CHAPTER 13

COMPLEX NUMBERS

13.1 NEED FOR COMPLEX NUMBERS

If a, b are natural numbers such that $a > b$, then the equation $x + a = b$ is not solvable in N, the set of natural numbers i.e. there is no natural number satisfying the equation $x + a = b$. So, the set of natural numbers is extended to form the set I of integers in which every equation of the form $x + a = b$; $a, b \in N$ is solvable. But, equations of the form $xa = b$, where $a, b \in I$, $a \neq 0$ are not solvable in I also. Therefore, the set I of integers is extended to obtain the set Q of all rational numbers in which every equation of the form $xa = b$, $a \neq 0$, $a, b \in I$ is uniquely solvable. The equations of the form $x^2 = 2$, $x^2 = 3$ etc. are not solvable in Q because there is no rational number whose square is 2. Such numbers are known as irrational numbers. The set Q of all rational numbers is extended to obtain the set R which includes both rational and irrational numbers. This set is known as the set of real numbers. The equations of the form $x^2 + 1 = 0$, $x^2 + 4 = 0$ etc. are not solvable in R i.e. there is no real number whose square is a negative real number. *Euler was the first mathematician to introduce the symbol i (iota) for the square root of -1 i.e. a solution of $x^2 + 1 = 0$ with the property $i^2 = -1$. He also called this symbol as the imaginary unit.*

13.2 INTEGRAL POWERS OF IOTA (i)

Positive integral powers of i: We have, $i = \sqrt{-1}$

$$\therefore \quad i^2 = -1,\ i^3 = i^2 \times i = -i,\ i^4 = (i^2)^2 = (-1)^2 = 1$$

In order to compute i^n for $n > 4$, we divide n by 4 and obtain the remainder r. Let m be the quotient when n is divided by 4. Then,

$$n = 4m + r, \text{ where } 0 \le r < 4$$

$$\Rightarrow \quad i^n = i^{4m+r} = (i^4)^m\, i^r = i^r$$

Thus, the value of i^n for $n > 4$ is i^r, where r is the remainder when n is divided by 4.

Negative integral powers of i: By the law of indices, we have

$$i^{-1} = \frac{1}{i} = \frac{i^3}{i^4} = i^3 = -i,\ i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{i}{i^4} = i, \qquad i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$$

If $n > 4$, then $i^{-n} = \dfrac{1}{i^n} = \dfrac{1}{i^r}$, where r is the remainder when n is divided by 4

NOTE *i^0 is defined as 1.*

The above discussion suggests the following algorithm to find integral exponents of i.

ALGORITHM

To find the value of i^n for $n \in Z$, we may follow the following steps.

<u>STEP I</u> *If $n = 0$, then write $i^n = 1$.*

<u>STEP II</u> *If $n > 0$, then*

$$i^n = \begin{cases} i, & \text{if } n = 1 \\ -1, & \text{if } n = 2 \\ -i, & \text{if } n = 3 \\ 1, & \text{if } n = 4 \\ i^r, & \text{if } n > 4, \textit{ where r is the remainder when n is divided by 4} \end{cases}$$

<u>STEP III</u> *If $n < 0$, then $n = -m$, where $m > 0$.*

$$\therefore \quad i^n = \begin{cases} i^{-1} = \dfrac{1}{i} = -i, & \text{if } n = -1 \\ i^{-2} = \dfrac{1}{i^2} = \dfrac{1}{-1} = -1, & \text{if } n = -2 \\ i^{-3} = \dfrac{1}{i^3} = \dfrac{i}{i^4} = i, & \text{if } n = -3 \\ i^{-4} = \dfrac{1}{i^4} = \dfrac{1}{1} = 1, & \text{if } n = -4 \\ i^{-m} = \dfrac{1}{i^m} = \dfrac{1}{i^r}, & \textit{where r is the remainder when m is divided by 4, if } n < -4. \end{cases}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Evaluate the following:*

(i) i^{135} (ii) i^{19} (iii) i^{-999} (iv) $(-\sqrt{-1})^{4n+3}, n \in N$

SOLUTION (i) 135 leaves remainder as 3 when it is divided by 4.

$\therefore \quad i^{135} = i^3 = -i$

(ii) The remainder is 3 when 19 is divided by 4.

$\therefore \quad i^{19} = i^3 = -i$

(iii) We have, $i^{-999} = 1/i^{999}$

On dividing 999 by 4, we obtain 3 as the remainder. Therefore, $i^{999} = i^3$.

Hence, $i^{-999} = \dfrac{1}{i^{999}} = \dfrac{1}{i^3} = \dfrac{i}{i^4} = \dfrac{i}{1} = i$

(iv) We have, $(-\sqrt{-1})^{4n+3} = (-i)^{4n+3} = (-i)^{4n}(-i)^3 = \{(-i)^4\}^n(-i^3) = 1 \times -i^3 = i$

EXAMPLE 2 *Show that:*

(i) $\left\{i^{19} + \left(\dfrac{1}{i}\right)^{25}\right\}^2 = -4$ (ii) $\left\{i^{17} - \left(\dfrac{1}{i}\right)^{34}\right\}^2 = 2i$

(iii) $\left\{i^{18} + \left(\dfrac{1}{i}\right)^{24}\right\}^3 = 0$ (iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$, *for all $n \in N$.*

SOLUTION (i) $\left\{i^{19} + \left(\dfrac{1}{i}\right)^{25}\right\}^2 = \left\{i^{19} + \dfrac{1}{i^{25}}\right\}^2 = \left\{i^3 + \dfrac{1}{i}\right\}^2 = \left\{-i + \dfrac{i^3}{i^4}\right\}^2$

$= [-i + i^3]^2 = (-i - i)^2 = 4i^2 = -4.$

(ii) $\left\{i^{17}-\left(\frac{1}{i}\right)^{34}\right\}^2 = \left\{i^{17}-\frac{1}{i^{34}}\right\}^2 = \left\{i-\frac{1}{i^2}\right\}^2 = \left\{i-\frac{1}{(-1)}\right\}^2 = (i+1)^2$

$= i^2 + 2i + 1 = -1 + 2i + 1 = 2i$

(iii) $\left\{i^{18}+\left(\frac{1}{i}\right)^{24}\right\}^3 = \left\{i^{18}+\frac{1}{i^{24}}\right\}^3 = \left(i^2+\frac{1}{1}\right)^3 = (-1+1)^3 = 0$

(iv) $i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n + i^n \times i + i^n \times i^2 + i^n \times i^3$

$= i^n(1 + i + i^2 + i^3)$

$= i^n(1 + i - 1 - i) = i^n(0) = 0$

LEVEL-2

EXAMPLE 3 *Evaluate* $\sum_{n=1}^{13}(i^n + i^{n+1})$*, where* $n \in N$.

SOLUTION $\sum_{n=1}^{13}(i^n + i^{n+1})$

$= \sum_{n=1}^{13}(i+1)\,i^n$

$= (i+1)\sum_{n=1}^{13} i^n$

$= (i+1)(i + i^2 + i^3 + \ldots + i^{13})$

$= (i+1) \times i\left(\frac{i^{13}-1}{i-1}\right)$

$= (i^2+i)\left(\frac{i-1}{i-1}\right)$ $[\because\ i^{13} = i]$

$= (-1+i)$

EXAMPLE 4 *Evaluate* $1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$.

SOLUTION Let $S = 1 + i^2 + i^4 + i^6 + \ldots + i^{2n}$. Then

$S = 1 + i^2 + (i^2)^2 + (i^2)^3 + \ldots + (i^2)^n$

$\Rightarrow\quad S = 1\left\{\frac{1-(i^2)^{n+1}}{1-i^2}\right\} = \frac{1-(i^2)^{n+1}}{1+1} = \frac{1}{2}\left\{1-(-1)^{n+1}\right\}$

$$\Rightarrow\quad S = \begin{cases} \frac{1}{2}(1-1) = 0, & \text{if } n \text{ is odd} \\ \frac{1}{2}(1+1) = 1, & \text{if } n \text{ is even} \end{cases}$$

EXERCISE 13.1

LEVEL-1

1. Evaluate the following:
   (i) i^{457} (ii) i^{528} (iii) $\frac{1}{i^{58}}$ (iv) $i^{37} + \frac{1}{i^{67}}$
   (v) $\left(i^{41} + \frac{1}{i^{257}}\right)^9$ (vi) $(i^{77} + i^{70} + i^{87} + i^{414})^3$ (vii) $i^{30} + i^{40} + i^{60}$
   (viii) $i^{49} + i^{68} + i^{89} + i^{110}$

2. Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.
3. Find the values of the following expressions:
   (i) $i^{49} + i^{68} + i^{89} + i^{110}$ (ii) $i^{30} + i^{80} + i^{120}$
   (iii) $i + i^2 + i^3 + i^4$ (iv) $i^5 + i^{10} + i^{15}$
   (v) $\dfrac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$ (vi) $1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}$
   (vii) $(1 + i)^6 + (1 - i)^3$

ANSWERS

1. (i) i (ii) 1 (iii) -1 (iv) $2i$ (v) 0 (vi) -8 (vii) 1 (viii) $2i$
3. (i) $2i$ (ii) 1 (iii) 0 (iv) -1 (v) -1 (vi) 1 (vii) $-2-10i$

13.3 IMAGINARY QUANTITIES

The square root of a negative real number is called an imaginary quantity or an imaginary number.

For example, $\sqrt{-3}, \sqrt{-4}, \sqrt{-9/4}$ etc. are imaginary quantities.

THEOREM *If a, b are positive real numbers, then* $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

PROOF We have,

$$\sqrt{-a} = \sqrt{-1 \times a} = \sqrt{-1} \times \sqrt{a} = i\sqrt{a} \text{ and, } \sqrt{-b} = \sqrt{-1 \times b} = \sqrt{-1} \times \sqrt{b} = i\sqrt{b}$$

$$\therefore \quad \sqrt{-a} \times \sqrt{-b} = (i\sqrt{a})(i\sqrt{b}) = i^2(\sqrt{a} \times \sqrt{b}) = -1(\sqrt{ab}) = -\sqrt{ab}$$

NOTE 1 *For any two real numbers* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is true only when at least one of a and b is either positive or zero. In other words,* $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ *is not valid if a and b both are negative.*

NOTE 2 *For any positive real number a, we have* $\sqrt{-a} = \sqrt{-1 \times a}\,\sqrt{-1} \times \sqrt{a} = i\sqrt{a}$.

ILLUSTRATION 1 *Compute the following:*

(i) $\sqrt{-144}$ (ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}}$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9}$

SOLUTION

(i) $\sqrt{-144} = \sqrt{-1 \times 144} = \sqrt{-1} \times \sqrt{144} = 12i$

(ii) $\sqrt{-4} \times \sqrt{\dfrac{-9}{4}} = (2i)\left(\dfrac{3i}{2}\right) = 3i^2 = -3$

(iii) $\sqrt{-25} + 3\sqrt{-4} + 2\sqrt{-9} = 5i + 6i + 6i = 17i$

ILLUSTRATION 2 *A student writes the formula* $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$. *Then he substitutes* $a = -1$ *and* $b = -1$ *and finds* $1 = -1$. *Explain where is he wrong?*

SOLUTION Since a and b both are negative. Therefore, $\sqrt{ab}$ cannot be written as $\sqrt{a}\sqrt{b}$. In fact, for a and b both negative, we have $\sqrt{a}\,\sqrt{b} = -\sqrt{ab}$.

ILLUSTRATION 3 *Is the following computation correct? If not give the correct computation:*

"$[\sqrt{(-2)} \,.\, \sqrt{(-3)}] = \sqrt{(-2)\,.\,(-3)} = \sqrt{6}$"

SOLUTION The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$$

13.4 COMPLEX NUMBERS

COMPLEX NUMBER *If a, b are two real numbers, then a number of the form $a + ib$ is called a complex number.*

For example, $7 + 2i, -1 + i, 3 - 2i, 0 + 2i, 1 + 0i$ etc. are complex numbers.

Real and imaginary parts of a complex number: If $z = a + ib$ is a complex number, then 'a' is called the real part of z and 'b' is known as the imaginary part of z. The real part of z is denoted by Re (z) and the imaginary part by Im (z).

If $z = 3 - 4i$, then $\text{Re}(z) = 3$ and $\text{Im}(z) = -4$.

Purely real and purely imaginary complex numbers: A complex number z is purely real if its imaginary part is zero i.e. $\text{Im}(z) = 0$ *and purely imaginary if its real part is zero i.e.* $\text{Re}(z) = 0$.

Set of complex numbers: The set of all complex numbers is denoted by C i.e. $C = \{a + ib : a, b \in R\}$.

Since a real number 'a' can be written as $a + 0i$. Therefore, every real number is a complex number. Hence, $R \subset C$, where R is the set of all real numbers.

13.5 EQUALITY OF COMPLEX NUMBERS

DEFINITION *Two complex numbers $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$.*

i.e. $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$.

Thus, $z_1 = z_2 \Leftrightarrow \text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$.

ILLUSTRATION 1 *If $z_1 = 2 - iy$ and $z_2 = x + 3i$ are equal, find x and y.*

SOLUTION We have,

$$z_1 = z_2 \Rightarrow 2 - iy = x + 3i \Rightarrow 2 = x \text{ and } -y = 3 \Rightarrow x = 2 \text{ and } y = -3.$$

ILLUSTRATION 2 *If $(a + b) - i(3a + 2b) = 5 + 2i$, find a and b.*

SOLUTION We have,

$$(a + b) - i(3a + 2b) = 5 + 2i \Rightarrow a + b = 5 \text{ and } -(3a + 2b) = 2 \Rightarrow a = -12, b = 17$$

13.6 ADDITION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$.*

It follows from this definition that the sum $z_1 + z_2$ is a complex number such that

$$\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2) \text{ and, } \text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$$

For example, If $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$, then $z_1 + z_2 = (2 + 3) + (3 - 2)i = 5 + i$

13.6.1 PROPERTIES OF ADDITION OF COMPLEX NUMBERS

(i) Addition is Commutative: For any two complex numbers z_1 and z_2

$$z_1 + z_2 = z_2 + z_1$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\begin{aligned} z_1 + z_2 &= (a_1 + a_2) + i(b_1 + b_2) && \text{[By definition of addition]} \\ &= (a_2 + a_1) + i(b_2 + b_1) && \text{[By commutativity of addition of real numbers]} \\ &= z_2 + z_1 && \text{[By definition of addition]} \end{aligned}$$

Thus, $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in C$.

Hence, addition of complex number is commutative.

(ii) Addition is Associative: For any three complex numbers z_1, z_2, z_3

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$, where a_1, a_2, a_3 and b_1, b_2, b_3 are real numbers. Then,

$$\begin{aligned}(z_1+z_2)+z_3 &= [(a_1+a_2)+i(b_1+b_2)]+(a_3+ib_3) && \text{[By definition of addition]}\\ &= [(a_1+a_2)+a_3]+i[(b_1+b_2)+b_3] && \text{[By definition of addition]}\\ &= [(a_1+(a_2+a_3)]+i[b_1+(b_2+b_3)] && \text{[By associativity of addition on } R]\\ &= (a_1+ib_1)+[(a_2+a_3)+i(b_2+b_3)] && \text{[By definition of addition]}\\ &= z_1+(z_2+z_3) && \text{[By definition of addition]}\end{aligned}$$

Thus, $(z_1+z_2)+z_3 = z_1+(z_2+z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, addition of complex numbers is associative.

(iii) Existence of Additive Identity: The complex number $0 = 0 + i\,0$ is the identity element for addition i.e. $z + 0 = z = 0 + z$ for all $z \in C$.

PROOF Let $z = a + ib$ be an arbitrary complex number. Then,

$$z+0 = (a+ib)+(0+i0) = (a+0)+i(b+0) = a+ib = z$$

and, $$0+z = (0+i0)+(a+ib) = (0+a)+i(0+b) = a+ib = z$$

Thus, $z+0 = z = 0+z$ for all $z \in C$

Hence, the complex number $0 = 0 + i\,0$ is the identity element for addition.

(iv) Existence of Additive Inverse: For any complex number $z = a + ib$, there exists $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

PROOF Let $z = a + ib$ be an arbitrary complex number. Then, $-z = (-a) + i(-b)$ is also a complex number such that

$$z+(-z) = (a+ib)+\{(-a)+i(-b)\} = \{a+(-a)\}+i\{b+(-b)\} = 0+i0 = 0$$

and $$(-z)+z = \{(-a)+i(-b)\}+(a+ib) = \{(-a)+a\}+i\{(-b)+b\} = 0+i0 = 0.$$

Thus, for each complex number $z = a + ib$, there exists a complex number $-z = (-a) + i(-b)$ such that $z + (-z) = 0 = (-z) + z$.

The complex number $-z$ is called the *additive inverse of z*.

13.7 SUBTRACTION OF COMPLEX NUMBERS

DEFINITION *Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.*

Thus, $z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

For example, If $z_1 = -2 + 3i$ and $z_2 = 4 + 5i$, then

$$z_1 - z_2 = (-2+3i)+(-4-5i) = (-2-4)+i(3-5) = -6-2i$$

13.8 MULTIPLICATION OF COMPLEX NUMBERS

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the multiplication of z_1 with z_2 is denoted by $z_1 z_2$ and is defined as the complex number $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$.

Thus, $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$\Rightarrow$ $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow$ $z_1 z_2 = [\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) - \mathrm{Im}(z_1)\,\mathrm{Im}(z_2)] + i[\mathrm{Re}(z_1)\,\mathrm{Im}(z_2) + \mathrm{Re}(z_2)\,\mathrm{Im}(z_1)]$

For example, If $z_1 = 3 + 2i$ and $z_2 = 2 - 3i$, then

$$z_1 z_2 = (3+2i)(2-3i) = (3\times 2 - 2\times(-3)) + i(3\times -3 + 2\times 2) = 12 - 5i$$

NOTE *The product $z_1 z_2$ can also be obtained if we actually carry out the multiplication $(a_1 + ib_1)(a_2 + ib_2)$ as given below:*

$$\begin{aligned}(a_1+ib_1)(a_2+ib_2) &= a_1 a_2 + i a_1 b_2 + i b_1 a_2 + i^2 b_1 b_2\\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) && [\because i^2 = -1]\end{aligned}$$

13.8.1 PROPERTIES OF MULTIPLICATION

(i) Multiplication is commutative: For any two complex numbers z_1 and z_2

$$z_1 z_2 = z_2 z_1$$

PROOF Let $z_1 = a_1 + i b_1$ and $z_2 = a_2 + i b_2$, where a_1, a_2, and b_1, b_2 are real numbers. Then,

$$z_1 z_2 = (a_1 + i b_1)(a_2 + i b_2) = (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)$$

and, $z_2 z_1 = (a_2 + i b_2)(a_1 + i b_1) = (a_2 a_1 - b_2 b_1) + i (b_2 a_1 + b_1 a_2)$

$= (a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)$ [By commutativity of mult. of real numbers]

$\therefore$ $z_1 z_2 = z_2 z_1$

Thus, $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in C$.

Hence, the multiplication of complex numbers is commutative on C.

(ii) Multiplication is associative: For any three complex numbers z_1, z_2, z_3

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

PROOF Let $z_1 = a_1 + i b_1, z_2 = a_2 + i b_2$ and $z_3 = a_3 + i b_3$ be any three complex numbers. Then,

$$\begin{aligned}
(z_1 z_2) z_3 &= \{(a_1 + i b_1)(a_2 + i b_2)\}(a_3 + i b_3) \\
&= \{(a_1 a_2 - b_1 b_2) + i (a_1 b_2 + a_2 b_1)\}(a_3 + i b_3) \\
&= \{(a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + a_2 b_1) b_3\} + i \{(a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + a_2 b_1) a_3\} \\
&= \{a_1 (a_2 a_3 - b_2 b_3) - b_1 (a_2 b_3 + a_3 b_2)\} + i \{b_1 (a_2 a_3 - b_2 b_3) + a_1 (a_3 b_2 + a_2 b_3)\} \\
&= (a_1 + i b_1)\{(a_2 a_3 - b_2 b_3) + i (a_2 b_3 + a_3 b_2)\} \\
&= z_1 (z_2 z_3)
\end{aligned}$$

Thus, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ for all $z_1, z_2, z_3 \in C$.

Hence, multiplication is associative on C.

(iii) Existence of identity element for multiplication: The complex number $1 = 1 + i0$ is the identity element for multiplication i.e. for every complex number z, $z \cdot 1 = z = 1 \cdot z$.

PROOF Let $z = a + i b$. Then,

$$z \cdot 1 = (a + ib)(1 + i0) = (a \times 1 - b \times 0) + i (a \times 0 + 1 \times b) = a + ib.$$

Similarly, we obtain $1 \cdot z = z$

Thus, $z \cdot 1 = z = 1 \cdot z$, for all $z \in C$.

Hence, $1 = 1 + 0i$ is the multiplicative identity in C.

(iv) Existence of multiplicative inverse: Corresponding to every non-zero complex number $z = a + ib$ there exists a complex number $z_1 = x + iy$ such that $z \cdot z_1 = 1 = z_1 \cdot z$.

PROOF Clearly,

$z \cdot z_1 = 1$

$\Rightarrow$ $(a + ib)(x + iy) = 1 + i0$

$\Rightarrow$ $(ax - by) + i (ay + bx) = 1 + i0$

$\Rightarrow$ $ax - by = 1$ and $ay + bx = 0$.

Solving these two equations, we get

$$x = \frac{a}{a^2 + b^2}, \quad y = -\frac{b}{a^2 + b^2} \qquad [\because a \neq 0, b \neq 0]$$

Thus, every non-zero complex number $z = a + i b$ possesses multiplicative inverse given by

$$\left\{\frac{a}{a^2 + b^2}\right\} + i \left\{\frac{-b}{a^2 + b^2}\right\}$$

NOTE *The multiplicative inverse of z is denoted by z^{-1}* or, $\frac{1}{z}$

ILLUSTRATION *Find the multiplicative inverse of $z = 3 - 2i$.*

SOLUTION Using the above formula, we have

$$z^{-1} = \frac{3}{3^2 + (-2)^2} + \frac{i(-(-2))}{3^2 + (-2)^2} = \frac{3}{13} + \frac{2}{13} i$$

(v) Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3

(i) $z_1(z_2+z_3)=z_1z_2+z_1z_3$ *(Left distributivity)*

(ii) $(z_2+z_3)z_1=z_2z_1+z_3z_1$ *(Right distributivity)*

PROOF Let $z_1=a_1+ib_1, z_2=a_2+ib_2$ and $z_3=a_3+ib_3$. Then,

$$\begin{aligned} z_1(z_2+z_3) &= (a_1+ib_1)\{(a_2+a_3)+i(b_2+b_3)\} \\ &= \{a_1(a_2+a_3)-b_1(b_2+b_3)\}+i\{a_1(b_2+b_3))+b_1(a_2+a_3)\} \\ &= [(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)]+[(a_1a_3-b_1b_3)+i(a_1b_3+a_3b_1)] \\ &= z_1z_2 + z_1z_3 \end{aligned}$$

Similarly, it can be established that $(z_2+z_3)z_1=z_2z_1+z_3z_1$.

13.9 DIVISION OF COMPLEX NUMBERS

The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Thus, $\frac{z_1}{z_2} = z_1z_2^{-1} = z_1\left\{\frac{1}{z_2}\right\}$

Let $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$. Then,

$$\frac{z_1}{z_2} = (a_1+ib_1)\left\{\frac{a_2}{a_2^2+b_2^2}+i\frac{(-b_2)}{a_2^2+b_2^2}\right\} \qquad \left[\because z=a+ib \Rightarrow \frac{1}{z}=\frac{a}{a^2+b^2}+\frac{i(-b)}{a^2+b^2}\right]$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+i\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right) \qquad \text{[By definition of multiplication]}$$

For example, If $z_1=2+3i$ and $z_2=1+2i$, then

$$\frac{z_1}{z_2}=z_1\times\frac{1}{z_2}=(2+3i)\times\frac{1}{1+2i}=(2+3i)\frac{1}{5}-\frac{2}{5}i=\left(\frac{2}{5}+\frac{6}{5}\right)+i\left(-\frac{4}{5}+\frac{3}{5}\right)=\frac{8}{5}-\frac{1}{5}i$$

13.10 CONJUGATE OF A COMPLEX NUMBER

DEFINITION *Let $z=a+ib$ be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to $a-ib$.*

Thus, $z=a+ib \Rightarrow \bar{z}=a-ib$

It follows from this definition that the conjugate of a complex number is obtained by replacing i by $-i$.

For example, if $z=3+4i$, then $\bar{z}=3-4i$.

13.10.1 PROPERTIES OF CONJUGATE

THEOREM *If z, z_1, z_2 are complex numbers, then*

(i) $(\bar{\bar{z}})=z$

(ii) $z+\bar{z}=2\,\mathrm{Re}(z)$

(iii) $z-\bar{z}=2i\,\mathrm{Im}(z)$

(iv) $z=\bar{z} \Leftrightarrow z$ *is purely real*

(v) $z+\bar{z}=0 \Rightarrow z$ *is purely imaginary*

(vi) $z\bar{z}=\{\mathrm{Re}(z)\}^2+\{\mathrm{Im}(z)\}^2$

(vii) $\overline{z_1+z_2}=\bar{z}_1+\bar{z}_2$

(viii) $\overline{z_1-z_2}=\bar{z}_1-\bar{z}_2$

(ix) $\overline{z_1z_2}=\bar{z}_1\bar{z}_2$

(x) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}$, $z_2\neq 0$.

