CHAPTER 20

GEOMETRIC PROGRESSIONS

20.1 GEOMETRIC PROGRESSION

A sequence of non-zero numbers is called a geometric progression (abbreviated as G.P.) if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the G.P.

In other words, a sequence, $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a geometric progression if $\frac{a_{n+1}}{a_n}$ = constant for all $n \in N$.

ILLUSTRATION 1 *The sequence 4, 12, 36, 108, ... is a G.P., because $\frac{12}{4} = \frac{36}{12} = \frac{108}{36} = \ldots = 3$, which is constant.*

Clearly, this sequence is a G.P. with first term 4 and common ratio 3.

ILLUSTRATION 2 *The sequence $\frac{1}{3}, -\frac{1}{2}, \frac{3}{4}, -\frac{9}{8}, \ldots$ is a G.P. with first term $\frac{1}{3}$ and common ratio equal to $\left(-\frac{1}{2}\right) \div \left(\frac{1}{3}\right) = -\frac{3}{2}$.*

ILLUSTRATION 3 *Show that the sequence given by $a_n = 3(2^n)$, for all $n \in N$, is a G.P. Also, find its common ratio.*

SOLUTION We have, $a_n = 3(2^n)$

$$\therefore \quad a_{n+1} = 3(2^{n+1})$$

So, $$\frac{a_{n+1}}{a_n} = \frac{3(2^{n+1})}{3(2^n)} = 2, \text{ which is constant for all } n \in N.$$

So, the given sequence is a G.P. with common ratio 2.

GEOMETRIC SERIES If $a_1, a_2, a_3, \ldots a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ *is called a geometric series.*

Note that the geometric series is finite or infinite according as the corresponding G.P. consists of finite or infinite number of terms.

20.2 GENERAL TERM OF A G.P.

THEOREM *Prove that the nth term of a G.P. with first term a and common ratio r is given by $a_n = ar^{n-1}$.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be the given G.P. Then,

$$a_1 = a \Rightarrow a_1 = ar^{1-1}$$

Since $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio r. Therefore,

$$\frac{a_2}{a_1} = r \Rightarrow a_2 = a_1 r \Rightarrow a_2 = ar \Rightarrow a_2 = ar^{2-1}$$

$$\frac{a_3}{a_2} = r \Rightarrow a_3 = a_2 r \Rightarrow a_3 = (ar)\,r \Rightarrow a_3 = ar^2 \Rightarrow a_3 = ar^{3-1}$$

$$\frac{a_4}{a_3} = r \Rightarrow a_4 = a_3 r \Rightarrow a_4 = (ar^2)\, r \Rightarrow a_4 = ar^3 \Rightarrow a_4 = ar^{4-1}$$

Continuing in this manner, we get $a_n = ar^{n-1}$ **Q.E.D.**

NOTE *It follows from the above discussion that if a is the first term and r is the common ratio of a G.P., then the G.P. can be written as $a, ar, ar^2, \ldots, ar^{n-1}$ or, $a, ar, ar^2, ar^3, ar^4, \ldots, ar^{n-1}, \ldots$ according as it is finite or infinite.*

20.2.1 nth TERM FROM THE END OF A FINITE G.P.

THEOREM 1 *Prove that the nth term from the end of a finite G.P. consisting of m terms is ar^{m-n}, where a is the first term and r is the common ratio of the G.P.*

PROOF Since the G.P. consists of m terms.

$\therefore$ nth term from the end = $(m-n+1)$th term from the beginning = ar^{m-n}

THEOREM 2 *Prove that the nth term from the end of a G.P. with last term l and common ratio r is given by $a_n = l\left(\frac{1}{r}\right)^{n-1}$.*

PROOF Clearly, when we look at the terms of a G.P. from the last term and move towards the beginning we find that the progression is a G.P. with common ratio $1/r$.

So, nth term from the end = $l\left(\frac{1}{r}\right)^{n-1}$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE INDICATED TERM OF A G.P. WHEN ITS FIRST TERM AND THE COMMON RATIO ARE GIVEN**

EXAMPLE 1 *Find the 9th term and the general term of the progression:* $\frac{1}{4}, -\frac{1}{2}, 1, -2, \ldots$

SOLUTION The given progression is clearly a G.P. with first term $a = 1/4$ and common ratio $r = -2$.

$\therefore$ 9th term $= a_9 = ar^{(9-1)} = ar^8 = \frac{1}{4}(-2)^8 = 64$

and, General term $= a_n = ar^{(n-1)} = \frac{1}{4}(-2)^{n-1} = (-1)^{n-1}\, 2^{n-3}$

EXAMPLE 2 *Find the 5th term of the progression*

$$1, \frac{(\sqrt{2}-1)}{2\sqrt{3}}, \left(\frac{3-2\sqrt{2}}{12}\right), \left(\frac{5\sqrt{2}-7}{24\sqrt{3}}\right), \ldots\ldots\ldots$$

SOLUTION Clearly, the given progression is a G.P. with first term $a = 1$ and common ratio $\frac{\sqrt{2}-1}{2\sqrt{3}}$. So, its 5th term is given by

$$a_5 = ar^{(5-1)} = 1 \times \left(\frac{\sqrt{2}-1}{2\sqrt{3}}\right)^4 = \frac{(\sqrt{2}-1)^4}{144}$$

EXAMPLE 3 *Find 4th term from the end of the G.P. 3, 6, 12, 24, ..., 3072.*

SOLUTION Clearly, the given progression is a G.P. with common ratio $r = 2$.

$\therefore$ 4th term from the end $= l\left(\frac{1}{r}\right)^{4-1} = (3072)\left(\frac{1}{2}\right)^{4-1} = 384$

Type II **ON FINDING THE POSITION OF A GIVEN TERM IN A GIVEN G.P.**

EXAMPLE 4 *Which term of the G.P.* $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$ *is* $\frac{1}{128}$?

SOLUTION Clearly, the given progression is a G.P. with first term $a = 2$ and common ratio $r = 1/2$. Let the nth term be $\frac{1}{128}$. Then,

$$a_n = \frac{1}{128} \Rightarrow ar^{n-1} = \frac{1}{128} \Rightarrow 2\left(\frac{1}{2}\right)^{n-1} = \frac{1}{128} \Rightarrow \left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^7 \Rightarrow n - 2 = 7 \Rightarrow n = 9$$

Thus, 9th term of the given G.P. is $\frac{1}{128}$.

EXAMPLE 5 *Which term of the G.P. 5, 10, 20, 40, ... is 5120?*

SOLUTION Clearly, the given G.P. has first term $a = 5$ and the common ratio $r = 2$. Let the nth term be 5120. Then,

$$a_n = 5120$$

$$\Rightarrow ar^{n-1} = 5120 \Rightarrow 5(2^{n-1}) = 5120 \Rightarrow 2^{n-1} = 1024 \Rightarrow 2^{n-1} = 2^{10} \Rightarrow n - 1 = 10 \Rightarrow n = 11$$

Thus, 11th term of the given G.P. is 5120.

EXAMPLE 6 *Which term of the G.P. 2, 8, 32, ... is 131072?* **[NCERT]**

SOLUTION Here, $a = 2$ and $r = 4$. Let the nth term be 131072. Then,

$$a_n = 131072$$

$$\Rightarrow ar^{n-1} = 131072 \Rightarrow 2 \times 4^{n-1} = 131072 \Rightarrow 4^{n-1} = 65536 \Rightarrow 4^{n-1} = 4^8 \Rightarrow n - 1 = 8 \Rightarrow n = 9$$

Hence, 131072 is the 9th term of the given G.P.

Type III **PROBLEMS BASED ON THE DEFINITION OF A G.P. AND THE FORMULA** $a_n = ar^{n-1}$

EXAMPLE 7 *The fourth, seventh and the last term of a G.P. are 10, 80 and 2560 respectively. Find the first term and the number of terms in the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the given G.P. Then,

$$a_4 = 10,\ a_7 = 80 \Rightarrow ar^3 = 10 \text{ and } ar^6 = 80 \Rightarrow \frac{ar^6}{ar^3} = \frac{80}{10} \Rightarrow r^3 = 8 \Rightarrow r = 2.$$

Putting $r = 2$ in $ar^3 = 10$, we get:

$$a(2)^3 = 10 \Rightarrow a = \frac{10}{8} = \frac{5}{4}.$$

Let there be n terms in the given G.P. Then,

$$a_n = 2560 \Rightarrow ar^{n-1} = 2560$$

$$\Rightarrow \quad \frac{5}{4}(2^{n-1}) = 2560 \Rightarrow 2^{n-4} = 256 \Rightarrow 2^{n-4} = 2^8 \Rightarrow n - 4 = 8 \Rightarrow n = 12.$$

EXAMPLE 8 *The first term of a G.P. is 1. The sum of the third and fifth terms is 90. Find the common ratio of the G.P.* **[NCERT]**

SOLUTION Let r be the common ratio of the G.P. It is given that the first term $a = 1$.

Now, $a_3 + a_5 = 90$

$$\Rightarrow \quad ar^2 + ar^4 = 90$$

$$\Rightarrow \quad r^2 + r^4 = 90$$

$\Rightarrow \quad r^4 + r^2 - 90 = 0$

$\Rightarrow \quad r^4 + 10r^2 - 9r^2 - 90 = 0 \Rightarrow (r^2 + 10)(r^2 - 9) = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow r = \pm 3.$

Hence, the common ratio of the given G.P. is 3 or – 3.

EXAMPLE 9 *If the 4th and 9th terms of a G.P. be 54 and 13122 respectively, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Then,

$a_4 = 54$ and $a_9 = 13122$

$\Rightarrow \quad ar^3 = 54$ and $ar^8 = 13122$

$$\Rightarrow \quad \frac{ar^8}{ar^3} = \frac{13122}{54} \Rightarrow r^5 = 243 \Rightarrow r^5 = 3^5 \Rightarrow r = 3$$

Putting $r = 3$ in $ar^3 = 54$, we get: $a(3)^3 = 54 \Rightarrow a = 2$

Thus, the given G.P. is $a, ar, ar^2, ar^3, \ldots$ i.e. 2, 6, 18, 54, ...

EXAMPLE 10 *Find a G.P. for which the sum of first two terms is –4 and the fifth term is 4 times the third term.* **[NCERT]**

SOLUTION Let a be the first term and r be the common ratio of the given G.P. It is given that
The sum of first two terms= –4. $\Rightarrow a_1 + a_2 = -4 \Rightarrow a + ar = -4$ ...(i)
It is also given that

$a_5 = 4a_3 \Rightarrow ar^4 = 4ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Putting $r = 2$ and -2 respectively in (i), we get $a = -\frac{4}{3}$ and $a = 4$ respectively.

Thus, the required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or $4, -8, 16, -32, \ldots$

EXAMPLE 11 *The third term of a G.P. is 4. Find the product of its first five terms.*

SOLUTION Let a be the first term and r the common ratio. Then,

$a_3 = 4 \Rightarrow ar^2 = 4$ ...(i)

$\therefore$ Product of first five terms $= a_1 a_2 a_3 a_4 a_5 = a(ar)(ar^2)(ar^3)(ar^4)$

$= a^5 r^{10} = (ar^2)^5 = 4^5$ [Using (i)]

EXAMPLE 12 *If the pth, qth and rth terms of a G.P. are a, b, c respectively, prove that:*
$a^{(q-r)} . b^{(r-p)} . c^{(p-q)} = 1.$ **[NCERT]**

SOLUTION Let A be the first term and R be the common ratio of the given G.P. Then,

$a = p$th term $= AR^{(p-1)}$, $b = q$th term $= AR^{(q-1)}$, and $c = r$th term $= AR^{(r-1)}$

Substituting the values of a, b and c, we get

$$a^{(q-r)} . b^{(r-p)} . c^{(p-q)}$$
$$= \left\{AR^{(p-1)}\right\}^{(q-r)} . \left\{AR^{(q-1)}\right\}^{(r-p)} . \left\{AR^{(r-1)}\right\}^{(p-q)}$$
$$= A^{(q-r)} R^{(p-1)(q-r)} . A^{(r-p)} R^{(q-1)(r-p)} . A^{(p-q)} R^{(r-1)(p-q)}$$
$$= A^{(q-r+r-p+p-q)} R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$$
$$= A^0 R^{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)} = A^0 R^0 = 1.$$

EXAMPLE 13 *If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of a G.P., show that*

$$(q-r)\log a+(r-p)\log b+(p-q)\log c=0.$$

SOLUTION Let A be the first term and R the common ratio of the given G.P. Then,

$$a=p\text{th term} \Rightarrow a=AR^{p-1} \Rightarrow \log a=\log A+(p-1)\log R \quad \text{...(i)}$$

$$b=q\text{th term} \Rightarrow b=AR^{q-1} \Rightarrow \log b=\log A+(q-1)\log R \quad \text{...(ii)}$$

$$c=r\text{th term} \Rightarrow c=AR^{r-1} \Rightarrow \log c=\log A+(r-1)\log R \quad \text{...(iii)}$$

Substituting the values of $\log a$, $\log b$ and $\log c$, we get

$$(q-r)\log a+(r-p)\log b+(p-q)\log c$$

$$=(q-r)\left\{\log A+(p-1)\log R\right\}+(r-p)\left\{\log A+(q-1)\log R\right\}$$
$$+(p-q)\left\{\log A+(r-1)\log R\right\}$$

$$=\log A\left\{(q-r)+(r-p)+(p-q)\right\}+\log R\left\{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)\right\}$$

$$=(\log A)\,0+\left\{p(q-r)+q(r-p)+r(p-q)-(q-r)-(r-p)-(p-q)\right\}\log R$$

$$=(\log A)\,0+(\log R)\,0=0.$$

EXAMPLE 14 *Find four numbers forming a geometric progression in which the third term is greater than the first terms by 9, and second term is greater than the 4th by 18.* **[NCERT]**

SOLUTION Let the four numbers in G.P. be a, ar, ar^2 and ar^3. It is given that

$$ar^2=a+9 \text{ and } ar=ar^3+18$$

$$\Rightarrow a(r^2-1)=9 \text{ and } ar(1-r^2)=18$$

$$\therefore \frac{ar(1-r^2)}{a(r^2-1)}=\frac{18}{9} \Rightarrow -r=2 \Rightarrow r=-2$$

Putting $r=-2$ in $a(r^2-1)=9$, we get

$$a(4-1)=9 \Rightarrow a=3$$

Hence, the numbers are: $3, 3(-2), 3(-2)^2, 3(-2)^3$ or, $3, -6, 12, -24$.

EXAMPLE 15 *The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?* **[NCERT]**

SOLUTION Clearly, number of bacteria at the end of different hours forms a G.P. with first term $a=30$ and common ratio $r=2$.

Number of bacteria present at the end of 2nd hour

$=$ Third term of the G.P. with first term $a=30$ and common ratio $r=2$

$$=ar^2=30\times 2^2=120$$

Number of bacteria present at the end of 4th hour

$=$ 5th term of the G.P. with first term $a=30$ and common ratio $r=2$

$$=ar^4=30\times 2^4=480$$

Number of bacteria present at the end of nth hour

$=(n+1)^{th}$ term of the G.P. with first term $a=30$ and common ratio $r=2$

$$=ar^n=30\times 2^n$$

EXAMPLE 16 *What will ₹ 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?* **[NCERT]**

SOLUTION We have,

P = Principal = ₹ 500, R = Rate of interest = 10%

$\therefore$ Amount at the end of one year $= ₹\left(P + \frac{PR}{100}\right) = ₹\, P\left(1 + \frac{R}{100}\right)$

Amount at the end of second year $= ₹\left\{P\left(1 + \frac{R}{100}\right) + P\left(1 + \frac{R}{100}\right)\frac{R}{100}\right\}$

$= ₹\, P\left(1 + \frac{R}{100}\right)\left(1 + \frac{R}{100}\right) = \text{Rs } P\left(1 + \frac{R}{100}\right)^2$

Amount at the end of the third year $= ₹\left\{P\left(1 + \frac{R}{100}\right)^2 + P\left(1 + \frac{R}{100}\right)^2 . \frac{R}{100}\right\}$

$= ₹\, P\left(1 + \frac{R}{100}\right)^3$

and so on.

Clearly, amounts at the end of various year form a G.P. with first term and common ratio $\left(1 + \frac{R}{100}\right)$.

$\therefore$ Amount at the end of 10th year = 11^{th} term of the G.P.

$$= ₹\, P\left(1 + \frac{R}{100}\right)^{10}$$

$$= ₹\, 500\left(1 + \frac{10}{100}\right)^{10} = ₹\, 500 \times \left(\frac{11}{10}\right)^{10} = ₹\, 500 \times (1.1)^{10}$$

EXAMPLE 17 *A manufacturer reckons that the value of a machine, which costs him ₹ 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.* **[NCERT]**

SOLUTION We have,

Initial value of the machine = V_0 = ₹ 15625 and, R = Rate of depreciation = 20%

$\therefore$ Depreciated value at the end of first year $= V_0 - \frac{V_0\, R}{100} = V_0\left(1 - \frac{R}{100}\right)$

Depreciated value at the end of second year $= V_1 - \frac{V_1\, R}{100} = V_1\left(1 - \frac{R}{100}\right) = V_0\left(1 - \frac{R}{100}\right)^2$

ans so on.

Clearly, depreciated values at the end of different years form a G.P. with first term V_0 and common ratio $\left(1 - \frac{R}{100}\right)$.

$\therefore$ Depriciated value at the end of 5 years

= 6th term of the G.P. with first term V_0 (= Rs 15625) and common ratio $r = \left(1 - \frac{R}{100}\right)$

$$= V_0\left(1 - \frac{R}{100}\right)^5 = ₹\left\{15625\left(1 - \frac{20}{100}\right)^5\right\} = ₹\left\{15625 \times \left(\frac{4}{5}\right)^5\right\} = ₹\, 5120$$

LEVEL-2

EXAMPLE 18 *In a G.P. of positive terms, if any terms is equal to the sum of next two terms, find the common ratio of the G.P.*

SOLUTION Let a be the first term and r be the common ratio of the G.P. By hypothesis

$$a_n = a_{n+1} + a_{n+2}$$

$$\Rightarrow \quad ar^{n-1} = ar^{n} + ar^{n+1}$$

$$\Rightarrow \quad 1 = r + r^2 \Rightarrow r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

But, $r > 0$. Therefore, $r = \dfrac{-1+\sqrt{5}}{2} = 2\left(\dfrac{\sqrt{5}-1}{4}\right) = 2 \sin 18°$

EXAMPLE 19 *If the first and the nth terms of a G.P. are a and b respectively and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.* **[NCERT]**

SOLUTION Let r be the common ratio of the given G.P. Then,

$$b = nth \text{ term} = ar^{n-1} \Rightarrow r^{n-1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n-1}}$$

Now,

$$P = \text{Product of the first } n \text{ terms}$$

$$\Rightarrow \quad P = a \,.\, ar \,.\, ar^2 \ldots ar^{n-1}$$

$$\Rightarrow \quad P = a^n \, r^{1+2+3+\ldots+(n-1)}$$

$$\Rightarrow \quad P = a^n \, r^{\frac{n(n-1)}{2}} \qquad \left[\because 1+2+3+\ldots+(n-1) = \left(\frac{n-1}{2}\right)\left(1+(n-1)\right) = \frac{n(n-1)}{2}\right]$$

$$\Rightarrow \quad P = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n-1}}\right\}^{\frac{n(n-1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2}\, b^{n/2} = (ab)^{n/2}$$

$$\therefore \quad P^2 = \left\{(ab)^{n/2}\right\}^2 = (ab)^n$$

EXAMPLE 20 *The $(m+n)$th and $(m-n)$th terms of a G.P. are p and q respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\dfrac{q}{p}\right)^{m/2n}$ respectively.*

SOLUTION Let a be the first term and r be the common ratio. Then,

$$a_{m+n} = p \text{ and } a_{m-n} = q$$

$$\Rightarrow \quad ar^{m+n-1} = p \text{ and } ar^{m-n-1} = q$$

$$\Rightarrow \quad \frac{ar^{m+n-1}}{ar^{m-n-1}} = \frac{p}{q} \Rightarrow r^{2n} = \frac{p}{q} \Rightarrow r = \left(\frac{p}{q}\right)^{1/2n} \Rightarrow \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}$$

Now, $a_m = ar^{m-1}$

$$\Rightarrow \quad a_m = ar^{(m+n-1)}\left(\frac{1}{r}\right)^n$$

$$\Rightarrow \quad a_m = a_{m+n}\left(\frac{1}{r}\right)^n \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{n/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

$$\Rightarrow \quad a_m = p\left(\frac{q}{p}\right)^{1/2} = \sqrt{pq}$$

and, $\quad a_n = ar^{n-1}$

$$\Rightarrow \quad a_n = ar^{m+n-1}\left(\frac{1}{r}\right)^m = a_{m+n}\left(\frac{1}{r}\right)^m \qquad [\because a_{m+n} = ar^{m+n-1}]$$

$$\Rightarrow \quad a_n = p\left(\frac{q}{p}\right)^{m/2n} \qquad \left[\because a_{m+n} = p \text{ and } \frac{1}{r} = \left(\frac{q}{p}\right)^{1/2n}\right]$$

EXAMPLE 21 *If pth, qth and rth terms of an A.P. as well as a G.P. are a, b and c respectively. Prove that*

$$a^{b-c}\, b^{c-a}\, c^{a-b} = 1.$$

SOLUTION Let A be the first term and d be the common difference of the A.P. It is given that a, b and c are p^{th}, q^{th} and r^{th} terms of the A.P. Therefore,

$$a = A + (p-1)\,d,\; b = A + (q-1)\,d,\; c = A + (r-1)\,d$$

$$\therefore \quad b - c = (q-r)\,d,\; c - a = (r-p)\,d \text{ and } a - b = (p-q)\,d.$$

Let α be the first term and R be the common ratio of the G.P. Then,

$$a = \alpha R^{p-1},\; b = \alpha R^{q-1}, \text{ and } c = \alpha R^{r-1}$$

$$a^{b-c}\, b^{c-a}\, c^{a-b} = \left\{\alpha R^{p-1}\right\}^{(q-r)d} \times \left\{\alpha R^{q-1}\right\}^{(r-p)d} \times \left\{\alpha R^{r-1}\right\}^{(p-q)d}$$

$$= \alpha^{(q-r)d + (r-p)d + (p-q)d}\, R^{(p-1)(q-r)d + (q-1)(r-p)d + (r-1)(p-q)d}$$

$$= \alpha^{(q-r+r-p+p-q)d}\, R^{\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\}d}$$

$$= \alpha^0 R^0 = 1.$$

EXAMPLE 22 *Find all sequences which are simultaneously A.P. and G.P.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_n \ldots$ be a sequence which is both an A.P. as well as a G.P.

