = g(y_2) \Rightarrow x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \Rightarrow y_1 = y_2$

$\therefore \quad g$ is an injective map.

Surjectivity of g: In order to prove that $g : B \to A$ is a surjection, we have to show that every element in A has its pre-image in B under funtion g. So, let x be an arbitrary element of A. Then,

$x \in A$

$\Rightarrow$ there exists $y \in B$ such that $f(x) = y$ [$\because$ f is a function from A to B]

$\Rightarrow$ there exists $y \in B$ such that $g(y) = x$ [$\because$ g is inverse of f]

Thus, for each $x \in A$, there exists $y \in B$ such that $g(y) = x$. So, g is a surjective map.

Hence, g is a bijection.

THEOREM 3 *If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are the identity functions on the sets A and B respectively.*

PROOF In order to prove that $gof = I_A$ and $fog = I_B$, we have to prove that

$(gof)(x) = x$ for all $x \in A$ and $(fog)(y) = y$ for all $y \in B$

Let x be an element of A such that $f(x) = y$. Then,

$g(y) = x$ [$\because$ g is inverse of f]

Now, $(gof)(x) = g(f(x)) = g(y) = x \Rightarrow (gof)(x) = x$ for all $x \in A \Rightarrow gof = I_A$.

We have,

$(fog)(y) = f(g(y)) = f(x) = y \Rightarrow fog(y) = y$ for all $y \in B \Rightarrow fog = I_B$.

Hence, $gof = I_A$ and $fog = I_B$.

May be skipped. Not from examination point of view.

REMARK *In the above property, if we have $B = A$. Then, we find that for every bijection $f : A \to A$ there exists a bijection $g : A \to A$ such that $fog = gof = I_A$.*

THEOREM 4 *If $f : A \to B$ and $g : B \to C$ are two bijections, then $gof : A \to C$ is a bijection and $(gof)^{-1} = f^{-1} og^{-1}$.* [NCERT]

PROOF It is given that

$$\left.\begin{array}{l} f : A \to B \text{ is a bijection} \\ g : B \to C \text{ is a bijection} \end{array}\right\} \Rightarrow gof : A \to C \text{ is a bijection} \Rightarrow (gof)^{-1} : C \to A \text{ exists.}$$

Now, $\left.\begin{array}{l} f : A \to B \text{ is a bijection} \Rightarrow f^{-1} : B \to A \text{ is a bijection} \\ g : B \to C \text{ is a bijection} \Rightarrow g^{-1} : C \to B \text{ is a bijection} \end{array}\right\} \Rightarrow f^{-1} og^{-1} : C \to A$

Let $x \in A, y \in B$ and $z \in C$ such that $f(x) = y$ and $g(y) = z$. Then,

$$(gof)(x) = g(f(x)) = g(y) = z \Rightarrow (gof)^{-1}(z) = x \quad \text{...(i)}$$

Now, $f(x) = y$ and $g(y) = z \Rightarrow f^{-1}(y) = x$ and, $g^{-1}(z) = y$

$$\therefore \quad (f^{-1}og^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x \quad \text{...(ii)}$$

From (i) and (ii), we obtain: $(gof)^{-1}(z) = (f^{-1}og^{-1})(z)$ for all $z \in C$.

Hence, $(gof)^{-1} = f^{-1}og^{-1}$.

THEOREM 5 *Let $f : A \to B$ and $g : B \to A$ be two functions such that $gof = I_A$ and $fog = I_B$. Then, f and g are bijections and $g = f^{-1}$.*

PROOF *f is one-one*: Let $x, y \in A$ such that $f(x) = f(y)$. Then,

$f(x) = f(y)$

$\Rightarrow \quad g(f(x)) = g(f(y))$

$\Rightarrow \quad (gof)(x) = (gof)(y) \Rightarrow I_A(x) = I_A(y) \Rightarrow x = y \quad [\because gof = I_A]$

$\therefore \quad f$ is a one-one map.

f is onto: Let $y \in B$ and let $g(y) = x$. Then,

$g(y) = x$

$\Rightarrow \quad f(g(y)) = f(x)$

$\Rightarrow \quad (fog)(y) = f(x)$

$\Rightarrow \quad I_B(y) = f(x) \quad [\because fog = I_B]$

$\Rightarrow \quad y = f(x) \quad [\because I_B(y) = y]$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$. So, f is onto. Hence, f is a bijection.

Similarly, it can be proved that g is a bijection. Now we shall show that $g = f^{-1}$.

Since $f : A \to B$ is a bijection. Therefore, f^{-1} exists.

Now, $fog = I_B$

$\Rightarrow \quad f^{-1} o(fog) = f^{-1} o I_B$

$\Rightarrow \quad (f^{-1}of) og = f^{-1} o I_B \quad$ [By associativity]

$\Rightarrow \quad I_A og = f^{-1} o I_B \quad [\because f^{-1} of = I_A]$

$\Rightarrow \quad g = f^{-1} \quad [\because I_A og = g \text{ and } f^{-1} o I_B = f^{-1}]$

Hence, $g = f^{-1}$

THEOREM 6 *Let $f : A \to B$ be an invertible function. Show that the inverse of f^{-1} is f. i.e., $(f^{-1})^{-1} = f$.* [NCERT]

SOLUTION Since inverse of a bijection is also a bijection. Therefore, $f^{-1} : B \to A$ is a bijection and hence invertible. As $f^{-1} : B \to A$ is a bijection. Therefore, $(f^{-1})^{-1} : A \to B$ is also a bijection.

Let x be an arbitrary element of A such that $f(x) = y$. Then,

$$f^{-1}(y) = x \qquad [\because f^{-1} \text{ is the inverse of } f]$$

$$\Rightarrow \quad (f^{-1})^{-1}(x) = y \qquad [\because (f^{-1})^{-1} \text{ is the inverse of } f^{-1}]$$

$$\Rightarrow \quad (f^{-1})^{-1}(x) = f(x) \qquad [\because f(x) = y]$$

Since x is an arbitrary element of A. Therefore, $(f^{-1})^{-1}(x) = f(x)$ for all $x \in A$.

