CHAPTER 23

THE STRAIGHT LINES

23.1 DEFINITION OF A STRAIGHT LINE

A straight line is a curve such that every point on the line segment joining any two points on it lies on it.

THEOREM *Every first degree equation in x, y represents a straight line.*

[NCERT EXEMPLAR]

PROOF Let $ax + by + c = 0$ be a first degree equation in x, y where a, b, c are constants. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points on the curve represented by $ax + by + c = 0$. Then,

$$ax_1 + by_1 + c = 0 \text{ and } ax_2 + by_2 + c = 0 \quad \text{...(i)}$$

Let R be any point on the line segment joining P and Q. Suppose R divides PQ in the ratio $\lambda : 1$. Then, the coordinates of R are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$. In order to prove that the curve represented by $ax + by + c = 0$ is a straight line, it is sufficient to show that R lies on it for all values of λ.

Now,

$$a\left(\frac{\lambda x_2 + x_1}{\lambda + 1}\right) + b\left(\frac{\lambda y_2 + y_1}{\lambda + 1}\right) + c = \frac{\lambda(ax_2 + by_2 + c) + (ax_1 + by_1 + c)}{\lambda + 1}$$

$$= \lambda 0 + 0 = 0 \quad \text{[Using (i)]}$$

$\therefore$ $R\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ lies on the curve represented by $ax + by + c = 0$.

Thus, every point on the line segment joining P and Q lies on $ax + by + c = 0$.

Hence, $ax + by + c = 0$ represents a straight line.

Q.E.D.

NOTE *When we say that a first degree equation in x, y i.e., $ax + by + c = 0$ represents a line, it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the condition $ax + by + c = 0$ where a, b, c are constants.*

It follows from the above discussion that $ax + by + c = 0$ is the general equation of a line.

It should be noted that there are only two unknowns in the equation of a straight line because equation of every straight line can be put in the form $ax + by + 1 = 0$ where a, b are two unknowns. Note that x, y are not unknowns. In fact these are the coordinates of any point on the line and are known as the current coordinates. Thus, to determine a line we will need two conditions to determine the two unknowns. In the further discussion on straight line you will find that whenever it will be asked to find a straight line there will always be two conditions connecting the two unknowns.

23.2 SLOPE (GRADIENT) OF A LINE

The trigonometrical tangent of the angle that a line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.

The slope of a line is generally denoted by m. Thus, $m = \tan \theta$.

Since a line parallel to x-axis makes an angle of $0°$ with x-axis, therefore its slope is $\tan 0° = 0$. A line parallel to y-axis i.e., perpendicular to x-axis makes an angle of $90°$ with x-axis, so its slope is

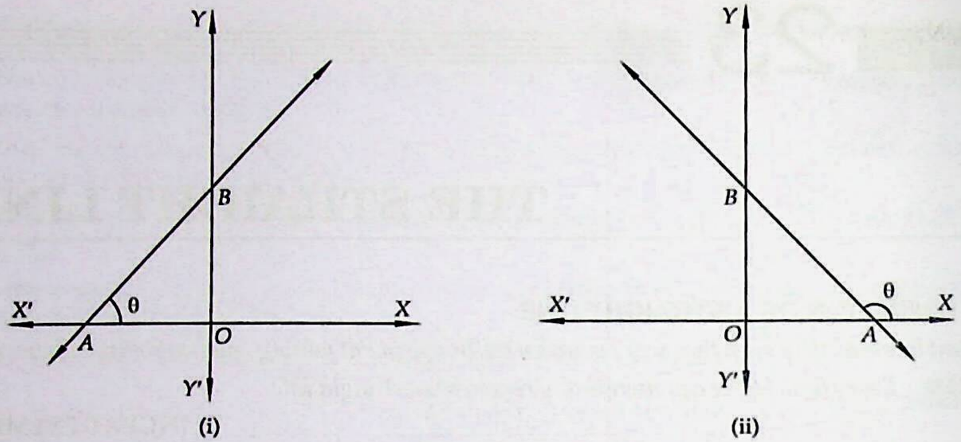

Fig. 23.1

tan $\pi/2 = \infty$. Also, the slope of a line equally inclined with axes is 1 or – 1 as it makes 45° or 135° angle with x-axis.

REMARK *The angle of inclination of a line with the positive direction of x-axis in anticlockwise sense always lies between 0° and 180°.*

ILLUSTRATION 1 *Find the slope of a line whose inclination to the positive direction of x-axis in anticlockwise sense is (i) 60° (ii) 0° (iii) 150° (iv) 120°.* [NCERT]

SOLUTION (i) Slope $= \tan 60° = \sqrt{3}$.

(ii) Slope $= \tan 0° = 0$.

(iii) Slope $= \tan 150° = -\cot 60° = -\frac{1}{\sqrt{3}}$.

(iv) Slope $= \tan 120° = -\cot 30° = -\sqrt{3}$.

ILLUSTRATION 2 *What can be said regarding a line if its slope is (i) positive (ii) zero (iii) negative?*

SOLUTION Let θ be the angle of inclination of the given line with the positive direction of x-axis in anticlockwise sense. Then, its slope is given by $m = \tan \theta$.

(i) If the slope of the line is positive, then

$$m = \tan \theta > 0 \Rightarrow \theta \text{ lies between } 0° \text{ and } 90° \Rightarrow \theta \text{ is an acute angle.}$$

Thus, a line of positive slope makes an acute angle with the positive direction of x-axis.

(ii) If the slope of the line is zero, then

$$m = \tan \theta = 0 \Rightarrow \theta = 0° \Rightarrow \text{either the line is } x\text{-axis or it is parallel to } x\text{-axis.}$$

Thus, a line of zero slope is parallel or coincident to x-axis.

(iii) If the slope of the line is negative, then

$$m = \tan \theta < 0 \Rightarrow \theta \text{ lies between } 90° \text{ and } 180° \Rightarrow \theta \text{ is an obtuse angle.}$$

Thus, a line of negative slope makes an obtuse angle with the positive direction of x-axis in anticlockwise direction.

23.2.1 SLOPE OF A LINE IN TERMS OF COORDINATES OF ANY TWO POINTS ON IT

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on a line making an angle θ with the positive direction of x-axis. Draw PL, QM perpendiculars on x-axis and $PN \perp$ on QM. Then,

$$PN = LM = OM - OL = x_2 - x_1 \text{ and, } QN = QM - NM = QM - PL = y_2 - y_1$$

In $\Delta\, PQN$, we have

$$\tan\theta = \frac{QN}{PN} = \frac{y_2 - y_1}{x_2 - x_1}$$

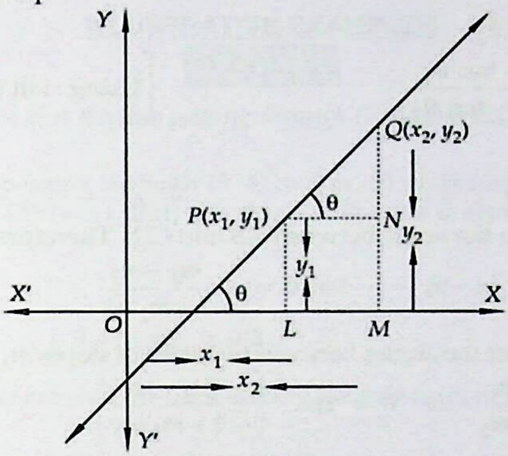

Fig. 23.2

Thus, if (x_1, y_1) and (x_2, y_2) are coordinates of any two points on a line, then its slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

ILLUSTRATION 3 *Find the slope of a line which passes through points (3, 2) and (– 1, 5).*

[NCERT]

SOLUTION We know that the slope of a line passing through two points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. Here, the line passes through (3, 2) and (–1, 5).

So, its slope is given by $m = \dfrac{5-2}{-1-3} = -\dfrac{3}{4}$.

23.3 ANGLE BETWEEN TWO LINES

THEOREM *The angle θ between the lines having slopes m_1 and m_2 is given by* $\tan\theta = \pm\dfrac{m_2 - m_1}{1 + m_1 m_2}$.

PROOF Let m_1 and m_2 be the slopes of two given lines AB and CD which intersect at a point P and make angles θ_1 and θ_2 respectively with the positive direction of x-axis. Then, $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$.

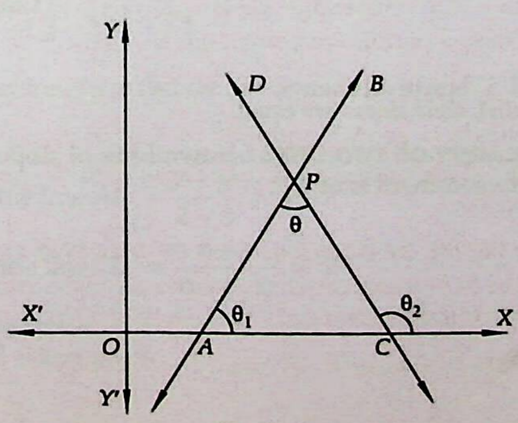

Fig. 23.3

Let $\angle APC = \theta$ be the angle between the given lines. Then,

$$\theta_2 = \theta + \theta_1 \qquad \text{[See Fig. 23.3]}$$

$$\Rightarrow \quad \theta = \theta_2 - \theta_1$$

$$\Rightarrow \quad \tan\theta = \tan(\theta_2 - \theta_1)$$

$$\Rightarrow \quad \tan\theta = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} \qquad \left[\text{Using} : \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right]$$

$$\Rightarrow \quad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{...(i)}$$

Since $\angle APD = \pi - \theta$ is also the angle between AB and CD. Therefore,

$$\tan \angle APD = \tan(\pi - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1 + m_1 m_2} \qquad \text{[Using (i)]} \quad \text{...(ii)}$$

From (i) and (ii), we find that the angles between two lines of slopes m_1 and m_2 are given by

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

The acute angle between the lines is given by $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Q.E.D.

ILLUSTRATION *If $A(-2, 1)$, $B(2, 3)$ and $C(-2, -4)$ are three points, find the angle between BA and BC.*

SOLUTION Let m_1 and m_2 be the slopes of BA and BC respectively. Then,

$$m_1 = \frac{3-1}{2-(-2)} = \frac{2}{4} = \frac{1}{2}, \quad \text{and} \quad m_2 = \frac{-4-3}{-2-2} = \frac{7}{4}$$

Let θ be the acute angle between BA and BC. Then,

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \left|\frac{\frac{7}{4} - \frac{1}{2}}{1 + \frac{7}{4} \times \frac{1}{2}}\right| = \left|\frac{\frac{10}{8}}{\frac{15}{8}}\right| = \frac{2}{3} \Rightarrow \theta = \tan^{-1}\left(\frac{2}{3}\right).$$

CONDITION OF PARALLELISM OF LINES If two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is of $0°$.

$$\therefore \quad \tan\theta = \tan 0° = 0$$

$$\Rightarrow \quad \frac{m_2 - m_1}{1 + m_1 m_2} = 0 \qquad \left[\text{Using} : \tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2}\right]$$

$$\Rightarrow \quad m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

CONDITION OF PERPENDICULARITY OF TWO LINES If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of $90°$.

From Fig. 23.3, we have

$$\theta_2 = \theta + \theta_1$$

$$\Rightarrow \quad \theta_2 = 90° + \theta_1 \qquad [\because \theta = 90°]$$

$$\Rightarrow \quad \tan\theta_2 = \tan(90° + \theta_1)$$

$$\Rightarrow \quad \tan\theta_2 = -\cot\theta_1$$

$\Rightarrow \quad \tan\theta_1 \tan\theta_2 = -1 \Rightarrow m_1 m_2 = -1$

Thus, when two lines are perpendicular, the product of their slopes is – 1. If m is the slope of a line, then the slope of a line perpendicular to it is – (1/m).

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Determine x so that the line passing through (3, 4) and (x, 5) makes 135° angle with the positive direction of x-axis.*

SOLUTION Since the line passing through (3, 4) and $(x, 5)$ makes an angle of 135° with x-axis. Therefore, its slope is $\tan 135° = -1$. But, the slope of the line is also equal to

$$\frac{5-4}{x-3} \qquad \left[\text{Using} : m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$$\therefore \quad -1 = \frac{5-4}{x-3} \Rightarrow -x+3 = 1 \Rightarrow x = 2.$$

EXAMPLE 2 *Find the angle between the lines joining the points (0, 0), (2, 3) and the points (2, – 2), (3, 5).*

SOLUTION Let θ be the angle between the given lines.

We have,

$$m_1 = \text{Slope of the line joining } (0, 0) \text{ and } (2, 3) = \frac{3-0}{2-0} = \frac{3}{2}$$

$$m_2 = \text{Slope of the line joining } (2, -2) \text{ and } (3, 5) = \frac{5+2}{3-2} = 7$$

$$\therefore \quad \tan\theta = \pm\frac{m_2 - m_1}{1 + m_1 m_2} = \pm\frac{7 - 3/2}{1 + 7\,(3/2)} = \pm\frac{11/2}{23/2} = \pm\frac{11}{23} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{11}{23}\right).$$

EXAMPLE 3 *Let A (6, 4) and B (2, 12) be two given points. Find the slope of a line perpendicular to AB.*

SOLUTION Let m be the slope of AB. Then ,

$$m = \frac{12-4}{2-6} = \frac{8}{-4} = -2$$

So, the slope of a line perpendicular to AB is $-\frac{1}{m} = \frac{1}{2}$

EXAMPLE 4 *Determine x so that 2 is the slope of the line through (2, 5) and (x, 3).*

SOLUTION The slope of the line through (2, 5) and $(x, 3)$ is $\frac{3-5}{x-2}$. But, the slope of the line is given as 2.

$$\therefore \quad \frac{3-5}{x-2} = 2 \Rightarrow 2x - 4 = -2 \Rightarrow x = 1$$

EXAMPLE 5 *What is the value of y so that the line through (3, y) and (2, 7) is parallel to the line through (–1, 4) and (0, 6)?*

SOLUTION Let $A\,(3, y)$, $B\,(2, 7)$, $C\,(-1, 4)$ and $D\,(0, 6)$ be the given points. Then,

$$m_1 = \text{Slope of the line } AB = \frac{7-y}{2-3} = y - 7$$

and, $$m_2 = \text{Slope of the line } CD = \frac{6-4}{0-(-1)} = 2$$

Since AB and CD are parallel.

$$\therefore \quad m_1 = m_2 \Rightarrow y - 7 = 2 \Rightarrow y = 9.$$

EXAMPLE 6 *Without using Pythagoras theorem, show that A (4, 4), B (3, 5) and C (–1, –1) are the vertices of a right-angled triangle.* **[NCERT]**

SOLUTION In $\Delta\, ABC$, we have,

$$m_1 = \text{Slope of } AB = \frac{4-5}{4-3} = -1 \text{ and, } m_2 = \text{Slope of } AC = \frac{4-(-1)}{4-(-1)} = 1$$

Clearly, $m_1 m_2 = -1$. This shows that AB is perpendicular to AC i.e. $\angle CAB = \pi/2$.
Hence, the given points are the vertices of a right-angled triangle.

EXAMPLE 7 *A quadrilateral has the vertices at the points* $(-4, 2), (2, 6), (8, 5)$ *and* $(9, -7)$. *Show that the mid-points of the sides of this quadrilateral are the vertices of a parallelogram.*

SOLUTION Let $A(-4, 2), B(2, 6), C(8, 5)$ and $D(9, -7)$ be the vertices of the given quadrilateral. Let P, Q, R and S be the mid-points of AB, BC, CD and DA respectively. Then, the coordinates of P, Q, R and S are $P(-1, 4)$, $Q(5, 11/2)$, $R(17/2, -1)$ and $S(5/2, -5/2)$ respectively. In order to prove that $PQRS$ is a parallelogram, it is sufficient to show that PQ is parallel to RS and $PQ = RS$. Let m_1 and m_2 be the slope of PQ and RS respectively. Then,

$$m_1 = \frac{11/2-4}{5-(-1)} = \frac{1}{4} \text{ and, } m_2 = \frac{-5/2+1}{5/2-17/2} = \frac{1}{4}$$

Clearly, $m_1 = m_2$. Therefore, PQ is parallel to RS.
Now,

$$PQ = \sqrt{(5+1)^2 + \left(\frac{11}{2}-4\right)^2} = \frac{\sqrt{153}}{2} \text{ and, } RS = \sqrt{\left(\frac{5}{2}-\frac{17}{2}\right)^2 + \left(-\frac{5}{2}+1\right)^2} = \frac{\sqrt{153}}{2}$$

$\therefore \quad PQ = RS$

Thus, $PQ \parallel RS$ and $PQ = RS$. Hence, $PQRS$ is a parallelogram.

EXAMPLE 8 *Prove that* $A(4, 3), B(6, 4), C(5, 6)$ *and* $D(3, 5)$ *are the angular points of a square.*

SOLUTION Clearly,

$$AB = \sqrt{(6-4)^2 + (4-3)^2} = \sqrt{5},\ BC = \sqrt{(6-4)^2 + (5-4)^2} = \sqrt{5},$$

$$CD = \sqrt{(5-6)^2 + (3-5)^2} = \sqrt{5} \text{ and, } DA = \sqrt{(5-3)^2 + (3-4)^2} = \sqrt{5}$$

$\therefore \quad AB = BC = CD = DA.$

Now, $m_1 = \text{Slope of } AB = \dfrac{4-3}{6-4} = \dfrac{1}{2}, \quad m_2 = \text{Slope of } BC = \dfrac{6-4}{5-6} = -2$

and, $m_3 = \text{Slope of } CD = \dfrac{5-6}{3-5} = \dfrac{1}{2}$

Clearly, $m_1 m_2 = (1/2)(-2) = -1$ and $m_1 = m_3$. Therefore, AB is perpendicular to BC and it is parallel to CD. Thus, $AB = BC = CA = AD$, $AB \perp BC$ and AB is parallel to CD. Hence, $ABCD$ is a square.

EXAMPLE 9 *If the angle between two lines is* $\dfrac{\pi}{4}$ *and slope of one of the line* $\dfrac{1}{2}$, *find the slope of the other line.* **[NCERT]**

SOLUTION We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \text{...(i)}$$

Let $m_1 = \dfrac{1}{2}$ and $m_2 = m$ = slope of the other line. It is given that $\theta = \dfrac{\pi}{4}$.

Substituting $m_1 = \dfrac{1}{2}$, $m_2 = m$ and $\theta = \dfrac{\pi}{4}$ in (i), we obtain

$$\therefore \quad \tan\frac{\pi}{4} = \left|\frac{m-\frac{1}{2}}{1+m\times\frac{1}{2}}\right|$$

$$\Rightarrow \quad 1 = \left|\frac{2m-1}{2+m}\right|$$

$$\Rightarrow \quad \frac{2m-1}{m+2} = \pm 1 \Rightarrow 2m-1 = m+2 \text{ or, } 2m-1 = -(m+2) \Rightarrow m = 3 \text{ or, } m = -\frac{1}{3}$$

Hence, the slope of the other line is 3 or, $-\frac{1}{3}$.

EXAMPLE 10 *If the points $P(h, k)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ lie on a line. Show that:*

$$(h-x_1)(y_2-y_1) = (k-y_1)(x_2-x_1)$$ **[NCERT]**

SOLUTION It is given that the points $P(h, k)$, $Q(x_1, y_1)$ and $R(x_2, y_2)$ are collinear.

