CHAPTER 27

STRAIGHT LINE IN SPACE

27.1 INTRODUCTION

We know that in space a straight line is uniquely determined if either (i) coordinates of one point on it and its direction are given or (ii) coordinates of two points on it are given.

In this chapter, we shall obtain the vector and Cartesian equations of a straight line under the above conditions.

27.2 VECTOR AND CARTESIAN EQUATIONS OF A LINE

THEOREM 1 *The vector equation of a straight line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is scalar.*

PROOF Let O be the origin and let A be the fixed point with position vector $\vec{a}$. Then, $\vec{OA} = \vec{a}$. Let $\vec{r}$ be the position vector of any point P on the line drawn through A and parallel to $\vec{b}$ as indicated in Fig. 27.1. Then, $\vec{OP} = \vec{r}$. Clearly, $\vec{AP}$ is parallel to $\vec{b}$. Therefore,

$$\vec{AP} = \lambda \vec{b} \text{ for some scalar } \lambda.$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda \vec{b} \Rightarrow \vec{r} = \vec{a} + \lambda \vec{b}$$

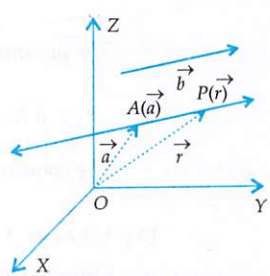

Fig. 27.1

Since every point on the line satisfies this equation and for each value of λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of a line is $\vec{r} = \vec{a} + \lambda \vec{b}$.

Q.E.D.

REMARK 1 *In the above equation $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore,* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

REMARK 2 *The position vector of any point on the line is taken as $\vec{a} + \lambda \vec{b}$.*

ILLUSTRATION *Find the vector equation of a line which passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + \hat{j} - 2\hat{k}$.*

SOLUTION Here, $\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$. So, the vector equation of the required line is

$$\vec{r} = \vec{a} + \lambda \vec{b} \quad \text{or,} \quad \vec{r} = (2\hat{i} - \hat{j} + 4\hat{k}) + \lambda(\hat{i} + \hat{j} - 2\hat{k}), \text{ where } \lambda \text{ is a scalar.}$$

THEOREM 2 *The Cartesian equations of a straight line passing through a fixed point* (x_1, y_1, z_1) *and having direction ratios proportional to a, b, c is given by*

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

PROOF We know that the vector equation of a line passing through a fixed point with position vector $\vec{a}$ and parallel to a given vector $\vec{m}$ is given by

$$\vec{r} = \vec{a} + \lambda \vec{m} \qquad \text{...(i)}$$

Here, $\vec{a} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$. Putting the values of $\vec{r}$, $\vec{a}$ and $\vec{m}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}) = (x_1 + \lambda a)\hat{i} + (y_1 + \lambda b)\hat{j} + (z_1 + \lambda c)\hat{k}$$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = x_1 + \lambda a, \; y = y_1 + \lambda b, \; z = z_1 + \lambda c \qquad \text{...(ii)}$$

Eliminating the parameter λ from (ii), we obtain the following Cartesians equation of the line

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \qquad \text{...(iii)}$$

REMARK 1 The above form of a line is known as the *symmetrical form* of a line.

Q.E.D.

REMARK 2 *The parametric equations of the line* $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ *are*

$x = x_1 + a\lambda, \; y = y_1 + b\lambda, \; z = z_1 + c\lambda,$ *where λ is the parameter.*

REMARK 3 *The coordinates of any point on the line* $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ *are*

$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, where $\lambda \in R$.

REMARK 4 *Since the direction cosines of a line are also its direction ratios. Therefore, equations of a line passing through* (x_1, y_1, z_1) *and having direction cosines l, m, n are*

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

REMARK 5 *Since x, y and z-axes pass through the origin and have direction cosines* 1, 0, 0; 0, 1, 0 *and* 0, 0, 1 *respectively. Therefore, their equations are*

x-axis : $\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ or, $y = 0$ and $z = 0$

y-axis : $\frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ or, $x = 0$ and $z = 0$

z-axis : $\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ or, $x = 0$ and $y = 0$

THEOREM 3 *The vector equation of a line passing through two points with position vectors* $\vec{a}$ *and* $\vec{b}$ *is* $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

PROOF Let O be the origin and A and B be the given points with position vectors $\vec{a}$ and $\vec{b}$ respectively. Let $\vec{r}$ be the position vector of any point P on the line passing through the points A and B. Then,

$$\vec{OP} = \vec{r},\ \vec{OA} = \vec{a} \text{ and } \vec{OB} = \vec{b}.$$

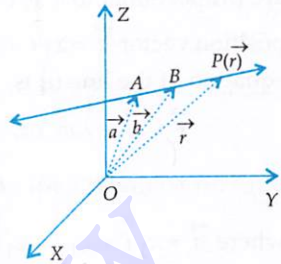

Fig. 27.2

Clearly, $\vec{AP}$ is collinear with $\vec{AB}$.

$\therefore \quad \vec{AP} = \lambda \vec{AB}$ for some scalar λ

$\Rightarrow \quad \vec{OP} - \vec{OA} = \lambda(\vec{OB} - \vec{OA})$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda(\vec{b} - \vec{a})$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$

Since every point on the line satisfies this equation for each value of λ, this equation gives the position vector of a point P on the line. Hence, the vector equation of the line is $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$. **Q.E.D.**

THEOREM 4 *The Cartesian equations of a line passing through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by*

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

PROOF We know that the vector equation of a line passing through two points with position vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \qquad \text{...(i)}$$

Here, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and, $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$. Since $\vec{r}$ is the position vector of any point $P(x, y, z)$ on the line. Therefore, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Putting the values of $\vec{r}$, $\vec{a}$ and $\vec{b}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda\{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{i} + (z_2 - z_1)\hat{k}\}$$

$\Rightarrow \quad (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k} = \lambda\{(x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}\}$

$\Rightarrow \quad x - x_1 = \lambda(x_2 - x_1),\ y - y_1 = \lambda(y_2 - y_1)$ and $z - z_1 = \lambda(z_2 - z_1)$

[On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$]

$\Rightarrow \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1} = \lambda$ [On eliminating λ]

Hence, the cartesian equations of the line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Q.E.D.

27.2.1 REDUCTION OF CARTESIAN FORM OF A LINE TO VECTOR FORM AND VICE-VERSA

The cartesian equations of a line can be reduced to vector form and vice-versa as discussed below.

CARTESIAN TO VECTOR *Let the cartesian equations of a line be*

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \quad \text{...(i)}$$

These are the equations of a line passing through the point $A(x_1, y_1, z_1)$ and its direction ratios are proportional to a, b, c. In vector form this means that the line passes through point having position vector $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and is parallel to the vector $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$. So, the vector equation of the line (i) is

$$\vec{r}=\vec{a}+\lambda\vec{m} \text{ or, } \vec{r}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k}), \text{ where } \lambda \text{ is a parameter.}$$

VECTOR TO CARTESIAN *Let the vector equation of a line be* $\vec{r}=\vec{a}+\lambda\vec{m}$ ...(ii)

where $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$, $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ and λ is a parameter.

In order to reduce equation (ii) to cartesian form we put $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and equate the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ as discussed below.

Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, $\vec{a}=x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$ and $\vec{m}=a\hat{i}+b\hat{j}+c\hat{k}$ in (ii), we obtain

$$x\hat{i}+y\hat{j}+z\hat{k}=(x_1\hat{i}+y_1\hat{j}+z_1\hat{k})+\lambda(a\hat{i}+b\hat{j}+c\hat{k})$$

On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x=x_1+a\lambda,\ y=y_1+b\lambda,\ z=z_1+c\lambda \Rightarrow \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$$

These are cartesian equations of the line.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE VECTOR EQUATION OF A LINE SATISFYING THE GIVEN CONDITIONS AND REDUCING IT TO CARTESIAN FORM**

Formulae to be used: (i) $\vec{r}=\vec{a}+\lambda\vec{b}$ (ii) $\vec{r}=\vec{a}+\lambda(\vec{b}-\vec{a})$.

EXAMPLE 1 *Find the vector equation of a line which passes through the point with position vector* $2\hat{i}-\hat{j}+4\hat{k}$ *and is in the direction of* $\hat{i}+\hat{j}-2\hat{k}$. *Also, reduce it to cartesian form.*

SOLUTION We know that the vector equation of a line passing through a point with position vector $\vec{a}$ and parallel to the vector $\vec{b}$ is $\vec{r}=\vec{a}+\lambda\vec{b}$.

Here, $\vec{a}=2\hat{i}-\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}+\hat{j}-2\hat{k}$. So, the vector equation of the required line is

$$\vec{r}=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k}) \quad \text{...(i)}$$

where λ is a parameter.

Reduction to cartesian form: Putting $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ in (i), we obtain

$$x\hat{i}+y\hat{j}+z\hat{k}=(2\hat{i}-\hat{j}+4\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k})=(2+\lambda)\hat{i}+(-1+\lambda)\hat{j}+(4-2\lambda)\hat{k}$$

On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x=2+\lambda,\ y=-1+\lambda,\ z=4-2\lambda \Rightarrow x-2=\lambda,\ y+1=\lambda,\ \frac{z-4}{-2}=\lambda$$

On eliminating λ, we obtain: $\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$

Hence, the cartesian form of equation (i) is $\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2}$.

ALITER The equation (i) represents a line passing through a point $(2, -1, 4)$ and has direction ratios proportional to $1, 1, -2$. So, the cartesian form of its equation is

$$\frac{x-2}{1} = \frac{y+1}{1} = \frac{z-4}{-2} \qquad \left[\text{Using}: \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}\right]$$

EXAMPLE 2 *Find the vector equation of the line through $A(3, 4, -7)$ and $B(1, -1, 6)$. Find also, its cartesian equations.*

SOLUTION We know that the vector equation of a line passing through the points having position vectors $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where λ is a scalar.

Here, $\vec{a} = 3\hat{i} + 4\hat{j} - 7\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + 6\hat{k}$. So, the vector equation of the line passing through $A(3, 4, -7)$ and $B(1, -1, 6)$ is

$$\vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda\left\{(\hat{i} - \hat{j} + 6\hat{k}) - (3\hat{i} + 4\hat{j} - 7\hat{k})\right\}$$

or, $$\vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k}) \qquad \text{...(i)}$$

where λ is a parameter.

Reduction to cartesian form: Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k}) = (3-2\lambda)\hat{i} + (4-5\lambda)\hat{j} + (-7+13\lambda)\hat{k}$$

On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$x = 3 - 2\lambda,\ y = 4 - 5\lambda,\ z = -7 + 13\lambda \Rightarrow \frac{x-3}{-2} = \lambda,\ \frac{y-4}{-5} = \lambda,\ \frac{z+7}{13} = \lambda$$

On eliminating λ, we obtain $\frac{x-3}{-2} = \frac{y-4}{-5} = \frac{z+7}{13}$ as the cartesian form of equation (i).

EXAMPLE 3 *The points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram ABCD. Find vector and cartesian equations for the sides AB and BC and find the coordinates of D.* **[CBSE 2010]**

SOLUTION The line AB passes through $A(4, 5, 10)$ and $B(2, 3, 4)$ having position vectors $\vec{a} = 4\hat{i} + 5\hat{j} + 10\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ respectively. So, vector equation of AB is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

or, $$\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda\left\{(2\hat{i} + 3\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + 10\hat{k})\right\}$$

or, $$\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(-2\hat{i} - 2\hat{j} - 6\hat{k})$$

or, $$\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \mu(\hat{i} + \hat{j} + 3\hat{k}),\ \text{where } \mu = -2\lambda \qquad \text{...(i)}$$

The Cartesian equations of line (i) are $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-10}{3}$

Since BC passes through the points $B(2, 3, 4)$ and $C(1, 2, -1)$ having position vector $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$ respectively. Therefore, vector equation of BC is

$$\vec{r} = \vec{b} + \lambda(\vec{c} - \vec{b})$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu\left\{(\hat{i} + 2\hat{j} - \hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})\right\}$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \mu(-\hat{i} - \hat{j} - 5\hat{k})$$

or, $$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + \nu(\hat{i} + \hat{j} + 5\hat{k}), \text{ where } \nu = -\mu \quad \text{...(ii)}$$

The Cartesian equations of line (ii) are $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{5}$

Suppose the coordinates of D are (x, y, z). Since $ABCD$ is a parallelogram, the diagonals AC and BD bisect each other. Therefore, AC and BD must have the same mid-point.

The coordinates of the mid-point of AC are $P\left(\dfrac{4+1}{2}, \dfrac{5+2}{2}, \dfrac{10-1}{2}\right)$ i.e. $P\left(\dfrac{5}{2}, \dfrac{7}{2}, \dfrac{9}{2}\right)$ and the coordinates of the mid-point of BD are $Q\left(\dfrac{2+x}{2}, \dfrac{3+y}{2}, \dfrac{4+z}{2}\right)$. Since P and Q coincide. Therefore,

$$\frac{2+x}{2} = \frac{5}{2}, \ \frac{3+y}{2} = \frac{7}{2}, \ \frac{4+z}{2} = \frac{9}{2} \Rightarrow x = 3, \ y = 4, \ z = 5.$$

Thus, the coordinates of D are (3, 4, 5).

EXAMPLE 4 *Find the vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$, and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. Also, find the cartesian equivalent of this equation.* **[CBSE 2003]**

SOLUTION Let A, B and C be the points with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$ respectively. We have to find the equation of a line passing through the point A and parallel to $\overrightarrow{BC}$. Clearly, $\overrightarrow{BC} = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}$

We know that the equation of a line passing through a point $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$.

Here, $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$. So, the vector equation of the required line is

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \quad \text{...(i)}$$

Reduction to cartesian form: Putting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in (i), we obtain

$$x\hat{i} + y\hat{j} + z\hat{k} = (2 + 2\lambda)\hat{i} + (-1 - 2\lambda)\hat{j} + (1 + \lambda)\hat{k}$$

$$\Rightarrow \quad x = 2 + 2\lambda, \ y = -1 - 2\lambda, \ z = 1 + \lambda$$

$$\Rightarrow \quad \frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1},$$ which is the cartesian equivalent of equation (i).

Type II ON FINDING THE CARTESIAN EQUATION OF A LINE SATISFYING THE GIVEN CONDITIONS AND REDUCING IT TO VECTOR FORM

Formulae to be used:

(i) $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$ (ii) $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

EXAMPLE 5 *Find the cartesian equation of a line passing through the points $A(2, -1, 3)$ and $B(4, 2, 1)$. Also, reduce it to vector form.*

SOLUTION We know that the equations of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) are given by

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

So, the equations of the required line are given by

$$\frac{x-2}{4-2} = \frac{y-(-1)}{2-(-1)} = \frac{z-3}{1-3} \text{ or, } \frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2} \quad \text{...(i)}$$

Reduction to vector form: We have,

$$\frac{x-2}{2} = \frac{y+1}{3} = \frac{z-3}{-2} = \lambda \text{ (say)} \Rightarrow x = 2\lambda + 2, y = 3\lambda - 1, z = -2\lambda + 3$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point on the line. Then,

$$\vec{r} = (2\lambda + 2)\hat{i} + (3\lambda - 1)\hat{j} + (-2\lambda + 3)\hat{k}$$

$\Rightarrow \quad \vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} - 2\hat{k})$, which is the required vector form.