Hence, $(f^{-1})^{-1} = f$. **Q.E.D.**

ALITER Since $f : A \to B$ is invertible and $f^{-1} : B \to A$ is inverse of f.

$$\therefore \quad f^{-1} o f = I_A \text{ and } fof^{-1} = I_B \Rightarrow f \text{ is inverse of } f^{-1} \Rightarrow f = (f^{-1})^{-1}$$ **Q.E.D.**

REMARK 1 *Sometimes $f : A \to B$ is one-one but not onto. In such a case f is not invertible. But, $f : A \to$ Range (f) is both one and onto. So, it is invertible and its inverse can be found.*

2.7.2 RELATION BETWEEN GRAPHS OF A FUNCTION AND ITS INVERSE

The graph of a bijection f and its inverse f^{-1} are closely related. If (a, b) is a point on the graph of f, then $b = f(a)$ and $a = f^{-1}(b)$. As $b \in$ Domain of f^{-1}, therefore $(b, f^{-1}(b))$ is a point on the graph of f^{-1}. But, $(b, f^{-1}(b)) = (b, a)$. Therefore, (b, a) is on the graph of f^{-1}. Thus, if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. But, (a, b) and (b, a) are reflections of one another in the line $y = x$. Thus, the graph of f^{-1} may be obtained by reflecting the graph of f in the line mirror $y = x$. That is the graphs of f and f^{-1} are mirror images of each other in the line mirror $y = x$ (see Fig. 2.42). It is also evident from the above discussion that if the graphs of $f(x)$ and $f^{-1}(x)$ intersect each other, their points of intersection lie on the line $y = x$. Consequently, solutions of the equation $f(x) = f^{-1}(x)$ are same as that of $f(x) = x$ or, $f^{-1}(x) = x$.

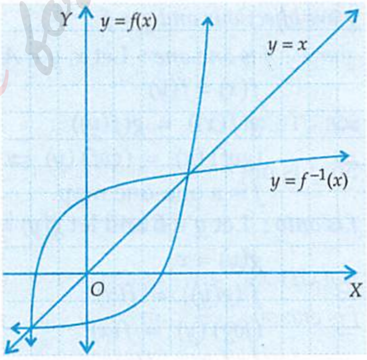

Fig. 2.42

REMARK 2 *Theorem 5 on page 2.51 suggests us an alternative method to prove the invertibility of a function. It states that if $f : A \to B$ and $g : B \to A$ are two functions such that $gof = I_A$ and $fog = I_B$, then f and g are inverse of each other.*

Theorem 5 on page 2.51 suggests the following algorithm to find the inverse of an invertible function.

ALGORITHM

Step I *Obtain the function and check its bijectivity.*

Step II *If f is a bijection, then it is invertible.*

In order to find the inverse of f, put $fof^{-1}(x) = x \Rightarrow f(f^{-1}(x)) = x$

Step III *Use the formula for $f(x)$ and replace x by $f^{-1}(x)$ in it to obtain the LHS of $f(f^{-1}(x)) = x$.*

Solve this equation for $f^{-1}(x)$ to get $f^{-1}(x)$.

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Prove that the function $f : R \to R$ defined as $f(x) = 2x - 3$ is invertible. Also, find f^{-1}.*

SOLUTION In order to prove that f is invertible, it is sufficient to show that f is a bijection.

f is one-one : Let $x, y \in R$. Then,

$$f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow x = y$$

Thus, $f(x) = f(y) \Rightarrow x = y$ for all $x, y \in R$. So, f is one-one.

f is onto : Let y be an arbitrary element in R (co-domain of f). Then,

$$f(x) = y \Rightarrow 2x - 3 = y \Rightarrow x = \frac{y+3}{2}$$

Clearly, $x = \frac{y+3}{2} \in R$ (domain) for all $y \in R$ (co-domain). Thus, for each $y \in R$ there exists $x \in R$ such that $f(x) = y$. So, f is onto.

Since $f : R \to R$ is one-one and onto both. So, it is a bijection and hence invertible.

Now,

$$fof^{-1}(x) = x \Rightarrow f(f^{-1}(x)) = x \Rightarrow 2f^{-1}(x) - 3 = x \quad [\because f(x) = 2x - 3]$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x+3}{2}$$

Thus, $f^{-1} : R \to R$ is given by $f^{-1}(x) = \frac{x+3}{2}$ for all $x \in R$.

EXAMPLE 2 *Show that the function $f : R \to R$ is given by $f(x) = x^2 + 1$ is not invertible.*

SOLUTION We have, $f(x) = x^2 + 1$. Clearly, $-2 \neq 2$ but, $f(-2) = f(2) = 5$.

So, f is not a one-one function. Hence, f is not invertible.

EXAMPLE 3 *Show that $f : R - \{-1\} \to R - \{1\}$ given by $f(x) = \frac{x}{x+1}$ is invertible. Also, find f^{-1}.*

SOLUTION In order to prove the invertibility of $f(x)$, it is sufficient to show that it is a bijection.

f is one-one : For any $x, y \in R - \{-1\}$

$$f(x) = f(y) \Rightarrow \frac{x}{x+1} = \frac{y}{y+1} \Rightarrow xy + x = xy + y \Rightarrow x = y.$$

So, f is one-one .

f is onto: Let $y \in R - \{1\}$. Then, $f(x) = y \Rightarrow \frac{x}{x+1} = y \Rightarrow x = \frac{y}{1-y}$

Clearly, $x \in R$ for all $y \in R - \{1\}$. Also, $x \neq -1$. Because,

$$x = -1 \Rightarrow \frac{y}{1-y} = -1 \Rightarrow y = -1 + y, \text{ which is not possible.}$$

Thus, for each $y \in R - \{1\}$ there exists $x = \frac{y}{1-y} \in R - \{-1\}$ such that $f(x) = \frac{x}{x+1} = \frac{\frac{y}{1-y}}{\frac{y}{1-y} + 1} = y$.

So, f is onto. Thus, f is both one-one and onto. Consequently it is invertible.

Now, $fof^{-1}(x) = x$ for all $x \in R - \{1\}$

$$\Rightarrow \quad f(f^{-1}(x)) = x \Rightarrow \frac{f^{-1}(x)}{f^{-1}(x) + 1} = x \Rightarrow f^{-1}(x) = \frac{x}{1-x} \text{ for all } x \in R - \{1\}.$$

EXAMPLE 4 *Show that $f:[-1,1] \to R$, given by $f(x)=\frac{x}{x+2}$ is one-one. Find the inverse of the function $f:[-1,1] \to$ Range (f).* **[NCERT]**

SOLUTION Let x, y be any two elements of $[-1, 1]$. Then,

$$f(x)=f(y) \Rightarrow \frac{x}{x+2}=\frac{y}{y+2} \Rightarrow xy+2x=xy+2y \Rightarrow x=y$$

So, $f:[-1,1] \to$ Range (f) is one-one.

