CHAPTER 16

INCREASING AND DECREASING FUNCTIONS

16.1 INTRODUCTION

In this chapter, we shall study monotonicity of functions. A function $f(x)$ is said to be a monotonically increasing function on $[a, b]$, if the values of $f(x)$ increase or decrease with the increase or decrease in x. If the values of $f(x)$ decrease with the increase in the values of x, then $f(x)$ is said to be a monotonically decreasing function. The monotonicity of functions in $[a, b]$ is strongly connected to the sign of its derivative in $[a, b]$. The relation between the two will be discused in section 16.4. In determining the intervals of monotonicity of a function in its domain, we shall be solving the inequations $f'(x) > 0$ and $f'(x) < 0$. So, we shall first discuss the procedure of solving inequations in the following section.

16.2 SOLUTION OF RATIONAL ALGEBRAIC INEQUATIONS

The following results are very useful in solving rational algebraic inequations:

(i) $ab > 0 \Rightarrow (a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$

(ii) $ab < 0 \Rightarrow (a > 0$ and $b < 0)$ or $(a < 0$ and $b > 0)$

(iii) $ab > 0$ and $a > 0 \Rightarrow b > 0$

(iv) $ab < 0$ and $a < 0 \Rightarrow b > 0$

If $P(x)$ and $Q(x)$ are polynomials, then the inequations $\frac{P(x)}{Q(x)} > 0$, $\frac{P(x)}{Q(x)} < 0$, $\frac{P(x)}{Q(x)} \geq 0$ and $\frac{P(x)}{Q(x)} \leq 0$ are known as rational algebraic inequations. These inequations can be solved by using the following algorithm.

ALGORITHM

Step I *Factorize $P(x)$ and $Q(x)$ into linear factors.*

Step II *Make coefficient of x positive in all factors.*

Step III *Equate all the factors to zero and find the corresponding values of x. These values are generally known as critical points.*

Step IV *Plot the critical points on the number line. Note that n critical points will divide the number line in $(n+1)$ regions.*

Step V *In the right most region, the expression will be positive and in other regions it will be alternatively negative and positive. So, mark positive sign in the right most region and than mark alternatively negative and positive signs in the remaining regions.*

Step VI *Obtain the solution set of the given inequation by selecting the appropriate regions in step V.*

Following illustrations will illustrate the above algorithm.

ILLUSTRATION 1 *Solve:* $4x^3 - 24x^2 + 44x - 24 > 0$.

SOLUTION We have,

$$4x^3 - 24x^2 + 44x - 24 > 0 \Rightarrow 4(x^3 - 6x^2 + 11x - 6) > 0$$

$\Rightarrow \quad x^3 - 6x^2 + 11x - 6 > 0 \qquad [\because 4 > 0$ and $ab > 0, a > 0 \Rightarrow b > 0]$

$\Rightarrow \quad (x-1)(x^2 - 5x + 6) > 0 \Rightarrow (x-1)(x-2)(x-3) > 0 \qquad \ldots$(i)

On equating all factors, on LHS of the inequation, to zero, we obtain $x = 1, 2, 3$ as critical points. Let us plot these critical points on the number line as shown in Fig. 16.1. These points divide the number line into four regions. In the right most region the expression on LHS of (i) bears positive sign and then alternatively negative and positive signs as marked in Fig. 16.1. Since the expression in (i) is positive. Therefore, solution set of inequation (i) is the union of the regions marked with + signs. Hence from Fig. 16.1, we obtain $(1, 2) \cup (3, \infty)$ as the solution set.

i.e. $\quad x^3 - 6x^2 + 11x - 6 > 0 \Rightarrow (x-1)(x-2)(x-3) > 0 \Rightarrow x \in (1, 2) \cup (3, \infty)$.

Hence, the solution set of the given inequality is $(1, 2) \cup (3, \infty)$.

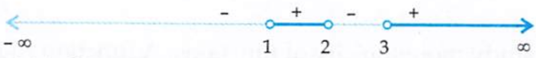

Fig. 16.1 Signs of $(x-1)(x-2)(x-3)$ for different values of x

ILLUSTRATION 2 *Solve :* $\dfrac{1}{x+1} - \dfrac{4}{(2+x)^2} > 0,\ x \neq -1, -2.$

SOLUTION We have,

$$\frac{1}{x+1} - \frac{4}{(2+x)^2} = \frac{(2+x)^2 - 4(x+1)}{(2+x)^2 (x+1)} = \frac{x^2}{(2+x)^2 (x+1)}$$

$$\therefore \quad \frac{1}{x+1} - \frac{4}{(2+x)^2} > 0$$

$$\Rightarrow \quad \frac{x^2}{(2+x)^2 (x+1)} > 0$$

$$\Rightarrow \quad \left(\frac{x}{2+x}\right)^2 \left(\frac{1}{x+1}\right) > 0$$

$$\Rightarrow \quad \frac{1}{x+1} > 0 \text{ and } x \neq 0 \qquad \left[\because \left(\frac{x}{2+x}\right)^2 > 0 \text{ and if } a > 0, \text{ then } ab > 0 \Rightarrow b > 0\right]$$

$$\Rightarrow \quad x + 1 > 0 \text{ and } x \neq 0 \qquad \left[\because \frac{a}{b} > 0 \text{ and } a > 0, \Rightarrow b > 0\right]$$

$$\Rightarrow \quad x > -1 \text{ and } x \neq 0 \Rightarrow x \in (-1, 0) \cup (0, \infty)$$

Hence, the solution set of the given inequality is $(-1, 0) \cup (0, \infty)$.

ILLUSTRATION 3 *Solve:* $\dfrac{1-x^2}{5x-6-x^2} < 0$

SOLUTION We have,

$$\frac{1-x^2}{5x-6-x^2} < 0 \Leftrightarrow \frac{-(x^2-1)}{-(x^2-5x+6)} < 0 \Leftrightarrow \frac{x^2-1}{x^2-5x+6} < 0 \Leftrightarrow \frac{(x-1)(x+1)}{(x-2)(x-3)} < 0 \qquad \text{...(i)}$$

Equating all the factors to zero, we obtain $x = 1, -1, 2, 3$ as the critical points.

Now, we plot these points on the number line as show in Fig. 16.2. These points divide the number line into 5 regions. In the right most region the expression in (i) bears '+' sign and in the other regions the expression bears alternate negative and positive signs as shown in Fig. 16.2.

Since the expression in (i) is negative, so solution set of the given inequation is the union of regions containing negative signs. Hence, from Fig. 16.2, we get $x \in (-1, 1) \cup (2, 3)$

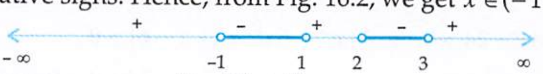

Fig. 16.2 Signs of $\dfrac{(x-1)(x+1)}{(x-2)(x-3)}$ for different values of x

i.e. $\dfrac{1-x^2}{5x-6-x^2} < 0 \Rightarrow x \in (-1, 1) \cup (2, 3)$

ILLUSTRATION 4 *Solve:* $\dfrac{8x^2+16x-51}{2x^2+5x-12} > 3.$

SOLUTION We have,

$$\frac{8x^2+16x-51}{2x^2+5x-12} > 3 \Leftrightarrow \frac{8x^2+16x-51}{2x^2+5x-12} - 3 > 0 \Leftrightarrow \frac{8x^2+16x-51-6x^2-15x+36}{2x^2+5x-12} > 0$$

$$\Leftrightarrow \frac{2x^2+x-15}{2x^2+5x-12} > 0 \Leftrightarrow \frac{2x^2+6x-5x-15}{2x^2+8x-3x-12} > 0 \Leftrightarrow \frac{(x+3)\ (2x-5)}{(x+4)\ (2x-3)} > 0 \qquad \text{...(i)}$$

Equating all factors to zero, we obtain $x = -4, -3, 3/2, 5/2$. Now, we plot these points on the number line as shown in Fig. 16.3. These points divide the number line into five regions. In the right most region the expression in (i) bears positive sign and in all other regions it bears alternate negative and positive signs as shown in Fig. 16.3.

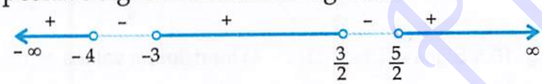

Fig. 16.3 Signs of $\dfrac{(x+3)(2x-5)}{(x+4)(2x-3)}$ for different values of x

Since the expression in (i) is positive, so the solution set of the given inequation is the union of regions containing '+' signs. Hence, from Fig 16.3, we get $x \in (-\infty, -4) \cup (-3, 3/2) \cup (5/2, \infty)$.

i.e. $\dfrac{8x^2+16x-51}{2x^2+5x-12} > 3 \Rightarrow x \in (-\infty, -4) \cup (-3, 3/2) \cup (5/2, \infty)$.

ILLUSTRATION 5 *Solve:* $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2}.$

SOLUTION We have,

$$\frac{x^2-2x+5}{3x^2-2x-5} > \frac{1}{2} \Leftrightarrow \frac{x^2-2x+5}{3x^2-2x-5} - \frac{1}{2} > 0 \Leftrightarrow \frac{2(x^2-2x+5)-(3x^2-2x-5)}{2(3x^2-2x-5)} > 0$$

$$\Leftrightarrow \frac{-x^2-2x+15}{2(3x^2-2x-5)} > 0 \Leftrightarrow \frac{-(x^2+2x-15)}{2(3x^2-2x-5)} > 0 \Leftrightarrow \frac{x^2+2x-15}{2(3x^2-2x-5)} < 0 \Leftrightarrow \frac{x^2+2x-15}{3x^2-2x-5} > 0$$

$$\Leftrightarrow \frac{(x+5)\,(x-3)}{(x+1)\,(3x-5)} < 0 \qquad \text{...(i)}$$

On equating all factors to zero, we get $x = -5, -1, 5/3, 3$. Plotting these points on number line and marking alternatively '+' and '–' signs, we obtain as shown in Fig. 16.4.

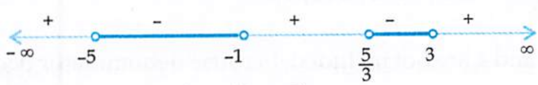

Fig. 16.4 Signs of $\dfrac{(x+5)(x-3)}{(x+1)(3x-5)}$ for different values of x

Since the expression in (i) is negative, so the solution set of the given inequation is the union of regions marked with '–' signs. Hence, from Fig. 16.4, we get $x \in (-5, -1) \cup (5/3, 3)$.

i.e. $\dfrac{x^2-2x+5}{3x^2-2x-5} > \dfrac{1}{2} \Rightarrow x \in (-5, -1) \cup (5/3, 3)$.

ILLUSTRATION 6 *Solve:* $\dfrac{x^2-2x+24}{x^2-3x+4} \le 4.$

SOLUTION We have,

$$\frac{x^2-2x+24}{x^2-3x+4} \le 4 \Leftrightarrow \frac{x^2-2x+24}{x^2-3x+4} - 4 \le 0 \Leftrightarrow \frac{(x^2-2x+24)-4(x^2-3x+4)}{x^2-3x+4} \le 0$$

$$\Leftrightarrow \frac{-3x^2+10x+8}{x^2-3x+4} \le 0 \Leftrightarrow \frac{3x^2-10x-8}{x^2-3x+4} \ge 0 \Leftrightarrow \frac{(3x+2)(x-4)}{(x^2-3x+4)} \ge 0$$

$$\Leftrightarrow (3x+2)(x-4) \ge 0 \qquad \left[\begin{array}{r} \because \text{Disc. of } x^2-3x+4 \text{ is } -\text{ve and coeff. of } x^2 \text{ is } +\text{ ve} \\ \therefore x^2-3x+4>0 \text{ for all } x \end{array}\right]$$

$$\Leftrightarrow x \le -\frac{2}{3} \text{ or } x \ge 4 \Leftrightarrow x \in (-\infty, -2/3] \cup [4, \infty) \qquad \text{[See Fig. 16.5]}$$

Thus, $\dfrac{x^2-2x+24}{x^2-3x+4} \le 4 \Rightarrow x \in (-\infty, -2/3 \cup [4, \infty)$.

Fig. 16.5 Signs of $(3x+2)(x-4)$ for different values of x.

ILLUSTRATION 7 *Solve:* $\dfrac{x^2-4x+7}{x^2-7x+12} \le \dfrac{2}{3}$.

SOLUTION We have,

$$\frac{x^2-4x+7}{x^2-7x+12} \le \frac{2}{3} \Leftrightarrow \frac{x^2-4x+7}{x^2-7x+12} - \frac{2}{3} \le 0$$

$$\Leftrightarrow \frac{3(x^2-4x+7)-2(x^2-7x+12)}{x^2-7x+12} \le 0 \Leftrightarrow \frac{x^2+2x-3}{x^2-7x+12} \le 0 \Leftrightarrow \frac{(x+3)(x-1)}{(x-3)(x-4)} \le 0 \qquad \text{...(i)}$$

On equating all factors in (i) to zero, we get $x = -3, 1, 3, 4$ as critical points. Plotting these points on the number line and marking alternatively '+' and '–' signs from the right most side, we obtain that the inequation in (i) has the signs as shown in Fig. 16.6. Since the expression in (i) is negative, so the solution set of the given in, equation is the union of the regions marked with '–' signs. Hence, from Fig 16.6, we get $x \in [-3, 1] \cup (3, 4)$.

