CHAPTER 25

SCALAR TRIPLE PRODUCT

25.1 INTRODUCTION

Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. By inserting dot and cross between $\vec{a}, \vec{b}$ and $\vec{c}$ in the same alphabetical order, we introduce the following products:

$$(\vec{a} \cdot \vec{b}) \cdot \vec{c}, (\vec{a} \cdot \vec{b}) \times \vec{c}, (\vec{a} \times \vec{b}) \cdot \vec{c} \text{ and } (\vec{a} \times \vec{b}) \times \vec{c}$$

In the product $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ we observe that $\vec{a} \cdot \vec{b}$ is a scalar quantity and $\vec{c}$ is a vector and dot product is defined between two vector quantities, therefore the product $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not meaningful. Similarly, the product $(\vec{a} \cdot \vec{b}) \times \vec{c}$ is not meaningful. But, $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is meaningful, because $\vec{a} \times \vec{b}$ is a vector and its dot product with $\vec{c}$ i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is a scalar quantity. This product is known as the *scalar triple product* of $\vec{a}, \vec{b}, \vec{c}$. The product $(\vec{a} \times \vec{b}) \times \vec{c}$ is also meaningful, because $\vec{a} \times \vec{b}$ is a vector and its cross-product with $\vec{c}$ i.e. $(\vec{a} \times \vec{b}) \times \vec{c}$ is also a vector. This product is known as the vector triple product of $\vec{a}, \vec{b}, \vec{c}$.

25.2 SCALAR TRIPLE PRODUCT

DEFINITION *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Then the scalar $(\vec{a} \times \vec{b}) \cdot \vec{c}$ is called the scalar triple product of $\vec{a}, \vec{b}$ and $\vec{c}$ and is denoted by $[\vec{a}\ \vec{b}\ \vec{c}]$.*

Thus, we have $[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$.

GEOMETRICAL INTERPRETATION OF SCALAR TRIPLE PRODUCT *Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors. Consider a parallelopiped having coterminous edges OA, OB and OC such that $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$. Then, $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$ as shown in Fig. 25.1. Let ϕ be the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$. If $\hat{\eta}$ is a unit vector along $\vec{a} \times \vec{b}$, then ϕ is also the angle between $\hat{\eta}$ and $\vec{c}$.*

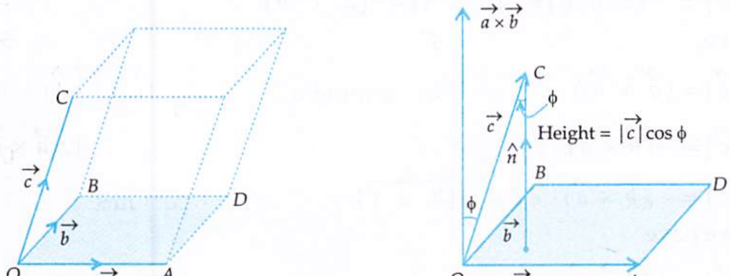

Fig. 25.1

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the parallelogram } OADB)\ \hat{\eta} \cdot \vec{c}$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the parallelogram } OADB)\ (\hat{\eta} \cdot \vec{c})$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the parallelogram } OADB)\ |\hat{\eta}|\ |\vec{c}| \cos\phi$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the parallelogram } OADB)\ (|\vec{c}| \cos\phi) \qquad [\because |\vec{\eta}| = 1]$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the parallelogram } OADB)\ (CL) \qquad [\because OC \cos\phi = CL]$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\text{Area of the base of the parallelopiped}) \times (\text{height})$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = \text{Volume of the parallelopiped with coterminous edges } \vec{a}, \vec{b}, \vec{c}$$

Thus, the scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ represents the volume of the parallelopiped whose coterminous edges $\vec{a}, \vec{b}, \vec{c}$ form a right handed system of vectors.

25.3 PROPERTIES OF SCALAR TRIPLE PRODUCT

PROPERTY I *If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same.*

i.e., $\quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

or, $\quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

PROOF Let $\vec{a}, \vec{b}, \vec{c}$ represent the coterminous edges of a parallelopiped such that they form a right handed system. Then, the volume V of the parallelopiped is given by

$$V = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

Clearly, vectors $\vec{b}, \vec{c}, \vec{a}$ as well as $\vec{c}, \vec{a}, \vec{b}$ form a right handed system of vectors and represent the coterminous edges of the same parallelopiped. Therefore,

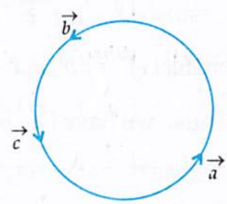

Fig. 25.2

$$\therefore \quad V = (\vec{b} \times \vec{c}) \cdot \vec{a} \text{ and } V = (\vec{c} \times \vec{a}) \cdot \vec{b}$$

Hence, $\quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

or, $\quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

PROPERTY II *The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.*

i.e. $\quad [\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

PROOF We have,

$$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -(\vec{b} \times \vec{a}) \cdot \vec{c} \qquad [\because \vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})]$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -\{(\vec{b} \times \vec{a}) \cdot \vec{c}\} = -[\vec{b}\ \vec{a}\ \vec{c}] \qquad \ldots(i)$$

By Property I, we have

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{b} \times \vec{c}) \cdot \vec{a}$ $\quad [\because [\vec{b}\ \vec{c}\ \vec{a}] = (\vec{b} \times \vec{c}) \cdot \vec{a}]$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -(\vec{c} \times \vec{b}) \cdot \vec{a}$ $\quad [\because \vec{b} \times \vec{c} = -(\vec{c} \times \vec{b})]$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -\{(\vec{c} \times \vec{b}) \cdot \vec{a}\} = -[\vec{c}\ \vec{b}\ \vec{a}]$ ...(ii)

Again, $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{c} \times \vec{a}) \cdot \vec{b}$ $\quad [\because [\vec{c}\ \vec{a}\ \vec{b}] = (\vec{c} \times \vec{a}) \cdot \vec{b}]$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -(\vec{a} \times \vec{c}) \cdot \vec{b}$ $\quad [\because \vec{c} \times \vec{a} = -(\vec{a} \times \vec{c})]$

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = -\{(\vec{a} \times \vec{c}) \cdot \vec{b}\} = -[\vec{a}\ \vec{c}\ \vec{b}]$ ...(iii)

From (i), (ii) and (iii), we obtain

$$[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}] = -[\vec{a}\ \vec{c}\ \vec{b}]$$

PROPERTY III *In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same.*

i.e., $\quad (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

PROOF We know that

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$ [By Property I]

$\Rightarrow \quad (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a}$

$\Rightarrow \quad (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$ [Using commutativity of dot product on RHS]

PROPERTY IV *The scalar triple product of three vectors is zero if any two of them are equal.*

PROOF Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors.

Case I *When* $\vec{a} = \vec{b}$*:* In this case,

$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{a} \times \vec{a}) \cdot \vec{c} = \vec{0} \cdot \vec{c} = 0$ $\quad [\because \vec{a} = \vec{b}]$

Case II *When* $\vec{b} = \vec{c}$*:* In this case,

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}]$ [By Property I]

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{b} \times \vec{b}) \cdot \vec{a} = \vec{0} \cdot \vec{a} = 0$ $\quad [\because \vec{b} = \vec{c}]$

Case III When $\vec{c} = \vec{a}$: In this case,

$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{c}\ \vec{a}\ \vec{b}]$ [By Property I]

$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{c} \times \vec{a}) \cdot \vec{b} = (\vec{a} \times \vec{a}) \cdot \vec{b} = \vec{0} \cdot \vec{b} = 0$ $\quad [\because \vec{c} = \vec{a}]$

Hence, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$, if $\vec{a} = \vec{b}$ or $\vec{b} = \vec{c}$ or $\vec{c} = \vec{a}$.