$\therefore$ Slope of PQ = Slope of QR

$$\Rightarrow \quad \frac{k-y_1}{h-x_1} = \frac{y_2-y_1}{x_2-x_1} \Rightarrow (k-y_1)(x_2-x_1) = (h-x_1)(y_2-y_1)$$

EXAMPLE 11 *In Fig. 23.4, time and distance graph of a linear motion is given. Two positions of time and distance recorded as, when $T = 0$, $D = 2$ and when $T = 3$, $D = 8$. Using the concept of slope, find law of motion i.e. how distance depends upon time.* **[NCERT]**

SOLUTION Let $P(T, D)$ be any point on the line, where D denotes the distance at any time T. Clearly, points $A(0, 2)$, $B(3, 8)$ and $P(T, D)$ are collinear.

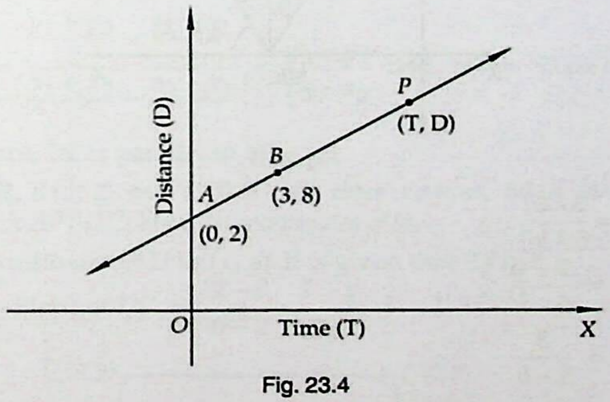

Fig. 23.4

$\therefore$ Slope of AB = Slope of BP

$$\Rightarrow \quad \frac{8-2}{3-0} = \frac{D-8}{T-3}$$

$$\Rightarrow \quad 2 = \frac{D-8}{T-3} \Rightarrow D = 2(T+1), \text{ which is the required relation.}$$

EXAMPLE 12 *If points $(a, 0)$, $(0, b)$ and (x, y) are collinear, using the concept of slope, prove that $\frac{x}{a} + \frac{y}{b} = 1$.*

SOLUTION Let $A(a, 0)$, $B(0, b)$ and $P(x, y)$ be the given collinear points. Then,

Slope of AB = Slope of BP

$$\Rightarrow \quad \frac{b-0}{0-a} = \frac{y-b}{x-0}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{y-b}{x}$$

$$\Rightarrow \quad -bx = ay - ab \Rightarrow bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \qquad \text{[On dividing both sides by } ab\text{]}$$

LEVEL-2

EXAMPLE 13 *A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the co-ordinates of A.*

[NCERT EXEMPLAR]

SOLUTION Let the coordinates of A be $(h, 0)$. Let AN be the normal at A. Then,

$$\angle PAN = \angle QAN = \theta \text{ (say)}$$

Clearly, AQ makes angle $90°-\theta$ with OX. Therefore, slope of AQ is $\tan(90°-\theta) = \cot\theta$. Also, the coordinates of A and Q arc $(h, 0)$ and $(5, 3)$ respectively.

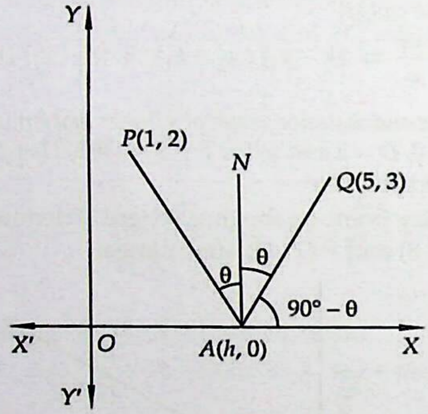

Fig. 23.5

$$\therefore \quad \text{Slope of } AQ = \frac{3-0}{5-h}$$

$$\text{Thus,} \quad \cot\theta = \frac{3-0}{5-h}$$

$$\Rightarrow \quad \cot\theta = \frac{3}{5-h} \qquad \text{...(i)}$$

AP makes angle $90°+\theta$ with OX. Therefore, slope of AP is $\tan(90°+\theta) = -\cot\theta$. Also, AP passes through $A(h, 0)$ and $P(1, 2)$. Therefore,

$$\text{Slope of } AP = \frac{2-0}{1-h}$$

$$\text{Thus} \quad -\cot\theta = \frac{2}{1-h}$$

$$\Rightarrow \quad \cot\theta = \frac{2}{h-1} \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$\frac{3}{5-h} = \frac{2}{h-1}$$

$$\Rightarrow \quad 3h - 3 = 10 - 2h$$

$$\Rightarrow \quad 5h = 13$$

$$\Rightarrow \qquad h = \frac{13}{5}$$

Hence, the coordinates of A are $(13/5\ ,\ 0)$.

EXAMPLE 14 *Prove that the line joining the mid-points of the two sides of a triangle is parallel to the third side.*

SOLUTION Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a $\Delta\ ABC$ and D and E be the mid-points of sides AB and AC respectively. Then, the coordinates of D and E are $\left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$ and $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)$ respectively.

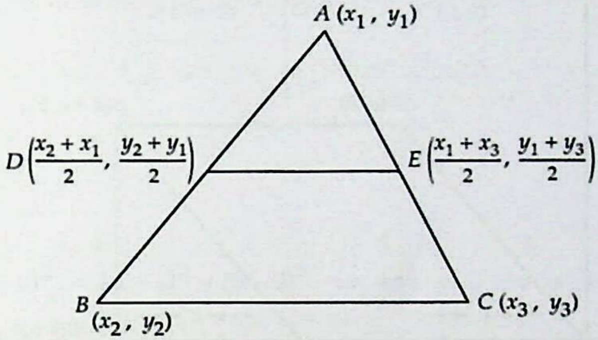

Fig. 23.6

$$\therefore\ m_1 = \text{Slope of } DE = \frac{\frac{y_1+y_3}{2} - \frac{y_2+y_1}{2}}{\frac{x_1+x_3}{2} - \frac{x_2+x_1}{2}} = \frac{y_3-y_2}{x_3-x_2} \text{ and, } m_2 = \text{Slope of } BC = \frac{y_3-y_2}{x_3-x_2}$$

Clearly, $m_1 = m_2$. Hence, DE is parallel to BC.

EXAMPLE 15 *If $A(2, 0)$, $B(0, 2)$ and $C(0, 7)$ are three vertices, taken in order, of an isosceles trapezium $ABCD$ in which $AB \| DC$. Find the coordinates of D.*

SOLUTION Let the coordinates of D be (x, y). It is given that $AB \| DC$.

$$\therefore \quad \text{Slope of } AB = \text{Slope of } DC \Rightarrow \frac{2-0}{0-2} = \frac{7-y}{0-x} \Rightarrow -1 = \frac{y-7}{x} \Rightarrow x+y-7=0 \qquad \text{...(i)}$$

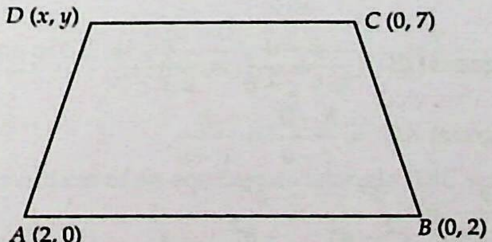

Fig. 23.7

Since $ABCD$ is an isosceles trapezium. Therefore,

$$AD = BC$$

$$\Rightarrow \quad AD^2 = BC^2$$

$$\Rightarrow \quad (x-2)^2 + (y-0)^2 = (0-0)^2 + (2-7)^2$$

$$\Rightarrow \quad (x-2)^2 + y^2 = 25$$

$\Rightarrow \quad (x-2)^2 + (7-x)^2 = 25$ [From (i), $y = 7 - x$]

$\Rightarrow \quad 2x^2 - 18x + 28 = 0 \Rightarrow x^2 - 9x + 14 = 0 \Rightarrow (x-2)(x-7) = 0 \Rightarrow x = 2, 7$

From (i), $x = 2 \Rightarrow y = 5$ and $x = 7 \Rightarrow y = 0$.

Hence, the coordinates of D are $(2, 5)$ or $(7, 0)$.

EXAMPLE 16 *By using the concept of slope, prove that the diagonals of a rhombus are at right angles.*

SOLUTION Let $OABC$ be a rhombus whose each side is of length a such that O is the origin and OA is along x-axis. Let b be the height of the rhombus. Let BL and CM be perpendiculars drawn from B and C respectively on x-axis.

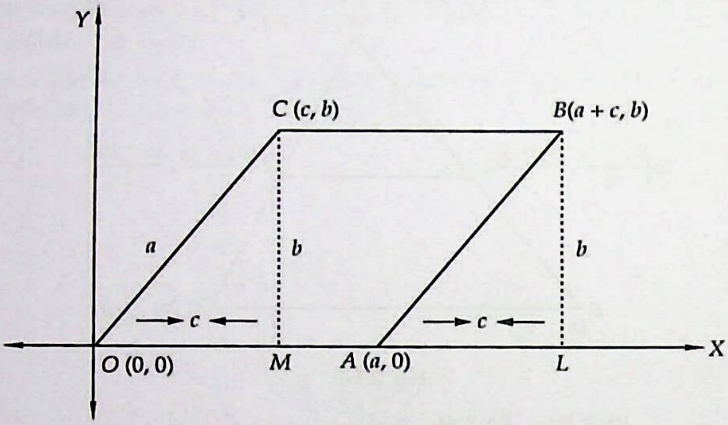

Fig. 23.8

Further, let $OM = c$.

Clearly, $\Delta OMC \cong \Delta ALB$

$\therefore \quad OM = AL \Rightarrow AL = c.$

Thus, we have

$$OM = c, CM = b, OA = a, OL = a + c \text{ and } LB = b$$

So, the coordinates of the vertices of the rhombus are $O(0, 0)$, $A(a, 0)$, $B(a+c, b)$ and $C(c, b)$

In right triangle OMC, we have

$$OC^2 = OM^2 + MC^2 \Rightarrow a^2 = c^2 + b^2$$

Now,

$$m_1 = \text{Slope of diagonal } OB = \frac{b-0}{a+c-0} = \frac{b}{a+c}$$

$$m_2 = \text{Slope of diagonal } AC = \frac{b-0}{c-a} = \frac{b}{c-a}$$

$$\therefore \quad m_1 m_2 = \frac{b}{a+c} \times \frac{b}{c-a} = \frac{b^2}{c^2 - a^2} = \frac{b^2}{-b^2} = -1 \qquad [\because a^2 = c^2 + b^2]$$

Hence, OB is perpendicular to AC.

EXAMPLE 17 *Using the concept of slope, prove that medians of an equilateral triangle are perpendicular to the corresponding sides.*

SOLUTION Let ABC be an equilateral triangle such that $AB = AC = BC = 2a$.

Let BC be along X-axis, mid-point of BC be the origin and a line passing through O and perpendicular to BC be Y-axis. Then, the coordinates of B and C are $(-a, 0)$ and $(a, 0)$ respectively. Let the coordinates of A be (α, β).

Since ABC is an equilateral triangle.

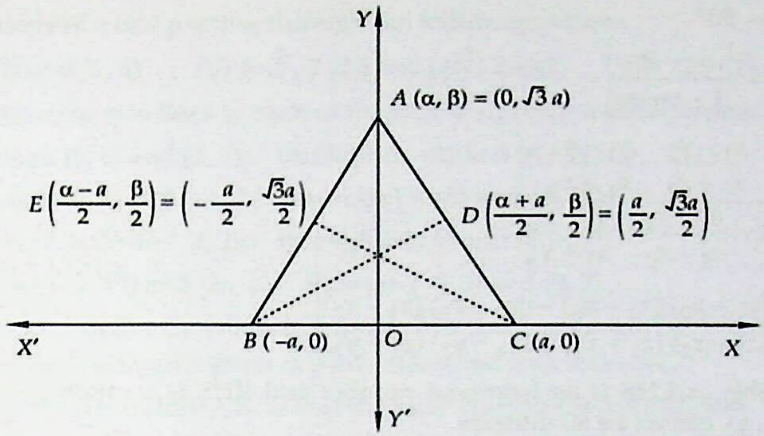

Fig. 23.9

$\therefore \quad AB = AC = BC$

Now, $\quad AB = AC$

$\Rightarrow \quad AB^2 = AC^2$

$\Rightarrow \quad (\alpha + a)^2 + (\beta - 0)^2 = (\alpha - a)^2 + (\beta - 0)^2 \Rightarrow 4a\alpha = 0 \Rightarrow \alpha = 0.$

In right triangle COA, we have

$$AC^2 = OA^2 + OC^2 \Rightarrow (2a)^2 = \beta^2 + a^2 \Rightarrow \beta = \sqrt{3}a$$

Thus, the coordinates of A are $(0, \sqrt{3}a)$. Consequently, A lies on y-axis.

$\therefore \quad OA \perp BC \qquad$ [$\because$ OA is along y-axis and BC is along x-axis]

Let D and E be the mid-points of AC and AB respectively. Then, the coordinates of D and E are $\left(\frac{\alpha + a}{2}, \frac{\beta}{2}\right) = \left(\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ and $\left(\frac{\alpha - a}{2}, \frac{\beta}{2}\right) = \left(-\frac{a}{2}, \frac{\sqrt{3}a}{2}\right)$ respectively.

Now, $\quad m_1 = \text{Slope of } AC = \dfrac{\sqrt{3}a - 0}{0 - a} = -\sqrt{3}$, $m_2 = \text{Slope of } BD = \dfrac{\frac{\sqrt{3}a}{2} - 0}{\frac{a}{2} + a} = \dfrac{1}{\sqrt{3}}$

Clearly, $m_1 m_2 = -1$. Therefore, $BD \perp AC$.

Also $\quad \text{Slope of } AB \times \text{Slope of } CE = \dfrac{\sqrt{3}a - 0}{0 + a} \times \dfrac{\frac{\sqrt{3}a}{2} - 0}{\frac{-a}{2} - a} = \sqrt{3} \times -\dfrac{\sqrt{3}}{3} = -1$

$\therefore \quad AB \perp CE$

Thus, AO, BD and CE are medians of an equilateral triangle ABC such that $AO \perp BC$, $BD \perp CA$ and $CE \perp AB$.

Hence, medians of an equilateral triangle are perpendicular to the corresponding sides.

EXAMPLE 18 *Prove that a triangle which has one of the angle as 30°, cannot have all vertices with integral coordinates.*

SOLUTION Let ABC be a triangle the coordinates of whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, where $x_1, x_2, x_3, y_1, y_2, y_3$ are integers. Let $\angle BAC = 30°$.

We have,

$$m_1 = \text{Slope of } AB = \frac{y_1 - y_2}{x_1 - x_2}, \text{ and } m_2 = \text{Slope of } AC = \frac{y_1 - y_3}{x_1 - x_3}$$

Now, $\angle BAC = 30^\circ$

$$\Rightarrow \quad \tan 30^\circ = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \left|\frac{\frac{y_1 - y_2}{x_1 - x_2} - \frac{y_1 - y_3}{x_1 - x_3}}{1 + \frac{y_1 - y_2}{x_1 - x_2} \times \frac{y_1 - y_3}{x_1 - x_3}}\right|$$

$$\Rightarrow \quad \frac{1}{\sqrt{3}} = \left|\frac{(y_1 - y_2)(x_1 - x_3) - (y_1 - y_3)(x_1 - x_2)}{(x_1 - x_2)(x_1 - x_3) + (y_1 - y_2)(y_1 - y_3)}\right|$$

This is not possible as LHS is an irrational number and RHS is a rational number. Hence, $x_1, x_2, x_3, y_1, y_2, y_3$ cannot be all integers.

EXAMPLE 19 *The vertices of a triangle are* $A(x_1, x_1 \tan \theta_1)$, $B(x_2, x_2 \tan \theta_2)$ *and* $C(x_3, x_3 \tan \theta_3)$. *If the circumcentre of* ΔABC *coincides with the origin and* $H(\bar{x}, \bar{y})$ *is the orthocentre, show that*

$$\frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

SOLUTION The circumcentre of ΔABC is at the origin O. Let the circum-radius be r. Then,

$$OA = OB = OC = r$$

$$\Rightarrow \quad OA^2 = OB^2 = OC^2 = r^2$$

$$\Rightarrow \quad x_1^2 + x_1^2 \tan^2 \theta_1 = x_2^2 + x_2^2 \tan^2 \theta_2 = x_3^2 + x_3^2 \tan^2 \theta_3 = r^2$$

$$\Rightarrow \quad x_1^2 \sec^2 \theta_1 = x_2^2 \sec^2 \theta_2 = x_3^2 \sec^2 \theta_3 = r^2$$

$$\Rightarrow \quad x_1 = r \cos \theta_1, \; x_2 = r \cos \theta_2, \; x_3 = r \cos \theta_3$$

So, the coordinates of the vertices of ΔABC are

$$A \equiv (x_1, x_1 \tan \theta_1) = (r \cos \theta_1, r \sin \theta_1), \; B \equiv (x_2, x_2 \tan \theta_2) = (r \cos \theta_2, r \sin \theta_2)$$

and, $C \equiv (x_3, x_3 \tan \theta_3) = (r \cos \theta_3, r \sin \theta_3)$.

So, the coordinates of the centroid G are

$$\left(\frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3}, \frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3}\right)$$

We know that the circumcentre (O), Centroid (G) and orthocentric (H) of a triangle are collinear.

$\therefore$ Slope of OH = Slope of OG

$$\Rightarrow \quad \frac{\bar{y} - 0}{\bar{x} - 0} = \frac{\frac{r \sin \theta_1 + r \sin \theta_2 + r \sin \theta_3}{3} - 0}{\frac{r \cos \theta_1 + r \cos \theta_2 + r \cos \theta_3}{3} - 0} \Rightarrow \frac{\bar{y}}{\bar{x}} = \frac{\sin \theta_1 + \sin \theta_2 + \sin \theta_3}{\cos \theta_1 + \cos \theta_2 + \cos \theta_3}$$

EXERCISE 23.1

LEVEL-1

1. Find the slopes of the lines which make the following angles with the positive direction of x-axis:

(i) $-\frac{\pi}{4}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{3\pi}{4}$ (iv) $\frac{\pi}{3}$

2. Find the slope of a line passing through the following points:
   (i) $(-3, 2)$ and $(1, 4)$ (ii) $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ (iii) $(3, -5)$, and $(1, 2)$
3. State whether the two lines in each of the following are parallel, perpendicular or neither:
   (i) Through $(5, 6)$ and $(2, 3)$; through $(9, -2)$ and $(6, -5)$
   (ii) Through $(9, 5)$ and $(-1, 1)$; through $(3, -5)$ and $(8, -3)$
   (iii) Through $(6, 3)$ and $(1, 1)$; through $(-2, 5)$ and $(2, -5)$
   (iv) Through $(3, 15)$ and $(16, 6)$; through $(-5, 3)$ and $(8, 2)$.
4. Find the slope of a line (i) which bisects the first quadrant angle (ii) which makes an angle of 30° with the positive direction of y-axis measured anticlockwise. [NCERT]
5. Using the method of slope, show that the following points are collinear:
   (i) $A(4, 8)$, $B(5, 12)$, $C(9, 28)$ (ii) $A(16, -18)$, $B(3, -6)$, $C(-10, 6)$
6. What is the value of y so that the line through $(3, y)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$?
7. What can be said regarding a line if its slope is
   (i) zero (ii) positive (iii) negative?
8. Show that the line joining $(2, -3)$ and $(-5, 1)$ is parallel to the line joining $(7, -1)$ and $(0, 3)$.
9. Show that the line joining $(2, -5)$ and $(-2, 5)$ is perpendicular to the line joining $(6, 3)$ and $(1, 1)$.
10. Without using Pythagoras theorem, show that the points $A(0, 4)$, $B(1, 2)$ and $C(3, 3)$ are the vertices of a right angled triangle.
11. Prove that the points $(-4, -1)$, $(-2, -4)$, $(4, 0)$ and $(2, 3)$ are the vertices of a rectangle.
12. If three points $A(h, 0)$, $P(a, b)$ and $B(0, k)$ lie on a line, show that: $\frac{a}{h} + \frac{b}{k} = 1$. [NCERT]
13. The slope of a line is double of the slope of another line. If tangents of the angle between them is $\frac{1}{3}$, find the slopes of the other line. [NCERT]
14. Consider the following population and year graph:
    Find the slope of the line AB and using it, find what will be the population in the year 2010. [NCERT]

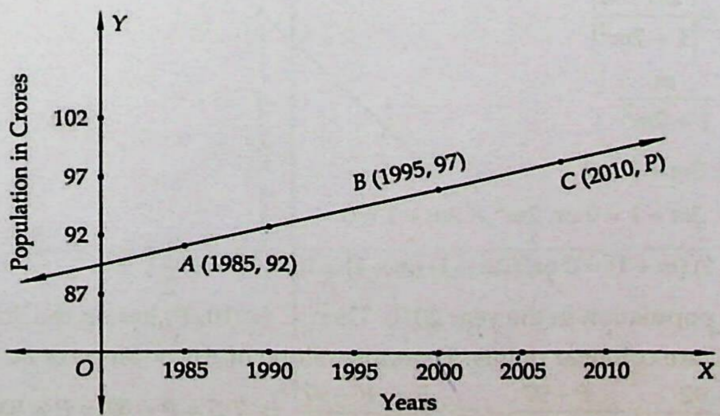

Fig. 23.10

15. Without using the distance formula, show that points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ are the vertices of a parallelogram.
16. Find the angle between the X-axis and the line joining the points $(3, -1)$ and $(4, -2)$. **[NCERT]**
17. Line through the points $(-2, 6)$ and $(4, 8)$ is perpendicular to the line through the points $(8, 12)$ and $(x, 24)$. Find the value of x. **[NCERT]**
18. Find the value of x for which the points $(x, -1)$, $(2, 1)$ and $(4, 5)$ are collinear. **[NCERT]**
19. Find the angle between X-axis and the line joining the points $(3, -1)$ and $(4, -2)$.
20. By using the concept of slope, show that the points $(-2, -1)$, $(4, 0)$, $(3, 3)$ and $(-3, 2)$ are the vertices of a parallelogram. **[NCERT]**
21. A quadrilateral has vertices $(4, 1)$, $(1, 7)$, $(-6, 0)$ and $(-1, -9)$. Show that the mid-points of the sides of this quadrilateral form a parallelogram.