Obviously, $f:[-1,1] \to$ Range (f) is onto and hence invertible. Let f^{-1} denote the inverse of f. Then,

$$fof^{-1}(x)=x \text{ for all } x \in \text{Range}(f)$$

$$\Rightarrow \quad f(f^{-1}(x))=x \Rightarrow \frac{f^{-1}(x)}{f^{-1}(x)+2}=x \Rightarrow f^{-1}(x)=x f^{-1}(x)+2x \Rightarrow f^{-1}(x)=\frac{2x}{1-x}$$

Hence, f^{-1}: Range $(f):[-1,1]$ is given by $f^{-1}(x)=\frac{2x}{1-x}$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Let $f:R \to R$ be defined as $f(x)=10x+7$. Find the function $g:R \to R$ such that $gof=fog=I_R$.* **[NCERT, CBSE 2011]**

SOLUTION We have, $fog = I_R$

$$\Rightarrow \quad fog(x) = I_R(x) \text{ for all } x \in R$$

$$\Rightarrow \quad f(g(x)) = x \text{ for all } x \in R \Rightarrow 10\,g(x)+7=x \text{ for all } x \in R \Rightarrow g(x)=\frac{x-7}{10} \text{ for all } x \in R$$

ALITER We have, $fog = gof = I_R \Rightarrow g$ is the inverse of f. Let $f(x) = y$. Then,

$$10x+7 = y \Rightarrow x = \frac{y-7}{10} \Rightarrow f^{-1}(y) = \frac{y-7}{10} \Rightarrow f^{-1}(x) = \frac{x-7}{10}$$

Hence, $g(x) = \frac{x-7}{10}$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 6 *If the function $f:[1,\infty) \to [1,\infty)$ defined by $f(x)=2^{x(x-1)}$ is invertible, find $f^{-1}(x)$.*

SOLUTION It is given that f is invertible with f^{-1} as its inverse.

$$\therefore \quad (fof^{-1})(x) = x \text{ for all } x \in [1,\infty)$$

$$\Rightarrow f(f^{-1}(x)) = x \Rightarrow 2^{f^{-1}(x)\{f^{-1}(x)-1\}} = x$$

$$\Rightarrow f^{-1}(x)\{f^{-1}(x)-1\} = \log_2 x \Rightarrow \{f^{-1}(x)\}^2 - f^{-1}(x) - \log_2 x = 0$$

$$\Rightarrow f^{-1}(x)=\frac{1 \pm \sqrt{1+4\log_2 x}}{2} \Rightarrow f^{-1}(x)=\frac{1+\sqrt{1+4\log_2 x}}{2} \quad [\because f^{-1}(x) \in [1,\infty) \therefore f^{-1}(x) \geq 1]$$

EXAMPLE 7 *Find the value of parameter α for which the function $f(x) = 1+\alpha x$, $\alpha \neq 0$ is the inverse of itself.*

SOLUTION Clearly, $f(x)$ is a bijection from R to itself.

Now, $fof^{-1}(x) = x \Rightarrow f(f^{-1}(x)) = x \Rightarrow 1+\alpha f^{-1}(x)=x \Rightarrow f^{-1}(x) = \frac{x-1}{\alpha}$

It is given that

$$f(x) = f^{-1}(x) \text{ for all } x \in R$$

$$\Rightarrow \quad 1+\alpha x = \frac{x-1}{\alpha} \text{ for all } x \in R$$

$\Rightarrow \quad \alpha x+1 = \left(\frac{1}{\alpha}\right)x+\left(\frac{-1}{\alpha}\right)$ for all $x \in R$

$\Rightarrow \quad \alpha = \frac{1}{\alpha}$ and $1 = -\frac{1}{\alpha} \Rightarrow \alpha^2 = 1$ and $\alpha = -1 \Rightarrow \alpha = -1.$

EXAMPLE 8 *Let $f : N \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3$ for some $x \in N\}$. Show that f is invertible. Find its inverse.* **[NCERT]**

SOLUTION In order to prove that f is invertible, it is sufficient to show that it is a bijection.

f is one-one: For any $x, y \in N$, we find that

$$f(x) = f(y) \Rightarrow 4x + 3 = 4y + 3 \Rightarrow x = y$$

So, $f : N \to Y$ is one-one.

f is onto: Let y be an arbitrary element of Y. Then, there exists $x \in N$ such that

$y = 4x + 3$ [By definition of Y]

$\Rightarrow \quad y = f(x)$

Thus, for each $y \in N$ there exists $x \in N$ such that $f(x) = y$. So, $f : N \to Y$ is onto.

Thus, $f : N \to Y$ is both one and onto. Consequently, it is invertible. Let f^{-1} be the inverse of f. Then,

$fof^{-1}(x) = x$ for all $x \in Y$

$\Rightarrow \quad f(f^{-1}(x)) = x$ for all $x \in Y$

$\Rightarrow \quad 4f^{-1}(x) + 3 = x$ for all $x \in Y$ [Using definition of f]

$\Rightarrow \quad f^{-1}(x) = \frac{x-3}{4}$ for all $x \in Y$

Hence, $f^{-1} : Y \to N$ is given by $f^{-1}(x) = \frac{x-3}{4}$ for all $x \in Y$.

EXAMPLE 9 *Let $Y = \{n^2 : n \in N\} \subset N$. Consider $f : N \to Y$ given by $f(n) = n^2$. Show that f is invertible. Find the inverse of f.* **[NCERT]**

SOLUTION In order to prove that f is invertible, it is sufficient to show that it is a bijection.

f is one-one: For any $n, m \in N$, we find that

$f(n) = f(m) \Rightarrow n^2 = m^2 \Rightarrow n = m$ [$\because$ $n, m \in N$]

So, $f : N \to Y$ is one-one.

f is onto: Let y be an arbitrary element of Y. Then there exists $n \in N$ such that

$y = n^2$ [By definition of Y]

$\Rightarrow \quad y = f(n)$

Thus, for each $y \in Y$ there exists $n \in N$ such that $y = f(n)$. So, $f : N \to Y$ is onto.

Hence, $f : N \to Y$ is a bijection. Consequently, it is invertible.

Let f^{-1} denote the inverse of f. Then,

$fof^{-1}(x) = x$ for all $x \in Y$

$\Rightarrow \quad f(f^{-1}(x)) = x$ for all $x \in Y$

$\Rightarrow \quad \{f^{-1}(x)\}^2 = x$ for all $x \in Y$ [Using the definition of f]

$\Rightarrow \quad f^{-1}(x) = \sqrt{x}$ for all $x \in Y$

Hence, $f^{-1} : Y \to N$ is given by $f^{-1}(x) = \sqrt{x}$ for all $x \in Y$.