Let a_n, a_{n+1}, a_{n+2} be three consecutive terms of the A.P. Then,

$$2\,a_{n+1} = a_n + a_{n+2},\; n \in N \qquad \ldots(i)$$

Let r be the common ratio of the sequence when it is considered a G.P. Then,

$$a_n = a_1 r^{n-1},\; a_{n+1} = a_1 r^n \text{ and } a_{n+2} = a_1 r^{n+1}$$

Putting these values in (i), we get

$$2\,a_1 r^n = a_1 r^{n-1} + a_1 r^{n+1} \Rightarrow 2r = 1 + r^2 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$$

Putting $r = 1$ in $a_1, a_2 = a_1 r, a_3 = a_1 r^2, a_4 = a_1 r^3, \ldots$, we obtain

$$a_1,\; a_2 = a_1,\; a_3 = a_1,\; a_4 = a_1 \ldots, \text{ which is a constant squance.}$$

Hence, the constant sequence is the only sequence which is both an A.P. as well as G.P.

EXAMPLE 23 *In a finite G.P. the product of the terms equidistant from the beginning and the end is always same and equal to the product of first and last term.*

SOLUTION Let $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ be a finite G.P. with common ratio r.
Now,

$$a_k = k\text{th term from the beginning} = a_1 r^{k-1}$$

and, $$a_{n-k+1} = k\text{th term from the end} = a_n \left(\frac{1}{r}\right)^{k-1}, \text{ where } 1 < k < n$$

$$\therefore \quad a_k\, a_{n-k+1} = \left(a_1 r^{k-1}\right) a_n \left(\frac{1}{r}\right)^{k-1} = a_1 a_n \quad \text{for all } k \text{ satisfying } 1 < k < n.$$

Hence, the product of terms equidistant from the beginning and the end is always equal to the product of first and last term.

EXAMPLE 24 *Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots, ar^{n-1}$ and $A, AR, AR^2, \ldots, AR^{n-1}$ form a G.P., and find the common ratio.* **[NCERT]**

SOLUTION The sequence formed by the products of the corresponding terms of the given sequences is

$$aA,\ aA\, rR,\ aA\, r^2 R^2, \ldots, aA\, r^{n-1} R^{n-1}$$

or, $$aA,\ aA\,(rR),\ aA\,(rR)^2,\ aA\,(rR)^3, \ldots, aA\,(rR)^{n-1}$$

Clearly, the ratio of any term and preceding term in the above squance is same equal to rR. So, it is a G.P. with common ratio rR.

EXAMPLE 25 *If p^{th}, q^{th}, r^{th} and s^{th} terms of an A.P. are in G.P., then show that $(p-q)$, $(q-r)$, $(r-s)$ are also in G.P.* **[NCERT]**

SOLUTION Let a be the first term and d be the common difference of the given A.P. Further, let a_p, a_q, a_r and a_s be its p^{th}, q^{th}, r^{th} and s^{th} terms respectively. Then,

$$a_p = a + (p-1)\,d,\ a_q = a + (q-1)\,d,\ a_r = a + (r-1)\,d \text{ and } a_s = a + (s-1)\,d.$$

$$\Rightarrow \quad a_p - a_q = (p-q)\,d,\ a_q - a_r = (q-r)\,d \text{ and } a_r - a_s = (r-s)\,d \qquad \ldots \text{(i)}$$

It is given that a_p, a_q, a_r and a_s are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$A = a_p,\ AR = a_q,\ AR^2 = a_r \text{ and } AR^3 = a_s.$$

$$\therefore \quad A - AR = a_p - a_q,\ AR - AR^2 = a_q - a_r \text{ and } AR^2 - AR^3 = a_r - a_s$$

$$\Rightarrow \quad A(1-R) = a_p - a_q,\ AR(1-R) = a_q - a_r \text{ and } AR^2(1-R) = a_r - a_s$$

$$\Rightarrow \quad (a_q - a_r)^2 = \left\{AR\,(1-R)\right\}^2 = \left\{A\,(1-R)\right\}\left\{AR^2(1-R)\right\} = (a_p - a_q)(a_r - a_s)$$

$$\Rightarrow \quad (q-r)^2 d^2 = \left\{(p-q)\,d\right\}\left\{(r-s)\,d\right\} \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad (q-r)^2 = (p-q)(r-s)$$

$$\Rightarrow \quad p-q,\ q-r,\ r-s \text{ are in G.P.}$$

EXERCISE 20.1

LEVEL-1

1. Show that each one of the following progressions is a G.P. Also, find the common ratio in each case:
   (i) $4, -2, 1, -1/2, \ldots$ (ii) $-2/3, -6, -54, \ldots$
   (iii) $a, \frac{3a^2}{4}, \frac{9a^3}{16}, \ldots$ (iv) $1/2, 1/3, 2/9, 4/27, \ldots$

2. Show that the sequence defined by $a_n = \frac{2}{3^n}, n \in N$ is a G.P.

3. Find:

(i) the ninth term of the G.P. 1, 4, 16, 64, ...

(ii) the 10th term of the G.P. $-\frac{3}{4}, \frac{1}{2}, -\frac{1}{3}, \frac{2}{9}, \ldots$

(iii) the 8th term of the G.P. 0.3, 0.06, 0.012, ...

(iv) the 12th term of the G.P. $\frac{1}{a^3 x^3}, ax, a^5 x^5, \ldots$

(v) nth term of the G.P. $\sqrt{3}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}}, \ldots$

(vi) the 10th term of the G.P. $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \ldots$

4. Find the 4th term from the end of the G.P. $\frac{2}{27}, \frac{2}{9}, \frac{2}{3}, \ldots, 162$.

5. Which term of the progression 0.004, 0.02, 0.1, ... is 12.5 ?

6. Which term of the G.P. :

(i) $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}, \ldots$ is $\frac{1}{512\sqrt{2}}$?

(ii) $2, 2\sqrt{2}, 4, \ldots$ is 128 ? [NCERT]

(iii) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729? [NCERT]

(iv) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27} \ldots$ is $\frac{1}{19683}$? [NCERT]

7. Which term of the progression 18, -12, 8, ... is $\frac{512}{729}$?

8. Find the 4th term from the end of the G.P. $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \ldots, \frac{1}{4374}$.

9. The fourth term of a G.P. is 27 and the 7th term is 729, find the G.P.

10. The seventh term of a G.P. is 8 times the fourth term and 5th term is 48. Find the G.P.

11. If the G.P.'s 5, 10, 20, ... and 1280, 640, 320, ... have their nth terms equal, find the value of n.

12. If 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s respectively, prove that $q^2 = ps$. [NCERT]

13. The 4th term of a G.P. is square of its second term, and the first term is -3. Find its 7^{th} term. [NCERT]

14. In a GP the 3^{rd} term is 24 and the 6^{th} term is 192. Find the 10^{th} term. [NCERT]

LEVEL-2

15. If a, b, c, d and p are different real numbers such that:
$(a^2 + b^2 + c^2)\, p^2 - 2\,(ab + bc + cd)\, p + (b^2 + c^2 + d^2) \le 0$, then show that a, b, c and d are in G.P. [NCERT]

16. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$ $(x \neq 0)$, then show that a, b, c and d are in G.P. [NCERT]

17. If the p^{th} and q^{th} terms of a G.P. are q and p respectively, show that $(p+q)^{th}$ term is $\left(\frac{q^p}{p^q}\right)^{\frac{1}{p-q}}$.

ANSWERS

1. (i) $-\frac{1}{2}$ (ii) 9 (iii) $\frac{3a}{4}$ (iv) $\frac{2}{3}$ 3. (i) 4^8 (ii) $\frac{1}{2}\left(\frac{2}{3}\right)^8$

(iii) $(0.3)(0.2)^7$ (iv) $(ax)^{41}$ (v) $\sqrt{3}\left(\frac{1}{3}\right)^{n-1}$ (vi) $\frac{1}{\sqrt{2}} \times \frac{1}{2^8}$

4. 6 5 6 6. (i) 11^{th} (ii) 13^{th} (iii) 12^{th} (iv) 9^{th} 7. 9 8. $\frac{1}{162}$

9. 1, 3, 9, ... 10. 3, 6, 12, .. 11. 5 13. – 2187 14. 3072

HINTS TO NCERT & SELECTED PROBLEMS

6. (ii) Let n^{th} term of the G.P. $2, 2\sqrt{2}, 4, \ldots$ be 128. Then,

$$2(\sqrt{2})^{n-1} = 128 \Rightarrow 2^{\frac{n+1}{2}} = 2^7 \Rightarrow \frac{n+1}{2} = 7 \Rightarrow n = 13$$

Thus, 13th term of the G.P. $2, 2\sqrt{2}, 4, \ldots$ is 128.

(iii) Let n^{th} term of the G.P. $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ be 729. Then,

$$\sqrt{3} \times (\sqrt{3})^{n-1} = 729 \Rightarrow 3^{\frac{n}{2}} = 3^6 \Rightarrow \frac{n}{2} = 6 \Rightarrow n = 12$$

Hence, 12^{th} term of the given G.P. is 729.

(iv) Let nth term of the G.P. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ be $\frac{1}{19683}$. Then,

$$\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1}{19683} \Rightarrow n^3 = 3^9 \Rightarrow n = 9$$

12. Let a be the first term and r the common ratio of the given G.P. It is given that

$p = 5^{th}$ term, $q = 8^{th}$ term, $s = 11^{th}$ term

$\Rightarrow \quad p = ar^4,\ q = ar^7,\ s = ar^{10}$

$\therefore \quad q^2 = a^2 r^{14}$ and $ps = a^2 r^{14}$

$\Rightarrow \quad q^2 = ps.$

13. Let the common ratio of the given G.P. be r. It is given that the fourth term is square of its second term.

$\therefore \quad (-3) r^3 = (-3r)^2 \Rightarrow -3r^3 = 9r^2 \Rightarrow r = -3$

Hence, 7^{th} term $= (-3) r^6 = -3(-3)^6 = -2187.$

14. Let the first term and common ratio of the given G.P. be a and r respectively.
It is given that 3^{rd} term $= 24$ and 6^{th} term $= 192$

$\Rightarrow \quad ar^2 = 24$ and $ar^5 = 192$

$$\Rightarrow \quad \frac{ar^5}{ar^2} = \frac{192}{24}$$

$\Rightarrow \quad r^3 = 8 \Rightarrow r = 2$

Putting $r = 2$ in $ar^2 = 24$, we get $a = 6$.

$\therefore \quad 10^{th}$ term $= ar^9 = 6 \times 2^9 = 3072$

15. It is given that

$$(a^2+b^2+c^2)\,p^2-2\,(ab+bc+cd)\,p+(b^2+c^2+d^2)\le 0$$

$$\Rightarrow \quad (a^2p^2-2\,abp+b^2)+(b^2p^2-2\,bcp+c^2)+(c^2p^2-2\,cdp+d^2)\le 0$$

$$\Rightarrow \quad (ap-b)^2+(bp-c)^2+(cp-d)^2\le 0$$

$$\Rightarrow \quad (ap-b)^2+(bp-c)^2+(cp-d)^2=0 \quad [\because (ap-b)^2+(bp-c)^2+(cp-d)^2 \text{ cannot be negative}]$$

$$\Rightarrow \quad ap-b=0,\ bp-c=0,\ cp-d=0$$

$$\Rightarrow \quad \frac{b}{a}=\frac{c}{b}=\frac{d}{c}=p$$

$\Rightarrow$ a, b, c, d are in G.P. with common ratio p.

16. We have,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$$

Now,

$$\frac{a+bx}{a-bx}=\frac{b+cx}{b-cx}$$

$$\Rightarrow \quad \frac{(a+bx)+(a-bx)}{(a+bx)-(a-bx)}=\frac{(b+cx)+(b-cx)}{(b+cx)-(b-cx)} \qquad \text{[Applying componendo-dividendo]}$$

$$\Rightarrow \quad \frac{a}{bx}=\frac{b}{cx}\Rightarrow \frac{b}{a}=\frac{c}{b}$$

Similarly, $\dfrac{b+cx}{b-cx}=\dfrac{c+dx}{c-dx} \Rightarrow \dfrac{c}{b}=\dfrac{d}{c}$

$\therefore \quad \dfrac{b}{a}=\dfrac{c}{b}=\dfrac{d}{c} \Rightarrow a, b, c, d$ are in G.P.

20.3 SELECTION OF TERMS IN G.P.

Sometimes it is required to select a finite number of terms in G.P. It is always convenient if we select the terms in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers are taken as $a, ar, ar^2, ar^3, \ldots$

The following examples illustrate the application of the above selections.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *If the sum of three numbers in G.P. is 38 and their product is 1728, find them.*

SOLUTION Let the numbers be $\frac{a}{r}$, a, ar. It is given that the product and sum of these numbers are 38 and 1728 respectively.

Now, Product = 1728 $\Rightarrow \frac{a}{r}(a)(ar)=1728 \Rightarrow a^3=1728 \Rightarrow a=12$

and, $\quad \text{Sum} = 38$

$\Rightarrow \quad \dfrac{a}{r} + a + ar = 38$

$\Rightarrow \quad a\left(\dfrac{1}{r} + 1 + r\right) = 38$

$\Rightarrow \quad 12\left(\dfrac{1 + r + r^2}{r}\right) = 38$

$\Rightarrow \quad 6 + 6r + 6r^2 = 19r \Rightarrow 6r^2 - 13r + 6 = 0 \Rightarrow (3r - 2)(2r - 3) = 0 \Rightarrow r = 3/2$ or, $r = 2/3$

Putting the values of a and r in $\dfrac{a}{r}, a, ar$, we find that the required numbers are 8, 12, 18 or 18, 12, 8.

EXAMPLE 2 *If the continued product of three numbers in G.P. is 216 and the sum of their products in pairs is 156, find the numbers.*

SOLUTION Let the three numbers be a/r, a, ar. Then,

$$\text{Product} = 216 \Rightarrow (a/r)\,.\,(a)\,.\,(ar) = 216 \Rightarrow a^3 = 6^3 \Rightarrow a = 6.$$

Sum of the products in pairs = 156

$\Rightarrow \quad \dfrac{a}{r}\,.\,a + a\,.\,ar + \dfrac{a}{r}\,.\,ar = 156$

$\Rightarrow \quad a^2\left(\dfrac{1}{r} + r + 1\right) = 156$

$\Rightarrow \quad 36\left(\dfrac{1 + r^2 + r}{r}\right) = 156$

$\Rightarrow \quad 3(r^2 + r + 1) = 13r \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = \dfrac{1}{3}$ or, $r = 3$

Putting the values of a and r, the required numbers are 18, 6, 2 or 2, 6, 18.

EXAMPLE 3 *Three numbers are in G.P. whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A.P. Find the numbers.*

SOLUTION Let the numbers be a, ar, ar^2. It is given that the sum of these numbers is 70.

$\therefore \quad a(1 + r + r^2) = 70 \quad$ ...(i)

It is also given that $4a$, $5ar$, $4ar^2$ are in A.P.

$\therefore \quad 2(5ar) = 4a + 4ar^2$

$\Rightarrow \quad 5r = 2 + 2r^2 \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2$ or, $r = 1/2$

Putting $r = 2$ in (i), we obtain $a = 10$. So, the numbers are 10, 20, 40

Putting $r = 1/2$ in (i), we get $a = 40$. So, the numbers are 40, 20, 10.

EXAMPLE 4 *Find three numbers in G.P. whose sum is 52 and the sum of whose products in pairs is 624.*

SOLUTION Let the required numbers be a, ar, ar^2. Then,

$\text{Sum} = 52 \Rightarrow a + ar + ar^2 = 52 \Rightarrow a(1 + r + r^2) = 52 \quad$ ...(i)

Sum of the products in pairs = 624

$\Rightarrow \quad a\,.\,ar + ar\,.\,ar^2 + a\,.\,ar^2 = 624$

$\Rightarrow \quad a^2 r(1 + r + r^2) = 624 \quad$ ...(ii)

Dividing (ii) by (i), we get

$$ar = 12 \Rightarrow a = \frac{12}{r} \qquad \text{....(iii)}$$

Putting $a = \frac{12}{r}$ in (i), we get

$$\frac{12}{r}(1 + r + r^2) = 52 \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = 1/3 \text{ or, } r = 3$$

Putting $r = 3$ in (iii), we obtain $a = 4$. So, the numbers are 4, 12, 36.

Putting $r = \frac{1}{3}$ in (iii), we get $a = 36$. So, the numbers are 36, 12, 4.

EXAMPLE 5 *The product of first three terms of a G.P. is 1000. If 6 is added to its second term and 7 added to its third term, the terms become in A.P. Find the G.P.*

SOLUTION Let first three terms of the given G.P. be $\frac{a}{r}$, a, ar. Then,

$$\text{Product} = 1000 \Rightarrow a^3 = 1000 \Rightarrow a = 10.$$

It is given that $\frac{a}{r}$, $a + 6$, $ar + 7$ are in A.P.

$$\therefore \quad 2(a + 6) = \frac{a}{r} + ar + 7$$

$$\Rightarrow \quad 32 = \frac{10}{r} + 10r + 7$$

$$\Rightarrow \quad 25 = \frac{10}{r} + 10r$$

$$\Rightarrow \quad 5 = \frac{2}{r} + 2r \Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0 \Rightarrow r = 2, 1/2$$

Hence, the G.P. is 5, 10, 20, ... or 20, 10, 5, ...

EXAMPLE 6 *The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.* **[NCERT]**

SOLUTION Let the numbers in G.P. be a, ar, ar^2. It is given that the sum of these numbers is 56.

$$\therefore \quad a + ar + ar^2 = 56 \qquad \text{...(i)}$$

It is also given that

$a - 1$, $ar - 7$ and $ar^2 - 21$ are in A.P.

$$2(ar - 7) = (a - 1) + (ar^2 - 21) \Rightarrow 2ar = a + ar^2 - 8 \Rightarrow a + ar^2 = 2ar + 8 \qquad \text{...(ii)}$$

From (i), we obtain

$$a + ar^2 = 56 - ar \qquad \text{...(iii)}$$

Substituting $a + ar^2 = 56 - ar$ on the LHS of (ii), we get

$$2ar + 8 = 56 - ar \Rightarrow 3ar = 48 \Rightarrow ar = 16 \Rightarrow r = \frac{16}{a}$$

Putting $r = \frac{16}{a}$ in (i), we get

$$a + 16 + \frac{256}{a} = 56$$

$$\Rightarrow \quad a^2 + 16a + 256 = 56a \Rightarrow a^2 - 40a + 256 = 0 \Rightarrow (a - 32)(a - 8) = 0 \Rightarrow a = 8, 32$$

Putting $a = 8$, in $r = \frac{16}{a}$ we get: $r = \frac{16}{8} = 2$

Putting $a = 32$, in $r = \frac{16}{a}$ we get: $r = \frac{16}{32} = \frac{1}{2}$

When $a = 8$ and $r = 2$, we obtain 8, 16 and 32 as the numbers in G.P.

When $a = 32$ and $r = \frac{1}{2}$, we obtain 32, 16, 8 as the numbers in G.P.

Hence, the numbers, in order, are 8, 16 and 32 or 32, 16 and 8.

