EXAMPLE 4 *Find the value of k if the straight line* $2x+3y+4+k(6x-y+12)=0$ *is perpendicular to the line* $7x+5y-4=0$.

SOLUTION The two lines are

$$x(2+6k)+y(3-k)+4+12k=0 \quad \text{...(i)} \quad \text{and} \quad 7x+5y-4=0 \quad \text{...(ii)}$$

Let m_1 and m_2 be the slopes of (i) and (ii) respectively. Then,

$$m_1=-\frac{2+6k}{3-k},\ m_2=-\frac{7}{5}$$

If lines (i) and (ii) are perpendicular. Then,

$$m_1 m_2=-1 \Rightarrow \left(-\frac{2+6k}{3-k}\right)\left(-\frac{7}{5}\right)=-1 \Rightarrow 14+42k=-15+5k \Rightarrow k=-\frac{29}{37}.$$

EXAMPLE 5 *A line passing through the points (a, 2 a) and (– 2, 3) is perpendicular to the line* $4x+3y+5=0$, *find the value of a.*

SOLUTION Let m_1 be the slope of the line joining $A(a, 2a)$ and $B(-2, 3)$. Then, $m_1=\frac{2a-3}{a+2}$

Let m_2 be the slope of the line $4x+3y+5=0$. Then, $m_2=-\frac{4}{3}$.

Since given lines are perpendicular. Therefore,

$$m_1 m_2=-1 \Rightarrow \frac{2a-3}{a+2}\times -\frac{4}{3}=-1 \Rightarrow 8a-12=3a+6 \Rightarrow a=18/5.$$

EXAMPLE 6 *Classify the following pairs of lines as coincident, parallel or intersecting:*

(i) $x+2y-3=0$ and $-3x-6y+9=0$ (ii) $x+2y+1=0$ and $2x+4y+3=0$

(iii) $3x-2y+5=0$ and $2x+y-9=0$

SOLUTION (i) The given lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where $a_1=1, b_1=2, c_1=-3, a_2=-3, b_2=-6$ and $c_2=9$.

Clearly, $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=-\frac{1}{3}$. So, the given lines are coincident.

(ii) The given lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where

$a_1=1,\ b_1=2,\ c_1=1,\ a_2=2, b_2=4$, and $c_2=3$.

Clearly, $\frac{a_1}{a_2}=\frac{b_1}{b_2}\neq\frac{c_1}{c_2}$. So, the given lines are parallel.

(iii) The given lines are $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$, where

$a_1=3, b_1=-2, c_1=5, a_2=2, b_2=1$, and $c_2=-9$.

Clearly, $\frac{a_1}{a_2}\neq\frac{b_1}{b_2}$. So, the given lines are intersecting.

EXAMPLE 7 *The hypotenuse of a right isosceles triangle has its ends at the points* (1, 3) *and* (– 4, 1). *Find the equations of the legs (perpendicular sides) of the triangle.* **[NCERT]**

SOLUTION Let ABC be the right triangle with diagonal AC. Let m be the slope of a line making 45° angle with AC.

Clearly, Slope of $AC=\frac{1-3}{-4-1}=\frac{2}{5}$. We observe that θ is the angle between AC and a line of slope m.

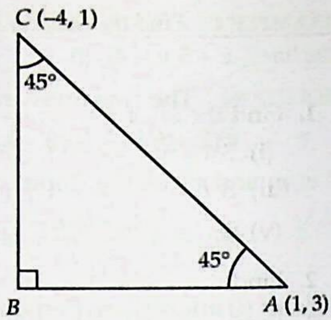

Fig. 23.79

$$\therefore \quad \tan 45^\circ = \left|\frac{m - \frac{2}{5}}{1 + \frac{2m}{5}}\right|$$

$$\Rightarrow \quad 2m + 5 = \pm(5m - 2)$$

$$\Rightarrow \quad 2m + 5 = 5m - 2 \text{ or, } 2m + 5 = -(5m - 2)$$

$$\Rightarrow \quad m = \frac{7}{3} \text{ or, } m = -\frac{3}{7}$$

Thus, the lines making 45° angle with AC having slopes $\frac{7}{3}$ or $-\frac{3}{7}$. So, the possible equations of 3 are

$$y - 3 = \frac{7}{3}(x - 1) \text{ and } y - 3 = -\frac{3}{7}(x - 1) \Rightarrow 7x - 3y + 2 = 0 \text{ and } 3x + 7y - 24 = 0$$

ssible equations of BC are

$$y - 1 = \frac{7}{3}(x + 4) \text{ and } y - 1 = -\frac{3}{7}(x + 4) \Rightarrow 7x - 3y + 31 = 0 \text{ and } 3x + 7y + 5 = 0$$

the equations of the sides are:

$$7x - 3y + 2 = 0 \text{ and } 3x + 7y + 5 = 0 \quad \text{or, } 7x - 3y + 31 = 0 \text{ and } 3x + 7y - 24 = 0$$

PLE 8 *If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$. Find alues of m.* **[NCERT]**

LUTION Let θ be the acute angle which the line $y = mx + 4$ makes with the lines $y = 3x + 1$ and $2y = x + 3$. Then,

$$\tan\theta = \left|\frac{m - 3}{1 + 3m}\right| \text{ and, } \tan\theta = \left|\frac{m - \frac{1}{2}}{1 + \frac{m}{2}}\right|$$

$$\Rightarrow \quad \left|\frac{m - 3}{1 + 3m}\right| = \left|\frac{2m - 1}{m + 2}\right|$$

$$\Rightarrow \quad \frac{m - 3}{3m + 1} = \pm\frac{2m - 1}{m + 2}$$

$$\Rightarrow \quad m^2 - m - 6 = \pm(6m^2 - m - 1)$$

$$\Rightarrow \quad 5m^2 + 5 = 0 \text{ or } 7m^2 - 2m - 7 = 0 \Rightarrow 7m^2 - 2m - 7 = 0 \Rightarrow m = \frac{1 \pm 5\sqrt{2}}{7}$$

EXAMPLE 9 *Find the slope of the lines which make an angle of 45° with the line $3x - y + 5 = 0$.*

SOLUTION Let m be the slope of the line which make an angle of 45° with the line $3x - y + 5 = 0$. Then,

$$\tan 45^\circ = \left|\frac{m - 3}{1 + 3m}\right| \qquad [\because \text{ Slope of } 3x - y + 5 = 0 \text{ is } 3]$$

$$\Rightarrow \quad 1 = \left|\frac{m - 3}{1 + 3m}\right|$$

$$\Rightarrow \quad |1 + 3m| = |m - 3|$$

$$\Rightarrow \quad 1 + 3m = \pm(m - 3) \Rightarrow 1 + 3m = m - 3,\ 1 + 3m = -m + 3 \Rightarrow m = -2, \frac{1}{2}$$

EXERCISE 23.13

LEVEL-1

1. Find the angles between each of the following pairs of straight lines:
   (i) $3x + y + 12 = 0$ and $x + 2y - 1 = 0$ (ii) $3x - y + 5 = 0$ and $x - 3y + 1 = 0$
   (iii) $3x + 4y - 7 = 0$ and $4x - 3y + 5 = 0$ (iv) $x - 4y = 3$ and $6x - y = 11$
   (v) $(m^2 - mn)\, y = (mn + n^2)\, x + n^3$ and $(mn + m^2)\, y = (mn - n^2)\, x + m^3$.
2. Find the acute angle between the lines $2x - y + 3 = 0$ and $x + y + 2 = 0$.
3. Prove that the points $(2, -1)$, $(0, 2)$, $(2, 3)$ and $(4, 0)$ are the coordinates of the vertices of a parallelogram and find the angle between its diagonals.
4. Find the angle between the line joining the points $(2, 0)$, $(0, 3)$ and the line $x + y = 1$.
5. If θ is the angle which the straight line joining the points (x_1, y_1) and (x_2, y_2) subtends at the origin, prove that $\tan\theta = \dfrac{x_2 y_1 - x_1 y_2}{x_1 x_2 + y_1 y_2}$ and $\cos\theta = \dfrac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2}}$.
6. Prove that the straight lines $(a + b)\, x + (a - b)\, y = 2ab$, $(a - b)\, x + (a + b)\, y = 2ab$ and $x + y = 0$ form an isosceles triangle whose vertical angle is $2\tan^{-1}\left(\dfrac{a}{b}\right)$.
7. Find the angle between the lines $x = a$ and $by + c = 0$.
8. Find the tangent of the angle between the lines which have intercepts 3, 4 and 1, 8 on the axes respectively.
9. Show that the line $a^2 x + ay + 1 = 0$ is perpendicular to the line $x - ay = 1$ for all non-zero real values of a.
10. Show that the tangent of an angle between the lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{a} - \dfrac{y}{b} = 1$ is $\dfrac{2ab}{a^2 - b^2}$.
    [NCERT EXEMPLAR]

ANSWERS

1. (i) $45°$ (ii) $\tan^{-1}\left(\dfrac{4}{3}\right)$ (iii) $90°$ (iv) $\tan^{-1}\left(\dfrac{23}{10}\right)$ (v) $\tan^{-1}\left(\dfrac{4m^2 n^2}{m^4 - n^4}\right)$

2. $\tan^{-1} 3$ 3. $-\tan^{-1}\dfrac{1}{2} - \dfrac{\pi}{2}$ 4. $\tan^{-1}\left(\dfrac{1}{5}\right)$ 7. $90°$ 8. $\dfrac{4}{7}$

23.12 POSITION OF TWO POINTS RELATIVE TO A LINE

In this section, we shall see how to check whether two given points are on the same side or opposite sides of a given line.

Let the equation of the given line be $ax + by + c = 0$ ...(i)

and let the coordinates of the two given points be $P(x_1, y_1)$ and $Q(x_2, y_2)$.