EXAMPLE 6 *The cartesian equations of a line are $6x - 2 = 3y + 1 = 2z - 2$. Find its direction ratios and also find vector equation of the line.* **[CBSE 2003]**

SOLUTION Recall that in the symmetrical form of a line the coefficients of x, y and z are unity. Therefore, to put the given line in symmetric form, we must make the coefficients of x, y and z as unity.

We have,

$$6x - 2 = 3y + 1 = 2z - 2$$

$$\Rightarrow \quad 6\left(x - \frac{1}{3}\right) = 3\left(y + \frac{1}{3}\right) = 2(z-1)$$

$$\Rightarrow \quad \frac{x - \frac{1}{3}}{1} = \frac{y + \frac{1}{3}}{2} = \frac{z-1}{3} \quad \text{[Dividing throughout by the l.c.m. of 6, 3, 2 i.e. 6]}$$

This shows that the given line passes through $(1/3, -1/3, 1)$ and has direction ratios proportional to 1, 2, 3. In vector form this means that the line passes through the point having position vector $\vec{a} = \frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}$ and is parallel to the vector $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. Therefore, its vector equation is

$$\vec{r} = \left(\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} + \hat{k}\right) + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}).$$

EXAMPLE 7 *Find the direction cosines of the line $\frac{x-2}{2} = \frac{2y-5}{-3}$, $z = -1$. Also, find the vector equation of the line.*

SOLUTION The Cartesian equations of the given line are $\frac{x-2}{2} = \frac{2y-5}{-3}$, $z = -1$

These equations can be re-written as

$$\frac{x-2}{2} = \frac{2y-5}{-3} = \frac{z+1}{0} \quad \text{or,} \quad \frac{x-2}{2} = \frac{y-5/2}{-3/2} = \frac{z+1}{0}$$

This shows that the given line passes through the point $(2, 5/2, -1)$ and has direction ratios proportional to $2, -3/2, 0$. So, its direction cosines are

$$\frac{2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}, \frac{-3/2}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}, \frac{0}{\sqrt{2^2+\left(-\frac{3}{2}\right)^2+0^2}}$$

or, $\frac{2}{5/2}, \frac{-3/2}{5/2}, 0$ or, $\frac{4}{5}, -\frac{3}{5}, 0$

Thus, given line passes through the point having position vector $\vec{a} = 2\hat{i} + \frac{5}{2}\hat{j} - \hat{k}$ and is parallel to the vector $\vec{b} = 2\hat{i} - \frac{3}{2}\hat{j} + 0\hat{k}$. So, its vector equation is

$$\vec{r} = \left(2\hat{i} + \frac{5}{2}\hat{j} - \hat{k}\right) + \lambda\left(2\hat{i} - \frac{3}{2}\hat{j} + 0\hat{k}\right)$$

Type III ON CHECKING COLLINEARITY OF THREE POINTS

EXAMPLE 8 *Show that the points whose position vectors are* $5\hat{i} + 5\hat{k}$, $2\hat{i} + \hat{j} + 3\hat{k}$ *and* $-4\hat{i} + 3\hat{j} - \hat{k}$ *are collinear.*

SOLUTION Let the given points be P, Q and R and let their position vectors be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then,

$$\vec{a} = 5\hat{i} + 5\hat{k}\ \vec{b} = 2\hat{i} + \hat{j} + 3\hat{k} \text{ and } \vec{c} = -4\hat{i} + 3\hat{j} - \hat{k}$$

The vector equation of the line passing through P and Q is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) \text{ or, } \vec{r} = (5\hat{i} + 5\hat{k}) + \lambda(-3\hat{i} + \hat{j} - 2\hat{k}) \quad \text{...(i)}$$

If points P, Q and R are collinear, then point R must satisfy equation (i). Replacing $\vec{r}$ by $\vec{c} = -4\hat{i} + 3\hat{j} - \hat{k}$ in (i), we get

$$-4\hat{i} + 3\hat{j} - \hat{k} = 5\hat{i} + 5\hat{k} + \lambda(-3\hat{i} + \hat{j} - 2\hat{k})$$

$$\Rightarrow \quad -4 = 5 - 3\lambda,\ 3 = \lambda \text{ and } -1 = 5 - 2\lambda \qquad [\text{On equating coefficients of } \hat{i}, \hat{j} \text{ and } \hat{k}]$$

These three equations are consistent i.e. they give the same value of λ. Hence, points P, Q and R are collinear.

ALITER We have, $\vec{PQ} = -3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{QR} = -6\hat{i} + 2\hat{j} - 4\hat{k}$

Clearly, $2\vec{PQ} = \vec{QR} \Rightarrow \vec{PQ} \parallel \vec{QR}$. Hence, points P, Q and R are collinear.

EXAMPLE 9 *If the points* $A(-1, 3, 2)$, $B(-4, 2, -2)$ *and* $C(5, 5, \lambda)$ *are collinear, find the value of* λ.

SOLUTION The equations of the line passing through $A(-1, 3, 2)$ and $B(-4, 2, -2)$ are

$$\frac{x+1}{-4+1} = \frac{y-3}{2-3} = \frac{z-2}{-2-2} \text{ or, } \frac{x+1}{-3} = \frac{y-3}{-1} = \frac{z-2}{-4} \text{ or, } \frac{x+1}{3} = \frac{y-3}{1} = \frac{z-2}{4} \quad \text{...(i)}$$

If the points $A(-1, 3, 2)$, $B(-4, 2, -2)$ and $C(5, 5, \lambda)$ are collinear, then the coordinates of C must satisfy equation (i). Therefore,

$$\frac{5+1}{3} = \frac{5-3}{1} = \frac{\lambda-2}{4} \Rightarrow \frac{\lambda-2}{4} = 2 \Rightarrow \lambda = 10.$$

Type IV ON FINDING A POINT ON A LINE

EXAMPLE 10 *Find the point on the line* $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ *at a distance of* $3\sqrt{2}$ *from the point* $(1, 2, 3)$. **[CBSE 2008]**

SOLUTION The coordinates of any point on the line $\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}$ are given by

$$\frac{x+2}{3}=\frac{y+1}{2}=\frac{z-3}{2}=\lambda$$

or, $x+2 = 3\lambda, y+1=2\lambda, z-3=2\lambda$ or, $x=3\lambda-2, y=2\lambda-1, z=2\lambda+3$ ...(i)

So, let the coordinates of the desired point are $(3\lambda-2, 2\lambda-1, 2\lambda+3)$. The distance between this point and $(1, 2, 3)$ is $3\sqrt{2}$.

$\therefore \quad \sqrt{(3\lambda-2-1)^2+(2\lambda-1-2)^2+(2\lambda+3-3)^2} = 3\sqrt{2}$

$\Rightarrow 9(\lambda-1)^2+(2\lambda-3)^2+4\lambda^2 = 18 \Rightarrow 17\lambda^2-30\lambda=0 \Rightarrow \lambda=0, \lambda=\frac{30}{17}$

Substituting the values of λ in (i), we obtain that the coordinates of the desired point are $(-2,-1,3)$ and $\left(\frac{56}{17},\frac{43}{17},\frac{111}{17}\right)$.

EXERCISE 27.1

BASIC

1. Find the vector and cartesian equations of the line through the point $(5, 2, -4)$ and which is parallel to the vector $3\hat{i}+2\hat{j}-8\hat{k}$. [NCERT]
2. Find the vector equation of the line passing through the points $(-1, 0, 2)$ and $(3, 4, 6)$. [NCERT]
3. Find the vector equation of a line which is parallel to the vector $2\hat{i}-\hat{j}+3\hat{k}$ and which passes through the point $(5, -2, 4)$. Also, reduce it to cartesian form.
4. A line passes through the point with position vector $2\hat{i}-3\hat{j}+4\hat{k}$ and is in the direction of $3\hat{i}+4\hat{j}-5\hat{k}$. Find equations of the line in vector and cartesian form.
5. *ABCD* is a parallelogram. The position vectors of the points *A*, *B* and *C* are respectively, $4\hat{i}+5\hat{j}-10\hat{k}$, $2\hat{i}-3\hat{j}+4\hat{k}$ and $-\hat{i}+2\hat{j}+\hat{k}$. Find the vector equation of the line *BD*. Also, reduce it to cartesian form.
6. Find in vector form as well as in cartesian form, the equation of the line passing through the points $A(1, 2, -1)$ and $B(2, 1, 1)$.
7. (i) Find the vector equation for the line which passes through the point $(1, 2, 3)$ and parallel to the vector $\hat{i}-2\hat{j}+3\hat{k}$. Reduce the corresponding equation in cartesian from.
   (ii) Write the cartesian equation of the line *PQ* passing through points $P(2, 2, 1)$ and $Q(5, 1, -2)$. Hence, find the *y*-coordinates of the point on the line *PQ* whose *z*-coordinates is –2. [CBSE 2022]
8. Find the vector equation of a line passing through $(2, -1, 1)$ and parallel to the line whose equations are $\frac{x-3}{2}=\frac{y+1}{7}=\frac{z-2}{-3}$.
9. The cartesian equations of a line are $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. Find a vector equation for the line. [NCERT]
10. Find the cartesian equation of a line passing through $(1, -1, 2)$ and parallel to the line whose equations are $\frac{x-3}{1}=\frac{y-1}{2}=\frac{z+1}{-2}$. Also, reduce the equation obtained in vector form.

11. Find the direction cosines of the line $\dfrac{4-x}{2} = \dfrac{y}{6} = \dfrac{1-z}{3}$. Also, reduce it to vector form.

BASED ON LOTS

12. The cartesian equations of a line are $x = ay + b, z = cy + d$. Find its direction ratios and reduce it to vector form.

13. Find the vector equation of a line passing through the point with position vector $\hat{i} - 2\hat{j} - 3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i} - \hat{j} + 4\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$. Also, find the cartesian equivalent of this equation.

14. Find the points on the line $\dfrac{x+2}{3} = \dfrac{y+1}{2} = \dfrac{z-3}{2}$ at a distance of 5 units from the point $P(1, 3, 3)$. [CBSE 2010, 2022]

15. Show that the points whose position vectors are $-2\hat{i} + 3\hat{j}$, $\hat{i} + 2\hat{j} + 3\hat{k}$ and $7\hat{i} + 9\hat{k}$ are collinear.

16. Find the cartesian and vector equations of a line which passes through the point (1, 2, 3) and is parallel to the line $\dfrac{-x-2}{1} = \dfrac{y+3}{7} = \dfrac{2z-6}{3}$. [CBSE 2004]

17. The cartesian equations of a line are $3x + 1 = 6y - 2 = 1 - z$. Find the fixed point through which it passes, its direction ratios and also its vector equation. [CBSE 2004]

18. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line $5x - 25 = 14 - 7y = 35z$. [CBSE 2017]

ANSWERS

1. $\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$; $\dfrac{x-5}{3} = \dfrac{y-2}{2} = \dfrac{z+4}{-8}$

2. $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$

3. $\vec{r} = 5\hat{i} - 2\hat{j} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} + 3\hat{k})$; $\dfrac{x-5}{2} = \dfrac{y+2}{-1} = \dfrac{z-4}{3}$

4. $\vec{r} = 2\hat{i} - 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k})$; $\dfrac{x-2}{3} = \dfrac{y+3}{4} = \dfrac{z-4}{-5}$

5. $\vec{r} = 2\hat{i} - 3\hat{j} + 4\hat{k} + \lambda(\hat{i} - 13\hat{j} + 17\hat{k})$; $\dfrac{x-2}{1} = \dfrac{y+3}{-13} = \dfrac{z-4}{17}$

6. $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + 2\hat{k})$; $\dfrac{x-1}{1} = \dfrac{y-2}{-1} = \dfrac{z+1}{2}$

7. (i) $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 2\hat{j} + 3\hat{k})$; $\dfrac{x-1}{1} = \dfrac{y-2}{-2} = \dfrac{z-3}{3}$ (ii) 1

8. $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \lambda(2\hat{i} + 7\hat{j} - 3\hat{k})$

9. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

10. $\dfrac{x-1}{1} = \dfrac{y+1}{2} = \dfrac{z-2}{-2}$; $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$

11. $-\dfrac{2}{7}, \dfrac{6}{7}, -\dfrac{3}{7}$; $\vec{r} = (4\hat{i} + 0\hat{j} + \hat{k}) + \lambda(-2\hat{i} + 6\hat{j} - 3\hat{k})$

12. DRS : $a, 1, c$; $\vec{r} = (b\hat{i} + 0\hat{j} + d\hat{k}) + \lambda(a\hat{i} + \hat{j} + c\hat{k})$

13. $\vec{r} = (\hat{i} - 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$; $\dfrac{x-1}{1} = \dfrac{y+2}{2} = \dfrac{z+3}{-2}$

14. (4, 3, 7), (−2, −1, 3)

16. $\dfrac{x-1}{-2} = \dfrac{y-2}{14} = \dfrac{z-3}{3}$; $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(-2\hat{i} + 14\hat{j} + 3\hat{k})$

17. $\left(-\dfrac{1}{3}, \dfrac{1}{3}, 1\right)$; $2, 1, -6$; $\vec{r} = -\dfrac{1}{3}\hat{i} + \dfrac{1}{3}\hat{j} + \hat{k} + \lambda(2\hat{i} + \hat{j} - 6\hat{k})$

18. $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$

27.3 ANGLE BETWEEN TWO LINES

VECTOR FORM Let the vector equations of the two lines be $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$.

These two lines are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, angle between these two lines is equal to the angle between $\vec{b_1}$ and $\vec{b_2}$. Thus, if θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$$

Condition of perpendicularity: If the lines $\vec{b_1}$ and $\vec{b_2}$ are perpendicular. Then, $\vec{b_1}\cdot\vec{b_2} = 0$.

Condition of parallelism: If the lines are parallel, then $\vec{b_1}$ and $\vec{b_2}$ are parallel.

$\therefore \quad \vec{b_1} = \lambda\vec{b_2}$ for some scalar λ

CARTESIAN FORM *Let the cartesian equations of the two lines be*

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \quad \text{...(i)} \quad \text{and,} \quad \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \quad \text{...(ii)}$$

Direction ratios of line (i) are proportional to a_1, b_1, c_1.

$\therefore \quad \vec{m_1}$ = Vector parallel to line (i) = $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

Direction ratios of line (ii) are proportional to a_2, b_2, c_2.