PROPERTY V *For any three vectors* $\vec{a}$, $\vec{b}$, $\vec{c}$ *and scalar* λ*, we have*

$$[\lambda \vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$$

PROOF We have,

$[\lambda \vec{a}\ \vec{b}\ \vec{c}] = (\lambda \vec{a} \times \vec{b}) \cdot \vec{c} = \lambda(\vec{a} \times \vec{b}) \cdot \vec{c}$ $\quad [\because \lambda \vec{a} \times \vec{b} = \lambda(\vec{a} \times \vec{b})]$

$\Rightarrow \quad [\lambda \vec{a}\ \vec{b}\ \vec{c}] = \lambda[(\vec{a} \times \vec{b}) \cdot \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$

PROPERTY VI *For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and any three scalars l, m, n*

$$[l\vec{a} \;\; m\vec{b} \;\; n\vec{c}] = lmn\,[\vec{a} \;\; \vec{b} \;\; \vec{c}]$$

PROOF Using definition of scalar triple product

$$[l\vec{a} \;\; m\vec{b} \;\; n\vec{c}] = (l\vec{a} \times m\vec{b}) \cdot n\vec{c} = lm(\vec{a} \times \vec{b}) \cdot n\vec{c} = lmn\{(\vec{a} \times \vec{b}) \cdot \vec{c}\} = lmn[\vec{a} \;\; \vec{b} \;\; \vec{c}]$$

PROPERTY VII *The scalar triple product of three vectors is zero if any two of them are parallel or collinear.*

PROOF *Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three vectors such that $\vec{a}$ is parallel or collinear to $\vec{b}$. Then, $\vec{a} = \lambda\vec{b}$ for some scalar λ.*

$$\therefore \quad [\vec{a}\;\vec{b}\;\vec{c}] = [\lambda\vec{b} \;\; \vec{b} \;\; \vec{c}] = \lambda[\vec{b} \;\; \vec{b} \;\; \vec{c}] = \lambda \times 0 = 0$$

PROPERTY VIII *If $\vec{a}$ $\vec{b}$ $\vec{c}$ $\vec{d}$, are four vectors, then $[\vec{a} + \vec{b} \;\; \vec{c} \;\; \vec{d}] = [\vec{a} \;\; \vec{c} \;\; \vec{d}] + [\vec{b} \;\; \vec{c} \;\; \vec{d}]$.*

PROOF We have,

$$[\vec{a} + \vec{b} \;\; \vec{c} \;\; \vec{d}] = \left\{(\vec{a} + \vec{b}) \times \vec{c}\right\} \cdot \vec{d}$$

$$= (\vec{a} \times \vec{c} + \vec{b} \times \vec{c}) \cdot \vec{d} \qquad \text{[By distributive law]}$$

$$= (\vec{a} \times \vec{c}) \cdot \vec{d} + (\vec{b} \times \vec{c}) \cdot \vec{d} \qquad \text{[By distributive law]}$$

$$= [\vec{a} \;\; \vec{c} \;\; \vec{d}] + [\vec{b} \;\; \vec{c} \;\; \vec{d}]$$

PROPERTY IX *The necessary and sufficient condition for three non-zero, non-collinear vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ to be coplanar is that $[\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$.*

i.e., $\quad \vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$

PROOF First let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero, non-collinear coplanar vectors. Then, we have to prove that their scalar triple product is zero.

We know that $\vec{a} \times \vec{b}$ is perpendicular to the plane of $\vec{a}$ and $\vec{b}$ and $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

$\therefore \quad \vec{a} \times \vec{b}$ is perpendicular to $\vec{c} \Rightarrow (\vec{a} \times \vec{b}) \cdot \vec{c} = 0 \Rightarrow [\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$

Thus, $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Rightarrow [\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$

Conversely, let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero, non-collinear vectors such that $[\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$. Then, we have to prove that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar.

Now, $\quad [\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0$

$\Rightarrow \quad (\vec{a} \times \vec{b}) \cdot \vec{c} = 0$

$\Rightarrow \quad \vec{a} \times \vec{b} = \vec{0}$ or, $\vec{c} = \vec{0}$ or, $(\vec{a} \times \vec{b}) \perp \vec{c}$

$\Rightarrow \quad (\vec{a} \times \vec{b}) \perp \vec{c} \qquad \left[\because \vec{c} \neq \vec{0} \text{ and } \vec{a} \times \vec{b} \neq 0 \text{ as } \vec{a}, \vec{b} \text{ are non-zero non-collinear vectors}\right]$

But, $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

$\therefore \quad (\vec{a} \times \vec{b}) \perp \vec{c} \Rightarrow \vec{c}$ lies in the plane of $\vec{a}$ and $\vec{b} \Rightarrow \vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar vectors

Thus, $\quad [\vec{a} \;\; \vec{b} \;\; \vec{c}] = 0 \Rightarrow \vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar vectors.

Hence, $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

PROPERTY X *(Scalar triple product in terms of components) Let* $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ *and* $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ *be three vectors. Then,*

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

PROOF We know that

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}$$

$$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = \{(a_2 b_3 - a_3 b_2)\hat{i} - (a_1 b_3 - a_3 b_1)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k}\} \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = (a_2 b_3 - a_3 b_2) c_1 - (a_1 b_3 - a_3 b_1) c_2 + (a_1 b_2 - a_2 b_1) c_3$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

ILLUSTRATION *If* $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ *and* $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, *find* $[\vec{a}\ \vec{b}\ \vec{c}]$.

SOLUTION We know that

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 2(-4-1) - 3(2+3) + 1(1-6) = -10 - 15 - 5 = -30$$

PROPERTY XI *(Distributivity of vector product over vector addition) For any three vectors* $\vec{a}, \vec{b}, \vec{c}$, *we have* $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

PROOF Let $\vec{r} = \vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}$, and let $\vec{d}$ be an arbitrary non-zero vector. Then,

$$\vec{d} \cdot \vec{r} = \vec{d} \cdot [\vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c}]$$

$$\Rightarrow \quad \vec{d} \cdot \vec{r} = \vec{d} \cdot \{\vec{a} \times (\vec{b} + \vec{c})\} - \vec{d} \cdot (\vec{a} \times \vec{b}) - \vec{d} \cdot (\vec{a} \times \vec{c}) \quad \left[\begin{array}{l}\text{By distributivity of dot} \\ \text{product over vector add.}\end{array}\right]$$

$$\Rightarrow \quad \vec{d} \cdot \vec{r} = (\vec{d} \times \vec{a}) \cdot (\vec{b} + \vec{c}) - (\vec{d} \times \vec{a}) \cdot \vec{b} - (\vec{d} \times \vec{a}) \cdot \vec{c} \quad \left[\begin{array}{l}\because \text{ In scalar triple product dot} \\ \text{and cross can be interchanged}\end{array}\right]$$

$$\Rightarrow \quad \vec{d} \cdot \vec{r} = (\vec{d} \times \vec{a}) \cdot \vec{b} + (\vec{d} \times \vec{a}) \cdot \vec{c} - (\vec{d} \times \vec{a}) \cdot \vec{b} - (\vec{d} \times \vec{a}) \cdot \vec{c} \quad \left[\begin{array}{l}\text{By distributivity of dot} \\ \text{product over vector add.}\end{array}\right]$$

$$\Rightarrow \quad \vec{d} \cdot \vec{r} = 0$$

Thus, $\vec{d} \cdot \vec{r} = 0 \Rightarrow$ either $\vec{r} = \vec{0}$ or, $\vec{d} \perp \vec{r}$