The coordinates of the point R which divides the line joining P and Q in the ratio $m : n$ are

$$\left(\frac{m x_2 + n x_1}{m + n}, \frac{m y_2 + n y_1}{m + n}\right) \quad \text{...(ii)}$$

If this point lies on (i), then

$$a\left(\frac{m x_2 + n x_1}{m + n}\right) + b\left(\frac{m y_2 + n y_1}{m + n}\right) + c = 0$$

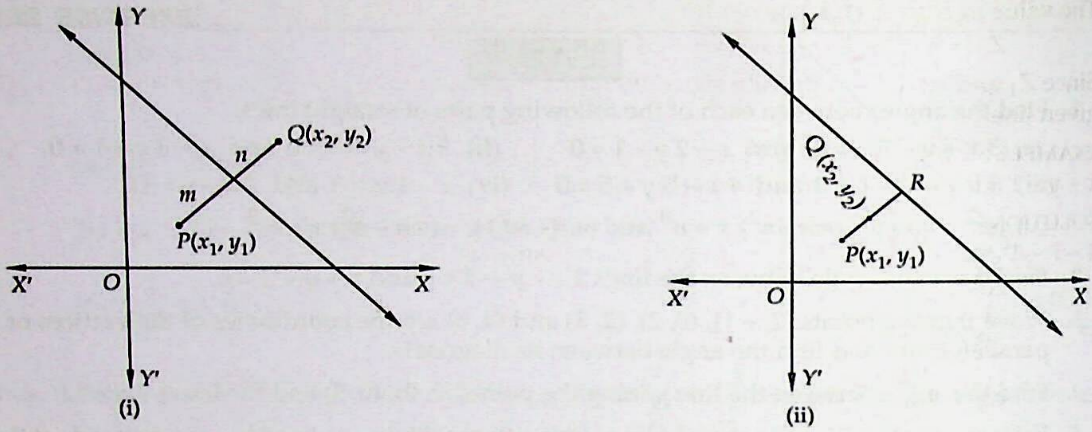

Fig. 23.80

$\Rightarrow \quad m(ax_2 + by_2 + c) + n(ax_1 + by_1 + c) = 0$

$$\Rightarrow \quad \frac{m}{n} = -\frac{a\,x_1 + by_1 + c}{ax_2 + by_2 + c} \qquad \text{...(iii)}$$

If the point R is between the points P and Q i.e. points P and Q are on the opposite sides of the given line, then the ratio $m : n$ is positive.

$$\therefore \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) > 0$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} < 0$$

$\Rightarrow \quad ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs

If the point R is not between P and Q i.e. point P and Q are on the same side of the given line, then the ratio $m : n$ is negative.

$$\therefore \quad -\left(\frac{ax_1 + by_1 + c}{ax_2 + by_2 + c}\right) < 0 \qquad \text{[From (iii)]}$$

$$\Rightarrow \quad \frac{ax_1 + by_1 + c}{ax_2 + by_2 + c} > 0$$

$\Rightarrow \quad ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of the same sign.

Thus, the two points. (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

REMARK 1 *A point (x_1, y_1) will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the same sign.*

REMARK 2 *A point (x_1, y_1) will lie on the opposite side of the origin relative to the line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the opposite signs.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Are the points (3, 4) and (2, – 6) on the same or opposite sides of the line $3x - 4y = 8$?*

SOLUTION Let $Z = 3x - 4y - 8$. Then, the value of Z at (3, 4) is given by

$$Z_1 = 3 \times 3 - 4 \times 4 - 8 = 9 - 16 - 8 = -15 < 0$$

The value of Z at (2, – 6) is given by

$$Z_2 = 3 \times 2 - 4 \times -6 - 8 = 6 + 24 - 8 = 22 > 0$$

Since Z_1 and Z_2 are of opposite signs, therefore the two points are on the opposite sides of the given line.

EXAMPLE 2 *If the points* $(4, 7)$ *and* $(\cos\theta, \sin\theta)$, *where* $0 < \theta < \pi$, *lie on the same side of the line* $x + y - 1 = 0$, *then prove that* θ *lies in the first quadrant.*

SOLUTION If the points (4, 7) and $(\cos\theta, \sin\theta)$ lie on the same side of $x + y - 1 = 0$, then $4 + 7 - 1$ and $\cos\theta + \sin\theta - 1$ must be of the same sign.

$$\therefore \quad \cos\theta + \sin\theta - 1 > 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\theta \sin\frac{\pi}{4} + \sin\theta\cos\frac{\pi}{4} > \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \sin\left(\theta + \frac{\pi}{4}\right) > \frac{1}{\sqrt{2}} \Rightarrow \frac{\pi}{4} < \theta + \frac{\pi}{4} < \frac{3\pi}{4} \Rightarrow 0 < \theta < \frac{\pi}{2} \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$$

Hence, θ lies in the first quadrant.

LEVEL-2

EXAMPLE 3 *Find the values of* β *so that the point* $(0, \beta)$ *lies on or inside the triangle having the sides* $3x + y + 2 = 0$, $2x - 3y + 5 = 0$ *and* $x + 4y - 14 = 0$.

SOLUTION Let *ABC* be the given triangle. The coordinates of the vertices of the triangle *ABC* are marked in Fig. 23.81. The point $P(0, \beta)$ will lie inside or on the triangle *ABC*, if the following three conditions hold simultaneously:

(i) *A* and *P* lie on the same side of *BC* (ii) *B* and *P* lie on the same side of *AC*,

(iii) *C* and *P* lie on the same side of *AB*.

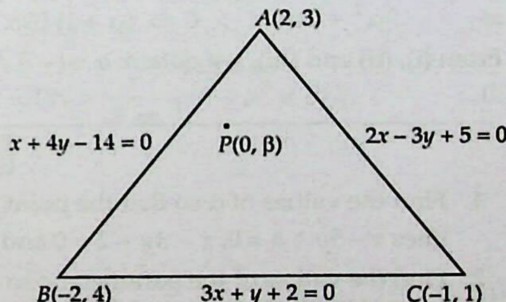

Fig. 23.81

Now,

A and *P* will lie on the same side of *BC*, if

$$(3 \times 2 + 3 + 2)(3 \times 0 + \beta + 2) > 0$$

$$\Rightarrow \quad 11(\beta + 2) \geq 0$$

$$\Rightarrow \quad \beta + 2 \geq 0$$

$$\Rightarrow \quad \beta \geq -2 \quad \text{...(i)}$$

B and *P* will lie on the same side of *AC*, if

$$(-2 \times 2 - 3 \times 4 + 5)(2 \times 0 - 3\beta + 5) \geq 0$$

$$\Rightarrow \quad -11(-3\beta + 5) \geq 0$$

$$\Rightarrow \quad 3\beta - 5 \geq 0$$

$$\Rightarrow \quad \beta \geq \frac{5}{3} \quad \text{...(ii)}$$

C and *P* will lie on the same side of *AB*, if

$$(-1 + 1 \times 4 - 14)(0 + 4\beta - 14) \geq 0$$

$$\Rightarrow \quad -22(2\beta - 7) \geq 0$$

$$\Rightarrow \quad 2\beta - 7 \leq 0$$

$$\Rightarrow \quad \beta \leq \frac{7}{2} \quad \text{...(iii)}$$

From (i), (ii) and (iii), we obtain that $\frac{5}{3} \leq \beta \leq \frac{7}{2}$ i.e., $\beta \in [5/3, 7/2]$

EXAMPLE 4 *Determine all values of α for which the point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0, x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$.*

SOLUTION Let ABC be the triangle, the equations of whose sides AB, BC and CA are respectively $2x + 3y - 1 = 0$ $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$. The coordinates of the vertices are $A(1/3, 1/9)$, $B(-7, 5)$ and $C(5/4, 7/8)$. If the point $P(\alpha, \alpha^2)$ lies inside the ΔABC, then

(i) A and P must lie on the same side of BC

(ii) B and P must lie on the same side of AC

(iii) C and P must lie on the same side of AB.

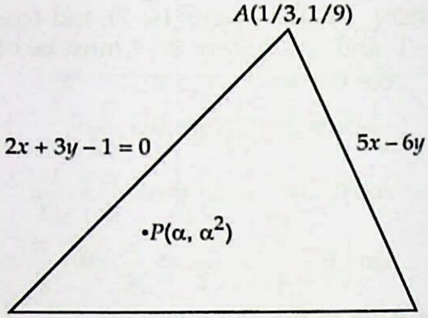

Fig. 23.82

Now,

A and P will lie on the same side of BC, if

$$\left(\frac{1}{3} + \frac{2}{9} - 3\right)\left(\alpha + 2\alpha^2 - 3\right) > 0$$

$$\Rightarrow \quad \alpha + 2\alpha^2 - 3 < 0$$

$$\Rightarrow \quad 2\alpha^2 + \alpha - 3 < 0$$

$$\Rightarrow \quad (\alpha - 1)(2\alpha + 3) < 0$$

$$\Rightarrow \quad \alpha \in (-3/2, 1) \qquad \text{...(i)}$$

B and P will lie on the same side of CA, if

$$(-35 - 30 - 1)(5\alpha - 6\alpha^2 - 1) > 0$$

$$\Rightarrow \quad 6\alpha^2 - 5\alpha + 1 > 0 \Rightarrow (3\alpha - 1)(2\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, 1/3) \cup (1/2, \infty) \qquad \text{...(ii)}$$

C and P will lie on the same side of AB, if

$$\left(\frac{5}{2} + \frac{21}{8} - 3\right)(2\alpha + 3\alpha^2 - 1) > 0$$

$$\Rightarrow \quad 3\alpha^2 + 2\alpha - 1 > 0 \Rightarrow (\alpha + 1)(3\alpha - 1) > 0 \Rightarrow \alpha \in (-\infty, -1) \cup (1/3, \infty) \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we obtain: $\alpha \in (-3/2, -1) \cup (1/2, 1)$

EXERCISE 23.14

LEVEL-2

1. Find the values of α so that the point $P(\alpha^2, \alpha)$ lies inside or on the triangle formed by the lines $x - 5y + 6 = 0$, $x - 3y + 2 = 0$ and $x - 2y - 3 = 0$.
2. Find the values of the parameter a so that the point $(a, 2)$ is an interior point of the triangle formed by the lines $x + y - 4 = 0$, $3x - 7y - 8 = 0$ and $4x - y - 31 = 0$.
3. Determine whether the point $(-3, 2)$ lies inside or outside the triangle whose sides are given by the equations $x + y - 4 = 0$, $3x - 7y + 8 = 0$, $4x - y - 31 = 0$.

ANSWERS

1. $\alpha \in [2, 3]$ 2. $a \in (22/3, 33/4)$ 3. Outside

23.13 DISTANCE OF A POINT FROM A LINE

THEOREM *Prove that the length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is*

$$\left|\frac{a x_1 + b y_1 + c}{\sqrt{a^2 + b^2}}\right|$$

PROOF The line $ax + by + c = 0$ meets x-axis at $y = 0$. Therefore, putting $y = 0$ in $ax + by + c = 0$, we get $x = -\frac{c}{a}$. Thus, the coordinates of the point A where the line $ax + by + c = 0$ meets x-axis are $(-c/a, 0)$. Similarly, the coordinates of B where the line cuts y-axis are $(0, -c/b)$.
Let $P(x_1, y_1)$ be the point. Draw $PN \perp AB$.
Now,

$$\text{Area of } \Delta PAB = \frac{1}{2}\left| x_1\left(0 + \frac{c}{b}\right) - \frac{c}{a}\left(-\frac{c}{b} - y_1\right) + 0\,(y_1 - 0)\right|$$

$$= \frac{1}{2}\left| \frac{cx_1}{b} + \frac{cy_1}{a} + \frac{c^2}{ab}\right| = \left|(ax_1 + by_1 + c)\,\frac{c}{2\,ab}\right| \quad \text{...(i)}$$

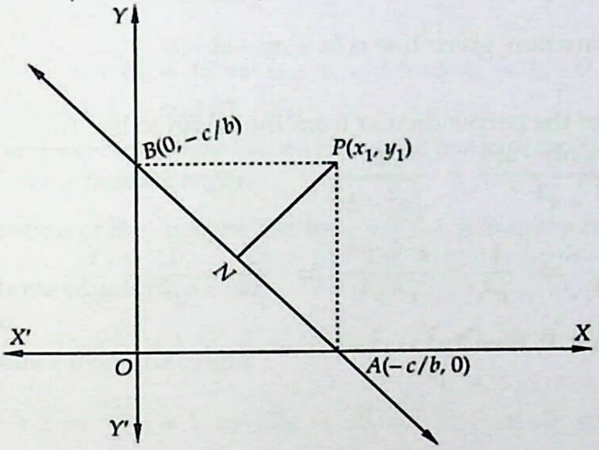

Fig. 23.83

Also,

$$\text{Area of } \Delta PAB = \frac{1}{2}\,AB \times PN = \frac{1}{2}\sqrt{\frac{c^2}{a^2} + \frac{c^2}{b^2}} \times PN = \frac{c}{2\,ab}\sqrt{a^2 + b^2} \times PN \quad \text{...(ii)}$$

From (i) and (ii), we get

$$\left|(ax_1 + by_1 + c)\,\frac{c}{2\,ab}\right| = \frac{c}{2\,ab}\sqrt{a^2 + b^2} \times PN$$

$$\Rightarrow \quad PN = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Q.E.D.