LEVEL-2

EXAMPLE 7 *Find three numbers in G.P. whose sum is 13 and the sum of whose squares is 91.*

SOLUTION Let the numbers be a, ar, ar^2. Then,

$$\text{Sum} = 13 \Rightarrow a + ar + ar^2 = 13 \Rightarrow a(1 + r + r^2) = 13 \quad \text{...(i)}$$

Sum of the squares = 91

$$\Rightarrow \quad a^2 + a^2 r^2 + a^2 r^4 = 91 \Rightarrow a^2 (1 + r^2 + r^4) = 91 \quad \text{...(ii)}$$

Now, $a(1 + r + r^2) = 13$

$$\Rightarrow \quad a^2 (1 + r + r^2)^2 = 169 \quad \text{[From (i)]}$$

$$\Rightarrow \quad a^2 (1 + r^2 + r^4) + 2a^2 r(1 + r + r^2) = 169$$

$$\Rightarrow \quad 91 + 2ar \left\{ a(1 + r + r^2) \right\} = 169$$

$$\Rightarrow \quad 91 + 2ar \times 13 = 169 \quad \text{[Using (ii)]}$$

$$\Rightarrow \quad ar = 3 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad a = \frac{3}{r} \quad \text{...(iii)}$$

Putting $a = \frac{3}{r}$ in (i), we get

$$\frac{3}{r}(1 + r + r^2) = 13$$

$$\Rightarrow \quad \frac{3}{r} + 3 + 3r = 13 \Rightarrow 3r^2 - 10r + 3 = 0 \Rightarrow (3r - 1)(r - 3) = 0 \Rightarrow r = 3 \text{ or, } r = \frac{1}{3}$$

Putting $r = 3$ in (iii), we get $a = 1$. So, the numbers are 1, 3, 9.

Putting $r = \frac{1}{3}$ in (iii), we get $a = 9$. So, the nmbers are 9, 3, 1.

Hence, the numbers are 1, 3, 9 or 9, 3, 1.

EXAMPLE 8 *Find four numbers in G.P. whose sum is 85 and product is 4096.*

SOLUTION Let the four numbers in G.P. be $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$.

It is given that

$$\text{Product} = 4096 \Rightarrow a^4 = 4096 \Rightarrow a^4 = 8^4 \Rightarrow a = 8$$

and, $\text{Sum} = 85$

$$\Rightarrow \quad a\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = 85$$

$$\Rightarrow \quad 8\left(r^3+\frac{1}{r^3}\right)+8\left(r+\frac{1}{r}\right) = 85$$

$$\Rightarrow \quad 8\left\{\left(r+\frac{1}{r}\right)^3-3\left(r+\frac{1}{r}\right)\right\}+8\left(r+\frac{1}{r}\right)=85$$

$$\Rightarrow \quad 8\left(r+\frac{1}{r}\right)^3-16\left(r+\frac{1}{r}\right)-85 = 0$$

$$\Rightarrow \quad 8x^3-16x-85 = 0, \text{ where } r+\frac{1}{r} = x$$

$$\Rightarrow \quad (2x-5)(4x^2+10x+17) = 0$$

$$\Rightarrow \quad 2x-5 = 0 \qquad [\because\ 4x^2+10x+17=0 \text{ has imaginary roots}]$$

$$\Rightarrow \quad x = \frac{5}{2}$$

$$\Rightarrow \quad r+\frac{1}{r} = \frac{5}{2} \Rightarrow 2r^2-5r+2=0 \Rightarrow (r-2)(2r-1)=0 \Rightarrow r=2 \text{ or, } r=\frac{1}{2}$$

Putting $a=8$ and $r=2$ or $r=\frac{1}{2}$, we obtain that the four numbers are either 1, 4, 16, 64 or, 64, 16, 4, 1.

EXERCISE 20.2

LEVEL-1

1. Find three numbers in G.P. whose sum is 65 and whose product is 3375.
2. Find three numbers in G.P. whose sum is 38 and their product is 1728.
3. The sum of first three terms of a G.P. is 13/12 and their product is – 1. Find the G.P.
4. The product of three numbers in G.P. is 125 and the sum of their products taken in pairs is 87½. Find them.
5. The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms. [NCERT]
6. The sum of three numbers in G.P. is 14. If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A.P. Find the numbers.
7. The product of three numbers in G.P. is 216. If 2, 8, 6 be added to them, the results are in A.P. Find the numbers.
8. Find three numbers in G.P. whose product is 729 and the sum of their products in pairs is 819.

LEVEL-2

9. The sum of three numbers in G.P. is 21 and the sum of their squares is 189. Find the numbers. [NCERT]

ANSWERS

1. 45, 15, 5 or 5, 15, 45
2. 8, 12, 18
3. $\frac{4}{3}, -1, \frac{3}{4}, \ldots$ or $\frac{3}{4}, -1, \frac{4}{3}, \ldots$
4. $10, 5, \frac{5}{2}$ or $\frac{5}{2}, 5, 10$

5. $\frac{2}{5}, 1, \frac{5}{2}$. 6. 2, 4, 8 or 8, 4, 2 7. 18, 6, 2 or 2, 6, 18

8. 1, 9, 81 or 81, 9, 9. 3, 6, 12

HINTS TO NCERT & SELECTED PROBLEMS

3. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{13}{12} \text{ and } \frac{a}{r} \times a \times ar = -1$$

$$\Rightarrow \quad a\left(\frac{r^2 + r + 1}{r}\right) = \frac{13}{12} \text{ and } a^3 = -1$$

$$\Rightarrow \quad a = -1 \text{ and } a\,(r^2 + r + 1) = \frac{13\,r}{12}$$

$$\Rightarrow \quad r^2 + r + 1 = -\frac{13\,r}{12}$$

$$\Rightarrow \quad 12\,r^2 + 25\,r + 12 = 0$$

$$\Rightarrow \quad 12\,r^2 + 16\,r + 9\,r + 12 = 0 \Rightarrow (3\,r + 4)\,(4\,r + 3) = 0 \Rightarrow r = -\frac{4}{3} \text{ or } -\frac{3}{4}$$

Hence, three numbers are $\frac{3}{4}, -1, \frac{4}{3}$ or, $\frac{4}{3}, -1, \frac{3}{4}$.

5. Let the terms of the G.P. be $\frac{a}{r}$, a, ar. It is given that

$$\frac{a}{r} + a + ar = \frac{39}{10} \text{ and } \frac{a}{r} \times a \times ar = 1$$

$$\Rightarrow \quad a\left(\frac{r^2 + r + 1}{r}\right) = \frac{39}{10} \text{ and } a^3 = 1$$

$$\Rightarrow \quad a = 1 \text{ and } a\,(r^2 + r + 1) = \frac{39r}{10}$$

$$\Rightarrow \quad 10\,(r^2 + r + 1) = 39r$$

$$\Rightarrow \quad 10r^2 - 29r + 10 = 0$$

$$\Rightarrow \quad (2r - 5)\,(5r - 2) = 0$$

$$\Rightarrow \quad r = \frac{5}{2} \text{ or } r = \frac{2}{5}$$

Hence, the numbers are $\frac{2}{5}, 1, \frac{5}{2}$ or $\frac{5}{2}, 1, \frac{2}{5}$

20.4 SUM OF THE TERMS OF A G.P.

THEOREM *Prove that the sum of n terms of a G.P. with first term 'a' and common ratio 'r' is given by*

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right) \quad \text{or,} \quad S_n = a\left(\frac{1 - r^n}{1 - r}\right), r \neq 1$$

PROOF Let S_n denote the sum of n terms of the G.P. with first term 'a' and common ratio r. Then,

$$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} \qquad \ldots\text{(i)}$$

Multiplying both sides by r, we get

$$r\,S_n = ar + ar^2 + \ldots + ar^{n-1} + a\,r^n \quad \ldots\text{(ii)}$$

On subtracting (ii) from (i), we get

$$S_n - r\,S_n = a - a\,r^n$$

$$\Rightarrow \quad S_n(1-r) = a(1-r^n)$$

$$\Rightarrow \quad S_n = a\left(\frac{1-r^n}{1-r}\right), \text{provided that } r \neq 1$$

or,
$$S_n = a\left(\frac{r^n-1}{r-1}\right)$$

Hence, $S_n = a\left(\dfrac{1-r^n}{1-r}\right)$ or, $S_n = a\left(\dfrac{r^n-1}{r-1}\right), r \neq 1$

Q.E.D

NOTE *Some authors state two different formulas for S_n viz.,*

$$S_n = a\left(\frac{1-r^n}{1-r}\right) \textit{ for } r<1 \quad \textit{and} \quad S_n = a\left(\frac{r^n-1}{r-1}\right) \textit{ for } r>1.$$

In fact these two are exactly identical. The only thing which must be noted is that the above formulae do not hold for $r=1$. For $r=1$, the sum of n terms of the G.P. is

$$S_n = a + a + a + \ldots + a \text{ (n times)} = na$$

REMARK 1 *If l is the last term of the G.P., then $l = a\,r^{n-1}$.*

$$\therefore \quad S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a-ar^n}{1-r} = \frac{a-(a\,r^{n-1})\,r}{1-r} = \frac{a-lr}{1-r}$$

Thus, $S_n = \dfrac{a-lr}{1-r}$ or, $S_n = \dfrac{lr-a}{r-1}$, $r \neq 1$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE SUM OF GIVEN NUMBER OF TERMS OF A GIVEN G.P.**

EXAMPLE 1 *Find the sum of 7 terms of the G.P. 3, 6, 12, ...*

SOLUTION Here, $a = 3, r = 2$ and $n = 7$.

$$\therefore \quad S_7 = a\left(\frac{r^7-1}{r-1}\right) = 3\left(\frac{2^7-1}{2-1}\right) = 3(128-1) = 381$$

EXAMPLE 2 *Find the sum of 10 terms of the G.P.* $1, 1/2, 1/4, 1/8 \ldots$

SOLUTION Here, $a = 1, r = 1/2$ and $n = 10$.

$$\therefore \quad S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$$

$$\Rightarrow \quad S_{10} = 1\left\{\frac{(1/2)^{10} - 1}{(1/2) - 1}\right\} = 2\left(1 - \frac{1}{2^{10}}\right) = 2\left(\frac{2^{10} - 1}{2^{10}}\right) = \frac{(1024 - 1)}{512} = \frac{1023}{512}$$

EXAMPLE 3 *Find the sum to 7 terms of the sequence*

$$\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}\right), \left(\frac{1}{5^4} + \frac{2}{5^5} + \frac{3}{5^6}\right), \left(\frac{1}{5^7} + \frac{2}{5^8} + \frac{3}{5^9}\right), \dots$$

SOLUTION The given sequence is

$$\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}\right), \frac{1}{5^3}\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}\right), \frac{1}{5^6}\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}\right), \dots$$

Clearly, this is a G.P. with first term $a = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} = \frac{38}{125}$ and common ratio $r = \frac{1}{5^3}$.

$$\therefore \quad S_7 = a\left(\frac{1 - r^7}{1 - r}\right) \Rightarrow S_7 = \frac{38}{125}\left\{\frac{1 - (1/5^3)^7}{1 - (1/5^3)}\right\} = \frac{38}{125}\left\{\frac{1 - 1/5^{21}}{1 - \frac{1}{125}}\right\} = \frac{19}{62}\left(1 - \frac{1}{5^{21}}\right)$$

EXAMPLE 4 *Sum the series:* $x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \dots$ *to n terms*

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = x(x + y) + x^2(x^2 + y^2) + x^3(x^3 + y^3) + \dots + x^n(x^n + y^n)$$

$$\Rightarrow \quad S_n = (x^2 + x^4 + x^6 + \dots + x^{2n}) + (xy + x^2y^2 + x^3y^3 + \dots + x^ny^n)$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n - 1}{x^2 - 1}\right\} + xy\left\{\frac{(xy)^n - 1}{xy - 1}\right\}$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{x^{2n} - 1}{x^2 - 1}\right\} + xy\left\{\frac{(xy)^n - 1}{xy - 1}\right\}$$

EXAMPLE 5 *Find the sum of the series* $2 + 6 + 18 + \dots + 4374$.

SOLUTION The given series is a geometric series in which $a = 2$, $r = 3$ and $l = 4374$.

$$\therefore \quad \text{Required sum} = \frac{(lr - a)}{(r - 1)} = \frac{4374 \times 3 - 2}{3 - 1} = 6560.$$

EXAMPLE 6 *Find the sum of the following series:*

(i) $5 + 55 + 555 + \dots$ to n terms (ii) $0.7 + 0.77 + 0.777 + \dots$ to n terms

SOLUTION (i) Let S be the sum of the series $5 + 55 + 555 + \dots$ to n terms. Then,

$$S = 5\{1 + 11 + 111 + \dots + \text{to } n \text{ terms}\}$$

$$\Rightarrow \quad S = \frac{5}{9}\{9 + 99 + 999 + \dots + \text{to } n \text{ terms}\}$$

$$\Rightarrow \quad S = \frac{5}{9}\{(10 - 1) + (10^2 - 1) + (10^3 - 1) + \dots + (10^n - 1)\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{(10 + 10^2 + 10^3 + \dots + 10^n) - \underbrace{(1 + 1 + 1 + \dots + 1)}_{n \text{ times}}\right\}$$

$$\Rightarrow \quad S = \frac{5}{9}\left\{10 \times \frac{(10^n - 1)}{10 - 1} - n\right\} = \frac{5}{9}\left\{\frac{10}{9}(10^n - 1) - n\right\} = \frac{5}{81}\left\{10^{n+1} - 10 - 9n\right\}$$

(ii) Let S be the sum $0.7 + 0.77 + 0.777 + \ldots$ to n terms. Then,

$$S = 7 \times 0.1 + 7 \times 0.11 + 7 \times 0.111 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S = 7\left\{0.1 + 0.11 + 0.111 + \ \ldots \ \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \ldots + \left(1 - \frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{n - \left(\frac{1}{10} + \frac{1}{10^2} + \ldots + \frac{1}{10^n}\right)\right\} = \frac{7}{9}\left[n - \frac{1}{10}\frac{\left\{1 - \left(\frac{1}{10}\right)^n\right\}}{\left(1 - \frac{1}{10}\right)}\right]$$

$$\Rightarrow \quad S = \frac{7}{9}\left\{n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right\} = \frac{7}{81}\left\{9n - 1 + \frac{1}{10^n}\right\}$$

EXAMPLE 7 *The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Find the sum of n terms of the G.P.* **[NCERT]**

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a + ar + ar^2 = 16 \quad \ldots \text{(i)} \quad \text{and,} \quad ar^3 + ar^4 + ar^5 = 128 \quad \ldots \text{(ii)}$$

$$\Rightarrow \quad a(1 + r + r^2) = 16 \quad \text{and,} \quad ar^3(1 + r + r^2) = 128$$

$$\Rightarrow \quad \frac{ar^3(1 + r + r^2)}{a(1 + r + r^2)} = \frac{128}{16} \Rightarrow r^3 = 8 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get: $a = \frac{16}{7}$.

$$\therefore \quad S_n = a\left(\frac{r^n - 1}{r - 1}\right) = \frac{16}{7}\left(\frac{2^n - 1}{2 - 1}\right) = \frac{16}{7}(2^n - 1)$$

EXAMPLE 8 *Find a G.P. for which the sum of the first two terms is – 4 and the fifth term is 4 times the third term.* **[NCERT]**

SOLUTION Let a be the first term and r be the common ratio of the G.P.

We have,

$$a_1 + a_2 = -4 \quad \text{and} \quad a_5 = 4a_3$$

$$\Rightarrow \quad a + ar = -4 \quad \text{and} \quad ar^4 = 4ar^2$$

$$\Rightarrow \quad a(1 + r) = -4 \quad \text{and} \quad r^2 = 4 \Rightarrow a(1 + r) = -4 \quad \text{and} \quad r = \pm 2$$

When $r = 2$,

$$a(1 + r) = -4 \Rightarrow a = -\frac{4}{3}$$

When $r = -2$,

$$a(1+r) = -4 \Rightarrow a = 4$$

Hence, required G.P. is $-\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}, \ldots$ or, $4, -8, 16, \ldots$

Type II **FINDING VALUE(S) OF *n*, *r* AND *a* WHEN THE SUM OF *n* TERMS OF A G.P. IS GIVEN**

EXAMPLE 9 *Determine the number of terms in G.P. $< a_n >$, if $a_1 = 3$, $a_n = 96$ and $S_n = 189$.*

SOLUTION Let r be the common ratio of the given G.P. Then,

$$a_n = 96 \Rightarrow a_1 r^{n-1} = 96 \Rightarrow 3r^{n-1} = 96 \Rightarrow r^{n-1} = 32 \qquad \ldots\text{(i)}$$

Now, $S_n = 189$

$$\Rightarrow \quad a_1\left(\frac{r^n-1}{r-1}\right) = 189$$

$$\Rightarrow \quad 3\left\{\frac{(r^{n-1})\,r-1}{r-1}\right\} = 189$$

$$\Rightarrow \quad 3\left(\frac{32r-1}{r-1}\right) = 189 \qquad \text{[Using (i)]}$$

$$\Rightarrow \quad 32r - 1 = 63r - 63 \Rightarrow 31r = 62 \Rightarrow r = 2$$

Putting $r = 2$ in (i), we get

$$2^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6.$$

EXAMPLE 10 *How many terms of the geometric series $1 + 4 + 16 + 64 + \ldots$ will make the sum 5461 ?*

SOLUTION Let the sum of n terms of the given series 5461.

Here, $a = 1$, $r = 4$ and $S_n = 5461$.

$\therefore \quad S_n = 5461$

$$\Rightarrow \quad a\left(\frac{r^n-1}{r-1}\right) = 5461$$

$$\Rightarrow \quad \frac{4^n-1}{4-1} = 5461 \qquad [\because a = 1 \text{ and } r = 4]$$

$$\Rightarrow \quad 4^n - 1 = 16383 \Rightarrow 4^n = 16384 \Rightarrow 4^n = 4^7 \Rightarrow n = 7$$

EXAMPLE 11 *The sum of some terms of a G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.* **[NCERT]**

SOLUTION Let there be n terms in the G.P. with first term $a = 5$ and common ratio $r = 2$. Then,

Sum of n terms $= 315$

$$\Rightarrow \quad a\left(\frac{r^n-1}{r-1}\right) = 315$$

$$\Rightarrow \quad 5\left(\frac{2^n-1}{2-1}\right) = 315$$

$$\Rightarrow \quad 2^n - 1 = 63 \Rightarrow 2^n = 64 = 2 \Rightarrow n = 6 \qquad [\because a = 5 \text{ and } r = 2]$$

$$\therefore \quad \text{Last term} = ar^{n-1} = 5 \times 2^{6-1} = 160$$

EXAMPLE 12 *In an increasing G.P., the sum of the first and the last term is 66, the product of the second and the last but one is 128 and the sum of the terms is 126. How many terms are there in the progression ?*

SOLUTION Let a be the first term and r the common ratio of the given G.P. Further, let there be n terms in the given G.P. It is given that the sum of the first and last term is 66.

i.e. $a_1 + a_n = 66$

$\Rightarrow \quad a + ar^{n-1} = 66$ ...(i)

It is also given that the product of second and the second last term is 128.

i.e. $a_2 \cdot a_{n-1} = 128 \Rightarrow ar \cdot ar^{n-2} = 128 \Rightarrow a^2 r^{n-1} = 128 \Rightarrow a \cdot (ar^{n-1}) = 128 \Rightarrow ar^{n-1} = \frac{128}{a}$

Putting this value of ar^{n-1} in (i), we get

$$a + \frac{128}{a} = 66 \Rightarrow a^2 - 66a + 128 = 0 \Rightarrow (a-2)(a-64) = 0 \Rightarrow a = 2, 64$$

Putting $a = 2$ in (i), we get

$$2 + 2 \cdot r^{n-1} = 66 \Rightarrow r^{n-1} = 32$$

Putting $a = 64$ in (i), we get

$$64 + 64r^{n-1} = 66 \Rightarrow r^{n-1} = \frac{1}{32}$$

We reject the second value as the G.P. is an increasing G.P. and therefore $r > 1$. Thus, we obtain $a = 2$ and $r^{n-1} = 32$.

Now, $S_n = 126$

$$\Rightarrow \quad 2\left(\frac{r^n - 1}{r - 1}\right) = 126$$

$$\Rightarrow \quad \frac{r^n - 1}{r - 1} = 63 \Rightarrow \frac{r^{n-1} r - 1}{r - 1} = 63 \Rightarrow \frac{32r - 1}{r - 1} = 63 \Rightarrow r = 2$$

$$\therefore \quad r^{n-1} = 32 \Rightarrow 2^{n-1} = 2^5 \Rightarrow n - 1 = 5 \Rightarrow n = 6$$

Hence, there are 6 terms in the progression.

EXAMPLE 13 *Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.* **[NCERT]**

SOLUTION If $a, ar, ar^2, \ldots$ and $A, AR, AR^2, \ldots$ are two geometric sequences, then the sequence having terms as the product of corresponding terms of the two sequences is also a geometric sequence with first term aA and common ratio rR.

Given sequences are geometric sequences with first terms 2 and 128 respectively and common ratios 2 and $\frac{1}{4}$ respectively. Therefore, the sequence formed by multiplying the corresponding terms of the given sequences is a G.P. with first term $a = 2 \times 128 = 256$ and common ratio $r = 2 \times \frac{1}{4} = \frac{1}{2}$.