$\therefore \quad \vec{m_2}$ = Vector parallel to line (ii) = $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$

Let θ be the angle between (i) and (ii). Then, θ is also the angle between $\vec{m_1}$ and $\vec{m_2}$.

$$\therefore \quad \cos\theta = \frac{\vec{m_1}\cdot\vec{m_2}}{|\vec{m_1}|\,|\vec{m_2}|} \Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of perpendicularity: If the lines are perpendicular, then

$$\vec{m_1}\cdot\vec{m_2} = 0 \Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$$

Condition of parallelism: If the lines are parallel, then $\vec{m_1}$ and $\vec{m_2}$ are parallel.

$$\therefore \quad \vec{m_1} = \lambda\vec{m_2} \text{ for some scalar } \lambda \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I **ON FINDING THE ANGLE BETWEEN TWO LINES**

Formula to be used: $\cos\theta = \dfrac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$ *or,* $\cos\theta = \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{{a_1}^2 + {b_1}^2 + {c_1}^2}\sqrt{{a_2}^2 + {b_2}^2 + {c_2}^2}}$

EXAMPLE 1 *Find the angle between the lines*

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \text{ and } \vec{r} = (5\hat{j} - 2\hat{k}) + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}).$$

SOLUTION Let θ be the angle between the given lines. The given lines are parallel to the vectors $\vec{b_1} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b_2} = 3\hat{i} + 2\hat{j} + 6\hat{k}$ respectively. So, the angle θ between them is given by

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|} = \frac{(\hat{i} + 2\hat{j} + 2\hat{k})\cdot(3\hat{i} + 2\hat{j} + 6\hat{k})}{|\hat{i} + 2\hat{j} + 2\hat{k}|\,|3\hat{i} + 2\hat{j} + 6\hat{k}|} = \frac{3+4+12}{\sqrt{1+4+4}\sqrt{9+4+36}} = \frac{19}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

EXAMPLE 2 *Find the angle between the lines*

$$\frac{x-2}{3} = \frac{y+1}{-2},\ z = 2 \text{ and } \frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}.$$

SOLUTION The given equations are not in the standard form. The equations of the given lines can be re-written as

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0} \quad ...(i) \quad \text{and,} \quad \frac{x-1}{1} = \frac{y+3/2}{3/2} = \frac{z+5}{2} \quad ...(ii)$$

Let $\vec{b_1}$ and $\vec{b_2}$ be vectors parallel to (i) and (ii) respectively. Then,

$$\vec{b_1} = 3\hat{i} - 2\hat{j} + 0\hat{k} \text{ and } \vec{b_2} = \hat{i} + \frac{3}{2}\hat{j} + 2\hat{k}.$$

If θ is the angle between the given lines, then

$$\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|} = \frac{(3)(1) + (-2)(3/2) + (0)(2)}{\sqrt{3^2 + (-2)^2 + 0^2}\sqrt{1^2 + (3/2)^2 + 2^2}} = 0 \Rightarrow \theta = \pi/2.$$

EXAMPLE 3 *Prove that the line $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$.*

SOLUTION The equations of the given lines are not in symmetrical form. Let us first write them in symmetrical form.

Equations of first line are $x = ay + b$, $z = cy + d$. These equations can be written as

$$\frac{x-b}{a} = y,\ \frac{z-d}{c} = y \text{ or, } \frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \quad ...(i)$$

Similarly equations $x = a'y + b'$, $z = c'y + d'$ can be written as

$$\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'} \quad ...(ii)$$

Lines (i) and (ii) are parallel to the vectors $\vec{m_1} = a\hat{i} + \hat{j} + c\hat{k}$ and $\vec{m_2} = a'\hat{i} + \hat{j} + c'\hat{k}$ respectively.

If lines (i) and (ii) are perpendicular, then vectors $\vec{m_1}$ and $\vec{m_2}$ are perpendicular.

$$\therefore \quad \vec{m_1}\cdot\vec{m_2} = 0 \Rightarrow (a\hat{i} + \hat{j} + c\hat{k})\cdot(a'\hat{i} + \hat{j} + c'\hat{k}) = 0 \Rightarrow aa' + 1 + cc' = 0.$$

EXAMPLE 4 *Find the angle between two lines whose direction ratios are proportional to 1, 1, 2 and $(\sqrt{3}-1), (-\sqrt{3}-1), 4$.*

SOLUTION Let $\vec{m_1}$ and $\vec{m_2}$ be vectors parallel to the two given lines. Then, angle between the two given lines is same as the angle between $\vec{m_1}$ and $\vec{m_2}$.

Now, $\vec{m_1}$ = Vector parallel to the line whose direction ratio are proportional to 1, 1, 2

and, $\vec{m_2}$ = Vector parallel to the line whose direction ratios are proportional to $\sqrt{3}-1, -\sqrt{3}-1, 4$

$\therefore \quad \vec{m}_1 = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{m}_2 = (\sqrt{3}-1)\hat{i} - (\sqrt{3}+1)\hat{j} + 4\hat{k}$

Let θ be the angle between the lines. Then,

$$\cos\theta = \frac{\vec{m}_1 \cdot \vec{m}_2}{|\vec{m}_1|\,|\vec{m}_2|} = \frac{(\sqrt{3}-1) - (\sqrt{3}+1) + 8}{\sqrt{1+1+4}\sqrt{(\sqrt{3}-1)^2 + (\sqrt{3}+1)^2 + 16}} = \frac{6}{\sqrt{6}\,\sqrt{24}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}.$$

Type II ON FINDING THE EQUATION OF A LINE PARALLEL TO A GIVEN LINE AND PASSING THROUGH A GIVEN POINT

Formulae to be used: (i) $\vec{r} = \vec{a} + \lambda \vec{b}$ (ii) $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

EXAMPLE 5 *Find the equation of a line passing through a point* $(2, -1, 3)$ *and parallel to the line* $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$.

SOLUTION The given line is parallel to the vector $2\hat{i} + \hat{j} - 2\hat{k}$ and the required line is parallel to the given line. So, required line is parallel to the vector $2\hat{i} + \hat{j} - 2\hat{k}$. It is given that the required line passes through the point $(2, -1, 3)$. So, the vector equation of the required line is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu\,(2\hat{i} + \hat{j} - 2\hat{k})$$

EXAMPLE 6 *Find the equation of a line passing through* $(1, -1, 0)$ *and parallel to the line*

$$\frac{x-2}{3} = \frac{2y+1}{2} = \frac{5-z}{1}$$

SOLUTION The equations of the given line are: $\dfrac{x-2}{3} = \dfrac{2y+1}{2} = \dfrac{5-z}{1}$.

These equations can be re-written as $\dfrac{x-2}{3} = \dfrac{y+1/2}{1} = \dfrac{z-5}{-1}$.

Clearly, direction ratios of this line are proportional to $3, 1, -1$. So, the direction ratios of the parallel line are also propcrtional to $3, 1, -1$.

The required line passes through $(1, -1, 0)$ and its direction ratios are proportional to $3, 1, -1$. So, its equations are:

$$\frac{x-1}{3} = \frac{y+1}{1} = \frac{z-0}{-1}.$$

Type III ON FINDING THE EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PERPENDICULAR TO TWO GIVEN LINES

Result to be used: A line passing through a point having position vector $\vec{\alpha}$ and perpendicular to the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\,\vec{b}_2$ is parallel to the vector $\vec{b}_1 \times \vec{b}_2$. So, its vector equation is

$$\vec{r} = \vec{\alpha} + \lambda\,(\vec{b}_1 \times \vec{b}_2)$$

Following algorithm may be used to find the equation of a line passing through a given point and perpendicular to the given lines.

ALGORITHM

Step I *Obtain the point through which the line passes. Let its position vector be* $\vec{\alpha}$.

Step II *Obtain the vectors parallel to the two given lines. Let the vectors be* $\vec{b}_1$ *and* $\vec{b}_2$.

Step III *Obtain* $\vec{b}_1 \times \vec{b}_2$

Step IV *The vector equation of the required line is* $\vec{r} = \vec{\alpha} + \lambda\,(\vec{b}_1 \times \vec{b}_2)$.

EXAMPLE 7 *Find the Cartesian equations of the line passing through the point* $(-1, 3, -2)$ *and perpendicular to the lines*

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ and } \frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$$ [CBSE 2005, 2012]

SOLUTION Let the direction ratios of the required line be proportional to a, b, c. Since it is perpendicular to the two given lines. Therefore,

$$a + 2b + 3c = 0 \quad ...(i)$$

and, $$-3a + 2b + 5c = 0 \quad ...(ii)$$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8} \text{ or, } \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = k \quad \text{(say)}$$

Thus, the required line passes through $(-1, 3, -2)$ and has direction ratios proportional to $2, -7, 4$. So, its Cartesian equations are

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4}$$

EXAMPLE 8 *A line passes through* $(2, -1, 3)$ *and is perpendicular to the line* $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ *and* $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. *Obtain its equation.* [CBSE 2012, 2014]

SOLUTION The required line is perpendicular to the lines which are parallel to vectors $\vec{b_1} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively. So, it is parallel to the vector $\vec{b} = \vec{b_1} \times \vec{b_2}$.

Now, $$\vec{b} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 6\hat{k}$$

Thus, the required line passes through the point $(2, -1, 3)$ and is parallel to the vector $\vec{b} = -6\hat{i} - 3\hat{j} + 6\hat{k}$. So, its vector equation is

$$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} + 6\hat{k}) \quad [\text{Using } \vec{r} = \vec{a} + \lambda\vec{b}]$$

or, $$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k}), \text{ where } \mu = -3\lambda.$$

Type IV ON PERPENDICULARITY OF TWO LINES

EXAMPLE 9 *Find the value of* λ *so that the lines*

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{z-3}{2} \text{ and } l_2 : \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5} \text{ are at right angle.}$$

Also, find the equations of a line passing through the point $(3, 2, -4)$ *and parallel to line* l_1.

[CBSE 2014, 2017, 2019, NCERT]

SOLUTION The equations of the given lines are

$$\frac{1-x}{3} = \frac{7y-14}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$$

These equations may be re-written in standard form as follows:

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2\lambda}{7}} = \frac{z-3}{2} \text{ and } \frac{x-1}{\frac{-3\lambda}{7}} = \frac{y-5}{7} = \frac{z-6}{-5}$$

If these lines are perpendicular, then

$$-3\times\frac{-3\lambda}{7}+\frac{2\lambda}{7}\times 1+2\times -5=0\Rightarrow\frac{9\lambda}{7}+\frac{2\lambda}{7}-10=0\Rightarrow\frac{11\lambda}{7}-10=0\Rightarrow\lambda=\frac{70}{11}.$$

The direction ratios of the l_1 are proportional to $-3, \frac{2\lambda}{7}, 2$ or, $-3, \frac{20}{11}, 2$.

So, the equations of a line passing through the point $(3, 2, -4)$ parallel to l_1 are

$$\frac{x-3}{-3}=\frac{y-2}{20/11}=\frac{z+4}{2}$$

EXERCISE 27.2

BASIC

1. Show that the three lines with direction cosines $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular. [NCERT]
2. Show that the line through the points $(1, -1, 2)$ and $(3, 4, -2)$ is perpendicular to the through the points $(0, 3, 2)$ and $(3, 5, 6)$. [NCERT]
3. Show that the line through the points $(4, 7, 8)$ and $(2, 3, 4)$ is parallel to the line through the points $(-1, -2, 1)$ and, $(1, 2, 5)$. [NCERT]
4. Find the cartesian equation of the line which passes through the point $(-2, 4, -5)$ and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$. [NCERT]
5. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other. [NCERT]
6. Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1)$ and $(4, 3, -1)$. [NCERT]
7. Find the equation of a line parallel to x-axis and passing through the origin. [NCERT]
8. Find the angle between the following pairs of lines:
   (i) $\vec{r}=(4\hat{i}-\hat{j})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$ and $\vec{r}=\hat{i}-\hat{j}+2\hat{k}-\mu(2\hat{i}+4\hat{j}-4\hat{k})$
   (ii) $\vec{r}=(3\hat{i}+2\hat{j}-4\hat{k})+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=(5\hat{j}-2\hat{k})+\mu(3\hat{i}+2\hat{j}+6\hat{k})$
   (iii) $\vec{r}=\lambda(\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=2\hat{j}+\mu\{(\sqrt{3}-1)\hat{i}-(\sqrt{3}+1)\hat{j}+4\hat{k}\}$
9. Find the angle between the following pairs of lines:
   (i) $\frac{x+4}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$
   (ii) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{-3}$ and $\frac{x+3}{-1}=\frac{y-5}{8}=\frac{z-1}{4}$
   (iii) $\frac{5-x}{-2}=\frac{y+3}{1}=\frac{1-z}{3}$ and $\frac{x}{3}=\frac{1-y}{-2}=\frac{z+5}{-1}$
   (iv) $\frac{x-2}{3}=\frac{y+3}{-2}, z=5$ and $\frac{x+1}{1}=\frac{2y-3}{3}=\frac{z-5}{2}$
   (v) $\frac{x-5}{1}=\frac{2y+6}{-2}=\frac{z-3}{1}$ and $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{5}$
   (vi) $\frac{-x+2}{-2}=\frac{y-1}{7}=\frac{z+3}{-3}$ and $\frac{x+2}{-1}=\frac{2y-8}{4}=\frac{z-5}{4}$ [CBSE 2011]

10. Find the angle between the pairs of lines with direction ratios proportional to
(i) 5, – 12, 13 and – 3, 4, 5 (ii) 2, 2, 1 and 4, 1, 8
(iii) 1, 2, – 2 and – 2, 2, 1 (iv) a, b, c and $b-c, c-a, a-b$. **[NCERT]**

11. Find the angle between two lines, one of which has direction ratios 2, 2, 1 while the other one is obtained by joining the points (3, 1, 4) and (7, 2, 12).

12. Find the equation of the line passing through the point (1, 2, – 4) and parallel to the line $\frac{x-3}{4}=\frac{y-5}{2}=\frac{z+1}{3}$.

13. Find the equations of the line passing through the point (– 1, 2, 1) and parallel to the line $\frac{2x-1}{4}=\frac{3y+5}{2}=\frac{2-z}{3}$.

14. Find the equation of the line passing through the point (2, – 1, 3) and parallel to the line $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\lambda(2\hat{i}+3\hat{j}-5\hat{k})$.

BASED ON LOTS

15. Find the equations of the line passing through the point (2, 1, 3) and perpendicular to the lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3} \text{ and } \frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$$ **[CBSE 2014]**

16. Find the equation of the line passing through the point $\hat{i}+\hat{j}-3\hat{k}$ and perpendicular to the lines $\vec{r}=\hat{i}+\lambda(2\hat{i}+\hat{j}-3\hat{k})$ and $\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\mu(\hat{i}+\hat{j}+\hat{k})$.

17. Find the equation of the line passing through the point (1, – 1, 1) and perpendicular to the lines joining the points (4, 3, 2), (1, – 1, 0) and (1, 2, – 1), (2, 1, 1).

18. Determine the equations of the line passing through the point (1, 2, – 4) and perpendicular to the two lines

$$\frac{x-8}{8}=\frac{y+9}{-16}=\frac{z-10}{7} \text{ and } \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$ **[CBSE 2012, 2016, 2017]**

19. Show that the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ are perpendicular to each other.

20. Find the vector equation of the line passing through the point (2, – 1, – 1) which is parallel to the line $6x-2=3y+1=2z-2$.

