CHAPTER 28

THE PLANE

28.1 PLANE

DEFINITION *A plane is a surface such that if any two points are taken on it, the line segment joining them lies completely on the surface.*

In other words, every point on the line segment joining any two points on a plane lies on the plane.

THEOREM *Prove that every first degree equation in x, y and z represents a plane.*

PROOF Let $ax + by + cz + d = 0$ be a first degree equation in x, y and z.

In order to prove that the equation $ax + by + cz + d = 0$ represents a plane, it is sufficient to show that every point on the line segment joining any two points on the surface represented by it lies on it. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on the surface represented by the equation $ax + by + cz + d = 0$. Then,

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{...(i)}$$

and, $$ax_2 + by_2 + cz_2 + d = 0 \quad \text{...(ii)}$$

Let R be an arbitrary point on the line segment joining P and Q. Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are

$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1}, \frac{z_1 + \lambda z_2}{\lambda + 1}\right), \text{ where } 0 \le \lambda \le 1.$$

We have to prove that R lies on the surface represented by the equation $ax + by + cz + d = 0$ for all values of λ satisfying $0 \le \lambda \le 1$. For this it is sufficient to show that the coordinates of R satisfy this equation.

Putting $x = \dfrac{x_1 + \lambda x_2}{\lambda + 1}$, $y = \dfrac{y_1 + \lambda y_2}{\lambda + 1}$ and $z = \dfrac{z_1 + \lambda z_2}{\lambda + 1}$ in $ax + by + cz + d$, we get

$$a\left(\frac{x_1 + \lambda x_2}{\lambda + 1}\right) + b\left(\frac{y_1 + \lambda y_2}{\lambda + 1}\right) + c\left(\frac{z_1 + \lambda z_2}{\lambda + 1}\right) + d$$

$$= \frac{1}{\lambda + 1}\left\{(ax_1 + by_1 + cz_1 + d) + \lambda(ax_2 + by_2 + cz_2 + d)\right\}$$

$$= \frac{1}{\lambda + 1}(0 + \lambda\, 0) = 0 \quad \text{[Using (i) and (ii)]}$$

Thus, the point $R\left(\dfrac{x_1 + \lambda x_2}{\lambda + 1}, \dfrac{y_1 + \lambda y_2}{\lambda + 1}, \dfrac{z_1 + \lambda z_2}{\lambda + 1}\right)$ lies on the surface represented by $ax + by + cz + d = 0$. Since R is an arbitrary point on the line segment joining P and Q. Therefore, every point on PQ lies on the surface represented by equation $ax + by + cz + d = 0$.

Hence, the equation $ax + by + cz + d = 0$ represents a plane.

Q.E.D.

REMARK *The general equation of a plane is $ax + by + cz + d = 0$. To determine a plane satisfying some given conditions we will have to find the values of constants a, b, c and d. It seems that there are four unknowns viz. a, b, c and d in the equation $ax + by + cz + d = 0$. But, there are only three unknowns, because the equation $ax + by + cz + d = 0$ can be written as*

$$\left(\frac{a}{d}\right)x+\left(\frac{b}{d}\right)y+\left(\frac{c}{d}\right)z+1 = 0 \quad \text{or,} \quad Ax+By+Cz+1=0.$$

Thus, to find a plane we must have three conditions to find the values of A, B and C.

28.2 EQUATIONS OF A PLANE PASSING THROUGH A GIVEN POINT

THEOREM *The general equation of a plane passing through a point (x_1, y_1, z_1) is*

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$, *where a, b and c are constants.*

PROOF The general equation of a plane is

$$ax+by+cz+d = 0 \quad \text{...(i)}$$

If it passes through (x_1, y_1, z_1), then

$$ax_1+by_1+cz_1+d=0 \Rightarrow d=-(ax_1+by_1+cz_1)$$

Substituting the value of d in (i), we obtain

$$ax+by+cz-(ax_1+by_1+cz_1) = 0 \Rightarrow a(x-x_1)+b(y-y_1)+c(z-z_1) = 0$$

This is the general equation of a plane passing through a given point (x_1, y_1, z_1)

Q.E.D.

In order to find the equation of a plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), we may use the following algorithm.

ALGORITHM

Step I *Write the equation of a plane passing through (x_1, y_1, z_1) as*

$$a(x-x_1)+b(y-y_1)+c(z-z_1) = 0 \quad \text{...(i)}$$

Step II *If the plane (i) passes through (x_2, y_2, z_2) and (x_3, y_3, z_3), then*

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1) = 0 \quad \text{...(ii)}$$

$$\text{and,} \quad a(x_3-x_1)+b(y_3-y_1)+c(z_3-z_1) = 0 \quad \text{...(iii)}$$

Step III *Solve equations (ii) and (iii), obtained in step II, by cross-multiplication.*

Step IV *Substitute the values of a, b and c, obtained in step III, in equation (i) in step I to get the required plane.*

REMARK *On eliminating a, b, c from equations (i), (ii) and (iii), we get*

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

as the equation of the plane passing through three given points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

EXAMPLE 1 *Find the equation of the plane through the points A (2, 2, −1), B (3, 4, 2) and C (7, 0, 6).*

SOLUTION The general equation of a plane passing through (2, 2, − 1) is

$$a(x-2)+b(y-2)+c(z+1) = 0 \quad \text{...(i)}$$

It will pass through B (3, 4, 2) and C (7, 0, 6), if

$$a(3-2)+b(4-2)+c(2+1) = 0 \Rightarrow a+2b+3c = 0 \quad \text{...(ii)}$$

$$\text{and,} \quad a(7-2)+b(0-2)+c(6+1) = 0 \Rightarrow 5a-2b+7c = 0 \quad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we obtain

$$\frac{a}{14+6}=\frac{b}{15-7}=\frac{c}{-2-10} \Rightarrow \frac{a}{5}=\frac{b}{2}=\frac{c}{-3}=\lambda \text{ (say)} \Rightarrow a=5\lambda,\ b=2\lambda \text{ and } c=-3\lambda$$

Substituting the values of a, b and c in (i), we get

$$5\lambda(x-2)+2\lambda(y-2)-3\lambda(z+1)=0 \Rightarrow 5(x-2)+2(y-2)-3(z+1)=0 \Rightarrow 5x+2y-3z=17,$$

which is the required equation of the plane.

ALITER The equation of the plane passing through points $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$ is given by

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix}=0 \text{ or, } \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix}=0$$

or, $(x-2)(14+6)-(y-2)(7-15)+(z+1)(-2-10)=0$

or, $20(x-2)+8(y-2)-12(z+1)=0$ or, $20x+8y-12z-68=0$ or, $5x+2y-3z=17$.

EXAMPLE 2 *Find the equation of the plane through the points $P(1, 1, 0)$, $Q(1, 2, 1)$ and $R(-2, 2, -1)$.*

SOLUTION The general equation of a plane passing through $P(1, 1, 0)$ is

$$a(x-1)+b(y-1)+c(z-0)=0 \qquad \text{...(i)}$$

It will pass through $Q(1, 2, 1)$ and $R(-2, 2, -1)$, if

$$a\times 0+b\times 1+c\times 1=0 \qquad \text{...(ii)}$$

and, $a(-3)+b\times 1+c(-1)=0$ ...(iii)

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(1)(-1)-(1)(1)}=\frac{b}{(1)(-3)-0(-1)}=\frac{c}{(0)(1)-(1)(-3)}$$

$$\Rightarrow \frac{a}{-2}=\frac{b}{-3}=\frac{c}{3}=\lambda \text{ (say)} \Rightarrow a=-2\lambda,\ b=-3\lambda \text{ and } c=3\lambda$$

Substituting the values of c, b and c in (i), we obtain

$$-2\lambda(x-1)-3\lambda(y-1)+3\lambda z=0 \Rightarrow -2(x-1)-3(y-1)+3z=0 \Rightarrow 2x+3y-3z-5=0,$$

which is the required equation.

EXAMPLE 3 *If from a point $P(a, b, c)$ perpendiculars PA and PB are drawn to yz and zx-planes, find the equation of the plane OAB.*

SOLUTION The coordinates of A and B are $(0, b, c)$ and $(a, 0, c)$ respectively.

The equation of the plane passing through $O(0, 0, 0)$, $A(0, b, c)$ and $B(a, 0, c)$ is given by

$$\begin{vmatrix} x-0 & y-0 & z-0 \\ 0-0 & b-0 & c-0 \\ a-0 & 0-0 & c-0 \end{vmatrix}=0 \Rightarrow bcx+acy-abz=0 \Rightarrow \frac{x}{a}+\frac{y}{b}-\frac{z}{c}=0$$

ALITER The equation of a plane passing through $O(0, 0, 0)$ is

$$P(x-0)+Q(y-0)+R(z-0)=0 \qquad \text{...(i)}$$

It passes through $A(0, b, c)$ and $B(a, 0, c)$.

$$P\times 0+Q\times b+R\times c=0$$
$$P\times a+Q\times 0+R\times c=0$$

Solving these equations by cross-multiplication, we obtain

$$\frac{P}{bc}=\frac{Q}{ac}=\frac{R}{-ab}=\lambda \text{ (say)} \Rightarrow P=\lambda bc,\ Q=\lambda ac,\ R=-\lambda ab$$

Substituting the values of P, Q and R in (i), we obtain

$$\lambda bc(x-0)+\lambda ac(y-0)-\lambda ab(z-0)=0$$

$\Rightarrow \quad bcx + acy - abz = 0 \Rightarrow \frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0$, which is the required equation of the plane.

Type II **ON PROVING COPLANARITY OF FOUR POINTS**

In order to prove the coplanarity of four points, we may use the following algorithm.

ALGORIGHM

Step I *Find the equation of a plane passing through any three out of given four points.*

Step II *Show that the coordinates of the fourth point satisfies the equation in Step I.*

EXAMPLE 4 *Show that the four points* $(0, -1, -1), (-4, 4, 4), (4, 5, 1)$ *and* $(3, 9, 4)$ *are coplanar. Find the equation of the plane containing them.*

SOLUTION The equation of a plane passing through $(0, -1, -1)$ is

$$a(x-0) + b(y+1) + c(z+1) = 0 \qquad \text{...(i)}$$

If it passes through $(-4, 4, 4)$ and $(4, 5, 1)$, then

$$a(-4) + b(5) + c(5) = 0 \qquad \text{...(ii)}$$

and, $\quad a(4) + b(6) + c(2) = 0$

or, $\quad a(2) + b(3) + c(1) = 0 \qquad \text{...(iii)}$

Solving (ii) and (iii) by cross-multiplication, we obtain

$$\frac{a}{5-15} = \frac{b}{10+4} = \frac{c}{-12-10} \Rightarrow \frac{a}{-5} = \frac{b}{7} = \frac{c}{-11} = \lambda \text{ (say)}$$

$\Rightarrow \quad a = -5\lambda, \ b = 7\lambda$ and $c = -11\lambda$

Substituting the values of a, b and c in (i), we obtain

$$-5\lambda x + 7\lambda(y+1) - 11\lambda(z+1) = 0 \Rightarrow -5x + 7y + 7 - 11z - 11 = 0 \Rightarrow 5x - 7y + 11z + 4 = 0 \quad \text{...(iv)}$$

which is the required equation of the plane.

Clearly, the fourth point viz. $(3, 9, 4)$ satisfies this equation. Hence, the given points are coplanar. The equation of the plane containing the given points is $5x - 7y + 11z + 4 = 0$.

EXERCISE 28.1

BASIC

1. Find the equation of the plane passing through the following points:
   (i) $(2, 1, 0), (3, -2, -2)$ and $(3, 1, 7)$ [NCERT EXEMPLAR]
   (ii) $(-5, 0, -6), (-3, 10, -9)$ and $(-2, 6, -6)$
   (iii) $(1, 1, 1), (1, -1, 2)$ and $(-2, -2, 2)$ (iv) $(2, 3, 4), (-3, 5, 1)$ and $(4, -1, 2)$
   (v) $(0, -1, 0), (3, 3, 0)$ and $(1, 1, 1)$ [CBSE 2004]
2. Show that the four point $(0, -1, -1), (4, 5, 1), (3, 9, 4)$ and $(-4, 4, 4)$ are coplanar and find the equation of the common plane.
3. Show that the following points are coplanar:
   (i) $(0, -1, 0), (2, 1, -1), (1, 1, 1)$ and $(3, 3, 0)$
   (ii) $(0, 4, 3), (-1, -5, -3), (-2, -2, 1)$ and $(1, 1, -1)$
4. Find the coordinates of the point P where the line through $A(3, -4, -5)$ and $B(2, -3, 1)$ crosses the plane passing through three points $L(2, 2, 1)$, $M(3, 0, 1)$ and $N(4, -1, 0)$. Also, find the ratio in which P divides the line segment AB. [CBSE 2016]

ANSWERS

1. (i) $7x + 3y - z = 17$, (ii) $2x - y - 2z - 2 = 0$, (iii) $x - 3y - 6z + 8 = 0$,
   (iv) $x + y - z = 1$, (v) $4x - 3y + 2z = 3$ 2. $5x - 7y + 11z + 4 = 0$ 4. $(1, -2, 7)$, $2:1$ externally

28.3 INTERCEPT FORM OF A PLANE

THEOREM *The equation of a plane intercepting lengths a, b and c with x-axis, y-axis and z-axis respectively is* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

PROOF Let O be the origin and let OX, OY and OZ be the coordinate axes. Suppose a plane meets the coordinate axes OX, OY and OZ at A, B and C respectively such that $OA=a, OB=b$ and $OC=c$. Then, the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The equation of a plane passing through $A(a, 0, 0)$ is

$$P(x-a)+Q(y-0)+R(z-0)=0 \quad \text{...(i)}$$

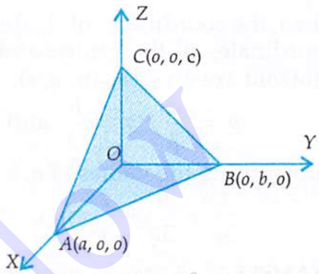


Fig. 28.1

If the plane in (i) passes through $B(0, b, 0)$ and $C(0, 0, c)$, then

$$P(0-a)+Q(b-0)+R(0-0)=0$$

$$\Rightarrow \quad P(-a)+Q(b)+R(0)=0 \quad \text{...(ii)}$$

and, $$P(0-a)+Q(0-0)+R(c-0)=0$$

$$\Rightarrow \quad P(-a)+Q(0)+R(c)=0 \quad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we have

$$\frac{P}{bc}=\frac{Q}{ac}=\frac{R}{ab}=\lambda \text{ (say)}$$

$$\Rightarrow \quad P=\lambda(bc),\ Q=\lambda(ac) \text{ and } R=\lambda(ab)$$

Substituting the values of P, Q and R in (i), we get the required equation of the plane as

$$\lambda(bc)(x-a)+\lambda(ac)(y-0)+\lambda(ab)(z-0)=0$$

$$\Rightarrow \quad bcx-abc+acy+abz=0 \Rightarrow bcx+acy+abz=abc \Rightarrow \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$ Q.E.D.

NOTE 1 *The above equation is known as the intercept form of the plane, because the plane intercepts lengths a, b and c with x, y and z-axes respectively.*

NOTE 2 *To determine the intercepts made by a plane with the coordinate axes we proceed as follows:*

For x-intercept: Put $y=0, z=0$ in the equation of the plane and obtain the value of x. The value of x is the intercept on x-axis.

For y-intercept: Put $x=0, z=0$ in the equation of the plane and obtain the value of y. The value of y is the intercept on y-axis

For z-intercept: Put $x=0, y=0$ in the equation of the plane and obtain the value of z. The value of z is the intercept on z-axis.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Write the equation of the plane whose intercepts on the coordinate axes are -4, 2 and 3 respectively.*

SOLUTION We know that the equation of a plane having a, b and c intercepts on the coordinate axes is given by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. Here, $a=-4$, $b=2$, and $c=3$. So, the equation of the required plane is

$$\frac{x}{-4}+\frac{y}{2}+\frac{z}{3}=1 \text{ or, } -3x+6y+4z=12.$$

EXAMPLE 2 *Reduce the equation of the plane $2x+3y-4z=12$ to intercept form and find its intercepts on the coordinate axes.*

SOLUTION The equation of the given plane is

$$2x+3y-4z=12 \Rightarrow \frac{2x}{12}+\frac{3y}{12}-\frac{4z}{12}=1 \Rightarrow \frac{x}{6}+\frac{y}{4}+\frac{z}{-3}=1$$

This is of the form $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. So, the intercepts made by the plane with the coordinate axes are 6, 4 and – 3 respectively.

EXAMPLE 3 *A plane meets the coordinate axes in A, B, C such that the centroid of triangle ABC is the point (p, q, r). Show that the equation of the plane is* $\frac{x}{p}+\frac{y}{q}+\frac{z}{r} = 3$.

SOLUTION Let the equation of the required plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(i)

Then, the coordinates of A, B and C are $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ respectively. So, the coordinates of the centroid of triangle ABC are $(a/3, b/3, c/3)$. But, the coordinates of the centroid are given as (p, q, r).

$$\therefore \quad p=\frac{a}{3},\ q=\frac{b}{3} \text{ and } r=\frac{c}{3} \Rightarrow a=3p,\ b=3q \text{ and } c=3r$$

Substituting the values of a, b and c in (i), we get

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3,$$ which is the equation of the required plane.

EXAMPLE 4 *A variable plane moves in such a way that the sum of the reciprocals of its intercepts on the three coordinate axes is constant. Show that the plane passes through a fixed point.*

SOLUTION Let the equation of the plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ ...(i)

This plane cuts intercepts of lengths a, b and c on the coordinate axes. It is given that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\lambda, \text{ where } \lambda \text{ is a constant}$$

$$\Rightarrow \quad \frac{1}{\lambda a}+\frac{1}{\lambda b}+\frac{1}{\lambda c}=1 \Rightarrow \frac{1}{a}\left(\frac{1}{\lambda}\right)+\frac{1}{b}\left(\frac{1}{\lambda}\right)+\frac{1}{c}\left(\frac{1}{\lambda}\right)=1$$

This shows that the point $(1/\lambda,\ 1/\lambda,\ 1/\lambda)$ lies on plane (i). Hence, the plane (i) passes through the fixed point $\left(\frac{1}{\lambda},\frac{1}{\lambda},\frac{1}{\lambda}\right)$.

EXERCISE 28.2

BASIC

1. Write the equation of the plane whose intercepts on the coordinate axes are 2, – 3 and 4.
2. Reduce the equations of the following planes in intercept form and find its intercepts on the coordinate axes:
   (i) $4x+3y-6z-12=0$ (ii) $2x+3y-z=6$ (iii) $2x-y+z=5$
3. Find the equation of a plane which meets the axes in A, B and C, given that the centroid of the triangle ABC is the point (α, β, γ).
4. Find the equation of the plane passing through the point (2, 4, 6) and making equal intercepts on the coordinate axes.
5. A plane meets the coordinate axes at A, B and C respectively such that the centroid of triangle ABC is (1, – 2, 3). Find the equation of the plane.

ANSWERS

1. $6x-4y+3z=12$
2. (i) $\frac{x}{3}+\frac{y}{4}+\frac{z}{-2}=1;\ 3, 4, -2$

(ii) $\frac{x}{3}+\frac{y}{2}+\frac{z}{-6}=1;\ 3, 2, -6$

(iii) $\frac{x}{5/2}+\frac{y}{-5}+\frac{z}{5}=1;\ \frac{5}{2}, -5, 5$

3. $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=3$
4. $x+y+z=12$
5. $6x-3y+2z=18$

28.4 VECTOR EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND NORMAL TO A GIVEN VECTOR

THEOREM *The vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n}=0$ or, $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$.*

PROOF Suppose the plane π passes through a point having a position vector $\vec{a}$ (see Fig. 28.2) and is normal to the vector $\vec{n}$. Let $\vec{r}$ be the position vector of an arbitrarily chosen point P on the plane π. Then, $\vec{OP}=\vec{r}$.

Clearly, $\vec{AP}$ lies in the plane and $\vec{n}$ is normal to the plane π.

$\therefore \quad \vec{AP}\perp\vec{n} \Rightarrow \vec{AP}\cdot\vec{n}=0 \Rightarrow (\vec{r}-\vec{a})\cdot\vec{n}=0$

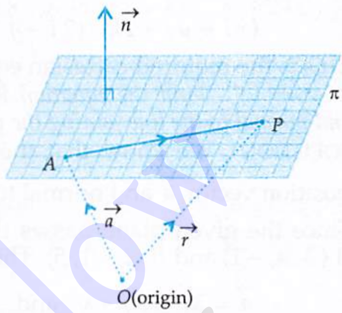


Fig. 28.2

Since $\vec{r}$ is the position vector of an arbitrary point on the plane. So, the vector equation of the plane is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \qquad \text{...(i)}$$

Equation (i) can also be written as

$$\vec{r}\cdot\vec{n}-\vec{a}\cdot\vec{n}=0 \text{ or, } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

NOTE 1 *It is to note here that vector equation of a plane means a relation involving the position vector $\vec{r}$ of an arbitrary point on the plane.*

NOTE 2 *The above equation can also be written as $\vec{r}\cdot\vec{n}=d$, where $d=\vec{a}\cdot\vec{n}$. This is known as the scalar product form of a plane.*

REDUCTION TO CARTESIAN FORM

If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{n}=n_1\hat{i}+n_2\hat{j}+n_3\hat{k}$

Then, $\quad \vec{r}-\vec{a}=(x-a_1)\hat{i}+(y-a_2)\hat{j}+(z-a_3)\hat{k}$

Substituting the values of $(\vec{r}-\vec{a})$ and $\vec{n}$ in equation $(\vec{r}-\vec{a})\cdot\vec{n}=0$, we get

$$[(x-a_1)\hat{i}+(y-a_2)\hat{j}+(z-a_3)\hat{k}]\cdot(n_1\hat{i}+n_2\hat{j}+n_3\hat{k})=0$$

$$\Rightarrow \quad (x-a_1)n_1+(y-a_2)n_2+(z-a_3)n_3=0 \qquad \text{...(ii)}$$

This is the cartesian equation of a plane passing through (a_1, a_2, a_3).

Note that the coefficients of x, y and z in equation (ii) are n_1, n_2 and n_3 respectively which are proportional to the direction ratios of vector $\vec{n}$ normal to the plane.

Thus, the coefficient of x, y and z in the cartesian equation of a plane are proportional to the direction ratios of normal to the plane.

For example, the direction ratios of a vector normal to the plane $2x+y-2z-5=0$ are proportional to 2, 1, –2 and hence a vector normal to the plane is $2\hat{i}+\hat{j}-2\hat{k}$.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the vector equation of a plane passing through a point having position vector $2\hat{i}+3\hat{j}-4\hat{k}$ and perpendicular to the vector $2\hat{i}-\hat{j}+2\hat{k}$. Also, reduce it to cartesian form.*

SOLUTION We know that the vector equation of a plane passing through a point $\vec{a}$ and normal to $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n}=0$ or, $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$. Here, $\vec{a}=2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{n}=2\hat{i}-\hat{j}+2\hat{k}$. So, the equation of the required plane is

$$\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=(2\hat{i}+3\hat{j}-4\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k}) \qquad [\text{Using: } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}]$$

$$\Rightarrow \quad \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=4-3-8 \Rightarrow \vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=-7 \qquad \text{...(i)}$$

Reduction to cartesian form: Since $\vec{r}$ denotes the position vector of an arbitrary point (x, y, z) on the plane. Therefore, putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in (i), we obtain

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k}) = -7 \Rightarrow 2x-y+2z = -7$$

This is the required cartesian equation of the plane.

EXAMPLE 2 *Find the equation in cartesian form of the plane passing through the point* $(3, -3, 1)$ *and normal to the line joining the points* $(3, 4, -1)$ *and* $(2, -1, 5)$.

SOLUTION We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to $\vec{n}$ is $(\vec{r}-\vec{a})\cdot\vec{n}=0$ or, $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$. ...(i)

Since the given plane passes through the point $(3, -3, 1)$ and is normal to the line joining $A(3, 4, -1)$ and $B(2, -1, 5)$. Therefore,

$$\vec{a}=3\hat{i}-3\hat{j}+\hat{k} \quad \text{and,} \quad \vec{n}=\vec{AB} = (2\hat{i}-\hat{j}+5\hat{k})-(3\hat{i}+4\hat{j}-\hat{k}) = -\hat{i}-5\hat{j}+6\hat{k}$$

Substituting $\vec{a}=3\hat{i}-3\hat{j}+\hat{k}$ and $\vec{n}=-\hat{i}-5\hat{j}+6\hat{k}$ in equation (i), we obtain

$$\vec{r}\cdot(-\hat{i}-5\hat{j}+6\hat{k}) = (3\hat{i}-3\hat{j}+\hat{k})\cdot(-\hat{i}-5\hat{j}+6\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(-\hat{i}-5\hat{j}+6\hat{k}) = -3+15+6$$

$\Rightarrow \quad \vec{r}\cdot(-\hat{i}-5\hat{j}+6\hat{k}) = 18$, which is the required vector equation of the plane.

The cartesian equation of this plane is given by

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(-\hat{i}-5\hat{j}+6\hat{k}) = 18 \qquad [\text{Putting } \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}]$$

$$\Rightarrow \quad -x-5y+6z = 18 \Rightarrow x+5y-6z+18 = 0$$

EXAMPLE 3 *The foot of perpendicular drawn from the origin to the plane is* $(4, -2, -5)$. *Find the equation of the plane.*

SOLUTION The required plane passes through the point $P(4, -2, -5)$ and is perpendicular to $\vec{OP}$. So, the plane passes through the point P having position vector $\vec{a}=4\hat{i}-2\hat{j}-5\hat{k}$ and is normal to vector $\vec{n}=\vec{OP}=4\hat{i}-2\hat{j}-5\hat{k}$.

The vector equation of the plane is

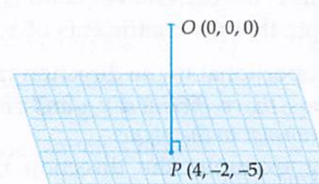


Fig. 28.3

$$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(4\hat{i}-2\hat{j}-5\hat{k}) = (4\hat{i}-2\hat{j}-5\hat{k})\cdot(4\hat{i}-2\hat{j}-5\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(4\hat{i}-2\hat{j}-5\hat{k}) = 16+4+25$$

$$\Rightarrow \quad \vec{r}\cdot(4\hat{i}-2\hat{j}-5\hat{k}) = 45. \qquad \text{...(i)}$$

Reduction to cartesian form: Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in (i), we obtain

$$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(4\hat{i}-2\hat{j}-5\hat{k}) = 45 \Rightarrow 4x-2y-5z = 45,$$

which is the required cartesian equation of the plane.

EXAMPLE 4 *If the line drawn from the point* $(-2, -1, -3)$ *meets a plane at right angle at the point* $(1, -3, 3)$, *find the equation of the plane.* **[NCERT EXEMPLAR]**

SOLUTION We observe that the required plane passes through the point $Q(1, -3, 3)$ and is perpendicular to $\vec{PQ}=(\hat{i}-3\hat{j}+3\hat{k})-(-2\hat{i}-\hat{j}-3\hat{k})=3\hat{i}-2\hat{j}+6\hat{k}$. Thus, the required plane passes through the point Q having position vector $\vec{a} = \hat{i} - 3\hat{j} + 3\hat{k}$ and is normal to $\vec{n} = \vec{PQ} = 3\hat{i} - 2\hat{j} + 6\hat{k}$. So, the vector equation of the plane is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

or, $\quad \vec{r} \cdot (3\hat{i} - 2\hat{j} + 6\hat{k}) = (\hat{i} - 3\hat{j} + 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 6\hat{k})$

or, $\quad \vec{r} \cdot (3\hat{i} - 2\hat{j} + 6\hat{k}) = 3 + 6 + 18$

or, $\quad \vec{r} \cdot (3\hat{i} - 2\hat{j} + 6\hat{k}) = 27$

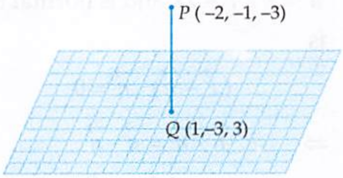


Fig. 28.4

The cartesian equation of the plane is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 2\hat{j} + 6\hat{k}) = 27$ or, $3x - 2y + 6z = 27$.

EXAMPLE 5 *Find the equation of the plane which bisects the line segment joining the points A (2, 3, 4) and B (4, 5, 8) at right angles.* **[NCERT EXEMPLAR]**

SOLUTION Let C be the mid-point of the line segment joining $A\,(2, 3, 4)$ and $B\,(4, 5, 8)$. Then, the coordinates of C are $(3, 4, 6)$.

Clearly, $\vec{AB} = (4\hat{i} + 5\hat{j} + 8\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) = 2\hat{i} + 2\hat{j} + 4\hat{k}$.

We observe that the required plane passes through the point $C\,(3, 4, 6)$ whose position vector is $\vec{a} = 3\hat{i} + 4\hat{j} + 6\hat{k}$ and is normal to $\vec{n} = \vec{AB} = 2\hat{i} + 2\hat{j} + 4\hat{k}$. So, the vector equation of the plane is

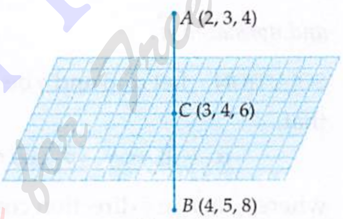


Fig. 28.5

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

or, $\quad \vec{r} \cdot (2\hat{i} + 2\hat{j} + 4\hat{k}) = (3\hat{i} + 4\hat{j} + 6\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 4\hat{k})$

or, $\quad \vec{r} \cdot (2\hat{i} + 2\hat{j} + 4\hat{k}) = 6 + 8 + 24$ or, $\vec{r} \cdot (2\hat{i} + 2\hat{j} + 4\hat{k}) = 38$ or, $\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 19$

The cartesian equation of the plane is $x + y + 2z = 19$.

EXAMPLE 6 *Find the vector equation of the plane whose cartesian form of equation is* $3x - 4y + 2z = 5$.

SOLUTION The equation of the given plane is

$$3x - 4y + 2z = 5$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$$

$\Rightarrow \quad \vec{r} \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$, which is the vector form of the equation of the given plane.

EXAMPLE 7 *Find a normal vector to the plane* $2x - y + 2z = 5$. *Also, find a unit vector normal to the plane.*

SOLUTION We know that the direction ratios of a vector normal to a plane are proportional to the coefficients of x, y and z respectively, in the cartesian equation of a plane. Therefore, direction ratios of a vector $\vec{n}$ normal to the given plane are proportional to 2, – 1, 2 and so $\vec{n} = 2\hat{i} - \hat{j} + 2\hat{k}$. Therefore, a unit vector normal to the plane is given by

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{4 + 1 + 4}} = \frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$$

EXAMPLE 8 *Find the equation of the plane passing through the point* (1, – 1, 2) *having* 2, 3, 2 *as direction ratios of normal to the plane.*

SOLUTION It is given that the required plane passes through the point having position vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and is normal to the vector $\vec{n} = 2\hat{i} + 3\hat{j} + 2\hat{k}$. So, the vector equation of the plane is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = (\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 2 - 3 + 4 \Rightarrow \vec{r} \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 3$$

The cartesian equation of the above plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 2\hat{k}) = 3 \text{ or, } 2x + 3y + 2z = 3 \quad [\text{Putting } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

EXAMPLE 9 *Let $\vec{n}$ be a vector of magnitude $2\sqrt{3}$ such that it makes equal acute angles with the coordinate axes. Find the vector and cartesian forms of the equation of a plane passing through $(1, -1, 2)$ and normal to $\vec{n}$.*

SOLUTION Let α, β and γ be the angles made by $\vec{n}$ with x, y and z-axes respectively. It is given that,

$$\alpha = \beta = \gamma \Rightarrow \cos\alpha = \cos\beta = \cos\gamma \Rightarrow l = m = n,$$

where l, m, n are direction cosines of $\vec{n}$.

$$\therefore \quad l^2 + m^2 + n^2 = 1 \Rightarrow l = m = n \Rightarrow 3l^2 = 1 \Rightarrow l = \frac{1}{\sqrt{3}} \quad [\because \alpha \text{ is acute} \therefore \cos\alpha = l > 0]$$

$$\text{Thus,} \quad \vec{n} = 2\sqrt{3}\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) = 2\hat{i} + 2\hat{j} + 2\hat{k} \quad [\text{Using } \vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})]$$

The required plane passes through a point $(1, -1, 2)$ having position vector $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and is normal to $\vec{n} = 2\hat{i} + 2\hat{j} + 2\hat{k}$. So, its vector equation is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 2\hat{j} + 2\hat{k}) = (\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (2\hat{i} + 2\hat{j} + 2\hat{k}) = 2 - 2 + 4 \Rightarrow \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2.$$

The cartesian equation of this plane is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ or, $x + y + z = 2$.

