CHAPTER 15

LINEAR INEQUATIONS

15.1 INTRODUCTION

In this chapter, we will study linear inequations in one and two variables. The knowledge of linear inequations is very helpful in solving problems in Science, Mathematics, Engineering, Linear Programming etc.

15.2 INEQUATIONS

In earlier classes, we have studied equations in one and two variables. An equation is defined as a statement involving variable (s) and the sign of equality (=). Similarly, we define the term inequation as follows:

INEQUATION *A statement involving variable (s) and the sign of inequality viz,* $>, <, \geq$ *or* $\leq$ *is called an inequation or an inequality.*

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Following are some examples of inequations:

(i) $3x-2<0$ (ii) $2x+3 \leq 0$ (iii) $5x-3>0$

(iv) $4x+5 \geq 0$ (v) $2x+3y<1$ (vi) $5x+4y \leq 3$

(vii) $4x-6y>5$ (viii) $2x+5y \geq 4$ (ix) $2x^2+3x+4>0$

(x) $x^2-3x+2 \geq 0$ (xi) $x^2+3x+2<0$ (xii) $x^2-5x+4 \leq 0$

(xiii) $x^3-6x^2+11x-6>0$ (xiv) $x^3+6x^2+11x+6 \leq 0$

LINEAR INEQUATION IN ONE VARIABLE *Let a be a non-zero real number and x be a variable. Then inequations of the form* $ax+b<0$, $ax+b \leq 0$, $ax+b>0$ *and* $ax+b \geq 0$ *are known as linear inequations in one variable x.*

For example, $9x-15>0$, $5x-4 \geq 0$, $3x+2<0$ and $2x-3 \leq 0$ are linear inequations in one variable.

LINEAR INEQUATIONS IN TWO VARIABLES *Let a, b be non-zero real numbers and x, y be variables. Then inequations of the form* $ax+by<c$, $ax+by \leq c$, $ax+by>c$ *and* $ax+by \geq c$ *are known as linear inequations in two variables x* and *y*.

For example, $2x+3y \leq 6$, $3x-2y \geq 12$, $x+y<4$, $2x+y \geq 6$ are linear inequations in two variables x and y.

QUADRATIC INEQUATION *Let a be a non-zero real number. Then an inequation of the form* $ax^2+bx+c<0$, *or* $ax^2+bx+c \leq 0$, *or* $ax^2+bx+c>0$, *or* $ax^2+bx+c \geq 0$ *is known as a quadratic inequation.*

For example, $x^2+x-6<0$, $x^2-3x+2 \geq 0$, $2x^2+3x+1>0$ and $x^2-5x+4 \leq 0$ are quadratic inequations.

In this chapter, we shall study linear inequations in one and two variables only.

15.3 SOLUTIONS OF AN INEQUATION

DEFINITION *A solution of an inequation is the value (s) of the variable (s) that makes it a true statement.*

Consider the inequation $\frac{3-2x}{5} < \frac{x}{3} - 4$.

Left hand side (LHS) of this inequation is $\frac{3-2x}{5}$ and right hand side (RHS) is $\frac{x}{3} - 4$.

We observe that:

For $x = 9$, we have

$$\text{LHS} = \frac{3-2\times 9}{5} = -3 \text{ and, RHS} = \frac{9}{3} - 4 = -1$$

Clearly, $-3 < -1$

$\Rightarrow$ LHS < RHS, which is true.

So, $x = 9$ is a solution of the given inequation.

For $x = 6$, we have

$$\text{LHS} = \frac{3-2\times 6}{5} = -\frac{9}{5} \text{ and RHS} = \frac{6}{3} - 4 = -2$$

Because, $-\frac{9}{5} < -2$ is not true. So, $x = 6$ is not a solution of the given inequation.

We can verify that any real number greater than 7 is a solution of the given inequation.

Let us now consider the inequation $x^2 + 1 < 0$.

We know that

$x^2 \geq 0$ for all $x \in R$

$\therefore$ $x^2 + 1 \geq 1$ for all $x \in R$

$\Rightarrow$ $x^2 + 1 \nless 0$ for any $x \in R$.

So, there is no real value of x which makes the given inequation a true statement. Hence, it has no solution.

It follows from the above discussion that an inequation may or may not have a solution. However, if an inequation has a solution it may have infinitely many solutions.

SOLVING AN INEQUATION *It is the process of obtaining all possible solutions of an inequation.*

SOLUTION SET *The set of all posible solutions of an inequation is known as its solution set.*

For example, the solution set of the inequation $x^2 + 1 \geq 0$ is the set R of all real numbers whereas the solution set of the inequation $x^2 + 1 < 0$ is the null set ϕ.

15.4 SOLVING LINEAR INEQUATIONS IN ONE VARIABLE

As mentioned in the previous section that solving an inequation is the process of obtaining its all possible solutions. In the process of solving an inequation, we use mathematical simplifications which are governed by the following rules:

<u>RULE 1</u> *Same number may be added to (or subtracted from) both sides of an inequation without changing the sign of inequality.*

<u>RULE 2</u> *Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.*

<u>RULE 3</u> *Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.*

A linear inequation in one variable is of the form

$$ax + b < 0 \text{ or, } ax + b \leq 0 \text{ or, } ax + b > 0 \text{ or, } ax + b \geq 0.$$

We follow the following algorithm to solve a linear inequation in one variable.

ALGORITHM

STEP I *Obtain the linear inequation.*

STEP II *Collect all terms involving the variable on one side of the inequation and the constant terms on the other side.*

STEP III *Simplify both sides of inequality in their simplest forms to reduce the inequation in the form* $ax < b$, or $ax \leq b$, or $ax > b$, or $ax \geq b$

STEP IV *Solve the inequation obtained in step III by dividing both sides of the inequation by the coefficient of the variable.*

STEP V *Write the solution set obtained in step IV in the form of an interval on the real line.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **SOLVING EQUATIONS OF THE FORM:** $ax + b > cx + d$, or, $ax + b \geq cx + d$, or, $ax + b < cx + d$ or, $ax + b \leq cx + d$

EXAMPLE 1 *Solve the following linear inequations:*

(i) $2x - 4 \leq 0$ (ii) $-3x + 12 < 0$ (iii) $4x - 12 \geq 0$ (iv) $7x + 9 > 30$

SOLUTION (i) We have,

$$2x - 4 \leq 0$$

$\Rightarrow \quad (2x - 4) + 4 \leq 0 + 4$ [Adding 4 on both sides]

$\Rightarrow \quad 2x \leq 4$

$\Rightarrow \quad \dfrac{2x}{2} \leq \dfrac{4}{2}$

$\Rightarrow \quad x \leq 2$

Hence, any real number less than or equal to 2 is a solution of the given inequation.

These solutions can be graphed on real line as shown in Fig. 15.1

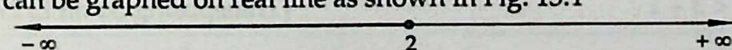

Fig. 15.1

The solution set of the given inequation is $(-\infty, 2]$

(ii) We have,

$$-3x + 12 < 0$$

$\Rightarrow \quad -3x < -12$ [Transposing 12 on right side]

$\Rightarrow \quad \dfrac{-3x}{-3} > \dfrac{-12}{-3}$ [Dividing both sides by -3]

$\Rightarrow \quad x > 4$

Thus, any real number greater than 4 is a solution of the given inequation.

Hence, the solution set of the given inequation is $(4, \infty)$. This solution set can be graphed on real line as shown in Fig. 15.2

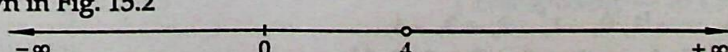

Fig. 15.2

(iii) We have,

$$4x - 12 \geq 0$$

$\Rightarrow \quad 4x \geq 12$ [Transposing 12 on RHS]

$\Rightarrow \quad \dfrac{4x}{4} \geq \dfrac{12}{4}$ [Dividing both sides by 4]

$\Rightarrow \quad x \geq 3$

$\Rightarrow \quad x \in [3, \infty)$

Hence, the solution set of the given inequation is $[3, \infty)$. This solution set can be graphed on real line as shown in Fig. 15.3

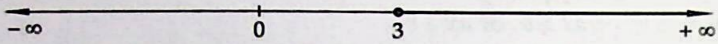

Fig. 15.3

(iv) We have,

$7x + 9 > 30$

$\Rightarrow \quad 7x > 30 - 9$

$\Rightarrow \quad 7x > 21$ [Transposing 9 on RHS]

$\Rightarrow \quad \dfrac{7x}{7} > \dfrac{21}{7}$

$\Rightarrow \quad x > 3$

$\Rightarrow \quad x \in (3, \infty)$

Hence, $(3, \infty)$ is the solution set of the given inequation. This can be graphed on real line as shown in Fig. 15.4.

Fig. 15.4

EXAMPLE 2 *Solve:* $5x - 3 < 3x + 1$ *when (i) x is a real number (ii) x is integer number (iii) x is a natural number.*

SOLUTION We have,

$5x - 3 < 3x + 1$

$\Rightarrow \quad 5x - 3x < 3 + 1$ [Transposing $3x$ on LHS and -3 on RHS]

$\Rightarrow \quad 2x < 4$

$\Rightarrow \quad \dfrac{2x}{2} < \dfrac{4}{2}$ $\left[\text{Multiplying both sides by } \dfrac{1}{2}\right]$

$\Rightarrow \quad x < 2$

(i) If $x \in R$, then

$x < 2 \Rightarrow x \in (-\infty, 2)$

Hence, the solution set is $(-\infty, 2)$ as shown in Fig. 15.5.

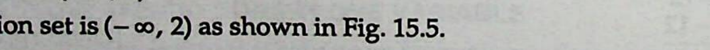

Fig. 15.5

(ii) If $x \in Z$, then

$x < 2 \Rightarrow x = 1, 0, -1, -2, -3, -4, \ldots$

So, the solution set is $\{\ldots -4, -3, -2, -1, 0, 1\}$

(iii) If $x \in N$, then

$x < 2 \Rightarrow x = 1$

So, the solution set is $\{1\}$.

EXAMPLE 3 *Solve the following equations:*

(i) $3x + 17 \leq 2(1 - x)$ (ii) $2(2x + 3) - 10 \leq 6(x - 2)$

SOLUTION (i) We have,

$$3x + 17 \leq 2(1 - x)$$
$$\Rightarrow \quad 3x + 17 \leq 2 - 2x$$
$$\Rightarrow \quad 3x + 2x \leq 2 - 17 \qquad \text{[Transposing } -2x \text{ to LHS and 17 to RHS]}$$
$$\Rightarrow \quad 5x \leq -15$$
$$\Rightarrow \quad \frac{5x}{5} \leq \frac{-15}{5}$$
$$\Rightarrow \quad x \leq -3$$
$$\Rightarrow \quad x \in (-\infty, -3]$$

Hence, the solution set of the given inequation is $(-\infty, -3]$, which can be graphed on real line as shown in Fig. 15.6.

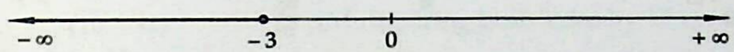

Fig. 15.6

(ii)We have,

$$2(2x + 3) - 10 \leq 6(x - 2)$$
$$\Rightarrow \quad 4x + 6 - 10 \leq 6x - 12$$
$$\Rightarrow \quad 4x - 4 \leq 6x - 12$$
$$\Rightarrow \quad 4x - 6x \leq -12 + 4 \qquad \text{[Transposing } -4 \text{ to RHS and } 6x \text{ to LHS]}$$
$$\Rightarrow \quad -2x \leq -8$$
$$\Rightarrow \quad \frac{-2x}{-2} \geq \frac{-8}{-2}$$
$$\Rightarrow \quad x \geq 4$$
$$\Rightarrow \quad x \in [4, \infty)$$

Hence, the solution set of the given inequation is $[4, \infty)$ which can be graphed on real line as shown in Fig. 15.7.