PROOF Let $z = a + ib, z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$.

(i) $z = a + ib \Rightarrow \bar{z} = a - ib \Rightarrow (\bar{\bar{z}}) = (\overline{a - ib}) = a + ib \Rightarrow (\bar{\bar{z}}) = z.$

(ii) $z + \bar{z} = (a + ib) + (a - ib) = 2a = 2\,\text{Re}(z)$

(iii) $z - \bar{z} = (a + ib) - (a - ib) = 2ib = 2i\,\text{Im}(z)$

(iv) $z = \bar{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0 \Leftrightarrow \text{Im}(z) = 0 \Rightarrow z$ is purely real

(v) $z + \bar{z} = 0 \Leftrightarrow (a + ib) + (a - ib) = 0 \Leftrightarrow 2a = 0 \Leftrightarrow a = 0 \Leftrightarrow \text{Re}(z) = 0 \Leftrightarrow z$ is purely imaginary

(vi) $z\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = \{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2$

(vii) We have,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$\therefore \quad \overline{z_1 + z_2} = (a_1 + a_2) - i(b_1 + b_2) = (a_1 - ib_1) + (a_2 - ib_2) = \overline{(a_1 + ib_1)} + \overline{(a_2 + ib_2)} = \bar{z}_1 + \bar{z}_2.$$

(viii) We have, $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$

$$\therefore \quad \overline{z_1 - z_2} = (a_1 - a_2) - i(b_1 - b_2) = (a_1 - ib_1) - (a_2 - ib_2) = \overline{(a_1 + ib_1)} - \overline{(a_2 + ib_2)} = \bar{z}_1 - \bar{z}_2$$

(ix) We have, $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$$\therefore \quad \overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1) = (a_1 - ib_1)(a_2 - ib_2) = \bar{z}_1 \bar{z}_2$$

(x) We have, $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \left(\dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$

$$\therefore \quad \overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) - i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right) \qquad \text{...(i)}$$

Now, $\dfrac{\bar{z}_1}{\bar{z}_2} = \bar{z}_1 \times \dfrac{1}{\bar{z}_2} = (a_1 - ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} + i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \left(\dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) - i\left(\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$ ... (ii)

From (i) and (ii), we get $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$.

13.11 MODULUS OF A COMPLEX NUMBER

DEFINITION *The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as*

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

Clearly, $|z| \geq 0$ for all $z \in C$.

For example, If $z_1 = 3 - 4i$, $z_2 = -5 + 2i$ and $z_3 = 1 + \sqrt{-3}$, then

$$|z_1| = \sqrt{3^2 + (-4)^2} = 5,\ |z_2| = \sqrt{(-5)^2 + 2^2} = \sqrt{29} \text{ and, } |z_3| = |1 + i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2} = 2.$$

REMARK *In the set C of all complex numbers, the order relation is not defined. As such $z_1 > z_2$ or, $z_1 < z_2$ has no meaning but $|z_1| > |z_2|$ or, $|z_1| < |z_2|$ has got its meaning as $|z_1|$ and $|z_2|$ are real numbers.*

13.11.1 PROPERTIES OF MODULUS

THEOREM *If $z, z_1, z_2 \in C$, then*

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$

(ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$; $-|z| \leq \text{Im}(z) \leq |z|$

(iv) $z\bar{z} = |z|^2$

(v) $|z_1 z_2| = |z_1||z_2|$

(vi) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$; $z_2 \neq 0$

(vii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1 \overline{z_2})$

(viii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\text{Re}(z_1 \overline{z_2})$

(ix) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

(x) $|a z_1 - b z_2|^2 + |b z_1 + a z_2|^2 = (a^2 + b^2)\left(|z_1|^2 + |z_2|^2\right)$, *where* $a, b \in R$.

[NCERT EXEMPLAR]

PROOF Let $z = a + ib$. Then,

(i) $|z| = 0 \Leftrightarrow \sqrt{a^2 + b^2} = 0 \Leftrightarrow a^2 + b^2 = 0 \Leftrightarrow a = 0$ and $b = 0 \Leftrightarrow \text{Re}(z) = \text{Im}(z) = 0$

(ii) Let $z = a + ib$. Then, $\bar{z} = a - ib$ and $-z = -a - ib$.

$\therefore$ $|z| = \sqrt{a^2 + b^2}, |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2}$ and, $|-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2}$

Clearly, $|z| = |\bar{z}| = |-z|$

(iii) Let $z = a + ib$. Then, $|z| = \sqrt{a^2 + b^2}$.

Clearly, $-\sqrt{a^2 + b^2} \leq a \leq \sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2} \leq b \leq \sqrt{a^2 + b^2}$

$\Rightarrow$ $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq \text{Im}(z) \leq |z|$

(iv) Let $z = a + ib$. Then, $\bar{z} = a - ib$.

$\therefore$ $z\bar{z} = (a + ib)(a - ib) = a^2 - i^2 b^2 = a^2 + b^2 = \left\{\sqrt{a^2 + b^2}\right\}^2 = |z|^2$

(v) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow$ $|z_1 z_2| = \sqrt{(a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2}$

$\Rightarrow$ $|z_1 z_2| = \sqrt{a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2}$

$\Rightarrow$ $|z_1 z_2| = \sqrt{a_1^2 (a_2^2 + b_2^2) + b_1^2 (a_2^2 + b_2^2)} = \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} = \sqrt{a_1^2 + b_1^2} \cdot \sqrt{a_2^2 + b_2^2}$

$\Rightarrow$ $|z_1 z_2| = |z_1||z_2|$

(vi) Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, where a_1, a_2 and b_1, b_2 are real numbers. Then,

$$\frac{z_1}{z_2} = z_1 \times \frac{1}{z_2} = (a_1 + ib_1)\left(\frac{a_2}{a_2^2 + b_2^2} + i\,\frac{(-b_2)}{a_2^2 + b_2^2}\right) = \left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right) + i\left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2}\right)^2 + \left(\frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}\right)^2}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1 a_2 + b_1 b_2)^2 + (a_2 b_1 - a_1 b_2)^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{a_1^2 a_2^2 + b_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2}{(a_2^2 + b_2^2)^2}}$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(a_1^2+b_1^2)(a_2^2+b_2^2)}{(a_2^2+b_2^2)^2}} = \sqrt{\frac{a_1^2+b_1^2}{a_2^2+b_2^2}} = \frac{\sqrt{a_1^2+b_1^2}}{\sqrt{a_2^2+b_2^2}} = \frac{|z_1|}{|z_2|}$$

(vii) Clearly,

$$|z_1+z_2|^2 = (z_1+z_2)\,\overline{(z_1+z_2)} \qquad [\because z\bar{z} = |z|^2]$$

$$\Rightarrow \quad |z_1+z_2|^2 = (z_1+z_2)(\overline{z_1}+\overline{z_2}) \qquad [\because \overline{z_1+z_2} = \overline{z_1}+\overline{z_2}]$$

$$\Rightarrow \quad |z_1+z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + z_2\overline{z_1} \qquad \text{[By distributivity of multiplication]}$$

$$\Rightarrow \quad |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + (\overline{z_1\overline{z_2}}) \qquad [\because (\overline{z_1\overline{z_2}}) = \overline{z_1}(\overline{\overline{z_2}}) = \overline{z_1}z_2 = z_2\overline{z_1}]$$

$$\Rightarrow \quad |z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) \qquad [\because z+\bar{z} = 2\,\mathrm{Re}(z)]$$

(viii) Clearly,

$$|z_1-z_2|^2 = (z_1-z_2)\,\overline{(z_1-z_2)} \qquad [\because z\bar{z} = |z|^2]$$

$$\Rightarrow \quad |z_1-z_2|^2 = (z_1-z_2)(\overline{z_1}-\overline{z_2}) \qquad [\because \overline{z_1+z_2} = \overline{z_1}-\overline{z_2}]$$

$$\Rightarrow \quad |z_1-z_2|^2 = z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1} \qquad \text{[By distributivity of multiplication]}$$

$$\Rightarrow \quad |z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - z_1\overline{z_2} - (\overline{z_1\overline{z_2}}) \qquad [\because (\overline{z_1\overline{z_2}}) = \overline{z_1}(\overline{\overline{z_2}}) = \overline{z_1}z_2]$$

$$\Rightarrow \quad |z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z_2})$$

(ix) Using (vii) and (viii), we get

$$|z_1+z_2|^2 + |z_1-z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z_2})$$

$$= 2(|z_1|^2 + |z_2|^2)$$

(x) We have,

$$|az_1 - bz_2|^2 = (az_1 - bz_2)\,\overline{(az_1 - bz_2)}$$

$$= (az_1 - bz_2)(a\overline{z_1} - b\overline{z_2})$$

$$= a^2 z_1\overline{z_1} - (az_1)(b\overline{z_2}) - (bz_2)(a\overline{z_1}) + b^2 z_2\overline{z_2}$$

$$= a^2|z_1|^2 - ab(z_1\overline{z_2} + \overline{z_1}z_2) + b^2|z_2|^2$$

$$= a^2|z_1|^2 - ab(z_1\overline{z_2} + (\overline{z_1\overline{z_2}})) + b^2|z_2|^2$$

$$= a^2|z_1|^2 - ab\{2\,\mathrm{Re}(z_1\overline{z_2})\} + b^2|z_2|^2 \qquad [\because z_1\overline{z_2} + (\overline{z_1\overline{z_2}}) = 2\,\mathrm{Re}(z_1\overline{z_2})]$$

$$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\overline{z_2}) + b^2|z_2|^2$$

Similarly, we obtain

$$|bz_1 + az_2|^2 = b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\mathrm{Re}(z_1\overline{z_2})$$

$$\therefore \quad |az_1 - bz_2|^2 + |bz_1 + az_2|^2$$

$$= a^2|z_1|^2 - 2ab\,\mathrm{Re}(z_1\overline{z_2}) + b^2|z_2|^2 + b^2|z_1|^2 + a^2|z_2|^2 + 2ab\,\mathrm{Re}(z_1\overline{z_2})$$

$$= |z_1|^2(a^2+b^2) + |z_2|^2(b^2+a^2)$$

$$= (a^2+b^2)\left(|z_1|^2 + |z_2|^2\right)$$

13.12 RECIPROCAL OF A COMPLEX NUMBER

Let $z = a + ib$ be a non-zero complex number. Then,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib}$$ [Multiplying numerator and denominator by conjugate of denominator]

$$\Rightarrow \quad \frac{1}{z} = \frac{a-ib}{a^2 - i^2 b^2} = \frac{a-ib}{a^2+b^2}$$

$$\Rightarrow \quad \frac{1}{z} = \frac{a}{a^2+b^2} + \frac{i(-b)}{a^2+b^2}$$

Clearly, $\frac{1}{z}$ is equal to the multiplicative inverse of z.

Also, $\frac{1}{z} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

Thus, the multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\text{Re}(z)}{|z|^2} + i\frac{(-\text{Im}(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **EXPRESSING A COMPLEX NUMBER IN THE STANDARD FORM a + ib**

In order to express a complex number in the standard form, we may follow the following algorithm.

ALGORITHM

STEP I *Write the complex number in the form $\frac{a+ib}{c+id}$ by using fundamental operations of addition, subtraction and multiplication.*

STEP II *Multiply the numerator and denominator by the conjugate of the denominator.*

EXAMPLE 1 *Express the following in the form a + ib :*

(i) $(-5i)\left(\frac{1}{8}i\right)$ [NCERT] (ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3$ [NCERT]

(iii) $(5i)\left(-\frac{3}{5}i\right)$ [NCERT] (iv) $i^9 + i^{19}$

(v) i^{-39} [NCERT] (vi) $(1-i)^4$ [NCERT]

SOLUTION (i) $(-5i)\left(\frac{1}{8}i\right) = -\frac{5}{8}i^2 = -\frac{5}{8} \times -1 = \frac{5}{8} = \frac{5}{8} + 0\,i$

(ii) $(-i)(2i)\left(-\frac{1}{8}i\right)^3 = -2i^2 \times -\frac{1}{512}i^3 = \frac{1}{256} \times i^2 \times i^3 = \frac{1}{256}i^5 = \frac{i}{256} = 0 + \frac{1}{256}i$

(iii) $(5i)\left(-\frac{3}{5}i\right) = -3i^2 = -3 \times -1 = 3 = 3 + 0\,i$

(iv) $i^9 + i^{19} = (i^4)^2\, i + (i^4)^4\, i^3 = i + i^3 = i - i = 0 = 0 + 0\,i$

(v) $i^{-39} = (i^4)^{-10}\, i = i = 0 + 1.i$

(vi) $(1-i)^4 = \left\{(1-i)^2\right\}^2 = (1 - 2i + i^2)^2 = (1 - 2i - 1)^2 = (-2i)^2 = 4i^2 = -4 = -4 + 0i$

EXAMPLE 2 *Express each of the following in the form a + ib:*

(i) $3(7+7i)+i(7+7i)$ [NCERT] (ii) $(1-i)-(-1+6i)$ [NCERT]

(iii) $\left(\frac{1}{5}+\frac{2}{5}i\right)-\left(4+\frac{5}{2}i\right)$ [NCERT] (iv) $\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)$ [NCERT]

SOLUTION (i) $3(7+7i)+i(7+7i)=21+21i+7i+7i^2=21+21i+7i-7=14+28i$

(ii) $(1-i)-(-1+6i)=1-i+1-6i=2-7i$

(iii) $\left(\frac{1}{5}+\frac{2}{5}\right)i-\left(4+\frac{5}{2}i\right)=\left(\frac{1}{5}-4\right)+\frac{2i}{5}-\frac{5i}{2}=-\frac{19}{5}-\frac{21}{10}i$

(iv) $\left\{\left(\frac{1}{3}+\frac{7}{3}i\right)+\left(4+\frac{1}{3}i\right)\right\}-\left(-\frac{4}{3}+i\right)=\left\{\left(\frac{1}{3}+4\right)+i\left(\frac{7}{3}+\frac{1}{3}\right)\right\}-\left(-\frac{4}{3}+i\right)$

$$=\left(\frac{13}{3}+\frac{8}{3}i\right)+\frac{4}{3}-i$$

$$=\left(\frac{13}{3}+\frac{4}{3}\right)+\left(\frac{8}{3}-1\right)i=\frac{17}{3}+\frac{5}{3}i$$

EXAMPLE 3 *Express each of the following in the form a + ib:*

(i) $\left(\frac{1}{3}+3i\right)^3$ [NCERT] (ii) $\left(-2-\frac{1}{3}i\right)^3$ [NCERT]

(iii) $(5-3i)^3$ [NCERT] (iv) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)$

SOLUTION (i) $\left(\frac{1}{3}+3i\right)^3=\left(\frac{1}{3}\right)^3+(3i)^3+3\times\frac{1}{3}\times 3i\left(\frac{1}{3}+3i\right)=\frac{1}{27}+27i^3+3i\left(\frac{1}{3}+3i\right)$

$$=\frac{1}{27}+27i^3+i+9i^2=\frac{1}{27}-27i+i-9=-\frac{242}{27}-26i$$

(ii) $\left(-2-\frac{1}{3}i\right)^3=(-2)^3+\left(-\frac{1}{3}i\right)^3+3\times-2\times-\frac{1}{3}i\left(-2-\frac{1}{3}i\right)=-8-\frac{1}{27}i^3+2i\left(-2-\frac{1}{3}i\right)$

$$=-8+\frac{1}{27}i-4i-\frac{2}{3}i^2=-8+\frac{1}{27}i-4i+\frac{2}{3}=-\frac{22}{3}-\frac{107}{27}i$$

(iii) $(5-3i)^3=5^3+(-3i)^3+3\times 25\times-3i+3\times 5\times(-3i)^2=125+27i-225i-135=-10-198i$

(v) $(-\sqrt{3}+\sqrt{-2})(2\sqrt{3}-i)=(-\sqrt{3}+i\sqrt{2})(2\sqrt{3}-i)=-6+\sqrt{3}i+2\sqrt{6}i-\sqrt{2}i^2$

$$=-6+(\sqrt{3}+2\sqrt{6})i+\sqrt{2}=(\sqrt{2}-6)+(\sqrt{3}+2\sqrt{6})i$$

EXAMPLE 4 *Express each one of the following in the standard form a + ib.*

(i) $\frac{1}{3-4i}$ (ii) $\frac{5+4i}{4+5i}$ (iii) $\frac{(1+i)^2}{3-i}$

(iv) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (v) $\frac{1}{-2+\sqrt{-3}}$ (vi) $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

(vii) $\frac{1}{1-\cos\theta+2i\sin\theta}$ (viii) $\frac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$ [NCERT]

SOLUTION (i) $\frac{1}{3-4i}=\frac{1}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i}{9-16i^2}=\frac{3+4i}{9+16}=\frac{3}{25}+\frac{4}{25}i$

(ii) $\dfrac{5+4i}{4+5i} = \dfrac{5+4i}{4+5i} \times \dfrac{4-5i}{4-5i} = \dfrac{(20+20)+i(16-25)}{16-25i^2} = \dfrac{40-9i}{41} = \dfrac{40}{41} - \dfrac{9}{41}i$

(iii) $\dfrac{(1+i)^2}{3-i} = \dfrac{1+2i+i^2}{3-i} = \dfrac{2i}{3-i} = \dfrac{2i}{3-i} \cdot \dfrac{3+i}{3+i} = \dfrac{6i+2i^2}{9-i^2} = \dfrac{-2+6i}{10} = -\dfrac{1}{5} + \dfrac{3}{5}i$

(iv) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \dfrac{(6+6)+i(-4+9)}{(2+2)+i(4-1)} = \dfrac{12+5i}{4+3i} = \dfrac{12+5i}{4+3i} \times \dfrac{4-3i}{4-3i}$

$= \dfrac{(48+15)+i(-36+20)}{16-9i^2} = \dfrac{63}{25} - \dfrac{16}{25}i$

(v) $\dfrac{1}{-2+\sqrt{-3}} = \dfrac{1}{-2+i\sqrt{3}} = \dfrac{1}{-2+i\sqrt{3}} \times \dfrac{-2-i\sqrt{3}}{-2-i\sqrt{3}} = \dfrac{-2-i\sqrt{3}}{4-3i^2} = -\dfrac{2}{7} - \dfrac{\sqrt{3}}{7}i$

(vi) $\left(\dfrac{1}{1-2i} + \dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right) = \dfrac{1+i+3-6i}{(1+2)+i(-2+1)} \times \dfrac{3+4i}{2-4i} = \dfrac{4-5i}{3-i} \times \dfrac{3+4i}{2-4i} = \dfrac{(12+20)+i(16-15)}{(6-4)+i(-2-12)}$

$= \dfrac{32+i}{2-14i} = \dfrac{32+i}{2-14i} \times \dfrac{2+14i}{2+14i} = \dfrac{(64-14)+i(2+448)}{4-196i^2} = \dfrac{50+450i}{200} = \dfrac{1}{4} + \dfrac{9}{4}i$

(vii) $\dfrac{1}{1-\cos\theta+2i\sin\theta} = \dfrac{1}{1-\cos\theta+2i\sin\theta} \times \dfrac{1-\cos\theta-2i\sin\theta}{1-\cos\theta-2i\sin\theta}$

$= \dfrac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2-4i^2\sin^2\theta} = \dfrac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta)^2+4\sin^2\theta}$

$= \dfrac{1-\cos\theta-2i\sin\theta}{1-2\cos\theta+\cos^2\theta+4\sin^2\theta} = \dfrac{1-\cos\theta-2i\sin\theta}{2-2\cos\theta+3\sin^2\theta}$

$= \left(\dfrac{1-\cos\theta}{2-2\cos\theta+3\sin^2\theta}\right) + i\left(\dfrac{-2\sin\theta}{2-2\cos\theta+3\sin^2\theta}\right)$

(viii) $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}\,i)-(\sqrt{3}-i\sqrt{2})} = \dfrac{(9-\sqrt{5}\times-\sqrt{5})+i(3\times-\sqrt{5}+3\sqrt{5})}{\sqrt{3}+\sqrt{2}\,i-\sqrt{3}+\sqrt{2}\,i}$

$= \dfrac{(9+5)+i\times 0}{2\sqrt{2}\,i} = \dfrac{14}{2\sqrt{2}\,i} = \dfrac{7}{\sqrt{2}i} = \dfrac{-7}{\sqrt{2}}i = 0 - \dfrac{7}{\sqrt{2}}i$

EXAMPLE 5 *Prove that the following complex numbers are purely real:*

(i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right)$ (ii) $\left(\dfrac{3+2i}{2-3i}\right)+\left(\dfrac{3-2i}{2+3i}\right)$

SOLUTION (i) $\left(\dfrac{2+3i}{3+4i}\right)\left(\dfrac{2-3i}{3-4i}\right) = \dfrac{(2+3i)(2-3i)}{(3+4i)(3-4i)} = \dfrac{4-9i^2}{9-16i^2} = \dfrac{13}{25}$, which is purely real.

(ii) $\left(\dfrac{3+2i}{2-3i}\right)+\left(\dfrac{3-2i}{2+3i}\right) = \dfrac{3+2i}{2-3i} \times \dfrac{2+3i}{2+3i} + \dfrac{3-2i}{2+3i} \times \dfrac{2-3i}{2-3i}$

$= \dfrac{(3+2i)(2+3i)}{4-9i^2} + \dfrac{(3-2i)(2-3i)}{4-9i^2}$

$= \dfrac{13i}{13} - \dfrac{13i}{13} = 0$, which is purely real.

EXAMPLE 6 *Express* $(1-2i)^{-3}$ *in the standard form* $a+ib$.

SOLUTION We have,

$$(1-2i)^{-3} = \frac{1}{(1-2i)^3} = \frac{1}{1-8i^3-6i+12i^2} = \frac{1}{1+8i-6i-12} = \frac{1}{-11+2i}$$

$$= \frac{1}{-11+2i} \times \frac{-11-2i}{-11-2i} = \frac{-11-2i}{(-11)^2-(2i)^2} = \frac{-11-2i}{125} = \frac{-11}{125} - \frac{2}{125}i$$

EXAMPLE 7 *Perform the indicated operation and find the result in the form a + ib.*

(i) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$ (ii) $\dfrac{3-\sqrt{-16}}{1-\sqrt{-9}}$

SOLUTION We have,

(i) $$\frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{2-5i}{1-4i} = \frac{2-5i}{1-4i} \times \frac{1+4i}{1+4i} = \frac{(2+20)+i(8-5)}{1-16i^2} = \frac{22+3i}{17} = \frac{22}{17} + \frac{3}{17}i$$

(ii) $$\frac{3-\sqrt{-16}}{1-\sqrt{-9}} = \frac{3-4i}{1-3i} = \frac{3-4i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(3+12)+i(-4+9)}{1-9i^2} = \frac{15}{10} + \frac{5}{10}i = \frac{3}{2} + \frac{1}{2}i$$

EXAMPLE 8 *If z_1, z_2 are $1-i, -2+4i$, respectively, find* $\mathrm{Im}\left(\dfrac{z_1 z_2}{\bar{z}_1}\right)$.

SOLUTION Clearly,

$$\frac{z_1 z_2}{\bar{z}_1} = \frac{(1-i)(-2+4i)}{(\overline{1-i})} = \frac{(-2+4)+i(2+4)}{1+i} = \frac{2+6i}{1+i}$$

$$= \frac{2+6i}{1+i} \times \frac{1-i}{1-i} = \frac{(2+6)+i(6-2)}{1+1} = 4+2i$$

$$\therefore \quad \mathrm{Im}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = 2$$

Type II ON EQUALITY OF COMPLEX NUMBERS

Recall that two complex numbers z_1 and z_2 are equal iff $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

EXAMPLE 9 *Find the real values of x and y, if*

(i) $(3x-7)+2iy = -5y+(5+x)i$ (ii) $(1-i)x+(1+i)y = 1-3i$

(iii) $(x+iy)(2-3i) = 4+i$ (iv) $\dfrac{x-1}{3+i} + \dfrac{y-1}{3-i} = i$

SOLUTION (i) We have

$$(3x-7)+2iy = -5y+(5+x)i$$
$$\Rightarrow \quad 3x-7 = -5y \text{ and } 2y = 5+x$$
$$\Rightarrow \quad 3x+5y = 7 \text{ and } x-2y = -5$$
$$\Rightarrow \quad x = -1, y = 2.$$

(ii) We have,

$$(1-i)x+(1+i)y = 1-3i$$
$$\Rightarrow \quad (x+y)+i(-x+y) = 1-3i$$
$$\Rightarrow \quad x+y = 1 \text{ and } -x+y = -3$$ [On equating real and imaginary parts]
$$\Rightarrow \quad x = 2, y = -1$$

(iii) We have,

$$(x+iy)(2-3i) = 4+i$$
$$\Rightarrow \quad (2x+3y)+i(-3x+2y) = 4+i$$
$$\Rightarrow \quad 2x+3y = 4 \text{ and } -3x+2y = 1$$ [On equating real and imaginary parts]

$\Rightarrow \quad x = \frac{5}{13}, y = \frac{14}{13}$

(iv) We have,

$$\frac{x-1}{3+i} + \frac{y-1}{3-i} = i$$

$$\Rightarrow \quad \frac{(x-1)(3-i)+(y-1)(3+i)}{(3+i)(3-i)} = i$$

$$\Rightarrow \quad \frac{(3x+3y-6)+i(y-x)}{9-i^2} = i$$

$$\Rightarrow \quad \left(\frac{3x+3y-6}{10}\right) + i\left(\frac{y-x}{10}\right) = 0+i$$

$$\Rightarrow \quad \frac{3x+3y-6}{10} = 0 \text{ and } \frac{y-x}{10} = 1 \qquad \text{[On equating real and imaginary parts]}$$

$\Rightarrow \quad x + y - 2 = 0$ and $y - x = 10$

$\Rightarrow \quad x = -4,\ y = 6.$

EXAMPLE 10 *Find real values of x and y for which the following equalities hold:*

(i) $(1+i)\,y^2 + (6+i) = (2+i)\,x$ (ii) $(x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$

SOLUTION (i) We have,

$$(1+i)\,y^2 + (6+i) = (2+i)\,x$$

$$\Rightarrow \quad (y^2+6) + i(y^2+1) = 2x + ix$$

$\Rightarrow \quad y^2 + 6 = 2x$ ...(i) and, $y^2 + 1 = x$ ...(ii)

From (i) and (ii), we get

$$y^2 + 6 = 2(y^2+1) \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$$

Substituting $y = \pm 2$ in (ii), we get $x = 5$.