Fig. 16.6 Signs of $\dfrac{(x+3)(x-1)}{(x-3)(x-4)}$ for different values of x

i.e. $\dfrac{x^2-4x+7}{x^2-7x+12} \le \dfrac{2}{3} \Rightarrow x \in [-3, 1] \in (3, 4)$.

It should be noted that 3 and 4 are not included, because denominator becomes zero at $x = 3$ and $x = 4$.

16.3 SOME DEFINITIONS

STRICTLY INCREASING FUNCTION *A function $f(x)$ is said to be a strictly increasing function on (a, b), if*

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

Thus, $f(x)$ is strictly increasing on (a, b) if the values of $f(x)$ increase with the increase in the values of x.

Graphically, $f(x)$ is increasing on (a, b) if the graph $y = f(x)$ moves up as x moves to the right. The graph in Fig. 16.7 is the graph of a strictly increasing function on (a, b).

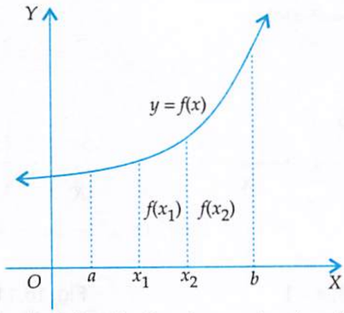

Fig. 16.7 Graph of an increasing function

ILLUSTRATION 1 *Show that the function $f(x) = 2x + 3$ is strictly increasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ and let $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow 2x_1 < 2x_2 \Rightarrow 2x_1 + 3 < 2x_2 + 3 \Rightarrow f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$. So, $f(x)$ is strictly increasing function on R.

This result is also evident from the graph of the function shown in Fig. 16.8.

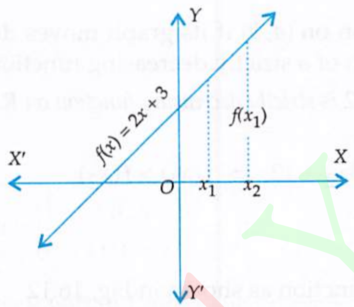

Fig. 16.8 Graph of $f(x) = 2x + 3$

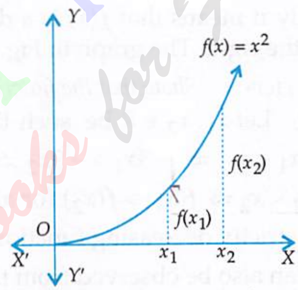

Fig. 16.9 Graph of $f(x) = x^2, x \geq 0$

ILLUSTRATION 2 *Show that the function $f(x) = x^2$ is strictly increasing function on $[0, \infty)$.*

SOLUTION Let $x_1, x_2 \in [0, \infty)$ such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow x_1^2 < x_1 x_2 \qquad \text{[Multiplying both sides by } x_1] \qquad ...(i)$$

again, $$x_1 < x_2 \Rightarrow x_1 x_2 < x_2^2 \qquad \text{[Multiplying both sides by } x_2] \qquad ...(ii)$$

From (i) and (ii), we get

$$x_1 < x_2 \Rightarrow x_1^2 < x_2^2 \Rightarrow f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in [0, \infty)$.

Hence, $f(x)$ is strictly increasing function on $[0, \infty)$ which is evident from the graph shown in Fig. 16.9.

ILLUSTRATION 3 *Show that the function $f(x) = a^x$, $a > 1$ is strictly increasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$$x_1 < x_2$$

$$\Rightarrow a^{x_1} < a^{x_2} \qquad [\because a > 1 \therefore x_1 < x_2 \Rightarrow a^{x_1} < a^{x_2}]$$

$$\Rightarrow f(x_1) < f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in R$.

Hence, $f(x)$ is strictly increasing function on R. This fact is also exhibited in the graph of this function as shown in Fig. 16.10.

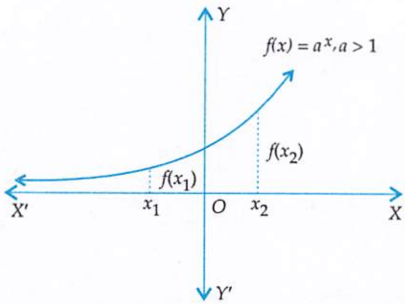

Fig. 16.10 Graph of $f(x) = a^x$, $a > 1$

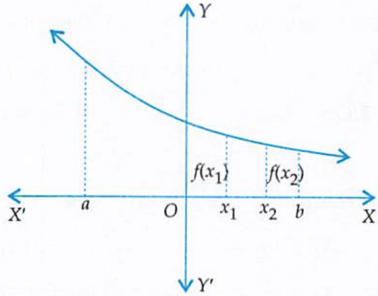

Fig. 16.11 Graph of decreasing function

REMARK *In the above example if we replace a by e (≈ 2.71), then we find that $f(x) = e^x$ is also increasing on R.*

STRICTLY DECREASING FUNCTION *A function $f(x)$ is said to be a strictly decreasing function on (a, b), if*

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \text{ for all } x_1, x_2 \in (a, b)$$

Thus, $f(x)$ is strictly decreasing on (a, b) if the values of $f(x)$ decrease with the increase in the values of x.

Graphically it means that $f(x)$ is a decreasing function on (a, b) if its graph moves down as x moves to the right. The graph in Fig. 16.11 is the graph of a strictly decreasing function.

ILLUSTRATION 4 *Show that the function $f(x) = -3x + 12$ is strictly decreasing function on R.*

SOLUTION Let $x_1, x_2 \in R$ be such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow -3x_1 > -3x_2 \Rightarrow -3x_1 + 12 > -3x_2 + 12 \Rightarrow f(x_1) > f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in R$.

So, $f(x)$ is strictly decreasing function on R.

This fact can also be observed from the graph of the function as shown in Fig. 16.12

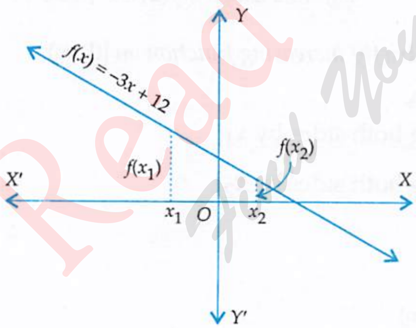

Fig. 16.12 Graph of $f(x) = -3x + 12$

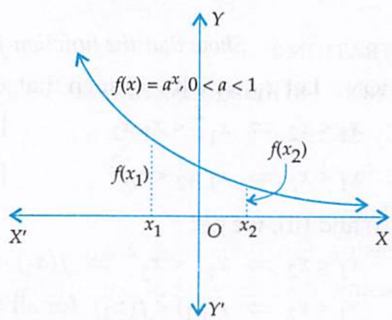

Fig. 16.13 Graph of $f(x) = a^x$, $0 < a < 1$

ILLUSTRATION 5 *Show that the function $f(x) = a^x$, $0 < a < 1$ is strictly decreasing on R.*

SOLUTION Let $x_1, x_2 \in R$ such that $x_1 < x_2$. Then,

$x_1 < x_2$

$\Rightarrow \quad a^{x_1} > a^{x_2}$ $\quad [\because 0 < a < 1 \therefore x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}]$

$\Rightarrow \quad f(x_1) > f(x_2)$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2, \in R$.

Hence, $f(x)$ is strictly decreasing function on R. This is also evident from the graph of $f(x)$ as shown in Fig. 16.13.

REMARK *Since* $0 < e^{-1} = \frac{1}{e} < 1$, *therefore* $f(x) = (e^{-1})^x = e^{-x}$ *is also a strictly decreasing function on R.*

ILLUSTRATION 6 *Show that the function* $f(x) = x^2$ *is a strictly decreasing function on* $(-\infty, 0]$.

SOLUTION Let $x_1, x_2 \in (-\infty, 0]$ be such that $x_1 < x_2$. Then,

$$x_1 < x_2 \Rightarrow x_1^2 > x_1 x_2 \quad \text{...(i)} \qquad \text{and,} \qquad x_1 < x_2 \Rightarrow x_1 x_2 > x_2^2 \quad \text{...(ii)}$$

From (i) and (ii), we obtain

$$x_1 < x_2 \Rightarrow x_1^2 > x_2^2 \Rightarrow f(x_1) > f(x_2)$$

Thus, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (-\infty, 0]$.

Hence, $f(x)$ is strictly decreasing on $(-\infty, 0]$. See also Fig. 16.14.

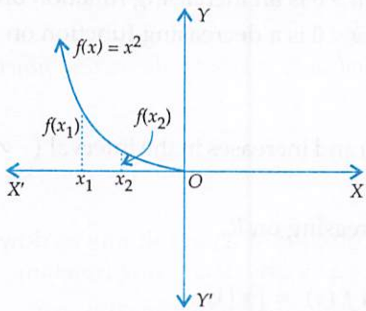

Fig. 16.14 Graph of $f(x) = x^2, x \leq 0$

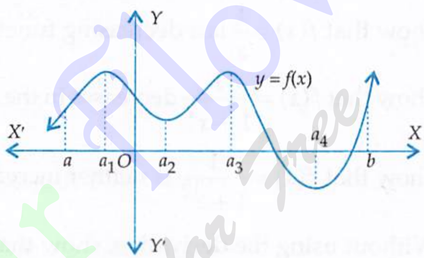

Fig. 16.15

Uptill now, we have been discussing about a strictly increasing or strictly decreasing functions. But, it is possible that a function may neither be strictly increasing nor strictly decreasing on a given interval. For example, $f(x)$ in Fig. 16.15 is neither strictly increasing nor strictly decreasing on (a, b). However, it is increasing on the sub-intervals (a, a_1), (a_2, a_3) and (a_4, b) and decreasing on the intervals (a_1, a_2) and (a_3, a_4).

ILLUSTRATION 7 *Show that the function* $f(x) = x^2$ *is neither strictly increasing nor strictly decreasing on R.*

SOLUTION In illustrations 3 and 6 we have seen that $f(x) = x^2$ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$. Hence, it is neither strictly increasing nor strictly decreasing on R i.e. $(-\infty, \infty)$.

Uptill now we were talking about strictly increasing and strictly decreasing functions. But, there can be functions which are increasing (decreasing) but not strictly increasing (decreasing). For example, consider the function whose graph is shown in Fig. 16.16. Clearly, $f(x)$ is increasing on (a, b) but it is strictly increasing only in the intervals (a, a_1) and (a_2, b). In this chapter, we shall be studying only strictly increasing and strictly decreasing function.

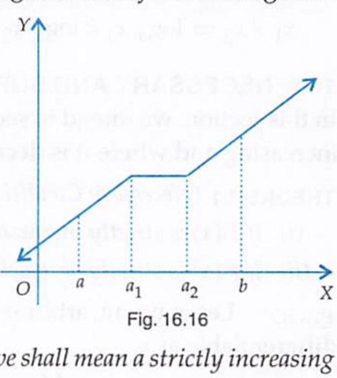

Fig. 16.16

NOTE *From now onwards, by an increasing or a decreasing function we shall mean a strictly increasing or a strictly decreasing function.*

MONOTONIC FUNCTION *A function* $f(x)$ *is said to be monotonic on an interval* (a, b) *if it is either increasing or decreasing on* (a, b).

DEFINITION *A function $f(x)$ is said to be increasing (decreasing) at a point x_0 if there is an interval $(x_0 - h, x_0 + h)$ containing x_0 such that $f(x)$ is increasing (decreasing) on $(x_0 - h, x_0 + h)$.*

DEFINITION *A function $f(x)$ is said to be increasing on $[a, b]$ if it is increasing (decreasing) on (a, b) and it is also increasing (decreasing) at $x = a$ and $x = b$.*

EXERCISE 16.1

BASIC

1. Prove that the function $f(x) = \log_e x$ is increasing on $(0, \infty)$.
2. Prove that the function $f(x) = \log_a x$ is increasing on $(0, \infty)$ if $a > 1$ and decreasing on $(0, \infty)$, if $0 < a < 1$.
3. Prove that $f(x) = ax + b$, where a, b are constants and $a > 0$ is an increasing function on R.
4. Prove that $f(x) = ax + b$, where a, b are constants and $a < 0$ is a decreasing function on R.
5. Show that $f(x) = \dfrac{1}{x}$ is a decreasing function on $(0, \infty)$.
6. Show that $f(x) = \dfrac{1}{1 + x^2}$ decreases in the interval $[0, \infty)$ and increases in the interval $(-\infty, 0]$.
7. Show that $f(x) = \dfrac{1}{1 + x^2}$ is neither increasing nor decreasing on R.
8. Without using the derivative, show that the function $f(x) = |x|$ is
   (a) strictly increasing in $(0, \infty)$ (b) strictly decreasing in $(-\infty, 0)$.
9. Without using the derivative show that the function $f(x) = 7x - 3$ is strictly increasing function on R.

HINTS TO SELECTED PROBLEMS

1. For any $x_1, x_2 \in (0, \infty)$, we have

   $x_1 < x_2 \Rightarrow \log_e x_1 < \log_e x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x)$ is increasing on $(0, \infty)$.

