CHAPTER 29

LINEAR PROGRAMMING

29.1 INTRODUCTION

The term 'programming' means planning and it refers to a particular plan of action from amongst several alternatives for maximizing profit or minimizing cost etc. Programming problems deal with determining optimal allocation of limited resources to meet the given objectives, such as least cost, maximum profit, highest margin or least time, when resources have alternative uses.

The term 'Linear' means that all inequations or equations used and the function to be maximized or minimized are linear. That is why linear programming deals with that class of problems for which all relations among the variables involved are linear.

Formally, linear programming deals with the optimization (maximization or minimization) of a linear function of a number of variables subject to a number of conditions on the variables, in the form of linear inequations or equations in variables involved.

In this chapter, we shall discuss mathematical formulation of linear programming problems that arise in trade, industry, commerce and military operations. We shall also discuss some elementary techniques to solve linear programming problems in two variables only.

29.2 LINEAR PROGRAMMING PROBLEMS

In this section, we shall discuss the general form of a linear programming problem. To give the general description of a linear programming problem, let us consider the following problem:

Suppose that a furniture dealer makes two products viz. chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 4 hours on machine A and 2 hours on machine B. There are 16 hours of time per day available on machine A and 20 hours on machine B. Profits gained by the manufacturer from a chair and a table are ₹ 300 and ₹ 500 respectively. The manufacturer is willing to know the daily product of each of the two products to maximize his profit.

The above data can be put in the following tabular form:

Item	*Chair*	*Table*	*Maximum available time*
Machine A	2 hrs	4 hrs	16 hrs
Machine B	6 hrs	2 hrs	20 hrs
Profit (in Rs)	₹ 300	₹ 500	

To maximize his profit, suppose that the manufacturer produces x chairs and y tables per day. It is given that a chair requires 2 hours on machine A and a table requires 4 hours on machine A. Hence, the total time taken by machine A to produce x chairs and y tables is $2x + 4y$. This must be less than or equal to the total hours available on machine A. Hence, $2x + 4y \le 16$. Similarly, for machine B, we have

$$6x + 2y \le 20.$$

The total profit for x chairs and y tables is $300x + 500y$. Since the number of chairs and tables is never negative. Therefore, $x \geq 0$ and $y \geq 0$.

Thus, we have to maximize

$$\text{Profit} = 300x + 500y$$

Subject to the constraints

$$2x + 4y \leq 16$$
$$6x + 2y \leq 20$$
$$x \geq 0, \quad y \geq 0$$

Out of all the points (x, y) in the solution set of the above linear constraints, the manufacturer has to choose that point, or those points for which the profit $300x + 500y$ has the maximum value.

In the above discussion if a chair costs ₹ 1250 and a table costs ₹ 3000 then the total cost of producing x chairs and y tables is $1250x + 3000y$. Now, the manufacturer will be interested to choose that point, or those points, in the solution set of the above linear constraints for which the cost $1250x + 3000y$ has the minimum value.

The two situations discussed above give the description of a type of linear programming problems. In the above discussion, the profit function $= 300x + 500y$ or the cost function $= 1250x + 3000y$ is known as the objective function. The inequations $2x + 4y \leq 16$, $6x + 2y \leq 20$ are known as the constraints and $x \geq 0$, $y \geq 0$ are known as the non-negativity restrictions.

The general mathematical description of a linear programming problem (LPP) is given below:

$$\text{Optimize } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \qquad \textit{(objective function)}$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n \ (\leq, =, \geq)\ b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots + a_{2n} x_n \ (\leq, =, \geq)\ b_2$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad \textit{(constraints)}$$
$$a_{m1} x_1 + a_{m2} x_2 + a_{m3} x_3 + \ldots + a_{mn} x_n \ (\leq, =, \geq)\ b_m$$
$$x_1, x_2, x_3, \ldots, x_n \geq 0 \qquad \textit{(non-negativity restrictions)}$$

where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

The above linear programming problem may also be written in the matrix form as follows :

$$\text{Optimize (maximize or minimize)} \quad Z = [c_1\ c_2\ \ldots\ c_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Subject to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \{\leq, =, \geq\} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_1, x_2, x_3, \ldots, x_n \geq 0$$

OR

Optimize (Maximize or Minimize) $Z = CX$

Subject to $AX \ (\leq, =, \geq)\ B$

$$X \geq 0,$$

where $C = [c_1 \;\; c_2 \ldots c_n]$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

29.3 SOME DEFINITIONS

In this section, we shall formally define various terms used in a linear programming problem.

As discussed in the previous section, the general form of a linear programming problem is

$$\text{Optimize (Maximize or Minimize) } Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n &\,(\le, =, \ge)\, b_1 \\ a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n &\,(\le, =, \ge)\, b_2 \\ \vdots \qquad \vdots \qquad\quad \vdots & \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n &\,\{\le, =, \ge\{\, b_m \\ x_1, x_2, \ldots, x_n &\ge 0 \end{aligned}$$

The definitions of various terms related to a LPP are as follows:

OBJECTIVE FUNCTION *If $c_1, c_2, \ldots, c_n$ are constants and $x_1, x_2, \ldots, x_n$ are variables, then the linear function $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ which is to be maximized or minimized is called the objective function.*

The objective function describes the primary purpose of the formulation of a linear programming problem and it is always non-negative. In business applications, the profit function which is to be maximized or the cost function which is to be minimized is called the objective function.

CONSTRAINTS *The inequations or equations in the variables of a LPP which describe the conditions under which the optimisation (maximization or minimization) is to be accomplished are called constraints.*

In the constraints given in the general form of a LPP there may be any one of the three signs $\le, =, \ge$.

Inequations in the form of greater than (or less than) indicate that the total use of the resources must be more than (or less than) the specified amount whereas equations in the constraints indicate that the resources described are to be fully used.

NON-NEGATIVITY RESTRICTIONS These are the constraints which describe that the variables involved in a LPP are non-negative.

29.4 MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING PROBLEMS

In the previous section, we have introduced the general form of a linear programming problem (LPP). In this section, we shall discuss the formulation of linear programming problems. Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form. There is not any set procedure to formulate linear programming problems. In fact, one can only learn the formulation with adequate practice. However, the following algorithm will be helpful in the formulation of linear programming problems.

ALGORITHM

Step I *In every LPP certain decisions are to be made. These decisions are represented by decision variables. These decision variables are those quantities whose values are to be determined. Identify the variables and denote them by $x_1, x_2, x_3, \ldots$*

Step II *Identify the objective function and express it as a linear function of the variables introduced in step I.*

Step III *In a LPP, the objective function may be in the form of maximizing profits or minimizing costs. So, after expressing the objective function as a linear function of the decision variables, we must find the type of optimization i.e. maximization or minimization. Identify the type of the objective function.*

Step IV *Identify the set of constraints, stated in terms of decision variables and express them as linear inequations or equations as the case may be..*

The following examples will illustrate the formulation of linear programming problems in various situations.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I OPTIMAL PRODUCT LINE PROBLEMS

EXAMPLE 1 *A factory produces two products P_1 and P_2. Each of the product P_1 requires 2 hrs for moulding, 3 hrs for grinding and 4 hrs for polishing, and each of the product P_2 requires 4 hrs for moulding, 2 hrs for grinding and 2 hrs for polishing. The factory has moulding machine available for 20 hrs, grinding machine for 24 hrs and polishing machine available for 13 hrs. The profit is ₹ 5 per unit of P_1 and ₹ 3 per unit of P_2 and the factory can sell all that it produces. Formulate the problem as a linear programing problem to maximize the profit.*

SOLUTION The given data may be put in the following tabular form:

Product / *Resources*	P_1	P_2	*Capacity*
Moulding	2	4	20
Grinding	3	2	24
Polishing	4	2	13
Profit in ₹	5	3	

Suppose x units of product P_1 and y units of product P_2 are produced to maximize the profit. Let Z denote the total profit.

Since each unit of product P_1 requires 2 hrs for moulding and each unit of product P_2 requires 4 hrs for moulding. Hence, the total hours required for moulding for x units of product P_1 and y units of product P_2 are $2x + 4y$. This must be less than or equal to the total hours available for moulding. Hence,

$$2x + 4y \leq 20$$

This is the first constraint.

The total hours required for grinding for x units of product P_1 and y units of product P_2 is $3x + 2y$. But, the maximum number of hours available for grinding is 24.

$\therefore \quad 3x + 2y \leq 24$

This is the second constraint.

Similarly, for polishing the constraint is $4x + 2y \leq 13$.

Since x and y are non-negative integers, therefore $x \geq 0, \; y \geq 0$.

The total profit for x units of product P_1 and y units of product P_2 is $5x + 3y$. Since we wish to maximize the profit, therefore the objective function is

$$\text{Maximize} \quad Z = 5x + 3y$$

Hence, the linear programming problem for the given problem is as follows:

$$\textit{Maximize} \quad Z = 5x + 3y$$

Subject to the constraints

$$2x + 4y \le 20$$
$$3x + 2y \le 24$$
$$4x + 2y \le 13$$

and, $x \ge 0,\ y \ge 0$

EXAMPLE 2 *A toy company manufactures two types of doll; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes profit of ₹ 3 and ₹ 5 per doll respectively on doll A and doll B; how many of each should be produced per day in order to maximize profit?*

SOLUTION Let x dolls of type A and y dolls of type B be produced per day. Then,

Total profit $= 3x + 5y$.

Since each doll of type B takes twice as long to produce as one of type A, therefore total time taken to produce x dolls of type A and y dolls of type B is $x + 2y$. But, the company has time to make a maximum of 2000 dolls per day

$\therefore \quad x + 2y \le 2000$

Since plastic is available to produce 1500 dolls only.

$\therefore \quad x + y \le 1500$

Also fancy dress is available for 600 dolls per day only

$\therefore \quad y \le 600$

Since the number of dolls cannot be negative. Therefore,

$$x \ge 0,\ y \ge 0$$

Hence, the linear programming problem for the given problem is as follows:

Maximize $Z = 3x + 5y$

Subject to the constraints

$$x + 2y \le 2000$$
$$x + y \le 1500$$
$$y \le 600$$

and, $x \ge 0,\ y \ge 0$

EXAMPLE 3 *A firm can produce three types of cloth, say C_1, C_2, C_3. Three kinds of wool are required for it, say red wool, green wool and blue wool. One unit of length C_1 needs 2 metres of red wool, 3 metres of blue wool; one unit of cloth C_2 needs 3 metres of red wool, 2 metres of green wool and 2 metres of blue wool; and one unit of cloth C_3 needs 5 metres of green wool and 4 metres of blue wool. The firm has only a stock of 16 metres of red wool, 20 metres of green wool and 30 metres of blue wool. It is assumed that the income obtained from one unit of length of cloth C_1 is ₹ 6, of cloth C_2 is ₹ 10 and of cloth C_3 is ₹ 8. Formulate the problem as a linear programming problem to maximize the income.*

SOLUTION The given information can be put in the following tabular form:

	Cloth C_1	*Cloth* C_2	*Cloth* C_3	*Total quality of wool available*
Red Wool	2	3	0	16
Green Wool	0	2	5	20
Blue Wool	3	2	4	30
Income (in ₹)	6	10	8	

Let x_1, x_2 and x_3 be the quantity produced in metres of the cloth of type C_1, C_2 and C_3 respectively.

Since 2 metres of red wool are required for one metre of cloth C_1 and x_1 metres of cloth C_1 are produced, therefore $2x_1$ metres of red wool will be required for cloth C_1. Similarly, cloth C_2 requires $3x_2$ metres of red wool and cloth C_3 does not require red wool. Thus, the total quantity of red wool required is $2x_1 + 3x_2 + 0x_3$.

But, the maximum available quantity of red wool is 16 metres.

$\therefore \quad 2x_1 + 3x_2 + 0x_3 \le 16$

Similarly, the total quantities of green and blue wool required are

$$0x_1 + 2x_2 + 5x_3 \text{ and } 3x_1 + 2x_2 + 4x_3 \text{ respectively.}$$

But, the total quantities of green and blue wool available are 20 metres and 30 metres respectively.

$\therefore \quad 0x_1 + 2x_2 + 5x_3 \le 20$ and $3x_1 + 2x_2 + 4x_3 \le 30$

Also, we cannot produce negative quantities, therefore

$$x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0$$

The total income is $Z = 6x_1 + 10x_2 + 8x_3$

Hence, the linear programming problem for the given problem is

Maximize $Z = 6x_1 + 10x_2 + 8x_3$

Subject to the constraints

$$2x_1 + 3x_2 + 0x_3 \le 16$$
$$0x_1 + 2x_2 + 5x_3 \le 20$$
$$3x_1 + 2x_2 + 4x_3 \le 30$$

and, $\quad x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0$

EXAMPLE 4 *A furniture firm manufactures chairs and tables, each requiring the use of three machines A, B and C. Production of one chair requires 2 hours on machine A, 1 hour on machine B, and 1 hour on machine C. Each table requires 1 hour each on machines A and B and 3 hours on machine C. The profit realized by selling one chair is ₹ 30 while for a table the figure is ₹ 60. The total time available per week on machine A is 70 hours, on machine B is 40 hours, and on machine C is 90 hours. How many chairs and tables should be made per week so as to maximize profit? Develop a mathematical formulation.*

SOLUTION The given data may be put in the following tabular form:

Machine	*Chair*	*Table*	*Available time per week (in hours)*
A	2	1	70
B	1	1	40
C	1	3	30
Profit per unit	₹ 30	₹ 60	

Let x chairs and y tables be produced per week to maximize the profit. Then, the total profit for x chairs and y tables is $30x + 60y$.

It is given that a chair requires 2 hours on machine A and a table requires 1 hour on machine A. Therefore, the total time taken by machine A to produce x chairs and y tables is $(2x + y)$ hours. This must be less than or equal to total hours available on machine A.

$\therefore \quad 2x + y \le 70$

Similarly, the total time taken by machine B to produce x chairs and y tables is $(x + y)$ hours. But, the total time available per week on machine B is 40 hours.

$\therefore \quad x + y \le 40$

Finally, the total time taken by machine C to produce x chairs and y tables is $x + 3y$ hours and the total time available per week on machine C is 90 hours.

$\therefore \quad x + 3y \leq 90$

Since the number of chairs and tables cannot be negative.

$\therefore \quad x \geq 0$ and $y \geq 0$

Let Z denote the total profit. Then,

$$Z = 30x + 60y$$

Hence, the mathematical form of the given LPP is as follows:

Maximize $Z = 30x + 60y$

Subject to

$$2x + y \leq 70$$
$$x + y \leq 40$$
$$x + 3y \leq 90$$

and, $\quad x \geq 0,\ y \geq 0$

EXAMPLE 5 *A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes one hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is ₹ 8 per bottle for A and ₹ 7 per bottle for B. Formulate this problem as a linear programming problem.*

SOLUTION Suppose the manufacturer produces x bottles of medicines A and y bottles of medicine B.

Since the profit is ₹ 8 per bottle for A and ₹ 7 per bottle for B. So, total profit in producing x bottles of medicine A and y bottles of medicine B is ₹ $(8x + 7y)$.

Let Z denote the total profit. Then,

$$Z = 8x + 7y$$

Since 1000 bottles of medicine A are prepared in 3 hours.

$\therefore \quad$ Time required to prepare x bottles of medicine $A = \dfrac{3x}{1000}$ hours.

It is given that 1000 bottles of medicine B are prepared in 1 hour.

$\therefore \quad$ Time required to prepare y bottles of medicine $B = \dfrac{y}{1000}$ hours.

Thus, total time required to prepare x bottles of medicine A and y bottles of medicine B is $\dfrac{3x}{1000} + \dfrac{y}{1000}$ hours. But, the total time available for this operation is 66 hours.

$\therefore \quad \dfrac{3x}{1000} + \dfrac{y}{1000} \leq 66$

$\Rightarrow \quad 3x + y \leq 66{,}000$

Since there are only 45,000 bottles into which the medicines can be put.

$\therefore \quad x + y \leq 45{,}000$

It is given that the ingredients are available for 20,000 bottles of A and 40,000 bottles of B.

$\therefore \quad x \leq 20{,}000$ and $y \leq 40{,}000$

Since the number of bottles can not be negative. Therefore, $x \geq 0$, $y \geq 0$.

Hence, the mathematical formulation of the given LPP is as follows:

Maximize $Z = 8x + 7y$

Subject to

$$3x + y \le 66{,}000$$
$$x + y \le 45{,}000$$
$$x \le 20{,}000$$
$$y \le 40{,}000$$

and, $x \ge 0,\ y \ge 0.$

EXAMPLE 6 *A resourceful home decorator manufactures two types of lamps say A and B. Both lamps go through two technicians, first a cutter, second a finisher. Lamp A requires 2 hours of the cutter's time and 1 hour of the finisher's time. Lamp B requires 1 hour of cutter's and 2 hours of finisher's time. The cutter has 104 hours and finisher has 76 hours of time available each month. Profit on one lamp A is ₹ 6.00 and on one lamp B is ₹ 11.00. Assuming that he can sell all that he produces, how many of each type of lamps should he manufacture to obtain the best return.*

SOLUTION The above information can be put in the following tabular form:

Lamp	*Cutter's time*	*Finisher's time*	*Profit in ₹*
A	2	1	6
B	1	2	11
Maximum time available	104	76	

Let the decorator manufacture x lamps of type A and y lamps of type B.

$\therefore$ Total profit = ₹ $(6x + 11y)$

Total time taken by the cutter in preparing x lamps of type A and y lamps of type B is $(2x + y)$ hours. But, the cutter has 104 hours only for each month.

$\therefore$ $2x + y \le 104$

Similarly, the total time taken by the finisher in preparing x lamps of type A and y lamps of type B is $(x + 2y)$ hours. But, the cutter has 76 hours only for each month.

$\therefore$ $x + 2y \le 76$

Since the number of lamps cannot be negative.

$\therefore$ $x \ge 0$ and $y \ge 0$

Let Z denote the total profit. Then, $Z = 6x + 11y$.

Since the profit is to be maximized. So, the mathematical formulation of the given LPP is as follows:

Maximize $Z = 6x + 11y$

Subject to

$$2x + y \le 104$$
$$x + 2y \le 76$$

and, $x \ge 0\ ,\ y \ge 0$

EXAMPLE 7 *A company makes two kinds of leather belts, A and B. Belt A is high quality belt, and B is of lower quality. The respective profits are ₹ 40 and ₹ 30 per belt. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle, and only 400 buckles per day are available. There are only 700 buckles available for belt B. What should be the daily production of each type of belt? Formulate the problem as a LPP.*

SOLUTION Suppose the company makes per day x belts of type A and y belts of type B.

$\therefore$ Profit $= 40x + 30y$.

Let Z denote the profit. Then, $Z = 40x + 30y$ and it is to be maximized.

It is given that 1000 belts of type B can be made per day and each belt of type A requires twice as much time as a belt of type B. So, 500 belts of type A can be made in a day.

So, total time taken in preparing x belts of type A and y belts of type B is $\left(\frac{x}{500}+\frac{y}{1000}\right)$. But the company is making x belts of type A and y belts of type B in a day.

$\therefore \quad \frac{x}{500}+\frac{y}{1000} \leq 1 \Rightarrow 2x+y \leq 1000$

Since the supply of leather is sufficient for only 800 belts per day.

$\therefore \quad x+y \leq 800$

It is given that only 400 fancy buckles for type A and 700 buckles for type B are available per day.

$\therefore \quad x \leq 400, \; y \leq 700$

Finally, the number of belts cannot be negative.

$\therefore \quad x \geq 0$ and $y \geq 0$

Thus, the mathematical formulation of the given LPP is as follows:

Maximize $Z = 40x+30y$

Subject to

$$2x+y \leq 1000$$
$$x+y \leq 800$$
$$x \leq 400$$
$$y \leq 700$$

and, $x \geq 0, \; y \geq 0.$

Type II DIET PROBLEMS

EXAMPLE 8 *A dietician whishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of Vitamin A and 10 units of vitamin C. Food 'I' contains 2 units per kg of vitamin A and 1 unit per kg of vitamin C while food 'II' contains 1 unit per kg of vitamin A and 2 units per kg of vitamin C. It costs ₹ 50.00 per kg to purchase food 'I' and ₹ 70.00 per kg to produce food 'II'. Formulate the above linear programming problem to minimize the cost of such a mixture.* **[CBSE 2011]**

SOLUTION The gives data may be put in the following tabular form:

Resources	Food		*Requirements*
	I	II	
Vitamin A	2	1	8
Vitamin C	1	2	10
Cost (in ₹)	50	70	

Let the dietician mix x kg of food 'I' and y kg of food 'II'. Clearly, $x \geq 0, y \geq 0$.

Since one kg of food 'I' costs ₹ 50 and one kg of food 'II' costs Rs. 70. Therefore, total cost of x kg of food 'I' and y kg of food 'II' is ₹ $(50x+70y)$.