Thus, $x = 5$ and $y = 2$ or, $x = 5$ and $y = -2$

(ii) We have,

$$(x^4 + 2xi) - (3x^2 + iy) = (3-5i) + (1+2iy)$$

$$\Rightarrow \quad (x^4 - 3x^2) + i(2x - y) = 4 + i(2y - 5)$$

$\Rightarrow \quad x^4 - 3x^2 = 4$ and, $2x - y = 2y - 5$ [On equating real and imaginary parts]

$\Rightarrow \quad x^4 - 3x^2 - 4 = 0,\ 2x - 3y + 5 = 0$

Now, $x^4 - 3x^2 - 4 = 0 \Rightarrow (x^2-4)(x^2+1) = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

Putting $x = \pm 2$ in $2x - 3y + 5 = 0$, we get

$y = 3$ when $x = 2$ and $y = 1/3$ when $x = -2$

Thus, $x = 2$ and $y = 3$ or, $x = -2$ and $y = 1/3$.

EXAMPLE 11 *If* $a + ib = \frac{c+i}{c-i}$, *where c is real, prove that:* $a^2 + b^2 = 1$ *and* $\frac{b}{a} = \frac{2c}{c^2-1}$.

SOLUTION We have,

$$a + ib = \frac{c+i}{c-i}$$

$$\Rightarrow \quad a + ib = \frac{(c+i)(c+i)}{(c-i)(c+i)}$$

$$\Rightarrow \quad a + ib = \frac{(c+i)^2}{c^2 - i^2}$$

$$\Rightarrow \quad a + ib = \frac{c^2 + 2ic + i^2}{c^2 - i^2}$$

$$\Rightarrow \quad a + ib = \frac{c^2 - 1}{c^2 + 1} + \frac{i\,2c}{c^2 + 1}$$

$$\Rightarrow \quad a = \frac{c^2 - 1}{c^2 + 1} \text{ and } b = \frac{2c}{c^2 + 1}$$

$$\Rightarrow \quad a^2 + b^2 = \left(\frac{c^2 - 1}{c^2 + 1}\right)^2 + \frac{4c^2}{(c^2 + 1)^2} \text{ and, } \frac{b}{a} = \left(\frac{2c}{c^2 + 1}\right) \div \left(\frac{c^2 - 1}{c^2 + 1}\right)$$

$$\Rightarrow \quad a^2 + b^2 = \frac{(c^2 + 1)^2}{(c^2 + 1)^2} = 1 \text{ and, } \frac{b}{a} = \frac{2c}{c^2 - 1}$$

EXAMPLE 12 *If $(x + iy)^{1/3} = a + ib$, $x, y, a\,b \in R$. Show that*

(i) $\frac{x}{a} + \frac{y}{b} = 4\,(a^2 - b^2)$ (ii) $\frac{x}{a} - \frac{y}{b} = -2\,(a^2 + b^2)$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$(x + iy)^{1/3} = a + ib$$

$$\Rightarrow \quad (x + iy) = (a + ib)^3 \qquad \text{[On cubing both sides]}$$

$$\Rightarrow \quad x + iy = a^3 + 3a^2\,i\,b + 3a\,i^2\,b^2 + i^3\,b^3$$

$$\Rightarrow \quad x + i\,y = (a^3 - 3ab^2) + i(3a^2\,b - b^3)$$

$$\Rightarrow \quad x = a^3 - 3ab^2 \text{ and } y = 3a^2\,b - b^3$$

$$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2 \text{ and } \frac{y}{b} = 3a^2 - b^2$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = a^2 - 3b^2 + 3a^2 - b^2 = 4(a^2 - b^2) \text{ and } \frac{x}{a} - \frac{y}{b} = (a^2 - 3b^2) - (3a^2 - b^2) = -2(a^2 + b^2)$$

Type III ON CONJUGATE OF A COMPLEX NUMBER

EXAMPLE 13 *Multiply $3 - 2i$ by its conjugate.*

SOLUTION The conjugate of $3 - 2i$ is $3 + 2i$.

$\therefore$ Required product $= (3 - 2i)\,(3 + 2i) = 9 - 4i^2 = 9 + 4 = 13$

ALITER Let $z = 3 - 2i$. Then, $\bar{z} = 3 + 2i$

$\therefore \quad z\bar{z} = |z|^2 \Rightarrow z\bar{z} = 3^2 + (-2)^2 = 13$

EXAMPLE 14 *Find the conjugate of $\frac{1}{3 + 4i}$.*

SOLUTION Let $z = \frac{1}{3 + 4i}$. Then,

$$z = \frac{1}{3 + 4i} = \frac{1}{3 + 4i} \times \frac{3 - 4i}{3 - 4i} = \frac{3 - 4i}{9 + 16} = \frac{3}{25} - \frac{4}{25}i$$

$$\therefore \quad \bar{z} = \frac{3}{25} + \frac{4}{25}i.$$

EXAMPLE 15 *Express the following complex numbers in the standard form. Also, find their conjugate:*

(i) $\dfrac{1-i}{1+i}$ (ii) $\dfrac{(1+i)^2}{3-i}$

(iii) $\dfrac{(2+3i)^2}{2-i}$ (iv) $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5-12i}-\sqrt{5-12i}}$ [NCERT EXEMPLAR]

SOLUTION (i) We have,

$$z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i+i^2}{1-i^2} = \frac{1-2i-1}{1+1} = 0-i$$

$\therefore \quad \bar{z} = 0+i$

(ii) We have,

$$z = \frac{(1+i)^2}{3-i} = \frac{1+2i+i^2}{3-i} \times \frac{3+i}{3+i} = \frac{2i}{3-i} \times \frac{3+i}{3+i} = \frac{6i+2i^2}{9-i^2} = \frac{6i-2}{10} = -\frac{1}{5}+\frac{3}{5}i$$

$\therefore \quad \bar{z} = -\frac{1}{5} - \frac{3}{5}i$

(iii) We have,

$$z = \frac{(2+3i)^2}{2-i} = \frac{4+12i+9i^2}{2-i} = \frac{4+12i-9}{2-i} \times \frac{2+i}{2+i} = \frac{-5+12i}{2-i} \times \frac{2+i}{2+i} = \frac{-22+19i}{4-i^2} = -\frac{22}{5}+\frac{19}{5}i$$

$\therefore \quad \bar{z} = -\frac{22}{5} - \frac{19}{5}i$

(iv) Let $\dfrac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5-12i}-\sqrt{5-12i}}$. Then,

$$z = \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}} \times \frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}+\sqrt{5-12i}} = \frac{\left(\sqrt{5+12i}+\sqrt{5-12i}\right)^2}{(5+12i)-(5-12i)}$$

$$\Rightarrow \quad z = \frac{5+12i+5-12i+2\sqrt{5+12i}\,\sqrt{5-12i}}{5+12i-5+12i} = \frac{10+2\sqrt{25+144}}{24i} = \frac{3}{2i} = -\frac{3}{2}i = 0-\frac{3}{2}i$$

$\therefore \quad \bar{z} = 0+\frac{3}{2}i$

EXAMPLE 16 *Find real values of x and y for which the complex numbers* $-3+ix^2y$ *and* x^2+y+4i *are conjugate of each other.*

SOLUTION Since $-3+ix^2y$ and x^2+y+4i are complex conjugates.

$\therefore \quad -3+ix^2y = \overline{x^2+y+4i}$

$\Rightarrow \quad -3+ix^2y = x^2+y-4i$

$\Rightarrow \quad -3 = x^2+y$ ...(i) and, $x^2y = -4$ ...(ii)

$\Rightarrow \quad -3 = x^2 - \frac{4}{x^2}$ [Putting $y=-4/x^2$ from (ii) in (i)]

$\Rightarrow \quad x^4+3x^2-4 = 0$

$\Rightarrow \quad (x^2+4)(x^2-1) = 0$

$\Rightarrow \quad x^2-1 = 0$ [$\because x^2+4 \neq 0$ for any real x]

$\Rightarrow \quad x = \pm 1$

From (ii), $y=-4$, when $x=\pm 1$.

Hence, $x=1, y=-4$ or, $x=-1, y=-4$

EXAMPLE 17 *Find the real numbers x and y, if $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.* **[NCERT]**

SOLUTION We have,

$$(x-iy)(3+5i)=(3x+5y)+i(5x-3y)$$

It is given that $(x-iy)(3+5i)$ is the conjugate of $-6-24i$.

$\therefore \quad (x-iy)(3+5i)=\overline{-6-24i}$

$\Rightarrow \quad (3x+5y)+i(5x-3y)=-6+24i$

$\Rightarrow \quad 3x+5y=-6$ and $5x-3y=24$ [On equating real and imaginary parts]

Solving these equations, we get $x=3$, $y=-3$.

Type IV ON FINDING THE MULTIPLICATIVE INVERSE OR RECIPROCAL OF A NON-ZERO COMPLEX NUMBER

EXAMPLE 18 *Find the multiplicative inverse of the following complex numbers:*

(i) $3+2i$ **[NCERT]** (ii) $(2+\sqrt{3}\,i)^2$

SOLUTION (i) Let $z=3+2i$. Then,

$$\frac{1}{z}=\frac{1}{3+2i}=\frac{3-2i}{(3+2i)(3-2i)}=\frac{3-2i}{9-4i^2}=\frac{3}{13}-\frac{2}{13}i$$

ALITER Let $z=3+2i$. Then,

$$\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{3-2i}{9+4}=\frac{3}{13}-\frac{2i}{13}$$

(ii) Let $z=(2+\sqrt{3}i)^2$. Then,

$$z=4+3i^2+4\sqrt{3}i=4-3+4\sqrt{3}i=1+4\sqrt{3}i$$

$$\therefore \quad \frac{1}{z}=\frac{1}{4+\sqrt{3}i}=\frac{1-4\sqrt{3}i}{(1+4\sqrt{3}i)(1-4\sqrt{3}i)}=\frac{1-4\sqrt{3}i}{1+48}=\frac{1}{49}-\frac{4\sqrt{3}i}{49}$$

Type V PROBLEMS BASED UPON CONJUGATE AND MODULUS OF A COMPLEX NUMBER

EXAMPLE 19 *If $\frac{a+ib}{c+id}=x+iy$, prove that $\frac{a-ib}{c-id}=x-iy$ and $\frac{a^2+b^2}{c^2+d^2}=x^2+y^2$.*

SOLUTION We have,

$$\frac{a+ib}{c+id}=x+iy$$

$$\Rightarrow \quad \overline{\left(\frac{a+ib}{c+id}\right)}=\overline{x+iy} \quad \text{[Taking Conjugate of both sides]}$$

$$\Rightarrow \quad \frac{\overline{a+ib}}{\overline{c+id}}=\overline{x+iy} \quad \left[\because \overline{\left(\frac{z_1}{z_2}\right)}=\frac{\bar{z}_1}{\bar{z}_2}\right]$$

$$\Rightarrow \quad \frac{a-ib}{c-id}=x-iy$$

Thus, we have $\frac{a+ib}{c+id}=x+iy$ and $\frac{a-ib}{c-id}=x-iy$

$$\Rightarrow \quad \frac{a+ib}{c+id}\times\frac{a-ib}{c-id}=(x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{(a+ib)(a-ib)}{(c+id)(c-id)}=(x+iy)(x-iy)$$

$$\Rightarrow \quad \frac{a^2+b^2}{c^2+d^2} = x^2+y^2 \qquad \text{[Using: } z\bar{z} = |z|^2\text{]}$$

EXAMPLE 20 *If* $\dfrac{(a+i)^2}{(2a-i)} = p+iq$, *show that:* $p^2+q^2 = \dfrac{(a^2+1)^2}{(4a^2+1)}$.

SOLUTION We have,

$$\frac{(a+i)^2}{(2a-i)} = (p+iq) \qquad \text{...(i)}$$

$$\Rightarrow \quad \overline{\left\{\frac{(a+i)^2}{(2a-i)}\right\}} = \overline{(p+iq)} \qquad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \frac{(\overline{a+i})^2}{(\overline{2a-i})} = \overline{(p+iq)}$$

$$\Rightarrow \quad \frac{(a-i)^2}{(2a+i)} = p-iq \qquad \text{...(ii)}$$

Multiplying (i) and (ii), we get

$$\frac{(a+i)^2}{(2a-i)} \times \frac{(a-i)^2}{(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \quad \frac{\{(a+i)(a-i)\}^2}{(2a-i)(2a+i)} = (p+iq)(p-iq)$$

$$\Rightarrow \quad \frac{(a^2+1)^2}{4a^2+1} = p^2+q^2 \qquad \text{[Using : } z\bar{z} = |z|^2\text{]}$$

EXAMPLE 21 *If* $a+ib = \dfrac{(x+i)^2}{2x^2+1}$, *prove that* $a^2+b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$. **[NCERT]**

SOLUTION We have,

$$a+ib = \frac{(x+i)^2}{2x^2+1} \qquad \text{...(i)}$$

$$\Rightarrow \quad \overline{a+ib} = \overline{\left\{\frac{(x+i)^2}{2x^2+1}\right\}} \qquad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \overline{a+ib} = \frac{(\overline{x+i})^2}{(2x^2+1)}$$

$$\Rightarrow \quad a-ib = \frac{(x-i)^2}{2x^2+1} \qquad \text{...(ii)}$$

Multiplying (i) and (ii), we get

$$(a+ib)(a-ib) = \frac{(x+i)^2 (x-i)^2}{(2x^2+1)(2x^2+1)}$$

$$\Rightarrow \quad a^2+b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2} \qquad \left[\because (x+i)(x-i) = x^2 - i^2 = x^2+1\right]$$

EXAMPLE 22 *If* $x+iy = \sqrt{\dfrac{a+ib}{c+id}}$, *prove that:* $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$. **[NCERT]**

SOLUTION We have,

$$x+iy = \sqrt{\frac{a+ib}{c+id}}$$

$$\Rightarrow \quad x-iy = \sqrt{\frac{a-ib}{c-id}} \qquad \text{[Taking conjugate of both sides]}$$

$$\therefore \quad (x+iy)(x-iy) = \sqrt{\frac{a+ib}{c+id}} \times \sqrt{\frac{a-ib}{c-id}} = \sqrt{\frac{a+ib}{c+id} \times \frac{a-ib}{c-id}}$$

$$\Rightarrow \quad x^2+y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

$$\Rightarrow \quad (x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$$

EXAMPLE 23 *Find the least positive value of n, if* $\left(\frac{1+i}{1-i}\right)^n = 1.$ **[NCERT]**

SOLUTION We have,

$$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i$$

$\therefore \quad \left(\frac{1+i}{1-i}\right)^n = 1 \Rightarrow i^n = 1 \Rightarrow n$ is a multiple of $4 \Rightarrow$ The smallest positive value of n is 4.

EXAMPLE 24 *Find real* θ *such that* $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ *is purely real.* **[NCERT]**

SOLUTION Clearly,

$$\frac{3+2i\sin\theta}{1-2i\sin\theta} = \frac{(3+2i\sin\theta)(1+2i\sin\theta)}{(1-2i\sin\theta)(1+2i\sin\theta)}$$

$$= \frac{(3-4\sin\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta}$$

$$= \frac{(3-4\sin\theta)+i(6\sin\theta+2\sin\theta)}{1+4\sin^2\theta} = \frac{3-4\sin\theta}{1+4\sin^2\theta} + \frac{i\,8\sin\theta}{1+4\sin^2\theta}$$

It is given that $\frac{3+2\sin\theta}{1-2i\sin\theta}$ is purely real. Therefore, its imaginary part is zero.

i.e. $\quad \frac{8\sin\theta}{1+4\sin^2\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = n\pi, n \in Z$

LEVEL-2

EXAMPLE 25 *The sum and product of two complex numbers are real if and only if they are conjugate of each other.*

SOLUTION First, let the two complex numbers be conjugate of each other. Let complex numbers be $z_1 = a+ib$ and $z_2 = a-ib$. Then,

$z_1 + z_2 = (a+ib)+(a-ib) = 2a,$ which is real.

And, $z_1 z_2 = (a+ib)(a-ib) = a^2 - i^2 b^2 = a^2+b^2$, which is also real.

Thus, if z_1 and z_2 are conjugate of each other. Then, Their sum z_1+z_2 and product z_1z_2 both are real.

Conversely, let z_1 and z_2 be two complex numbers such that their sum z_1+z_2 and product $z_1 z_2$ both are real. Then, we have to prove that z_1 and z_2 are conjugate of each other.

Let $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$. Then,

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2) \text{ and } z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$$

Now, $z_1 + z_2$ and $z_1 z_2$ are real

$\Rightarrow$ $(a_1 + a_2) + i(b_1 + b_2)$ and, $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ are real

$\Rightarrow$ $b_1 + b_2 = 0$ and $a_1 b_2 + a_2 b_1 = 0$ [$\because$ z is real $\Leftrightarrow$ Im (z) = 0]

$\Rightarrow$ $b_2 = -b_1$ and $a_1 b_2 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $-a_1 b_1 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $(a_2 - a_1) b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $a_2 - a_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $a_2 = a_1$

$\Rightarrow$ $z_2 = a_2 + i\,b_2 = a_1 - i\,b_1 \Rightarrow z_2 = \bar{z}_1$

$\Rightarrow$ z_1 and z_2 are conjugate of each other

EXAMPLE 26 *If $(1+i)(1+2i)(1+3i)\ldots(1+ni) = (x+iy)$, show that:* $2.5.10 \ldots (1+n^2) = x^2 + y^2$.

SOLUTION We have,

$$(1+i)(1+2i)(1+3i)\ \ldots\ (1+ni) = x + iy$$

$\Rightarrow$ $|(1+i)(1+2i)\ldots(1+ni)| = |x+iy|$ [Taking modulus of both sides]

$\Rightarrow$ $|1+i|\,|1+2i|\ldots|1+ni| = |x+iy|$ [$\because |z_1 z_2 \ldots z_n| = |z_1|\,|z_2|\ldots|z_n|$]

$\Rightarrow$ $\sqrt{1+1}\ \sqrt{1+4}\ldots\sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow$ $2.5.10\ldots(1+n^2) = (x^2+y^2)$ [On squaring both sides]

EXAMPLE 27 *If $(a+ib)(c+id)(e+if)(g+ih) = A + iB$, prove that*

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2 + B^2$$

SOLUTION We have,

$$(a+ib)(c+id)(e+if)(g+ih) = A + iB$$

$\Rightarrow$ $|(a+ib)(c+id)(e+if)(g+ih)| = |A+iB|$ [Taking modulus of both sides]

$\Rightarrow$ $|a+ib|\ |c+id|\ |e+if|\ |g+ih| = |A+iB|$ [Using: $|z_1 z_2 \ldots z_n| = |z_1|\ |z_2| \ldots |z_n|$]

$\Rightarrow$ $\sqrt{a^2+b^2}\ \sqrt{c^2+d^2}\ \sqrt{c^2+f^2}\ \sqrt{g^2+h^2} = \sqrt{A^2+B^2}$

$\Rightarrow$ $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2 + B^2$ [On squaring both sides]

EXAMPLE 28 *If z_1, z_2 are complex numbers such that $\frac{2z_1}{3z_2}$ is purely imaginary number, then find $\left|\frac{z_1 - z_2}{z_1 + z_2}\right|$.*

SOLUTION It is given that $\frac{2z_1}{3z_2}$ is purely imaginary. Therefore,

$$\frac{2z_1}{3z_2} = \lambda i \text{ for some } \lambda \in R.$$

$$\Rightarrow \frac{z_1}{z_2} = \frac{3\lambda}{2} i$$

Now, $\left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{\frac{z_1}{z_2} - 1}{\frac{z_1}{z_2} + 1}\right| = \left|\frac{\frac{3}{2}\lambda i - 1}{\frac{3}{2}\lambda i + 1}\right|$ $\left[\because \frac{z_1}{z_2} = \frac{3\lambda}{2} i\right]$

$$\Rightarrow \quad \left|\frac{z_1 - z_2}{z_1 + z_2}\right| = \left|\frac{3\lambda i - 2}{3\lambda i + 2}\right| = \left|\frac{-2 + 3\lambda i}{2 + 3\lambda i}\right| = \frac{|-2 + 3\lambda i|}{|2 + 3\lambda i|} = \frac{\sqrt{4 + 9\lambda^2}}{\sqrt{4 + 9\lambda^2}} = 1$$

Type VI **ON FINDING THE VALUE OF A POLYNOMIAL FOR A GIVEN VALUE OF THE VARIABLE**

EXAMPLE 29 *If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.*

SOLUTION We have,

$$x = -5 + 2\sqrt{-4}$$
$$\Rightarrow \quad x + 5 = 4i$$
$$\Rightarrow \quad (x+5)^2 = 16i^2 \Rightarrow x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$$

Now, $x^4 + 9x^3 + 35x^2 - x + 4$

$$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) + 4(x^2 + 10x + 41) - 160$$
$$= x^2(0) - x(0) + 4(0) - 160 = -160 \qquad [\because x^2 + 10x + 41 = 0]$$

Thus, the value of the given polynomial for $x = -5 + 2\sqrt{-4}$ is -160.

EXAMPLE 30 *Find the value of $x^3 + 7x^2 - x + 16$, when $x = 1 + 2i$.*

SOLUTION We have,

$$x = 1 + 2i \Rightarrow x - 1 = 2i \Rightarrow (x-1)^2 = 4i^2 \Rightarrow x^2 - 2x + 1 = -4 \Rightarrow x^2 - 2x + 5 = 0$$
$$\therefore \quad x^3 + 7x^2 - x + 16 = x(x^2 - 2x + 5) + 9(x^2 - 2x + 5) + (12x - 29)$$
$$= x(0) + 9(0) + 12x - 29 \qquad [\because x^2 - 2x + 5 = 0]$$
$$= 12(1 + 2i) - 29 \qquad [\because x = 1 + 2i]$$
$$= -17 + 24i$$

Hence, the value of the given polynomial when $x = 1 + 2i$ is $-17 + 24i$.