ANSWERS

1. (i) -1 (ii) $-\sqrt{3}$ (iii) -1 (iv) $\sqrt{3}$
2. (i) $\frac{1}{2}$ (ii) $\frac{2}{t_2 + t_1}$ (iii) $-\frac{7}{2}$
3. (i) parallel (ii) parallel (iii) perpendicular (iv) neither.
4. (i) 1 (ii) $-\sqrt{3}$ 6. 9
7. (i) The line is either x-axis or it is parallel to x-axis.
   (ii) The line makes an acute angle with positive direction of x-axis.
   (iii) The line makes an obtuse angle with the positive direction of x-axis.

13. $1, \frac{1}{2}$ 14. 104.50 crores 16. 135° 17. 4 18. 1 19. $\frac{3\pi}{4}$

HINTS TO NCERT & SELECTED PROBLEMS

12. It is given that points $A(h, 0)$, $B(0, k)$ and $P(a, b)$ are collinear. Therefore,

Slope of PA = Slope of PB

$$\Rightarrow \quad \frac{b-0}{a-h} = \frac{b-k}{a-0}$$

$$\Rightarrow \quad ab = (a-h)(b-k) \Rightarrow ab = ab - ak - bh + hk \Rightarrow hk = ak + bh \Rightarrow \frac{a}{h} + \frac{b}{k} = 1$$

13. Let m be the slop of first line. Then the slope of the second line is $2m$. Let θ be the angle between the lines. Then,

$$\tan\theta = \left|\frac{2m - m}{1 + 2m^2}\right|$$

$$\Rightarrow \quad \frac{1}{3} = \pm\frac{m}{1 + 2m^2}$$

$$\Rightarrow \quad 2m^2 \pm 3m + 1 = 0$$

$$\Rightarrow \quad 2m^2 + 3m + 1 = 0 \text{ or, } 2m^2 - 3m + 1 = 0$$

$$\Rightarrow \quad (2m+1)(m+1) = 0 \text{ or, } (2m-1)(m-1) = 0 \Rightarrow m = \pm\frac{1}{2}, \pm 1.$$

14. Let P be the population in the year 2010. Then, $C(2010, P)$ lies on the line.

Since A, B, C are collinear points. Therefore, Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{97-92}{1995-1985} = \frac{P-97}{2010-1995} \Rightarrow \frac{5}{10} = \frac{P-97}{15} \Rightarrow 7.5 = P - 97 = P = 104.5$$

16. Let θ be the angle between the line joining the points $(3, -1)$ and $(4, -2)$ and x-axis. Then,

$$\tan\theta = \text{Slope of the line} \Rightarrow \tan\theta = \frac{-2+1}{4-3} \Rightarrow \tan\theta = -1 \Rightarrow \theta = 135°.$$

17. It is given that the line through the points $A(-2, 6)$ and $B(4, 8)$ is perpendicular to the line through the points $C(8, 12)$ and $D(x, 24)$. Therefore,

Slope of $AB \times$ Slope of $CD = -1$

$$\Rightarrow \quad \frac{8-6}{4+2} \times \frac{24-12}{x-8} = -1 \Rightarrow \frac{1}{3} \times \frac{12}{x-8} = -1 \Rightarrow -4 = x-8 \Rightarrow x = 4$$

18. It is given that points $A(x, -1)$, $B(2, 1)$ and $C(4, 5)$ are collinear.

$\therefore$ Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{1+1}{2-x} = \frac{5-1}{4-2} \Rightarrow \frac{2}{2-x} = 2 \Rightarrow 2-x = 1 \Rightarrow x = 1.$$

19. Let the required angle be θ. Then,

$$\tan\theta = \text{Slope of the line} \Rightarrow \tan\theta = \frac{-2-(-1)}{4-3} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

20. Given points are $A(-2, -1)$, $B(4, 0)$, $C(3, 3)$ and $D(-3, 2)$. Therefore,

$$m_1 = \text{Slope of } AB = \frac{0+1}{4+2} = \frac{1}{6}, \quad m_2 = \text{Slope of } CD = \frac{2-3}{-3-2} = \frac{1}{6}$$

$$m_3 = \text{Slope of } BC = \frac{3-0}{3-4} = -3 \text{ and, } m_4 = \text{Slope of } AD = \frac{2+1}{-3+2} = -3$$

Clearly, $m_1 = m_2$ and $m_3 = m_4$. Therefore, $AB \parallel CD$ and $BC \parallel AD$.

Hence, $ABCD$ is a parallelogram.

23.4 INTERCEPTS OF A LINE ON THE AXES

If a straight line cuts x-axis at A and the y-axis at B, then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.

The intercepts are positive or negative according as the line meets with positive or negative directions of the coordinate axes.

In Fig. 23.11, OA = x-intercept, OB = y-intercept.

OA is positive or negative according as A lies on OX or OX' respectively. Similarly, OB is positive or negative according as B lies on OY or OY' respectively.

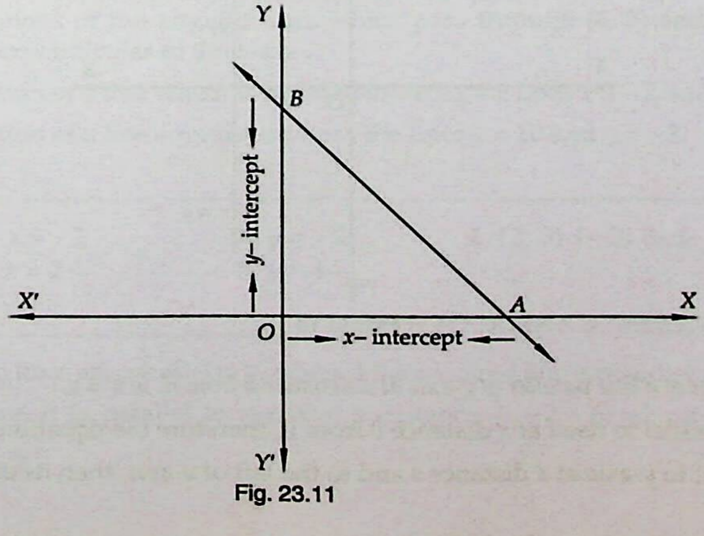

Fig. 23.11

23.5 EQUATIONS OF LINES PARALLEL TO THE COORDINATE AXES

23.5.1 EQUATION OF A LINE PARALLEL TO x-AXIS

Let AB be a straight line parallel to x-axis at a distance b from it. Then, clearly the ordinate of each point on AB is b. Thus, AB can be considered as the locus of a point at a distance b from x-axis. Thus, if $P(x, y)$ is any point on AB, then $y = b$. (See Fig. 23.12).

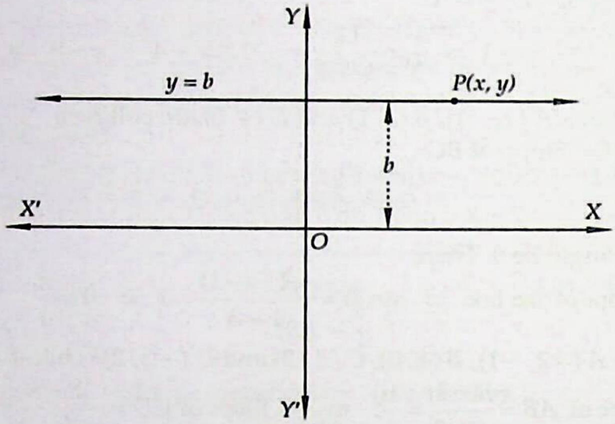

Fig. 23.12

Hence, the equation of a line parallel to x-axis at a distance b from it is $y = b$.

Since x-axis is a parallel to itself at a distance 0 from it, therefore the equation of x-axis is $y = 0$.

If a line is parallel to x-axis at a distance b and below x-axis, then its equation is $y = -b$.

23.5.2 EQUATION OF A LINE PARALLEL TO y-AXIS

Let AB be a line parallel to y-axis and at a distance a from it. Then, the abscissa of every point on AB is a. So it can be treated as the locus of a point a distance a from y-axis. Thus, if $P(x, y)$ is any point on AB, then $x = a$. (See Fig. 23.13)

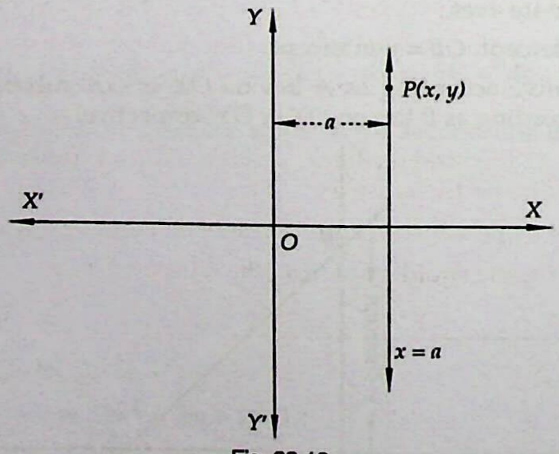

Fig. 23.13

Hence, the equation of a line parallel to y-axis at a distance a from it, is $x = a$.

Since y-axis is parallel to itself at a distance 0 from it, therefore the equation of y-axis is $x = 0$.

If a line is parallel to y-axis at a distance a and to the left of y-axis, then its equation is $x = -a$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Write down the equations of the following lines:*

(i) *x-axis* (ii) *y-axis* **[NCERT]**

(iii) *A line parallel to x-axis at a distance of 3 units below x-axis.*

(iv) *A line parallel to y-axis at a distance of 5 units on the left hand side of it.*

SOLUTION (i) $y = 0$ (ii) $x = 0$ (iii) $y = -3$ (iv) $x = -5$.

EXAMPLE 2 *Find the equation of a line which is parallel to x-axis and passes through (3, –5).*

SOLUTION The equation of a line parallel to x-axis is $y = b$. Since it passes through $(3, -5)$. So, $-5 = b \Rightarrow b = -5$. Hence, the equation of the required line is $y = -5$.

ALITER Since y-coordinate of every point on a line parallel to x-axis is always same, it follows that the equation of the required line is $y = -5$.

EXAMPLE 3 *Find the equation of a line which is parallel to y-axis and passes through (–4, 3).*

SOLUTION The equation of a line parallel to y-axis is $x = a$. Since, it passes through $(-4, 3)$. So $-4 = a \Rightarrow a = -4$. Hence, the equation of the required line is $x = -4$.

ALITER Since the abscissa of every point on a line parallel to *y-axis* is always same. So, the equation of the required line is $x = -4$.

EXAMPLE 4 *Find the equation of a line which is equidistant from the lines $x = -4$ and $x = 8$.*

SOLUTION Since the given lines are both parallel to y-axis and the required line is equidistant from these lines, so it is also parallel to y-axis and its distance from y-axis is $\frac{1}{2}(-4 + 8) = 2$ units.

Hence, its equation is $x = 2$.

EXERCISE 23.2

LEVEL-1

1. Find the equation of the line parallel to x-axis and passing through $(3, -5)$.
2. Find the equation of the line perpendicular to x-axis and having intercept -2 on x-axis.
3. Find the equation of the line parallel to x-axis and having intercept -2 on y-axis.
4. Draw the lines $x = -3$, $x = 2$, $y = -2$, $y = 3$ and write the coordinates of the vertices of the square so formed.
5. Find the equations of the straight lines which pass through $(4, 3)$ and are respectively parallel and perpendicular to the x-axis.
6. Find the equation of a line which is equidistant from the lines $x = -2$ and $x = 6$.
7. Find the equation of a line equidistant from the lines $y = 10$ and $y = -2$.

ANSWERS

1. $y = -5$ 2. $x = -2$ 3. $y = -2$ 4. $(2, 3), (-3, 3), (-3, -2), (2, -2)$

5. $y = 3, x = 4$ 6. $x = 2$ 7. $y = 4$

HINTS TO NCERT & SELECTED PROBLEMS

6. Since the given lines are parallel to y-axis and the required line is equidistant from the given lines. Therefore, it is parallel to y-axis at a distance $\frac{1}{2}(-2 + 6) = 2$ units from it. So its equation is $x = 2$.

23.6 DIFFERENT FORMS OF THE EQUATION OF A STRAIGHT LINE

In section 23.1, we have seen that a first degree equation in x, y represents a straight line. The equation of a straight line can be written in different forms depending on the data given. In this section, we shall learn about these forms.

23.6.1 SLOPE INTERCEPT FORM OF A LINE

THEOREM *The equation of a line with slope m and making an intercept c on y-axis is* $y = mx + c$.

PROOF Let the given line intersects y-axis at Q and makes an angle θ with x-axis. Then, $m = \tan\theta$. Let $P(x, y)$ be any point on the line. Draw PL perpendicular to x-axis and $QM \perp PL$.

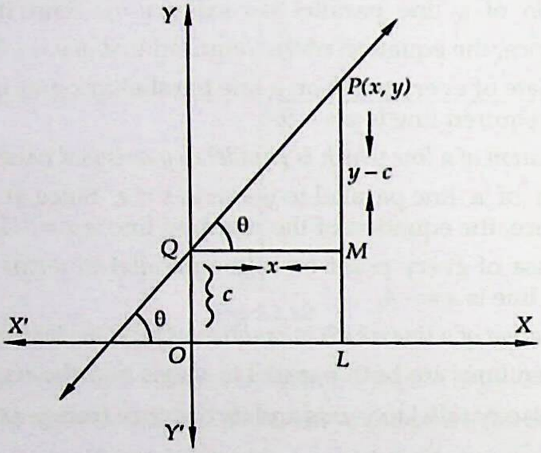

Fig. 23.14

Clearly, $\angle MQP = \theta$, $QM = OL = x$

and, $PM = PL - ML = PL - OQ = y - c$.

From triangle PMQ, we have

$$\tan\theta = \frac{PM}{QM} = \frac{y-c}{x}$$

$$\Rightarrow \quad m = \frac{y-c}{x}$$

$\Rightarrow \quad y = mx + c$, which is the required equation of the line. Q.E.D.

REMARK 1 *If the line passes through the origin, then* $0 = m0 + c \Rightarrow c = 0$. *Therefore, the equation of a line passing through the origin is* $y = mx$, *where m is the slope of the line.*

REMARK 2 *If the line is parallel to x-axis, then* $m = 0$, *therefore the equation of a line parallel to x-axis is* $y = c$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of a line with slope* -1 *and cutting off an intercept of 4 units on negative direction of y-axis.*

SOLUTION Here, $m = -1$ and $c = -4$

Substituting these values in $y = mx + c$, we obtain that the equation of the line is

$$y = -x - 4 \text{ or, } x + y + 4 = 0$$

EXAMPLE 2 *Find the equation of a straight line which cuts off an intercept of 5 units on negative direction of y-axis and makes an angle of* $120°$ *with the positive direction of x-axis.*

SOLUTION Here, $m = \tan 120° = \tan(90 + 30°) = -\cot 30° = -\sqrt{3}$ and $c = -5$.

Substituting these values in $y = mx + c$, we obtain that the equation of the line is

$$y = -\sqrt{3}x - 5 \quad \text{or,} \quad \sqrt{3}\,x + y + 5 = 0$$

EXAMPLE 3 *Find the equation of a straight line cutting off an intercept – 1 from y-axis and being equally inclined to the axes.*

SOLUTION Since the required line is equally inclined with the coordinate axes, therefore it makes either an angle of 45° or 135° with the x-axis.

So, its slope is either $m = \tan 45°$ or, $m = \tan 135°$ i.e. $m = 1$ or, -1. It is given that $c = -1$.

Substituting these values in $y = mx + c$, we obtain that the equations of the lines are $y = x - 1$ and $y = -x - 1$.

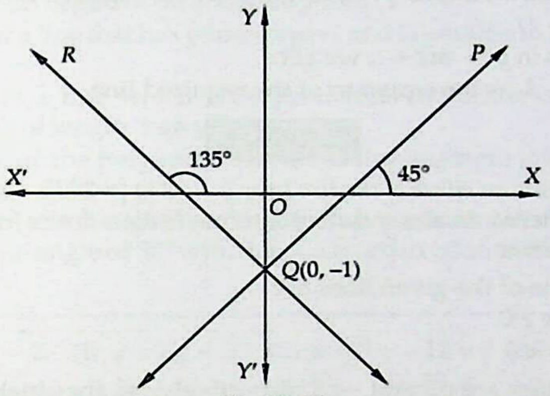

Fig. 23.15

EXAMPLE 4 *Find the equation of a straight line which makes an angle of* $\tan^{-1}\sqrt{2}$ *with the x-axis and cuts off an intercept of* $-\frac{3}{\sqrt{2}}$ *with the y-axis.*

SOLUTION Here, $m = \tan\theta = \tan(\tan^{-1}\sqrt{2}) = \sqrt{2}$ and $c = -\frac{3}{\sqrt{2}}$.

Substituting these values in $y = mx + c$, we obtain that the equation of the required line is

$$y = \sqrt{2}\,x - \frac{3}{\sqrt{2}} \quad \text{or,} \quad \sqrt{2}y = 2x - 3$$

EXAMPLE 5 *Find the equation of a straight line which cuts off an intercept of length 3 on y-axis and is parallel to the line joining the points* $(3, -2)$ *and* $(1, 4)$.