EXAMPLE 10 *Find the angle between the normals to the planes $2x - y + z = 6$ and $x + y + 2z = 7$.*

SOLUTION Let $\vec{n_1}$ and $\vec{n_2}$ be vectors normal to the planes $2x - y + z = 6$ and $x + y + 2z = 7$.

The direction ratios of normal to $2x - y + z = 6$ are proportional to 2, − 1, 1. Therefore, $\vec{n_1} = 2\hat{i} - \hat{j} + \hat{k}$.

The direction ratios of normal to $x + y + 2z = 7$ are proportional to 1, 1, 2. Therefore, $\vec{n_2} = \hat{i} + \hat{j} + 2\hat{k}$

Let θ be the angle between the normals $\vec{n_1}$ and $\vec{n_2}$. Then,

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} \Rightarrow \cos\theta = \frac{2\times 1 + (-1)\times 1 + 1\times 2}{\sqrt{2^2+(-1)^2+1^2}\,\sqrt{1^2+1^2+2^2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 11 *Show that the normals to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=3$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}-\hat{k})+5=0$ are perpendicular to each other.*

SOLUTION Let $\vec{n}_1$ and $\vec{n}_2$ be vectors normal to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=3$ and, $\vec{r}\cdot(3\hat{i}+2\hat{j}-\hat{k})=0$ respectively. Then, $\vec{n}_1=\hat{i}-\hat{j}+\hat{k}$ and $\vec{n}_2=3\hat{i}+2\hat{j}-\hat{k}$.

Now, $\vec{n}_1\cdot\vec{n}_2=(\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}+2\hat{j}-\hat{k}) = 3-2-1 = 0 \Rightarrow \vec{n}_1 \perp \vec{n}_2$.

Hence, normals to the given planes are perpendicular to each other.

EXAMPLE 12 *Find the angles at which the normal vector to the plane $4x+8y+z=5$ is inclined to the coordinate axes.*

SOLUTION Let $\vec{n}$ be a vector normal to the plane. Since direction ratios of normal to the plane are proportional to 4, 8, 1. Therefore, $\vec{n}=4\hat{i}+8\hat{j}+\hat{k}$. The direction cosines of $\vec{n}$ are

$$\frac{4}{\sqrt{4^2+8^2+1^2}},\ \frac{8}{\sqrt{4^2+8^2+1^2}},\ \frac{1}{\sqrt{4^2+8^2+1^2}} \text{ or, } \frac{4}{9},\ \frac{8}{9},\ \frac{1}{9}$$

Let α, β, γ be the angles made by $\vec{r}$ with x-axis, y-axis and z-axis respectively. Then,

$$\cos\alpha=\frac{4}{9},\ \cos\beta=\frac{8}{9} \text{ and } \cos\gamma=\frac{1}{9} \Rightarrow \alpha=\cos^{-1}\left(\frac{4}{9}\right),\ \beta=\cos^{-1}\left(\frac{8}{9}\right) \text{ and } \gamma=\cos^{-1}\left(\frac{1}{9}\right).$$

EXAMPLE 13 *A vector $\vec{n}$ of magnitude 8 units is inclined to x-axis at $45°$, y-axis at $60°$ and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2},-1,1)$ and is normal to $\vec{n}$, find its equation in vector form.*

SOLUTION Let γ be the angle made by $\vec{n}$ with z-axis. Then, the direction cosines of $\vec{n}$ are

$$l = \cos 45° = \frac{1}{\sqrt{2}},\ m = \cos 60° = \frac{1}{2} \text{ and } n = \cos\gamma.$$

But, $l^2+m^2+n^2 = 1$

$$\Rightarrow \left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+n^2=1 \Rightarrow n^2=\frac{1}{4} \Rightarrow n=\frac{1}{2} \qquad [\because \gamma \text{ is acute } \therefore n=\cos\gamma>0]$$

Now, $\vec{n}=|\vec{n}|(l\hat{i}+m\hat{j}+n\hat{k}) \Rightarrow \vec{n}=8\left(\frac{1}{\sqrt{2}}\hat{i}+\frac{1}{2}\hat{j}+\frac{1}{2}\hat{k}\right)=4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k}$ $\left[\because |\vec{n}|=8 \text{ (given)}\right]$

The required plane passes through the point $(\sqrt{2},-1,1)$ whose position vector is $\vec{a}=\sqrt{2}\hat{i}-\hat{j}+\hat{k}$ and is normal to $\vec{n}=4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k}$. So, its vector equation is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

$$\Rightarrow \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

$$\Rightarrow \vec{r}\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k}) = (\sqrt{2}\hat{i}-\hat{j}+\hat{k})\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})$$

$$\Rightarrow \vec{r}\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})=8-4+4 \Rightarrow \vec{r}\cdot(4\sqrt{2}\hat{i}+4\hat{j}+4\hat{k})=8 \Rightarrow \vec{r}\cdot(\sqrt{2}\hat{i}+\hat{j}+\hat{k})=2.$$

EXERCISE 28.3

BASIC

1. Find the vector equation of a plane passing through a point having position vector $2\hat{i} - \hat{j} + \hat{k}$ and perpendicular to the vector $4\hat{i} + 2\hat{j} - 3\hat{k}$.
2. Find the cartesian form of equation of a plane whose vector equation is
   (i) $\vec{r} \cdot (12\hat{i} - 3\hat{j} + 4\hat{k}) + 5 = 0$ (ii) $\vec{r} \cdot (-\hat{i} + \hat{j} + 2\hat{k}) = 9$
3. Find the vector equations of the coordinates planes.
4. Find the vector equation of each one of following planes:
   (i) $2x - y + 2z = 8$ (ii) $x + y - z = 5$ (iii) $x + y = 3$
5. Find the vector and cartesian equations of a plane passing through the point $(1, -1, 1)$ and normal to the line joining the points (1, 2, 5) and (–1, 3, 1).
6. If $\vec{n}$ is a vector of magnitude $\sqrt{3}$ and is equally inclined with an acute angle with the coordinate axes. Find the vector and cartesian forms of the equation of a plane which passes through $(2, 1, -1)$ and is normal to $\vec{n}$.
7. The coordinates of the foot of the perpendicular drawn from the origin to a plane are $(12, -4, 3)$. Find the equation of the plane.
8. Find the equation of the plane passing through the point (2, 3, 1) given that the direction ratios of normal to the plane are proportional to 5, 3, 2.
9. If the axes are rectangular and P is the point $(2, 3, -1)$, find the equation of the plane through P at right angles to OP.
10. Find the intercepts made on the coordinate axes by the plane $2x + y - 2z = 3$ and find also the direction cosines of the normal to the plane.
11. A plane passes through the point $(1, -2, 5)$ and is perpendicular to the line joining the origin to the point $3\hat{i} + \hat{j} - \hat{k}$. Find the vector and cartesian forms of the equation of the plane.
12. Find the equation of the plane that bisects the line segment joining points (1, 2, 3) and (3, 4, 5) and is at right angle to it.
13. Show that the normals to the following pairs of planes are perpendicular to each other:
    (i) $x - y + z - 2 = 0$ and $3x + 2y - z + 4 = 0$ (ii) $\vec{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 5$ and $\vec{r} \cdot (2\hat{i} - 2\hat{j} - 2\hat{k}) = 5$
14. Show that the normal vector to the plane $2x + 2y + 2z = 3$ is equally inclined with the coordinate axes.
15. Find a vector of magnitude 26 units normal to the plane $12x - 3y + 4z = 1$.
16. If the line drawn from $(4, -1, 2)$ meets a plane at right angles at the point $(-10, 5, 4)$, find the equation of the plane.
17. Find the equation of the plane which bisects the line segment joining the points $(-1, 2, 3)$ and $(3, -5, 6)$ at right angles.
18. Find the vector and cartesian equations of the plane which passes through the point $(5, 2, -4)$ and perpendicular to the line with direction ratios $2, 3, -1$. [NCERT]
19. If O be the origin and the coordinates of P be $(1, 2, -3)$, then find the equation of the plane passing through P and perpendicular to OP. [NCERT]
20. If O is the origin and the coordinates of A are (a, b, c). Find the direction cosines of OA and the equation of the plane through A at right angles to OA. [NCERT EXEMPLAR]
21. Find the vector equation of the plane with intercepts 3, – 4 and 2 on x, y and z axes respectively. [CBSE 2016]

ANSWERS

1. $\vec{r} \cdot (4\hat{i} + 2\hat{j} - 3\hat{k}) = 3$ 2. (i) $12x - 3y + 4z + 5 = 0$ (ii) $-x + y + 2z = 9$

3. $\vec{r}\cdot\hat{i} = 0$, $\vec{r}\cdot\hat{j} = 0$, $\vec{r}\cdot\hat{k} = 0$
4. (i) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 8$ (ii) $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k}) = 5$ (iii) $\vec{r}\cdot(\hat{i}+\hat{j}) = 3$
5. $\vec{r}\cdot(2\hat{i}-\hat{j}+4\hat{k}) = 7$, $2x-y+4z = 7$
6. $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 2$, $x+y+z = 2$
7. $12x-4y+3z = 169$
8. $5x+3y+2z = 21$
9. $2x+3y-z = 14$
10. $\frac{3}{2}, 3, -\frac{3}{2}; \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$
11. $\vec{r}\cdot(3\hat{i}+\hat{j}-\hat{k}) = -4$, $3x+y-z = -4$
12. $x+y+z = 9$
15. $24\hat{i}-6\hat{j}+8\hat{k}$
16. $7x-3y-z+89 = 0$
17. $4x-7y+3z-28 = 0$
18. $2x+3y-z = 20$
19. $x+2y-3z = 14$
20. $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$, $ax+by+cz = a^2+b^2+c^2$
21. $\vec{r}\cdot(4\hat{i}-3\hat{j}+6\hat{k}) = 12$

HINTS TO SELECTED PROBLEMS

3. XY-plane passes through the origin and is perpendicular to z-axis. So, its vector equation is $(\vec{r}-\vec{0})\cdot\hat{k} = 0$ or, $\vec{r}\cdot\hat{k} = 0$.

   Similarly, the equations of YZ and XZ-planes are $\vec{r}\cdot\hat{i} = 0$ and $\vec{r}\cdot\hat{j} = 0$ respectively.

18. The required plane passes through the point with position vector $\vec{a} = 5\hat{i}+2\hat{j}-4\hat{k}$ and perpendicular to the vector $\vec{n} = 2\hat{i}+3\hat{j}-\hat{k}$. So, its vector equation is

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0$$

or, $\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$

or, $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k}) = (5\hat{i}+2\hat{j}-4\hat{k})\cdot(2\hat{i}+3\hat{j}-\hat{k})$

or, $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k}) = 10+6+4$ or, $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k}) = 20$

The cartesian equation of the above plane is $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(2\hat{i}+3\hat{j}-\hat{k}) = 20$

or, $2x+3y-z = 20$.

19. The equation of the plane passing through $P(1, 2, -3)$ and perpendicular to $\vec{OP} = \hat{i}+2\hat{j}-3\hat{k}$ is $\{\vec{r}-(\hat{i}+2\hat{j}-3\hat{k})\}\cdot(\hat{i}+2\hat{j}-3\hat{k}) = 0$ or, $\vec{r}\cdot(\hat{i}+2\hat{j}-3\hat{k})-14 = 0$

28.5 EQUATION OF A PLANE IN NORMAL FORM

28.5.1 VECTOR FORM

THEOREM 1 *The vector equation of a plane normal to unit vector $\hat{n}$ and at a distance d from the origin is $\vec{r}\cdot\hat{n} = d$.*

PROOF Let O be the origin and let ON be the perpendicular from O to the given plane π such that $ON = d$. Let $\hat{n}$ be a unit vector along ON. Then, $\vec{ON} = d\hat{n}$. So, the position vector of N is $d\hat{n}$. Let $\vec{r}$ be the position vector of an arbitrary point P on the plane. Then,

$$\vec{NP} \perp \vec{ON}$$

$$\Rightarrow \quad \vec{NP} \perp \hat{n}$$

$$\Rightarrow \quad \vec{NP} \cdot \hat{n} = 0$$

$$\Rightarrow \quad (\vec{r} - d\hat{n}) \cdot \hat{n} = 0 \qquad [\because \vec{NP} = \vec{r} - d\hat{n}]$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d(\hat{n} \cdot \hat{n}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d|\hat{n}|^2 = 0$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} - d = 0 \qquad [\because |\hat{n}| = 1]$$

$$\Rightarrow \quad \vec{r} \cdot \hat{n} = d.$$

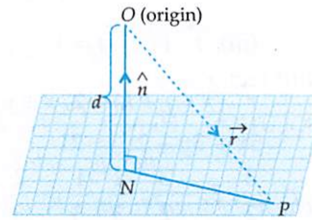


Fig. 28.6

Since $\vec{r}$ denotes the position vector of an arbitrary point on the plane π. Thus, the required equation of the plane is $\vec{r} \cdot \hat{n} = d$.

REMARK 1 *The vector equation of ON is $\vec{r} = \vec{0} + \lambda\hat{n}$ and the position vector of N is $d\hat{n}$ as it is at a distance d from the origin O.*

28.5.2 CARTESIAN FORM

THEOREM 2 *If l, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $lx + my + nz = p$.*

PROOF We know that the vector equation of a plane at a distance p from the origin and normal to unit vector $\hat{n}$ is $\vec{r} \cdot \hat{n} = p$. Here, $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$. So, the cartesian equation of the plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p \text{ or, } lx + my + nz = p$$

NOTE 1 *The equation $\vec{r} \cdot \vec{n} = d$ is in normal form if $\vec{n}$ is a unit vector and in such a case d on the right hand side denotes the distance of the plane from the origin. If $\vec{n}$ is not a unit vector, then to reduce the equation $\vec{r} \cdot \vec{n} = d$ to normal form divide both sides by $|\vec{n}|$ to obtain*

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{d}{|\vec{n}|} \Rightarrow \vec{r} \cdot \hat{n} = \frac{d}{|\vec{n}|}$$

In order to reduce the cartesian equation $ax + by + cz + d = 0$ of a plane to normal form, we may use the following algorithm.

ALGORITHM

Step I *Keep terms containing x, y and z on LHS and shift the constant term d on the RHS.*

Step II *If the constant term on RHS is not positive make it positive by multiplying both sides by –1.*

Step III *Divide each term on two sides by*

$$\sqrt{a^2 + b^2 + c^2} = \sqrt{(\text{Coeff. of } x)^2 + (\text{Coeff. of } y)^2 + (\text{Coeff. of } z)^2}.$$

The coefficients of x, y and z in the equation so obtained will be the direction cosines of the normal to the plane and the RHS will be the distance of the plane from the origin.

REMARK *The cartesian equations of the normal to the plane $lx + my + nz = p$ drawn from the origin are $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and the coordinates of the foot N of the perpendicular drawn from the origin O are given by $\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = p$ i.e. (lp, mp, np).*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the vector equation of a plane at a distance of 5 units from the origin and has $\hat{i}$ as the unit vector normal to it.*

SOLUTION Here, $\hat{n} = \hat{i}$ and $d = 5$. So, the vector equation of the plane is $\vec{r} \cdot \hat{i} = 5$.

EXAMPLE 2 *Find the vector equation of a plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$.*

SOLUTION Here, $d = 8$ and $\vec{n} = 2\hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \quad \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, the required equation of the plane is

$$\vec{r} \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 8 \text{ or, } \vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24. \qquad [\text{Using } \vec{r} \cdot \hat{n} = d]$$

EXAMPLE 3 *Reduce the equation $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5$ to normal form and hence find the length of perpendicular from the origin to the plane.*

SOLUTION The given equation is

$$\vec{r} \cdot (3\hat{i} - 4\hat{j} + 12\hat{k}) = 5 \text{ or, } \vec{r} \cdot \vec{n} = 5, \quad \text{where } \vec{n} = 3\hat{i} - 4\hat{j} + 12\hat{k}.$$

Since $|\vec{n}| = \sqrt{3^2 + (-4)^2 + 12^2} = 13 \neq 1$. Therefore, the given equation is not in normal form. To reduce it to normal form, we divide both sides by $|\vec{n}|$ to obtain

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|} \Rightarrow \vec{r} \cdot \left(\frac{3}{13}\hat{i} - \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k}\right) = \frac{5}{13} \qquad [\because |\vec{n}| = 13]$$

This is the normal form of the equation of given plane. The length of the perpendicular from the origin is $\frac{5}{13}$.

EXAMPLE 4 *Reduce the equation of the plane $x - 2y - 2z = 12$ to normal form and hence find the length of the perpendicular from the origin to the plane. Also, find the direction cosines of the normal to the plane.*

SOLUTION The given equation of the plane is $x - 2y - 2z = 12$ ...(i)

Dividing (i) throughout by $\sqrt{1^2 + (-2)^2 + (-2)^2}$ i.e. by 3, we get

$$\frac{x}{3} - \frac{2}{3}y - \frac{2}{3}z = 4$$

This is the normal form of the equation of the given plane. Clearly, the length of the perpendicular from the origin to the plane is 4 units and the direction cosines of the normal to the plane are $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$.

EXAMPLE 5 *Find the vector equation of a plane which is at a distance of 6 units from the origin and has 2, –1, 2 as the direction ratios of a normal to it. Also, find the coordinates of the foot of the normal drawn from the origin.*

SOLUTION Let $\vec{n}$ be a vector normal to the plane. It is given that the direction ratios of $\vec{n}$ are proportional to 2, – 1, 2.

$\therefore \quad \vec{n} = 2\hat{i} - \hat{j} + 2\hat{k} \Rightarrow |\vec{n}| = \sqrt{2^2 + (-1)^2 + 2^2} = 3$ and, $\hat{n} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{2}{3}\hat{i} - \dfrac{1}{3}\hat{j} + \dfrac{2}{3}\hat{k}$

The required plane is at a distance of 6 units from the origin. So, its vector equation is

$$\vec{r} \cdot \hat{n} = 6 \text{ or, } \vec{r} \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 6$$

The position vector of the foot of the normal drawn from the origin is

$$d\hat{n} = 6\left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

Consequently, the coordinates of the foot of the normal are $(4, -2, 4)$.

EXAMPLE 6 *Find the coordinates of the foot of the perpendicular drawn from the origin to the plane* $2x - 3y + 4z - 6 = 0$.

SOLUTION The equation of the plane is

$$2x - 3y + 4z - 6 = 0$$

$$\Rightarrow \quad 2x - 3y + 4z = 6$$

$$\Rightarrow \quad \frac{2}{\sqrt{29}}x - \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{6}{\sqrt{29}} \quad \left[\text{Dividing through out by } \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{29}\right]$$

This is the normal form of the given plane. It is evident from this equation that the direction cosines of the normal drawn from the origin to the given plane are

$$l = \frac{2}{\sqrt{29}}, m = -\frac{3}{\sqrt{29}}, n = \frac{4}{\sqrt{29}}$$

and the plane is at a distance of $d = \dfrac{6}{\sqrt{29}}$ units from the origin.

The coordinates of the foot of the perpendicular drawn from the origin are

$$(ld, md, nd) \text{ i.e., } \left(\frac{12}{29}, -\frac{18}{29}, \frac{24}{29}\right).$$

EXAMPLE 7 *Find the direction cosines of perpendicular from the origin to the plane* $\vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) + 5 = 0$.

SOLUTION In order to find the direction cosines of perpendicular from the origin to the given plane, let us first reduce it to the normal form. The vector equation of the given plane is

$$\vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) + 5 = 0 \Rightarrow \vec{r} \cdot (2\hat{i} - 3\hat{j} - 6\hat{k}) = -5 \Rightarrow \vec{r} \cdot (-2\hat{i} + 3\hat{j} + 6\hat{k}) = 5$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = 5, \text{ where } \vec{n} = -2\hat{i} + 3\hat{j} + 6\hat{k} \quad \text{...(i)}$$

$$\therefore \quad |\vec{n}| = \sqrt{(-2)^2 + 3^2 + 6^2} = 7$$

Dividing (i) throughout by $|\vec{n}| = 7$, we get

$$\vec{r} \cdot \frac{\vec{n}}{|\vec{n}|} = \frac{5}{|\vec{n}|} \text{ or, } \vec{r} \cdot \left(-\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) = \frac{5}{7}, \text{ which is in normal form.}$$

So, the direction cosines of the normal to the plane are $-\dfrac{2}{7}, \dfrac{3}{7}, \dfrac{6}{7}$.

EXAMPLE 8 *Find the equation of the plane passing through the point* $(-1, 2, 1)$ *and perpendicular to the line joining the points* $(-3, 1, 2)$ *and* $(2, 3, 4)$. *Find also the perpendicular distance of the origin from this plane.*

SOLUTION The required plane passes through the point (– 1, 2, 1) having position vector $\vec{a} = -\hat{i} + 2\hat{j} + \hat{k}$ and is perpendicular to the line segment joining the points A (– 3, 1, 2) and B (2, 3, 4). Therefore, $\vec{AB}$ is a vector normal to the plane.

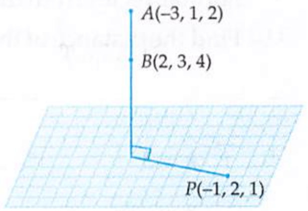


Fig. 28.7

Clearly, $\vec{n} = \vec{AB} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - (-3\hat{i} + \hat{j} + 2\hat{k}) = 5\hat{i} + 2\hat{j} + 2\hat{k}$

We know that the vector equation of a plane passing through a point having position vector $\vec{a}$ and normal to vector $\vec{n}$ is given by $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$ or, $\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$.

Therefore, the equation of the required plane is

$$\vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = (-\hat{i} + 2\hat{j} + \hat{k}) \cdot (5\hat{i} + 2\hat{j} + 2\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = -5 + 4 + 2 \Rightarrow \vec{r} \cdot (5\hat{i} + 2\hat{j} + 2\hat{k}) = 1. \qquad \text{...(i)}$$

To find the distance of this plane from the origin, we reduce its equation to normal form.

We have $\vec{n} = 5\hat{i} + 2\hat{j} + 2\hat{k}$. Therefore, $|\vec{n}| = \sqrt{5^2 + 2^2 + 2^2} = \sqrt{33}$

Dividing (i) throughout by $|\vec{n}| = \sqrt{33}$, we get

$$\vec{r} \cdot \left(\frac{5}{\sqrt{33}}\hat{i} + \frac{2}{\sqrt{33}}\hat{j} + \frac{2}{\sqrt{33}}\hat{k} \right) = \frac{1}{\sqrt{33}}$$, which is normal form of plane (i).

So, the perpendicular distance of the origin from the plane is $\frac{1}{\sqrt{33}}$.

EXERCISE 28.4

BASIC

1. Find the vector equation of a plane which is at a distance of 3 units from the origin and has $\hat{k}$ as the unit vector normal to it.
2. Find the vector equation of a plane which is at a distance of 5 units from the origin and which is normal to the vector $\hat{i} - 2\hat{j} - 2\hat{k}$.
3. Reduce the equation $2x - 3y - 6z = 14$ to the normal form and hence find the length of perpendicular from the origin to the plane. Also, find the direction cosines of the normal to the plane.
4. Reduce the equation $\vec{r} \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) + 6 = 0$ to normal form and hence find the length of perpendicular from the origin to the plane.
5. Write the normal form of the equation of the plane $2x - 3y + 6z + 14 = 0$.
6. The direction ratios of the perpendicular from the origin to a plane are 12, – 3, 4 and the length of the perpendicular is 5. Find the equation of the plane.
7. Find a unit normal vector to the plane $x + 2y + 3z - 6 = 0$.
8. Find the equation of a plane which is at a distance of $3\sqrt{3}$ units from the origin and the normal to which is equally inclined with the coordinate axes.
9. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). Find also the perpendicular distance of the origin from this plane. [NCERT EXEMPLAR]

10. Find the vector equation of the plane which is at a distance of $\frac{6}{\sqrt{29}}$ from the origin and its normal vector from the origin is $2\hat{i} - 3\hat{j} + 4\hat{k}$. Also, find its cartesian form. [NCERT]

11. Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin. [NCERT]

ANSWERS

1. $\vec{r} \cdot \hat{k} = 3$
2. $\vec{r} \cdot \left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = 5$
3. $2; \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}$
4. $\vec{r} \cdot \left(-\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{2}{3}\hat{k}\right) = 2; 2$
5. $-\frac{2}{7}x + \frac{3}{7}y - \frac{6}{7}z = 2$
6. $\frac{12}{13}x - \frac{3}{13}y + \frac{4}{13}z = 5$
7. $\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$
8. $x + y + z = 9$
9. $x - y + 3z - 2 = 0, \frac{2}{\sqrt{11}}$
10. $\vec{r} \cdot \left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$, $2x - 3y + 4z = 6$
11. $\frac{6}{\sqrt{29}}$

HINTS TO SELECTED PROBLEMS

10. A vector normal to the given plane is $\vec{n} = 2\hat{i} - 3\hat{j} + 4\hat{k}$. Therefore, a unit vector normal to the plane is $\hat{n} = \frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}$. The plane is at a distance of $\frac{6}{\sqrt{29}}$ units from the origin, so its vector equation in normal form is $\vec{r} \cdot \left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$.

    or, $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 6$. The cartesian form of this plane is $2x - 3y + 4z = 6$.

11. The equation of the given plane is $2x - 3y + 4z - 6 = 0$. The vector equation of this plane is $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 6$. Its normal form is $\vec{r} \cdot \left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$

    Hence, the distance of the plane form the origin is $\frac{6}{\sqrt{29}}$.

28.6 VECTOR EQUATION OF A PLANE PASSING THROUGH THREE GIVEN POINTS

In section 28.2, we have learnt about the method of finding the cartesian equation of a plane passing through three given points. In this section, we will learn about the procedure for finding the vector equation of a plane passing through three points.

Let A, B, and C be three points on a plane π having their position vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then, vectors $\vec{AB}$ and $\vec{AC}$ are in the plane π. Therefore, $\vec{AB} \times \vec{AC}$ is a vector perpendicular to the plane π. Let $\vec{n} = \vec{AB} \times \vec{AC}$. Then,

$$\vec{n} = \vec{AB} \times \vec{AC} = (\vec{b} - \vec{a}) \times (\vec{c} - \vec{a}) = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

Thus, the plane π passes through the point A with position vector $\vec{a}$ and is normal to vector

$$\vec{n} = \vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$$

So, the vector equation of the plane π is

$(\vec{r}-\vec{a}).\vec{n}=0$

or, $(\vec{r}-\vec{a}).(\vec{AB}\times\vec{AC})=0$

or, $(\vec{r}-\vec{a}).(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=0$

or, $\vec{r}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})=\vec{a}\cdot(\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a})$

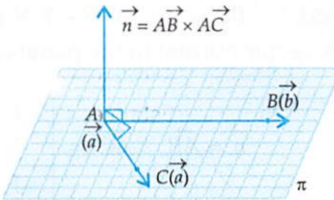


Fig. 28.8

or, $\vec{r}\cdot(\vec{a}\times\vec{b})+\vec{r}\cdot(\vec{b}\times\vec{c})+\vec{r}\cdot(\vec{c}\times\vec{a})=\vec{a}\cdot(\vec{a}\times\vec{b})+\vec{a}\cdot(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{c}\times\vec{a})$

or, $[\vec{r}\ \vec{a}\ \vec{b}]+[\vec{r}\ \vec{b}\ \vec{c}]+[\vec{r}\ \vec{c}\ \vec{a}]=[\vec{a}\ \vec{b}\ \vec{c}]$ $[\because \vec{a}\cdot(\vec{a}\times\vec{b})=0$ and $\vec{a}\cdot(\vec{c}\times\vec{a})=0]$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the vector equation of the plane passing through the points $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Also, find the cartesian equation of the plane.* **[CBSE 2019]**

SOLUTION The required plane passes through the point $A(2, 2, -1)$ whose position vector is $\vec{a}=2\hat{i}+2\hat{j}-\hat{k}$ and is normal to the vector $\vec{n}$ given by $\vec{n}=\vec{AB}\times\vec{AC}$.

Clearly, $\vec{AB}=(3\hat{i}+4\hat{j}+2\hat{k})-(2\hat{i}+2\hat{j}-\hat{k})=\hat{i}+2\hat{j}+3\hat{k}$

and, $\vec{AC}=(7\hat{i}+0\hat{j}+6\hat{k})-(2\hat{i}+2\hat{j}-\hat{k})=5\hat{i}-2\hat{j}+7\hat{k}$

$$\therefore \quad \vec{n}=\vec{AB}\times\vec{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 5 & -2 & 7\end{vmatrix}=(14+6)\hat{i}-(7-15)\hat{j}+(-2-10)\hat{k}=20\hat{i}+8\hat{j}-12\hat{k}$$

The vector equation of the required plane is

$\vec{r}.\vec{n}=\vec{a}.\vec{n}$

$\Rightarrow \quad \vec{r}.(20\hat{i}+8\hat{j}-12\hat{k})=(2\hat{i}+2\hat{j}-\hat{k})\cdot(20\hat{i}+8\hat{j}-12\hat{k})$

$\Rightarrow \quad \vec{r}.(20\hat{i}+8\hat{j}-12\hat{k})=40+16+12$

$\Rightarrow \quad \vec{r}.(20\hat{i}+8\hat{j}-12\hat{k})=68 \Rightarrow \vec{r}.(5\hat{i}+2\hat{j}-3\hat{k})=17$

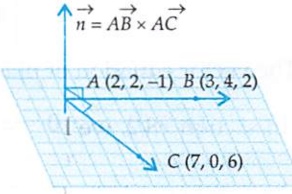


Fig. 28.9

The cartesian equation of the plane is given by

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(5\hat{i}+2\hat{j}-3\hat{k})=17$ or, $5x+2y-3z=17$.

EXAMPLE 2 *Find the vector equation of the plane passing through the points having position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$.*

SOLUTION Let A, B, C be the points with position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$, respectively.

Then, $\vec{AB}=$ P.V. of $B-$ P.V. of $A=(2\hat{i}-\hat{j}+\hat{k})-(\hat{i}+\hat{j}-2\hat{k})=\hat{i}-2\hat{j}+3\hat{k}$

and, $\quad \vec{BC} = \text{P.V. of } C - \text{P.V. of } B = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + \hat{k}) = -\hat{i} + 3\hat{j} + 0\hat{k}$

A vector normal to the plane containing points A, B, and C is

$$\vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix} = -9\hat{i} - 3\hat{j} + \hat{k}$$

The required plane passes through the point having position vector $\vec{a} = \hat{i} + \hat{j} - 2\hat{k}$ and is normal to the vector $-9\hat{i} - 3\hat{j} + \hat{k}$. So, its vector equation is

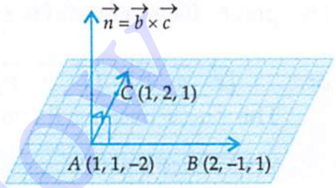


Fig. 28.10

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \vec{r} \cdot (-9\hat{i} - 3\hat{j} + \hat{k}) = (\hat{i} + \hat{j} - 2\hat{k}) \cdot (-9\hat{i} - 3\hat{j} + \hat{k})$$

$$\Rightarrow \vec{r} \cdot (-9\hat{i} - 3\hat{j} + \hat{k}) = -9 - 3 - 2 \Rightarrow \vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *If from a point $P(a, b, c)$ perpendiculars PA and PB are drawn to YZ and ZX-planes, find the vector equation of the plane OAB.*

SOLUTION The coordinates A and B are $(0, b, c)$ and $(a, 0, c)$ respectively.