Type VII **MISCELLANEOUS PROBLEMS**

EXAMPLE 31 *Prove that: $x^4 + 4 = (x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$.*

SOLUTION We have,

$$(x + 1 + i)(x + 1 - i)(x - 1 + i)(x - 1 - i)$$
$$= \{(x+1)^2 - i^2\}\{(x-1)^2 - i^2\}$$
$$= \{(x+1)^2 + 1\}\{(x-1)^2 + 1\}$$
$$= \{x^2 + 2x + 2\}\{x^2 - 2x + 2\}$$
$$= \{x^2 + 2 + 2x\}\{x^2 + 2 - 2x\} = (x^2 + 2)^2 - (2x)^2 = x^4 + 4x^2 + 4 - 4x^2 = x^4 + 4$$

EXAMPLE 32 *If z is a complex number such that $|z| = 1$, prove that $\frac{z-1}{z+1}$ is purely imaginary. What will be your conclusion if $z = 1$?*

SOLUTION Let $z = x + iy$. Then,

$$|z| = 1 \Rightarrow \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$$

Now,

$$\frac{z-1}{z+1} = \frac{x + iy - 1}{x + iy + 1} = \frac{(x-1) + iy}{(x+1) + iy} = \frac{(x-1) + iy}{(x+1) + iy} \times \frac{(x+1) - iy}{(x+1) - iy}$$
$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{(x^2 - 1 + y^2) + i(xy + y - xy + y)}{(x+1)^2 + y^2} = \frac{(x^2 + y^2 - 1) + 2iy}{(x+1)^2 + y^2}$$

$$\Rightarrow \quad \frac{z-1}{z+1} = \frac{2iy}{(x+1)^2+y^2} \qquad [\because x^2+y^2=1]$$

$$\therefore \quad \frac{z-1}{z+1} = \frac{2iy}{(x+1)^2+y^2}, \text{ which is purely imaginary.}$$

Again, when $z = 1$, then $x + iy = 1 + i.0 \Rightarrow x = 1$ and $y = 0$.

$$\therefore \quad \frac{z-1}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{1+i0-1}{1+i0+1} = 0, \text{ which is purely real}$$

EXAMPLE 33 *If $z = x + iy$ and $w = \dfrac{1-iz}{z-i}$, show that $|w| = 1 \Rightarrow z$ is purely real.*

SOLUTION We have,

$$|w| = 1$$

$$\Rightarrow \quad \left|\frac{1-iz}{z-i}\right| = 1$$

$$\Rightarrow \quad \frac{|1-iz|}{|z-i|} = 1$$

$$\Rightarrow \quad |1-iz| = |z-i|$$

$$\Rightarrow \quad |1-i(x+iy)| = |x+iy-i)|, \text{ where } z = x+iy$$

$$\Rightarrow \quad |1+y-ix| = |x+i(y-1)|$$

$$\Rightarrow \quad \sqrt{(1+y)^2+(-x)^2} = \sqrt{x^2+(y-1)^2}$$

$$\Rightarrow \quad (1+y)^2+x^2 = x^2+(y-1)^2$$

$$\Rightarrow \quad y = 0$$

$$\Rightarrow \quad z = x+i0 = x, \text{ which is purely real}$$

EXAMPLE 34 *If $z = 2 - 3i$, show that $z^2 - 4z + 13 = 0$ and hence find the value of $4z^3 - 3z^2 + 169$.*

SOLUTION We have,

$$z = 2-3i \Rightarrow z-2 = -3i \Rightarrow (z-2)^2 = (-3i)^2 \Rightarrow z^2-4z+4 = 9i^2 \Rightarrow z^2-4z+13 = 0$$

$$\therefore \quad 4z^3 - 3z^2 + 169 = 4z(z^2-4z+13) + 13(z^2-4z+13) = 4z(0) + 13(0) = 0 \qquad [\because z^2-4z+13=0]$$

EXAMPLE 35 *Show that a real value of x will satisfy the equation $\dfrac{1-ix}{1+ix} = a - ib$ if $a^2 + b^2 = 1$, where a, b are real.*

SOLUTION We have,

$$\frac{1-ix}{1+ix} = \frac{a-ib}{1}$$

$$\Rightarrow \quad \frac{(1-ix)+(1+ix)}{(1-ix)-(1+ix)} = \frac{a-ib+1}{a-ib-1} \qquad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{2}{-2ix} = \frac{1+a-ib}{-(1-a+ib)}$$

$$\Rightarrow \quad ix = \frac{1-a+ib}{1+a-ib}$$

$$\Rightarrow \quad ix = \frac{(1-a+ib)}{(1+a-ib)} \times \frac{(1+a+ib)}{(1+a+ib)}$$

$$\Rightarrow \quad i\,x = \frac{1-a^2-b^2+2i\,b}{(1+a)^2-i^2\,b^2}$$

$$\Rightarrow \quad i\,x = \frac{1-a^2-b^2+2i\,b}{(1+a)^2+b^2}$$

$$\Rightarrow \quad i\,x = \frac{2i\,b}{(1+a)^2+b^2}, \text{ if } a^2+b^2 = 1$$

$$\Rightarrow \quad x = \frac{2b}{(1+a)^2+b^2}, \text{ which is real}$$

EXAMPLE 36 *If* α and β *are different complex numbers with* $|\beta| = 1$, *find* $\left|\dfrac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right|$. **[NCERT]**

SOLUTION Clearly,

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right|^2 = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\overline{\left(\frac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right)} = \left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\;\bar{\beta}}\right) = \frac{(\beta-\alpha)(\bar{\beta}-\bar{\alpha})}{(1-\bar{\alpha}\,\beta)(1-\alpha\;\bar{\beta})}$$

$$\Rightarrow \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right|^2 = \frac{\beta\,\bar{\beta}-\beta\,\bar{\alpha}-\alpha\,\bar{\beta}+\alpha\,\bar{\alpha}}{1-\alpha\,\bar{\beta}\;-\bar{\alpha}\,\beta+\bar{\alpha}\,\beta\,\alpha\,\bar{\beta}} = \frac{|\beta|^2-\alpha\,\bar{\beta}-\bar{\alpha}\,\beta+|\alpha|^2}{1-\alpha\,\bar{\beta}\;-\bar{\alpha}\,\beta+(\alpha\,\bar{\alpha})(\beta\,\bar{\beta})}$$

$$\Rightarrow \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right|^2 = \frac{|\alpha|^2-\alpha\,\bar{\beta}-\bar{\alpha}\,\beta+|\beta|^2}{1\;-\alpha\,\bar{\beta}-\bar{\alpha}\,\beta+|\alpha|^2\,|\beta|^2} = \frac{|\alpha|^2-\alpha\,\bar{\beta}-\bar{\alpha}\,\beta+1}{1-\alpha\,\bar{\beta}\;-\bar{\alpha}\,\beta+|\alpha|^2} = 1 \qquad [\because |\beta| = 1]$$

$$\therefore \quad \left|\frac{\beta-\alpha}{1-\bar{\alpha}\,\beta}\right| = 1.$$

EXAMPLE 37 *If* $|z_1| = |z_2| = \ldots = |z_n| = 1$, *prove that* $|z_1+z_2+z_3+\ldots+z_n| = \left|\dfrac{1}{z_1}+\dfrac{1}{z_2}+\dfrac{1}{z_3}+\ldots+\dfrac{1}{z_n}\right|$.

SOLUTION Clearly, **[NCERT EXEMPLAR]**

$$|z_1+z_2+z_3+\ldots.+z_n| = \left|\frac{z_1\bar{z}_1}{\bar{z}_1}+\frac{z_2\bar{z}_2}{\bar{z}_2}+\frac{z_3\bar{z}_3}{\bar{z}_3}+\ldots+\frac{z_n\bar{z}_n}{\bar{z}_n}\right|$$

$$= \left|\frac{|z_1|^2}{\bar{z}_1}+\frac{|z_2|^2}{\bar{z}_2}+\frac{|z_3|^2}{\bar{z}_3}+\ldots+\frac{|z_n|^2}{\bar{z}_n}\right|$$

$$= \left|\frac{1}{\bar{z}_1}+\frac{1}{\bar{z}_2}+\frac{1}{\bar{z}_3}+\ldots+\frac{1}{\bar{z}_n}\right| \qquad [\because |z_1| = |z_2| = \ldots = |z_n| = 1]$$

$$= \left|\overline{\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right)}\right| = \left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right| \qquad [\because |z| = |\bar{z}|]$$

EXAMPLE 38 *Find non-zero integral solutions of* $|1-i|^x = 2^x$. **[NCERT]**

SOLUTION We have,

$$|1-i|^x = 2^x$$

$$\Rightarrow \quad (\sqrt{2})^x = 2^x \Rightarrow 2^{x/2} = 2^x \Rightarrow 2^{x/2} = 1 \Rightarrow 2^{x/2} = 2^0 \Rightarrow \frac{x}{2} = 0 \Rightarrow x = 0.$$

Hence, the given equation has no non-zero integral solution.

EXAMPLE 39 *Find all non-zero complex numbers z satisfying $\bar{z} = i z^2$.*

SOLUTION Let $z = x + iy$. Then,

$$\bar{z} = iz^2$$

$$\Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$$

$$\Rightarrow \quad x - iy = i(x^2 - y^2) - 2xy$$

$$\Rightarrow \quad (x + 2xy) - i(x^2 - y^2 + y) = 0$$

$$\Rightarrow \quad x + 2xy = 0 \quad \text{...(i)} \qquad \text{and, } x^2 - y^2 + y = 0 \quad \text{...(ii)}$$

Now,

$$x + 2xy = 0 \Rightarrow x(1 + 2y) = 0 \Rightarrow x = 0 \text{ or, } 1 + 2y = 0 \Rightarrow x = 0 \text{ or, } y = -\frac{1}{2}$$

CASE I When $x = 0$:

Putting $x = 0$ in (ii), we have

$$\Rightarrow \quad -y^2 + y = 0 \Rightarrow y(y-1) = 0 \Rightarrow y = 0, y = 1$$

Thus, we have the following pairs of values of x and y:

$$x = 0, y = 0; x = 0, y = 1$$

$$\therefore \quad z = 0 + i0 = 0, \ z = 0 + 1i = i$$

CASE II When $y = -\frac{1}{2}$:

Putting $y = -\frac{1}{2}$ in (ii), we get

$$x^2 - y^2 + y = 0 \Rightarrow x^2 - \frac{1}{4} - \frac{1}{2} = 0 \Rightarrow x^2 - \frac{3}{4} = 0 \Rightarrow x = \pm\frac{\sqrt{3}}{2}$$

Thus, we have the following pairs of values of x and y:

$$x = \frac{\sqrt{3}}{2}, y = \frac{-1}{2} \text{ and, } x = \frac{-\sqrt{3}}{2}, y = \frac{-1}{2}$$

$$\therefore \quad z = \frac{\sqrt{3}}{2} - \frac{1}{2}i, \ z = \frac{-\sqrt{3}}{2} - \frac{1}{2}i$$

Hence, $z = 0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

EXAMPLE 40 *If $iz^3 + z^2 - z + i = 0$, then show that $|z| = 1$.*

SOLUTION We have,

$$iz^3 + z^2 - z + i = 0$$

$$\Rightarrow \quad z^3 - iz^2 + iz + 1 = 0 \qquad \text{[Dividing both sides by } i\text{]}$$

$$\Rightarrow \quad z^2(z - i) + i(z - i) = 0 \Rightarrow (z - i)(z^2 + i) = 0 \Rightarrow z = i \text{ or, } z^2 = -i$$

Now,

$$z = i \Rightarrow |z| = |i| = 1$$

and, $z^2 = -i$

$$\Rightarrow \quad |z^2| = |-i| = 1 \Rightarrow |z|^2 = 1 \Rightarrow |z| = 1.$$

Hence, in either case, we have $|z| = 1$.

EXAMPLE 41 *Solve the equation $z^2 + |z| = 0$, where z is a complex number.*

SOLUTION Let $z = x + iy$. Then,

$$z^2 + |z| = 0$$

$$\Rightarrow \quad (x+iy)^2 + \sqrt{x^2+y^2} = 0$$

$$\Rightarrow \quad (x^2 - y^2) + \sqrt{x^2+y^2} + 2i\,xy = 0$$

$$\Rightarrow \quad x^2 - y^2 + \sqrt{x^2+y^2} = 0 \quad ...(i) \qquad \text{and,} \quad 2xy = 0 \quad ...(ii)$$

Now,

$$2xy = 0 \Rightarrow xy = 0 \Rightarrow x = 0 \text{ or, } y = 0$$

<u>CASE I</u> *When* $y = 0$

Putting $y = 0$ in (i), we get

$$x^2 + \sqrt{x^2} = 0 \Rightarrow x^2 + |x| = 0$$

Clearly, $x^2 + |x| > 0$ for all $x > 0$. So, let $x < 0$.

In this case, we have

$$x^2 + |x| = 0$$

$$\Rightarrow \quad x^2 - x = 0 \qquad [\because x < 0 \therefore |x| = -x]$$

$$\Rightarrow \quad x(x-1) = 0$$

$$\Rightarrow \quad x = 0, x = 1$$

But, $x < 0$. So, the equation $x^2 + |x| = 0$ has no solution for $x < 0$.

Clearly, $x = 0$ satisfies the equation $x^2 + |x| = 0$.

Thus, we have $x = 0,\ y = 0$.

$$\therefore \quad z = 0$$

<u>CASE II</u> *When* $x = 0$

Putting $x = 0$ in (i), we get

$$-y^2 + \sqrt{y^2} = 0 \Rightarrow -y^2 + |y| = 0$$

If $y > 0$, then $|y| = y$.

$$\therefore \quad -y^2 + |y| = 0$$

$$\Rightarrow \quad -y^2 + y = 0$$

$$\Rightarrow \quad y = 0, y = 1$$

$$\Rightarrow \quad y = 1 \qquad [\because y > 0]$$

If $y < 0$, then $|y| = -y$.

$$-y^2 + |y| = 0$$

$$\Rightarrow \quad -y^2 - y = 0$$

$$\Rightarrow \quad y = 0, -1$$

$$\Rightarrow \quad y = -1 \qquad [\because y < 0]$$

Thus, we obtain $x = 0, y = 1$ or, $x = 0, y = -1$.

$$\therefore \quad z = 0 + i \text{ or, } z = 0 - i$$

Hence, $z = 0, i$ and $-i$ are solutions of $z^2 + |z| = 0$.

EXAMPLE 42 *Solve the equation* $z^2 = \bar{z}$. **[NCERT EXEMPLAR]**

SOLUTION Let $z = x + i\,y$. Then,

$$z^2 = \bar{z}$$

$$\Rightarrow \quad (x+iy)^2 = x - iy$$

$\Rightarrow \quad x^2 + 2ixy + (iy)^2 = x - iy$

$\Rightarrow \quad (x^2 - y^2) + 2ixy = x - iy$

$\Rightarrow \quad x^2 - y^2 = x \quad$ ...(i) and, $2xy = -y$ ...(ii)

Now, $2xy = -y \Rightarrow (2x+1)y = 0 \Rightarrow 2x+1 = 0$ or $y = 0 \Rightarrow x = -\frac{1}{2}$ or $y = 0$

Following cases arise :

<u>CASE I</u> *When* $y = 0$

Putting $y = 0$ in (i), we obtain

$$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0 \text{ or, } x = 1$$

Thus, we obtain ($x = 0$ and $y = 0$) or ($x = 1$ and $y = 0$)

$\therefore \quad z = 0 + i0 = 0$ or $z = 1 + i0$

<u>CASE II</u> *When* $x = -\frac{1}{2}$

Putting $x = -\frac{1}{2}$ in (i), we obtain

$$\frac{1}{4} - y^2 = -\frac{1}{2} \Rightarrow y^2 = \frac{3}{4} \Rightarrow y = \pm\frac{\sqrt{3}}{2}$$

Thus, we obtain $\left(x = -\frac{1}{2} \text{ and } y = \frac{\sqrt{3}}{2}\right)$ or $\left(x = -\frac{1}{2} \text{ and } y = -\frac{\sqrt{3}}{2}\right)$

$\therefore \quad z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ or $z = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Hence the values of z satisfying the given equation are

$$z = 0 + i0, z = 1 + i0, z = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ and } z = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

EXAMPLE 43 *Solve the equation* $|z+1| = z + 2(1+i)$. **[NCERT EXEMPLAR]**

SOLUTION Let $z = x + iy$. Then, $z + 1 = (x+1) + iy$

$\therefore \quad |z+1| = \sqrt{(x+1)^2 + y^2}$

Now, $|z+1| = z + 2(1+i)$

$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = (x+iy) + 2(1+i)$

$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} + 0i = (x+2) + (y+2)i$

$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = x + 2$ and $y + 2 = 0$ [On equating real and imaginary parts]

$\Rightarrow \quad (x+1)^2 + y^2 = (x+2)^2$ and $y = -2$

$\Rightarrow \quad y^2 = 2x + 3$ and $y = -2$

$\Rightarrow \quad 4 = 2x + 3$ and $y = -2$

$\Rightarrow \quad x = \frac{1}{2}$ and $y = -2$

Hence, $z = \frac{1}{2} - 2i$

EXAMPLE 44 *If* $|z^2 - 1| = |z|^2 + 1$, *then show that z lies on the imaginary axis.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$.

$\therefore \quad |z^2 - 1| = |z|^2 + 1$

$\Rightarrow \quad \left|(x^2-y^2)+2i\,xy\right| = x^2+y^2+1$

$\Rightarrow \quad \sqrt{(x^2-y^2-1)^2+4x^2y^2} = x^2+y^2+1$

$\Rightarrow \quad (x^2-y^2-1)^2+4x^2y^2 = (x^2+y^2+1)^2$

$\Rightarrow \quad x^4+y^4+1-2x^2+2y^2-2x^2y^2+4x^2y^2 = x^4+y^4+1+2x^2y^2+2x^2+2y^2$

$\Rightarrow \quad 4x^2 = 0$

$\Rightarrow \quad x = 0$

$\therefore \quad z = x+iy = 0+iy$

Thus, z is purely imaginary and hence it lies on y-axis.

EXAMPLE 45 *If the imaginary part of $\frac{2z+1}{iz+1}$ is –2, then show that the locus of the point representing z in the argand plane is a straight line.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x+i\,y$. Then,

$$\frac{2z+1}{iz+1} = \frac{2\,(x+iy)+1}{i\,(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+ix}$$

$$= \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix} = \frac{(2x+1-y)+i\,(2y-2y^2-2x^2-x)}{(1-y)^2+x^2}$$

$$= \left\{\frac{2x+1-y}{x^2+(1-y)^2}\right\} + i\left\{\frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}\right\}$$

$$\therefore \quad \text{Im}\left(\frac{2z+1}{iz+1}\right) = \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2}$$

But, it is given that Im $(z) = -2$.

$$\therefore \quad \frac{2y-2y^2-2x^2-x}{x^2+(1-y)^2} = -2$$

$\Rightarrow \quad 2y-2y^2-2x^2-x = -2x^2-2\,(1-y)^2$

$\Rightarrow \quad x+2y-2 = 0$, which is a straight line.

Hence, the locus of z is a starlight line.

EXAMPLE 46 *If the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4, then show that the locus of the point representing z in the complex plane is a circle.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x+i\,y$. Then, $\bar{z} = x-iy$

$$\therefore \quad \frac{\bar{z}+2}{\bar{z}-1} = \frac{x-iy+2}{x-iy-1} = \frac{(x+2)-iy}{(x-1)-iy}$$

$$= \frac{(x+2)-iy}{(x-1)-iy} \times \frac{(x-1)+iy}{(x-1)+iy}$$

$$= \frac{(x^2+y^2+x-2)+3iy}{(x-1)^2+y^2} = \left\{\frac{x^2+y^2-x-2}{(x-1)^2+y^2}\right\} + i\left\{\frac{3y}{(x-1)^2+y^2}\right\}$$

It is given that the real part of $\frac{\bar{z}+2}{\bar{z}-1}$ is 4.

$$\therefore \quad \frac{x^2+y^2-x-y}{(x-1)^2+y^2} = 4$$

$\Rightarrow \quad 3x^2+3y^2-7x+y+4 = 0$, which represents a circle.

EXAMPLE 47 *If $z = x + iy$, then show that $z\bar{z} + 2(z + \bar{z}) + a = 0$, where $a \in R$, represents a circle.* **[NCERT EXEMPLAR]**

SOLUTION We have,

$$z = x + iy \Rightarrow \bar{z} = x - iy$$

$$\therefore \quad z\bar{z} + 2(z + \bar{z}) + a = 0$$

$$\Rightarrow \quad (x + iy)(x - iy) + 2(x + iy + x - iy) + a = 0$$

$$\Rightarrow \quad x^2 + y^2 + 4x + a = 0$$

$$\Rightarrow \quad (x + 2)^2 + (y - 0)^2 = (\sqrt{4 - a})^2,$$ which represents a circle for all $a \le 4$.

EXAMPLE 48 *Show that $\left|\frac{z-2}{z-3}\right| = 2$ represents a circle. Find its centre and radius.*

SOLUTION Let $z = x + iy$. Then,

$$\left|\frac{z-2}{z-3}\right| = 2$$

$$\Rightarrow \quad \left|\frac{(x-2)+iy}{(x-3)+iy}\right| = 2$$

$$\Rightarrow \quad |(x-2)+iy| = 2|(x-3)+iy|$$

$$\Rightarrow \quad \sqrt{(x-2)^2+y^2} = 2\sqrt{(x-3)^2+y^2}$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 4\{(x-3)^2 + y^2\}$$

$$\Rightarrow \quad 3x^2 + 3y^2 - 20x + 32 = 0$$

$$\Rightarrow \quad x^2 + y^2 - \frac{20}{3}x + \frac{32}{3} = 0$$

$$\Rightarrow \quad \left(x - \frac{10}{3}\right)^2 + (y-0)^2 = \left(\frac{2}{3}\right)^2,$$ which represents a circle with centre at $(10/3, 0)$ and radius $2/3$

EXAMPLE 49 *Find a complex number z satisfying the equation $z + \sqrt{2}|z + 1| + i = 0$.* **[NCERT EXEMPLAR]**

SOLUTION Let $z = x + iy$. Then,

$$z + \sqrt{2}|z+1| + i = 0$$

$$\Rightarrow \quad x + iy + \sqrt{2}|(x+1)+iy| + i = 0$$

$$\Rightarrow \quad x + \sqrt{2}\sqrt{(x+1)^2+y^2} + (y+1)i = 0$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } (y+1) = 0$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2y^2} = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x + \sqrt{2(x+1)^2 + 2} = 0 \text{ and } y = -1$$

$$\Rightarrow \quad \sqrt{2(x+1)^2 + 2} = -x \text{ and } y = -1$$

$$\Rightarrow \quad 2(x+1)^2 + 2 = x^2 \text{ and } y = -1$$

$$\Rightarrow \quad x^2 + 4x + 4 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad (x+2)^2 = 0 \text{ and } y = -1$$

$$\Rightarrow \quad x = -2 \text{ and } y = -1$$

Hence, $z = x + iy = -2 - i$.

EXAMPLE 50 *Let z_1 and z_2 be two complex numbers such that*

$$\left|1-\bar{z}_1 z_2\right|^2-\left|z_1-z_2\right|^2=k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$$

Find the value of k.

SOLUTION We have,

$$|1-\bar{z}_1 z_2|^2-|z_1-z_2|^2$$
$$=(1-\bar{z}_1 z_2)\,\overline{(1-\bar{z}_1 z_2)}-(z_1-z_2)\,\overline{(z_1-z_2)}$$
$$=(1-\bar{z}_1 z_2)(1-z_1\bar{z}_2)-(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$=(1-\bar{z}_1 z_2-z_1\bar{z}_2+\bar{z}_1 z_2 z_1\bar{z}_2)-(z_1\bar{z}_1-z_1\bar{z}_2-\bar{z}_1 z_2+z_2\bar{z}_2)$$
$$=1-\bar{z}_1 z_2-z_1\bar{z}_2+(z_1\bar{z}_1)(z_2\bar{z}_2)-\left(|z_1|^2-z_1\bar{z}_2-\bar{z}_1 z_2+|z_2|^2\right)$$
$$=1-\bar{z}_1 z_2-z_1\bar{z}_2+|z_1|^2|z_2|^2-|z_1|^2+z_1\bar{z}_2+\bar{z}_1 z_2-|z_2|^2$$
$$=1-|z_1|^2-|z_2|^2+|z_1|^2|z_2|^2=(1-|z_1|^2)(1-|z_2|^2)$$

$\therefore \quad |1-\bar{z}_1 z_2|^2-|z_1-z_2|^2=k(1-|z|^2)(1-|z_2|^2)$

$\Rightarrow \quad \left(1-|z_1|^2\right)\left(1-|z_2|^2\right)=k\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)$

$\Rightarrow \quad k=1.$

EXERCISE 13.2

LEVEL-1

1. Express the following complex numbers in the standard form $a+ib$:
   (i) $(1+i)(1+2i)$
   (ii) $\dfrac{3+2i}{-2+i}$
   (iii) $\dfrac{1}{(2+i)^2}$
   (iv) $\dfrac{1-i}{1+i}$
   (v) $\dfrac{(2+i)^3}{2+3i}$
   (vi) $\dfrac{(1+i)(1+\sqrt{3}i)}{1-i}$
   (vii) $\dfrac{2+3i}{4+5i}$
   (viii) $\dfrac{(1-i)^3}{1-i^3}$
   (ix) $(1+2i)^{-3}$
   (x) $\dfrac{3-4i}{(4-2i)(1+i)}$
   (xi) $\left(\dfrac{1}{1-4i}-\dfrac{2}{1+i}\right)\left(\dfrac{3-4i}{5+i}\right)$ [NCERT]
   (xii) $\dfrac{5+\sqrt{2}i}{1-\sqrt{2}i}$ [NCERT]
2. Find the real values of x and y, if
   (i) $(x+iy)(2-3i)=4+i$
   (ii) $(3x-2iy)(2+i)^2=10(1+i)$
   (iii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$
   (iv) $(1+i)(x+iy)=2-5i$
3. Find the conjugates of the following complex numbers:
   (i) $4-5i$
   (ii) $\dfrac{1}{3+5i}$

(iii) $\frac{1}{1+i}$ (iv) $\frac{(3-i)^2}{2+i}$

(v) $\frac{(1+i)(2+i)}{3+i}$ (vi) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ [NCERT]

4. Find the multiplicative inverse of the following complex numbers:

(i) $1-i$ (ii) $(1+i\sqrt{3})^2$

(iii) $4-3i$ (iv) $\sqrt{5}+3i$

5. If $z_1 = 2-i, z_2 = 1+i$, find $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right|$. [NCERT]

6. If $z_1 = 2-i, z_2 = -2+i$, find

(i) $\text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right)$ (ii) $\text{Im}\left(\frac{1}{z_1 \bar{z}_1}\right)$ [NCERT]

7. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ [NCERT]

8. If $x+iy = \frac{a+ib}{a-ib}$, prove that $x^2+y^2=1$ [NCERT]

9. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.