SOLUTION Let m be the slope of the required line. Since the required line is parallel to the line joining the points $A(3, -2)$ and $B(1, 4)$.

$$\therefore \quad m = \text{Slope of the line } AB = \frac{4 - (-2)}{1 - 3} = -3$$

It is given that $c = 3$. Substituting these values in $y = mx + c$, we obtain that the equation of the required line is $y = -3x + 3$ or, $3x + y - 3 = 0$.

EXAMPLE 6 *Find the equation of a line that has y-intercept 4 and is perpendicular to the line joining* $(2, -3)$ *and* $(4, 2)$.

SOLUTION Let m be the slope of the required line. Since the required line is perpendicular to the line joining $A(2, -3)$ and $B(4, 2)$.

$$\therefore \quad m \times \text{Slope of } AB = -1 \Rightarrow m \times \frac{2 + 3}{4 - 2} = -1 \Rightarrow m = -\frac{2}{5}$$

The required line cuts off an intercept of length 4 on y-axis. So, $c = 4$

Substituting these values in $y = mx + c$, we obtain that the equation of the required line is

$$y = -\frac{2}{5}x + 4 \quad \text{or,} \quad 2x + 5y - 20 = 0.$$

EXAMPLE 7 *Find the equation of the straight line which makes angle of 15° with the positive direction of x-axis and which cuts an intercept of length 4 on the negative direction of Y-axis.*

SOLUTION Let m be the slope of the line. Then,

$$m = \tan 15° = \tan (45° - 30°) = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$\Rightarrow \quad m = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

It is given that the line cuts an intercept of length 4 on the negative direction of y-axis.

$\therefore \quad c = -4$

Substituting these values in $y = mx + c$, we get

$y = (2 - \sqrt{3})\, x - 4$ as the equation of the required line.

LEVEL-2

EXAMPLE 8 *P_1, P_2 are points on either of the two lines $y - \sqrt{3}\,|x| = 2$ at a distance of 5 units from their point of intersection. Find the coordinates of the foot of perpendiculars drawn from P_1, P_2 on the bisector of the angle between the given lines.* **[NCERT EXEMPLAR]**

SOLUTION The equations of the given lines are

$\therefore \quad y - \sqrt{3}x = 2$ for $x \geq 0$ ... (i)

and, $\quad y + \sqrt{3}x = 2$ for $x < 0$ ... (ii)

The slopes of these two lines are $\sqrt{3}$ and $-\sqrt{3}$ respectively. So, they make angles of 60° and 120° respectively with x-axis. Consequently, each makes 30° angles with the positive direction of y-axis as shown in Fig. 23.16.

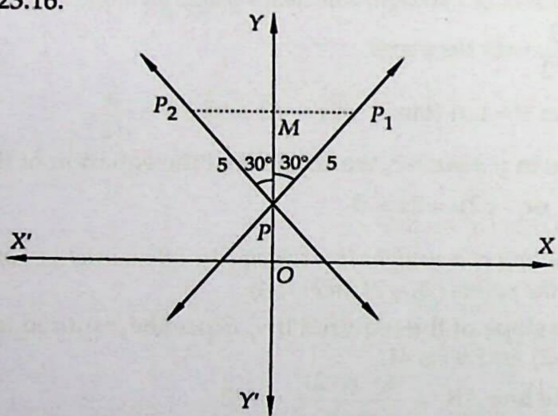

Fig. 23.16

Clearly, lines (i) and (ii) intersect at P (0, 2) and y-axis is the bisector of the acute angle between them. It is given that $PP_1 = 5 = PP_2$.

Let M be the foot of the perpendiculars drawn from P_1 and P_2 on y-axis.

In right triange PMP_1, we have

$$\cos 30° = \frac{PM}{PP_1} \Rightarrow \frac{\sqrt{3}}{2} = \frac{PM}{5} \Rightarrow PM = \frac{5\sqrt{3}}{2}$$

$$\therefore \quad OM = OP + PM = 2 + \frac{5\sqrt{3}}{2} = \frac{4 + 5\sqrt{3}}{2}$$

Hence, the coordinates of M are $\left(\frac{4 + 5\sqrt{3}}{2}, 0\right)$.

EXERCISE 23.3

LEVEL-1

1. Find the equation of a line making an angle of 150° with the x-axis and cutting off an intercept 2 from y-axis.
2. Find the equation of a straight line:
   (i) with slope 2 and y-intercept 3; (ii) with slope $-1/3$ and y-intercept -4.
   (iii) with slope -2 and intersecting the x-axis at a distance of 3 units to the left of origin.
3. Find the equations of the bisectors of the angles between the coordinate axes.
4. Find the equation of a line which makes an angle of $\tan^{-1}(3)$ with the x-axis and cuts off an intercept of 4 units on negative direction of y-axis.
5. Find the equation of a line that has y-intercept -4 and is parallel to the line joining $(2, -5)$ and $(1, 2)$.
6. Find the equation of a line which is perpendicular to the line joining (4, 2) and (3, 5) and cuts off an intercept of length 3 on y-axis.
7. Find the equation of the perpendicular to the line segment joining (4, 3) and $(-1, 1)$ if it cuts off an intercept -3 from y-axis.
8. Find the equation of the straight line intersecting y-axis at a distance of 2 units above the origin and making an angle of 30° with the positive direction of the x-axis. [NCERT]

ANSWERS

1. $x+\sqrt{3}\,y=2\sqrt{3}$ 2. (i) $y=2x+3$ (ii) $x+3y+12=0$ (iii) $2x+y+6=0$
3. $x \pm y=0$ 4. $y=3x-4$ 5. $7x+y+4=0$ 6. $x-3y+9=0$
7. $5x+2y+6=0$ 8. $x-\sqrt{3}y+2\sqrt{3}=0$

HINTS TO NCERT & SELECTED PROBLEM

8. Clearly, $c = y$-intercept $= 2$ and, $m = \tan 30° = \dfrac{1}{\sqrt{3}}$.

   So, the equation of the line is $y=\dfrac{1}{\sqrt{3}}x+2$ or, $x-\sqrt{3}y+2\sqrt{3}=0$.

23.6.2 POINT-SLOPE FORM OF A LINE

THEOREM *The equation of a line which passes through the point (x_1, y_1) and has the slope 'm' is* $y-y_1=m(x-x_1)$.

PROOF Let the line pass through the point $Q(x_1, y_1)$ and let $P(x, y)$ be any point on the line. Then,

Slope of the line is $=\dfrac{y-y_1}{x-x_1}$

But, the slope of the line is m.

$$\therefore \quad m=\frac{y-y_1}{x-x_1} \Rightarrow y-y_1=m(x-x_1)$$

Hence, $y-y_1=m(x-x_1)$ is the required equation of the line. Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of a line passing through $(2, -3)$ and inclined at an angle of 135° with the positive direction of x-axis.*

SOLUTION Here, m = Slope of the line $= \tan 135° = \tan(90°+45°) = -\cot 45° = -1$.

$x_1 = 2, y_1 = -3$

So, the equation of the line is $y - y_1 = m(x - x_1)$

i.e. $y - (-3) = -1(x-2) \Rightarrow y + 3 = -x + 2 \Rightarrow x + y + 1 = 0.$

EXAMPLE 2 *Determine the equation of line through the point $(-4, -3)$ and parallel to x-axis.*

SOLUTION Here, m = Slope = 0, $x_1 = -4$, $y_1 = -3$.

So, the equation of the line is $y - y_1 = m(x - x_1)$

or, $y + 3 = 0(x + 4) \Rightarrow y + 3 = 0.$

EXAMPLE 3 *Find the equation of the line passing through (1, 2) and making angle of 30° with y-axis.* **[NCERT EXEMPLAR]**

SOLUTION The required line makes 30° with the positive direction of y-axis as shown in Fig. 23.17. So, it makes 60° with the positive direction of x-axis. Therefore, its slope m is given by $m = \tan 60° = \sqrt{3}$.

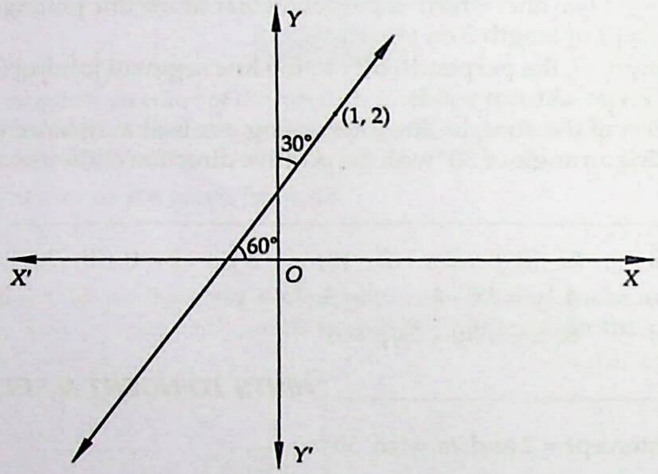

Fig. 23.17

Thus, the required line passes through (1, 2) and has $\sqrt{3}$ as its slope.
Hence, its equation is

$$y - 2 = \sqrt{3}(x - 1) \text{ or } \sqrt{3}x - y + 2 - \sqrt{3} = 0$$

EXAMPLE 4 *Find the equation of the perpendicular bisector of the line segment joining the points A (2, 3) and B (6, −5).*

SOLUTION The slope of AB is given by

$$m = \frac{-5-3}{6-2} = -2 \qquad \left[\text{Using: } m = \frac{y_2 - y_1}{x_2 - x_1}\right]$$

$\therefore$ Slope of a line perpendicular to $AB = -\frac{1}{m} = \frac{1}{2}$

Let P be the mid-point of AB. Then, the coordinates of P are $\left(\frac{2+6}{2}, \frac{3-5}{2}\right)$ i.e. $(4, -1)$.

Thus, the required line passes through $P(4, -1)$ and has slope $\frac{1}{2}$. So its equation is

$$y + 1 = \frac{1}{2}(x - 4) \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad x - 2y - 6 = 0.$$

EXAMPLE 5 *Find the equation of the line for which* $\tan\theta = \frac{1}{2}$, *where* θ *is the inclination of the line and*

(i) x-intercept equal to 4. (ii) y-intercept is $-\frac{3}{2}$. **[NCERT]**

SOLUTION (i) Clearly, the line passes through (4, 0) and has slope $= \frac{1}{2}$.

So, the equation of the line is

$$y - 0 = \frac{1}{2}(x-4) \qquad \left[\text{Putting } x_1 = 4, y_1 = 0 \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1)\right]$$

$$\Rightarrow \quad x - 2y - 4 = 0$$

(ii) The line passes through $\left(0, -\frac{3}{2}\right)$ and has slope $= \frac{1}{2}$.

So, it equation is

$$y - \left(-\frac{3}{2}\right) = \frac{1}{2}(x-0) \qquad \left[\text{Putting } x_1 = 0, y_1 = -\frac{3}{2} \text{ and } m = \frac{1}{2} \text{ in } y - y_1 = m(x - x_1)\right]$$

$$\Rightarrow \quad 2y + 3 = x \text{ or, } x - 2y - 3 = 0$$

EXAMPLE 6 *The perpendicular from the origin to a line meets it at the point* $(-2, 9)$, *find the equation of the line.* **[NCERT]**

SOLUTION We have,

$$m_1 = \text{Slope of } OP = \frac{9-0}{-2-0} = -\frac{9}{2}$$

Let m be the slope of the line AB. Then,

Slope of $AB \times$ Slope of $OP = -1$

$$\Rightarrow \quad m \times -\frac{9}{2} = -1$$

$$\Rightarrow \quad m = \frac{2}{9}$$

The equation of AB is

$$y - 9 = \frac{2}{9}\{x - (-2)\}$$

$$\Rightarrow \quad 9y - 81 = 2x + 4$$

$$\Rightarrow \quad 2x - 9y + 85 = 0$$

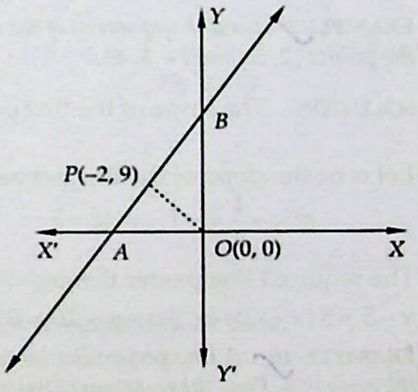

Fig. 23.18

EXAMPLE 7 *Find the equation of the line passing through the point (0, 2) making an angle* $\frac{2\pi}{3}$ *with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.* **[NCERT]**

SOLUTION The equation of the line passing through (0, 2) and making an angle $\frac{2\pi}{3}$ with the positive x-axis is

$$y - 2 = \tan\frac{2\pi}{3}(x - 0) \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad y - 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y - 2 = 0$$

A line parallel to this line crosses y-axis at a distance of 2 units below the origin. So, it passes through $(0, -2)$ and makes an angle $\frac{2\pi}{3}$ with the x-axis. Hence, its equation is

$$y + 2 = \tan\frac{2\pi}{3}(x - 0) \Rightarrow y + 2 = -\sqrt{3}x \Rightarrow \sqrt{3}x + y + 2 = 0$$

EXAMPLE 8 *Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2, find the equation of the other line.* **[NCERT]**

SOLUTION Let the slope of the other line be m. It is given that the angle between the two lines is 60°.

$$\therefore \quad \tan 60^\circ = \left|\frac{m-2}{1+2m}\right|$$

$$\Rightarrow \quad \sqrt{3} = \left|\frac{m-2}{1+2m}\right|$$

$$\Rightarrow \quad \frac{m-2}{1+2m} = \pm\sqrt{3}$$

$$\Rightarrow \quad m - 2 = \pm\sqrt{3} \pm 2\sqrt{3}m$$

$$\Rightarrow \quad m(1 + -2\sqrt{3}) = 2 \pm \sqrt{3}$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{3}}{1 + -2\sqrt{3}} \Rightarrow m = \frac{2+\sqrt{3}}{1-2\sqrt{3}}, \frac{2-\sqrt{3}}{1+2\sqrt{3}} \Rightarrow m = -\frac{2+\sqrt{3}}{2\sqrt{3}-1}, \frac{2-\sqrt{3}}{2\sqrt{3}+1}$$

Substituting $x_1 = 2$, $y_1 = 3$ and the values of m in $y - y_1 = m(x - x_1)$, we obtain that the equations of the required lines are

$$y - 3 = -\frac{2+\sqrt{3}}{2\sqrt{3}-1}(x-2) \text{ and } y - 3 = \frac{2-\sqrt{3}}{2\sqrt{3}+1}(x-2)$$

EXAMPLE 9 *Find the equation of the line passing through (– 3, 5) and perpendicular to the line through the points (2, 5) and (– 3, 6).* **[NCERT]**

SOLUTION The slope of the line passing through (2, 5) and (– 3, 6) is $\frac{6-5}{-3-2} = \frac{-1}{5}$

Let m be the slope of the line perpendicular to the line passing through (2, 5) and (–3, 6). Then,

$$m \times -\frac{1}{5} = -1 \Rightarrow m = 5$$

The required line passes through (– 3, 5) and has slope $m = 5$. So, its equation is

$y - 5 = 5(x + 3)$ or, $5x - y + 20 = 0$

EXAMPLE 10 *A line perpendicular to the line segment joining the points* (1, 0) *and* (2, 3) *divides it in the ratio* 1 : *n*. *Find the equation of the line.* **[NCERT]**

SOLUTION The slope of the line joining $A(1, 0)$ and $B(2, 3)$ is $\frac{3-0}{2-1} = 3$ and the coordinates of the point dividing it in the ratio $1 : n$ are $\left(\frac{n+2}{n+1}, \frac{3}{n+1}\right)$. The slope of the line perpendicular to the line segment AB is $-\frac{1}{3}$.

Hence, the equation of the required line is

$$y - \frac{3}{n+1} = -\frac{1}{3}\left(x - \frac{n+2}{n+1}\right) \text{ or, } (n+1)x + 3(n+1)y = n + 11$$

EXAMPLE 11 *Find the equation of a line which divides the join of (1, 0) and (3, 0) in the ratio 2 : 1 and perpendicular to it.*

SOLUTION Let C be the point which divides the join of $A(1, 0)$ and $B(3, 0)$ in the ratio 2 : 1. Then, the coordinates of C are

$$\left(\frac{2\times 3+1\times 1}{2+1}, \frac{2\times 0+1\times 0}{2+1}\right) = \left(\frac{7}{3}, 0\right)$$

Since AB is along x-axis, therefore a line perpendicular to AB is parallel to y-axis. As it passes through $C\ (7/3, 0)$, therefore its equation is

$$x = \frac{7}{3} \Rightarrow 3x = 7$$

Hence, the equation of the required line is $3x = 7$.

EXAMPLE 12 *The vertices of a triangle are A (10, 4), B (− 4, 9) and C (− 2, − 1). Find the equation of its altitudes. Also, find its orthocentre.*

SOLUTION Let AD, BE and CF be three altitudes of ΔABC. Clearly, $AD \perp BC$, $BE \perp CA$ and $CF \perp AB$.

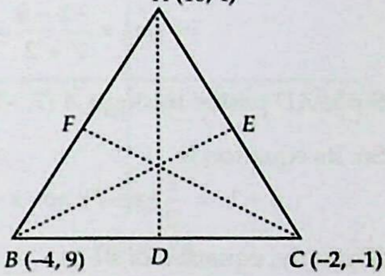

Fig. 23.19

We have,

$$\text{Slope of } BC = \frac{-1-9}{-2+4} = -5$$

$$\therefore \quad \text{Slope of } AD = \frac{1}{5} \qquad [\because AD \perp BC]$$

Since AD passes through A (10, 4). Therefore, equation of AD is

$$y - 4 = \frac{1}{5}(x - 10) \Rightarrow x - 5y + 10 = 0$$

...(i)

$$\text{Slope of } AC = \frac{4+1}{10+2} = \frac{5}{12}$$

$$\therefore \quad \text{Slope of } BE = -\frac{12}{5} \qquad [\because BE \perp AC]$$

Clearly, BE passes through $B\,(-4, 9)$ and has slope $-12/5$. So, the equation of BE is

$$y - 9 = -\frac{12}{5}(x + 4) \Rightarrow 12x + 5y + 3 = 0 \qquad \text{...(ii)}$$

$$\text{Slope of } AB = \frac{4-9}{10+4} = -\frac{5}{14} \Rightarrow \text{Slope of } CF = \frac{14}{5} \qquad [\because CF \perp AB]$$

Clearly, CF passes through $C\,(-2, -1)$ and has slope $\frac{14}{5}$. So, the equation of CF is

$$y + 1 = \frac{14}{5}(x + 2) \Rightarrow 14x - 5y + 23 = 0 \qquad \text{...(iii)}$$

Thus, the altitudes of ΔABC are

$$x - 5y + 10 = 0, 12x + 5y + 3 = 0 \text{ and, } 14x - 5y + 23 = 0.$$

The orthocentre of ΔABC is the point of intersection of its altitudes.

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{-65} = \frac{y}{117} = \frac{1}{65} \Rightarrow x = -1,\ y = \frac{9}{5}$$

Hence, the coordinates of the orthocentre are $(-1, 9/5)$.

EXAMPLE 13 *Find the equations of the altitudes of the triangle whose vertices are* $A(7,-1)$, $B(-2,8)$ *and* $C(1,2)$.