Since each sequence contains 5 terms. Therefore, the sequence formed by the products of the corresponding terms has 5 terms.

$$\text{Hence, required sum} = 256\left\{\frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}}\right\} = 256\left\{\frac{\left(1 - \frac{1}{32}\right)}{\frac{1}{2}}\right\} = 512\left(1 - \frac{1}{32}\right) = 512 \times \frac{31}{32} = 496$$

ALITER Required sum $= 2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \frac{1}{2}$

$$= 256 + 128 + 64 + 32 + 16$$

$$= 256 \left\{ \frac{1 - \left(\frac{1}{2}\right)^5}{1 - \frac{1}{2}} \right\} = 512\left(1 - \frac{1}{32}\right) = 512 \times \frac{31}{32} = 496$$

Type III **ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM OF n TERMS OF A G.P.**

EXAMPLE 14 *If S_1, S_2 and S_3 be respectively the sum of n, $2n$ and $3n$ terms of a G.P., prove that*

$$S_1 (S_3 - S_2) = (S_2 - S_1)^2$$

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S_1 = a\left(\frac{r^n - 1}{r - 1}\right), S_2 = a\left(\frac{r^{2n} - 1}{r - 1}\right) \text{ and } S_3 = a\left(\frac{r^{3n} - 1}{r - 1}\right)$$

Now,

$$S_1 (S_3 - S_2) = a\left(\frac{r^n - 1}{r - 1}\right)\left\{a\left(\frac{r^{3n} - 1}{r - 1}\right) - a\left(\frac{r^{2n} - 1}{r - 1}\right)\right\}$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1) \left\{(r^{3n} - 1) - (r^{2n} - 1)\right\}$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1)(r^{3n} - r^{2n})$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \frac{a^2}{(r-1)^2} (r^n - 1)\, r^{2n} (r^n - 1)$$

$$\Rightarrow \quad S_1 (S_3 - S_2) = \left\{ar^n\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

and, $$(S_2 - S_1)^2 = \left\{a\left(\frac{r^{2n} - 1}{r - 1}\right) - a\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

$$\Rightarrow \quad (S_2 - S_1)^2 = \frac{a^2}{(r-1)^2}\left\{(r^{2n} - 1) - (r^n - 1)\right\}^2$$

$$\Rightarrow \quad (S_2 - S_1)^2 = \frac{a^2}{(r-1)^2}\left\{r^n (r^n - 1)\right\}^2 = \left\{a\, r^n\left(\frac{r^n - 1}{r - 1}\right)\right\}^2$$

Hence, $S_1 (S_3 - S_1) = (S_2 - S_1)^2$

EXAMPLE 15 *If S be the sum, P the product and R the sum of the reciprocals of n terms of a G.P., prove that* $\left(\frac{S}{R}\right)^n = P^2$. **[NCERT]**

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$S = a + ar + ar^2 + \dots + ar^{n-1} = a\left(\frac{r^n - 1}{r-1}\right) \quad \dots\text{(i)}$$

$$P = a \,.\, ar\, ar^2 \dots ar^{n-1} = a^n r^{1+2+3+\dots+(n-1)} = a^n r^{\frac{n(n-1)}{2}} \quad \dots\text{(ii)}$$

and, $$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \dots + \frac{1}{ar^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\frac{\left\{(1/r)^n - 1\right\}}{\left\{(1/r) - 1\right\}} = \frac{1}{a}\left(\frac{1-r^n}{1-r}\right)\frac{1}{r^{n-1}}$$

$$\Rightarrow \quad R = \frac{1}{a}\left(\frac{r^n-1}{r-1}\right)\frac{1}{r^{n-1}} \quad \dots\text{(iii)}$$

$$\therefore \quad \frac{S}{R} = a\left(\frac{r^n-1}{r-1}\right) . \, a\left(\frac{r-1}{r^n-1}\right) r^{n-1} = a^2 r^{n-1}$$

$$\Rightarrow \quad \left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = \left\{a^n r^{\frac{n(n-1)}{2}}\right\}^2 = P^2 \quad \text{[Using (ii)]}$$

Hence, $\left(\frac{S}{R}\right)^n = P^2$

EXAMPLE 16 *A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed* **[NCERT]**

SOLUTION Amount spent on mailing one letter = ₹ $\frac{1}{2}$

$\therefore$ Amount spent when first set of 4 letters is mailed = ₹ 2

Amount spent when second set of $4 \times 4 = 16$ letters is mailed = ₹ $(2 \times 4) = 8$

Amount spent when third set of $4 \times 4 \times 4 = 64$ letters is mailed = ₹ $(8 \times 4) = 32$

Clearly, 2, 8, 32, ... is a G.P. with first term 2 and common ratio 4.

$\therefore$ Total amount spent when 8th set of letters is mailed = Sum of 8 terms of the G.P.

$$= a\left(\frac{r^8 - 1}{r - 1}\right)$$

$$= ₹\left\{2\left(\frac{4^8 - 1}{4 - 1}\right)\right\} \quad [\because\ a = ₹\ 2 \text{ and } r = 4]$$

$$= ₹\left\{2 \times \left(\frac{65536 - 1}{3}\right)\right\} = ₹\ (2 \times 21845)$$

$$= ₹\ 43690$$

LEVEL-2

EXAMPLE 17 *Find the sum to n terms of the sequence*

$$\left(x+\frac{1}{x}\right)^2, \left(x^2+\frac{1}{x^2}\right)^2, \left(x^3+\frac{1}{x^3}\right)^2, \ldots$$

SOLUTION Let S_n denote the sum to n terms of the given sequence. Then,

$$S_n = \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \ldots + \left(x^n+\frac{1}{x^n}\right)^2$$

$$\Rightarrow \quad S_n = \left(x^2+\frac{1}{x^2}+2\right) + \left(x^4+\frac{1}{x^4}+2\right) + \left(x^6+\frac{1}{x^6}+2\right) + \ldots + \left(x^{2n}+\frac{1}{x^{2n}}+2\right)$$

$$\Rightarrow \quad S_n = (x^2+x^4+x^6+\ldots+x^{2n}) + \left(\frac{1}{x^2}+\frac{1}{x^4}+\frac{1}{x^6}+\ldots+\frac{1}{x^{2n}}\right) + \underbrace{(2+2+\ldots)}_{n \text{ times}}$$

$$\Rightarrow \quad S_n = x^2\left\{\frac{(x^2)^n-1}{x^2-1}\right\} + \frac{1}{x^2}\left\{\frac{(1/x^2)^n-1}{(1/x^2)-1}\right\} + 2n$$

$$\Rightarrow \quad S_n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^{2n}}\left(\frac{1-x^{2n}}{1-x^2}\right) + 2n$$

$$\Rightarrow \quad S_n = x^2\left(\frac{x^{2n}-1}{x^2-1}\right) + \frac{1}{x^{2n}}\left(\frac{x^{2n}-1}{x^2-1}\right) + 2n$$

$$\Rightarrow \quad S_n = \left(\frac{x^{2n}-1}{x^2-1}\right)\left(x^2+\frac{1}{x^{2n}}\right) + 2n$$

EXAMPLE 18 *Find the sum to n terms of the sequence given by* $a_n = 2^n + 3n, n \in N$.

SOLUTION Let S_n denote the sum to terms of the given sequence. Then,

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n$$

$$\Rightarrow \quad S_n = (2^1 + 3\times 1) + (2^2 + 3\times 2) + (2^3 + 3\times 3) + \ldots + (2^n + 3\times n)$$

$$\Rightarrow \quad S_n = (2^1 + 2^2 + 2^3 + \ldots + 2^n) + (3\times 1 + 3\times 2 + 3\times 3 + \ldots + 3\times n)$$

$$\Rightarrow \quad S_n = (2^1 + 2^2 + 2^3 + \ldots + 2^n) + 3\,(1 + 2 + 3 + \ldots + n)$$

$$\Rightarrow \quad S_n = 2\left(\frac{2^n-1}{2-1}\right) + 3\left\{\frac{n}{2}(1+n)\right\} = 2\,(2^n-1) + \frac{3n}{2}(n+1)$$

EXAMPLE 19 *Prove that the sum to n terms of the series:* $11 + 103 + 1005 + \ldots$ *is* $\frac{10}{9}(10^n-1)+n^2$.

SOLUTION Let S_n denote the sum to n terms of the given series. Then,

$$S_n = 11 + 103 + 1005 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = (10+1) + (10^2+3) + (10^3+5) + \ldots + \{10^n + (2n-1)\}$$

$$\Rightarrow \quad S_n = (10 + 10^2 + \ldots + 10^n) + \left\{1 + 3 + 5 + \ldots + (2n-1)\right\}$$

$$\Rightarrow \quad S_n = \frac{10\,(10^n-1)}{(10-1)} + \frac{n}{2}(1+2n-1) = \frac{10}{9}(10^n-1) + n^2$$

EXAMPLE 20 *Find the least value of n for which the sum $1 + 3 + 3^2 + \ldots$ to n terms is greater than 7000.*

SOLUTION We have,

$$S_n = 1 + 3 + 3^2 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = 1 \times \left(\frac{3^n - 1}{3 - 1}\right) = \frac{3^n - 1}{2}$$

Now, $S_n > 7000$

$$\Rightarrow \quad \frac{3^n - 1}{2} > 7000$$

$$\Rightarrow \quad 3^n - 1 > 14000$$

$$\Rightarrow \quad 3^n > 14001 \Rightarrow n \log 3 > \log 14001 \Rightarrow n > \frac{\log 14001}{\log 3} \Rightarrow n > \frac{4.1461}{0.4771} = 8.69$$

Hence, the least value of n is 9.

EXAMPLE 21 *If f is a function satisfying $f(x + y) = f(x) f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.* **[NCERT]**

SOLUTION We have,

$$f(x + y) = f(x) f(y) \text{ for all } x, y \in N$$

$$\therefore \quad f(x) = f(\underbrace{1 + 1 + 1 + \ldots + 1}_{x\text{-times}}) = \underbrace{f(1) f(1) f(1) \ldots f(1)}_{x\text{-times}} = [f(1)]^x \text{ for all } x \in N$$

$$\Rightarrow \quad f(x) = 3^x \text{ for all } x \in N \qquad [\because f(1) = 3]$$

Now,

$$\sum_{x=1}^{n} f(x) = 120$$

$$\Rightarrow \quad \sum_{x=1}^{n} 3^x = 120$$

$$\Rightarrow \quad 3 + 3^2 + \ldots + 3^n = 120$$

$$\Rightarrow \quad 3\left(\frac{3^n - 1}{3 - 1}\right) = 120 \Rightarrow 3^n - 1 = 80 \Rightarrow 3^n = 81 \Rightarrow 3^n = 3^4 \Rightarrow n = 4.$$

EXAMPLE 22 *Find the natural number a for which $\sum_{k=1}^{n} f(a + k) = 16(2^n - 1)$, where the function f statisfies $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$.* **[NCERT]**

SOLUTION Proceding as in Example 20, we obtain $f(x) = (f(1))^x = 2^x$ for all $x \in N$.

$$\therefore \quad \sum_{k=1}^{n} f(a + k) = 16(2^n - 1)$$

$$\Rightarrow \quad \sum_{k=1}^{n} 2^{a+k} = 16(2^n - 1)$$

$$\Rightarrow \quad 2^a\left(\sum_{k=1}^{n} 2^k\right) = 16(2^n - 1)$$

$$\Rightarrow \quad 2^a(2 + 2^2 + 2^3 + \ldots + 2^n) = 16(2^n - 1)$$

$$\Rightarrow \quad 2^a\left\{2\left(\frac{2^n-1}{2-1}\right)\right\}=16\,(2^n-1)$$

$$\Rightarrow \quad 2^{a+1}(2^n-1)=16\,(2^n-1)$$

$$\Rightarrow \quad 2^{a+1}=2^4$$

$$\Rightarrow \quad a+1=4 \Rightarrow a=3.$$

EXERCISE 20.3

LEVEL-1

1. Find the sum of the following geometric progressions:
   (i) 2, 6, 18, ... to 7 terms
   (ii) 1, 3, 9, 27, ... to 8 terms
   (iii) $1, -1/2, 1/4, -1/8, \ldots$
   (iv) $(a^2-b^2), (a-b), \left(\frac{a-b}{a+b}\right)$, ... to n terms
   (v) 4, 2, 1, 1/2 ... to 10 terms.
2. Find the sum of the following geometric series:
   (i) 0.15 + 0.015 + 0.0015 + ... to 8 terms;
   (ii) $\sqrt{2}+\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{2}}+\ldots$ to 8 terms;
   (iii) $\frac{2}{9}-\frac{1}{3}+\frac{1}{2}-\frac{3}{4}+\ldots$ to 5 terms;
   (iv) $(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\ldots$ to n terms;
   (v) $\frac{3}{5}+\frac{4}{5^2}+\frac{3}{5^3}+\frac{4}{5^4}+\ldots$ to $2n$ terms;
   (vi) $\frac{a}{1+i}+\frac{a}{(1+i)^2}+\frac{a}{(1+i)^3}+\ldots+\frac{a}{(1+i)^n}$.
   (vii) $1, -a, a^2, -a^3, \ldots$ to n terms $(a \neq 1)$ [NCERT]
   (viii) $x^3, x^5, x^7, \ldots$ to n terms [NCERT]
   (ix) $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$ to n terms [NCERT]
3. Evaluate the following:
   (i) $\sum_{n=1}^{11}(2+3^n)$ [NCERT]
   (ii) $\sum_{k=1}^{n}(2^k+3^{k-1})$
   (iii) $\sum_{n=2}^{10}4^n$
4. Find the sum of the following series:
   (i) 5 + 55 + 555 + ... to n terms. [NCERT]
   (ii) 7 + 77 + 777 + ... to n terms. [NCERT]
   (iii) 9 + 99 + 999 + ... to n terms.
   (iv) 0.5 + 0.55 + 0.555 + ... to n terms.
   (v) 0.6 + 0.66 + 0.666 + ... to n terms. [NCERT]
5. How many terms of the G.P. 3, 3/2, 3/4, ... be taken together to make $\frac{3069}{512}$?
6. How many terms of the series 2 + 6 + 18 + ... must be taken to make the sum equal to 728?
7. How many terms of the sequence $\sqrt{3}$, 3, $3\sqrt{3}$, ... must be taken to make the sum $39+13\sqrt{3}$?
8. The sum of n terms of the G.P. 3, 6, 12, ... is 381. Find the value of n.
9. The common ratio of a G.P. is 3 and the last term is 486. If the sum of these terms be 728, find the first term.

10. The ratio of the sum of first three terms is to that of first 6 terms of a G.P. is 125 : 152. Find the common ratio.

11. The 4th and 7th terms of a G.P. are $\frac{1}{27}$ and $\frac{1}{729}$ respectively. Find the sum of n terms of the G.P.

12. Find the sum : $\sum_{n=1}^{10} \left\{ \left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{5}\right)^{n+1} \right\}$.

13. The fifth term of a G.P. is 81 whereas its second term is 24. Find the series and sum of its first eight terms.

14. If S_1, S_2, S_3 be respectively the sums of n, $2n$, $3n$ terms of a G.P., then prove that $S_1^2 + S_2^2 = S_1(S_2 + S_3)$.

15. Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$. **[NCERT]**

16. If a and b are the roots of $x^2 - 3x + p = 0$ and c, d are the roots $x^2 - 12x + q = 0$, where a, b, c, d form a G.P. Prove that $(q + p) : (q - p) = 17 : 15$. **[NCERT]**

17. How many terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}$ ..... are needed to give the sum $\frac{3069}{512}$? **[NCERT]**

18. A person has 2 parents, 4 grandparents, 8 great grand parents, and so on. Find the number of his ancestors during the ten generations preceding his own. **[NCERT]**

LEVEL-2

19. If $S_1, S_2, \ldots, S_n$ are the sums of n terms of n G.P.'s whose first term is 1 in each and common ratios are 1, 2, 3, ... , n respectively, then prove that

$$S_1 + S_2 + 2S_3 + 3S_4 + \ldots (n-1) S_n = 1^n + 2^n + 3^n + \ldots + n^n.$$

20. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of the terms occupying the odd places. Find the common ratio of the G.P. **[NCERT]**

21. Let a_n be the nth term of the G.P. of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$. Prove that the common ratio of the G.P. is α/β.

22. Find the sum of $2n$ terms of the series whose every even term is 'a' times the term before it and every odd term is 'c' times the term before it, the first term being unity.

ANSWERS

1. (i) 2186 (ii) 3280 (iii) $\frac{171}{256}$ (iv) $\frac{a-b}{(a+b)^{n-2}} \left\{ \frac{(a+b)^n - 1}{(a+b) - 1} \right\}$ (v) $8\left(1 - \frac{1}{1024}\right)$

2. (i) $\frac{1}{6}\left(1 - \frac{1}{10^8}\right)$ (ii) $\frac{255\sqrt{2}}{128}$ (iii) $\frac{55}{72}$ (iv) $\frac{1}{x-y}\left\{ x^2\left(\frac{x^n - 1}{x-1}\right) - y^2\left(\frac{y^n - 1}{y-1}\right) \right\}$

(v) $\frac{19}{24}\left(1 - \frac{1}{5^{2n}}\right)$ (vi) $-ai\,\{1 - (1+i)^{-n}\}$ (vii) $\frac{1-(-a)^n}{1+a}$ (viii) $x^3 \frac{(x^{2n} - 1)}{x^2 - 1}$

(xi) $\sqrt{7}\left(\dfrac{3^{n/2}-1}{\sqrt{3}-1}\right)$ 3. (i) 265741 (ii) $\dfrac{1}{2}(2^{n+2}+3^n-5)$ (iii) $\dfrac{16}{3}(4^9-1)$

4. (i) $\dfrac{5}{81}[10^{n+1}-9n-10]$ (ii) $\dfrac{7}{81}[10^{n+1}-9n-10]$ (iii) $\dfrac{1}{9}[10^{n+1}-9n-10]$

(iv) $\dfrac{5}{9}\left\{n-\dfrac{1}{9}\left(1-\dfrac{1}{10^n}\right)\right\}$ (v) $\dfrac{6}{9}\left\{n-\dfrac{1}{9}\left(1-\dfrac{1}{10^n}\right)\right\}$ 5. 10 6. 6

7. 6 8. 7 9. 2 10. $\dfrac{3}{5}$ 11. $\dfrac{3}{2}\left(1-\dfrac{1}{3^n}\right)$

12. $\dfrac{2^{10}-1}{2^9}+\dfrac{5^{10}-1}{4\times 5^{11}}$ 13. $a=16,\ r=\dfrac{3}{2},\ S_8=\dfrac{65\times 97}{8}$ 17. 10

18. 2046 20. 4 22. $(a+1)\left\{\dfrac{(ac)^n-1}{ac-1}\right\}$

HINTS TO NCERT & SELECTED PROBLEMS

2. (vii) Let S_n denote the sum of n terms of the G.P. $1-a, a^2, -a^3$. ... Then,

$$S_n = 1\left\{\frac{(-a)^n-1}{-a-1}\right\}=\frac{1-(-1)^n a^n}{1+a}$$

(viii) Let S_n be the sum of n terms of the G.P. x^3, x^5, x^7, ... Then,

$$S_n = x^3\left\{\frac{(x^2)^n-1}{x^2-1}\right\}=x^3\left(\frac{x^{2n}-1}{x^2-1}\right).$$

(ix) Let S_n denote the sum of n terms of the G.P. $\sqrt{7}, \sqrt{21}, 3\sqrt{7}$, ... Then,

$$S_n=\sqrt{7}\left\{\frac{(\sqrt{3})^n-1}{\sqrt{3}-1}\right\}=\sqrt{7}\left(\frac{3^{n/2}-1}{3^{1/2}-1}\right)$$

3. (i) $\sum\limits_{n=1}^{11}(2+3^n)=\sum\limits_{n=1}^{11}2+\sum\limits_{n=1}^{11}3^n=2\times 11+3\left(\dfrac{3^{11}-1}{3-1}\right)=22+\dfrac{3}{2}(3^{11}-1)=265741.$

4. (i) Let $S_n = 5+55+555+\dots$ to n terms. Then,

$S_n = 5(1+11+111+\dots$ to n terms)

$\Rightarrow\ S_n = \dfrac{5}{9}(9+99+999+\dots$ to n terms)

$\Rightarrow\ S_n = \dfrac{5}{9}\left\{(10-1)+(10^2-1)+(10^3-1)+\dots+(10^n-1)\right\}$

$\Rightarrow\ S_n = \dfrac{5}{9}\left\{(10+10^2+10^3+\dots+10^n)-n\right\}$

$\Rightarrow\ S_n = \dfrac{5}{9}\left\{10\left(\dfrac{10^n-1}{10-1}\right)-n\right\}$

$\Rightarrow\ S_n = \dfrac{5}{9}\left\{\dfrac{10}{9}(10^n-1)-n\right\} = \dfrac{5}{81}(10^{n+1}-9n-10).$

(ii) Proceed as in (i)

(v) Let $S_n = 0.6 + 0.66 + 0.666 + \ldots$ to n terms. Then,

$$S_n = 0.6 + 0.66 + 0.666 + \ldots \text{ to } n \text{ terms}$$

$$\Rightarrow \quad S_n = \frac{6}{9}(0.1 + 0.11 + 0.111 + \ldots \text{ to } n \text{ terms})$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{0.9 + 0.99 + 0.999 + \ldots \text{ to } n \text{ terms}\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots+\left(1-\frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{n-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^n}\right)\right\}$$

$$\Rightarrow \quad S_n = \frac{6}{9}\left\{n-\frac{1}{10}\frac{\left(1-\left(\frac{1}{10}\right)^n\right)}{1-\frac{1}{10}}\right\} = \frac{6}{9}\left\{n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\right\}$$

15. Required ratio $= \dfrac{a_1 + a_2 + \ldots + a_n}{a_{n+1} + a_{n+2} + \ldots + a_{2n}} = \dfrac{a + ar + \ldots + ar^{n-1}}{ar^n + ar^{n+1} + \ldots + ar^{2n-1}}$

$$= \frac{a\left(\frac{1-r^n}{1-r}\right)}{ar^n\left(\frac{1-r^n}{1-r}\right)} = \frac{1}{r^n}$$

16. We have, $a + b = 3$, $ab = p$, $c + d = 12$ and $cd = q$ Let $b = ar$, $c = ar^2$ and $d = ar^3$. Then,

$$a + b = 3 \text{ and } c + d = 12$$

$$\Rightarrow \quad a(1+r) = 3 \text{ and } ar^2(1+r) = 12$$

$$\Rightarrow \quad \frac{ar^2(1+r)}{a(1+r)} = \frac{12}{3} \Rightarrow r = 2$$

$$\therefore \quad a(1+r) = 3 \Rightarrow a = 1$$

Now, $p = ab = a.ar = 2$, $q = cd = ar^2 \times ar^3 = 2^5 = 32$

$$\therefore \quad \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$

17. Let the sum of n terms of the G.P. $3, \frac{3}{2}, \frac{3}{4}, \ldots$ be $\frac{3069}{512}$. Then,

$$3\left\{\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right\} = \frac{3069}{512} \Rightarrow 1 - \frac{1}{2^n} = \frac{1023}{1024} \Rightarrow \frac{1}{2^n} = \frac{1}{2^{10}} \Rightarrow n = 10$$

Hence, the sum of 10 terms of the given G.P. is $\frac{3069}{512}$.