EXAMPLE 10 *Let $f : N \to R$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f : N \to Range(f)$ is invertible. Find the inverse of f.* **[CBSE 2010]**

SOLUTION In order to prove that f is invertible, it is sufficient to show that $f : N \to$ Range (f) is a bijection.

f is one-one: For any $x, y \in N$, we find that

$f(x) = f(y)$

$\Rightarrow \quad 4x^2 + 12x + 15 = 4y^2 + 12y + 15$

$\Rightarrow \quad 4(x^2 - y^2) + 12(x - y) = 0$

$\Rightarrow \quad (x - y)(4x + 4y + 3) = 0$

$\Rightarrow \quad x - y = 0 \qquad [\because 4x + 4y + 3 \neq 0$ for any $x, y \in N]$

$\Rightarrow \quad x = y$

So, $f : N \to$ Range (f) is one-one.

Obviously, $f : N \to$ Range (f) is onto. Hence, $f : N \to$ Range (f) is invertible. Let f^{-1} denote the inverse of f. Then,

$fof^{-1}(x) = x$ for all $x \in$ Range (f)

$\Rightarrow \quad f(f^{-1}(x)) = x$ for all $x \in$ Range (f)

$\Rightarrow \quad 4\{f^{-1}(x)\}^2 + 12 f^{-1}(x) + 15 = x$ for all $x \in$ Range (f)

$\Rightarrow \quad 4\{f^{-1}(x)\}^2 + 12 f^{-1}(x) + 15 - x = 0$

$$\Rightarrow \quad f^{-1}(x) = \frac{-12 \pm \sqrt{144 - 16(15 - x)}}{8} = \frac{-12 \pm \sqrt{16x - 96}}{8} = \frac{-3 \pm \sqrt{x - 6}}{2}$$

$$\Rightarrow \quad f^{-1}(x) = \frac{-3 + \sqrt{x - 6}}{2} \qquad [\because f^{-1}(x) \in N \therefore f^{-1}(x) > 0]$$

EXERCISE 2.4

BASIC

1. State with reasons whether following functions have inverse:
   (i) $f : \{1, 2, 3, 4\} \to \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$
   (ii) $g : \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$
   (iii) $h : \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$
2. Find f^{-1} if it exists : $f : A \to B$ where
   (i) $A = \{0, -1, -3, 2\}$; $B = \{-9, -3, 0, 6\}$ and $f(x) = 3x$.
   (ii) $A = \{1, 3, 5, 7, 9\}$; $B = \{0, 1, 9, 25, 49, 81\}$ and $f(x) = x^2$.
3. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ and $g : \{a, b, c\} \to$ {apple, ball, cat} defined as $f(1) = a$, $f(2) = b$, $f(3) = c$, $g(a) =$ apple, $g(b) =$ ball and $g(c) =$ cat. Show that f, g and gof are invertible. Find f^{-1}, g^{-1} and $(gof)^{-1}$ and show that $(gof)^{-1} = f^{-1} o g^{-1}$. **[NCERT]**
4. Let $A = \{1, 2, 3, 4\}$; $B = \{3, 5, 7, 9\}$; $C = \{7, 23, 47, 79\}$ and $f : A \to B, g : B \to C$ be defined as $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Express $(gof)^{-1}$ and $f^{-1} o g^{-1}$ as the sets of ordered pairs and verify that $(gof)^{-1} = f^{-1} o g^{-1}$.
5. Show that the function $f : Q \to Q$ defined by $f(x) = 3x + 5$ is invertible. Also, find f^{-1}.
6. Consider $f : R \to R$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f. **[NCERT]**
7. Consider $f : R^+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with inverse f^{-1} of f given by $f^{-1}(x) = \sqrt{x - 4}$, where R^+ is the set of all non-negative real numbers. **[NCERT, CBSE 2013]**

8. If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$ for all $x \neq \dfrac{2}{3}$. What is the inverse of f? **[CBSE 2012, 2013, 2020, NCERT]**

9. If $f : R \to R$ be defined by $f(x) = x^3 - 3$, then prove that f^{-1} exists and find a formula for f^{-1}. Hence, find $f^{-1}(24)$ and $f^{-1}(5)$.

10. A function $f : R \to R$ is defined as $f(x) = x^3 + 4$. Is it a bijection or not? In case it is a bijection, find $f^{-1}(3)$. **[CBSE 2020]**

11. If $f : Q \to Q$, $g : Q \to Q$ are two functions defined by $f(x) = 2x$ and $g(x) = x + 2$, show that f and g are bijective maps. Verify that $(gof)^{-1} = f^{-1} og^{-1}$.

12. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \dfrac{x-2}{x-3}$. Show that f is one-one and onto and hence find f^{-1}. **[CBSE 2012, 2014]**

13. Let $f : N \to N$ be a function defined as $f(x) = 9x^2 + 6x - 5$. Show that $f : N \to S$, where S is the range of f, is invertible. Find the inverse of f and hence find $f^{-1}(43)$ and $f^{-1}(163)$. **[CBSE 2016, 2017]**

14. Let $f : R - \left\{-\dfrac{4}{3}\right\} \to R$ be a function defined as $f(x) = \dfrac{4x}{3x+4}$. Show that $f : R - \left\{-\dfrac{4}{3}\right\} \to \text{Rang}(f)$ is one-one and onto. Hence, find f^{-1}. **[CBSE 2017, 2020]**

15. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B$ is a function defined by $f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Find f^{-1}. **[CBSE 2019]**

BASED ON LOTS

16. Consider $f : R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(x) = \dfrac{\sqrt{x+6} - 1}{3}$. **[NCERT]**

17. Consider the function $f : R^+ \to [-9, \infty)$ given by $f(x) = 5x^2 + 6x - 9$. Prove that f is invertible with $f^{-1}(y) = \dfrac{\sqrt{54 + 5y} - 3}{5}$. **[CBSE 2015]**

18. Show that the function $f : N \to N$ defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find the inverse of $f : N \to S$, where S is range of f. **[CBSE 2019]**

19. Consider the bijective function $f : R^+ \to (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where R^+ is the set of positive real numbers. Find the inverse function of f. **[CBSE 2020]**

BASED ON HOTS

20. Let $f : [-1, \infty) \to [-1, \infty)$ is given by $f(x) = (x+1)^2 - 1$. Show that f is invertible. Also, find the set $S = \{x : f(x) = f^{-1}(x)\}$.