SOLUTION Let AD, BE and CF be three altitudes of triangle ABC. Let m_1, m_2 and m_3 be the slopes of AD, BE and CF respectively. Then,

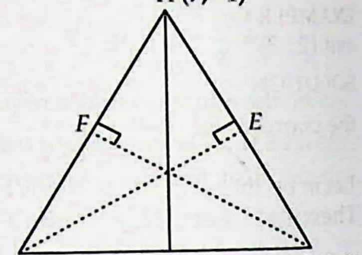

Fig. 23.20

$$AD \perp BC \Rightarrow \text{Slope of } AD \times \text{Slope of } BC = -1$$

$$\Rightarrow m_1 \times \left(\frac{2-8}{1+2}\right) = -1 \Rightarrow m_1 = \frac{1}{2}$$

$$BE \perp AC \Rightarrow \text{Slope of } BE \times \text{Slope of } AC = -1$$

$$\Rightarrow m_2 \times \left(\frac{-1-2}{7-1}\right) = -1 \Rightarrow m_2 = 2$$

and, $CF \perp AB \Rightarrow \text{Slope of } CF \times \text{Slope of } AB = -1$

$$\Rightarrow m_3 \times \frac{-1-8}{7+2} = -1 \Rightarrow m_3 = 1.$$

Since AD passes through $A(7,-1)$ and has slope $m_1 = \frac{1}{2}$.

So, its equation is

$$y+1 = \frac{1}{2}(x-7) \Rightarrow x-2y-9 = 0$$

Similarly, equation of BE is

$$y-8 = 2(x+2) \Rightarrow 2x-y+12 = 0$$

Equation of CF is $y-2 = 1(x-1) \Rightarrow x-y+1 = 0$

EXAMPLE 14 *The mid-points of the sides of a triangle are* $(2,1)$, $(-5,7)$ *and* $(-5,-5)$. *Find the equations of the sides of the triangle.*

SOLUTION Let $D(2,1)$, $E(-5,7)$ and $F(-5,-5)$ be the mid-points of sides BC, CA and AB respectively of $\Delta\, ABC$.

We know that the line joining the mid-points of two sides of a triangle is parallel to the third two side.

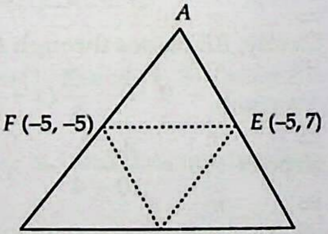

Fig. 23.21

$\therefore$ $DE \parallel AB$, $EF \parallel BC$ and $DF \parallel AC$

$\therefore$ Slope of AB = Slope of DE

Slope of BC = Slope of EF and, Slope of AC = Slope of DF

Let m_1, m_2 and m_3 be the slopes of AB, BC and CA respectively. Then,

$$m_1 = \text{Slope of } AB = \text{Slope of } DE = \frac{7-1}{-5-2} = \frac{-6}{7}$$

$$m_2 = \text{Slope of } BC = \text{Slope of } EF = \frac{7+5}{-5+5} \text{ (Undefined)}$$

$$m_3 = \text{Slope of } CA = \text{Slope of } DF = \frac{1+5}{2+5} = \frac{6}{7}$$

Side AB passes through $F(-5,-5)$ and has slope $m_1 = \frac{-6}{7}$. So, its equation is

$$y+5 = \frac{-6}{7}(x+5) \Rightarrow 6x+7y+65 = 0$$

Side BC is parallel to Y-axis and passes through $D(2,1)$. So, its equation is $x = k$. As it passes through $(2,1)$.

$\therefore$ $2 = k$.

Hence, equation of BC is $x = 2$.

Side CA passes through $E(-5, 7)$ and has slope $m_3 = \frac{6}{7}$. So, its equation is

$$y - 7 = \frac{6}{7}(x+5) \Rightarrow 6x - 7y + 79 = 0$$

EXAMPLE 15 *Find the equation of the perpendicular bisector of the line segment joining the points* (1, 1) *and* (2, 3). **[NCERT]**

SOLUTION Let P be the mid-point of the line segment joining points $A(1, 1)$ and $B(2, 3)$. Then, the coordinates of P are $\left(\frac{3}{2}, 2\right)$.

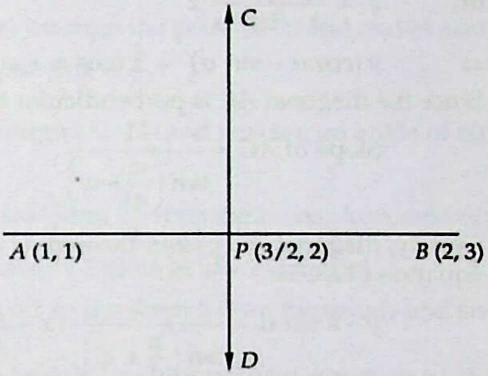

Fig. 23.22

Let m be the slope of perpendicular bisector of AB. Then,

$$m \times \text{Slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{3-1}{2-1} = -1$$

$$\Rightarrow \quad m = \frac{-1}{2}$$

Clearly, perpendicular bisector of AB passes through $P\left(\frac{3}{2}, 2\right)$ and has slope $m = -\frac{1}{2}$. So, its equation is

$$y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right) \text{ or, } 2x + 4y - 11 = 0.$$

EXAMPLE 16 *Show that the perpendicular drawn from the point* (4, 1) *on the line segment joining* (6, 5) *and* (2, −1) *divides it internally in the ratio* 8 : 5.

SOLUTION Suppose perpendicular drawn from $P(4, 1)$ on the line joining $A(6, 5)$ and $B(2, -1)$ meets AB at M. Let m be the slope of PM. Then,

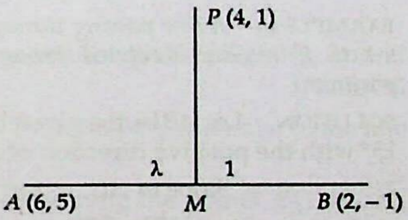

Fig. 23.23

$$PM \perp AB$$

$$\Rightarrow \quad m \times \text{Slope of } AB = -1$$

$$\Rightarrow \quad m \times \frac{-1-5}{2-6} = -1$$

$$\Rightarrow \quad m \times \frac{3}{2} = -1$$

$$\Rightarrow \quad m = -\frac{2}{3}$$

Clearly, PM passes through $P(4, 1)$ and has slope $m = -\frac{2}{3}$. So, its equation is

$$y - 1 = -\frac{2}{3}(x-4) \text{ or, } 2x + 3y - 11 = 0 \qquad \text{...(i)}$$

Suppose M divides line segment AB in the ratio $\lambda : 1$. Then, coordinates of M are

$$\left(\frac{2\lambda+6}{\lambda+1}, \frac{-\lambda+5}{\lambda+1}\right)$$

Since M lies on line PM whose equation is $2x + 3y - 11 = 0$

$$\therefore \quad 2\left(\frac{2\lambda+6}{\lambda+1}\right) + 3\left(\frac{-\lambda+5}{\lambda+1}\right) - 11 = 0$$

$$\Rightarrow \quad 4\lambda + 12 - 3\lambda + 15 - 11\lambda - 11 = 0 \Rightarrow -10\lambda + 16 = 0 \Rightarrow \lambda = \frac{8}{5}$$

Hence, M divides AB internally in the ratio 8 : 5.

LEVEL-2

EXAMPLE 17 *One side of a square makes an angle α with x-axis and one vertex of the square is at the origin. Prove that the equations of its diagonals are $x(\sin\alpha + \cos\alpha) = y(\cos\alpha - \sin\alpha)$ and $x(\cos\alpha - \sin\alpha) + y(\sin\alpha + \cos\alpha) = a$, where a is the length of the side of the square.*

SOLUTION Let $OABC$ be the square such that its side OA makes an angle α with x-axis. Since $OA = a$, therefore coordinates of A are $(a\cos\alpha, a\sin\alpha)$. Clearly, the diagonal OB makes an angle $(\pi/4 + \alpha)$ with x-axis and passes through $(0, 0)$. So, equation of OB is

$$y - 0 = \tan(\pi/4 + \alpha)(x - 0)$$

or, $$y = \frac{1 + \tan\alpha}{1 - \tan\alpha}x$$

$$\Rightarrow \quad y(\cos\alpha - \sin\alpha) = x(\cos\alpha + \sin\alpha) \qquad \text{...(i)}$$

Since the diagonal AC is perpendicular to OB. Therefore,

$$\text{Slope of } AC = \frac{-1}{\tan\left(\frac{\pi}{4} + \alpha\right)}.$$

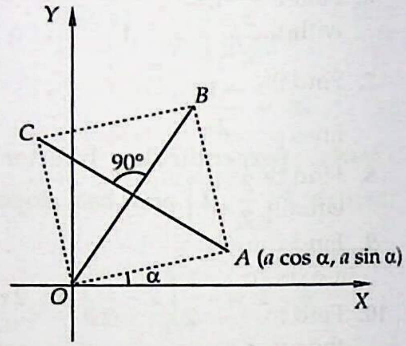

Fig. 23.24

Clearly, diagonal AC passes through $(a\cos\alpha, a\sin\alpha)$. So, equation of AC is

$$y - a\sin\alpha = \frac{-1}{\tan\left(\frac{\pi}{4} + \alpha\right)}(x - a\cos\alpha)$$

$$\Rightarrow \quad y - a\sin\alpha = -\frac{1 - \tan\alpha}{1 + \tan\alpha}(x - a\cos\alpha)$$

$$\Rightarrow \quad y - a\sin\alpha = -\left(\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}\right)(x - a\cos\alpha)$$

$$\Rightarrow \quad x(\cos\alpha - \sin\alpha) + y(\cos\alpha + \sin\alpha) = a.$$

EXAMPLE 18 *A line passing through the point $A(3, 0)$ makes $30°$ angle with the positive direction of x-axis. If this line is rotated through an angle of $15°$ in clockwise direction, find its equation in new position.* **[NCERT EXEMPLAR]**

SOLUTION Let AB be the given line and AC be its new position. Clearly, AC makes an angle of $15°$ with the positive direction of X-axis.

$$\therefore \quad m = \text{Slope of } AC = \tan 15°$$

$$\Rightarrow \quad m = \tan(45° - 30°)$$

$$\Rightarrow \quad m = \frac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$$

$$\Rightarrow \quad m = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$\Rightarrow \quad m = \frac{(\sqrt{3} - 1)^2}{(\sqrt{3} + 1)(\sqrt{3} - 1)}$$

$$= \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

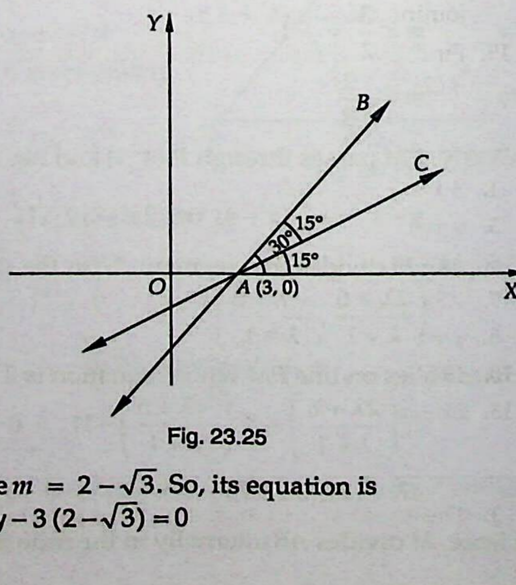

Fig. 23.25

Clearly, AC passes through $A(3, 0)$ and has slope $m = 2 - \sqrt{3}$. So, its equation is

$$y - 0 = (2 - \sqrt{3})(x - 3) \text{ or, } (2 - \sqrt{3})x - y - 3(2 - \sqrt{3}) = 0$$

EXERCISE 23.4

LEVEL-1

1. Find the equation of the straight line passing through the point (6, 2) and having slope – 3.
2. Find the equation of the straight line passing through (– 2, 3) and inclined at an angle of 45° with the x-axis.
3. Find the equation of the line passing through (0, 0) with slope m. **[NCERT]**
4. Find the equation of the line passing through $(2, 2\sqrt{3})$ and inclined with x-axis at an angle of 75°. **[NCERT]**
5. Find the equation of the straight line which passes through the point (1, 2) and makes such an angle with the positive direction of x-axis whose sine is $\frac{3}{5}$.
6. Find the equation of the straight line passing through (3, – 2) and making an angle of 60° with the positive direction of y-axis.
7. Find the lines through the point (0, 2) making angles $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with the x-axis. Also, find the lines parallel to them cutting the y-axis at a distance of 2 units below the origin.
8. Find the equations of the straight lines which cut off an intercept 5 from the y-axis and are equally inclined to the axes.
9. Find the equation of the line which intercepts a length 2 on the positive direction of the x-axis and is inclined at an angle of 135° with the positive direction of y-axis.
10. Find the equation of the straight line which divides the join of the points (2, 3) and (–5, 8) in the ratio 3 : 4 and is also perpendicular to it.
11. Prove that the perpendicular drawn from the point (4, 1) on the join of (2, –1) and (6, 5) divides it in the ratio 5 : 8.
12. Find the equations to the altitudes of the triangle whose angular points are A (2, – 2), B (1, 1) and C (–1, 0).
13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (– 1, 2). **[NCERT]**
14. Find the equation of the line passing through the point (– 3, 5) and perpendicular to the line joining (2, 5) and (– 3, 6).
15. Find the equation of the right bisector of the line segment joining the points A (1, 0) and B (2, 3).

ANSWERS

1. $3x + y - 20 = 0$
2. $x - y + 5 = 0$
3. $y = mx$
4. $(2 + \sqrt{3})x - y - 4 = 0$
5. $3x - 4y + 5 = 0$
6. $x - \sqrt{3}y - 3 - 2\sqrt{3} = 0$
7. $\sqrt{3}x - y + 2 = 0$, $\sqrt{3}x + y - 2 = 0$, $\sqrt{3}x + y + 2 = 0$
8. $y = x + 5$ or $x + y = 5$
9. $x - y - 2 = 0$
10. $49x - 35y + 229 = 0$
12. $2x + y - 2 = 0$, $3x - 2y - 1 = 0$, $x - 3y + 1 = 0$
13. $2x + y = 5$
14. $5x - y + 20 = 0$
15. $x + 3y - 6 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

3. The equation of the line passing through (0, 0) and slope m is $y - 0 = m(x - 0)$ or, $y = mx$.

4. The equation of the required line is

$$y - 2\sqrt{3} = \tan 75^\circ (x - 2)$$

or, $$y - 2\sqrt{3} = \frac{\sqrt{3}+1}{\sqrt{3}-1}(x - 2) \qquad \left[\because \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}\right]$$

or, $$y - 2\sqrt{3} = (2 + \sqrt{3})(x - 2)$$

13. The right bisector of the segment joining $A(3, 4)$ and $B(-1, 2)$ passes through the mid-point $C(1, 3)$ of AB and is perpendicular to AB. Let m be the slope of AB. Then,

$$m = \frac{2-4}{-1-3} = \frac{1}{2}$$

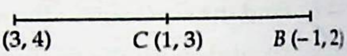

Fig. 23.25

So, m_1 = Slope of a line perpendicular to $AB = -2$.

Hence, the equation of the right bisector of AB is $y - 3 = -2(x - 1)$ or, $2x + y - 5 = 0$.

23.6.3 TWO-POINT FORM OF A LINE

THEOREM *The equation of a line passing through two points* (x_1, y_1) and (x_2, y_2) *is*

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$

PROOF Let m be the slope of the line passing through (x_1, y_1) and (x_2, y_2). Then,

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the line is

$$y - y_1 = m(x - x_1) \qquad \text{[Using point-slope form]}$$

Substituting the value of m, we obtain

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This is the required equation of the line in two point form. Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line joining the points* $(-1, 3)$ *and* $(4, -2)$.

SOLUTION Here, the two points are $(x_1, y_1) \equiv (-1, 3)$ and $(x_2, y_2) \equiv (4, -2)$.

So, the equation of the line in two-point form is

$$y - 3 = \frac{3 - (-2)}{-1 - 4}(x + 1) \Rightarrow y - 3 = -x - 1 \Rightarrow x + y - 2 = 0.$$

EXAMPLE 2 *Find the equation of the line joining the points* $(at_1^2, 2at_1)$ *and* $(at_2^2, 2at_2)$.

SOLUTION Here, $x_1 = at_1^2,\ y_1 = 2at_1,\ x_2 = at_2^2,\ y_2 = 2at_2$.

So, the equation of the required line is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\Rightarrow \quad y(t_1 + t_2) - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$

$$\Rightarrow \quad y(t_1 + t_2) = 2x + 2at_1t_2.$$

EXAMPLE 3 *Find the equations of the medians of the triangle ABC whose vertices are A (2, 5), B (− 4, 9) and C (− 2, − 1).*

SOLUTION Let D, E, F be the mid-points of BC, CA and AB respectively. Then, the coordinates of these points are $D(-3, 4)$, $E(0, 2)$ and $F(-1, 7)$ respectively. The median AD passes through points $A(2, 5)$ and $D(-3, 4)$.

So, equation of AD is

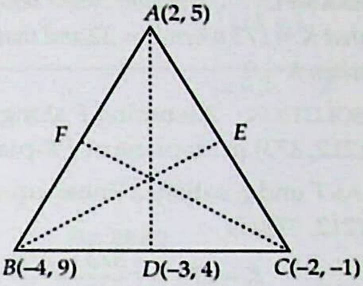

Fig. 23.26

$$y - 5 = \frac{4-5}{-3-2}(x-2)$$

$$\Rightarrow \quad y - 5 = \frac{1}{5}(x-2)$$

$$\Rightarrow \quad x - 5y + 23 = 0$$

The median BE passes through points $B(-4, 9)$ and $E(0, 2)$.

So, equation of median BE is

$$(y-9) = \left(\frac{2-9}{0+4}\right)(x+4) \Rightarrow 7x + 4y - 8 = 0.$$

Similarly, the equation of the median CF is

$$(y+1) = \frac{7+1}{-1+2}(x+2) \Rightarrow 8x - y + 15 = 0.$$

EXAMPLE 4 *In what ratio is the line joining the points (2, 3) and (4, 1) divides the segment joining the points (1, 2) and (4, 3)?*

SOLUTION The equation of the line joining the points (2, 3) and (4, 1) is

$$y - 3 = \frac{1-3}{4-2}(x-2) \Rightarrow y - 3 = -x + 2 \Rightarrow x + y - 5 = 0 \qquad \text{...(i)}$$

Suppose the line joining (2, 3) and (4, 1) divides the segment joining (1, 2) and (4, 3) at point P in the ratio $\lambda : 1$. Then, the coordinates of P are $\left(\frac{4\lambda+1}{\lambda+1}, \frac{3\lambda+2}{\lambda+1}\right)$.

Clearly P lies on line (i).

$$\therefore \quad \frac{4\lambda+1}{\lambda+1} + \frac{3\lambda+2}{\lambda+1} - 5 = 0 \Rightarrow \lambda = 1.$$

Hence, the required ratio is $\lambda : 1$ i.e., 1 : 1.

EXAMPLE 5 *In what ratio, the line joining (− 1, 1) and (5, 7) is divided by the line $x + y = 4$?*

SOLUTION Suppose the line $x + y = 4$ divides the join of $A(-1, 1)$ and $B(5, 7)$ in the ratio $\lambda : 1$. The coordinates of the point of division are $\left(\frac{5\lambda+1}{\lambda+1}, \frac{7\lambda+1}{\lambda+1}\right)$. It lies on $x + y = 4$.

$$\therefore \quad \frac{5\lambda-1}{\lambda+1} + \frac{7\lambda+1}{\lambda+1} = 4 \Rightarrow 5\lambda - 1 + 7\lambda + 1 = 4(\lambda+1) \Rightarrow 12\lambda = 4\lambda + 4 \Rightarrow 8\lambda = 4 \Rightarrow \lambda = \frac{1}{2}$$

Hence, the required ratio are 1 : 2.