But, $\vec{d}$ is an arbitrary non-zero vector which is not necessarily perpendicular to $\vec{r}$.

$$\therefore \quad \vec{r} = \vec{0} \Rightarrow \vec{a} \times (\vec{b} + \vec{c}) - \vec{a} \times \vec{b} - \vec{a} \times \vec{c} = \vec{0} \Rightarrow \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

PROPERTY XII *If $\vec{a}, \vec{b}, \vec{c}$ are three non-coplanar vectors and $\vec{u}, \vec{v}, \vec{w}$ are three vectors such that*

$$\vec{u} = x_1 \vec{a} + y_1 \vec{b} + z_1 \vec{c},\ \vec{v} = x_2 \vec{a} + y_2 \vec{b} + z_2 \vec{c} \text{ and, } \vec{w} = x_3 \vec{a} + y_3 \vec{b} + z_3 \vec{c}$$

Then, $$[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

PROOF See Author's book on Objective Mathematics.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I ON FINDING THE SCALAR TRIPLE PRODUCT

EXAMPLE 1 *Evaluate: $[\hat{i}\ \hat{j}\ \hat{k}]$. Also, interpret it geometrically.*

SOLUTION We have, $[\hat{i}\ \hat{j}\ \hat{k}] = (\hat{i} \times \hat{j}) . \hat{k} = \hat{k} . \hat{k} = 1$

Geometrical interpretation: $[\hat{i}\ \hat{j}\ \hat{k}]$ represents the volume of a cube of edge 1 unit whose three coterminous edges are along the coordinate axes. Clearly, volume of such cube is 1 cubic unit.

$\therefore \quad [\hat{i}\ \hat{j}\ \hat{k}] = 1$

EXAMPLE 2 *Evaluate* $[\hat{i}\ \hat{j}\ \hat{k}] + [\hat{i}\ \hat{k}\ \hat{j}]$.

SOLUTION We find that

$$[\hat{i}\ \hat{j}\ \hat{k}] + [\hat{i}\ \hat{k}\ \hat{j}] = (\hat{i} \times \hat{j}) \cdot \hat{k} + (\hat{i} \times \hat{k}) \cdot \hat{j} = \hat{k} \cdot \hat{k} + (-\hat{j}) \cdot \hat{j} = \hat{k} \cdot \hat{k} - \hat{j} \cdot \hat{j} = 1 - 1 = 0$$

EXAMPLE 3 *Find $[\vec{a}\ \vec{b}\ \vec{c}]$, when $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.*

SOLUTION We find that

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 2(4-1) - (-3)(2+3) + 4(-1-6) = -7$$

Type II ON FINDING THE VOLUME OF A PARALLELOPIPED WHOSE THREE COTERMINOUS EDGES ARE GIVEN

EXAMPLE 4 *Find the volume of a parallelopiped whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$, $-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.*

SOLUTION Let $\vec{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $\vec{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$ and $\vec{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$. We know that the volume of a parallelopiped whose three adjacent edges are $\vec{a}, \vec{b}, \vec{c}$ is equal to $|[\vec{a}\ \vec{b}\ \vec{c}]|$.

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} = -3(-21-15) - 7(15+21) + 5(25-49) = -264$$

$\therefore$ Volume of the parallelopiped $= |[\vec{a}\ \vec{b}\ \vec{c}]| = |-264| = 264$ cubic units

Type III ON COPLANARITY OF THREE VECTORS

EXAMPLE 5 *Show that the vectors $\vec{a} = -2\hat{i} - 2\hat{j} + 4\hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} - 2\hat{k}$ and $\vec{c} = 4\hat{i} - 2\hat{j} - 2\hat{k}$ are coplanar.*

SOLUTION We know that three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar iff their scalar triple product is zero i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

Here, $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = -2(-8-4) + 2(4+8) + 4(4-16) = 24 + 24 - 48 = 0$

Hence, the given vectors are coplanar.

EXAMPLE 6 *Find λ so that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{c} = 3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.*

SOLUTION We know that vector $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar iff $[\vec{a}\ \vec{b}\ \vec{c}] = 0$. It is given that $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix} = 0 \Rightarrow 2(10 + 3\lambda) + 1(5+9) + (\lambda - 6) = 0 \Rightarrow 7\lambda + 28 = 0 \Rightarrow \lambda = -4$

EXAMPLE 7 *Determine α such that a vector $\vec{r}$ is at right angles to each of the vectors*

$$\vec{a} = \alpha\hat{i} + \hat{j} + 3\hat{k},\ \vec{b} = 2\hat{i} + \hat{j} - \alpha\hat{k},\ \vec{c} = -2\hat{i} + \alpha\hat{j} + 3\hat{k}$$

SOLUTION Since $\vec{r}$ is at right angles to each of the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$. Therefore, $\vec{a}$, $\vec{b}$, $\vec{c}$ must be coplanar vectors.

$\therefore \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Rightarrow \quad \begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ -2 & \alpha & 3 \end{vmatrix} = 0 \Rightarrow \alpha^3 + 11\alpha = 0 \Rightarrow \alpha(\alpha^2 + 11) = 0 \Rightarrow \alpha = 0 \qquad [\because \alpha^2 + 11 \neq 0]$

Type IV ON COPLANARITY OF FOUR POINTS

EXAMPLE 8 *Show that four points whose position vectors are $6\hat{i} - 7\hat{j}$, $16\hat{i} - 29\hat{j} - 4\hat{k}$, $3\hat{j} - 6\hat{k}$, $2\hat{i} + 5\hat{j} + 10\hat{k}$ are coplanar.*

SOLUTION Let A, B, C, D be the given points. The given points will be coplanar iff any one of the following triads of vectors are coplanar:

$$\vec{AB},\ \vec{AC},\ \vec{AD};\ \vec{AB},\ \vec{BC},\ \vec{CD};\ \vec{BC},\ \vec{BA},\ \vec{BD} \text{ etc.}$$

In order to show that $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ are coplanar, we will have to show that their scalar triple product i.e. $[\vec{AB}\ \vec{AC}\ \vec{AD}] = 0$. Using $\vec{PQ}$ = Position vector Q – Position vector of P, we obtain

$$\vec{AB} = (16\hat{i} - 29\hat{j} - 4\hat{k}) - (6\hat{i} - 7\hat{j}) = 10\hat{i} - 22\hat{j} - 4\hat{k}$$

$$\vec{AC} = (3\hat{j} - 6\hat{k}) - (6\hat{i} - 7\hat{j}) = -6\hat{i} + 10\hat{j} - 6\hat{k}$$

and, $\vec{AD} = (2\hat{i} + 5\hat{j} + 10\hat{k}) - (6\hat{i} - 7\hat{j}) = -4\hat{i} + 12\hat{j} + 10\hat{k}$

$\therefore \quad [\vec{AB}\ \vec{AC}\ \vec{AD}] = \begin{vmatrix} 10 & -22 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 10(100 + 72) + 22(-60 - 24) - 4(-72 + 40) = 0$

Hence, the given points are coplanar.

EXAMPLE 9 *Find the value of λ for which the four points with position vector $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$ are coplanar.*

SOLUTION Let A, B, C, D be the given points. Then,

$$\overrightarrow{AB} = (2\hat{i}+3\hat{j}-4\hat{k})-(3\hat{i}-2\hat{j}-\hat{k}) = -\hat{i}+5\hat{j}-3\hat{k}$$

$$\overrightarrow{AC} = (-\hat{i}+\hat{j}+2\hat{k})-(3\hat{i}-2\hat{j}-\hat{k}) = -4\hat{i}+3\hat{j}+3\hat{k}$$

and, $$\overrightarrow{AD} = (4\hat{i}+5\hat{j}+\lambda\hat{k})-(3\hat{i}-2\hat{j}-\hat{k}) = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$$

The given points are coplanar iff vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar.