COROLLARY *The length of the perpendicular from the origin to the line $ax + by + c = 0$ is $\frac{|c|}{\sqrt{a^2 + b^2}}$.*

We may use the following algorithm for finding the length of the perpendicular from a point (x_1, y_1) to the line $ax + by + c = 0$.

ALGORITHM

STEP I *Write the equation of the line in the form $ax + by + c = 0$.*

STEP II *Substitute the coordinates x_1 and y_1 of the point in place of x and y respectively in the expression.*

STEP III *Divide the result obtained in step II by the square root of the sum of the squares of the coefficients of x and y.*

STEP IV *Take the modulus of the expression obtained in step III.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the line $12x - 5y + 9 = 0$ and the point (2, 1).*

SOLUTION Required distance $= \left| \dfrac{12 \times 2 - 5 \times 1 + 9}{\sqrt{12^2 + (-5)^2}} \right| = \dfrac{|24 - 5 + 9|}{13} = \dfrac{28}{13}$

EXAMPLE 2 *If p is the length of the perpendicular from the origin to the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$, then prove that*

(i) $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$ (ii) $a^4 + b^4 = 0$, if a^2, p^2, b^2 are in A.P. [NCERT]

SOLUTION (i) The equation given line is $bx + ay - ab = 0$ ...(i)

It is given that

p = Length of the perpendicular from the origin to line (i)

$$\Rightarrow \quad p = \frac{|b(0) + a(0) - ab|}{\sqrt{b^2 + a^2}} = \frac{ab}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \quad p^2 = \frac{a^2 b^2}{a^2 + b^2} \Rightarrow \frac{1}{p^2} = \frac{a^2 + b^2}{a^2 b^2} \Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

(ii) If a^2, p^2, b^2 are in A.P, then $2p^2 = a^2 + b^2$.

Now, $\dfrac{1}{p^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$ [from (i)]

$$\Rightarrow \quad p^2(a^2 + b^2) = a^2 b^2$$

$$\Rightarrow \quad \left(\frac{a^2 + b^2}{2}\right)(a^2 + b^2) = a^2 b^2 \qquad [\text{Using } 2p^2 = a^2 + b^2]$$

$$\Rightarrow \quad (a^2 + b^2)^2 = 2a^2 b^2 \Rightarrow a^4 + b^4 = 0.$$

EXAMPLE 3 *If p and p' be the perpendicular from the origin upon the straight lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x \cos\theta - y \sin\theta = a \cos 2\theta$. Prove that: $4p^2 + p'^2 = a^2$.* [NCERT]

SOLUTION We have,

p = Length of the perpendicular from (0, 0) to $x \sec\theta + y \operatorname{cosec}\theta - a = 0$

$$\Rightarrow \quad p = \left| \frac{0 \sec\theta + 0 \operatorname{cosec}\theta - a}{\sqrt{\sec^2\theta + \operatorname{cosec}^2\theta}} \right| = \frac{a \cos\theta \sin\theta}{\sqrt{\sin^2\theta + \cos^2\theta}} = a \sin\theta \cos\theta$$

and, p' = Length of the perpendicular from (0, 0) to $x \cos\theta - y \sin\theta - a \cos 2\theta = 0$

$$\Rightarrow \quad p' = \left| \frac{0 \cos\theta - 0 \sin\theta - a \cos 2\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} \right| = a \cos 2\theta.$$

Now, $4p^2 + p'^2 = 4a^2 \sin^2\theta \cos^2\theta + a^2 \cos^2 2\theta$

$$\Rightarrow \quad 4p^2 + p'^2 = a^2 (2 \sin\theta \cos\theta)^2 + a^2 \cos^2 2\theta = a^2 (\sin^2 2\theta + \cos^2 2\theta) = a^2.$$

EXAMPLE 4 *What are the points on x-axis whose perpendicular distance from the line $4x + 3y = 12$ is 4?*

SOLUTION Let the required point be $P(\alpha, 0)$. Then, Length of the perpendicular from $P(\alpha, 0)$ on $4x + 3y - 12 = 0$ is 4.

$$\therefore \quad \left|\frac{4\alpha + 3\times 0 - 12}{\sqrt{4^2+3^2}}\right| = 4$$

$$\Rightarrow \quad \left|\frac{4\alpha - 12}{5}\right| = 4 \Rightarrow |4\alpha - 12| = 20 \Rightarrow |\alpha - 3| = 5 \Rightarrow \alpha - 3 = \pm 5 \Rightarrow \alpha = 8, -2$$

Hence, the required points are (8, 0) and (– 2, 0).

EXAMPLE 5 *Find the points on y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.*

SOLUTION Let te required point be $P(0, \alpha)$. It is given that the length of the perpendicular from $P(0, \alpha)$ on $4x - 3y - 12 = 0$ is 3.

$$\therefore \quad \left|\frac{4\times 0 - 3\alpha - 12}{\sqrt{4^2+(-3)^2}}\right| = 3$$

$$\Rightarrow \quad |3\alpha + 12| = 15 \Rightarrow |\alpha + 4| = 15 \Rightarrow \alpha + 4 = \pm 5 \Rightarrow \alpha = 1, -9$$

Hence, the required points are (0, 1) and (0, – 9).

EXAMPLE 6 *Find the equation of the straight line which cuts off intercept on X-axis which is twice that on Y-axis and is at a unit distance from the origin.* **[NCERT EXEMPLAR]**

SOLUTION Let the equation of the straight line be $\frac{x}{a} + \frac{y}{b} = 1$. It is given that $a = 2b$.

Putting $a = 2b$ in the above equation, we get

$$x + 2y - 2b = 0 \qquad \text{...(i)}$$

This line is at a unit distance from the origin.

$$\therefore \quad \left|\frac{0 + 2\times 0 - 2b}{\sqrt{1^2+2^2}}\right| = 1 \Rightarrow \frac{|2b|}{\sqrt{5}} = 1 \Rightarrow |2b| = \sqrt{5} \Rightarrow 2b = \pm\sqrt{5} \Rightarrow b = \pm\frac{\sqrt{5}}{2}$$

Substituting the value of b in (i), we obtain $x + 2y \pm \sqrt{5} = 0$ as the equations of the required line.

EXAMPLE 7 *The equation of the base of an equilateral triangle is $x + y - 2 = 0$ and the opposite vertex has coordinates $(2, -1)$. Find the area of the triangle.* **[NCERT EXEMPLAR]**

SOLUTION Let p be the altitude of the given triangle and 'a' be the length of each side. Then,

p = Length of perpendicular from $(2, -1)$ on $x + y - 2 = 0$

$$\Rightarrow p = \left|\frac{2-1-2}{\sqrt{1^2+1^2}}\right| = \frac{1}{\sqrt{2}}$$

In ΔABD, we have

$$\sin 60° = \frac{p}{a} \Rightarrow \frac{\sqrt{3}}{2} = \frac{p}{a} \Rightarrow a = \frac{2p}{\sqrt{3}} \Rightarrow a = \frac{2}{\sqrt{3}}\times\frac{1}{\sqrt{2}} = \sqrt{\frac{2}{3}}$$

$$\therefore \text{ Area of the triangle} = \frac{\sqrt{3}a^2}{4} = \frac{\sqrt{3}}{4}\times\left(\sqrt{\frac{2}{3}}\right)^2 = \frac{1}{2\sqrt{3}} \text{ sq. units}$$

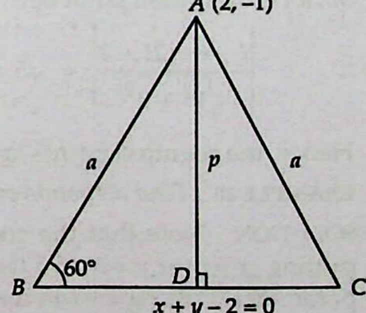

Fig. 23.84

EXAMPLE 8 *Prove that the length of perpendiculars from points $P(m^2, 2m)$, $Q(mn, m+n)$ and $R(n^2, 2n)$ to the line $x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$ are in G.P.*

SOLUTION Let a, b and c denote the lengths of perpendiculars drawn from P, Q and R respectively on the line $x\cos^2\theta + y\sin\theta\cos\theta + \sin^2\theta = 0$. Then,

$$a = \left|\frac{m^2 \cos^2\theta + 2m \sin\theta \cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(m\cos\theta + \sin\theta)^2}{\cos\theta}\right| \quad ...(i)$$

$$b = \left|\frac{mn \cos^2\theta + (m+n) \sin\theta \cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(m\cos\theta + \sin\theta)(n\cos\theta + \sin\theta)}{\cos\theta}\right| \quad ...(ii)$$

and, $$c = \left|\frac{n^2 \cos^2\theta + 2n \sin\theta \cos\theta + \sin^2\theta}{\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}}\right| = \left|\frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}\right| \quad ...(iii)$$

$$\therefore \quad b^2 = \frac{(m\cos\theta + \sin\theta)^2 (n\cos\theta + \sin\theta)^2}{\cos^2\theta} = \frac{(m\cos\theta + \sin\theta)^2}{\cos\theta} \times \frac{(n\cos\theta + \sin\theta)^2}{\cos\theta}$$

$\Rightarrow \quad b^2 = ac$

$\Rightarrow \quad a, b, c$ are in G.P.

LEVEL-2

EXAMPLE 9 *Find the coordinates of a point on $x + y + 3 = 0$, whose distance from $x + 2y + 2 = 0$ is $\sqrt{5}$.*

SOLUTION Let the required point be (x_1, y_1). Since it lies on $x + y + 3 = 0$.

$\therefore \quad x_1 + y_1 + 3 = 0 \quad ...(i)$

Now,

Length of the perpendicular from (x_1, y_1) to $x + 2y + 2 = 0$ is $\sqrt{5}$.

$$\Rightarrow \quad \left|\frac{x_1 + 2y_1 + 2}{\sqrt{1^2 + 2^2}}\right| = \sqrt{5} \Rightarrow x_1 + 2y_1 + 2 = \pm 5 \quad ...(ii)$$

Solving equation (i) and (ii), we get: $x_1 = -9$, $y_1 = 6$ and $x_1 = 1$, $y_1 = -4$.

Hence, the required points are $(-9, 6)$ and $(1, -4)$.

ALITER Putting $x = t$ in $x + y + 3 = 0$, we get: $y = -3 - t$.