2. Case I When $a > 1$: For any $x_1, x_2 \in (0, \infty)$

   $x_1 > x_2 \Rightarrow \log_a x_1 > \log_a x_2 \Rightarrow f(x_1) > f(x_2) \Rightarrow f(x)$ is increasing on $(0, \infty)$.

   Case II When $a < 1$: For any $x_1, x_2 \in (0, \infty)$

   $x_1 > x_2 \Rightarrow \log_a x_1 < \log_a x_2 \Rightarrow f(x_1) < f(x_2) \Rightarrow f(x)$ is decreasing on $(0, \infty)$

16.4 NECESSARY AND SUFFICIENT CONDITIONS FOR MONOTONICITY

In this section, we intend to see how we can use derivative of a function to determine where it is increasing and where it is decreasing.

THEOREM 1 (Necessary Condition) *Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b).*

(i) If $f(x)$ is strictly increasing on (a, b), then $f'(x) > 0$ for all $x \in (a, b)$.

(ii) If $f(x)$ is strictly decreasing on (a, b), then $f'(x) < 0$ for all $x \in (a, b)$.

PROOF Let x be an arbitrary point in (a, b). Since $f(x)$ is differentiable on (a, b). So, it is differentiable at x.

$$\therefore \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, h > 0 \text{ exists.}$$

(i) If $f(x)$ is strictly increasing on (a, b), then

$$f(x+h) > f(x) \text{ for all } h > 0$$

$$\Rightarrow \quad \frac{f(x+h)-f(x)}{h} > 0 \text{ for all } h>0 \Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} > 0 \Rightarrow f'(x) > 0.$$

Since x is an arbitrary point of (a, b). Therefore, $f'(x) > 0$ for all $x \in (a,b)$.

(ii) If $f(x)$ is strictly decreasing on (a, b), then

$$f(x+h) < f(x) \text{ for all } h>0$$

$$\Rightarrow \quad \frac{f(x+h)-f(x)}{h} < 0 \text{ for all } h>0 \Rightarrow \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} < 0 \Rightarrow f'(x) < 0$$

Since x is an arbitrary point of (a, b). Therefore, $f'(x) < 0$ for all $x \in (a, b)$.

Q.E.D.

REMARK *If $f(x)$ is an increasing function on (a, b), then as shown in Fig. 16.17, the tangent at every point on the curve $y = f(x)$ makes an acute angle θ with the positive direction of x-axis.*

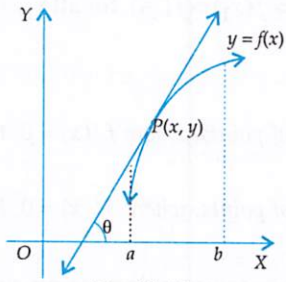

Fig. 16.17

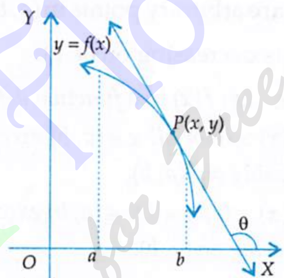

Fig. 16.18

$$\therefore \quad \tan\theta > 0 \Rightarrow \frac{dy}{dx} > 0 \text{ or, } f'(x) > 0 \text{ for all } x \in (a, b)$$

If $f(x)$ is a decreasing function on (a, b), then as shown in Fig. 16.18, the tangent at every point on the curve $y = f(x)$ makes an obtuse angle θ with x-axis.

$$\therefore \quad \tan\theta < 0 \Rightarrow \frac{dy}{dx} < 0 \text{ or, } f'(x) < 0 \text{ for all } x \in (a, b).$$

THEOREM 2 *(Sufficient Condition) Let f be a differentiable real function defined on an open interval (a, b).*

(i) If $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is increasing on (a, b).

(ii) If $f'(x) < 0$ for all $x \in (a, b)$, then $f(x)$ is decreasing on (a, b).

PROOF Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$. Consider the sub-interval $[x_1, x_2]$. Since $f(x)$ is differentiable on (a, b) and $[x_1, x_2] \subset (a, b)$. Therefore, $f(x)$ is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the Lagrange's mean value theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2-x_1} \quad \text{...(i)}$$

(i) Since $f'(x) > 0$ for all $x \in (a, b)$, so in particular, $f'(c) > 0$.

Now, $f'(c) > 0$

$$\Rightarrow \quad \frac{f(x_2)-f(x_1)}{x_2-x_1} > 0 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad f(x_2)-f(x_1) > 0 \quad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$

$\Rightarrow \quad f(x_2) > f(x_1)$ or, $f(x_1) < f(x_2)$

Since x_1, x_2 are arbitrary points in (a, b). Therefore, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$.

Hence, $f(x)$ is increasing on (a, b).

(ii) Since $f'(x) < 0$ for all $x \in (a, b)$, so in particular, $f'(c) < 0$.

Now, $\quad f'(c) < 0$

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad f(x_2) - f(x_1) < 0 \qquad [\because x_2 - x_1 > 0 \text{ when } x_1 < x_2]$$

$$\Rightarrow \quad f(x_2) < f(x_1) \Rightarrow f(x_1) > f(x_2)$$

Since x_1, x_2 are arbitrary points in (a, b). Therefore, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all $x_1, x_2 \in (a, b)$.

Hence, $f(x)$ is decreasing on (a, b).

COROLLARY *Let $f(x)$ be a function defined on (a, b).*

(i) *If $f'(x) > 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is increasing on ((a, b).*

(ii) *If $f'(x) < 0$ for all $x \in (a, b)$ except for a finite number of points, where $f'(x) = 0$, then $f(x)$ is decreasing on (a, b).*

In order to find the interval in which a given function is increasing or decreasing, we may use the following algorithm.

ALGORITHM

Step I *Obtain the function and put it equal to $f(x)$.*

Step II *Find $f'(x)$.*

Step III *Put $f'(x) > 0$ and solve this inequation.*

For the values of x obtained in step III $f(x)$ is increasing and for the remaining points in its domain it is decreasing.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE INTERVALS IN WHICH A FUNCTION IS INCREASING OR DECREASING

EXAMPLE 1 *Find the intervals in which $f(x) = -x^2 - 2x + 15$ is increasing or decreasing.*

SOLUTION We have,

$$f(x) = -x^2 - 2x + 15 \Rightarrow f'(x) = -2x - 2 = -2(x + 1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad -2(x + 1) > 0$$

$$\Rightarrow \quad x + 1 < 0 \qquad [\because -2 < 0 \text{ and } ab > 0, a < 0 \Rightarrow b < 0]$$

$$\Rightarrow \quad x < -1 \Rightarrow x \in (-\infty, -1)$$

Thus, $f(x)$ is increasing on the interval $(-\infty, -1)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$
$\Rightarrow \quad -2(x+1) < 0$
$\Rightarrow \quad x+1 > 0 \qquad [\because -2 < 0 \text{ and } ab < 0, a < 0 \Rightarrow b > 0]$
$\Rightarrow \quad x > -1 \Rightarrow x \in (-1, \infty)$

So, $f(x)$ is decreasing on $(-1, \infty)$.

EXAMPLE 2 *Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is (i) increasing, (ii) decreasing:* **[CBSE 2010, 2011]**

SOLUTION We have,

$$f(x) = 2x^3 - 9x^2 + 12x + 15 \Rightarrow f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2)$$

(i) For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$
$\Rightarrow \quad 6(x^2 - 3x + 2) > 0$
$\Rightarrow \quad x^2 - 3x + 2 > 0 \qquad [\because 6 > 0 \therefore 6(x^2 - 3x + 2) > 0 \Rightarrow x^2 - 3x + 2 > 0]$
$\Rightarrow \quad (x-1)(x-2) > 0$
$\Rightarrow \quad x < 1 \text{ or } x > 2 \Rightarrow x \in (-\infty, 1) \cup (2, \infty)$. [See Fig. 16.19]

So, $f(x)$ is increasing on $(-\infty, 1) \cup (2, \infty)$.

Fig. 16.19 Signs of $f'(x)$ for different values of x

(ii) For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$
$\Rightarrow \quad 6(x^2 - 3x + 2) < 0$
$\Rightarrow \quad x^2 - 3x + 2 < 0 \qquad [\because 6 > 0 \therefore 6(x^2 - 3x + 2) < 0 \Rightarrow x^2 - 3x + 2 < 0]$
$\Rightarrow \quad (x-1)(x-2) < 0 \Rightarrow 1 < x < 2 \Rightarrow x \in (1, 2)$ [See Fig. 16.20]

So, $f(x)$ is decreasing on $(1, 2)$.

Fig. 16.20 Signs of $f'(x)$ for different values of x

EXAMPLE 3 *Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is (i) increasing; (ii) decreasing:*

SOLUTION We have,

$$f(x) = 2x^3 + 9x^2 + 12x + 20 \Rightarrow f'(x) = 6x^2 + 18x + 12 = 6(x^2 + 3x + 2)$$

(i) For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$
$\Rightarrow \quad 6(x^2 + 3x + 2) > 0$
$\Rightarrow \quad (x^2 + 3x + 2) > 0 \qquad [\because 6 > 0 \text{ and } 6(x^2 + 3x + 2) > 0 \therefore x^2 + 3x + 2 > 0]$
$\Rightarrow \quad (x+1)(x+2) > 0$ [See Fig. 16.21]
$\Rightarrow \quad x < -2 \text{ or } x > -1$
$\Rightarrow \quad x \in (-\infty, -2) \cup (-1, \infty)$

So, $f(x)$ is increasing on $(-\infty, -2) \cup (-1, \infty)$

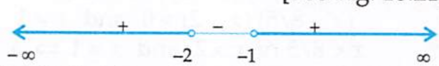

Fig. 16.21 Signs of $f'(x)$ for different values of x

(ii) For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$\Rightarrow \quad 6(x^2 + 3x + 2) < 0$

$\Rightarrow \quad x^2 + 3x + 2 < 0 \qquad [\because 6 > 0 \text{ and } 6(x^2 + 3x + 2) < 0 \;\therefore\; x^2 + 3x + 2 < 0]$

$\Rightarrow \quad (x+1)(x+2) < 0$ [See Fig. 16.22]

$\Rightarrow \quad -2 < x < -1$

So, $f(x)$ is decreasing on $(-2, -1)$.

Fig. 16.22 Signs of $f'(x)$ for different values of x

EXAMPLE 4 *Find the intervals in which $f(x) = (x+1)^3 (x-3)^3$ is increasing or decreasing.*

[NCERT, CBSE 2001C, 2011]

SOLUTION We have,

$f(x) = (x+1)^3 (x-3)^3$

$\Rightarrow \quad f'(x) = \left\{3(x+1)^2 \frac{d}{dx}(x+1)\right\}(x-3)^3 + (x+1)^3 \left\{3(x-3)^2 \frac{d}{dx}(x-3)\right\}$

$\Rightarrow \quad f'(x) = 3(x+1)^2 (x-3)^3 + 3(x+1)^3 (x-3)^2 = 3(x+1)^2 (x-3)^2 (x+1+x-3)$

$\Rightarrow \quad f'(x) = 6(x+1)^2 (x-3)^2 (x-1)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow \quad 6(x+1)^2 (x-3)^2 (x-1) > 0$

$\Rightarrow \quad x - 1 > 0 \text{ and } x \neq -1, 3 \qquad [\because 6(x+1)^2 (x-3)^2 > 0 \text{ for all } x \neq -1, 3]$

$\Rightarrow \quad x > 1 \text{ and } x \neq -1, 3 \Rightarrow x \in (1, 3) \cup (3, \infty)$

So, $f(x)$ is increasing on $(1, 3) \cup (3, \infty)$.

For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow \quad 6(x+1)^2 (x-3)^2 (x-1) < 0$

$\Rightarrow \quad x - 1 < 0 \text{ and } x \neq -1, 3 \qquad [\because 6(x+1)^2 (x-3)^2 > 0 \text{ for all } x \neq -1, 3]$

$\Rightarrow \quad x < 1 \text{ and } x \neq -1, 3 \Rightarrow x \in (-\infty, -1) \cup (-1, 1)$

So, $f(x)$ is decreasing on $(-\infty, -1) \cup (-1, 1)$.

EXAMPLE 5 *Find the intervals in which $f(x) = (x-1)^3 (x-2)^2$ is increasing or decreasing.*

SOLUTION We have,

$f(x) = (x-1)^3 (x-2)^2$

$\Rightarrow \quad f'(x) = 3(x-1)^2 \left\{\frac{d}{dx}(x-1)\right\} (x-2)^2 + (x-1)^3 \, 2 \times (x-2) \frac{d}{dx}(x-2)$

$\Rightarrow \quad f'(x) = 3(x-1)^2 (x-2)^2 + 2(x-1)^3 (x-2) = (x-1)^2 (x-2)(3x - 6 + 2x - 2)$

$\Rightarrow \quad f'(x) = (x-1)^2 (x-2)(5x-8)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow \quad (x-1)^2 (x-2)(5x-8) > 0$

$\Rightarrow \quad (x-2)(5x-8) > 0 \text{ and } x \neq 1 \qquad [\because (x-1)^2 > 0 \text{ for all } x \neq 1]$

$\Rightarrow \quad 5(x - 8/5)(x-2) > 0 \text{ and } x \neq 1$

$\Rightarrow \quad (x - 8/5)(x-2) > 0 \text{ and } x \neq 1 \qquad [\because 5 > 0]$

$\Rightarrow \quad x < 8/5 \text{ or } x > 2 \text{ and } x \neq 1 \Rightarrow x \in (-\infty, 1) \cup (1, 8/5) \cup (2, \infty)$ [See Fig. 16.23]

So, $f(x)$ is increasing on $(-\infty, 1) \cup (1, 8/5) \cup (2, \infty)$.