Let Z denote the total cost. Then,

$$Z = 5x+7y$$

Since one kg of food 'I' contains 2 units of vitamin A. Therefore, x kg of food 'I' contain $2x$ units of vitamin A. One kg of food 'II' contains one unit of vitamin A. So, y kg of food 'II' contains y units of vitamin A. Thus, x kg of food 'I' and y kg of food 'II' contain $2x+y$ units of vitamin A. But, the minimum requirement of vitamin A is 8 units.

$\therefore \quad 2x+y \geq 8$

Similarly, total amount of vitamin C supplied by x units of food 'I' and y units of food 'II' is $(x+2y)$ units and the minimum requirement of vitamin C is 10 units.

$\therefore \quad x+2y \geq 10$

Hence, the mathematical model of the LPP is as follows:

$$\text{Minimize } Z = 5x + 7y$$

Subject to

$$2x + y \geq 8$$
$$x + 2y \geq 10$$

and, $x, y \geq 0.$

EXAMPLE 9 *A diet is to contain at least 400 units of carbohydrate, 500 units of fat, and 300 units of protein. Two foods are available : F_1, which costs ₹ 2 per unit, and F_2, which costs ₹ 4 per unit. A unit of food F_1 contains 10 units of carbohydrate, 20 units of fat, and 15 units of protein; a unit of food F_2 contains 25 units of carbohydrate, 10 units of fat, and 20 units of protein. Find the minimum cost forA diet that consists of A mixture of these two foods and also meets the minimum nutrition requirements. Formulate the problem as A linear programming problem.*

SOLUTION The given data may be put in the following tabular form:

Food	*Carbohydrate*	*Fat*	*Protein*	*Cost per unit*
F_1	10	20	15	₹ 2
F_2	25	10	20	₹ 4
Minimum requirement	400	500	300	

Suppose the diet contains x units of food F_1 and y units of food F_2.

Since one unit of food F_1 costs ₹ 2 and one unit of food F_2 costs ₹ 4. Therefore, total cost of x units of food F_1 and y units of food F_2 is ₹ $(2x + 4y)$.

Let Z denote the total cost. Then, $Z = 2x + 4y$.

Since each unit of food F_1 contains 10 units of carbohydrate. Therefore, x units of food F_1 contain $10x$ units of carbohydrate. A unit of food F_2 contains 25 units of carbohydrate. So, y units of food F_2 contain $25y$ units of carbohydrate.

Thus, x units of food F_1 and y units of food F_2 contain $10x + 25y$ units of carbohydrate. But, the minimum requirement of carbohydrate is 400 units.

$$\therefore \quad 10x + 25y \geq 400$$

Similarly, the total amount of fat supplied by x units of Food F_1 and y units of food F_2 is $20x + 10y$ and the minimum requirement is of 500 units.

$$\therefore \quad 20x + 10y \geq 500$$

Finally, the total amount of protein supplied by x units of food F_1 and y units of food F_2 is $15x + 20y$. But, the minimum requirement of protein is of 300 units.

$$\therefore \quad 15x + 20y \geq 300$$

Clearly, $x \geq 0$ and $y \geq 0$.

Since we have to minimize the total cost $Z = 2x + 4y$.

Thus, the mathematical form of the given LPP is as follows:

$$\text{Minimize } Z = 2x + 4y$$

Subject to

$$10x + 25y \geq 400$$
$$20x + 10y \geq 500$$
$$15x + 20y \geq 300$$
$$x, y \geq 0.$$

EXAMPLE 10 *The objective of A diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at minimum cost. The consideration is limited to milk, beaf and eggs, and to vitamins A, B, C. The number of milligrams of each of these vitamins contained within A unit of each food is given below:*

Vitamin	*Litre of milk*	*Kg of beaf*	*Dozen of eggs*	*Minimum daily requirements*
A	1	1	10	1 mg
B	100	10	10	50 mg
C	10	100	10	10 mg
Cost	₹ 1.00	₹ 1.10	₹ 0.50	

What is the linear programming formulation for this problem?

SOLUTION Let the daily diet consists of x litres of milk, y kgs of beaf and z dozens of eggs. Then,

Total cost per day = ₹ $(x + 1.10y + 0.50z)$.

Let Z denote the total cost in ₹. Then, $Z = x + 1.10y + 0.50z$

Total amount of vitamin A in the daily diet is $(x + y + 10z)$ mg

But, the minimum requirement is 1 mg of vitamin A.

$\therefore \quad x + y + 10z \geq 1$

Similarly, total amounts of vitamins B and C in the daily diet are $(100x + 10y + 10z)$ mg and $(10x + 100y + 10z)$ mg respectively and their minimum requirements are of 50 mg and 10 mg respectively.

$\therefore \quad 100x + 10y + 10z \geq 50$ and, $10x + 100y + 10z \geq 10$

Finally, the quantity of milk, kgs of beaf and dozens of eggs cannot assume negative values.

$\therefore \quad x \geq 0,\ y \geq 0,\ z \geq 0$

Hence, the mathematical formulation of the given LPP is

Minimize $Z = x + 1.10y + 0.50z$

Subject to

$$x + y + 10z \geq 1$$
$$100x + 10y + 10z \geq 50$$
$$10x + 100y + 10z \geq 10$$

and, $x \geq 0,\ y \geq 0,\ z \geq 0.$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type III TRANSPORTATION PROBLEMS

EXAMPLE 11 *There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are 8 and 6 units respectively. The cost of transportation per unit is given below.*

From \ *To*	*Cost (in ₹)*		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate the above as a linear programming problem.

SOLUTION The given information can be exhibited diagrammatically as follows:

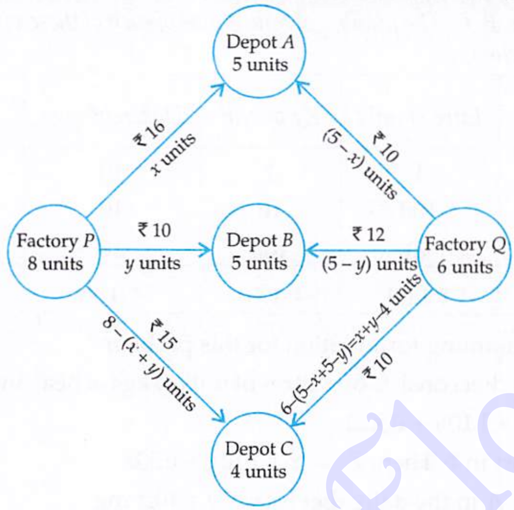

Fig. 29.1

Let the factory at P transports x units of commodity to depot at A and y units to depot at B. Since the factory at P has the capacity of 8 units of the commodity. Therefore, the left out $(8 - x - y)$ units will be transported to depot at C.

Since the requirements are always non-negative quantities. Therefore,

$$x \geq 0, y \geq 0 \text{ and } 8 - x - y \geq 0 \Rightarrow x \geq 0, y \geq 0 \text{ and } x + y \leq 8$$

Since the weekly requirement of the depot at A is 5 units of the commodity and x units are transported from the factory at P. Therefore, the remaining $(5 - x)$ units are to be transported from the factory at Q. Similarly, $5 - y$ units of the commodity will be transported from the factory at Q to the depot at B. But the factory at Q has the capacity of 6 units only, therefore the remaining $6 - (5 - x + 5 - y) = x + y - 4$ units will be transported to the depot at C. As the requirements at the depots at A, B and C are always non-negative.

$$\therefore \quad 5 - x \geq 0, 5 - y \geq 0 \text{ and } x + y - 4 \geq 0 \Rightarrow x \leq 5, y \leq 5 \text{ and } x + y \geq 4.$$

The transportation cost from the factory at P to the depots at A, B and C are respectively ₹$16x$, $10y$ and $15(8 - x - y)$. Similarly, the transportation cost from the factory at Q to the depots at A, B and C are respectively ₹ $10(5 - x)$, $12(5 - y)$ and $10(x + y - 4)$. Therefore, the total transportation cost Z is given by

$$Z = 16x + 10y + 15(8 - x - y) + 10(5 - x) + 12(5 - y) + 10(x + y - 4) = x - 7y + 190$$

Hence, the above LPP can be stated mathematically as follows:

Find x and y which

$$\text{Minimize } Z = x - 7y + 190$$

Subject to

$$x + y \leq 8$$
$$x + y \geq 4$$
$$x \leq 5$$
$$y \leq 5$$

and, $\quad x \geq 0, y \geq 0$

EXAMPLE 12 *A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in ₹ of transporting 1000 bricks to the builders from the depots are given below :*

From \ To	P	Q	R
A	40	20	30
B	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum? Formulate the above linear programming problem.

SOLUTION The given information can be exhibited diagrammatically as shown in Fig. 29.2.

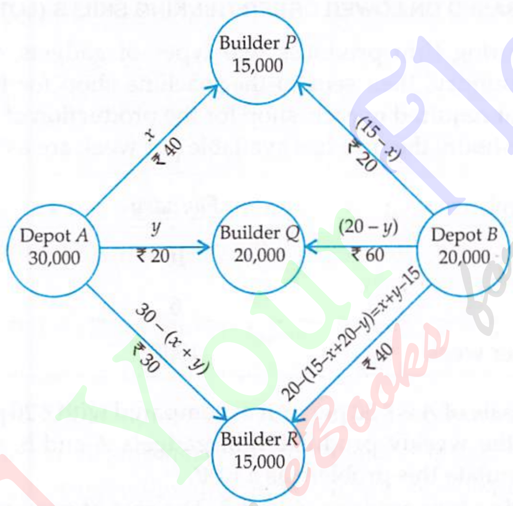

Fig. 29.2

Let the depot A transport x thousands bricks to builders P, y thousands bricks to builder Q. Since the depot A has stock of 30,000 bricks. Therefore, the remaining bricks i.e. $30-(x+y)$ thousands bricks will be transported to the builder R.

Since the number of bricks is always a non-negative real number. Therefore,

$$x \geq 0,\ y \geq 0 \text{ and } 30-(x+y) \geq 0 \Rightarrow x \geq 0,\ y \geq 0 \text{ and } x+y \leq 30$$

Now, the requirement of the builder P is of 15000 bricks and x thousand bricks are transported from the depot A. Therefore, the remaining $(15-x)$ thousands bricks are to be transported from the depot at B. The requirement of the builder Q is of 20,000 bricks and y thousand bricks are transported from depot A. Therefore, the remaining $(20-y)$ thousand bricks are to be transported from depot B.

Now, depot B has $20-(15-x+20-y)=x+y-15$ thousand bricks which are to be transported to the builder R.

Also, $15-x \geq 0,\ 20-y \geq 0$ and $x+y-15 \geq 0 \Rightarrow x \leq 15,\ y \leq 20$ and $x+y \geq 15$

The transportation cost from the depot A to the builders P, Q and R are respectively ₹ $40x$, $20y$ and $30(30-x-y)$. Similarly, the transportation cost from the depot B to the builders P, Q and R are respectively ₹ $20(15-x)$, $60(20-y)$ and $40(x+y-15)$ respectively. Therefore, the total transportation cost Z is given by

$$Z = 40x + 20y + 30(30-x-y) + 20(15-x) + 60(20-y) + 40(x+y-15)$$

$$\Rightarrow \quad Z = 30x - 30y + 1800$$

Hence, the above LPP can be stated mathematically as follows:

Find x and y in thousands which

$$\text{Minimize } Z = 30x - 30y + 1800$$

Subject to

$$x + y \le 30$$
$$x \le 15$$
$$y \le 20$$
$$x + y \ge 15$$

and, $x \ge 0, y \ge 0$

EXERCISE 29.1

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

1. A small manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry, then sent to the machine shop for finishing. The number of man-hours of labour required in each shop for the production of each unit of A and B, and the number of man-hours the firm has available per week are as follows:

Gadget	*Foundry*	*Machine-shop*
A	10	5
B	6	4
Firm's capacity per week	1000	600

The profit on the sale of A is ₹ 30 per unit as compared with ₹ 20 per unit of B. The problem is to determine the weekly production of gadgets A and B, so that the total profit is maximized. Formulate this problem as a LPP.

2. A company is making two products A and B. The cost of producing one unit of products A and B are ₹ 60 and ₹ 80 respectively. As per the agreement, the company has to supply at least 200 units of product B to its regular customers. One unit of product A requires one machine hour whereas product B has machine hours available abundantly within the company. Total machine hours available for product A are 400 hours. One unit of each product A and B requires one labour hour each and total of 500 labour hours are available. The company wants to minimize the cost of production by satisfying the given requirements. Formulate the problem as a LPP.

3. A firm manufactures 3 products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product:

Machine	*Products*		
	A	B	C
M_1	4	3	5
M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's. Set up a LPP to maximize the profit.

4. A firm manufactures two types of products A and B and sells them at a profit of ₹ 2 on type A and ₹ 3 on type B. Each product is processed on two machines M_1 and M_2. Type A requires one minute of processing time on M_1 and two minutes of M_2 ; type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hours 40 minutes while machine M_2 is available for 10 hours during any working day. Formulate the problem as a LPP.
5. A rubber company is engaged in producing three types of tyres A, B and C. Each type requires processing in two plants, Plant I and Plant II. The capacities of the two plants, in number of tyres per day, are as follows:

Plant	*A*	*B*	C
I	50	100	100
II	60	60	200

The monthly demand for tyre A, B and C is 2500, 3000 and 7000 respectively. If plant I costs ₹ 2500 per day, and plant II costs ₹ 3500 per day to operate, how many days should each be run per month to minimize cost while meeting the demand? Formulate the problem as LPP.
6. A company sells two different products A and B. The two products are produced in a common production process and are sold in two different markets. The production process has a total capacity of 45000 man-hours. It takes 5 hours to produce a unit of A and 3 hours to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 7000 and that of B is 10,000. If the profit is ₹ 60 per unit for the product A and ₹ 40 per unit for the product B, how many units of each product should be sold to maximize profit? Formulate the problem as LPP.
7. To maintain his health a person must fulfil certain minimum daily requirements for several kinds of nutrients. Assuming that there are only three kinds of nutrients — calcium, protein and calories and the person's diet consists of only two food items, I and II, whose price and nutrient contents are shown in the table below:

	Food I (per lb)	*Food II (per lb)*	*Minimum daily requirement for the nutrient*
Calcium	10	5	20
Protein	5	4	20
Calories	2	6	13
Price (₹)	60	100	

What combination of two food items will satisfy the daily requirement and entail the least cost? Formulate this as a LPP.
8. A manufacturer can produce two products, A and B, during a given time period. Each of these products requires four different manufacturing operations: grinding, turning, assembling and testing. The manufacturing requirements in hours per unit of products A and B are given below.

	A	*B*
Grinding	1	2
Turning	3	1
Assembling	6	3
Testing	5	4

The available capacities of these operations in hours for the given time period are: grinding 30; turning 60, assembling 200; testing 200. The contribution to profit is ₹ 20 for each unit of

A and ₹ 30 for each unit of *B*. The firm can sell all that it produces at the prevailing market price. Determine the optimum amount of *A* and *B* to produce during the given time period. Formulate this as a LPP.

9. Vitamins *A* and *B* are found in two different foods F_1 and F_2. One unit of food F_1 contains 2 units of vitamin *A* and 3 units of vitamin *B*. One unit of food F_2 contains 4 units of vitamin *A* and 2 units of vitamin *B*. One unit of food F_1 and F_2 cost ₹ 50 and 25 respectively. The minimum daily requirements for a person of vitamin *A* and *B* is 40 and 50 units respectively. Assuming that any thing in excess of daily minimum requirement of vitamin *A* and *B* is not harmful, find out the optimum mixture of food F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamin *A* and *B*. Formulate this as a LPP.

10. An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop *A*, which performs the basic assembly operation, must work 5 man-days on each truck but only 2 man-days on each automobile. Shop *B*, which performs finishing operations, must work 3 man-days for each automobile or truck that it produces. Because of men and machine limitations, shop *A* has 180 man-days per week available while shop *B* has 135 man-days per week. If the manufacturer makes a profit of ₹ 30000 on each truck and ₹ 2000 on each automobile, how many of each should he produce to maximize his profit? Formulate this as a LPP.

11. Two tailors *A* and *B* earn ₹ 150 and ₹ 200 per day respectively. A can stitch 6 shirts and 4 pants per day while *B* can stitch 10 shirts and 4 pants per day. Form a linear programming problem to minimize the labour cost to produce at least 60 shirts and 32 pants. **[CBSE 2005]**

12. An airline agrees to charter planes for a group. The group needs at least 160 first class seats and at least 300 tourist class seats. The airline must use at least two of its model 314 planes which have 20 first class and 30 tourist class seats. The airline will also use some of its model 535 planes which have 20 first class seats and 60 tourist class seats. Each flight of a model 314 plane costs the company ₹ 100,000 and each flight of a model 535 plane costs ₹ 150,000. How many of each type of plane should be used to minimize the flight cost? Formulate this as a LPP.

13. Amit's mathematics teacher has given him three very long lists of problems with the instruction to submit not more than 100 of them (correctly solved) for credit. The problem in the first set are worth 5 points each, those in the second set are worth 4 points each, and those in the third set are worth 6 points each. Amit knows from experience that he requires on the average 3 minutes to solve a 5 point problem, 2 minutes to solve a 4 point problem, and 4 minutes to solve a 6 point problem. Because he has other subjects to worry about, he can not afford to devote more than $3\frac{1}{2}$ hours altogether to his mathematics assignment. Moreover, the first two sets of problems involve numerical calculations and he knows that he cannot stand more than $2\frac{1}{2}$ hours work on this type of problem. Under these circumstances, how many problems in each of these categories shall he do in order to get maximum possible credit for his efforts? Formulate this as a LPP.

14. A farmer has a 100 acre farm. He can sell the tomatoes, lettuce, or radishes he can raise. The price he can obtain is ₹ 1 per kilogram for tomatoes, ₹ 0.75 a head for lettuce and ₹ 2 per kilogram for radishes. The average yield per acre is 2000 kgs for radishes, 3000 heads of lettuce and 1000 kilograms of radishes. Fertilizer is available at ₹ 0.50 per kg and the amount required per acre is 100 kgs each for tomatoes and lettuce and 50 kilograms for radishes. Labour required for sowing, cultivating and harvesting per acre is 5 man-days for tomatoes and radishes and 6 man-days for lettuce. A total of 400 man-days of labour are available at ₹ 20 per man-day. Formulate this problem as a LPP to maximize the farmer's total profit.

15. A firm has to transport at least 1200 packages daily using large vans which carry 200 packages each and small vans which can take 80 packages each. The cost of engaging each large van is ₹ 400 and each small van is ₹ 200. Not more than ₹ 3000 is to be spent daily on the job and the number of large vans cannot exceed the number of small vans. Formulate this problem as a LPP given that the objective is to minimize cost. **[CBSE 2017]**

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

16. A firm manufactures two products, each of which must be processed through two departments, 1 and 2. The hourly requirements per unit for each product in each department, the weekly capacities in each department, selling price per unit, labour cost per unit, and raw material cost per unit are summarized as follows:

	Product A	*Product B*	*Weekly capacity*
Department 1	3	2	130
Department 2	4	6	260
Selling price per unit	₹ 25	₹ 30	
Labour cost per unit	₹ 16	₹ 20	
Raw material cost per unit	₹ 4	₹ 4	

The problem is to determine the number of units to produce each product so as to maximize total contribution to profit. Formulate this as a LPP.