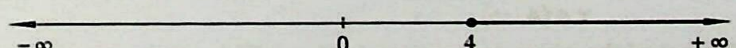

Fig. 15.7

EXAMPLE 4 *Solve the following inequations:*

(i) $\frac{2x - 3}{4} + 9 \geq 3 + \frac{4x}{3}$ (ii) $\frac{5x - 2}{3} - \frac{7x - 3}{5} > \frac{x}{4}$

(iii) $\frac{1}{2}\left(\frac{3}{5}x + 4\right) \geq \frac{1}{3}(x - 6)$ (iv) $\frac{3(x - 2)}{5} \geq \frac{5(2 - x)}{3}$

SOLUTION (i) We have,

$$\frac{2x - 3}{4} + 9 \geq 3 + \frac{4x}{3}$$
$$\Rightarrow \quad \frac{2x - 3}{4} - \frac{4x}{3} \geq 3 - 9 \qquad \left[\text{Transposing } \frac{4x}{3} \text{ to LHS and 9 to RHS}\right]$$
$$\Rightarrow \quad \frac{3(2x - 3) - 16x}{12} \geq -6$$
$$\Rightarrow \quad \frac{6x - 9 - 16x}{12} \geq -6$$

$$\Rightarrow \quad \frac{-9-10x}{12} \geq -6$$

$$\Rightarrow \quad -9-10x \geq -72 \qquad \text{[Multiplying both sides by 12]}$$

$$\Rightarrow \quad -10x \geq -72+9$$

$$\Rightarrow \quad -10x \geq -63$$

$$\Rightarrow \quad \frac{-10x}{-10} \leq \frac{-63}{-10}$$

$$\Rightarrow \quad x \leq \frac{63}{10}$$

$$\Rightarrow \quad x \in (-\infty, 63/10]$$

Hence, the solution set of the given inequation is $(-\infty, 63/10]$. This can be graphed on real line as shown in Fig. 15.8.

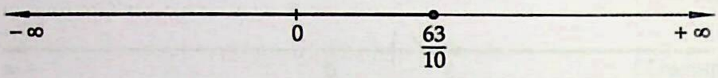

Fig. 15.8

(ii) We have,

$$\frac{5x-2}{3} - \frac{7x-3}{5} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{5(5x-2)-3(7x-3)}{15} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{25x-10-21x+9}{15} > \frac{x}{4}$$

$$\Rightarrow \quad \frac{4x-1}{15} > \frac{x}{4}$$

$$\Rightarrow \quad 4(4x-1) > 15x \qquad \text{[Multiplying both sides by 60 i.e. lcm of 15 and 4]}$$

$$\Rightarrow \quad 16x-4 > 15x$$

$$\Rightarrow \quad 16x-15x > 4 \qquad \text{[Transposing } 15x \text{ to LHS and } -4 \text{ to RHS]}$$

$$\Rightarrow \quad x > 4$$

$$\Rightarrow \quad x \in (4, \infty)$$

Hence, the solution set of the given inequation is $(4, \infty)$. This can be graphed on the real line as shown in Fig. 15.9.

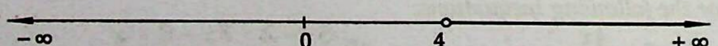

Fig. 15.9

(iii) We have,

$$\frac{1}{2}\left(\frac{3}{5}x+4\right) \geq \frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{3x+20}{5}\right) \geq \frac{1}{3}(x-6)$$

$$\Rightarrow \quad \frac{3x+20}{10} \geq \frac{x-6}{3}$$

$$\Rightarrow \quad 3(3x+20) \geq 10(x-6) \qquad \text{[Multiplying both sides by 30 i.e. the lcm of 3 and 10]}$$

$$\Rightarrow \quad 9x+60 \geq 10x-60$$

$$\Rightarrow \quad 9x-10x \geq -60-60 \qquad \text{[Transposing } 10x \text{ on LHS and 60 on RHS]}$$

$$\Rightarrow \quad -x \geq -120$$

$$\Rightarrow \quad x \leq 120 \qquad \text{[Multiplying both sides by } -1\text{]}$$

$$\Rightarrow \quad x \in (-\infty, 120]$$

Hence, the solution set of the given in equation is $(-\infty, 120]$ which can be graphed on real line as shown in Fig. 15.10.

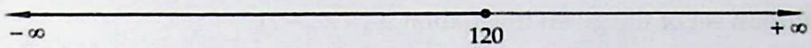

Fig. 15.10

(iv) We have,

$$\frac{3(x-2)}{5} \geq \frac{5(2-x)}{3}$$

$$\Rightarrow \quad \frac{3x-6}{5} \geq \frac{10-5x}{3}$$

$$\Rightarrow \quad 3(3x-6) \geq 5(10-5x) \qquad \text{[Multiplying both sides by 15 i.e. the lcm of 5 and 3]}$$

$$\Rightarrow \quad 9x - 18 \geq 50 - 25x$$

$$\Rightarrow \quad 9x + 25x \geq 50 + 18 \qquad \text{[Transposing } -25x \text{ to LHS and 18 to RHS]}$$

$$\Rightarrow \quad 34x \geq 68$$

$$\Rightarrow \quad \frac{34x}{34} \geq \frac{68}{34}$$

$$\Rightarrow \quad x \geq 2$$

$$\Rightarrow \quad x \in [2, \infty)$$

Hence, $[2, \infty)$ is the solution set of the given inequation. This solution set can be graphed on real line as shown in Fig. 15.11.

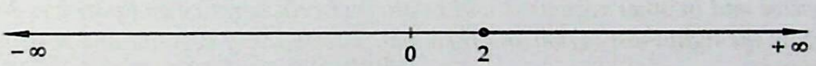

Fig. 15.11

EXAMPLE 5 *Solve the following inequations:*

(i) $\dfrac{1}{x-2} < 0$ (ii) $\dfrac{x+1}{x+2} \geq 1$

SOLUTION (i) We have,

$$\frac{1}{x-2} < 0$$

$$\Rightarrow \quad x - 2 < 0 \qquad \left[\because \frac{a}{b} < 0 \text{ and } a > 0 \Rightarrow b < 0\right]$$

$$\Rightarrow \quad x < 2$$

$$\Rightarrow \quad x \in (-\infty, 2)$$

Hence, the solution set of the given inequation is $(-\infty, 2)$.

(ii) We have,

$$\frac{x+1}{x+2} \geq 1$$

$$\Rightarrow \quad \frac{x+1}{x+2} - 1 \geq 0$$

$$\Rightarrow \quad \frac{x+1-x-2}{x+2} \geq 0$$

$$\Rightarrow \quad \frac{-1}{x+2} \geq 0$$

$$\Rightarrow \quad x+2<0 \qquad \left[\because \frac{a}{b}>0 \text{ and } a<0 \Rightarrow b<0\right]$$

$$\Rightarrow \quad x<-2$$

$$\Rightarrow \quad x \in(-\infty,-2)$$

Hence, the solution set of the given inequation is $(-\infty,-2)$.

Type II **EQUATIONS OF THE FORM**

$$\frac{ax+b}{cx+d}>k, \text{ or } \frac{ax+b}{cx+d} \geq k, \text{ or } \frac{ax+b}{cx+d}<k, \text{ or } \frac{ax+b}{cx+d} \leq k$$

In order to solve this type of inequation, we use the following algorithm.

ALGORITHM

STEP I *Obtain the inequation.*

STEP II *Transpose all terms on LHS.*

STEP III *Simplify LHS of the inequation obtained in step II to obtain an inequation of the form*

$$\frac{px+q}{rx+s}>0, \text{ or } \frac{px+q}{rx+s} \geq 0, \text{ or } \frac{px+q}{rx+s}<0, \text{ or } \frac{px+q}{rx+s} \leq 0.$$

STEP IV *Make coefficient x positive in numerator and denominator if they are not.*

STEP V *Equate numerator and denominator separately to zero and obtain the values of x. These values of x are generally called critical points.*

STEP VI *Plot the critical points obtained in step V on real line. These points will divide the real line in three regions.*

STEP VII *In the right most region the expression on LHS of the inequation obtained in step IV will be positive and in other regions it will be alternatively negative and positive. So, mark positive sign in the right most region and then mark alternatively negative and positive signs in other regions.*

STEP VIII *Select appropriate region on the basis of the sign of the inequation obtained in step IV. Write these regions in the form of intervals to obtain the desired solution sets of the given inequation.*

EXAMPLE 6 *Solve the following linear inequations:*

(i) $\frac{x-3}{x-5}>0$ (ii) $\frac{x-2}{x+5}>2$

SOLUTION (i) We have,

$$\frac{x-3}{x-5}>0 \quad \text{...(i)}$$

Equating $x-3$ and $x-5$ to zero, we obtain $x=3,5$ as critical points. Plot these points on real line as shown in Fig. 15.12. The real line is divided into three regions. In the right most region the expression on LHS of (i) is positive and in the remaining two regions it is alternatively negative and positive as shown in Fig. 15.12.

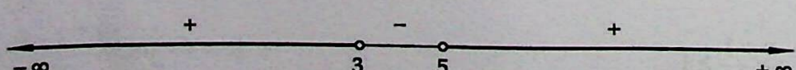

Fig. 15.12

Since the expression in (i) is positive, so the solution set of the given inequation is the union of regions containing positive signs. Hence, from Fig. 15.12

$$\frac{x-3}{x-5}>0 \Rightarrow x \in(-\infty, 3) \cup(5, \infty)$$

Hence, the solution set of the given inequation is $(-\infty, 3) \cup(5, \infty)$ as shown in Fig. 15.12.

(ii) We have,

$$\frac{x-2}{x+5} > 2$$

$$\Rightarrow \quad \frac{x-2}{x+5} - 2 > 0$$

$$\Rightarrow \quad \frac{x-2-2(x+5)}{x+5} > 0$$

$$\Rightarrow \quad \frac{x-2-2x-10}{x+5} > 0$$

$$\Rightarrow \quad \frac{-x-12}{x+5} > 0$$

$$\Rightarrow \quad \frac{x+12}{x+5} < 0 \qquad \left[\begin{array}{l}\text{Multiplying by } -1 \text{ to make coefficient of} \\ x \text{ positive in the expression in numerator}\end{array}\right] \quad \text{...(i)}$$

On equating $x + 12$ and $x + 5$ to zero, we obtain $x = -12, -5$ as critical points. These points are plotted on number line as shown in Fig. 15.12. The real line is divided into three regions and the signs of LHS of inequation (i) are marked. Since the inequation in (i) possesses less than sign which means that LHS of the inequation is negative. So, the solution set of the given inequation is the union of the regions ontaining negative sign in Fig. 15.13. Hence, the solution set of the given inequation is $(-12, -5)$.

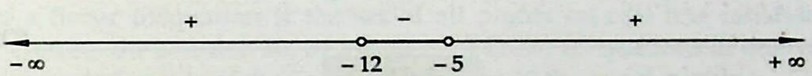

Fig. 15.13

EXAMPLE 7 *Solve the following inequations:*

(i) $\dfrac{2x+4}{x-1} \geq 5$ (ii) $\dfrac{x+3}{x-2} \leq 2$

SOLUTION (i) We have,

$$\frac{2x+4}{x-1} \geq 5$$

$$\Rightarrow \quad \frac{2x+4}{x-1} - 5 \geq 0$$

$$\Rightarrow \quad \frac{2x+4-5(x-1)}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{2x+4-5x+5}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{-3x+9}{x-1} \geq 0$$

$$\Rightarrow \quad \frac{3x-9}{x-1} \leq 0 \qquad \text{[Multiplying both sides by } -1]$$

$$\Rightarrow \quad \frac{3(x-3)}{(x-1)} \leq 0$$

$$\Rightarrow \quad \frac{x-3}{x-1} \leq 0 \qquad \text{[Dividing both sides by 3]}$$

$$\Rightarrow \quad 1 < x \leq 3 \qquad \text{[See Fig. 15.14]}$$

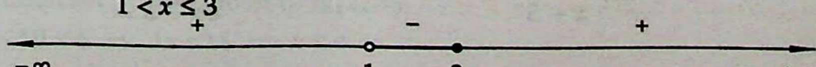

Fig. 15.14

$$\Rightarrow \quad x \in (1, 3]$$

Hence, the solution set of the given inequation is $(1, 3]$.

(ii) We have,

$$\frac{x+3}{x-2} \leq 2$$

$$\Rightarrow \quad \frac{x+3}{x-2} - 2 \leq 0$$

$$\Rightarrow \quad \frac{x+3-2x+4}{x-2} \leq 0$$

$$\Rightarrow \quad \frac{-x+7}{x-2} \leq 0$$

$$\Rightarrow \quad \frac{x-7}{x-2} \geq 0$$ [Multiplying both sides by -1]

$$\Rightarrow \quad x \in (-\infty, 2) \cup [7, \infty)$$ [See Fig. 15.15]

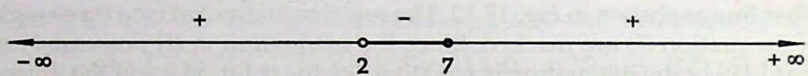

Fig. 15.15

Hence, the solution set of the given inequation is $(-\infty, 2) \cup [7, \infty)$.