For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow \quad (x-1)^2 (x-2)(5x-8) < 0$

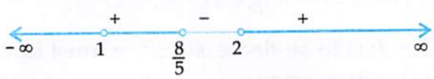

Fig. 16.23 Signs of $f'(x)$ for different values of x

$\Rightarrow \quad (x-2)(5x-8) < 0$ and $x \neq 1$ $\quad [\because (x-1)^2 > 0$ for all $x \neq 1]$

$\Rightarrow \quad 5(x-2)(x-8/5) < 0$ and $x \neq 1$

$\Rightarrow \quad (x-8/5)(x-2) < 0$ and $x \neq 1$ $\quad [\because 5 > 0]$

$\Rightarrow \quad x \in (8/5, 2)$ and $x \neq 1 \Rightarrow x \in (8/5, 2)$ $\quad$ [See Fig. 16.24]

So, $f(x)$ is decreasing on $(8/5, 2)$.

Fig. 16.24 Signs of $f'(x)$ for different values of x

EXAMPLE 6 *Find the intervals in which the function* $f(x) = x^4 - \dfrac{x^3}{3}$ *is increasing or decreasing.*

[NCERT EXEMPLAR]

SOLUTION We have,

$$f(x) = x^4 - \frac{x^3}{3} \Rightarrow f'(x) = 4x^3 - x^2 = x^2(4x-1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow x^2(4x-1) > 0 \Rightarrow 4x-1 > 0 \text{ and } x \neq 0 \qquad [\because x^2 > 0]$$

$$\Rightarrow \quad 4x > 1 \text{ and } x \neq 0 \Rightarrow x > \frac{1}{4} \Rightarrow x \in (1/4, \infty)$$

So, $f(x)$ is increasing on $(1/4, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow x^2(4x-1) > 0 \Rightarrow 4x-1 < 0 \text{ and } x \neq 0 \qquad [\because x^2 > 0 \text{ for all } x \neq 0]$$

$$\Rightarrow \quad 4x < 1 \text{ and } x \neq 0 \Rightarrow x < 1/4 \text{ and } x \neq 0 \Rightarrow x \in (-\infty, 0) \cup (0, 1/4)$$

So, $f(x)$ is decreasing on $(-\infty, 0) \cup (0, 1/4)$.

EXAMPLE 7 *Find the intervals in which the function* $f(x) = \ln(1+x) - \dfrac{2x}{2+x}$ *is increasing or decreasing.*

[CBSE 2012, NCERT]

SOLUTION We have, $f(x) = \ln(1+x) - \dfrac{2x}{2+x}$. Clearly, $f(x)$ is defined for all x satisfying $x+1 > 0$ i.e. $x > -1$. So, domain $(f) = (-1, \infty)$.

Now,

$$f(x) = \ln(1+x) - \frac{2x}{2+x}$$

$$\Rightarrow \quad f'(x) = \frac{1}{1+x}\frac{d}{dx}(x+1) - \frac{(2+x)\times 2 - 2x(0+1)}{(2+x)^2} = \frac{1}{1+x} - \frac{4}{(2+x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{(2+x)^2 - 4(1+x)}{(2+x)^2(1+x)} = \frac{x^2}{(2+x)^2(1+x)} = \left(\frac{x}{2+x}\right)^2 \frac{1}{x+1}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad \left(\frac{x}{2+x}\right)^2 \frac{1}{x+1} > 0$$

$$\Rightarrow \quad \frac{1}{x+1} > 0 \text{ and } x \neq 0 \qquad \left[\because \left(\frac{x}{2+x}\right)^2 > 0 \text{ for all } x \neq 0\right]$$

$$\Rightarrow \quad x+1 > 0 \text{ and } x \neq 0 \Rightarrow x > -1 \text{ and } x \neq 0 \Rightarrow x \in (-1, 0) \cup (0, \infty)$$

So, $f(x)$ is increasing on $(-1, 0) \cup (0, \infty)$.

EXAMPLE 8 *Find the intervals in which* $f(x)=\dfrac{4x^2+1}{x}$ *is increasing or decreasing.* **[CBSE 2004]**

SOLUTION We have, $f(x) = \dfrac{4x^2+1}{x} = 4x+\dfrac{1}{x} \Rightarrow f'(x) = 4-\dfrac{1}{x^2} = \dfrac{4x^2-1}{x^2}$.

For $f(x)$ to be increasing, we must have

$$f'(x)>0$$

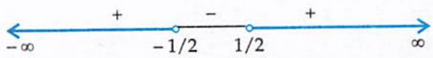

Fig. 16.25 Signs of $f(x)$ for different values of x

$$\Rightarrow \frac{4x^2-1}{x^2}>0$$

$$\Rightarrow 4x^2-1>0$$

$$\Rightarrow (2x-1)(2x+1)>0$$

$$\Rightarrow (x-1/2)(x+1/2)>0 \Rightarrow x<-1/2 \text{ or, } x>1/2 \Rightarrow x\in(-\infty,-1/2)\cup(1/2,\infty) \quad \text{[See Fig. 16.25]}$$

So, $f(x)$ is increasing on $(-\infty,-1/2)\cup(1/2,\infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x)<0$$

$$\Rightarrow \frac{4x^2-1}{x^2}<0$$

$$\Rightarrow 4x^2-1<0 \quad [\because x^2>0]$$

$$\Rightarrow (2x-1)(2x+1)>0 \Rightarrow -\frac{1}{2}<x<\frac{1}{2} \Rightarrow x\in(-1/2,\ 1/2) \quad \text{[See Fig. 16.26]}$$

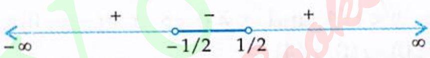

Fig. 16.26 Signs of $f(x)$ for different values of x

But, domain $(f)=R-\{0\}$. So, $f(x)$ is decreasing on $(-1/2,0)\cup(0,1/2)$.

EXAMPLE 9 *Determine the intervals in which the function* $f(x)=x^4-8x^3+22x^2-24x+21$ *is decreasing or increasing.*

SOLUTION We have,

$$f(x) = x^4-8x^3+22x^2-24x+21$$

$$\Rightarrow f'(x) = 4x^3-24x^2+44x-24 = 4(x^3-6x^2+11x-6) = 4(x-1)(x^2-5x+6)$$

For $f(x)$ to be increasing, we must have

$$f'(x)>0$$

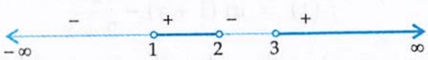

Fig. 16.27 Signs of $f(x)$ for different values of x

$$\Rightarrow 4(x-1)(x^2-5x+6)>0$$

$$\Rightarrow (x-1)(x^2-5x+6)>0 \quad [\because 4>0]$$

$$\Rightarrow (x-1)(x-2)(x-3)>0 \Rightarrow 1<x<2 \text{ or, } 3<x<\infty \Rightarrow x\in(1,2)\cup(3,\infty) \quad \text{[See Fig. 16.27]}$$

So, $f(x)$ is increasing on $(1,2)\cup(3,\infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x)<0$$

$$\Rightarrow 4(x-1)(x^2-5x+6)<0$$

$$\Rightarrow (x-1)(x^2-5x+6)<0 \quad [\because 4>0]$$

$$\Rightarrow (x-1)(x-2)(x-3)<0 \Rightarrow 2<x<3 \text{ or, } x<1 \Rightarrow x\in(2,3)\cup(-\infty,1) \quad \text{[See Fig. 16.28]}$$

So, $f(x)$ is decreasing on $(2,3)\cup(-\infty,1)$.

Fig. 16.28 Signs of $f(x)$ for different values of x

EXAMPLE 10 *Find the intervals for which $f(x) = x^4 - 2x^2$ is increasing or decreasing.*

SOLUTION We have, $f(x) = x^4 - 2x^2 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2 - 1)$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad 4x(x^2 - 1) > 0$$

$$\Rightarrow \quad x(x^2 - 1) > 0 \qquad [\because 4 > 0]$$

Fig. 16.29 Signs of $f'(x)$ for different values of x

$\Rightarrow \quad x(x-1)(x+1) > 0 \Rightarrow -1 < x < 0$ or, $x > 1 \Rightarrow x \in (-1, 0) \cup (1, \infty)$ [See Fig. 16.29]

So, $f(x)$ is increasing on $(-1, 0) \cup (1, \infty)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \quad 4x(x^2 - 1) < 0$$

$$\Rightarrow \quad x(x^2 - 1) < 0 \qquad [\because 4 > 0]$$

Fig. 16.30 Signs of $f'(x)$ for different values of x

$\Rightarrow \quad x(x-1)(x+1) < 0 \Rightarrow x < -1$ or, $0 < x < 1 \Rightarrow x \in (-\infty, -1) \cup (0, 1)$ [See Fig. 16.30]

So, $f(x)$ is decreasing on $(-\infty, -1) \cup (0, 1)$.

EXAMPLE 11 *Determine the values of x for which $f(x) = \dfrac{x-2}{x+1}$, $x \neq -1$ is increasing or decreasing.*

SOLUTION We have, $f(x) = \dfrac{x-2}{x+1}$, $x \neq -1$.

$$\Rightarrow \quad f'(x) = \frac{(x+1) \times 1 - (x-2) \times 1}{(x+1)^2} = \frac{3}{(x+1)^2}, x \neq -1.$$

Clearly, $f'(x) = \dfrac{3}{(x+1)^2} > 0$ for all $x \in R - \{-1\}$. So, $f(x)$ is increasing on $R - \{-1\}$.

EXAMPLE 12 *Find the intervals in which $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$, $x \neq 0$ is increasing or decreasing.*

SOLUTION We have,

$$f(x) = \frac{x}{2} + \frac{2}{x} \Rightarrow f'(x) = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad \frac{x^2 - 4}{2x^2} > 0$$

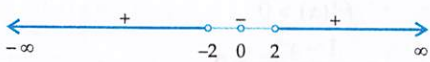

Fig. 16.31 Signs of $f'(x)$ for different values of x

$\Rightarrow \quad x^2 - 4 > 0 \Rightarrow (x-2)(x+2) > 0 \Rightarrow x < -2$ or, $x > 2 \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

So, $f(x)$ is increasing on $(-\infty, -2) \cup (2, \infty)$. [See Fig. 16.31]

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \quad \frac{x^2 - 4}{2x^2} < 0$$

Fig. 16.32 Signs of $f'(x)$ for different values of x

$\Rightarrow \quad x^2 - 4 < 0 \Rightarrow (x-2)(x+2) < 0 \Rightarrow x \in (-2, 2)$ [See Fig. 16.32]

But, domain $(f) = R - \{0\}$. So, $f(x)$ is decreasing on $(-2, 0) \cup (0, 2)$.

EXAMPLE 13 *Find the intervals in which the function f given by $f(x) = x^3 + \dfrac{1}{x^3}$, $x \neq 0$ is*

(i) *increasing* (ii) *decreasing* **[NCERT, CBSE 2009]**

SOLUTION Clearly, domain $(f) = R - \{0\}$.

Now, $f(x) = x^3 + \dfrac{1}{x^3}$

$$\Rightarrow \quad f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3}{x^4}(x^6 - 1) = \frac{3}{x^4}(x^2 - 1)(x^4 + x^2 + 1) = 3\left(\frac{x^4 + x^2 + 1}{x^4}\right)(x^2 - 1)$$

(i) For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad 3\left(\frac{x^4 + x^2 + 1}{x^4}\right)(x^2 - 1) > 0$$

Fig. 16.33 Signs of $f'(x)$ for different values of x

$$\Rightarrow \quad (x^2 - 1) > 0 \qquad \left[\because\ 3\left(\frac{x^4 + x^2 + 1}{x^4}\right) > 0,\ x \neq 0\right]$$

$$\Rightarrow \quad (x - 1)(x + 1) > 0 \Rightarrow x \in (-\infty, -1) \cup (1, \infty) \qquad \text{[See Fig. 16.33]}$$

So, $f(x)$ is increasing on $(-\infty, -1) \cup (1, \infty)$

(ii) For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

Fig. 16.34 Signs of $f'(x)$ for different values of x

$$\Rightarrow \quad 3\left(\frac{x^4 + x^2 + 1}{x^4}\right)(x^2 - 1) < 0$$

$$\Rightarrow \quad x^2 - 1 < 0 \qquad \left[\because\ 3\left(\frac{x^4 + x^2 + 1}{x^4}\right) < 0\right]$$

$$\Rightarrow \quad (x - 1)(x + 1) < 0 \Rightarrow x \in (-1, 0) \cup (0, 1) \qquad \text{[See Fig. 16.34]} \qquad [\because\ x \neq 0]$$

Hence, $f(x)$ is decreasing an $(-1, 0) \cup (0, 1)$.