ANSWERS

1. *Max.* $Z = 30x + 20y$
Subject to
$10x + 6y \le 1000$
$5x + 4y \le 600$
$x, y \ge 0$

2. *Mini.* $Z = 60x + 80y$
Subject to
$x + y \le 500$
$x \le 400$
$y \ge 200$
$x \ge 0, y \ge 0$

3. *Max.* $Z = 3x + 2y + 4z$
Subject to
$4x + 3y + 5z \le 2000$
$2x + 2y + 4z \le 2500$
$100 \le x \le 150$
$y \ge 200$
$z \ge 50$
$x \ge 0,\ y \ge 0,\ z \ge 0.$

4. *Max.* $Z = 2x + 3y$
Subject to
$x + y \le 400$
$2x + y \le 600$
$x \ge 0, y \ge 0$

5. *Mini.* $Z = 2500x + 3500y$
Subject to
$50x + 60y \ge 2500$
$100x + 60y \ge 3000$
$100x + 200y \ge 7000$
$x, y \ge 0.$

6. *Max.* $Z = 60x + 40y$
Subject to
$5x + 3y \le 45000$
$x \le 7000$
$y \le 10{,}000$
$x, y \ge 0$

7. *Mini* $Z = 60x + 100y$
Subject to
$10x + 5y \ge 20$
$5x + 4y \ge 20$
$2x + 6y \ge 13$
$x, y \ge 0.$

8. *Max.* $Z = 20x + 30y$
Subject to
$x + 2y \le 30$
$3x + y \le 60$
$6x + 3y \le 200$
$5x + 4y \le 200$
$x, y \ge 0$

9. *Mini.* $Z = 50x + 25y$
Subject to
$2x + 4y \ge 40$
$3x + 2y \ge 50$
$x \ge 0, y \ge 0$

10. *Max.* $Z = 30000x + 2000y$
Subject to
$5x + 2y \le 180$
$3x + 3y \le 135$
$x \ge 0, y \ge 0$

11. *Mini.* $Z = 150x + 200y$
Subject to
$6x + 10y \ge 60$
$4x + 4y \ge 32$

12. *Mini.* $Z = 100{,}000x + 150{,}000y$
Subject to
$20x + 20y \ge 160$
$30x + 60y \ge 300$

$x, y \geq 0$

13. $\text{Max. } Z = 5x_1 + 4x_2 + 6x_3$

Subject to

$$x_1 + x_2 + x_3 \leq 100$$
$$3x_1 + 2x_2 + 4x_3 \leq 210$$
$$3x_1 + 2x_2 \leq 150.$$
$$x_1, x_2, x_3 \geq 0$$

15. $\text{Minimize } Z = 400x + 200y$

Subject to

$$400x + 200y \leq 3000$$
$$200x + 80y \geq 1200$$
$$x \leq y$$
$$x \geq 0, y \geq 0$$

$x \geq 0, y \geq 0$

14. $\text{Max. } Z = 1850x + 2080y + 1875z$

Subject to

$$x + y + z \leq 100$$
$$5x + 6y + 5z \leq 400$$
$$x, y, z \geq 0$$

16. $\text{Max. } Z = 5x + 6y$

Subject to

$$3x + 2y \leq 130$$
$$4x + 6y \leq 260$$
$$x \geq 0, y \geq 0.$$

HINTS TO SELECTED PROBLEMS

2. Let x units of product A and y units of product B be manufactured. Then, the mathematical formulation of the LPP is

Minimize $Z = 60x + 80y$

Subject to

$x + y \leq 500$ (Labour hours constraint)

$x \leq 400$ (Machine hours constraint)

$y \geq 200$ (Agreement constraint)

$x \geq 0, \ y \geq 0$

16. Suppose x units of product A and y units of product B are produced to maximize the profit. Then,

Profit $= (25 - 16 - 4)\, x + (30 - 20 - 4)\, y = 5x + 6y.$

$3x + 2y \leq 130$ (Capacity constraint of Department 1)

$4x + 6y \leq 260$ (Capacity constraint of Department 2)

and $x \geq 0, \ y \geq 0$

29.5 SOME DEFINITIONS AND RESULTS

In this section, we shall discuss some definitions related to the solution of linear programming problems.

The general form of a LPP is as given below:

Maximize (or *minimize*) $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ (objective function)

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \{\leq, =, \geq\} b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \{\leq, =, \geq\} b_2 \quad \text{(Constraints)}$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \{\leq, =, \geq\} b_m$$
$$x_1, x_2, \ldots, x_n \geq 0 \quad \text{(Non-negativity restrictions)}$$

The following are some definitions related to a LPP.

SOLUTION *A set of values of variables $x_1, x_2, \ldots, x_n$ is called a solution of a LPP, if it satisfies the constraints of the LPP.*

ILLUSTRATION 1 *Consider the following LPP:*

Maximize $Z = 4x + 5y$

Subject to

$$x + 2y \leq 6$$

$$3x + y \le 12$$
$$x \ge 0, y \ge 0$$

Clearly, $x = 1$, $y = 2$; $x = -2$, $y = 3$; $x = -1$, $y = -2$; $x = 2$, $y = -3$ etc. are solutions of this LPP as they satisfy the constraints $x + 2y \le 6$ and $3x + y \le 12$. Note that $x = 2$, $y = 4$ is not a solution, because it does not satisfy $x + 2y \le 6$.

FEASIBLE SOLUTION *A set of values of the variables $x_1, x_2, \ldots, x_n$ is called a feasible solution of a LPP, if it satisfies the constraints and non-negativity restrictions of the problem.*

In other words, a solution that also satisfies the non-negativity restrictions of a LPP, is called a feasible solution.

INFEASIBLE SOLUTION *A solution of a LPP is an infeasible solution, if it does not satisfy the non-negativity restrictions.*

ILLUSTRATION 2 *Consider the following a LPP:*

Maximize $Z = 6x + 8y$

Subject to

$$3x + 2y \le 30$$
$$x + 2y \le 22$$
$$x, y \ge 0$$

We observe that $x = 2$, $y = 3$; $x = 5$, $y = 0$; $x = -2$, $y = -1$; $x = 0$, $y = -2$ etc. are solutions of this LPP. Out of these solutions $x = 2$, $y = 3$ and $x = 5$, $y = 0$ are feasible solutions, because these solutions also satisfy non-negativity restrictions. Remaining solutions given above are infeasible solutions.

FEASIBLE REGION *The common region determined by all the constraints of a LPP is called the feasible region and every point in this region is a feasible solution of the given LPP.*

OPTIMAL FEASIBLE SOLUTION *A feasible solution of a LPP is said to be an optimal feasible solution, if it also optimizes (maximizes or minimizes) the objective function.*

Now, we shall discuss some definitions and results related to the feasible solutions of a LPP.

CONVEX SET *A set is a convex set, if every point on the line segment joining any two points in it lies in it.*

In Figs. 29.3 to 29.4 the polygons are convex sets whereas polygon in Fig. 29.5 is not a convex set.

Fig. 29.3

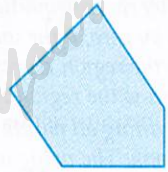

Fig. 29.4

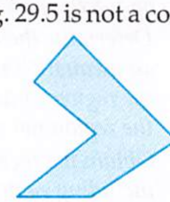

Fig. 29.5

THEOREM *The set of all feasible solutions of a LPP is a convex set.*

The proof of the above theorem is beyond the scope of the syllabus for CBSE class XII.

It follows from the above theorem that the set of all feasible solutions of a LPP is a convex polygon. When we are asked to solve a linear programming problem, it always means that we have to find its optimal solution. It is known from the general mathematical theory of linear programming that a LPP may or may not attain an optimal solution. However, if it attains an optimal solution, then one of the corner points (vertices) of the convex polygon of all feasible solutions gives the optimal solution as stated in the following theorem.

FUNDAMENTAL EXTREME POINT THEOREM *An optimal solution of a LPP, if it exists, occurs at one of the extreme (corner) points of the convex polygon of the set of all feasible solutions.*

It may happen that the two vertices of the corner polygon give the optimal value of the objective function. In such a case all points on the line segment joining these two vertices give the optimal value and the LPP is said to have infinitely many solutions. Sometimes, the convex polygon is an empty set. In such a case, we say that the LPP has no solution.

If the feasible region for a linear programming problem is bounded i.e., it can be enclosed within a circle, then the objective function has both a maximum and a minimum value and each of these values occurs at a corner point of the feasible region.

If the feasible region of a linear programming problem is unbounded i.e., it extends indefinitely in any direction, then a maximum or a minimum value of the objective function may not exist. However, if it exists, it occurs at a corner point of the feasible region.

29.6 GRAPHICAL METHODS OF SOLVING LINEAR PROGRAMMING PROBLEMS

There are two graphical methods for the solution of linear programming problems. These methods are suitable for solving linear programming problems containing two variables only. If a LPP contains more than two variables, these graphical methods are not suitable to solve them. Such type of problems are solved by simplex method which is beyond the scope of our discussion. We shall, therefore, be concerned only with the graphical methods involving two variables x and y.

The following methods are used to solve linear programming problems graphically:

(i) Corner-Point Method (ii) Iso-profit or iso-cost method.

We shall now apply these two methods for solving linear programming problems.

29.7 CORNER-POINT METHOD

This method is based on the Fundamental extreme point theorem which is stated in the earlier section.

Following algorithm can be used to solve a LPP in two variables graphically by using the corner-point method.

ALGORITHM

Step I *Formulate the given LPP in mathematical form if it is not so.*

Step II *Convert all inequations into equations and draw their graphs. To draw the graph of a linear equation, put $y = 0$ in it and obtain a point on x-axis. Similarly, by putting $x = 0$ obtain a point on y-axis. Join these two points to obtain the graph representing the equation.*

Step III *Determine the region represented by each inequation. To determine the region represented by an inequation replace x and y both by zero, if the inequation reduces to a valid statement, then the region containing the origin is the region represented by the given inequation. Otherwise, the region not containing the origin is the region represented by the given inequation.*

Step IV *Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the feasible region and is known as the convex polygon of the set of all feasible solutions of the LPP.*

Step V *Determine the coordinates of the vertices (corner points) of the convex polygon obtained in Step II. These vertices are known as the extreme points of the set of all feasible solutions of the LPP.*

Step VI *Obtain the values of the objective function at each of the vertices of the convex polygon. The point where the objective function attains its optimum (maximum or minimum) value is the optimal solution of the given LPP.*

REMARK 1 *If the feasible region of a LPP is bounded i.e., it is a convex polygon. Then, the objective function $Z = ax + by$ has both a maximum value M and a minimum value m and each of these values occurs at a corner point of the convex polygon.*

REMARK 2 *Sometimes the feasible region of a LPP is not a bounded convex polygon. That is, it extends indefinitely in any direction. In such cases, we say that the feasible region is unbounded. The above algorithm is applicable when the feasible region is bounded. If the feasible region is unbounded, then we find the values of the objective function $Z = ax + by$ at each corner point of the feasible region. Let M and m respectively denote the largest and smallest values of Z at these points. In order to check whether Z has maximum and minimum values as M and m respectively, we proceed as follows:*

(i) *Draw the line $ax + by = M$ and find the open half plane $ax + by > M$. If the open half-plane represented by $ax + by > M$ has no point common with the unbounded feasible region, then M is the maximum value of Z. Otherwise Z has no maximum value.*

(ii) *Draw the line $ax + by = m$ and find the open half plane represented by $ax + by < m$. If the open half-plane $ax + by < m$ has no point common with the unbounded feasible region, then m is the minimum value of Z. Otherwise, Z has no minimum value.*

Following examples illustrate the above algorithm.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Solve the following LPP graphically:*

Maximize $Z = 5x + 3y$

Subject to

$$3x + 5y \leq 15$$
$$5x + 2y \leq 10$$

and, $x, y \geq 0$

SOLUTION Converting the given inequations into equations, we obtain the following equations:

$$3x + 5y = 15, 5x + 2y = 10, x = 0 \text{ and } y = 0$$

Region represented by $3x + 5y \leq 15$: The line $3x + 5y = 15$ meets the coordinate axes at A_1 (5, 0) and B_1 (0, 3) respectively. Join these points to obtain the line $3x + 5y = 15$. Clearly, (0, 0) satisfies the inequation $3x + 5y \leq 15$. So, the region containing the origin represents the solution set of the inequation $3x + 5y \leq 15$.

Region Represented by $5x + 2y \leq 10$: The line $5x + 2y = 10$ meets the coordinate axes at A_2 (2, 0) and B_2 (0, 5) respectively. Join these points to obtain the graph of the line $5x + 2y = 10$. Clearly, (0, 0) satisfies the inequation $5x + 2y \leq 10$. So, the region containing the origin represents the solution set of this inequation.

Region represented by $x \geq 0$ *and* $y \geq 0$: Since every point in the first quadrant satisfies these inequations. So, the first quadrant is the region represented by the inequations $x \geq 0$ and $y \geq 0$.

The shaded region $OA_2\ PB_1$ in Fig. 29.6 represents the common region of the above inequations. This region is the feasible region of the given LPP.

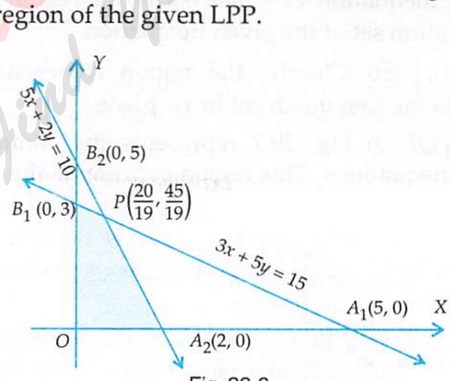

Fig. 29.6

The coordinates of the vertices (corner-points) of the shaded feasible region are O (0, 0), A_2 (2, 0), P (20/19, 45/19) and B_1 (0, 3).

These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	Value of the objective function $Z = 5x + 3y$
$O(0, 0)$	$Z = 5 \times 0 + 3 \times 0 = 0$
$A_2(2, 0)$	$Z = 5 \times 2 + 3 \times 0 = 10$
$P\left(\frac{20}{19}, \frac{45}{19}\right)$	$Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19} = \frac{235}{19}$
$B_1(0, 3)$	$Z = 5 \times 0 + 3 \times 3 = 9$

Clearly, Z is maximum at $P(20/19, 45/19)$. Hence, $x = 20/19$, $y = 45/19$ is the optimal solution of the given LPP and the optimal value of Z is $235/19$.

EXAMPLE 2 *Solve the following LPP by graphical method:*

Minimize $\quad Z = 20x + 10y$

Subject to

$$x + 2y \leq 40$$
$$3x + y \geq 30$$
$$4x + 3y \geq 60$$

and, $\quad x, y \geq 0$

SOLUTION Converting the given inequations into equations, we obtain the following equations:

$$x + 2y = 40,\ 3x + y = 30,\ 4x + 3y = 60,\ x = 0 \text{ and } y = 0$$

Region represented by $x + 2y \leq 40$: The line $x + 2y = 40$ meets the coordinate axes at $A_1(40, 0)$ and $B_1(0, 20)$ respectively. Join these points to obtain the line $x + 2y = 40$. Clearly, $(0, 0)$ satisfies the inequation $x + 2y \leq 40$. So, the region in xy-plane that contains the origin represents the solution set of the given inequation.

Region represented by $3x + y \geq 30$: The line $3x + y = 30$ meets x and y axes at $A_2(10, 0)$ and $B_2(0, 30)$ respectively. Join these points to obtain this line. We find that the point $O(0, 0)$ does not satisfy the inequation $3x + y \geq 30$. So, that region in xy–plane which does not contain the origin is the solution set of this inequation.

Region represented by $4x + 3y \geq 60$: The line $4x + 3y = 60$ meets x and y axes at $A_3(15, 0)$ and $B_1(0, 20)$ respectively. Join these points to obtain the line $4x + 3y = 60$. We observe that the point $O(0, 0)$ does not satisfy the inequation $4x + 3y \geq 60$. So, the region not containing the origin in xy–plane represents the solution set of the given inequation.

Region represented by $x \geq 0, y \geq 0$: Clearly, the region represented by the non-negativity restrictions $x \geq 0$ and $y \geq 0$ is the first quadrant in xy–plane.

The shaded region $A_3 A_1 QP$ in Fig. 29.7 represents the common region of the regions represented by the above inequations. This region expresents the feasible region of the given LPP.

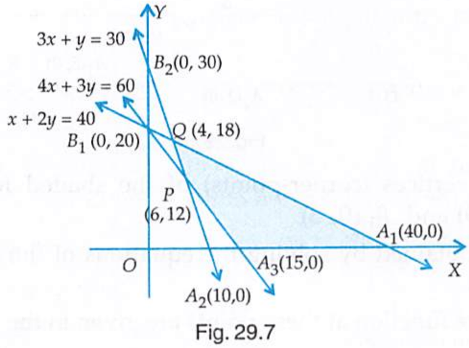

Fig. 29.7

The coordinates of the corner-points of the shaded feasible region are A_3 (15, 0), A_1 (40, 0), Q (4, 18) and P (6, 12). These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 20x + 10y$
A_3 (15, 0)	$Z = 20 \times 15 + 10 \times 0 = 300$
A_1 (40, 0)	$Z = 20 \times 40 + 10 \times 0 = 800$
Q (4, 18)	$Z = 20 \times 4 + 10 \times 18 = 260$
P (6, 12)	$Z = 20 \times 6 + 10 \times 12 = 240$

Out of these values of Z, the minimum value is 240 which is attained at point P (6, 12). Hence, $x = 6$, $y = 12$ is the optimal solution of the given LPP and the optimal value of Z is 240.

EXAMPLE 3 *Solve the following LPP graphically:*

Minimize and Maximize $Z = 5x + 2y$

Subject to

$$\begin{aligned} -2x - 3y &\le -6 \\ x - 2y &\le 2 \\ 3x + 2y &\le 12 \\ -3x + 2y &\le 3 \\ x, y &\ge 0 \end{aligned}$$

SOLUTION Converting the given inequations into equations, we get

$2x + 3y = 6$, $x - 2y = 2$, $3x + 2y = 12$, $-3x + 2y = 3$, $x = 0$ and $y = 0$

Region represented by $-2x - 3y \le -6$: The line $-2x - 3y = -6$ or, $2x + 3y = 6$ cuts OX and OY at A_1 (3, 0) and B_1 (0, 2) respectively. Join these points to obtain the line $2x + 3y = 6$.

Since O (0, 0) does not satisfy the inequation $-2x - 3y \le -6$. So, the region represented by $-2x - 3y \le -6$ is that part of XOY–plane which does not contain the origin.

Region represented by $x - 2y \le 2$: The line $x - 2y = 2$ meets the coordinate axes at A_2 (2, 0) and B_2 (0, −1). Join these points to obtain $x - 2y = 2$. Since (0, 0) satisfies the inequation $x - 2y \le 2$, so the region containing the origin represents the solution set of this inequation.

Region represented by $3x + 2y \le 12$: The line $3x + 2y \le 12$ interesects OX and OY at A_3 (4, 0) and B_3 (0, 6). Join these points to obtain the line $3x + 2y = 12$. Clearly, (0, 0) satisfies the inequation $3x + 2y \le 12$. So, the region containing the origin is the solution set of the given inequations.

Region represented by $-3x + 2y \le 3$: The line $-3x + 2y = 3$ intersects OX and OY at A_4 (−1, 0) and B_4 (0, 3/2). Join these points to obtain the line $-3x + 2y = 3$. Clearly, (0, 0) satisfies this inequation. So, the region containing the origin represents the solution set of the given inequation.

Region represented by $x \ge 0$, $y \ge 0$: Clearly, XOY quadrant represents the solution set of these two inequations.

The shaded region shown in Fig. 29.8 represents the common solution set of the above inequations. This region is the feasible region of the given LPP.

The coordinates of the corner-points (vertices) of the shaded feasible region $P_1 P_2 P_3 P_4$ are $P_1\left(\frac{18}{7}, \frac{2}{7}\right)$, $P_2\left(\frac{7}{2}, \frac{3}{4}\right)$, $P_3\left(\frac{3}{2}, \frac{15}{4}\right)$ and $P_4\left(\frac{3}{13}, \frac{24}{13}\right)$. These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

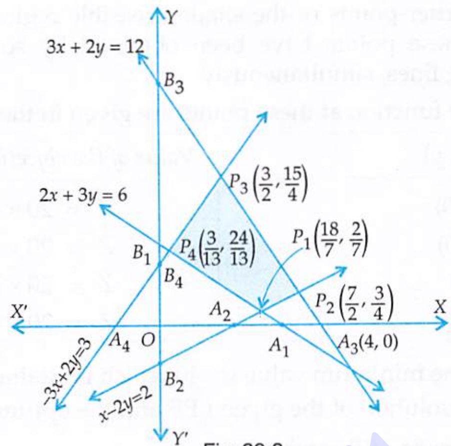

Fig. 29.8

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function Z = 5x + 2y*
$P_1\left(\frac{18}{7}, \frac{2}{7}\right)$	$Z = 5\times\frac{18}{7}+2\times\frac{2}{7}=\frac{94}{7}$
$P_2\left(\frac{7}{2}, \frac{3}{4}\right)$	$Z = 5\times\frac{7}{2}+2\times\frac{3}{4}=19$
$P_3\left(\frac{3}{2}, \frac{15}{4}\right)$	$Z = 5\times\frac{3}{2}+2\times\frac{15}{4}=15$
$P_4\left(\frac{3}{13}, \frac{24}{13}\right)$	$Z = 5\times\frac{3}{13}+2\times\frac{24}{13}=\frac{63}{13}$

Clearly, Z is minimum at $x=\frac{3}{13}$ and $y=\frac{24}{13}$ and maximum at $x=\frac{7}{2}$ and $y=\frac{3}{4}$. The minimum and maximum values of Z are $\frac{63}{13}$ and 19 respectively.

EXAMPLE 4 *Solve the following LPP graphically:*

Maximize and Minimize $Z = 3x+5y$

Subject to $3x-4y+12 \geq 0$

$2x-y+2 \geq 0$

$2x+3y-12 \geq 0$

$0 \leq x \leq 4$

$y \geq 2$

SOLUTION The given LPP can be re-written as:

Maximize or *Minimize* $Z = 3x+5y$

Subject to $3x-4y \geq -12$

$2x-y \geq -2$

$2x+3y \geq 12$

$x \leq 4$

$y \geq 2$

$x \geq 0$

Converting the inequations into equations, we obtain the following equations $3x-4y=-12$, $2x-y=-2$, $2x+3y=12$, $x=4$, $y=2$ and $x=0$.