21. Let $A = \{x \in R \mid -1 \leq x \leq 1\}$ and let $f : A \to A$, $g : A \to A$ be two functions defined by $f(x) = x^2$ and $g(x) = \sin \dfrac{\pi x}{2}$. Show that g^{-1} exists but f^{-1} does not exist. Also, find g^{-1}.

22. Let f be a function from R to R such that $f(x) = \cos(x + 2)$. Is f invertible? Justify your answer.

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23. If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Define any four bijections from A to B. Also, give their inverse functions.

24. Let A and B be two sets each with finite number of elements. Assume that there is an injective map from A to B and that there is an injective map from B to A. Prove that there is a bijection from A to B.

25. If $f : A \to A$, $g : A \to A$ are two bijections, then prove that
(i) fog is an injection (ii) fog is a surjection.

ANSWERS

1. (i) No, f is many-one (ii) No, g is many-one (iii) Yes, h is a bijection
2. (i) $f^{-1} = \{(-9, -3), (-3, -1), (0, 0), (6, 2)\}$ (ii) f^{-1} does not exist as f is not surjective.
3. $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$, $g^{-1} = \{(\text{apple}, a), (\text{ball}, b), (\text{cat}, c)\}$
and, $(gof)^{-1} = \{(\text{apple}, 1), (\text{ball}, 2) (\text{cat}, 3)\}$
4. $(gof)^{-1} = f^{-1}og^{-1} = \{(7, 1), (23, 2), (47, 3), (79, 4)\}$ 5. $f^{-1}(x) = \dfrac{x-5}{3}$
6. $f^{-1}(x) = \dfrac{x-3}{4}$ 8. $f^{-1}(x) = \dfrac{4x+3}{6x-4}$ 9. $f^{-1}(x) = (3+x)^{1/3}$, $f^{-1}(24) = 3$, $f^{-1}(5) = 2$
10. Bijection, $f^{-1}(3) = -1$ 12. $f^{-1}(x) = \dfrac{3x-2}{x-1}$ 13. $f^{-1}(43) = 2$, $f^{-1}(163) = 4$
14. $f^{-1}(x) = \dfrac{4x}{4-3x}$ 15. $f^{-1}(x) = \dfrac{2x-1}{x-1}$ 18. $f^{-1}(x) = \dfrac{-1+\sqrt{4x-3}}{2}$
19. $f^{-1}(x) = \dfrac{-3+\sqrt{x+2}}{4}$ 20. $S = \{0, -1\}$ 21. $g^{-1}(x) = \left(\dfrac{2}{\pi}\right)\sin^{-1} x$
22. Not invertible
23. $f_1 = \{(1, a), (2, b), (3, c), (4, d)\}$, $f_1^{-1} = \{(a, 1), (b, 2), (c, 3), (d, 4)\}$
$f_2 = \{(1, a), (2, c), (3, b), (4, d)\}$, $f_2^{-1} = \{(a, 1), (c, 2), (b, 3), (d, 4)\}$
$f_3 = \{(1, d), (3, b), (2, a), (4, c)\}$, $f_3^{-1} = \{(d, 1), (b, 3), (a, 2), (c, 4)\}$ etc.

HINTS TO SELECTED PROBLEMS

3. $f : \{1, 2, 3\} \to \{a, b, c\}$ is given by $f(1) = a$, $f(2) = b$, $f(3) = c$. Clearly, it is a bijection. Similarly, $g : \{a, b, c\} \to \{\text{apple, ball, cat}\}$ given by $g(a) = \text{apple}$, $g(b) = \text{ball}$, $g(c) = \text{cat}$ is also a bijection. Since composition of two bijection is a bijection.
So, $gof : \{1, 2, 3\} \to \{\text{apple, ball, cat}\}$ is a bijection.
It is given that: $f = \{(1, a), (2, b), (3, c)\}$ and $g = \{(a, \text{apple}), (b, \text{ball}), (c, \text{cat})\}$
$\therefore$ $gof = \{(1, \text{apple}), (2, \text{ball}), (3, \text{cat})\}$
Clearly, $f^{-1} = \{(a, 1), (b, 2), (c, 3)\}$ and $g^{-1} = \{(\text{apple}, a), (\text{ball}, b), (\text{cat}, c)\}$
$\therefore$ $(gof)^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$ ...(i)
and, $f^{-1}og^{-1} = \{(\text{apple}, 1), (\text{ball}, 2), (\text{cat}, 3)\}$ ...(ii)
From (i) and (ii), we get $(gof)^{-1} = f^{-1}og^{-1}$.

6. It is given that $f : R \to R$ such that $f(x) = 4x + 3$.
f is an injection: Let $x, y \in R$ be such that $f(x) = f(y) \Rightarrow 4x + 3 = 4y + 3 \Rightarrow x = y$. So, f is an injection.
f is a surjection: Let y be an arbitrary element of R (Co-domain) such that $f(x) = y$. Then,

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$$f(x) = y \Rightarrow 4x + 3 = y \Rightarrow x = \frac{y-3}{4}$$

Thus, for any $y \in R$ there exists $x = \dfrac{y-3}{4} \in R$ such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$$

So, $f : R \to R$ is a bijection and hence invertible. Let f^{-1} denote the inverse of f. Then,

$$fof^{-1}(x) = x \text{ for all } x \in R$$

$$\Rightarrow \quad f(f^{-1}(x)) = x \text{ for all } x \in R$$

$$\Rightarrow \quad 4f^{-1}(x) + 3 = x \text{ for all } x \in R \Rightarrow f^{-1}(x) = \frac{x-3}{4} \text{ for all } x \in R$$

7. We have, $f : R^+ \to [4, \infty)$ such that $f(x) = x^2 + 4$.

f is an injection: Let $x, y \in R^+$ such that

$$f(x) = f(y) \Rightarrow x^2 + 4 = y^2 + 4 \Rightarrow x = y \qquad [\because x, y \in R^+]$$

So, f is an injective map.

f is onto: Let $y \in [4, \infty)$. Then, $f(x) = y \Rightarrow x^2 + 4 = y \Rightarrow x = \sqrt{y-4}$ $\qquad [\because x \in R^+]$

And, $\quad y \in [4, \infty) \Rightarrow y - 4 > 0 \Rightarrow \sqrt{y-4} > 0 \Rightarrow x = \sqrt{y-4} \in R^+$

Thus, for each $y \in [4, \infty)$ there exists $x = \sqrt{y-4} \in R^+$ such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y.$$

So, $f : R^+ \to [4, \infty)$ is onto. Thus, $f : R^+ \to [4, \infty)$ is a bijection and hence invertible.