21. If the lines $\frac{x-1}{-3}=\frac{y-2}{2\lambda}=\frac{z-3}{2}$ and $\frac{x-1}{3\lambda}=\frac{y-1}{1}=\frac{z-6}{-5}$ are perpendicular, find the value of λ. **[NCERT, CBSE 2009]**

22. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD. **[NCERT]**

23. Find the value of λ so that the following lines are perpendicular to each other.

$$\frac{x-5}{5\lambda+2}=\frac{2-y}{5}=\frac{1-z}{-1},\quad \frac{x}{1}=\frac{2y+1}{4\lambda}=\frac{1-z}{-3}$$ **[CBSE 2009]**

24. Find the direction cosines of the line $\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$. Also, find the vector equation of the line through the point A (–1, 2, 3) and parallel to the given line. **[CBSE 2014]**

25. Find the value of k so that the lines $x=-y=kz$ and $x-2=2y+1=-z+1$ are perpendicular to each other. **[CBSE 2020]**

26. Find the vector and cartesian equations of the line which is perpendicular to the lines with equations $\frac{x+2}{1}=\frac{y-3}{2}=\frac{z+1}{4}$ and $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and passes through the point $(1, 1, 1)$. Also, find the angle between the given lines. **[CBSE 2020]**

ANSWERS

4. $\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6}$ **7.** $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$

8. (i) $0°$ (ii) $\cos^{-1}\left(\frac{19}{21}\right)$ (iii) $\frac{\pi}{3}$

9. (i) $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$ (ii) $\cos^{-1}\left(\frac{10}{9\sqrt{22}}\right)$ (iii) $\cos^{-1}\left(\frac{11}{14}\right)$ (iv) $\frac{\pi}{2}$ (v) $\frac{\pi}{2}$ (vi) $\frac{\pi}{2}$

10. (i) $\cos^{-1}\left(\frac{1}{65}\right)$ (ii) $\cos^{-1}\left(\frac{2}{3}\right)$ (iii) $\frac{\pi}{2}$ (iv) $\frac{\pi}{2}$

11. $\cos^{-1}\left(\frac{2}{3}\right)$ **12.** $\frac{x-1}{4}=\frac{y-2}{2}=\frac{z+4}{3}$ **13.** $\frac{x+1}{2}=\frac{y-2}{2/3}=\frac{z-1}{-3}$

14. $\vec{r}=(2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-5\hat{k})$ **15.** $\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}$

16. $\vec{r}=(\hat{i}+\hat{j}-3\hat{k})+\lambda(4\hat{i}-5\hat{j}+\hat{k})$ **17.** $\frac{x-1}{10}=\frac{y+1}{-4}=\frac{z-1}{-7}$

18. $\frac{x-1}{24}=\frac{y-2}{61}=\frac{z+4}{112}$ **20.** $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\lambda(\hat{i}+2\hat{j}+3\hat{k})$

21. $\lambda=-\frac{10}{7}$ **22.** 0 **23.** 1 **24.** $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$; $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{-6}$

25. $k=2$ **26.** $\vec{r}=\hat{i}+\hat{j}+\hat{k}+\lambda(-4\hat{i}+4\hat{j}-\hat{k})$, $\frac{x-1}{-4}=\frac{y-1}{4}=\frac{z-1}{-1}$, $\cos^{-1}\left(\frac{24}{\sqrt{609}}\right)$

HINTS TO SELECTED PROBLEMS

1. The direction cosines of three lines are

$$l_1=\frac{12}{13}, m_1=\frac{-3}{13}, n_1=\frac{-4}{13};\ l_2=\frac{4}{13}, m_2=\frac{12}{13}, n_2=\frac{3}{13}$$

$$l_3=\frac{3}{13}, m_3=\frac{-4}{13}, n_3=\frac{12}{13}$$

$$\therefore\ l_1l_2+m_1m_2+n_1n_2=\frac{48-36-12}{169}=0,\ l_2l_3+m_2m_3+n_2n_3=\frac{12-48+36}{169}=0$$

$$l_1l_3+m_1m_3+n_1n_3=\frac{36+12-48}{169}=0$$

Hence, given lines are mutually perpendicular lines.

2. The direction ratios of the line through points $A(1, -1, 2)$ and $B(3, 4, -2)$ are proportional to $a_1=2, b_1=5, c_1=-4$.

The direction ratios of the line through the points $P(0, 3, 2)$ and $Q(3, 5, 6)$ are proportional to $a_2=3, b_2=2, c_2=4$.

Clearly, $a_1a_2+b_1b_2+c_1c_2=6+10-16=0$. Hence, AB is perpendicular to PQ.

3. The direction ratios of the line through the points $P(4, 7, 8)$ and $Q(2, 3, 4)$ are proportional to $a_1=-2, b_1=-4\ c_1=-4$.

The direction ratios of the line through the points $L(-1, -2, 1)$ and $M(1, 2, 5)$ are proportional to $a_2=2, b_2=4, c_2=4$.

Clearly, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$. Hence, PQ and LM are parallel.

4. The direction ratios of $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ are proportional to 3, 5, 6. So, direction ratios of the parallel line are also proportional to 3, 5, 6. It passes through $(-2, 4, -5)$. So, its cartesian equations are $\frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6}$.

5. The direction ratios of the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are proportional to $a_1 = 7, b_1 = -5, c_1 = 1$ and $a_2 = 1, b_2 = 2, c_2 = 3$ respectively.
Let θ the be angle between these lines. Then,

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} = \frac{7 - 10 + 3}{\sqrt{49+25+1}\sqrt{1+4+9}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

6. The direction ratios of the line joining the origin $O(0, 0, 0)$ and $A(2, 1, 1)$ are proportional to $a_1 = 2, b_1 = 1, c_1 = 1$.
The line determined by the points $C(3, 5, -1)$ and $D(4, 3, -1)$ has direction ratios proportional to $a_2 = 1, b_2 = -2, c_2 = 0$.
Clearly, $a_1 a_2 + b_1 b_2 + c_1 c_2 = 2 - 2 + 0 = 0$. Hence, OA is perpendicular to CD.

7. The direction ratios of the line parallel to x-axis are proportional to 1, 0, 0. So, the equation of the line passing through the origin and parallel to x-axis is

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$

10. (iv) The direction ratios of the given lines are proportional to a, b, c and $b-c, c-a, a-b$ respectively. Let θ be the angle between the lines. Then,

$$\cos\theta = \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2+b^2+c^2}\sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

21. The equations of the given lines can be written as

$$\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2} \text{ and } \frac{x-1}{3\lambda} = \frac{y-1}{1} = \frac{z-6}{-5}$$

These lines will be perpendicular, if

$$-3 \times 3\lambda + 2\lambda + 2 \times -5 = 0 \Rightarrow -7\lambda - 10 = 0 \Rightarrow \lambda = -\frac{10}{7}$$

22. The direction ratios of AB and CD are proportional to 3, 3, 4 and 6, 6, 8 respectively.
Let θ be the angle between AB and CD. Then,

$$\cos\theta = \frac{6\times 3 + 6\times 3 + 8\times 4}{\sqrt{9+9+16}\sqrt{36+36+64}} = \frac{68}{\sqrt{34}\sqrt{136}} = 1 \Rightarrow \theta = 0°$$

27.4 INTERSECTION OF TWO LINES

The following algorithm may be used to check whether two given lines intersect or not and in case they intersect, how to find their point of intersection.

ALGORITHM

Let the two lines be

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \quad \text{...(i)} \qquad \text{and,} \qquad \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \quad \text{...(ii)}$$

Step I *Write the coordinates of general points on (i) and (ii). The coordinates of general points on (i) and (ii) are given by*

$$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}=\lambda \text{ and } \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}=\mu \text{ respectively,}$$

i.e. $(a_1\lambda+x_1, b_1\lambda+y_1, c_1\lambda+z_1)$ *and* $(a_2\mu+x_2, b_2\mu+y_2, c_2\mu+z_2)$

Step II *If the line (i) and (ii) intersect, then they have a common point.*

$$\therefore \quad a_1\lambda+x_1 = a_2\mu+x_2,\ b_1\lambda+y_1 = b_2\mu+y_2 \text{ and } c_1\lambda+z_1=c_2\mu+z_2$$

Step III *Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the lines (i) and (ii) intersect. Otherwise they do not intersect.*

Step IV *To obtain the coordinates of the point of intersection, substitute the value λ (or μ) in the coordinates of general point(s) obtained in step I.*

ALGORITHM FOR VECTOR FORM

Let the two lines be

$$\vec{r}=(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}) \quad \text{...(i)}$$

and, $$\vec{r}=(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k}) \quad \text{...(ii)}$$

Step I *Since $\vec{r}$ in the equation of a line denotes the position vector of an arbitrary point on it. Therefore, position vectors of arbitrary points on (i) and (ii) are given by*

$$(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})$$

and, $(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k})$ *respectively*

Step II *If the lines (i) and (ii) intersect, then they have a common point. So,*

$$(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})+\lambda(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})=(a'_1\hat{i}+a'_2\hat{j}+a'_3\hat{k})+\mu(b'_1\hat{i}+b'_2\hat{j}+b'_3\hat{k})$$

$$\Rightarrow \quad (a_1+\lambda b_1)\hat{i}+(a_2+\lambda b_2)\hat{j}+(a_3+\lambda b_3)\hat{k}=(a'_1+\mu b'_1)\hat{i}+(a'_2+\mu b'_2)\hat{j}+(a'_3+\mu b'_3)\hat{k}$$

$$\Rightarrow \quad a_1+\lambda b_1 = a'_1+\mu b'_1,\ a_2+\lambda b_2 = a'_2+\mu b'_2 \text{ and } a_3+\lambda b_3 = a'_3+\mu b'_3$$

Step III *Solve any two of the equations in λ and μ obtained in step II. If the values of λ and μ satisfy the third equation, then the two lines intersect. Otherwise they do not.*

Step IV *To obtain the position vector of the point of intersection, substitute the value of λ(or μ) in (i) (or (ii)).*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Show that the line* $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ *and* $\frac{x-4}{5}=\frac{y-1}{2}=z$ *intersect. Find their point of intersection.* **[CBSE 2004, 2005, NCERT EXEMPLAR]**

SOLUTION The coordinates of any point on first line are given by

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda \quad \text{(say)}$$

or, $x=2\lambda+1, y=3\lambda+2$ and $z=4\lambda+3$

So, the coordinates of a general point on first line are $(2\lambda+1, 3\lambda+2, 4\lambda+3)$.

The coordinates of any point on second line are given by

$$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1}=\mu \text{ (say) or, } x=5\mu+4, y=2\mu+1, z=\mu$$

So, the coordinates of a general point on second line are $(5\mu+4, 2\mu+1, \mu)$.

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$2\lambda+1 = 5\mu+4,\ 3\lambda+2 = 2\mu+1 \text{ and } 4\lambda+3 = \mu$$

or, $\quad 2\lambda-5\mu = 3,\ 3\lambda-2\mu = -1,\ 4\lambda-\mu = -3.$

Solving first two of these two equations, we get: $\lambda=-1$ and $\mu=-1$.

Clearly, $\lambda=-1$ and $\mu=-1$ satisfy the third equation. So, the given lines intersect.

Putting $\lambda=-1$ in $(2\lambda+1,\ 3\lambda+2,\ 4\lambda+3)$, the coordinates of the required point of intersection are $(-1,-1,-1)$.

EXAMPLE 2 *Show that the lines* $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ *and* $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ *do not intersect.*

[CBSE 2002]

SOLUTION The coordinates of any point on first line are given by

$$\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}=\lambda \text{ (say) or, } x=3\lambda+1,\ y=2\lambda-1,\ z=5\lambda+1.$$

So, the coordinates of any point on this line are: $(3\lambda+1,\ 2\lambda-1,\ 5\lambda+1)$.

The coordinates of any point on the second line are given by

$$\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}=\mu. \text{ (say) or, } x = 4\mu-2,\ y = 3\mu+1,\ z = -2\mu-1$$

So, the coordinates of any point on second line are: $(4\mu-2,\ 3\mu+1,\ -2\mu-1)$.

If the line intersect, then they have a common point. So, for some values of λ and μ, we must have

$$3\lambda+1 = 4\mu-2,\ 2\lambda-1 = 3\mu+1 \text{ and } 5\lambda+1 = -2\mu-1.$$

$$\Rightarrow \quad 3\lambda-4\mu = -3 \ \text{...(i)} \quad 2\lambda-3\mu = 2 \ \text{...(ii)} \quad \text{and,} \quad 5\lambda+2\mu = -2 \quad \text{...(iii)}$$

Solving (i) and (ii), we obtain $\lambda=-17$ and $\mu=-12$. These values of λ and μ do not satisfy the third equation. Hence, the given lines do not intersect.

EXAMPLE 3 *Show that the lines* $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ *and* $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ *intersect. Find their point of intersection.*

[CBSE 2014]

SOLUTION The position vectors of arbitrary points on the given lines are

$$(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})=(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k}$$

and, $\quad (4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})=(2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$ respectively.

If the lines intersect, then they have a common point. So, for some values of λ and μ, we must have

$$(3\lambda+1)\hat{i}+(1-\lambda)\hat{j}-\hat{k} = (2\mu+4)\hat{i}+0\hat{j}+(3\mu-1)\hat{k}$$

$\Rightarrow \quad 3\lambda+1 = 2\mu+4,\ 1-\lambda=0$ and $-1=3\mu-1$. [On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$]

Solving last two of these two equations, we get $\lambda=1$ and $\mu=0$. These values of λ and μ satisfy the third equation. So, the given lines intersect.

Putting $\lambda=1$ in first line, we get $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+(3\hat{i}-\hat{j})=4\hat{i}+0\hat{j}-\hat{k}$ as the position vector of the point of interaction.

Thus, the coordinates of the point of intersection are $(4, 0, -1)$.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 4 *Find the equations of the two lines through the origin which intersect the line* $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ *at angle of* $\frac{\pi}{3}$ *each.* **[NCERT EXEMPLAR]**

SOLUTION The coordinates of any point on the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ are given by $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}=\lambda$. So, let the coordinates of A be $(2\lambda+3, \lambda+3, \lambda)$.

Let the line through O (0, 0, 0) and making an angle $\frac{\pi}{3}$ with the given line be along OA. Then, its direction ratios are proportional to

$$2\lambda+3-0,\ \lambda+3-0,\ \lambda-0 \text{ or, } 2\lambda+3,\ \lambda+3,\ \lambda$$

The direction ratios of the given line are proportional to 2, 1, 1. It is given that the angle between the given line and the line along OA is $\frac{\pi}{3}$.

$$\therefore \quad \cos\frac{\pi}{3}=\frac{(2\lambda+3)\times 2+(\lambda+3)\times 1+\lambda\times 1}{\sqrt{(2\lambda+3)^2+(\lambda+3)^2+\lambda^2}\ \sqrt{2^2+1^2+1^2}}=\frac{6\lambda+9}{\sqrt{6\lambda^2+18\lambda+18}\ \sqrt{6}}$$

$$\Rightarrow \quad \frac{1}{2}=\frac{3\,(2\lambda+3)}{6\sqrt{\lambda^2+3\lambda+3}}$$

$$\Rightarrow \quad 2\lambda+3=\sqrt{\lambda^2+3\lambda+3}$$

$$\Rightarrow \quad (2\lambda+3)^2=\lambda^2+3\lambda+3$$

$$\Rightarrow \quad 3\lambda^2+9\lambda+6=0 \Rightarrow \lambda^2+3\lambda+2=0 \Rightarrow \lambda=-1,-2$$

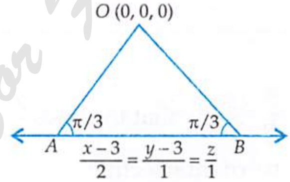

Fig. 27.3

Putting these values of λ in the coordinates of A i.e. $(2\lambda+3, \lambda+3, \lambda)$, we find the coordinates of A and B are $A\,(1, 2, -1)$ and $B\,(-1, 1, -2)$.