Now, $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$ are coplanar

$$\Leftrightarrow \quad [\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$$

$$\Leftrightarrow \quad \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad -1(3\lambda+3-21)-5(-4\lambda-4-3)-3(-28-3) = 0$$

$$\Leftrightarrow \quad -3\lambda+18+20\lambda+35+93 = 0 \Leftrightarrow 17\lambda+146 = 0 \Rightarrow \lambda = \frac{-146}{17}$$

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type V ON PROVING RESULTS ON SCALAR TRIPLE PRODUCT

EXAMPLE 10 *For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$, prove that $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$.* [CBSE 2014]

SOLUTION We have,

$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= \{(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})\}.(\vec{c}+\vec{a})$ [By definition]

$= (\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\cdot(\vec{c}+\vec{a})$ [By distributive law]

$= (\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c})\cdot(\vec{c}+\vec{a})$ $[\because \vec{b}\times\vec{b}=\vec{0}]$

$= (\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{a}\times\vec{b})\cdot\vec{a}+(\vec{a}\times\vec{c})\cdot\vec{c}+(\vec{a}\times\vec{c})\cdot\vec{a}+(\vec{b}\times\vec{c})\cdot\vec{c}+(\vec{b}\times\vec{c})\cdot\vec{a}$

[By distributive law]

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{a}\ \vec{b}\ \vec{a}]+[\vec{a}\ \vec{c}\ \vec{c}]+[\vec{a}\ \vec{c}\ \vec{a}]+[\vec{b}\ \vec{c}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$

$= [\vec{a}\ \vec{b}\ \vec{c}]+[\vec{b}\ \vec{c}\ \vec{a}]$ [$\because$ Scalar triple product when any two vectors are equal is zero]

$= 2[\vec{a}\ \vec{b}\ \vec{c}]$ $[\because [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]]$

Hence, $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

ALITER We have

$$[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = \left[\vec{a}+\vec{b}+0\vec{c}\ \ 0\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+0\vec{b}+\vec{c}\right]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$ [By property XII]

$$= \left\{1(1-0) - 1(0-1) + 0(0-1)\right\} [\vec{a}\ \vec{b}\ \vec{c}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$$

EXAMPLE 11 *Simplify:* $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$

SOLUTION We have,

$$[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$$

$$= \{(\vec{a}-\vec{b})\times(\vec{b}-\vec{c})\}\cdot(\vec{c}-\vec{a}) \qquad \text{[By definition]}$$

$$= (\vec{a}\times\vec{b} - \vec{a}\times\vec{c} - \vec{b}\times\vec{b} + \vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a}) \qquad \text{[By distributive law]}$$

$$= (\vec{a}\times\vec{b} + \vec{c}\times\vec{a} + \vec{b}\times\vec{c})\cdot(\vec{c}-\vec{a}) \qquad [\because\ \vec{b}\times\vec{b} = \vec{0}]$$

$$= (\vec{a}\times\vec{b})\cdot\vec{c} - (\vec{a}\times\vec{b})\cdot\vec{a} + (\vec{c}\times\vec{a})\cdot\vec{c} - (\vec{c}\times\vec{a})\cdot\vec{a} + (\vec{b}\times\vec{c})\cdot\vec{c} - (\vec{b}\times\vec{c})\cdot\vec{a}$$

[By distributive law]

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{a}] + [\vec{c}\ \vec{a}\ \vec{c}] - [\vec{c}\ \vec{a}\ \vec{a}] + [\vec{b}\ \vec{c}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}] \qquad [\because \text{ Scalar triple product when any two vectors are equal is zero}]$$

$$= [\vec{a}\ \vec{b}\ \vec{c}] - [\vec{a}\ \vec{b}\ \vec{c}] = 0 \qquad [\because\ [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{a}\ \vec{b}\ \vec{c}]]$$

EXAMPLE 12 *Show that vectors* $\vec{a}, \vec{b}, \vec{c}$ *are coplanar iff* $\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ *are coplanar.* [CBSE 2013, 2014, 2016]

SOLUTION $\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Leftrightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Leftrightarrow \quad 2[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$\Leftrightarrow \quad [\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}] = 0$ [See Example 10]

$\Leftrightarrow \quad \vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}$ are coplanar

EXAMPLE 13 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$ *show that* $\vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ *are coplanar.*

SOLUTION From Example 11, we have

$[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}] = 0 \Rightarrow \vec{a}-\vec{b}, \vec{b}-\vec{c}, \vec{c}-\vec{a}$ are coplanar.

EXAMPLE 14 *If the vectors* $\vec{\alpha} = a\hat{i} + a\hat{j} + c\hat{k}$, $\vec{\beta} = \hat{i} + \hat{k}$ *and* $\vec{\gamma} = c\hat{i} + c\hat{j} + b\hat{k}$ *are coplanar, then prove that c is the geometric mean of a and b.*

SOLUTION If $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are coplanar vectors, then

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = 0$$

$$\Rightarrow \quad \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$$\Rightarrow \quad a(0-c) - a(b-c) + c(c-0) = 0$$

$$\Rightarrow \quad -ac - ab + ac + c^2 = 0 \Rightarrow c^2 = ab \Rightarrow c \text{ is the geometric mean of } a \text{ and } b.$$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 15 *For any three vectors* $\vec{a}, \vec{b}, \vec{c}$, *show that* $[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}] = 0$.

SOLUTION We have,

$$[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}]$$

$$= \{\vec{a}\times(\vec{b}+\vec{c})\}.(\vec{a}+\vec{b}+\vec{c})$$
$$= (\vec{a}\times\vec{b}+\vec{a}\times\vec{c}).(\vec{a}+\vec{b}+\vec{c})$$
$$= (\vec{a}\times\vec{b}).\vec{a}+(\vec{a}\times\vec{b}).\vec{b}+(\vec{a}\times\vec{b}).\vec{c}+(\vec{a}\times\vec{c}).\vec{a}+(\vec{a}\times\vec{c}).\vec{b}+(\vec{a}\times\vec{c}).\vec{c}$$
$$= 0+0+[\vec{a}\ \vec{b}\ \vec{c}\ ]+0+[\vec{a}\ \vec{c}\ \vec{b}]+0 = [\vec{a}\ \vec{b}\ \vec{c}]-[\vec{a}\ \vec{b}\ \vec{c}]=0$$

ALITER 1 Since $\vec{a}+\vec{b}+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ i.e. $\vec{a}+\vec{b}+\vec{c}$ is expressible as the linear combination of the other two vectors. Therefore, $\vec{a}$, $\vec{a}+\vec{b}$, $\vec{a}+\vec{b}+\vec{c}$ are coplanar vectors.

Hence, $[\vec{a}\ \ \vec{a}+\vec{b}\ \ \vec{a}+\vec{b}+\vec{c}]=0$

ALITER 2 We have,

$$[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}] = [\vec{a}+0\vec{b}+0\vec{c}\ \ 0\vec{a}+\vec{b}+\vec{c}\ \ \vec{a}+\vec{b}+\vec{c}]$$
$$= \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 1 & 1 & 1\end{vmatrix}[\vec{a}\ \vec{b}\ \vec{c}] \qquad \text{[By property XII]}$$
$$= 0\times[\vec{a}\ \vec{b}\ \vec{c}] = 0$$

EXAMPLE 16 *Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, prove that $[\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2\,|\vec{b}|^2$.*

SOLUTION Since $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Therefore, it is parallel to $\vec{a}\times\vec{b}$.