18. Number of ancestors during the ten generations preceding his own generation

$= $ Sum of 10 terms of the G.P. 2, 4, 8, ...

$$= 2\left(\frac{2^{10}-1}{2-1}\right) = 2046.$$

20. Let there be $2n$ terms in the G.P. with first term a and common ratio r. Then,

Sum of all the terms $=5$ (Sum of the terms occupying the odd places)

$$\Rightarrow \quad a_1 + a_2 + \dots + a_{2n} = 5\,(a_1 + a_3 + a_5 + \dots + a_{2n-1})$$

$$\Rightarrow \quad a + ar + \dots + ar^{2n-1} = 5\,(a + ar^2 + \dots + ar^{2n-2})$$

$$\Rightarrow \quad a\left\{\frac{1-r^{2n}}{1-r}\right\} = 5a\left\{\frac{1-(r^2)^n}{1-r^2}\right\} \Rightarrow 1 + r = 5 \Rightarrow r = 4$$

21. Let a be the first term and r be the common ratio of the G.P. Then,

$$\sum_{n=1}^{100} a_{2n} = \alpha \text{ and } \sum_{n=1}^{100} a_{2n-1} = \beta$$

$$\Rightarrow \quad a_2 + a_4 + \dots + a_{200} = \alpha \text{ and } a_1 + a_3 + \dots + a_{199} = \beta$$

$$\Rightarrow \quad ar + ar^3 + \dots + ar^{199} = \alpha \text{ and } a + ar^2 + \dots + ar^{198} = \beta$$

$$\Rightarrow \quad ar\left\{\frac{1-(r^2)^{100}}{1-r^2}\right\} = \alpha, \text{ and } a\left\{\frac{1-(r^2)^{100}}{1-r^2}\right\} = \beta$$

$$\Rightarrow \quad ar\left(\frac{1-r^{200}}{1-r^2}\right) = \alpha, \text{ and } a\left(\frac{1-r^{200}}{1-r^2}\right) = \beta$$

$$\Rightarrow \quad r = \frac{\alpha}{\beta}$$

22. Let $a_1 + a_2 + a_3, \dots + a_{2n}$ be the given series. It is given that

$$a_1 = 1,\ a_2 = a\,a_1,\ a_3 = ca_2,\ a_4 = aa_3,\ a_5 = ca_4 \text{ and so on.}$$

$$\Rightarrow \quad a_1 = 1,\ a_2 = a,\ a_3 = ac,\ a_4 = a^2 c,\ a_5 = a^2 c^2,\ a_6 = a^3 c^2, \dots$$

$\therefore$ Required sum $= a_1 + a_2 + a_3 + \dots + a_{2n}$

$$= 1 + a + ac + a^2 c + a^2 c^2 + \dots \text{ to } 2n \text{ term}$$

$$= (1 + a) + ac\,(1 + a) + a^2 c^2 (1 + a) + \dots \text{ to } n \text{ terms}$$

$$= (1 + a)\left\{\frac{1-(ac)^n}{1-ac}\right\} = (1 + a)\left\{\frac{(ac)^n - 1}{ac - 1}\right\}$$

20.5 SUM OF AN INFINITE G.P.

THEOREM *The sum of an infinite G.P. with first term a and common ratio* r $(-1 < r < 1$ i.e., $|r| < 1)$ is $S = \dfrac{a}{1-r}$.

<u>PROOF</u> Consider an infinite G.P. with first term a and common ratio r, where $-1 < r < 1$ i.e. $|r| < 1$. The sum of n terms of this G.P. is given by

$$S_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \text{...(i)}$$

Since $-1 < r < 1$, therefore r^n decreases as n increases and tends to zero as n tends to infinity

i.e. $\quad r^n \to 0$ as $n \to \infty$.

$$\therefore \quad \frac{a\,r^n}{1-r} \to 0 \text{ as } n \to \infty.$$

Hence, from (i), the sum of an infinite G.P. is given by

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(\frac{a}{1-r} - \frac{ar^n}{1-r} \right) = \frac{a}{1-r}, \text{ if } |r| < 1$$

NOTE *If $r \geq 1$, then the sum of an infinite G.P. tends to infinity.* **Q.E.D.**

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **FINDING THE SUM TO INFINITY OF A G.P. OR A GEOMETRIC SERIES**

EXAMPLE 1 *Find the sum to infinity of the G.P.* $-\frac{5}{4}, \frac{5}{16}, -\frac{5}{64}, \ldots$

SOLUTION The given G.P. has first term $a = -5/4$ and the common ratio $r = -1/4$. Also, $|r| < 1$. Hence, the sum S to infinity is given by

$$S = \frac{a}{1-r} = \frac{-5/4}{1-(-1/4)} = -1$$

EXAMPLE 2 *Sum the following geometric series to infinity:*

(i) $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \ldots \infty$ (ii) $\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \infty$

SOLUTION (i) The given series is a geometric series with first term $a = \sqrt{2}+1$ and the common ratio r given by

$$r = \frac{1}{\sqrt{2}+1} = \frac{\sqrt{2}-1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2}-1$$

Hence, the sum S to infinity is given by

$$S = \frac{a}{1-r} = \frac{\sqrt{2}+1}{1-(\sqrt{2}-1)} = \frac{\sqrt{2}+1}{2-\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}(\sqrt{2}-1)}$$

$$\Rightarrow \quad S = \frac{(\sqrt{2}+1)^2}{\sqrt{2}(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{3+2\sqrt{2}}{\sqrt{2}} = \frac{4+3\sqrt{2}}{2}$$

(ii) We have,

$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \text{to } \infty$$

$$= \left(\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \ldots \right) + \left(\frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \ldots \right)$$

$$= \left(\text{An infinite G.P. with } a = \frac{1}{2}, r = \frac{1}{2^2} \right) + \left(\text{An infinite G.P. with } a = \frac{1}{3^2}, r = \frac{1}{3^2} \right)$$

$$= \left\{ \frac{(1/2)}{1-(1/2^2)} \right\} + \left\{ \frac{(1/3^2)}{1-(1/3^2)} \right\} = \frac{2}{3} + \frac{1}{8} = \frac{19}{24}$$

EXAMPLE 3 *Prove that:* $6^{1/2} \times 6^{1/4} \times 6^{1/8} \ldots\ldots \infty = 6.$

SOLUTION Clearly,

$$6^{1/2} \times 6^{1/4} \times 6^{1/8} \dots \infty = 6^{\{1/2 + 1/4 + 1/8 + \dots \infty\}}$$

$$= 6^{\{(1/2)/(1-1/2)\}} \quad \left[\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \text{ to } \infty = \frac{1/2}{1-1/2} = 1\right]$$

$$= 6^1 = 6$$

Type II **ON PROVING RESULTS BASED UPON THE FORMULA FOR THE SUM TO INFINITY OF A G.P.**

EXAMPLE 4 *If* $b = a + a^2 + a^3 + \dots \infty$, *prove that* $a = \dfrac{b}{1+b}$.

SOLUTION We have,

$$b = a + a^2 + a^3 + \dots \infty$$

Clearly, RHS is a geometric series with first term 'a' and common ratio 'a'

$$\therefore \quad b = \frac{a}{1-a} \Rightarrow b - ab = a \Rightarrow a = \frac{b}{1+b}$$

EXAMPLE 5 *If* $x = a + \dfrac{a}{r} + \dfrac{a}{r^2} + \dots \infty$, $y = b - \dfrac{b}{r} + \dfrac{b}{r^2} - \dots \infty$ *and,* $z = c + \dfrac{c}{r^2} + \dfrac{c}{r^4} + \dots \infty$, *prove that* $\dfrac{xy}{z} = \dfrac{ab}{c}$.

SOLUTION Clearly, x, y and z are the sums of infinite geometric progressions.

$$\therefore \quad x = \frac{a}{1-\frac{1}{r}} = \frac{ar}{r-1}, \quad y = \frac{b}{1-\left(-\frac{1}{r}\right)} = \frac{br}{1+r} \quad \text{and,} \quad z = \frac{c}{1-\frac{1}{r^2}} = \frac{cr^2}{r^2-1}$$

$$\Rightarrow \quad xy = \left(\frac{ar}{r-1}\right)\left(\frac{br}{r+1}\right) = \frac{abr^2}{r^2-1}$$

$$\Rightarrow \quad \frac{xy}{z} = \left\{\left(\frac{abr^2}{r^2-1}\right) \div \frac{cr^2}{(r^2-1)}\right\} = \frac{ab}{c}$$

EXAMPLE 6 *If* $x = 1 + a + a^2 + \dots \infty$, *where* $|a| < 1$ *and* $y = 1 + b + b^2 + \dots \infty$, *where* $|b| < 1$. *Prove that:*

$$1 + ab + a^2b^2 + \dots \infty = \frac{xy}{x+y-1}$$

SOLUTION We have,

$$x = 1 + a + a^2 + \dots \infty$$

$$\therefore \quad x = \frac{1}{1-a} \Rightarrow 1 - a = \frac{1}{x} \Rightarrow a = 1 - \frac{1}{x} \qquad \dots(i)$$

$$\text{and,} \quad y = 1 + b + b^2 + b^3 + \dots\dots \infty$$

$$\Rightarrow \quad y = \frac{1}{1-b} \Rightarrow 1 - b = \frac{1}{y} \Rightarrow b = 1 - \frac{1}{y} \qquad \dots(ii)$$

$$\therefore \quad 1 + ab + (ab)^2 + (ab)^3 + \dots \infty$$

$$= \frac{1}{1-ab} = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \qquad \text{[Using (i) and (ii)]}$$

$$= \frac{xy}{x+y-1}$$

EXAMPLE 7 *If $A = 1 + r^a + r^{2a} + \ldots$ to ∞ and $B = 1 + r^b + r^{2b} + \ldots\ \infty$, prove that*

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

SOLUTION We have,

$$A = 1 + r^a + r^{2a} + \ldots\ \infty \quad \text{and,} \quad B = 1 + r^b + r^{2b} + \ldots \infty$$

$$\Rightarrow \quad A = \frac{1}{1-r^a} \text{ and, } B = \frac{1}{1-r^b}$$

$$\Rightarrow \quad 1 - r^a = \frac{1}{A} \text{ and, } 1 - r^b = \frac{1}{B}$$

$$\Rightarrow \quad r^a = 1 - \frac{1}{A} \text{ and, } r^b = 1 - \frac{1}{B}$$

$$\Rightarrow \quad l = \left(\frac{A-1}{A}\right)^{1/a} \text{ and, } r = \left(\frac{B-1}{B}\right)^{1/b}$$

Hence, $\quad r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$

Type III **FINDING REQUIRED UNKNOWN WHEN THE SUM OF AN INFINITE G.P. IS GIVEN**

EXAMPLE 8 *The first term of a G.P. is 2 and the sum to infinity is 6. Find the common ratio.*

SOLUTION Let r be the common ratio of the given G.P. It is given that, $a = 2$ and $S_\infty = 6$.

Now, $\quad S_\infty = 6 \Rightarrow \frac{a}{1-r} = 6 \Rightarrow \frac{2}{1-r} = 6 \Rightarrow 6 - 6r = 2 \Rightarrow r = 2/3.$

EXAMPLE 9 *The sum of an infinite G.P. is 8, its second term is 2, find the first term.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$S_\infty = 8 \quad \text{and} \quad ar = 2$$

$$\Rightarrow \quad \frac{a}{1-r} = 8 \quad \text{and} \quad r = \frac{2}{a}$$

$$\Rightarrow \quad \frac{a}{1-(2/a)} = 8 \qquad \text{[Eliminating } r\text{]}$$

$$\Rightarrow \quad a^2 - 8a + 16 = 0 \Rightarrow (a-4)^2 = 0 \Rightarrow a = 4.$$

EXAMPLE 10 *The sum of an infinite G.P. is 57 and the sum of their cubes is 9747, find the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. Then,

$$\text{Sum} = 57 \Rightarrow \frac{a}{1-r} = 57 \qquad \text{...(i)}$$

Sum of the cubes $= 9747$

$$\Rightarrow \quad a^3 + a^3r^3 + a^3r^6 + \ldots = 9747 \Rightarrow \frac{a^3}{1-r^3} = 9747 \qquad \text{...(ii)}$$

Dividing the cube of (i) by (ii), we get

$$\frac{a^3}{(1-r)^3} \times \frac{(1-r^3)}{a^3} = \frac{(57)^3}{9747}$$

$$\Rightarrow \quad \frac{1-r^3}{(1-r)^3} = 19$$

$$\Rightarrow \quad \frac{1+r+r^2}{(1-r)^2} = 19$$

$$\Rightarrow \quad 18r^2 - 39r + 18 = 0$$

$$\Rightarrow \quad (3r-2)(6r-9) = 0$$

$$\Rightarrow \quad r = 2/3 \text{ or, } r = 3/2$$

$$\Rightarrow \quad r = 2/3 \qquad [\because r \neq 3/2, \text{ because } -1 < r < 1 \text{ for an infinite G.P.}]$$

Putting $r = 2/3$ in (i), we get

$$\frac{a}{1-(2/3)} = 57 \Rightarrow a = 19$$

Hence, the G.P. is 19, 38/3, 76/9, ...

Type IV **FINDING A RATIONAL NUMBER WHOSE DECIMAL EXPANSION IS GIVEN**

EXAMPLE 11 *Which is the rational number having the decimal expansion* $0.3\overline{56}$ *?*

SOLUTION We have,

$$0.3\overline{56} = 0.3 + 0.056 + 0.00056 + 0.0000056 + \ldots \infty$$

$$= 0.3 + \left\{\frac{56}{10^3} + \frac{56}{10^5} + \frac{56}{10^7} + \ldots \infty\right\}$$

$$= \frac{3}{10} + \frac{\frac{56}{10^3}}{1-\frac{1}{10^2}} = \frac{3}{10} + \frac{56}{990} = \frac{353}{990}$$

EXAMPLE 12 *Use geometric series to express* $0.555\ldots = 0.\overline{5}$ *as a rational number.*

SOLUTION We have,

$$0.\overline{5} = 0.5555\ldots$$

$$= 0.5 + 0.05 + 0.005 + \ldots \infty$$

$$= \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \ldots \infty$$

$$= \frac{(5/10)}{1-(1/10)} = \frac{5}{9}$$

Type V **ON APPLICATIONS OF INFINITE G.P.**

EXAMPLE 13 *A square is drawn by joining the mid-points of the sides of a square. A third square is drawn inside the second square in the same way and the process is continued indefinitely. If the side of the square is 10 cm, find the sum of the areas of all the squares so formed.*

SOLUTION Let $A_1A_2A_3A_4$ be the first square with each side equal to 10 cm. Let B_1, B_2, B_3, B_4 be the mid-points of its sides. Then,

$$B_1B_2 = \sqrt{A_2B_1^2 + A_2B_2^2} = \sqrt{5^2+5^2} = 5\sqrt{2} \text{ cm.}$$

Let C_1, C_2, C_3, C_4 be the mid-points of the sides of the square $B_1B_2B_3B_4$. Then,

$$C_1C_2 = \sqrt{B_1C_2^2 + B_1C_1^2} = \sqrt{\left(\frac{5\sqrt{2}}{2}\right)^2 + \left(\frac{5\sqrt{2}}{2}\right)^2} = 5 \text{ cm}$$

Similarly, the side of fourth square is $\frac{5}{\sqrt{2}}$ cm and so on.

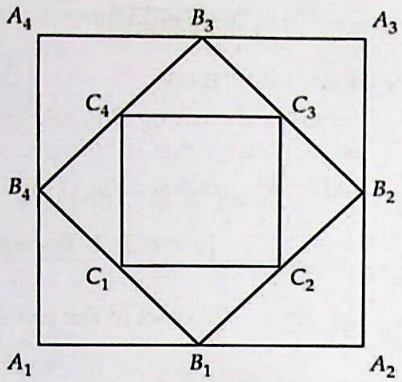

Fig. 20.1

$\therefore$ Sum of the areas of all the squares so formed

$$= \left\{10^2 + (5\sqrt{2})^2 + (5)^2 + \left(\frac{5}{\sqrt{2}}\right)^2 + \dots \infty\right\} \text{ sq. cm.} \qquad [\because \text{ Area} = (\text{Side})^2]$$

$$= \left\{100 + 50 + 25 + \frac{25}{2} + \dots \infty\right\} = \frac{100}{1-(1/2)} = 200 \text{ sq. cm.}$$

EXAMPLE 14 *After striking a floor a certain ball rebounds* $\left(\frac{4}{5}\right)^{th}$ *of the height from which it has fallen. Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 metres.*

SOLUTION Initially the ball falls from a height of 120 metres. After striking the floor it rebounds and goes to a height of $\frac{4}{5}(120)$ metres. Now, it falls from a height of $\frac{4}{5}(120)$ metres and after rebounding again it goes to a height of $\frac{4}{5}\left(\frac{4}{5}(120)\right)$ metres. This process is continued till the ball comes to rest.

$\therefore$ The total distance traveled $= 120 + 2\left\{\frac{4}{5}(120) + \left(\frac{4}{5}\right)^2(120) + \dots \infty\right\}$

$$= 120 + 2 \times \left\{\frac{\frac{4}{5}(120)}{1-\frac{4}{5}}\right\} = 120 + 960 = 1080 \text{ metres.}$$

EXAMPLE 15 *The inventor of the chess board suggested a reward of one grain of wheat for the first square, 2 grains for the second, 4 grains for the third and so on, doubling the number of the grains for subsequent squares. How many grains would have to be given to inventor ? (There are 64 squares in the chess board).*

SOLUTION Clearly, required number of grains is the sum of an infinite G.P. with first term 1 and common ratio 2.

$\therefore$ Number of grains $= 1 + 2 + 2^2 + 2^3 + \dots$ to 64 terms $= 1\left(\frac{2^{64}-1}{2-1}\right) = 2^{64} - 1.$

LEVEL-2

Type VI ON FINDING THE SUM OF AN INFINITE G.P.

EXAMPLE 16 *Find the sum of an infinitely decreasing G.P. whose first term is equal to* $b+2$ *and the common ratio to* $2/c$, *where* b *is the least value of the product of the roots of the equation* $(m^2+1)x^2-3x+(m^2+1)^2=0$, *and* c *is the greatest value of the sum of its roots.*

SOLUTION We have,

$$(m^2+1)x^2-3x+(m^2+1)^2=0$$

$\therefore$ Sum of the roots $=\dfrac{3}{m^2+1}$ and, Product of the roots $=(m^2+1)$

Now, $b=$ Least value of the product of roots

$b=$ Least value of (m^2+1)

$\Rightarrow$ $b=1$ $[\because m^2+1>1$ for all $m]$

$c=$ Greatest value of the sum of the roots

$\Rightarrow$ $c=$ Greatest value of $\dfrac{3}{m^2+1}$

Clearly, $\dfrac{3}{m^2+1}$ is greatest when m^2+1 is least and the least value of m^2+1 is 1.

$\therefore$ $c=\dfrac{3}{1}=3$

So, first term of the infinite G.P. is $b+2=1+2=3$ and, the common ratio is $\dfrac{2}{c}=\dfrac{2}{3}$.

Hence, the sum S of the infinite G.P. is given by

$$S=\frac{3}{1-\frac{2}{3}}=9 \qquad \left[\text{Using : } S=\frac{a}{1-r}\right]$$

Type VII ON PROVING RESULTS BASED UPON SUM OF AN INFINITE G.P.

EXAMPLE 17 *If* $x=\sum_{n=0}^{\infty}\cos^{2n}\theta$, $y=\sum_{n=0}^{\infty}\sin^{2n}\phi$, $z=\sum_{n=0}^{\infty}\cos^{2n}\theta\sin^{2n}\phi$, *where* $0<\theta,\phi<\pi/2$ *then prove that* $xz+yz-z=xy$.