$\therefore \quad \vec{OA} = b\hat{j} + c\hat{k}$ and $\vec{OB} = a\hat{i} + c\hat{k}$

The plane OAB passes through $O(\vec{0})$ and is perpendicular to $\vec{n} = \vec{OA} \times \vec{OB}$.

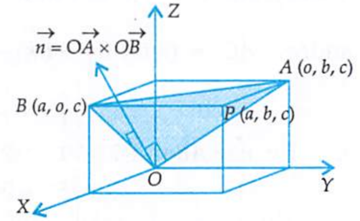


Fig. 28.11

Now, $\quad \vec{n} = \vec{OA} \times \vec{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & b & c \\ a & 0 & c \end{vmatrix} = bc\,\hat{i} + ac\,\hat{j} - ab\,\hat{k}$

The vector equation of plane OAB is

$$(\vec{r} - \vec{0}) \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot \vec{n} = 0 \Rightarrow \vec{r} \cdot (bc\,\hat{i} + ca\,\hat{j} - ab\,\hat{k}) = 0.$$

EXERCISE 28.5

BASIC

1. Find the vector equation of the plane passing through the points $(1, 1, 1)$, $(1, -1, 1)$ and $(-7, -3, -5)$.
2. Find the vector equation of the plane passing through the points $P(2, 5, -3)$, $Q(-2, -3, 5)$ and $R(5, 3, -3)$. [NCERT]
3. Find the vector equation of the plane passing through three points whose position vectors are $-\hat{j}$, $3\hat{i} + 3\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$. [CBSE 2022]
4. Find the vector equation of the plane passing through the points $(1, 1, -1)$, $(6, 4, -5)$ and $(-4, -2, 3)$. [NCERT]

BASED ON LOTS

5. Find the vector equation of the plane passing through points $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Reduce it to normal form. If plane ABC is at a distance p from the origin, prove that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

ANSWERS

1. $\vec{r}.(3\hat{i} - 4\hat{k}) + 1 = 0$ 2. $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$ 3. $\vec{r} \cdot (4\hat{i} - 3\hat{j} + 2\hat{k}) = 3$
4. These points are collinear. There will be infinite number of planes passing through these points. Their equations are given $a(x-1) + b(y-1) + c(z+1) = 0$, where $5a + 3b - 4c = 0$.

HINTS TO SELECTED PROBLEMS

2. The plane passing through $P(2, 5, -3)$, $Q(-2, -3, 5)$ and $R(5, 3, -3)$ is normal to the vector $\vec{n}$ given by

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 16\hat{i} + 24\hat{j} + 32\hat{k}$$

Clearly, required plane passes $P(2, 5, -3)$ and is normal to $\vec{n} = 16\hat{i} + 24\hat{j} + 32\hat{k}$. So, its vector equation is

$$\{\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})\} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 0$$

or, $\vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) - (32 + 120 - 96) = 0$

or, $\vec{r} \cdot (16\hat{i} + 24\hat{j} + 32\hat{k}) = 56$ or, $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$

4. The equation of a plane passing through the point $(1, 1, -1)$ is

$$a(x-1) + b(y-1) + c(z+1) = 0 \quad \text{...(i)}$$

This passes through points $(6, 4, -5)$ and $(-4, -2, 3)$.

$\therefore$ $5a + 3b - 4c = 0$ and, $-5a - 3b + 4c = 0 \Rightarrow a = \frac{4c - 3b}{5}$

Substituting the value of a in (i), we get

$$\left(\frac{4c - 3b}{5}\right)(x-1) + b(y-1) + c(z+1) = 0 \text{ or, } c(4x + 5z + 1) + b(-3x + 5y + 2) = 0$$

This represents a family of planes passing through the intersection of planes $4x + 5z + 1 = 0$ and $-3x + 5y + 2 = 0$.

28.7 ANGLE BETWEEN TWO PLANES

DEFINITION *The angle between two planes is defined as the angle between their normals.*

28.7.1 VECTOR FORM

THEOREM 1 *The angle θ between the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is given by* $\cos\theta = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}|\,|\vec{n_2}|}$.

PROOF Let θ be the angle between the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$. Then, θ is the angle between their normals $\vec{n_1}$ and $\vec{n_2}$.

$$\therefore \qquad \cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$$

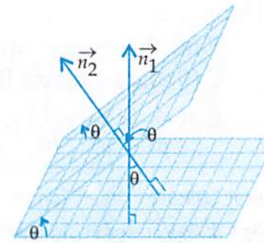


Fig. 28.12

Condition of perpendicularity: If the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular.

$$\therefore \qquad \vec{n}_1 \cdot \vec{n}_2 = 0$$

Condition of parallelism: If the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ are parallel, then $\vec{n}_1$ and $\vec{n}_2$ are parallel. Therefore, there exists a scalar λ such that $\vec{n}_1 = \lambda \vec{n}_2$.

28.7.2 CARTESIAN FORM

THEOREM 2 *The angle θ between the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by*

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

PROOF Let $\vec{n}_1$ and $\vec{n}_2$ be the vectors normal to the planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$. Then, $\vec{n}_1 = a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}$ and $\vec{n}_2 = a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}$.

The angle θ between $\vec{n}_1$ and $\vec{n}_2$ is given by

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} \Rightarrow \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of perpendicularity: If the planes are perpendicular, then $\vec{n}_1$ and $\vec{n}_2$ are perpendicular.

$$\therefore \qquad \vec{n}_1 \cdot \vec{n}_2 = 0 \Rightarrow (a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) \cdot (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}) = 0 \Rightarrow a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

Condition of parallelism: If the planes are parallel, then $\vec{n}_1$ and $\vec{n}_2$ are parallel.

$$\therefore \qquad \vec{n}_1 = \lambda \vec{n}_2 \text{ for some scalar } \lambda$$

$$\Rightarrow \qquad (a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}) = \lambda (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}) \Rightarrow a_1 = \lambda a_2,\ b_1 = \lambda b_2 \text{ and } c_1 = \lambda c_2$$

$$\Rightarrow \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE ANGLE BETWEEN TWO PLANES**

EXAMPLE 1 *Find the angle between the planes* $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 6$ *and* $\vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 5$.

SOLUTION We know that the angle between the planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}$$

Here, $\vec{n_1} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{n_2} = \hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \quad \cos\theta = \frac{(2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k})}{|2\hat{i} - \hat{j} + \hat{k}|\ |\hat{i} + \hat{j} + 2\hat{k}|} = \frac{2 - 1 + 2}{\sqrt{4+1+1}\ \sqrt{1+1+4}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 2 *Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$.*

SOLUTION We know that the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Therefore, the angle between $x + y + 2z = 9$ and $2x - y + z = 15$ is given by

$$\cos\theta = \frac{(1)(2) + (1)(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2}\ \sqrt{2^2 + (-1)^2 + 1^2}} = \frac{2 - 1 + 2}{\sqrt{6}\ \sqrt{6}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

EXAMPLE 3 *Show that the planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$ are at right angles.*

SOLUTION We know that the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are at right angles, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$. Here, $a_1 = 2, b_1 = 6, c_1 = 6, a_2 = 3, b_2 = 4, c_2 = -5$

$\therefore \quad a_1a_2 + b_1b_2 + c_1c_2 = (2)(3) + (6)(4) + (6)(-5) = 0.$

Therefore, planes $2x + 6y + 6z = 7$ and $3x + 4y - 5z = 8$ are at right angles.

EXAMPLE 4 *If the planes $\vec{r} \cdot (2\hat{i} - \hat{j} + \lambda\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} + 2\hat{j} + 2\hat{k}) = 4$ are perpendicular. Find the value of λ.*

SOLUTION We know that the planes $\vec{r} \cdot \vec{n_1} = d$ and $\vec{r} \cdot \vec{n_2} = d_2$ are perpendicular, if $\vec{n_1} \cdot \vec{n_2} = 0$.

Here, $\vec{n_1} = 2\hat{i} - \hat{j} + \lambda\hat{k}$ and $\vec{n_2} = 3\hat{i} + 2\hat{j} + 2\hat{k}$. Therefore, given planes will be perpendicular to each other, iff $\vec{n_1} \cdot \vec{n_2} = 0$.

Now, $\vec{n_1} \cdot \vec{n_2} = 0 \Rightarrow (2\hat{i} - \hat{j} + \lambda\hat{k}) \cdot (3\hat{i} + 2\hat{j} + 2\hat{k}) = 0 \Rightarrow 6 - 2 + 2\lambda = 0 \Rightarrow \lambda = -2.$

Type II ON FINDING A PLANE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO TWO GIVEN PLANES

The normal to the plane passing through a point having position vector $\vec{a}$ and perpendicular to the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ is perpendicular to the vectors $\vec{n_1}$ and $\vec{n_2}$. So, it is parallel to $\vec{n_1} \times \vec{n_2}$. We may use the following algorithm to find the plane passing through a given point and perpendicular to two planes.

ALGORITHM

Step I *Obtain the position vector of the given point say, $\vec{a}$.*

Step II *Obtain the normal vectors to two planes. Let the normal vectors be $\vec{n_1}$ and $\vec{n_2}$.*

Step III *Compute $\vec{n} = \vec{n_1} \times \vec{n_2}$. Clearly, $\vec{n}$ is normal to the required plane.*

Step IV *Write the equation of the desired plane as*

$$(\vec{r} - \vec{a}) \cdot (\vec{n_1} \times \vec{n_2}) = 0 \quad \text{or,} \quad \vec{r} \cdot (\vec{n_1} \times \vec{n_2}) = \vec{a} \cdot (\vec{n_1} \times \vec{n_2}) \quad \text{or,} \quad [\vec{r}\ \vec{n_1}\ \vec{n_2}] = [\vec{a}\ \vec{n_1}\ \vec{n_2}]$$

EXAMPLE 5 *Find the equation of the plane passing through the point $(1, 1, -1)$ and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.* **[CBSE 2003]**

SOLUTION The equation of any plane passing through (1, 1, – 1) is

$$a(x-1)+b(y-1)+c(z+1) = 0 \quad ...(i)$$

If plane (i) is perpendicular to each one of the planes $x+2y+3z-7=0$ and $2x-3y+4z=0$, then

$$a+2b+3c = 0 \quad ...(ii)$$

and, $$2a-3b+4c = 0 \quad ...(iii)$$

On solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(2)(4)-(3)(-3)} = \frac{b}{(3)(2)-(1)(4)} = \frac{c}{(1)(-3)-(2)(2)}$$

$$\Rightarrow \quad \frac{a}{17} = \frac{b}{2} = \frac{c}{-7} = \lambda \text{ (say)} \Rightarrow a = 17\lambda,\ b = 2\lambda \text{ and } c = -7\lambda$$

Putting $a = 17\lambda$, $b = 2\lambda$ and $c = -7\lambda$ in (i), we get

$$17\lambda(x-1)+2\lambda(y-1)-7\lambda(z+1) = 0 \text{ or, } 17x+2y-7z = 26,$$

which is the required equation of the plane.

ALITER The required plane passes through the point having position vector $\vec{a} = \hat{i}+\hat{j}-\hat{k}$. Let the normal vector to the required plane be $\vec{n}$. Then, $\vec{n}$ is perpendicular to the normals to the planes $x+2y+3z-7=0$ and $2x-3y+4z=0$ i.e. to the vectors $\vec{n_1} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{n_2} = 2\hat{i}-3\hat{j}+4\hat{k}$.

$$\therefore \quad \vec{n} = \vec{n_1}\times\vec{n_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -3 & 4 \end{vmatrix} = 17\hat{i}+2\hat{j}-7\hat{k}$$

So, required equation of the plane is

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(17\hat{i}+2\hat{j}-7\hat{k}) = (\hat{i}+\hat{j}-\hat{k})\cdot(17\hat{i}+2\hat{j}-7\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(17\hat{i}+2\hat{j}-7\hat{k}) = 17+2+7 \Rightarrow \vec{r}\cdot(17\hat{i}+2\hat{j}-7\hat{k}) = 26$$

Type III ON FINDING A PLANE PASSING THROUGH TWO GIVEN POINTS AND PERPENDICULAR TO A GIVEN PLANE

The normal to the plane passing through two points P and Q having their position vectors $\vec{a}$ and $\vec{b}$ respectively and perpendicular to the plane $\vec{r}\cdot\vec{n_1} = d_1$ is perpendicular to the vectors $\vec{PQ}$ and $\vec{n_1}$. So, the normal vector $\vec{n}$ to the plane is parallel to the vector $\vec{PQ}\times\vec{n_1}$. Thus, we may use the following algorithm to find a plane passing through two given points and perpendicular to a given plane.

ALGORITHM

Step I *Obtain the position vectors of the given points. Let the positions vectors of the given points P and Q be $\vec{a}$ and $\vec{b}$ respectively.*

Step II *Obtain the equation of the plane perpendicular to the required plane. Let its equation be $\vec{r}\cdot\vec{n_1} = d$.*

Step III *Let $\vec{n}$ be the normal vector to the required plane. Then, $\vec{n}$ is perpendicular to both $\vec{n_1}$ and $\vec{PQ}$. So, compute $\vec{n} = \vec{n_1}\times\vec{PQ}$.*

Step IV *Write the equation of the required plane as* $(\vec{r}-\vec{a}).\vec{n}=0$ *or,* $\vec{r}.\vec{n}=\vec{a}.\vec{n}$.

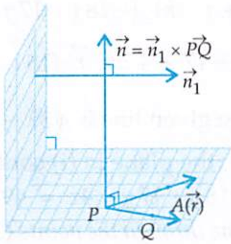


Fig. 28.13

EXAMPLE 6 *Find the equation of the plane through the points* $(2,1,-1)$ *and* $(-1,3,4)$ *and perpendicular to the plane* $x-2y+4z=10$. *Also, show that the plain thus obtained contains the line* $\vec{r}=-\hat{i}+3\hat{j}+4\hat{k}+\lambda(3\hat{i}-2\hat{j}-5\hat{k})$. **[CBSE 2012, NCERT EXEMPLAR]**

SOLUTION The equation of any plane passing through $(2,1,-1)$ is

$$a(x-2)+b(y-1)+c(z+1)=0 \quad \text{...(i)}$$

If it passes through $(-1,3,4)$, then

$$a(-1-2)+b(3-1)+c(4+1)=0 \Rightarrow -3a+2b+5c=0 \quad \text{...(ii)}$$

If plane (i) is perpendicular to the plane $x-2y+4z=10$, then

$$a-2b+4c=0 \quad \text{...(iii)}$$

Solving (ii) and (iii) by the method of cross-multiplication, we obtain

$$\frac{a}{8+10}=\frac{b}{5+12}=\frac{c}{6-2} \Rightarrow \frac{a}{18}=\frac{b}{17}=\frac{c}{4}=\lambda\text{ (say)} \Rightarrow a=18\lambda,\ b=17\lambda \text{ and } c=4\lambda.$$

Putting $a=18\lambda$, $b=17\lambda$ and $c=4\lambda$ in (i), we obtain

$$18\lambda(x-2)+17\lambda(y-1)+4\lambda(z+1)=0 \Rightarrow 18x+17y+4z=49 \quad \text{...(iv)}$$

This is the required equation of the plane.

The coordinates of any point on the line $\vec{r}=-\hat{i}+3\hat{j}+4\hat{k}+\lambda(3\hat{i}-2\hat{j}-5\hat{k})$ are $(3\lambda-1,-2\lambda+3,-5\lambda+4)$. Substituting $x=3\lambda-1$, $y=-2\lambda+3$, $z=-5\lambda+4$ in (iv), we obtain

$$\text{LHS} = 18(3\lambda-1)+17(-2\lambda+3)+4(-5\lambda+4) = 49 = \text{RHS}$$

So, $(3\lambda-1,-2\lambda+3,-5\lambda+4)$ lies on plane (iv). Hence, plane in (iv) contains the given line.

ALITER The required plane passes through the points $P(2,1,-1)$ and $Q(-1,3,4)$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of points P and Q respectively. Then, $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{b}=-\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{PQ}=\vec{b}-\vec{a}=-3\hat{i}+2\hat{j}+5\hat{k}$. The required plane is perpendicular to the plane $x-2y+4z=10$. Let $\vec{n_1}$ be the normal vector to this plane. Then, $\vec{n_1}=\hat{i}-2\hat{j}+4\hat{k}$.

Let $\vec{n}$ be the normal vector to the desired plane. Then,

$$\vec{n}=\vec{n_1}\times\vec{PQ}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 4\\ -3 & 2 & 5\end{vmatrix}=-18\hat{i}-17\hat{j}-4\hat{k}$$

The required plane passes through a point having position vector $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$ and is normal to the vector $\vec{n}=-18\hat{i}-17\hat{j}-4\hat{k}$. So, its vector equation is

$$\vec{r}\cdot\vec{n} = \vec{a}\cdot\vec{n}$$

$$\Rightarrow \quad \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k}) = (2\hat{i}+\hat{j}-\hat{k})\cdot(-18\hat{i}-17\hat{j}-4\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(-18\hat{i}-17\hat{j}-4\hat{k}) = -36-17+4 \Rightarrow \vec{r}\cdot(18\hat{i}+17\hat{j}+4\hat{k}) = 49 \qquad \text{...(i)}$$

The position vector of any point on the given line is $(3\lambda-1)\hat{i}+(3-2\lambda)\hat{j}+(4-5\lambda)\hat{k}$

Clearly, it satisfies equation (i). Hence, the plane in (i) contains given line.

The cartesian equation of the plane is $18x+17y+4z = 49$.

EXAMPLE 7 *Find the equation of the plane through the points* (3, 4, 2) *and* (7, 0, 6) *and is perpendicular to the plane* $2x-5y=15$. [CBSE 2012]

SOLUTION The equation of a plane passing through (3, 4, 2) is

$$a(x-3)+b(y-4)+c(z-2) = 0 \qquad \text{...(i)}$$

This passes through the point (7, 0, 6).

$$\therefore \quad a(7-3)+b(0-4)+c(6-2) = 0 \Rightarrow 4a-4b+4c = 0 \Rightarrow a-b+c = 0 \qquad \text{...(ii)}$$

The plane (i) is perpendicular to the plane $2x-5y+0z=15$.

$$\therefore \quad 2a+(-5)b+(0)c = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{5} = \frac{b}{2} = \frac{c}{-3} = \lambda\,(\text{say}) \Rightarrow a = 5\lambda,\ b = 2\lambda,\ c = -3\lambda$$

Substituting the values of a, b, c in (i), we get

$$5\lambda(x-3)+2\lambda(y-4)-3\lambda(z-2) = 0 \Rightarrow 5x+2y-3z-17 = 0$$

This is the equation of the required plane.

EXERCISE 28.6

BASIC

1. Find the angle between the planes:
   (i) $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(-\hat{i}+\hat{j})=4$
   (ii) $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=6$ and $\vec{r}\cdot(3\hat{i}+6\hat{j}-2\hat{k})=9$
   (iii) $\vec{r}\cdot(2\hat{i}+3\hat{j}-6\hat{k})=5$ and $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})=9$
2. Find the angle between the planes:
   (i) $2x-y+z=4$ and $x+y+2z=3$ (ii) $x+y-2z=3$ and $2x-2y+z=5$
   (iii) $x-y+z=5$ and $x+2y+z=9$ (iv) $2x-3y+4z=1$ and $-x+y=4$
   (v) $2x+y-2z=5$ and $3x-6y-2z=7$ [NCERT]
3. Show that the following planes are at right angles:
   (i) $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=5$ and $\vec{r}\cdot(-\hat{i}-\hat{j}+\hat{k})=3$
   (ii) $x-2y+4z=10$ and $18x+17y+4z=49$
4. Determine the value of λ for which the following planes are perpendicular to each other.
   (i) $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=7$ and $\vec{r}\cdot(\lambda\hat{i}+2\hat{j}-7\hat{k})=26$
   (ii) $2x-4y+3z=5$ and $x+2y+\lambda z=5$
   (iii) $3x-6y-2z = 7$ and $2x+y-\lambda z = 5$
5. Find the equation of a plane passing through the point $(-1,-1,2)$ and perpendicular to the planes $3x+2y-3z=1$ and $5x-4y+z=5$. [CBSE 2004, 08]

6. Obtain the equation of the plane passing through the point $(1, -3, -2)$ and perpendicular to the planes $x + 2y + 2z = 5$ and $3x + 3y + 2z = 8$. [CBSE 2009]
7. Find the equation of the plane passing through the origin and perpendicular to each of the planes $x + 2y - z = 1$ and $3x - 4y + z = 5$.
8. Find the equation of the plane passing through the points $(1, -1, 2)$ and $(2, -2, 2)$ and which is perpendicular to the plane $6x - 2y + 2z = 9$. [CBSE 2005]
9. Find the equation of the plane passing through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 1$.
10. Find the equation of the plane passing through the points whose coordinates are $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$.
11. Find the equation of the plane with intercept 3 on the y-axis and parallel to ZOX plane. [NCERT]
12. Find the equation of the plane that contains the point $(1, -1, 2)$ and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. [CBSE 2014, NCERT]
13. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$. [CBSE 2014, NCERT]
14. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$. [NCERT]
15. Find the vector equation of the plane through the points $(2, 1, -1)$ and $(-1, 3, 4)$ and perpendicular to the plane $x - 2y + 4z = 10$. [CBSE 2013]

ANSWERS

1. (i) $\cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$ (ii) $\cos^{-1}\left(-\frac{4}{21}\right)$ (iii) $\cos^{-1}\left(-\frac{16}{21}\right)$
2. (i) $\frac{\pi}{3}$ (ii) $\cos^{-1}\left(-\frac{2}{3\sqrt{6}}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\cos^{-1}\left(-\frac{5}{\sqrt{58}}\right)$ (v) $\cos^{-1}\left(\frac{4}{21}\right)$
4. (i) 17 (ii) 2 (iii) 0
5. $5x + 9y + 11z - 8 = 0$
6. $2x - 4y + 3z - 8 = 0$
7. $x + 2y + 5z = 0$
8. $x + y - 2z + 4 = 0$
9. $3x + 4y - 5z = 9$
10. $2x + 2y - 3z + 3 = 0$
11. $y = 3$
12. $5x - 4y - z = 7$
13. $x + y + z = a + b + c$
14. $7x - 8y + 3z + 25 = 0$
15. $18x + 17y + 4z - 49 = 0$

HINTS TO SELECTED PROBLEMS

11. The equation of a plane parallel to ZOX plane is $y = b$, where b is a constant. It passes through $(0, 3, 0)$. Therefore, $b = 3$. Hence, the equation of the plane is $y = 3$.
12. The equation of a plane passing through $(1, -1, 2)$ is

$$a(x - 1) + b(y + 1) + c(z - 2) = 0 \quad \text{...(i)}$$

It is perpendicular to the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$. Therefore,

$$2a + 3b - 2c = 0 \text{ and, } a + 2b - 3c = 0$$

Solving these two equations by cross-multiplication, we get $\frac{a}{-5} = \frac{b}{4} = \frac{c}{1}$

Substituting $a = -5, b = 4$ and $c = 1$ in (i), we get $-5x + 4y + z = -7$ as the equation of the required plane.

13. The equation of a plane parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ is $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda$. If it passes through (a, b, c), then $(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda \Rightarrow \lambda = a + b + c$.

So, equation of the required plane is $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$.

14. Proceed as in Q. No. 12.

28.8 EQUATION OF A PLANE PASSING THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN VECTORS OR LINES

28.8.1 PARAMETRIC FORM

THEOREM 1 *The equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$, where λ and μ are scalars.*

PROOF Let O be the origin and π be a plane passing through a point A having position vector $\vec{a}$. $\vec{a}$. Let the plane π be parallel to vectors $\vec{b}$ and $\vec{c}$ respectively. Through the point A draw two vectors $\vec{AB}$ and $\vec{AC}$ such that $\vec{AB} = \vec{b}$ and $\vec{AC} = \vec{c}$. Let the position vector of an arbitrary point P on the plane be $\vec{r}$. Complete the parallelogram $ALPM$. Since $\vec{AL}$ and $\vec{AM}$ are parallel to $\vec{b}$ and $\vec{c}$ respectively. Therefore, $\vec{AL} = \lambda \vec{b}$ and $\vec{AM} = \mu \vec{c}$ for some scalars λ and μ.

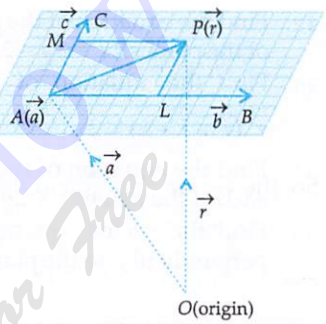


Fig. 28.14

In ΔAPL, we have

$$\vec{AP} = \vec{AL} + \vec{LP} \Rightarrow \vec{r} - \vec{a} = \lambda \vec{b} + \mu \vec{c} \Rightarrow \vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$$

Since $\vec{r}$ denotes the position vector of any point P on the plane. Therefore, the equation of the plane is $\vec{r} = \vec{a} + \lambda \vec{b} + \mu \vec{c}$, where λ and μ are scalars.

NOTE *In the above equation λ and μ are variable scalars, because for different points on the plane the values of λ and μ are different. That is why, it is called the parametric form of the plane passing through a given point and parallel to given vectors.*

28.8.2 NON-PARAMETRIC FORM

THEOREM 2 *Prove that the equation of the plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is*

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0 \text{ or, } \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c}) \text{ or, } [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

PROOF It is evident from Fig. 28.14, the plane is parallel to vectors $\vec{AB} = \vec{b}$ and $\vec{AC} = \vec{c}$. Therefore, it is perpendicular to the vector $\vec{n} = \vec{AB} \times \vec{AC} = \vec{b} \times \vec{c}$.

Since the plane passes through a point A having position vector $\vec{a}$. Therefore, the equation of the plane in scalar product form is

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0 \qquad [\text{Using} : (\vec{r} - \vec{a}) \cdot \vec{n} = 0]$$

$$\Rightarrow \quad \vec{r} \cdot (\vec{b} \times \vec{c}) - \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

$$\Rightarrow \quad \vec{r} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow \quad [\vec{r}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]$$

Q.E.D.

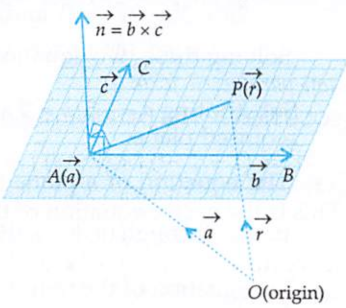


Fig. 28.15

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the vector equation of the following plane in scalar product form:*

$$\vec{r} = (\hat{i} - \hat{j}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k}).$$

SOLUTION We know that the equation $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$ represents a plane passing through a point having position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$.

Here, $\vec{a} = \hat{i} - \hat{j}$, $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + 3\hat{k}$.

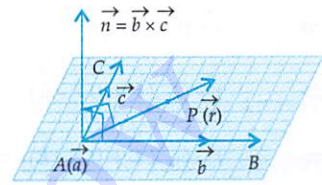


Fig. 28.16

The given plane is perpendicular to the vector

$$\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{i} - 2\hat{j} - 3\hat{k}$$

So, the vector equation of the plane in scalar product form is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = (\hat{i} - \hat{j}) \cdot (5\hat{i} - 2\hat{j} - 3\hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = 5 + 2 + 0 \text{ or, } \vec{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = 7.$$

EXAMPLE 2 *Find the cartesian form of the equation of the plane* $\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}$.

SOLUTION The equation of the given plane is

$$\vec{r} = (s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k} \text{ or, } \vec{r} = 3\hat{j} + s(\hat{i} + 2\hat{k}) + t(-2\hat{i} - \hat{j} + \hat{k})$$

Clearly, it represents a plane passing through a point having position vector $\vec{a} = 3\hat{j}$ and parallel to vectors $\vec{b} = \hat{i} + 2\hat{k}$ and $\vec{c} = -2\hat{i} - \hat{j} + \hat{k}$. So, it is perpendicular to $\vec{n}$ given by $\vec{n} = \vec{b} \times \vec{c}$.

Now, $$\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2 \\ -2 & -1 & 1 \end{vmatrix} = 2\hat{i} - 5\hat{j} - \hat{k}$$

The equation of the plane in scalar product form is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \Rightarrow \vec{r} \cdot (2\hat{i} - 5\hat{j} - \hat{k}) = (3\hat{j}) \cdot (2\hat{i} - 5\hat{j} - \hat{k}) \Rightarrow \vec{r} \cdot (2\hat{i} - 5\hat{j} - \hat{k}) = -15$$

The cartesian form of the equation of this plane is

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - 5\hat{j} - \hat{k}) = -15 \text{ or, } 2x - 5y - z = -15$$

EXAMPLE 3 *Find the vector equation of the plane* $\vec{r} = (1 + s - t)\hat{i} + (2 - s)\hat{j} + (3 - 2s + 2t)\hat{k}$ *in non-parametric form.*

SOLUTION The parametric equation of the given plane is

$$\vec{r} = (1 + s - t)\hat{i} + (2 - s)\hat{j} + (3 - 2s + 2t)\hat{k} \text{ or, } \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + s(\hat{i} - \hat{j} - 2\hat{k}) + t(-\hat{i} + 2\hat{k})$$

This is the vector equation of the plane passing through the point $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and parallel to vectors $\vec{b} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{c} = -\hat{i} + 2\hat{k}$. So, it is perpendicular to the vector $\vec{n}$ given by $\vec{n} = \vec{b} \times \vec{c}$.

Now, $\vec{n} = \vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 0 & 2 \end{vmatrix} = -2\hat{i} + 0\hat{j} - \hat{k}$

The vector equation of the plane in non-parametric form is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0$$

$$\Rightarrow \quad \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}$$

$$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 0\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j} + 3\hat{k}) \cdot (-2\hat{i} + 0\hat{j} - \hat{k})$$

$$\Rightarrow \quad \vec{r} \cdot (-2\hat{i} + 0\hat{j} - \hat{k}) = -2 + 0 - 3 \Rightarrow \vec{r} \cdot (2\hat{i} + 0\hat{j} + \hat{k}) = 5$$

EXERCISE 28.7

BASIC

1. Find the vector equation of the following planes in scalar product form ($\vec{r} \cdot \vec{n} = d$):
   (i) $\vec{r} = (2\hat{i} - \hat{k}) + \lambda\hat{i} + \mu(\hat{i} - 2\hat{j} - \hat{k})$
   (ii) $\vec{r} = (1 + s - t)\hat{i} + (2 - s)\hat{j} + (3 - 2s + 2t)\hat{k}$
   (iii) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$
   (iv) $\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(4\hat{i} - 2\hat{j} + 3\hat{k})$
2. Find the cartesian form of the equation of the following planes:
   (i) $\vec{r} = (\hat{i} - \hat{j}) + s(-\hat{i} + \hat{j} + 2\hat{k}) + t(\hat{i} + 2\hat{j} + \hat{k})$
   (ii) $\vec{r} = (1 + s + t)\hat{i} + (2 - s + t)\hat{j} + (3 - 2s + 2t)\hat{k}$
3. Find the vector equation of the following planes in non-parametric form:
   (i) $\vec{r} = (\lambda - 2\mu)\hat{i} + (3 - \mu)\hat{j} + (2\lambda + \mu)\hat{k}$
   (ii) $\vec{r} = (2\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}) + \mu(5\hat{i} - 2\hat{j} + 7\hat{k})$

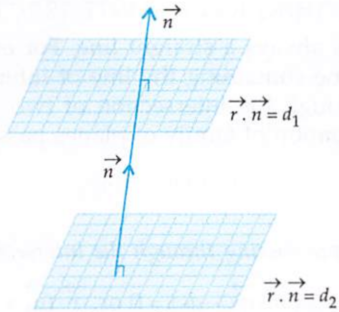
ANSWERS

1. (i) $\vec{r} \cdot (\hat{j} - 2\hat{k}) = 2$ (ii) $\vec{r} \cdot (2\hat{i} + \hat{k}) = 5$ (iii) $\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$
   (iv) $\vec{r} \cdot (5\hat{i} + \hat{j} - 6\hat{k}) = 4$
2. (i) $x - y + z = 2$ (ii) $2y - z = 1$
3. (i) $\vec{r} \cdot (2\hat{i} - 5\hat{j} - \hat{k}) + 15 = 0$ (ii) $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$

28.9 EQUATION OF A PLANE PARALLEL TO A GIVEN PLANE

28.9.1 VECTOR FORM

Since parallel planes have the common normal, therefore, equation of a plane parallel to the plane $\vec{r} \cdot \vec{n} = d_1$ is $\vec{r} \cdot \vec{n} = d_2$, where d_2 is a constant determined by the given condition.