So, the equations of OA and OB are

$$\frac{x-0}{1-0}=\frac{y-0}{2-0}=\frac{z-0}{-1-0} \text{ and } \frac{x-0}{-1-0}=\frac{y-0}{1-0}=\frac{z-0}{-2-0} \text{ or, } \frac{x}{1}=\frac{y}{2}=\frac{z}{-1} \text{ and } \frac{x}{-1}=\frac{y}{1}=\frac{z}{-2} \text{ respectively}$$

Hence, the equations of the required lines are $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ and $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$.

EXAMPLE 5 $\vec{AB}=3\hat{i}-\hat{j}+\hat{k}$ *and* $\vec{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ *are two vectors. The position vectors of the points A and C are* $6\hat{i}+7\hat{j}+4\hat{k}$ *and* $-9\hat{j}+2\hat{k}$ *respectively. Find the position vector of a point P on the line AB and a point Q on the line CD such that* $\vec{PQ}$ *is perpendicular to* $\vec{AB}$ *and* $\vec{CD}$ *both.* **[NCERT EXEMPLAR]**

SOLUTION The equation of line AB passing through $A\,(6\hat{i}+7\hat{j}+4\hat{k})$ and parallel to $\vec{AB}=3\hat{i}-\hat{j}+\hat{k}$ is

$$\vec{r}=(6\hat{i}+7\hat{j}+4\hat{k})+\lambda\,(3\hat{i}-\hat{j}+4\hat{k}) \quad \text{...(i)}$$

The equation of line CD passing through $C\,(-9\hat{j}+2\hat{k})$ and parallel to $\vec{CD}=-3\hat{i}+2\hat{j}+4\hat{k}$ is

$$\vec{r}=(-9\hat{j}+2\hat{k})+\mu\,(-3\hat{i}+2\hat{j}+4\hat{k}) \quad \text{...(ii)}$$

The points P and Q are chosen on AB and CD respectively. So, let their position vectors be

$$(6\hat{i}+7\hat{j}+4\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})=(6+3\lambda)\hat{i}+(7-\lambda)\hat{j}+(4+\lambda)\hat{k}$$

and $(-9\hat{j}+2\hat{k})+\mu(-3\hat{i}+2\hat{j}+4\hat{k})=-3\mu\hat{i}+(2\mu-9)\hat{j}+(4\mu+2)\hat{k}$ respectively

$\therefore$ $\vec{PQ}=\{-3\mu\hat{i}+(2\mu-9)\hat{j}+(4\mu+2)\hat{k}\}-\{(6+3\lambda)\hat{i}+(7-\lambda)\hat{j}+(4+\lambda)\hat{k}\}$

$\Rightarrow$ $\vec{PQ}=(-3\mu-3\lambda-6)\hat{i}+(2\mu+\lambda-16)\hat{j}+(4\mu-\lambda-2)\hat{k}$

It is given that $\vec{PQ}$ is perpendicular to both $\vec{AB}$ and $\vec{CD}$.

$\therefore$ $\vec{PQ}\cdot\vec{AB}=0$ and $\vec{PQ}\cdot\vec{CD}=0$

$\Rightarrow$ $3(-3\mu-3\lambda-6)-(2\mu+\lambda-16)+(4\mu-\lambda-2)=0$

and $-3(-3\mu-3\lambda-6)+2(2\mu+\lambda-16)+4(4\mu-\lambda-2)=0$

$\Rightarrow$ $-7\mu-11\lambda-4=0$ and $29\mu+7\lambda-22=0 \Rightarrow \lambda=-1, \mu=1$

Substituting the values of λ and μ in the position vectors of P and Q, we find that

P.V of $P=3\hat{i}+8\hat{j}+3\hat{k}$ and P.V. of $Q=-3\hat{i}-7\hat{j}+6\hat{k}$.

EXERCISE 27.3

BASIC

1. Show that the lines $\frac{x}{1}=\frac{y-2}{2}=\frac{z+3}{3}$ and $\frac{x-2}{2}=\frac{y-6}{3}=\frac{z-3}{4}$ intersect and find their point of intersection.
2. Show that the lines $\frac{x-1}{3}=\frac{y+1}{2}=\frac{z-1}{5}$ and $\frac{x+2}{4}=\frac{y-1}{3}=\frac{z+1}{-2}$ do not intersect.
3. Show that the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$ and $\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Find their point of intersection. [CBSE 2014]
4. Prove that the lines through $A(0,-1,-1)$ and $B(4,5,1)$ intersects the line through $C(3,9,4)$ and $D(-4,4,4)$. Also, find their point of intersection. [NCERT EXEMPLAR, CBSE 2016]
5. Prove that the line $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$ intersect and find their point of intersection.
6. Determine whether the following pair of lines intersect or not:
   (i) $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+\mu(\hat{i}+\hat{j}-\hat{k})$
   (ii) $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=\frac{y-2}{1}; z=2$
   (iii) $\frac{x-1}{3}=\frac{y-1}{-1}=\frac{z+1}{0}$ and $\frac{x-4}{2}=\frac{y-0}{0}=\frac{z+1}{3}$
   (iv) $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$ [CBSE 2002]
7. Show that the lines $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$ and $\vec{r}=5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$ are intersecting. Hence, find their point of intersection. [CBSE 2013]

ANSWERS

1. $(2, 6, 3)$ 3. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$ 4. $(10, 14, 4)$ 5. $(4, 0, -1)$

6. (i) No (ii) No (iii) Yes (iv) Yes 7. $(-1, -6, -12)$

27.5 PERPENDICULAR DISTANCE OF A LINE FROM A POINT

CARTESIAN FORM *Let $P(\alpha, \beta, \gamma)$ be a given point and let $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be a given line.*

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Let the coordinates of L be $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$.

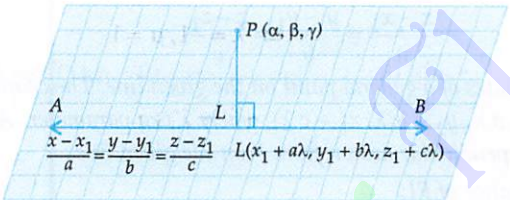

Fig. 27.4

Then, direction ratios of PL are proportional to $x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$
Direction ratio of AB are proportional to a, b, c.
Since PL is perpendicular to AB. Therefore,

$$(x_1 + a\lambda - \alpha)\, a + (y_1 + b\lambda - \beta)\, b + (z_1 + c\lambda - \gamma)\, c = 0 \Rightarrow \lambda = -\frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2}$$

Putting this value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L.
Now, using distance formula we can find the length PL.

VECTOR FORM *Let $P(\vec{\alpha})$ be a point and $\vec{r} = \vec{a} + \lambda \vec{b}$ be the vector equation of a line.*

Let L be the foot of the perpendicular drawn from $P(\vec{\alpha})$ on the line $\vec{r} = \vec{a} + \lambda \vec{b}$.

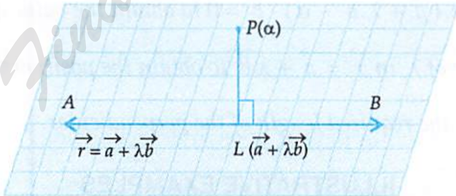

Fig. 27.5

Since $\vec{r}$ denotes the position vector of any point on the line $\vec{r} = \vec{a} + \lambda \vec{b}$. So, let the position vector of L be $\vec{a} + \lambda \vec{b}$. Then,

$$\vec{PL} = \vec{a} + \lambda \vec{b} - \vec{\alpha} = \vec{a} - \vec{\alpha} + \lambda \vec{b}$$

Since $\vec{PL}$ is perpendicular to the line which is parallel to $\vec{b}$. Therefore,

$$\vec{PL} \cdot \vec{b} = 0 \Rightarrow (\vec{a} - \vec{\alpha} + \lambda \vec{b}) \cdot \vec{b} = 0 \Rightarrow (\vec{a} - \vec{\alpha}) \cdot \vec{b} + \lambda (\vec{b} \cdot \vec{b}) = 0 \Rightarrow \lambda = -\frac{(\vec{a} - \vec{\alpha}) \cdot \vec{b}}{|\vec{b}|^2}$$

Substituting this value of λ in $\vec{a}+\lambda\vec{b}$ and $\vec{PL}=\vec{\alpha}-\vec{a}+\lambda\vec{b}$, we obtain the position vector of L and vector $\vec{PL}$. The magnitude of $\vec{PL}$ gives the length of perpendicular.

Since PL passes through P and L. Therefore, equation of the perpendicular line PL is

$$\vec{r}=\vec{\alpha}+\mu\left\{\vec{a}-\left(\frac{(\vec{a}-\alpha)\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}-\vec{\alpha}\right\} \text{ or, } \vec{r}=\vec{\alpha}+\mu\left\{(\vec{a}-\vec{\alpha})-\left(\frac{(\vec{a}-\vec{\alpha})\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}\right\}$$

In order to find the coordinates of the foot of the perpendicular and the length of the perpendicular drawn from a given point on a given line we may use the following algorithm.

ALGORITHM

CARTESIAN FORM *Let $P(\alpha,\beta,\gamma)$ be the given point, and let the given line be*

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

Step I *Write the coordinates of a general point on the given line. The coordinates of general point on the line are $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$, where λ is a parameter. Assume that this point L is the foot of the perpendicular drawn from P on the given line.*

Step II *Write direction ratios of PL.*

Step III *Apply the condition of perpendicularity of the given line and PL.*

Step IV *Obtain the value of λ from step III.*

Step V *Substitute λ in $(x_1+a\lambda, y_1+b\lambda, z_1+c\lambda)$ to obtain the coordinates of L.*

Step VI *Obtain PL by using distance formula.*

VECTOR FORM *Let $P(\vec{\alpha})$ be the given point, and let $\vec{r}=\vec{a}+\lambda\vec{b}$ be the given line.*

Step I *Write the position vector of a general point on the given line. The position vector of a general point on $\vec{r}=\vec{a}+\lambda\vec{b}$ is $\vec{a}+\lambda\vec{b}$, where λ is a parameter. Assume that this point L is required foot of the perpendicular from P on the given line.*

Step II *Obtain $\vec{PL}$ = Position vector of L – Position vector of P $=\vec{a}+\lambda\vec{b}-\vec{\alpha}$.*

Step III *Put $\vec{PL}\cdot\vec{b}=0$ i.e. $(\vec{a}+\lambda\vec{b}-\vec{\alpha})\cdot\vec{b}=0$ to obtain the value of λ.*

Step IV *Substitute the value of λ in $\vec{r}=\vec{a}+\lambda\vec{b}$ to obtain the position vector of L.*

Step V *Find $|\vec{PL}|$ to obtain the required length of the perpendicular.*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the foot of the perpendicular from the point (0, 2, 3) on the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$. Also, find the length of the perpendicular.*

SOLUTION Let L be the foot of the perpendicular drawn from the point $P(0, 2, 3)$ to the given line. The coordinates of a general point on the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$ are given by

$$\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}=\lambda$$

or, $x=5\lambda-3,\ y=2\lambda+1,\ z=3\lambda-4.$

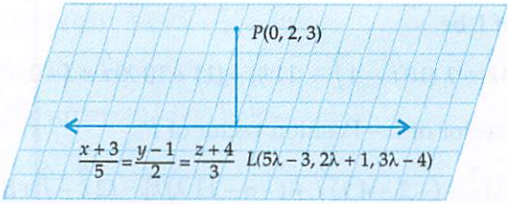

Fig. 27.6

Let the coordinates of L be $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$. Therefore, direction ratios of PL are proportional to

$$5\lambda - 3 - 0,\ 2\lambda + 1 - 2,\ 3\lambda - 4 - 3 \text{ i.e. } 5\lambda - 3,\ 2\lambda - 1,\ 3\lambda - 7.$$

Direction ratios of the given line are proportional to 5, 2, 3.

But, PL is perpendicular to the given line.

$$\therefore \quad 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0 \Rightarrow \lambda = 1$$

Putting $\lambda = 1$ in $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$, the coordinates of L are $(2, 3, -1)$.

$$\therefore \quad PL = \sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} = \sqrt{21} \text{ units.}$$

Hence, length of the perpendicular from P on the given line is $PL = \sqrt{21}$ units.

EXAMPLE 2 *Find the length of the perpendicular from the point (1, 2, 3) to the line* $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$.

SOLUTION Let L be the foot of the perpendicular drawn from the point $P(1, 2, 3)$ to the given line. The coordinated of a general point on $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ are given by

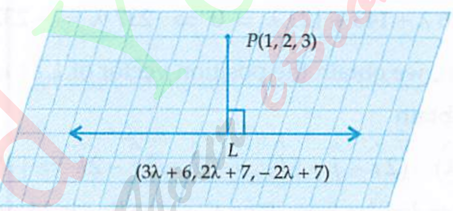

Fig. 27.7

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2} = \lambda \text{ or, } x = 3\lambda + 6,\ y = 2\lambda + 7,\ z = -2\lambda + 7$$

Let the coordinates of L be $(3\lambda + 6, 2\lambda + 7, -2\lambda + 7)$ ...(i)

The direction ratios of PL are proportional to $3\lambda + 6 - 1,\ 2\lambda + 7 - 2,\ -2\lambda + 7 - 3$

or, $3\lambda + 5, 2\lambda + 5, -2\lambda + 4$. The direction ratios of the given line are proportional to 3, 2, −2.

Since PL is perpendicular to the given line. Therefore,

$$\therefore \quad 3(3\lambda + 5) + 2(2\lambda + 5) + (-2)(-2\lambda + 4) = 0 \Rightarrow \lambda = -1.$$

Putting $\lambda = -1$ in (i), we obtain the coordinates of L as (3, 5, 9).

$$\therefore \quad PL = \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2} = 7 \text{ unit}$$

Hence, the required length of the perpendicular is 7 units.

EXAMPLE 3 *Find the foot of the perpendicular drawn from the point* $2\hat{i} - \hat{j} + 5\hat{k}$ *to the line* $\vec{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$. *Also, find the length of the perpendicular.*

SOLUTION Let L be the foot of the perpendicular drawn from $P(2\hat{i} - \hat{j} + 5\hat{k})$ on the line

$$\vec{r} = 11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}).$$

Let the position vector of L be

$$\vec{r} = 11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}) = (11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k} \quad \text{...(i)}$$

Then, $\vec{PL}$ = Position vector of L – Position vector of P

$$= \{(11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}\} - (2\hat{i} - \hat{j} + 5\hat{k})$$

$$= (9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k} \quad \text{...(ii)}$$

Since $\vec{PL}$ is perpendicular to the given line which is parallel to $\vec{b} = 10\hat{i} - 4\hat{j} - 11\hat{k}$.