Now,

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \left\{(\vec{a}\times\vec{b}).\vec{c}\right\}^2$$
$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]^2 = \left\{|\vec{a}\times\vec{b}|\,|\vec{c}|\cos 0°\right\}^2 \qquad [\because \vec{c}\parallel(\vec{a}\times\vec{b})]$$
$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]^2 = |\vec{a}\times\vec{b}|^2\,|\vec{c}|^2$$
$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]^2 = \left\{|\vec{a}|\ |\vec{b}|\sin\frac{\pi}{6}\right\}^2 \qquad \left[\because |\vec{c}|=1 \text{ and the angle between } \vec{a} \text{ and } \vec{b} \text{ is } \pi/6\right]$$
$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2\,|\vec{b}|^2$$

EXAMPLE 17 *Prove that: $\vec{a}\cdot(\vec{b}+\vec{c})\times(\vec{a}+2\vec{b}+3\vec{c})=[\vec{a}\ \vec{b}\ \vec{c}]$.*

SOLUTION We have,

$$\vec{a}.(\vec{b}+\vec{c})\times(\vec{a}+2\vec{b}+3\vec{c})$$
$$= \vec{a}.\left\{(\vec{b}+\vec{c})\times(\vec{a}+2\vec{b}+3\vec{c})\right\}$$
$$= \vec{a}.\left\{\vec{b}\times\vec{a}+2(\vec{b}\times\vec{b})+3(\vec{b}\times\vec{c})+\vec{c}\times\vec{a}+2(\vec{c}\times\vec{b})+3(\vec{c}\times\vec{c})\right\}$$
$$= \vec{a}.\left\{\vec{b}\times\vec{a}+3(\vec{b}\times\vec{c})+\vec{c}\times\vec{a}-2(\vec{b}\times\vec{c})\right\} = \vec{a}.\left\{-(\vec{a}\times\vec{b})+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right\}$$
$$= -\vec{a}.(\vec{a}\times\vec{b})+\vec{a}.(\vec{b}\times\vec{c})+\vec{a}\cdot(\vec{c}\times\vec{a}) = 0+[\vec{a}\ \vec{b}\ \vec{c}]+0=[\vec{a}\ \vec{b}\ \vec{c}]$$

ALITER We have,

$$[\vec{a}\ \ \vec{b}+\vec{c}\ \ \vec{a}+2\vec{b}+3\vec{c}\,]$$

$$= [\vec{a}+0\vec{b}+0\vec{c}\ \ \ 0\vec{a}+\vec{b}+\vec{c}\ \ \ \vec{a}+2\vec{b}+3\vec{c}\,]$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}\,] \qquad \text{[Using property XII]}$$

$$= (3-2)\ [\vec{a}\ \vec{b}\ \vec{c}\,] = [\vec{a}\ \vec{b}\ \vec{c}\,]$$

EXAMPLE 18 *Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three non-zero vectors. If $\vec{a}\cdot(\vec{b}\times\vec{c})=0$ and $\vec{b}$ and $\vec{c}$ are not parallel, then prove that $\vec{a}=\lambda\vec{b}+\mu\vec{c}$, where λ and μ are some scalars.*

SOLUTION We have,

$$\vec{a}\cdot(\vec{b}\times\vec{c})=0 \Rightarrow \vec{a}\perp\vec{b}\times\vec{c} \qquad [\because \vec{a}\neq\vec{0} \text{ and } \vec{b}\times\vec{c}\neq\vec{0}]$$

But, $\vec{b}\times\vec{c}$ is a vector perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\therefore$ $\vec{a}\perp\vec{b}\times\vec{c}$

$\Rightarrow$ $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$

$\Rightarrow$ $\vec{a}$ can be expressed as a linear combination of $\vec{b}$ and $\vec{c}$

$\Rightarrow$ There exist scalars λ, μ such that $\vec{a}=\lambda\vec{b}+\mu\vec{c}$.

EXAMPLE 19 *If the vectors $\vec{\alpha}=a\hat{i}+\hat{j}+\hat{k}$, $\vec{\beta}=\hat{i}+b\hat{j}+\hat{k}$ and $\vec{\gamma}=\hat{i}+\hat{j}+c\hat{k}$ are coplanar, then prove that $\dfrac{1}{1-a}+\dfrac{1}{1-b}+\dfrac{1}{1-c}=1$, where $a\neq 1, b\neq 1$ and $c\neq 1$.*

SOLUTION It is given that $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ are coplanar vectors.

$$\therefore \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}\,] = 0$$

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow abc-a-c+1+1-b=0 \Rightarrow abc=a+b+c-2 \qquad \text{...(i)}$$

Now,

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\frac{(1-b)(1-c)+(1-c)(1-a)+(1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$=\frac{3-2(a+b+c)+(ab+bc+ca)}{1-(a+b+c)+(ab+bc+ca)-abc}$$

$$=\frac{3-2(a+b+c)+(ab+bc+ca)}{1-(a+b+c)+(ab+bc+ca)-(a+b+c-2)} \qquad \text{[Using (i)]}$$

$$=\frac{3-2(a+b+c)+ab+bc+ca}{3-2(a+b+c)+ab+bc+ca}=1$$

EXAMPLE 20 *If a is a non-zero real number, then prove that the vectors*

$$\vec{\alpha}=a\hat{i}+2a\hat{j}-3a\hat{k},\ \vec{\beta}=(2a+1)\hat{i}+(2a+3)\hat{j}+(a+1)\hat{k} \text{ and, } \vec{\gamma}=(3a+5)\hat{i}+(a+5)\hat{j}+(a+2)\hat{k}$$

are never coplanar.

SOLUTION We have,

$$[\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}\,]=\begin{vmatrix} a & 2a & -3a \\ 2a+1 & 2a+3 & a+1 \\ 3a+5 & a+5 & a+2 \end{vmatrix}$$

$$\Rightarrow \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = a\left\{(2a+3)(a+2)-(a+1)(a+5)\right\} - 2a\left\{(2a+1)(a+2)-(a+1)(3a+5)\right\} - 3a\left\{(2a+1)(a+5)-(2a+3)(3a+5)\right\}$$

$$\Rightarrow \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = a(2a^2+7a+6-a^2-6a-5) - 2a(2a^2+5a+2-3a^2-8a-5) - 3a(2a^2+11a+5-6a^2-19a-15)$$

$$\Rightarrow \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = a(a^2+a+1) - 2a(-a^2-3a-3) - 3a(-4a^2-8a-10)$$

$$\Rightarrow \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = a(a^2+a+1) + a(2a^2+6a+6) + a(12a^2+24a+30)$$

$$\Rightarrow \quad [\vec{\alpha}\ \vec{\beta}\ \vec{\gamma}] = a(15a^2+31a+37) = 15a\left\{\left(a+\frac{31}{30}\right)^2 + \frac{1259}{900}\right\} \neq 0 \text{ for all non-zero } a.$$

Hence, the given vectors are non-coplanar.

EXAMPLE 21 *Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non-zero non-coplanar vectors. Prove that:*

(i) $\vec{a}-2\vec{b}+3\vec{c}$, $-2\vec{a}+3\vec{b}-4\vec{c}$ *and* $\vec{a}-3\vec{b}+5\vec{c}$ *are coplanar vectors*

(ii) $2\vec{a}-\vec{b}+3\vec{c}$, $\vec{a}+\vec{b}-2\vec{c}$ *and* $\vec{a}+\vec{b}-3\vec{c}$ *are non-coplanar vectors.*

SOLUTION Since $\vec{a}$, $\vec{b}$, $\vec{c}$ are non-zero non-coplanar vectors. Therefore, $[\vec{a}\ \vec{b}\ \vec{c}] \neq 0$.