Parallel planes have common normal

Fig. 28.17

ILLUSTRATION *Find the equation of plane passing through the point* $\hat{i}+\hat{j}+\hat{k}$ *and parallel to the plane* $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$.

SOLUTION The equation of a plane parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$ is

$$\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = d \quad \text{...(i)}$$

If it passes through $\hat{i}+\hat{j}+\hat{k}$, then

$$(\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k})=d \Rightarrow 2-1+2=d \Rightarrow d=3.$$

Putting $d=3$ in (i), we obtain $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=3$ as the equation of the required plane.

ALITER The required planes passes through the point $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and is parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$. So, it is normal to the vector $\vec{n}=2\hat{i}-\hat{j}+2\hat{k}$ which is normal to the given plane. Hence, the equation of the required plane is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0$$

or, $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

or, $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = (\hat{i}+\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+2\hat{k})$

or, $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 2-1+2$ or, $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k}) = 3$.

28.9.2 CARTESIAN FORM

Let $ax+by+cz+d=0$ be the cartesian equation of a plane. Then, direction ratios of its normal are proportional to a, b, c. Since parallel planes have common normal. Therefore, the direction ratios of the normal to the parallel plane are also proportional to a, b, c. Thus, the equation of a plane parallel to the plane $ax+by+cz+d=0$ is $ax+by+cz+k=0$, where k is an arbitrary constant and is determined by the given condition.

ILLUSTRATION 1 *Find the equation of the plane through the point* $(1, 4, -2)$ *and parallel to the plane* $-2x+y-3z=7$.

SOLUTION Let the equation of a plane parallel to the plane $-2x+y-3z=7$ be

$$-2x+y-3z+k = 0 \quad \text{...(i)}$$

If it passes through $(1, 4, -2)$ then,

$$(-2)(1)+4-3(-2)+k = 0 \Rightarrow -2+4+6+k = 0 \Rightarrow k = -8.$$

Putting $k=-8$ in (i), we obtain $-2x+y-3z-8 = 0$ or, $2x-y+3z+8 = 0$ as the equation of the required plane.

28.10 EQUATION OF A PLANE THROUGH THE INTERSECTION OF TWO PLANES

The intersection of two planes is always a straight line. For example, *xy*-plane and *xz*-plane intersect to form *x*-axis. The plane containing the line of intersection of two given planes is known as the plane passing through the intersection of two given planes. In the following discussion we will obtain the equation of family of planes passing through the intersection of two given planes.

28.10.1 VECTOR FORM

THEOREM 1 *The equation of a plane passing through the intersection of the planes* $\vec{r} \cdot \vec{n_1} = d_1$ *and* $\vec{r} \cdot \vec{n_2} = d_2$ *is given by* $(\vec{r} \cdot \vec{n_1} - d_1) + \lambda(\vec{r} \cdot \vec{n_2} - d_2) = 0$ or, $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$, *where* λ *is an arbitrary constant.*

PROOF The equation $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ is of the form $\vec{r} \cdot \vec{n} = d$. So, it represents a plane. In order to prove that it represents a plane passing through the line of intersection of the planes $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$, it is sufficient to show that every point on the line of intersection of the plane $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$ lies on $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$.

Let $\vec{r_1}$ be the position vector of any point on the line of intersection of $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$. Then,

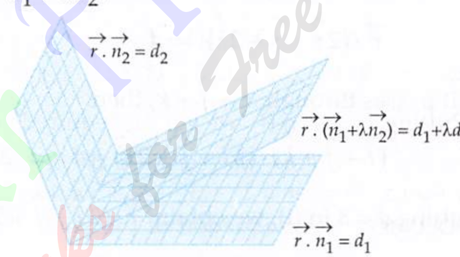


Fig. 28.18

$$\vec{r_1} \cdot \vec{n_1} = d_1 \text{ and } \vec{r_1} \cdot \vec{n_2} = d_2$$

$$\Rightarrow \quad \vec{r_1} \cdot \vec{n_1} - d_1 = 0 \text{ and } \vec{r_1} \cdot \vec{n_2} - d_2 = 0$$

$$\Rightarrow \quad (\vec{r_1} \cdot \vec{n_1} - d_1) + \lambda(\vec{r_1} \cdot \vec{n_2} - d_2) = 0 + \lambda 0 = 0$$

$$\Rightarrow \quad \vec{r_1} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2 \Rightarrow \vec{r_1} \text{ lies on the plane } \vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2.$$

Hence, equation $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = d_1 + \lambda d_2$ represents a plane passing through the intersection of $\vec{r} \cdot \vec{n_1} = d_1$ and $\vec{r} \cdot \vec{n_2} = d_2$.

Q.E.D.

28.10.2 CARTESIAN FORM

THEOREM 2 *The equation of a plane passing through the intersection of planes* $a_1x + b_1y + c_1z + d_1 = 0$ *and* $a_2x + b_2y + c_2z + d_2 = 0$ *is* $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$, *where* λ *is a constant.*

PROOF Consider the equation

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0 \quad \text{...(i)}$$

or, $$x(a_1 + \lambda a_2) + y(b_1 + \lambda b_2) + z(c_1 + \lambda c_2) + d_1 + \lambda d_2 = 0$$

This is a first degree equation in *x*, *y*, *z*. So, it represents a plane.

In order to prove that equation (i) represents a plane passing through the intersection of planes

$$a_1x + b_1y + c_1z + d_2 = 0 \quad \text{...(ii)}$$

and, $$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{...(iii)}$$

It is sufficient to show that every point on the line of intersection of (ii) and (iii) is a point on the plane (i). Let (α, β, γ) be a point on the line of intersection of (ii) and (iii). Then,

$$a_1\alpha + b_1\beta + c_1\gamma + d_1 = 0 \text{ and } a_2\alpha + b_2\beta + c_2\gamma + d_2 = 0$$

$$\Rightarrow \quad (a_1\alpha + b_1\beta + c_1\gamma + d_1) + \lambda(a_2\alpha + b_2\beta + c_2\gamma + d_2) = 0 + \lambda 0 = 0$$

$$\Rightarrow \quad (\alpha, \beta, \gamma) \text{ lies on plane (i).}$$

Thus, $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$ represents a plane passing through the intersection of the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO GIVEN PLANES AND A GIVEN POINT**

EXAMPLE 1 *Find the equation of a plane through the intersection of the planes* $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=5$ *and* $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=3$ *and passing through the point* $(2,1,-2)$. [CBSE 2022]

SOLUTION The equation of a plane through the intersection of $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=5$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=3$ is

$$[\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})-5]+\lambda[\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})-3]=0$$

$$\Rightarrow \quad \vec{r}\cdot[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-5-3\lambda=0 \quad \text{...(i)}$$

If plane in (i) passes through $(2,1,-2)$, then the vector $2\hat{i}+\hat{j}-2\hat{k}$ should satisfy it.

$$\therefore \quad (2\hat{i}+\hat{j}-2\hat{k})\cdot[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-(5+3\lambda)=0$$

$$\Rightarrow \quad 2(1+2\lambda)+1(3-\lambda)-2(-1+\lambda)-(5+3\lambda)=0 \Rightarrow -2\lambda+2=0 \Rightarrow \lambda=1$$

Putting $\lambda=1$ in (i), we get the required equation of the plane as $\vec{r}\cdot(3\hat{i}+2\hat{j}+0\hat{k})=8$.

EXAMPLE 2 *Find the equation of the plane containing the line of intersection of the plane* $x+y+z-6=0$ *and* $2x+3y+4z+5=0$ *and passing through the point* $(1,1,1)$.

SOLUTION The equation of the plane through the line of intersection of the given planes is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \quad \text{...(i)}$$

If (i) passes through $(1,1,1)$, then

$$-3+14\lambda=0 \Rightarrow \lambda=3/14$$

Putting $\lambda=\frac{3}{14}$ in (i), we obtain the equation of the required plane as

$$(x+y+z-6)+\frac{3}{14}(2x+3y+4z+5)=0 \text{ or, } 20x+23y+26z-69=0.$$

EXAMPLE 3 *Find the direction ratios of the normal to the plane passing through the point* $(2,1,3)$ *and the line of intersection of the planes* $x+2y+z=3$ *and* $2x-y-z=5$.

SOLUTION The equation of the plane passing through the line of intersection of the planes $x+2y+z=3$ and $2x-y-z=5$ is given by

$$(x+2y+z-3)+\lambda(2x-y-z-5)=0 \Rightarrow x(2\lambda+1)+y(2-\lambda)+z(1-\lambda)-3-5\lambda=0 \quad \text{...(i)}$$

It passes through $(2,1,3)$, then

$$2(2\lambda+1)+(2-\lambda)+3(1-\lambda)-3-5\lambda=0$$

$$\Rightarrow \quad 4\lambda+2+2-\lambda+3-3\lambda-3-5\lambda=0 \Rightarrow 4-5\lambda=0 \Rightarrow \lambda=\frac{4}{5}$$

Substituting $\lambda=\frac{4}{5}$ in (i), we get $13x+6y+z-35=0$ as the equation of the required plane.

Clearly, direction ratios of normal to this plane are proportional to 13, 6, 1.

Type II **EQUATION OF A PLANE PASSING THROUGH THE INTERSECTION OF TWO PLANES AND PERPENDICULAR TO A GIVEN PLANE**

EXAMPLE 4 *Find the equation of the plane which is perpendicular to the plane* $5x+3y+6z+8=0$ *and which contains the line of intersection of the planes* $x+2y+3z-4=0$ *and* $2x+y-z+5=0$.

[CBSE 2007, NCERT EXEMPLAR]

SOLUTION The equation of a plane through the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ is

$$(x+2y+3z-4)+\lambda(2x+y-z+5)=0 \Rightarrow x(1+2\lambda)+y(2+\lambda)+z(3-\lambda)-4+5\lambda=0 \quad ...(i)$$

If it is perpendicular to the plane $5x+3y+6z+8=0$, then

$$5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0 \Rightarrow \lambda=-29/7 \quad [\text{Using}: a_1a_2+b_1b_2+c_1c_2=0]$$

Putting $\lambda=-29/7$ in (i), we obtain the equation of the required plane as

$$-51x-15y+50z-173=0 \text{ or, } 51x+15y-50z+173=0.$$

EXAMPLE 5 *Find the equation of the plane through the line of intersection of* $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ *and* $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ *and perpendicular to* $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0.$ **[CBSE 2017]**

SOLUTION The equation of any plane through the line of intersection of the given planes is

$$[\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})-1]+\lambda[\vec{r}\cdot(\hat{i}-\hat{j})+4]=0 \text{ or, } \vec{r}\cdot[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}]=1-4\lambda \quad ...(i)$$

If plane (i) is perpendicular to $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$, then

$$[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}]\cdot(2\hat{i}-\hat{j}+\hat{k})=0 \quad [\text{Using } \vec{n_1}\cdot\vec{n_2}=0]$$

$$\Rightarrow \quad 2(2+\lambda)+(3+\lambda)+4=0 \Rightarrow 3\lambda+11=0 \Rightarrow \lambda=-\frac{11}{3}$$

Putting $\lambda=-\dfrac{11}{3}$ in (i), we obtain the equation of the required plane as

$$\vec{r}\cdot\left\{\left(2-\frac{11}{3}\right)\hat{i}-\left(3-\frac{11}{3}\right)\hat{j}+4\hat{k}\right\}=1+\frac{44}{3} \text{ or, } \vec{r}\cdot(-5\hat{i}+2\hat{j}+12\hat{k})=47.$$

EXAMPLE 6 *Find the equation of the plane passing through the intersection of the planes* $2x-3y+z-4=0$ *and* $x-y+z+1=0$ *and perpendicular to the plane* $x+2y-3z+6=0.$

SOLUTION The equation of a plane the intersection of the planes $x-y+z+1=0$ and $2x-3y+z-4=0$ is

$$(2x-3y+z-4)+\lambda(x-y+z+1)=0$$

or, $$x(2+\lambda)-y(3+\lambda)+z(1+\lambda)-4+\lambda=0 \quad ...(i)$$

This is perpendicular to the plane $x+2y-3z+6=0$.

$$\therefore \quad 1(2+\lambda)-2(3+\lambda)-3(1+\lambda)=0 \quad [\text{Using: } a_1a_2+b_1b_2+c_1c_2=0]$$

$$\Rightarrow \quad 2+\lambda-6-2\lambda-3-3\lambda=0 \Rightarrow -4\lambda-7=0 \Rightarrow \lambda=-\frac{7}{4}$$

Putting $\lambda=-\dfrac{7}{4}$ in (i), we obtain

$$x\left(2-\frac{7}{4}\right)-y\left(3-\frac{7}{4}\right)+z\left(1-\frac{7}{4}\right)-4-\frac{7}{4}=0 \text{ or, } x-5y-3z-23=0,$$

which is the required equation of the plane.

EXAMPLE 7 *Find the cartesian as well as vector equations of the planes through the intersection of the planes* $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ *and* $\vec{r}.(3\hat{i}-\hat{j}+4\hat{k})=0$ *which are at a unit distance from the origin.*

[CBSE 2005, 2013, NCERT EXEMPLAR]

SOLUTION The equation of the planes through the intersection of the planes $\vec{r}\cdot(2\hat{i}+6\hat{j})+12=0$ and $\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})=0$ is

$$[\vec{r}.(2\hat{i}+6\hat{j})+12]+\lambda[\vec{r}\cdot(3\hat{i}-\hat{j}+4\hat{k})]=0 \text{ or, } \vec{r}.\{(2+3\lambda)\hat{i}+(6-\lambda)\hat{j}+4\lambda\hat{k}\}+12=0 \; ...(i)$$

This equation can be re-written as

$$\vec{r}\cdot\left\{(-2-3\lambda)\,\hat{i}+(\lambda-6)\,\hat{j}+(-4\lambda)\,\hat{k}\right\}=12$$

or, $$\frac{\vec{r}\cdot\left\{(-2-3\lambda)\,\hat{i}+(\lambda-6)\,\hat{j}+(-4\lambda)\,\hat{k}\right\}}{\sqrt{(2+3\lambda)^2+(\lambda-6)^2+(4\lambda)^2}}=\frac{12}{\sqrt{(2+3\lambda)^2+(\lambda-6)^2+(4\lambda)^2}}$$

This is the normal form of plane (i) and its distance from the origin is $\dfrac{12}{\sqrt{(2+3\lambda)^2+(\lambda-6)^2+(4\lambda)^2}}$.

It is given that the plane (i) is at a unit distance from the origin.

$$\therefore\quad \frac{12}{\sqrt{(2+3\lambda)^2+(\lambda-6)^2+(4\lambda)^2}}=1$$

$$\Rightarrow\quad 144=(2+3\lambda)^2+(\lambda-6)^2+(4\lambda)^2 \Rightarrow 26\lambda^2=104 \Rightarrow \lambda^2=4 \Rightarrow \lambda=\pm 2$$

Putting the values of λ in (i), we obtain

$$\vec{r}\cdot(8\,\hat{i}+4\,\hat{j}+8\,\hat{k})=12=0 \quad\text{and}\quad \vec{r}\cdot(-4\,\hat{i}+8\,\hat{j}-8\,\hat{k})+12=0$$

as the equations of the required planes.

These equations can also be written as

$$\vec{r}\,.(2\,\hat{i}+\hat{j}+2\,\hat{i})+3=0 \quad\text{and}\quad \vec{r}\,.(-\,\hat{i}+2\,\hat{j}-2\hat{k})+3=0$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 8 *The plane $lx+my=0$ is rotated through an angle α about its line of intersection with the plane $z=0$. Prove that the equation of the plane in its new position is $lx+my\pm(\sqrt{l^2+m^2}\ \tan\alpha)z=0$.*

SOLUTION The equation of any plane passing through the line of intersection of the planes $lx+my=0$ and $z=0$ is

$$lx+my+\lambda z=0 \qquad \text{...(i)}$$

It is given that the angle between the plane $lx+my=0$ and the plane (i) is α.

$$\therefore\quad \cos\alpha=\frac{l^2+m^2+0\,\lambda}{\sqrt{l^2+m^2}\,\sqrt{l^2+m^2+\lambda^2}}=\sqrt{\frac{l^2+m^2}{l^2+m^2+\lambda^2}}$$

$$\Rightarrow\quad (l^2+m^2+\lambda^2)\cos^2\alpha=l^2+m^2 \Rightarrow \lambda^2\cos^2\alpha=(l^2+m^2)\sin^2\alpha \Rightarrow \lambda=\pm\sqrt{l^2+m^2}\ \tan\alpha.$$

Substituting the value of λ in (i), we get $lx+my\pm(\sqrt{l^2+m^2}\ \tan\alpha)\,z=0$ as the required equation of the plane.

EXAMPLE 9 *The plane $x-2y+3z=0$ is rotated through a right angle about the line of intersection with the plane $2x+3y-4z-5=0$, find the equation of the plane in its new position.*

SOLUTION The equation of any plane through the intersection of the planes $x-2y+3z=0$ and $2x+3y-4z-5=0$ is

$$(x-2y+3z)+\lambda(2x+3y-4z-5)=0 \ \text{ or, }\ (1+2\lambda)x+(3\lambda-2)y+(3-4\lambda)z-5\lambda=0 \qquad \text{...(i)}$$

It is given that the angle between the planes $x-2y+3z=0$ and the plane (i) is a right angle.

$$(1+2\lambda)\times 1-2\times(3\lambda-2)+(3-4\lambda)\times 3=0$$

$$\Rightarrow\quad 1+2\lambda-6\lambda+4+9-12\lambda=0 \Rightarrow 16\lambda=14 \Rightarrow \lambda=7/8.$$

Substituting the value of λ in (i), we obtain $22x+5y-4z-35=0$ as the required equation of the plane.

EXERCISE 28.8

BASIC

1. Find the equation of the plane which is parallel to $2x - 3y + z = 0$ and which passes through $(1, -1, 2)$.
2. Find the equation of the plane through $(3, 4, -1)$ which is parallel to the plane $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$.
3. Find the equation of the plane passing through the line of intersection of the planes $2x - 7y + 4z - 3 = 0$, $3x - 5y + 4z + 11 = 0$ and the point $(-2, 1, 3)$.
4. Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 4 = 0$ and $x - y + z + 3 = 0$ and passing through the origin. [CBSE 2020]
5. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$ and the point $(1, 1, 1)$. [CBSE 2014, NCERT]
6. Find the equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + \hat{j} + 3\hat{k}) = 7$, $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and the point $(2, 1, 3)$. [CBSE 2007, 2022]
7. Find the equation of the plane through the intersection of the planes $3x - y + 2z = 4$ and $x + y + z = 2$ and the point $(2, 2, 1)$. [NCERT]
8. Find the equation of the plane passing through (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$.

BASED ON LOTS

9. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$, $2x + y - z + 5 = 0$.
10. Find the equation of the plane through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) + 6 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, which is at a unit distance from the origin. [CBSE 2010, 2013, 2020]
11. Find the equation of the plane passing through the intersection of the planes $2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z - 4 = 0$.
12. Find the equation of the plane that contains the line of intersection of the planes $\vec{r} . (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\vec{r} . (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r} . (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$. [CBSE 2011, 2013, 2019, NCERT]
13. Find the equation of the plane through the point $2\hat{i} + \hat{j} - \hat{k}$ and passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 0$ and $\vec{r} \cdot (\hat{j} + 2\hat{k}) = 0$.
14. Find the equation of the plane passing through the line of intersection of the planes $2x - y = 0$ and $3z - y = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$.
15. Find the equation of the plane which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ and which is perpendicular to the plane $5x + 3y - 6z + 8 = 0$. [NCERT EXEMPLAR]
16. Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. [CBSE 2014, NCERT]
17. Find the vector equation (in scalar product form) of the plane containing the line of intersection of the planes $x - 3y + 2z - 5 = 0$ and $2x - y + 3z - 1 = 0$ and passing through $(1, -2, 3)$.

18. Find the equation of the plane which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$ and whose x-intercept is twice its z-intercept. Hence, write the equation of the plane passing through the point $(2, 3, -1)$ and parallel to the plane obtained above. [CBSE 2016]

19. Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ and twice of its y-intercept is equal to three times its z-intercept. [CBSE 2017]

ANSWERS

1. $2x-3y+z=7$
2. $\vec{r}\cdot(2\hat{i}-3\hat{j}+5\hat{k})+11=0$
3. $15x-47y+28z=7$
4. $x-10y-5z=0$
5. $\vec{r}.(20\hat{i}+23\hat{j}+26\hat{k})=69$
6. $\vec{r}\cdot(2\hat{i}-13\hat{j}+3\hat{k})=0$
7. $7x-5y+4z = 8$
8. $x+y+z=a+b+c$
9. $51x+15y-50z+173=0$
10. $\vec{r}\cdot(-2\hat{i}+4\hat{j}+4\hat{k})+6=0$ or, $\vec{r}\cdot(4\hat{i}+2\hat{j}-4\hat{k})+6=0$
11. $7x+13y+4z-9=0$
12. $33x+45y+50z-41=0$
13. $\vec{r}\cdot(\hat{i}+9\hat{j}+11\hat{k})=0$
14. $28x-17y+9z=0$
15. $33x+45y+50z-41=0$
16. $\vec{r}\cdot(\hat{i}-\hat{k})+2=0$
17. $\vec{r}\cdot(\hat{i}+7\hat{j})+13=0$
18. $7x+11y+14z-15=0$, $\vec{r}\cdot(7\hat{i}+11\hat{j}+14\hat{k})=33$, $13x+14y+11z=0$
19. $x+2y+3z=4$

HINTS TO SELECTED PROBLEMS

7. The equation of the family of planes through the intersection of the planes $3x-y+2z=4$ and $x+y+z=2$ is

$$(3x-y+2z-4)+\lambda(x+y+z-2) = 0 \qquad \text{...(i)}$$

If it passes through (2, 2, 1), then

$$(6-2+2-4)+\lambda(2+2+1-2) = 0 \Rightarrow 2+3\lambda = 0 \Rightarrow \lambda = -\frac{2}{3}$$

Substituting $\lambda=-\frac{2}{3}$ in (i), we obtain $7x-5y+4z=8$ as the equation of the required plane.

8. The equation of the plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=6$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})=-5$ is

$$[\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-6]+\lambda[\vec{r}\cdot(2\hat{i}+3\hat{j}+4\hat{k})+5] = 0$$

or, $\vec{r}\cdot\{(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}\}+5\lambda-6=0$ ...(i)

It passes through the point (1, 1, 1) whose position vector is $\hat{i}+\hat{j}+\hat{k}$. So, $\vec{r}=\hat{i}+\hat{j}+\hat{k}$ satisfies equation (i).

$$\therefore\quad (\hat{i}+\hat{j}+\hat{k})\cdot\{(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1+4\lambda)\hat{k}\}+5\lambda-6 = 0$$

$$\Rightarrow\quad 1+2\lambda+1+3\lambda+1+4\lambda+5\lambda-6 = 0 \Rightarrow 14\lambda = 3 \Rightarrow \lambda = \frac{3}{14}$$

Putting $\lambda=\frac{3}{14}$ in (i), we obtain $\vec{r}\cdot(20\hat{i}+23\hat{j}+26\hat{k}) = 69$ as the equation of the desired plane.

12. The equation of the family of planes containing the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=4$ and $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ is

$$\{\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4\}+\lambda\{\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5\}=0$$

or, $\vec{r}\cdot\{(2\lambda+1)\hat{i}\ \ (\lambda+2)\hat{j}+(3-\lambda)\hat{k}\}=4-5\lambda$ ...(i)

This is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$

$\therefore\quad (5\hat{i}+3\hat{j}-6\hat{k})\cdot\{(2\lambda+1)\hat{i}+(\lambda+2)\hat{j}+(3-\lambda)\hat{k}\}=0$

$\Rightarrow\quad 10\lambda+5+3\lambda+6-18+6\lambda=0 \Rightarrow \lambda=\dfrac{7}{19}$

Putting $\lambda=\dfrac{7}{19}$ in (i), we obtain $\vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})=41$ as the required vector equation of the plane and its cartesian equation is $33x+45y+50z=41$.

16. The equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ is

$(x+y+z-1)+\lambda(2x+3y+4z-5)=0$ ...(i)

or, $(2\lambda+1)x+(3\lambda+1)y+(4\lambda+1)z=5\lambda+1$

It is perpendicular to the plane $x-y+z=0$.

$\therefore\quad (2\lambda+1)\times 1+(3\lambda+1)\times(-1)+(4\lambda+1)(1)=0 \Rightarrow 2\lambda+1-3\lambda-1+4\lambda+1=0 \Rightarrow \lambda=-\dfrac{1}{3}$

Putting $\lambda=-\dfrac{1}{3}$ in (i), we obtain $x-z+2=0$ as the equation of the required plane and its vector equation is $\vec{r}\cdot(\hat{i}-\hat{k})+2=0$.

28.11 DISTANCE OF A POINT FROM A PLANE

In this section, we shall find the perpendicular distance of a point from a plane in both cartesian and vector forms.

28.11.1 VECTOR FORM

THEOREM 1 *Prove that the length of the perpendicular from a point having position vector $\vec{a}$ to the plane $\vec{r}\cdot\vec{n}=d$ is $p=\dfrac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$.*

PROOF Let π be the given plane and $P(\vec{a})$ be the given point. Let PM be the length of the perpendicular from P on the plane π. Since line PM passes through $P(\vec{a})$ and is parallel to the vector $\vec{n}$ which is normal to the plane π. So, vector equation of line PM is

$\vec{r}=\vec{a}+\lambda\vec{n},$ ...(i)

where λ is a scalar.

Clearly, point M is the intersection of line (i) and the given plane. Therefore, for point M, we must have

$(\vec{a}+\lambda\vec{n})\cdot\vec{n}=d$ [On substituting $\vec{r}=\vec{a}+\lambda\vec{n}$ in the equation of the plane]

$\Rightarrow\quad \vec{a}\cdot\vec{n}+\lambda\,\vec{n}\cdot\vec{n}=d \Rightarrow \lambda=\dfrac{d-(\vec{a}\cdot\vec{n})}{\vec{n}\cdot\vec{n}} \Rightarrow \lambda=\dfrac{d-(\vec{a}\cdot\vec{n})}{|\vec{n}|^2}$

Putting this value of λ in (i), we obtain the position vector of M as $\vec{r} = \vec{a} + \left(\dfrac{d - (\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right)\vec{n}$

Now, $\quad \vec{PM} = \text{P.V. of } M - \text{P.V. of } P = \vec{a} + \left\{\dfrac{d - (\vec{a}\cdot\vec{n})}{|\vec{n}|^2}\right\}\vec{n} - \vec{a} = \dfrac{\{d - (\vec{a}\cdot\vec{n})\}\,\vec{n}}{|\vec{n}|^2}$

$$\therefore \quad PM = |\vec{PM}| = \left|\frac{[d - (\vec{a}\cdot\vec{n})]\,\vec{n}}{|\vec{n}|^2}\right|$$

$$\Rightarrow \quad PM = \frac{|d - (\vec{a}\cdot\vec{n})|\;|\vec{n}|}{|\vec{n}|^2}$$

$$\Rightarrow \quad PM = \frac{|(\vec{a}\cdot\vec{n}) - d|}{|\vec{n}|}$$

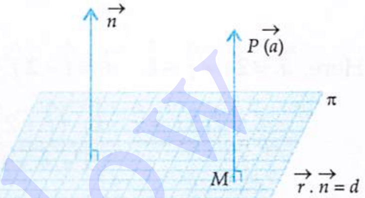


Fig. 28.19

Thus, the length of the perpendicular from a point having position vector $\vec{a}$ on the plane $\vec{r}\cdot\vec{n} = d$ is given by $\dfrac{|(\vec{a}\cdot\vec{n}) - d|}{|\vec{n}|}$. **Q.E.D.**

28.11.2 CARTESIAN FORM

THEOREM 2 *Prove that the length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by*

$$p = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

PROOF Let M be the foot of the perpendicular from $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$. Clearly, line PM passes through the point $P(x_1, y_1, z_1)$ and is normal to the plane. So, its direction ratios are proportional to a, b, c. Consequently, the equation of PM is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \qquad \text{...(i)}$$

The coordinates of any point on this line are $(x_1 + ar, y_1 + br, z_1 + cr)$, where r is a real number.

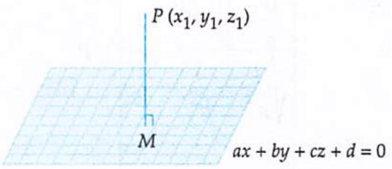


Fig. 28.20

This point coincides with M iff it lies on the plane.

$$\therefore \quad a(x_1 + ar) + b(y_1 + br) + c(z_1 + cr) + d = 0 \Rightarrow r = -\frac{(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2} \qquad \text{...(ii)}$$

Now, $\quad PM = \sqrt{(x_1 + ar - x_1)^2 + (y_1 + br - y_1)^2 + (z_1 + cr - z_1)^2}$

$$\Rightarrow \quad PM = \sqrt{a^2 + b^2 + c^2}\;|r| = \sqrt{a^2 + b^2 + c^2}\left|\frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}\right| \qquad \text{[From (ii)]}$$

$$\Rightarrow \quad PM = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the distance of the point* $2\hat{i}+\hat{j}-\hat{k}$ *from the plane* $\vec{r}\cdot(\hat{i}-2\hat{j}+4\hat{k})=9$.

SOLUTION We know that the perpendicular distance of a point P with position vector $\vec{a}$ from the plane $\vec{r}\cdot\vec{n}=d$ is given by $p=\dfrac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}$.

Here, $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$, $\vec{n}=\hat{i}-2\hat{j}+4\hat{k}$ and $d=9$. So, required distance p is given by

$$p=\frac{|(2\hat{i}+\hat{j}-\hat{k})\cdot(\hat{i}-2\hat{j}+4\hat{k})-9|}{\sqrt{1+4+16}}=\frac{|2-2-4-9|}{\sqrt{21}}=\frac{13}{\sqrt{21}}$$

EXAMPLE 2 *Find the distance of the point* $(2, 1, 0)$ *from the plane* $2x+y+2z+5=0$.

SOLUTION We know that the distance of the point (x_1, y_1, z_1) from the plane $ax+by+cz+d=0$ is given by $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

So, required distance $=\dfrac{|2\times2+1+2\times0+5|}{\sqrt{2^2+1^2+2^2}}=\dfrac{10}{3}$.