SOLUTION We have,

$$x=\sum_{n=0}^{\infty}\cos^{2n}\theta=1+\cos^2\theta+\cos^4\theta+\ldots\infty$$

$$\Rightarrow\quad x=\frac{1}{1-\cos^2\theta}\Rightarrow\sin^2\theta=\frac{1}{x}$$

$$\Rightarrow\quad y=\sum_{n=0}^{\infty}\sin^{2n}\phi=1+\sin^2\phi+\sin^4\phi+\ldots\infty$$

$$\Rightarrow\quad y=\frac{1}{1-\sin^2\phi}\Rightarrow\cos^2\phi=\frac{1}{y}$$

and, $$z=\sum_{n=0}^{\infty}\cos^{2n}\theta,\sin^{2n}\phi=1+\cos^2\theta\sin^2\phi+\cos^4\theta\sin^4\phi+\ldots\infty$$

$$\Rightarrow \quad z = \frac{1}{1-\cos^2\theta\sin^2\phi}$$

$$\Rightarrow \quad z = \frac{1}{1-(1-\sin^2\theta)(1-\cos^2\phi)}$$

$$\Rightarrow \quad z = \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)} \Rightarrow z = \frac{1}{\frac{1}{x}+\frac{1}{y}-\frac{1}{xy}} \Rightarrow z = \frac{xy}{x+y-1} \Rightarrow xz+yz-z = xy$$

EXAMPLE 18 *If* $|x|<1$ and $|y|<1$, *find the sum to infinity of the following series:*

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\dots$$

SOLUTION We have,

$$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+\dots\infty$$

$$= \frac{1}{x-y}\left\{(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\dots \text{ to } \infty\right\}$$

$$\left[\because \frac{x^n-a^n}{x-a} = x^{n-1}+x^{n-2}a^1+x^{n-3}a^2+\dots+a^{n-1}, n\in N\right]$$

$$= \frac{1}{x-y}\left\{(x^2+x^3+x^4+\dots \text{ to } \infty)-(y^2+y^3+y^4+\dots \text{ to } \infty)\right\}$$

$$= \frac{1}{x-y}\left\{\frac{x^2}{1-x}-\frac{y^2}{1-y}\right\}$$

$$= \frac{1}{x-y}\,\frac{\left\{x^2-x^2y-y^2+y^2x\right\}}{(1-x)(1-y)} = \frac{1}{(x-y)}\,\frac{\left\{(x^2-y^2)-xy(x-y)\right\}}{(1-x)(1-y)} = \frac{x+y-xy}{(1-x)(1-y)}$$

Type VIII ON FINDING REQUIRED UNKNOWN WHEN SUM OF AN INFINITE G.P. IS GIVEN

EXAMPLE 19 *The sum of an infinite geometric series is 15 and the sum of the squares of these terms is 45. Find the series.*

SOLUTION Let a be the first term and r be the common ratio of the infinite geometric series.

$$\text{Sum} = 15 \Rightarrow \frac{a}{1-r} = 15 \qquad \text{...(i)}$$

Sum of the squares = 45

$$\Rightarrow \quad (a^2+a^2r^2+a^2r^4+\dots\infty) = 45 \Rightarrow \frac{a^2}{1-r^2} = 45 \qquad \text{...(ii)}$$

Dividing the square of (i), by (ii), we get

$$\frac{a^2}{(1-r)^2}\times\frac{1-r^2}{a^2} = \frac{(15)^2}{45} \Rightarrow \frac{1+r}{1-r} = 5 \Rightarrow 6r = 4 \Rightarrow r = \frac{2}{3}$$

Putting $r=\frac{2}{3}$ in (i), we get

$$\frac{a}{1-2/3} = 15 \Rightarrow a = 5$$

Hence, the required series is $5+\frac{10}{3}+\frac{20}{9}+\frac{40}{27}+\dots\infty$.

EXAMPLE 20 *If each term of an infinite G.P. is twice the sum of the terms following it, then find the common ratio of the G.P.*

SOLUTION Let a be the first term and r the common ratio of the G.P. It is given that

$$a_n = 2\,[a_{n+1} + a_{n+2} + a_{n+3} + \ldots \infty] \text{ for all } n \in N$$

$$\Rightarrow \quad ar^{n-1} = 2\,[ar^n + ar^{n+1} + \ldots \infty]$$

$$\Rightarrow \quad ar^{n-1} = \frac{2\,ar^n}{1-r} \Rightarrow 1 = \frac{2r}{1-r} \Rightarrow r = \frac{1}{3}$$

EXERCISE 20.4

LEVEL-1

1. Find the sum of the following series to infinity:

 (i) $1 - \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3} + \frac{1}{3^4} + \ldots \infty$ (ii) $8 + 4\sqrt{2} + 4 + \ldots \infty$

 (iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + \ldots \infty$. (iv) $10 - 9 + 8.1 - 7.29 + \ldots \infty$

 (v) $\frac{1}{3} + \frac{1}{5^2} + \frac{1}{3^3} + \frac{1}{5^4} + \frac{1}{3^5} + \frac{1}{5^6} + \ldots \infty$ **[NCERT]**

2. Prove that: $(9^{1/3} . 9^{1/9} . 9^{1/27} \ldots \infty) = 3$.

3. Prove that: $(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \ldots \infty) = 2$.

4. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \ldots$ to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \ldots$ to ∞, prove that $S_p + s_p = 2\ S_{2p}$.

5. Find the sum of the terms of an infinite decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to 32/81.

6. Express the recurring decimal 0.125125125 ... as a rational number.

7. Find the rational number whose decimal expansion is $0.4\overline{23}$.

8. Find the rational numbers having the following decimal expansions:

 (i) $0.\overline{3}$ (ii) $0.\overline{231}$ (iii) $3.5\overline{2}$ (iv) $0.6\overline{8}$ **[NCERT]**

9. One side of an equilateral triangle is 18 cm. The mid-points of its sides are joined to form another triangle whose mid-points, in turn, are joined to form still another triangle. The process is continued indefinitely. Find the sum of the (i) perimeters of all the triangles. (ii) areas of all triangles.

LEVEL-2

10. Find an infinite G.P. whose first term is 1 and each term is the sum of all the terms which follow it.

11. The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of the succeeding terms. Find the G.P.

12. Show that in an infinite G.P. with common ratio r ($|r| < 1$), each term bears a constant ratio to the sum of all terms that follow it.

13. If S denotes the sum of an infinite G.P. and S_1 denotes the sum of the squares of its terms, then prove that the first term and common ratio are respectively $\frac{2SS_1}{S^2 + S_1}$ and $\frac{S^2 - S_1}{S^2 + S_1}$.

ANSWERS

1. (i) $\frac{3}{4}$ (ii) $8(2+\sqrt{2})$ (iii) $\frac{13}{24}$ (iv) 5.263 (v) $\frac{5}{12}$ 5. $6, \frac{12}{3-2\sqrt{2}}$
6. $\frac{125}{999}$ 7. $\frac{419}{990}$ 8. (i) $\frac{1}{3}$ (ii) $\frac{231}{999}$ (iii) $\frac{317}{90}$ (iv) $\frac{31}{45}$
9. (i) 108 cm (ii) $108\sqrt{3}$ square cm 10. $1, \frac{1}{2}, \frac{1}{4}, \dots$ 11. $4, 1, \frac{1}{4}, \frac{1}{16}, \dots$

HINTS TO NCERT & SELECTED PROBLEM

9. Sum of the perimeters $= 3\left\{18 + \frac{18}{2} + \frac{18}{4} + \dots \infty\right\}$

Sum of the areas $= \frac{\sqrt{3}}{4}\left\{18^2 + \left(\frac{18}{2}\right)^2 + \left(\frac{18}{4}\right)^2 + \dots \infty\right\}$

20.6 PROPERTIES OF GEOMETRIC PROGRESSIONS

In this section, we shall discuss some important properties of geometric progressions and geometric series.

PROPERTY I *If all the terms of a G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.*

PROOF Let $a_1, a_2, a_3, \dots, a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r, \text{ for all } n \in N \quad \text{...(i)}$$

Let k be a non-zero constant. Multiplying all the terms of the given G.P. by k, we obtain the new sequence: $ka_1, ka_2, ka_3, \dots ka_n, \dots$

Clearly, $\frac{k\,a_{n+1}}{k\,a_n} = \frac{a_{n+1}}{a_n} = r$ for all $n \in N$ [Using (i)]

Hence, the new sequence also forms a G.P. with common ratio r.

PROPERTY II *The reciprocals of the terms of a given G.P. form a G.P.*

PROOF Let $a_1, a_2, a_3, \dots, a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad \text{...(i)}$$

The sequence formed by the reciprocals of the terms of the given G.P. is

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots \frac{1}{a_n}, \dots$$

For this sequence the ratio of a term and the preceding term is given by

$$\frac{1/a_{n+1}}{1/a_n} = \frac{a_n}{a_{n+1}} = \frac{1}{r} \quad \text{[Using (i)]}$$

So, the new sequence is a G.P. with common ratio $1/r$.

PROPERTY III *If each term of a G.P. be raised to the same power, the resulting sequence also forms a G.P.*

PROOF Let $a_1, a_2, a_3, \dots a_n, \dots$ be a G.P. with common ratio r. Then,

$$\frac{a_{n+1}}{a_n} = r \text{ for all } n \in N \quad \text{...(i)}$$

Let k be a non-zero real number.

Consider the sequence whose terms are k^{th} powers of the terms of the given sequence

i.e. $a_1^k, a_2^k, a_3^k, \ldots, a_n^k, \ldots$

For this sequence, we have

$$\frac{a_{n+1}^k}{a_n^k} = \left(\frac{a_{n+1}}{a_n}\right)^k = r^k \text{ for all } n \in N \qquad \text{[Using (i)]}$$

Hence, $a_1^k, a_2^k, a_3^k, \ldots a_n^k, \ldots$ is a G.P. with common ratio r^k.

PROPERTY IV *In a finite G.P the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.*

PROOF Let $a_1, a_2, a_3, \ldots, a_n$ be a finite G.P. with common ratio r. Then,

kth term from the beginning $= a_k = a_1 r^{k-1}$

kth term from the end $= (n-k+1)$*th* term from the beginning $= a_{n-k+1} = a_1 r^{n-k}$

$\therefore$ (*kth* term from the beginning) (*kth* term from the end)

$$= a_k\, a_{n-k+1} = a_1 r^{k-1} a_1 r^{n-k} = a_1^2 r^{n-1} = a_1 . a_1 r^{n-1} = a_1 a_n \quad \text{for all } k = 2, 3, \ldots, n-1$$

Hence, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last term.

PROPERTY V *Three non-zero numbers a, b, c are in G.P. iff* $b^2 = ac$

PROOF Clearly,

$$a, b, c \text{ are in G.P.} \Leftrightarrow \frac{b}{a} = \frac{c}{b} = (\text{ common ratio}) \Leftrightarrow b^2 = ac$$

NOTE *When a, b, c are in G.P., then b is known as the Geometric mean of a and c.*

PROPERTY VI *If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.*

PROPERTY VII *If* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *is a G.P. of non-zero non-negative terms, then* $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ *is an A.P. and vice-versa.*

PROOF *Let* $a_1, a_2, a_3, \ldots, a_n, \ldots$ *be a G.P. of non-zero non-negative terms with common ratio* r. *Then,*

$$a_n = a_1 r^{n-1}, \text{ for all } n \in N$$

$$\Rightarrow \quad \log a_n = \log a_1 + (n-1) \log r, \quad \text{for all } n \in N$$

Let $b_n = \log a_n = \log a_1 + (n-1) \log r, \quad$ for all $n \in N$

Then, $b_{n+1} - b_n = [\log a_1 + n \log r] - [\log a_1 + (n-1) \log r] = \log r \quad$ for all $n \in N$

Clearly, $b_{n+1} - b_n = \log r =$ Constant for all $n \in N$.

Hence, $b_1, b_2, \ldots, b_n, \ldots$ i.e. $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ is an A.P. with common difference $\log r$.

Conversely, let $\log a_1, \log a_2, \ldots, \log a_n, \ldots$ be an A.P. with common difference d. Then,

$$\log a_{n+1} - \log a_n = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \log\left(\frac{a_{n+1}}{a_n}\right) = d \quad \text{for all } n \in N.$$

$$\Rightarrow \quad \frac{a_{n+1}}{a_n} = e^d \text{ (a constant) for all } n \in N.$$

$\Rightarrow \quad a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P. with common ratio e^d.

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED UPON FOLLOWING RESULTS:**

(i) a, b, c *are in G.P. iff* $b^2 = ac$ (ii) a, b, c *are in A.P. iff* $2b = a + c$.

EXAMPLE 1 *If p, q, r are in A.P., show that the pth, qth and rth terms of any G.P. are in G.P.*

SOLUTION Let A be the first term and R the common ratio of a G.P. Then,

$$a_p = AR^{p-1},\ a_q = AR^{q-1} \text{ and } a_r = AR^{r-1}$$

We have to prove that a_p, a_q, a_r are in G.P. For this it is sufficient to show that

$$(a_q)^2 = a_p \cdot a_r$$

Now, $(a_q)^2 = (AR^{q-1})^2$

$\Rightarrow \quad (a_q)^2 = A^2 R^{2q-2}$

$\Rightarrow \quad (a_q)^2 = A^2 R^{p+r-2}$ $\quad [\because\ p, q, r \text{ are in A.P.} \therefore 2q = p + r]$

$\Rightarrow \quad (a_q)^2 = (AR^{p-1})(AR^{r-1}) = a_p \cdot a_r$

Hence, a_p, a_q, a_r are in G.P.

EXAMPLE 2 *If a, b, c are in G.P., then prove that* $\log a^n, \log b^n, \log c^n$ *are in A.P.*

SOLUTION It is given that a, b, c are in G.P.

$\therefore \quad b^2 = ac$

$\Rightarrow \quad (b^2)^n = (ac)^n$

$\Rightarrow \quad b^{2n} = a^n c^n$

$\Rightarrow \quad \log b^{2n} = \log (a^n c^n)$

$\Rightarrow \quad \log (b^n)^2 = \log a^n + \log c^n$

$\Rightarrow \quad 2 \log b^n = \log a^n + \log c^n$

$\Rightarrow \quad \log a^n, \log b^n, \log c^n$ are in A.P

EXAMPLE 3 *Three numbers whose sum is 15 are in A.P. If 1, 4, 19 be added to them respectively, then they are in G.P. Find the numbers.*

SOLUTION Let the three numbers be $a - d, a, a + d$. Then,

$$\text{Sum} = 15 \Rightarrow (a-d) + a + (a+d) = 15 \Rightarrow a = 5.$$

So, the numbers are $5 - d, 5, 5 + d$. Adding 1, 4, 19 respectively to these numbers, we get $6 - d,\ 9,\ 24 + d$. These numbers are in G.P.

$\therefore \quad 9^2 = (6-d)(24+d) \Rightarrow d^2 + 18d - 63 = 0 \Rightarrow (d+21)(d-3) = 0 \Rightarrow d = -21$ or, $d = 3$.

Hence, the numbers are 26, 5, –16 or 2, 5, 8.

Type II **PROBLEMS BASED UPON PROPERTIES OF G.P.**

EXAMPLE 4 *If a, b, c, d are in G.P., show that:*

(i) $(b-c)^2 + (c-a)^2 + (d-b)^2 = (a-d)^2$

(ii) $(ab + bc + cd)^2 = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$ **[NCERT]**

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar$, $c = ar^2$ and $d = ar^3$.

(i) $\quad \text{LHS} = (b-c)^2 + (c-a)^2 + (d-b)^2$

$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= a^2r^2 (1 - r)^2 + a^2 (r^2 - 1^2) + a^2r^2 (r^2 - 1)^2$$
$$= a^2 (r^6 - 2r^3 + 1) = a^2 (1 - r^3)^2 = (a - ar^3)^2 = (a - d)^2 = \text{RHS.}$$

(ii) $\text{LHS} = (ab + bc + cd)^2 = (a \times ar + ar \times ar^2 + ar^2 \times ar^3)^2 = a^4r^2 (1 + r^2 + r^4)^2$

$\text{RHS} = (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$$= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$$
$$= a^2 (1 + r^2 + r^4)\, a^2r^2 (1 + r^2 + r^4) = a^4r^2 (1 + r^2 + r^4)^2$$

$\therefore$ LHS = RHS.

EXAMPLE 5 *If a, b, c, d are in G.P., prove that $a + b, b + c, c + d$ are also in G.P.*

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar,\ c = ar^2$ and $d = ar^3$

$\therefore$ $a + b = a + ar = a(1 + r)$, $b + c = ar + ar^2 = ar(1 + r)$ and $c + d = ar^2 + ar^3 = ar^2 (1 + r)$

Now, $(b + c)^2 = \left\{ar(1 + r)\right\}^2 = a^2r^2 (1 + r)^2 = \left\{a(1 + r)\right\}\left\{ar^2 (1 + r)\right\}$

$= (a + b)(c + d)$ $\qquad$ [$\because$ $a + b = a(1 + r)$, and $c + d = ar^2 (1 + r)$]

Hence, $a + b, b + c, c + d$ are in G.P.

EXAMPLE 6 *If a, b, c, d are in G.P., prove that $a^n + b^n, b^n + c^n, c^n + d^n$ are also in G.P.* **[NCERT]**

SOLUTION Let r be the common ratio of the G.P. a, b, c, d. Then, $b = ar, c = ar^2$ and $d = ar^3$.

$\therefore$ $a^n + b^n = a^n + a^n r^n = a^n (1 + r^n)$

$b^n + c^n = a^n r^n + a^n r^{2n} = a^n r^n (1 + r^n)$, $c^n + d^n = a^n r^{2n} + a^n r^{3n} = a^n r^{2n} (1 + r^n)$

Clearly, $(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$.

Hence, $a^n + b^n, b^n + c^n, c^n + d^n$ are in G.P.

LEVEL-2

EXAMPLE 7 *If a, b, c are in A.P. and x, y, z are in G.P., then show that $x^{b-c} . y^{c-a} . z^{a-b} = 1$.*

SOLUTION It is given that

a, b, c are in A.P. $\Rightarrow$ $2b = a + c$ $\qquad$ ...(i)

x, y, z are in G.P. $\Rightarrow$ $y^2 = xz$ $\qquad$ ...(ii)

$\therefore$ $x^{b-c}\, y^{c-a}\, z^{a-b} = x^{b-c} (\sqrt{xz})^{c-a}\, z^{a-b}$ $\qquad$ [Using (ii)]

$$= x^{b-c}\, x^{\frac{c-a}{2}}\, z^{\frac{c-a}{2}}\, z^{a-b}$$
$$= x^{b-c+\frac{c-a}{2}}\, z^{a-b+\frac{c-a}{2}}$$
$$= x^{\frac{2b-(a+c)}{2}}\, z^{\frac{(c+a)-2b}{2}} = x^0 z^0 = 1 \qquad \text{[Using (i)]}$$

EXAMPLE 8 *If mth, nth and pth terms of a G.P. form three consecutive terms of a G.P. Prove that m, n and p form three consecutive terms of an arithmetic sequence.*

SOLUTION Let a be the first term and r be the common ratio the G.P. Then,

$a_m = ar^{m-1}$, $a_n = ar^{n-1}$ and $a_p = ar^{p-1}$

It is given that a_m, a_n, a_p are in GP.

$$\begin{aligned}
\therefore \quad & (a_n)^2 = a_m\, a_p \\
\Rightarrow \quad & (ar^{n-1})^2 = (ar^{m-1} \times ar^{p-1}) \\
\Rightarrow \quad & a^2\, r^{2n-2} = a^2\, r^{m+p-2} \\
\Rightarrow \quad & r^{2n-2} = r^{m+p-2} \\
\Rightarrow \quad & 2n-2 = m+p-2 \\
\Rightarrow \quad & 2n = m+p \\
\Rightarrow \quad & m, n, p \text{ are in AP.}
\end{aligned}$$

EXAMPLE 9 *If a, b, c are in G.P. and x, y are the arithmetic means of a, b and b, c respectively, then prove that:*

$$\frac{a}{x} + \frac{c}{y} = 2 \text{ and } \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$$

SOLUTION It is given that

a, b, c are in G.P. $\Rightarrow b^2 = ac$ ...(i)

x is the A.M. of a and $b \Rightarrow x = \frac{a+b}{2}$ ...(ii)

and, y is the A.M. of b and $c \Rightarrow y = \frac{b+c}{2}$ ...(iii)

Now, $\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c} = \frac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$ $\left[\because x = \frac{a+b}{2} \text{ and } y = \frac{b+c}{2}\right]$

$\Rightarrow \frac{a}{x} + \frac{c}{y} = \frac{2(ab+2ac+bc)}{(ab+ac+b^2+bc)} = \frac{2(ab+2ac+bc)}{(ab+2ac+bc)} = 2$ [Using (i)]

And, $\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c} = \frac{2(a+c+2b)}{(ab+b^2+ac+bc)} = \frac{2(a+c+2b)}{(ab+2b^2+bc)}$ [Using (i)]

$\Rightarrow \frac{1}{x} + \frac{1}{y} = \frac{2(a+c+2b)}{b(a+2b+c)} = \frac{2}{b}$

EXAMPLE 10 *If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, prove that x, y, z are in A.P.* **[NCERT]**

SOLUTION We have,

$a^{1/x} = b^{1/y} = c^{1/z} = \lambda$ (say) $\Rightarrow a = \lambda^x, b = \lambda^y$ and $c = \lambda^z$

Now, a, b, c are in G.P.