10. Find the real values of θ for which the complex number $\frac{1+i\cos\theta}{1-2i\cos\theta}$ is purely real.

11. Find the smallest positive integer value of n for which $\frac{(1+i)^n}{(1-i)^{n-2}}$ is a real number.

12. If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x+iy$, find (x, y) [NCERT EXEMPLAR]

13. If $\frac{(1+i)^2}{2-i} = x+iy$, find $x+y$. [NCERT EXEMPLAR]

14. If $\left(\frac{1-i}{1+i}\right)^{100} = a+ib$, find (a, b). [NCERT EXEMPLAR]

15. If $a = \cos\theta + i\sin\theta$, find the value of $\frac{1+a}{1-a}$. [NCERT EXEMPLAR]

LEVEL-2

16. Evaluate the following:

(i) $2x^3+2x^2-7x+72$, when $x = \frac{3-5i}{2}$

(ii) $x^4-4x^3+4x^2+8x+44$, when $x = 3+2i$

(iii) $x^4+4x^3+6x^2+4x+9$, when $x = -1+i\sqrt{2}$

(iv) $x^6+x^4+x^2+1$, when $x = \frac{1+i}{\sqrt{2}}$.

(v) $2x^4+5x^3+7x^2-x+41$, when $x = -2-\sqrt{3}i$ [NCERT EXEMPLAR]

17. For a positive integer n, find the value of $(1-i)^n \left(1-\frac{1}{i}\right)^n$. [NCERT EXEMPLAR]

18. If $(1+i)z=(1-i)\bar{z}$, then show that $z=-i\bar{z}$. **[NCERT EXEMPLAR]**
19. Solve the system of equations $\text{Re}(z^2)=0, |z|=2$. **[NCERT EXEMPLAR]**
20. If $\frac{z-1}{z+1}$ is purely imaginary number $(z \neq -1)$, find the value of $|z|$. **[NCERT EXEMPLAR]**
21. If z_1 is a complex number other than -1 such that $|z_1|=1$ and $z_2=\frac{z_1-1}{z_1+1}$, then show that the real parts of z_2 is zero.
22. If $|z+1|=z+2(1+i)$, find z.
23. Solve the equation $|z|=z+1+2i$. **[NCERT EXEMPLAR]**
24. What is the smallest positive integer n for which $(1+i)^{2n}=(1-i)^{2n}$?
25. If z_1, z_2, z_3 are complex numbers such that $|z_1|=|z_2|=|z_3|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|=1$, then find the value of $|z_1+z_2+z_3|$. **[NCERT EXEMPLAR]**
26. Find the number of solutions of $z^2+|z|^2=0$. **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $-1+3i$ (ii) $-\frac{4}{5}-\frac{7}{5}i$ (iii) $\frac{3}{25}-\frac{4}{25}i$ (iv) $-i$
   (v) $\frac{37}{13}+\frac{16}{13}i$ (vi) $-\sqrt{3}+i$ (vii) $\frac{23}{41}+\frac{2}{41}i$ (viii) $-2+0i$
   (ix) $\frac{-11}{125}+\frac{2i}{125}$ (x) $\frac{1}{4}-\frac{3}{4}i$ (xi) $\frac{307}{442}+i\frac{599}{442}$ (xii) $1+2\sqrt{2}i$
2. (i) $x=\frac{5}{13}, y=\frac{14}{13}$ (ii) $x=\frac{14}{15}, y=\frac{1}{5}$ (iii) $x=3, y=-1$ (iv) $x=-\frac{3}{2}, y=-\frac{7}{2}$
3. (i) $4+5i$ (ii) $\frac{1}{34}(3+5i)$ (iii) $\frac{1}{2}+\frac{1}{2}i$ (iv) $2+4i$
   (v) $\frac{3}{5}-\frac{4}{5}i$ (vi) $\frac{63}{25}+\frac{16}{25}i$
4. (i) $\frac{1}{2}+\frac{1}{2}i$ (ii) $-\frac{1}{8}-i\frac{\sqrt{3}}{8}$ (iii) $\frac{4}{25}+\frac{3}{25}i$ (iv) $\frac{\sqrt{5}}{14}-\frac{3i}{14}$
5. $\frac{4}{\sqrt{2}}$ 6. (i) $-\frac{2}{5}$ (ii) 0 7. 2 9. $n=2$
10. $\theta=2n\pi\pm\frac{\pi}{2}, n\in Z$ 11. 1 12. $(0,-2)$ 13. $\frac{2}{5}$ 14. $(1,0)$
15. $i\cot\frac{\theta}{2}$ 16. (i) 4 (ii) 5 (iii) 12 (iv) 0 (v) 6
17. 2^n 19. $\sqrt{2}(1\pm i), \sqrt{2}(-1\pm i)$ 20. 1 22. $\frac{1}{2}-2i$
23. $\frac{3}{2}-2i$ 24. $n=2$ 25. 1
26. Infinitely many solutions of the form $z=0+iy,\ y\in R$.

HINTS TO NCERT & SELECTED PROBLEMS

1. (xi) $\left(\frac{1}{1-4i}-\frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)=\frac{1+i-2(1-4i)}{(1-4i)(1+i)}\times\frac{3-4i}{5+i}=\frac{-1+9i}{5-3i}\times\frac{3-4i}{5+i}$

$$= \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)} = \frac{33+31i}{28-10i} = \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} = \frac{614+1198i}{784+100} = \frac{307}{442} + i\frac{599}{442}$$

(xii) $\frac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i} = \frac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i} \times \frac{1+\sqrt{2}\,i}{1+\sqrt{2}\,i} = \frac{3+6\sqrt{2}\,i}{1+2} = 1+2\sqrt{2}\,i$

3. (vi) $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i$

5. We have, $z_1 = 2-i$ and $z_2 = 1+i$

$\therefore \quad z_1 + z_2 = 3$ and $z_1 - z_2 = 1-2i$

So, $\left|\frac{z_1+z_2+1}{z_1-z_2+i}\right| = \left|\frac{3+1}{1-2i+i}\right| = \frac{4}{|1-i|} = \frac{4}{\sqrt{2}}$

6. (i) We have, $z_1 = 2-i$ and $z_2 = -2+i$

$\therefore \quad z_1 z_2 = (2-i)(-2+i) = -3+4i$

$\Rightarrow \quad \frac{z_1 z_2}{\bar{z}_1} = \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} = \frac{-2+11i}{4+1} = \frac{-2}{5} + \frac{11}{5}i$

$\therefore \quad \text{Re}\left(\frac{z_1 z_2}{\bar{z}_1}\right) = \frac{-2}{5}$

(ii) $z_1 = 2-i \Rightarrow \bar{z}_1 = 2+i$

$\therefore \quad z_1 \bar{z}_1 = (2-i)(2+i) = 5$

So, $\text{Im}\left(\frac{1}{z_1 \bar{z}_1}\right) = \text{Im}\left(\frac{1}{5} + 0i\right) = 0$

7. $z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} = \frac{4i}{1-i^2} = \frac{4i}{1+1} = 2i$

$\therefore \quad |z| = 2$

8. We have, $x + iy = \frac{a+ib}{a-b}$

$\Rightarrow \quad \overline{x+iy} = \overline{\left(\frac{a+ib}{a-ib}\right)} \Rightarrow x - iy = \frac{a-ib}{a+ib}$

$\therefore \quad (x+iy)(x-iy) = \frac{a+ib}{a-ib} \times \frac{a-ib}{a+ib}$

$\Rightarrow \quad x^2 + y^2 = \frac{a^2+b^2}{a^2+b^2} \Rightarrow x^2 + y^2 = 1.$

11. We have,

$$\frac{(1+i)^n}{(1-i)^{n-2}} = \frac{(1+i)^n}{(1-i)^n}(1-i)^2 = \left(\frac{1+i}{1-i}\right)^n (1-2i+i^2) = -2i\left\{\frac{(1+i)^2}{(1+i)(1-i)}\right\}^n$$

$$= -2i\left(\frac{1+2i+i^2}{1-i^2}\right)^n = -2i\left(\frac{1+2i-1}{2}\right)^n = -2i\,(i)^n = -2i^{n+1}$$

$\therefore \quad \frac{(1+i)^n}{(1-i)^{n-2}}$ is real $\Rightarrow -2i^{n+1}$ is real $\Rightarrow n+1 = 2, 4, 6 \Rightarrow n = 1, 3, 5, \ldots$

Hence, the least value of n is 1.

12. We have,

$$\left(\frac{1+i}{1-i}\right)^3-\left(\frac{1-i}{1+i}\right)^3=x+iy$$

$$\Rightarrow i^3-(-i)^3=x+iy \qquad \left[\because \frac{1+i}{1-i}=\frac{(1+i)^2}{1-i^2}=\frac{2i}{2}=i \therefore \frac{1-i}{1+i}=\frac{1}{i}=-i\right]$$

$$\Rightarrow 2i^3=x+iy \Rightarrow 0-2i=x+iy \Rightarrow x=0, y=-2$$

13. We have,

$$\frac{(1+i)^2}{2-i}=x+iy$$

$$\Rightarrow \frac{1+2i+i^2}{2-i}=x+iy$$

$$\Rightarrow \frac{2i}{2-i}=x+iy$$

$$\Rightarrow \frac{2i(2+i)}{(2-i)(2+i)}=x+iy$$

$$\Rightarrow \frac{4i+2i^2}{4-i^2}=x+iy$$

$$\Rightarrow -\frac{2}{5}+\frac{4}{5}i=x+iy \Rightarrow x=-\frac{2}{5} \text{ and } y=\frac{4}{5} \Rightarrow x+y=\frac{2}{5}$$

14. We have,

$$\left(\frac{1-i}{1+i}\right)^{100}=a+ib$$

$$\Rightarrow (-i)^{100}=a+ib \qquad \left[\because \frac{1-i}{1+i}=\frac{(1-i)^2}{1-i^2}=\frac{1-2i+i^2}{2}=-i\right]$$

$$\Rightarrow 1=a+ib \Rightarrow 1+0i=a+ib \Rightarrow a=1, b=0$$

15. We have,

$$a=\cos\theta+i\sin\theta$$

$$\therefore \frac{1+a}{1-a}=\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}=\left(\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta}\right)\times\left(\frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}\right)$$

$$\Rightarrow \frac{1+a}{1-a}=\frac{(1-\cos^2\theta-\sin^2\theta)+2i\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}=\frac{i\sin\theta}{1-\cos\theta}=\frac{2i\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}=i\cot\frac{\theta}{2}$$

17. $(1-i)^n\left(1-\frac{1}{i}\right)^n=(1-i)^n(1+i)^n=\{(1-i)(1+i)^n\}=(1-i^2)^n=2^n$

18. We have,

$$(1+i)z=(1-i)\bar{z}$$

$$\Rightarrow \frac{z}{\bar{z}}=\frac{1-i}{1+i}\times\frac{1-i}{1-i}=\frac{1-2i+i^2}{1+i^2}=-i$$

$$\Rightarrow z=-i\bar{z}$$

19. Let $z=x+iy$. Then, $z^2=x^2-y^2+2ixy$ and $|z|=\sqrt{x^2+y^2}$.

$\therefore$ $\text{Re}(z^2)=0$ and $|z|=2$

$\Rightarrow x^2-y^2=0$ and $x^2+y^2=4$

$\Rightarrow x^2=y^2=2$

$\Rightarrow x=\pm\sqrt{2}, y=\pm\sqrt{2}$

$\therefore \quad z = \pm\sqrt{2} \pm \sqrt{2}i$

20. Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

If $\frac{z-1}{z+1}$ is purely imaginary, then

$$\text{Re}\left(\frac{z-1}{z+1}\right) = 0 \Rightarrow \frac{x^2+y-1}{(x+1)^2+y^2} = 0 \Rightarrow x^2+y^2=1 \Rightarrow |z|=1$$

21. Let $z_1 = x + iy$. Then,

$$z_2 = \frac{z_1-1}{z_1+1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\Rightarrow \text{Re}(z_2) = \frac{x^2+y^2-1}{(x+1)^2+y^2} = 0 \qquad [\because |z_1| = 1 \Rightarrow x^2+y^2=1]$$

22. Let $z = x + iy$. Then, $z + 1 = (x+1) + iy$ and $z + 2(1+i) = (x+2) + i(y+2)$

$\therefore \quad |z+1| = z + 2(1+i)$

$\Rightarrow \quad \sqrt{(x+1)^2+y^2} = (x+2) + i(y+2)$

$\Rightarrow \quad \sqrt{(x+1)^2+y^2} = x+2$ and $y+2=0$

$\Rightarrow \quad (x+1)^2+y^2 = (x+2)^2$ and $y=-2$

$\Rightarrow \quad y^2 = 2x+3$ and $y=-2$

$\Rightarrow \quad 4 = 2x+3$ and $y=-2$

$\Rightarrow \quad x = \frac{1}{2}$ and $y=-2$

$\therefore \quad z = x+iy = \frac{1}{2} - 2i$

23. Let $z = x+iy$. Then, $|z| = \sqrt{x^2+y^2}$ and $z+1+2i = (x+1)+i(y+2)$.

$\therefore \quad |z| = z+1+2i$

$\Rightarrow \quad \sqrt{x^2+y^2} = (x+1)+i(y+2)$

$\Rightarrow \quad \sqrt{x^2+y^2} = x+1$ and $y+2=0$

$\Rightarrow \quad x^2+y^2 = (x+1)^2$ and $y=-2$

$\Rightarrow \quad x^2+4 = (x+1)^2$ and $y=-2$

$\Rightarrow \quad x = \frac{3}{2}$ and $y=-2$

Hence, $z = x+iy = \frac{3}{2} - 2i$

24. $(1+i)^{2n} = (1-i)^{2n}$.

$\Rightarrow \quad \{(1+i)^2\}^n = \{(1-i)^2\}^n$

$\Rightarrow \quad (1+2i+i^2)^n = (1-2i+i^2)^n$

$\Rightarrow \quad (2i)^n = (-2i)^n$

$\Rightarrow \quad i^n = (-1)^n i^n$

$\Rightarrow \quad (-1)^n = 1 \Rightarrow n$ is a multiple of 2.

25. Proceed as in Example No. 37.

26. Let $z = x + iy$. Then, $z^2 = x^2 - y^2 + 2ixy$ and $|z|^2 = x^2 + y^2$.

$\therefore \quad z^2 + |z|^2 = 0$

$\Rightarrow \quad x^2 - y^2 + 2i\,xy + x^2 + y^2 = 0$

$\Rightarrow \quad 2x^2 + 2ixy = 0 \Rightarrow 2x^2 = 0$ and $2xy = 0 \Rightarrow x = 0$ and $y \in R$

$\therefore \quad z = 0 + iy$, where $y \in R$.

13.13 SQUARE ROOTS OF A COMPLEX NUMBER

Let $a + ib$ be a complex number such that $\sqrt{a + ib} = x + iy$, where x and y are real numbers. Then,

$$\sqrt{a+ib} = x + iy$$

$$\Rightarrow \quad (a + ib) = (x + iy)^2$$

$$\Rightarrow \quad a + ib = (x^2 - y^2) + 2\,i\,xy$$

On equating real and imaginary parts, we get

$$x^2 - y^2 = a \qquad \text{...(i)}$$

and, $$2\,xy = b \qquad \text{...(ii)}$$

Now, $$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4\,x^2\,y^2$$

$$\Rightarrow \quad (x^2 + y^2)^2 = a^2 + b^2$$

$$\Rightarrow \quad (x^2 + y^2) = \sqrt{a^2 + b^2} \qquad [\because x^2 + y^2 > 0] \qquad \text{...(iii)}$$

Solving the equations (i) and (ii), we get

$$x^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\} \text{ and } y^2 = \frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}} \text{ and } y = \pm\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}}$$

If b is positive, then by equation (ii), x and y are of the same sign.

$$\therefore \quad \sqrt{a + ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}} + i\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}}\right]$$

If b is negative, then by equation (ii), x and y are of different signs.

$$\therefore \quad \sqrt{a + ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} + a\right\}} - i\sqrt{\frac{1}{2}\left\{\sqrt{a^2 + b^2} - a\right\}}\right]$$

REMARK *It is evident from the above discussion that for any complex number z, we have*

(i) $\sqrt{z} = \pm\left\{\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} + i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right\}$, if $\text{Im}(z) > 0$

(ii) $\sqrt{z} = \pm\left\{\sqrt{\dfrac{|z| + \text{Re}(z)}{2}} - i\sqrt{\dfrac{|z| - \text{Re}(z)}{2}}\right\}$, if $\text{Im}(z) < 0$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the square roots of the following:*

(i) $7 - 24i$ (ii) $5 + 12i$

SOLUTION (i) Let $\sqrt{7 - 24\,i} = x + i\,y$. Then,

$$\sqrt{7 - 24\,i} = x + iy$$

$\Rightarrow \quad 7 - 24i = (x + iy)^2$

$\Rightarrow \quad 7 - 24i = (x^2 - y^2) + 2ixy$

$\Rightarrow \quad x^2 - y^2 = 7 \quad$ ...(i) $\quad$ and, $\quad 2xy = -24 \quad$ ...(ii)

Now, $\quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow \quad (x^2 + y^2)^2 = 49 + 576 = 625$

$\Rightarrow \quad x^2 + y^2 = 25 \quad [\because x^2 + y^2 > 0] \quad$ ...(iii)

On solving (i) and (iii), we get

$x^2 = 16$ and $y^2 = 9 \Rightarrow x = \pm 4$ and $y = \pm 3$

From (ii) we observe that $2xy$ is negative. So, x and y are of opposite signs.

$\therefore \quad (x = 4$ and $y = -3)$ or, $(x = -4$ and $y = 3)$

Hence, $\sqrt{7 - 24i} = \pm(4 - 3i)$

<u>ALITER</u> Let $z = 7 - 24i$. Then, $\text{Re}(z) = 7$ and $|z| = \sqrt{49 + 576} = 25$

$$\therefore \quad \sqrt{7 - 24i} = \pm\left\{\sqrt{\frac{|z| + \text{Re}(z)}{2}} - i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right\} \quad [\because \text{Im}(z) < 0]$$

$$\Rightarrow \quad \sqrt{7 - 24i} = \pm\left\{\sqrt{\frac{25 + 7}{2}} - i\sqrt{\frac{25 - 7}{2}}\right\} = \pm(4 - 3i)$$

(ii) Let $\sqrt{5 + 12i} = x + iy$. Then,

$\sqrt{5 + 12i} = x + iy$

$\Rightarrow \quad 5 + 12i = (x + iy)^2$

$\Rightarrow \quad 5 + 12i = (x^2 - y^2) + 2ixy$

$\Rightarrow \quad x^2 - y^2 = 5 \quad$ ...(i) $\quad$ and, $\quad 2xy = 12 \quad$ ...(ii)

Now, $\quad (x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$

$\Rightarrow \quad (x^2 + y^2)^2 = 5^2 + 12^2 = 169$

$\Rightarrow \quad x^2 + y^2 = 13 \quad [\because x^2 + y^2 > 0] \quad$ ...(iii)

On solving (i) and (iii), we get

$x^2 = 9$ and $y^2 = 4 \Rightarrow x = \pm 3$ and $y = \pm 2$

From (ii) we observe that $2xy$ is positive. So, x and y are of the same sign.

$\therefore \quad (x = 3$ and $y = 2)$ or, $(x = -3$ and $y = -2)$

Hence, $\sqrt{5 + 12i} = \pm(3 + 2i)$.

<u>ALITER</u> Let $z = 5 + 12i$. Then, $\text{Re}(z) = 5$, and $|z| = \sqrt{25 + 144} = 13$

$$\therefore \quad \sqrt{5 + 12i} = \pm\left\{\sqrt{\frac{|z| + \text{Re}(z)}{2}} + i\sqrt{\frac{|z| - \text{Re}(z)}{2}}\right\} \quad [\because \text{Im}(z) > 0]$$

$$\Rightarrow \quad \sqrt{5 + 12i} = \pm\left\{\sqrt{\frac{13 + 5}{2}} + i\sqrt{\frac{13 - 5}{2}}\right\} = \pm(3 + 2i)$$

EXAMPLE 2 *Find the square roots of* $-15 - 8i$.

SOLUTION Let $\sqrt{-15 - 8i} = x + iy$. Then,

$\sqrt{-15 - 8i} = x + iy$

$\Rightarrow \quad -15 - 8i = (x + iy)^2$

$\Rightarrow \quad -15-8i = (x^2-y^2)+2i\,xy$

$\Rightarrow \quad -15 = x^2-y^2 \quad \ldots(i) \quad$ and, $\quad 2xy = -8 \quad \ldots(ii)$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = (-15)^2+64 = 289$

$\Rightarrow \quad x^2+y^2 = 17 \quad \ldots(iii)$

On solving (i) and (iii), we get

$$x^2 = 1 \text{ and } y^2 = 16 \Rightarrow x = \pm 1 \text{ and } y = \pm 4$$

From (ii), we observe that $2xy$ is negative. So, x and y are of opposite signs.

$\therefore \quad (x = 1 \text{ and } y = -4)$ or, $(x = -1 \text{ and } y = 4)$

Hence, $\sqrt{-15-8i} = \pm(1-4i)$

EXAMPLE 3 *Find the square root of i.*

SOLUTION Let $\sqrt{i} = x+iy$. Then,

$\quad \sqrt{i} = x+iy$

$\Rightarrow \quad i = (x+iy)^2$

$\Rightarrow \quad (x^2-y^2)+2i\,xy = 0+i$

$\Rightarrow \quad x^2-y^2 = 0 \quad \ldots(i) \quad$ and, $\quad 2xy = 1 \quad \ldots(ii)$

Now, $\quad (x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2$

$\Rightarrow \quad (x^2+y^2)^2 = 0+1 = 1$

$\Rightarrow \quad x^2+y^2 = 1 \quad [\because x^2+y^2>0] \quad \ldots(iii)$

Solving (i) and (iii), we get

$$x^2 = 1/2 \text{ and } y^2 = 1/2 \Rightarrow x = \pm 1/\sqrt{2} \text{ and } y = \pm 1/\sqrt{2}$$

From (ii) we observe that we find that $2xy$ is positive. So, x and y are of same sign.

$$\therefore \quad \left(x = \frac{1}{\sqrt{2}} \text{ and } y = \frac{1}{\sqrt{2}}\right) \text{ or, } \left(x = -\frac{1}{\sqrt{2}} \text{ and } y = -\frac{1}{\sqrt{2}}\right)$$

$$\text{Hence, } \sqrt{i} = \pm\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(1+i)$$

ALITER Let $z = i$. Then, $\text{Re}(z) = 0$ and $|z| = 1$.

$$\therefore \quad \sqrt{i} = \pm\left\{\sqrt{\frac{|z|+\text{Re}(z)}{2}}+i\sqrt{\frac{|z|-\text{Re}(z)}{2}}\right\} \quad [\because \text{Im}(z)>0]$$

$$\Rightarrow \quad \sqrt{i} = \pm\left\{\sqrt{\frac{1+0}{2}}+i\sqrt{\frac{1-0}{2}}\right\} = \pm\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(1+i)$$

EXERCISE 13.3

LEVEL-1

1. Find the square root of the following complex numbers:
   (i) $-5+12i$ (ii) $-7-24i$ (iii) $1-i$
   (iv) $-8-6i$ (v) $8-15i$ (vi) $-11-60\sqrt{-1}$
   (vii) $1+4\sqrt{-3}$ (viii) $4i$ (ix) $-i$

ANSWERS

1. (i) $\pm(2+3i)$ (ii) $\pm(3-4i)$ (iii) $\pm\left\{\left(\sqrt{\frac{\sqrt{2}+1}{2}}\right)-\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right)i\right\}$

(iv) $\pm(1-3i)$ (v) $\pm\frac{1}{\sqrt{2}}(5-3i)$ (vi) $\pm(5-6i)$ (vii) $\pm(2+\sqrt{3}i)$

(viii) $\pm\sqrt{2}(1+i)$ (ix) $\pm\frac{1}{\sqrt{2}}(1-i)$

13.14 REPRESENTATIONS OF A COMPLEX NUMBER

A complex number can be represented in the following forms:

(i) Geometrical form (ii) Vectorial form (iii) Trigonometrical form or, Polar form

In this section, we shall learn about these three representations of a complex number.

13.14.1 GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z=x+iy$ can be represented by a point (x, y) on the plane which is known as the Argand plane. To represent $z=x+iy$ geometrically we take two mutually perpendicular straight lines $X'OX$ and $Y'OY$. Now plot a point whose x and y coordinates are respectively the real and imaginary parts of z. This point $P(x, y)$ represents the complex number $z=x+iy$.

If a complex number is purely real, then its imaginary part is zero. Therefore, a purely real number is represented by a point on x-axis. A purely imaginary complex number is represented by a point on y-axis. That is why x-axis is known as the *real axis* and y-axis, as the imaginary axis.

Conversely, *if* $P(x, y)$ is a point in the plane, then the point $P(x, y)$ represents a complex number $z=x+iy$. The complex number $z=x+iy$ is known as the *affix* of the point P.

Thus, there exists a one-one correspondence between the points of the plane and the members (elements) of the set C of all complex numbers, *i.e.*, for every complex number $z=x+iy$ there exists uniquely a point (x, y) on the plane and for every point (x, y) of the plane there exists uniquely a complex number $z=x+iy$.