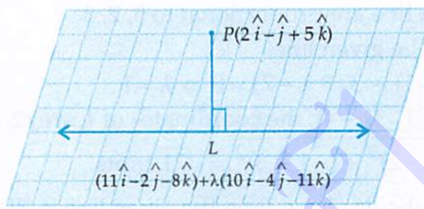

Fig. 27.8

$\therefore \quad \vec{PL} \cdot \vec{b} = 0$

$\Rightarrow \quad \left\{(9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k}\right\} \cdot (10\hat{i} - 4\hat{j} - 11\hat{k}) = 0$

$\Rightarrow \quad 10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) = 0$

$\Rightarrow \quad 90 + 100\lambda + 4 + 16\lambda + 143 + 121\lambda = 0 \Rightarrow 237\lambda = -237 \Rightarrow \lambda = -1$

Putting the value of λ in (i), we obtain the position vector of L as $\hat{i} + 2\hat{j} + 3\hat{k}$.

Putting $\lambda = -1$ in (ii), we obtain

$$\vec{PL} = (\hat{i} + 2\hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} + 5\hat{k}) = -\hat{i} + 3\hat{j} - 2\hat{k} \Rightarrow |\vec{PL}| = \sqrt{1 + 9 + 4} = \sqrt{14}.$$

Hence, length of the perpendicular from P on the give line is 14 units.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the image of the point (1, 6, 3) in the line* $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. *Also, write the equation of the line joining the given point and its image and find the length of the segment joining the given point and its image.* **[CBSE 2010, NCERT EXEMPLAR]**

SOLUTION Let Q be the image of point P (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ and M be the foot of perpendicular drawn from P to this line. Then, $PM = MQ$.

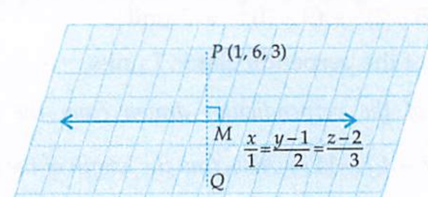

Fig. 27.9

Let the coordinates of M be given by $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r$. Let the coordinates of M be $(r, 2r+1, 3r+2)$.

The direction ratios of PM are proportional to $r-1, 2r-5, 3r-1$.

Since PM is perpendicular to the given line. Therefore,

$$1(r-1) + 2(2r-5) + 3(3r-1) = 0 \Rightarrow 14r - 14 = 0 \Rightarrow r = 1.$$

So, the coordinates of M are $(1, 3, 5)$.

Let (x_1, y_1, z_1) be the coordinates of Q. Since M is the mid-point of PQ.

$$\therefore \quad \frac{x_1+1}{2} = 1, \frac{y_1+6}{2} = 3, \frac{z_1+3}{2} = 5 \Rightarrow x_1 = 1, y_1 = 0, z_1 = 7$$

Thus, the coordinates of Q are $(1, 0, 7)$. So, the cartesian equations of PQ are

$$\frac{x-1}{1-1} = \frac{y-6}{0-6} = \frac{z-3}{7-3} \text{ or, } \frac{x-1}{0} = \frac{y-6}{-6} = \frac{z-3}{4}$$

and, $\quad PQ = \sqrt{(1-1)^2 + (6-0)^2 + (3-7)^2} = 2\sqrt{13}$

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

EXAMPLE 5 *Show that the distance d from point P to the line l having equation* $\vec{r} = \vec{a} + \lambda \vec{b}$ *is given by*

$$d = \frac{|\vec{b} \times \vec{PQ}|}{|\vec{b}|}, \text{ where } Q \text{ is any point on the line } l.$$

SOLUTION Let PM be perpendicular from P to line l and Q be pint on it such that PQ makes an angle θ with l.

In right triangle PMQ, we obtain

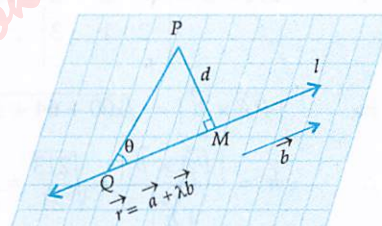

Fig. 27.10

$$\sin\theta = \frac{PM}{PQ}$$

$$\Rightarrow \quad \sin\theta = \frac{d}{PQ}$$

$$\Rightarrow \quad d = PQ \sin\theta$$

$$\Rightarrow \quad d|\vec{b}| = |\vec{PQ}|\,|\vec{b}| \sin\theta \qquad \text{[Multiplying both sides by } |\vec{b}|]$$

$$\Rightarrow \quad d|\vec{b}| = |\vec{b} \times \vec{PQ}| \qquad \left[\because \vec{b} \text{ is parallel to line } l. \text{ So, angle between } \vec{b} \text{ and } \vec{PQ} \text{ is also } \theta\right]$$

$$\Rightarrow \quad d = \frac{|\vec{b} \times \vec{PQ}|}{|\vec{b}|}$$

EXAMPLE 6 *Find the distance from the point P* $(3, -8, 1)$ *to the line* $\frac{x-3}{3} = \frac{y+7}{-1} = \frac{z+2}{5}$ *by using the formula derived in Example 5.*

SOLUTION In order to find the required distance we need to find a point Q on the line. We see that the line passes through the point $Q(3, -7, -2)$. So, let us take this point as the required point. Also, line is parallel to the vector $\vec{b} = 3\hat{i} - \hat{j} + 5\hat{k}$.

Now, $\quad \vec{PQ} = (3\hat{i} - 7\hat{j} - 2\hat{k}) - (3\hat{i} - 8\hat{j} + \hat{k}) = \hat{j} - 3\hat{k}$

$$\therefore \quad \vec{b} \times \vec{PQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 5 \\ 0 & 1 & -3 \end{vmatrix} = 2\hat{i} - 9\hat{j} - 3\hat{k} \Rightarrow |\vec{b} \times \vec{PQ}| = \sqrt{4+81+9} = \sqrt{94}$$

Hence, $d = \dfrac{|\vec{b} \times \vec{PQ}|}{|\vec{b}|} = \dfrac{\sqrt{94}}{\sqrt{9+1+25}} = \sqrt{\dfrac{94}{35}}$

EXAMPLE 7 *Vertices B and C of ΔABC lie along the line $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4}$. Find the area of the triangle given that A has coordinates $(1, -1, 2)$ and line segment BC has length 5.*

SOLUTION Clearly, height h of ΔABC is the length of perpendicular from $A\ (1, -1, 2)$ to the line $\dfrac{x+2}{2} = \dfrac{y-1}{1} = \dfrac{z-0}{4}$ which passes through $P\ (-2, 1, 0)$ and is parallel to $\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$.

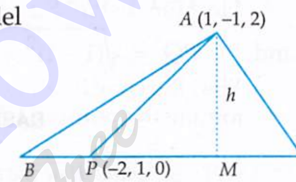

Fig. 27.11

$$\therefore \quad h = \frac{|\vec{PA} \times \vec{b}|}{|\vec{b}|}$$

Now, $\vec{PA} = -3\hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 4\hat{k}$

$$\therefore \quad \vec{PA} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & -2 \\ 2 & 1 & 3 \end{vmatrix} = 10\hat{i} + 8\hat{j} - 7\hat{k} \text{ and } |\vec{b}| = \sqrt{4+1+16} = \sqrt{21}$$

$$\Rightarrow \quad |\vec{PA} \times \vec{b}| = \sqrt{100+64+49} = 213$$

$$\therefore \quad h = \frac{|\vec{PA} \times \vec{b}|}{|\vec{b}|} = \sqrt{\frac{213}{21}} = \sqrt{\frac{71}{7}}$$

It is given that the length of BC is 5 units.

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}(BC \times h) = \frac{1}{2} \times 5 \times \sqrt{\frac{71}{7}} = \sqrt{\frac{1775}{28}} \text{ sq units.}$$

EXERCISE 27.4

BASED ON LOTS

1. Find the perpendicular distance of the point (1, 0, 0) from the line $\dfrac{x-1}{2} = \dfrac{y+1}{-3} = \dfrac{z+10}{8}$. Also, find the coordinates of the foot of the perpendicular and the equation of the perpendicular. **[CBSE 2005, 2011]**
2. $A\ (1, 0, 4)$, $B\ (0, -11, 3)$, $C\ (2, -3, 1)$ are three points and D is the foot of perpendicular from A on BC. Find the coordinates of D.
3. Find the foot of perpendicular from the point (2, 3, 4) to the line $\dfrac{4-x}{2} = \dfrac{y}{6} = \dfrac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line. **[NCERT EXEMPLAR]**
4. Find the equation of the perpendicular drawn from the point $P\ (2, 4, -1)$ to the line $\dfrac{x+5}{1} = \dfrac{y+3}{4} = \dfrac{z-6}{-9}$.

   Also, write down the coordinates of the foot of the perpendicular from P.

5. Find the length of the perpendicular drawn from the point (5, 4, −1) to the line $\vec{r} = \hat{i} + \lambda(2\hat{i} + 9\hat{j} + 5\hat{k})$.

6. Find the foot of the perpendicular drawn from the point $\hat{i} + 6\hat{j} + 3\hat{k}$ to the line $\vec{r} = \hat{j} + 2\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k})$. Also, find the length of the perpendicular

7. Find the equation of the perpendicular drawn from the point $P(-1, 3, 2)$ to the line $\vec{r} = (2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} + 3\hat{k})$. Also, find the coordinates of the foot of the perpendicular from P.

8. Find the equation of line passing through the points $A(0, 6, -9)$ and $B(-3, -6, 3)$. If D is the foot of perpendicular drawn from a point $C(7, 4, -1)$ on the line AB, then find the coordinates of the point D and the equation of line CD. **[CBSE 2010]**

9. Find the distance of the point (2, 4, −1) from the line $\dfrac{x+5}{1} = \dfrac{y+3}{4} = \dfrac{z-6}{-9}$. **[NCERT EXEMPLAR]**

10. Find the coordinates of the foot of perpendicular drawn from the point $A(1, 8, 4)$ to the line joining the points $B(0, -1, 3)$ and $C(2, -3, -1)$. **[NCERT EXEMPLAR]**

ANSWERS

1. $2\sqrt{6}, (3, -4, -2), \vec{r} = \hat{i} + \lambda(\hat{i} - 2\hat{j} - \hat{k})$
2. $\left(\dfrac{22}{9}, -\dfrac{11}{9}, \dfrac{5}{9}\right)$
3. $\left(\dfrac{170}{49}, \dfrac{78}{49}, \dfrac{10}{49}\right), 3\sqrt{\dfrac{101}{49}}$
4. $(-4, 1, -3); \dfrac{x-2}{-6} = \dfrac{y-4}{-3} = \dfrac{z+1}{-2}$
5. $\sqrt{\dfrac{2109}{110}}$
6. $\hat{i} + 3\hat{j} + 5\hat{k}, \sqrt{13}$
7. $\vec{r} = (-\hat{i} + 3\hat{j} + 2\hat{k}) + \lambda(3\hat{i} - 9\hat{j} + \hat{k})$; $(-4/7, 12/7, 15/7)$
8. $\dfrac{x}{1} = \dfrac{y-6}{4} = \dfrac{z+9}{-4}; D(-1, 2, -5); \dfrac{x-7}{4} = \dfrac{y-4}{1} = \dfrac{z+1}{2}$
9. 7
10. $(-5/3, 2/3, 19/3)$

27.6 SHORTEST DISTANCE BETWEEN TWO STRAIGHT LINES

Let us recall that two lines in a plane must intersect if they are not parallel. However, two lines in space i.e. R^3 may neither be parallel nor intersecting. The three different situations that can occur are shown in Fig. 27.12.

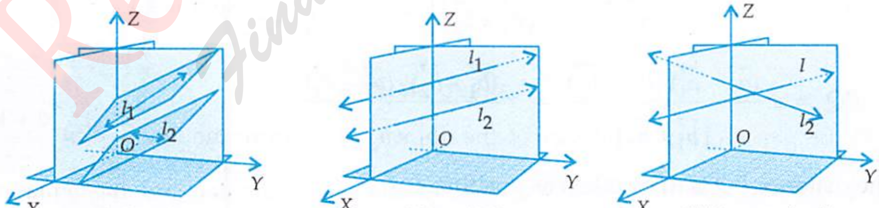

Fig. 27.12

If two lines are not parallel and do not intersect, we call them skew lines as defined below.

SKEW LINES *Two straight line in space which are neither parallel nor intersecting are called skew lines.*

Thus, the skew lines are those lines which do not lie in the same plane as shown in Fig. 27.12 (i).

LINE OF SHORTEST DISTANCE *If l_1 and l_2 are two skew-lines, then there is one and only one line perpendicular to each of lines l_1 and l_2 which is known as the line of shortest distance.*

SHORTEST DISTANCE *The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q where the lines of shortest distance intersects the two given lines.*

If two lines intersect then the shortest distance between them is zero. If two lines are parallel then the shortest distance between them is the distance between the two lines.

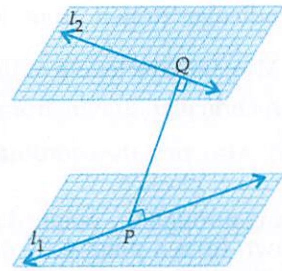

Fig. 27.13

27.6.1 SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Vector Form)

Let l_1 and l_2 be two lines having vector equations

$$l_1 : \vec{r} = \vec{a_1} + \lambda \vec{b_1} \text{ and } l_2 : \vec{r} = \vec{a_2} + \mu \vec{b_2} \text{ respectively.}$$

Clearly, l_1 and l_2 pass through the points A and B with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively and are parallel to the vectors $\vec{b_1}$ and $\vec{b_2}$ respectively. Let $\vec{PQ}$ be the shortest distance vector between l_1 and l_2. Then, $\vec{PQ}$ is perpendicular to both l_1 and l_2 which are parallel to $\vec{b_1}$ and $\vec{b_2}$ respectively. Therefore, $\vec{PQ}$ is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$. But, $\vec{b_1} \times \vec{b_2}$ is perpendicular to both $\vec{b_1}$ and $\vec{b_2}$. Therefore, $\vec{PQ}$ is parallel to the vector $\vec{b_1} \times \vec{b_2}$. Let $\hat{\eta}$ be a unit vector along $\vec{PQ}$.