EXAMPLE 3 *Prove that if a plane has the intercepts a, b, c and is at a distance of p units from the origin, then* $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}=\dfrac{1}{p^2}$.

SOLUTION The equation of the plane having intercepts a, b and c on the coordinate axes is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

It is given that this plane is at a distance of p units from the origin.

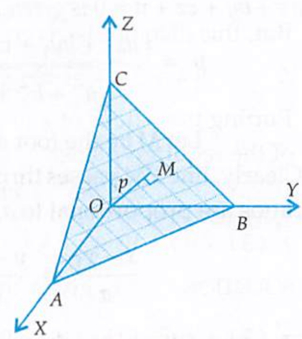


Fig. 28.21

$$\therefore\quad \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=p$$

$$\Rightarrow\quad \left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=p\Rightarrow\frac{1}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}=p^2\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$$

EXAMPLE 4 *Show that the points* $\hat{i}-\hat{j}+3\hat{k}$ *and* $3(\hat{i}+\hat{j}+\hat{k})$ *are equidistant from the plane* $\vec{r}\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9=0$ *and lie on opposite sides of it.* **[NCERT EXEMPLAR]**

SOLUTION Using the formula for the perpendicular distance of a plane from a point, we have

$$PM=\left|\frac{(\hat{i}-\hat{j}+3\hat{k})\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9}{\sqrt{25+4+49}}\right|=\frac{|5-2-21+9|}{\sqrt{78}}=\frac{9}{\sqrt{78}}$$

and, $\quad QM = \left| \dfrac{(3\hat{i} + 3\hat{j} + 3\hat{k})\cdot(5\hat{i} + 2\hat{j} - 7\hat{k}) + 9}{\sqrt{25+4+49}} \right| = \dfrac{|15+6-21+9|}{\sqrt{78}} = \dfrac{9}{\sqrt{78}}$

Clearly, $PM = QM$.

So, P and Q are equidistant from the given plane. The position vector of M, the mid point of PQ, is $2\hat{i} + \hat{j} + 3\hat{k}$. We observe that

$$(2\hat{i} + \hat{j} + 3\hat{k})\cdot(5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 10 + 2 - 21 + 9 = 0$$

i.e. the position vector of M satisfies the equation of the plane.

So, M lies on the given plane.

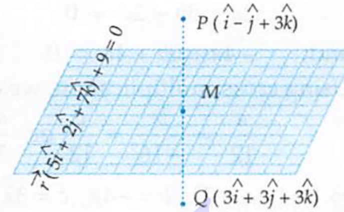


Fig. 28.22

Hence, P and Q are equidistant from the given plane and lie on opposite side of it.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 5 *Find the equations of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ which is at a unit distance from the point* (1, 2, 3).

SOLUTION The equation of a plane parallel to the plane $x - 2y + 2z - 3 = 0$ is

$$x - 2y + 2z + \lambda = 0 \qquad \text{...(i)}$$

Distance of plane (i) from point (1, 2, 3) is given by $\left| \dfrac{1 - 2\times 2 + 2\times 3 + \lambda}{\sqrt{1^2 + (-2)^2 + 2^2}} \right| = \left| \dfrac{\lambda + 3}{3} \right|$

But, this distance is given to be unity.

$\therefore \quad |\lambda + 3| = 3 \Rightarrow \lambda + 3 = \pm 3 \Rightarrow \lambda = 0$ or, $\lambda = -6$

Putting the values of λ in (i), we obtain $x - 2y + 2z = 0$ and $x - 2y + 2z - 6 = 0$ as the equations of the required planes.

EXAMPLE 6 *If the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ be equidistant from the plane $\vec{r}\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, find the value of λ.* **[NCERT]**

SOLUTION It is given that the points $(1, 1, \lambda)$ and $(-3, 0, 1)$ are equidistant from the plane $\vec{r}\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$.

$$\therefore \quad \left| \frac{(\hat{i} + \hat{j} + \lambda\hat{k})\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13}{\sqrt{9+16+144}} \right| = \left| \frac{(-3\hat{i} + 0\hat{j} + \hat{k})\cdot(3\hat{i} + 4\hat{j} - 12\hat{k}) + 13}{\sqrt{9+16+144}} \right|$$

$$\Rightarrow \quad \left| \frac{3 + 4 - 12\lambda + 13}{13} \right| = \left| \frac{-9 + 0 - 12 + 13}{13} \right|$$

$$\Rightarrow \quad |20 - 12\lambda| = 8$$

$$\Rightarrow \quad 20 - 12\lambda = \pm 8$$

$$\Rightarrow \quad 20 - 12\lambda = 8 \text{ or, } 20 - 12\lambda = -8 \Rightarrow 12\lambda = 12 \text{ or, } 12\lambda = 28 \Rightarrow \lambda = 1 \text{ or, } \lambda = \frac{7}{3}.$$

EXAMPLE 7 *Find the distance between the point $P(6, 5, 9)$ and the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$.* **[NCERT, CBSE 2010, 2012]**

SOLUTION The equation of a plane passing through $A(3, -1, 2)$ is

$$a(x-3) + b(y+1) + c(z-2) = 0 \quad \text{...(i)}$$

If this plane passes through $B(5, 2, 4)$ and $C(-1, -1, 6)$. Then,

$$2a + 3b + 2c = 0$$

and, $\quad -4a + 0b + 4c = 0$

Using cross-multiplication, we obtain

$$\frac{a}{12} = \frac{b}{-16} = \frac{c}{12} \text{ or, } \frac{a}{3} = \frac{b}{-4} = \frac{c}{3} = \lambda \text{ (say)}$$

$\Rightarrow \quad a = 3\lambda,\ b = -4\lambda,\ c = 3\lambda$

Substituting the values of a, b, c in (i), we obtain $3(x-3) - 4(y+1) + 3(z-2) = 0$ or, $3x - 4y + 3z = 19$ as the equation of the plane passing through A, B and C.

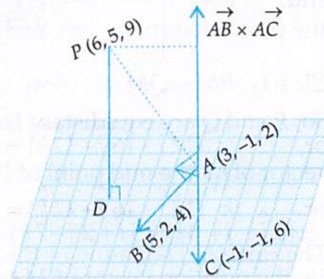


Fig. 28.23

The distance of $P(6, 5, 9)$ from this plane is given by

$$p = \left| \frac{18 - 20 + 27 - 19}{\sqrt{9 + 16 + 9}} \right| = \frac{6}{\sqrt{34}}$$

ALITER Let D be the foot of the perpendicular drawn from P to the plane passing through the points A, B and C. Then, PD is the required distance. Clearly $\vec{AB} \times \vec{AC}$ is normal to the plane.

$$\therefore \quad PD = \text{Projection of } \vec{AP} \text{ on } \vec{AB} \times \vec{AC} = \left| \frac{\vec{AP} \cdot (\vec{AB} \times \vec{AC})}{|\vec{AB} \times \vec{AC}|} \right|$$

$$\text{Now, } \quad \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k} \text{ and, } \vec{AP} = 3\hat{i} + 6\hat{j} + 7\hat{k}$$

$$\therefore \quad \vec{AP} \cdot (\vec{AB} \times \vec{AC}) = (3\hat{i} + 6\hat{j} + 7\hat{k}) \cdot (12\hat{i} - 16\hat{j} + 12\hat{k}) = 36 - 96 + 84 = 24$$

$$\text{and, } \quad |\vec{AB} \times \vec{AC}| = \sqrt{144 + 256 + 144} = \sqrt{544} = 4\sqrt{34}$$

$$\therefore \quad PD = \frac{24}{4\sqrt{34}} = \frac{6}{\sqrt{34}}$$

EXAMPLE 8 *Find the equation of a plane passing through the point $P(6, 5, 9)$ and parallel to the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$. Also, find the distance of this plane from the point A.* [CBSE 2015]

SOLUTION A vector $\vec{n}$ normal to the plane determined by the points $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$ is given by $\vec{n} = \vec{AB} \times \vec{AC}$. We have, $\vec{AB} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\vec{AC} = -4\hat{i} + 0\hat{j} + 4\hat{k}$

$$\therefore \quad \vec{n} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 12\hat{i} - 16\hat{j} + 12\hat{k}$$

Clearly, $\vec{n} = 12\hat{i} - 16\hat{j} + 12\hat{k}$ is also normal to the plane passing through $P(6, 5, 9)$ and parallel to the plane determined by point A, B and C. So, its equation is

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n}, \text{ where } \vec{a} = 6\hat{i} + 5\hat{j} + 9\hat{k}$$

or, $\vec{r} \cdot (12\hat{i} - 16\hat{j} + 12\hat{k}) = (12\hat{i} - 16\hat{j} + 12\hat{k}) \cdot (6\hat{i} + 5\hat{j} + 9\hat{k})$

or, $\vec{r} \cdot (12\hat{i} - 16\hat{j} + 12\hat{k}) = 72 - 80 + 108$ or, $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 3\hat{k}) = 25$

The cartesian equation of this plane is $3x - 4y + 3z = 25$.

The distance d of this plane from the point $A(3, -1, 2)$ is given by

$$d = \left| \frac{3 \times 3 - 4 \times -1 + 3 \times 2 - 25}{\sqrt{9 + 16 + 9}} \right| = \frac{6}{\sqrt{34}}$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 9 *Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a′, b′, c′ respectively, prove that*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$ [NCERT EXEMPLAR]

SOLUTION The equation of the plane with reference to two systems of rectangular axes are

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

and, $$\frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \quad \text{...(ii)}$$

It is given that the two systems of rectangular axes have the same origin O. Therefore, distances of planes (i) and (ii) from the origin O are same. That is

$$\left| \frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \right| = \left| \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} \right|$$

$$\Rightarrow \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

EXAMPLE 10 *A variable plane which remains at a constant distance 3p from the origin cut the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is* $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

SOLUTION Let the equation of the plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ...(i)

where a, b, c are variables.

This meets X, Y and Z axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of triangle ABC. Then,

$$\alpha = \frac{a + 0 + 0}{3} = \frac{a}{3}, \beta = \frac{0 + b + 0}{3} = \frac{b}{3}, \gamma = \frac{0 + 0 + c}{3} = \frac{c}{3}. \quad \text{...(ii)}$$

The plane (i) is at a distance $3p$ from the origin.

$\therefore \quad 3p$ = Length of perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow \quad 3p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \quad 3p = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{9p^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad ...(iii)$$

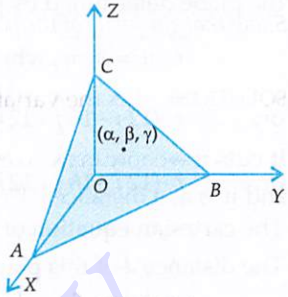


Fig. 28.24

From (ii), we obtain: $a = 3\alpha, b = 3\beta$ and $c = 3\gamma$.

Substituting the values of a, b, c in (iii), we obtain

$$\frac{1}{9p^2} = \frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2} \Rightarrow \frac{1}{p^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

Hence, the locus of (α, β, γ) is $\frac{1}{p^2} = \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$ or, $x^{-2}+y^{-2}+z^{-2} = p^{-2}$.

EXAMPLE 11 *A variable plane is at a constant distance p from the origin and meets the coordinate axes in A, B, C. Show that the locus of the centroid of the tetrahedron OABC is* $x^{-2}+y^{-2}+z^{-2} = 16p^{-2}$.

SOLUTION Let the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ ...(i)

This meets the coordinate axes at $A\ (a, 0, 0)$, $B\ (0, b, 0)$ and $C\ (0, 0, c)$ respectively.

Let (α, β, γ) be the coordinates of the centroid of tetrahedron $OABC$. Then,

$$\alpha = \frac{0+a+0+0}{4} = \frac{a}{4},\ \beta = \frac{0+0+b+0}{4} = \frac{b}{4}, \gamma = \frac{0+0+0+c}{4} = \frac{c}{4} \quad ...(ii)$$

The plane in (i) is at a constant distance p from the origin. Therefore,

p = Length perpendicular from (0, 0, 0) to plane (i)

$$\Rightarrow \quad p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \quad p = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad ...(iii)$$

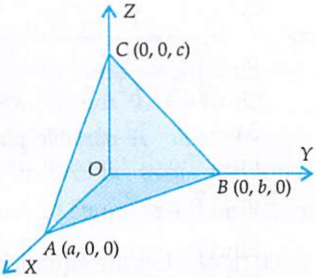


Fig. 28.25

Eliminating variables a, b, c from (ii) and (iii), we obtain

$$\frac{1}{p^2} = \frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2} \Rightarrow 16p^{-2} = \alpha^{-2}+\beta^{-2}+\gamma^{-2}$$

Hence, the locus of (α, β, γ) is $16p^{-2} = x^{-2}+y^{-2}+z^{-2}$.

EXAMPLE 12 *If a variable plane at a constant distance p from the origin meets the coordinate axes in points A, B and C respectively. Through these points, planes are drawn parallel to the coordinate planes. Show that the locus of the point of intersection is* $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.

SOLUTION Let the variable plane be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

It cuts the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and it is at a distance p from the origin.

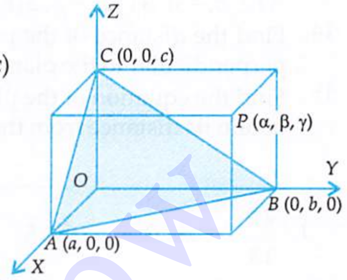


Fig. 28.26

$$\therefore \quad \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = p$$

$$\Rightarrow \quad \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = p \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \quad \text{...(i)}$$

Let $P(\alpha, \beta, \gamma)$ be the point of intersection of planes through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and parallel to yz, zx and xy-planes respectively.

The equations of planes passing through $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$ and parallel to respectively yz, zx and xy-planes are $x = a$, $y = b$ and $z = c$ respectively.

These three planes intersect at point (a, b, c).

$$\therefore \quad \alpha = a, \beta = b \text{ and } \gamma = c \quad \text{...(ii)}$$

Eliminating a, b, c between (i) and (ii), we get: $\alpha^{-2}+\beta^{-2}+\gamma^{-2}=p^{-2}$

Hence, the locus of $P(\alpha, \beta, \gamma)$ is $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

EXERCISE 28.9

BASIC

1. Find the distance of the point $2\hat{i}-\hat{j}-4\hat{k}$ from the plane $\vec{r}\cdot(3\hat{i}-4\hat{j}+12\hat{k})-9=0$.
2. Show that the points $\hat{i}-\hat{j}+3\hat{k}$ and $3\hat{i}+3\hat{j}+3\hat{k}$ are equidistant from the plane $\vec{r}\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9=0$.
3. Find the distance of the point $(2, 3, -5)$ from the plane $x+2y-2z-9=0$.
4. Show that the points $(1, 1, 1)$ and $(-3, 0, 1)$ are equidistant from the plane $3x+4y-12z+13=0$.
5. Find the distance of the point $(2, 3, 5)$ from the xy-plane.
6. Find the distance of the point $(3, 3, 3)$ from the plane $\vec{r}\cdot(5i+2j-7k)+9=0$.
7. Find the values of λ, for which the distance of point $(2, 1, \lambda)$ from the plane $3x+5y+4z=11$ is $2\sqrt{2}$ units. [CBSE 2022]
8. If the product of distances of the point $(1, 1, 1)$ from the origin and the plane $x-y+z+\lambda=0$ be 5, find the value of λ.
9. A plane makes intercepts $-6, 3, 4$ respectively on the coordinate axes. Find the length of the perpendicular from the origin on it. [CBSE 2014]

BASED ON LOTS

10. Find the equations of the planes parallel to the plane $x+2y-2z+8=0$ which are at distance of 2 units from the point $(2, 1, 1)$.

11. Find the equations of the planes parallel to the plane $x-2y+2z-3=0$ and which are at a unit distance from the point (1, 1, 1).
12. Find an equation for the set of all points that are equidistant from the planes $3x-4y+12z=6$ and $4x+3z=7$.
13. Find the distance between the point (7, 2, 4) and the plane determined by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $(5, 3, -3)$. **[CBSE 2014]**
14. Find the distance of the point $(1, -2, 4)$ from plane passing through the point (1, 2, 2) and perpendicular to the planes $x-y+2z=3$ and $2x-2y+z+12=0$. **[CBSE 2017]**
15. Find the equation of the plane passing through points $(2, 1, 0)$, $(3, -2, -2)$ and $(1, 1, -7)$. Also, obtain its distance from the origin. **[CBSE 2022]**

ANSWERS

1. $\frac{47}{13}$ 3. 3 5. 5 6. $\frac{9}{\sqrt{78}}$ 7. ± 5 8. 4 9. $\frac{12}{\sqrt{29}}$

10. $x+2y-2z+4=0$ or $x+2y-2z-8=0$ 11. $x-2y+2z+2=0$, $x-2y+2z-4=0$

12. $37x+20y-21z=61$, $67x-20y+99z=121$ 13. $\sqrt{29}$ 14. $2\sqrt{2}$ 15. $\frac{17}{\sqrt{59}}$

28.12 DISTANCE BETWEEN THE PARALLEL PLANES

Let $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$ be two parallel planes. In order to find the distance between them, we may follow the following algorithm.

ALGORITHM

Step I *Take an arbitrary point $P(x_1, y_1, z_1)$ on one of the planes, say $ax+by+cz+d_1=0$.*

Step II *Find length of the perpendicular 'd' drawn from $P(x_1, y_1, z_1)$ on the other plane i.e. $ax+by+cz+d_2=0$. Clearly,*

$$d=\left|\frac{ax_1+by_1+cz_1+d_2}{\sqrt{a^2+b^2+c^2}}\right|$$

Step III *As $P(x_1, y_1, z_1)$ lies on the plane $ax+by+cz+d_1=0$.*

$$\therefore \quad ax_1+by_1+cz_1+d_1=0 \Rightarrow ax_1+by_1+cz_1=-d_1$$

Step IV *Substitute $ax_1+by_1+cz_1=-d_1$ in the expression for d obtained in step II to get $d=\frac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$, which gives the required distance.*

REMARK 1 *We may use the formula $d=\frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$ to find the distance between parallel planes $ax+by+cz+d_1=0$ and $ax+by+cz+d_2=0$.*

REMARK 2 *Instead of taking an arbitrary point on one of the parallel planes, we may take a specific point by giving values to any two variables randomly and computing the third variable form the equation of the plane.*

REMARK 3 *The distance between parallel planes $ax+by+cz+d_1=0$ and $\lambda(ax+by+cz)+d_2=0$ is given by $d=\frac{|d_1-d_2/\lambda|}{\sqrt{a^2+b^2+c^2}}$.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the distance between the parallel planes $x+y-z+4=0$ and $x+y-z+5=0$.*

SOLUTION Let $P(x_1, y_1, z_1)$ be any point on $x+y-z+4=0$. Then,

$$x_1+y_1-z_1+4=0 \quad \text{...(i)}$$

Let d be the distance between the given planes. Then,

$$d = \text{Length of the perpendicular from } P(x_1, y_1, z_1) \text{ to } x+y-z+5=0$$

$$\Rightarrow \quad d = \left|\frac{x_1+y_1-z_1+5}{\sqrt{1^2+1^2+(-1)^2}}\right| = \frac{|-4+5|}{\sqrt{3}} = \frac{1}{\sqrt{3}} \qquad \text{[Using (i)]}$$

Hence, the distance between the given parallel planes is $\frac{1}{\sqrt{3}}$.

ALITER Putting $x=0$, $y=0$ in $x+y-z+4=0$, we get $z=4$. So, the coordinates of a point on the plane $x+y-z+4=0$ are $(0, 0, 4)$. Let d be the distance between the given planes. Then,

$$d = \text{Length of the perpendicular from } (0, 0, 4) \text{ on the plane } x+y-z+5=0$$

$$\Rightarrow \quad d = \left|\frac{0+0-4+5}{\sqrt{1^2+1^2+(-1)^2}}\right| = \frac{1}{\sqrt{3}}$$

EXAMPLE 2 *Find the distance between the parallel planes* $2x-y+2z+3=0$ *and* $4x-2y+4z+5=0$.

SOLUTION Let $P(x_1, y_1, z_1)$ be any point on $2x-y+2z+3=0$. Then,

$$2x_1-y_1+2z_1+3 = 0 \qquad \text{...(i)}$$

Let d be the distance between the given planes. Then,

$$d = \text{Length of the perpendicular from } P(x_1, y_1, z_1) \text{ to } 4x-2y+4z+5=0$$

$$\Rightarrow \quad d = \left|\frac{4x_1-2y_1+4z_1+5}{\sqrt{4^2+(-2)^2+4^2}}\right| = \left|\frac{2(2x_1-y_1+2z_1)+5}{\sqrt{36}}\right| = \frac{|2(-3)+5|}{6} = \frac{1}{6} \qquad \text{[Using (i)]}$$

Hence, the distance between the given parallel planes is $\frac{1}{6}$.

ALITER Putting $x=0$, $z=0$ in $2x-y+2z+3=0$, we get $y=3$. So, the coordinates of a point on the plane $2x-y+2z+3=0$ are $(0, 3, 0)$. Let d be the distance between the given planes. Then,

$$d = \text{Length of perpendicular from } (0, 3, 0) \text{ on the plane } 4x-2y+4z+5=0$$

$$\Rightarrow \quad d = \left|\frac{4\times 0-2\times 3+4\times 0+5}{\sqrt{4^2+(-2)^2+4^2}}\right| = \frac{1}{6}.$$

EXAMPLE 3 *Find the distance between the parallel planes,* $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=5$ *and* $\vec{r}\cdot(6\hat{i}-9\hat{j}+18\hat{k})+20=0$.

SOLUTION Let $\vec{a}$ be the position vector of any point P on the plane $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=5$. Then,

$$\vec{a}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=5 \qquad \text{...(i)}$$

Let d be the distance between the given planes. Then,

$$d = \text{Length of the perpendicular from } P(\vec{a}) \text{ to } \vec{r}\cdot(6\hat{i}-9\hat{j}+18\hat{k})+20=0$$

$$\Rightarrow d = \left|\frac{\vec{a}\cdot(6\hat{i}-9\hat{j}+18\hat{k})+20}{\sqrt{6^2+(-9)^2+(18)^2}}\right| = \left|\frac{3[\vec{a}\cdot(2\hat{i}-3\hat{j}+6\hat{k})]+20}{\sqrt{36+81+324}}\right| = \left|\frac{3(5)+20}{\sqrt{441}}\right| = \frac{35}{21} = \frac{5}{3} \qquad \text{[Using (i)]}$$

EXERCISE 28.10

BASIC

1. Find the distance between the parallel planes $2x - y + 3z - 4 = 0$ and $6x - 3y + 9z + 13 = 0$.
2. Find the equation of the plane which passes through the point $(3, 4, -1)$ and is parallel to the plane $2x - 3y + 5z + 7 = 0$. Also, find the distance between the two planes.
3. Find the distance between the planes $\vec{r}\cdot(\hat{i} + 2\hat{j} + 3\hat{k}) + 7 = 0$ and $\vec{r}\cdot(2\hat{i} + 4\hat{j} + 6\hat{k}) + 7 = 0$.

BASED ON LOTS

4. Find the equation of the plane mid-parallel to the planes $2x - 2y + z + 3 = 0$ and $2x - 2y + z + 9 = 0$.

ANSWERS

1. $\frac{25}{3\sqrt{14}}$ 2. $2x - 3y + 5z + 11 = 0, \frac{4}{\sqrt{38}}$ 3. $\frac{7}{\sqrt{56}}$ 4. $2x - 2y + z + 6 = 0$

HINTS TO SELECTED PROBLEMS

4. Let the equation of the plane be $2x - 2y + z + k = 0$. This plane is equidistant from the given planes.
$\therefore \ |-k+3| = |-k+9| \Rightarrow -k+3 = -(-k+9) \Rightarrow 2k = 12 \Rightarrow k = 6.$

28.13 LINE AND A PLANE

28.13.1 UNSYMMETRICAL FORM OF A LINE

From elementary solid geometry, we know that two non-parallel planes intersect in a straight line. We have seen that every first degree equation in x, y, z represents a plane, therefore a line in space can be represented by two equations of first degree in x, y and z. Thus, if $u = a_1x + b_1y + c_1z + d_1 = 0$ and $v = a_2x + b_2y + c_2z + d_2 = 0$ are equations of two non-parallel planes, then these two equations taken together represent a line because any point on the line will lie on the two planes and conversely any point which lies on two planes will also lie on the straight line which can be written as $u = 0 = v$. This form is called unsymmetrical form of a line.

For example, x-axis is the intersection of zx-plane i.e., $y = 0$ and xy-plane i.e., $z = 0$. So, the equations of x-axis are $y = 0 = z$. Similarly, the equations of y and z axes are $x = 0 = z$ and $x = 0 = y$ respectively.

28.13.2 REDUCTION OF UNSYMMETRICAL FORM TO SYMMETRICAL FORM

Let the unsymmetrical form of a line be

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{aligned}\right\} \qquad \text{...(i)}$$

where $a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$.

To transform the equations to symmetrical form, we require

(a) Direction ratios of the line, and (b) Coordinates of a point on the line.

(a) Let the direction ratios of the line (i) be proportional to a, b, c. Then, as this line lies in both the planes, it must be perpendicular to the normals to these planes. We, therefore, have

$$aa_1 + bb_1 + cc_1 = 0 \qquad \text{...(ii)}$$

and, $$aa_2 + bb_2 + cc_2 = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we obtain

$$\frac{a}{b_1c_2 - b_2c_1} = \frac{b}{c_1a_2 - c_2a_1} = \frac{c}{a_1b_2 - a_2b_1}$$

So, direction ratios of the line (i) are proportional to

$$b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1.$$

(b) There are infinitely many points on the line (i) from which we have to choose a point. For this, we put $z = 0$ (we may put $y = 0$ or $x = 0$) in the equations of the planes to obtain

$$a_1x + b_1y + d_1 = 0$$
$$a_2x + b_2y + d_2 = 0$$

Solving these two equations, we obtain

$$\frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

So, the coordinates of a point on line (i) are $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1},\ \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1},\ 0\right)$

Thus, the symmetrical form of the line (i) is given by

$$\frac{x - \left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}\right)}{b_1c_2 - b_2c_1} = \frac{y - \left(\dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}\right)}{c_1a_2 - c_2a_1} = \frac{z - 0}{a_1b_2 - a_2b_1}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Reduce in symmetrical form, the equations of the line $x - y + 2z = 5$, $3x + y + z = 6$.*

SOLUTION Let the direction ratios of the required line be proportional to a, b, c. Since the required line lies in both the given planes, we must have

$$a - b + 2c = 0 \text{ and } 3a + b + c = 0$$

Solving these two equations by cross-multiplication, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3} \Leftrightarrow \frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$$

In order to find a point on the required line, we put $z = 0$ in the two given equations to obtain

$$x - y = 5,\ 3x + y = 6$$

Solving these two equations, we obtain $x = \dfrac{11}{4}$, $y = -\dfrac{9}{4}$. Therefore, coordinates of a point on the required line are $\left(\dfrac{11}{4}, -\dfrac{9}{4}, 0\right)$. Hence, the equations of the required line are

$$\frac{x - \frac{11}{4}}{-3} = \frac{y - \left(-\frac{9}{4}\right)}{5} = \frac{z-0}{4} \text{ or, } \frac{4x-11}{-12} = \frac{4y+9}{20} = \frac{z-0}{4} \text{ or, } \frac{4x-11}{-3} = \frac{4y+9}{5} = \frac{z-0}{1}.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 2 *Reduce in symmetrical form, the equations of the line $x = ay + b$, $z = cy + d$.*

SOLUTION Let the direction ratios of the required line be proportional to l, m, n.. Since the required line lies in both the given planes, we must have

$$l + m(-a) + n\,0 = 0 \text{ and } l \cdot 0 + m(-c) + n = 0$$

Solving these two equations, we obtain

$$\frac{l}{-a} = \frac{m}{-1} = \frac{n}{-c} \text{ or, } \frac{l}{a} = \frac{m}{1} = \frac{n}{c}$$

In order to obtain the coordinates of a point on the required line let us put $y = 0$ in the given equations to obtain

$$x = b,\ z = d$$

So, the coordinates of a point on the required line are $(b, 0, d)$. Therefore, its equations are

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 3 *Find the angle between the lines* $x - 2y + z = 0 = x + 2y - 2z$ *and* $x + 2y + z = 0 = 3x + 9y + 5z$.

SOLUTION Let the direction ratios of the line $x - 2y + z = 0$ and $x + 2y - 2z = 0$ be proportional to a_1, b_1, c_1. Since it lies in both the planes, therefore, it is perpendicular to their normals.

$$\therefore \quad a_1 - 2b_1 + c_1 = 0$$
$$a_1 + 2b_1 - 2c_1 = 0$$

Solving these two equations by cross-multiplication, we obtain

$$\frac{a_1}{4-2} = \frac{b_1}{1+2} = \frac{c_1}{2+2} \Rightarrow \frac{a_1}{2} = \frac{b_1}{3} = \frac{c_1}{4}.$$

Let the direction ratios of the line $x + 2y + z = 0 = 3x + 9y + 5z$ be proportional to a_2, b_2, c_2. As it lies in both the planes.

$$\therefore \quad a_2 + 2b_2 + c_2 = 0 \ \ 3a_2 + 9b_2 + 5c_2 = 0 \Rightarrow \frac{a_2}{10-9} = \frac{b_2}{3-5} = \frac{c_2}{9-6} \Rightarrow \frac{a_2}{1} = \frac{b_2}{-2} = \frac{c_2}{3}$$

Let θ be the angle between the given lines. Then,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{(1)(2) + (-2)(3) + (3)(4)}{\sqrt{2^2 + 3^2 + 4^2}\ \sqrt{1^2 + (-2)^2 + (3)^2}} = \frac{8}{\sqrt{406}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{8}{\sqrt{406}}\right).$$

28.13.3 ANGLE BETWEEN A LINE AND A PLANE

The angle between a line and a plane is the complement of the angle between the line and normal to the plane.

VECTOR FORM

THEOREM 1 *Prove that the angle θ between a line $\vec{r} = a + \lambda\vec{b}$ and the plane $\vec{r}\cdot\vec{n} = d$ is given by*

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}$$

PROOF The line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the vector $\vec{b}$ and the plane is normal to the vector $\vec{n}$. Therefore, if θ is the angle between the given line and the given plane, then the angle between $\vec{b}$ and $\vec{n}$ is $\left(\frac{\pi}{2} - \theta\right)$.

$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|} \Rightarrow \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|}$$

Q.E.D.

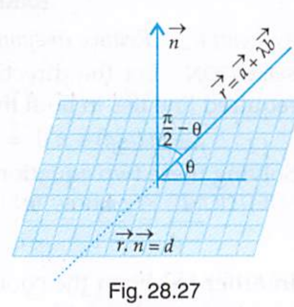


Fig. 28.27

Condition of perpendicularity: If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore, $\vec{b}$ and $\vec{n}$ are parallel. Consequently, $\vec{b} \times \vec{n} = 0$ or, $\vec{b} = \lambda \vec{n}$ for some scalar λ.

Condition of parallelism: If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore, $\vec{b}$ and $\vec{n}$ are perpendicular. Consequently $\vec{b} \cdot \vec{n} = 0$.

CARTESIAN FORM

THEOREM 2 *Prove that is the angle θ between the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ and the plane $ax + by + cz + d = 0$ is given by*

$$\sin\theta = \frac{al + bm + cn}{\sqrt{a^2+b^2+c^2}\ \sqrt{l^2+m^2+n^2}}.$$

PROOF Clearly, the given line is parallel to the vector $\vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$ and the given plane is normal to the vector $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$. Therefore, if θ is the angle between the line and the plane, then the angle between $\vec{b}$ and $\vec{n}$ is $\left(\frac{\pi}{2} - \theta\right)$.