(i) Let $\vec{u} = \vec{a}-2\vec{b}+3\vec{c}$, $\vec{v} = -2\vec{a}+3\vec{b}-4\vec{c}$ and $\vec{w} = \vec{a}-3\vec{b}+5\vec{c}$. Then,

$$\vec{v}\times\vec{w} = (-2\vec{a}+3\vec{b}-4\vec{c})\times(\vec{a}-3\vec{b}+5\vec{c})$$

$$\Rightarrow \quad \vec{v}\times\vec{w} = -2(\vec{a}\times\vec{a}) + 6(\vec{a}\times\vec{b}) - 10(\vec{a}\times\vec{c}) + 3(\vec{b}\times\vec{a}) - 9(\vec{b}\times\vec{b}) + 15(\vec{b}\times\vec{c}) - 4(\vec{c}\times\vec{a}) + 12(\vec{c}\times\vec{b}) - 20(\vec{c}\times\vec{c})$$

$$\Rightarrow \quad \vec{v}\times\vec{w} = 6(\vec{a}\times\vec{b}) + 10(\vec{c}\times\vec{a}) - 3(\vec{a}\times\vec{b}) + 15(\vec{b}\times\vec{c}) - 4(c\times\vec{a}) - 12(\vec{b}\times\vec{c})$$

$$\Rightarrow \quad \vec{v}\times\vec{w} = 3(\vec{a}\times\vec{b}) + 3(\vec{b}\times\vec{c}) + 6(\vec{c}\times a)$$

$$\Rightarrow \quad \vec{v}\times\vec{w} = 3\{\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + 2(\vec{c}\times\vec{a})\}$$

$$\therefore \quad \vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{a}-2\vec{b}+3\vec{c}).\,3\left\{\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + 2(\vec{c}\times\vec{a})\right\}$$

$$\Rightarrow \quad \vec{u}\cdot(\vec{v}\times\vec{w}) = 3\left[(\vec{a}-2\vec{b}+3\vec{c})\cdot\left\{\vec{a}\times\vec{b} + \vec{b}\times\vec{c} + 2(\vec{c}\times\vec{a})\right\}\right]$$

$$\Rightarrow \quad \vec{u}\cdot(\vec{v}\times\vec{w}) = 3\left[\vec{a}\cdot(\vec{a}\times\vec{b}) + \vec{a}\cdot(\vec{b}\times\vec{c}) + 2\vec{a}\cdot(\vec{c}\times\vec{a}) - 2\vec{b}\cdot(\vec{a}\times\vec{b}) - 2\vec{b}\cdot(\vec{b}\times\vec{c}) - 4\vec{b}\cdot(\vec{c}\times\vec{a}) + 3\vec{c}\cdot(\vec{a}\times\vec{b}) + 3\vec{c}\cdot(\vec{b}\times\vec{c}) + 6\vec{c}\cdot(\vec{c}\times\vec{a})\right]$$

$$\Rightarrow \quad \vec{u}\cdot(\vec{v}\times\vec{w}) = 3\left[[\vec{a}\ \vec{b}\ \vec{c}] - 4[\vec{b}\ \vec{c}\ \vec{a}] + 3[\vec{c}\ \vec{a}\ \vec{b}]\right]$$

$$\Rightarrow \quad \vec{u}\cdot(\vec{v}\times\vec{w}) = 3\times 0 = 0 \qquad \left[\because [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]\right]$$

$$\Rightarrow \quad [\vec{u}\ \vec{v}\ \vec{w}] = 0$$

Hence, $\vec{u}$, $\vec{v}$, $\vec{w}$ are coplanar vectors.

(ii) Let $\vec{u} = 2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{v} = \vec{a} + \vec{b} - 2\vec{c}$ and $\vec{w} = \vec{a} + \vec{b} - 3\vec{c}$. Then,

$$\begin{aligned}
\vec{v} \times \vec{w} &= (\vec{a} + \vec{b} - 2\vec{c}) \times (\vec{a} + \vec{b} - 3\vec{c}) \\
&= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - 3(\vec{a} \times \vec{c}) + \vec{b} \times \vec{a} + \vec{b} \times \vec{b} - 3(\vec{b} \times \vec{c}) - 2(\vec{c} \times \vec{a}) - 2(\vec{c} \times \vec{b}) + 6(\vec{c} \times \vec{c}) \\
&= \vec{a} \times \vec{b} + 3(\vec{c} \times \vec{a}) - \vec{a} \times \vec{b} - 3(\vec{b} \times \vec{c}) - 2(\vec{c} \times \vec{a}) + 2(\vec{b} \times \vec{c}) \\
&= -\vec{b} \times \vec{c} + \vec{c} \times \vec{a}
\end{aligned}$$

$$\therefore \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = (2\vec{a} - \vec{b} + 3\vec{c}) \cdot (-\vec{b} \times \vec{c} + \vec{c} \times \vec{a})$$

$$\Rightarrow \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = -2\,\vec{a} \cdot (\vec{b} \times \vec{c}) + 2\,\vec{a} \cdot (\vec{c} \times \vec{a}) + \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{c} \times \vec{a}) - 3\,\vec{c} \cdot (\vec{b} \times \vec{c}) + 3\,\vec{c} \cdot (\vec{c} \times \vec{a})$$

$$\Rightarrow \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = -2[\vec{a}\ \vec{b}\ \vec{c}] - [\vec{b}\ \vec{c}\ \vec{a}]$$

$$\Rightarrow \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = -3[\vec{a}\ \vec{b}\ \vec{c}] \neq 0 \qquad [\because [\vec{a}\ \vec{b}\ \vec{c}] \neq 0]$$

$$\Rightarrow \quad [\vec{u}\ \vec{v}\ \vec{w}] \neq 0$$

Hence, $\vec{u}$, $\vec{v}$, $\vec{w}$ are non-coplanar vectors.

ALITER (i) Let $\vec{u} = \vec{a} - 2\vec{b} + 3\vec{c}$, $\vec{v} = -2\vec{a} + 3\vec{b} - 4\vec{c}$ and $\vec{w} = \vec{a} - 3\vec{b} + 5\vec{c}$. Then,

$$[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow \quad [\vec{u}\ \vec{v}\ \vec{w}] = [1(15-12) + 2(-10+4) + 3(6-3)]\,[\vec{a}\ \vec{b}\ \vec{c}] = 0 \times [\vec{a}\ \vec{b}\ \vec{c}] = 0$$

Hence, $\vec{u}$, $\vec{v}$, $\vec{w}$ are coplanar vectors.

(ii) Let $\vec{u} = 2\vec{a} - \vec{b} + 3\vec{c}$, $\vec{v} = \vec{a} + \vec{b} - 2\vec{c}$ and $\vec{w} = \vec{a} + \vec{b} - 3\vec{c}$. Then,

$$[\vec{u}\ \vec{v}\ \vec{w}] = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & -3 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow \quad [\vec{u}\ \vec{v}\ \vec{w}] = [2(-3+2) + 1(-3+2) + 3(1-1)]\,[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\Rightarrow \quad [\vec{u}\ \vec{v}\ \vec{w}] = -3[\vec{a}\ \vec{b}\ \vec{c}] \neq 0 \qquad [\because [\vec{a}\ \vec{b}\ \vec{c}] \neq 0]$$

$\Rightarrow$ $\vec{u}$, $\vec{v}$, $\vec{w}$ are non-coplanar vectors.