EXAMPLE 14 *For which values of x, the function* $f(x) = \dfrac{x}{x^2 + 1}$ *is increasing and for which values of x, it is decreasing.*

SOLUTION We have, $f(x) = \dfrac{x}{x^2 + 1} \Rightarrow f'(x) = \dfrac{(x^2 + 1) \times 1 - x(2x + 0)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2}$.

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

Fig. 16.35 Signs of $f'(x)$ for different values of x

$$\Rightarrow \quad \frac{1 - x^2}{(x^2 + 1)^2} > 0$$

$$\Rightarrow \quad 1 - x^2 > 0 \qquad [\because\ (x^2 + 1)^2 > 0]$$

$$\Rightarrow \quad -(x^2 - 1) > 0 \Rightarrow x^2 - 1 < 0 \Rightarrow (x - 1)(x + 1) < 0 \Rightarrow -1 < x < 1 \qquad \text{[See Fig. 16.35]}$$

$$\Rightarrow \quad x \in (-1, 1)$$

So, $f(x)$ is increasing on $(-1, 1)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

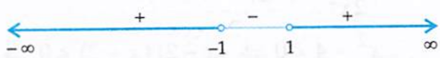

Fig. 16.36 Signs of $f'(x)$ for different values of x

$$\Rightarrow \quad \frac{1 - x^2}{(x^2 + 1)^2} < 0$$

$$\Rightarrow \quad 1 - x^2 < 0 \qquad [\because\ (x^2 + 1)^2 > 0]$$

$$\Rightarrow \quad -(x^2 - 1) < 0 \Rightarrow x^2 - 1 > 0 \Rightarrow (x - 1)(x + 1) > 0 \Rightarrow x < -1 \text{ or } x > 1 \qquad \text{[See Fig. 16.36]}$$

So, $f(x)$ is decreasing on $(-\infty, -1) \cup (1, \infty)$.

EXAMPLE 15 *Find the intervals in which $f(x) = 2 \ln (x-2) - x^2 + 4x + 1$ is increasing or decreasing.*

SOLUTION Clearly, $f(x)$ is defined for all $x > 2$.

Now, $f(x) = 2 \ln (x-2) - x^2 + 4x + 1$

$$\Rightarrow \quad f'(x) = \frac{2}{x-2} - 2x + 4 = \frac{2 - 2x(x-2) + 4(x-2)}{x-2} = \frac{-2x^2 + 8x - 6}{x-2}$$

$$\Rightarrow \quad f'(x) = \frac{-2(x^2 + 4x - 3)}{x-2} = \frac{-2(x-1)(x-3)}{x-2}$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow \frac{-2(x-1)(x-3)}{x-2} > 0 \Rightarrow \frac{(x-1)(x-3)}{x-2} < 0$$

$\Rightarrow \quad x - 3 < 0$ $\quad [\because x \in \text{Domain}(f) \Rightarrow x > 2 \Rightarrow x - 1 > 0 \text{ and } x - 2 > 0]$

$\Rightarrow \quad x < 3 \Rightarrow x \in (2, 3)$ $\quad [\because x > 2]$

So, $f(x)$ is increasing on $(2, 3)$.

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow \frac{-2(x-1)(x-3)}{x-2} < 0 \Rightarrow \frac{(x-1)(x-3)}{x-2} > 0$$

$\Rightarrow \quad x - 3 > 0$ $\quad [\because \text{For } x > 2,\ x - 2 > 0 \text{ and } x - 1 > 0]$

$\Rightarrow \quad x > 3 \Rightarrow x \in (3, \infty)$

So, $f(x)$ is decreasing on $(3, \infty)$.

EXAMPLE 16 *Separate $[0, \pi/2]$ into subintervals in which $f(x) = \sin 3x$ is increasing or decreasing.* [NCERT]

SOLUTION We have, $f(x) = \sin 3x$. Therefore, $f'(x) = 3 \cos 3x$.

Now, $x \in D(f) \Rightarrow 0 < x < \pi/2 \Rightarrow 0 < 3x < 3\pi/2$.

Since cosine function is positive in first quadrant and negative in the second and third quadrants. Therefore, we consider the following cases.

Case I *When* $0 < 3x < \pi/2$ i.e. $0 < x < \pi/6$: In this case, we have

$$0 < 3x < \pi/2 \Rightarrow \cos 3x > 0 \Rightarrow 3 \cos 3x > 0 \Rightarrow f'(x) > 0$$

Thus, $f'(x) > 0$ for $0 < 3x < \pi/2$ i.e. $0 < x < \pi/6$. So, $f(x)$ is increasing on $(0, \pi/6)$.

Case II *When* $\pi/2 < 3x < 3\pi/2$ i.e. $\pi/6 < x < \pi/2$: In this case, we have

$$\pi/2 < 3x < 3\pi/2 \Rightarrow \cos 3x < 0 \Rightarrow 3 \cos 3x < 0 \Rightarrow f'(x) < 0$$

Thus, $f'(x) < 0$ for $\pi/2 < 3x < 3\pi/2$ i.e. $\pi/6 < x < \pi/2$. So, $f(x)$ is decreasing on $(\pi/6, \pi/2)$.

Hence, $f(x)$ is increasing on $(0, \pi/6)$ and decreasing on $(\pi/6, \pi/2)$.

EXAMPLE 17 *Find the intervals in which the function f given by*

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x},\ 0 \le x \le 2\pi$$ [NCERT]

is (i) increasing (ii) decreasing

SOLUTION We have,

$$f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$$

$$\Rightarrow \quad f'(x) = \frac{(2+\cos x)(4\cos x - 2 - \cos x + x\sin x) + (4\sin x - 2x - x\cos x)\sin x}{(2+\cos x)^2}$$

$$\Rightarrow \quad f'(x) = \frac{\cos x\,(4-\cos x)}{(2+\cos x)^2}$$

(i) For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow \frac{\cos x\,(4-\cos x)}{(2+\cos x)^2} > 0 \Rightarrow \cos x > 0 \qquad \left[\because \frac{4-\cos x}{(2+\cos x)^2} > 0\right]$$

$$\Rightarrow \quad x \in (0, \pi/2) \cup (3\pi/2, 2\pi)$$

Hence, $f(x)$ is increasing on $(0, \pi/2) \cup (3\pi/2, 2\pi)$.

(ii) For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow \frac{\cos x\,(4-\cos x)}{(2+\cos x)^2} < 0 \Rightarrow \cos x < 0 \Rightarrow x \in (\pi/2, 3\pi/2)$$

Hence, $f(x)$ is decreasing on $(\pi/2, 3\pi/2)$.

EXAMPLE 18 *Separate the interval* $[0, \pi/2]$ *into sub-intervals in which* $f(x) = \sin^4 x + \cos^4 x$ *is increasing or decreasing.* **[CBSE 2000 C]**

SOLUTION We have, $f(x) = \sin^4 x + \cos^4 x$

$$\Rightarrow \quad f'(x) = 4\sin^3 x\cos x - 4\cos^3 x\sin x = -4\sin x\cos x(\cos^2 x - \sin^2 x)$$

$$\Rightarrow \quad f'(x) = -2(2\sin x\cos x)(\cos 2x) = -2\sin 2x\cos 2x = -\sin 4x$$

We have, $0 < x < \pi/2 \Rightarrow 0 < 4x < 2\pi$.

Since sine function is positive in the first and second quadrants and negative in the third and fourth quadrants. So, we consider the following cases:

Case I *When* $0 < 4x < \pi$ i.e. $0 < x < \pi/4$: In this case, we have

$$\sin 4x > 0 \Rightarrow -\sin 4x < 0 \Rightarrow f'(x) < 0$$

$\therefore$ $f'(x) < 0$ for $0 < 4x < \pi$ i.e. $0 < x < \pi/4$. So, $f(x)$ is decreasing on $[0, \pi/4]$.

Case II *When,* $\pi < 4x < 2\pi$ i.e. $\pi/4 < x < \pi/2$: In this case, we have

$$\sin 4x < 0 \Rightarrow -\sin 4x > 0 \Rightarrow f'(x) > 0$$

$\therefore$ $f'(x) > 0$ for $\pi < 4x < 2\pi$ i.e. $\pi/4 < x < \pi/2$. So, $f(x)$ is increasing on $[\pi/4, \pi/2]$.

EXAMPLE 19 *Find the intervals in which the function f given by* $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ *is increasing or decreasing.* **[NCERT, CBSE 2009, 2017, 2020]**

SOLUTION We have,

$$f(x) = \sin x + \cos x$$

$$\Rightarrow \quad f'(x) = \cos x - \sin x = \sqrt{2}\left(\cos x\sin\frac{\pi}{4} - \sin x\cos\frac{\pi}{4}\right) = \sqrt{2}\sin\left(\frac{\pi}{4} - x\right) = -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

Now, $0 \le x \le 2\pi \Rightarrow 0 - \frac{\pi}{4} < x - \frac{\pi}{4} < 2\pi - \frac{\pi}{4} \Rightarrow -\frac{\pi}{4} < x - \frac{\pi}{4} < \frac{7\pi}{4}$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad -\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) > 0$$

$\Rightarrow \quad \sin\left(x-\frac{\pi}{4}\right)<0$

$\Rightarrow \quad -\frac{\pi}{4}<x-\frac{\pi}{4}<0$ or, $\pi<x-\frac{\pi}{4}<\frac{7\pi}{4}$

$\Rightarrow \quad 0<x<\frac{\pi}{4}$ or, $\frac{5\pi}{4}<x<2\pi \Rightarrow x\in\left(0,\frac{\pi}{4}\right)$ or, $x\in\left(\frac{5\pi}{4},2\pi\right) \Rightarrow x\in(0,\pi/4)\cup(5\pi/4, 2\pi)$

Hence, $f(x)$ is increasing on $(0,\pi/4)\cup(5\pi/4, 2\pi)$.

For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow -\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)<0 \Rightarrow \sin\left(x-\frac{\pi}{4}\right)>0 \Rightarrow 0<x-\frac{\pi}{4}<\pi \Rightarrow \frac{\pi}{4}<x<\frac{5\pi}{4} \Rightarrow x\in\left(\frac{\pi}{4},\frac{5\pi}{4}\right).$

Hence, $f(x)$ is decreasing on $(\pi/4, 5\pi/4)$.

EXAMPLE 20 *Find the intervals in which $f(x)=\sin 3x-\cos 3x$, $0<x<\pi$, is strictly increasing or decreasing.* [CBSE 2016]

SOLUTION We have,

$f(x) = \sin 3x-\cos 3x$

$\Rightarrow f'(x)=3(\cos 3x+\sin 3x)$

$\Rightarrow f'(x)=3\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos 3x+\frac{1}{\sqrt{2}}\sin 3x\right)=3\sqrt{2}\left(\sin\frac{\pi}{4}\cos 3x+\cos\frac{\pi}{4}\sin 3x\right)=3\sqrt{2}\sin\left(3x+\frac{\pi}{4}\right)$

It is given that

$$0<x<\pi \Rightarrow 0<3x<3\pi \Rightarrow \frac{\pi}{4}<3x+\frac{\pi}{4}<3\pi+\frac{\pi}{4} \Rightarrow \frac{\pi}{4}<3x+\frac{\pi}{4}<\frac{13\pi}{4}$$

(i) For $f(x)$ to be strictly increasing, we must have

$f'(x)>0$

$\Rightarrow 3\sqrt{2}\sin\left(3x+\frac{\pi}{4}\right)>0$

$\Rightarrow \sin\left(3x+\frac{\pi}{4}\right)>0$

$\Rightarrow \frac{\pi}{4}<3x+\frac{\pi}{4}<\pi$ or, $2\pi<3x+\frac{\pi}{4}<3\pi$

$\Rightarrow 0<3x<\frac{3\pi}{4}$ or, $\frac{7\pi}{4}<3\pi<\frac{11\pi}{4} \Rightarrow 0<x<\frac{\pi}{4}$ or, $\frac{7\pi}{12}<x<\frac{11\pi}{12} \Rightarrow x\in\left(0,\frac{\pi}{4}\right)\cup\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$

So, $f(x)$ is strictly increasing on $\left(0,\frac{\pi}{4}\right)\cup\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$.

(ii) For $f(x)$ to be strictly decreasing we must have

$f'(x)<0$

$\Rightarrow 3\sqrt{2}\sin\left(3x+\frac{\pi}{4}\right)<0$

$\Rightarrow \sin\left(3x+\frac{\pi}{4}\right)<0$

$\Rightarrow \pi<3x+\frac{\pi}{4}<2\pi$ or, $3\pi<3x+\frac{\pi}{4}<\frac{13\pi}{4}$

$\Rightarrow \frac{3\pi}{4}<3x<\frac{7\pi}{4}$ or, $\frac{11\pi}{4}<3x<3\pi \Rightarrow \frac{\pi}{4}<x<\frac{7\pi}{12}$ or, $\frac{11\pi}{12}<x<\pi \Rightarrow x\in\left(\frac{\pi}{4},\frac{7\pi}{12}\right)\cup\left(\frac{11\pi}{12},\pi\right)$

So, $f(x)$ is strictly decreasing on $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right) \cup \left(\frac{11\pi}{12}, \pi\right)$.