EXERCISE 15.1

LEVEL-1

Solve the following linear inequations in R.

1. Solve: $12x < 50$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
2. Solve: $-4x > 30$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
3. Solve: $4x - 2 < 8$, when
   (i) $x \in R$ (ii) $x \in Z$ (iii) $x \in N$
4. $3x - 7 > x + 1$
5. $x + 5 > 4x - 10$
6. $3x + 9 \geq -x + 19$
7. $2(3-x) \geq \frac{x}{5} + 4$
8. $\frac{3x-2}{5} \leq \frac{4x-3}{2}$
9. $-(x-3) + 4 < 5 - 2x$
10. $\frac{x}{5} < \frac{3x-2}{4} - \frac{5x-3}{5}$
11. $\frac{2(x-1)}{5} \leq \frac{3(2+x)}{7}$
12. $\frac{5x}{2} + \frac{3x}{4} \geq \frac{39}{4}$
13. $\frac{x-1}{3} + 4 < \frac{x-5}{5} - 2$
14. $\frac{2x+3}{4} - 3 < \frac{x-4}{3} - 2$
15. $\frac{5-2x}{3} < \frac{x}{6} - 5$
16. $\frac{4+2x}{3} \geq \frac{x}{2} - 3$
17. $\frac{2x+3}{5} - 2 < \frac{3(x-2)}{5}$
18. $x - 2 \leq \frac{5x+8}{3}$
19. $\frac{6x-5}{4x+1} < 0$
20. $\frac{2x-3}{3x-7} > 0$
21. $\frac{3}{x-2} < 1$
22. $\frac{1}{x-1} \leq 2$
23. $\frac{4x+3}{2x-5} < 6$
24. $\frac{5x-6}{x+6} < 1$
25. $\frac{5x+8}{4-x} < 2$
26. $\frac{x-1}{x+3} > 2$
27. $\frac{7x-5}{8x+3} > 4$
28. $\frac{x}{x-5} > \frac{1}{2}$

ANSWERS

1. (i) $(-\infty, 25/6)$	(ii) $\{\ldots -3, -2, -1, 0, 1, 2, 3, 4\}$	(iii) $\{1, 2, 3, 4\}$
2. (ii) $(-\infty, -15/2)$	(ii) $\{\ldots\ldots\ldots, -9, -8\}$	(iii) ϕ
3. (i) $(-\infty, 5/2)$	(ii) $\{\ldots\ldots, -2, -1, 0, 1, 2\}$	(iii) $\{1, 2\}$
4. $(4, \infty)$	5. $(-\infty, 5)$	6. $[5/2, \infty)$
7. $(-\infty, 10/11]$	8. $[11/14, \infty)$	9. $(-\infty, -2)$
10. $(-\infty, 2/9)$	11. $[-44, \infty)$	12. $[3, \infty)$
13. $(-\infty, -50)$	14. $(-\infty, -13/2)$	15. $(8, \infty)$
16. $[-26, \infty)$	17. $(-1, \infty)$	18. $[-7, \infty)$
19. $(-1/4, 5/6)$	20. $(-\infty, 3/2) \cup (7/3, \infty)$	21. $(-\infty, 2) \cup (5, \infty)$
22. $(-\infty, 1) \cup [3/2, \infty)$	23. $(-\infty, 5/2) \cup (33/8, \infty)$	24. $(-6, 3)$
25. $(-\infty, 0) \cup (4, \infty)$	26. $(-7, -3)$	27. $(-17/25, -3/8)$
28. $(-\infty, -5) \cup (5, \infty)$		

15.5 SOLUTION OF SYSTEM OF LINEAR INEQUATIONS IN ONE VARIABLE

In the previous section, we have learnt how to solve a linear inequation in one variable. In this section, we shall use it to solve a system of linear inequations in one variable. Recall that the solution set of a linear inequation is the set of all points on real line satisfying the given inequation. Therefore, the solution set of a system of linear inequations in one variable is the intersection of the solution sets of the linear inequations in the given system.

We use the following algorithm to solve a system of linear inequations in one variable.

ALGORITHM

STEP I *Obtain the system of linear inequations.*

STEP II *Solve each inequation and obtain their solution sets. Also, represent them on real time.*

STEP III *Find the intersection of the solution sets obtained in step II by taking the help of the graphical representation of the solution sets in step II.*

STEP IV *The set obtained in step III is the required solution set of the given system of inequations.*

Following examples will illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following system of linear inequations:*

$$3x - 6 \geq 0$$
$$4x - 10 \leq 6$$

SOLUTION The given system of inequations is

$$3x - 6 \geq 0 \qquad \ldots(i)$$
$$4x - 10 \leq 6 \qquad \ldots(ii)$$

Now, $3x - 6 \geq 0 \Rightarrow 3x \geq 6 \Rightarrow \frac{3x}{3} \geq \frac{6}{3} \Rightarrow x \geq 2$

$\therefore$ Solution set of inequation (i) is $[2, \infty)$

and, $4x - 10 \leq 6 \Rightarrow 4x \leq 16 \Rightarrow x \leq 4$

$\therefore$ Solution set of inequation (ii) is $(-\infty, 4]$

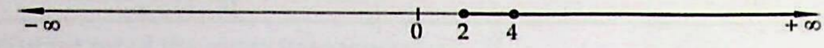

Fig. 15.16(i)

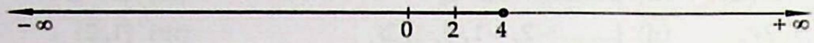

Fig. 15.16(ii)

The solution sets of inequations (i) and (ii) are represented graphically on real line in Figs. 15.16 (i) and (ii) respectively.

Clearly, the intersection of these solution sets is the set [2, 4].

Hence, the solution set of the given system of inequations is the interval [2, 4].

EXAMPLE 2 *Solve the following system of inequations:*

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$$

$$\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$$

SOLUTION The given system of inequation is

$$\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8} \quad \text{...(i)}$$

$$\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4} \quad \text{...(ii)}$$

Now, $\frac{5x}{4} + \frac{3x}{8} > \frac{39}{8}$

$\Rightarrow \frac{10x + 3x}{8} > \frac{39}{8}$

$\Rightarrow 13x > 39$

$\Rightarrow x > 3$

$\Rightarrow x \in (3, \infty)$

So, the solution set of inequation (i) is the interval $(3, \infty)$.

and, $\frac{2x-1}{12} - \frac{x-1}{3} < \frac{3x+1}{4}$

$\Rightarrow \frac{(2x-1) - 4(x-1)}{12} < \frac{3x+1}{4}$

$\Rightarrow \frac{-2x+3}{12} < \frac{3x+1}{4}$

$\Rightarrow -2x + 3 < 3(3x+1)$ [Multiplying both sides by 12 i.e. the l.c.m. of 12 and 4]

$\Rightarrow -2x + 3 < 9x + 3$

$\Rightarrow -2x - 9x < 3 - 3$

$\Rightarrow -11x < 0$

$\Rightarrow x > 0$

$\Rightarrow x \in (0, \infty)$

So, the solution set of inequation (ii) is the interval $(0, \infty)$. Let us now represent the solution sets of inequations (i) and (ii) on real line. These solution sets are graphed on real line in Figs. 15.17 (i) and 15.17 (ii) respectively.

From Figs. 15.17 (i) and (ii), we observe that the intersection of the solution sets of inequations (i) and (ii) is interval $(3, \infty)$ represented by common thick line.

Hence, the solution set of the given system of inequations is the interval $(3, \infty)$.

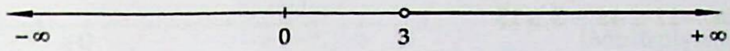

Fig. 15.17(i)

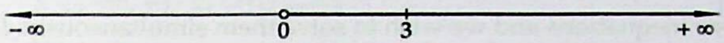

Fig. 15.17(ii)

EXAMPLE 3 *Solve the following system of inequations:* $2(2x+3)-10<6(x-2)$

$$\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3}$$

SOLUTION The given system of inequations is

$$2(2x+3)-10<6(x-2) \qquad \text{...(i)}$$

$$\frac{2x-3}{4}+6 \geq 2+\frac{4x}{3} \qquad \text{...(ii)}$$

Now, $2(2x+3)-10<6(x-2)$

$\Rightarrow \quad 4x+6-10<6x-12$

$\Rightarrow \quad 4x-4<6x-12$

$\Rightarrow \quad 4x-6x<4-12$

$\Rightarrow \quad -2x<-8$

$\Rightarrow \quad x>4$

$\Rightarrow \quad x \in (4, \infty)$

So, the solution set of the first inequation is the interval $(4, \infty)$.

and, $\dfrac{2x-3}{4}+6 \geq 2+\dfrac{4x}{3}$

$\Rightarrow \quad \dfrac{2x-3+24}{4} \geq \dfrac{6+4x}{3}$

$\Rightarrow \quad \dfrac{2x+21}{4} \geq \dfrac{4x+6}{3}$

$\Rightarrow \quad 3(2x+21) \geq 4(4x+6)$

$\Rightarrow \quad 6x+63 \geq 16x+24$

$\Rightarrow \quad 6x-16x \geq 24-63$

$\Rightarrow \quad -10x \geq -39$

$\Rightarrow \quad x \leq \dfrac{39}{10}$

$\Rightarrow \quad x \leq 3.9$

$\Rightarrow \quad x \in (-\infty, 3.9]$

So, the solution set of inequation (ii) is the interval $(-\infty, 3.9]$.

The solution sets of inequations (i) and (ii) are graphed on real line in Figs. 15.18 (i) and (ii) respectively.

We observe that there is no common solution of the two inequations. So, the given system of inequations has no solution.

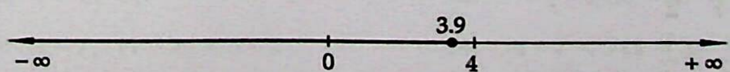

Fig. 15.18 (i)

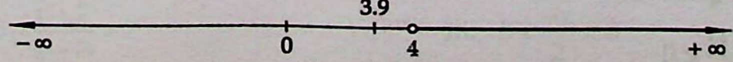

Fig. 15.18 (ii)

EXAMPLE 4 *Solve:* $-11 \le 4x - 3 \le 13$

SOLUTION We have,

$$-11 \ge 4x - 3 \ge 13 \quad \Leftrightarrow \quad -11 \ge 4x - 3 \text{ and } 4x - 3 \ge 13$$

Thus, we have two inequations and we wish to solve them simultaneously. Instead of solving these inequations by using the method discussed in first three examples, let us solve them directly in a different way as given below.

We have,

$$-11 \le 4x - 3 \le 13$$

$$\Rightarrow \quad -11 + 3 \le 4x - 3 + 3 \le 13 + 3 \qquad \text{[Adding 3 throughout]}$$

$$\Rightarrow \quad -8 \le 4x \le 16$$

$$\Rightarrow \quad \frac{-8}{4} \le x \le \frac{16}{4} \qquad \text{[Dividing by 4 throughout]}$$

$$\Rightarrow \quad -2 \le x \le 4$$

$$\Rightarrow \quad x \in [-2, 4]$$

Hence, the interval $[-2, 4]$ is the solution set of the given system of inequations.

EXAMPLE 5 *Solve:* $-5 \le \dfrac{2 - 3x}{4} \le 9$ **[NCERT EXEMPLAR]**

SOLUTION We have,

$$-5 \le \frac{2 - 3x}{4} \le 9$$

$$\Rightarrow \quad -5 \times 4 \le \frac{2 - 3x}{4} \times 4 \le 9 \times 4 \qquad \text{[Multiplying throughout by 4]}$$

$$\Rightarrow \quad -20 \le 2 - 3x \le 36$$

$$\Rightarrow \quad -20 - 2 \le 2 - 3x - 2 \le 36 - 2 \qquad \text{[Subtracting 2 throughout]}$$

$$\Rightarrow \quad -22 \le -3x \le 34$$

$$\Rightarrow \quad \frac{-22}{-3} \ge \frac{-3x}{-3} \ge \frac{34}{-3} \qquad \text{[Dividing throughout by } -3\text{]}$$

$$\Rightarrow \quad \frac{22}{3} \ge x \ge \frac{-34}{3}$$

$$\Rightarrow \quad \frac{-34}{3} \le x \le \frac{22}{3}$$

$$\Rightarrow \quad x \in [-34/3, 22/3]$$

Hence, the interval $[-34/3, 22/3]$ is the solution set of the given system of inequations.