Let f^{-1} denote the inverse of f. Then,

$$fof^{-1}(x) = x \text{ for all } x \in [4, \infty) \Rightarrow f(f^{-1}(x)) = x \text{ for all } x \in [4, \infty)$$

$$\Rightarrow \quad (f^{-1}(x))^2 + 4 = x \text{ for all } x \in [4, \infty) \Rightarrow f^{-1}(x) = \sqrt{x-4} \text{ for all } x \in [4, \infty)$$

8. We have, $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$

$$\therefore \quad fof(x) = f(f(x)) = f\left(\frac{4x+3}{67x-4}\right) = \frac{4\left(\dfrac{4x+3}{6x-4}\right) + 3}{6\left(\dfrac{4x+3}{6x-4}\right) - 4} = \frac{16x + 12 + 18x - 12}{24x + 18 - 24x + 16} = x$$

$$\Rightarrow \quad (fof(x) = x \text{ for all } x \neq \frac{2}{3} \Rightarrow fof = I \Rightarrow f \text{ is inverse of itself}$$

Hence, $f^{-1}(x) = f(x) = \dfrac{4x+3}{6x-4}$

16. We have, $f : R^+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$.

f is an injection: For any $x, y \in R^+$

$$f(x) = f(y)$$

$$\Rightarrow \quad 9x^2 + 6x - 5 = 9y^2 + 6y - 5 \Rightarrow 9(x^2 - y^2) + 6(x - y) = 0$$

$$\Rightarrow \quad 3(x - y)\{3(x + y) + 2\} = 0$$

$$\Rightarrow \quad x - y = 0 \Rightarrow x = y \qquad [3(x + y) + 2 \neq 0 \text{ as } x, y \in R^+]$$

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So, f is an injection.

f is a surjection: Let y be an arbitrary element of $[-5, \infty)$. Then,

$$f(x) = y \Rightarrow 9x^2 + 6x - 5 = y \Rightarrow (3x+1)^2 = y + 6 \Rightarrow 3x + 1 = \sqrt{y+6} \Rightarrow x = \frac{-1+\sqrt{y+6}}{3}$$

Now, $y \in [-5, \infty)$

$\Rightarrow y \geq -5 \Rightarrow y + 6 \geq 1 \Rightarrow \sqrt{y+6} \geq 1 \Rightarrow -1 + \sqrt{y+6} \geq 0 \Rightarrow \dfrac{-1+\sqrt{y+6}}{3} \geq 0$

$\Rightarrow x \geq 0 \Rightarrow x \in R^+$

Thus, for each $y \in [-5, \infty)$ there exists $x = \dfrac{-1+\sqrt{y+6}}{3} \in R^+$ such that $f(x) = y$.

So, $f : R^+ \to [-5, \infty)$ is onto.

Thus, $f : R^+ \to [-5, \infty)$ is a bijection and hence invertible. Let f^{-1} denote the inverse of f. Then,

$$(fof^{-1})(x) = x \text{ for all } x \in [-5, \infty) \Rightarrow f\left(f^{-1}(x)\right) = x \text{ for all } x \in [-5, \infty)$$

$$\Rightarrow \quad 9\left\{f^{-1}(x)\right\}^2 + 6\left\{f^{-1}(x)\right\} - 5 = x \text{ for all } x \in [-5, \infty)$$

$$\Rightarrow \quad \left\{3f^{-1}(x) + 1\right\}^2 = 6 + x \text{ for all } x \in [-5, \infty)$$

$$\Rightarrow \quad 3f^{-1}(x) + 1 = \sqrt{6+x} \text{ for all } x \in [-5, \infty) \Rightarrow f^{-1}(x) = \frac{\sqrt{x+6}-1}{3} \text{ for all } x \in [-5, \infty)$$

20. Let $f(x) = y$. Then, $f(x) = y \Rightarrow x = \sqrt{y+1} - 1 \Rightarrow f^{-1}(y) = \sqrt{y+1} - 1$.

Now, $f(x) = f^{-1}(x)$

$\Rightarrow \quad (x+1)^2 - 1 = \sqrt{x+1} - 1$

$\Rightarrow \quad \sqrt{x+1}\,\{(x+1)^{3/2} - 1\} = 0 \Rightarrow x + 1 = 0$ or, $(x+1)^{3/2} = 1 \Rightarrow x = 0, -1$

22. f is neither one-one nor onto. So, f is not a bijection. Hence, it is not invertible.

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The total number of functions from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{a, b, c\}$ is ....
2. The total number of one-one functions from the set $A = \{a, b, c\}$ to the set $B = \{x, y, z, t\}$ is ....
3. The total number of onto functions from the set $A = \{1, 2, 3, 4, 5\}$ to the set $B = \{x, y\}$ is ..... .
4. The domain of the real function $f(x) = \sqrt{16 - x^2}$ is ....
5. The domain of the real function $f(x) = \dfrac{x}{\sqrt{9 - x^2}}$ is ....
6. The range of the function $f : R \to R$ given by $f(x) = x + \sqrt{x^2}$ is ....
7. The range of the function $f : R - \{-2\} \to R$ given by $f(x) = \dfrac{x+2}{|x+2|}$ is ....
8. If $f : C \to C$ is defined by $f(x) = 8x^3$, then $f^{-1}(8) =$ ....
9. If $f : R \to R$ is defined by $f(x) = 8x^3$ then, $f^{-1}(8) =$ ....
10. If $f : R - \{0\} \to R - \{0\}$ is defined as $f(x) = \dfrac{2}{3x}$, then $f^{-1}(x) =$ ....
11. If $f : R \to R$ is defined by $f(x) = 6 - (x-9)^3$, then $f^{-1}(x) =$ ....