$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\,|\vec{n}|}$$

$$\Rightarrow \sin\theta = \frac{(l\hat{i} + m\hat{j} + n\hat{k}) \cdot (a\hat{i} + b\hat{j} + c\hat{k})}{\sqrt{l^2+m^2+n^2}\ \sqrt{a^2+b^2+c^2}} = \frac{al + bm + cn}{\sqrt{l^2+m^2+n^2}\ \sqrt{a^2+b^2+c^2}}$$

Q.E.D.

Condition of perpendicularity: If the line is perpendicular to the plane, then it is parallel to its normal. Therefore, $\vec{n} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ are parallel. Consequently, we obtain

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

Condition of parallelism: If the line is parallel to the plane, then it is perpendicular to its normal. Therefore, $\vec{b} = l\hat{i} + m\hat{j} + n\hat{k}$ and $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$ are perpendicular. Consequently, we get

$$\vec{b} \cdot \vec{n} = 0 \Rightarrow al + bm + cn = 0.$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the angle between the line $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$.*

SOLUTION We know that the angle θ between the line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = d$ is given by $\sin\theta = \frac{\vec{b} \cdot \vec{n}}{|\vec{b}|\,|\vec{n}|}$. Here, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$.

$$\therefore \quad \sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|} = \frac{(\hat{i}-\hat{j}+\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{\sqrt{1^2+(-1)^2+1^2}\,\sqrt{2^2+(-1)^2+1^2}} = \frac{2+1+1}{\sqrt{3}\,\sqrt{6}} = \frac{4}{3\sqrt{2}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

EXAMPLE 2 *Find the angle between the line* $\frac{x+1}{3}=\frac{y-1}{2}=\frac{z-2}{4}$ *and the plane* $2x+y-3z+4=0$.

SOLUTION The given line is parallel to the vector $\vec{b}=3\hat{i}+2\hat{j}+4\hat{k}$ and the given plane is normal to the vector $\vec{n}=2\hat{i}+\hat{j}-3\hat{k}$. Therefore, the angle θ between the given line and given plane is given by

$$\sin\theta = \frac{\vec{b}\cdot\vec{n}}{|\vec{b}|\,|\vec{n}|} = \frac{(3\hat{i}+2\hat{j}+4\hat{k})\cdot(2\hat{i}+\hat{j}-3\hat{k})}{\sqrt{3^2+2^2+4^2}\,\sqrt{2^2+(1)^2+(-3)^2}} = \frac{6+2-12}{\sqrt{29}\,\sqrt{14}} = \frac{-4}{\sqrt{406}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{-4}{\sqrt{406}}\right)$$

EXAMPLE 3 *If the line* $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ *is parallel to the plane* $\vec{r}\cdot(3\hat{i}-2\hat{j}+m\hat{k})=14$, *find the value of m.*

SOLUTION The given line is parallel to the vector $\vec{b}=2\hat{i}+\hat{j}+2\hat{k}$ and the given plane is normal to the vector $\vec{n}=3\hat{i}-2\hat{j}+m\hat{k}$.

If the line is parallel to the plane, then normal to the plane is perpendicular to the line i.e. $\vec{b}\perp\vec{n}$.

$$\therefore \quad \vec{b}\cdot\vec{n}=0 \Rightarrow (2\hat{i}+\hat{j}+2\hat{k})\cdot(3\hat{i}-2\hat{j}+m\hat{k}) = 0 \Rightarrow 6-2+2m = 0 \Rightarrow m = -2$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Show that the line whose vector equation is* $\vec{r}=(2\hat{i}-2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+4\hat{k})$ *is parallel to the plane whose vector equation is* $\vec{r}\cdot(\hat{i}+5\hat{j}+\hat{k})=5$. *Also, find the distance between them.*

[CBSE 2001C, 2004]

SOLUTION The given line passes through the point having position vector $\vec{a}=2\hat{i}-2\hat{j}+3\hat{k}$ and is parallel to the vector $\vec{b}=\hat{i}-\hat{j}+4\hat{k}$. The given plane is normal to the vector $\vec{n}=\hat{i}+5\hat{j}+\hat{k}$.

Now, $\vec{b}\cdot\vec{n}=(\hat{i}-\hat{j}+4\hat{k})\cdot(\hat{i}+5\hat{j}+\hat{k})=1-5+4=0$

So, $\vec{b}$ perpendicular to $\vec{n}$. Hence, the given line is parallel to the given plane.

The distance between the line and the parallel plane is the distance between any point on the line and the given plane. Since the line passes through the point $\vec{a}=2\hat{i}-2\hat{j}+3\hat{k}$. Therefore, if '*d*' is the distance between the given line and given plane. Then,

d = Length of perpendicular from $\vec{a} = 2\hat{i} - 2\hat{j} + 3\hat{k}$ to the given plane

$$\vec{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{i})$$

$$\Rightarrow \quad d = \frac{|(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5|}{\sqrt{1^2 + 5^2 + 1^2}} = \frac{|(2 - 10 + 3) - 5|}{\sqrt{27}} = \frac{10}{\sqrt{27}}$$

EXAMPLE 5 *Find the vector equation of the line passing through the point with position vector* $2\hat{i} - 3\hat{j} - 5\hat{k}$ *and perpendicular to the plane* $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$.

SOLUTION The required line is perpendicular to the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} + 5\hat{k}) + 2 = 0$. Therefore, it is parallel to the normal $\vec{n} = 6\hat{i} - 3\hat{j} + 5\hat{k}$. Thus, the required line passes through the point with position vector $\vec{a} = 2\hat{i} - 3\hat{j} - 5\hat{k}$ and is parallel to the vector $\vec{n} = 6\hat{i} - 3\hat{j} + 5\hat{k}$.

So, its vector equation is

$$\vec{r} = \vec{a} + \lambda\,\vec{n} \quad \text{or,} \quad \vec{r} = (2\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda(6\hat{i} - 3\hat{j} + 5\hat{k})$$

EXAMPLE 6 *Find the equations of the line passing through the point* (3, 0, 1) *and parallel to the planes* $x + 2y = 0$ *and* $3y - z = 0$. [CBSE 2012, NCERT EXEMPLAR]

SOLUTION Let the direction ratios of the required line be proportional to a, b, c. As it passes through (3, 0, 1). So, its equations are

$$\frac{x-3}{a} = \frac{y-0}{b} = \frac{z-1}{c} \qquad \text{...(i)}$$

It is given that the line (i) is parallel to the planes $x + 2y + 0z = 0$ and $0x + 3y - z = 0$.

$\therefore \quad a(1) + b(2) + c(0) = 0$ and $a(0) + b(3) + c(-1) = 0$

Solving these two equations by cross-multiplication, we obtain

$$\frac{a}{(2)(-1) - (0)(3)} = \frac{b}{(0)(0) - (1)(-1)} = \frac{c}{(1)(3) - (0)(2)}$$

$$\Rightarrow \quad \frac{a}{-2} = \frac{b}{1} = \frac{c}{3} = \lambda \text{ (say)} \Rightarrow a = -2\lambda,\ b = \lambda,\ c = 3\lambda$$

Substituting the values of a, b, c in (i), we obtain that the equations of the required line are

$$\frac{x-3}{-2} = \frac{y-0}{1} = \frac{z-1}{3}$$

ALITER The required line passes through the point (3, 0, 1) whose position vector is $\vec{a} = 3\hat{i} + \hat{k}$ and is parallel to the planes $x + 2y = 0$ and $3y - z = 0$. So, it is perpendicular to their normals $\vec{n_1} = \hat{i} + 2\hat{j}$ and $\vec{n_2} = 3\hat{j} - \hat{k}$ respectively. Consequently, it is parallel to the vector

$$\vec{b} = \vec{n_1} \times \vec{n_2} = -2\hat{i} + \hat{j} + 3\hat{k}.$$

Hence, the vector equation of the required line is

$$\vec{r} = \vec{a} + \lambda\vec{b} \quad \text{or,} \quad \vec{r} = (3\hat{i} + \hat{k}) + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$$

EXAMPLE 7 *Find the equation of the plane passing through the line of intersection of the planes* $2x + y - z = 3$, $5x - 3y + 4z + 9 = 0$ *and parallel to the line* $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z-5}{5}$.

[CBSE 2011, 2015]

SOLUTION The equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ is

$$(2x + y - z - 3) + \lambda(5x - 3y + 4z + 9) = 0 \Rightarrow x(2 + 5\lambda) + y(1 - 3\lambda) + z(4\lambda - 1) + 9\lambda - 3 = 0 \quad ...(i)$$

The plane in (i) is parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$.

$$\therefore \quad 2(2 + 5\lambda) + 4(1 - 3\lambda) + 5(4\lambda - 1) = 0 \Rightarrow 18\lambda + 3 = 0 \Rightarrow \lambda = -\frac{1}{6}$$

Putting the value of λ in (i), we obtain $x\left(2 - \frac{5}{6}\right) + y\left(1 + \frac{3}{6}\right) + z\left(-\frac{4}{6} - 1\right) - \frac{9}{6} - 3 = 0$

or, $\quad 7x + 9y - 10z - 27 = 0$, which is the required equation of the plane.

EXAMPLE 8 *Find the equation of the plane passing through the intersection of the planes* $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 1$ *and* $\vec{r}\cdot(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ *and parallel to x-axis.* **[NCERT, CBSE 2011]**

SOLUTION The equation of a plane passing through the intersection of the planes $\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r}\cdot(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ is

$$\left\{\vec{r}\cdot(\hat{i} + \hat{j} + \hat{k}) - 1\right\} + \lambda\left\{\vec{r}\cdot(2\hat{i} + 3\hat{j} - \hat{k}) + 4\right\} = 0$$

or, $$\vec{r}\cdot\left\{(2\lambda + 1)\hat{i} + (3\lambda + 1)\hat{j} + (1 - \lambda)\hat{k}\right\} + (4\lambda - 1) = 0 \quad ...(i)$$

It is given that the plane (i) is parallel to x-axis i.e. the vector $\hat{i}$.

$$\therefore \quad \left\{(2\lambda + 1)\hat{i} + (3\lambda + 1)\hat{j} + (1 - \lambda)\hat{k}\right\}.\hat{i} = 0 \Rightarrow 2\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2}$$

Putting $\lambda = -\frac{1}{2}$ in (i), we get

$$\vec{r}.\left(-\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}\right) - 3 = 0 \text{ or, } \vec{r}.(-\hat{j} + 3\hat{k}) = 6,$$ which is the required equation of the plane.

EXAMPLE 9 *Find the equation of the plane through the points* $(1, 0, -1)$, $(3, 2, 2)$ *and parallel to the line* $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3}$.

SOLUTION The equation of a plane passing through $(1, 0, -1)$ is

$$a(x - 1) + b(y - 0) + c(z + 1) = 0 \quad ...(i)$$

This passes through the point $(3, 2, 2)$.

$$\therefore \quad a(3 - 1) + b(2 - 0) + c(2 + 1) = 0 \Rightarrow 2a + 2b + 3c = 0 \quad ...(ii)$$

The plane in (i) is parallel to the line $\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3}$.

Therefore, normal to the plane is perpendicular to the line.

$$\therefore \quad a(1) + b(-2) + c(3) = 0 \quad ...(iii)$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(2)(3) - (3)(-2)} = \frac{b}{(1)(3) - (2)(3)} = \frac{c}{(2)(-2) - (2)(1)}$$

$$\Rightarrow \quad \frac{a}{12}=\frac{b}{-3}=\frac{c}{-6} \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}=\lambda \text{ (say)} \Rightarrow a=4\lambda,\ b=-\lambda,\ c=-2\lambda$$

Substituting the values of a, b, c in (i), we obtain $4x-y-2z-6=0$ as the required equation of plane.

ALITER The required plane passes through the point $A(1, 0, -1)$ and $B(3, 2, 2)$. Let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B respectively. Let $\vec{n}$ be the normal to the required plane. Then, $\vec{n}$ is perpendicular to $\overrightarrow{AB}$. Also, the required plane is parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

So, its normal vector $\vec{n}$ is perpendicular to the vector $\vec{b_1}=\hat{i}-2\hat{j}+3\hat{k}$ which is parallel to the given line.

Thus, $\vec{n}$ is perpendicular to both $\overrightarrow{AB}$ and $\vec{b_1}=\hat{i}-2\hat{j}+3\hat{k}$. Therefore, $\vec{n}$ is parallel to the vector $\overrightarrow{AB}\times\vec{b_1}$.

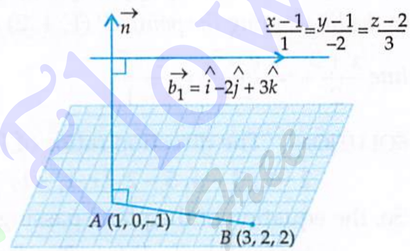


Fig. 28.28

Now, $\quad \vec{n}=\overrightarrow{AB}\times\vec{b_1}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 2 & 3\\ 1 & -2 & 3\end{vmatrix}=12\hat{i}-3\hat{j}-6\hat{k}$

Hence, vector equation of the required plane is $\vec{r}\,.\,\vec{n}=\vec{a}\,.\,\vec{n}$

or, $\quad \vec{r}\,.(12\hat{i}-3\hat{j}-6\hat{k})=(\hat{i}-\hat{k})\,.(12\hat{i}-3\hat{j}-6\hat{k}) \Rightarrow \vec{r}\,.(12\hat{i}-3\hat{j}-6\hat{k})=18 \Rightarrow \vec{r}\,.(4\hat{i}-\hat{j}-2\hat{k})=6$

EXAMPLE 10 *State when the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n}=d$. Show that the line $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ is parallel to the plane $\vec{r}\cdot(-2\hat{i}+\hat{k})=5$. Also, find the distance between the line and the plane.*

SOLUTION The line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n}=d$, if the normal to the plane is perpendicular to the line i.e., $\vec{b}\perp\vec{n}$. Here, $\vec{a}=\hat{i}+\hat{j}$, $\vec{b}=2\hat{i}+\hat{j}+4\hat{k}$ and $\vec{n}=-2\hat{i}+\hat{k}$

$\therefore \quad \vec{b}\cdot\vec{n}=(2\hat{i}+\hat{j}+4\hat{k})\cdot(-2\hat{i}+0\hat{j}+\hat{k})=-4+0+4=0 \Rightarrow \vec{b}\perp\vec{n}$

So, the given line is parallel to the given plane. Let d be the distance between the line and the plane. Then,

d = Length of the perpendicular from the point $\vec{a}=\hat{i}+\hat{j}$ on the plane $\vec{r}\cdot\vec{n}=d$.

$$\Rightarrow \quad d=\frac{|\vec{a}\cdot\vec{n}-d|}{|\vec{n}|}=\frac{|(\hat{i}+\hat{j})\cdot(-2\hat{i}+\hat{k})-5|}{\sqrt{(-2)^2+(0)^2+1^2}}=\frac{|-2+0+0-5|}{\sqrt{5}}=\frac{7}{\sqrt{5}}.$$

EXAMPLE 11 *Find the equation of the plane passing through the intersection of the planes $4x-y+z=10$ and $x+y-z=4$ and parallel to the line with direction ratios proportional to 2, 1, 1. Find also the perpendicular distance of point (1, 1, 1) from this plane.*

SOLUTION The equation of a plane passing through the intersection of the given planes is

$(4x-y+z-10)+\lambda(x+y-z-4)=0$, where λ is a parameter.

or, $\quad x(4+\lambda)+y(\lambda-1)+z(1-\lambda)-10-4\lambda=0 \qquad$ ...(i)

This plane is parallel to the line with direction ratios proportional to 2, 1, 1.

$\therefore \quad 2(4+\lambda)+1(\lambda-1)+1(1-\lambda) = 0 \Rightarrow \lambda = -4$

Putting $\lambda = -4$ in (i), we obtain $5y - 5z - 6 = 0$ ...(ii)

This is the required equation of the plane.

Let d be the length of the perpendicular from (1, 1, 1) on plane (ii).Then,

$$d = \left| \frac{5\times 1 - 5\times 1 - 6}{\sqrt{5^2+(-5)^2}} \right| = \frac{3\sqrt{2}}{5}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 12 *Find the equation of the plane passing through the point A (1, 2, 1) and perpendicular to the line joining the points P (1, 4, 2) and Q (2, 3, 5). Also, find the distance of this plane from the line* $\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-7}{-1}$. [CBSE 2010, 2011]

SOLUTION The direction ratios of PQ are proportional to

$2-1, 3-4, 5-2$ *i.e.* $1, -1, 3$

So, the equation of the plane passing through A (1, 2, 1) and perpendicular to PQ is

$$1\times(x-1)+(-1)(y-2)+3\times(z-1) = 0 \text{ or, } x-y+3z = 2 \qquad \text{...(i)}$$

The given line is parallel to the vector $\vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and the plane (i) is normal to the vector $\vec{n} = \hat{i} - \hat{j} + 3\hat{k}$ such that $\vec{b}\cdot\vec{n} = 0$. So, given line is parallel to the plane (i).

The distance between the plane (i) and the given line is the distance of any point on the line from the plane (i). The line passes through the point $(-3, 5, 7)$.

So, required distance = Length of perpendicular from $(-3, 5, 7)$ on plane (i).

$$= \left| \frac{-3-5+21-2}{\sqrt{1^2+(-1)^2+3^2}} \right| = \frac{11}{\sqrt{11}} = \sqrt{11}$$

EXAMPLE 13 *Find an equation for the line that passes through the point P (2, 3, 1) and is parallel to the line of intersection of the planes* $x+2y-3z = 4$ *and* $x-2y+z = 0$

SOLUTION Let the direction ratios of the required line be proportional to a, b, c. As it lies on both the planes. So, their normals are perpendicular to it.

$\therefore \quad a + 2b - 3c = 0$

$\quad a - 2b + c = 0$

Using cross multiplication, we get

$$\frac{a}{2-6} = \frac{b}{-3-1} = \frac{c}{-2-2} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

Thus, required line passes through the point P (2, 3, 1) and its direction ratios are proportional to 1, 1, 1. Hence, its equations are

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-1}{1}$$

EXAMPLE 14 *Find the plane passing through* $(4, -1, 2)$ *and parallel to the lines*

$$\frac{x+2}{3} = \frac{y-2}{-1} = \frac{z+1}{2} \text{ and } \frac{x-2}{1} = \frac{y-3}{2} = \frac{z-4}{3}$$

SOLUTION The equation of a plane passing through $(4, -1, 2)$ is

$$a(x-4)+b(y+1)+c(z-2)=0 \quad \text{...(i)}$$

It is parallel to the lines $\frac{x+2}{3}=\frac{y-2}{-1}=\frac{z+1}{2}$ and $\frac{x-2}{1}=\frac{y-3}{2}=\frac{z-4}{3}$

$\therefore \quad 3a-b+2c=0$

$a+2b+3c=0$

Using cross-multiplication, we get

$$\frac{a}{-3-4}=\frac{b}{2-9}=\frac{c}{6+1} \Rightarrow \frac{a}{1}=\frac{b}{1}=\frac{c}{-1}=\lambda\,(\text{say}) \Rightarrow a=\lambda,\ b=\lambda,\ c=-\lambda$$

Substituting the values of a, b, c in (i), we obtain

$(x-4)+(y+1)-(z-2)=0$ or, $x+y-z-1=0$ as the equation of the required plane.

EXERCISE 28.11

BASIC

1. Find the angle between the line $\vec{r}=(2\hat{i}+3\hat{j}+9\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=5$.
2. Find the angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $10x+2y-11z=3$. [NCERT]
3. Find the angle between the line joining the points $(3,-4,-2)$ and $(12, 2, 0)$ and the plane $3x-y+z=1$.
4. The line $\vec{r}=\hat{i}+\lambda(2\hat{i}-m\hat{j}-3\hat{k})$ is parallel to the plane $\vec{r}\cdot(m\hat{i}+3\hat{j}+\hat{k})=4$. Find m.
5. Show that the line whose vector equation is $\vec{r}=2\hat{i}+5\hat{j}+7\hat{k}+\lambda(\hat{i}+3\hat{j}+4\hat{k})$ is parallel to the plane whose vector equation is $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=7$. Also, find the distance between them.
6. Find the vector equation of the line through the origin which is perpendicular to the plane $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})=3$.
7. Find the equation of the plane through $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to x-axis.

BASED ON LOTS

8. Find the equation of a plane passing through the points $(0, 0, 0)$ and $(3, -1, 2)$ and parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$. [CBSE 2010]
9. Find the vector and cartesian equations of the line passing through $(1, 2, 3)$ and parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+2\hat{k})=6$ [CBSE 2012]
10. Prove that the line of section of the planes $5x+2y-4z+2=0$ and $2x+8y+2z-1=0$ is parallel to the plane $4x-2y-5z-2=0$.
11. Find the vector equation of the line passing through the point $(1, -1, 2)$ and perpendicular to the plane $2x-y+3z-5=0$.
12. Find the equation of the plane through the points $(2, 2, -1)$ and $(3, 4, 2)$ and parallel to the line whose direction ratios are $7, 0, 6$.
13. Find the value of λ such that the line $\frac{x-2}{6}=\frac{y-1}{\lambda}=\frac{z+5}{-4}$ is perpendicular to the plane $3x-y-2z=7$. [CBSE 2010]

14. Find the equation of the plane passing through the intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 8$ and parallel to the line with direction ratios proportional to 1, 2, 1. Find also the perpendicular distance of (1, 1, 1) from this plane. [CBSE 2005]

15. State when the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r}\cdot\vec{n} = d$. Show that the line $\vec{r} = \hat{i} + \hat{j} + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ is parallel to the plane $\vec{r}\cdot(2\hat{j} + \hat{k}) = 3$. Also, find the distance between the line and the plane.

16. Show that the plane whose vector equation is $\vec{r}\cdot(\hat{i} + 2\hat{j} - \hat{k}) = 1$ and the line whose vector equation is $\vec{r} = (-\hat{i} + \hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$ are parallel. Also, find the distance between them.

17. Find the equation of the plane through the intersection of the planes $3x - 4y + 5z = 10$ and $2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

18. Find the vector and cartesian forms of the equation of the plane passing through the point $(1, 2, -4)$ and parallel to the lines $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = (\hat{i} - 3\hat{j} + 5\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$. Also, find the distance of the point $(9, -8, -10)$ from the plane thus obtained. [CBSE 2014]

19. Find the equation of the plane passing through the points (3, 4, 1) and (0, 1, 0) and parallel to the line $\frac{x+3}{2} = \frac{y-3}{7} = \frac{z-2}{5}$. [CBSE 2008]

20. Find the coordinates of the point where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersects the plane $x - y + z - 5 = 0$. Also, find the angle between the line and the plane. [CBSE 2013]

21. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane $\vec{r}\cdot(\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$. [NCERT]

22. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}\cdot(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}\cdot(3\hat{i} + \hat{j} + \hat{k}) = 6$. [NCERT]

23. Find the equation of the plane passing through the points $(-1, 2, 0)$, $(2, 2, -1)$ and parallel to the line $\frac{x-1}{1} = \frac{2y+1}{2} = \frac{z+1}{-1}$. [CBSE 2015]

24. Find the equation of the plane that contains the point $A(2, 1, -1)$ and is perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Also, find the angle between the plane thus obtained and the y-axis. [CBSE 2020]

ANSWERS

1. $\sin^{-1}\left(\frac{3\sqrt{3}}{\sqrt{29}}\right)$
2. $\sin^{-1}\left(\frac{8}{21}\right)$
3. $\sin^{-1}\left(\frac{23}{11\sqrt{11}}\right)$
4. -3
5. $\frac{7}{\sqrt{3}}$
6. $\vec{r} = \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$
7. $7y + 4z - 5 = 0$
8. $x - 19y - 11z = 0$
9. $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$; $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$
11. $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$
12. $12x + 15y - 14z - 68 = 0$

13. -2 14. $9x-8y+7z=21, \frac{13}{\sqrt{194}}$ 15. $\vec{b}\cdot\vec{n}=0$, Distance $=\frac{1}{\sqrt{5}}$

16. $\frac{1}{\sqrt{6}}$ 17. $x-20y+27z=14$ 18. $\vec{r}\cdot(-9\hat{i}+8\hat{j}-\hat{k})=11, -9x+8y-z=11, \sqrt{146}$

19. $8x-13y+15z+13=0$ 20. $(2,-1,2)$, $\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$

21. $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}+2\hat{j}-5\hat{k})$ 22. $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})$

23. $x+y+3z=1$ 24. $x-y+z=0$, $\sin^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

HINTS TO SELECTED PROBLEMS

13. Let θ be the acute angle between the line $\frac{x+1}{2}=\frac{y}{3}=\frac{z-3}{6}$ and the plane $10x+2y-11z=3$.

The line is parallel to the vector $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$ and the plane is normal to the vector $\vec{b}=10\hat{i}+2\hat{j}-11\hat{k}$.

$$\therefore \quad \sin\theta=\left|\frac{\vec{b}\cdot\vec{n}}{|\vec{b}||\vec{n}|}\right|=\left|\frac{20+6-66}{\sqrt{4+9+36}\sqrt{100+4+121}}\right|=\frac{40}{7\times 15}=\frac{8}{21}\Rightarrow\theta=\sin^{-1}\left(\frac{8}{21}\right)$$

21. The required line passes through (1, 2, 3) and is parallel to the normal to the plane $\vec{r}\cdot(\hat{i}+2\hat{j}-5\hat{k})+9=0$ i.e. the vector $\vec{n}=\hat{i}+2\hat{j}-5\hat{k}$. So, its vector equation is

$$\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}+2\hat{j}-5\hat{k}).$$

22. The vector equation of a line passing through (1, 2, 3) and parallel to $\vec{b}$ is

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda\vec{b} \quad \text{...(i)}$$

It is parallel to the planes $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5$ and $\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6$. Therefore, $\vec{b}$ is perpendicular to vectors $\vec{n_1}=\hat{i}-\hat{j}+2\hat{k}$ and $\vec{n_2}=3\hat{i}+\hat{j}+\hat{k}$ and hence parallel to $\vec{n_1}\times\vec{n_2}$.

$$\text{Now,}\quad \vec{n_1}\times\vec{n_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 2\\ 3 & 1 & 1\end{vmatrix}=-3\hat{i}+5\hat{j}+4\hat{k}.\ \text{So, } \vec{b}=\lambda_1(-3\hat{i}+5\hat{j}+4\hat{k})$$

Substituting the value of $\vec{b}$ in (i), we obtain:

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda_2(-3\hat{i}+5\hat{j}+4\hat{k}),\ \text{where } \lambda_2=\lambda\lambda_1$$

24. The required plane is perpendicular to the line of intersection of given planes. So, vector $\vec{n}$ normal to the plane is parallel to the line. Therefore, $\vec{n}=\vec{n_1}\times\vec{n_2}$ where $\vec{n_1}$ and $\vec{n_2}$ are normals to the given planes. The equation of required plane is $\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$ where $\vec{a}=2\hat{i}+\hat{j}-\hat{k}$.

28.13.4 INTERSECTION OF A LINE AND A PLANE

Let the equation of a line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and that of a plane be $ax+by+cz+d=0$.

The coordinates of any point on the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ are given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r\text{ (say) or, }(x_1+lr,\, y_1+mr,\, z_1+nr) \qquad \text{...(i)}$$

If it lies on the plane $ax+by+cz+d=0$, then

$$a(x_1+lr)+b(y_1+mr)+c(z_1+nr)+d = 0$$

$$\Rightarrow \quad (ax_1+by_1+cz_1+d)+r(al+bm+cn) = 0$$

$$\Rightarrow \quad r = -\frac{(ax_1+by_1+cz_1+d)}{al+bm+cn}$$

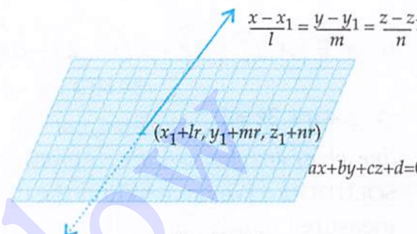


Fig. 28.29

Substituting the value of r in (i), we obtain the coordinates of the required point of intersection.

In order to find the coordinates of the point of intersection of a line and a plane, we may use the following algorithm.

ALGORITHM

Step I *Write the coordinates of any point on the line in terms of some parameters r (say).*

Step II *Substitute these coordinates in the equation of the plane to obtain the value of r.*

Step III *Put the value of r in the coordinates of the point in step I.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLES 1 *Find the coordinates of the point where the line through the points $A(3,4,1)$ and $B(5,1,6)$ crosses the XY-plane.* [CBSE 2012, NCERT]

SOLUTION The equation of the line passing through A and B

$$\frac{x-3}{5-3}=\frac{y-4}{1-4}=\frac{z-1}{6-1}\text{ or, }\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}$$

The coordinates of any point on this line are given by

$$\frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5}=\lambda \Rightarrow x = 2\lambda+3,\ y = -3\lambda+4,\ z = 5\lambda+1$$

So, $(2\lambda+3, -3\lambda+4, 5\lambda+1)$ are coordinates of any point on the line passing through A and B. If it lies on XY-plane i.e. $z=0$. Then,

$$5\lambda+1 = 0 \Rightarrow \lambda = -\frac{1}{5}$$

Substituting the value of λ, we find that the coordinates of required point are

$$\left(2\times-\frac{1}{5}+3,\ -3\times-\frac{1}{5}+4,\ 5\times-\frac{1}{5}+1\right)\text{ i.e., }\left(\frac{13}{5},\frac{23}{5},0\right)$$

EXAMPLE 2 *Find the distance between the point with position vector $-\hat{i}-5\hat{j}-10\hat{k}$ and the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ with the plane $x-y+z=5$.* [CBSE 2018]

SOLUTION The coordinates of any point on the line

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r \text{ (say) are } (3r+2, 4r-1, 12r+2) \quad \text{...(i)}$$

If it lies on the plane $x - y + z = 5$, then

$$3r + 2 - 4r + 1 + 12r + 2 = 5 \Rightarrow 11r = 0 \Rightarrow r = 0$$

Putting $r = 0$ in (i), we obtain $(2, -1, 2)$ as the coordinates of the point of intersection of the given line and plane.

$\therefore$ Required distance = Distance between points $(-1, -5, -10)$ and $(2, -1, 2)$

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = 13.$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 3 *Find the distance of the point* $(1, -2, 3)$ *from the plane* $x - y + z = 5$ *measured parallel to the line whose direction cosines are proportional to* $2, 3, -6$. [CBSE 2015]

SOLUTION We have to find the distance PQ of point $P(1, -2, 3)$ from the plane $x - y + z - 5 = 0$ measured parallel to the line passing through A and B whose direction cosines are proportional to $2, 3, -6$. Clearly, direction cosines of PQ are also proportional to $2, 3, -6$. So, equations of PQ are given by

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}$$

The coordinates of Q are given by

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ or, } (2r+1, 3r-2, -6r+3).$$

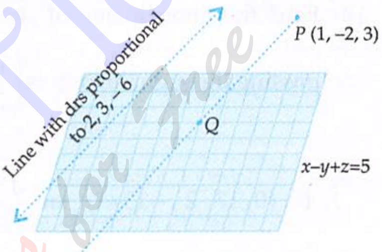


Fig. 28.30

We observe that the point Q lies on the plane $x - y + z = 5$.

$$\therefore \quad 2r + 1 - 3r + 2 - 6r + 3 = 5 \Rightarrow -7r = -1 \Rightarrow r = \frac{1}{7}$$

Putting $r = \frac{1}{7}$ in $(2r+1, 3r-2, -6r+3)$, we obtain that the coordinates of Q are $\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

Hence, required distance $= PQ = \sqrt{\left(\frac{9}{7}-1\right)^2 + \left(-\frac{11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$ unit.