EXAMPLE 6 *Solve the system of inequations:* $\dfrac{x}{2x+1} \ge \dfrac{1}{4}, \quad \dfrac{6x}{4x-1} < \dfrac{1}{2}$

SOLUTION The given system of inequations is

$$\frac{x}{2x+1} \ge \frac{1}{4} \qquad \text{...(i)}$$

$$\frac{6x}{4x-1} < \frac{1}{2} \qquad \text{...(ii)}$$

Now, $$\frac{x}{2x+1} \ge \frac{1}{4}$$

$$\Rightarrow \quad \frac{x}{2x+1} - \frac{1}{4} \ge 0$$

$$\Rightarrow \quad \frac{4x - (2x+1)}{4(2x+1)} \ge 0$$

$\Rightarrow \quad \dfrac{2x-1}{2x+1} \geq 0$ [Multiplying both sides by 4]

$\Rightarrow \quad x \in (-\infty, -1/2) \cup [1/2, \infty)$ [See Fig. 15.19 (i)]

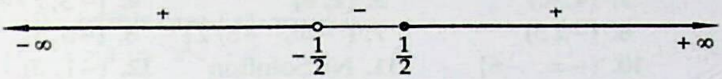

Fig. 15.19(i)

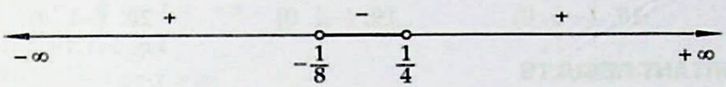

Fig. 15.19(ii)

Thus, the solution set of inequation (i) is $(-\infty, -1/2) \cup [1/2, \infty)$ ... (iii)

And, $\dfrac{6x}{4x-1} < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{6x}{4x-1} - \dfrac{1}{2} < 0$

$\Rightarrow \quad \dfrac{12x-(4x-1)}{2(4x-1)} < 0$

$\Rightarrow \quad \dfrac{8x+1}{2(4x-1)} < 0$

$\Rightarrow \quad \dfrac{8x+1}{4x-1} < 0$ [Multiplying both sides by 2]

$\Rightarrow \quad x \in (-1/8, 1/4)$ [See Fig. 15.19 (ii)]

Thus, the solution set of inequation (ii) is

$$(-1/8, 1/4) \qquad \text{...(iv)}$$

It is evident from Fig. 15.19 that the intersection of (iii) and (iv) is the null set.

Hence, the given system of equations has no solution.

EXERCISE 15.2

LEVEL-1

Solve each of the following system of equations in R.

1. $x+3>0,\ 2x<14$
2. $2x-7>5-x,\ 11-5x \leq 1$
3. $x-2>0,\ 3x<18$
4. $2x+6 \geq 0,\ 4x-7<0$
5. $3x-6>0,\ 2x-5>0$
6. $2x-3<7,\ 2x>-4$
7. $2x+5 \leq 0,\ x-3 \leq 0$
8. $5x-1<24,\ 5x+1>-24$
9. $3x-1 \geq 5,\ x+2>-1$
10. $11-5x>-4,\ 4x+13 \leq -11$
11. $4x-1 \leq 0,\ 3-4x<0$
12. $x+5>2(x+1),\ 2-x<3(x+2)$
13. $2(x-6)<3x-7,\ 11-2x<6-x$
14. $5x-7<3(x+3),\ 1-\dfrac{3x}{2} \geq x-4$
15. $\dfrac{2x-3}{4}-2 \geq \dfrac{4x}{3}-6,\ 2(2x+3)<6(x-2)+10$
16. $\dfrac{7x-1}{2}<-3,\ \dfrac{3x+8}{5}+11<0$
17. $\dfrac{2x+1}{7x-1}>5,\ \dfrac{x+7}{x-8}>2$
18. $0<\dfrac{-x}{2}<3$
19. $10 \leq -5(x-2)<20$
20. $-5<2x-3<5$

21. $\frac{4}{x+1} \le 3 \le \frac{6}{x+1}, x > 0$ **[NCERT EXEMPLAR]**

ANSWERS

1. $(-3, 7)$ 2. $(4, \infty)$ 3. $(2, 6)$ 4. $[-3, 7/4)$
5. $(5/2, \infty)$ 6. $(-2\,5)$ 7. $(-\infty, -5/2]$ 8. $(-5, 5)$
9. $[2, \infty)$ 10. $(-\infty, -6]$ 11. No Solution 12. $(-1, 3)$
13. $(5, \infty)$ 14. $(-\infty, 2]$ 15. No Solution 16. $(-\infty, -21)$
17. No Solution 18. $(-6, 0)$ 19. $(-2, 0]$ 20. $(-1, 4)$ 21. $[1/3, 1]$

15.5.1 SOME IMPORTANT RESULTS

In this sub-section, let us discuss some results on inequations involving modulus of the variable. We state and prove these results as theorems.

THEOREM 1 *If a is a positive real number, then*

(i) $|x| < a \Leftrightarrow -a < x < a$ i.e. $x \in (-a, a)$
(ii) $|x| \le a \Leftrightarrow -a \le x \le a$ i.e. $x \in [-a, a]$

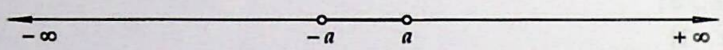

Fig. 15.20(i)

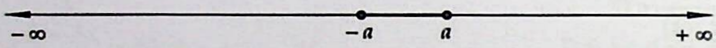

Fig. 15.20(ii)

PROOF (i) We know that: $|x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x < 0 \end{cases}$

So, we consider the following cases:

CASE I When $x \ge 0$: In this case, $|x| = x$.

$\therefore$ $|x| < a \Rightarrow x < a$

Thus, in this case the solution set of the given inequation is given by

$x \ge 0$ and $x < a \Rightarrow 0 \le x < a$ ...(i)

CASE II When $x < 0$: In this case, $|x| = -x$.

$\therefore$ $|x| < a \Rightarrow -x < a \Rightarrow x > -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x > -a \Rightarrow -a < x < 0$ ...(ii)

Combining (i) and (ii), we get

$|x| < a \Leftrightarrow -a < x < 0$ or, $0 \le x < a \Leftrightarrow -a < x < a$.

(ii) Proceeding exactly as in (i), we get

$|x| \le a \Rightarrow -a \le x \le a$.

THEOREM 2 *If a is a positive real number, then*

(i) $|x| > a \Leftrightarrow x < -a$ or $x > a$

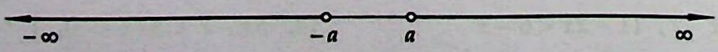

Fig. 15.21(i)

(ii) $|x| \ge a \Leftrightarrow x \le -a$ or $x \ge a$

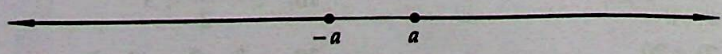

Fig. 15.21(ii)

PROOF CASE I When $x > 0$: In this case, $|x| = x$

$\therefore \quad |x| > a \Rightarrow x > a$

Thus, in this case the solution set of the inequation $|x| > a$ is given by

$x > 0$ and $x > a \Rightarrow x > a$ $\quad [\because a > 0]$ ...(i)

CASE II *When* $x < 0$: In this case, $|x| = -x$

$\therefore \quad |x| > a \Rightarrow -x > a \Rightarrow x < -a$

Thus, in this case the solution set of the given inequation is given by

$x < 0$ and $x < -a \Rightarrow x < -a$ $\quad [\because a > 0]$ ...(ii)

Combining (i) and (ii), we get

$|x| > a \Leftrightarrow x < -a$ or $x > a$

(ii) Proceeding as in (i), we get

$|x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$.

THEOREM 3 *Let r be a positive real number and a be a fixed real number. Then,*

(i) $|x - a| < r \Leftrightarrow a - r < x < a + r$ i.e. $x \in (a - r, a + r)$

(ii) $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$ i.e. $x \in [a - r, a + r]$

(iii) $|x - a| > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) $|x - a| \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

PROOF (i) Using Theorem 1, we obtain

$|x - a| < r \Leftrightarrow -r < x - a < r \Leftrightarrow a - r < x - a + a < a + r \Leftrightarrow a - r < x < a + r$

(ii) Using Theorem 1 (ii), we obtain

$|x - a| \leq r \Leftrightarrow -r \leq x - a \leq r \Leftrightarrow a - r \leq x - a + a \leq a + r \Leftrightarrow a - r \leq x \leq a + r$

(iii) Using Theorem 2(i), we obtain

$|x - a| > r \Leftrightarrow x - a < r$, or $x - a > r \Leftrightarrow x < a - r$, or $x > a + r$

(iv) Using Theorem 2 (ii), we obtain,

$|x - a| \geq r \Leftrightarrow x - a \leq -r$, or $x - a \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

NOTE: *These results may be used directly for solving linear inequations involving absolute values.*

THEOREM 4 *Let a, b be positive real numbers. Then*

(i) $a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a, b)$

(ii) $a \leq |x| \leq b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(iii) $a \leq |x - c| \leq b \Leftrightarrow x \in [-b + c, -a + c] \cup [a + c, b + c]$

(iv) $a < |x - c| < b \Leftrightarrow x \in (-b + c, -a + c) \cup (a + c, b + c)$

PROOF (i) $a < |x| < b \Leftrightarrow |x| > a$ and $|x| < b \Leftrightarrow (x < -a$ or $x > a)$ and $(-b < x < b)$

$\Leftrightarrow x \in (-b, -a) \cup (a, b)$

Similarly, we can prove other results.

ILLUSTRATIVE EXAMPLES

LEVEL-2

EXAMPLE 1 $|3x - 2| \leq \frac{1}{2}$

SOLUTION We know that: $|x - a| \leq r \Leftrightarrow a - r \leq x \leq a + r$

$\therefore \quad |3x - 2| \leq \frac{1}{2} \Leftrightarrow 2 - \frac{1}{2} \leq 3x \leq 2 + \frac{1}{2} \Leftrightarrow \frac{3}{2} \leq 3x \leq \frac{5}{2} \Leftrightarrow \frac{1}{2} \leq x \leq \frac{5}{6} \Leftrightarrow x \in [1/2, 5/6]$

Hence, the solution set of the given inequation is the interval [1/2, 5/6].

EXAMPLE 2 *Solve:* $|x - 2| \geq 5$

SOLUTION We know that: $|x - a| \geq r \Leftrightarrow x \leq a - r$, or $x \geq a + r$

$\therefore \quad |x - 2| \geq 5$

$$\Leftrightarrow \quad x \le 2-5, \text{ or } x \ge 2+5$$

$$\Leftrightarrow \quad x \le -3 \text{ or } x \ge 7 \Leftrightarrow x \in (-\infty, -3] \text{ or } x \in [7, \infty) \Leftrightarrow x \in (-\infty, -3] \cup [7, \infty)$$

Hence the solution set of the given inequation is $(-\infty, -3] \cup [7, \infty)$

EXAMPLE 3 *Solve* : $1 \le |x - 2| \le 3$

SOLUTION We know that :

$$a \le |x - c| \le b \Leftrightarrow x \in [-b + c, \ -a + c] \cup [a + c, \ b + c]$$

$$\therefore \quad 1 \le |x - 2| \le 3 \Leftrightarrow x \in [-3 + 2, -1 + 2] \cup [1 + 2, 3 + 2] \Leftrightarrow x \in [-1, 1] \cup [3, 5]$$

Hence, the solution set of the given inequation is $[-1, 1] \cup [3, 5]$.

EXAMPLE 4 *Solve the following system of inequations:* $|x - 1| \le 5, \ \ |x| \ge 2$ **[NCERT EXEMPLAR]**

SOLUTION The given system of inequations is

$$|x - 1| \le 5 \qquad \text{...(i)}$$

$$|x| \ge 2 \qquad \text{...(ii)}$$

Now, $|x - 1| \le 5$

$$\Rightarrow \quad 1 - 5 \le x \le 1 + 5 \qquad [\because |x - a| \le r \Leftrightarrow a - r \le x \le a + r]$$

$$\Rightarrow \quad -4 \le x \le 6 \Rightarrow x \in [-4, 6]$$

Thus, the solution set of (i) is the interval $x \in [-4, 6]$.

and, $|x| \ge 2 \Leftrightarrow x \le -2, \text{ or } x \ge 2 \qquad [\because |x| \ge a \Rightarrow x \le -a \text{ or } x \le a]$

$$\Leftrightarrow x \in (-\infty, -2] \cup [2, \infty)$$

Thus, the solution set of (ii) is $(-\infty, -2] \cup [2, \infty)$.