So, let the required point be $(t, -3-t)$. This point is at a distance of $\sqrt{5}$ units from $x + 2y + 2 = 0$.

$$\therefore \quad \left|\frac{t - 6 - 2t + 2}{\sqrt{1^2 + 2^2}}\right| = \sqrt{5} \Rightarrow \left|\frac{-t-4}{\sqrt{5}}\right| = \sqrt{5} \Rightarrow t + 4 = \pm 5 \Rightarrow t = 1, -9$$

Hence, the required points are $(1, -4)$ and $(-9, 6)$.

EXAMPLE 10 *Find all points on $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.*

SOLUTION Note that the coordinates of an arbitrary point on $x + y = 4$ can be obtained by putting $x = t$ (or $y = t$)and then obtaining y (or x) from the equation of the line, where t is a parameter. Putting $x = t$ in the equation $x + y = 4$ of the given line, we obtain $y = 4 - t$.

So, coordinates of an arbitrary point on the given line are $P(t, 4-t)$.

Let $P(t, 4-t)$ be the required point. Then, distance of P from the line $4x + 3y - 10 = 0$ is unity.

$$\therefore \quad \left|\frac{4t + 3(4-t) - 10}{\sqrt{4^2 + 3^2}}\right| = 1 \Rightarrow |t + 2| = 5 \Rightarrow t + 2 = \pm 5 \Rightarrow t = -7 \text{ or}, t = 3$$

Hence, required points are $(-7, 11)$ and $(3, 1)$.

EXAMPLE 11 *Find the equations of lines passing through the point* $(1, 0)$ *and at a distance* $\frac{\sqrt{3}}{2}$ *from the origin.* **[NCERT EXEMPLAR]**

SOLUTION Let m be the slope of a line passing through $(1, 0)$. Then, its equation is

$$y-0=m(x-1) \text{ or, } mx-y-m=0 \quad \ldots(i)$$

It is given that line (i) is at a distance $\frac{\sqrt{3}}{2}$ from the origin.

$$\therefore \quad \left|\frac{m\times 0-0-m}{\sqrt{m^2+(-1)^2}}\right|=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad \frac{|m|}{\sqrt{m^2+1}}=\frac{\sqrt{3}}{2} \Rightarrow 4m^2=3m^2+3 \Rightarrow m^2=3 \Rightarrow m=\pm\sqrt{3}$$

Substituting the values of m in (i), we obtain

$\sqrt{3}x-y+\sqrt{3}=0$ and $\sqrt{3}x+y+\sqrt{3}=0$ as the equations of the line.

EXAMPLE 12 *Find the locus of a point which moves in such away that the square of its distnace from the point* $(3, -2)$ *is numerically equal to its distance from the line* $5x-12y=13$. **[NCERT EXEMPLAR]**

SOLUTION Let $P(h, k)$ be a variable point moving in such a way that the square of its distance from $A(3, -2)$ is numerically equal to its distance from the line $5x-12y=13$.

$$\therefore \quad (h-3)^2+(k+2)^2=\frac{|5h-12k+13|}{\sqrt{5^2+(-12)^2}}$$

$$\Rightarrow \quad 13\left\{(h-3)^2+(k+2)^2\right\}=\pm(5h-12k+13)$$

$$\Rightarrow \quad 13(h^2+k^2)-83h+64k+182=0 \text{ or, } 13(h^2+k^2)-73h+40k+156=0$$

Hence, the locus of (h, k) is

$$13(x^2+y^2)-83x+64y+182=0 \text{ or, } 13(x^2+y^2)-73x+40y+156=0$$

EXAMPLE 13 *A point moves such that its distance from the point* $(4, 0)$ *is half that of its distance from the line* $x=16$, *find its locus.* **[NCERT EXEMPLAR]**

SOLUTION Let $P(h, k)$ be the variable point.

By hypothesis

$$\sqrt{(h-4)^2+(k-0)^2}=\frac{1}{2}\left|\frac{h-16}{\sqrt{1^2+0^2}}\right|$$

$$\Rightarrow \quad 4\left\{(h-4)^2+k^2\right\}=(h-16)^2 \Rightarrow 3h^2+4k^2=192$$

Hence, the locus of (h, k) is $3x^2+4y^2=192$.

EXERCISE 23.15

LEVEL-1

1. Find the distance of the point $(4, 5)$ from the straight line $3x-5y+7=0$.
2. Find the perpendicular distance of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ from the origin.
3. Find the length of the perpendicular from the origin to the straight line joining the two points whose coordinates are $(a\cos\alpha, a\sin\alpha)$ and $(a\cos\beta, a\sin\beta)$.

4. Show that the perpendiculars let fall from any point on the straight line $2x+11y-5=0$ upon the two straight lines $24x+7y=20$ and $4x-3y-2=0$ are equal to each other.
5. Find the distance of the point of intersection of the lines $2x+3y=21$ and $3x-4y+11=0$ from the line $8x+6y+5=0$.
6. Find the length of the perpendicular from the point $(4,-7)$ to the line joining the origin and the point of intersection of the lines $2x-3y+14=0$ and $5x+4y-7=0$.
7. What are the points on X-axis whose perpendicular distance from the straight line $\frac{x}{a}+\frac{y}{b}=1$ is a?
8. Show that the product of perpendiculars on the line $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ from the points $(\pm\sqrt{a^2-b^2}, 0)$ is b^2. [NCERT]
9. Find the perpendicular distance from the origin of the perpendicular from the point $(1,2)$ upon the straight line $x-\sqrt{3}\,y+4=0$.
10. Find the distance of the point $(1,2)$ from the straight line with slope 5 and passing through the point of intersection of $x+2y=5$ and $x-3y=7$. [NCERT]
11. What are the points on y-axis whose distance from the line $\frac{x}{3}+\frac{y}{4}=1$ is 4 units?
12. In the triangle ABC with vertices $A(2,3)$, $B(4,-1)$ and $C(1,2)$, find the equation and the length of the altitude from the vertex A. [NCERT]
13. Show that the path of a moving point such that its distances from two lines $3x-2y=5$ and $3x+2y=5$ are equal is a straight line. [NCERT]
14. If sum of perpendicular distances of a variable point $P(x,y)$ from the lines $x+y-5=0$ and $3x-2y+7=0$ is always 10. Show that P must move on a line. [NCERT]

LEVEL-2

15. If the length of the perpendicular from the point $(1,1)$ to the line $ax-by+c=0$ be unity, show that $\frac{1}{c}+\frac{1}{a}-\frac{1}{b}=\frac{c}{2ab}$.

ANSWERS

1. $\frac{6}{\sqrt{34}}$ 2. $\cos\left(\frac{\theta-\phi}{2}\right)$ 3. $a\cos\left(\frac{\alpha-\beta}{2}\right)$ 5. $\frac{59}{10}$

6. 1 7. $\left\{\frac{a}{b}(b\pm\sqrt{a^2+b^2}), 0\right\}$ 9. $\frac{1}{2}(2+\sqrt{3})$

10. $\frac{132}{\sqrt{650}}$ 11. $(0, 32/3), (0, -8/3)$ 12. $x-y+1=0, \sqrt{2}$

HINTS TO NCERT & SELECTED PROBLEMS

2. The equation of the line joining $(\cos\theta, \sin\theta)$ and $(\cos\theta, \sin\phi)$ is

$$x\cos\left(\frac{\theta+\phi}{2}\right)+y\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

$\therefore$ Required distance $=\left|\dfrac{0\cos\left(\frac{\theta+\phi}{2}\right)+0\sin\left(\frac{\theta+\phi}{2}\right)-\cos\left(\frac{\theta+\phi}{2}\right)}{\sqrt{\cos^2\left(\frac{\theta+\phi}{2}\right)+\sin^2\left(\frac{\theta+\phi}{2}\right)}}\right|=\cos\left(\frac{\theta-\phi}{2}\right)$

8. Let p_1 and p_2 be the lengths of perpendiculars drawn from points $P(\sqrt{a^2-b^2}, 0)$ and $Q(-\sqrt{a^2-b^2}, 0)$ on the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. Then,

$$p_1 = \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta + \frac{0}{b}\sin\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|, \quad p_2 = \left|\frac{-\frac{\sqrt{a^2-b^2}}{a}\cos\theta + \frac{0}{b}\sin\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$$

$$\therefore \quad p_1 p_2 = \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right| \times \left|\frac{\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}\right|$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta - 1\right)\left(\frac{\sqrt{a^2-b^2}}{a}\cos\theta + 1\right)}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2-\theta}{b^2}}\right|$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{\frac{a^2-b^2}{a^2}\cos^2\theta - 1}{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}\right| = \left|\frac{(a^2-b^2)\cos^2\theta - a^2}{b^2\cos^2\theta + a^2\sin^2\theta}\right| b^2$$

$$\Rightarrow \quad p_1 p_2 = \left|\frac{-b^2\cos^2\theta - a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right| b^2 = b^2$$

10. Given lines $x + 2y = 5$ and $x - 3y = 7$ intersect at $\left(\frac{29}{5}, -\frac{2}{5}\right)$.

The equation of the line of slope 5 passing through this point is

$$y + \frac{2}{5} = 5\left(x - \frac{29}{5}\right) \text{ or, } 25x - 5y - 147 = 0$$

The distance d of the point (1, 2) from this line is

$$d = \left|\frac{25 - 10 - 147}{\sqrt{625 + 25}}\right| = \frac{132}{\sqrt{650}}$$

11. Let the required point on y-axis be $(0, a)$. Then,

$$\left|\frac{4 \times 0 + 3 \times a - 12}{\sqrt{4^2 + 3^2}}\right| = 4 \Rightarrow \left|\frac{3a - 12}{5}\right| = 4$$

$$\Rightarrow \quad 3a - 12 = \pm 20 \Rightarrow 3a = 32, -8 \Rightarrow a = \frac{32}{3}, -\frac{8}{3}$$

12. The equation of BC is

$$y+1 = \frac{2+1}{1-4}(x-4) \text{ or, } y+1 = -x+4$$

or, $$x+y-3 = 0$$

$\therefore$ $$AL = \left|\frac{2+3-3}{\sqrt{1+1}}\right| = \sqrt{2}$$

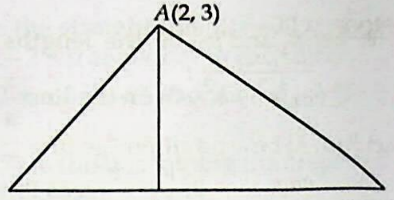

Fig. 23.85

Clearly, slope of BC having equation $x+y-3=0$ is -1. So, Slope of AL is 1. As it passes through A (2, 3). So, its equation is $y-3=1(x-2)$ or $x-y+1=0$

13. Let $P(h, k)$ be a moving point such that it is equidistant from the lines $3x-2y-5=0$ and $3x+2y-5=0$. Then,

$$\left|\frac{3h-2k-5}{\sqrt{9+4}}\right| = \left|\frac{3h+2k-5}{\sqrt{9+4}}\right|$$

$\Rightarrow$ $$|3h-2k-5| = |3h+2k-5|$$

$\Rightarrow$ $$3h-2k-5 = \pm(3h+2k-5)$$

$\Rightarrow$ $$4k = 0 \text{ or}, 6h-10 = 0 \Rightarrow k = 0 \text{ or}, 3h = 5$$

Hence, the locus of (h, k) is $y=0$ or $3x=5$, which are straight lines.