Type II ON PROVING THE MONOTONICITY OF A FUNCTION ON A GIVEN INTERVAL

EXAMPLE 21 *Prove that the function $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing on R.* [NCERT]

SOLUTION We have,

$$f(x) = x^3 - 3x^2 + 3x - 100 \Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$$

Now, $x \in R \Rightarrow (x-1)^2 \geq 0 \Rightarrow f'(x) \geq 0$.

Thus, $f'(x) \geq 0$ for all $x \in R$. Hence, $f(x)$ is increasing on R.

EXAMPLE 22 *Let I be an interval disjointed from $[-1, 1]$. Prove that the function $f(x) = x + \frac{1}{x}$ is increasing on I.* [NCERT]

SOLUTION We have,

$$f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Now, $x \in I \Rightarrow x \notin [-1, 1] \Rightarrow x < -1 \text{ or } x > 1 \Rightarrow x^2 > 1 \Rightarrow x^2 - 1 > 0 \Rightarrow \frac{x^2 - 1}{x^2} > 0 \Rightarrow f'(x) > 0$

Thus, $f'(x) > 0$ for all $x \in I$. Hence, $f(x)$ increasing on I.

EXAMPLE 23 *Show that the function $f(x) = \frac{3}{x} + 7$ is decreasing for $x \in R - \{0\}$.*

SOLUTION We have, $f(x) = \frac{3}{x} + 7 \Rightarrow f'(x) = -\frac{3}{x^2}$

Now, $x \in R, x \neq 0 \Rightarrow \frac{1}{x^2} > 0 \Rightarrow -\frac{3}{x^2} < 0 \Rightarrow f'(x) < 0$.

Hence, $f(x)$ is decreasing for $x \in R, x \neq 0$.

EXAMPLE 24 *Show that the function $x + 1/x$ is increasing for $x > 1$.*

SOLUTION Let $f(x) = x + \frac{1}{x}$. Then, $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$

Now, $x > 1 \Rightarrow x^2 > 1 \Rightarrow \frac{x^2 - 1}{x^2} > 0 \Rightarrow f'(x) > 0$. Hence, $f(x)$ is increasing for $x > 1$.

EXAMPLE 25 *Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function on the interval $(0, \pi/4)$.* [NCERT EXEMPLAR]

SOLUTION We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\Rightarrow \quad f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times \frac{d}{dx}(\sin x + \cos x) = \frac{1}{1 + (\sin + \cos x)^2} \times (\cos x - \sin x)$$

$$\Rightarrow \quad f'(x) = \frac{\cos x (1 - \tan x)}{1 + (\sin x + \cos x)^2}$$

Now, $0 < x < \pi/4$

$$\Rightarrow \quad \cos x > 0, \frac{1}{1 + (\sin x + \cos x)^2} \text{ and } 1 - \tan x > 0 \qquad [\because\ 0 < \tan x < 1 \text{ for } 0 < x < \pi/4]$$

$$\Rightarrow \quad \frac{\cos x(1-\tan x)}{1+(\sin x+\cos x)^2} > 0 \Rightarrow f'(x) > 0$$

Thus, $f'(x) > 0$ for all $x \in (0, \pi/4)$. Hence, $f(x)$ is increasing on $(0, \pi/4)$.

EXAMPLE 26 *Prove that* $f(\theta) = \dfrac{4\sin\theta}{2+\cos\theta} - \theta$ *is an increasing function of* θ *in* $\left[0, \dfrac{\pi}{2}\right]$.

[NCERT, CBSE 2011]

SOLUTION We have,

$$f(\theta) = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

$$\Rightarrow f'(\theta) = \frac{(2+\cos\theta)(4\cos\theta)+4\sin^2\theta}{(2+\cos\theta)^2} - 1 = \frac{8\cos\theta+4}{(2+\cos\theta)^2} - 1 = \frac{4\cos\theta-\cos^2\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow f'(\theta) = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} > 0 \text{ for all } \theta \in \left(0, \frac{\pi}{2}\right). \quad [\because \cos\theta > 0,\ 4-\cos\theta > 0 \text{ and } 2+\cos\theta > 0]$$

Hence, $f(\theta)$ is increasing on $[0, \pi/2]$.

EXAMPLE 27 *Prove that the function* $f(x) = \tan x - 4x$ *is strictly decreasing on* $(-\pi/3, \pi/3)$.

[NCERT EXEMPLAR]

SOLUTION We have, $f(x) = \tan x - 4x$

$$\Rightarrow \quad f'(x) = \sec^2 x - 4 = \frac{1-4\cos^2 x}{\cos^2 x} = \frac{4}{\cos^2 x}\left(\frac{1}{4}-\cos^2 x\right) = \frac{4}{\cos^2 x}\left(\frac{1}{2}+\cos x\right)\left(\frac{1}{2}-\cos x\right)$$

Now,

$$x \in (-\pi/3, \pi/3) \Rightarrow -\frac{\pi}{3} < x < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \cos x < 1 \Rightarrow \frac{1}{2} < \cos x \text{ and } \frac{1}{2}+\frac{1}{2} < \frac{1}{2}+\cos x < \frac{1}{2}+1$$

$$\Rightarrow \quad \frac{1}{2}-\cos x < 0 \text{ and } 1 < \frac{1}{2}+\cos x < \frac{3}{2} \Rightarrow \frac{1}{2}-\cos x < 0 \text{ and } \frac{1}{2}+\cos x > 0$$

$$\Rightarrow \quad \left(\frac{1}{2}-\cos x\right)\left(\frac{1}{2}+\cos x\right) < 0 \Rightarrow \frac{4}{\cos^2 x}\left(\frac{1}{2}-\cos x\right)\left(\frac{1}{2}+\cos x\right) < 0 \Rightarrow f'(x) < 0$$

Hence, f is strictly decreasing on $(-\pi/3, \pi/3)$.

EXAMPLE 28 *Show that* $f(x) = 2x + \cot^{-1} x + \ln\left(\sqrt{1+x^2} - x\right)$ *is incresing on* R.

[NCERT EXEMPLAR]

SOLUTION We have,

$$f(x) = 2x + \cot^{-1} x + \ln\left(\sqrt{1+x^2} - x\right)$$

$$\Rightarrow \quad f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}-x}\left(\frac{x}{\sqrt{1+x^2}} - 1\right)$$

$$\Rightarrow \quad f'(x) = 2 - \frac{1}{1+x^2} + \frac{1}{\sqrt{1+x^2}-x}\left(\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right) = 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \quad f'(x) = \left(1 - \frac{1}{1+x^2}\right) + \left(1 - \frac{1}{\sqrt{1+x^2}}\right) > 0 \text{ for all } x \in R \Rightarrow f'(x) > 0 \text{ for all } x \in R,\ x \neq 0.$$

Hence, $f(x)$ is increasing on R.

EXAMPLE 29 *Test whether the function $f(x) = x^3 - 8$ is increasing on* $[1, 2]$.

SOLUTION We have,

$$f(x) = x^3 - 8 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(x) > 0 \text{ for all } x \in [1, 2]$$

So, $f(x)$ is increasing on $[1, 2]$.

EXAMPLE 30 *Which of the following functions are decreasing on* $(0, \pi/2)$?

(i) $\cos x$ (ii) $\cos 2x$ (iii) $\tan x$ (iv) $\cos 3x$ [NCERT]

SOLUTION (i) We have, $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

Now, $x \in (0, \pi/2) \Rightarrow \sin x > 0 \Rightarrow -\sin x < 0 \Rightarrow f'(x) < 0$

So, $f(x)$ is decreasing on $(0, \pi/2)$.

(ii) Let $f(x) = \cos 2x$. Then, $f'(x) = -2\sin 2x$.

Now, $x \in (0, \pi/2) \Rightarrow 0 < x < \pi/2 \Rightarrow 0 < 2x < \pi \Rightarrow \sin 2x > 0 \Rightarrow -2\sin 2x < 0 \Rightarrow f'(x) < 0$

So, $f(x)$ is decreasing on $(0, \pi/2)$.

(iii) Let $f(x) = \tan x$. Then, $f'(x) = \sec^2 x$.

Now, $x \in (0, \pi/2) \Rightarrow \sec^2 x > 0 \Rightarrow f'(x) > 0$. So, $f(x)$ is increasing on $(0, \pi/2)$.

(iv) Let $f(x) = \cos 3x$. Then, $f'(x) = -3\sin 3x$

Now, $x \in (0, \pi/2) \Rightarrow 0 < x < \pi/2 \Rightarrow 0 < 3x < 3\pi/2 \Rightarrow \sin 3x$ can be positive as well as negative

$\Rightarrow f'(x) = -3\sin 3x$ can be positive as well as negative

So, $f(x)$ is neither increasing nor decreasing on $(0, \pi/2)$.

EXAMPLE 31 *Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing on* $(-1, 1)$.

[CBSE 2014, NCERT]

SOLUTION We have, $f(x) = x^2 - x + 1 \Rightarrow f'(x) = 2x - 1 = 2(x - 1/2)$.

Now, $-1 < x < 1/2 \Rightarrow (x - 1/2) < 0 \Rightarrow 2(x - 1/2) < 0 \Rightarrow f'(x) < 0$

and, $1/2 < x < 1 \Rightarrow x - 1/2 > 0 \Rightarrow 2(x - 1/2) > 0 \Rightarrow f'(x) > 0$.

Thus, $f'(x)$ does not have the same sign throughout the interval $(-1, 1)$. Hence, $f(x)$ is neither increasing or decreasing on $(-1, 1)$.

EXAMPLE 32 *On which of the following intervals is the function $f(x) = x^{100} + \sin x - 1$ increasing?*

(i) $(0, \pi/2)$ (ii) $(\pi/2, \pi)$ (iii) $(0, 1)$ (iv) $(-1, 1)$. [NCERT]

SOLUTION We have, $f(x) = x^{100} + \sin x - 1 \Rightarrow f'(x) = 100x^{99} + \cos x$

(i) $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow 0 < x < \frac{\pi}{2} \Rightarrow x^{99} > 0$ and $\cos x > 0 \Rightarrow 100x^{99} + \cos x > 0 \Rightarrow f'(x) > 0$.

Thus, $f(x)$ is increasing on $(0, \pi/2)$.

(ii) $x \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \frac{\pi}{2} < x < \pi \Rightarrow x^{99} > 1$ $\left[\because \frac{\pi}{2} < x < \pi \Rightarrow \frac{22}{14} < x < \frac{22}{7}\right]$

$\Rightarrow 100x^{99} > 100$ ...(i)

Again, $x \in (\pi/2, \pi) \Rightarrow -1 < \cos x < 0 \Rightarrow 0 > \cos x > -1$ ...(ii)

From (i) and (ii), we find that for $x \in (\pi/2, \pi)$

$100x^{99} > 100$ and $\cos x > -1$

$\Rightarrow 100x^{99} + \cos x > 100 - 1 \Rightarrow 100x^{99} + \cos x > 0 \Rightarrow f'(x) > 0$

Thus, $f(x)$ is increasing on $(\pi/2, \pi)$.

(iii) $x \in (0, 1) \Rightarrow x^{99} > 0 \Rightarrow 100x^{99} > 0$

Again, $x \in (0, 1)$

$\Rightarrow$ x lies between 0 and 1 radian

$\Rightarrow$ x lies between $0°$ and $57°$ $[\because 1° \simeq 57°]$

$\Rightarrow$ x lies in first quadrant $\Rightarrow \cos x > 0$

$\therefore$ $x \in (0, 1) \Rightarrow 100\, x^{99} > 0$ and $\cos x > 0 \Rightarrow 100x^{99} + \cos x > 0 \Rightarrow f'(x) > 0$

Thus, $f(x)$ is increasing on $(0, 1)$.

(iv) We have seen in (iii) that $f'(x) > 0$ for $0 < x < 1$. But, $f'(x)$ can be positive as well as negative when $-1 < x < 0$. So, $f'(x)$ can be positive as well as negative for $x \in (-1, 1)$. Hence, $f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type I ON FINDING THE INTERVAL IN WHICH A FUNCTION IS INCREASING OR DECREASING

EXAMPLE 33 *Determine the values of x for which $f(x) = x^x$, $x > 0$ is increasing or decreasing.*

SOLUTION Clearly, $f(x) = x^x$ is defined for $x > 0$. So, domain $f = (0, \infty)$.

Now, $f(x) = x^x = e^{x \log x}$

$\Rightarrow$ $f'(x) = e^{x \log x} \dfrac{d}{dx}(x \log_e x) \Rightarrow f'(x) = x^x (1 + \log_e x)$

For $f(x)$ to be increasing, we must have

$f'(x) > 0$

$\Rightarrow$ $x^x (1 + \log_e x) > 0$

$\Rightarrow$ $1 + \log_e x > 0$ $[\because x^x > 0 \text{ for } x > 0]$

$\Rightarrow$ $\log_e x > -1$

$\Rightarrow$ $x > e^{-1}$ $[\because \log_a x > N \Rightarrow x > a^N$ for $a > 1$. Here, $e > 1$. So, $\log_e x > -1 \Rightarrow x > e^{-1}]$

$\Rightarrow$ $x \in (1/e, \infty)$

Thus, $f(x)$ is increasing on $(1/e, \infty)$.