The plane in which we represent a complex number geometrically is known as the **complex plane** or **Argand plane** or the **Gaussian plane**. The point P, plotted on the Argand plane, is called the Argand diagram.

The length of the line segment OP is called the *modulus* of z and is denoted by $|z|$.

From Fig. 13.1, we have

$$OP^2 = OM^2 + MP^2$$

$$\Rightarrow \quad OP^2 = x^2+y^2$$

$$\Rightarrow \quad OP = \sqrt{x^2+y^2}$$

Thus, $|z|=\sqrt{x^2+y^2} = \sqrt{\{\text{Re}(z)\}^2+\{\text{Im}(z)\}^2}$

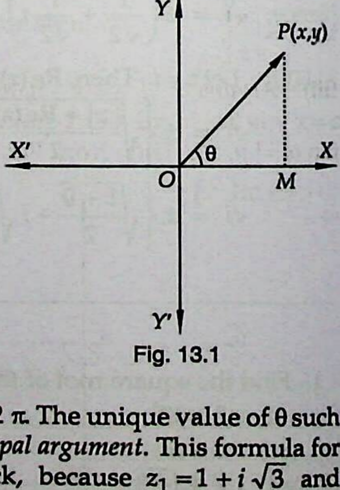

Fig. 13.1

The angle θ which OP makes with positive direction of x-axis in anticlockwise sense is called the *argument or amplitude of* z and is denoted by arg(z) or amp(z).

From Fig. 13.1, we have

$$\tan\theta = \frac{PM}{OM} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)} \Rightarrow \theta = \tan^{-1}\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right)$$

This angle θ has infinitely many values differing by multiples of 2π. The unique value of θ such that $-\pi<\theta\le\pi$ is called the *principal value of the amplitude or principal argument*. This formula for determining the argument of $z=x+iy$ has severe drawback, because $z_1=1+i\sqrt{3}$ and

$z_2 = -1 - i\sqrt{3}$ are two distinct complex numbers represented by two distinct points in the Argand plane but their arguments seem to be $\tan^{-1}\sqrt{3} = \pi/3$ or $4\pi/3$ which is not correct. In fact the argument is the common solution of the simultaneous trigonometric equations

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \text{ and, } \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Since the above system of equations has infinitely many solutions. Therefore, there can be infinitely many arguments of $z = x + iy$. The argument θ which satisfies the inequality $-\pi < \theta \le \pi$ is usually known as the *principal argument* of z. The argument of z depends upon the quadrant in which the point P lies as discussed below.

13.14.2 ARGUMENT OR AMPLITUDE OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

(i) *Argument of $z = x + iy$ when $x > 0$ and $y > 0$:* Since x and y both are positive, therefore the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the first quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 13.2 that $\theta = \alpha$.

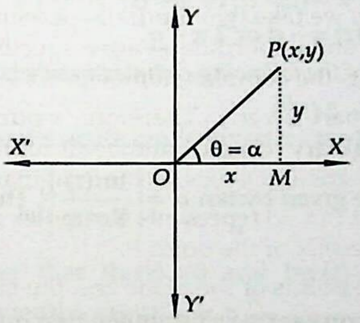

Fig. 13.2

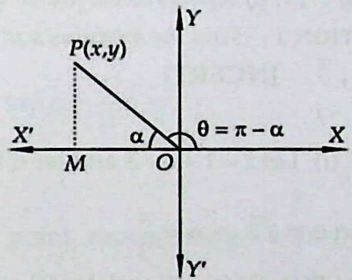

Fig. 13.3

Thus, if x and y both are positive, then the argument of $z = x + iy$ is the acute angle given by $\tan\alpha = \dfrac{y}{x}$.

(ii) *Argument of $z = x + iy$ when $x < 0$ and $y > 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ in the Argand plane lies in the second quadrant. Let θ be the argument of z and let α be the acute angle satisfying $\tan\alpha = |y/x|$. Then it is evident from Fig. 13.3 that $\theta = \pi - \alpha$.

Thus, if $x < 0$ and $y > 0$, then the argument of $z = x + iy$ is $\pi - \alpha$, where α is the acute angle given by $\tan\alpha = \left|\dfrac{y}{x}\right|$.

(iii) *Argument of $z = x + iy$ when $x < 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the third quadrant. Let θ be the argument of z and α be the acute angle given by $\tan\alpha = |y/x|$. Then from Fig. 13.4, we obtain $\theta = -(\pi - \alpha) = \alpha - \pi$.

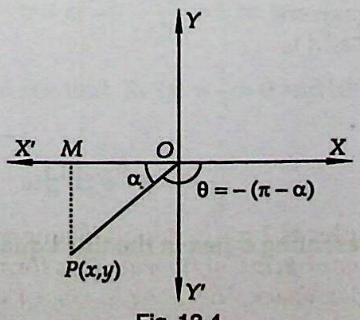

Fig. 13.4

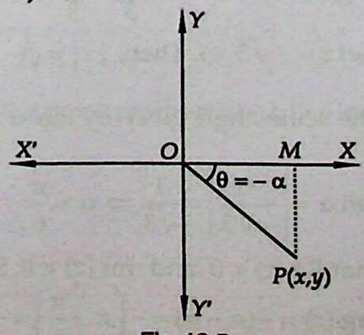

Fig. 13.5

Thus, if $x < 0$ and $y < 0$ then the argument of $z = x + iy$ is $\alpha - \pi$ where α is the acute angle given by $\tan \alpha = |y/x|$.

(iv) *Argument of $z = x + iy$ when $x > 0$ and $y < 0$:* In this case, the point $P(x, y)$ representing $z = x + iy$ lies in the fourth quadrant. Let θ be the argument of z and let α be the acute angle given by $\tan \alpha = |y/x|$. Then from Fig. 13.5, we obtain $\theta = -\alpha$.

Thus, if $x > 0$ and $y < 0$, then the argument of $z = x + iy$ is $-\alpha$ where α is the acute angle given by $\tan \alpha = |y/x|$.

The above discussion suggests us the following algorithm for finding the argument of a complex number $z = x + iy$.

ALGORITHM

STEP I *Find the acute angle α given by $\tan \alpha = |y/x|$.*

STEP II *Determine quadrant in which the point $P(x, y)$ lies.*
If $P(x, y)$ belongs to the first quadrant, then $\arg(z) = \alpha$.
If $P(x, y)$ belongs to the second quadrant, then $\arg(z) = \pi - \alpha$.
If $P(x, y)$ belongs to the third quadrant, the $\arg(z) = -(\pi - \alpha)$ or $\pi + \alpha$.
If $P(x, y)$ belongs to the fourth quadrant, then $\arg(z) = -\alpha$ or $2\pi - \alpha$.

ILLUSTRATION 1 *Find the modulus and argument of each of the following complex numbers:*

(i) $1 + i\sqrt{3}$ [NCERT] (ii) $-2 + 2i\sqrt{3}$

(iii) $-\sqrt{3} - i$ (iv) $2\sqrt{3} - 2i$

SOLUTION (i) Let $z = 1 + i\sqrt{3}$ and let α be the acute angle given by $\tan \alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan \alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

We observe that $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the first quadrant.

$$\therefore \quad \arg(z) = \alpha = \frac{\pi}{3}.$$

Also, $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

(ii) Let $z = -2 + 2\sqrt{3}\, i$. Then, $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$.

Let α be the angle given by $\tan \alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$.

$$\tan \alpha = \left|\frac{2\sqrt{3}}{-2}\right| = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Clearly, $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$. So, the point representing z lies in the second quadrant.

$$\therefore \quad \arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

(iii) Let $z = -\sqrt{3} - i$. Then, $|z| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$.

Let α be the acute angle given by $\tan \alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$.

$$\therefore \quad \tan \alpha = \left|\frac{-1}{-\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

We find that $\text{Re}(z) < 0$ and $\text{Im}(z) < 0$. So, the point representing z lies in the third quadrant.

$$\therefore \quad \arg(z) = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$$

(iv) Let $z = 2\sqrt{3} - 2i$. Then, $|z| = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$.

Let α be the acute angle given by $\tan\alpha \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-2}{2\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$$

We observe that Re $(z) > 0$ and Im $(z) < 0$. So, the point representing z lies in the fourth quadrant.

$\therefore \quad \arg(z) = -\alpha = -\pi/6$

ILLUSTRATION 2 *Find the modulus and argument of the following complex numbers:*

(i) $\frac{1+i}{1-i}$ [NCERT] (ii) $\frac{1}{1+i}$ [NCERT]

SOLUTION (i) Let $z = \frac{1+i}{1-i}$. Then,

$$z = \frac{1+i}{1-i} = \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = i = 0 + i$$

$\therefore \quad |z| = \sqrt{0^2 + 1^2} = 1$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then, $\tan\alpha = \frac{1}{0} = \infty$

$$\tan\alpha = \infty \Rightarrow \alpha = \frac{\pi}{2}$$

We find that Re $(z) = 0$ and Im $(z) = 1 > 0$. So the point representing z lies on y-axis. Consequently, $\arg(z) = \alpha = \frac{\pi}{2}$.

Hence, $|z| = 1$ and $arg(z) = \frac{\pi}{2}$.

(ii) Let $z = \frac{1}{1+i}$. Then,

$$z = \frac{1}{1+i} = \frac{1}{1+i} \times \frac{1-i}{1-i} = \frac{1-i}{1-i^2} = \frac{1-i}{1+1} = \frac{1}{2} - \frac{1}{2}i$$

$$\therefore \quad |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We observe that Re $(z) = \frac{1}{2} > 0$ and Im $(z) = -\frac{1}{2} < 0$. So, the point representing z lies in the fourth quadrant.

$\therefore \quad \arg(z) = -\alpha = -\frac{\pi}{4}$

Hence, $|z| = \frac{1}{\sqrt{2}}$ and $arg(z) = -\frac{\pi}{4}$.

13.14.3 VECTORIAL REPRESENTATION OF A COMPLEX NUMBER

A complex number $z = x + iy$ can be represented by the position vector OP of point $P(x, y)$ in a two dimensional plane because a complex number depends on two things viz. (i) its modulus and (ii) its argument which are also the requirements of a vector on a plane.

In Fig. 13.6, the complex number $z = x + i\,y$ is represented by the vector $\overrightarrow{OP}$ and in such a case $|z|$ is the length OP and arg (z) is the angle which the directed line OP makes with the positive direction of x-axis.

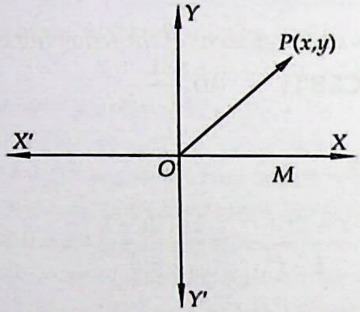

Fig. 13.6

13.14.4 POLAR OR TRIGONOMETRICAL FORM OF A COMPLEX NUMBER

Let $z = x + i\,y$ be a complex number represented by a point $P(x, y)$ in the Argand plane. Then by the geometrical representation of $z = x + i\,y$, we obtain

$$OP = |z| \text{ and, } \angle POX = \theta = \arg(z)$$

In $\Delta\, POM$, we obtain

$$\cos\theta = \frac{OM}{OP} = \frac{x}{|z|} \Rightarrow x = |z|\cos\theta$$

and, $$\sin\theta = \frac{PM}{OP} = \frac{y}{|z|} \Rightarrow y = |z|\sin\theta$$

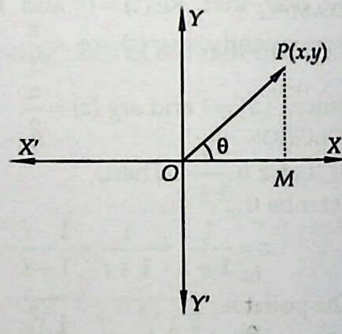

Fig. 13.7

$\therefore$ $z = x + i\,y$

$\Rightarrow$ $z = |z|\cos\theta + i|z|\sin\theta$

$\Rightarrow$ $z = |z|(\cos\theta + i\sin\theta)$

$\Rightarrow$ $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg(z)$

This form of z is called a polar form of z. *If* we use the general value of the argument of θ, then the polar form of z is given by

$$z = r\,[\cos(2n\pi + \theta) + i\sin(2n\pi + \theta)], \text{ where } r = |z|,\ \theta = \arg(z) \text{ and } n \text{ is an integer.}$$

13.14.5 MULTIPLICATION OF A COMPLEX NUMBER BY IOTA

Let $z = x + i\,y$ be a complex number represented by a point $P(x, y)$ in the argand plane. Let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\arg(z) = \theta$.

Now, $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = i\,r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = r(-\sin\theta + i\cos\theta)$

$\Rightarrow$ $iz = r\{\cos(\pi/2 + \theta) + \sin(\pi/2 + \theta)\}$

This shows that iz is a complex number such that

$$|iz| = r = |z| \text{ and } \arg(iz) = \pi/2 + \theta = \pi/2 + \arg(z).$$

Thus, multiplication of a complex number by i results in rotating the vector joining the origin to point representing z through a right angle.

13.14.6 POLAR FORM OF A COMPLEX NUMBER FOR DIFFERENT SIGNS OF REAL AND IMAGINARY PARTS

Let $|z| = r$ and α be the acute angle given by $\tan\alpha = \left|\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$. Let θ be the argument of z.

CASE I *Polar form of $z = x + iy$ when $x > 0$ and $y > 0$:* In this case, we have $\theta = \alpha$.
So, the polar form of $z = x + iy$ is $z = r(\cos\alpha + i\sin\alpha)$

CASE II *Polar form of $z = x\ iy$ when $x < 0$ and $y > 0$:* In this case, we have $\theta = \pi - \alpha$.
So, the polar form of $z = x + iy$ is

$$z = r[\cos(\pi - \alpha) + i\sin(\pi - \alpha)] = r(-\cos\alpha + i\sin\alpha)$$

CASE III *Polar form of $z = x + iy$ when $x < 0$ and $y < 0$:* In this case, we have $\theta = -(\pi - \alpha)$.
So, the polar form of z is given by

$$z = r[\cos(\pi - \alpha) + i\sin(-(\pi - \alpha))] = r(-\cos\alpha - i\sin\alpha)$$

CASE IV *Polar form of $z = x + iy$ when $x > 0$ and $y < 0$:* In this case, we have $\theta = -\alpha$.
So, the polar form of z is

$$z = r[\cos(-\alpha) + i\sin(-\alpha)] = r(\cos\alpha - i\sin\alpha)$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write the following complex numbers in the polar form:*

(i) $-3\sqrt{2} + 3\sqrt{2}\,i$ (ii) $1 + i$

(iii) $-1 - i$ [NCERT] (iv) $1 - i$ [NCERT EXEMPLAR]

SOLUTION (i) Let $z = -3\sqrt{2} + 3\sqrt{2}i$. Then, $r = |z| = \sqrt{(-3\sqrt{2})^2 + (3\sqrt{2})^2} = 6$.

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$. Then,

$$\tan\alpha = 1 \Rightarrow \alpha = \pi/4$$

The point representing z lies in the second quadrant. So, the argument θ of z is given by

$$\theta = \pi - \alpha = \pi - (\pi/4) = 3\pi/4.$$

Hence, the polar form of $z = -3\sqrt{2} + 3\sqrt{2}\,i$ is

$$z = r(\cos\theta + i\sin\theta) = 6\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

(ii) Let $z = 1 + i$. Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right|$. Then,

$$\tan\alpha = \frac{1}{1} = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that the point (1, 1) representing z lies in first quadrant. Therefore, the argument of z is given by $\theta = \alpha = \dfrac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(iii) Let $z = -1 - i$. Then, $r = |z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Clearly, the point $(-1, -1)$ representing z lies in the third quadrant. Therefore, the argument of z is given by

$$\theta = -(\pi - \alpha) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}.$$

Hence, the polar form of $z = -1 - i$ is

$$z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right)$$

(iv) Let $z = 1 - i$. Then, $|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

We find that the point $(1, -1)$ representing z lies in the fourth quadrant. Therefore, the argument of z is given by $\theta = -\alpha = -\frac{\pi}{4}$.

Hence, the polar form of $z = 1 - i$ is

$$r(\cos\theta + i\sin\theta) = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\} = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

EXAMPLE 2 *Find the modulus and principal argument of* $(1 + i)$ *and hence express it in the polar form.* **[NCERT]**

SOLUTION Let $z = 1 + i$. Then, $|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}.$$

Clearly, the point $(1, 1)$ representing $z = 1 + i$ lies in first quadrant. Therefore, $\theta = \arg(z) = \frac{\pi}{4}$.

Hence, the polar form of $z = 1 + i$ is $z = |z|(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$.

EXAMPLE 3 *Find the modulus and principal argument of* $-2i$.

SOLUTION Let $z = -2i = 0 + (-2)i$. Then, $|z| = \sqrt{0 + (-2)^2} = 2$.

Clearly, the point $(0, -2)$ representing $z = -2i$ lies on the negative side of imaginary axis. Therefore, principal argument of is $-\frac{\pi}{2}$.

EXAMPLE 4 *Find the modulus and principal argument of* -4. **[NCERT]**

SOLUTION Let $z = -4 + 0i$. Then, $|z| = \sqrt{(-4)^2 + 0} = 4$.

Clearly, the point $(-4, 0)$ representing $z = -4 + 0i$ lies on the negative side of real axis. Therefore, principal argument of z is π.

EXAMPLE 5 *Express the following complex numbers in the polar form:*

(i) $\dfrac{1+i}{1-i}$ (ii) $\dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$

SOLUTION (i) Let $z = \dfrac{1+i}{1-i}$. and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Now, $z = \dfrac{1+i}{1-i} = \dfrac{(1+i)(1+i)}{(1-i)(1+i)} = \dfrac{1-2i+i^2}{1-i^2} = \dfrac{1+2i-1}{1+1} = i = 0 + 1\,i.$

$\therefore \quad r = |z| = \sqrt{0+1} = 1.$

Clearly, the point $(0, 1)$ representing $z = 0 + i$ lies on positive direction of imaginary axis. Therefore, $\arg(z) = \pi/2$.

Hence, the polar form of z is $z = 1\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$

(ii) Let $z = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i}$ and, let $r(\cos\theta + i\sin\theta)$ be the polar form of z. Then, $r = |z|$ and $\theta = \arg(z)$.

Clearly, $z = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} = \dfrac{2+6\sqrt{3}\,i}{5+\sqrt{3}\,i} \cdot \dfrac{(5-\sqrt{3}\,i)}{(5-\sqrt{3}\,i)} = \dfrac{28+28\sqrt{3}\,i}{28} = 1 + i\sqrt{3}$

$\therefore \quad r = |z| = \sqrt{1+3} = 2.$

Let α be acute angle given by

$\tan\alpha \dfrac{|\text{Im}(z)|}{|\text{Re}(z)|}$. Then,

$\tan\alpha = \dfrac{\sqrt{3}}{1} = \sqrt{3} \Rightarrow \alpha = \dfrac{\pi}{3}$

Clearly, the point $(1, \sqrt{3})$ representing z lies in first quadrant. Therefore, $\theta \arg(z) = \alpha = \pi/3$.

Hence, the polar form of z is $2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$.

EXAMPLE 6 *Put the complex number* $\dfrac{1+7i}{(2-i)^2}$ *in the form* $r(\cos\theta + i\sin\theta)$, *where* r *is a positive real number and* $-\pi < \theta \le \pi$. **[NCERT]**

SOLUTION Let $z = \dfrac{1+7i}{(2-i)^2}$. Then,

$$z = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{3-4i} = \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{-25+25i}{25} = -1+i$$

$\therefore \quad r = |z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

Let α be the acute angle given by $\tan\alpha = \left|\dfrac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|-\frac{1}{1}\right| = 1 \Rightarrow \alpha = \pi/4$$

Clearly, the point $(-1, 1)$ representing z lies in the second quadrant. Therefore,

$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$

Hence, z in the polar form is given by $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

EXAMPLE 7 *Find the modulus and argument of the following complex numbers and convert them in polar form:*

(i) $\frac{1+2i}{1-3i}$ **[NCERT]** (ii) $\frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$ **[NCERT]**

(iii) $\frac{1+3i}{1-2i}$ **[NCERT]**

SOLUTION (i) Let $z = \frac{1+2i}{1-3i}$. Then,

$$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{(1-6) + i(2+3)}{1+9} = -\frac{1}{2} + \frac{1}{2}i$$

$$\therefore \quad r = |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$$

Let α be the acute angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{-1/2}{1/2}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that $\text{Re}(z) = -\frac{1}{2} < 0$ and $\text{Im}(z) = \frac{1}{2} > 0$. So, the point representing z lies in the second quadrant.

$\therefore \quad \theta = \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

(ii) Let $z = \frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$. Then,

$$z = \frac{i-1}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \frac{2(-1+i)}{1+i\sqrt{3}} \times \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{2\{(-1+\sqrt{3}) + i(1+\sqrt{3})\}}{1+3} = \left(\frac{\sqrt{3}-1}{2}\right) + i\left(\frac{\sqrt{3}+1}{2}\right)$$

$$\therefore \quad |z| = \sqrt{\left(\frac{\sqrt{3}-1}{2}\right)^2 + \left(\frac{\sqrt{3}+1}{2}\right)^2} = \sqrt{\frac{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2}{4}} = \sqrt{\frac{2(3+1)}{4}} = \sqrt{2}$$

Let α be the acture angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right| = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{1+\frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}} = \frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \tan\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \tan\frac{5\pi}{12}$$

$\therefore \qquad \alpha = \frac{5\pi}{12}$

Clearly, Re (z) $= \frac{\sqrt{3}-1}{2} > 0$ and, Im (z) $= \frac{\sqrt{3}+1}{2} > 0$. So, the point representing z lies in the first quadrant. Therefore, $\theta = \arg(z) = \frac{5\pi}{12}$.

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

(iii) Let $z = \frac{1+3i}{1-2i}$. Then,

$$z = \frac{1+3i}{1-2i} = \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{(1-6)+i(3+2)}{1+4} = -1+i$$

$$\therefore \qquad r = |z| = \sqrt{(-1^2 + 1^2} = \sqrt{2}$$

Let α be the acture angle given by $\tan\alpha = \left|\frac{\text{Im}(z)}{\text{Re}(z)}\right|$. Then,

$$\tan\alpha = \left|\frac{1}{-1}\right| = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

We find that Re (z) < 0 and Im (z) > 0. So, the point representing z lies in the second quadrant.

$$\therefore \qquad \arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Hence, the polar form of z is $r(\cos\theta + i\sin\theta) = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

LEVEL-2

EXAMPLE 8 *For any complex number z, prove that* $|\text{Re}(z)| + |\text{Im}(z) \le \sqrt{2}|z|$.

SOLUTION Let $z = r(\cos\theta + i\sin\theta)$. Then, $|z| = r$ and $\arg(z) = \theta$.

Now,

$$|\text{Re}(z)| + \text{Im}(z)| = |r\cos\theta| + |r\sin\theta|$$

$$\Rightarrow \quad |\text{Re}(z)| + \text{Im}(z)| = r\left\{|\cos\theta| + |\sin\theta|\right\} \qquad [\because r = |z| > 0]$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{|\cos\theta| + |\sin\theta|\right\}^2$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{1 + |2\sin\theta\cos\theta|\right\}$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 = r^2\left\{1 + |\sin 2\theta|\right\}$$

$$\Rightarrow \quad \left\{|\text{Re}(z) + |\text{Im}(z)|\right\}^2 \le r^2(1+1) \qquad [\because |\sin 2\theta| \le 1]$$

$$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}\, r$$

$$\Rightarrow \quad |\text{Re}(z)| + |\text{Im}(z)| \le \sqrt{2}|z|$$

EXAMPLE 9 *If z and w are two complex number such that* $|zw| = 1$ *and* $\arg(z) - \arg(w) = \frac{\pi}{2}$, *then show that* $\bar{z}w = -i$. **[NCERT EXEMPLAR]**

SOLUTION Let $|z| = r$ and $\arg(z) = \theta$. Then, $z = r(\cos\theta + \sin\theta)$.

Now, $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$

$\Rightarrow \quad |z|\,|w| = 1$ and $\arg(w) = \arg(z) - \dfrac{\pi}{2}$

$\Rightarrow \quad |w| = \dfrac{1}{r}$ and $\arg(w) = \theta - \dfrac{\pi}{2}$

Thus, $\quad w = |w|\,\{\cos(\arg w) + i\sin(\arg w)\}$

$\Rightarrow \quad w = \dfrac{1}{r}\left\{\cos\left(\theta - \dfrac{\pi}{2}\right) + i\sin\left(\theta - \dfrac{\pi}{2}\right)\right\}$

$\Rightarrow \quad w = \dfrac{1}{r}\{\sin\theta - i\cos\theta\} = -\dfrac{i}{r}(\cos\theta + i\sin\theta)$

$\therefore \quad \bar{z}w = r(\cos\theta - i\sin\theta) \times -\dfrac{i}{r}(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \bar{z}w = -i(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad \bar{z}w = -i.$

EXAMPLE 10 *What is the locus of z, if amplitude of $(z - 2 - 3i)$ is $\dfrac{\pi}{4}$?*

SOLUTION Let $z = x + iy$. Then,

$$z - 2 - 3i = (x + iy) - 2 - 3i = (x + 2) + i(y - 3)$$

Let θ be the amplitude of $(x - 2) + i(y - 3)$. Then,

$$\tan\theta = \frac{y-3}{x-2}$$

$\Rightarrow \quad \tan\dfrac{\pi}{4} = \dfrac{y-3}{x-2}$ $\qquad \left[\because \theta = \dfrac{\pi}{4}\right]$

$\Rightarrow \quad 1 = \dfrac{y-3}{x-2}$

$\Rightarrow \quad x - y + 1 = 0$, which is a straight line.