EXAMPLE 6 *Prove that the points (5, 1), (1, − 1) and (11, 4) are collinear. Also ,find the equation of the straight line on which these points lie.*

SOLUTION Let the given points be $A(5, 1)$, $B(1, -1)$ and $C(11, 4)$. Then, the equation of the line passing through A and B is

$$y - 1 = \frac{-1-1}{1-5}(x-5) \Rightarrow x - 2y - 3 = 0$$

Clearly, point C (11, 4) satisfies the equation $x - 2y - 3 = 0$. Hence, the given points lie on the same straight line, whose equation is $x - 2y - 3 = 0$.

EXAMPLE 7 *The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.* **[NCERT]**

SOLUTION Assuming F along X-axis and K along Y-axis, we have two points (32, 273) and (212, 373) in xy-plane or FK-plane.

As F and K satisfy a linear equation. The equation of the line passing through (32, 273) and (212, 373) is

$$K - 273 = \frac{373-273}{212-32}(F-32)$$

$$\Rightarrow \quad K - 273 = \frac{100}{180}(F-32)$$

$$K = \frac{5}{9}(F-32) + 273 \qquad \text{...(i)}$$

ting $K = 0$ in (i), we get

$$0 = \frac{5}{9}(F-32) + 273 \Rightarrow F - 32 = -\frac{273 \times 9}{5} \Rightarrow F = 32 - 491.4 \Rightarrow F = -459.4$$

LEVEL-2

EXAMPLE 8 *Find the equation of the internal bisector of angle BAC of the triangle ABC whose vertices A, B, C are* (5, 2), (2, 3) *and* (6, 5) *respectively.*

SOLUTION We have,

$$AB = \sqrt{(5-2)^2 + (2-3)^2} = \sqrt{10}$$

and, $$AC = \sqrt{(5-6)^2 + (2-5)^2} = \sqrt{10}$$

$$\therefore \quad AB : AC = \sqrt{10} : \sqrt{10} = 1:1$$

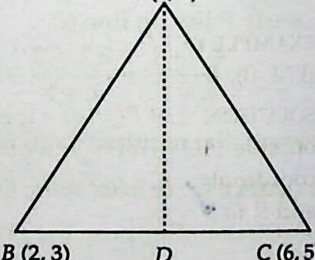

Fig. 23.27

The internal bisector AD of $\angle BAC$ divides BC in the ratio $AB : AC$ i.e. 1 : 1. So, coordinates of D are

$$\left(\frac{2+6}{2}, \frac{3+5}{4}\right) = (4, 4).$$

Equation of AD is $y - 2 = \frac{4-2}{4-5}(x-5)$ or, $2x + y - 12 = 0$

EXAMPLE 9 *A rectangle has two opposite vertices at the points* (1, 2) *and* (5, 5). *If the other vertices lie on the line* $x = 3$, *find the equations of the sides of the rectangle.*

SOLUTION Let $ABCD$ be a rectangle whose two opposite vertices are A (1, 2) and C (5, 5).

Let the coordinates of other two vertices B and D of rectangle $ABCD$ be $B(3, y_1)$ and $D(3, y_2)$.

Since diagonals AC and BD bisect each other. Therefore, mid-points of AC and BD are same.

$\therefore \quad \dfrac{y_1 + y_2}{2} = \dfrac{2+5}{2} \Rightarrow y_1 + y_2 = 7 \quad ...(i)$

Since $ABCD$ is a rectangle.

$\therefore \quad AC = BD$

$\Rightarrow \quad AC^2 = BD^2$

$\Rightarrow \quad (1-5)^2 + (2-5)^2 = (3-3)^2 + (y_1 - y_2)^2$

$\Rightarrow \quad 16 + 9 = (y_1 - y_2)^2$

$\Rightarrow \quad y_1 - y_2 = \pm 5 \quad ...(ii)$

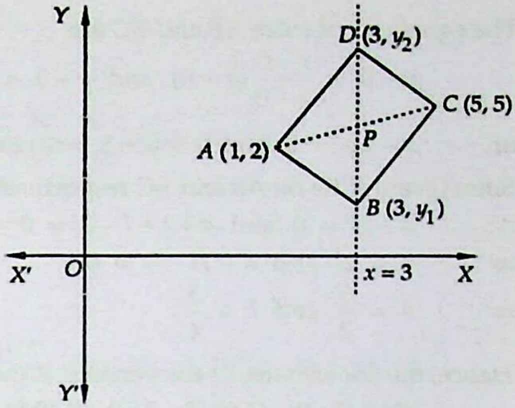

Fig. 23.28

Solving (i) and (ii), we get

$$y_1 = 6 \text{ and } y_2 = 1 \text{ or, } y_1 = 1 \text{ and } y_2 = 6$$

Thus, the coordinates of B and D are $B\,(3, 1)$ and $D\,(3, 6)$.

The equation of side AB is

$$y - 2 = \frac{1-2}{3-1}(x-1) \text{ or, } y - 2 = -\frac{1}{2}(x-1) \text{ or, } x + 2y - 5 = 0$$

The equation of side BC is

$$y - 1 = \frac{5-1}{5-3}(x-3) \text{ or, } y - 1 = 2(x-3) \text{ or, } 2x - y - 5 = 0$$

The equation of side CD is

$$y - 5 = \frac{6-5}{3-5}(x-5) \text{ or, } y - 5 = -\frac{1}{2}(x-5) \text{ or, } x + 2y - 15 = 0$$

The equation of side AD is

$$y - 2 = \frac{6-2}{3-1}(x-1) \text{ or, } y - 2 = 2(x-1) \text{ or, } 2x - y = 0$$

EXAMPLE 10 *Find the coordinates of the vertices of a square inscribed in the triangle with vertices $A\,(0, 0)$, $B\,(2, 1)$ and $C\,(3, 0)$; given that two of its vertices are on the side AC.*

SOLUTION Let $PQRS$ be the square inscribed in the triangle ABC such that its vertices P and S lie on side AC which is along X-axis. Let the length of each side of the square be l and the coordinates of P be $(a, 0)$. Then, the coordinates of other vertices are $P\,(a, 0)$, $S\,(a + l, 0)$, $Q\,(a, l)$ and $R\,(a + l, l)$.

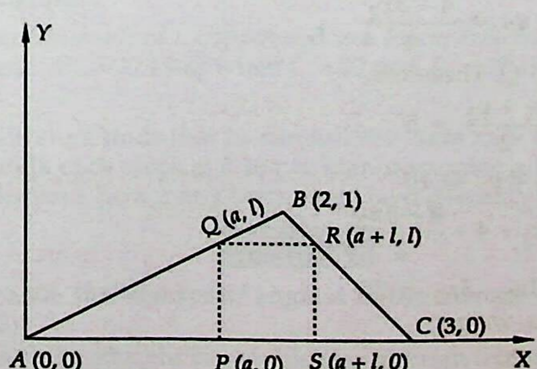

Fig. 23.29

The equations of sides AB and BC are

$$y - 0 = \frac{1-0}{2-0}(x-0) \text{ and } y - 0 = \frac{1-0}{2-3}(x-3) \text{ respectively}$$

or, $\quad x - 2y = 0$ and $x + y - 3 = 0$ respectively.

Since Q and R lie on AB and BC respectively.

$\therefore \quad a - 2l = 0$ and $a + l + l - 3 = 0$

$\Rightarrow \quad a = 2l$ and $a + 2l - 3 = 0$

$\Rightarrow \quad a = \frac{3}{2}$ and $l = \frac{3}{4}$

Hence, the coordinates of the vertices of the square are

$$P(3/2, 0), Q(3/2, 3/4), R(9/4, 3/4) \text{ and } S(9/4, 0)$$

EXAMPLE 11 *A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.* **[NCERT]**

SOLUTION Suppose the required line intersects the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ at $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively. Clearly, $P(x_1, y_1)$ lies on $5x - y + 4 = 0$ and $Q(x_2, y_2)$ lies on $3x + 4y - 4 = 0$.

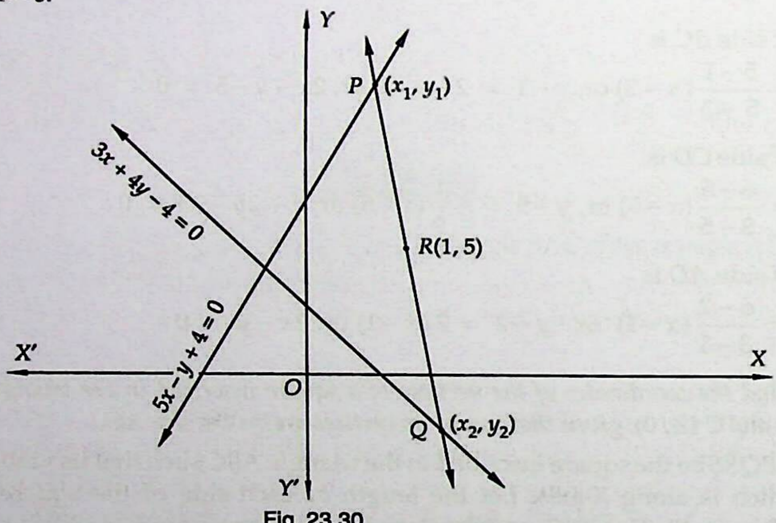

Fig. 23.30

$\therefore \quad 5x_1 - y_1 + 4 = 0$ and, $3x_2 + 4y_2 - 4 = 0$

$\Rightarrow \quad y_1 = 5x_1 + 4$ and $y_2 = \frac{4 - 3x_2}{4}$ ...(i)

Since R is the mid-point of PQ. Therefore,

$$\frac{x_1 + x_2}{2} = 1 \text{ and } \frac{y_1 + y_2}{2} = 5$$

$\Rightarrow \quad x_1 + x_2 = 2$ and $y_1 + y_2 = 10$

$\Rightarrow \quad x_1 + x_2 = 2$ and $5x_1 + 4 + \frac{4 - 3x_2}{4} = 10$ [Using (i)]

$\Rightarrow \quad x_1 + x_2 = 2$ and $20x_1 - 3x_2 = 20$

Solving these two equations, we get

$$x_1 = \frac{26}{23} \text{ and } x_2 = \frac{20}{23}$$

Substituting these values in (i), we get

$$y_1 = \frac{222}{23} \text{ and } y_2 = \frac{8}{23}$$

Thus, the coordinates of P and Q are $\left(\frac{26}{23}, \frac{222}{23}\right)$ and $\left(\frac{20}{23}, \frac{8}{23}\right)$ respectively.

Hence, the equation of PQ is

$$y - \frac{222}{23} = \frac{\frac{8}{23} - \frac{222}{23}}{\frac{20}{23} - \frac{26}{23}}\left(x - \frac{26}{23}\right)$$

$$\Rightarrow \quad 23y - 222 = \frac{-214}{-6}(23x - 26) \Rightarrow 23y - 222 = \frac{107}{3}(23x - 26) \Rightarrow 107x - 3y - 92 = 0.$$

EXERCISE 23.5

LEVEL-1

1. Find the equation of the straight lines passing through the following pair of points:
   (i) $(0, 0)$ and $(2, -2)$
   (ii) (a, b) and $(a + c \sin \alpha, b + c \cos \alpha)$
   (iii) $(0, -a)$ and $(b, 0)$
   (iv) (a, b) and $(a + b, a - b)$
   (v) $(at_1, a/t_1)$ and $(at_2, a/t_2)$
   (vi) $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$
2. Find the equations to the sides of the triangles the coordinates of whose angular points are respectively: (i) $(1, 4), (2, -3)$ and $(-1, -2)$ (ii) $(0, 1), (2, 0)$ and $(-1, -2)$.
3. Find the equations of the medians of a triangle, the coordinates of whose vertices are $(-1, 6), (-3, -9)$ and $(5, -8)$.
4. Find the equations to the diagonals of the rectangle the equations of whose sides are $x = a$, $x = a'$, $y = b$ and $y = b'$.
5. Find the equation of the side BC of the triangle ABC whose vertices are $A(-1, -2)$, $B(0, 1)$ and $C(2, 0)$ respectively. Also, find the equation of the median through $A(-1, -2)$.
6. By using the concept of equation of a line, prove that the three points $(-2, -2), (8, 2)$ and $(3, 0)$ are collinear. [NCERT]
7. Prove that the line $y - x + 2 = 0$ divides the join of points $(3, -1)$ and $(8, 9)$ in the ratio $2:3$.
8. Find the equation to the straight line which bisects the distance between the points $(a, b), (a', b')$ and also bisects the distance between the points $(-a, b)$ and $(a', -b')$.
9. In what ratio is the line joining the points $(2, 3)$ and $(4, -5)$ divided by the line passing through the points $(6, 8)$ and $(-3, -2)$. [NCERT]
10. The vertices of a quadrilateral are $A(-2, 6)$, $B(1, 2)$, $C(10, 4)$ and $D(7, 8)$. Find the equations of its diagonals.
11. The length L (in centimeters) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if $L = 124.942$ when $C = 20$ and $L = 125.134$ when $C = 110$, express L interms of C. [NCERT]
12. The owner of a milk store finds that he can sell 980 liters milk each week at Rs 14 per liter and 1220 liters of milk each week at ₹ 16 per liter. Assuming a linear relationship between selling price and demand, how many liters could he sell weekly at Rs 17 per liter. [NCERT]

LEVEL-2

13. Find the equation of the bisector of angle A of the triangle whose vertices are $A(4, 3)$, $B(0, 0)$ and $C(2, 3)$.
14. Find the equations to the straight lines which go through the origin and trisect the portion of the straight line $3x + y = 12$ which is intercepted between the axes of coordinates.

15. Find the equations of the diagonals of the square formed by the lines $x=0$, $y=0$, $x=1$ and $y=1$. **[NCERT EXEMPLAR]**

ANSWERS

1. (i) $y=-x$ (ii) $y-b=\cot\alpha\,(x-a)$ (iii) $ax-by=ab$
   (iv) $(a-2b)\,x-by+b^2+2\,ab-a^2=0$ (v) $t_1\,t_2\,y+x=a\,(t_1+t_2)$
   (vi) $x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos\left(\frac{\alpha-\beta}{2}\right)$
2. (i) $x+3\,y+7=0,\ y-3\,x=1,\ y+7\,x=11$ (ii) $2\,x-3\,y=4,\ y-3\,x=1,\ x+2\,y=2.$
3. $29\,x+4\,y+5=0,\ 8\,x-5\,y-21=0,\ 13\,x+14\,y+47=0$
4. $y\,(a'-a)-x\,(b'-b)=a'\,b-ab',\ y\,(a'-a)+x\,(b'-b)=a'\,b'-ab$
5. $x+2y-2=0$, Median: $5x-4y-3=0$
6. $2\,ay-2\,b'\,x=ab-a'\,b'$
9. $5:97$
10. $x+6y-34=0,\ x-y+1=0$
11. $L=\frac{4}{1875}C+124.899$
12. 1340 liters
13. $x-3\,y+5=0$
14. $y=6\,x,\ 2\,y=3\,x$
15. $y=x,\ x+y=1$

HINTS TO NCERT & SELECTED PROBLEMS

6. The equation of the line passing through points $(-2,-2)$ and $(8,2)$ is

$$y+2=\frac{2+2}{8+2}(x+2) \text{ or, } 2x-5y-6=0$$

Clearly, $(3,0)$ satisfies this equation which means that the line passing through $(-2,-2)$ and $(8,2)$ also passes through $(3,0)$. Hence, these points are collinear.

9. The equation of the line passing through $(6,8)$ and $(-3,-2)$ is

$$y+2=\frac{8+2}{6+3}(x+3) \text{ or, } 10x-9y+12=0 \qquad \text{...(i)}$$

Suppose this line divides the line segment joining $(2,3)$ and $(4,-5)$ in the ratio $\lambda:1$, then the point of division $\left(\frac{4\lambda+2}{\lambda+1},\frac{-5\lambda+3}{\lambda+1}\right)$ lies on (i).

$$\therefore \quad 10\left(\frac{4\lambda+2}{\lambda+1}\right)-9\left(\frac{-5\lambda+3}{\lambda+1}\right)+12=0$$

$$\Rightarrow \quad 40\lambda+20+45\lambda-27+12\lambda+12=0 \Rightarrow 97\lambda+5=0 \Rightarrow \lambda=\frac{-5}{97}.$$

Hence, the required ratio is $5:97$ externally.

11. The equation of the line passing through $(C_1=20,\ L_1=124.942)$ and $(C_2=110,\ L_2=125.134)$ is

$$L-124.942=\frac{125.134-124.942}{110-20}(C-20) \Rightarrow L=\frac{4}{1875}C+124.899$$

12. Let x denote the price per liter and y denote the quantity of the milk sold at this price. Since there is linear relationship between the price per liter and quantity solved. So, the line representing the relationship passes through $(14,980)$ and $(16,1220)$. So, its equation is

$$y-980=\frac{1220-980}{16-14}(x-14) \Rightarrow y-980=120(x-14) \Rightarrow 120x-y-700=0$$

When $x=17$, we obtain

$$120\times17-y-700 \Rightarrow y=1340$$

23.6.4 THE INTERCEPT FORM OF A LINE

THEOREM *The equation of a line which cuts off intercepts a and b respectively from the x and y-axes is* $\frac{x}{a}+\frac{y}{b}=1.$

PROOF Let AB be the line which cuts off intercepts $OA=a$ and $OB=b$ on the x and y axes respectively. Let $P\,(x, y)$ be any point on the line. Draw $PL \perp OX$. Then, $OL = x$ and $PL = y$.

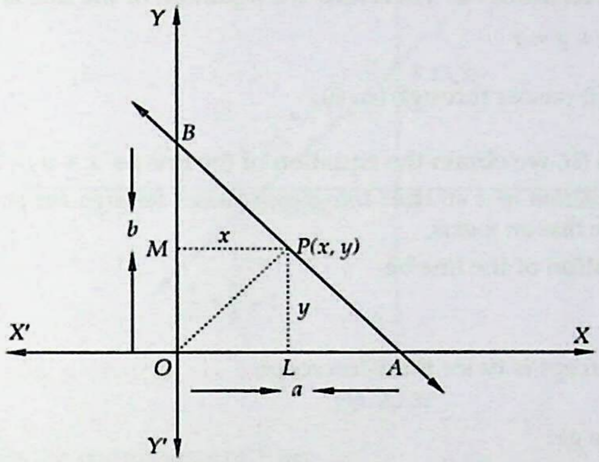

Fig. 23.31

Clearly,

$$\text{Area of } \Delta OAB = \text{Area of } \Delta OPA + \text{ Area of } \Delta OPB$$

$$\Rightarrow \quad \frac{1}{2} OA \cdot OB = \frac{1}{2} OA \cdot PL + \frac{1}{2} OB \cdot PM$$

$$\Rightarrow \quad \frac{1}{2} ab = \frac{1}{2} ay + \frac{1}{2} bx$$

$$\Rightarrow \quad ab = ay + bx$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1$$

This is the equation of the line in the intercept form.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which cuts off an intercept 4 on the positive direction of x-axis and an intercept 3 on the negative direction of y-axis.*

SOLUTION Here $a = 4,\ b = -\ 3$. So, the equation of the line is

$$\frac{x}{a}+\frac{y}{b}=1 \text{ or, } \frac{x}{4}+\frac{y}{-3}=1 \text{ or, } 3x-4y=12.$$

EXAMPLE 2 *Find the equation of the straight line which makes equal intercepts on the axes and passes through the point (2, 3).* **[NCERT]**

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$. Since it makes equal intercepts on the coordinate axes, therefore $a = b$. So, the equation of the line is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or, } x+y=a \qquad \text{... (i)}$$

This passes through the point (2, 3).

$\therefore \quad 2 + 3 = a \Rightarrow a = 5.$

Thus, the equation of the required line is $x + y = 5$. [Putting $a = 5$ in (i)]

EXAMPLE 3 *Find the equation of the line which cuts off equal and positive intercepts from the axes and passes through the point* (α, β).