EXERCISE 28.12

BASIC

1. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the
   (i) yz-plane (ii) zx-plane.
   Also, find the angle which this line makes with these planes. [NCERT, CBSE 2016]
2. Find the coordinates of the point where the line through $(3, -4, -5)$ and $(2, -3, 1)$ crosses the plane $2x + y + z = 7$. [NCERT, CBSE 2012]
3. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. [NCERT, CBSE 2014, 22]
4. Find the distance of the point $(2, 12, 5)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$. [CBSE 2014, 2020]
5. Find the distance of the point $P(-1, -5, -10)$ from the point of intersection of the line joining the points $A(2, -1, 2)$ and $B(5, 3, 4)$ with the plane $x - y + z = 5$. [CBSE 2014, 2015, 22]
6. Find the distance of the point $P(3, 4, 4)$ from the point, where the line joining the points $A(3, -4, -5)$ and $B(2, -3, 1)$ intersects the plane $2x + y + z = 7$. [CBSE 2015]

7. (i) Find the coordinates of the point where the line $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}$ cuts the xy-plane. [CBSE 2020]

(ii) Find the coordinates of the point where the line through the points $(-1, 1, -8)$ and $(5, -2, 10)$ crosses the ZX-plane. [CBSE 2022]

8. (i) Find the distance of the point $P(-2, -4, 7)$ from the point of intersection Q of the line $\vec{r}=(3\hat{i}-2\hat{j}+6\hat{k})+\lambda(2\hat{i}-\hat{j}+2\hat{k})$ and the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=6$. Also, write the vector equation of the line PQ. [CBSE 2020]

(ii) Find the distance between the point (3, 4, 5) and the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ meets the plane $x+y+z=17$. [CBSE 2022]

BASED ON LOTS

9. Find the distance of the point $(1, -5, 9)$ from the plane $x-y+z=5$ measured along the line $x=y=z$. [CBSE 2017]

10. Find the coordinates of the point where the line $\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}$ crosses the plane passing through the points $\left(\frac{7}{2}, 0, 0\right)(0, 7, 0)$ and $(0, 0, 7)$. [CBSE 2022]

ANSWERS

1. (i) $(0, 17/2, -13/2)$, $\sin^{-1}\frac{2}{\sqrt{38}}$ (ii) $(17/3, 0, 23/3)$, $\sin^{-1}\left(\frac{-3}{\sqrt{38}}\right)$ 2. $(1, -2, 7)$

3. 13 units 4. 13 units 5. 13 units 6. 7 units 7. (i) $(7, 10, 0)$ (ii) $(1, 0, -2)$

8. (i) $3\sqrt{3}$, $\vec{r}=(-2\hat{i}-4\hat{j}+7\hat{k})+\lambda(-3\hat{i}-3\hat{j}+3\hat{k})$ (ii) 3 units 9. $10\sqrt{3}$ units 10. $(1, -2, 7)$

HINTS TO SELECTED PROBLEMS

1. The cartesian equations of the line through (5, 1, 6) and (3, 4, 1) is

$$\frac{x-5}{3-5}=\frac{y-1}{4-1}=\frac{z-6}{1-6} \text{ or, } \frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}$$

The coordinates of any point on this line are given by

$$\frac{x-5}{-2}=\frac{y-1}{3}=\frac{z-6}{-5}=\lambda \text{ or, } (-2\lambda+5, 3\lambda+1, -5\lambda+6)$$

(i) If $(-2\lambda+5, 3\lambda+1, -5\lambda+6)$ lies on yz-plane i.e. $x=0$, then $-2\lambda+5=0 \Rightarrow \lambda=5/2$
So, coordinates of the required point are $(0, 17/2, -13/2)$

(ii) If $(-2\lambda+5, 3\lambda+1, -5\lambda+6)$ lies on zx-plane i.e. $y=0$, then $3\lambda+1=0 \Rightarrow \lambda=-1/3$
So, coordinates of the required point are $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.

2. The equations of the line passing through $(3, -4, -5)$ and $(2, -3, 1)$ are

$$\frac{x-3}{2-3}=\frac{y+4}{-3+4}=\frac{z+5}{1+5} \text{ or, } \frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6}=\lambda$$

The coordinates of any point on this line are $(-\lambda+3, \lambda-4, 6\lambda-5)$. If it lies on the plane $2x+y+z=7$, then $-2\lambda+6+\lambda-4+6\lambda-5=7 \Rightarrow 5\lambda=10 \Rightarrow \lambda=2$.
So, coordinates of the desired point are $(1, -2, 7)$.

3. The position vector of any point on the line $\vec{r}=2\hat{i}-\hat{j}+2\hat{k}+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ is $(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}+(2\lambda+2)\hat{k}$. If it lies on the plane $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$, then

$$\{(3\lambda+2)\hat{i}+(4\lambda-1)\hat{j}+(2\lambda+2)\hat{k}\}\cdot(\hat{i}-\hat{j}+\hat{k})=5 \Rightarrow (3\lambda+2)-(4\lambda-1)+(2\lambda+2)=5 \Rightarrow \lambda=0$$

So, the position vector of the point of intersection of the given line and the plane is $2\hat{i}-\hat{j}+2\hat{k}$ and so the coordinates of the point of intersection are $(2, -1, 2)$.

The distance between $(-1, -5, -10)$ and $(2, -1, 2)$ is

$$\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2} = \sqrt{9+16+144} = 13$$

28.13.5 CONDITION FOR A LINE TO LIE IN A PLANE

THEOREM 1 *(Vector form) If the line $\vec{r}=\vec{a}+\lambda\vec{b}$ lies in the plane $\vec{r}\cdot\vec{n}=d$, then*

(i) $\vec{b}\cdot\vec{n}=0$ and, (ii) $\vec{a}\cdot\vec{n}=d$.

PROOF If the line $\vec{r}=\vec{a}+\lambda\vec{b}$ lies in the plane $\vec{r}\cdot\vec{n}=d$, then every point on the line lies on the plane. The position vector of any point on the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is $\vec{a}+\lambda\vec{b}$. This point lies on the plane, if

$$(\vec{a}+\lambda\vec{b})\cdot\vec{n}=d \quad \text{for all values of } \lambda$$

$$\Rightarrow \quad (\vec{a}\cdot\vec{n}-d)+\lambda(\vec{b}\cdot\vec{n})=0 \quad \text{for all values of } \lambda.$$

$$\Rightarrow \quad \vec{a}\cdot\vec{n}-d=0 \quad ...(i) \quad \text{and,} \quad \vec{b}\cdot\vec{n}=0 \quad ...(ii)$$

The conditions (i) and (ii) also confirm the geometrical fact that a line will lie in a plane, if (i) any point on the line lies on the plane (ii) the normal to the plane is perpendicular to the line.

Q.E.D.

THEOREM 2 *(Cartesian form) If the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ lies in the plane $ax+by+cz+d=0$, then*

(i) $ax_1+by_1+cz_1+d=0$ and, (ii) $al+bm+cn=0$.

PROOF The coordinates of any point on the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda \text{ are } (x_1+l\lambda,\ y_1+m\lambda,\ z_1+n\lambda)$$

If the line lies on the plane, then every point on the line lies on the plane. Therefore, $(x_1+l\lambda, y_1+m\lambda, z_1+n\lambda)$ lies on the plane $ax+by+cz+d=0$ for all values of λ.

$$\therefore \quad a(x_1+l\lambda)+b(y_1+m\lambda)+c(z_1+n\lambda)+d = 0 \quad \text{for all values } \lambda$$

$$\Rightarrow \quad (ax_1+by_1+cz_1+d)+\lambda(al+bm+cn) = 0 \quad \text{for all values } \lambda$$

$$\Rightarrow \quad ax_1+by_1+cz_1+d = 0 \text{ and } al+bm+cn = 0$$

Q.E.D.

28.13.6 CONDITION OF COPLANARITY OF TWO LINES AND EQUATION OF THE PLANE CONTAINING THEM

THEOREM 1 *(Vector form) If the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, then*

$$\vec{r_1}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_2}\cdot(\vec{b_1}\times\vec{b_2}) \quad \text{or,} \quad [\vec{r}\ \vec{b_1}\ \vec{b_2}]=[\vec{a_2}\ \vec{b_1}\ \vec{b_2}]$$

and the equation of the plane containing them is

$$\vec{r}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) \quad \text{or,} \quad \vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

PROOF If the given lines are coplanar, their common plane should be parallel to each of the vectors $\vec{a_1}-\vec{a_2}$, $\vec{b_1}$ and $\vec{b_2}$ i.e., these vectors should be coplanar and the condition for the same is

$$(\vec{a_1}-\vec{a_2})\cdot(\vec{b_1}\times\vec{b_2})=0 \Rightarrow \vec{a_1}\cdot(\vec{b_1}\times\vec{b_1})-\vec{a_2}\cdot(\vec{b_1}\times\vec{b_1})=0 \Rightarrow \vec{a_1}\cdot(\vec{b_1}\times\vec{b_1})=\vec{a_2}\cdot(\vec{b_1}\times\vec{b_1})$$

The plane containing the given lines passes through $\vec{a_1}$ and $\vec{a_2}$ and is normal to the vector $\vec{b_1}\times\vec{b_2}$. So, its equation is

$$(\vec{r}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})=0 \text{ or, } (\vec{r}-\vec{a_2})\cdot(\vec{b_1}\times\vec{b_2})=0$$

$$\Rightarrow \quad \vec{r}\cdot(\vec{b_1}\times\vec{b_2}) = \vec{a_1}\cdot(\vec{b_1}\times\vec{b_2}) \text{ or, } \vec{r}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$$

$$\Rightarrow \quad [\vec{r}\ \vec{b_1}\ \vec{b_2}] = [\vec{a_1}\ \vec{b_1}\ \vec{b_2}] \text{ or, } [\vec{r}\ \vec{b_1}\ \vec{b_2}]=[\vec{a_2}\ \vec{b_1}\ \vec{b_2}]$$

Q.E.D.

THEOREM 2 *(Cartesian form) If the line* $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ *and* $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ *are coplanar, then*

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ and the equation of the plane containing them is}$$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ or, } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

PROOF The equations of two straight lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad ...(i) \qquad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(ii)$$

Let $\quad ax+by+cz+d = 0 \quad ...(iii)$

be the plane containing the line (i). Then, (x_1, y_1, z_1) lies on the plane and the normal to the plane is perpendicular to the line.

$\therefore \quad ax_1+by_1+cz_1+d = 0 \quad ...(iv) \quad$ and, $\quad al_1+bm_1+cn_1 = 0 \quad ...(v)$

Substituting the value of d obtained from (iv) in equation (iii), we get

$$a(x-x_1)+b(y-y_1)+c(z-z_1) = 0 \quad ...(vi)$$

If the lines (i) and (ii) are coplanar then line (ii) lies in the plane (vi), i.e., the point (x_2, y_2, z_2) lies in the plane (vi) and line (ii) is perpendicular to the normal to the plane (vi).

$\therefore \quad a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0 \quad ...(vii) \quad$ and, $\quad al_2+bm_2+cn_2 = 0 \quad ...(viii)$

Eliminating a, b, c from the equations (v), (vii) and (viii), we get

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0, \text{ which is the required condition.}$$

The equating of the required plane is obtained by eliminating a, b and c from the equations (vi), (vii) and (viii).

$$\therefore \quad \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \text{ is the required plane.}$$

Q.E.D.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that the lines* $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ *and* $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ *are coplanar. Also, find the plane containing these two lines.*

SOLUTION We know that the lines $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ are coplanar, if

$\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_2}\cdot(\vec{b_1}\times\vec{b_2})$ and the equation of the plane containing them is

$\vec{r}\cdot(\vec{b_1}\times\vec{b_2})=\vec{a_1}\cdot(\vec{b_1}\times\vec{b_2})$

Here, $\vec{a_1} = \hat{i} + \hat{j} - \hat{k}$, $\vec{b_1} = 3\hat{i} - \hat{j}$, $\vec{a_2} = 4\hat{i} + 0\hat{j} - \hat{k}$ and $\vec{b_2} = 2\hat{i} + 0\hat{j} + 3\hat{k}$.

$$\therefore \quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = -3\hat{i} - 9\hat{j} + 2\hat{k}$$

$$\Rightarrow \quad \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} + \hat{j} - \hat{k}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = -3 - 9 - 2 = -14$$

$$\text{and,} \quad \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2}) = (4\hat{i} + 0\hat{j} - \hat{k}) \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = -12 + 0 - 2 = -14$$

Clearly, $\vec{a_1} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_2} \cdot (\vec{b_1} \times \vec{b_2})$. Hence, the given lines are coplanar.

The equation of the plane containing the given lines is

$$\vec{r} \cdot (\vec{b_1} \times \vec{b_2}) = \vec{a_1} \cdot (\vec{b_1} \times \vec{b_2})$$

$$\Rightarrow \quad \vec{r} \cdot (\vec{b_1} \times \vec{b_2}) = 14$$

$$\Rightarrow \quad \vec{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = -14 \text{ or, } \vec{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14. \quad [\because \vec{b_1} \times \vec{b_2} = -3\hat{i} - 9\hat{j} + 2\hat{k}]$$

EXAMPLE 2 *Prove that the lines* $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ *and* $\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$ *are coplanar. Also, find the plane containing these two lines.*

SOLUTION We know that the lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and, } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

are coplanar, if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing these two lines is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Here, $x_1 = -1, y_1 = -3, z_1 = -5, x_2 = 2, y_2 = 4, z_2 = 6, l_1 = 3, m_1 = 5, n_1 = 7, l_2 = 1, m_2 = 4, n_2 = 7$.

$$\therefore \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 3 & 7 & 11 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 21 - 98 + 77 = 0$$

So, the given lines are coplanar. The equation of the plane containing the lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x+1)(35 - 28) - (y+3)(21 - 7) + (z+5)(12 - 5) = 0 \Rightarrow x - 2y + z = 0$$

EXAMPLE 3 *Show that the lines*

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta} \text{ and, } \frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar. **[NCERT]**

SOLUTION We know that the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and, } \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \text{ are coplanar, if}$$

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

The equations of the given lines are

$$\frac{x-(a-d)}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-(a+d)}{\alpha+\delta} \text{ and, } \frac{x-(b-c)}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-(b+c)}{\beta+\gamma}$$

Here, $x_1=a-d,\ y_1=a,\ z_1=a+d,\ x_2=b-c,\ y_2=b,\ z_2=b+c$

$l_1=\alpha-\delta,\ m_1=\alpha,\ n_1=\alpha+\delta;\ l_2=\beta-\gamma,\ m_2=\beta, n_2=\beta+\gamma$

$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} b-c-a+d & b-a & b+c-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}$$

$$= \begin{vmatrix} 2(b-a) & b-a & b+c-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix} \quad \text{Applying } C_1 \to C_1+C_3$$

$$=0 \qquad [\because C_1 \text{ and } C_2 \text{ are proportional}]$$

Hence, given lines are coplanar.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the vector equation of the plane that contains the lines* $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ *and* $\vec{r}=(\hat{i}+\hat{j})+\mu(-\hat{i}+\hat{j}-2\hat{k})$. *Also, find the length of the perpendicular drawn from the point* (2, 1, 4) *to the plane thus obtained.* **[CBSE 2012]**

SOLUTION The two given lines pass through the point having position vector $\vec{a}=\hat{i}+\hat{j}$ and are parallel to the vectors $\vec{b_1}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{b_2}=-\hat{i}+\hat{j}-2\hat{k}$ respectively. Therefore, the plane containing the given lines also passes through the point with position vector $\vec{a}=\hat{i}+\hat{j}$. Since the plane contains the lines which are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, the plane is normal to the vector

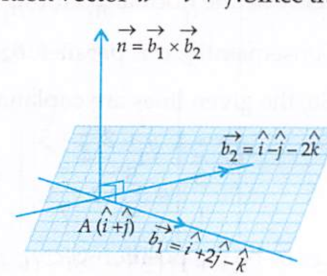


Fig. 28.31

$$\vec{n}=\vec{b_1}\times\vec{b_2}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix} = -3\hat{i}+3\hat{j}+3\hat{k}$$

Thus, the vector equation of the required planes is

$$(\vec{r}-\vec{a})\cdot\vec{n}=0 \text{ or, } \vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$$

or, $$\vec{r}\cdot(-3\hat{j}+3\hat{j}+3\hat{k})=(\hat{i}+\hat{j})\cdot(-3\hat{i}+3\hat{j}+3\hat{k})$$

$$\Rightarrow \quad \vec{r}\cdot(-3\hat{i}+3\hat{j}+3\hat{k}) = -3+3 \Rightarrow \vec{r}\cdot(-\hat{i}+\hat{j}+\hat{k}) = 0.$$

The length of perpendicular from $P(2, 1, 4)$ to the above plane is given by

$$d=\left|\frac{(2\hat{i}+\hat{j}+4\hat{k})\cdot(-\hat{i}+\hat{j}+\hat{k})}{\sqrt{(-1)^2+1^2+1^2}}\right|=\frac{|-2+1+4|}{\sqrt{3}}=\sqrt{3}$$

EXAMPLE 5 *Find the equation of the plane passing through the point* $(0, 7, -7)$ *and containing the line* $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$.

SOLUTION Let the equation of a plane passing through $(0, 7, -7)$ be

$$a(x-0)+b(y-7)+c(z+7) = 0 \qquad \text{...(i)}$$

The line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ passes through the point $(-1, 3, -2)$ and its direction ratios are proportional to $-3, 2, 1$. If plane (i) contains this line, it must pass through $(-1, 3, -2)$ and must be parallel to the line.

$$\therefore \quad a(-1)+b(3-7)+c(-2+7) = 0 \text{ and, } 3\times a+2\times b+1\times c=0$$

$$\Rightarrow \quad a(-1)+b(-4)+c(5) = 0 \quad \text{...(ii)} \quad \text{and,} \quad -3a+2b+1c = 0 \qquad \text{...(iii)}$$

On solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{-14} = \frac{b}{-14} = \frac{c}{-14} \Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1} = \lambda(\text{say}) \Rightarrow a = \lambda, b = \lambda, c = \lambda.$$

Putting the values of a, b, c in (i), we obtain

$\lambda(x-0)+\lambda(y-7)+\lambda(z+7) = 0$ or, $x+y+z=0$ as the equation of the required plane.

ALITER The required plane passes through the point $A(0, 7, -7)$ and contains the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ which passes through $B(-1, 3, -2)$ and is parallel to the vector $\vec{b}=-3\hat{i}+2\hat{j}+\hat{k}$. Thus, required plane passes through two points $A(0, 7, -7)$ and $B(-1, 3, -2)$ and is parallel to the vector $\vec{b}=-3\hat{i}+2\hat{j}+\hat{k}$.

Let $\vec{n}$ be the normal vector to the required plane. Then, $\vec{n}$ is perpendicular to both $\vec{b}$ and $\vec{AB}$. Consequently, it is parallel to $\vec{AB}\times\vec{b}$. Let $\vec{n_1} = \vec{AB}\times\vec{b}$. Then,

$$\vec{n_1} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -4 & 5 \\ -3 & 2 & 1 \end{vmatrix} = -14\hat{i}-14\hat{j}-14\hat{k}$$

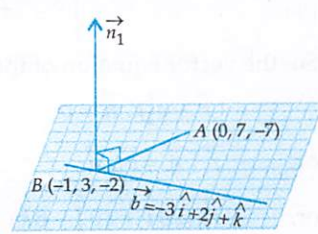


Fig. 28.32

Let $\vec{\alpha}$ be the position vector of A. Then, $\vec{\alpha}=7\hat{j}-7\hat{k}$.

Clearly, the required plane passes through $\vec{\alpha} = 7\hat{j} - 7\hat{k}$ and is perpendicular to $\vec{n_1} = -14\hat{i} - 14\hat{j} - 14\hat{k}$. So, its vector equation is

$$(\vec{r} - \vec{\alpha}) \cdot \vec{n_1} = 0$$

or, $\vec{r} \cdot \vec{n_1} = \vec{\alpha} \cdot \vec{n_1}$

or, $\vec{r} \cdot (-14\hat{i} - 14\hat{j} - 114\hat{k}) = (7\hat{j} - 7\hat{k}) \cdot (-14\hat{i} - 14\hat{j} - 14\hat{k})$

$\Rightarrow \quad \vec{r} \cdot (-14\hat{i} - 14\hat{j} - 14\hat{k}) = -98 + 98 \Rightarrow \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 0$

The cartesian equation of the above plane is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = 0$ or, $x + y + z = 0$.

EXAMPLE 6 *Show that the plane whose vector equation is $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the line whose vector equation is $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$.*

SOLUTION The line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$ passes through a point with position vector $\vec{a} = \hat{i} + \hat{j}$ and is parallel to the vector $\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$. If the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$ contains the given line, then (i) it should pass through the point $\hat{i} + \hat{j}$ and, (ii) it should be parallel to the line.

Now, $(\hat{i} + \hat{j}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 1 + 2 = 3$. So, the plane passes through the point $\hat{i} + \hat{j}$.

The normal vector to the given plane is $\vec{n} = \hat{i} + 2\hat{j} - \hat{k}$.

We observe that $\vec{b} \cdot \vec{n} = (2\hat{i} + \hat{j} + 4\hat{k}) \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 2 + 2 - 4 = 0$. Therefore, the plane is parallel to the line. Hence, the given plane contains the given line.

EXAMPLE 7 *Find the vector and cartesian equations of the plane containing the two lines*

$$\vec{r} = 2\hat{i} + \hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} + 5\hat{k}) \text{ and, } \vec{r} = 3\hat{i} + 3\hat{j} + 2\hat{k} + \mu(3\hat{i} - 2\hat{j} + 5\hat{k})$$

SOLUTION Given lines pass through points having position vectors $\vec{a_1} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\vec{a_2} = 3\hat{i} + 3\hat{j} + 2\hat{k}$ respectively and are parallel to the vectors $\vec{b_1} = \hat{i} + 2\hat{j} + 5\hat{k}$ and $\vec{b_2} = 3\hat{i} - 2\hat{j} + 5\hat{k}$ respectively. Therefore, the plane containing these two lines passes through points having position vectors $\vec{a_1}$ and $\vec{a_2}$ and is perpendicular to the vector $\vec{n} = \vec{b_1} \times \vec{b_2}$.

Now,

$$\vec{n} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 3 & -2 & 5 \end{vmatrix} = 20\hat{i} + 10\hat{j} - 8\hat{k}$$

So, the vector equation of the required plane is

$$(\vec{r} - \vec{a_1}) \cdot \vec{n} = 0$$

or, $\vec{r} \cdot \vec{n} = \vec{a_1} \cdot \vec{n}$

or, $\vec{r} \cdot (20\hat{i} + 10\hat{j} - 8\hat{k}) = (2\hat{i} + \hat{j} - 3\hat{k}) \cdot (20\hat{i} + 10\hat{j} - 8\hat{k})$

or, $\quad \vec{r}.(20\hat{i}+10\hat{j}-8\hat{k}) = 40+10+24$ or, $\vec{r}.(10\hat{i}+5\hat{j}-4\hat{k}) = 37$

The cartesian equation of the above plane is $(x\hat{i}+y\hat{j}+z\hat{k})\cdot(10\hat{i}+5\hat{j}-4\hat{k}) = 37$

or, $\quad 10x+5y-4z=37.$

EXAMPLE 8 *If $4x+4y-\lambda z=0$ is the equation of the plane through the origin that contains the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}$, find the value of λ.*

SOLUTION The line passes through the point $(1, -1, 0)$ and is parallel to the vector $\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$. A vector normal to the plane $4x+4y-\lambda z=0$ is $\vec{n}=4\hat{i}+4\hat{j}-\lambda\hat{k}$.

If the plane $4x+4y-\lambda z=0$ contains the line $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z}{4}$, then normal $\vec{n}$ to the plane must be perpendicular to $\vec{b}$.

Now, $\vec{b}\perp\vec{n} \Rightarrow \vec{b}\cdot\vec{n}=0 \Rightarrow 2\times4+3\times4-4\times\lambda=0 \Rightarrow 20-4\lambda=0 \Rightarrow \lambda=5.$

EXAMPLE 9 *If the lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then find the value of k and hence find the equation of the plane containing these lines.* **[CBSE 2015]**

SOLUTION We know that the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar, if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0 \text{ or, } \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Here, $\quad x_1=1, y_1=-1, z_1=1, x_2=3, y_2=k, z_2=0, l_1=2, m_1=3, n_1=4, l_2=1, m_2=2, n_2=1$

If given lines intersect, then they must be coplanar.

$$\therefore \quad \begin{vmatrix} 3-1 & k+1 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad 2(3-8)-(k+1)(2-4)-1(4-3)=0 \Rightarrow -10+2k+2-1=0 \Rightarrow 2k-9=0 \Rightarrow k=\frac{9}{2}$$

The equation of the plane containing the given lines is

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow \quad (x-1)(3-8)-(y+1)(2-4)+(z-1)(4-3)=0 \Rightarrow -5+5+2y+2+z-1=0 \Rightarrow 5x-2y-z=6$$

EXAMPLE 10 *Show that the line of intersection of planes $x+2y+3z=8$ and $2x+3y+4z=11$ is coplanar with the line $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$. Also, find the equation of the plane containing them.*

[CBSE 2019]

SOLUTION The equation of the plane containing the line of intersection of the planes $x+2y+3z=8$ and $2x+3y+4z=11$ is

$$(x+2y+3z-8)+\lambda(2x+3y+4z-11)=0, \text{ where } \lambda \text{ is a parameter}$$

or, $$x(2\lambda+1)+y(3\lambda+2)+(4\lambda+3)z-(8+11\lambda)=0 \quad \ldots(i)$$

If it is parallel to the line $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$, then

$$(2\lambda+1)\times 1+2(3\lambda+2)+3(4\lambda+3)=0 \Rightarrow 20\lambda+14=0 \Rightarrow \lambda=\frac{-7}{10}$$

Putting $\lambda=-\frac{7}{10}$ in (i), we obtain: $-4x-y+2z-3=0$ ...(ii)

The coordinates of any point on the line $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ are given by

$$\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}=\mu \Rightarrow x=\mu-1, y=2\mu-1, z=3\mu-1$$

Clearly, $(\mu-1, 2\mu-1, 3\mu-1)$ lies on plane given in equation (ii).

Hence, the line of intersection of the given planes is coplanar with the given line and $-4x-y+2z-3=0$ is the plane containing them.

ALITER Let the direction ratios of the line of intersection of the planes $x+2y+3z=8$ and $2x+3y+4z=11$ be proportional to l, m, n. Then,

$$\left.\begin{array}{l} l+2m+3n=0 \\ 2l+3m+4n=0 \end{array}\right\} \Rightarrow \frac{l}{8-9}=\frac{m}{6-4}=\frac{n}{3-4} \Rightarrow \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}$$

Putting $z=0$ in $x+2y+3z=8$ and $2x+3y+4z=11$, we obtain: $x+2y=8$ and $2x+3y=11$

Solving these two equations, we obtain: $x=-2, y=5$.

So, the coordinates of a point on the line of intersection of given planes are $(-2, 5, 0)$ and hence the equations of the line are

$$\frac{x+2}{-1}=\frac{y-5}{2}=\frac{z-0}{-1} \quad \ldots(i)$$

We have to show that this line is coplanar with the line

$$\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3} \quad \ldots(ii)$$

Now, $$\begin{vmatrix} -2-(-1) & 5-(-1) & 0-(-1) \\ -1 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 6 & 1 \\ -1 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = -8+12-4=0$$

Hence, the lines (i) and (ii) are coplanar.

The equation of the plane containing lines (i) and (ii) is

$$\begin{vmatrix} x+2 & y-5 & z-0 \\ -1 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix}=0 \text{ or, } 8(x+2)+2(y-5)-4z=0 \Rightarrow 4x+y-2z+3=0$$

EXERCISE 28.13

BASED ON LOTS

1. Show that the lines $\vec{r}=(2\hat{j}-3\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{r}=(2\hat{i}+6\hat{j}+3\hat{k})+\mu(2\hat{i}+3\hat{j}+4\hat{k})$ are coplanar. Also, find the equation of the plane containing them.

2. Show that the lines $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ are coplanar. Also, find the equation of the plane containing them.

3. Find the equation of the plane containing the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ and the point $(0, 7, -7)$ and show that the line $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ also lies in the same plane.

4. Find the equation of the plane which contains two parallel lines
$$\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5} \text{ and } \frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}.$$

5. Show that the lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x-2y+z+5=0=2x+3y+4z-4$ intersect. Find the equation of the plane in which they lie and also their point of intersection.

6. Show that the plane whose vector equation is $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k})=3$ contains the line whose vector equation is $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$.

7. Find the equation of the plane determined by the intersection of the lines $\frac{x+3}{3}=\frac{y}{-2}=\frac{z-7}{6}$ and $\frac{x+6}{1}=\frac{y+5}{-3}=\frac{z-1}{2}$,

8. Find the vector equation of the plane passing through the points (3, 4, 2) and (7, 0, 6) and perpendicular to the plane $2x-5y-15=0$. Also, show that the plane thus obtained contains the line $\vec{r}=\hat{i}+3\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$. [CBSE 2012]

9. If the lines $\frac{x-1}{-3}=\frac{y-2}{-2k}=\frac{z-3}{2}$ and $\frac{x-1}{k}=\frac{y-2}{1}=\frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of the plane containing these lines. [CBSE 2012]

10. Find the coordinates of the point where the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$ intersect the plane $x-y+z-5=0$. Also, find the angle between the line and the plane.

11. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also, find the coordinates of the point of intersection of this plane and the line $\vec{r}=3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$. [CBSE 2013]

12. Show that the lines $\frac{5-x}{-4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{2y-8}{2}=\frac{z-5}{3}$ are coplanar. [CBSE 2014]

13. Find the equation of a plane which passes through the point $(3, 2, 0)$ and contains the line $\frac{x-3}{1}=\frac{y-6}{5}=\frac{z-4}{4}$. [CBSE 2015]

14. Show that the lines $\frac{x+3}{-3}=\frac{y-1}{1}=\frac{z-5}{5}$ and $\frac{x+1}{-1}=\frac{y-2}{2}=\frac{z-5}{5}$ are coplanar. Hence, find the equation of the plane containing these lines. [CBSE 2017]

15. If the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane $lx+my-z=9$, then find the value of l^2+m^2.

16. Find the values of λ for which the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{\lambda^2}$ and $\frac{x-3}{1}=\frac{y-2}{\lambda^2}=\frac{z-1}{2}$ are coplanar.

17. If the lines $x=5, \frac{y}{3-\alpha}=\frac{z}{-2}$ and $x=\alpha, \frac{y}{-1}=\frac{z}{2-\alpha}$ are coplanar, find the values of α.

18. If the straight lines $\frac{x-1}{2}=\frac{y+1}{k}=\frac{z}{2}$ and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{k}$ are coplanar, find the equations of the planes containing them.

19. Find the vector equation of the plane that contains the lines $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ and the point $(-1, 3, -4)$. Also, find the length of the perpendicular drawn from the point $(2, 1, 4)$ to the plane, thus obtained. **[CBSE 2019]**

ANSWERS

1. $\vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})+7=0$ 2. $x+y+z=0$ 3. $x+y+z=0$
4. $11x-y-3z=35$ 5. $(2, 4, -3)$, $45x-17y+25z+53=0$
7. $2x-z+13=0$ 8. $\vec{r}\cdot(5\hat{i}+2\hat{j}-3\hat{k})=17$ 9. $k=2, -22x+19y+5z=31$
10. $(2, -1, 2)$, $\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$ 11. $\vec{r}\cdot(9\hat{i}+3\hat{j}-\hat{k})=14$, $(1, 1, -2)$ 13. $x-y+z-1=0$
14. $x-2y+z=0$ 15. 2 16. $0, \pm\sqrt{2}$ 17. 1, 4, 5
18. $y\pm z+1=0$ 19. $\vec{r}\cdot(\hat{i}-\hat{j}-\hat{k})=0, \sqrt{3}$

28.13.7 ALTERNATIVE METHOD FOR SHORTEST DISTANCE BETWEEN SKEW LINES

Recall from Remark in section 28.6.2 that two skew-lines l_1 and l_2 (say) do not intersect and are not parallel, but do lie in parallel planes. So, the shortest distance 'd' between l_1 and l_2 is same as the distance between parallel planes P_1 and P_2 containing l_1 and l_2 respectively. The shortest distance 'd' is also equal to the distance between any point on one of the lines l_1 (say) and the plane P_2 containing the other line l_2 and parallel to plane P_1.
This suggests us the following algorithm for finding shortest distance between two lines.