Hence, the locus of z is a straight line.

EXAMPLE 11 *Show that the complex number z, satisfying $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$ lies on a circle.*

[NCERT EXEMPLAR]

SOLUTION Let $z = x + iy$. Then,

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2-1+y^2)+2iy}{(x+1)^2+y^2}$$

$\Rightarrow \quad \dfrac{z-1}{z+1} = \left\{\dfrac{x^2+y^2-1}{(x+1)^2+y^2}\right\} + i\left\{\dfrac{2y}{(x+1)^2+y^2}\right\}$

Let θ be the argument of $\dfrac{z-1}{z+1}$. Then,

$$\tan\theta = \frac{2y/(x+1)^2+y^2}{(x^2+y^2-1)/(x+1)^2+y^2} = \frac{2y}{x^2+y^2-1}$$

But, it is given that $\arg\left(\dfrac{z-1}{z+1}\right)$ is $\dfrac{\pi}{4}$ i.e. $\theta = \dfrac{\pi}{4}$.

$\therefore \quad \tan\dfrac{\pi}{4} = \dfrac{2y}{x^2+y^2-1}$

$\Rightarrow \quad x^2 + y^2 - 1 = 2y$

$\Rightarrow \quad x^2 + y^2 - 2y - 1 = 0$

$\Rightarrow \quad (x-0)^2+(y-1)^2=(\sqrt{2})^2$, which represents a circle.

EXAMPLE 12 *If arg (z − 1) = arg (z + 3i), then find (x − 1) : y, where z = x + iy.*

SOLUTION We have, $z = x + iy$. **[NCERT EXEMPLAR]**

$\therefore \quad z-1=(x-1)+iy$ and $z+3i=x+i(y+3)$

Let θ_1 and θ_2 be the arguments of $z-1$ and $z+3i$. Then,

$$\tan\theta_1=\frac{y}{x-1} \text{ and } \tan\theta_2=\frac{y+3}{x}$$

It is given that $\arg(z-1)=\arg(z+3i)$ i.e. $\theta_1=\theta_2$.

$\therefore \quad \tan\theta_1=\tan\theta_2$

$\Rightarrow \quad \dfrac{y}{x-1}=\dfrac{y+3}{x}$

$\Rightarrow \quad 3x-y-3=0$

$\Rightarrow \quad 3(x-1)=y$

$\Rightarrow \quad \dfrac{x-1}{y}=\dfrac{1}{3}$

$\Rightarrow \quad (x-1):y=1:3$

EXAMPLE 13 *If for complex numbers z_1 and z_2, arg (z_1) − arg $(z_2)=0$, then show that $|z_1-z_2|=||z_1|-|z_2||$.* **[NCERT EXEMPLAR]**

SOLUTION Let $|z|=r_1$ and $|z_2|=r_2$. It is given that $\arg(z_1)-\arg(z_2)=0$

i.e. $\quad \arg(z_1)=\arg(z_2)=\theta$ (say)

$\therefore \quad z_1=r_1(\cos\theta+i\sin\theta)$ and $z_2=r_2(\cos\theta+i\sin\theta)$

$\Rightarrow \quad z_1-z_2=(r_1-r_2)\cos\theta+i(r_1-r_2)\sin\theta$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2\cos^2\theta+(r_1-r_2)^2\sin^2\theta$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2(\cos^2\theta+\sin^2\theta)$

$\Rightarrow \quad |z_1-z_2|^2=(r_1-r_2)^2$

$\Rightarrow \quad |z_1-z_2|=|r_1-r_2|$

$\Rightarrow \quad |z_1-z_2|=||z_1|-|z_2||$

EXAMPLE 14 *If z, z_1 and z_2 are complex numbers, prove that:*

(i) $\arg(\bar{z})=-\arg(z)$. In general, $\arg(\bar{z})=2n\pi-\arg(z)$

(ii) $\arg(z_1 z_2)=\arg(z_1)+\arg(z_2)$

(iii) $\arg(z_1\bar{z_2})=\arg(z_1)-\arg(z_2)$

(iv) $\arg\left(\dfrac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$

SOLUTION (i) Let $z=r(\cos\theta+i\sin\theta)$ be the polar form of z. Then, $|z|=r$ and $\arg(\bar{z})=\theta$.

Now, $\quad z=r(\cos\theta\, i+i\sin\theta)$

$\Rightarrow \quad \bar{z}=r(\cos\theta-i\sin\theta)$

$\Rightarrow \quad \bar{z}=r\{\cos(-\theta)+i\sin(-\theta)\}$

$\Rightarrow \quad |\bar{z}|=r$ and $\arg(\bar{z})=-\theta$

Since $\cos\theta$ and $\sin\theta$ are periodic functions with period 2π. Therefore, in general

$$\arg(\bar{z})=2n\pi-\arg(z)$$

(ii) Let $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$ be two complex numbers in their polar forms. Then,

$$|z_1|=r_1, |z_2|=r_2, \arg(z_1)=\theta_1 \text{ and } \arg(z_2)=\theta_2$$

$\therefore \quad z_1 z_2 = r_1 (\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\}$

$\Rightarrow \quad z_1 z_2 = r_1 r_2 \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}$

$\Rightarrow \quad |z_1 z_2| = r_1 r_2$ and, $\arg(z_1 z_2) = \theta_1 + \theta_2$

$\Rightarrow \quad |z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

<u>REMARK</u> *It follows from the above result that*

$$|z_1 z_2 \ldots z_n| = |z_1||z_2|\ldots|z_n|$$

and, $\quad \arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n)$

Replacing $z_1, z_2, z_3, \ldots, z_n$ by z, we get

$$|z^n| = |z|^n \text{ and } \arg(z^n) = n\arg(z)$$

(iii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$\overline{z_2} = r_2(\cos\theta_2 - i\sin\theta_2) = r_2\{\cos(-\theta_2) + i\sin(-\theta_2)\}$

$\therefore \quad z_1\overline{z_2} = r_1 r_2[\cos\{\theta_1 + (-\theta_2)\} + i\sin\{\theta_1 + (-\theta_2)\}]$ [Using (ii)]

$\Rightarrow \quad z_1\overline{z_2} = r_1 r_2\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1-\theta_2)\}$

$\Rightarrow \quad \arg(z_1\overline{z_2}) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

(iv) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$|z_1| = r_1, |z_2| = r_2, \arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$$\therefore \quad \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right\}$$

$$\Rightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\left\{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right\}$$

$$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$$

EXAMPLE 15 *Let* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ *and* $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ *be two complex numbers. Then, prove that*

(i) $|z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2\cos(\theta_1 - \theta_2)$

or, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1-\theta_2)$

(ii) $|z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_1 - \theta_2)$

or, $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1-\theta_2)$

SOLUTION We have,

$z_1 = r_1\cos\theta_1 + i\sin\theta_1$ and, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore \quad |z_1| = r_1, |z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

(i) We have,

$z_1 + z_2 = (r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)$

$\therefore \quad |z_1 + z_2|^2 = (r_1 \cos\theta_1 + r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 + r_2 \sin\theta_2)^2$

$\Rightarrow \quad |z_1 + z_2|^2 = r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

(ii) We have,

$z_1 - z_2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i(r_1 \sin\theta_1 - r_2 \sin\theta_2)$

$\therefore \quad |z_1 - z_2|^2 = (r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2$

$\Rightarrow \quad |z_1 - z_2|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \cos(\theta_1 - \theta_2)$

EXAMPLE 16 *For any two complex numbers* z_1 and z_2 *, prove that :*

(i) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2} \Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary.*

(ii) $|z_1 + z_2| = |z_1||z_2| \Leftrightarrow \arg(z_1) = \arg(z_2) \Leftrightarrow \frac{z_1}{z_2}$ *is purely real.*

(iii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ *is purely imaginary*

SOLUTION Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then, $|z_1| = r_1$, $|z_2| = r_2$, $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$.

(i) We have,

$|z_1 + z_2| = |z_1 - z_2|$

$\Leftrightarrow \quad |z_1 + z_2|^2 = |z_1 - z_2|^2$

$\Leftrightarrow \quad r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$

$\Leftrightarrow \quad 4r_1 r_2 \cos(\theta_1 - \theta_2) = 0$

$\Leftrightarrow \quad \cos(\theta_1 - \theta_2) = 0$

$\Leftrightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$ i.e. $\arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$

$\Leftrightarrow \quad \frac{z_1}{z_2}$ is purely imaginary.

(ii) We have,

$|z_1 + z_2| = |z_1| + |z_2|$

$\Leftrightarrow \quad |z_1 + z_2|^2 = (r_1 + r_2)^2 \qquad [\because |z_1| = r_1 \text{ and } |z_2| = r_2]$

$\Leftrightarrow \quad r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2) = r_1^2 + r_2^2 + 2r_1 r_2$

$\Leftrightarrow \quad \cos(\theta_1 - \theta_2) = 1$

$\Leftrightarrow \quad \theta_1 - \theta_2 = 0$ i.e. $\arg(z_1) - \arg(z_2) = 0$ or, $\arg(z_1) = \arg(z_2)$

$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = 0 \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\right]$

$\Leftrightarrow \quad \frac{z_1}{z_2}$ is purely real

(iii) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2$$

$$\Leftrightarrow \quad r_1^{\,2}+r_2^{\,2}+2r_1r_2\cos(\theta_1-\theta_2) = r_1^{\,2}+r_2^{\,2}$$

$$\Leftrightarrow \quad 2r_1r_2\cos(\theta_1-\theta_2) = 0$$

$$\Leftrightarrow \quad \cos(\theta_1-\theta_2) = 0$$

$$\Leftrightarrow \quad \theta_1-\theta_2 = \frac{\pi}{2}$$

$$\Leftrightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} \qquad \left[\because \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1)-\arg(z_2) = \theta_1-\theta_2\right]$$

$\Leftrightarrow$ $\frac{z_1}{z_2}$ is purely imaginary.

EXAMPLE 17 *For any two complex numbers z_1 and z_2, prove that:*

(i) $|z_1+z_2| \le |z_1|+|z_2|$

ii) $|z_1-z_2| \le |z_1|+|z_2|$ *(Triangle inequalities)*

) $|z_1+z_2| \ge |z_1|-|z_2|$

) $|z_1-z_2| \ge |z_1|-|z_2|$

JTION (i) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$$\cos(\theta_1-\theta_2) \le 1$$

$$\Rightarrow \quad 2|z_1||z_2|\cos(\theta_1-\theta_2) \le 2|z_1||z_2| \qquad \text{[Multiplying both sides by } 2|z_1||z_2|]$$

$$\Rightarrow \quad |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2) \le |z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\Rightarrow \quad |z_1+z_2|^2 \le \left(|z_1|+|z_2|\right)^2$$

$$\Rightarrow \quad |z_1+z_2| \le |z_1|+|z_2|$$

(ii) We have,

$$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$$\because \quad -1 \le \cos(\theta_1-\theta_2) \le 1$$

$$\therefore \quad -1 \le -\cos(\theta_1-\theta_2) \le 1$$

$$\Rightarrow \quad -\cos(\theta_1-\theta_2) \le 1$$

$$\Rightarrow \quad -2|z_1||z_2|\cos(\theta_1-\theta_2) \le 2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2) \le |z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\Rightarrow \quad |z_1-z_2|^2 \le \left(|z_1|+|z_2|\right)^2$$

$$\Rightarrow \quad |z_1-z_2| \le |z_1|+|z_2|$$

(iii) We have,

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$$

$$\because \quad -1 \le \cos(\theta_1-\theta_2) \le 1$$

$$\Rightarrow \quad \cos(\theta_1-\theta_2) \ge -1$$

$$\Rightarrow \quad 2|z_1||z_2|\cos(\theta_1-\theta_2) \ge -2|z_1||z_2|$$

$$\Rightarrow \quad |z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2) \ge |z_1|^2+|z_2|^2-2|z_1||z_2|$$

$\Rightarrow \quad |z_1+z_2|^2 \geq \left(|z_1|-|z_2|\right)^2$

$\Rightarrow \quad |z_1+z_2| > |z_1|-|z_2|$

(iv) We have,

$|z_1-z_2|^2 = |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

$\therefore \quad -1 \leq \cos(\theta_1-\theta_2) \leq 1$

$\Rightarrow \quad \cos(\theta_1-\theta_2) \leq 1$

$\Rightarrow \quad -\cos(\theta_1-\theta_2) \geq -1$

$\Rightarrow \quad -2|z_1||z_2|\cos(\theta_1-\theta_2) \geq -2|z_1||z_2|$

$\Rightarrow \quad |z_1|^2+|z_2|^2-2|z_1||z_2|\cos(\theta_1-\theta_2) \geq |z_1|^2+|z_2|^2-2|z_1||z_2|$

$\Rightarrow \quad |z_1+z_2|^2 \geq \left(|z_1|-|z_2|\right)^2$

$\Rightarrow \quad |z_1-z_2| \geq |z_1|-|z_2|$

EXAMPLE 18 *If* $z_r = \cos\left(\frac{\pi}{3^r}\right)+i\sin\left(\frac{\pi}{3^r}\right), r=1, 2, 3, \ldots,$ *prove that* $z_1 z_2 z_3 \ldots z_\infty = i.$

SOLUTION We know that, if $z_1 = r_1(\cos\theta_1+i\sin\theta_1), z_2 = r_2(\cos\theta_2+i\sin\theta_2), \ldots$ are complex numbers, then

$$z_1 z_2 z_3 \ldots z_n = r_1 r_2 r_3 \ldots r_n \{\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n)\}$$

Here, $z_r = \cos\frac{\pi}{3^r}+i\sin\frac{\pi}{3^r}, r=1, 2, 3, \ldots$

$\therefore \quad |z_r| = \sqrt{\cos^2\frac{\pi}{3^r}+\sin^2\frac{\pi}{3^r}} = 1, r=1, 2, 3, \ldots$ and, $\arg(z_r) = \frac{\pi}{3^r}, r=1, 2, 3, \ldots$

$$\therefore \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\ldots\frac{\pi}{3^n}\right\}+i\sin\left\{\frac{\pi}{3}+\frac{\pi}{3^2}+\frac{\pi}{3^3}+\ldots+\frac{\pi}{3^n}\right\}$$

$$\Rightarrow \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\frac{\pi}{3}\left(1-\frac{1}{3^n}\right)}{\left(1-\frac{1}{3}\right)}\right\}+i\sin\left\{\frac{\frac{\pi}{3}\left(1-\frac{1}{3^n}\right)}{\left(1-\frac{1}{3}\right)}\right\}$$

$$\Rightarrow \quad z_1 z_2 z_3 \ldots z_n = \cos\left\{\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right\}+i\sin\left\{\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right\}$$

Hence, $z_1 z_2 z_3 \ldots z_\infty = \lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n)$

$$= \lim_{n\to\infty}\left[\cos\left\{\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right\}+i\sin\left\{\frac{\pi}{2}\left(1-\frac{1}{3^n}\right)\right\}\right]$$

$$= \cos\frac{\pi}{2}+i\sin\frac{\pi}{2} = i \qquad \left[\because \lim_{n\to\infty}\frac{1}{3^n}=0\right]$$

EXAMPLE 19 *If* $x_n = \cos\frac{\pi}{2^n}+i\sin\frac{\pi}{2^n}$, *prove that* $x_1 x_2 x_3 \ldots x_\infty = -1.$

SOLUTION We have,

$$x_1 x_2 \ldots x_n = \left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^2}+i\sin\frac{\pi}{2^3}\right)\ldots\left(\cos\frac{\pi}{2^n}+i\sin\frac{\pi}{2^n}\right)$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\dots+\frac{\pi}{2^n}\right\}+i\sin\left\{\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3}+\dots+\frac{\pi}{2^n}\right\}$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\frac{\frac{\pi}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}\right\}+i\sin\left\{\frac{\frac{\pi}{2}\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)}\right\}$$

$$\Rightarrow \quad x_1 x_2 \dots x_n = \cos\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}+i\sin\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}$$

$$\therefore \quad x_1 x_2 x_3 \dots x_\infty = \lim_{n\to\infty}(x_1 x_2 x_3 \dots x_n) = \lim_{n\to\infty} \cos\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}+i\sin\left\{\pi\left(1-\frac{1}{2^n}\right)\right\}$$

$$= \cos\pi + i\sin\pi = -1$$

EXAMPLE 20 *Let z_1 and z_2 be two complex numbers such that $\overline{z}_1 + i\,\overline{z}_2 = 0$ and $\arg(z_1 z_2) = \pi$. Then, find $\arg(z_1)$.* **[NCERT EXEMPLAR]**

SOLUTION It is given that

$$\overline{z}_1 + i\,\overline{z}_2 = 0$$

$$\Rightarrow \quad \overline{z}_1 = -i\,\overline{z}_2$$

$$\Rightarrow \quad \overline{\left(\overline{z}_1\right)} = \overline{\left(-i\overline{z}_2\right)} \qquad \text{[Taking conjugate of both sides]}$$

$$\Rightarrow \quad \overline{\left(\overline{z}_1\right)} = \overline{\left(-i\overline{z}_2\right)}$$

$$\Rightarrow \quad z_1 = i z_2$$

$$\Rightarrow \quad z_2 = -i z_1$$

$$\Rightarrow \quad \arg(z_2) = \arg(-i z_1)$$

$$\Rightarrow \quad \arg(z_2) = \arg(-i) + \arg(z_1)$$

$$\Rightarrow \quad \arg(z_2) = -\frac{\pi}{2} + \arg(z_1) \qquad \dots(i)$$

It is also given that

$$\arg(z_1 z_2) = \pi$$

$$\Rightarrow \quad \arg(z_1) + \arg(z_2) = \pi$$

$$\Rightarrow \quad \arg(z_1) - \frac{\pi}{2} + \arg(z_1) = \pi \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 2\arg(z_1) = \frac{3\pi}{2}$$

$$\Rightarrow \quad \arg(z_1) = \frac{3\pi}{4}$$

EXAMPLE 21 *If z_1 and z_2 both satisfy $z + \overline{z} = 2|z-1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, then find $\mathrm{Im}(z_1 + z_2)$.* **[NCERT EXEMPLAR]**

SOLUTION Let $z_1 = x_1 + iy$ and $z_2 = x_2 + i\,y_2$.

It is given that z_1 and z_2 satisfy $z + \overline{z} = 2|z-1|$.

$$\therefore \quad z_1 + \overline{z}_1 = 2|z_1 - 1| \text{ and } z_2 + \overline{z}_2 = 2|z_2 - 1|$$

$$\Rightarrow \quad 2x_1 = 2|(x_1 - 1) + i\,y_1| \text{ and } 2x_2 = 2|(x_2 - 1) + i\,y_2|$$

$$\Rightarrow \quad x_1 = \sqrt{(x_1-1)^2 + y_1^2} \quad \text{and } x_2 = \sqrt{(x_2-1)^2 + y_2^2}$$

$\Rightarrow \quad x_1^2 = (x_1 - 1)^2 + y_1^2$ and $x_2^2 = (x_2 - 1)^2 + y_2^2$

$\Rightarrow \quad 2x_1 = 1 + y_1^2$ ...(i) and $2x_2 = 1 + y_2^2$ ...(ii)

$\Rightarrow \quad 2(x_1 - x_2) = y_1^2 - y_2^2$ [Subtracting (ii) from (i)]

$\Rightarrow \quad 2\left(\dfrac{x_1 - x_2}{y_1 - y_2}\right) = y_1 + y_2$ ...(iii)

Now, $z_1 = x_1 + i\,y_1$ and $z_2 = x_2 + i\,y_2$

$\Rightarrow \quad z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

It is given that $\arg(z_1 - z_2) = \dfrac{\pi}{4}$.

$$\arg(z_1 - z_2) = \frac{\pi}{4}$$

$\therefore \quad \tan\dfrac{\pi}{4} = \dfrac{y_1 - y_2}{x_1 - x_2}$

$\Rightarrow \quad 1 = \dfrac{y_1 - y_2}{x_1 - x_2}$

$\Rightarrow \quad \dfrac{x_1 - x_2}{y_1 - y_2} = 1$ ...(iv)

From (iii) and (iv), we obtain

$$2 = y_1 + y_2 \Rightarrow \text{Im}(z_1 + z_2) = 2.$$

EXERCISE 13.4

LEVEL-1

1. Find the modulus and argument of the following complex numbers and hence express each of them in the polar form:
   (i) $1 + i$
   (ii) $\sqrt{3} + i$ [NCERT]
   (iii) $1 - i$ [NCERT]
   (iv) $\dfrac{1 - i}{1 + i}$
   (v) $\dfrac{1}{1 + i}$
   (vi) $\dfrac{1 + 2i}{1 - 3i}$
   (vii) $\sin 120° - i \cos 120°$
   (viii) $\dfrac{-16}{1 + i\sqrt{3}}$ [NCERT]
2. Write $(i^{25})^3$ in polar form. [NCERT EXEMPLAR]

LEVEL-2

3. Express the following complex numbers in the form $r(\cos\theta + i\sin\theta)$:
   (i) $1 + i\tan\alpha$
   (ii) $\tan\alpha - i$
   (iii) $1 - \sin\alpha + i\cos\alpha$
   (iv) $\dfrac{1 - i}{\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}}$ [NCERT EXEMPLAR]
4. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi$, then show that $z_1 = -\overline{z}_2$. [NCERT EXEMPLAR]
5. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, prove that
   $\arg\left(\dfrac{z_1}{z_4}\right) + \arg\left(\dfrac{z_2}{z_3}\right) = 0.$ [NCERT EXEMPLAR]

6. Express $\sin\frac{\pi}{5}+i\left(1-\cos\frac{\pi}{5}\right)$ in polar form. **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $\sqrt{2}\,(\cos \pi/4 + i \sin \pi/4)$ (ii) $2\,(\cos \pi/6 + i\sin \pi/6)$

(iii) $\sqrt{2}\,(\cos \pi/4 - i \sin \pi/4)$ (iv) $(\cos \pi/2 - i \sin \pi/2)$

(v) $\frac{1}{\sqrt{2}}(\cos \pi/4 - i \sin \pi/4)$ (vi) $\frac{1}{\sqrt{2}}(\cos 3\,\pi/4 + i \sin 3\,\pi/4)$

(vii) $\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ (viii) $8\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

2. $\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$

3. (i) $1+i\tan\alpha=\begin{cases}\sec\alpha\,(\cos\alpha+i\sin\alpha), & 0\le\alpha\frac{\pi}{2}\\ -\sec\alpha\,\{\cos(\alpha-\pi)+i\sin(\alpha-\pi)\}, & \frac{\pi}{2}<\alpha\le\pi\end{cases}$

(ii) $\tan\alpha-i=\begin{cases}\sec\alpha\left\{\cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)\right\}, & 0\le\alpha<\frac{\pi}{2}\\ -\sec\alpha\left\{\cos\left(\frac{\pi}{2}+\alpha\right)+i\sin\left(\frac{\pi}{2}+\alpha\right)\right\}, & \frac{\pi}{2}<\alpha\le\pi\end{cases}$

(iii) $(1-\sin\alpha)+i\cos\alpha=\begin{cases}\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}, & \text{if } 0\le\alpha<\frac{\pi}{2}\\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{\pi}{2}<\alpha<\frac{3\pi}{2}\\ -\sqrt{2}\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}, & \text{if } \frac{3\pi}{2}<\alpha<2\pi\end{cases}$

(iv) $\dfrac{1-i}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}=\sqrt{2}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right)$

6. $2\sin\frac{\pi}{10}\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)$

HINTS TO NCERT & SELECTED PROBLEMS

1. (ii) Let $z=\sqrt{3}+i$. Then, $|z|\;=\;\sqrt{(\sqrt{3})^2+(1)^2}=2.$

Let θ be the argument of z and α be the acute angle given by $\tan\alpha=\dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\alpha=\frac{1}{\sqrt{3}}\Rightarrow\alpha=\frac{\pi}{6}$$

Clearly, z lies in the first quadrant. So, $\arg(z)=\alpha=\frac{\pi}{6}$.

(iii) Let $z=1-i$. Then, $|z|=\sqrt{1^2+(-1)^2}=\sqrt{2}$

Let α be the acute angle given by $\tan\alpha=\dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\alpha=\frac{|-1|}{|1|}=1\Rightarrow\alpha=\frac{\pi}{4}$$

Clearly, z lies in the fourth quadrant. Therefore, $\arg(z) = -\alpha = -\dfrac{\pi}{4}$.

(viii) Let $z = \dfrac{-16}{1+i\sqrt{3}} = \dfrac{-16(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \dfrac{-16(1-i\sqrt{3})}{1+3} = -4+4i\sqrt{3}$. Then,

$$|z| = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{16+48} = 8$$

Let α the acute angle given by $\tan\alpha = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\alpha = \frac{|4\sqrt{3}|}{|-4|} = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

Clearly, z lies in the second quadrant. Therefore, $\arg(z) = \pi - \alpha = \pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$.