SOLUTION Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ which cuts off intercepts a and b with the coordinate axes. It is given that $a = b$. Therefore, the equation of the line is

$$\frac{x}{a} + \frac{y}{a} = 1 \Rightarrow x + y = a \quad \text{...(i)}$$

It is given that the line (i) passes through (α, β).

$\therefore \quad \alpha + \beta = a.$

Putting the value of a in (i), we obtain the equation of the line as $x + y = \alpha + \beta$.

EXAMPLE 4 *Find the equation of a straight line which passes through the point* $(4, -2)$ *and whose intercept on y-axis is twice that on x-axis.*

SOLUTION Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{...(i)}$$

It is given that its y-intercept is twice the x-intercept.

$\therefore \quad b = 2a$

Putting $b = 2a$ in (i), we get

$$\frac{x}{a} + \frac{y}{2a} = 1 \text{ or, } 2x + y = 2a \quad \text{...(ii)}$$

It passes through the point $(4, -2)$. Therefore, putting $x = 4$, $y = -2$ in (ii), we get

$$8 - 2 = 2a \Rightarrow a = 3.$$

Substituting $a = 3$ in (ii), we get

$2x + y = 6$ as the required equation of the line.

EXAMPLE 5 *Find the equation of the straight line whose intercepts on X-axis and Y-axis are respectively twice and thrice of those by the line* $3x + 4y = 12$.

SOLUTION The equation of the given line is $3x + 4y = 12$. This can be written as $\frac{x}{4} + \frac{y}{3} = 1$.

Clearly, its intercepts on X and Y-axes are 4 and 3 respectively.

$\therefore$ x-intercept of the required line $= 2 \times 4 = 8$ and, y-intercept of the required line $= 3 \times 3 = 9$

Hence, the equation of the required line is $\frac{x}{8} + \frac{y}{9} = 1$ or, $9x + 8y = 72$.

EXAMPLE 6 *Find the equation of the line through* $(2, 3)$ *so that the segment of the line intercepted between the axes is bisected at this point.*

SOLUTION Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ which meets the x and y axes at $A(a, 0)$ and $B(0, b)$ respectively. The coordinates of the mid-point of AB are $(a/2, b/2)$. It is given that the point $(2, 3)$ bisects AB.

$$\therefore \quad \frac{a}{2} = 2 \text{ and } \frac{b}{2} = 3 \Rightarrow a = 4 \text{ and } b = 6.$$

Puting $a = 4$ and $b = 6$ in $\frac{x}{a} + \frac{y}{b} = 1$, we obtain

$$\frac{x}{4} + \frac{y}{6} = 1 \text{ or, } 3x + 2y = 12$$

Hence, the equation of the required line is $3x + 2y = 12$.

EXAMPLE 7 *If the intercept of a line between the coordinate axes is divided by the point (–5, 4) in the ratio 1 : 2, then find the equation of the line.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$. It meets the coordinate axes at $A\,(a, 0)$ and $B\,(0, b)$. It is given that $P\,(-5, 4)$ divides AB in the ratio 1 : 2.

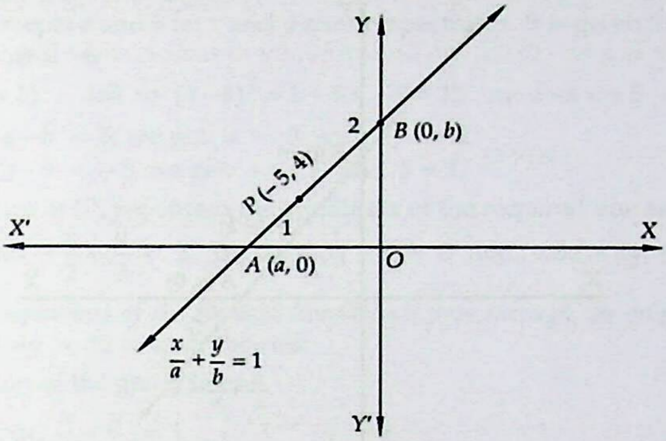

Fig. 23.32

Using section formula the coordinates of P are

$$\left(\frac{1\times 0+2\times a}{1+2}, \frac{1\times b+2\times 0}{1+2}\right)=\left(\frac{2a}{3}, \frac{b}{3}\right).$$

$$\therefore \quad -5=\frac{2a}{3}, 4=\frac{b}{3} \Rightarrow a=-\frac{15}{2}, b=12.$$

Substituting the values of a and b in $\frac{x}{a}+\frac{y}{b}=1$, we obtain

$$-\frac{2x}{15}+\frac{y}{12}=1 \text{ or, } -8x+5y=60 \text{ or, } 8x-5y+60=0 \text{ as the equation of the line.}$$

EXAMPLE 8 *A straight line cuts intercepts from the axes of coordinates the sum of whose reciprocals is a constant. Show that it always passes through a fixed point.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

Its intercepts on x and y axes are a and b respectively. It is given that

$$\frac{1}{a}+\frac{1}{b}=\text{Constant}=k \text{ (say)}$$

$$\therefore \quad \frac{1}{ka}+\frac{1}{kb}=1$$

$$\Rightarrow \quad \frac{\frac{1}{k}}{a}+\frac{\frac{1}{k}}{b}=1$$

$$\Rightarrow \quad \left(\frac{1}{k}, \frac{1}{k}\right) \text{ satisfies the equation } \frac{x}{a}+\frac{y}{b}=1$$

Hence, line (i) passes through the fixed point $\left(\frac{1}{k}, \frac{1}{k}\right)$.

EXAMPLE 9 *A line passes through the point (3, – 2). Find the locus of the middle point of the portion of the line intercepted between the axes.*

SOLUTION Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

It passes through (3, − 2).

$\therefore \quad \frac{3}{a} - \frac{2}{b} = 1$ ...(ii)

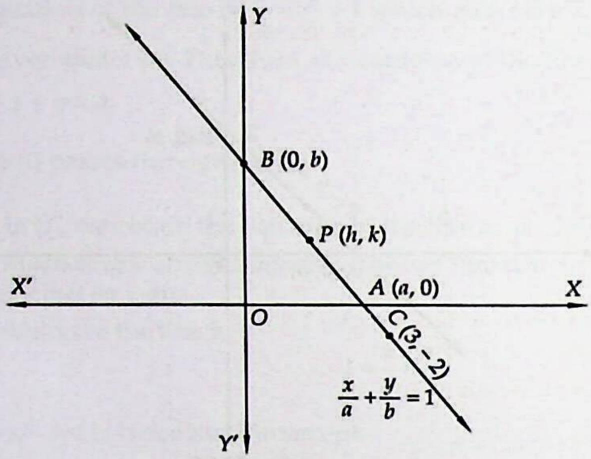

Fig. 23.33

The line (i) cuts the coordinate axes at $A(a, 0)$ and $B(0, b)$. Let $P(h, k)$ be the mid-point of the portion AB. Then,

$$h = \frac{a+0}{2}, k = \frac{0+b}{2} \Rightarrow a = 2h \text{ and } b = 2k$$

Substituting the values of a and b in (ii), we get

$$\frac{3}{2h} - \frac{2}{2k} = 1$$

Hence, locus of $P(h, k)$ is $\frac{3}{2x} - \frac{1}{y} = 1$ or, $3y - 2x = 2xy$.

LEVEL-2

EXAMPLE 10 *Find the equation of the line which passes through the point (3, 4) and the sum of its intercepts on the axes is 14.*

SOLUTION Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$ ...(i)

This passes through (3, 4).

$\therefore \quad \frac{3}{a} + \frac{4}{b} = 1$ ...(ii)

It is given that $a + b = 14$

$\therefore \quad b = 14 - a$

Putting $b = 14 - a$ in (ii), we get

$$\frac{3}{a} + \frac{4}{14-a} = 1 \Rightarrow 3(14-a) + 4a = a(14-a) \Rightarrow a^2 - 13a + 42 = 0 \Rightarrow (a-7)(a-6) = 0 \Rightarrow a = 7, 6$$

When $a = 7, b = 14 - a \Rightarrow b = 14 - 7 = 7$ and for $a = 6$, $b = 14 - a \Rightarrow b = 14 - 6 = 8$.

Thus, we obtain

$$a = 7, b = 7 \text{ or, } a = 6, b = 8$$

Putting the values of a and b in (i), we obtain that the equations of the lines are

$$\frac{x}{7} + \frac{y}{7} = 1 \text{ and } \frac{x}{6} + \frac{y}{8} = 1 \text{ or, } x + y = 7 \text{ and } 4x + 3y = 24.$$

EXAMPLE 11 *Find the equations of the lines which cut-off intercepts on the axes whose sum and product are 1 and – 6 respectively.* **[NCERT]**

SOLUTION Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \quad \text{...(i)}$$

Clearly, it cuts off intercepts a and b on x and y-axes respectively. It is given that

$$a+b=1 \text{ and } ab=-6$$

$\therefore \quad (a-b)^2=(a+b)^2-4ab \Rightarrow (a-b)^2=1-4\times-6=25 \Rightarrow a-b=\pm 5$

Solving $a+b=1$ and $a-b=5$, we get: $a=3$ and $b=-2$

Solving $a+b=1$ and $a-b=-5$, we get: $a=-2$ and $b=3$.

Substituting these values in (i), we obtain the equations of the required line as

$$\frac{x}{3}-\frac{y}{2}=1 \text{ and } -\frac{x}{2}+\frac{y}{3}=1 \text{ or, } 2x-3y-6=0 \text{ and } -3x+2y-6=0$$

EXAMPLE 12 *Find the equations of the straight lines which pass through the origin and trisect the intercept of the line* $3x+4y=12$ *between the axes.*

SOLUTION The equation of the given line is

$$3x+4y=12 \text{ or, } \frac{x}{4}+\frac{y}{3}=1$$

It cuts the coordinate axes at $A(4, 0)$ and $B(0, 3)$.

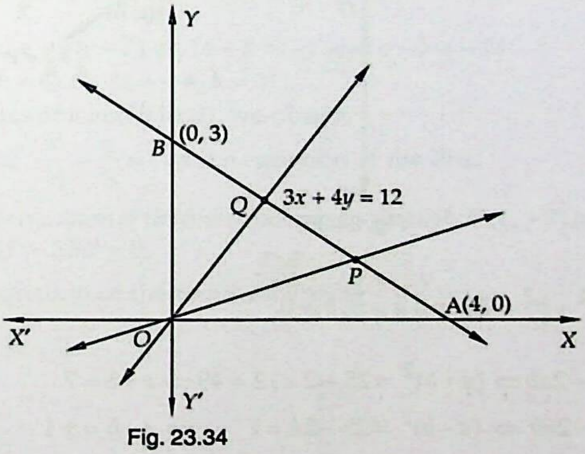

Fig. 23.34

The portion AB of the given line intercepted between the axes is trisected by points P and Q.

$$\therefore \quad \frac{AP}{PB}=\frac{1}{2} \text{ and } \frac{AQ}{QB}=\frac{2}{1}$$

$\Rightarrow$ P and Q divide AB internally in the ratio 1:2 and 2:1 respectively.

So, coordinates P and Q are

$$P\left(\frac{1\times 0+2\times 4}{1+2}, \frac{1\times 3+2\times 0}{1+2}\right)\equiv P\left(\frac{8}{3},1\right),\quad Q\left(\frac{2\times 0+1\times 4}{2+1}, \frac{2\times 3+1\times 0}{2+1}\right)\equiv Q\left(\frac{4}{3},2\right)$$

Hence, the equation of OQ is

$$y-0=\frac{2-0}{\frac{4}{3}-0}(x-0) \text{ or, } 3x-2y=0.$$

EXAMPLE 13 *The area of the triangle formed by the coordinates axes and a line is 6 square units and the length of the hypotenuse is 5 units. Find the equation of the line.*

SOLUTION Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ ...(i)

It cuts the coordinates axes at $A(a, 0)$ and $B(0, b)$ such that area of ΔOAB is 6 square units and $AB=5$ units.

Now,

Area of $\Delta OAB = 6$ sq.units

$\Rightarrow \quad \frac{1}{2}(OA \times OB) = 6$

$\Rightarrow \quad |a||b|=12 \Rightarrow |ab|=12 \Rightarrow ab=\pm 12$ ...(ii)

and,

$AB=5 \Rightarrow AB^2=25 \Rightarrow OA^2+OB^2=25 \Rightarrow a^2+b^2=25$ ...(iii)

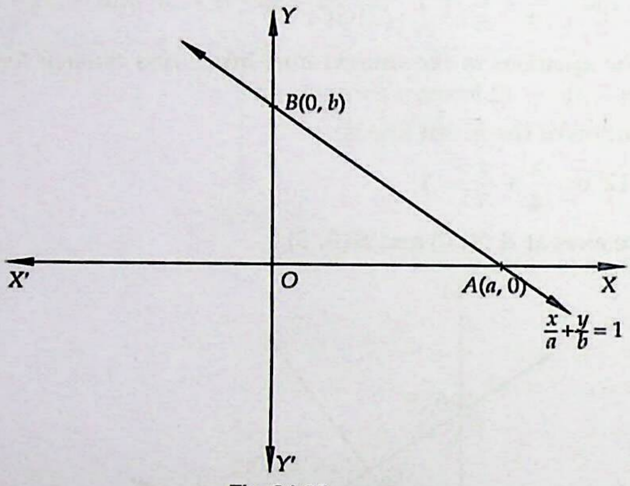

Fig. 24.35

Following cases arise:

<u>CASE I</u> When $ab=12$, $a^2+b^2=25$ and $a>0, b>0$.

In this case,

$(a+b)^2=a^2+b^2+2ab \Rightarrow (a+b)^2=25+2\times 12=49 \Rightarrow a+b=7$

and, $(a-b)^2=a^2+b^2-2ab \Rightarrow (a-b)^2=25-24=1 \quad \Rightarrow a-b=\pm 1$

Thus, we have

$(a+b=7$ and $a-b=1)$ or, $(a+b=7$ and $a-b=-1)$

$\Rightarrow \quad (a=4, b=3)$ or, $(a=3, b=4)$

Substituting the values of a and b in (i), we obtain

$\frac{x}{4}+\frac{y}{3}=1$ or $\frac{x}{3}+\frac{y}{4}=1$ as the equation of the line.

<u>CASE II</u> *When $ab=12$, $a^2+b^2=25$ and $a<0, b<0$:*

In this case, we have

$(a+b)^2=a^2+b^2+2ab=25+24=49 \Rightarrow a+b=-7$ $[\because a<0, b<0]$

and, $(a-b)^2=a^2+b^2-2ab=25-24=1 \Rightarrow a-b=\pm 1$

Thus, we have

$(a+b=-7$ and $a-b=1)$ or $(a+b=-7$ and $a-b=-1)$

$\Rightarrow \quad (a=-3, b=-4)$ or $(a=-4, b=-3)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{-3}+\frac{y}{-4}=1 \text{ or } \frac{x}{-4}+\frac{y}{-3}=1 \text{ as the equation of the line.}$$

<u>CASE III</u> *When* $ab=-12$, $a^2+b^2=25$ *and* $a>0, b<0$:

In this case, we have

$$(a+b)^2=a^2+b^2+2ab=25-24=1 \Rightarrow a+b=\pm 1$$

and, $\quad (a-b)^2=a^2+b^2-2ab=25+24=49 \Rightarrow a-b=7 \qquad [\because a>0, b<0 \therefore a-b>0]$

Thus, we have

$$(a+b=1 \text{ and } a-b=7) \text{ or } (a+b=-1 \text{ and } a-b=7)$$

$\Rightarrow \quad (a=4, b=-3)$ or $(a=3, b=-4)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{4}-\frac{y}{3}=1 \text{ or } \frac{x}{3}-\frac{y}{4}=1 \text{ as the equation of the line.}$$

<u>CASE IV</u> *When* $ab=-12$, $a^2+b^2=25$ *and* $a<0, b>0$:

In this case, we have

$$(a+b)^2=a^2+b^2+2ab=25-24=1 \Rightarrow a+b=\pm 1$$

and, $\quad (a-b)^2=a^2+b^2-2ab=25+24=49 \Rightarrow a-b=-7 \qquad [a<0, b>0 \therefore a-b<0]$

Thus, we have

$$(a+b=1 \text{ and } a-b=-7) \text{ or, } (a+b=-1 \text{ and } a-b=-7)$$

$\Rightarrow \quad (a=-3 \text{ and } b=4)$ or $(a=-4, b=3)$

Substituting the values of a and b in (i), we obtain

$$\frac{x}{-3}+\frac{y}{4}=1 \text{ or } \frac{x}{-4}-\frac{y}{3}=1 \text{ as the equation of the line.}$$

EXAMPLE 14 *Find the equation of the line which passes through* $P(1, -7)$ *and meets the axes at A and B respectively so that* $4AP-3BP=0$.

SOLUTION Let the equation of the required line be $\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots(i)$

It passes through $P(1, -7)$.

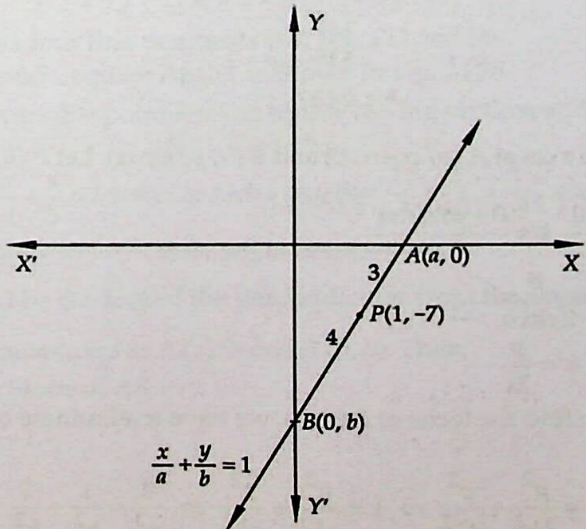

Fig. 24.36

$\therefore \quad \frac{1}{a} - \frac{7}{b} = 1$ ...(ii)

It is given that the point $P(1, -7)$ divides segment AB in such a way that

$$4AP - 3BP = 0 \text{ i.e. } \frac{AP}{BP} = \frac{3}{4} \text{ or, } AP : BP = 3 : 4$$

This means that P divides AB internally in the ratio 3 : 4. So, the coordinates of P are

$$\left(\frac{3 \times 0 + 4 \times a}{3+4}, \frac{3 \times b + 4 \times 0}{3+4}\right) = \left(\frac{4a}{7}, \frac{3b}{7}\right)$$

But, the coordinates of P are given as $(1, -7)$.