For $f(x)$ to be decreasing, we must have

$f'(x) < 0$

$\Rightarrow$ $x^x (1 + \log_e x) < 0$

$\Rightarrow$ $1 + \log_e x < 0$ $[\because x^x > 0 \text{ for } x > 0]$

$\Rightarrow$ $\log_e x < -1 \Rightarrow x < e^{-1} \Rightarrow x \in (0, 1/e)$

Thus, $f(x)$ is decreasing on $(0, 1/e)$.

Hence, $f(x)$ is increasing on $(1/e, \infty)$ and decreasing on $(0, 1/e)$.

EXAMPLE 34 *Find the intervals in which $f(x) = \dfrac{x}{\log x}$ is increasing or decreasing.*

SOLUTION Note that the domain of $f(x)$ is the set of all positive real numbers other than unity i.e. $(0, 1) \cup (1, \infty)$.

Now, $f(x) = \dfrac{x}{\log x} \Rightarrow f'(x) = \dfrac{\log x - 1}{(\log x)^2}$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad \frac{\log x - 1}{(\log x)^2} > 0$$

$$\Rightarrow \quad \log x - 1 > 0 \qquad [\because (\log x)^2 > 0 \text{ for } x > 0,\ x \neq 1]$$

$$\Rightarrow \quad \log x > 1$$

$$\Rightarrow \quad x > e^1 \qquad [\because \log_a x > N \Rightarrow x > a^N \text{ for } a > 1. \text{ Here, } e > 1 \therefore \log_e x > 1 \Rightarrow x > e^1]$$

$$\Rightarrow \quad x \in (e, \infty).$$

So, $f(x)$ is increasing on (e, ∞).

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \quad \frac{\log x - 1}{(\log x)^2} < 0$$

$$\Rightarrow \quad \log x - 1 < 0 \qquad [\because (\log x)^2 > 0 \text{ for } x > 0,\ x \neq 1]$$

$$\Rightarrow \quad \log x < 1 \Rightarrow x < e^1 \Rightarrow x \in (0, e) - \{1\} \qquad [\because f(x) \text{ is defined for } x > 0,\ x \neq 1]$$

So, $f(x)$ is decreasing on $(0, e) - \{1\}$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 35 *If a, b, c are real numbers, then find the intervals in which* $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$ *is increasing or decreasing.*

SOLUTION We have,

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = (x+b^2)(x+c^2) - b^2 c^2 + (x+a^2)(x+c^2) - a^2 c^2 + (x+a)^2 (x+b)^2 - a^2 b^2$$

$$\Rightarrow \quad f'(x) = 3x^2 + 2x(a^2+b^2+c^2).$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$

$$\Rightarrow \quad 3x^2 + 2x(a^2+b^2+c^2) > 0$$

$$\Rightarrow \quad x\left\{3x + 2(a^2+b^2+c^2)\right\} > 0$$

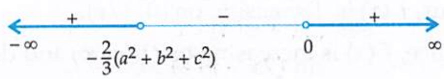

Fig. 16.37 Signs of $f'(x)$ for different values of x

$$\Rightarrow \quad x < -\frac{2}{3}(a^2+b^2+c^2) \text{ or, } x > 0 \qquad \text{[See Fig. 16.37]}$$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{2}{3}(a^2+b^2+c^2)\right) \cup (0, \infty).$$

So, $f(x)$ is increasing on $\left(-\infty, -\frac{2}{3}(a^2+b^2+c^2)\right) \cup (0, \infty)$

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \quad 3x^2 + 2x(a^2+b^2+c^2) < 0$$

$$\Rightarrow \quad x\left\{3x + 2(a^2+b^2+c^2)\right\} < 0$$

$$\Rightarrow \quad -\frac{2}{3}(a^2+b^2+c^2) < x < 0 \qquad \text{[See Fig. 16.38]}$$

$$\Rightarrow \quad x \in \left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$$

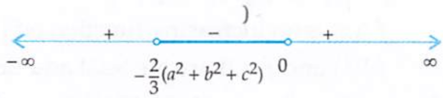

Fig. 16.38 Signs of $f'(x)$ for different values of x

So, $f(x)$ is decreasing on $\left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$.

Hence, $f(x)$ is increasing on $\left(-\infty, -\frac{2}{3}(a^2+b^2+c^2)\right) \cup (0, \infty)$ and decreasing on $\left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$.

Type II ON PROVING MONOTONICITY OF A FUNCTION ON A GIVEN INTERVAL

EXAMPLE 36 *Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing on R.*

SOLUTION We have,

$$f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$$

$$\Rightarrow \quad f'(x) = \sqrt{3}\cos x + \sin x - 2a$$

$$\Rightarrow \quad f'(x) = 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right) - 2a$$

$$\Rightarrow \quad f'(x) = 2\left(\sin\frac{\pi}{3}\cos x + \cos\frac{\pi}{3}\sin x\right) - 2a = 2\sin\left(x + \frac{\pi}{3}\right) - 2a = 2\left\{\sin\left(x+\frac{\pi}{3}\right) - a\right\}$$

$$\Rightarrow \quad f'(x) < 0 \text{ for all } x \in R \qquad \left[\because \sin\left(x + \frac{\pi}{3}\right) \leq 1 \text{ and } a \geq 1 \text{ for all } x \in R\right]$$

Hence, $f(x)$ is decreasing on R.

EXAMPLE 37 *Show that $f(x) = \cos(2x + \pi/4)$ is an increasing function on $(3\pi/8, 7\pi/8)$.*

SOLUTION We have, $f(x) = \cos(2x + \pi/4) \Rightarrow f'(x) = -2\sin(2x + \pi/4)$

Now,

$$x \in (3\pi/8, 7\pi/8) \Rightarrow 3\pi/8 < x < 7\pi/8 \Rightarrow 3\pi/4 < 2x < 7\pi/4$$

$$\Rightarrow \quad \pi/4 + 3\pi/4 < 2x + \pi/4 < 7\pi/4 + \pi/4 \Rightarrow \pi < 2x + \pi/4 < 2\pi$$

$$\Rightarrow \quad \sin(2x + \pi/4) < 0 \qquad [\because \text{sine function is negative in third and fourth quadrants}]$$

$$\Rightarrow \quad -2\sin(2x + \pi/4) > 0 \Rightarrow f'(x) > 0$$

Hence, $f(x)$ is increasing on $(3\pi/8, 7\pi/8)$.

EXAMPLE 38 *Find the least value of 'a' such that the function $f(x) = x^2 + ax + 1$ is increasing on* [1, 2]. *Also, find the greatest value of 'a' for which $f(x)$ is decreasing on* [1, 2]. **[NCERT]**

SOLUTION We have, $f(x) = x^2 + ax + 1$

$$\therefore \quad f'(x) = 2x + a \text{ and } f''(x) = 2 \text{ for all } x.$$

Now, $f''(x) = 2$ for all $x \in (1, 2)$

$\Rightarrow$ $f''(x) > 0$ for all $x \in [1, 2]$

$\Rightarrow$ $f'(x)$ is an increasing function on $[1, 2]$

$\Rightarrow$ $f'(1)$ and $f'(2)$ are the least and the greatest values of $f'(x)$ on $[1, 2]$.

As $f(x)$ is increasing on $[1, 2]$. Therefore, $f'(x) > 0$ for all $x \in [1, 2]$. This is possible when least value of $f'(x)$ i.e. $f'(1) > 0$.

Now, $f'(1) > 0 \Rightarrow 2 + a > 0 \Rightarrow a > -2$

Thus, the least value of a is -2.

If $f(x)$ is decreasing on $[1, 2]$, then

$f'(x) < 0$ for all $x \in [1, 2]$

$\Rightarrow$ Greatest value of $f'(x) < 0$ for $x \in [1, 2]$

$\Rightarrow$ $f'(2) < 0$ $\quad$ [$\because f'(x)$ is increasing on $[1, 2]$ $\therefore f'(2)$ is the greatest value of $f(x)$]

$\Rightarrow$ $4 + a < 0 \Rightarrow a < -4$.

So, the greatest value of a is -4.

NOTE (i) $ax^2 + bx + c > 0$ *for all* $x \Rightarrow a > 0$ *and* $b^2 - 4ac < 0$

(ii) $ax^2 + bx + c < 0$ *for all* $x \Rightarrow a < 0$ *and* $b^2 - 4ac < 0$

(iii) *If the least value of* $f(x)$ *defined on* $[a, b]$ *is positive, then* $f(x) > 0$ *for all* $x \in [a, b]$.

(iv) *If the greatest value of* $f(x)$ *defined on* $[a, b]$ *is negative, then* $f(x) < 0$ *for all* $x \in [a, b]$.

EXAMPLE 39 *Find the values 'a' for which the function* $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ *decreases for all real values of* x.

SOLUTION We have,

$$f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1 \Rightarrow f'(x) = 3(a + 2)x^2 - 6ax + 9a$$

Since $f(x)$ is decreasing for all real values of x. Therefore,

$f'(x) < 0$ for all $x \in R$

$\Rightarrow$ $3(a + 2)x^2 - 6ax + 9a < 0$ for all $x \in R$

$\Rightarrow$ $(a + 2)x^2 - 2ax + 3a < 0$ for all $x \in R$

$\Rightarrow$ $a + 2 < 0$ and $4a^2 - 4 \times (a + 2) \times 3a < 0$ [$\because ax^2 + bx + c < 0$ for all $x \in R \Rightarrow a < 0$ and Disc < 0]

$\Rightarrow$ $a < -2$ and $a^2 - 3a^2 - 6a < 0$

$\Rightarrow$ $a < -2$ and $-2a^2 - 6a < 0$

$\Rightarrow$ $a < -2$ and $-2a(a + 3) < 0$

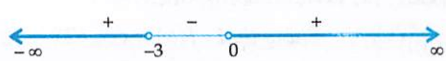

Fig. 16.39 Signs of $a(a + 3)$ for different values of x

Now,

$-2a(a + 3) < 0$

$\Rightarrow$ $a(a + 3) > 0$

$\Rightarrow$ $a < -3$ or, $a > 0$ [See Fig. 16.39]

$\Rightarrow$ $a \in (-\infty, -3) \cup (0, \infty)$

$\therefore$ $a < -2$ and $-2a(a + 3) < 0 \Rightarrow a < -2$ and $a \in (-\infty, -3) \cup (0, \infty) \Rightarrow a \in (-\infty, -3)$.

Hence, $f(x)$ decreases for all $x \in R$, if $a \in (-\infty, -3)$.

EXAMPLE 40 *Find the values of* k *for which* $f(x) = kx^3 - 9kx^2 + 9x + 3$ *is increasing on* R.

SOLUTION It is given that $f(x)$ is increasing on R. Therefore,

$f'(x) > 0$ for all $x \in R$

$\Rightarrow \quad 3kx^2 - 18kx + 9 > 0$ for all $x \in R$

$\Rightarrow \quad kx^2 - 6kx + 3 > 0$ for all $x \in R$

$\Rightarrow \quad k > 0$ and $36k^2 - 12k < 0$ $\quad [\because ax^2 + bx + c > 0$ for all $x \in R \Rightarrow a > 0$ and Disc $< 0]$

$\Rightarrow \quad k > 0$ and $12k(3k-1) < 0$

$\Rightarrow \quad k > 0$ and $k(3k-1) < 0$

$\Rightarrow \quad 3k - 1 < 0$ $\quad [\because k > 0]$

$\Rightarrow \quad k < \frac{1}{3} \Rightarrow k \in (0, 1/3)$.

Hence, $f(x)$ is increasing on R, if $k \in (0, 1/3)$.

EXERCISE 16.2

BASIC

1. Find the intervals in which the following functions are increasing or decreasing.
   (i) $f(x) = 10 - 6x - 2x^2$ [NCERT]
   (ii) $f(x) = x^2 + 2x - 5$ [NCERT]
   (iii) $f(x) = 6 - 9x - x^2$ [NCERT]
   (iv) $f(x) = 2x^3 - 12x^2 + 18x + 15$
   (v) $f(x) = 5 + 36x + 3x^2 - 2x^3$
   (vi) $f(x) = 8 + 36x + 3x^2 - 2x^3$
   (vii) $f(x) = 5x^3 - 15x^2 - 120x + 3$
   (viii) $f(x) = x^3 - 6x^2 - 36x + 2$
   (ix) $f(x) = 2x^3 - 15x^2 + 36x + 1$ [CBSE 2005, 2010]
   (x) $f(x) = 2x^3 + 9x^2 + 12x + 20$ [CBSE 2011]
   (xi) $f(x) = 2x^3 - 9x^2 + 12x - 5$
   (xii) $f(x) = 6 + 12x + 3x^2 - 2x^3$
   (xiii) $f(x) = 2x^3 - 24x + 107$
   (xiv) $f(x) = -2x^3 - 9x^2 - 12x + 1$ [NCERT]
   (xv) $f(x) = (x-1)(x-2)^2$
   (xvi) $f(x) = x^3 - 12x^2 + 36x + 17$ [CBSE 2001]
   (xvii) $f(x) = 2x^3 - 24x + 7$
   (xviii) $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ [NCERT]
   (xix) $f(x) = x^4 - 4x$
   (xx) $f(x) = \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{5}{2}x^2 - 6x + 7$
   (xxi) $f(x) = x^4 - 4x^3 + 4x^2 + 15$
   (xxii) $f(x) = 5x^{3/2} - 3x^{5/2}, x > 0$
   (xxiii) $f(x) = x^8 + 6x^2$
   (xxiv) $f(x) = x^3 - 6x^2 + 9x + 15$ [CBSE 2000, 04]
   (xxv) $f(x) = \{x(x-2)\}^2$ [NCERT, CBSE 2010, 14]
   (xxvi) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ [CBSE 2014]
   (xxvii) $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ [CBSE 2014]
   (xxviii) $f(x) = \log(2+x) - \frac{2x}{2+x}, x \in R$ [CBSE 2014]
   (xxix) $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ [CBSE 2018]
2. Determine the values of x for which the function $f(x) = x^2 - 6x + 9$ is increasing or decreasing. Also, find the coordinates of the point on the curve $y = x^2 - 6x + 9$ where the normal is parallel to the line $y = x + 5$.