3. (i) Let $z = 1 + i\tan\alpha$. Clearly, z is meaningful for $\alpha \neq (2n-1)\dfrac{\pi}{2}, n \in Z$. Also, $\tan\alpha$ is a periodic function with period π. So, let us take α lying in the interval $[0, \pi/2) \cup (\pi/2, \pi]$.

Following cases arise:

<u>CASE I</u>: When $\alpha \in [0, \pi/2)$

We have, $z = 1 + i\tan\alpha$

$\therefore \quad |z| = \sqrt{1+\tan^2\alpha} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha \qquad \left[\because \dfrac{\pi}{2} < \alpha < \pi \;\therefore\; \sec\alpha < 0\right]$

Let β be an acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = |\tan\alpha| = \tan\alpha \Rightarrow \beta = \alpha$$

As z is represented by a point lying in first quadrant. Therefore, $\arg(z) = \beta = \alpha$.

So, the polar form of z is $\sec\alpha(\cos\alpha + i\sin\alpha)$

<u>CASE II</u>: When $\alpha \in (\pi/2, \pi]$

We have, $z = 1 + i\tan\alpha$

$\therefore \quad |z| = \sqrt{1+\tan^2\alpha} = |\sec\alpha| = -\sec\alpha \qquad \left[\because \dfrac{\pi}{2} < \alpha < \pi \;\therefore \sec\alpha < 0\right]$

Let β be an acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$\tan\beta = |\tan\alpha| = -\tan\alpha \qquad [\because \alpha \in (\pi/2, \pi]]$

$\Rightarrow \quad \tan\beta = \tan(\pi - \alpha)$

$\Rightarrow \quad \beta = \pi - \alpha$

We observe that z is represented by a point in fourth quadrant. Therefore,

$\arg(z) = -\beta = \alpha - \pi$.

Thus, z in polar form is $-\sec\alpha\{\cos(\alpha-\pi) + i\sin(\alpha-\pi)\}$.

(ii) Let $z = \tan\alpha - i$. Since $\tan\alpha$ is periodic with period π. So, let us take $\alpha \in [0, \pi/2) \cup (\pi/2, \pi)]$.

<u>CASE I</u>: When $\alpha \in [0, \pi/2)$

We have, $z = \tan\alpha - i$

$\therefore \quad |z| = \sqrt{\tan^2\alpha + 1} = |\sec\alpha| = \sec\alpha$

Let β be the acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan(\pi/2-\alpha)$$

$$\Rightarrow \quad \beta = \frac{\pi}{2} - \alpha$$

Clearly, Re (z) > 0 and Im (z) < 0. So, z lies in the fourth quadrant.

$$\therefore \quad \arg(z) = -\beta = \alpha - \frac{\pi}{2}$$

Thus, z in polar form is given by

$$z = \sec\alpha\left\{\cos\left(\alpha-\frac{\pi}{2}\right)+i\sin\left(\alpha-\frac{\pi}{2}\right)\right\}$$

<u>CASE II</u>: When $\alpha \in (\pi/2, \pi]$

$$z = \tan\alpha - i \Rightarrow |z| = \sqrt{\tan^2\alpha+1} = |\sec\alpha| = -\sec\alpha$$

Let β be the acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha-\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \beta = \alpha - \frac{\pi}{2}$$

Clearly, Re (z) < 0 and Im (z) < 0. So, z lies in third quadrant.

$$\therefore \quad \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha$$

Thus, the polar form of z is $-\sec\alpha\left\{\cos\left(\dfrac{\pi}{2}+\alpha\right)+i\sin\left(\dfrac{\pi}{2}+\alpha\right)\right\}$.

(iii) Let $z = (1-\sin\alpha) + i\cos\alpha$. Since sine and cosine functions are periodic functions with period 2π. So, let us take α lying in the interval $[0, 2\pi]$.

Now, $z = 1 - \sin\alpha + i\cos\alpha$

$$\Rightarrow \quad |z| = \sqrt{(1-\sin\alpha)^2+\cos^2\alpha} = \sqrt{2-2\sin\alpha} = \sqrt{2}\sqrt{1-\sin\alpha}$$

$$\Rightarrow \quad |z| = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left|\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right|$$

Let β be the acute angle given by $\tan\beta = \dfrac{|\operatorname{Im}(z)|}{|\operatorname{Re}(z)|}$. Then,

$$\tan\beta = \frac{|\cos\alpha|}{|1-\sin\alpha|} = \left|\frac{\cos\alpha}{1-\sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \quad \tan\beta = \left|\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right|$$

Following cases arise:

CASE I: When $0 \le \alpha < \frac{\pi}{2}$

In this case, we have

$$\cos\frac{\alpha}{2} > \sin\frac{\alpha}{2} \text{ and } \frac{\pi}{4}+\frac{\alpha}{2} \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$$

$$\therefore \quad |z| = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \Rightarrow \beta = \frac{\pi}{4}+\frac{\alpha}{2}$

Clearly, z lies in the first quadrant. Therefore, $\arg(z) = \frac{\pi}{4}+\frac{\alpha}{2}$.

Hence, the polar form of z is

$$\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\}$$

CASE II: When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

In this case, we have

$$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} \text{ and } \frac{\pi}{4}+\frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = -\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = \tan\left\{\pi - \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\} = \tan\left(\frac{3\pi}{4}-\frac{\alpha}{2}\right)$

$$\Rightarrow \quad \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$$

Since $1 - \sin\alpha > 0$ and $\cos\alpha < 0$. So, z lies in fourth quadrant.

$$\therefore \quad \arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

Hence, the polar form of z is

$$-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)\right\}$$

CASE III: When $\frac{3\pi}{2} < \alpha < 2\pi$

In this case, we have

$$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2} \text{ and } \left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \in \left(\pi, \frac{5\pi}{4}\right)$$

$$\therefore \quad |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = -\tan\left\{\pi - \left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\right\} = \tan\left(\frac{\alpha}{2}-\frac{3\pi}{4}\right)$

$$\Rightarrow \quad \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$$

Clearly, Re $(z) < 0$ and Im $(z) > 0$. So, z lies in the first quadrant.

$\therefore \quad \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

Hence, the polar form of z is

$$-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left\{\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right\}$$

4. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then,

$$|z_1| = r_1,\ \arg(z_1) = \theta_1,\ |z_2| = \theta_2 \text{ and } \arg(z_2) = \theta_2$$

It is given that

$$|z_2| = |z_1| \text{ and } \arg(z_1) + \arg(z_2) = \pi$$

$\Rightarrow \quad r_1 = r_2$ and $\theta_1 + \theta_2 = \pi$

$\Rightarrow \quad r_1 = r_2$ and $\theta_1 = \pi - \theta_2$

$\therefore \quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$\Rightarrow \quad z_1 = r_2\{\cos(\pi - \theta_2) + i\sin(\pi - \theta_2)\} = r_2(-\cos\theta_2 + i\sin\theta_2)$

$\Rightarrow \quad z_1 = -r_2(\cos\theta_2 - i\sin\theta_2) = -\overline{z}_2$

5. Let $\arg(z_1) = \theta_1$ and $\arg(z_3) = \theta_2$

It is given that $z_2 = \overline{z}_1$ and $z_4 = \overline{z}_3$.

$\therefore \quad \arg(z_2) = -\theta_1$ and $\arg(z_4) = -\theta_2$

Hence,

$$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg(z_1) - \arg(z_4) + \arg(z_2) - \arg(z_3) = \theta_1 + \theta_2 - \theta_1 - \theta_2 = 0.$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the values of the square root of i.
2. Write the values of the square root of $-i$.
3. If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then write the value of $(x^2 + y^2)^2$.
4. If $\pi < \theta < 2\pi$ and $z = 1 + \cos\theta + i\sin\theta$, then write the value of $|z|$.
5. If n is any positive integer, write the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$.
6. Write the value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}}$.
7. Write $1 - i$ in polar form.
8. Write $-1 + i\sqrt{3}$ in polar form.
9. Write the argument of $-i$. **[NCERT]**
10. Write the least positive integral value of n for which $\left(\frac{1+i}{1-i}\right)^n$ is real.
11. Find the principal argument of $(1 + i\sqrt{3})^2$.
12. Find z, if $|z| = 4$ and $\arg(z) = \frac{5\pi}{6}$.
13. If $|z - 5i| = |z + 5i|$, then find the locus of z.

14. If $\dfrac{(a^2+1)^2}{2a-i} = x+iy$, find the value of x^2+y^2.
15. Write the value of $\sqrt{-25} \times \sqrt{-9}$.
16. Write the sum of the series $i + i^2 + i^3 + \ldots$upto 1000 terms.
17. Write the value of arg (z) + arg $(\bar{z})$.
18. If $|z+4| \leq 3$, then find the greatest and least values of $|z+1|$.
19. For any two complex numbers z_1 and z_2 and any two real numbers a, b, find the value of $|az_1 - bz_2|^2 + |az_2 + bz_1|^2$.
20. Write the conjugate of $\dfrac{2-i}{(1-2i)^2}$.
21. If $n \in N$, then find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$.
22. Find the real value of a for which $3i^3 - 2ai^2 + (1-a)\,i + 5$ is real.
23. If $|z| = 2$ and arg $(z) = \dfrac{\pi}{4}$, find z.
24. Write the argument of $(1+\sqrt{3})\,(1+i)\,(\cos\theta + i\sin\theta)$.

ANSWERS

1. $\pm \dfrac{1}{\sqrt{2}}(1+i)$
2. $\pm \dfrac{1}{\sqrt{2}}(1-i)$
3. $\dfrac{a^2+b^2}{c^2+d^2}$
4. $-2\cos\dfrac{\theta}{2}$
5. i
6. -2
7. $\sqrt{2}\left(\cos\dfrac{\pi}{4} - i\sin\dfrac{\pi}{4}\right)$
8. $\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$
9. $\dfrac{3\pi}{2}$ or $-\dfrac{\pi}{2}$
10. 2
11. $\dfrac{2\pi}{3}$
12. $-2\sqrt{3}+2i$
13. Real axis
14. $\dfrac{(a^2+1)^4}{4a^2+1}$
15. -15
16. 0
17. 0
18. 6 and 0
19. $(a^2+b^2)\,(|z_1|^2+|z_2|^2)$
20. $-\dfrac{2}{25} - \dfrac{11}{25}i$
21. 0
22. $a=2$
23. $\sqrt{2}\,(1+i)$
24. $\dfrac{7\pi}{12}+\theta$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The value of $(1+i)\,(1+i^2)\,(1+i^3)\,(1+i^4)$ is
   (a) 2 (b) 0 (c) 1 (d) i
2. If $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is a real number and $0 < \theta < 2\pi$, then $\theta =$
   (a) π (b) $\dfrac{\pi}{2}$ (c) $\dfrac{\pi}{3}$ (d) $\dfrac{\pi}{6}$
3. If $(1+i)\,(1+2i)\,(1+3i) \ldots (1+ni) = a+ib$, then $2 \times 5 \times 10 \times \ldots \times (1+n^2)$ is equal to
   (a) $\sqrt{a^2+b^2}$ (b) $\sqrt{a^2-b^2}$ (c) a^2+b^2 (d) a^2-b^2 (e) $a+b$
4. If $\sqrt{a+ib} = x+iy$, then possible value of $\sqrt{a-ib}$ is

(a) $x^2 + y^2$ (b) $\sqrt{x^2 + y^2}$ (c) $x + i y$ (d) $x - i y$ (e) $\sqrt{x^2 - y^2}$

5. If $z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{6}$, then

(a) $|z| = 1$, $\arg(z) = \frac{\pi}{4}$ (b) $|z| = 1$, $\arg(z) = \frac{\pi}{6}$

(c) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z) = \frac{5\pi}{24}$ (d) $|z| = \frac{\sqrt{3}}{2}$, $\arg(z) = \tan^{-1} \frac{1}{\sqrt{2}}$

6. The polar form of $(i^{25})^3$ is

(a) $\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$ (b) $\cos \pi + i \sin \pi$ (c) $\cos \pi - i \sin \pi$ (d) $\cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$

7. If $i^2 = -1$, then the sum $i + i^2 + i^3 + \ldots$ upto 1000 terms is equal to

(a) 1 (b) -1 (c) i (d) 0

8. If $z = \frac{-2}{1 + i\sqrt{3}}$, then the value of arg(z) is

(a) π (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$

9. If $a = \cos \theta + i \sin \theta$, then $\frac{1+a}{1-a} =$

(a) $\cot \frac{\theta}{2}$ (b) $\cot \theta$ (c) $i \cot \frac{\theta}{2}$ (d) $i \tan \frac{\theta}{2}$

10. If $(1 + i)(1 + 2i)(1 + 3i) \ldots\ldots (1 + ni) = a + ib$, then $2 . 5 . 10 . 17 \ldots\ldots (1 + n^2) =$

(a) $a - ib$ (b) $a^2 - b^2$ (c) $a^2 + b^2$ (d) none of these

11. If $\frac{(a^2+1)^2}{2a - i} = x + iy$, then $x^2 + y^2$ is equal to

(a) $\frac{(a^2+1)^4}{4a^2+1}$ (b) $\frac{(a+1)^2}{4a^2+1}$ (c) $\frac{(a^2-1)^2}{(4a^2-1)^2}$ (d) none of these

12. The principal value of the amplitude of $(1 + i)$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{12}$ (c) $\frac{3\pi}{4}$ (d) π

13. The least positive integer n such that $\left(\frac{2i}{1+i}\right)^n$ is a positive integer, is

(a) 16 (b) 8 (c) 4 (d) 2

14. If z is a non-zero complex number, then $\left|\frac{|\bar{z}|^2}{z\bar{z}}\right|$ is equal to

(a) $\left|\frac{\bar{z}}{z}\right|$ (b) $|z|$ (c) $|\bar{z}|$ (d) none of these

15. If $a = 1 + i$, then a^2 equals

(a) $1 - i$ (b) $2i$ (c) $(1 + i)(1 - i)$ (d) $i - 1$.

16. If $(x + iy)^{1/3} = a + ib$, then $\frac{x}{a} + \frac{y}{b} =$

(a) 0 (b) 1 (c) -1 (d) none of these

17. $(\sqrt{-2})(\sqrt{-3})$ is equal to

(a) $\sqrt{6}$ (b) $-\sqrt{6}$ (c) $i\sqrt{6}$ (d) none of these

18. The argument of $\dfrac{1-i\sqrt{3}}{1+i\sqrt{3}}$ is

(a) 60° (b) 120° (c) 210° (d) 240°.

19. If $z=\left(\dfrac{1+i}{1-i}\right)$, then z^4 equals

(a) 1 (b) -1 (c) 0 (d) none of these

20. If $z=\dfrac{1+2i}{1-(1-i)^2}$, then arg (z) equals

(a) 0 (b) $\dfrac{\pi}{2}$ (c) π (d) none of these

21. If $z=\dfrac{1}{(2+3i)^2}$, then $|z|=$

(a) $\dfrac{1}{13}$ (b) $\dfrac{1}{5}$ (c) $\dfrac{1}{12}$ (d) none of these

22. If $z=\dfrac{1}{(1-i)(2+3i)}$, then $|z|=$

(a) 1 (b) $1/\sqrt{26}$ (c) $5/\sqrt{26}$ (d) none of these

23. If $z=1-\cos\theta+i\sin\theta$, then $|z|=$

(a) $2\sin\dfrac{\theta}{2}$ (b) $2\cos\dfrac{\theta}{2}$ (c) $2\left|\sin\dfrac{\theta}{2}\right|$ (d) $2\left|\cos\dfrac{\theta}{2}\right|$

24. If $x+iy=(1+i)(1+2i)(1+3i)$, then $x^2+y^2=$

(a) 0 (b) 1 (c) 100 (d) none of these

25. If $z=\dfrac{1}{1-\cos\theta-i\sin\theta}$, then Re $(z)=$

(a) 0 (b) $\dfrac{1}{2}$ (c) $\cot\dfrac{\theta}{2}$ (d) $\dfrac{1}{2}\cot\dfrac{\theta}{2}$

26. If $x+iy=\dfrac{3+5i}{7-6i}$, then $y=$

(a) $9/85$ (b) $-9/85$ (c) $53/85$ (d) none of these

27. If $\dfrac{1-ix}{1+ix}=a+ib$, then $a^2+b^2=$

(a) 1 (b) -1 (c) 0 (d) none of these

28. If θ is the amplitude of $\dfrac{a+ib}{a-ib}$, then $\tan\theta=$

(a) $\dfrac{2a}{a^2+b^2}$ (b) $\dfrac{2ab}{a^2-b^2}$ (c) $\dfrac{a^2-b^2}{a^2+b^2}$ (d) none of these

29. If $z=\dfrac{1+7i}{(2-i)^2}$, then

(a) $|z|=2$ (b) $|z|=\dfrac{1}{2}$ (c) amp $(z)=\dfrac{\pi}{4}$ (d) amp $(z)=\dfrac{3\pi}{4}$

30. The amplitude of $\dfrac{1}{i}$ is equal to

(a) 0 (b) $\dfrac{\pi}{2}$ (c) $-\dfrac{\pi}{2}$ (d) π.

31. The argument of $\frac{1-i}{1+i}$ is

(a) $-\frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{2}$

32. The amplitude of $\frac{1+i\sqrt{3}}{\sqrt{3}+i}$ is

(a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $-\frac{\pi}{6}$

33. The value of $(i^5+i^6+i^7+i^8+i^9)/(1+i)$ is

(a) $\frac{1}{2}(1+i)$ (b) $\frac{1}{2}(1-i)$ (c) 1 (d) $\frac{1}{2}$

34. $\frac{1+2i+3i^2}{1-2i+3i^2}$ equals

(a) i (b) -1 (c) $-i$ (d) 4

35. The value of $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}-1$ is

(a) -1 (b) -2 (c) -3 (d) -4

36. The value of $(1+i)^4+(1-i)^4$ is

(a) 8 (b) 4 (c) -8 (d) -4

37. If $z=a+ib$ lies in third quadrant, then $\frac{\bar{z}}{z}$ also lies in the third quadrant if

(a) $a>b>0$ (b) $a<b<0$ (c) $b<a<0$ (d) $b>a>0$

38. If $f(z)=\frac{7-z}{1-z^2}$, where $z=1+2i$, then $|f(z)|$ is

(a) $\frac{|z|}{2}$ (b) $|z|$ (c) $2|z|$ (d) none of these

39. A real value of x satisfies the equation $\frac{3-4ix}{3+4ix}=a-ib\ (a, b \in R)$, if $a^2+b^2=$

(a) 1 (b) -1 (c) 2 (d) -2

40. The complex number z which satisfies the condition $\left|\frac{i+z}{i-z}\right|=1$ lies on

(a) circle $x^2+y^2=1$ (b) the x-axis (c) the y-axis (d) the line $x+y=1$

41. If z is a complex number, then

(a) $|z|^2>|z|^2$ (b) $|z|^2=|z|^2$ (c) $|z|^2<|z|^2$ (d) $|z|^2\geq|z|^2$

42. Which of the following is correct for any two complex numbers z_1 and z_2?

(a) $|z_1 z_2|=|z_1||z_2|$
(b) $\arg(z_1 z_2)=\arg(z_1)\arg(z_2)$
(c) $|z_1+z_2|=|z_1|+|z_2|$
(d) $|z_1+z_2|\geq|z_1|+|z_2|$

43. If the complex number $z=x+iy$ satisfies the condition $|z+1|=1$, then z lies on

(a) x-axis
(b) circle with centre $(-1, 0)$ and radius 1
(c) y-axis
(d) none of these

ANSWERS

1. (b)	2. (a)	3. (c)	4. (d)	5. (d)	6. (d)	7. (d)	8. (c)
9. (c)	10. (c)	11. (a)	12. (a)	13. (b)	14. (a)	15. (b)	16. (d)
17. (b)	18. (d)	19. (a)	20. (a)	21. (a)	22. (b)	23. (c)	24. (c)
25. (b)	26. (c)	27. (a)	28. (b)	29. (d)	30. (c)	31. (a)	32. (c)
33. (a)	34. (c)	35. (b)	36. (c)	37. (c)	38. (a)	39. (a)	40. (b)
41. (b)	42. (a)	43. (b)					

SUMMARY

1. $\sqrt{-1}$ is an imaginary quantity and is denoted by i which has the following properties:
   $i^2 = -1,\ i^3 = -i,\ i^4 = 1$ and, $i^{\pm n} = i^{\pm k}, n \in N$
   where k is the remainder when n is denoted by 4.
2. For any positive real number a, $\sqrt{-a} = i\sqrt{a}$.
3. For any two real numbers a and b, we have
   $$\sqrt{a}\,\sqrt{b} = \begin{cases} \sqrt{ab}, & \text{if at least one of a and b is positive} \\ -\sqrt{ab}, & \text{if } a<0, b<0. \end{cases}$$
4. If a, b are real numbers, then a number $z = a + ib$ is called a complex number, real number a is known as the real part of z and b is known as its imaginary part. We write $a = \text{Re}(z)$, $b = \text{Im}(z)$.
   A complex number z is purely real iff $\text{Im}(z) = 0$ and z is purely imaginary iff $\text{Re}(z) = 0$
5. For any two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, we define
   Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
   Subtraction: $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
   Multiplication: $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
   Reciprocal: $\dfrac{1}{z_1} = \dfrac{a_1}{a_1^2 + b_1^2} - i\dfrac{b_1}{a_1^2 + b_1^2}$
   Division: $\dfrac{z_1}{z_2} = z_1\left(\dfrac{1}{z_2}\right) = (a_1 + ib_1)\left(\dfrac{a_2}{a_2^2 + b_2^2} - i\dfrac{b_2}{a_2^2 + b_2^2}\right) = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$
   Addition is commutative and associative. Complex number $0 = 0 + i0$ is the identity element for addition and every complex number $z = a + ib$ has its additive inverse $-z = -a - ib$.
   Multiplication is also commutative and associative. Complex number $1 = 1 + 0i$ is the identity element for multiplication. Every non-zero complex number $z = a + ib$ has its multiplicative inverse $1/z$ (also known as reciprocal of z) such that $\dfrac{1}{z} = \dfrac{a - ib}{a^2 + b^2} = \dfrac{\bar{z}}{|z|^2}$.
6. The conjugate of a complex number $z = a + ib$ is denoted by $\bar{z}$ and is equal to $a - ib$.
   For any three complex numbers z, z_1, z_2, we have
   (i) $\overline{(\bar{z})} = z$ (ii) $z + \bar{z} = 2\,Re(z)$
   (iii) $z - \bar{z} = 2i\,Im(z)$ (iv) $z = \bar{z} \Leftrightarrow z$ is purely real
   (v) $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary (vi) $z\bar{z} = \{Re(z)\}^2 + \{Im(z)\}^2 = |z|^2$
   (vii) $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$ (viii) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(ix) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

7. The modulus of a complex number $z = a + ib$ is denoted by $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$$

If z, z_1, z_2 are three complex numbers, then

(i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{Im}(z) = 0$ (ii) $|z| = |\bar{z}| = |-z|$

(iii) $-|z| \leq \text{Re}(z) \leq |z|$; $-|z| \leq \text{Im}(z) \leq |z|$ (iv) $z\bar{z} = |z|^2$

(v) $|\text{Im}(z^n)| \leq n\,|\text{Im}(z)|\,|z|^{n-1}, n \in N$ (vi) $|\text{Re}(z)| + |\text{Im}(z)| \leq \sqrt{2}\,|z|$

8. A complex number $z = x + iy$ can be represented by a point $P(x, y)$ (see Fig. 13.8) on the plane which is known as the Argand or Gaussian or Complex plane. The length of the line segment OP is called the modulus of z and is denoted by $|z|$.

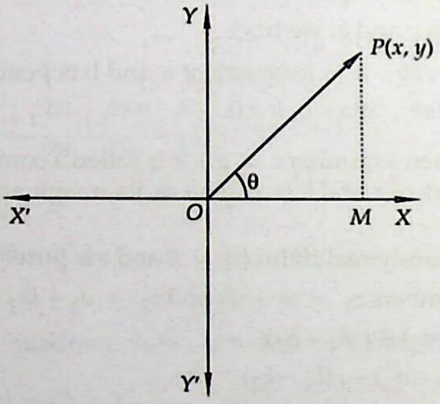

Fig. 13.8

Clearly, $|z| = \sqrt{x^2 + y^2} = \sqrt{\{\text{Re}(z)\}^2 + \{\text{Im}(z)\}^2}$

The angle θ which OP makes with the positive direction of x-axis in anti-clockwise sense is called the argument or amplitude of z and is denoted by *arg* (z) or amp (z).

Clearly, $\tan\theta = \dfrac{\text{Im}(z)}{\text{Re}(z)}$.

Let $OP = r$ and $\angle XOP = \theta$. Then, $x = r\cos\theta$ and $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$

This is known as the polar form of complex number z. The Euler's notations are

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore \quad z = r(\cos\theta + i\sin\theta)$

or, $\quad z = re^{i\theta}$, which is known as the Eulerian form of z.