These lines are drawn on suitable scale. The shaded region $P_1\,P_2\,P_3\,P_4\,P_5$ shown in Fig. 29.9 represents the feasible region of the given LPP.

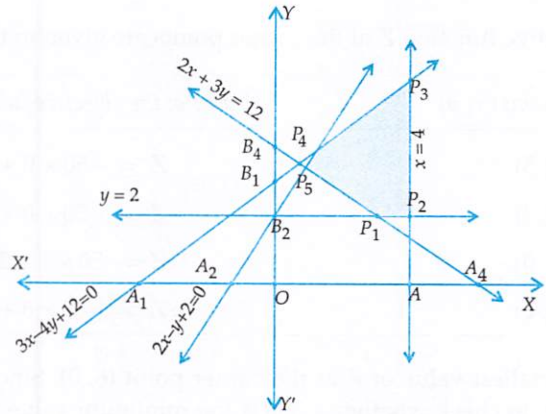

Fig. 29.9

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 3x + 5y$
$P_1\,(3, 2)$	$Z = 3 \times 3 + 5 \times 2 = 19$
$P_2\,(4, 2)$	$Z = 3 \times 4 + 2 \times 5 = 22$
$P_3\,(4, 6)$	$Z = 3 \times 4 + 5 \times 6 = 42$
$P_4\left(\frac{4}{5}, \frac{18}{5}\right)$	$Z = 3 \times \frac{4}{5} + 5 \times \frac{18}{5} = \frac{102}{5}$
$P_5\left(\frac{3}{4}, \frac{7}{2}\right)$	$Z = 3 \times \frac{3}{4} + 5 \times \frac{7}{2} = \frac{79}{4}$

Clearly Z assumes its minimum value 19 at $x = 3$ and $y = 2$. The maximum value of Z is 42 at $x = 4$ and $y = 6$.

EXAMPLE 5 *Determine graphically the minimum value of the objective function* $Z = -50x + 20y$ *Subject to constraints:*

$$2x - y \geq -5$$
$$3x + y \geq 3$$
$$2x - 3y \leq 12$$
$$x \geq 0,\ y \geq 0$$

SOLUTION The feasible region of the system of inequations given in constraints is shown in Fig. 29.10. We observe that the feasible region is unbounded.

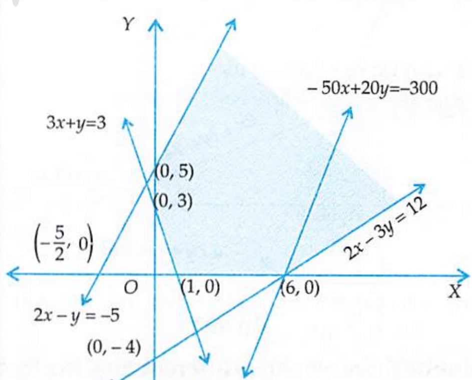

Fig. 29.10

The values of the objective function Z at the corner points are given in the following table:

Corner point (x, y)	*Value of the objective function* $Z = -50x + 20y$
(0, 5)	$Z = -50 \times 0 + 20 \times 5 = 100$
(0, 3)	$Z = -50 \times 0 + 20 \times 3 = 60$
(1, 0)	$Z = -50 \times 1 + 20 \times 0 = -50$
(6, 0)	$Z = -50 \times 6 + 20 \times 0 = -300$

Clearly, – 300 is the smallest value of Z at the corner point (6, 0). Since the feasible region is unbounded. Therefore, to check whether – 300 is the minimum value of Z, we draw the line $-300 = -50x + 20y$ and check whether the open half plane $-50x + 20y < -300$ has points in common with the feasible region or not. From Fig. 29.10, we find that the open half plane represented by $-50x + 20y < -300$ has points in common with the feasible region. Therefore, $Z = -50x + 20y$ has no minimum value subject to the given constraints.

29.8 ISO-PROFIT OR ISO COST METHOD

Consider the following LPP

Maximize $Z = 10x + 6y$

Subject to

$$3x + y \leq 12$$
$$2x + 5y \leq 34$$
$$x, y \geq 0$$

The convex set of all feasible solutions of this LPP is the set of all points in the shaded region of Fig. 29.11. Any point in this region is a feasible solution of the above LPP and only the points in this region are feasible solutions of the above LPP. In order to solve the above LPP, we have to find the point or points in the shaded region which give the largest value of the objective function. For any fixed value of Z, $Z = 10x + 6y$ or, $10x + 6y = Z$ is a straight line. For example, for $Z = 5$, $10x + 6y = 5$ is a straight line. Any point on the line $Z = 10x + 6y$ will give the same value of Z. So, it is known as an *iso-profit* line.

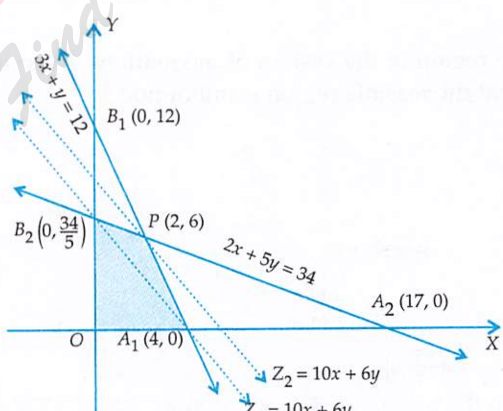

Fig. 29.11

Also, for each different value of Z, we obtain a different line. In other words, for different values of Z, equation $Z = 10x + 6y$ gives a family of parallel straight lines of slope $-\frac{10}{6} = -\frac{5}{2}$ and any point on the line $Z = 10x + 6y$, for given value of Z, gives the same value of Z. These lines are iso-profit lines. In order to maximize the objective function $Z = 10x + 6y$, we have to find the line with the largest value of Z which has at least one point in common with the shaded region. In other words, to maximize the objective function find the line parallel to $Z = 10x + 6y$ which is farthest from the origin O and which has at least one point in common with the shaded region. Clearly, $10x + 6y = Z_1$ is not farthest from the origin. However, $10x + 6y = Z_2$ is farthest from the origin and has a point $P(2, 6)$ common with the shaded region.

Thus, we see that Z_2 is the maximum value of Z, and the feasible solution which gives this value of Z is the corner $P(2, 6)$ of the shaded region. The values of the variables for the optimal solution are $x = 2$, $y = 6$. Substituting these values in $Z = 10x + 6y$, we get $Z = 56$ as the optimal value.

Now, consider the LPP

$$\begin{aligned} \textit{Minimize} \quad Z &= 18x + 10y \\ \textit{Subject to} \quad 4x + y &\geq 20 \\ 2x + 3y &\geq 30 \\ x, y &\geq 0 \end{aligned}$$

The convex set of all feasible solutions of this LPP is the set of all points in the shaded region of Fig. 29.12. In order to solve this LPP, we have to find the points in the shaded region which give the smallest value of the objective function. We observe that for any fixed value of Z, equation $18x + 10y = Z$ is a straight line and any point on this line gives the same value of Z. So, for some value of Z say Z_1, if the line $18x + 10y = Z_1$ has some points common with the feasible region of the LPP, then all these points give the same value of Z equal to Z_1 i.e. for every point in the feasible region lying on $18x + 10y = Z_1$, we obtain the same value of Z equal to Z_1. The line $18x + 10y = Z_1$ is known as iso-cost line. Thus, $18x + 10y = Z$ gives a family of parallel lines of slope $-\frac{18}{10}$ in xy-plane. In order to find the minimum value of Z, we have to find the line nearest

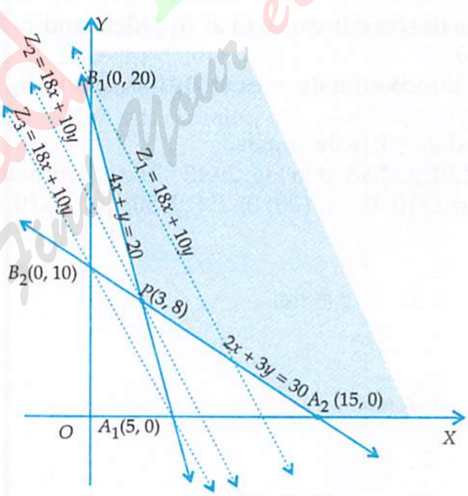

Fig. 29.12

to the origin and having at least one point common with the shaded region. Clearly, $18x + 10y = Z_2$ is nearest to the origin and has a common point $P(3, 8)$ with the shaded region. The line $18x + 10y = Z_3$ is more closer to the origin than the line $18x + 10y = Z_2$, but it does not have any point common to the feasible region. Thus, Z_2 is the minimum value of Z, and the feasible solution which gives this value of Z is the corner $P(3, 8)$ of the shaded region. The values of the variables for the optimal solution are $x = 3$, $y = 8$. Substituting these values in $Z = 18x + 10y$, we get $Z = 128$ as the optimal value of Z.

The above discussion suggests the following algorithm to solve a LPP by using iso-profit (iso-cost) lines.

ALGORITHM

Step I *Formulate the given LPP in mathematical form, if it is not given so.*

Step II *Obtain the region in xy-plane containing all points that simultaneously satisfy all constraints including non-negativity restrictions. The polygonal region so obtained is the convex set of all feasible solutions of the given LPP and it is also known as the feasible region.*

Step III *Determine the coordinates of the vertices (corner points) of the feasible region obtained in step II.*

Step IV *Give some convenient value to Z and draw the line so obtained in xy-plane.*

Step V *If the objective function is of maximization type, then draw lines parallel to the line in step IV and obtain a line which is farthest from the origin and has at least one point common to the feasible region.*

If the objective function is of minimization type, then draw lines parallel to the line in step IV and obtain a line which is nearest to the origin and has at least one point common to the feasible region.

Step VI *Find the coordinates of the common point (s) obtained in step V. The point (s) so obtained determine the optimal solution (s) and the value (s) of the objective function at these point (s) give the optimal solution.*

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *Solve the following linear programming problem graphically:*

$$\text{Maximize } Z = 50x + 15y$$

Subject to

$$5x + y \le 100$$
$$x + y \le 60$$
$$x, y \ge 0$$

SOLUTION We first convert the inequations into equations to obtain the lines $5x + y = 100$, $x + y = 60$, $x = 0$ and $y = 0$.

The line $5x + y = 100$ meets the coordinate axes at A_1 (20, 0) and B_1 (0, 100). Join these points to obtain the line $5x + y = 100$.

The line $x + y = 60$ meets the coordinate axes at A_2 (60, 0) and B_2 (0, 60). Join these points to obtain the line $x + y = 60$.

Also, $x = 0$ is the y-axis and $y = 0$ is the x-axis.

The feasible region of the LPP is shaded in Fig. 29.13. The coordinates of the corner-points of the feasible region OA_1PB_2 are O (0, 0), A_1 (20, 0), P (10, 50) and B_2 (0, 60).

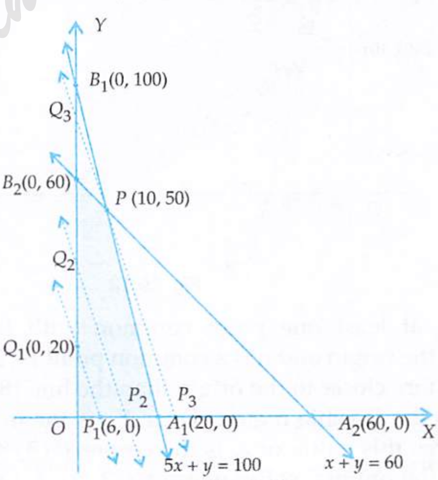

Fig. 29.13

Now, we take a constant value, say 300 (i.e. 2 times the l.c.m. of 50 and 15) for Z. Then,

$$300 = 50x + 15y$$

This line meets the coordinate axes at P_1 (6, 0) and Q_1 (0, 20). Join these points by a dotted line. Now, move this line parallel to itself in the increasing direction i.e. away from the origin. $P_2 Q_2$ and $P_3 Q_3$ are such lines. Out of these lines locate a line which is farthest from the origin and has at least one point common to the feasible region.

Clearly, $P_3 Q_3$ is such line and it passes through the vertex $P(10, 50)$ the convex polygon $OA_1 PB_2$. Hence, $x = 10$ and $y = 50$ will give the maximum value of Z. The maximum value of Z is given by

$$Z = 50 \times 10 + 15 \times 50 = 1250.$$

EXAMPLE 2 *Solve the following LPP graphically:*

$$\text{Maximize} \quad Z = 5x + 7y$$

Subject to

$$\begin{aligned} x + y &\le 4 \\ 3x + 8y &\le 24 \\ 10x + 7y &\le 35 \\ x, y &\ge 0 \end{aligned}$$

SOLUTION Converting the inequations into equations, we obtain the following equations:

$$x + y = 4,\ 3x + 8y = 24,\ 10x + 7y = 35,\ x = 0 \text{ and } y = 0.$$

These equations represent straight lines in XOY-plane.

The line $x + y = 4$ meets the coordinate axes at A_1 (4, 0) and B_1 (0, 4). Join these points to obtain the line $x + y = 4$.

The line $3x + 8y = 24$ meets the coordinate axes at A_2 (8, 0) and B_2 (0, 3). Join these points to obtain the line $3x + 8y = 24$.

The line $10x + 7y = 35$ cuts the coordinates axes at A_3 (3.5, 0) and B_3 (0, 5). These points are joined to obtain the line $10x + 7y = 35$.

Also, $x = 0$ is the y-axis and $y = 0$ is the x-axis.

The feasible region of the LPP is shaded in Fig. 29.14. The coordinates of the corner points of the feasible region $OA_3\ PQB_2$ are O (0, 0), A_3 (3.5, 0), $P\left(\frac{7}{3}, \frac{5}{3}\right)$, $Q\left(\frac{8}{5}, \frac{12}{5}\right)$ and B_2 (0, 3).

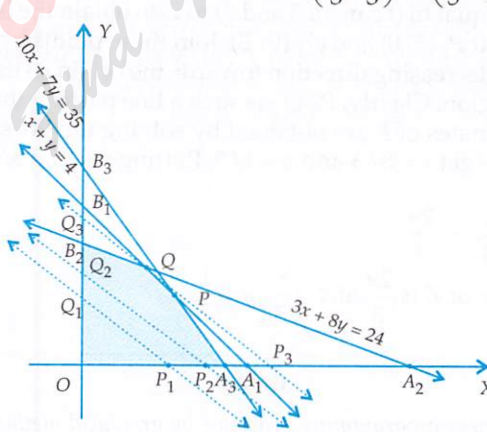

Fig. 29.14

Now, we take a constant value, say 10 for Z. Putting $Z = 10$ in $Z = 5x + 7y$, we obtain the line $5x + 7y = 10$. This line meets the coordinate axes at P_1 (2, 0) and $Q_1\left(0, \frac{10}{7}\right)$. Join these points by a dotted line. Now, move this line parallel to itself in the increasing direction away from the origin. P_2Q_2 and P_3Q_3 are such lines. Out of these lines locate a line farthest from the origin and has at least one common point to the feasible region OA_3PQB_2. Clearly, P_3Q_3 is such line and it passes through the vertex Q (8/5, 12/5) of the feasible region. Hence $x = 8/5$ and $y = 12/5$ gives the maximum value of Z. The maximum value of Z is given by

$$Z = 5 \times \frac{8}{5} + 7 \times \frac{12}{5} = 24.8.$$

EXAMPLE 3 *Solve the following LPP graphically:*

Minimize $Z = 3x + 5y$

Subject to

$$\begin{aligned} -2x + y &\le 4 \\ x + y &\ge 3 \\ x - 2y &\le 2 \\ x, y &\ge 0 \end{aligned}$$

SOLUTION Converting the inequations into equations, we obtain the lines $-2x + y = 4$, $x + y = 3$, $x - 2y = 2$, $x = 0$ and $y = 0$.

These lines are drawn on a suitable scale and the feasible region of the LPP is shaded in Fig. 29.15.

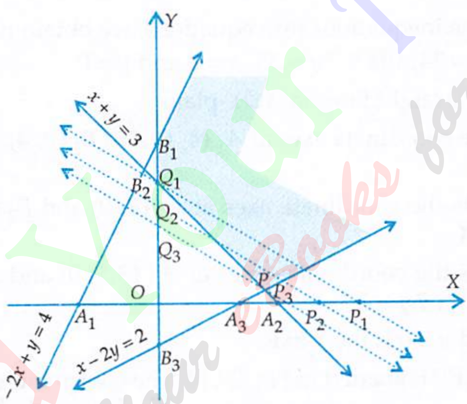

Fig. 29.15

Now, give a value, say 15 equal to (l.c.m. of 3 and 5) to Z to obtain the line $3x + 5y = 15$. This line meets the coordinate axes at P_1 (5, 0) and Q_1 (0, 3). Join these points by a dotted line. Move this line parallel to itself in the decreasing direction towards the origin so that it passes through only one point of the feasible region. Clearly, P_3Q_3 is such a line passing through the vertex P of the feasible region. The coordinates of P are obtained by solving the lines $x - 2y = 2$ and $x + y = 3$. Solving these equations, we get $x = 8/3$ and $y = 1/3$. Putting $x = 8/3$ and $y = 1/3$ in $Z = 3x + 5y$, we get

$$Z = 3 \times \frac{8}{3} + 5 \times \frac{1}{3} = \frac{29}{3}$$

Hence, the minimum value of Z is $\frac{29}{3}$ at $x = \frac{8}{3}$, $y = \frac{1}{3}$.

EXERCISE 29.2

BASIC

Solve each of the following linear programming problems by graphical method.

1. *Maximize* $Z = 5x + 3y$

   Subject to

   $$\begin{aligned} 3x + 5y &\le 15 \\ 5x + 2y &\le 10 \\ x, y &\ge 0 \end{aligned}$$

2. *Maximize* $Z = 9x + 3y$

   Subject to

   $$\begin{aligned} 2x + 3y &\le 13 \\ 3x + y &\le 5 \\ x, y &\ge 0 \end{aligned}$$

3. Minimize $Z = 18x + 10y$
Subject to
$$4x + y \geq 20$$
$$2x + 3y \geq 30$$
$$x, y \geq 0$$

4. Maximize $Z = 50x + 30y$
Subject to
$$2x + y \leq 18$$
$$3x + 2y \leq 34$$
$$x, y \geq 0$$

5. Maximize $Z = 4x + 3y$
Subject to
$$3x + 4y \leq 24$$
$$8x + 6y \leq 48$$
$$x \leq 5$$
$$y \leq 6$$
$$x, y \geq 0$$

6. Maximize $Z = 15x + 10y$
Subject to
$$3x + 2y \leq 80$$
$$2x + 3y \leq 70$$
$$x, y \geq 0$$

7. Maximize $Z = 10x + 6y$
Subject to
$$3x + y \leq 12$$
$$2x + 5y \leq 34$$
$$x, y \geq 0$$

8. Maximize $Z = 3x + 4y$
Subject to
$$2x + 2y \leq 80$$
$$2x + 4y \leq 120$$

9. Maximize $Z = 7x + 10y$
Subject to
$$x + y \leq 30000$$
$$y \leq 12000$$
$$x \geq 6000$$
$$x \geq y$$
$$x, y \geq 0$$

10. Minimize $Z = 2x + 4y$
Subject to
$$x + y \geq 8$$
$$x + 4y \geq 12$$
$$x \geq 3, y \geq 2$$

11. Minimize $Z = 5x + 3y$
Subject to
$$2x + y \geq 10$$
$$x + 3y \geq 15$$
$$x \leq 10$$
$$y \leq 8$$
$$x, y \geq 0$$

12. Minimize $Z = 30x + 20y$
Subject to
$$x + y \leq 8$$
$$x + 4y \geq 12$$
$$5x + 8y = 20$$
$$x, y \geq 0$$

13. Maximize $Z = 4x + 3y$
Subject to
$$3x + 4y \leq 24$$
$$8x + 6y \leq 48$$
$$x \leq 5$$
$$y \leq 6$$
$$x, y \geq 0$$

14. Minimize $Z = x - 5y + 20$
Subject to
$$x - y \geq 0$$
$$-x + 2y \geq 2$$
$$x \geq 3$$
$$y \leq 4$$
$$x, y \geq 0$$
[CBSE 2004]

15. Maximize $Z = 3x + 5y$
Subject to
$$x + 2y \leq 20$$
$$x + y \leq 15$$
$$y \leq 5$$
$$x, y \geq 0$$

16. Minimize $Z = 3x_1 + 5x_2$
Subject to
$$x_1 + 3x_2 \geq 3$$
$$x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$
[CBSE 2005]

17. Maximize $Z = 2x + 3y$
Subject to
$$x + y \geq 1$$
$$10x + y \geq 5$$
$$x + 10y \geq 1$$
$$x, y \geq 0$$

18. Maximize $Z = -x_1 + 2x_2$
Subject to
$$-x_1 + 3x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

19. *Maximize* $Z = x + y$

Subject to

$$-2x + y \le 1$$
$$x \le 2$$
$$x + y \le 3$$
$$x, y \ge 0$$

20. *Maximize* $Z = 3x_1 + 4x_2$, if possible,

Subject to the constraints

$$x_1 - x_2 \le -1$$
$$-x_1 + x_2 \le 0$$
$$x_1, x_2 \ge 0$$

21. Maximize $Z = 3x + 3y$, if possible,

Subject to the constraints

$$x - y \le 1$$
$$x + y \ge 3$$
$$x, y \ge 0$$

22. Show the solution zone of the following inequalities on a graph paper:

$$5x + y \ge 10$$
$$x + y \ge 6$$
$$x + 4y \ge 12$$
$$x \ge 0, y \ge 0$$

Find x and y for which $3x + 2y$ is minimum subject to these inequalities. Use a graphical method.