EXAMPLE 22 *If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three non-coplanar vectors, prove that*

$$\left[\vec{a} + \vec{b} + \vec{c} \quad \vec{a} + \vec{b} \quad \vec{a} + \vec{c}\right] = -[\vec{a}\ \vec{b}\ \vec{c}]$$

SOLUTION Using property XII, we have

$$\begin{aligned}
[\vec{a} + \vec{b} + \vec{c}\quad \vec{a} + \vec{b}\quad \vec{a} + \vec{c}] &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}] \\
&= \{1 \times (1-0) - 1 \times (1-0) + 1 \times (0-1)\}\,[\vec{a}\ \vec{b}\ \vec{c}] \\
&= (1-1-1)\,[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{b}\ \vec{c}]
\end{aligned}$$

EXAMPLE 23 *Find the altitude of a parallelopiped determined by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, if the base is taken as the parallelogram determined by $\vec{a}$ and $\vec{b}$, and if $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+3\hat{k}$.*

SOLUTION Let V be the volume of the parallelopiped determined by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. Then,

$$V=[\vec{a}\ \vec{b}\ \vec{c}]=\begin{vmatrix}1 & 1 & 1\\ 2 & 4 & -1\\ 1 & 1 & 3\end{vmatrix}=(12+1)-(6+1)+(2-4)=4 \text{ cubic units.} \quad ...(i)$$

Let A be the area of the base of the parallelopiped. Then, $A=|\vec{a}\times\vec{b}|$

Now,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2 & 4 & -1\end{vmatrix}=-5\hat{i}+3\hat{j}+2\hat{k} \Rightarrow A=|\vec{a}\times\vec{b}|=\sqrt{25+9+4}=\sqrt{38}$$

We know that: Volume of the parallelopiped $=$ Area of the base $\times$ Altitude

i.e. $V=A\times$ Altitude $\Rightarrow$ Altitude $=\dfrac{V}{A}=\dfrac{4}{\sqrt{38}}$ units

EXAMPLE 24 *What can you conclude about four non-zero vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$, given that $|(\vec{a}\times\vec{b})\cdot\vec{c}|+|(\vec{b}\times\vec{c})\cdot\vec{d}|=0$*

SOLUTION We have,

$$|(\vec{a}\times\vec{b})\cdot\vec{c}|+|(\vec{b}\times\vec{c})\cdot\vec{d}|=0$$

$$\Rightarrow \quad |[\vec{a}\ \vec{b}\ \vec{c}]|+|[\vec{b}\ \vec{c}\ \vec{d}]|=0$$

$$\Rightarrow \quad [\vec{a}\ \vec{b}\ \vec{c}]=0 \text{ and } [\vec{b}\ \vec{c}\ \vec{d}]=0 \qquad [\because |x|+|y|=0 \Leftrightarrow x=0, y=0]$$

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}$ and $\vec{b}, \vec{c}, \vec{d}$, are coplanar triads of vectors.

$\Rightarrow$ $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors.

EXAMPLE 25 *If $\begin{vmatrix}a & a^2 & 1+a^3\\ b & b^2 & 1+b^3\\ c & c^2 & 1+b^3\end{vmatrix}=0$ and the vectors $\vec{A}=\hat{i}+a\hat{j}+a^2\hat{k}$, $\vec{B}=\hat{i}+b\hat{j}+b^2\hat{k}$, $\vec{C}=\hat{i}+c\hat{j}+c^2\hat{k}$ are non-coplanar, then prove that $abc=-1$.*

SOLUTION It is given that $\vec{A}, \vec{B}, \vec{C}$ are non-coplanar vectors.

$$\therefore \quad [\vec{A}\ \vec{B}\ \vec{C}]\neq 0 \Rightarrow \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\neq 0 \quad ...(i)$$

Now, $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+b^3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$

$\Rightarrow \quad -\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ c & c & c^2 \end{vmatrix} = 0$ $\left[\begin{array}{l}\text{Applying } C_2 \leftrightarrow C_3 \text{ in 1st det. and taking } a, b, c \\ \text{common from } C_1, C_2, C_3 \text{ of 2nd det.}\end{array}\right]$

$\Rightarrow \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$ [Applying $C_2 \leftrightarrow C_1$ in first determinant]

$\Rightarrow \quad (1 + abc) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$\Rightarrow \quad (1 + abc) \, [\vec{A} \; \vec{B} \vec{C}] = 0$ [Using (i)]

$\Rightarrow \quad 1 + abc = 0 \Rightarrow abc = -1$ $[\because [\vec{A} \; \vec{B} \; \vec{C}] \neq 0]$

EXERCISE 25.1

BASIC

1. Evaluate the following:

(i) $[\hat{i} \; \hat{j} \; \hat{k}] + [\hat{j} \; \hat{k} \; \hat{i}] + [\hat{k} \; \hat{i} \; \hat{j}]$ (ii) $[2\hat{i} \; \hat{j} \; \hat{k}] + [\hat{i} \; \hat{k} \; \hat{j}] + [\hat{k} \; \hat{j} \; 2\hat{i}]$

2. Find $[\vec{a} \; \vec{b} \; \vec{c}]$, when

(i) $\vec{a} = 2\hat{i} - 3\hat{j}$, $\vec{b} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{k}$

(ii) $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = \hat{j} + \hat{k}$

(iii) $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$ **[CBSE 2019]**

3. Find the volume of the parallelopiped whose coterminous edges are represented by the vectors:

(i) $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

(ii) $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{c} = 3\hat{i} - \hat{j} - 2\hat{k}$

(iii) $\vec{a} = 11\hat{i}$, $\vec{b} = 2\hat{j}$, $\vec{c} = 13\hat{k}$

(iv) $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

4. Show that each of the following triads of vectors are coplanar:

(i) $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} + 7\hat{k}$, $\vec{c} = 5\hat{i} + 6\hat{j} + 5\hat{k}$

(ii) $\vec{a} = -4\hat{i} - 6\hat{j} - 2\hat{k}$, $\vec{b} = -\hat{i} + 4\hat{j} + 3\hat{k}$, $\vec{c} = -8\hat{i} - \hat{j} + 3\hat{k}$

(iii) $\hat{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\hat{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$, $\hat{c} = \hat{i} - 3\hat{j} + 5\hat{k}$

5. Find the value of λ so that the following vectors are coplanar:

(i) $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{c} = \lambda\hat{i} - \hat{j} + \lambda\hat{k}$

(ii) $\vec{a} = 2\hat{i} - \hat{j} + \hat{k},\ \vec{b} = \hat{i} + 2\hat{j} - 3\hat{k},\ \vec{c} = \lambda\hat{i} + \lambda\hat{j} + 5\hat{k}$

(iii) $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k},\ \vec{b} = 3\hat{i} + \lambda\hat{j} + \hat{k},\ \vec{c} = \hat{i} + 2\hat{j} + 2\hat{k}$

(iv) $\vec{a} = \hat{i} + 3\hat{j},\ \vec{b} = 5\hat{k},\ \vec{c} = \lambda\hat{i} - \hat{j}$

6. Show that four points whose position vectors are $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k},\ 3\hat{i} - 6\hat{k},\ 2\hat{i} - 5\hat{j} + 10\hat{k}$ are not coplanar.