14. It is given that the sum of the perpendicular distances of a variable point $P(x, y)$ from the lines $x+y-5=0$ and $3x-2y+7=0$ is always 10.

$\therefore$ $$\frac{(x+y-5)}{\sqrt{2}} + \frac{(3x-2y+7)}{\sqrt{9+4}} = 10$$

$\Rightarrow$ $$(3\sqrt{2}+\sqrt{13})x + (\sqrt{13}-2\sqrt{2})y + (7\sqrt{2}-5\sqrt{13}-10\sqrt{26}) = 0$$

Clearly, it is a straight line.

23.14 DISTANCE BETWEEN PARALLEL LINES

If two lines are parallel, then they have the same distance between them throughout. Therefore to find the distance between two parallel lines choose an arbitrary point on one of them and find the length of the perpendicular on the other. In order to choose a point on a line, we give an arbitrary value to x or y and find the value of the other variable.

We may use the following algorithm to find the distance between two parallel lines.

ALGORITHM

Let the two parallel lines be $ax+by+c_1=0$ and $ax+by+c_2=0$. To find the distance between these two lines we proceed as follows:

<u>STEP I</u> *Choose a point on any one of the two lines by giving a particular value to x or y of your choice.*

<u>STEP II</u> *Find the length of the perpendicular from the chosen point in step I to the other line.*

<u>STEP III</u> *The length obtained in step II is the required distance between the parallel lines.*

THEOREM *Prove that the distance between two parallel lines* $ax+by+c_1 = 0$ *and* $ax+by+c_2 = 0$ *is given by* $\dfrac{|c_1-c_2|}{\sqrt{a^2+b^2}}$.

PROOF Given lines are

$$ax + by + c_1 = 0 \quad \text{...(i)}$$

$$ax + by + c_2 = 0 \quad \text{...(ii)}$$

Let $P(h, k)$ be a point on the line $ax + by + c_1 = 0$. Then,

$$ah + bk + c_1 = 0 \quad \text{...(iii)}$$

Clearly, distance 'd' between parallel lines (i) and (ii) is equal to the length of perpendicular from P on line (ii).

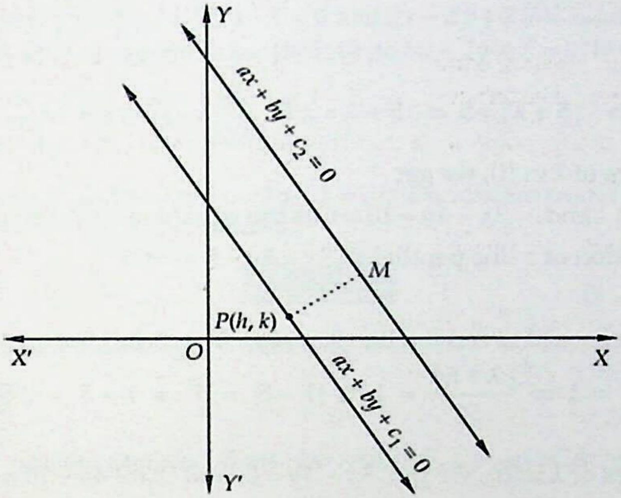

Fig. 23.86

$\therefore \quad d = PM$

$$\Rightarrow \quad d = \left|\frac{ah + bk + c_2}{\sqrt{a^2 + b^2}}\right| = \left|\frac{-c_1 + c_2}{\sqrt{a^2 + b^2}}\right| \quad \text{[From (iii): } ah + bk = -c_1]$$

$$\Rightarrow \quad d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}} \quad \text{Q.E.D.}$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the distance between the parallel lines* $3x - 4y + 9 = 0$ *and* $6x - 8y - 15 = 0$.

SOLUTION Putting $y = 0$ in $3x - 4y + 9 = 0$, we get $x = -3$. Thus, $(-3, 0)$ is a point on the line $3x - 4y + 9$.

Length of the perpendicular from $(-3, 0)$ to $6x - 8y - 15 = 0$ is given by

$$d = \frac{|-3 \times 6 - 8 \times 0 - 15|}{\sqrt{6^2 + (-8)^2}} = \frac{33}{10}$$

Hence, the distance between the given lines is $\frac{33}{10}$ units.

ALITER Given lines are

$$3x - 4y + 9 = 0 \quad \text{...(i)} \quad \text{and,} \quad 6x - 8y - 15 = 0 \text{ or, } 3x - 4y - \frac{15}{2} = 0 \quad \text{...(ii)}$$

$$\therefore \quad \text{Required distance} = \frac{\left|9 - \left(\frac{-15}{2}\right)\right|}{\sqrt{3^2 + (-4)^2}} = \frac{9 + \frac{15}{2}}{5} = \frac{33}{10}$$

EXAMPLE 2 *Find the equations of lines parallel to $3x-4y-5=0$ at a unit distance from it.*

SOLUTION Equation of any line parallel to $3x-4y-5=0$ is

$$3x-4y+\lambda=0 \qquad \text{...(i)}$$

Putting $x=-1$ in $3x-4y-5=0$, we get $y=-2$. Therefore, $(-1,-2)$ is a point on $3x-4y-5=0$.

Since the distance between the two lines is one unit. Therefore, the length of the perpendicular from $(-1,-2)$ to $3x-4y+\lambda=0$ is one unit.

i.e. $$\frac{|3\times-1-4\times-2+\lambda|}{\sqrt{3^2+(-4)^2}}=1$$

$$\Rightarrow \quad \frac{|5+\lambda|}{5}=1 \Rightarrow |5+\lambda|=5 \Rightarrow 5+\lambda=\pm5 \Rightarrow \lambda=0 \text{ or } -10.$$

Substituting the values of λ in (i), we get

$3x-4y=0$ and, $3x-4y-10=0$ as the equations of the required lines.

ALITER Let the equation of a line parallel to $3x-4y-5=0$ be

$$3x-4y+\lambda=0 \qquad \text{...(i)}$$

It is given that the distance between the line $3x-4y-5=0$ and line (i) is 1 unit.

$$\therefore \quad \frac{|\lambda-(-5)|}{\sqrt{3^2+(-4)^2}}=1 \Rightarrow \frac{|\lambda+5|}{5}=1 \Rightarrow |\lambda+5|=5 \Rightarrow \lambda+5=\pm5 \Rightarrow \lambda=0,-10.$$

Substituting the values of λ in (i), we get $3x-4y=0$ and $3x-4y-10=0$ as the equations of required lines.

EXAMPLE 3 *Two sides of a square lie on the lines $x+y=1$ and $x+y+2=0$. What is its area?*

SOLUTION clearly, the length of the side of the square is equal to the distance between the parallel lines

$$x+y-1=0 \quad \text{...(i)} \qquad \text{and} \qquad x+y+2=0 \qquad \text{...(ii)}$$

Putting $x=0$ in (i), we get $y=1$. So $(0, 1)$ is a point on line (i).

$\therefore$ Distance between the parallel lines

$$= \{\text{Length of the perpendicular from } (0,1) \text{ to } x+y+2=0\} = \frac{|0+1+2|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}.$$

Thus, the length of the side of the square is $\frac{3}{\sqrt{2}}$ and hence its area is $\left(\frac{3}{\sqrt{2}}\right)^2=\frac{9}{2}$ square units

ALITER The equations of parallel sides of the square are $x+y-1=0$ and $x+y+2=0$.

$\therefore$ Length of the side of the square $=$ Distance between parallel side $=\frac{|2-(-1)|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}$

Hence, Area of the square $=(\text{Side})^2=\left(\frac{3}{\sqrt{2}}\right)^2=\frac{9}{2}$ sq. units.

EXAMPLE 4 *Prove that the line $5x-2y-1=0$ is mid-parallel to the lines $5x-2y-9=0$ and $5x-2y+7=0$.*

SOLUTION Clearly, the slope of each of the given lines is same equal to $5/2$. Hence, the line $5x-2y-1=0$ is parallel to each of the given lines.

In order to prove that the line $5x-2y-1=0$ is mid-parallel to the given lines it is sufficient to show that the line $5x-2y-1=0$ is equidistant from the given lines.

Putting $y=0$ in $5x-2y-1=0$, we get $x=1/5$. So, the coordinates of a point on $5x-2y-1=0$ are $(1/5, 0)$.

The distance d_1 between the lines $5x-2y-1=0$ and $5x-2y-9=0$ is given by

d_1 = Length of the perpendicular from $(1/5, 0)$ to $5x-2y-9=0$

$$\Rightarrow \quad d_1 = \left|\frac{5\times(1/5)-2\times 0-9}{\sqrt{5^2+(-2)^2}}\right| = \frac{8}{\sqrt{29}}$$

The distance d_2 between the lines $5x-2y-1=0$ and $5x-2y+7=0$ is given by

d_2 = Length of the perpendicular $(1/5, 0)$ to $5x-2y+7=0$

$$\Rightarrow \quad d_2 = \left|\frac{5\times(1/5)-2\times 0+7}{\sqrt{5^2+(-2)^2}}\right| = \frac{8}{\sqrt{29}}$$

Clearly, $d_1=d_2$. Consequently the line $5x-2y-1=0$ is equidistant from the lines $5x-2y-9=0$ and $5x-2y+7=0$. Hence, the result follows.

LEVEL-2

EXAMPLE 5 *Prove that the parallelogram formed by the lines* $\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1, \frac{x}{a}+\frac{y}{b}=2$ *and* $\frac{x}{b}+\frac{y}{a}=2$ *is a rhombus.*

SOLUTION Let the given straight lines be AB, BC, CD and CA whose equations are respectively

$$\frac{x}{a}+\frac{y}{b}=1 \quad ...(i) \qquad \frac{x}{b}+\frac{y}{a}=1 \quad ...(ii) \qquad \frac{x}{a}+\frac{y}{b}=2 \quad ...(iii) \quad \text{and} \quad \frac{x}{b}+\frac{y}{a}=2 \quad ...(iv)$$

Putting $y=0$ in (i) and (ii), we get $x=a$ and $x=b$ respectively. So, the coordinate points on lines (i) and (ii) are (a, 0) and (b, 0) respectively.

Now, d_1 = Distance between the parallel lines (i) and (iii)

$\Rightarrow$ d_1 = Length of the perpendicular drawn from $(a, 0)$ upon the line (iii)

$$\Rightarrow \quad d_1 = \left|\frac{\frac{a}{a}+\frac{0}{b}-2}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2+b^2}}$$

and, d_2 = Distance between the parallel lines (ii) and (iv)

$\Rightarrow$ d_2 = Length of the perpendicular drawn from $(b, 0)$ upon the line (iv)

$$\Rightarrow \quad d_2 = \left|\frac{\frac{b}{b}+\frac{0}{a}-2}{\sqrt{\frac{1}{b^2}+\frac{1}{a^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2+b^2}}$$

Clearly, $d_1=d_2$ i.e. the distances between the pairs of parallel lines are equal. Hence, $ABCD$ is a rhombus.