23. Find the maximum and minimum value of $2x + y$ subject to the constraints:

$x + 3y \ge 6$, $x - 3y \le 3$, $3x + 4y \le 24$, $-3x + 2y \le 6$, $5x + y \ge 5$, $x, y \ge 0$.

24. Find the minimum value of $3x + 5y$ subject to the constraints

$-2x + y \le 4$, $x + y \ge 3$, $x - 2y \le 2$, $x, y \ge 0$.

25. Solved the following linear programming problem graphically:

Maximize $Z = 60x + 15y$

Subject to constraints

$$x + y \le 50$$
$$3x + y \le 90$$
$$x, y \ge 0$$

[CBSE 2005]

26. Find graphically, the maximum value of $z = 2x + 5y$, subject to constraints given below:

$$2x + 4y \le 8$$
$$3x + y \le 6$$
$$x + y \le 4$$
$$x \ge 0, y \ge 0$$

[CBSE 2015]

27. Solve the following LPP graphically:

Maximize $Z = 20x + 10y$

Subject to the following constraints

$$x + 2y \le 28$$
$$3x + y \le 24$$
$$x \ge 2$$
$$x, y \ge 0$$

[CBSE 2017]

28. Solve the following linear programming problem graphically:

Minimize $z = 6x + 3y$

Subject to the constraints:

$$4x + y \ge 80$$
$$x + 5y \ge 115$$
$$3x + 2y \le 150$$
$$x \ge 0, y \ge 0$$

[CBSE 2017]

ANSWERS

1. $x=\frac{20}{19}, y=\frac{45}{19}, Z=\frac{235}{19}$ 2. $x=\frac{2}{7}, y=\frac{29}{7}, Z=15$ or, $x=\frac{5}{3}, y=0, Z=15$
3. $x=3, y=8$, Min. $Z=134$ 4. $x=10, y=2, Z=560$
5. $x=\frac{24}{7}, y=\frac{27}{7}, Z=24$ or $x=5, y=\frac{4}{3}, Z=24$
6. $x=\frac{80}{3}, y=0, Z=400$ or, $x=20, y=10, Z=400$ 7. $x=1, y=6, Z=56$
8. $x=20, y=20, Z=140$ 9. $x=18000, y=12000, Z=246000$
10. $x=4, y=2, Z=16$ 11. $x=3, y=4, Z=27$ 12. $x=\frac{4}{7}, y=\frac{15}{7}, Z=60$
13. $x=\frac{24}{7}, y=\frac{24}{7}, Z=24$ 14. $x=4, y=4, Z=4$ 15. $x=10, y=5, Z=55$
16. 7 17. 2 18. $\frac{20}{3}$ 19. 3 20. Does not exist
21. Max value is infinity i.e. the solution is unbounded
22. $x=1, y=5, Z=13$ 23. Max. $=\frac{43}{3}$ at $x=\frac{84}{13}, y=\frac{15}{3}$
25. $x=20, y=30, Z=1650$ 26. $x=\frac{8}{5}, y=\frac{6}{5}, Z=\frac{46}{5}$ 28. $x=15, y=20$

29.9 DIFFERENT TYPES OF LINEAR PROGRAMMING PROBLEMS

In this section, we shall discuss formulation and solution of some important types of linear programming problems viz. Diet problems, Optimal product line problems and Transportation problems.

29.9.1 DIET PROBLEMS

In this type of problems, we have to find the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimize the cost of the desired diet such that it contains a certain minimum amount of each constituent/nutrient.

Following are some examples on this type of problems.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *A house wife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C.*

The vitamin contents of one kg of food are given below:

	Vitamin A	*Vitamin B*	*Vitamin C*
Food X:	1	2	3
Food Y:	2	2	1

One kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10. Find the least cost of the mixture which will produce the diet. [CBSE 2003]

SOLUTION Let x kg of food X and y kg of food Y are mixed together to make the mixture.

Since one kg of food X contains one unit of vitamin A and one kg of food Y contains 2 units of vitamin A. Therefore, x kg of food X and y kg of food Y will contain $x+2y$ units of vitamin A. But the mixture should contain at least 10 units of vitamin A. Therefore,

$$x+2y \geq 10$$

Similarly, x kg of food X and y kg of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C. But the minimum requirements of vitamins B and C are respectively of 12 and 8 units.

$\therefore \quad 2x + 2y \geq 12$ and $3x + y \geq 8$

Since the quantity of food X and food Y cannot be negative.

$\therefore \quad x \geq 0, y \geq 0$

It is given that one kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10. So, x kg of food X and y kg of food Y will cost ₹ $(6x + 10y)$.

Thus, the given linear programming problem is

$$\text{Minimize } Z = 6x + 10y$$

Subject to $x + 2y \geq 10$

$$2x + 2y \geq 12$$

$$3x + y \geq 8$$

and, $\quad x \geq 0, y \geq 0$

To solve this LPP, we draw the lines

$$x + 2y = 10, 2x + 2y = 12 \text{ and } 3x + y = 8.$$

The feasible region of the LPP is shaded in Fig. 29.16.

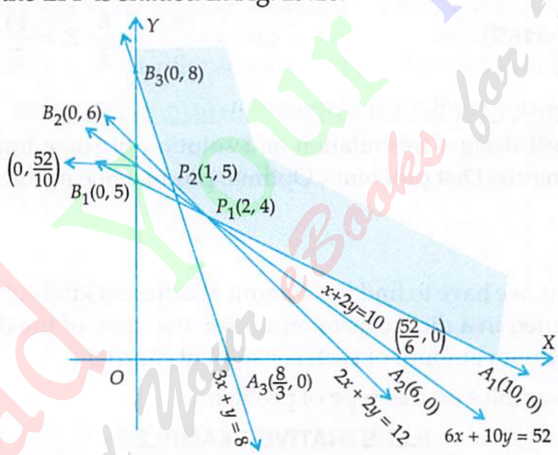

Fig. 29.16

The coordinates of the vertices (Corner-points) of shaded feasible region $A_1 P_1 P_2 B_3$ are A_1 (10, 0), P_1 (2, 4), P_2 (1, 5) and B_3 (0, 8). These points have been obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 6x + 10y$
A_1 (10, 0)	$Z = 6 \times 10 + 10 \times 0 = 60$
A_2 (2, 4)	$Z = 6 \times 2 + 10 \times 4 = 52$
P_2 (1, 5)	$Z = 6 \times 1 + 10 \times 5 = 56$
B_3 (0, 8)	$Z = 6 \times 0 + 10 \times 8 = 80$

Clearly, Z is minimum at $x = 2$ and $y = 4$. The minimum value of Z is 52.

We observe that the open half-plane represented by $6x + 10y < 52$ does not have points in common with the feasible region. So, Z has minimum value equal to 52.

Hence, the least cost of the mixture is ₹ 52.

EXAMPLE 2 *A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5.00 per kg to purchase food 'I' and ₹ 7.00 per kg to produce food 'II'. Determine the minimum cost to such a mixture. Formulate the above as a LPP and solve it.* **[CBSE 2012]**

SOLUTION Let the dietician mix x kg of food 'I' with y kg of food 'II'. Then, the mathematical model of the LPP is as follows:

$$\text{Minimize} \quad Z = 5x + 7y$$

Subject to $2x + y \geq 8$

$x + 2y \geq 10$ [See Example 8, page 29.10]

and, $x, y \geq 0$

To solve this LPP graphically, we first convert the inequations into equations to obtain the following lines.

$$2x + y = 8, x + 2y = 10, x = 0, y = 0$$

The line $2x + y = 8$ meets the coordinate axes at A_1 (4, 0) and B_1 (0, 8). Join these points to obtain the line represented by $2x + y = 8$. The region not containing the origin is represented by $2x + y \geq 8$.

The line $x + 2y = 10$ meets the coordinate axes at A_2 (10, 0) and B_2 (0, 5). Join these points to obtain the line represented by $x + 2y = 10$. Clearly, O (0, 0) does not satisfy the inequation $x + 2y \geq 10$. So, the region not containing the origin is represented by this inequation.

Clearly, $x \geq 0$ and, $y \geq 0$ represent the first quadrant.

Thus, the shaded region in Fig. 29.17 is the feasible region of the LPP. The coordinates of the corner-points of this region are A_2 (10, 0), P (2, 4) and B_1 (0, 8).

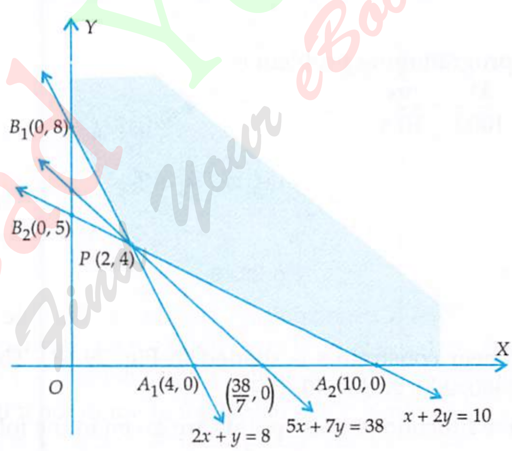

Fig. 29.17

The point P (2, 4) is obtained by solving $2x + y = 8$ and $x + 2y = 10$ simultaneously.

The values of the objective function $Z = 5x + 7y$ at the corner points of the feasible region are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 5x + 7y$
A_2 (10, 0)	$Z = 5 \times 10 + 7 \times 0 = 50$
P (2, 4)	$Z = 5 \times 2 + 7 \times 4 = 38$
B_1 (0, 8)	$Z = 5 \times 0 + 7 \times 8 = 56$

Clearly, Z is minimum at $x = 2$ and $y = 4$. The minimum value of Z is 38.

We observe that open half plane represented by $5x + 7y < 38$ does not have points in common with the feasible region. So, Z has minimum value equal to 38 at $x = 2$ and $y = 4$.

Hence, the optimal mixing strategy for the dietician will be to mix 2 kg of food 'I' and 4 kg of food 'II'. In this case, his cost will be minimum and the minimum cost will be ₹ 38.00.

EXAMPLE 3 *Every gram of wheat provides 0.1 gm of proteins and 0.25 gm of carbohydrates. The corresponding values of rice are 0.05 gm and 0.5 gm respectively. Wheat costs ₹ 4 per kg and rice ₹ 6. The minimum daily requirements of proteins and carbohydrates for an average child are 50 gms and 200 gms respectively. In what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost?*

SOLUTION Suppose x gms of wheat and y grams of rice are mixed in the daily diet.

Since every gram of wheat provides 0.1 gm of proteins and every gram of rice gives 0.05 gm of proteins. Therefore, x gms of wheat and y grams of rice will provide $0.1x + 0.05y$ gms of proteins. But the minimum daily requirement of proteins is of 50 gms.

$$\therefore \quad 0.1x + 0.05y \geq 50 \Rightarrow \frac{x}{10} + \frac{y}{20} \geq 50$$

Similarly, x gms of wheat and y gms of rice will provide $0.25x + 0.5y$ gms of carbohydrates and the minimum daily requirement of carbohydrates is of 200 gms.

$$\therefore \quad 0.25x + 0.5y \geq 200 \Rightarrow \frac{x}{4} + \frac{y}{2} \geq 200$$

Since the quantities of wheat and rice cannot be negative. Therefore,

$$x \geq 0, \quad y \geq 0$$

It is given that wheat costs ₹ 4 per kg and rice ₹ 6 per kg. So, x gms of wheat and y gms of rice will cost ₹ $\frac{4x}{1000} + \frac{6y}{1000}$

Hence, the given linear programming problem is

$$\textit{Minimize } Z = \frac{4x}{1000} + \frac{6y}{1000}$$

Subject to the constraints

$$\frac{x}{10} + \frac{y}{20} \geq 50,$$

$$\frac{x}{4} + \frac{y}{2} \geq 200,$$

and, $x \geq 0, \quad y \geq 0$

The solution set of the linear constraints is shaded in Fig. 29.18. The vertices of the shaded region are A_2 (800,0), P (400, 200) and B_1 (0, 1000).

The values of the objective function at these points are given in the following table.

Point (x_1, x_2)	*Value of objective function* $Z = \frac{4x}{1000} + \frac{6y}{1000}$
A_2 (800, 0)	$Z = \frac{4}{1000} \times 800 + \frac{8}{1000} \times 0 = 3.2$
P (400, 200)	$Z = \frac{4}{1000} \times 400 + \frac{6}{1000} \times 200 = 2.8$
B_1 (0, 1000)	$Z = \frac{4}{1000} \times 0 + \frac{6}{1000} \times 1000 = 6$

Clearly, Z is minimum for $x = 400$, $y = 200$ and the minimum value of Z is 2.8.

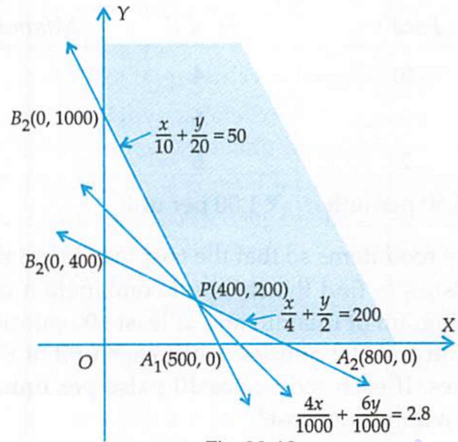

Fig. 29.18

We observe that the open half plane represented by $\frac{4x}{1000}+\frac{6y}{1000}<2.8$ does not have points is common with the feasible region. So, Z has minimum value 2.8 at $x = 400$ and $y = 200$.

Hence, the diet cost is minimum when $x = 400$ and $y = 200$. The minimum diet cost is ₹ 2.8.

EXERCISE 29.3

BASED ON LOTS

1. A diet of two foods F_1 and F_2 contains nutrients thiamine, phosphorous and iron. The amount of each nutrient in each of the food (in milligrams per 25 gms) is given in the following table:

Nutrients \ Food	F_1	F_2
Thiamine	0.25	0.10
Phosphorous	0.75	1.50
Iron	1.60	0.80

The minimum requirement of the nutrients in the diet are 1.00 mg of thiamine, 7.50 mg of phosphorous and 10.00 mg of iron. The cost of F_1 is 20 paise per 25 gms while the cost of F_2 is 15 paise per 25 gms. Find the minimum cost of diet.

2. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 of calories. Two foods A and B, are available at a cost of ₹ 4 and ₹ 3 per unit respectively. If one unit of A contains 200 units of vitamin, 1 unit of mineral and 40 calories and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost? [CBSE 2004]

3. To maintain one's health, a person must fulfil certain minimum daily requirements for the following three nutrients: calcium, protein and calories. The diet consists of only items I and II whose prices and nutrient contents are shown below:

	Food I	*Food II*	*Minimum daily requirement*
Calcium	10	4	20
Protein	5	6	20
Calories	2	6	12
Price	₹ 0.60 per unit	₹ 1.00 per unit	

Find the combination of food items so that the cost may be minimum.

4. A hospital dietician wishes to find the cheapest combination of two foods, A and B, that contains at least 0.5 milligram of thiamin and at least 600 calories. Each unit of A contains 0.12 milligram of thiamin and 100 calories, while each unit of B contains 0.10 milligram of thiamin and 150 calories. If each food costs 10 paise per unit, how many units of each should be combined at a minimum cost?

5. A dietician mixes together two kinds of food in such a way that the mixture contains at least 6 units of vitamin A, 7 units of vitamin B, 11 units of vitamin C and 9 units of vitamin D. The vitamin contents of 1 kg of food X and 1 kg of food Y are given below:

	Vitamin A	*Vitamin B*	*Vitamin C*	*Vitamin D*
Food X	1	1	1	2
Food Y	2	1	3	1

One kg of food X costs ₹ 5, whereas one kg of food Y costs ₹ 8. Find the least cost of the mixture which will produce the desired diet.

6. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available. Food F_1 costs ₹ 4 per unit and F_2 costs ₹ 6 per unit one unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem and find graphically the minium cost for diet that consists of mixture of these foods and also meets the mineral nutritional requirements. [CBSE 2009]

7. Kellogg is a new cereal formed of a mixture of bran and rice that contains at least 88 grams of protein and at least 36 milligrams of iron. Knowing that bran contains 80 grams of protein and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs ₹ 5 per kg and rice costs ₹ 4 per kg. [CBSE 2002]

8. A wholesale dealer deals in two kinds, A and B (say) of mixture of nuts. Each kg of mixture A contains 60 grams of almonds, 30 grams of cashew nuts and 30 grams of hazel nuts. Each kg of mixture B contains 30 grams of almonds, 60 grams of cashew nuts and 180 grams of hazel nuts. The remainder of both mixtures is per nuts. The dealer is contemplating to use mixtures A and B to make a bag which will contain at least 240 grams of almonds, 300 grams of cashew nuts and 540 grams of hazel nuts. Mixture A costs ₹ 8 per kg. and mixture B costs ₹ 12 per kg. Assuming that mixtures A and B are uniform, use graphical method to determine the number of kg. of each mixture which he should use to minimise the cost of the bag.

9. One kind of cake requires 300 gm of flour and 15 gm of fat, another kind of cake requires 150 gm of flour and 30 gm of fat. Find the maximum number of cakes which can be made from 7.5 kg of flour and 600 gm of fat, assuming that there is no shortage of the other ingradients used in making the cakes. Make it as an LPP and solve it graphically. [CBSE 2010]

10. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs ₹ 60 kg and Food Q costs ₹ 80 kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.
11. One kind of cake requires 200 g of flour and 25 g of fat, and another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.
12. A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?
13. A farmer mixes two brands P and Q of cattle feed. Brand P, costing ₹ 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing ₹ 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?
14. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	*Vitamin A*	*Vitamin B*	*Vitamin C*
X	1	2	3
Y	2	2	1

One kg of food X costs ₹ 16 and one kg of food Y costs ₹ 20. Find the least cost of the mixture which will produce the required diet?

15. A fruit grower can use two types of fertilizer in his garden, brand P and Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine.

	kg per bag	
	Brand P	*Brand Q*
Nitrogen	3	3.5
Phosphoric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

If the grower wants to minimize the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

ANSWERS

1. 125/2 gm of food F_1, 375/4 gm of food F_2; Min cost 425/4 Paise
2. 5 units of food A and 30 units of food B
3. Food I : 3 units, Food II : 1 unit, Min cost ₹ 2.80
4. 1.875 units of A and 2.75 units of B
5. ₹ 41
6. ₹ 104
7. ₹ 4.6
8. 2 kg of A and 4 kg of B ; Min cost = ₹ 64
9. 20, 10
10. ₹ 160 at all points on the line segment joining points (8/3, 0) and (2, 1/2).
11. 30 cakes of one kind and 10 cakes of second kind.
12. Food P = 15 packets, Food Q = 20 packets, Minimum amount of vitamin A = 150 units.
13. Bags of brand P = 3, Bags of brand Q = 15, Minimum cost = ₹ 1950
14. Food X = 2 kg, Food Y = 4 kg, Least cost = ₹ 112.
15. Brand P = 40 bags, Brand Q = 50 bags, Minimum amount of Nitrogen = 470 kg.