7. Show that the points $A(-1, 4, -3)$, $B(3, 2, -5)$, $C(-3, 8, -5)$ and $D(-3, 2, 1)$ are coplanar.

8. Show that four points whose position vectors are $6\hat{i} - 7\hat{j}, 16\hat{i} - 19\hat{j} - 4\hat{k},\ 3\hat{i} - 6\hat{k},\ 2\hat{i} - 5\hat{j} + 10\hat{k}$ are coplanar. **[CBSE 2014]**

9. Find the value of λ for which the four points with position vectors $-\hat{j} - \hat{k},\ 4\hat{i} + 5\hat{j} + \lambda\hat{k},\ 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar.

BASED ON LOTS

10. Prove that: $(\vec{a} - \vec{b}) \cdot \{(\vec{b} - \vec{c}) \times (\vec{c} - \vec{a})\} = 0$

11. $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the position vectors of points A, B and C respectively, prove that: $\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is a vector perpendicular to the plane of triangle ABC.

12. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k},\ \vec{b} = \hat{i}$ and $\hat{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then,

(i) If $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$, and $\vec{c}$ coplanar.

(ii) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar. **[CBSE 2017]**

13. Find λ for which the points $A(3, 2, 1)$, $B(4, \lambda, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar. **[CBSE 2019]**

14. If four points A, B, C and D with position vectors $4\hat{i} + 3\hat{j} + 3\hat{k}, 5\hat{i} + x\hat{j} + 7\hat{k}, 5\hat{i} + 3\hat{j}$ and $7\hat{i} + 6\hat{j} + \hat{k}$ respectively are coplanar, then find the value of x. **[CBSE 2017]**

15. Find the volume of the parallelopiped whose adjacent edges are represented by $2\vec{a}, -\vec{b}$ and $3\vec{c}$, where $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$. **[CBSE 2020]**

ANSWERS

1. (i) 3 (ii) -1 2. (i) 4 (ii) 12 (iii) -30 3. (i) 37 (ii) 35 (iii) 286 (iv) 4

5. (i) 1 (ii) $-\frac{25}{8}$ (iii) 6 (iv) $-\frac{1}{3}$ 9. $\lambda = 1$ 12. (i) $c_3 = 2$ 13. $\lambda = 5$ 14. $x = 6$ 15. 48

HINTS TO SELECTED PROBLEMS

11. In order to prove the desired result, it is sufficient to prove that the vector

$\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}$ is perpendicular to each of the vectors $\vec{AB}$, $\vec{BC}$ and $\vec{CA}$.

i.e. $(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) \cdot \vec{AB} = 0,\ (\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) \cdot \vec{BC} = 0,$

and, $(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}) \cdot \vec{CA} = 0.$

15. Volume $= |[2\vec{a} - \vec{b}\ 3\vec{c}]| = |-6[\vec{a}\ \vec{b}\ \vec{c}]| = 6|[\vec{a}\ \vec{b}\ \vec{c}]|$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. $[\hat{i}\ \hat{k}\ \hat{j}]+[\hat{k}\ \hat{j}\ \hat{i}]+[\hat{j}\ \hat{k}\ \hat{i}]=$ ....
2. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $[\vec{a}\ \vec{b}\ \vec{c}]=10$, then $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=$.............. .
3. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $\dfrac{(\vec{b}\times\vec{c}).(\vec{a}+\vec{b}+\vec{c})}{[\vec{a}\ \vec{b}\ \vec{c}]}=$.......................... .
4. For any three vectors $\vec{a}, \vec{b}, \vec{c}$, the value of $\vec{a}.\{(\vec{b}+\vec{c})\times(\vec{a}+\vec{b}+\vec{c})\}$ is ....
5. The value of $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$, where $|\vec{a}|=1, |\vec{b}|=5$ and $|\vec{c}|=3$, is ....
6. For any two vectors $[\vec{a}\ \vec{b}\ \ \vec{a}\times\vec{b}]+(\vec{a}.\vec{b})=$........................
7. If non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ form a parallelopiped of volume 6 cubic units, then the values of $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$ are ....
8. If three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ form a parallelopiped of volume 8 cubic units, then the values of $[3\vec{a}\ \ 4\vec{b}\ \ 5\vec{c}]$ are ....
9. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then vectors $\vec{a}-\vec{b}, \vec{b}-\vec{c}$ and $\vec{c}-\vec{a}$ form a parallelopiped whose volume is ....
10. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors such that $[\vec{a}\ \vec{b}\ \vec{c}]=24$ and $|\vec{a}\times\vec{b}|=6$. Then the height of the parallelopiped formed by $\vec{a}, \vec{b}, \vec{c}$ with $\vec{a}$ and $\vec{b}$ as two adjacent edges of the base, is ....
11. If $\vec{r}\cdot\vec{a}=0$, $\vec{r}\cdot\vec{b}=0$ and $\vec{r}\cdot\vec{c}=0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a}\cdot(\vec{b}\times\vec{c})$ is .... **[NCERT EXEMPLAR]**

ANSWERS

1. -1	2. 20	3. 1	4. 0	5. 0	
6. $\lvert\vec{a}\rvert^2\lvert\vec{b}\rvert^2$	7. ± 12	8. ± 480	9. 0	10. 4	11. 0

VERY SHORT ANSWER TYPE QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the value of $[2\hat{i}\ \ 3\hat{j}\ \ 4\hat{k}]$.
2. Write the value of $[\hat{i}+\hat{j}\ \ \hat{j}+\hat{k}\ \ \hat{k}+\hat{i}]$
3. Write the value of $[\hat{i}-\hat{j}\ \ \hat{j}-\hat{k}\ \ \hat{k}-\hat{i}]$.
4. Find the values of 'a' for which the vectors $\vec{\alpha}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{\beta}=a\hat{i}+\hat{j}+2\hat{k}$ and $\vec{\gamma}=\hat{i}+2\hat{j}+a\hat{k}$ are coplanar.
5. Find the volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j}$, $\hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$.

6. If $\vec{a}, \vec{b}$ are non-collinear vectors, then find the value of $[\vec{a}\ \vec{b}\ \hat{i}]\hat{i} + [\vec{a}\ \vec{b}\ \hat{j}]\hat{j} + [\vec{a}\ \vec{b}\ \hat{k}]\hat{k}$.

7. If the vectors $(\sec^2 A)\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + (\sec^2 B)\hat{j} + \hat{k}$, $\hat{i} + \hat{j} + (\sec^2 C)\hat{k}$ are coplanar, then find the value of $\operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$.

8. For any two vectors $\vec{a}$ and $\vec{b}$ of magnitudes 3 and 4 respectively, write the value of $[\vec{a}\ \vec{b}\ \vec{a}\times\vec{b}] + (\vec{a}\cdot\vec{b})^2$.

9. If $[3\vec{a} + 7\vec{b}\ \ \vec{c}\ \ \vec{d}] = \lambda[\vec{a}\ \ \vec{c}\ \ \vec{d}] + \mu\ [\vec{b}\ \ \vec{c}\ \ \vec{d}]$, then find the value of $\lambda + \mu$.

10. If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then find the value of $\dfrac{\vec{a}\cdot(\vec{b}\times\vec{c})}{(\vec{c}\times\vec{a})\cdot\vec{b}} + \dfrac{\vec{b}\cdot(\vec{a}\times\vec{c})}{\vec{c}\cdot(\vec{a}\times\vec{b})}$

11. Find $\vec{a}\cdot(\vec{b}\times\vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\vec{i} + 2\vec{j} + \vec{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$. [CBSE 2014]

ANSWERS

1. 24 2. 2 3. 0 4. $a = 1, \dfrac{1}{2}$ 5. π cubic units 6. $\vec{a}\times\vec{b}$

7. 2 8. 144 9. 10 10. 0