3. Find the intervals in which $f(x) = \sin x - \cos x$, where $0 < x < 2\pi$ is increasing or decreasing.
4. Show that $f(x) = e^{2x}$ is increasing on R. [CBSE 2000, 2010]
5. Show that $f(x) = e^{1/x}, x \neq 0$ is a decreasing function for all $x \neq 0$.
6. Show that $f(x) = \ln_a x, 0 < a < 1$ is a decreasing function for all $x > 0$.
7. Show that $f(x) = \sin x$ is increasing on $(0, \pi/2)$ and decreasing on $(\pi/2, \pi)$ and neither increasing nor decreasing in $(0, \pi)$. [NCERT]
8. Show that $f(x) = \ln \sin x$ is increasing on $(0, \pi/2)$ and decreasing on $(\pi/2, \pi)$. [NCERT]
9. Show that $f(x) = x - \sin x$ is increasing for all $x \in R$.
10. Show that $f(x) = x^3 - 15x^2 + 75x - 50$ is an increasing function for all $x \in R$.
11. Show that $f(x) = \cos^2 x$ is a decreasing function on $(0, \pi/2)$.
12. Show that $f(x) = \sin x$ is an increasing function on $(-\pi/2, \pi/2)$.
13. Show that $f(x) = \cos x$ is a decreasing function on $(0, \pi)$, increasing in $(-\pi, 0)$ and neither increasing nor decreasing in $(-\pi, \pi)$.
14. Show that $f(x) = \tan x$ is an increasing function on $(-\pi/2, \pi/2)$.
15. Show that $f(x) = (x-1)e^x + 1$ is an increasing function for all $x > 0$.
16. Prove that the function $f(x) = x^3 - 6x^2 + 12x - 18$ is increasing on R. [CBSE 2002C]
17. State when a function $f(x)$ is said to be increasing on an interval $[a, b]$. Test whether the function $f(x) = x^2 - 6x + 3$ is increasing on the interval $[4, 6]$.
18. Show that $f(x) = \sin x - \cos x$ is an increasing function on $(-\pi/4, \pi/4)$.
19. Show that $f(x) = \tan^{-1} x - x$ is a decreasing function on R.
20. Determine whether $f(x) = -\frac{x}{2} + \sin x$ is increasing or decreasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.
21. Find the intervals in which $f(x) = \ln(1+x) - \frac{x}{1+x}$ is increasing or decreasing. [CBSE 2000 C]
22. Find the intervals in which $f(x) = (x+2)e^{-x}$ is increasing or decreasing. [CBSE 2000 C]
23. Show that the function f given by $f(x) = 10^x$ is increasing for all x.
24. Prove that the function f given by $f(x) = x - [x]$ is increasing in $(0, 1)$.
25. Prove that the following functions are increasing on R:
    (i) $f(x) = 3x^5 + 40x^3 + 240x$ (ii) $f(x) = 4x^3 - 18x^2 + 27x - 27$ [CBSE 2017]
26. Prove that the function f given by $f(x) = \ln \cos x$ is strictly increasing on $(-\pi/2, 0)$ and strictly decreasing on $(0, \pi/2)$. [NCERT]
27. Prove that the function f given by $f(x) = x^3 - 3x^2 + 4x$ is strictly increasing on R. [NCERT]
28. Prove that the function $f(x) = \cos x$ is:
    (i) strictly decreasing in $(0, \pi)$ (ii) strictly increasing in $(\pi, 2\pi)$
    (iii) neither increasing nor decreasing in $(0, 2\pi)$ [NCERT]

BASED ON LOTS

29. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is a decreasing function on the interval $(\pi/4, \pi/2)$.
30. Show that the function $f(x) = \sin(2x + \pi/4)$ is decreasing on $(3\pi/8, 5\pi/8)$.

31. Show that the function $f(x) = \cot^{-1}(\sin x + \cos x)$ is decreasing on $(0, \pi/4)$ and increasing on $(\pi/4, \pi/2)$.

32. Show that the function $x^2 - x + 1$ is neither increasing nor decreasing on $(0, 1)$.

33. Show that $f(x) = x^9 + 4x^7 + 11$ is an increasing function for all $x \in R$.

BASED ON HOTS

34. Show that $f(x) = x^2 - x \sin x$ is an increasing function on $(0, \pi/2)$.

35. Find the value(s) of a for which $f(x) = x^3 - ax$ is an increasing function on R.

36. Find the values of b for which the function $f(x) = \sin x - bx + c$ is a decreasing function on R.

37. Show that $f(x) = x + \cos x - a$ is an increasing function on R for all values of a.

38. Let f defined on $[0, 1]$ be twice differentiable such that $|f''(x)| \le 1$ for all $x \in [0, 1]$. If $f(0) = f(1)$, then show that $|f'(x)| < 1$ for all $x \in [0, 1]$. **[NCERT]**

39. Find the intervals in which $f(x)$ is increasing or decreasing: **[CBSE 2014]**

(i) $f(x) = x|x|, x \in R$

(ii) $f(x) = \sin x + |\sin x|, 0 < x \le 2\pi$

(iii) $f(x) = \sin x(1 + \cos x), 0 < x < \frac{\pi}{2}$

ANSWERS

1.

	Increasing	Decreasing		Increasing	Decreasing
(i)	$(-\infty, -3/2)$	$(-3/2, \infty)$	(ii)	$(-1, \infty)$	$(-\infty, -1)$
(iii)	$(-\infty, -9/2)$	$(-9/2, \infty)$	(iv)	$(-\infty, 1) \cup (3, \infty)$	$(1, 3)$
(v)	$(-2, 3)$	$(-\infty, -2) \cup (3, \infty)$	(vi)	$(-2, 3)$	$(-\infty, -2) \cup (3, \infty)$
(vii)	$(-\infty, -2) \cup (4, \infty)$	$(-2, 4)$	(viii)	$(-\infty, -2) \cup (6, \infty)$	$(-2, 6)$
(ix)	$(-\infty, 2) \cup (3, \infty)$	$(2, 3)$	(x)	$(-\infty, -2) \cup (-1, \infty)$	$(-2, -1)$
(xi)	$(-\infty, 1) \cup (2, \infty)$	$(1, 2)$	(xii)	$(-1, 2)$	$(-\infty, -1) \cup (2, \infty)$
(xiii)	$(-\infty, -2) \cup (2, \infty)$	$(-2, 2)$	(xiv)	$(-2, -1)$	$(-\infty, -2) \cup (-1, \infty)$
(xv)	$(-\infty, 4/3) \cup (2, \infty)$	$(4/3, 2)$	(xvi)	$(-\infty, 2) \cup (6, \infty)$	$(2, 6)$
(xvii)	$(-\infty, -2) \cup (2, \infty)$	$(-2, 2)$	(xviii)	$(-2, 1) \cup (3, \infty)$	$(-\infty, -2) \cup (1, 3)$
(xix)	$(1, \infty)$	$(-\infty, 1)$	(xx)	$(-3, -1) \cup (2, \infty)$	$(-\infty, -3) \cup (-1, 2)$
(xxi)	$(0, 1) \cup (2, \infty)$	$(-\infty, 0) \cup (1, 2)$	(xxii)	$(0, 1)$	$(1, \infty)$
(xxiii)	$(0, \infty)$	$(-\infty, 0)$	(xxiv)	$(-\infty, -1) \cup (3, \infty)$	$(1, 3)$
(xxv)	$(0, 1) \cup (2, \infty)$	$(-\infty, 0) \cup (1, 2)$	(xxvi)	$(-1, 0) \cup (2, \infty)$	$(-\infty, -1) \cup (0, 2)$
(xxvii)	$(-3, 0) \cup (5, \infty)$	$(-\infty, -3) \cup (0, 5)$	(xxviii)	$(2, \infty)$	$(-\infty, 2)$
(xxix)	$(-3, 2) \cup (4, \infty)$	$(-\infty, -3) \cup (2, 4)$			

2. Increasing on $(3, \infty)$ and decreasing on $(-\infty, 3)$; $(5/2, 1/4)$

3. Increasing on $(0, 3\pi/4) \cup (7\pi/4, 2\pi)$; decreasing on $(3\pi/4, 7\pi/4)$

17. Increasing

20. Increasing

21. Increasing on $(0, \infty)$; decreasing on $(-1, 0)$

22. Increasing on $(-\infty, -1)$; decreasing on $(-1, \infty)$

26. Increasing on $(-\infty, -1)$; decreasing on $(-1, \infty)$

35. $a \le 0$

36. $b \ge 1$

39. (i) Increasing for all $x \in R$

(ii) Increasing on $(0, \pi/2)$, decreasing on $(\pi/2, \pi)$, neither increasing nor decreasing on $(\pi, 2\pi)$

(iii) Increasing on $(0, \pi/3)$, decreasing on $(\pi/3, \pi/2)$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. The values of 'a' for which the function $f(x) = \sin x - ax + b$ increases on R are ....
2. The function $f(x) = \dfrac{2x^2 - 1}{x^4}$, $x > 0$, decreases in the interval ....
3. The function $g(x) = x + \dfrac{1}{x}$, $x \neq 0$ decreases in the closed interval ....
4. The largest open interval in which the function $f(x) = \dfrac{1}{1 + x^2}$ decreases is ....
5. The set of values of x for which $f(x) = \tan x - x$ is increasing is ....
6. The set of values of 'a' for which the function $f(x) = \sin x - \cos x - ax + b$ decreases for all real values of x, is .... **[NCERT EXEMPLAR]**
7. The set of values of 'a' for which the $f(x) = ax + b$ is strictly increasing for all real x, is ....
8. If $k\pi$ is the length of the largest interval in which the function $f(x) = 3 \sin x - 4 \sin^3 x$ is increasing, then $k =$ ....
9. The set of values of λ for which the function $f(x) = \dfrac{\lambda \sin x + 6 \cos x}{2 \sin x + 3 \cos x}$ is strictly increasing, is....................... .
10. The largest interval in which $f(x) = x^{1/x}$ is strictly increasing is ....

ANSWERS

1. $(-\infty, -1)$
2. $(1, \infty)$
3. $[-1, 1]$
4. $(0, \infty)$
5. $R = \left\{(2n-1)\dfrac{\pi}{2}, n \in Z\right\}$
6. $(\sqrt{2}, \infty)$
7. $(0, \infty)$
8. $\dfrac{1}{3}$
9. $(4, \infty)$
10. $(1, e)$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. What are the values of 'a' for which $f(x) = a^x$ is increasing on R?
2. What are the values of 'a' for which $f(x) = a^x$ is decreasing on R?
3. Write the set of values of 'a' for which $f(x) = \log_a x$ is increasing in its domain.
4. Write the set of values of 'a' for which $f(x) = \log_a x$ is decreasing in its domain.
5. Find 'a' for which $f(x) = a(x + \sin x) + a$ is increasing on R.
6. Find the values of 'a' for which the function $f(x) = \sin x - ax + 4$ is increasing function on R.
7. Find the set of values of 'b' for which $f(x) = b(x + \cos x) + 4$ is decreasing on R.
8. Find the set of values of 'a' for which $f(x) = x + \cos x + ax + b$ is increasing on R.
9. Write the set of values of k for which $f(x) = kx - \sin x$ is increasing on R.

10. If $g(x)$ is a decreasing function on R and $f(x) = \tan^{-1}\{g(x)\}$. State whether $f(x)$ is increasing or decreasing on R.
11. Write the set of values of a for which the function $f(x) = ax + b$ is decreasing for all $x \in R$.
12. Write the interval in which $f(x) = \sin x + \cos x$, $x \in [0, \pi/2]$ is increasing.
13. State whether $f(x) = \tan x - x$ is increasing or decreasing its domain.
14. Write the set of values of a for which $f(x) = \cos x + a^2 x + b$ is strictly increasing on R.

ANSWERS

1. $a > 1$
2. $0 < a < 1$
3. $a > 1$
4. $0 < a < 1$
5. $a \in (0, \infty)$
6. $a \in (-\infty, -1)$
7. $b \in (-\infty, 0)$
8. $a \in (0, \infty)$
9. $k \in (1, \infty)$
10. Decreasing
11. $a \in (-\infty, 0)$
12. $[0, \pi/4]$
13. Increasing
14. $a \in (-\infty, -1] \cup [1, \infty)$