HINTS TO SELECTED PROBLEMS

1. Let 25 x gms of food F_1 and 25 y gms of food F_2 be used to fulfil the minimum requirements of thiamine, phosphorous and iron. Then, the LLP is

$$\begin{aligned} &\text{Minimize} && Z = 20x + 15y \\ &\text{Subject to} && 0.25x + 0.10y \geq 1 \\ &&& 0.75x + 1.50y \geq 7.50 \\ &&& 1.60x + 0.80y \geq 10 \\ &\text{and,} && x, y \geq 0 \end{aligned}$$

2. Let x units of food A any y units of food B are used. Then, the LPP is

$$\begin{aligned} &\text{Minimize} && Z = 4x + 3y \\ &\text{Subject to} && 200x + 100y \geq 4000 \\ &&& x + 2y \geq 50 \\ &&& 40x + 40y \geq 1400 \\ &\text{and,} && x, y \geq 0 \end{aligned}$$

3. Let x units of food I and y units of food II are used to fulfil minimum daily requirements. Then, the LPP is

$$\begin{aligned} &\text{Minimize} && Z = 0.60x + y \\ &\text{Subject to} && 10x + 4y \geq 20 \\ &&& 5x + 6y \geq 20 \\ &&& 2x + 6y \geq 12 \\ &&& x, y \geq 0 \end{aligned}$$

4. Let x units of food A and y units of food B are combined. The LPP is

$$\begin{aligned} &\text{Minimize} && Z = 0.1x + 0.1y \\ &\text{Subject to} && 0.12x + 0.10y \geq 0.5 \\ &&& 100x + 150y \geq 600 \\ &\text{and,} && x, y \geq 0 \end{aligned}$$

5. Let x kg of food X and y kg of food Y are mixed to produce the desired diet. The LPP is

$$\begin{aligned} &\text{Minimize} && Z = 5x + 8y \\ &\text{Subject to} && x + 2y \geq 6 \\ &&& x + y \geq 7 \\ &&& x + 3y \geq 11 \\ &&& 2x + y \geq 9 \\ &\text{and,} && x, y \geq 0 \end{aligned}$$

7. Let the cereal contain x kg of bran and y kg of rice. Then, the *LPP* is

$$\text{Minimize } Z = 5x + 4y$$

$$\text{Subject to } x \times \frac{80}{1000} + y \times \frac{100}{1000} \geq \frac{88}{1000} \text{ or, } 20x + 25y \geq 22$$

$$x \times \frac{40}{100000} + y \times \frac{30}{100000} \geq \frac{36}{100000} \text{ or, } 20x + 15y \geq 18$$

$$x, y \geq 0$$

29.9.2 OPTIMAL PRODUCT LINE PROBLEMS

In this type of problems, we have to determine the number of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of the product, ware house space per unit of output, etc. in order to make maximum profit.

Following are some examples on this type of problems.

ILLUSTRATIVE EXAMPLES

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 1 *A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machin B to produce a package of nuts while it takes 3 hours on macine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 2.50 per package of nuts and Re 1.00 per package of bolts. How many packages of each should he produce each day so as to maximize his profit, if he operates his machines for at most 12 hours a day? Formulate this mathematically and then solve it.* **[CBSE 2012, 2019, NCERT]**

SOLUTION The given information can be summarized in the following tabular form:

Machines	*Time required to produce products*		*Max. Machine hours available*
	Nut	Bolt	
A	1	3	12
B	3	1	12
Profit (in ₹)	2.50	1.00	

Let the manufacturer produce x packages of nuts and y packages of bolts each day.

Since machine A takes one hour to produce one package of nuts and 3 hours to produce one package of bolts. Therefore, the total time required by machine A to produce x packages of nuts and y packages of bolts is $(x + 3y)$ hours. But machine A operates for at most 12 hours.

$\therefore \quad x + 3y \leq 12$

Similarly, the total time required by machine B to produce x packages of nuts and y packages of bolts is $(3x + y)$ hours. But machine B operates for at most 12 hours.

$\therefore \quad 3x + y \leq 12$

Since the profit on one package of nuts is ₹ 2.50 and on one package of bolts the profit is ₹ 1. Therefore, profit on x packages of nuts and y packages of bolts is of ₹ $(2.50x + y)$. Let Z denote the total profit. Then, $Z = 2.50x + y$.

Clearly, $x \geq 0$ and $y \geq 0$

Thus, the above LPP can be stated mathematically as follows:

$$\text{Maximize } Z = 2.50\,x + y$$

$$\text{Subject to } x + 3y \leq 12$$

$$3x + y \leq 12$$

$$\text{and, } \quad x, y \geq 0$$

To solve this LPP graphically, we first convert the inequations into equations to obtain the following equations.

$$x + 3y = 12,\ 3x + y = 12,\ x = 0,\ y = 0$$

The line $x + 3y = 12$ meets the coordinate axes at A_1 (12, 0) and B_1 (0, 4). Join these two points to obtain the line represented by $x + 3y = 12$. The region represented by the inequation $x + 3y \leq 12$ is the region containing the origin as $x = 0$, $y = 0$ satisfies the inequation $x + 3y \leq 12$.

The line $3x + y = 12$ meets the coordinate axes at A_2 (4, 0) and B_2 (0, 12). Join these points to obtain the line represented by $3x + y = 12$. Since $x = 0$, $y = 0$ satisfies the inequation $3x + y \leq 12$. So, the region containing the origin and below the line $3x + y = 12$ represents the region represented by $3x + y \leq 12$.

Clearly, $x \geq 0$ and $y \geq 0$ represent all points in first quadrant.

Thus, the shaded region $OA_2\ PB_1$ in Fig. 29.19 represents the feasible region of the given LPP.

The coordinates of the corner-points of the feasible region $OA_2\ PB_1$ are O (0, 0), A_2 (4, 0), P (3, 3) and B_1 (0, 4). These points are obtained by solving the corresponding intersecting lines simultaneously.

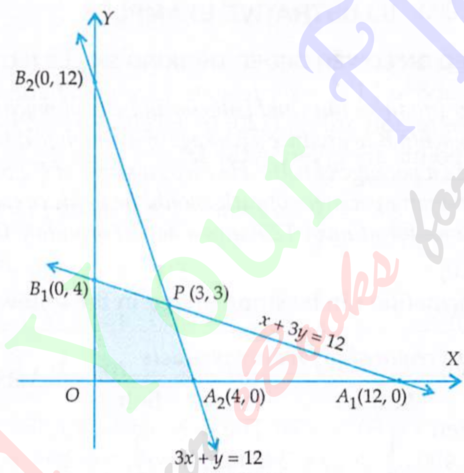

Fig. 29.19

The values of the objective function at the corner-points of the feasible region are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 2.50x + y$
O (0, 0)	$Z = 2.50 \times 0 + 1 \times 0 = 0$
A_2 (4, 0)	$Z = 2.50 \times 4 + 1 \times 0 = 10$
P (3, 3)	$Z = 2.50 \times 3 + 1 \times 3 = 10.50$
B_1 (0, 4)	$Z = 2.50 \times 0 + 1 \times 4 = 4$

Clearly, Z is maximum at $x = 3$, $y = 3$ and the maximum value of Z is 10.50.

Hence, the optimal production strategy for the manufacturer will be to manufacture 3 packages each of nuts and bolts daily and in this case his maximum profit will be ₹ 10.50.

EXAMPLE 2 *An oil company requires 12,000, 20,000 and 15,000 barrels of high-grade, medium grade and low grade oil, respectively. Refinery A produces 100, 300 and 200 barrels per day of high-grade, medium-grade and low-grade oil, respectively, while refinery B produces 200, 400 and 100 barrels per day of high-grade, medium-grade and low-grade oil, respectively. If refinery A costs ₹ 400 per day and refinery B costs ₹ 300 per day to operate, how many days should each be run to minimize costs while satisfying requirements.* **[CBSE 2004]**

SOLUTION The given data may be put in the following tabular form:

Refinery	*High-grade*	*Medium-grade*	*Low-grade*	*Cost per day*
A	100	300	200	₹ 400
B	200	400	100	₹ 300
Minimum Requirement	12,000	20,000	15,000	

Suppose refineries A and B should run for x and y days respectively to minimize the total cost.

The mathematical form of the above LPP is

$$\text{Minimize} \quad Z = 400x + 300y$$

Subject to

$$100x + 200y \geq 12{,}000$$
$$300x + 400y \geq 20{,}000$$
$$200x + 100y \geq 15{,}000$$

and, $\quad x, y \geq 0$

The feasible region of the above LPP is represented by the shaded region in Fig. 29.20. The corner points of the feasible region are A_2 (120, 0), P (60, 30) and B_3 (0, 150). The value of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = 400x + 300y$
A_2 (120, 0)	$Z = 400 \times 120 + 300 \times 0 = 48000$
P (60, 30)	$Z = 400 \times 60 + 300 \times 30 = 33000$
B_3 (0, 150)	$Z = 400 \times 0 + 300 \times 150 = 45000$

Clearly, Z is minimum when $x = 60$, $y = 30$. The feasible region is unbounded. So, we find the half-plane represented by $400x + 300y < 33000$. Clearly, the half-plane does not have points common with the feasible region. So, Z is minimum at $x = 60$, $y = 30$.

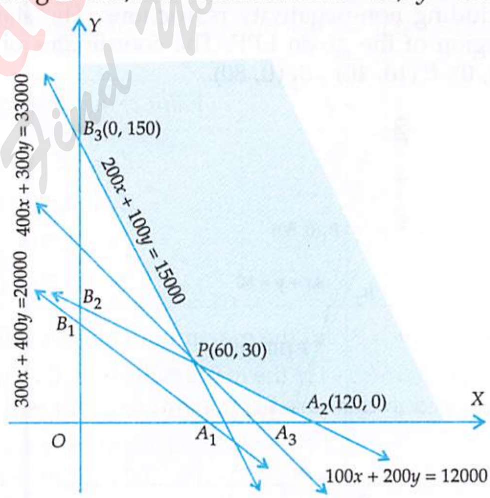

Fig. 29.20

Hence, the machine A should run for 60 days and the machine B should run for 30 days to minimize the cost while satisfying the constraints.

EXAMPLE 3 *A company produces soft drinks that has a contract which requires that a minimum of 80 units of the chemical A and 60 units of the chemical B to go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier S has a mix of 4 units of A and 2 units of B that costs ₹ 10, the supplier T has a mix of 1 unit of A and 1 unit of B that costs ₹ 4. How many mixes from S and T should the company purchase to honour contract requirement and yet minimize cost?*

[CBSE 2012]

SOLUTION The given data may be put in the following tabular form:

Supplier / *Chemical*	*S*	*T*	*Minimum Requirement*
A	4	1	80
B	2	1	60
Cost per unit	₹ 10	₹ 4	

Suppose x units of mix are purchased from supplier S and y units are purchased from supplier T.

Total cost $Z = 10x + 4y$.

Units of chemical A per bottle $= 4x + y$. But the minimum requirement of chemical A per bottle is 80 units.

$\therefore \quad 4x + y \geq 80$

Similarly, $\quad 2x + y \geq 60$. Clearly, $\quad x \geq 0, y \geq 0$

Thus, the mathematical formulation of the given LPP is

Minimize $\quad Z = 10x + 4y$

Subject to

$$4x + y \geq 80$$
$$2x + y \geq 60$$

and, $\quad x \geq 0, y \geq 0$

Now, we find the feasible region which is the set of all points whose coordinates simultaneously satisfy all constraints including non-negativity restrictions. The shaded region in Fig. 29.21 represents the feasible region of the given LPP. The coordinates of the corner points of the feasible region are A_2 (30, 0), P (10, 40), B_1 (0, 80).

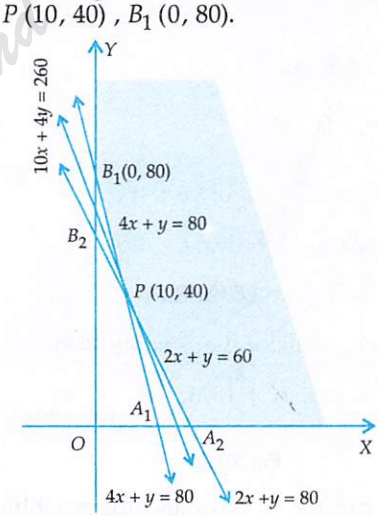

Fig. 29.21

These points are obtained by solving the equations of the corresponding intersecting lines, simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of objective function* $Z = 10x + 4y$
$A_2\,(30, 0)$	$Z = 10 \times 30 + 4 \times 0 = 300$
$P\,(10, 40)$	$Z = 10 \times 10 + 40 \times 4 = 260$
$B_1\,(0, 80)$	$Z = 10 \times 0 + 4 \times 80 = 320$

Clearly, Z is minimum at $(10, 40)$. The feasible region is unbounded and the open half plane represented by $10x + 4y < 260$ does not have points in common with the feasible region. So, Z is minimum at $x = 10$, $y = 40$. Hence, $x = 10$, $y = 40$ is the optimal solution of the given LPP.

Hence, the cost per bottle is minimum when the company purchases 10 mixes from supplier S and 40 mixes from supplier T.

EXAMPLE 4 *A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760.00 to invest and has space for at most 20 items. A fan costs him ₹ 360.00 and a sewing machine ₹ 240.00. His expectation is that he can sell a fan at a profit of ₹ 22.00 and a sewing machine at a profit of ₹ 18.00. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ? Translate this problem mathematically and then solve it.* **[CBSE 2001C, 2002C]**

SOLUTION Suppose the dealer buys x fans and y sewing machines. Since the dealer has space for at most 20 items. Therefore,

$$x + y \le 20$$

A fan costs ₹ 360 and a sewing machine costs ₹ 240. Therefore, total cost of x fans and y sewing machines is ₹ $(360x + 240y)$. But the dealer has only ₹ 5760 to invest. Therefore,

$$360x + 240y \le 5760$$

Since the dealer can sell all the items that he can buy and the profit on a fan is of ₹ 22 and on a sewing machine the profit is of ₹ 18. Therefore, total profit on selling x fans and y sewing machines is of ₹ $(22x + 18y)$.

Let Z denote the total profit. Then, $Z = 22x + 18y$.

Clearly, $x, y \ge 0$.

Thus, the mathematical formulation of the given problem is

$$\textit{Maximize} \quad Z = 22x + 18y$$

Subject to

$$x + y \le 20$$

$$360x + 240y \le 5760$$

and, $\quad x \ge 0, y \ge 0$

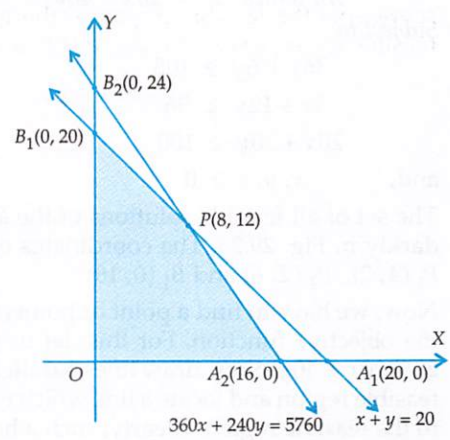

Fig. 29.22

To solve this LPP graphically, we first convert the inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.22. The corner points of the feasible region $OA_2\,PB_1$ are $O\,(0, 0)$, $A_2\,(16, 0)$, $P\,(8, 12)$ and $B_1\,(0, 20)$.

These points have been obtained by solving the corresponding intersecting lines, simultaneously.

The values of the objective function Z at corner-points of the feasible region are given in the following table.

Point (x, y)	*Value of the objective function* $Z = 22x + 18y$
$O\ (0\ 0)$	$Z = 22 \times 0 + 18 \times 0 = 0$
$A_2\ (16, 0)$	$Z = 22 \times 16 + 18 \times 0 = 352$
$P\ (8, 12)$	$Z = 22 \times 8 + 18 \times 12 = 392$
$B_1\ (0, 20)$	$Z = 22 \times 0 + 20 \times 18 = 360$

Clearly, Z is maximum at $x = 8$ and $y = 12$. The maximum value of Z is 392.

Hence, the dealer should purchase 8 fans and 12 sewing machines to obtain the maximum profit under given conditions.

EXAMPLE 5 *A farm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it is necessary to buy two additional products, say, A and B. One unit of product A contains 36 units of X, 3 units of Y, and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs ₹ 20 per unitand product B costs ₹ 40 per unit. Formulate the above as a linear programming problem to minimize the total cost, and solve the problem by using graphical method.*

SOLUTION The data given in the problem can be summarized in the following tabular form:

Product	*Nutrient constituent*			*Cost in ₹*
	X	*Y*	*Z*	
A	36	3	20	20
B	6	12	10	40
Minimum Requirement	108	36	100	

Let x units of product A and y units of product B are bought to fulfill the minimum requirement of X, Y and Z and to minimize the cost.

The mathematical formulation of the above problem is as follows:

$$\text{Minimize } Z = 20x + 40y$$

Subject to

$$36x + 6y \geq 108$$
$$3x + 12y \geq 36$$
$$20x + 10y \geq 100$$

and, $x, y, z \geq 0$

The set of all feasible solutions of the above LPP is represented by the feasible region shaded darkly in Fig. 29.23. The coordinates of the corner points of the feasible region are $A_2\ (12, 0)$, $P_1\ (4, 2)$, $P_2\ (2, 6)$ and $B_1\ (0, 18)$.

Now, we have to find a point or points in the feasible region which give the minimum value of the objective function. For this, let us give some value to Z, say 20, and draw a dotted line $20 = 20x + 40y$. Now, draw lines parallel to this line which have at least one point common to the feasible region and locate a line which is nearest to the origin and has at least one point common to the feasible region. Clearly, such a line is $Z_1 = 20x + 40y$ and it has a point $P_1\ (4, 2)$ common with the feasible region. Thus, $Z_1 = 20x + 40y$ is the minimum value of Z, and the feasible solution which gives this value of Z is the corner $P_1\ (4, 2)$ of the shaded region. The values of the variables for the optimal solution are $x = 4$, $y = 2$. Substituting these values in $Z = 20x + 40y$, we get $Z = 160$ as the optimal value of Z.

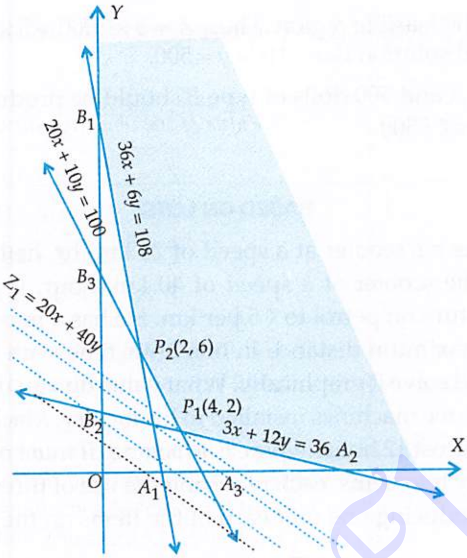

Fig. 29.23

Hence, 2 units of product A and 4 units of product B are sufficient to fulfill the minimum requirement at a minimum cost of ₹ 160.

EXAMPLE 6 *A toy manufacturer produces two types of dolls; a basic version doll A and a deluxe version doll B. Each doll of type B takes twice as long to produce as one doll of type A. The company have time to make a maximum of 2000 dolls of type A per day, the supply of plastic is sufficient to produce 1500 dolls per day and each type requires equal amount of it. The deluxe version, i.e. type B requires a fancy dress of which there are only 600 per day available. If the company makes a profit of ₹ 3 and ₹ 5 per doll, respectively, on doll A and B; how many of each should be produced per day in order to maximize profit? Solve it by graphical method.*

SOLUTION Let x dolls of type A and y dolls of type B be produced per day to maximize the profit.

The mathematical form of the given LPP is as follows:

$$\text{Maximize } Z = 3x + 5y$$

$$\text{Subject to } x + 2y \leq 2000$$
$$x + y \leq 1500$$
$$y \leq 600$$
$$\text{and, } x, y \geq 0$$

(See Ex.2 on page 29.5)

The set of all feasible solutions of the given LPP is represented by the feasible region shaded darkly in Fig. 29.24. The coordinates of the corner points of the feasible region are O (0, 0), A_2 (1500, 0) P (1000, 500), Q (800, 600) and R (0, 600).

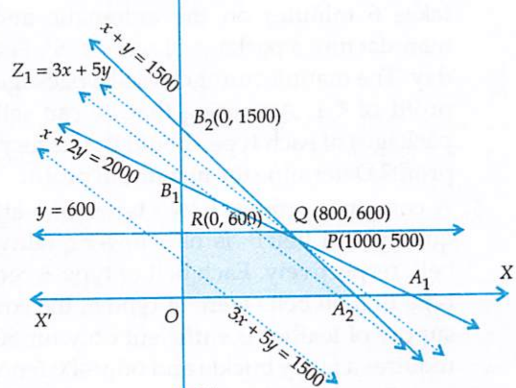

Fig. 29.24

Now, to find a point or points in the feasible region which give the maximum value of the objective function $Z = 3x + 5y$, let us give some value to Z, say 1500 and draw the dotted line $3x + 5y = 1500$ as shown in Fig. 29.23.

Now, draw lines parallel to the line $3x + 5y = 1500$ and obtain a line which is farthest from the origin and have at least one point common to the feasible region. Clearly, $Z_1 = 3x + 5y$ is such a line. This line has only one point

$P(1000, 500)$ common to the feasible region. Thus, $Z = 3 \times 1000 + 5 \times 500 = 5500$ is the maximum value of Z and the optimal solution is $x = 1000, y = 500$.