Then, $\hat{n} = \pm \dfrac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|}$. From Fig. 27.14, it is evident that

$$PQ = \text{Projection of } \vec{AB} \text{ on } \vec{PQ}$$

$$\Rightarrow \quad PQ = \vec{AB} \cdot \hat{\eta} = \pm (\vec{a_2} - \vec{a_1}) \cdot \left\{ \frac{\vec{b_1} \times \vec{b_2}}{|\vec{b_1} \times \vec{b_2}|} \right\}$$

$$\Rightarrow \quad PQ = \pm \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} = \pm \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}$$

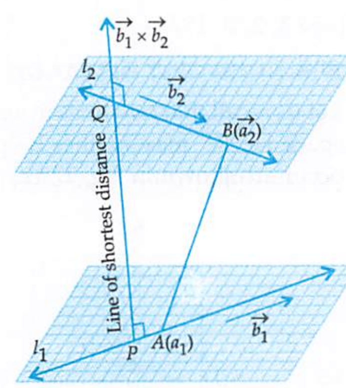

Fig. 27.14

Since the distance PQ is to be taken as positive.

$$\therefore \quad PQ = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Thus, the shortest (S.D.) between two non-parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by

$$\text{S.D.} = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|.$$

Condition for two given lines to intersect: If the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ intersect, then the shortest distance between them is zero.

$$\therefore \quad \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right| = 0 \Rightarrow (\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$$

27.6.2 SHORTEST DISTANCE BETWEEN TWO SKEW LINES (Cartesian Form)

Let the Cartesian equations of two skew lines be

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

The vector equations of these two lines are $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ respectively, where

$$\vec{a_1} = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}, \; \vec{a_2} = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k},$$

$$\vec{b_1} = l_1 \hat{i} + m_1 \hat{j} + n_1 \hat{k} \text{ and, } \vec{b_2} = l_2 \hat{i} + m_2 \hat{j} + n_2 \hat{k}.$$

Now,

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1 n_2 - m_2 n_1) \hat{i} + (l_2 n_1 - l_1 n_2) \hat{j} + (l_1 m_2 - l_2 m_1) \hat{k}$$

$$\therefore \quad \left| \vec{b_1} \times \vec{b_2} \right| = \sqrt{(m_1 n_2 - m_2 n_1)^2 + (l_2 n_1 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}$$

$$\text{and,} \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$\therefore \quad \text{Shortest distance (S.D.)} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_2} \times \vec{b_2}|} \right|$$

$$\Rightarrow \quad \text{S.D.} = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

Condition for two given lines to intersect: If the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n}$ and $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ intersect, then the shortest distance between them is zero.

$$\therefore \quad \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_1 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}} = 0 \Rightarrow \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

REMARK *As two skew lines do not intersect and are not parallel, but do lie in parallel planes. So shortest distance d between them is same as the distance between parallel planes that contain them which is same as the distance between a point on one line and the plane containing the other line and parallel to first.* *(See 28.13.7 on page 28.71)*

27.6.3 SHORTEST DISTANCE BETWEEN TWO PARALLEL LINES

Let l_1 and l_2 be two parallel lines whose vector equations are $l_1 : \vec{r} = \vec{a_1} + \lambda \vec{b}$ and $l_2 : \vec{r} = \vec{a_2} + \mu \vec{b}$ respectively. Clearly, l_1 and l_2 pass through the points A and B with position vectors $\vec{a_1}$ and $\vec{a_2}$ respectively and both are parallel to the vector $\vec{b}$.

Let BM be perpendicular from B on l_1. Then, BM is the shortest distance between l_1 and l_2.

Let θ be the angle between AB and line l_1. Then, angle between $\vec{AB}$ and $\vec{b}$ is $\pi - \theta$.

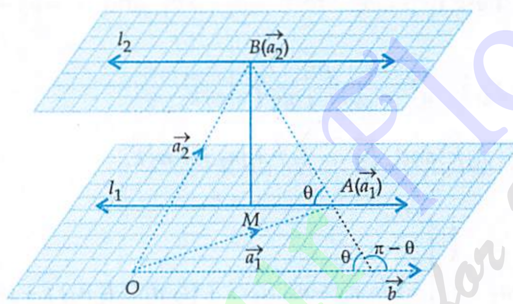

Fig. 27.15

In triangle ABM, we obtain

$$\sin\theta = \frac{BM}{AB} \Rightarrow BM = AB \sin\theta = |\vec{AB}| \sin\theta \quad \text{...(i)}$$

Now,

$$|\vec{AB} \times \vec{b}| = |\vec{AB}|\,|\vec{b}| \sin(\pi - \theta) = |\vec{AB}|\,|\vec{b}| \sin\theta = (|\vec{AB}| \sin\theta)\,|\vec{b}| = BM\,|\vec{b}| \quad \text{[Using (i)]}$$

$$\therefore \quad BM = \frac{|\vec{AB} \times \vec{b}|}{|\vec{b}|} = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$$

Thus, the shortest distance 'd' between the parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is given by

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \times \vec{b}}{|\vec{b}|} \right|$$

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

EXAMPLE 1 *Find the shortest distance between the lines*

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \text{ and } \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k}) \quad \text{[CBSE 2018]}$$

SOLUTION We know that the shortest distance between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is given by

$$d = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

Comparing the given equations with the equations $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ respectively, we obtain

$$\vec{a_1} = 4\hat{i} - \hat{j},\ \vec{a_2} = \hat{i} - \hat{j} + 2\hat{k},\ \vec{b_1} = \hat{i} + 2\hat{j} - 3\hat{k} \text{ and } \vec{b_2} = 2\hat{i} + 4\hat{j} - 5\hat{k}$$

$$\therefore \quad \vec{a_2} - \vec{a_1} = -3\hat{i} + 0\hat{j} + 2\hat{k} \text{ and, } \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0\hat{k}$$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (-3\hat{i} + 0\hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 0\hat{k}) = -6 + 0 + 0 = -6$$

and, $\quad |\vec{b_1} \times \vec{b_2}| = \sqrt{4 + 1 + 0} = \sqrt{5}$

$$\therefore \quad \text{Shortest distance} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{-6}{\sqrt{5}} \right| = \frac{6}{\sqrt{5}}$$

EXAMPLE 2 *Find the shortest distance between the lines*

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \ \textit{and} \ \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

SOLUTION The equations of two given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{...(i)} \quad \text{and} \quad \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5} \quad \text{...(ii)}$$

Line (i) passes through (1, 2, 3) and has direction ratios proportional to 2, 3, 4. So, its vector equation is

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \quad \text{...(iii)}$$

where, $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Line (ii) passes through (2, 4, 5) and has direction ratio proportional to 3, 4, 5. So, its vector equation is

$$\vec{r} = \vec{a_2} + \mu \vec{b_2} \quad \text{...(iv)}$$

where, $\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$.

The shortest distance between the lines (iii) and (iv) is given by

$$\text{S.D.} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| \quad \text{...(v)}$$

Now, $\quad \vec{a_2} - \vec{a_1} = (2\hat{i} + 4\hat{j} + 5\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} + 2\hat{j} + 2\hat{k}$

$$\text{and,} \quad \vec{b_2} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k} \Rightarrow |\vec{b_1} \times \vec{b_2}| = \sqrt{1 + 4 + 1} = \sqrt{6}$$

and, $\quad (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = (\hat{i} + 2\hat{j} + 2\hat{k}) \cdot (-\hat{i} + 2\hat{j} - \hat{k}) = -1 + 4 - 2 = 1.$

Substituting the values of $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})$ and $|\vec{b_1}\times\vec{b_2}|$ in (v), we obtain

$$\text{S.D.} = \left|\frac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \frac{1}{\sqrt{6}}$$

EXAMPLE 3 *By computing the shortest distance determine whether the following pairs of lines intersect or not:*

(i) $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$; $\vec{r}=2\hat{i}-\hat{j}+\mu(\hat{i}-\hat{j}-\hat{k})$

(ii) $\dfrac{x-1}{2}=\dfrac{y+1}{3}=z$; $\dfrac{x+1}{5}=\dfrac{y-2}{1}$; $z=2$ [CBSE 2020]

SOLUTION (i) Let the vector equations of two given lines be $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$ and $\vec{r}=\vec{a_2}+\mu\vec{b_2}$ respectively. Then,

$$\vec{a_1}=\hat{i}-\hat{j},\ \vec{b_1}=2\hat{i}+\hat{k},\ \vec{a_2}=2\hat{i}-\hat{j} \text{ and } \vec{b_2}=\hat{i}-\hat{j}-\hat{k}.$$

$$\therefore\quad \vec{a_2}-\vec{a_1} = (2\hat{i}-\hat{j})-(\hat{i}-\hat{j})=\hat{i} \quad\text{and,}\quad \vec{b_1}\times\vec{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 1\\ 1 & -1 & -1\end{vmatrix} = \hat{i}+3\hat{j}-2\hat{k}$$

So, $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = \hat{i}\cdot(\hat{i}+3\hat{j}-2\hat{k}) = 1+0+0 = 1$

Clearly, $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})\neq 0$. So, the given lines do not intersect.

(ii) The equations of given lines can be re-written as

$$\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-0}{1} \quad \text{...(i)} \qquad \text{and,} \quad \frac{x+1}{5}=\frac{y-2}{1}=\frac{z-2}{0} \quad \text{...(ii)}$$

Line (i) passes through the point $(1,-1,0)$ and has direction ratios proportional to 2, 3, 1. So, its vector equation is $\vec{r}=\vec{a_1}+\lambda\vec{b_1}$, where $\vec{a_1}=\hat{i}-\hat{j}$ and $\vec{b_1}=2\hat{i}+3\hat{j}+\hat{k}$.

Line (ii) passes through the point $(-1,2,2)$ and has direction ratios proportional to 5, 1, 0. So, its vector equation is $\vec{r}=\vec{a_2}+\mu\vec{b_2}$, where $\vec{a_2}=-\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b_2}=5\hat{i}+\hat{j}+0\hat{k}$.

$$\therefore\quad \vec{a_2}-\vec{a_1} = (-\hat{i}+2\hat{j}+2\hat{k})-(\hat{i}-\hat{j}) = -2\hat{j}+3\hat{j}+2\hat{k}$$

$$\text{and,}\quad \vec{b_1}\times\vec{b_2} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 1\\ 5 & 1 & 0\end{vmatrix} = -\hat{i}+5\hat{j}-13\hat{k}$$

$$\therefore\quad (\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = (-2\hat{i}+3\hat{j}+2\hat{k})\cdot(-\hat{i}+5\hat{j}-13\hat{k}) = 2+15-26 = -9 \neq 0$$

Hence, given lines do not intersect.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

EXAMPLE 4 *Find the shortest distance between the lines whose vector equations are*

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k}) \textit{ and, } \vec{r}=(2\hat{i}+4\hat{j}+5\hat{k})+\mu(4\hat{i}+6\hat{j}+8\hat{k})$$

[CBSE 2008, 2015]

SOLUTION The vector equations of given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{...(i)}$$

and, $$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + 2\mu(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{...(ii)}$$

Equation (ii) can re-written as

$$\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu'(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \text{...(iii)}$$

where $\mu' = 2\mu$.

These two lines passes through the points having position vectors $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$ respectively and both are parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$. So, the shortest distance between them is given by

$$\text{S.D} = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|} \quad \text{...(iv)}$$

Now, $$(\vec{a_2} - \vec{a_1}) \times \vec{b} = (\hat{i} + 2\hat{j} + 2\hat{k}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 2\hat{i} - 0\hat{j} - \hat{k}$$

$$\therefore \quad |(\vec{a_2} - \vec{a_1}) \times \vec{b}| = \sqrt{4 + 0 + 1} = \sqrt{5} \text{ and } |\vec{b}| = \sqrt{4 + 9 + 16} = \sqrt{29}$$

Substituting the values of $|(\vec{a_2} - \vec{a_1}) \times \vec{b}|$ and $|\vec{b}|$ in (iv), we obtain: S.D. $= \frac{\sqrt{5}}{\sqrt{29}}$.

EXERCISE 27.5

BASIC

1. Find the shortest distance between the following pairs of lines whose vector equations are:

(i) $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$

(ii) $\vec{r} = (3\hat{i} + 5\hat{j} + 7\hat{k}) + \lambda(\hat{i} - 2\hat{j} + 7\hat{k})$ and $\vec{r} = -\hat{i} - \hat{j} - \hat{k} + \mu(7\hat{i} - 6\hat{j} + \hat{k})$

(iii) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$

(iv) $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - t)\hat{k}$ and $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$

[NCERT, CBSE 2002, 2011]

(v) $\vec{r} = (\lambda - 1)\hat{i} + (\lambda + 1)\hat{j} - (1 + \lambda)\hat{k}$ and $\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$

(vi) $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \lambda(2\hat{i} - 5\hat{j} + 2\hat{k})$ and, $\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$ [CBSE 2008]

(vii) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and, $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

[CBSE 2014, NCERT]

(viii) $\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k}$ and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

[NCERT EXEMPLAR]

2. Find the shortest distance between the following pairs of lines whose Cartesian equations are:

(i) $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-5}{5}$ **[CBSE 2005]**

(ii) $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{3}=\frac{y-2}{1}; z=2$

(iii) $\frac{x-1}{-1}=\frac{y+2}{1}=\frac{z-3}{-2}$ and $\frac{x-1}{1}=\frac{y+1}{2}=\frac{z+1}{-2}$

(iv) $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ **[CBSE 2008, 2014]**

(v) $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ and $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z-1}{3}$ **[CBSE 2022]**

3. By computing the shortest distance determine whether the following pairs of lines intersect or not:

(i) $\vec{r}=(\hat{i}-\hat{j})+\lambda(2\hat{i}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j})+\mu(\hat{i}+\hat{j}-\hat{k})$

(ii) $\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(3\hat{i}-\hat{j})$ and $\vec{r}=(4\hat{i}-\hat{k})+\mu(2\hat{i}+3\hat{k})$

(iii) $\frac{x-1}{2}=\frac{y+1}{3}=z$ and $\frac{x+1}{5}=\frac{y-2}{1}; z=2$

(iv) $\frac{x-5}{4}=\frac{y-7}{-5}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-7}{1}=\frac{z-5}{3}$

4. Find the shortest distance between the following pairs of parallel lines whose equations are:

(i) $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}-\hat{j}-\hat{k})+\mu(-\hat{i}+\hat{j}-\hat{k})$

(ii) $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}-\hat{j}+\hat{k})$ and $\vec{r}=(2\hat{i}+\hat{j}-\hat{k})+\mu(4\hat{i}-2\hat{j}+2\hat{k})$

(iii) $\vec{r}=2\hat{i}+\hat{j}-\hat{k}+\lambda(\hat{i}+\hat{j}-\hat{k})$ and $\vec{r}=(\hat{i}-2\hat{j}+\hat{k})+\mu(\hat{i}+\hat{j}-\hat{k})$ **[CBSE 2022]**

5. Find the equations of the lines joining the following pairs of vertices and then find the shortest distance between the lines

(i) (0, 0, 0) and (1, 0, 2) (ii) (1, 3, 0) and (0, 3, 0)

6. Write the vector equations of the following lines and hence determine the distance between them

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z+4}{6} \text{ and } \frac{x-3}{4}=\frac{y-3}{6}=\frac{z+5}{12}$$ **[CBSE 2010]**

7. Find the shortest distance between the lines

(i) $\vec{r}=(\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k})$ and, $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$ **[NCERT]**

(ii) $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ **[NCERT]**

(iii) $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

[NCERT, CBSE 2014]

(iv) $\vec{r}=6\hat{i}+2\hat{j}+2\hat{k}+\lambda(\hat{i}-2\hat{j}+2\hat{k})$ and $\vec{r}=-4\hat{i}-\hat{k}+\mu(3\hat{i}-2\hat{j}-2\hat{k})$ **[NCERT]**

BASED ON LOTS

8. Find the distance between the lines l_1 and l_2 given by
$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and, $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$ **[NCERT, CBSE 2014]**

9. Find the vector equation of a line passing through the point (2, 3, 2) and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also find the distance between these lines. **[CBSE 2019]**