EXAMPLE 6 *Find the equation of the line mid-way between the parallel lines* $9x+6y-7=0$ *and* $3x+2y+6=0$. **[NCERT]**

SOLUTION The equations of the lines are

$$3x+2y-\frac{7}{3}=0 \qquad ...(i)$$

$$3x+2y+6=0 \qquad ...(ii)$$

Let the equation of the line mid-way between the parallel lines (i) and (ii) be

$$3x + 2y + \lambda = 0 \quad \text{...(iii)}$$

Then,

Distance between lines (i) and (iii) = Distance between lines (ii) and (iii)

$$\Rightarrow \quad \frac{\left|\lambda + \frac{7}{3}\right|}{\sqrt{9+4}} = \frac{|\lambda - 6|}{\sqrt{9+4}}$$

$$\Rightarrow \quad \left|\lambda + \frac{7}{3}\right| = |\lambda - 6| \Rightarrow \lambda + \frac{7}{3} = -\lambda + 6 \Rightarrow 2\lambda = 6 - \frac{7}{3} \Rightarrow 2\lambda = \frac{11}{3} \Rightarrow \lambda = \frac{11}{6}$$

Hence, the equation of the required line is $3x + 2y + \frac{11}{6} = 0$.

EXERCISE 23.16

LEVEL-1

1. Determine the distance between the following pair of parallel lines :
   (i) $4x - 3y - 9 = 0$ and $4x - 3y - 24 = 0$
   (ii) $8x + 15y - 34 = 0$ and $8x + 15y + 31 = 0$
   (iii) $y = mx + c$ and $y = mx + d$
   (iv) $4x + 3y - 11 = 0$ and $8x + 6y = 15$
2. The equations of two sides of a square are $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$. Find the area of the square.
3. Find the equation of two straight lines which are parallel to $x + 7y + 2 = 0$ and at unit distance from the point $(1, -1)$.
4. Prove that the lines $2x + 3y = 19$ and $2x + 3y + 7 = 0$ are equidistant from the line $2x + 3y = 6$.
5. Find the equation of the line mid-way between the parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.
6. Find the ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$. [NCERT EXEMPLAR]

ANSWERS

1. (i) 3 units (ii) $\frac{65}{17}$ units (iii) $\frac{|c - d|}{\sqrt{1 + m^2}}$ (iv) $\frac{7}{10}$ units

2. 49 sq. units 3. $x + 7y + 6 \pm 5\sqrt{2} = 0$ 5. $18x + 12y + 11 = 0$ 6. $3 : 7$

23.15 AREA OF A PARALLELOGRAM

Let $ABCD$ be a parallelogram the equations of whose sides AB, BC, CD and DA are $a_1 x + b_1 y + c_1 = 0$, $a_2 x + b_2 y + c_2 = 0$, $a_1 x + b_1 y + d_1 = 0$ and $a_2 x + b_2 y + d_2 = 0$. Let p_1 and p_2 be the distances between the pairs of parallel sides of the parallelogram. In Δs ALD and AMB, we obtain

$$\sin\theta = \frac{p_1}{AD} \text{ and, } \sin\theta = \frac{p_2}{AB} \text{ respectively.}$$

$$\Rightarrow \quad AD = \frac{p_1}{\sin\theta} \text{ and } AB = \frac{p_2}{\sin\theta}$$

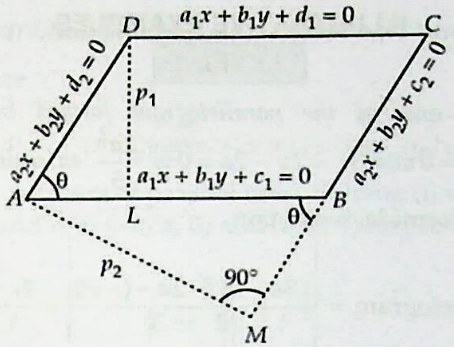

Fig. 23.87

Now,

$$\text{Area of parallelogram } ABCD = AB \times p_1 = \frac{p_1 p_2}{\sin\theta} \qquad \left[\because AB = \frac{p_2}{\sin\theta}\right]$$

Also,

$$\text{Area of parallelogram } ABCD = AD \times p_2 = \frac{p_1 p_2}{\sin\theta} \qquad \left[\because AD = \frac{p_1}{\sin\theta}\right]$$

Thus, area of a parallelogram is $\dfrac{p_1 p_2}{\sin\theta}$, *where* p_1 *and* p_2 *are the distances between pairs of parallel sides and* θ *is the angle between two adjacent sides.*

Let m_1 and m_2 be the slopes of sides AB and AD respectively. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and, } m_2 = -\frac{a_2}{b_2}$$

The angle θ between AB and AD is given by

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \quad \tan\theta = \frac{-\dfrac{a_1}{b_1} + \dfrac{a_2}{b_2}}{1 + \dfrac{a_1 a_2}{b_1 b_2}}$$

$$\Rightarrow \quad \tan\theta = \frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2} \Rightarrow \sin\theta = \frac{a_2 b_1 - a_1 b_2}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}$$

We have,

$$p_1 = \text{Distance between parallel sides } AB \text{ and } AD = \frac{|c_1 - d_1|}{\sqrt{{a_1}^2 + {b_1}^2}}$$

and,

$$p_2 = \text{Distance between parallel sides } AD \text{ and } BC = \frac{|c_2 - d_2|}{\sqrt{{a_2}^2 + {b_2}^2}}$$

$$\therefore \quad \text{Area of parallelogram } ABCD = \frac{p_1 p_2}{\sin\theta} = \frac{|c_1 - d_1||c_2 - d_2|}{|a_2 b_1 - a_1 b_2|} = \left|\frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\right|$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Show that the area of the parallelogram formed by the lines* $2x-3y+a=0$, $3x-2y-a=0$, $2x-3y+3a=0$ *and* $3x-2y-2a=0$ *is* $\frac{2a^2}{5}$ *sq. units.*

SOLUTION Using the above formula, we obtain

$$\text{Area of given parallelogram} = \left|\frac{(3a-a)\{-2a-(-a)\}}{\begin{vmatrix}2 & -3\\ 3 & -2\end{vmatrix}}\right| = \frac{2a^2}{5} \text{ sq. units}$$

EXAMPLE 2 *Prove that the area of the parallelogram formed by the lines* $x\cos\alpha + y\sin\alpha = p$, $x\cos\alpha + y\sin\alpha = q$, $x\cos\beta + y\sin\beta = r$ *and* $x\cos\beta + y\sin\beta = s$ *is* $\pm(p-q)(r-s)\operatorname{cosec}(\alpha-\beta)$.

SOLUTION The equations of the sides of the parallelogram are:

$x\cos\alpha + y\sin\alpha - p = 0$, $x\cos\alpha + y\sin\alpha - q = 0$, $x\cos\beta + y\sin\beta - r = 0$

and, $x\cos\beta + y\sin\beta - s = 0$

$$\therefore \quad \text{Area of the parallelogram} = \left|\frac{\{(-p)-(-q)\}\{(-r)-(-s)\}}{\begin{vmatrix}\cos\alpha & \sin\alpha\\ \cos\beta & \sin\beta\end{vmatrix}}\right|$$

$$= \left|\frac{(q-p)(s-r)}{(\cos\alpha\sin\beta - \sin\alpha\cos\beta)}\right| = \pm\frac{(p-q)(r-s)}{\sin(\alpha-\beta)}$$

LEVEL-2

EXAMPLE 3 *Prove that the four straight lines* $\frac{x}{a}+\frac{y}{b}=1$, $\frac{x}{b}+\frac{y}{a}=1$, $\frac{x}{a}+\frac{y}{b}=2$ *and* $\frac{x}{b}+\frac{y}{a}=2$ *form a rhombus. Find its area.*

SOLUTION The equations of the four sides are

$\frac{x}{a}+\frac{y}{b} = 1$ ...(i) $\qquad \frac{x}{b}+\frac{y}{a} = 1$ ...(ii)

$\frac{x}{a}+\frac{y}{b} = 2$ ...(iii) $\qquad \frac{x}{b}+\frac{y}{a} = 2$ ...(iv)

Clearly, (i), (iii) and (ii), (iv) form two sets of parallel lines. So, the four lines form a parallelogram.

Let p_1 be the distance between parallel lines (i) and (iii) and p_2 be the distance between (ii) and (iv). Then,

$$p_1 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}} \text{ and, } p_2 = \left|\frac{2-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right| = \frac{ab}{\sqrt{a^2+b^2}} \quad \left[\text{Using}: d = \left|\frac{c_1-c_2}{\sqrt{a^2+b^2}}\right|\right]$$

Clearly, $p_1 = p_2$. So, the given lines form a rhombus.

$$\therefore \quad \text{Area of the rhombus} = \left|\frac{(2-1)(2-1)}{\begin{vmatrix}1/a & 1/b\\ 1/b & 1/a\end{vmatrix}}\right| = \left|\frac{1}{(1/a^2-1/b^2)}\right| = \frac{a^2b^2}{|b^2-a^2|}$$

EXAMPLE 4 *Show that the four lines $ax \pm by \pm c = 0$ enclose a rhombus whose area is $2c^2/ab$.*

SOLUTION The four lines are

$$ax + by + c = 0 \quad \text{...(i)} \qquad ax + by - c = 0 \quad \text{...(ii)}$$

$$ax - by + c = 0 \quad \text{...(iii)} \qquad ax - by - c = 0 \quad \text{...(iv)}$$

Clearly, (i), (ii) and (iii), (iv) are pairs of parallel lines. Solving (i) with (iii) and (ii) with (iv), we obtain the coordinates of C and A as $(-c/a, 0)$ and $(c/a, 0)$ respectively.

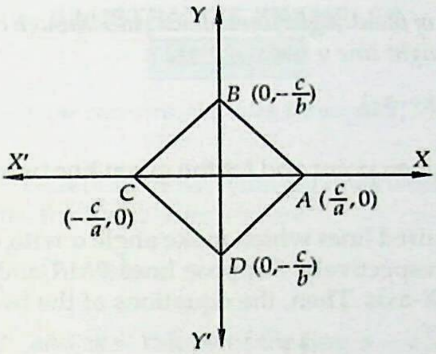

Fig. 23.88

Solving (ii) with (iii) and (i) with (iv), we obtain the coordinates of B and D as $(0, c/b)$ and $(0, -c/b)$ respectively.

Thus, the vertices of the parallelogram $ABCD$ are

$A\,(c/a, 0)$, $B\,(0, c/b)$, $C\,(-c/a, 0)$ and $D\,(0, -c/b)$

This shows that the vertices of the parallelogram are on the coordinate axes such that one diagonal is along x-axis and other along y-axis. Since, the diagonals are at right angles. Hence, $ABCD$ is a rhombus.

$$\text{Area of the rhombus} = \frac{1}{2} AC \times BD = \frac{1}{2}\left(\frac{2c}{a} \times \frac{2c}{b}\right) = \frac{2c^2}{ab}.$$

EXERCISE 23.17

LEVEL-1

1. Prove that the area of the parallelogram formed by the lines $a_1 x + b_1 y + c_1 = 0, a_1 x + b_1 y + d_1 = 0,\ a_2 x + b_2 y + c_2 = 0,\ a_2 x + b_2 y + d_2 = 0$ is $\left|\frac{(d_1 - c_1)(d_2 - c_2)}{a_1 b_2 - a_2 b_1}\right|$ sq. units.