Hence, 1000 dolls of type A and 500 dolls of type B should be produced to maximize the profit and the maximum profit is ₹ 5500.

EXERCISE 29.4

BASED ON LOTS

1. If a young man drives his scooter at a speed of 25 km/hr, he has to spend ₹ 2 per km on petrol. If he drives the scooter at a speed of 40 km/hour, it produces air polution and increases his expenditure on petrol to ₹ 5 per km. He has a maximum of ₹ 100 to spend on petrol and travel a maximum distance in one hour time with less polution. Express this problem as an LPP and solve it graphically. What value do you find here? **[CBSE 2013]**
2. A manufacturer has three machines installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas Machine III must operate at least for 5 hours a day. He produces only two items, each requiring the use of three machines. The number of hours required for producing one unit each of the items on the three machines is given in the following table:

Item	*Number of hours required by the machine*		
	I	II	III
A	1	2	1
B	2	1	5/4

He makes a profit of ₹ 6.00 on item A and ₹ 4.00 on item B. Assuming that he can sell all that he produces, how many of each item should he produce so as to maximize his profit? Determine his maximum profit. Formulate this LPP mathematically and then solve it.

3. Two tailors, A and B earn ₹ 15 and ₹ 20 per day respectively. A can stitch 6 shirts and 4 pants while B can stitch 10 shirts and 4 pants per day. How many days shall each work if it is desired to produce (at least) 60 shirts and 32 pants at a minimum labour cost?
4. A factory manufactures two types of screws, A and B, each type requiring the use of two machines — an automatic and a hand-operated. It takes 4 minute on the automatic and 6 minutes on the hand-operated machines to manufacture a package of screws 'A', while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a package of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a package of screws 'A' at a profit of 70 P and screws 'B' at a profit of ₹ 1. Assuming that he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit. **[CBSE 2018, NCERT]**
5. A company produces two types of leather belts, say type A and B. Belt A is a superior quality and belt B is of a lower quality. Profits on each type of belt are ₹ 2 and ₹ 1.50 per belt, respectively. Each belt of type A requires twice as much time as required by a belt of type B. If all belts were of type B, the company could produce 1000 belts per day. But the supply of leather is sufficient only for 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 fancy buckles are available for this per day. For belt of type B, only 700 buckles are available per day.

   How should the company manufacture the two types of belts in order to have a maximum overall profit?

6. A small manufacturer has employed 5 skilled men and 10 semi-skilled men and makes an article in two qualities deluxe model and an ordinary model. The making of a deluxe model requires 2 hrs. work by a skilled man and 2 hrs. work by a semi-skilled man. The ordinary model requires 1 hr by a skilled man and 3 hrs. by a semi-skilled man. By union rules no man may work more than 8 hrs per day. The manufacturers clear profit on deluxe model is ₹ 15 and on an ordinary model is ₹ 10. How many of each type should be made in order to maximize his total daily profit. **[CBSE 2019]**

7. A manufacturer makes two types A and B of tea-cups. Three machines are needed for the manufacture and the time in minutes required for each cup on the machines is given below:

	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each cup A is 75 paise and that on each cup B is 50 paise, show that 15 tea-cups of type A and 30 of type B should be manufactured in a day to get the maximum profit. **[CBSE 2003, 2008]**

8. A factory owner purchases two types of machines, A and B, for his factory. The requirements and limitations for the machines are as follows:

	Area occupied by the machine	*Labour force for each machine*	*Daily output in units*
Machine A	1000 sq. m	12 men	60
Machine B	1200 sq. m	8 men	40

He has an area of 7600 sq.m available and 72 skilled men who can operate the machines. How many machines of each type should he buy to maximize the daily output? **[CBSE 2003, 2008]**

9. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requires 3 gm of silver and 1 gm of gold while that of type B requires 1 gm of silver and 2 gm of gold. The company can produce 9 gm of silver and 8 gm of gold. If each unit of type A brings a profit of ₹ 120 and that of type B ₹ 150, find the number of units of each type that the company should produce to maximize the profit. What is the maximum profit? **[CBSE 2020]**

10. A manufacturer of Furniture makes two products : chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hrs on machine A and 6 hrs on machine B. A table requires 4 hrs on machine A and 2 hrs on machine B. There are 16 hrs of time per day available on machine A and 30 hrs on machine B. Profit gained by the manufacturer from a chair and a table is ₹ 3 and ₹ 5 respectively. Find with the help of graph what should be the daily production of each of the two products so as to maximize his profit.

11. A furniture manufacturing company plans to make two products : chairs and tables. From its available resources which consists of 400 square feet of teak wood and 450 man hours. It is known that to make a chair requires 5 square feet of wood and 10 man-hours and yields a profit of ₹ 45, while each table uses 20 square feet of wood and 25 man-hours and yields a profit of ₹ 80. How many items of each product should be produced by the company so that the profit is maximum?

12. A firm manufactures two products A and B. Each product is processed on two machines M_1 and M_2. Product A requires 4 minutes of processing time on M_1 and 8 min. on M_2; product B requires 4 minutes on M_1 and 4 min. on M_2. The machine M_1 is available for not more than 8 hrs 20 min. while machine M_2 is available for 10 hrs. during any working day. The products A and B are sold at a profit of ₹ 3 and ₹ 4 respectively.
Formulate the problem as a linear programming problem and find how many products of each type should be produced by the firm each day in order to get maximum profit.

13. A firm manufacturing two types of electric items, A and B, can make a profit of ₹ 20 per unit of A and ₹ 30 per unit of B. Each unit of A requires 3 motors and 4 transformers and each unit of B requires 2 motors and 4 transformers. The total supply of these per month is restricted to 210 motors and 300 transformers. Type B is an export model requiring a voltage stabilizer which has a supply restricted to 65 units per month. Formulate the linear programing problem for maximum profit and solve it graphically.

14. A factory uses three different resources for the manufacture of two different products, 20 units of the resources A, 12 units of B and 16 units of C being available. 1 unit of the first product requires 2, 2 and 4 units of the respective resources and 1 unit of the second product requires 4, 2 and 0 units of respective resources. It is known that the first product gives a profit of 2 monetary units per unit and the second 3. Formulate the linear programming problem. How many units of each product should be manufactured for maximizing the profit? Solve it graphically.

15. A publisher sells a hard cover edition of a text book for ₹ 72.00 and a paperback edition of the same ext for ₹ 40.00. Costs to the publisher are ₹ 56.00 and ₹ 28.00 per book respectively in addition to weekly costs of ₹ 9600.00. Both types require 5 minutes of printing time, although hardcover requires 10 minutes binding time and the paperback requires only 2 minutes. Both the printing and binding operations have 4,800 minutes available each week. How many of each type of book should be produced in order to maximize profit?

16. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine; size B contains 1 grain of aspirin, 8 grains of bicarbonate and 66 grains of codeine. It has been found by users that it requires at least 12 grains of aspirin, 7.4 grains of bicarbonate and 24 grains of codeine for providing immediate effects. Determine graphically the least number of pills a patient should have to get immediate relief. Determine also the quantity of codeine consumed by patient.

17. A chemical company produces two compounds, A and B. The following table gives the units of ingredients, C and D per kg of compounds A and B as well as minimum requirements of C and D and costs per kg of A and B. Find the quantities of A and B which would give a supply of C and D at a minimum cost.

	Compound		*Minimum requirement*
	A	*B*	
Ingredient C	1	2	80
Ingredient D	3	1	75
Cost (in ₹) per kg	4	6	

18. A company manufactures two types of novelty Souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours available for assembling. The profit is 50 paise each for type A and 60 paise each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize the profit?
[NCERT, CBSE 2020]

19. A manufacturer makes two products A and B. Product A sells at ₹ 200 each and takes 1/2 hour to make. Product B sells at ₹ 300 each and takes 1 hour to make. There is a permanent order for 14 of product A and 16 of product B. A working week consists of 40 hours of production and weekly turnover must not be less than ₹ 10000. If the profit on each of product A is ₹ 20 and on product B is ₹ 30, then how many of each should be produced so that the profit is maximum. Also, find the maximum profit.

20. A manufacturer produces two types of steel trunks. He has two machines A and B. For completing, the first types of the trunk requires 3 hours on machine A and 3 hours on machine B, whereas the second type of the trunk requires 3 hours on machine A and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 and ₹ 25 per trunk of the first type and the second type respectively. How many trunks of each type must he make each day to make maximum profit? **[CBSE 2001, 2005, 2012]**

21. A manufacturer of patent medicines is preparing a production plan on medicines, A and B. There are sufficient raw materials available to make 20000 bottles of A and 40000 bottles of B, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes 1 hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is ₹ 8 per bottle for A and ₹ 7 per bottle for B. How should the manufacturer schedule his production in order to maximize his profit?

22. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 600 is made on each economy class ticket. The airline reserves at least 20 seats of first class. However, at least 4 times as many passengers prefer to travel by economy class to the first class. Determine how many each type of tickets must be sold in order to maximize the profit for the airline. What is the maximum profit.

23. A gardener has supply of fertilizer of type I which consists of 10% nitrogen and 6% phosphoric acid and type II fertilizer which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosporic acid for his crop. If the type I fertilizer costs 60 paise per kg and type II fertilizer costs 40 paise per kg, determine how many kilograms of each fertilizer should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost? **[CBSE 2002, 2008]**

24. Anil wants to invest at most ₹ 12000 in Saving Certificates and National Saving Bonds. According to rules, he has to invest at least ₹ 2000 in Saving Certificates and at least ₹ 4000 in National Saving Bonds. If the rate of interest on saving certificate is 8% per annum and the rate of interest on National Saving Bond is 10% per annum, how much money should he invest to earn maximum yearly income ? Find also his maximum yearly income.

25. A man owns a field of area 1000 sq.m. He wants to plant fruit trees in it. He has a sum of ₹ 1400 to purchase young trees. He has the choice of two types of trees. Type A requires 10 sq.m of ground per tree and costs ₹ 20 per tree and type B requires 20 sq.m of ground per tree and costs ₹ 25 per tree. When fully grown, type A produces an average of 20 kg of fruit which can be sold at a profit of ₹ 2.00 per kg and type B produces an average of 40 kg of fruit which can be sold at a profit of ₹ 1.50 per kg. How many of each type should be planted to achieve maximum profit when the trees are fully grown ? What is the maximum profit?

26. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp while it takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at most 20 hours and the grinding/cutting machine for at most 12 hours. The profit from the sale of a lamp is ₹ 5.00 and a shade is ₹ 3.00. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximize his profit? **[NCERT, CBSE 2013]**

27. A producer has 30 and 17 units of labour and capital respectively which he can use to produce two type of goods X and Y. To produce one unit of X, 2 units of labour and 3 units of capital are required. Similarly, 3 units of labour and 1 unit of capital is required to produce one unit of Y. If X and Y are priced at ₹ 100 and ₹ 120 per unit respectively, how should be producer use his resources to maximize the total revenue? Solve the problem graphically. **[CBSE 2000]**

28. A firm manufactures two types of products A and B and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2, whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product should the firm produce a day in order to maximize the profit. Solve the problem graphically. **[CBSE 2000]**

29. A small firm manufacturers items A and B. The total number of items A and B that it can manufacture in a day is at the most 24. Item A takes one hour to make while item B takes only half an hour. The maximum time available per day is 16 hours. If the profit on one unit of item A be ₹ 300 and one unit of item B be ₹ 160, how many of each type of item be produced to maximize the profit? Solve the problem graphically. **[CBSE 2001, 2004]**

30. A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. The profit is ₹ 50 each on type A and ₹ 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? **[CBSE 2001]**

31. A company manufactures two articles A and B. There are two departments through which these articles are processed: (i) assembly and (ii) finishing departments. The maximum capacity of the first department is 60 hours a week and that of other department is 48 hours per week. The product of each unit of article A requires 4 hours in assembly and 2 hours in finishing and that of each unit of B requires 2 hours in assembly and 4 hours in finishing. If the profit is ₹ 6 for each unit of A and ₹ 8 for each unit of B, find the number of units of A and B to be produced per week in order to have maximum profit. **[CBSE 2003]**

32. A firm makes items A and B and the total number of items it can make in a day is 24. It takes one hour to make an item of A and half an hour to make an item of B. The maximum time available per day is 16 hours. The profit on an item of A is ₹ 300 and on one item of B is ₹ 160. How many items of each type should be produced to maximize the profit? Solve the problem graphically. **[CBSE 2004]**

33. A company sells two different products, A and B. The two products are produced in a common production process, which has a total capacity of 500 man-hours. It takes 5 hours to produce a unit of A and 3 hours to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 70 and that for B is 125. If the profit is ₹ 20 per unit for the product A and ₹ 15 per unit for the product B, how many units of each product should be sold to maximize profit?

34. A box manufacturer makes large and small boxes from a large piece of cardboard. The large boxes require 4 sq. metre per box while the small boxes require 3 sq. metre per box. The manufacturer is required to make at least three large boxes and at least twice as many small boxes as large boxes. If 60 sq. metre of carboard is in stock, and if the profits on the large and small boxes are ₹ 3 and ₹ 2 per box, how many of each should be made in order to maximize the total profit?

35. A manufacturer makes two products, A and B. Product A sells at ₹ 200 each and takes 1/2 hour to make. Product B sells at ₹ 300 each and takes 1 hour to make. There is a permanent order for 14 units of product A and 16 units of product B. A working week consists of 40 hours of production and the weekly turn over must not be less than ₹ 10000. If the profit on each of product A is ₹ 20 and an product B is ₹ 30, then how many of each should be produced so that the profit is maximum? Also find the maximum profit.

BASED ON HOTS

36. If a young man drives his vehicle at 25 km/hr, he has to spend ₹ 2 per km on petrol. If he drives it at a faster speed of 40 km/hr, the petrol cost increases to ₹ 5/per km. He has ₹ 100 to spend on petrol and travel within one hour. Express this as an LPP and solve the same. **[CBSE 2007]**

37. An oil company has two depots, A and B, with capacities of 7000 litres and 4000 litres respectively. The company is to supply oil to three petrol pumps, D, E, F whose requirements are 4500, 3000 and 3500 litres respectively. The distance (in km) between the depots and petrol pumps is given in the following table: **[NCERT]**

To \ From	Distance (in km) A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km is ₹ 1.00 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

38. A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is ₹ 300 and that on a chain is ₹ 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as an LPP and solve it graphically. **[CBSE 2010]**

39. A library has to accommodate two different types of books on a shelf. The books are 6 cm and 4 cm thick and weigh 1 kg and $1\frac{1}{2}$ kg each respectively. The shelf is 96 cm long and atmost can support a weight of 21 kg. How should the shelf be filled with the books of two types in order to include the greatest number of books? Make it as an LPP and solve it graphically. **[CBSE 2010]**

40. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftman's time. If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find the number of tennis rackets and cricket bats that the factory must manufacture to earn the maximum profit. Make it as an LPP and solve it graphically. **[CBSE 2011]**

41. A merchant plans to sell two types of personal computers a desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs and his profit on the desktop model is ₹ 4500 and on the portable model is ₹ 5000. Make an LPP and solve it graphically. **[CBSE 2011]**

42. A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as ₹ 10,500 and ₹ 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so at to maximise the total profit of the society? **[CBSE 2013]**

43. A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of ₹ 8000 on each piece of model A and ₹ 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?
44. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
    (i) What number of rackets and bats must be made if the factory is to work at full capacity?
    (ii) If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find the maximum profit of the factory when it works at full capacity.
45. A merchant plans to sell two types of personal computers a desktop model and a portable model that will cost ₹ 25000 and ₹ 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs and if his profit on the desktop model is ₹ 4500 and on portable model is ₹ 5000.
46. A toy company manufactures two types of dolls, A and B. Market tests and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of ₹ 12 and ₹ 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?
47. There are two types of fertilisers F_1 and F_2. F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs ₹ 6/kg and F_2 costs ₹ 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?
48. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines.

    The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	*Number of hours required on machines*		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of ₹ 600 and ₹ 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

49. There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

To / From	Cost (in ₹)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

50. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

Types of Toys	Machines		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum fo 6 hours per day. If the profit on each toy of type A is ₹ 7.50 and that on each toy of type B is ₹ 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

51. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 1000 is made on each executive class ticket and a profit of ₹ 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit?

52. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at ₹ 100 and ₹ 120 per unit respectively, how should he use his resources to maximize the total revenue? Form the above as an LPP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate? **[CBSE 2013]**

53. A manufacturer produces two products A and B. Both the products are processed on two different machines. The available capacity of first machine is 12 hours and that of second machine is 9 hours per day. Each unit of product A requires 3 hours on both machines and each unit of product B requires 2 hours on first machine and 1 hour on second machine. Each unit of product A is sold at ₹ 7 profit and that of B at a profit of ₹ 4. Find the production level per day for maximum profit graphically. **[CBSE 2016]**

54. There are two types of fertilisers 'A' and 'B'. 'A' consists of 12% nitrogen and 5% phosphoric acid whereas 'B' consists of 4% nitrogen and 5% phosphoric acid. After testing the soil conditions, farmer finds that he needs at least 12 kg of nitrogen and 12 kg of phosphoric acid for his crops. If 'A' costs ₹ 10 per kg and 'B' cost ₹ 8 per kg, then graphically determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. **[CBSE 2016]**

55. A small firm manufactures necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half on hour to make necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹300. Formulate an LPP for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced. **[CBSE 2017]**

ANSWERS

1. $\frac{50}{3}$ km with the speed of 25 km/hr, $\frac{40}{3}$ km with the speed of 40 km/hr
2. 4 units of A, 4 units of B, ₹ 40.00
3. A : 5 days, B : 3 days
4. 30 packages of screw 'A' and 20 packages of screw 'B', ₹ 41.00
5. 200 Belts of type A and 600 belts of type B; Max. profit = ₹ 1300
6. 20 ordinary models, 10 deluxe models; Max profit = ₹ 400
8. Either 4 machines of type A and 3 machines of type

   or

   6 Machines of type A and no Machine of type B
9. 2 units of A, 3 units of B, Profit = ₹ 690
10. $\frac{22}{5}$ chairs and $\frac{9}{5}$ tables ; Max profit = ₹ 22.2
11. 24 chairs and 14 tables; Max profit = ₹ 2200
12. 25 units of product A, 100 units of product B; Max profit = ₹ 475
13. 30 type A and 60 type B ; Max profit = ₹ 2400
14. 2 units of first product, 4 units of second product; Max profit = 16 monetary units
15. 360 hard cover edition, 600 paper back edition, Max. profit = ₹ 2880
16. 2 pills of size A, 8 pills of size B ; Quantity of codeine = 50 grains
27. 19 kg, 13 kg; ₹ 254
18. 8 type A, 20 type B, Max profit = ₹ 16
19. 48 units of product A, 16 units of product B , Max. profit = ₹ 1440
20. 3, Trunks of type A, 3 Trunks of Type B, Max profit = ₹ 165
21. 10500 bottles of A, 34500 bottles of B, max profit = ₹ 325500
22. First class tickets = 40, Economy class tickets = 160, Profit = ₹ 64000.00
23. Type I Fertilizer 100 kg, Type II Fertilizer 80 kg, Cost = ₹ 92
24. ₹ 2000 in Saving Certificates, ₹ 10,000 in National Saving Bonds, Income = ₹ 1160 per month
25. Type A : 20 trees Type B : 40 trees, Max profit = ₹ 2200
26. 4 pedestal lamps, 4 wooden shades
27. 3 units of X and 8 units of Y
28. 60 units of type A and 240 units of type B
29. 8 items of type A and 16 items of type B
30. 12 toys of type A and 15 toys of type B
31. 12 units of product A and 6 units of product B
32. 8 units of A and 16 units of B.
33. Product $A \rightarrow 25$ units, Product $B \rightarrow 125$ units, Maximum profit = ₹ 2375
34. Large Box = 6, Small box = 12, Maximum profit = ₹ 42
35. Product A : 48 units, Product B : 16 units, Maximum profit = ₹ 1440
36. At 25 km/hr : 50/3 km, the 40 km/hr : 40/3 km , Maximum Distance = 30 km
37. From A : 500 litres, 3000 litres, 3500 litres to D, E, F respectively

    From B : 400 litres, 0 litres, 0 litres to D, E, F respectively
38. 8 rings and 16 chains
39. 12, 6
40. LPP is maximize $Z = 20x + 10y$, subject to $1.5x + 3y \leq 42$, $3x + y \leq 24$ and, $x, \ y \geq 0$

    Number of rackets = 4 , Number of bats = 12
41. 200 Desktop model, 50 portable model
42. 30 hectares for crop X, 20 hectares for crop Y, Total profit = ₹ 4, 95,000.
43. 12 pieces of Model A, 6 pieces of Model B, Profit = ₹ 1,68,000
44. (i) Tennis rackets = 4; Cricket bats = 12; Max. Profit = ₹ 200.
45. 200 units of desktop model and 50 units of portable model, Max. Profit = ₹ 1150000.
46. 800 dolls of type A, 400 dolls of type B, Max. Profit = ₹ 16000.