$\Rightarrow b^2 = ac$

$\Rightarrow (\lambda^y)^2 = \lambda^x \times \lambda^z \Rightarrow \lambda^{2y} = \lambda^{x+z} \Rightarrow 2y = x+z \Rightarrow x, y, z$ are in A.P.

EXAMPLE 11 *If a^2+b^2, $ab+bc$ and b^2+c^2 are in G.P., prove that a, b, c are also in G.P.*

SOLUTION It is given that

$a^2+b^2, ab+bc, b^2+c^2$ are in G.P.

$\Rightarrow (ab+bc)^2 = (a^2+b^2)(b^2+c^2)$

$\Rightarrow a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$

$\Rightarrow b^4 + a^2c^2 - 2ab^2c = 0 \Rightarrow (b^2-ac)^2 = 0 \Rightarrow b^2 = ac \Rightarrow a, b, c$ are in G.P.

EXERCISE 20.5

LEVEL-1

1. If a, b, c are in G.P., prove that $\log a, \log b, \log c$ are in A.P.
2. If a, b, c are in G.P., prove that $\frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m}$ are in A.P.
3. Find k such that $k+9, k-6$ and 4 form three consecutive terms of a G.P.
4. Three numbers are in A.P. and their sum is 15. If 1, 3, 9 be added to them respectively, they form a G.P. Find the numbers.
5. The sum of three numbers which are consecutive terms of an A.P. is 21. If the second number is reduced by 1 and the third is increased by 1, we obtain three consecutive terms of a G.P. Find the numbers.
6. The sum of three numbers a, b, c in A.P. is 18. If a and b are each increased by 4 and c is increased by 36, the new numbers form a G.P. Find a, b, c.
7. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an A.P. Find the numbers.
8. If a, b, c are in G.P., prove that:
   (i) $a(b^2+c^2) = c(a^2+b^2)$ (ii) $a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) = a^3+b^3+c^3$
   (iii) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$ (iv) $\frac{1}{a^2-b^2}+\frac{1}{b^2} = \frac{1}{b^2-c^2}$
   (v) $(a+2b+2c)(a-2b+2c) = a^2+4c^2$.
9. If a, b, c, d are in G.P., prove that:
   (i) $\frac{ab-cd}{b^2-c^2} = \frac{a+c}{b}$ (ii) $(a+b+c+d)^2 = (a+b)^2+2(b+c)^2+(c+d)^2$
   (iii) $(b+c)(b+d) = (c+a)(c+d)$
10. If a, b, c are in G.P., prove that the following are also in G.P.:
    (i) a^2, b^2, c^2 (ii) a^3, b^3, c^3 (iii) $a^2+b^2, ab+bc, b^2+c^2$
11. If a, b, c, d are in G.P., prove that:
    (i) $(a^2+b^2), (b^2+c^2), (c^2+d^2)$ are in G.P.
    (ii) $(a^2-b^2), (b^2-c^2), (c^2-d^2)$ are in G.P.
    (iii) $\frac{1}{a^2+b^2}, \frac{1}{b^2+c^2}, \frac{1}{c^2+d^2}$ are in G.P.
    (iv) $(a^2+b^2+c^2), (ab+bc+cd), (b^2+c^2+d^2)$ are in G.P.
12. If $(a-b), (b-c), (c-a)$ are in G.P., then prove that $(a+b+c)^2 = 3(ab+bc+ca)$
13. If a, b, c are in G.P. then prove that: $\frac{a^2+ab+b^2}{bc+ca+ab} = \frac{b+a}{c+b}$
14. If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z respectively. Prove that x, y, z are in G.P. **[NCERT]**

LEVEL-2

15. If a, b, c are in A.P. and a, b, d are in G.P., then prove that $a, a-b, d-c$ are in G.P.
16. If pth, qth, rth and sth terms of an A.P. be in G.P., then prove that $p-q, q-r, r-s$ are in G.P. **[NCERT]**

17. If $\frac{1}{a+b}, \frac{1}{2b}, \frac{1}{b+c}$ are three consecutive terms of an A.P., prove that a, b, c are the three consecutive terms of a G.P.

18. If $x^a = x^{b/2} z^{b/2} = z^c$, then prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

19. If a, b, c are in A.P. b, c, d are in G.P. and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P., prove that a, c, e are in G.P.

20. If a, b, c are in A.P. and a, x, b and b, y, c are in G.P., show that x^2, b^2, y^2 are in A.P.

21. If a, b, c are in A.P. and a, b, d are in G.P., show that $a, (a-b), (d-c)$ are in G.P.

22. If a, b, c are three distinct real numbers in G.P. and $a+b+c = xb$, then prove that either $x < -1$ or $x > 3$.

23. If $p^{\text{th}}, q^{\text{th}}$ and r^{th} terms of an A.P. and G.P. are both a, b and c respectively, show that $a^{b-c} b^{c-a} c^{a-b} = 1$.

ANSWERS

3. 0 or 16 6. $a = -2,\ b = 6,\ c = 14$ or $a = 46,\ b = 6,\ c = -34$ 7. 8, 16, 32
8. 15, 5, –5 or 3, 5, 7 9. 12, 7, 2 or 3, 7, 11 14. 2046

HINTS TO NCERT & SELECTED PROBLEMS

1. a, b, c are in G.P. $\Rightarrow b^2 = ac \Rightarrow \log b^2 = \log ac \Rightarrow 2 \log b = \log a + \log c$

2. a, b, c are in G.P.

$$\therefore \quad b^2 = ac = \log_m b^2 = \log_m ac$$

$$\Rightarrow \quad 2 \log_m b = \log_m a + \log_m c$$

$$\Rightarrow \quad \frac{2}{\log_b m} = \frac{1}{\log_a m} + \frac{1}{\log_c m} \Rightarrow \frac{1}{\log_a m}, \frac{1}{\log_b m}, \frac{1}{\log_c m} \text{ are in A.P.}$$

3. It is given that $k+9, k-6, 4$ are in G.P.

$$\Rightarrow \quad (k-6)^2 = (k+9) \times 4 \Rightarrow k = 0, 16.$$

14. Let the first term and common ratio of the G.P. be a and r respectively. It is given that

$$x = ar^3, y = ar^9 \text{ and } z = ar^{15}$$

$$\therefore \quad y^2 = a^2 r^{18} \text{ and } xz = a^2 r^{18}$$

$$\Rightarrow \quad y^2 = xz$$

$$\Rightarrow \quad x, y, z \text{ are in G.P.}$$

16. Let the first term and the common difference of the AP be a and d respectively.
It is given that its $p^{\text{th}}, r^{\text{th}}$ and s^{th} terms are in G.P. Let A be the first term and R be the common ratio of the G.P. Then,

$$a + (p-1)\, d = A \qquad \text{...(i)}$$

$$a + (q-1)\, d = AR \qquad \text{...(ii)}$$

$$a + (r-1)\, d = AR^2 \qquad \text{...(iii)}$$

$$a + (s-1)\, d = AR^3 \qquad \text{...(iv)}$$

Subtracting (ii) from (i), we get

$$\left\{a + (p-1)\, d\right\} - \left\{a + (q-1)\, d\right\} = A - AR$$

$$\Rightarrow \quad (p-q)\, d = A(1-R) \qquad \text{...(v)}$$

Subtracting (iii) from (ii), we get

$$\left\{a + (q-1)\, d\right\} - \left\{a + (r-1)\, d\right\} = AR - AR^2$$

$$\Rightarrow \quad (q-r)\, d = AR\,(1-R) \qquad \text{...(vi)}$$

Subtracting (iv) from (iii), we get

$$\left\{a + (r-1)\, d\right\} - \left\{a + (s-1)\, d\right\} = AR^2 - AR^3$$

$$\Rightarrow \quad (r-s)\, d = AR^2\,(1-R) \qquad \text{...(vii)}$$

From (v), (vi) and (vii), we obtain that

$$(q-r)^2\, d^2 = (p-q)\, d\, (r-s)\, d$$

$$\Rightarrow \quad (q-r)^2 = (p-q)\,(r-s)$$

$\Rightarrow \quad (p-q), (q-r), (r-s)$ are in G.P.

17. It is given that $\dfrac{1}{a+b}, \dfrac{1}{2b}, \dfrac{1}{b+c}$ are in A.P.

$$\therefore \quad \frac{2}{2b} = \frac{1}{a+b} + \frac{1}{b+c} \Rightarrow b^2 = ac.$$

19. We have,

$$2b = a + c \quad \text{...(i)} \qquad c^2 = bd \quad \text{...(ii)} \qquad \text{and,} \quad \frac{2}{d} = \frac{1}{c} + \frac{1}{e} \qquad \text{...(iii)}$$

We have to eliminate b and d from these relations. Substitute b and d obtained from (i) and (iii) in (ii) to get $c^2 = ae$.

22. Let r be the common ratio of the G.P. Then, $b = ar$ and $c = ar^2$.

Now, $a + b + c = xb \Rightarrow a + ar + ar^2 = xar \Rightarrow r^2 + (1-x)\, r + 1 = 0$.

But, r is real.

$\therefore \quad \text{Disc} \geq 0 \Rightarrow x^2 - 2x - 3 > 0 \Rightarrow x < -1$ or $x > 3$

20.7 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

GEOMETRIC MEANS *Let a and b be two given numbers. If n numbers $G_1, G_2, \ldots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Then the numbers $G_1, G_2, \ldots, G_n$ are known as n geometric means (G.M.'s) between a and b.*

GEOMETRIC MEAN *If a single geometric mean G is inserted between two given numbers a and b, then G is known as the geometric mean between a and b.*

Thus,

G is the G.M. between a and b. $\Leftrightarrow a, G, b$ are in G.P. $\Leftrightarrow G^2 = ab \Leftrightarrow G = \sqrt{ab}$.

The geometric mean G between 4 and 9 is given by $G = \sqrt{4 \times 9} = 6$.

The geometric mean G between –9 and –4 is given by $G = \sqrt{-9 \times -4} = -6$.

<u>NOTE</u> *If a and b are two numbers of opposite signs, then geometric mean between them does not exist.*

20.7.1 INSERTION OF GEOMETRIC MEANS BETWEEN TWO GIVEN NUMBERS

Let $G_1, G_2, \ldots, G_n$ be n geometric means between two given numbers a and b. Then,

$a, G_1, G_2, \ldots, G_n, b$ is a G.P. consisting of $(n + 2)$ terms. Let r be the common ratio of this G.P. Then,

$$b = (n+2)th \text{ term } = ar^{n+1}$$

$$\Rightarrow \quad r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

$$\therefore \quad G_1 = ar = a\left(\frac{b}{a}\right)^{1/(n+1)}, G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/(n+1)} \ldots, G_n = ar^n = a\left(\frac{b}{a}\right)^{n/n+1}.$$

THEOREM *If n geometric means are inserted between two quantities, then the product of n geometric means is the nth power of the single geometric mean between the two quantities.*

PROOF Let $G_1, G_2, G_3, \ldots, G_n$ be n geometric means between two quantities a and b. Then, $a, G_1, G_2, \ldots, G_n, b$ is a G.P. Let r be the common ratio of this G.P. Then,

$$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ and, } G_1 = ar, G_2 = ar^2, G_3 = ar^3, \ldots, G_n = ar^n.$$

$$\therefore \quad G_1 . G_2 . G_3 \ldots . G_n = (ar)(ar^2)(ar^3)\ldots(ar^n) = a^n r^{1+2+3+\ldots+n}$$

$$= a^n r^{\frac{n(n+1)}{2}} = a^n \left\{\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right\}^{\frac{n(n+1)}{2}} = a^n \left(\frac{b}{a}\right)^{n/2} = a^{n/2} b^{n/2}$$

$$= \left\{\sqrt{ab}\right\}^n$$

$= G^n$, where $G = \sqrt{ab}$ is the single geometric mean between a and b.

Q.E.D.

20.7.2 SOME IMPORTANT PROPERTIES OF ARITHMETIC AND GEOMETRIC MEANS

THEOREM 1 *If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then $A > G$.*

PROOF We have,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

$$\therefore \quad A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 > 0$$

$$\Rightarrow \quad A > G.$$

Q.E.D.

THEOREM 2 *If A and G are respectively arithmetic and geometric means between two positive quantities a and b, then the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$.*

PROOF We have,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}$$

The equation having a and b as its roots is

$$x^2 - x(a+b) + ab = 0 \text{ or, } x^2 - 2Ax + G^2 = 0 \qquad \left[\because A = \frac{a+b}{2} \text{ and } G = \sqrt{ab}\right]$$

Q.E.D.

THEOREM 3 *If A and G be the A.M. and G.M. between two positive numbers, then the numbers are* $A \pm \sqrt{A^2 - G^2}$. **[NCERT]**

PROOF The equation having its roots as the given numbers is

$$x^2 - 2Ax + G^2 = 0 \Rightarrow x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2} \qquad \text{Q.E.D.}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **INSERTION OF GEOMETRIC MEANS BETWEEN TWO NUMBERS**

EXAMPLE 1 *Insert 5 geometric means between 576 and 9.*

SOLUTION Let G_1, G_2, G_3, G_4, G_5 be 5 geometric means between $a = 576$ and $b = 9$. Then, $576, G_1, G_2, G_3, G_4, G_5, 9$ is a G.P. with common ratio r given by

$$r = \left(\frac{9}{576}\right)^{\frac{1}{5+1}} = \left(\frac{1}{64}\right)^{\frac{1}{6}} = \frac{1}{2}. \qquad \left[\text{Using: } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

$$\therefore \quad G_1 = ar = 576 \times \frac{1}{2} = 288, \qquad G_2 = ar^2 = 576 \times \frac{1}{4} = 144,$$

$$G_3 = ar^3 = 576 \times \frac{1}{8} = 72, \qquad G_4 = ar^4 = 576 \times \frac{1}{16} = 36 \text{ and, } G_5 = ar^5 = 576 \times \frac{1}{32} = 18$$

Hence, 288, 144, 72, 36, 18 are the required geometric means between 576 and 9.

Type II **PROBLEMS BASED UPON ARITHMETIC AND GEOMETRIC MEANS**

EXAMPLE 2 *Find the value of n so that* $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ *may be the geometric mean between a and b.* **[NCERT]**

SOLUTION It is given that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the G.M. between a and b.

$$\therefore \quad \frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$$

$$\Rightarrow \quad a^{n+1} + b^{n+1} = a^{\left(n + \frac{1}{2}\right)} b^{1/2} + a^{1/2} b^{\left(n + \frac{1}{2}\right)}$$

$$\Leftrightarrow \quad a^{n+1} - a^{\left(n + \frac{1}{2}\right)} b^{1/2} = a^{1/2} b^{\left(n + \frac{1}{2}\right)} - b^{n+1}$$

$$\Leftrightarrow \quad a^{\left(n + \frac{1}{2}\right)} (a^{1/2} - b^{1/2}) = b^{\left(n + \frac{1}{2}\right)} (a^{1/2} - b^{1/2})$$

$$\Leftrightarrow \quad a^{\left(n + \frac{1}{2}\right)} = b^{\left(n + \frac{1}{2}\right)} \qquad [\because \ a^{1/2} - b^{1/2} \neq 0, \text{ as } a \neq b]$$

$$\Leftrightarrow \quad \left(\frac{a}{b}\right)^{\left(n + \frac{1}{2}\right)} = 1 \Leftrightarrow \left(\frac{a}{b}\right)^{\left(n + \frac{1}{2}\right)} = \left(\frac{a}{b}\right)^0 \Leftrightarrow n + \frac{1}{2} = 0 \Leftrightarrow n = -\frac{1}{2}$$

EXAMPLE 3 *Find two numbers whose arithmetic mean is 34 and the geometric mean is 16.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that AM and GM of a and b are 34 and 16 respectively.

i.e. $\frac{a+b}{2} = 34$ and $\sqrt{ab} = 16$

$\Rightarrow$ $a+b = 68$ and $ab = 256$ ...(i)

$\therefore$ $(a-b)^2 = (a+b)^2 - 4ab$

$\Rightarrow$ $(a-b)^2 = (68)^2 - 4 \times 256 = 3600$

$\Rightarrow$ $a-b = 60$ $[\because a > b \therefore a-b > 0]$

Solving $a+b = 68$ and $a-b = 60$ simultaneously, we get $a = 64$ and $b = 4$.

Hence, the required numbers are 64 and 4.

ALITER Here, $A = 34$ and $G = 16$.

So, the numbers are $A + \sqrt{A^2 - G^2}$ and $A - \sqrt{A^2 - G^2}$

i.e. $34 + \sqrt{34^2 - 16^2}$ and $34 - \sqrt{34^2 - 16^2}$ or, 64 and 4.

EXAMPLE 4 *If the A.M. and G.M. between two numbers are in the ratio $m : n$, then prove that the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.* [NCERT]

SOLUTION Let the two numbers be a and b. Let A and G be respectively the arithmetic and geometric means between a and b. Then,

$$A = \frac{a+b}{2} \text{ and } G = \sqrt{ab} \Rightarrow a+b = 2A \text{ and } G^2 = ab \quad \text{...(i)}$$

The equation having a and b as its roots is

$$x^2 - (a+b)x + ab = 0$$

or, $$x^2 - 2Ax + G^2 = 0 \quad \text{[Using (i)]}$$

$$\Rightarrow \quad x = \frac{2A \pm \sqrt{4A^2 - 4G^2}}{2} \Rightarrow x = A \pm \sqrt{A^2 - G^2}$$

So, the two numbers are $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$.

It is given that

$$A : G = m : n \Rightarrow A = \lambda m \text{ and } G = \lambda n \text{ for some } \lambda$$

Substituting the values of A and G in $a = A + \sqrt{A^2 - G^2}$ and $b = A - \sqrt{A^2 - G^2}$, we get

$$\frac{a}{b} = \frac{\lambda m + \sqrt{\lambda^2 m^2 - \lambda^2 n^2}}{\lambda m - \sqrt{\lambda^2 m^2 - \lambda^2 n^2}} \Rightarrow \frac{a}{b} = \frac{m + \sqrt{m^2 - n^2}}{m - \sqrt{m^2 - n^2}} \Rightarrow a : b = \left(m + \sqrt{m^2 - n^2}\right) : \left(m - \sqrt{m^2 - n^2}\right)$$

LEVEL-2

Type III ON GEOMETRIC AND ARITHMETIC MEANS

EXAMPLE 5 *Find two positive numbers whose difference is 12 and whose A.M. exceeds the G.M. by 2.*

SOLUTION Let the two numbers be a and b such that $a > b$. It is given that

$$a - b = 12 \quad \text{...(i)}$$

It is also given that

$$AM - GM = 2$$

$$\Rightarrow \quad \frac{a+b}{2} - \sqrt{ab} = 2 \quad \left[\because AM = \frac{a+b}{2} \text{ and } GM = \sqrt{ab}\right]$$

$$\Rightarrow \quad a+b-2\sqrt{ab} = 4$$

$$\Rightarrow \quad (\sqrt{a}-\sqrt{b})^2 = 4$$

$$\Rightarrow \quad \sqrt{a}-\sqrt{b} = 2 \quad \text{...(ii)}$$

Now, $a-b = 12$

$$\Rightarrow \quad (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = 12$$

$$\Rightarrow \quad (\sqrt{a}+\sqrt{b}) \times (2) = 12 \quad \text{...(iii)}$$

$$\Rightarrow \quad \sqrt{a}+\sqrt{b} = 6 \quad \text{[Using (ii)]}$$

Solving (ii) and (iii), we get $a = 16$, $b = 4$. Hence, the required numbers are 16 and 4.