   Deduce the condition for these lines to form a rhombus.

2. Prove that the area of the parallelogram formed by the lines $3x - 4y + a = 0$, $3x - 4y + 3a = 0, 4x - 3y - a = 0$ and $4x - 3y - 2a = 0$ is $\frac{2a^2}{7}$ sq. units.

LEVEL-2

3. Show that the diagonals of the parallelogram whose sides are $lx + my + n = 0$, $lx + my + n' = 0, mx + ly + n = 0$ and $mx + ly + n' = 0$ include an angle $\pi/2$.

HINTS TO SELECTED PROBLEMS

1. Area $= \dfrac{p_1 p_2}{\sin\theta}$, where p_1, p_2 are the distance between the pairs of parallel lines and θ is he angle between two adjacent sides. For, rhombus use $p_1 = p_2$.
3. Use: $p_1 = p_2$.

23.16 EQUATIONS OF LINES PASSING THROUGH A GIVEN POINT AND MAKING A GIVEN ANGLE WITH A LINE

THEOREM *Prove the equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are*

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

PROOF Let $P(x_1, y_1)$ be the given point and let the given line be LMN, making an angle θ with the axis of x. Then, $m = \tan\theta$.

Let PMR and PNS be two required lines which make angle α with the given line. Let these lines meet the axis of X at R and S respectively. Suppose lines PMR and PNS make angles θ_1 and θ_2 with the positive direction of X-axis. Then, the equations of the two required lines are

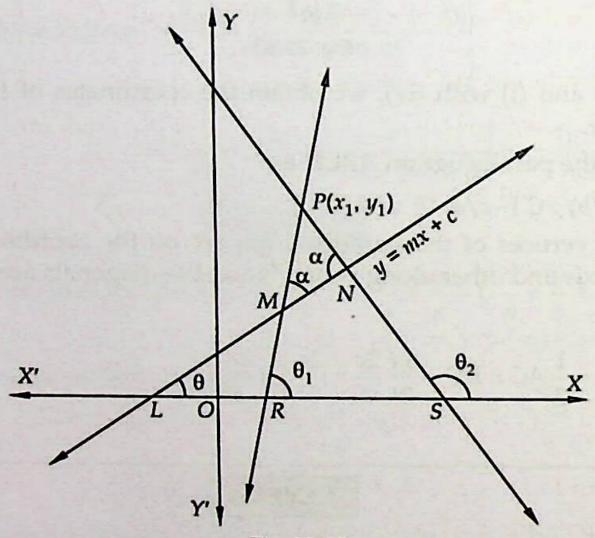

Fig. 23.89

$$y - y_1 = \tan\theta_1 (x - x_1) \quad \text{...(i)}$$

and, $$y - y_2 = \tan\theta_2 (x - x_2) \quad \text{...(ii)}$$

In ΔLMR, we have

$$\theta_1 = \theta + \alpha.$$

In ΔLNS, we have

$$\theta_2 = \theta + 180 - \alpha$$

Now, $\theta_1 = \theta + \alpha$

$\Rightarrow \quad \tan\theta_1 = \tan(\theta + \alpha) = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \dfrac{m + \tan\alpha}{1 - m\tan\alpha}$

and, $\theta_2 = \theta + 180° - \alpha$

$\Rightarrow \quad \tan\theta_2 = \tan(180 + \theta - \alpha) = \tan(\theta - \alpha)$

$$\Rightarrow \quad \tan\theta_2 = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha} = \frac{m - \tan\alpha}{1 + m\tan\alpha}$$

On substituting the values of $\tan\theta_1$ and, $\tan\theta_2$ in (i) and (ii), we get

$$y - y_1 = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - x_1) \quad \text{and} \quad y - y_1 = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - x_1)$$

These are the equations of the required lines.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equations of the two straight lines through (7, 9) and making an angle of 60° with the line $x - \sqrt{3}\,y - 2\sqrt{3} = 0$.*

SOLUTION We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle α with the line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here, $x_1 = 7, y_1 = 9, \alpha = 60°$ and $m =$ (Slope of the line $x - \sqrt{3}\,y - 2\sqrt{3} = 0$) $= \frac{1}{\sqrt{3}}$

So, equations of required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60°}{1 - \frac{1}{\sqrt{3}}\tan 60°}(x - 7) \quad \text{and} \quad y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60°}{1 + \frac{1}{\sqrt{3}}\tan 60°}(x - 7)$$

or, $\quad (y - 9)\left(1 - \frac{1}{\sqrt{3}}\tan 60°\right) = \left(\frac{1}{\sqrt{3}} + \tan 60°\right)(x - 7)$

and, $\quad (y - 9)\left(1 + \frac{1}{\sqrt{3}}\tan 60°\right) = \left(\frac{1}{\sqrt{3}} - \tan 60°\right)(x - 7)$

or, $\quad 0 = \left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)(x - 7)$, and $(y - 9)(2) = \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)(x - 7)$

or, $\quad x - 7 = 0$ and, $x + \sqrt{3}\,y = 7 + 9\sqrt{3}$.

Hence, the required lines are $x = 7$ and $x + \sqrt{3}\,y = 7 + 9\sqrt{3}$.

EXAMPLE 2 *Show that the equations of the straight lines passing through the point (3, – 2) and inclined at 60° to the line $\sqrt{3}\,x + y = 1$ are $y + 2 = 0$ and $y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0$.* **[NCERT EXEMPLAR]**

SOLUTION The equations of two straight lines passing through a point (x_1, y_1) and making an angle α with $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Here, $x_1 = 3, y_1 = -2, \alpha = 60°$ and $m =$ (Slope of $\sqrt{3}\,x + y = 1$) $= -\sqrt{3}$.

So, the equations of the required lines are

$$y + 2 = \frac{-\sqrt{3} - \tan 60°}{1 - \sqrt{3}\tan 60°}(x - 3) \quad \text{and} \quad y + 2 = \frac{-\sqrt{3} + \tan 60°}{1 + \sqrt{3}\tan 60°}(x - 3)$$

or, $\quad y + 2 = \sqrt{3}(x - 3) \quad$ and $\quad y + 2 = 0$

or, $\quad y - \sqrt{3}\,x + 2 + 3\sqrt{3} = 0 \quad$ and $\quad y + 2 = 0$.

EXAMPLE 3 *Find the equations of the straight lines through (3, 2) which make acute angle of 45° with the line $x - 2y - 3 = 0$.* **[NCERT]**

SOLUTION Here, $x_1 = 3, y_1 = 2, \alpha = 45°$ and, $m =$ (Slope of the line $x - 2y - 3 = 0) = \frac{1}{2}$

So, equations of required lines are

$$y - 2 = \frac{\frac{1}{2} - \tan 45°}{1 + \frac{1}{2}\tan 45°}(x - 3) \text{ and, } y - 2 = \frac{\frac{1}{2} + \tan 45°}{1 - \frac{1}{2}\tan 45°}(x - 3)$$

$$\Rightarrow \quad y - 2 = -\frac{1}{3}(x - 3) \quad \text{and } y - 2 = 3(x - 3)$$

$$\Rightarrow \quad x + 3y = 9 \text{ and } 3x - y = 7.$$

ALITER The equation of any line through (3, 2) is

$$y - 2 = m(x - 3) \qquad \text{...(i)}$$

where m is the slope of the line and is to be determined.

It is given that the line (i) makes an acute angle of 45° with the line $x - 2y - 3 = 0$. Therefore,

$$\tan 45° = \pm \frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} \qquad \left[\text{Using: } \tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}\right]$$

$$\Rightarrow \quad 1 = \pm\left(\frac{2m - 1}{2 + m}\right) \Rightarrow 2 + m = \pm(2m - 1) \Rightarrow m = 3, -\frac{1}{3}$$

Putting the values of m in (i), equations of required lines are

$$y - 2 = 3(x - 3) \text{ and } y - 2 = -\frac{1}{3}(x - 3) \text{ or, } 3x - y = 7 \text{ and } x + 3y = 9.$$

EXAMPLE 4 *A vertex of an equilateral triangle is (2, 3) and the opposite side is $x + y = 2$. Find the equations of other sides.* **[NCERT EXEMPLAR]**

SOLUTION Let A (2, 3) be one vertex and $x + y = 2$ be the opposite side of an equilateral triangle. Clearly, remaining two sides pass through the point A (2, 3) and make an angle of 60° with $x + y = 2$.

Let m be the slope of $x + y = 2$. Then, $m = -1$. Fig. 23.90 the equations of the other two sides are

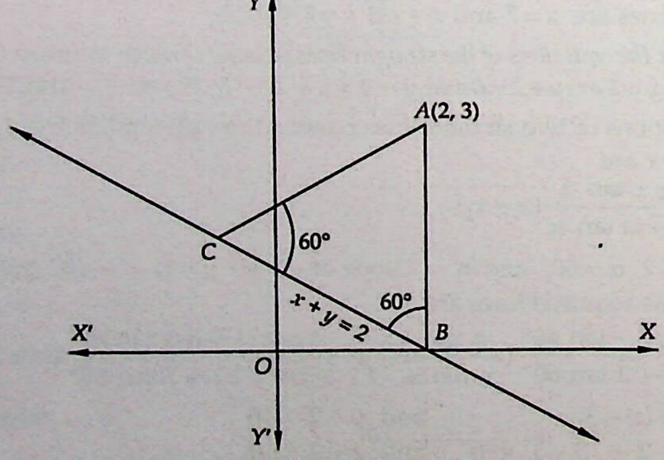

Fig. 23.90

$$y - 3 = \frac{-1 - \tan 60°}{1 - \tan 60°}(x-2) \text{ and } y - 3 = \frac{-1 + \tan 60°}{1 + \tan 60°}(x-2)$$

$$\Rightarrow \quad y - 3 = \frac{-(1+\sqrt{3})}{1-\sqrt{3}}(x-2) \quad \text{and} \quad y - 3 = \frac{\sqrt{3}-1}{\sqrt{3}+1}(x-2)$$

$$\Rightarrow \quad y - 3 = (2+\sqrt{3})(x-2) \quad \text{and,} \quad (y-3)(2+\sqrt{3}) = x - 2$$

$$\Rightarrow \quad (2+\sqrt{3})x - y = 1 + 2\sqrt{3} \quad \text{and,} \quad (2-\sqrt{3})x - y = 1 - 2\sqrt{3}.$$

EXAMPLE 5 *Show that the equation of the straight line through the origin making angle ϕ with the line $y = mx + b$ is $\frac{y}{x} = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}$.* **[NCERT]**

SOLUTION Let m_1 be the slope of the desired line. Then, its equation is

$$(y - 0) = m_1(x - 0) \qquad [\text{Using : } y - y_1 = m(x - x_1)]$$

$$\Rightarrow \quad y = m_1 x \qquad \text{...(i)}$$

If $y = mx + b$ makes angle ϕ with line (i), then

$$\tan\phi = \pm\frac{m - m_1}{1 + mm_1} \Rightarrow m_1 = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}$$

Putting the values of m_1 in (i), we get: $y = \frac{m \pm \tan\phi}{1 \mp m\tan\phi}x$ as the required equations of two lines.