The solution sets of inequations (i) and (ii) are represented graphically in Figures 15.22 (i) and 15.22 (ii) respectively. The intersection of these two is $[-4, -2] \cup [2, 6]$

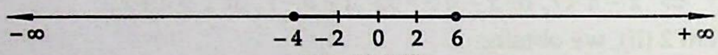

Fig. 15.22(i)

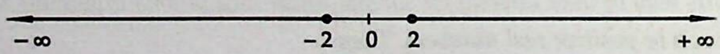

Fig. 15.22(ii)

Hence, the solution set of the given system of inequations is $[-4, -2] \cup [2, 6]$.

EXAMPLE 5 *Solve* : $\dfrac{|x| - 1}{|x| - 2} \ge 0, \ x \in R, \ x \ne \pm 2.$

SOLUTION We have,

$$\frac{|x| - 1}{|x| - 2} \ge 0$$

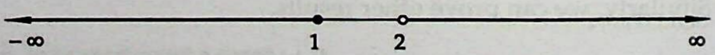

Fig. 15.23

$$\Rightarrow \quad \frac{y - 1}{y - 2} \ge 0, \text{ where } y = |x|$$

$$\Rightarrow \quad y \le 1 \text{ or } y > 2 \qquad \text{[See Fig. 15.23]}$$

$$\Rightarrow \quad |x| \le 1 \text{ or } |x| > 2$$

$$\Rightarrow \quad (-1 \le x \le 1) \text{ or } (x < -2 \text{ or } x > 2)$$

$$\Rightarrow \quad x \in [-1, 1] \text{ or } x \in (-\infty, -2) \cup (2, \infty)$$

$$\Rightarrow \quad x \in [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$$

Hence, the solution set of the given inequation is $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$.

EXAMPLE 6 *Solve:* $\dfrac{-1}{|x| - 2} \ge 1$, *where* $x \in R, x \ne \pm 2$

SOLUTION We have, $\dfrac{-1}{|x|-2} \geq 1$

$\Rightarrow \quad \dfrac{-1}{|x|-2} - 1 \geq 0$

$\Rightarrow \quad \dfrac{-1-(|x|-2)}{|x|-2} \geq 0$

$\Rightarrow \quad \dfrac{1-|x|}{|x|-2} \geq 0$

$\Rightarrow \quad \dfrac{|x|-1}{|x|-2} \leq 0$

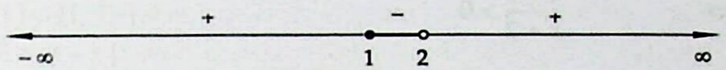

Fig. 15.24

$\Rightarrow \quad \dfrac{y-1}{y-2} \leq 0$, where $y = |x|$

$\Rightarrow \quad 1 \leq y < 2$ [See Fig. 15.24]

$\Rightarrow \quad 1 \leq |x| < 2$ $[\because \ y = |x|]$

$\Rightarrow \quad x \in (-2, -1] \cup [1, 2)$ $[\because \ a < |x| \leq b \Leftrightarrow x \in [-b, -a) \cup (a, b]]$

Hence, the solution set of the given inequation is $(-2, -1] \cup [1, 2)$

EXAMPLE 7 *Solve the inequation:* $\left|\dfrac{2}{x-4}\right| > 1, \ x \neq 4.$

SOLUTION We have,

$\left|\dfrac{2}{x-4}\right| > 1 \quad x \neq 4$

$\Rightarrow \quad \dfrac{2}{|x-4|} > 1$ $\left[\because \ \left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}\right]$

$\Rightarrow \quad 2 > |x-4|$ $[\because \ |x-4| > 0 \text{ for all } x \neq 4]$

$\Rightarrow \quad 4-2 < x < 4+2$ $[\because \ |x-a| < r \Leftrightarrow a-r < x < a+r]$

$\Rightarrow \quad 2 < x < 6$

$\Rightarrow \quad x \in (2, 6)$

But, $\quad x \neq 4.$

Hence, the solution set of the given inequation is $(2, 4) \cup (4, 6)$.

EXAMPLE 8 *Solve:* $\dfrac{|x+3|+x}{x+2} > 1$ **[NCERT EXEMPLAR]**

SOLUTION We have, $\dfrac{|x+3|+x}{x+2} > 1.$

Clearly, LHS of this inequation is meaningful for $x \neq -2$.

Now, $\quad \dfrac{|x+3|+x}{x+2} > 1$

$\Rightarrow \quad \dfrac{|x+3|+x}{x+2} - 1 > 0$

$\Rightarrow \quad \dfrac{|x+3|+x-x-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{|x+3|-2}{x+2} > 0.$

Now two cases arise:

CASE I When $x+3 \geq 0$ *i.e.* $x \geq -3$: In this case, $|x+3| = x+3$.

$\therefore \quad \dfrac{|x+3|-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{x+3-2}{x+2} > 0$

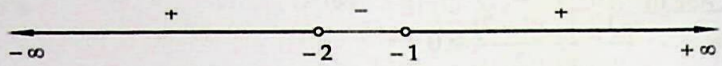

Fig. 15.25

$\Rightarrow \quad \dfrac{x+1}{x+2} > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (-1, \infty)$ [See Fig. 15.25]

But, $x \geq -3$. Therefore, the solution set of the given inequation in this case is $[-3, -2) \cup (-1, \infty)$.

CASE II *When* $x+3 < 0$ *i.e.* $x < -3$: In this case, $|x+3| = -(x+3)$.

$\therefore \quad \dfrac{|x+3|-2}{x+2} > 0$

$\Rightarrow \quad \dfrac{-(x+3)-2}{x+2} > 0$

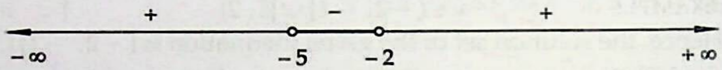

Fig. 15.26

$\Rightarrow \quad \dfrac{-(x+5)}{x+2} > 0$

$\Rightarrow \quad \dfrac{x+5}{x+2} < 0$

$\Rightarrow \quad x \in (-5 - 2)$ [See Fig. 15.26]

But, $x < -3$. Therefore, the solution set of the given inequation in this case is the interval $(-5, -3)$.

From Case I and Case II, we obtain that the solution set of the given inequation is

$$[-3, -2) \cup (-1, \infty) \cup (-5, -3) = (-5, -2) \cup (-1, \infty).$$

EXAMPLE 9 *Solve:* $|x-1| + |x-2| \geq 4$

SOLUTION On the LHS of the given inequation there are two terms both containing modulus. By equating the expressions within the modulus to zero, we get $x = 1, 2$ as critical points. These points divide real line in three parts viz. $(-\infty, 1]$, $[1, 2]$ and $[2, \infty)$. So, we consider the following three cases.

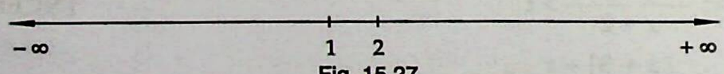

Fig. 15.27

CASE I *When* $-\infty < x < 1$: In this case, we have $|x-1| = -(x-1)$ and $|x-2| = -(x-2)$

$\therefore \quad |x-1| + |x-2| \geq 4$

$\Rightarrow \quad -(x-1) - (x-2) \geq 4$

$\Rightarrow \quad -2x + 3 \geq 4$

$\Rightarrow \quad -2x \geq 1$

$\Rightarrow \quad x \leq -\dfrac{1}{2}$

But, $-\infty < x < 1$. Therefore, in this case the solution set of the given inequation is $(-\infty, -1/2]$

CASE II When $1 \leq x < 2$: In this case, we have $|x-1| = (x-1)$ and $|x-2| = -(x-2)$

$\therefore \quad |x-1| + |x-2| \geq 4$

$\Rightarrow \quad x-1-(x-2) \geq 4$

$\Rightarrow \quad 1 \geq 4$, which is an absurd result.

So, the given inequation has no solution for $x \in [1, 2)$.

<u>CASE III</u> When $x \geq 2$: In this case, we have $|x-1| = x-1$ and $|x-2| = x-2$

$$\therefore \quad |x-1| + |x-2| \geq 4$$

$$\Rightarrow \quad x-1+x-2 \geq 4$$

$$\Rightarrow \quad 2x-3 \geq 4$$

$$\Rightarrow \quad 2x \geq 7$$

$$\Rightarrow \quad x \geq \frac{7}{2}$$

But, $x > 2$. Therefore, in this case the solution set of the given inequation is $[7/2, \infty)$.

Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -1/2] \cup [7/2, \infty)$$

EXAMPLE 10 *Solve:* $\dfrac{|x-1|}{x+2} < 1$.

SOLUTION We have,

$$\frac{|x-1|}{x+2} < 1 \Rightarrow \frac{|x-1|}{x+2} - 1 < 0 \Rightarrow \frac{|x-1|-(x+2)}{x+2} < 0$$

Now the following cases arise.

<u>CASE I</u> When $x-1 \geq 0$ *i.e.* $x \geq 1$: In this case, we have $|x-1| = x-1$

$$\therefore \quad \frac{|x-1|-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad \frac{(x-1)-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad \frac{-3}{x+2} < 0$$

$$\Rightarrow \quad x+2 > 0 \qquad \left[\because \frac{a}{b} < \text{ and } a < 0 \Rightarrow b > 0\right]$$

$$\Rightarrow \quad x > -2$$

But, $x \geq 1$. Therefore, $x > -2$ and $x \geq 1$. implies that $x \geq 1$. Thus, in this case the solution set of the given inequation is $[1, \infty)$.

<u>CASE II</u> When $x-1 < 0$ *i.e.* $x < 1$: In this case, we have $|x-1| = -(x-1)$.

$$\therefore \quad \frac{|x-1|-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad \frac{-(x-1)-(x+2)}{x+2} < 0$$

$$\Rightarrow \quad -\frac{2x+1}{x+2} < 0$$

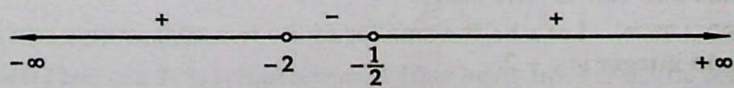

Fig. 15.28

$$\Rightarrow \quad \frac{2x+1}{x+2} > 0$$

$$\Rightarrow \quad x \in (-\infty \quad -2) \cup (-1/2, \infty)$$ [See Fig. 15.28]

But, $x < 1$. Therefore, $x \in (-\infty, -2) \cup (-1/2, \infty)$ and $x < 1$ implies that $x \in (-\infty, -2) \cup \left(-\frac{1}{2}, 1\right)$.

Thus, in this case the solution set of the given inequation is $(-\infty, -2) \cup (-1/2, 1)$.
Combining Case I and Case II, we obtain that the solution set of the given inequation is

$$(-\infty, -2) \cup (-1/2, \infty)$$

EXERCISE 15.3

LEVEL-2

Solve each of the following system of equations in R.

1. $\left|x + \frac{1}{3}\right| > \frac{8}{3}$
2. $|4 - x| + 1 < 3$
3. $\left|\frac{3x-4}{2}\right| \le \frac{5}{12}$
4. $\frac{|x-2|}{x-2} > 0$
5. $\frac{1}{|x|-3} < \frac{1}{2}$
6. $\frac{|x+2|-x}{x} < 2$
7. $\left|\frac{2x-1}{x-1}\right| > 2$
8. $|x-1| + |x-2| + |x-3| \ge 6$
9. $\frac{|x-2|-1}{|x-2|-2} \le 0$ [NCERT EXEMPLAR]
10. $\frac{1}{|x|-3} \le \frac{1}{2}$ [NCERT EXEMPLAR]
11. $|x+1| + |x| > 3$ [NCERT EXEMPLAR]
12. $1 \le |x-2| \le 3$
13. $|3-4x| \ge 9$ [NCERT EXEMPLAR]

ANSWERS

1. $(-\infty, -3) \in (7/3, \infty)$
2. $(2, 6)$
3. $[19/18, 29/18]$
4. $(2, \infty)$
5. $(-\infty, -5) \cup (-3, 3) \cup (5, \infty)$
6. $(-\infty, 0) \cup (1, \infty)$
7. $(3/4, 1) \cup (1, \infty)$
8. $(-\infty, 0] \cup [4, \infty)$
9. $(0, 1] \cup [3, 4)$
10. $(-\infty, -5] \cup (-3, 3) \cup [5, \infty)$
11. $(-\infty, -2) \cup (1, \infty)$
12. $[-1, 1] \cup [3, 5]$
13. $(-\infty, -3/2] \cup [3, \infty)$

15.6 SOME APPLICATIONS OF LINEAR IN EQUATIONS IN ONE VARIABLE

In this section, we shall utilize the knowledge of solving linear in equations in one variable in solving different problems from various fields such as science, engineering, economics etc. Following examples will illustrate the same

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find all pairs of consecutive odd positive integers, both of which are smaller than 18, such that their sum is more than 20.*

SOLUTION Let x be the smaller of the two consecutive odd positive integers. Then, the other odd integer is $x + 2$.