EXAMPLE 6 *If a, b, c are in G.P and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P.* **[NCERT]**

SOLUTION It is given that a, b, c are in G.P. Therefore, $b^2 = ac$

Now, $ax^2 + 2bx + c = 0$

$$\Rightarrow \quad ax^2 + 2\sqrt{ac}\, x + c = 0 \Rightarrow (\sqrt{a}x + \sqrt{c})^2 = 0 \Rightarrow \sqrt{a}x + \sqrt{c} = 0 \Rightarrow x = -\sqrt{\frac{c}{a}}$$

It is given that the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root and the equation $ax^2 + 2bx + c = 0$ has equal roots both equal to $-\sqrt{\frac{c}{a}}$.

$$\therefore \quad -\sqrt{\frac{c}{a}} \text{ is a root of the equation } dx^2 + 2ex + f = 0$$

$$\Rightarrow \quad d\frac{c}{a} - 2e\sqrt{\frac{c}{a}} + f = 0$$

$$\Rightarrow \quad \frac{d}{a} - 2e\sqrt{\frac{1}{ac}} + \frac{f}{c} = 0 \quad \text{[Dividing through out by } c\text{]}$$

$$\Rightarrow \quad \frac{d}{a} - \frac{2e}{b} + \frac{f}{c} = 0 \quad [\because b^2 = ac]$$

$$\Rightarrow \quad 2\frac{e}{b} = \frac{d}{a} + \frac{f}{c} \quad \Rightarrow \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in A.P.}$$

EXAMPLE 7 *Let x be the arithmetic mean and y, z be two geometric means between any two positive numbers. Then, prove that $\frac{y^3 + z^3}{xyz} = 2$.*

SOLUTION Let a and b be two positive numbers. Then,

$$x = \text{A.M. of } a \text{ and } b \Rightarrow x = \frac{a+b}{2} \quad \text{...(i)}$$

It is given that y and z are two geometric means between a and b. Then, a, y, z, b is a G.P. with common ratio $r = \left(\frac{b}{a}\right)^{\frac{1}{2+1}} = \left(\frac{b}{a}\right)^{1/3}$

$$\therefore \quad y = ar \Rightarrow y = a\left(\frac{b}{a}\right)^{1/3} \Rightarrow y = b^{1/3}a^{2/3} \text{ and, } z = ar^2 \Rightarrow z = a\left(\frac{b}{a}\right)^{2/3} \Rightarrow z = b^{2/3}a^{1/3}$$

$\therefore \quad y^3 + z^3 = (b^{1/3} a^{2/3})^3 + (b^{2/3} a^{1/3})^3 = ba^2 + b^2 a = ab(a+b)$

and, $\quad yz = (b^{1/3} a^{2/3})(b^{2/3} a^{1/3}) = ab.$

Now, $\quad y^3 + z^3 = ab(a+b)$ and $yz = ab$

$\Rightarrow \quad y^3 + z^3 = yz(a+b)$

$\Rightarrow \quad y^3 + z^3 = yz(2x)$ [Using (i)]

$$\Rightarrow \quad \frac{y^3+z^3}{xyz} = 2.$$

EXAMPLE 8 *If a is the A.M. of b and c and the two geometric means are G_1 and G_2, then prove that $G_1^3 + G_2^3 = 2abc$.*

SOLUTION It is given that a is the A.M. of b and c.

$$\therefore \quad a = \frac{b+c}{2} \Rightarrow b + c = 2a \quad \text{...(i)}$$

Since G_1 and G_2 are two geometric means between b and c. Therefore, b, G_1, G_2, c is a G.P. with common ratio $r = \left(\frac{c}{b}\right)^{1/3}$.

$$\therefore \quad G_1 = br = b\left(\frac{c}{b}\right)^{1/3} = c^{1/3} b^{2/3} \text{ and } G_2 = br^2 = b\left(\frac{c}{b}\right)^{2/3} = b^{1/3} c^{2/3}$$

$\Rightarrow \quad G_1^3 = b^2 c$ and $G_2^3 = bc^2$

$\Rightarrow \quad G_1^3 + G_2^3 = b^2 c + bc^2 = bc(b+c) = 2abc$ [Using (i)]

EXAMPLE 9 *If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.*

SOLUTION Let a and b be two given quantities. It is given that G is the geometric mean of a and b.

$$\therefore \quad G = \sqrt{ab} \Rightarrow G^2 = ab \quad \text{...(i)}$$

It is also given that A_1, A_2 are two arithmetic means between a and b. Therefore,, a, A_1, A_2, b is an A.P. with common difference $d = \frac{b-a}{3}$.

$$\therefore \quad A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}, \quad A_2 = a + 2d = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$$

$$\text{So,} \quad 2A_1 - A_2 = 2\left(\frac{2a+b}{3}\right) - \left(\frac{a+2b}{3}\right) = a \text{ and } 2A_2 - A_1 = 2\left(\frac{a+2b}{3}\right) - \left(\frac{2a+b}{3}\right) = b$$

$\therefore \quad (2A_1 - A_2)(2A_2 - A_1) = ab$

$\Rightarrow \quad (2A_1 - A_2)(2A_2 - A_1) = G^2$ [Using (i)]

Type III **PROBLEMS ON A.M. > G.M.**

EXAMPLE 10 *If x, y, z are distinct positive numbers, then prove that $(x+y)(y+z)(z+x) > 8xyz$.*

SOLUTION Using A.M. > G.M., we obtain **[NCERT EXEMPLAR]**

$$\frac{x+y}{2} > \sqrt{xy}, \quad \frac{y+z}{2} > \sqrt{yz} \text{ and } \frac{z+x}{2} > \sqrt{zx}$$

$\Rightarrow \quad x + y > 2\sqrt{xy}, \; y + z > 2\sqrt{yz}$ and $z + x > 2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 2\sqrt{xy} \times 2\sqrt{yz} \times 2\sqrt{zx}$

$\Rightarrow \quad (x+y)(y+z)(z+x) > 8xyz.$

EXAMPLE 11 *If $x \in R$, find the minimum value of the expression $3^x + 3^{1-x}$.* **[NCERT EXEMPLAR]**

SOLUTION We know that A.M. > G.M.

$\therefore \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3^x \times 3^{1-x}}$ for all $x \in R$

$\Rightarrow \quad \frac{3^x + 3^{1-x}}{2} \geq \sqrt{3}$ for all $x \in R$

$\Rightarrow \quad 3^x + 3^{1-x} \geq 2\sqrt{3}$ for all $x \in R$

Hence, the minimum value of $3^x + 3^{1-x}$ for any $x \in R$ is $2\sqrt{3}$.

EXAMPLE 12 *If a, b, c, d are four distinct positive numbers in A.P. then show that bc > ad.*

SOLUTION It is given that a, b, c, d are in A.P. Therefore, a, b, c are in A.P.

$\Rightarrow \quad b$ is the A.M. of a and c

The G.M. of a and c is $\sqrt{ac}$.

$\therefore \quad$ A.M. of a and c > G.M. of a and c

$\Rightarrow \quad b > \sqrt{ac}$

$\Rightarrow \quad b^2 > ac \quad$ ...(i)

Again, a, b, c, d are in A.P.

$\Rightarrow \quad b, c, d$ are in A.P.

$\Rightarrow \quad c$ is the A.M of b and d.

The G.M. of b and d is $\sqrt{bd}$.

$\therefore \quad$ A.M. of b and d > G.M. of b and d

$\Rightarrow \quad c > \sqrt{bd}$

$\Rightarrow \quad c^2 > bd \quad$ ...(ii)

From (i) and (ii), we obtain

$$b^2c^2 > (ac)(bd) \Rightarrow bc > ad.$$

EXAMPLE 13 *If a, b, c, d are four distinct positive numbers in G.P. then show that a + d > b + c.* **[NCERT EXEMPLAR]**

SOLUTION It is given that a, b, c, d are in G.P.

$\therefore \quad a, b, c$ are in G.P.

$\Rightarrow \quad b$ is the G.M of a and c

But, A.M. of a and c is $\frac{a+c}{2}$.

$\therefore \quad$ A.M. of a and c > G.M. of a and c

$\Rightarrow \quad \frac{a+c}{2} > b$

$\Rightarrow \quad a + c > 2b \quad$ ...(i)

Again, a, b, c, d are in G.P.

$\Rightarrow \quad b, c, d$ are in G.P.

$\Rightarrow \quad c$ is the G.M. of b and d.

But, A.M. of b and d is $\frac{b+d}{2}$

$\therefore \quad$ A.M. of b and d > G.M. of b and d

$\Rightarrow \quad \frac{b+d}{2} > c$

$\Rightarrow \quad b+d>2c$ ...(ii)

Adding (i) and (ii), we obtain

$$a+c+b+d>2b+2c$$

$$\Rightarrow \quad a+d>b+c$$

EXERCISE 20.6

LEVEL-1

1. Insert 6 geometric means between 27 and $\frac{1}{81}$.
2. Insert 5 geometric means between 16 and $\frac{1}{4}$.
3. Insert 5 geometric means between $\frac{32}{9}$ and $\frac{81}{2}$.
4. Find the geometric means of the following pairs of numbers:
   (i) 2 and 8 (ii) a^3b and ab^3 (iii) -8 and -2
5. If a is the G.M. of 2 and $\frac{1}{4}$, find a.
6. Find the two numbers whose A.M. is 25 and GM is 20.
7. Construct a quadratic in x such that A.M. of its roots is A and G.M. is G.
8. The sum of two numbers is 6 times their geometric means, show that the numbers are in the ratio $(3+2\sqrt{2}):(3-2\sqrt{2})$. [NCERT]
9. If AM and GM of roots of a quadratic equation are 8 and 5 respectively, then obtain the quadratic equation. [NCERT]
10. If AM and GM of two positive numbers a and b are 10 and 8 respectively, find the numbers [NCERT]

LEVEL-2

11. Prove that the product of n geometric means between two quantities is equal to the nth power of a geometric mean of those two quantities.
12. If the A.M. of two positive numbers a and b $(a>b)$ is twice their geometric mean. Prove that: $a:b = (2+\sqrt{3}):(2-\sqrt{3})$.
13. If one A.M., A and two geometric means G_1 and G_2 inserted between any two positive numbers, show that $\frac{G_1^2}{G_2}+\frac{G_2^2}{G_1} = 2A$.

ANSWERS

1. $9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$ 2. $8, 4, 2, 1, \frac{1}{2}$ 3. $\frac{16}{3}, 8, 12, 18, 27$

4.(i) 4 (ii) a^2b^2 (iii) -4 5. $\frac{1}{\sqrt{2}}$ 6. 40, 7. $x^2-2Ax+G^2=0$

9. $x^2-16x+25 = 0$ 10. 4, 16 or 16, 4

HINTS TO NCERT & SELECTED PROBLEMS

8. Let the numbers be a and b. Further, let A and G denote their arithmetic and geometric means respectively. It is given that

$$a+b = 6G \Rightarrow \frac{a+b}{2} = 3G \Rightarrow A = 3G.$$

Numbers a and b are roots of the quadratic equation

$$x^2 - x(a+b) + ab = 0$$

or, $$x^2 - 2Ax + G^2 = 0$$

or, $$x^2 - 6Gx + G^2 = 0 \qquad [\because A = 3G]$$

$$\Rightarrow \quad x = \frac{6G \pm \sqrt{36G^2 - 4G^2}}{2}$$

$$\Rightarrow \quad x = 3G \pm 2\sqrt{2}\,G$$

$$\Rightarrow \quad x = 3G \pm 2\sqrt{2}\,G$$

$$\Rightarrow \quad x = (3 \pm 2\sqrt{2})G$$

$$\Rightarrow \quad a = (3 + 2\sqrt{2})G \text{ and } b = (3 - 2\sqrt{2})G$$

Hence, $a : b = (3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$

9. Let a and b be the roots of the quadratic equation. Then, the quadratic equation is

$$x^2 - (a+b)x + ab = 0 \qquad \text{...(i)}$$

It is given that AM = 8 and GM = 5.

i.e. $\frac{a+b}{2} = 8$ and $\sqrt{ab} = 5 \Rightarrow a + b = 16$ and $ab = 25$

Substituting these values in (i), we obtain $x^2 - 16x + 25 = 0$ as the required equation.

10. We have,

$$\frac{a+b}{2} = 10 \text{ and } \sqrt{ab} = 8 \Rightarrow a + b = 20 \text{ and } ab = 64$$

Clearly, a and b are roots of the equation

$$x^2 - (a+b)x + ab = 0$$

or, $$x^2 - 20x + 64 = 0$$

$$\Rightarrow \quad (x-16)(x-4) = 0 \Rightarrow x = 4, 16 \Rightarrow a = 4, b = 16 \text{ or } a = 16, b = 4.$$

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If the fifth term of a G.P. is 2, then write the product of its 9 terms.
2. If $(p+q)^{th}$ and $(p-q)^{th}$ terms of a G.P. are m and n respectively, then write its pth term.
3. If $\log_x a$, $a^{x/2}$ and $\log_b x$ are in G.P., then write the value of x.
4. If the sum of an infinite decreasing G.P. is 3 and the sum of the squares of its term is $\frac{9}{2}$, then write its first term and common difference.
5. If p^{th}, q^{th} and r^{th} terms of a G.P. are x, y, z respectively, then write the value of $x^{q-r}\, y^{r-p}\, z^{p-q}$.
6. If A_1, A_2 be two AM's and G_1, G_2 be two GM's between a and b, then find the value of $\frac{A_1 + A_2}{G_1 G_2}$.
7. If second, third and sixth terms of an A.P. are consecutive terms of a G.P., write the common ratio of the G.P.

8. Write the quadratic equation the arithmetic and geometric means of whose roots are A and G respectively.
9. Write the product of n geometric means between two numbers a and b.
10. If $a = 1 + b + b^2 + b^3 + \ldots$ to ∞, then write b in terms of a given that $|b| < 1$.

ANSWERS

1. 512 2. $\sqrt{mn}$ 3. $\log_a (\log_b a)$ 4. $a = 2, r = \frac{1}{3}$ 5. 1 6. $\frac{a+b}{ab}$
7. 3 8. $x^2 - 2Ax + G^2 = 0$ 9. $(ab)^{n/2}$ 10. $\frac{a-1}{a}$

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. If in an infinite G.P., first term is equal to 10 times the sum of all successive terms, then its common ratio is
   (a) 1/10 (b) 1/11 (c) 1/9 (d) 1/20
2. If the first term of a G.P. $a_1, a_2, a_3, \ldots$ is unity such that $4a_2 + 5a_3$ is least, then the common ratio of G.P. is
   (a) $-2/5$ (b) $-3/5$ (c) 2/5 (d) none of these
3. If a, b, c are in A.P. and x, y, z are in G.P., then the value of $x^{b-c} y^{c-a} z^{a-b}$ is
   (a) 0 (b) 1 (c) xyz (d) $x^a y^b z^c$
4. The first three of four given numbers are in G.P. and their last three are in A.P. with common difference 6. If first and fourth numbers are equal, then the first number is
   (a) 2 (b) 4 (c) 6 (d) 8
5. If a, b, c are in G.P. and $a^{1/x} = b^{1/y} = c^{1/z}$, then xyz are in
   (a) AP (b) GP (c) HP (d) none of these
6. If S be the sum, P the product and R be the sum of the reciprocals of n terms of a GP, then P^2 is equal to
   (a) S/R (b) R/S (c) $(R/S)^n$ (d) $(S/R)^n$
7. The fractional value of $2.\dot{3}5\dot{7}$ is
   (a) 2355/1001 (b) 2379/997 (d) 2355/999 (d) none of these
8. If pth, qth and rth terms of an A.P. are in G.P., then the common ratio of this G.P. is
   (a) $\frac{p-q}{q-r}$ (b) $\frac{q-r}{p-q}$ (c) pqr (d) none of these
9. The value of $9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots$ to ∞, is
   (a) 1 (b) 3 (c) 9 (d) none of these
10. The sum of an infinite G.P. is 4 and the sum of the cubes of its terms is 92. The common ratio of the original G.P. is
    (a) 1/2 (b) 2/3 (c) 1/3 (d) $-1/2$
11. If the sum of first two terms of an infinite GP is 1 and every term is twice the sum of all the successive terms, then its first term is
    (a) 1/3 (b) 2/3 (c) 1/4 (d) 3/4

12. The nth term of a G.P. is 128 and the sum of its n terms is 225. If its common ratio is 2, then its first term is

(a) 1 (b) 3 (c) 8 (d) none of these

13. If second term of a G.P. is 2 and the sum of its infinite terms is 8, then its first term is

(a) 1/4 (b) 1/2 (c) 2 (d) 4

14. If a, b, c are in G.P. and x, y are AM's between a, b and b, c respectively, then

(a) $\frac{1}{x}+\frac{1}{y}=2$ (b) $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$ (c) $\frac{1}{x}+\frac{1}{y}=\frac{2}{a}$ (d) $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$

15. If A be one A.M. and p, q be two G.M.'s between two numbers, then 2 A is equal to

(a) $\frac{p^3+q^3}{pq}$ (b) $\frac{p^3-q^3}{pq}$ (c) $\frac{p^2+q^2}{2}$ (d) $\frac{pq}{2}$

16. If p, q be two A.M.'s and G be one G.M. between two numbers, then $G^2 =$

(a) $(2p-q)(p-2q)$ (b) $(2p-q)(2q-p)$ (c) $(2p-q)(p+2q)$ (d) none of these

17. If x is positive, the sum to infinity of the series $\frac{1}{1+x}-\frac{1-x}{(1+x)^2}+\frac{(1-x)^2}{(1+x)^3}-\frac{(1-x)^3}{(1+x)^4}+\ldots$ is

(a) 1/2 (b) 3/4 (c) 1 (d) none of these

18. If $(4^3)(4^6)(4^9)(4^{12})\ldots(4^{3x})=(0.0625)^{-54}$, the value of x is

(a) 7 (b) 8 (c) 9 (d) 10

19. Given that $x>0$, the sum $\sum_{n=1}^{\infty}\left(\frac{x}{x+1}\right)^{n-1}$ equals

(a) x (b) $x+1$ (c) $\frac{x}{2x+1}$ (d) $\frac{x+1}{2x+1}$

20. In a G.P. of even number of terms, the sum of all terms is five times the sum of the odd terms. The common ratio of the G.P. is

(a) $-\frac{4}{5}$ (b) $\frac{1}{5}$ (c) 4 (d) none of these

21. Let x be the A.M. and y, z be two G.M.s between two positive numbers. Then, $\frac{y^3+z^3}{xyz}$ is equal to

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) none of these

22. The product $(32), (32)^{1/6} (32)^{1/36} \ldots$ to ∞ is equal to

(a) 64 (b) 16 (c) 32 (d) 0

23. The two geometric means between the numbers 1 and 64 are

(a) 1 and 64 (b) 4 and 16 (c) 2 and 16 (d) 8 and 16

24. In a G.P. if the $(m+n)^{th}$ term is p and $(m-n)^{th}$ term is q, then its m^{th} term is

(a) 0 (b) pq (c) $\sqrt{pq}$ (d) $\frac{1}{2}(p+q)$

25. Let S be the sum, P be the product and R be the sum of the reciprocals of 3 terms of a G.P. then $P^2R^3 : S^3$ is equal to

(a) 1 : 1 (b) (Common ratio)^n : 1

(c) (First term)^2 (Common ratio)^2 (d) None of these

ANSWERS

1. (b)	2. (a)	3. (b)	4. (d)	5. (a)	6. (d)	7. (c)	8. (b)
9. (b)	10. (a)	11. (d)	12. (a)	13. (d)	14. (d)	15. (a)	16. (b)
17. (a)	18. (b)	19. (b)	20. (c)	21. (b)	22. (a)	23. (b)	24. (c)
25. (a)							

SUMMARY

1. A sequence of non-zero numbers is called a geometric progression if the ratio of a term and the term preceding to it is always a constant quantity. The constant ratio is called the common ratio of the G.P.
2. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ is a G.P., then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called a geometric series.
3. The nth term of a G.P. with first term 'a' and common ratio 'r' is given by $a_n = a\,r^{n-1}$.
4. If a G.P. consists of m terms, then n^{th} term from the end is $(m-n+1)^{\text{th}}$ term from the beginning and is given by ar^{m-n}.

   If l is the last term of a G.P., then nth term from the end is given by $l\left(\dfrac{1}{r}\right)^{n-1}$.
5. In a G.P., the product of the terms equidistant from the beginning and the end is always same and is equal to the product of first and last term.
6. It is always convenient to select the terms of a G.P. in the following manner:

No. of terms	*Terms*	*Common ratio*
3	$\dfrac{a}{r}, a, ar$	r
4	$\dfrac{a}{r^3}, \dfrac{a}{r}, ar, ar^3$	r^2
5	$\dfrac{a}{r^2}, \dfrac{a}{r}, a, ar, ar^2$	r

7. If sum of n terms of a G.P. with first term 'a' and common ratio is given by

$$S_n = a\left(\frac{r^n - 1}{r-1}\right) \text{ or, } S_n = a\left(\frac{1-r^n}{1-r}\right), \text{ if } r \neq 1$$

$$S_n = n, \text{ if } r = 1$$

Also, $S_n = \dfrac{a - lr}{1-r}$ or, $S_n = \dfrac{lr - a}{r-1}$, where l is the last term.

8. If all the terms of G.P. be multiplied or divided by the same non-zero constant, then it remains a G.P. with the same common ratio.

9. The reciprocals of the terms of a given G.P. form a G.P.
10. If each term of a G.P. be raised to the same power the resulting sequence also forms a G.P.
11. Three numbers a, b, c are in G.P. iff $b^2 = ac$. If a, b, c are in G.P., then b is known as the geometric mean of a and c.
12. If the terms of a given G.P. are chosen at regular intervals, then the new sequence so formed also forms a G.P.
13. Let a and b be two given numbers. If n numbers $G_1, G_2, G_3, \dots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \dots, G_n, b$ is a G.P., then the numbers $G_1, G_2, G_3, \dots, G_n$ are known as n geometric means between a and b.

    The common ratio of the G.P. is given by $r = \left(\frac{b}{a}\right)^{1/n+1}$.
14. The geometric mean of a and b is given by $\sqrt{ab}$.
15. If n geometric means are inserted between two quantities, then the product of n geometric means is n^{th} power of the single geometric mean between the two quantities.
16. If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then
    (i) $A > G$
    (ii) the quadratic equation having a, b as its roots is $x^2 - 2Ax + G^2 = 0$
    (iii) $a : b = \left(A + \sqrt{A^2 - G^2}\right) : \left(A - \sqrt{A^2 - G^2}\right)$
17. If AM and GM between two numbers are in the ratio $m : n$, then the numbers are in the ratio $m + \sqrt{m^2 - n^2} : m - \sqrt{m^2 - n^2}$.