12. Let $A = \{1, 2, 3, 4\}$ and $f : A \to A$ be given by $f = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$. Then $f^{-1} =$ ....
13. If $f : R \to R$ be defined by $f(x) = (2 - x^5)^{1/5}$, then $fof(x) =$ ....
14. Let $A = \{1, 2, 3, 4, 5, \ldots, 10\}$ and $f : A \to A$ be an invertible function. Then, $\sum_{r=1}^{10} (f^{-1}of)(r) =$ ....
15. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{2, 4, 6, 8, 10, 12\}$. If $f : A \to B$ is given by $f(x) = 2x$, then f^{-1} as set of ordered pairs, is ....
16. Let $f = \{(0, -1), (-1, 3), (2, 3), (3, 5)\}$ be a function from Z to Z defined by $f(x) = ax + b$. Then, $(a, b) =$ ....
17. Let $f : R \to R$ and $g : R \to R$ be functions defined by $f(x) = 5 - x^2$ and $g(x) = 3x - 4$. Then the value of $fog\,(-1)$ is ....
18. Let f be the greatest integer function defined as $f(x) = [x]$ and g be the modules function defined as $g(x) = |x|$, then the value of $gof\left(-\frac{5}{4}\right)$ is ....
19. If $f(x) = \cos[e]\,x + \cos[-e]\,x$, then $f(\pi) =$ ....
20. Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$ be two sets. Then the number of constant functions from A to B is ....
21. If $f(x) = \cos[\pi^2]\,x + \cos[-\pi^2]\,x$, then $f\left(\frac{\pi}{2}\right) =$ ....
22. The number of onto functions from $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$ is.............................. .
23. If $f(0, \infty) \to R$ is given by $f(x) = \log_{10} x$, then $f^{-1}(x) =$ ....
24. If $f : R^+ \to R$ is defined as $f(x) = \log_3 x$, then $f^{-1}(x) =$ ....
25. If $f : R \to R$, $g : R \to R$ are defined by $f(x) = 5x - 3$, $g(x) = x^2 + 3$, then $(gof^{-1})(3) =$ ....
26. If $f : R \to R$ is given by $f(x) = 2x + |x|$, then $f(2x) + f(-x) + 4x =$ ....
27. If $f(x) = \frac{1 - x}{1 + x}$, then $fof(\cos 2\theta) =$ ....
28. Let $f(x) = \frac{x}{x - 1}$ and $\frac{f(\alpha)}{f(\alpha + 1)} = f(\alpha^k)$, then $k =$ ....
29. If $f(f(x)) = x + 1$ for all $x \in R$ and if $f(0) = \frac{1}{2}$, then $f(1) =$ ....
30. If $f(x) = 3x + 10$ and $g(x) = x^2 - 1$, then $(fog)^{-1}$ is equal to ....
31. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$. Then, $gof =$ .... and $fog =$ ....

ANSWERS

1. 81 **2.** 24 **3.** 30 **4.** $[-4, 4]$ **5.** $(-3, 3)$
6. $[0, \infty)$ **7.** $\{-1, 1\}$ **8.** $\{1, \omega, \omega^2\}$ **9.** $\{1\}$ **10.** $f^{-1}(x) = \frac{2}{3x}$
11. $f^{-1}(x) = 9 + (6 - x)^{1/3}$ **12.** $f^{-1} = \{(4, 1), (3, 2), (2, 3), (1, 4)\}$ **13.** x
14. 55 **15.** $f^{-1} = \{(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6)\}$ **16.** $(2, -1)$
17. -44 **18.** 2 **19.** 0 **20.** 2 **21.** -1
22. 0 **23.** 10^x **24.** 3^x **25.** 3 **26.** $2f(x)$
27. $\cos 2\theta$ **28.** 2 **29.** $\frac{3}{2}$ **30.** $\left(\frac{x - 7}{3}\right)^{1/2}$

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31. $gof = \{(1, 3), (3, 1), (4, 3)\}$ $fog = \{(2, 5), (5, 2), (1, 5)\}$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Which one of the following graphs represent a function?
2. Which one of the following graphs represent a one-one function?

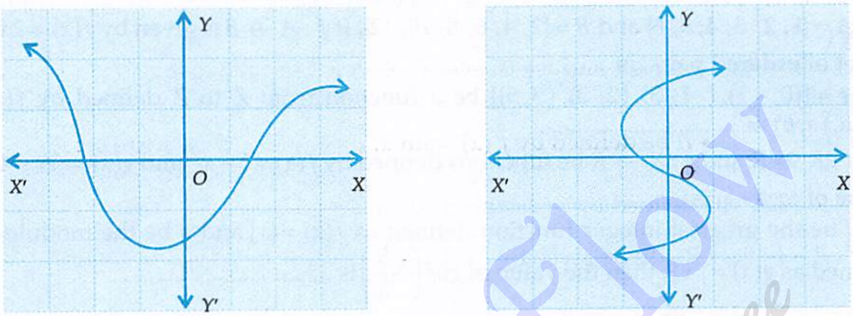

Fig. 2.43

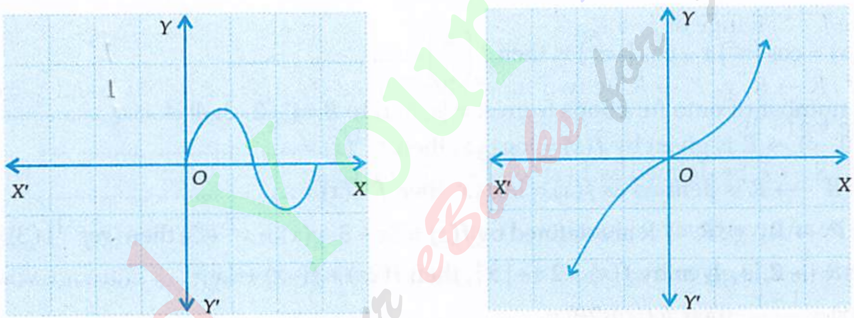

Fig. 2.44

3. If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, write total number of functions from A to B.
4. If $A = \{a, b, c\}$ and $B = \{-2, -1, 0, 1, 2\}$, write total number of one-one functions from A to B.
5. Write total number of one-one functions from set $A = \{1, 2, 3, 4\}$ to set $B = \{a, b, c\}$.
6. If $f : R \to R$ is defined by $f(x) = x^2$, write $f^{-1}(25)$.
7. If $f : C \to C$ is defined by $f(x) = x^2$, write $f^{-1}(-4)$. Here, C denotes the set of all complex numbers.
8. If $f : R \to R$ is given by $f(x) = x^3$, write $f^{-1}(1)$.
9. Let C denote the set of all complex numbers. A function $f : C \to C$ is defined by $f(x) = x^3$. Write $f^{-1}(1)$.
10. Let f be a function from C (set of all complex numbers) to itself given by $f(x) = x^3$. Write $f^{-1}(-1)$.
11. Let $f : R \to R$ be defined by $f(x) = x^4$, write $f^{-1}(1)$.
12. If $f : C \to C$ is defined by $f(x) = x^4$, write $f^{-1}(1)$.
13. If $f : R \to R$ is defined by $f(x) = x^2$, find $f^{-1}(-25)$.