It is given that both the integers are smaller than 18 and their sum is more than 20. Therefore,

$$x + 2 < 18 \text{ and, } x + (x+2) > 20$$

$$\Rightarrow \quad x < 16 \text{ and } 2x + 2 > 20$$

$$\Rightarrow \quad x < 16 \text{ and } 2x > 18$$

$$\Rightarrow \quad x < 16 \text{ and } x > 9 \Rightarrow 9 < x < 16 \Rightarrow x = 11, 13, 15 \qquad [\because x \text{ is an odd integer}]$$

Hence, the required pairs of odd integers are (11, 13), (13, 15) and (15, 17).

EXAMPLE 2 *Find all pairs of consecutive even positive integers, both of which are larger than 8, such that their sum is less than 25.*

SOLUTION Let x be the smaller of the two consecutive even positive integers. Then, the other even integer is $x + 2$.

It is sgiven that both the integers are larger than 8 and their sum is less than 25. Therefore,

$$x > 8 \text{ and } x + x + 2 < 25$$

$$\Rightarrow \quad x > 8 \text{ and } 2x + 2 < 25$$

$$\Rightarrow \quad x > 8 \text{ and } 2x < 23$$

$$\Rightarrow \quad x > 8 \text{ and } x < \frac{23}{2} \Rightarrow 8 < x < \frac{23}{2} \Rightarrow x = 10 \qquad [\because x \text{ is an even integer}]$$

Hence, the required pair of even integers is (10, 12).

EXAMPLE 3 *The cost and revenue functions of a product are given by* $C(x) = 2x + 400$ *and* $R(x) = 6x + 20$ *respectively, where x is the number of items produced by the manufacturer. How many items the manufacturer must sell to realize some profit?*

SOLUTION We know that: Profit = Revenue – Cost. Therefore, to earn some profit, we must have

$$\text{Revenue} > \text{Cost}$$

$$\Rightarrow \quad 6x + 20 > 2x + 400$$

$$\Rightarrow \quad 6x - 2x > 400 - 20 \Rightarrow 4x > 380 \Rightarrow x > \frac{380}{4} = 95$$

Hence, the manufacturer must sell more than 95 items to realize some profit.

EXAMPLE 4 *IQ of a person is given by the formula:* $IQ = \frac{MA}{CA} \times 100$, *where MA is mental age and CA is chronological age. If* $80 \le IQ \le 140$ *for a group of 12 year children, find the range of their mental age.*

SOLUTION We have: $CA = 12$ years

$$\therefore \quad IQ = \frac{MA}{CA} \times 100 \Rightarrow IQ = \frac{MA}{12} \times 100 = \frac{25}{3} MA$$

Now, $80 \le IQ \le 140$

$$\Rightarrow \quad 80 \le \frac{25}{3} MA \le 140$$

$$\Rightarrow \quad 240 \le 25\, MA \le 420 \Rightarrow \frac{240}{25} \le MA \le \frac{420}{25} \Rightarrow 9.6 \le MA \le 16.8$$

EXAMPLE 5 *In the first four papers each of 100 marks, Rishi got 95, 72, 73, 83 marks. If he wants an average of greater than or equal to 75 marks and less than 80 marks, find the range of marks he should score in the fifth paper.*

SOLUTION Suppose scores x marks in the fifth paper. Then,

$$75 \le \frac{95 + 72 + 73 + 83 + x}{5} < 80$$

$$\Rightarrow \quad 75 \le \frac{323 + x}{5} < 80 \Rightarrow 375 < 323 + x < 400 \Rightarrow 52 < x < 77$$

Hence, Rishi must score between 52 and 77 marks.

EXAMPLE 6 *A manufacturer has 600 litres of a 12% solution of acid. How many litres of a 30% acid solution must be added to it so that acid content in the resulting mixture will be more than 15% but less than 18% ?*

SOLUTION Let x litres of 30% acid solution be added to 600 litres of 12% solution of acid. Then,

Total quantity of mixture = $(600 + x)$ litres

Total acid content in the $(600 + x)$ litres of mixture = $\frac{30x}{100} + \frac{12}{100} \times 600$

It is given that acid content in the resulting mixture must be more than 15% and less than 18%.

$\therefore \quad 15\%$ of $(600 + x) < \left(\frac{30x}{100} + \frac{12}{100} \times 600\right) < 18\%$ of $(600 + x)$

$\Rightarrow \quad \frac{15}{100} \times (600 + x) < \frac{30x}{100} + \frac{12}{100} \times 600 < \frac{18}{100} \times (600 + x)$

$\Rightarrow \quad 15(600 + x) < 30x + 12 \times 600 < 18(600 + x)$ [Multiplying through out by 100]

$\Rightarrow \quad 9000 + 15x < 30x + 7200 < 10800 + 18x$

$\Rightarrow \quad 9000 + 15x < 30x + 7200$ and $30x + 7200 < 10800 + 18x$

$\Rightarrow \quad 9000 - 7200 < 30x - 15x$ and $30x - 18x < 10800 - 7200$

$\Rightarrow \quad 1800 < 15x$ and $12x < 3600$

$\Rightarrow \quad 15x > 1800$ and $12x < 3600$

$\Rightarrow \quad x > 120$ and $x < 300$

$\Rightarrow \quad 120 < x < 300$

Hence, the number of litres of the 30% solution of acid must be more than 120 but less than 300.

EXAMPLE 7 *A man wants to cut three lengths from a single piece of board of length 91 cm. The second length is to be 3 cm longer than the shortest and third length is to be twice as long as the shortest. What are the possible lengths for the shortest board if third piece is to be at least 5 cm longer than the second?*

SOLUTION Let the length of the shortest piece be x cm. Then, the lengths of the second and third piece are $x + 3$ cm and $2x$ cm respectively. Then,

$x + (x + 3) + 2x \leq 91$ and $2x \geq (x + 3) + 5$

$\Rightarrow \quad 4x + 3 \leq 91$ and $2x \geq x + 8$

$\Rightarrow \quad 4x \leq 88$ and $x \geq 8 \Rightarrow x \leq 22$ and $x \geq 8 \Rightarrow 8 \leq x \leq 22.$

Hence, the shortest piece must be at least 8 cm long but not more than 22 cm long.

EXERCISE 15.4

LEVEL-1

1. Find all pairs of consecutive odd positive integers, both of which are smaller than 10, such that their sum is more than 11.
2. Find all pairs of consecutive odd natural number, both of which are larger than 10, such that their sum is less than 40.
3. Find all pairs of consecutive even positive integers, both of which are larger than 5, such that their sum is less than 23.
4. The marks scored by Rohit in two tests were 65 and 70. Find the minimum marks he should score in the third test to have an average of at least 65 marks.
5. A solution is to be kept between 86° and 95°F. What is the range of temperature in degree Celsius, if the Celsius (C)/Fahrenheit (F) conversion formula is given by $F = \frac{9}{5}C + 32$.
6. A solution is to be kept between 30°C and 35°C. What is the range of temperature in degee Fahrenheit?
7. To receive grade 'A' in a cource, one must obtain an average of 90 marks or more in five papers each of 100 marks. If Shikha scored 87, 95, 92 and 94 marks in first four papers, find the minimum marks that she must score in the last paper to get grade 'A' in the course.
8. A company manufactures cassettes and its cost and revenue functions for a week are $C = 300 + \frac{3}{2}x$ and $R = 2x$ respectively, where x is the number of cassettes produced and sold in a week. How many cassettes must be sold for the company to realize a profit?

9. The longest side of a triasngle is three times the shortest side and the third side is 2 cm shorter than the longest side if the perimeter of the triangles at least 61 cm, Find the minimum length of the shortest-side.
10. How many litres of water will have to be added to 1125 litres of the 45% solution of acid so that the resulting mixture will contain more than 25% but less than 30% acid content?
11. A solution of 8% boric acid is to be diluted by adding a 2% boric acid solution to it. The resulting mixture is to be more than 4% but less than 6% boric acid. If there are 640 litres of the 8% solution, how many litres of 2% solution will have to be added?
12. The water acidity in a pool is considered normal when the average pH reading of three daily measurements is between 7.2 and 7.8. If the first two pH reading are 7.48 and 7.85, find the range of pH value for the third reading that will result in the acidity level being normal.

ANSWERS

1. (5,7), (7, 9)
2. (11, 13), (13, 15), (15, 17), (17, 19)
3. (6, 8), (8, 10), (10, 12)
4. 60
5. Between 30°C and 35°C
6. Between 86°F and 95°F
7. 82 marks
8. More than 600
9. 9 cm
10. More than 562.5 litres but less than 900 litres
11. More than 320 litres but less than 1280 litres
12. Between 6.27 and 8.07

15.7 GRAPHICAL SOLUTION OF LINEAR INEQUATIONS IN TWO VARIABLES

If a, b, c are real numbers, then the equation $ax + by + c = 0$ is called a linear equation in two variables x and y whereas the inequalities $ax + by \leq c$, $ax + by \geq c$, $ax + by < c$ and $ax + by > c$ are called linear inequations in two variables x and y.

We have studied in coordinate geometry that the graph of the equation $ax + by = c$ is a straight line which divides the xy-plane into two parts which are represented by $ax + by \leq c$ and $ax + by \geq c$. These two parts are known as the closed half-spaces. The regions represented by $ax + by < c$ and $ax + by > c$ are known as the open half spaces. In set theoretical notations, the set $\{(x, y) : ax + by = c\}$ is the straight line, sets $\{(x, y) : ax + by \leq c\}$ and $\{(x, y) : ax + by \geq c\}$ are closed half spaces and the sets $\{(x, y) : ax + by < c\}$ and $\{(x, y) : ax + by > c\}$ are open half-spaces. These half spaces are also known as the solution sets of the corresponding inequations.

In order to find the solution set of a linear inequation in two variables, we follow the following algorithm.

ALGORITHM

STEP I *Convert the given inequation, say $ax + by \leq c$, into the equation $ax + by = c$ which represents a straight line in xy-plane.*

STEP II *Put $y = 0$ in the equation obtained in step I to get the point where the line meets with x-axis. Similarly, put $x = 0$ to obtain a point where the line meets with y-axis.*

STEP III *Join the points obtained in step II to obtain the graph of the line obtained from the given inequation. In case of a strict inequality i.e. $ax + by < c$ or $ax + by > c$, draw the dotted line, otherwise mark it thick line.*

STEP IV *Choose a point, if possible (0, 0), not lying on this line : Substitute its coordinates in the inequation. If the inequation is satisfied, then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain the chosen point.*

STEP V *The shaded region obtained in step IV represents the desired solution set.*

REMARK *In case of the inequalities $ax + by \leq c$ and $ax + by > c$ points on the line are also a part of the shaded region while in case of inequalities $ax + by < c$ and $ax + by > c$ points on the line $ax + by = c$ are not in the shaded region.*

The following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Solve the following inequations graphically:*

(i) $2x+3y \le 6$ (ii) $2x-y \ge 1$ (iii) $x \ge 2$ (iv) $y \le -3$

SOLUTION (i) Converting the given inequation into equation, we obtain $2x+3y=6$. Putting $y=0$ and $x=0$ respectively in this equation, we get $x=3$ and $y=2$. So, this line meets x-axis at $A(3, 0)$ and y-axis at $B(0, 2)$. We plot these points and join them by a thick line. This line divides the xy-plane in two parts. To determine the region represented by the given inequality consider the point $O(0, 0)$. Clearly, $(0, 0)$ satisfies the inequality. So, the region containing the origin is represented by the given inequation as shown in Fig. 15.29. This region represents the solution set of the given inequations.