ALGORITHM

Step I *Obtain the equations of two lines. Let the lines be*
$$l_1: \frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ and } l_2: \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

Step II *Take a point P on one of the lines. Clearly, line l_1 passes through (x_1, y_1, z_1). So, take point P as (x_1, y_1, z_1).*

Step III *Find the plane P_2 containing l_2 and parallel to l_1 as follows. The equation of a plane containing l_2 is*
$$a(x-x_2)+b(y-y_2)+c(z-z_2) = 0 \quad \text{...(i)}$$
where,
$$a\,l_2+bm_2+cn_2 = 0 \quad \text{...(ii)}$$
If it is parallel to l_1, then
$$al_1+bm_1+cn_1 = 0 \quad \text{...(iii)}$$
Solving (ii) and (iii) by cross-multiplication and substituting in (i), we obtain the equation of plane P_2.

Step IV *Find the distance between point P and plane P_2 obtained in step III. This is the required shortest distance between the given lines.*

Following examples will illustrate the same.

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I ON FINDING THE SHORTEST DISTANCE BETWEEN TWO LINES IN SYMMETRIC FORM

EXAMPLE 1 *Find the shortest distance between the skew-lines*

$$l_1 : \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{4} \text{ and } l_2 : \frac{x+2}{4} = \frac{y-0}{-3} = \frac{z+1}{1}$$

SOLUTION Clearly, line l_1 passes through the point $P(1, -1, 2)$. The equation of a plane containing line l_2 is

$$a(x+2) + b(y-0) + c(z+1) = 0 \qquad \text{...(i)}$$

where $4a - 3b + c = 0$ ...(ii)

If it is parallel to line l_1, then

$$2a + b + 4c = 0 \qquad \text{...(iii)}$$

Solving (ii) and (iii) by cross-multiplication, we get

$$\therefore \quad \frac{a}{-13} = \frac{b}{-14} = \frac{c}{10}$$

Substituting values of a, b, c in (i), we obtain

$$-13(x+2) - 14y + 10(z+1) = 0 \text{ or, } 13x + 14y - 10z + 16 = 0 \qquad \text{...(iv)}$$

This is the equation of the plane containing line l_2 and parallel to line l_1.

Let d be the shortest distance between the given lines. Then,

d = Distance between point $P(1, -1, 2)$ and plane (iv)

$$\Rightarrow \quad d = \left| \frac{13 - 14 - 20 + 16}{\sqrt{13^2 + 14^2 + (-10)^2}} \right| = \frac{5}{\sqrt{465}}$$

EXAMPLE 2 *Find the distance between the line* $\vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(\hat{i} - 2\hat{j})$ *and the line passing through* $(0, -1, 2)$ *and* $(1, -2, 3)$.

SOLUTION The vector equation of the line passing through points $(0, -1, 2)$ and $(1, -2, 3)$ is

$$\vec{r} = (-\hat{j} + 2\hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$$

Thus, the equations of two lines are

$$l_1 : \vec{r} = (-\hat{i} + 3\hat{k}) + \lambda(\hat{i} - 2\hat{j}) \quad \text{...(i)} \qquad l_2 : \vec{r} = (-\hat{j} + 2\hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k}) \quad \text{...(ii)}$$

Clearly, line l_1 passes through the point $\vec{a} = -\hat{i} + 3\hat{k}$. The plane containing line l_2 and parallel to line l_1 is normal to the vector $\vec{n}$ given by

$$\vec{n} = (\hat{i} - 2\hat{j}) \times (\hat{i} - \hat{j} + \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = -2\hat{i} - \hat{j} + \hat{k}$$

Also, it passes through the point $\vec{a} = -\hat{j} + 2\hat{k}$. Thus, the equation of the plane containing l_2 and parallel to l_1 is

$$(\vec{r} - \vec{a}) \cdot \vec{n} = 0 \text{ or, } \vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \text{ or, } \vec{r} \cdot (-2\hat{i} - \hat{j} + \hat{k}) = 3 \qquad \text{...(iii)}$$

Let d be the shortest distance between the lines l_1 and l_2. Then,

$d =$ Length of the perpendicular from $\vec{a} = -\hat{i} + 3\hat{k}$ on the plane (iii)

$$\Rightarrow \quad d = \frac{|(-\hat{i} + 3\hat{k}) \cdot (-2\hat{i} - \hat{j} + \hat{k}) - 3|}{\sqrt{4+1+1}} = \frac{|2+3-3|}{\sqrt{6}} = \frac{2}{\sqrt{6}}$$

Type II ON FINDING THE SHORTEST DISTANCE BETWEEN TWO LINES WHEN ONE IS IN SYMMETRIC FORM AND OTHER IN UN-SYMMETRIC FORM

EXAMPLE 3 *Find the shortest distance between the lines* $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-0}{4}$ *and* $2x + 3y - 5z - 6 = 0 = 3x - 2y - z + 3$.

SOLUTION The equation of a plane containing the line $2x + 3y - 5z - 6 = 0 = 3x - 2y - z + 3$ is

$$(2x + 3y - 5z - 6) + \lambda(3x - 2y - z + 3) = 0 \text{ or, } x(3\lambda + 2) + y(3 - 2\lambda) - z(5 + \lambda) + 3\lambda - 6 = 0 \quad \ldots\text{(i)}$$

If it is parallel to the line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-0}{4}$, then

$$2(3\lambda + 2) + 3(3 - 2\lambda) - 4(5 + \lambda) = 0 \Rightarrow -4\lambda - 7 = 0 \Rightarrow \lambda = -\frac{7}{4}$$

Putting $\lambda = -\frac{7}{4}$ in (i), we obtain: $13x - 26y + 13z + 45 = 0$ ...(ii)

as the equation of the plane containing the second line and parallel to the first line.

Clearly, the line $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-0}{4}$ passes through the point $(2, -1, 0)$. So, the shortest distance 'd' between the given lines is equal to the length of perpendicular from $(2, -1, 0)$ on plane (ii). Hence,

$$d = \left| \frac{13 \times 2 - 26 \times (-1) + 13 \times 0 + 45}{\sqrt{13^2 + (-26)^2 + 13^2}} \right| = \frac{97}{13\sqrt{7}}$$

EXAMPLE 4 *Find the equation of the plane through the line* $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$ *and parallel to the line* $\frac{x+1}{2} = \frac{1-y}{4} = \frac{z+2}{1}$. *Hence, find the line shortest distance between the lines.*

SOLUTION The equation of a plane containing the line $\frac{x-1}{3} = \frac{y-4}{2} = \frac{z-4}{-2}$ is

$$a(x-1) + b(y-4) + c(z-4) = 0 \quad \ldots\text{(i)}$$

where $\quad 3a + 2b - 2c = 0$ ...(ii)

If (i) is parallel to the line $\frac{x+1}{2} = \frac{1-y}{4} = \frac{z+2}{1}$, then

$$2a - 4b + c = 0 \quad \ldots\text{(iii)}$$

From (ii) and (iii), we obtain

$$\frac{a}{-6} = \frac{b}{-7} = \frac{c}{-16} \text{ or, } \frac{a}{6} = \frac{b}{7} = \frac{c}{16} \Rightarrow a = 6\lambda, b = 7\lambda, c = 16\lambda$$

Substituting the values of a, b and c in (i), we obtain: $6x + 7y + 16z = 98$ ...(iv)

The shortest distance between the given lines is equal to the length of the perpendicular from any point on $\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1}$ to the plane (iv). The coordinates of a point on the line $\frac{x+1}{2} = \frac{y-1}{-4} = \frac{z+2}{1}$ are $(-1, 1, -2)$.

$\therefore$ Shortest distance = Length of the perpendicular from $(-1, 1, -2)$ to plane (iv)

$$= \frac{|-6+7-32-98|}{\sqrt{36+49+256}} = \frac{129}{\sqrt{341}}$$

EXERCISE 28.14

BASED ON HOTS

1. Find the shortest distance between the lines $\frac{x-2}{-1} = \frac{y-5}{2} = \frac{z-0}{3}$ and $\frac{x-0}{2} = \frac{y+1}{-1} = \frac{z-1}{2}$.
2. Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$.
3. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z+2}{1}$ and $3x - y - 2z + 4 = 0 = 2x + y + z + 1$.

ANSWERS

1. $\frac{65}{\sqrt{122}}$ 2. $2\sqrt{29}$ 3. $\frac{8}{\sqrt{14}}$

28.14 IMAGE OF A POINT IN A PLANE

DEFINITION *Let P and Q be two points and let π be a plane such that (i) line PQ is perpendicular to the plane π and, (ii) mid-point of PQ lies on the plane π.*

Then, either of the point is the image of the other in the plane π.

In order to find the image of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$, we may use the following algorithm.

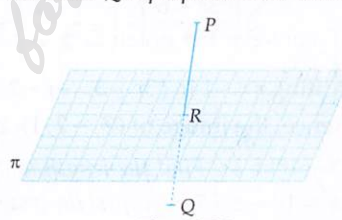


Fig. 28.33

ALGORITHM

Step I *Write the equations of the line passing through P and normal to the given plane as* $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Step II *Write the coordinates of image Q as* $(x_1 + ar, y_1 + br, z_1 + cr)$.

Step III *Find the coordinates of the mid-point R of PQ.*

Step IV *Obtain the value of r by substituting the coordinates of R in the equation of the plane.*

Step V *Put the value of r in the coordinates of Q.*

The above algorithm is illustrated in the following examples.

REMARK 1 *The coordinates (u, v, w) of the foot of the perpendicular from the point (x_1, y_1, z) to the plane $ax + by + cz + d = 0$ are given by* $\frac{u-x_1}{a} = \frac{v-y_1}{b} = \frac{w-z_1}{c} = -\left(\frac{ax_1+by_1+cz_1}{a^2+b^2+c^2}\right)$.

REMARK 2 *The coordinates (α, β, γ) of the image of the point (x_1, y_1, z_1) in the plane $ax + by + cz + d = 0$ are given by* $\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{\gamma - z_1}{c} = -\frac{2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Find the image of the point* $(3, -2, 1)$ *in the plane* $3x - y + 4z = 2$.

SOLUTION Let Q be the image of the point $P(3, -2, 1)$ in the plane $3x - y + 4z = 2$. Then, PQ is normal to the plane. Therefore, direction ratios of PQ are proportional to $3, -1, 4$. Since PQ passes through $P(3, -2, 1)$ and has direction ratios proportional to $3, -1, 4$. Therefore, equation of PQ is

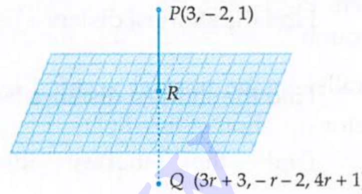


Fig. 28.34

$$\frac{x-3}{3} = \frac{y+2}{-1} = \frac{z-1}{4} = r \quad \text{(say)}$$

Let the coordinates of Q be $(3r + 3, -r - 2, 4r + 1)$. Let R be the mid-point of PQ. Then, R lies on the plane $3x - y + 4z = 2$. The coordinates of R are

$$\left(\frac{3r+3+3}{2}, \frac{-r-2-2}{2}, \frac{4r+1+1}{2}\right) \text{ or, } \left(\frac{3r+6}{2}, \frac{-r-4}{2}, 2r+1\right)$$

Since R lies on the plane $3x - y + 4z = 2$.

$$3\left(\frac{3r+6}{2}\right) - \left(\frac{-r-4}{2}\right) + 4(2r+1) = 2 \Rightarrow 13r = -13 \Rightarrow r = -1$$

Putting $r = -1$ in $(3r + 3, \ -r - 2, 4r + 1)$, we obtain the coordinates of Q as $(0, -1, -3)$.

Hence, the image of $(3, -2, 1)$ in the plane $3x - y + 4z = 2$ is $(0, -1, -3)$.

EXAMPLE 2 *Find the length and the foot of the perpendicular from the point* $(7, 14, 5)$ *to the plane* $2x + 4y - z = 2$. *Also, find the image of the point P in the plane.* **[CBSE 2012]**

SOLUTION Let M be the foot of the perpendicular from P on the plane $2x + 4y - z = 2$. Then, PM is normal to the plane. So, its direction ratios are proportional to $2, 4, -1$. Since PM passes through $P(7, 14, 5)$. Therefore, its equation is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = r \qquad \text{(say)}$$

Let the coordinates of M be $(2r + 7, 4r + 14, -r + 5)$.

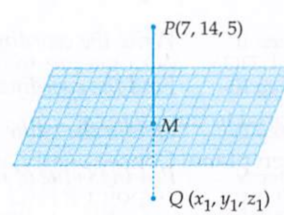


Fig. 28.35

Since M lies on the plane $2x + 4y - z = 2$. Therefore,

$$2(2r + 7) + 4(4r + 14) - (-r + 5) = 2 \Rightarrow 21r + 63 = 0 \Rightarrow r = -3$$

So, the coordinates of M are $(1, 2, 8)$.

$$\therefore \qquad PM = \sqrt{(7-1)^2 + (14-2)^2 + (5-8)^2} = 3\sqrt{21}.$$

Hence, Length of the perpendicular form $P = 3\sqrt{21}$.

Let $Q(x_1, y_1, z_1)$ be the image of P in the given plane. Then, the coordinates of M are $\left(\frac{x_1+7}{2}, \frac{y_1+14}{2}, \frac{z_1+5}{2}\right)$. But, the coordinates of M are given as $(1, 2, 8)$.

$$\therefore \qquad \frac{x_1+7}{2} = 1, \ \frac{y_1+14}{2} = 2, \ \frac{z_1+5}{2} = 8 \Rightarrow x_1 = -5, \ y_1 = -10, \ z_1 = 11$$

Hence, coordinates of Q are $(-5, -10, 11)$.

EXAMPLE 3 *Find the image of the point having position vector* $\hat{i}+3\hat{j}+4\hat{k}$ *in the plane* $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$. **[NCERT EXEMPLAR, CBSE 2022]**

SOLUTION Let Q be the image of the point $P\,(\hat{i}+3\hat{j}+4\hat{k})$ in the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$. Then, PQ is normal to the plane. Since PQ passes through P and is normal to the given plane. So, it is parallel to the normal vector $2\hat{i}-\hat{j}+\hat{k}$. Therefore, vector equation of line PQ is

$$\vec{r}=(\hat{i}+3\hat{j}+4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k}).$$

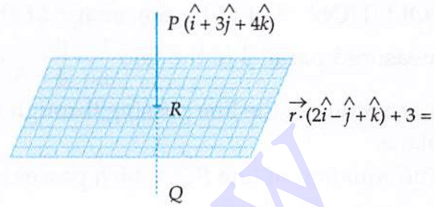


Fig. 28.36

As Q lies on line PQ so, let the position vector of Q be

$$(\hat{i}+3\hat{j}+4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})=(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}. \quad \ldots\text{(i)}$$

Since R is the mid-point of PQ. Therefore, position vector of R is

$$\frac{[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(4+\lambda)\hat{k}]+[\hat{i}+3\hat{j}+4\hat{k}]}{2}=(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}$$

Clearly, R lies on the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$.

$$\therefore \quad \left\{(\lambda+1)\hat{i}+\left(3-\frac{\lambda}{2}\right)\hat{j}+\left(4+\frac{\lambda}{2}\right)\hat{k}\right\}\cdot(2\hat{i}-\hat{j}+\hat{k})+3=0$$

$$\Rightarrow \quad 2\lambda+2-3+\frac{\lambda}{2}+4+\frac{\lambda}{2}+3=0 \Rightarrow \lambda=-2.$$

Putting $\lambda=-2$ in (i), we obtain the position vector of Q as

$$(\hat{i}+3\hat{j}+4\hat{k})-2(2\hat{i}-\hat{j}+\hat{k})=-3\hat{i}+5\hat{j}+2\hat{k}.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find the image of line* $\dfrac{x-1}{3}=\dfrac{y-3}{1}=\dfrac{z-4}{-5}$ *in the plane* $2x-y+z+3=0$.

SOLUTION We observe that the line is parallel to the vector $\vec{b}=3\hat{i}+\hat{j}-5\hat{k}$ and the plane is normal to the vector $\vec{n}=2\hat{i}-\hat{j}+\hat{k}$ such that $\vec{b}\cdot\vec{n}=6-1-5=0$. So, the given line is parallel to the given plane. Consequently, its image in the plane $2x-y+z+3=0$ will pass through the image $Q\,(x_1, y_1, z_1)$ of point $P\,(1, 3, 4)$ in the given plane. Also, the image is parallel to the given line. The co-ordinates of Q are given by

$$\frac{x_1-1}{2}=\frac{y_1-3}{-1}=\frac{z_1-4}{1}=-2\left(\frac{2\times1-3+4+3}{2^2+(-1)^2+1^2}\right)$$

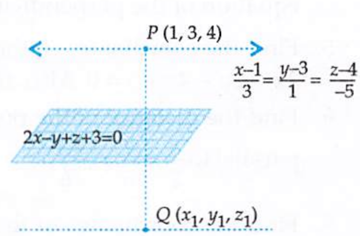


Fig. 28.37

$$\Rightarrow \quad \frac{x_1-1}{2}=\frac{y_1-3}{-1}=\frac{z_1-4}{1}=-2$$

$$\Rightarrow \quad \frac{x_1-1}{2}=-2,\ \frac{y_1-3}{-1}=-2,\ \frac{z_1-4}{1}=-2$$

$$\Rightarrow \quad x_1=-3,\ y_1=5,\ z_1=2$$

The required line passes through (– 3, 5, 2) and is parallel to the given line. So, its equation is

$$\frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$$

EXAMPLE 5 *Find the image of the point $P(1,-2,3)$ in the plane $2x+3y-4z+22=0$ measured parallel to the line $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$. Also, find the distance between point P and its image.*

SOLUTION Let Q be the image of the point $P(1,-2,3)$ in the plane $2x+3y-4z+22=0$ measured parallel to the line $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ and R be the point of intersection of the line passing through P and Q and the given plane.

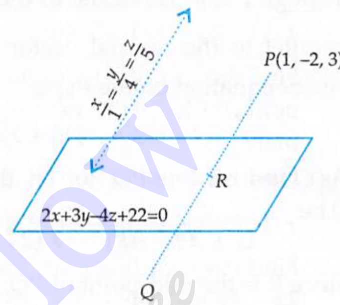


Fig. 28.38

The equation of line PQ, which passes through P and is parallel to $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$, is

$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}$$

Clearly, R is on this line. So, let the co-ordinates of R be given by

$$\frac{x-1}{1}=\frac{y+2}{4}=\frac{z-3}{5}=\lambda \text{ or, } (\lambda+1,4\lambda-2,5\lambda+3)$$

Point $R(\lambda+1,4\lambda-2,5\lambda+3)$ lies on the plane $2x+3y-4z+22=0$.

$\therefore \quad 2(\lambda+1)+3(4\lambda-2)-4(5\lambda+3)+22=0 \Rightarrow 6\lambda=6 \Rightarrow \lambda=1.$

So, the co-ordinates of R are (2, 2, 8). Let (x_1,y_1,z_1) be the co-ordinates of Q. Then,

$$\frac{x_1+1}{2}=2,\ \frac{y_1-2}{2}=2,\ \frac{z_1+3}{2}=8 \Rightarrow x_1=3,\ y_1=6, z_1=13 \qquad [\because R \text{ is the mid-point of } PQ]$$

Hence, the co-ordinates of Q are (3, 6, 13) and $PQ=\sqrt{(3-1)^2+(6+2)^2+(13-3)^2}=\sqrt{168}$.

EXERCISE 28.15

BASED ON LOTS

1. Find the image of the point (0, 0, 0) in the plane $3x+4y-6z+1=0$.
2. Find the reflection of the point (1, 2, – 1) in the plane $3x-5y+4z=5$.
3. Find the coordinates of the foot of the perpendicular drawn from the point (5, 4, 2) to the line $\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}$. Hence or otherwise deduce the length of the perpendicular.
4. Find the image of the point with position vector $3\hat{i}+\hat{j}+2\hat{k}$ in the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=4$. Also, find the position vectors of the foot of the perpendicular and the equation of the perpendicular line through $3\hat{i}+\hat{j}+2\hat{k}$.
5. Find the coordinates of the foot of the perpendicular from the point (1, 1, 2) to the plane $2x-2y+4z+5=0$. Also, find the length of the perpendicular.
6. Find the distance of the point (1, – 2, 3) from the plane $x-y+z=5$ measured along a line parallel to $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$. [CBSE 2008]
7. Find the coordinates of the foot of the perpendicular from the point (2, 3, 7) to the plane $3x-y-z=7$. Also, find the length of the perpendicular.

8. Find the image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$.

9. Find the distance of the point with position vector $-\hat{i} - 5\hat{j} - 10\hat{k}$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ with the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. **[CBSE 2011]**

10. Find the length and the foot of the perpendicular from the point (1, 1, 2) to the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) + 5 = 0$. **[CBSE 2002C]**

11. Find the coordinates of the foot of the perpendicular and the perpendicular distance of the point P (3, 2, 1) from the plane $2x - y + z + 1 = 0$. Find also the image of the point in the plane. **[CBSE 2010, 2012, 2019]**

12. Find the direction cosines of the unit vector perpendicular to the plane $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ passing through the origin. **[NCERT]**

13. Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z - 6 = 0$. **[NCERT]**

14. Find the length and the foot of perpendicular from the point (1, 3/2, 2) to the plane $2x - 2y + 4z + 5 = 0$. **[NCERT EXEMPLAR]**

15. Find the position vector of the foot of the perpendicular and the perpendicular distance from the point P with position vector $2\hat{i} + 3\hat{j} + 4\hat{k}$ to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} + 3\hat{k}) - 26 = 0$. Also, find the image of P in the plane. **[CBSE 2016]**

ANSWERS

1. $(-6/61, -8/61, 12/61)$ 2. $(73/25, -6/5, 39/25)$ 3. $(1, 6, 0), 2\sqrt{6}$

4. $(1, 2, 1), 2\hat{i} + \frac{3}{2}\hat{j} + \frac{3}{2}\hat{k}, \vec{r} = 3\hat{i} + \hat{j} + 2\hat{k} + \lambda(2\hat{i} - \hat{j} + \hat{k})$

5. $(-1/12, 25/12, -1/6), 13/\sqrt{24}$ 6. 1   7. $(5, 2, 6); \sqrt{11}$

8. $(-3, 5, 2)$ 9. 13   10. $\frac{13}{12}\sqrt{6}, (-1/12, 25/12, -2/12)$

11 $(1, 3, 0), \sqrt{6}$ 12. $-6/7, 3/7, 2/7$ 13. $(12/29, -18/29, 24/29)$ 14. $\sqrt{6}; (0, 5/2, 0)$

15. $3\hat{i} + \frac{7}{2}\hat{j} + \frac{11}{2}\hat{k}, \sqrt{\frac{7}{2}}, -2\hat{j} - \hat{k}$

HINTS TO SELECTED PROBLEMS

12. The equation of the plane is $\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$. To find the direction cosines of the normal through the origin, we must write the equation in normal form as follows.

We have,

$$\vec{r} \cdot (6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$$

$$\Rightarrow \vec{r} \cdot (-6\hat{i} + 3\hat{j} + 2\hat{k}) = 1$$

$$\Rightarrow \vec{r} \cdot \frac{(-6\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{36 + 9 + 4}} = \frac{1}{\sqrt{36 + 9 + 4}} \Rightarrow \vec{r} \cdot \left(-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = \frac{1}{7}$$

Hence, direction cosines of the normal vector through the origin are $-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}$.

13. The direction ratios of OP are proportional to 2, – 3, 4.

So, cartesian equations of OP are $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{z}{4}$

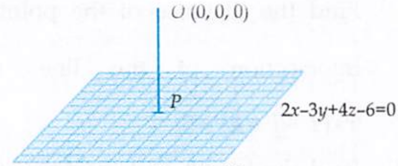


Fig. 28.39

The coordinates of P are given by $\dfrac{x}{2} = \dfrac{y}{-3} = \dfrac{z}{4} = r.$

So, the coordinates of P are $(2r, -3r, 4r)$.

It lies on $2x - 3y + 4z - 6 = 0$.

$\therefore \quad 4r + 9r + 16r - 6 = 0 \Rightarrow r = \dfrac{6}{29}$

Hence, the coordinates of P are $\left(\dfrac{12}{29}, \dfrac{-18}{29}, \dfrac{24}{29}\right)$.

FILL IN THE BLANK TYPE QUESTIONS (FBQs)

1. If the plane $x + 2y - 2z = d,\ d > 0$ is at a 5 unit distance from the point $(1, -2, 1)$, then $d =$ ....
2. The planes $3x - 6y - 2z = 7$ and $2x + y - \lambda z = 5$ are perpendicular. Then the value of λ is ....
3. The equation of the plane passing through (1, 2, 3) and parallel to the plane $2x + 3y - 4z = 0$ is.................. .
4. The equation of a line passing through (1, 2, 3) and normal to the plane $2x - 3y + 6z = 11$ is............... .
5. The sum of the intercepts of the plane $2x + 3y - 4kz = 24$ on the coordinate is 8. then the value of k is.................. .
6. If the distance of the point (1, 1, 1) from the origin is half its distance from the plane $x + y + z + k = 0$, then the values of k are.................. .
7. If the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 3$ meets the coordinates axes at A, B and C, then the coordinates of the centroid of ΔABC are ....
8. If O is the origin, then the equation of the plane passing through $P(a, b, c)$ and perpendicular to OP is ....
9. A variable plane moves so that the sum of the reciprocals of its intercepts on the coordinate axis is $\dfrac{1}{2}$. The plane passes through the point................. .
10. The value of α for which the plane $x + \alpha y + z = 5$ cuts equal intercepts on the axes, is............ .
11. If the line $\dfrac{2x-1}{2} = \dfrac{2-y}{2} = \dfrac{z+1}{k}$ is parallel to the plane $2x - y + z = 3$, then $k =$ ....
12. If the line $\dfrac{x-2}{3} = \dfrac{y-1}{-5} = \dfrac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + 7 = 0$, then $\alpha =$ ....
13. The line $\dfrac{x+1}{\lambda} = \dfrac{y-1}{1} = \dfrac{z+2}{-4}$ is perpendicular to the plane $2x + 2y - 8z + 5 = 0$. Then the value of λ is ....
14. A plane passes through the points (2, 0, 0), (0, 3, 0) and (0, 0, 4). The equation of the plane is....................... . **[NCERT EXEMPLAR]**
15. The cartesian equation of the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ is ....**[NCERT EXEMPLAR]**
16. The intercepts made by the plane $2x - 3y + 5z + 4 = 0$ on the coordinate axes are .... **[NCERT EXEMPLAR]**

17. The angle between the line $\vec{r}=(5\hat{i}-\hat{j}-4\hat{k})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and the plane $\vec{r}\cdot(3\hat{i}-4\hat{j}-\hat{k})+5=0$ is ....
18. The foot of the perpendicular from the origin to a plane has the coordinates $(5,-3,-2)$. The equation of the plane is ....
19. The distance between the parallel planes $2x+y-2z-6=0$ and $4x+2y-4z=0$ is .... [CBSE 2020]
20. If $P(1, 0, -3)$ is the foot of the perpendicular from the origin to the plane, then the cartesian equation of the plane is .... [CBSE 2020]

ANSWERS

1. 10
2. 0
3. $2x+3y-4z+4=0$
4. $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$
5. $k=\frac{1}{2}$
6. $3,-9$
7. (a,b,c)
8. $ax+by+cz=a^2+b^2+c^2$
9. $(2, 2, 2)$
10. 1
11. -4
12. -6
13. 1
14. $6x+4y+3z=12$
15. $x+y+z=2$
16. $-2, \frac{4}{3}, \frac{-4}{5}$
17. $\sin^{-1}\left(\frac{9}{\sqrt{156}}\right)$
18. $\vec{r}\cdot(5\hat{i}-3\hat{j}-2\hat{k})=38$
19. 3
20. $x-3z=10$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the equation of the plane parallel to XOY-plane and passing through the point $(2,-3,5)$.
2. Write the equation of the plane parallel to YOZ-plane and passing through $(-4, 1, 0)$.
3. Write the equation of the plane passing through points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.
4. Write the general equation of a plane parallel to X-axis.
5. Write the value of k for which the planes $x-2y+kz=4$ and $2x+5y-z=9$ are perpendicular.
6. Write the intercepts made by the plane $2x-3y+4z=12$ on the coordinate axes.
7. Write the ratio in which the plane $4x+5y-3z=8$ divides the line segment joining points $(-2,1,5)$ and $(3, 3, 2)$.
8. Write the distance between the parallel planes $2x-y+3z=4$ and $2x-y+3z=18$.
9. Write the plane $\vec{r}\cdot(2\hat{i}+3\hat{j}-6\hat{k})=14$ in normal form.
10. Write the distance of the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=12$ from the origin.
11. Write the equation of the plane $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ in scalar product form.
12. Write a vector normal to the plane $\vec{r}=l\vec{b}+m\vec{c}$.
13. Write the equation of the plane passing through $(2,-1,1)$ and parallel to the plane $3x+2y-z=7$.

14. Write the equation of the plane containing the lines $\vec{r} = \vec{a} + \lambda \vec{b}$ and $\vec{r} = \vec{a} + \mu \vec{c}$.

15. Write the position vector of the point where the line $\vec{r} = \vec{a} + \lambda \vec{b}$ meets the plane $\vec{r} \cdot \vec{n} = 0$.

16. Write the value of k for which the line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{k}$ is perpendicular to the normal to the plane $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 4$.

17. Write the angle between the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{-2}$ and the plane $x + y + 4 = 0$.

18. Write the intercept cut off by the plane $2x + y - z = 5$ on x-axis. [CBSE 2011]

19. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. [CBSE 2013]

20. Write the vector equation of the line passing through the point $(1, -2, -3)$ and normal to the plane $\vec{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 5$. [CBSE 2013]

21. Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$. [CBSE 2014]

22. Find the vector equation of a plane which is at a distance of 5 units from the origin and its normal vector is $2\hat{i} - 3\hat{j} + 6\hat{k}$. [CBSE 2016]

23. Write the equation of a plane which is at a distance of $5\sqrt{3}$ units from the origin and the normal to which is equally inclined to coordinate axes. [CBSE 2016]

24. Find the acute angle between the planes $\vec{r} \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = 1$ and $\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = 0$. [CBSE 2019]

25. Find the values of λ, for which the distance of point $(2, 1, \lambda)$ from the plane $3x + 5y + 4z - 11 = 0$ is $2\sqrt{2}$ units. [CBSE 2022]

ANSWERS

1. $z = 5$ 2. $x = -4$ 3. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ 4. $by + cz + d = 0$

5. -8 6. $6, -4, 3$ 7. $2:1$ 8. $\sqrt{14}$

9. $\vec{r} \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right) = 2$ 10. 4 11. $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$

12. $\vec{b} \times \vec{c}$ 14. $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{c}) = 0$ 15. $\vec{a} - \left(\frac{\vec{a} \cdot \vec{n}}{\vec{b} \cdot \vec{n}}\right)\vec{b}$ 16. $\frac{-13}{4}$

17. $45°$ 18. $\frac{5}{2}$ 19. 3 20. $\vec{r} = \hat{i} - 2\hat{j} - 3\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$

21. $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$ 22. $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 35$ 23. $x + y + z = 15$

24. $\cos^{-1}\left(\frac{11}{21}\right)$ 25. ± 5