ANSWERS

1. (i) $\sqrt{270}$ (ii) $\frac{512}{\sqrt{3968}}$ (iii) $\frac{1}{\sqrt{6}}$ (iv) $\frac{3}{\sqrt{2}}$ (v) $\frac{5}{\sqrt{2}}$
(vi) $\frac{3}{\sqrt{2}}$ (vii) $\frac{10}{\sqrt{59}}$ (viii) 14

2. (i) $\frac{1}{\sqrt{6}}$ (ii) $\frac{3}{\sqrt{59}}$ (iii) $\frac{8}{\sqrt{29}}$ (iv) $2\sqrt{29}$ (v) $3\sqrt{\frac{3}{7}}$

3. (i) No (ii) Yes (iii) No (iv) No

4. (i) $\sqrt{26}$ (ii) $\frac{\sqrt{11}}{\sqrt{6}}$ (iii) $\sqrt{2}$

5. $\frac{x}{1} = \frac{y}{0} = \frac{z}{2}$; $\frac{x-1}{-1} = \frac{y-3}{0} = \frac{z}{0}$; 3 units

6. $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$; $\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$, S.D. $= \frac{\sqrt{293}}{7}$ units

7. (i) $\frac{3}{\sqrt{2}}$ (ii) $2\sqrt{29}$ (iii) $\frac{3}{\sqrt{19}}$ (iv) 9

8. $\frac{\sqrt{293}}{7}$ units

HINTS TO SELECTED PROBLEMS

1. (iv) The equations of the lines are

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - \hat{k}) \quad \text{and,} \quad \vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$$

These lines pass through points $\vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{a_2} = \hat{i} - \hat{j} - \hat{k}$ and are parallel to vectors $\vec{b_1} = -\hat{i} + \hat{j} - \hat{k}$ and $\vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$ respectively.

$$\therefore \quad \vec{a_2} - \vec{a_1} = \hat{j} - 4\hat{k} \quad \text{and,} \quad \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = 0\hat{i} - 3\hat{j} - 3\hat{k}$$

So, $(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0 - 3 + 12 = 9$

$$\text{Hence, S.D.} = \left| \frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|} \right| = \left| \frac{-3 + 12}{\sqrt{0 + 9 + 9}} \right| = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}}$$

(vii) Given lines are

$$\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k}) \text{ and } \vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$$

or, $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$

where $\vec{a_1} = \hat{i} + \hat{j}$, $\vec{b_1} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{a_2} = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{b_2} = 3\hat{i} - 5\hat{j} + 2\hat{k}$

$$\therefore \quad \vec{a_2}-\vec{a_1} = \hat{i}+0\hat{j}-\hat{k} \quad \text{and,} \quad \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & -5 & 2 \end{vmatrix} = 3\hat{i}-\hat{j}-7\hat{k}$$

So, $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = 3+0+7 = 10$, $|\vec{b_1}\times\vec{b_2}| = \sqrt{9+1+49} = \sqrt{59}$

Hence, $\text{S.D.} = \left|\dfrac{(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})}{|\vec{b_1}\times\vec{b_2}|}\right| = \dfrac{10}{\sqrt{59}}$

7. (i) Given lines are:

$$l_1 : \vec{r} = (\hat{i}+2\hat{j}+\hat{k})+\lambda(\hat{i}-\hat{j}+\hat{k}) \text{ and, } l_2 : \vec{r} = (2\hat{i}-\hat{j}-\hat{k})+\mu(2\hat{i}+\hat{j}+2\hat{k})$$

These two lines pass through points with position vectors $\vec{a_1} = \hat{i}+2\hat{j}+\hat{k}$ and $\vec{a_2} = 2\hat{i}-\hat{j}-\hat{k}$ respectively and parallel to vectors $\vec{b_1} = \hat{i}-\hat{j}+\hat{k}$ and $\vec{b_2} = 2\hat{i}+\hat{j}+2\hat{k}$ respectively.

$$\therefore \quad \vec{a_2}-\vec{a_1} = \hat{i}-3\hat{j}-2\hat{k} \text{ and, } \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -3\hat{i}+0\hat{j}+3\hat{k}$$

Now, $(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = -3+0-6 = -9$ and $|\vec{b_1}\times\vec{b_2}| = \sqrt{9+0+9} = 3\sqrt{2}$

So, the shortest distance d between l_1 and l_2 is given by

$$d = \frac{|(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})|}{|\vec{b_1}\times\vec{b_2}|} = \frac{9}{3\sqrt{2}} = \frac{3}{\sqrt{2}}$$

(ii) Given lines are: $\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$ and, $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$

These lines pass through points with position vectors $\vec{a_1} = -\hat{i}-\hat{j}-\hat{k}$ and $\vec{a_2} = 3\hat{i}+5\hat{j}+7\hat{k}$ respectively and parallel to vectors $\vec{b_1} = 7\hat{i}-6\hat{j}+\hat{k}$ and $\vec{b_2} = \hat{i}-2\hat{j}+\hat{k}$ respectively.

$$\text{Now, } \vec{a_2}-\vec{a_1} = 4\hat{i}+6\hat{j}+8\hat{k} \text{ and } \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} = -4\hat{i}-6\hat{j}-8\hat{k}$$

$\therefore \quad (\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2}) = -16-36-64 = -116$ and $|\vec{b_1}\times\vec{b_2}| = \sqrt{16+36+64} = \sqrt{116}$

Hence, $\text{S.D.} = \dfrac{|(\vec{a_2}-\vec{a_1})\cdot(\vec{b_1}\times\vec{b_2})|}{|\vec{b_1}\times\vec{b_2}|} = \dfrac{116}{\sqrt{116}} = \sqrt{116} = 2\sqrt{29}$

(iii) Proceed as in 7 (i) (iv) Proceed as in 7 (ii)

8. Given lines are

$$l_1 : \vec{r} = \hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k}) \text{ and, } l_2 : \vec{r} = 3\hat{i}+3\hat{j}-5\hat{k}+\mu(2\hat{i}+3\hat{j}+6\hat{k})$$

Line l_1 passes through $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$ and is parallel to $\vec{b_1} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

Line l_2 passes through $\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$ and is parallel to $\vec{b_2} = 2\hat{i} + 3\hat{j} + 6\hat{k}$.

Clearly, $\vec{b_1} = \vec{b_2}$. So, lines l_1 and l_2 are parallel. Consequently, the shortest distance between them is given by

$$d = \frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$$

Now, $\vec{a_2} - \vec{a_1} = 2\hat{i} + \hat{j} - \hat{k}$ and, $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$

$$\therefore \quad (\vec{a_2} - \vec{a_1}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}$$

$$\Rightarrow \quad \left|(\vec{a_2} - \vec{a_1}) \times \vec{b}\right| = \sqrt{81 + 196 + 16} = \sqrt{293} \text{ and, } |\vec{b}| = \sqrt{4 + 9 + 36} = 7$$

Hence, $d = \dfrac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|} = \dfrac{\sqrt{293}}{7}$

FILL IN THE BLANK TYPE QUESTIONS (FBQs)

1. The vector equations of OX, OY and OZ are ....
2. The vector equation of the line $\dfrac{5-x}{3} = \dfrac{y+4}{7} = \dfrac{z-6}{2}$ is ....
3. The vector equation of the line passing through the points $(3, 4, -7)$ and $(1, -1, 6)$ is ....
4. If l, l, l are direction cosines of a line, then $l =$ ....
5. If the line $\dfrac{x-1}{l} = \dfrac{y-2}{m} = \dfrac{z+1}{n}$ passes through the point $(-1, 0, 1)$, then its direction cosines l, m, n are ....
6. The equations of a line passing through the point $(-2, 3, 4)$ and equally inclined with the coordinate axes OX, OY and OZ are ....
7. The angle between the lines whose direction ratios are proportional to a, b, c and $\dfrac{1}{bc}, \dfrac{1}{ca}, \dfrac{1}{ab}$ is ....
8. The equation of the straight line passing through (a, b, c) and $(a-b, b-c, c-a)$ are.................. .
9. The equations of straight line passing through (a, b, c) and parallel to z-axis are ....
10. If the lines $\dfrac{x-1}{-3} = \dfrac{y-2}{2k} = \dfrac{z-3}{2}, \dfrac{x-1}{3k} = \dfrac{5-y}{-1} = \dfrac{6-z}{5}$ are at right angle, then $k =$ ....
11. The Cartesian equations of the straight line passing through the point $(2, 1, -1)$ and making angles $\dfrac{\pi}{3}, \dfrac{\pi}{3}$ and $\dfrac{\pi}{4}$ with the positive directions of the coordinate axes are ....
12. The equations of x-axis in unsymmetrical form are: ....
13. The equation of y-axis in symmetrical form is $\dfrac{x}{0} = \dfrac{y}{1} = \dfrac{z}{0}$. ....

14. If the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = 2\vec{a}_2 + \lambda \vec{b}_2$ are coplanar, then $\dfrac{[\vec{a}_1\ \vec{b}_1\ \vec{b}_2]}{[\vec{a}_2\ \vec{b}_1\ \vec{b}_2]} =$ ....

15. If the lines $\vec{r} = 2\vec{a}_1 - 3\vec{a}_2 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are coplanar and $[\vec{a}_1\ \vec{b}_1\ \vec{b}_2] = k\,[\vec{a}_2\ \vec{b}_1\ \vec{b}_2]$, then $k =$.................. .

16. The equation of x-axis in symmetrical form is.................. .

17. The equation of x-axis in unsymmetrical form is.................. .

18. The vector equation of the line though the points $(3, 4, -7)$ and $(1, -1, 6)$ is ... . **[NCERT EXEMPLAR, CBSE 2020]**

19. The vector equation of the line $\dfrac{x-5}{3} = \dfrac{y+4}{7} = \dfrac{z-6}{2}$ is .... **[NCERT EXEMPLAR]**

20. The line of shortest distance between two skew-lines is .... to both the lines. **[CBSE 2020]**

ANSWERS

1. $\vec{r} = \lambda \hat{i},\ \vec{r} = \mu \hat{j},\ \vec{r} = \nu \hat{k}$
2. $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(-3\hat{i} + 7\hat{j} + 2\hat{k})$
3. $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$
4. $\pm \dfrac{1}{\sqrt{3}}$
5. $l = -\dfrac{1}{\sqrt{3}}, m = -\dfrac{1}{\sqrt{3}}, n = \dfrac{1}{\sqrt{3}}$
6. $\dfrac{x+2}{1} = \dfrac{y-3}{1} = \dfrac{z-4}{1}$
7. $0°$
8. $\dfrac{x-a}{b} = \dfrac{y-b}{c} = \dfrac{z-c}{a}$
9. $\dfrac{x-a}{0} = \dfrac{y-b}{0} = \dfrac{z-c}{1}$
10. $\dfrac{-10}{7}$
11. $\dfrac{x-2}{1} = \dfrac{y-1}{1} = \dfrac{z+1}{\sqrt{2}}$
12. $y = 0, z = 0$
14. 2
15. 2
16. $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z-0}{0}$
17. $y = 0, z = 0$
18. $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$
19. $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$
20. perpendicular

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write the cartesian and vector equations of X-axis.
2. Write the cartesian and vector equations of Y-axis.
3. Write the cartesian and vector equations of Z-axis.
4. Write the vector equation of a line passing through a point having position vector $\vec{\alpha}$ and parallel to vector $\vec{\beta}$.
5. Cartesian equations of a line AB are $\dfrac{2x-1}{2} = \dfrac{4-y}{7} = \dfrac{z+1}{2}$. Write the direction ratios of a line parallel to AB.
6. Write the direction cosines of the line whose cartesian equations are $6x - 2 = 3y + 1 = 2z - 4$.
7. Write the direction cosines of the line $\dfrac{x-2}{2} = \dfrac{2y-5}{-3}, z = 2$.
8. Write the coordinate axis to which the line $\dfrac{x-2}{3} = \dfrac{y+1}{4} = \dfrac{z-1}{0}$ is perpendicular.

9. Write the angle between the lines $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z-2}{1}$ and $\frac{x-1}{1}=\frac{y}{2}=\frac{z-1}{3}$.

10. Write the direction cosines of the line whose cartesian equations are $2x = 3y = -z$.

11. Write the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$.

12. Write the value of λ for which the lines $\frac{x-3}{-3}=\frac{y+2}{2\lambda}=\frac{z+4}{2}$ and $\frac{x+1}{3\lambda}=\frac{y-2}{1}=\frac{z+6}{-5}$ are perpendicular to each other.

13. Write the formula for the shortest distance between the lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$.

14. Write the condition for the lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ to be intersecting.

15. The cartesian equations of a line AB are $\frac{2x-1}{\sqrt{3}}=\frac{y+2}{2}=\frac{z-3}{3}$. Find the direction cosines of a line parallel to AB. **[CBSE 2008]**

16. If the equations of a line AB are $\frac{3-x}{1}=\frac{y+2}{-2}=\frac{z-5}{4}$, write the direction ratios of a line parallel to AB. **[CBSE 2011]**

17. Write the vector equation of a line given by $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$. **[CBSE 2011]**

18. The equations of a line are given by $\frac{4-x}{3}=\frac{y+3}{3}=\frac{z+2}{6}$. Write the direction cosines of a line parallel to this line. **[CBSE 2012]**

19. Find the Cartesian equations of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3}=\frac{4-y}{5}=\frac{z+8}{6}$. **[CBSE 2013]**

20. Find the angle between the lines $\vec{r} = (2\hat{i} - 5\hat{j} + \hat{k}) + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and $\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. **[CBSE 2014]**

21. Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$. **[CBSE 2015]**

22. Find the vector equation of a line passing through the point $(3, 4, 5)$ and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. **[CBSE 2019]**

ANSWERS

	Cartesian equation	Vector equation		Cartesian equation	Vector equation
1.	$\frac{x-0}{1}=\frac{y-0}{0}-\frac{z-0}{0}$	$\vec{r} = \lambda\hat{i}$	2.	$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{0}$,	$\vec{r} = \lambda\hat{j}$
3.	$\frac{x-0}{0}=\frac{y-0}{0}=\frac{z-0}{1}$,	$\vec{r} = \lambda\hat{k}$			

4. $\vec{r} = \vec{\alpha} + \lambda\vec{\beta}$

5. $\frac{1}{\sqrt{54}}, -\frac{7}{\sqrt{54}}, \frac{2}{\sqrt{54}}$

6. $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$

7. $\frac{4}{5}, -\frac{3}{5}, 0$

8. Z-axis

9. $90°$

10. $\frac{3}{7}, \frac{2}{7}, -\frac{6}{7}$

11. $90°$

12. $\frac{-10}{7}$

13. $\frac{|(\vec{a_2} - \vec{a_1}) \times \vec{b}|}{|\vec{b}|}$

14. $(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$ 15. $\frac{\sqrt{3}}{\sqrt{55}}, \frac{4}{\sqrt{55}}, \frac{6}{\sqrt{55}}$ 16. $-1, 2, -4$

17. $r = (4\hat{i} - 3\hat{j} - 2\hat{k}) + \lambda(-3\hat{i} + 3\hat{j} + 6\hat{k})$ 18. $-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ 19. $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$

20. $\cos^{-1}\left(\frac{19}{21}\right)$ 21. $\frac{\pi}{2}$ 22. $\vec{r} = 3\hat{i} + 4\hat{j} + 5\hat{k} + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$