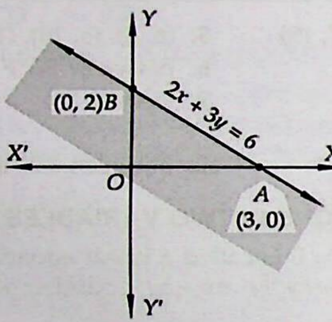

Fig. 15.29

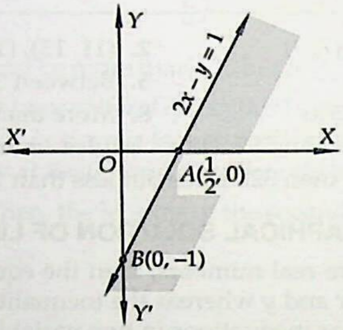

Fig. 15.30

(ii) Converting the given inequation into equation we obtain $2x-y=1$. This line meets x and y-axes at $A(1/2, 0)$ and $B(0, -1)$ respectively. Joining these points by a thick line we obtain the line passing through A and B as shown in Fig. 15.30. This line divides the xy-plane into two regions viz. one lying above it and the other lying below it. Consider the point $O(0, 0)$. Clearly, $(0, 0)$ does not satisfy the inequation $2x-y \ge 1$. So, the region not containing the origin is represented by the given inequation as shown in Fig. 15.30. Clearly it represents the solution set of the given inequation.

(iii) We have $x \ge 2$.Converting the inequation into equation, we obtain $x=2$. Clearly, it is a line parallel to y-axis at a distance of 2 units from it. This line divides the xy-plane into two parts viz. one part on the LHS of $x=2$ and the other on its RHS. We find that the point $(0, 0)$ does not satisfy the inequation $x \ge 2$. So, the region represented by the given equation is the shaded region shown in Fig. 15.31. The shaded region is the required solution set of the given inequation.

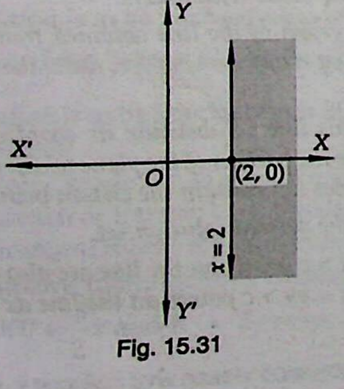

Fig. 15.31

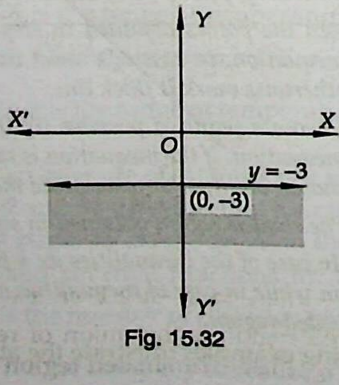

Fig. 15.32

(iv) We have $y \le -3$. Converting the given inequation into equation we obtain $y = -3$. Clearly, it is a line parallel to x-axis at a distance of 3 units below it. The line $y = -3$ divides the xy-plane into two regions one below it and the other above it. Consider the point O (0, 0). We find that (0, 0) does not satisfy the inequation $y \le -3$. So, the region represented by the given inequation is the region not containing the origin as show in Fig. 15.32. Clearly, it is the solution set of the given inequation.

EXAMPLE 2 *Solve the following inequations graphically:*

(i) $|x| \le 3$ (ii) $|y - x| \le 3$ (iii) $|x - y| \ge 1$

SOLUTION (i) Converting the given inequatino into equation, we obtain $x = 3$. This equation represents a line parallel to y-axis at a distance of 3 units from it. The line given by $x = 3$ divides the xy-plane into two regions. Clearly, the point O (0, 0) satisfies $x \le 3$. So, the graph of $x \le 3$ is as shown in Fig. 15.33. The shaded region represents the solution set of this inequation.

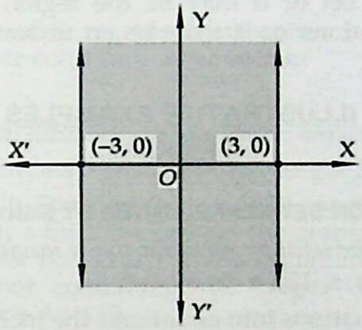

Fig. 15.33

(ii) We have, $|y - x| \le 3$. This inequation is equivalent to

$$-3 \le y - x \le 3 \qquad [\because\ |x| \le a \Leftrightarrow -a \le x \le a]$$

$$\Leftrightarrow \quad -3 \le y - x \text{ and } y - x \le 3$$

$$\Leftrightarrow \quad x - y - 3 \le 0 \text{ and } x - y + 3 \ge 0$$

The region represented by $|y - x| \le 3$ is the region common to the regions represented by $x - y - 3 \le 0$ and $x - y + 3 \ge 0$ as shown in Fig. 15.34. This shaded region represents the solution set of the given inequation.

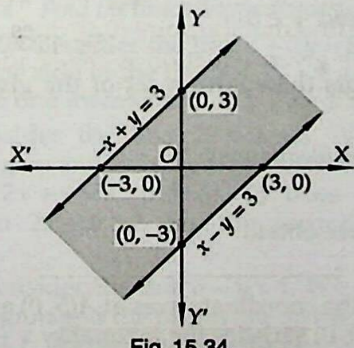

Fig. 15.34

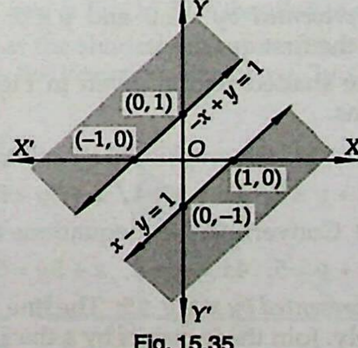

Fig. 15.35

(iii) We have,

$$|x - y| \ge 1 \Leftrightarrow x - y \ge 1 \textit{ or } x - y \le -1 \Leftrightarrow x - y - 1 > 0 \textit{ or } x - y + 1 \le 0$$

The required region is the union of regions represented by $x - y - 1 \ge 0$ and $x - y + 1 \le 0$ as shown in Fig. 15.35. The shaded region represents the solution set of the given inequation.

EXERCISE 15.5

LEVEL-1

Represent to solution set of each of the following inequations graphically in two dimensional plane:

1. $x + 2y - y \le 0$
2. $x + 2y \ge 6$
3. $x + 2 \ge 0$
4. $x - 2y < 0$
5. $-3x + 2y \le 6$
6. $x \le 8 - 4y$
7. $0 \le 2x - 5y + 10$
8. $3y > 6 - 2x$
9. $y > 2x - 8$
10. $3x - 2y \le x + y - 8$

15.8 SOLUTION OF SIMULTANEOUS LINEAR INEQUATIONS IN TWO VARIABLE

In this section, we will discuss the technique of finding the solution set of simultaneous linear inequations. Solving simultaneous linear inequations means finding the set of points (x, y) for which all the constraints are satisfied. Note that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight line boundaries.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **ON FINDING THE SOLUTION SET REPRESENTED BY SIMULTANEOUS LINEAR INEQUATIONS**

EXAMPLE 1 *Exhibit graphically the solution set of the linear inequations*

$$3x + 4y \le 12, \quad 4x + 3y \le 12, \quad x \ge 0, \quad y \ge 0$$

SOLUTION Converting the inequations into equations, the inequations reduce to

$$3x + 4y = 12, 4x + 3y = 12, x = 0 \text{ and } y = 0.$$

Region Represented by $3x + 4y \le 12$: The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B (0, 3). Draw a thick line joining A and B. We find that (0, 0) satisfies inequation $3x + 4y \le 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \le 12$.

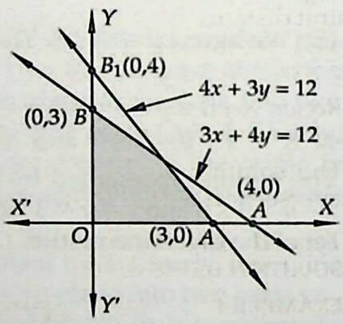

Fig. 15.36

Region Represented by $4x + 3y \le 12$: The line $4x + 3y = 12$ meets the x and y-axes at $A_1(3, 0)$ and $B_1(0, 4)$ respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \le 12$.

Region Represented by $x \ge 0$ and $y \ge 0$: Clearly, $x \ge 0$ and $y \ge 0$ represent the first quadrant.

Hence, the shaded region given in Fig. 15.36 represents the solution set of the given linear inequations.

EXAMPLE 2 *Exhibit graphically the solution set of the linear inequations*

$$x + y \le 5, \quad 4x + y \ge 4, \quad x + 5y \ge 5, \quad x \le 4, \quad y \le 3$$

SOLUTION Converting the inequations into equations, we obtain

$$x + y = 5, \quad 4x + y = 4, \quad x + 5y = 5, \quad x = 4, \quad y = 3$$

Region Represented by $x + y \le 5$: The line $x + y = 5$ meets the coordinate axes at $A(5, 0)$ and $B(0, 5)$ respectively. Join these points by a thick line. Clearly, (0, 0) satisfies the inequality $x + y \le 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \le 5$.

Region Represented by $4x + y \ge 4$: The line $4x + y = 4$ meets the coordinate axes at $A_1(1, 0)$ and B_1 (0, 4)respectively. Join these points by a thick line. Clearly, (0, 0) does not satisfy the inequation $4x + y \ge 4$. So, the portion not containing the origin is represented by the inequation $4x + y \ge 4$.

Region Represented by $x + 5y \geq 5$: The line $x + 5y \geq 5$ meets the coordinate axes at A (5, 0) and B_2 (0, 1) respectively. Join these two points by a thick line. We find that (0, 0) does not satisfy the inequation $x + 5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

Region Represented by $x \leq 4$: Clearly, $x = 4$ is a line parallel to y-axis at a distance of 4 units from the origin. Since (0, 0) satisfies the inequation $x \leq 4$. So, the portion lying on the left side of $x = 4$ is the region represented by $x \leq 4$.

Region Represented by $y \leq 3$: Clearly, $y = 3$ is a line parallel to x-axis at a distance 3 from it. Since (0, 0) satisfies $y \leq 3$. So, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear constaints as shown in Fig. 15.37.

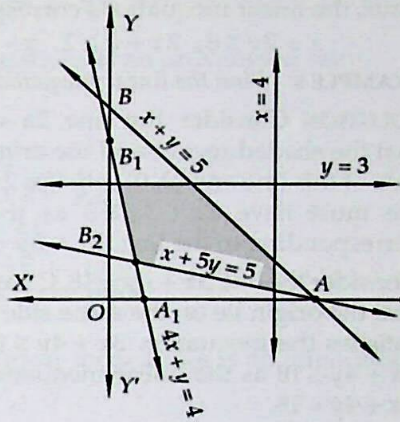

Fig. 15.37

EXAMPLE 3 *Draw the diagram of the solution set of the linear inequations* $3x + 4y \geq 12$, $y \geq 1$, $x \geq 0$.

SOLUTION Converting the inequations into equations, we get $3x + 4y = 12$, $y = 1$, $x = 0$

Region Represented by $3x + 4y \geq 12$: The line $3x + 4y = 12$ meets the coordinate axes at A (4, 0) and B(0, 3) joining these points by a thick line we get the graph of $3x + 4y = 12$. Since (0, 0) does not satisfy the inequation $3x + 4y \geq 12$. So, the portion not containing the origin is represented by the inequation $3x + 4y \geq 12$.

Region Represented by $y \geq 1$: The line $y = 1$ is parallel to x-axis at a unit distance from it. Since (0, 0) does not satisfy the inequation $y \geq 1$. So, the region lying above the line $y = 1$ is represented by $y \geq 1$.

Region Represented by $x \geq 0$: Clearly, $x \geq 0$ represents the region lying on the right side of y-axis.

The solution set of the given linear constraints is the intersection of the above regions as shown in Fig. 15.38.

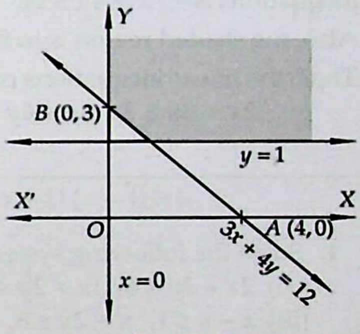

Fig. 15.38

Type II ON FINDING THE LINEAR INEQUATIONS WHEN THEIR SOLUTION SET IS GIVEN

EXAMPLE 4 *Find the linear inequations for which the shaded area in Fig. 15.39 is the solution set.*

SOLUTION Consider the line $x + 2y = 8$. We observe that the shaded region and the origin are on the same side of the line $x + 2y = 8$ and (0, 0) satisfies the linear constraint $x + 2y \leq 8$. So, we must have one inequations as $x + 2y \leq 8$.