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14. If $f : C \to C$ is defined by $f(x) = (x-2)^3$, write $f^{-1}(-1)$.
15. If $f : R \to R$ is defined by $f(x) = 10x - 7$, then write $f^{-1}(x)$.
16. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be a function defined by $f(x) = \cos[x]$. Write range (f).
17. If $f : R \to R$ defined by $f(x) = 3x - 4$ is invertible then write $f^{-1}(x)$. [CBSE 2010]
18. If $f : R \to R$, $g : R \to R$ are given by $f(x) = (x+1)^2$ and $g(x) = x^2 + 1$, then write the value of $fog(-3)$.
19. Let $A = \{x \in R : -4 \le x \le 4 \text{ and } x \ne 0\}$ and $f : A \to R$ be defined by $f(x) = \frac{|x|}{x}$. Write the range of f.
20. Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to A$ be defined by $f(x) = \sin x$. If f is a bijection, write set A.
21. Let $f : R \to R^+$ be defined by $f(x) = a^x$, $a > 0$ and $a \ne 1$. Write $f^{-1}(x)$.
22. Let $f : R - \{-1\} \to R - \{1\}$ be given by $f(x) = \frac{x}{x+1}$. Write $f^{-1}(x)$.
23. Let $f : R - \left\{-\frac{3}{5}\right\} \to R$ be a function defined as $f(x) = \frac{2x}{5x+3}$.

    Write f^{-1} : Range of $f \to R - \left\{-\frac{3}{5}\right\}$.
24. Let $f : R \to R, g : R \to R$ be two functions defined by $f(x) = x^2 + x + 1$ and $g(x) = 1 - x^2$. Write $fog(-2)$.
25. Let $f : R \to R$ be defined as $f(x) = \frac{2x-3}{4}$. Write $fof^{-1}(1)$.
26. Let f be an invertible real function. Write $(f^{-1} of)(1) + (f^{-1} of)(2) + \ldots + (f^{-1} of)(100)$.
27. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$ be two sets. Write total number of onto functions from A to B.
28. Write the domain of the real function $f(x) = \sqrt{x - [x]}$.
29. Write the domain of the real function $f(x) = \sqrt{[x] - x}$.
30. Write the domain of the real function $f(x) = \frac{1}{\sqrt{|x| - x}}$.
31. Write whether $f : R \to R$ given by $f(x) = x + \sqrt{x^2}$ is one-one, many-one, onto or into.
32. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$, write fog (7).
33. What is the range of the function $f(x) = \frac{|x-1|}{x-1}$? [CBSE 2010]
34. If $f : R \to R$ be defined by $f(x) = (3 - x^3)^{1/3}$, then find $fof(x)$. [CBSE 2010]
35. If $f : R \to R$ is defined by $f(x) = 3x + 2$, find $f(f(x))$. [CBSE 2010]
36. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. State whether f is one-one or not. [CBSE 2011]
37. If $f : \{5, 6\} \to \{2, 3\}$ and $g : \{2, 3\} \to \{5, 6\}$ are given by $f = \{(5, 2), (6, 3)\}$ and $g = \{(2, 5), (3, 6)\}$, find fog. [NCERT EXEMPLAR]
38. Let $f : R \to R$ be the function defined by $f(x) = 4x - 3$ for all $x \in R$. Then write f^{-1}.

    [NCERT EXEMPLAR]

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39. Which one the following relations on $A = \{1, 2, 3\}$ is a function?
$f = \{(1, 3), (2, 3), (3, 2)\}$, $g = \{(1, 2), (1, 3), (3, 1)\}$ **[NCERT EXEMPLAR]**

40. Write the domain of the real function f defined by $f(x) = \sqrt{25 - x^2}$. **[NCERT EXEMPLAR]**

41. Let $A = \{a, b, c, d\}$ and $f : A \to A$ be given by $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}. **[NCERT EXEMPLAR]**

42. Let $f, g : R \to R$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ for all $x \in R$, respectively. Then, find gof. **[NCERT EXEMPLAR]**

43. If the mapping $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$, given by $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$, write fog. **[NCERT EXEMPLAR]**

44. If a function $g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ is described by $g(x) = \alpha x + \beta$, find the values of α and β. **[NCERT EXEMPLAR]**

45. If $f(x) = 4 - (x - 7)^3$, write $f^{-1}(x)$. **[NCERT EXEMPLAR]**

ANSWERS

1. (a) **2.** (b) **3.** 8 **4.** 60 **5.** 0 **6.** $\{-5, 5\}$

7. $\{2i, -2i\}$ **8.** $\{1\}$ **9.** $\{1, w, w^2\}$ **10.** $\{-1, -w, -w^2\}$

11. $\{-1, 1\}$ **12.** $\{-1, -i, 1, i\}$ **13.** ϕ **14.** $\{1, 2 - w, 2 - w^2\}$

15. $f^{-1}(x) = \dfrac{x + 7}{10}$ **16.** $\{1, \cos 1, \cos 2\}$ **17.** $f^{-1}(x) = \dfrac{x + 4}{3}$

18. 121 **19.** $\{-1, 1\}$ **20.** $A = [-1, 1]$ **21.** $\log_a x$

22. $f^{-1}(x) = \dfrac{x}{1 - x}$ **23.** $f^{-1}(x) = \dfrac{3x}{2 - 5x}$ **24.** 7 **25.** 1

26. 5050 **27.** 14 **28.** R **29.** ϕ **30.** $(-\infty, 0)$ **31.** Many one-into

32. 7 **33.** $\{-1, 1\}$ **34.** $fof(x) = x$ **35.** $f(f(x)) = 9x + 9$

36. Yes **37.** $fog = \{(2, 2), (3, 3)\}$ **38.** $f^{-1}(x) = \dfrac{x + 3}{4}$ **39.** f

40. $[-5, 5]$ **41.** $f^{-1} = \{(b, a), (d, b), (a, c), (c, d)\}$ **42.** $gof(x) = 4x^2 + 4x - 1$

43. $fog = \{(2, 5), (5, 2), (1, 5)\}$ **44.** $\alpha = 2, \beta = -1$ **45.** $f^{-1}(x) = 7 + (4 - x)^{1/3}$

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