$\therefore \quad \frac{4a}{7} = 1 \text{ and } \frac{3b}{7} = -7 \Rightarrow a = \frac{7}{4} \text{ and } b = -\frac{49}{3}$

Substituting the values of a and b in (i), we obtain

$$\frac{4x}{7} - \frac{3y}{49} = 1 \text{ or } 28x - 3y = 49 \text{ as the required equation.}$$

EXAMPLE 15 *Show that the locus of the mid-point of the segment intercepted between the axes of the variable line $x\cos\alpha + y\sin\alpha = p$ is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$, where p is a constant.* **[NCERT EXEMPLAR]**

SOLUTION The given equation is $x\cos\alpha + y\sin\alpha = p$ or, $\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1$ ...(i)

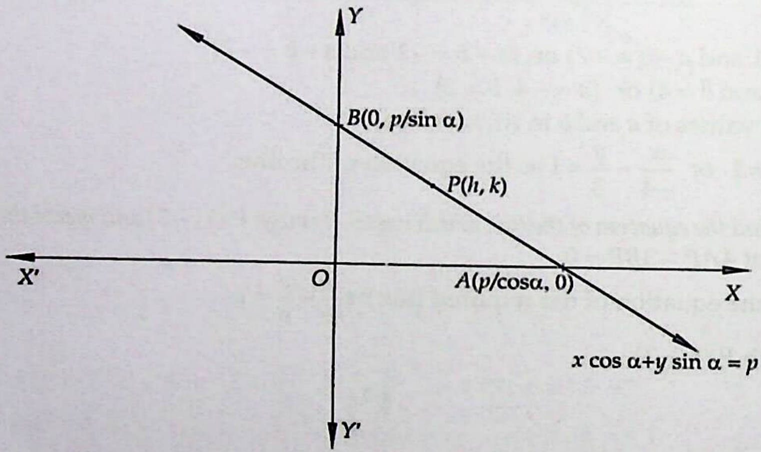

Fig. 24.37

This cuts the coordinate axes at $A(p/\cos\alpha, 0)$ and $B(0, p/\sin\alpha)$. Let $P(h, k)$ be the mid-point of the intercept AB. Then,

$$h = \frac{p/\cos\alpha + 0}{2}, k = \frac{0 + p/\sin\alpha}{2}$$

$$\Rightarrow \quad h = \frac{p}{2\cos\alpha}, k = \frac{p}{2\sin\alpha}$$

$$\Rightarrow \quad \cos\alpha = \frac{p}{2h}, \sin\alpha = \frac{p}{2k} \quad \text{...(i)}$$

Here, α is a variable. To find the locus of $P(h, k)$, we have to eliminate α.
From (i), we obtain

$$\cos^2\alpha + \sin^2\alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow 1 = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} \Rightarrow \frac{4}{p^2} = \frac{1}{h^2} + \frac{1}{k^2}$$

Hence, the locus of (h, k) is $\frac{1}{x^2}+\frac{1}{y^2}=\frac{4}{p^2}$.

EXAMPLE 16 *If the sum of the distances of a moving point in a plane from the axes is 1, then find the locus of the point.* **[NCERT EXEMPLAR]**

SOLUTION Let $P(h, k)$ be a moving point in the xy-plane. Let PL and PM be perpendiculars from P on OX and OY respectively. Then, $PL=|k|$ and $PM=|h|$.

It is given that $P(h, k)$ moves in the xy-plane such that

$$PL+PM=1 \Rightarrow |k|+|h|=1$$

Hence, the locus of $P(h, k)$ is $|y|+|x|=1$ or, $|x|+|y|=1$.

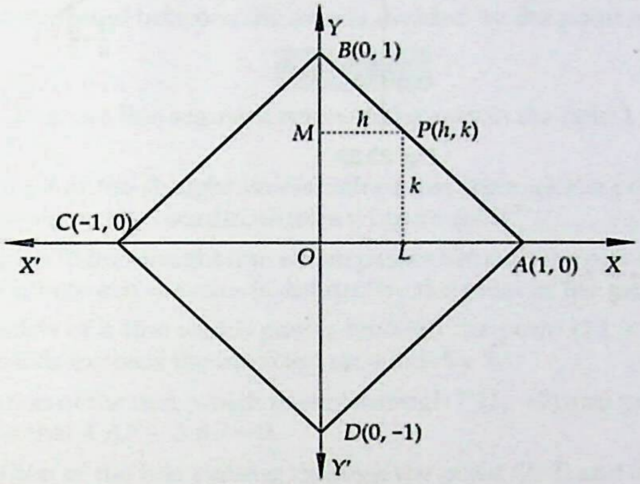

Fig. 24.38

Now,

$$|x|+|y|=1 \Rightarrow 0 \leq |x| \leq 1, 0 \leq |y| \leq 1$$

Also,

$$|x|+|y|=1 \Rightarrow \begin{cases} x+y=1, \text{ if } 0 \leq x \leq 1, 0 \leq y \leq 1 \\ -x+y=1, \text{ if } -1 \leq x < 0, 0 \leq y \leq 1 \\ -x-y=1, \text{ if } -1 < x \leq 0, -1 < y \leq 0 \\ x-y=1, \text{ if } 0 \leq x \leq 1, -1 < y \leq 0 \end{cases}$$

Thus, $|x|+|y|=1$ gives four line segments AB, BC, CD and DA.

These line segments form a square $ABCD$ as shown in Fig. 24.38.

Thus, the locus of the variable point P is the square having vertices at $A(1, 0)$, $B(0, 1)$, $C(-1, 0)$ and $D(0, -1)$.

EXAMPLE 17 *The line* $\frac{x}{a}+\frac{x}{b}=1$ *moves in such a way that* $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, *where c is a constant. Find the locus of the foot of the perpendicular from the origin on the given line.*

SOLUTION Let $P(h, k)$ be the foot of the perpendicular from the origin O on the line $\frac{x}{a}+\frac{y}{b}=1$ which cuts the coordinates axes at $A(a, 0)$ and $B(0, b)$. Then,

$$\text{Slope of } OP \times \text{Slope of } AB=-1$$

$$\Rightarrow \quad \frac{k-0}{h-0} \times \frac{b-0}{0-a}=-1$$

$$\Rightarrow \quad bk=ah$$

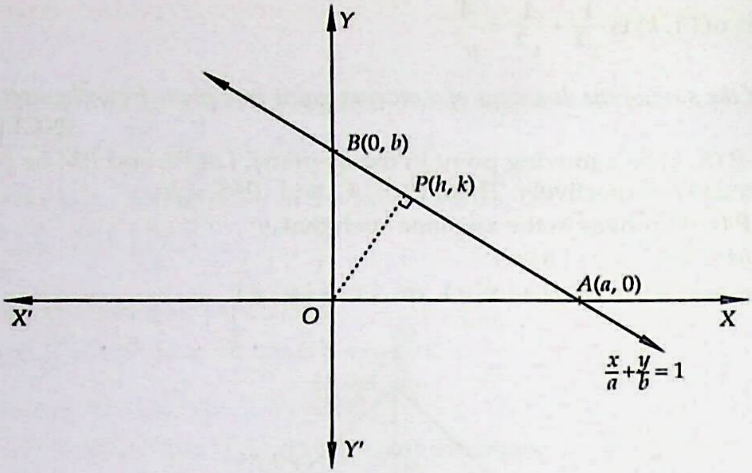

Fig. 23.39

$$\Rightarrow \quad b=\frac{ah}{k} \qquad \text{...(i)}$$

Also, $P(h, k)$ lies on $\frac{x}{a}+\frac{y}{b}=1$.

$$\therefore \quad \frac{h}{a}+\frac{k}{b}=1 \qquad \text{...(ii)}$$

$$\Rightarrow \quad \frac{h}{a}+\frac{k^2}{ah}=1 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad a=\frac{h^2+k^2}{h}$$

Substituting this values of a in (i), we obtain

$$b=\frac{h^2+k^2}{k}$$

Substituting the values of a and b in $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$, we obtain

$$\frac{h^2}{(h^2+k^2)^2}+\frac{k^2}{(h^2+k^2)^2}=\frac{1}{c^2} \text{ or, } h^2+k^2=c^2$$

Hence, the locus of (h, k) is $x^2+y^2=c^2$.

EXERCISE 23.6

LEVEL-1

1. Find the equation to the straight line
   (i) cutting off intercepts 3 and 2 from the axes.
   (ii) cutting off intercepts -5 and 6 from the axes.
2. Find the equation of the straight line which passes through $(1, -2)$ and cuts off equal intercepts on the axes [NCERT EXEMPLAR]
3. Find the equation to the straight line which passes through the point (5, 6) and has intercepts on the axes
   (i) equal in magnitude and both positive. (ii) equal in magnitude but opposite in sign.
4. For what values of a and b the intercepts cut off on the coordinate axes by the line $ax+by+8=0$ are equal in length but opposite in signs to those cut off by the line $2x-3y+6=0$ on the axes. [NCERT EXEMPLAR]

5. Find the equation to the straight line which cuts off equal positive intercepts on the axes and their product is 25.
6. Find the equation of the line which passes through the point $(-4, 3)$ and the portion of the line intercepted between the axes is divided internally in the ratio $5:3$ by this point. **[NCERT EXEMPLAR]**
7. A straight line passes through the point (α, β) and this point bisects the portion of the line intercepted between the axes. Show that the equation of the straight line is $\frac{x}{2\alpha} + \frac{y}{2\beta} = 1$. **[NCERT]**
8. Find the equation of the line which passes through the point $(3, 4)$ and is such that the portion of it intercepted between the axes is divided by the point in the ratio $2:3$.

LEVEL-2

9. Point $R(h, k)$ divides a line segment between the axes in the ratio $1:2$. Find the equation of the line. **[NCERT]**
10. Find the equation of the straight line which passes through the point $(-3, 8)$ and cuts off positive intercepts on the coordinate axes whose sum is 7.
11. Find the equation to the straight line which passes through the point $(-4, 3)$ and is such that the portion of it between the axes is divided by the point in the ratio $5:3$.
12. Find the equation of a line which passes through the point $(22, -6)$ and is such that the intercept on x-axis exceeds the intercept on y-axis by 5.
13. Find the equation of the line, which passes through $P(1, -7)$ and meets the axes at A and B respectively so that $4\,AP - 3\,BP = 0$.
14. Find the equation of the line passing through the point $(2, 2)$ and cutting off intercepts on the axes whose sum is 9. **[NCERT]**
15. Find the equation of the straight line which passes through the point $P(2, 6)$ and cuts the coordinate axes at the point A and B respectively so that $\frac{AP}{BP} = \frac{2}{3}$.
16. Find the equations of the straight lines each of which passes through the point $(3, 2)$ and cuts off intercepts a and b respectively on x and y-axes such that $a - b = 2$.
17. Find the equations of the straight lines which pass through the origin and trisect the portion of the straight line $2x + 3y = 6$ which is intercepted between the axes.
18. Find the equation of the straight line passing through the point $(2, 1)$ and bisecting the portion of the straight line $3x - 5y = 15$ lying between the axes.
19. Find the equation of the straight line passing through the origin and bisecting the portion of the line $ax + by + c = 0$ intercepted between the coordinate axes.

ANSWERS

1. (i) $2x + 3y = 6$ (ii) $-6x + 5y = 30$ 2. $x + y = -1$
3. (i) $x + y = 11$ (ii) $x - y = -1$
4. $a = -\frac{8}{3}, b = 4$ 5. $x + y = 5$ 6. $9x - 20y + 96 = 0$ 8. $2x + y = 10$
9. $2kx + hy = 3hk$ 10. $4x + 3y = 12$ 11. $9x - 20y + 96 = 0$
12. $6x + 11y - 66 = 0$ or $x + 2y - 10 = 0$ 13. $28x - 3y = 49$
14. $x + 2y - 6 = 0, 2x + y - 6 = 0$ 15. $9x + 2y = 30$
16. $2x + 3y = 12,\ x - y = 1$ 17. $x - 3y = 0,\ 4x - 3y = 0$
18. $5x + y = 11$ 19. $ax - by = 0$

HINTS TO NCERT & SELECTED PROBLEMS

2. The equation of a line cutting off equal intercepts 'a' on the coordinate axes is

$$\frac{x}{a}+\frac{y}{a}=1 \text{ or, } x+y=a \qquad ...(i)$$

If it passes through $(1,-2)$, then $1-2=a \Rightarrow a=-1$.

Substituting $a=1$ in (i), we get $x+y=-1$ as the equation of the line.

7. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(i)$$

This line cuts the coordinate axes at $A(a, 0)$ and $B(0, b)$. It is given that (α, β) bisects the segment AB.

$$\therefore \qquad \alpha=\frac{a+0}{2},\ \beta=\frac{0+b}{2} \Rightarrow a=2\alpha,\ b=2\beta$$

Substituting these values in (i), we get

$$\frac{x}{2\alpha}+\frac{y}{2\beta}=1$$

9. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(i)$$

It cuts the axes at $A(a, 0)$ and $B(0, b)$.

It is given that the point $R(h, k)$ divides segment AB in the ratio $1:2$.

$$\therefore \qquad h=\frac{2a+0}{3} \text{ and } k=\frac{0+b}{3} \Rightarrow a=\frac{3h}{2}, b=3k$$

Substituting these values in (i), we obtain $\frac{2x}{h}+\frac{y}{k}=3$ or, $2kx+hy=3hk$ as the equation of the line.

14. Let the equation of the line be

$$\frac{x}{a}+\frac{y}{b}=1 \qquad ...(i)$$

It passes through $(2, 2)$ and the sum of the intercepts on the axes is 9. Therefore,

$$\frac{2}{a}+\frac{2}{b}=1 \text{ and } a+b=9$$

$$\Rightarrow \qquad 2b+2a=ab \text{ and } a+b=9$$

$$\Rightarrow \qquad 2(9-a)+2a=a(9-a) \qquad \text{[On eliminating } b\text{]}$$

$$\Rightarrow \qquad a^2-9a+18=0$$

$$\Rightarrow \qquad (a-6)(a-3)=0$$

$$\Rightarrow \qquad a=3, 6.$$

When $a=3$, $a+b=9$ gives $b=6$. When $a=6$, $a+b=9$ gives $b=3$.

Hence, the equations of the line are

$$\frac{x}{3}+\frac{y}{6}=1 \text{ and, } \frac{x}{6}+\frac{y}{3}=1 \text{ or, } 2x+y=6 \text{ and } x+2y=6$$

23.6.5 NORMAL FORM OR PERPENDICULAR FORM OF A LINE

THEOREM *The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with x-axis is $x\cos\alpha+y\sin\alpha=p$.*

PROOF Let the line AB be such that the length of the perpendicular OQ from the origin O to the line be p and $\angle XOQ=\alpha$. Let $P(x, y)$ be any point on the line. Draw $PL \perp OX$, $LM \perp OQ$ and $PN \perp LM$. Then, $OL=x$ and $LP=y$.

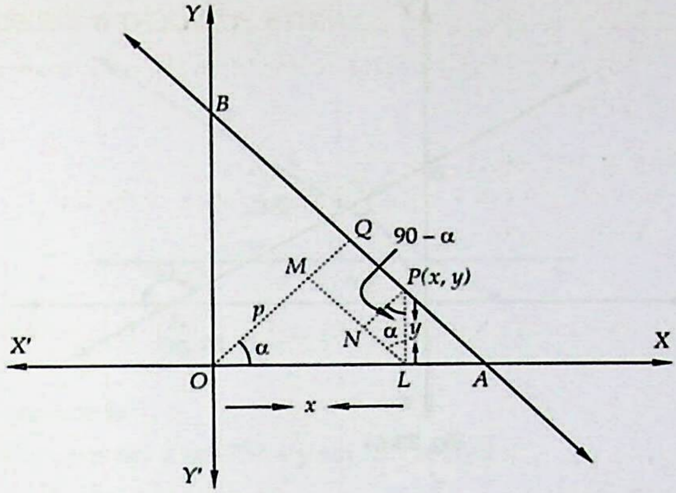

Fig. 23.40

In $\Delta\, OLM$, we have

$$\cos\alpha = \frac{OM}{OL}$$

$$\Rightarrow \quad OM = OL\cos\alpha = x\cos\alpha.$$

In $\Delta\, PNL$, we have

$$\sin\alpha = \frac{PN}{PL}$$

$$\Rightarrow \quad PN = PL\sin\alpha = y\sin\alpha$$

$$\Rightarrow \quad MQ = PN = y\sin\alpha$$

Now, $p = OQ = OM + MQ = x\cos\alpha + y\sin\alpha$

Hence, the equation of the required line is $x\cos\alpha + y\sin\alpha = p$.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which is at a distance 3 from the origin and the perpendicular from the origin to the line makes an angle of 30° with the positive direction of the x-axis.*

SOLUTION Here, $p = 3$, $\alpha = 30°$.

The equation of the line in the normal form is

$$x\cos 30° + y\sin 30° = 3 \Rightarrow x\,\frac{\sqrt{3}}{2} + \frac{y}{2} = 3 \Rightarrow \sqrt{3}\,x + y = 6.$$

EXAMPLE 2 *Find the equation of the straight line on which the length of the perpendicular from the origin is 4 units and the line makes an angle of 120° with positive direction of x-axis.*

[NCERT EXEMPLAR]

SOLUTION It is given that $\angle XAB = 120°$. Therefore, $\angle AOP = 30°$.

Thus, we have

$$p = 4 \text{ and } \alpha = 30°$$

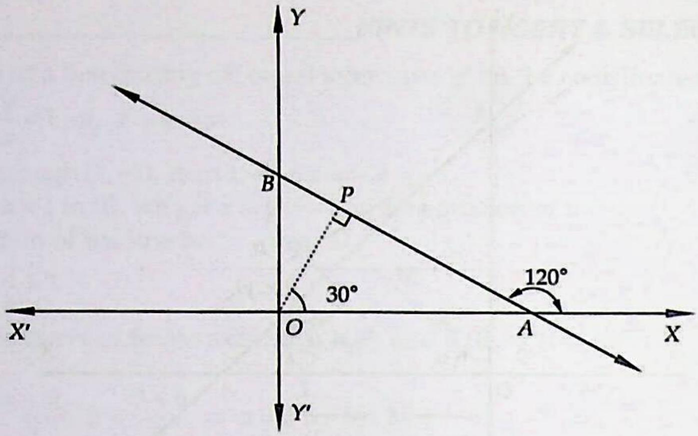

Fig. 23.41

So, the equation of the line is

$$x\cos\alpha + y\sin\alpha = p \text{ or, } x\cos 30°+y\sin 30°=4 \text{ or, } \sqrt{3}x+y=8$$

EXAMPLE 3 *The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.*

SOLUTION It is evident from the Figure 23.42 that the perpendicular OQ from the origin on the line makes 30° angle with x-axis. Therefore, $\alpha = 30°$. It is given that $OQ = 7$. Therefore, $p = 7$.

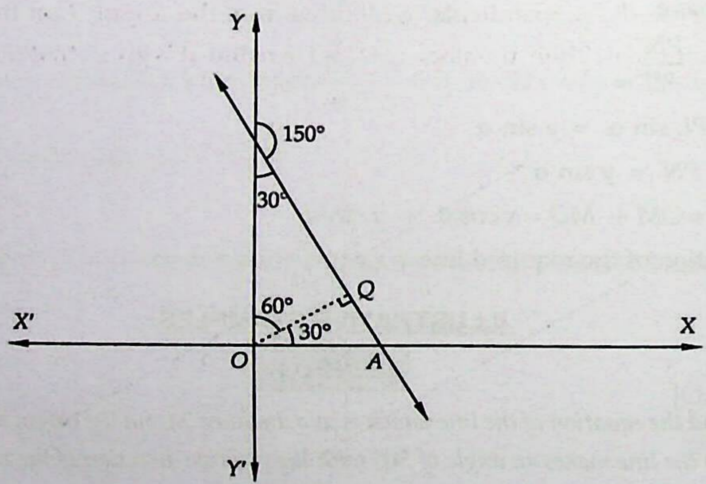

Fig. 23.42

So, the equation of the required line is

$$x\cos\alpha + y\sin\alpha = p \text{ or, } x\cos 30° + y\sin 30° = 7$$

$$\Rightarrow \quad \frac{\sqrt{3}x}{2} + \frac{y}{2} = 7 \Rightarrow \sqrt{3}\,x + y = 14.$$

EXAMPLE 4 *Find the equation of the straight line upon which the length of perpendicular from origin is $3\sqrt{2}$ units and this perpendicular makes an angle of 75° with the positive direction of x-axis.*

SOLUTION Let OL be the perpendicular from the origin on the required line. It is given that $OL = 3\sqrt{2}$ and $\angle XOL = 75°$ i.e. $p = 3\sqrt{2}$ and $\alpha = 75°$.