Now consider the line $2x + y = 2$. We find that the shaded region and the origin are on the opposite sides of the line $2x + y = 2$ and (0, 0) does not satisfy the inequation $2x + y \geq 2$. So, the second inequations is $2x + y \geq 2$.

Finally, consider the line $x - y = 1$. We observe that the shaded region and the origin are on the same side of the line $x - y = 1$. We observe that the shaded region and the origin are on the same side of the line $x - y = 1$ and (0, 0) satisfies $x - y \leq 1$. So, the third constraint is $x - y \leq 1$.

We also notice that the shaded region is above x-axis and is on the right side of y-axis. So, we must have $x \geq 0$ and $y \geq 0$.

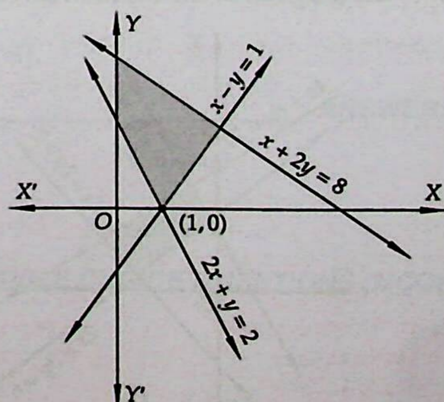

Fig. 15.39

Thus, the linear inequations corresponding to the given solution set are

$$x + 2y \leq 8,\ \ 2x + y \geq 2,\ \ x - y \leq 1,\ \ x \geq 0,\ \ y \geq 0$$

EXAMPLE 5 *Find the linear inequations for which the shaded region in Fig. 15.40 is the solution set.*

SOLUTION Consider the line $2x + 3y = 3$. We observe that the shaded region and the origin lie on the opposite side of this line and (0, 0) satisfies $2x + 3y \leq 3$. Therefore, we must have $2x + 3y \geq 3$ as the linear inequations corresponding to the line $2x + 3y = 3$.

Consider the line $3x + 4y = 18$. Clearly, the shaded region and the origin lie on the same side of this line and (0, 0) satisfies the inequation $3x + 4y \leq 18$. So, we must have $3x + 4y \leq 18$ as the linear inequations corresponding to $3x + 4y = 18$.

Consider the line $x - 6y = 3$. It is evident from the figure that the origin and the shaded region lie on the same side of this line and (0, 0) satisfies $x - 6y \leq 3$. So, $x - 6y \leq 3$ is the corresponding inequations.

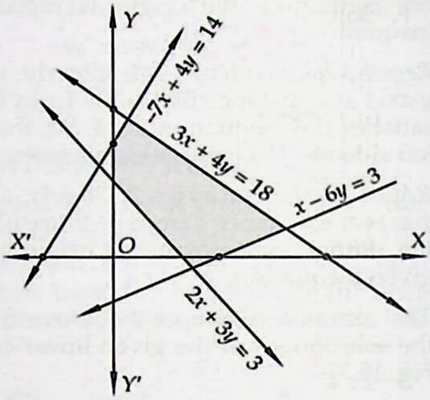

Fig. 15.40

Consider the line $-7x + 4y = 14$. We find that the shaded region and the origin are on the same side of this line and (0, 0) satisfies the inequations $-7x + 4y \leq 14$. So, the corresponding linear inequations is $-7x + 4y \leq 14$.

Also, the shaded region is in first quadrant only. So, we must have $x \geq 0$ and $y \geq 0$.

Thus, the linear inequations comprising the given solution set are

$$2x + 3y \geq 3,\ \ 3x + 4y \leq 18,\ \ -7x + 4y \leq 14,\ \ x - 6y \leq 3,\ \ x \geq 0,\ \ y \geq 0$$

EXERCISE 15.6

LEVEL-1

1. Solve the following systems of linear inequations graphically:
   (i) $2x + 3y \leq 6,\ 3x + 2y \leq 6,\ x \geq 0,\ y \geq 0$ (ii) $2x + 3y \leq 6,\ x + 4y \leq 4,\ x \geq 0,\ y \geq 0$
   (iii) $x - y \leq 1,\ x + 2y \leq 8,\ 2x + y \geq 2,\ x \geq 0,\ y \geq 0$
   (iv) $x + y \geq 1,\ 7x + 9y \leq 63,\ x \leq 6,\ y \leq 5,\ x \geq 0,\ y \geq 0$
   (v) $2x + 3y \leq 35,\ y \geq 3,\ x \geq 2,\ x \geq 0,\ y \geq 0$
2. Show that the solution set of the following linear inequations is empty set :
   (i) $x - 2y \geq 0,\ 2x - y \leq -2,\ x \geq 0,\ y \geq 0$ (ii) $x + 2y \leq 3,\ 3x + 4y \geq 12,\ y \geq 1,\ x \geq 0,\ y \geq 0$
3. Find the linear inequations for which the shaded area in Fig. 15.41 is the solution set. Draw the diagram of the solution set of the linear inequations:

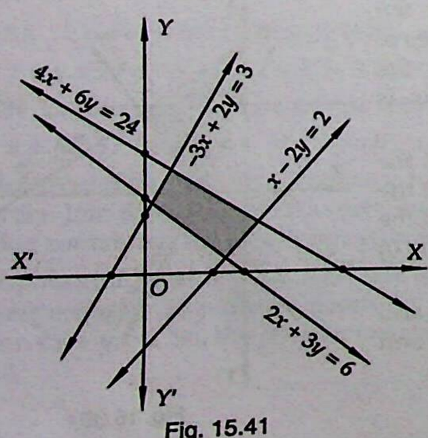

Fig. 15.41

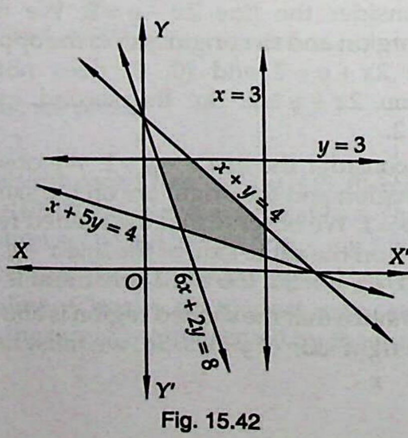

Fig. 15.42

4. Find the linear inequations for which the solution set is the shaded region given in Fig. 15.42.
5. Show that the solution set of the following linear in equations is an unbounded set:
$x + y \geq 9,\ 3x + y \geq 12,\ x \geq 0,\ y \geq 0.$
6. Solve the following systems of inequations graphically:
(i) $2x + y \geq 8,\ x + 2y \geq 8,\ x + y \leq 6$
(ii) $12x + 12y \leq 840,\ 3x + 6y \leq 300,\ 8x + 4y \leq 480\ \ x \geq 0, y \geq 0$
(iii) $x + 2y \leq 40,\ 3x + y \geq 30,\ 4x + 3y \geq 60,\ x \geq 0,\ y \geq 0$
(iv) $5x + y \geq 10,\ 2x + 2y \geq 12,\ x + 4y \geq 12,\ x \geq 0,\ y \geq 0$
7. Show that the following system of linear equations has no solution:
$x + 2y \leq 3,\ 3x + 4y \geq 12,\ x \geq 0,\ y \geq 1.$
8. Show that the solution set of the following system of linear inequalities is an unbounded region $2x + y \geq 8,\ x + 2y \geq 10,\ x \geq 0,\ y \geq 0.$

ANSWERS

3. $2x + 3y \geq 6, 4x + 6y \leq 24, -3x + 2y \leq 3, x - 2y \leq 2, x \geq 0, y \geq 0,$
4. $x + y \leq 4, y \leq 3, x \leq 3, x + 5y \geq 4, 6x + 2y \geq 8, x \geq 0, y \geq 0$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the solution set of the inequation $\frac{x^2}{x-2} > 0.$
2. Write the solution set of the inequation $x + \frac{1}{x} \geq 2.$
3. Write the set of values of x satisfying the inequation $(x^2 - 2x + 1)(x - 4) \geq 0.$
4. Write the solution set of the equation $|2 - x| = x - 2.$
5. Write the set of values of x satisfying $|x - 1| \leq 3$ and $|x - 1| \leq 1.$
6. Write the solution set of the inequation $\left|\frac{1}{x} - 2\right| < 4.$
7. Write the number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}.$
8. Write the set of values of x satisfying the inequations $5x + 2 < 3x + 8$ and $\frac{x+2}{x-1} < 4.$
9. Write the solution set of $\left|x + \frac{1}{x}\right| > 2.$
10. Write the solution set of the inequation $|x - 1| \geq |x - 3|.$

ANSWERS

1. $[2, \infty)$	2. $(0, \infty)$	3. $(-\infty, 4)$	4. $(2, \infty)$
5. $[2, 4]$	6. $(-\infty, -1/2) \cup (1/6, \infty)$		7. 3
8. $(-\infty, 1) \cup (2, 3)$	9. $R - \{-1, 0, 1\}$	10. $[2, \infty)$	

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If $x < 7$, then
(a) $-x < -7$ (b) $-x \leq -7$ (c) $-x > -7$ (d) $-x \geq -7$

2. If $-3x+17<-13$, then
(a) $x\in(10,\infty)$ (b) $x\in[10,\infty)$ (c) $x\in(-\infty,10]$ (d) $x\in[-10,10)$

3. Given that x, y and b are real numbers and $x<y$, $b>0$, then
(a) $\frac{x}{b}<\frac{y}{b}$ (b) $\frac{x}{b}\le\frac{y}{b}$ (c) $\frac{x}{b}>\frac{y}{b}$ (d) $\frac{x}{b}\ge\frac{y}{b}$

4. If x is a real number and $|x|<5$, then
(a) $x\ge5$ (b) $-5<x<5$ (c) $x\le-5$ (d) $-5\le x\le5$

5. If x and a are real numbers such that $a>0$ and $|x|>a$, then
(a) $x\in(-a,\infty)$ (b) $x\in[-\infty,a]$ (c) $x\in(-a,a)$ (d) $x\in(-\infty,-a)\cup(a,\infty)$

6. If $|x-1|>5$, then
(a) $x\in(-4,6)$ (b) $x\in[-4,6]$
(c) $x\in(-\infty,-4)\cup(6,\infty)$ (d) $x\in(-\infty,-4)\cup[6,\infty)$

7. If $|x+2|\le9$, then
(a) $x\in(-7,11)$ (b) $x\in[-11,7]$
(c) $x\in(-\infty,-7)\cup(11,\infty)$ (d) $x\in(-\infty,-7)\cup[11,\infty)$

8. The inequality representing the following graph is
(a) $|x|<3$ (b) $|x|\le3$ (c) $|x|>3$ (d) $|x|\ge3$

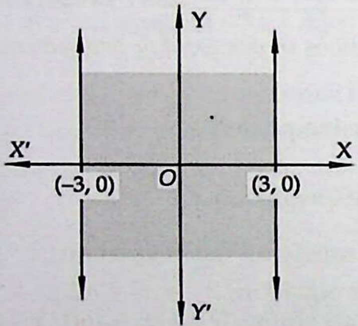

Fig. 15.43

9. The linear inequality representing the solution set given in Fig. 15.44 is
(a) $|x|<5$ (b) $|x|>5$ (c) $|x|\ge5$ (d) $|x|\le5$

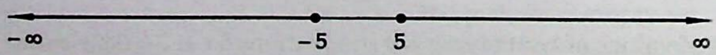

Fig. 15.44

10. The solution set of the inequation $|x+2|\le5$ is
(a) $(-7,5)$ (b) $[-7,3]$ (c) $[-5,5]$ (d) $(-7,3)$

11. If $\frac{|x-2|}{x-2}\ge0$, then
(a) $x\in[2,\infty)$ (b) $x\in(2,\infty)$ (c) $x\in(-\infty,2)$ (d) $x\in(-\infty,2]$

12. If $|x+3|\ge10$, then
(a) $x\in(-13,7]$ (b) $x\in(-13,7)$
(c) $x\in(-\infty,-13)\cup(7,\infty)$ (d) $x\in(-\infty,-13]\cup[7,\infty)$

ANSWERS

1. (c) 2. (a) 3. (a) 4. (b) 5. (d) 6. (c) 7. (b) 8. (b)
9. (c) 10. (b) 11. (b) 12. (d)