47. Fertiliser $F_1 = 100$ kg, Fertiliser $F_2 = 80$ kg, Minimum cost = ₹ 1000.
48. Item $M = 4$, Item $N = 4$, Profit = ₹ 4000.
49.

Factory at	*Depot*		
	A	*B*	*C*
P	0	5	3
Q	5	0	1

Cost = ₹ 1550.

50. 400 tickets of executive class, 160 tickets of economy class, Profit = ₹ 136000.
52. Let x units of worker and y units of capital are required to maximize the total revenue. Then, the LPP is

$$\text{Maximize } Z = 100x + 120y$$
$$\text{Subject to: } 2x + 3y \le 30$$
$$3x + y \le 17$$
$$x \ge 0,\ y \ge 0.$$

Revenue is maximum when $x = 3$, $y = 8$ and Maximum revenue is ₹ 1260.
Yes, because the efficiency of a person does not depend on sex (male or female).
53. Product A : 2 units, Product B : 3 units
54. Fertilize A : 0.3 kg, Fertilizer B : 0.21 kg
55. Necklace = 16, Bracelet = 8

HINTS TO SELECTED PROBLEMS

1. *Maximum* $Z = x + y$

Subject to $2x + 5y \le 100$
$$8x + 5y \le 200$$
and, $x, y \ge 0$

2. Suppose the manufacturer produces x units of item A and y units of item B. Then, the mathematical form of the given LPP is

Maximize $Z = 6x + 4y$

Subject to $x + 2y \le 12$
$$2x + y \le 12$$
$$x + \frac{5}{4}y \ge 5$$
and, $x, y \ge 0$

3. We have to minimize the labour cost. This means that the profit is to be maximized. For this, suppose the tailors A and B work for x and y days respectively. Then the LPP is

Maximize $Z = 15x + 20y$

Subject to $6x + 10y \ge 60$
$$4x + 4y \ge 32$$
and, $x, y \ge 0$

4. Suppose the manufacturer produces x packages of screws A and y packages of screws B in a day. The LPP is

Maximize $Z = 0.7x + y$

Subject to $4x + 6y \le 240$
$$6x + 3y \le 240$$
and, $x, y \ge 0$

5. Suppose the company produces x belts of type A and y belts of type B. Then,
Profit $= 2x + 1.5y$
Since the rate of production of belts of type B is 1000 per day. Therefore, time taken to produce y belts of type B is $\frac{y}{1000}$. Also, since each belt of type A requires twice as much time as a belt of type B, the rate of production of belts of type A is 500 per day and consequently total time taken to produce x belts of type A is $\frac{x}{500}$. Thus, we have

$$\frac{x}{500} + \frac{y}{1000} \le 1 \Rightarrow 2x + y \le 1000$$

The supply of leather is sufficient only for 800 belts per day.

$\therefore \quad x + y \le 800$

Since 400 buckles are available for belt A and 700 buckles are available for belt B per day.

$\therefore \quad x \le 400, y \le 700$

Thus, the mathematical formulation of the LPP is

Maximize $Z = 2x + 1.5y$

Subject to $2x + y \le 1000$

$x + y \le 800$

$x \le 400$

$y \le 700$

and, $x, y \ge 0$

6. If the manufacturer makes x_1 deluxe model articles and x_2 ordinary model, then

Maximize $Z = 15x_1 + 10x_2$

Subject to $2x_1 + x_2 \le 40$

$2x_1 + 3x_2 \le 80$

and, $x_1, x_2 \ge 0$

7. Let x tea-cups of type A and y tea-cups of type B are manufactured per day. Then, the LPP is

Maximize $Z = 0.75x + 0.50y$

Subject to $12x + 6y \le 360$

$18x + 0y \le 360$

$6x + 9y \le 360$

and, $x, y \ge 0$

8. Let x machines of type A and y machines of type B are bought to maximize the daily output. Then, the LPP is

Maximize $Z = 60x + 40y$

Subject to $1000x + 1200y \le 7600$

$12x + 8y \le 72$

and, $x, y \ge 0$

9. Suppose the company produces x goods of type A and y goods of type B. The mathematical form of the LPP is as follows:

Maximize $Z = 120x + 150y$

Subject to $3x + y \le 9$

$x + 2y \le 8$

and, $x, y \ge 0$

12. The LPP is as follows:

Maximum $Z = 3x_1 + 4x_2$

$4x_1 + 4x_2 \le 500$

$8x_1 + 4x_2 \le 600$

$x_1, x_2 \ge 0$

17. Let x kg of compound A and y kg of compound B are produced. Then, the mathematical formulation of the LPP is as follows:

Minimize $Z = 4x + 6y$

Subject to $x + 2y \ge 80$

$3x + y \ge 75$

$x, y \ge 0$

18. Let x souvenirs of type A and y souvenirs of type B are manufactured. Then, the LPP is as follows:

Maximize $Z = 50x + 60y$

Subject to $5x + 8y \leq 200$

$10x + 8y \leq 240$

and, $x, y \geq 0$

19. Let x units of product A and y units of product B be produced. Then the mathematical form of the LPP is as follows:

Maximize $Z = 20x + 30y$

Subject to $200x + 300y \geq 10{,}000$

$x \geq 14$

$y \geq 16$

$\frac{x}{2} + y \leq 40$

and, $x, y \geq 0$

20. Suppose x trunks of type A and y trunks of type B are manufactured per day. Then, the mathematical form of the LPP is as follows:

Maximize $Z = 30x + 25y$

Subject to $3x + 3y \leq 18$

$3x + 2y \leq 15$

and, $x, y \geq 0$

21. Let the manufacturer produce x bottles of medicine A and y bottles of medicine B. Then the mathematical form of the LPP is as follows:

Maximize $Z = 8x + 7y$

Subject to $x \leq 20000$

$y \leq 40000$

$x + y \leq 45000$

$\frac{3x}{1000} + \frac{y}{1000} \leq 66$

and, $x, y \geq 0$

22. Let x first-class tickets and y economy class tickets are sold. Then the mathematical form of the LPP is

Maximize $Z = 400x + 600y$

Subject to $x + y \leq 2000$

$x \geq 20$

$y \geq 4x$

and, $x, y \geq 0$

23. Let x kg of fertilizer I and y kg of fertilizer II are used. Then the mathematical form of the LPP is as follows:

Minimize $Z = 60x + 40y$

Subject to $\frac{10x}{100} + \frac{5y}{100} \geq 14$

$\frac{6x}{100} + \frac{10x}{100} \geq 14$

and, $x, y \geq 0$

24. Suppose Anil invests ₹ x in Saving Certicate and ₹ y in National Saving Bonds. Then, the mathematical formulation of the LPP is as follows:

Maximize $Z = \frac{8x}{100} + \frac{10y}{100}$

Subject to $x + y \le 12000$

$x \ge 2000$

$y \ge 4000$

and, $x, y \ge 0$

25. Let x trees of type A and y trees of type B are planted. Then, the mathematical formulation of the LPP is as follows:

Maximize $Z = 40x + 60y - (20x + 25y)$

Subject to $20x + 25y \le 1400$

$10x + 20y \le 1000$

and, $x, y \ge 0$

26. Let x lamps and y shades be manufactured by the manufacturer. Then, the mathematical formulation of the LPP is

Maximize $Z = 5x + 3y$

Subject to $2x + y \le 12$

$3x + 2y \le 20$

and, $x, y \ge 0$

27. Let x units of X and y units of Y be produced to maximize the revenue. Then, the LPP is

Maximize $Z = 100x + 120y$

Subject to $2x + 3y \le 30$

$3x + y \le 17$

$x \ge 0, y \ge 0$

31. Let x units of A and y units of B be produced per week for maximum profit. Then, the LPP is

Maximize $Z = 6x + 8y$

Subjected to $4x + 2y \le 60$

$2x + 4y \le 48$

$x, y \ge 0$

36. Suppose he travels x km with the speed of 25 km/hr and y km with the speed of 40 km/hr. Then, the LPP is

Maximize $Z = x + y$

Subjected to $2x + 5y \le 100$

$$\frac{x}{25} + \frac{y}{40} \le 1$$

$x, y \ge 0$

29.9.3 TRANSPORTATION PROBLEMS

In this type of problems, we have to determine transportation schedule for a commodity from different plants or factories situated at different locations to different markets at different locations in such a way that the total cost of transportation is minimum, subject to the limitations (constraints) as regards the demand of each market and supply from each plant or factory.

Following are some examples on this type of problems:

ILLUSTRATIVE EXAMPLES

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 1 *There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of these depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:*

From \ To	Cost (in ₹)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. Formulate the above LPP mathematically and then solve it.

SOLUTION For the formulation see Example 11 in section 29.4 on page 29.11.

Let the factory at P transports x units of commodity to depot at A and y units to depot at B. Then, as discussed in Example 11 on page 29.11, the mathematical model of the LPP is as follows:

$$\text{Minimize } Z = x - 7y + 190$$

$$\text{Subject to } x + y \le 8$$
$$x + y \ge 4$$
$$x \le 5$$
$$y \le 5$$
$$\text{and, } x \ge 0, y \ge 0$$

To solve this LPP graphically, we first convert the inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.25.

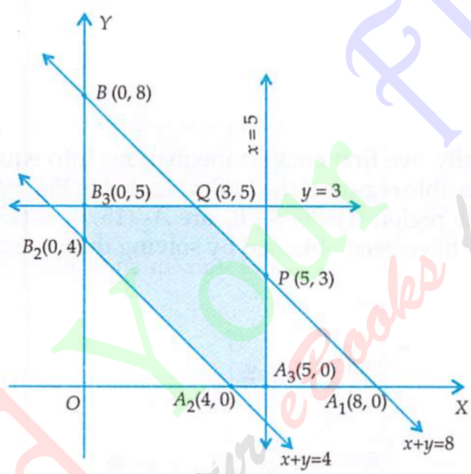

Fig. 29.25

The coordinates of the corner points of the feasible region $A_2\,A_3\,PQ\,B_3\,B_2$ are $A_2\,(4, 0)$, $A_3\,(5, 0)$, $P\,(5, 3)$, $Q\,(3, 5)$, $B_3\,(0, 5)$ and $B_2\,(0, 4)$. These points have been obtained by solving the corresponding intersecting lines simultaneously.

The values of the objective function at these points are given in the following table:

Point (x, y)	*Value of the objective function* $Z = x - 7y + 190$
$A_2\,(4, 0)$	$Z = 4 - 7 \times 0 + 190 = 194$
$A_3\,(5, 0)$	$Z = 5 - 7 \times 0 + 190 = 195$
$P\,(5, 3)$	$Z = 5 - 7 \times 3 + 190 = 174$
$Q\,(3, 5)$	$Z = 3 - 7 \times 5 + 190 = 158$
$B_3\,(0, 5)$	$Z = 0 - 7 \times 5 + 190 = 155$
$B_2\,(0, 4)$	$Z = 0 - 7 \times 4 + 190 = 162$

Clearly, Z is minimum at $x = 0$, $y = 5$. The minimum value of Z is 155.

Thus, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 unit from the factory at Q to the depots at A, B and C respectively. The minimum transportation cost in this case is ₹ 155.

EXAMPLE 2 *A brick manufacturer has two depots, A and B, with stocks of 30,000 and 20,000 bricks respectively. He receives orders from three builders P, Q and R for 15,000, 20,000 and 15,000 bricks respectively. The cost in ₹ transporting 1000 bricks to the builders from the depots are given below:*

From \ To	*P*	*Q*	*R*
A	40	20	30
B	20	60	40

How should the manufacturer fulfill the orders so as to keep the cost of transportation minimum?

SOLUTION The formulation of this LPP is discussed in Ex. 12 in section 29.4 on page 29.13.

Let the depot A transport x thousand bricks to builder P and y thousand bricks to builder Q. Then, the above LPP can be stated mathematically as follows:

$$\text{Minimize } Z = 30x - 30y + 1800$$

$$\text{Subject to } x + y \leq 30$$
$$x \leq 15$$
$$y \leq 20$$
$$x + y \geq 15$$
$$\text{and, } x \geq 0, y \geq 0$$

To solve this LPP graphically, we first convert inequations into equations and then draw the corresponding lines. The feasible region of the LPP is shaded in Fig. 29.26. The coordinates of the corner points of the feasible region A_2 PQ B_3 B_2 are A_2 (15, 0), P (15, 15), Q (10, 20), B_3 (0, 20) and B_2 (0, 15). These points have been obtained by solving the corresponding intersecting lines simultaneously.

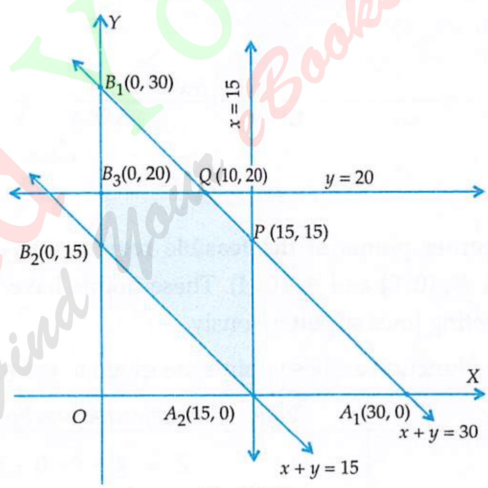

Fig. 29.26

The values of the objective function at the corner points of the feasible region are given in the following table

Point (x, y)	*Value of the objective function* $Z = 30x - 30y + 1800$
A_2 (15, 0)	$Z = 30 \times 15 - 30 \times 0 + 1800 = 2250$
P (15, 15)	$Z = 30 \times 15 - 30 \times 15 + 1800 = 1800$
Q (10, 20)	$Z = 30 \times 10 - 30 \times 20 + 1800 = 1500$
B_3 (0, 20)	$Z = 30 \times 0 - 30 \times 20 + 1800 = 1200$
B_2 (0, 15)	$Z = 30 \times 0 - 30 \times 15 + 1800 = 1350$

Clearly, Z is minimum at $x = 0$, $y = 20$ and the minimum value of Z is 1200.

Thus, the manufacturer should supply 0, 20 and 10 thousand bricks to builders P, Q and R from depot A and 15, 0 and 5 thousand bricks to builders P, Q and R from depot B respectively. In this case the minimum transportation cost will be ₹ 1200.

EXERCISE 29.5

BASED ON HOTS

1. Two godowns, A and B, have grain storage capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table: [NCERT]

	Transportation cost per quintal (in ₹)	
From / *To*	*A*	*B*
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum ?

2. A medical company has factories at two places, A and B. From these places, supply is made to each of its three agencies situated at P, Q and R. The monthly requirements of the agencies are respectively 40, 40 and 50 packets of the medicines, while the production capacity of the factories, A and B, are 60 and 70 packets respectively. The transportation cost per packet from the factories to the agencies are given below:

	Transportation cost per packet (in ₹)	
From / *To*	*A*	*B*
P	5	4
Q	4	2
R	3	5

How many packets from each factory be transported to each agency so that the cost of transportation is minimum ? Also find the minimum cost?

ANSWERS

1. From A : 10 quintals, 50 quintals and 40 quintals to D, E, F respectively.
   From B : 50 quintals, 0 quintal and 0 quintal to D, E, F respectively.
2. From A : 10 packets, 0 packets and 50 packets to P, Q and R respectively.
   From B : 30 packets, 40 packets and 0 packets to P, Q and R respectively.
   Minimum cost = ₹ 400.

HINTS TO SELECTED PROBLEMS

1. Suppose godown A supplies x quintals of grain to the ration shop D and y quintals to ration shop E. Then, the mathematical formulation of the LPP is as follows:
   Minimize $Z = 6x + 3y + \frac{5}{2}(100 - x - y) + 4(60 - x) + 2(50 - y) + 3(x + y - 60)$

$$\begin{aligned} \textit{Subject to } x+y &\le 100\\ x &\le 60\\ y &\le 50\\ x+y &\ge 60\\ \text{and,}\quad x, y &\ge 0 \end{aligned}$$

2. Let x and y packets be transported from the factory A to the agencies P and Q respectively. Then, the mathematical formulation of the LPP is as follows:

$$\textit{Minimize } Z = 5x+4y+3(60-x-y)+4(40-x)+2(40-y)+5(x+y-30)$$

$$\begin{aligned} \textit{Subject to}\quad x+y &\le 60\\ x &\le 40\\ y &\le 40\\ x+y &\ge 30\\ \text{and,}\quad x \ge 0, y &\ge 0 \end{aligned}$$

29.10 SOME EXCEPTIONAL CASES

Uptill now we have been discussing linear programming problems having finite unique solutions. In this section, we shall discuss some problems which either do not have solutions or they have unbounded solutions. Consider the following linear programming problem:

$$\textit{Maximize } Z = 2x+5y$$

Subject to the constraints

$$\begin{aligned} x+y &\le 4\\ 3x+3y &\ge 18\\ x, y &\ge 0 \end{aligned}$$

The graphical representation of constraints and non-negativity restrictions is shown in Fig. 29.27. We observe that the constraints do not have any common feasible solution. So, the LPP does not have any solution.

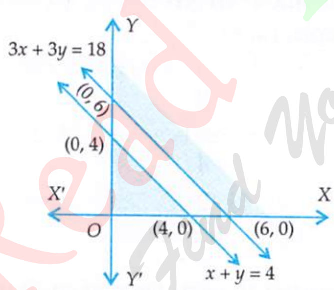

Fig. 29.27

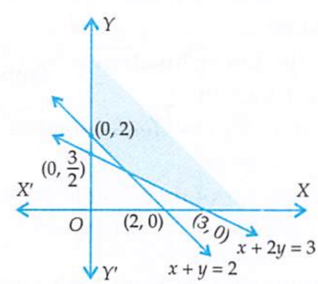

Fig. 29.28

In some linear programming problems, the common feasible region may not be bounded and the variables can take any value in the unbounded feasible region. Such type of problems are said to have unbounded solutions. Consider the following linear programming problem.

$$\textit{Maximize } Z = 2x+3y$$

Subject to the constraints

$$\begin{aligned} x+y &\ge 2\\ x+2y &\ge 3\\ x, y &\ge 0 \end{aligned}$$

The common feasible solution set of this LPP is the set of all points in the shaded region of Fig. 29.28. We observe that the feasible region is unbounded. So, x and y can take arbitrary large values. Consequently, the objective function can be made as large as we please. Thus, we say that the LPP has unbounded solution.

FILL IN THE BLANK TYPE QUESTIONS (FBQs)

1. In a LPP, the objective function is always ....
2. The feasible region for an LPP is always a .... polygon.
3. If the feasible region for an LPP is ...., then the optimal value of the objective function $z = ax + by$ may or may not exist.
4. A feasible region of a system of linear inequalities is said to be .... if it can be enclosed within a circle.
5. A corner point of a feasible region is a point in the region which is the.................. of two boundary lines.
6. The maximum value of $z = 4x + 2y$ subject to the constraints $2x + 3y \le 18$, $x + y \le 10$, $x \ge 0$, $y \ge 0$, is ....
7. The point which provides the optimal solution of the linear programming problem
$$\max z = 45x + 55y$$
$$6x + 4y \le 120$$
$$3x + 10y \le 180$$
$$x \ge 0,\ y \ge 0$$
has the coordinates ....
8. The maximum value of $z = 3x + 4y$ subject to the constraints $x + y \le 40$, $x + 2y \le 60$ $x, y \ge 0$, is ....
9. The minimum of the objective function $Z = 2x + 10y$ for linear constraints $x - y \ge 0$, $x - 5y \le -5$, $x \ge 0$, $y \ge 0$, is ....
10. The coordinates of the point for minimum value of $Z = 7x - 8y$ subject to the conditions $x + y \le 20$, $y \ge 5$, $x \ge 0$, $y \ge 0$ are ....
11. In a LPP, the linear inequalities or restrictions on the variables are called ....
12. In a LPP, if the objective function $Z = ax + by$ has the same maximum value on two corner points of the feasible region, then every point on the line segment joining these two points give the same .... value.
13. In a LPP, the linear function which has to be maximised or minimized is called a linear .... function.
14. The common region determined by all the linear constraints of a LPP is called the .... region.
15. The corner points of the feasible region of a LPP are (0, 0), (0, 8), (2, 7), (5, 4) and (6, 0). The maximum profit $P = 3x + 2y$ occurs at the point.................. [CBSE 2020]

ANSWERS

1. linear	2. convex	3. unbounded	4. bounded	5. intersection
6. 36	7. (10, 15)	8. 140	9. 15	10. (0, 20)
11. linear constraints	12. maximum	13. objective	14. feasible	15. (5, 4)