EXAMPLE 6 *The opposite angular points of a square are (3, 4) and (1, – 1). Find the coordinates of the other two vertices.*

SOLUTION We have,

$$\text{Slope of } AC = \frac{-1-4}{1-3} = \frac{5}{2}$$

Clearly, AB and AD pass through A (3, 4) and make angle of 45° with AC whose slope is 5/2. Therefore, equations of AB and AD are given by

$$y - 4 = \frac{\frac{5}{2} \mp \tan 45°}{1 \pm \frac{5}{2}\tan 45°}(x-3)$$

$$\Rightarrow \quad y - 4 = \frac{5 \mp 2}{2 \pm 5}(x-3)$$

$$\Rightarrow \quad y - 4 = \frac{3}{7}(x-3) \quad \text{and} \quad y - 4 = -\frac{7}{3}(x-3)$$

$$\Rightarrow \quad 3x - 7y + 19 = 0 \text{ and } 7x + 3y - 33 = 0.$$

Thus, equations of AB and AD are $3x - 7y + 19 = 0$ and $7x + 3y - 33 = 0$ respectively. Since BC is a line parallel to AD. Therefore, equation of BC is $7x + 3y + \lambda = 0$. This passes through (1, –1).

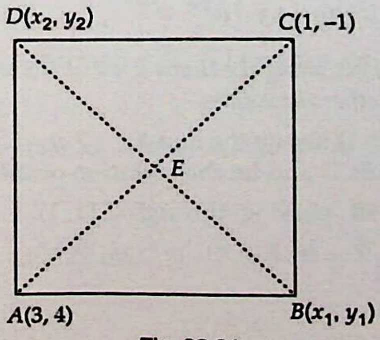

Fig. 23.91

$\therefore \quad 7-3+\lambda=0 \Rightarrow \lambda=-4.$

So, the equation of BC is $7x+3y-4=0$.

Since B is the point of intersection of AB and BC. Therefore, solving equations of AB and BC by cross-multiplication, we get

$$\frac{x}{-29}=\frac{y}{155}=\frac{1}{58} \Rightarrow x=-\frac{1}{2},\ y=\frac{5}{2}$$

So, the coordinates of B are $(-1/2, 5/2)$.

Let the coordinates of D be (x_2, y_2). Then, the coordinates of the mid-point of BD are

$$\left(\frac{x_2-\frac{1}{2}}{2}, \frac{y_2+\frac{5}{2}}{2}\right)$$

The coordinates of the mid-point of AC are $(2, 3/2)$.

Since the diagonals AC and BD bisect each other.

$$\therefore \quad \frac{x_2-\frac{1}{2}}{2}=2 \text{ and } \frac{y_2+\frac{5}{2}}{2}=\frac{3}{2} \Rightarrow x_2=\frac{9}{2} \text{ and } y_2=\frac{1}{2}$$

So, the coordinates of D are $(9/2, 1/2)$. Hence, the other two vertices are $(-1/2, 5/2)$ and $(9/2, 1/2)$.

LEVEL-2

EXAMPLE 7 *If one diagonal of a square is along the line $8x-15y=0$ and one of its vertex is at $(1, 2)$, then find the equations of sides of the square passing through this vertex.* **[NCERT EXEMPLAR]**

SOLUTION Let $ABCD$ be the given square whose one vertex is at $A\ (1, 2)$ and the diagonal BD is along the line $8x-15y=0$.

We observe that the sides AB and AD pass through the vertex $A\ (1, 2)$ and make 45° angle with the diagonal BD of slope $m=8/15$. Therefore, equations of AB and AD are given by

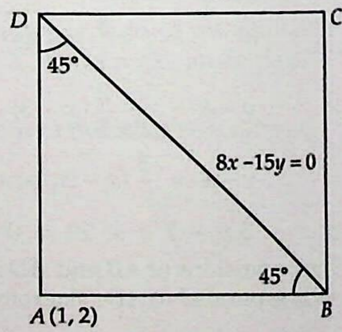

Fig. 23.92

$$y-2=\frac{\frac{8}{15}\pm\tan 45^\circ}{1\mp\frac{8}{15}\tan 45^\circ}(x-1)$$

or, $$y-2=\frac{8\pm 15}{15\mp 8}(x-1)$$

or, $$y-2=\frac{23}{7}(x-1) \text{ and } y-2=-\frac{7}{23}(x-1)$$

or, $$23x-7y-9=0 \text{ and } 7x+23y-53=0$$

EXAMPLE 8 *One side of a rectangle lies along the line $4x+7y+5=0$. Two of its vertices are $(-3, 1)$ and $(1, 1)$. Find the equations of the other three sides.*

SOLUTION Clearly, the point $(-3, 1)$ lies on the line $4x+7y+5=0$. So let $A\ (-3, 1)$ and $C\ (1, 1)$ be two vertices of the rectangle $ABCD$ and let the equation of the side AB be $4x+7y+5=0$.

Clearly, DC is a line parallel to AB passing through $C\ (1, 1)$.

Let the equation of DC be $4x+7y+\lambda=0$ ...(i)

This passes through $(1, 1)$.

$\therefore \quad 4+7+\lambda=0 \Rightarrow \lambda=-11.$

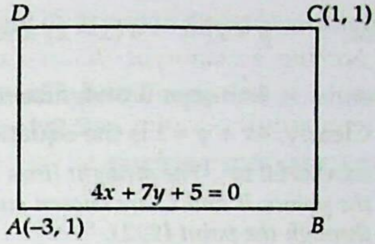

Fig. 23.93

Putting the value of λ in (i), we get

$$4x + 7y - 11 = 0.$$

This is the equation of the side DC.

Since AD is a line through A perpendicular to AB, therefore equation of AD is

$$7x - 4y + \lambda_1 = 0 \qquad \text{...(ii)}$$

This will pass through $(-3, 1)$, if

$$-21 - 4 + \lambda_1 = 0 \Rightarrow \lambda_1 = 25.$$

Putting the value of λ_1 in the equation of AD is $7x - 4y + 25 = 0$.

Clearly, BC is a line perpendicular to AB passing through $C(1, 1)$. Let the equation of BC be

$$7x - 4y + \mu = 0 \qquad \text{...(iii)}$$

This will pass through $(1, 1)$ if

$$7 - 4 + \mu = 0 \Rightarrow \mu = -3.$$

Putting the value of μ in (iii), we get $7x - 4y - 3 = 0$. This is the equation of BC.

EXAMPLE 9 *A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B. Find the equation to the line AC so that $AB = AC$.*

SOLUTION The lines AB and BC meet at a point B. Let α be the angle between them. Then,

$$\tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}, \text{ where, } m_1 = \text{Slope of } AB = -4,\ m_2 = \text{Slope of } BC = \frac{3}{4}$$

$$\Rightarrow \quad \tan\alpha = \frac{-4 - 3/4}{1 + (-4) \times 3/4} = \frac{19}{8}$$

It is given that $AB = AC$. Therefore, triangle ABC is an isosceles triangle.

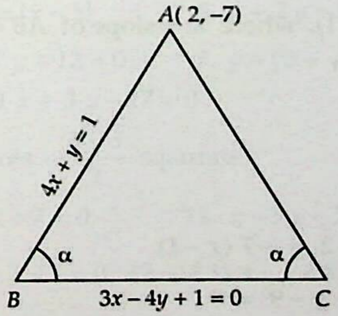

Fig. 23.94

Clearly, AB and AC both pass through $A(2, -7)$ and are equally inclined to $3x - 4y + 1 = 0$. So, their equations are given by

$$(y + 7) = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - 2), \text{ where, } m = \text{Slope of } BC = \frac{3}{4} \text{ and } \tan\alpha = \frac{19}{8}$$

$$\text{or,} \quad (y + 7) = \frac{\frac{3}{4} + \frac{19}{8}}{1 - \frac{3}{4} \times \frac{19}{8}}(x - 2) \text{ and } y + 7 = \frac{\frac{3}{4} - \frac{19}{8}}{1 + \frac{3}{4} \times \frac{19}{8}}(x - 2)$$

or, $\quad y+7 = -4(x-2)$ and $y+7 = -\frac{52}{89}(x-2)$

$\Rightarrow \quad 4x+y = 1$ and $52x+89y+519 = 0$

Clearly, $4x+y=1$ is the equation of AB. So, equation of AC is $52x+89y+519=0$.

EXAMPLE 10 *The straight lines $3x+4y=5$ and $4x-3y=15$ intersect at the point A. On these lines, the points B and C are chosen such that AB = AC. Find the possible equations of the line BC passing through the point (1, 2).*

SOLUTION Let m_1 and m_2 be the slopes of the lines $3x+4y=5$ and $4x-3y=15$ respectively. Then, $m_1 = -\frac{3}{4}$ and $m_2 = \frac{4}{3}$. Clearly, $m_1 . m_2 = -1$. So, lines AB and AC are at right angle.

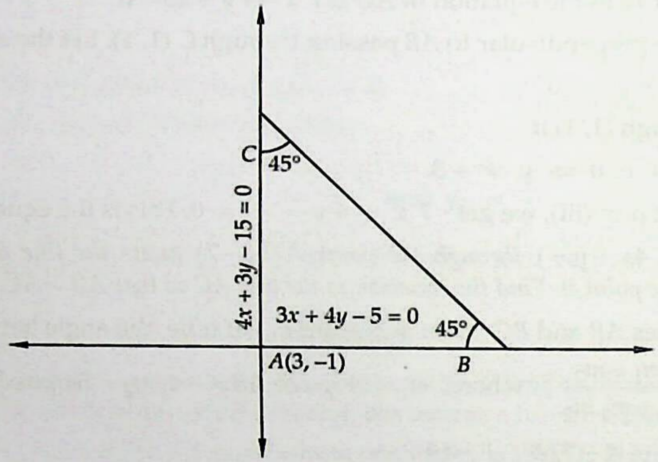

Fig. 23.95

Thus the triangle ABC is a right angled isosceles triangle. Hence, the line BC through (1, 2) will make an angle of 45° with the given lines. So, possible equations of BC are

$$(y-2) = \frac{m \pm \tan 45°}{1 \mp m \tan 45°}(x-1), \text{ where } m = \text{slope of } AB = -\frac{3}{4}$$

$$\Rightarrow \quad y-2 = \frac{-3/4 \pm 1}{1 \mp (-3/4)}(x-1)$$

$$\Rightarrow \quad (y-2) = \frac{-3 \pm 4}{4 \pm 3}(x-1)$$

$$\Rightarrow \quad y-2 = \frac{1}{7}(x-1) \text{ and } y-2 = -7(x-1)$$

$$\Rightarrow \quad x-7y+13 = 0 \text{ and } 7x+y-9 = 0$$

EXERCISE 23.18

LEVEL-1

1. Find the equation of the straight lines passing through the origin and making an angle of 45° with the straight line $\sqrt{3}\,x+y=11$.
2. Find the equations to the straight lines which pass through the origin and are inclined at an angle of 75° to the straight line $x+y+\sqrt{3}(y-x)=a$.
3. Find the equations of the straight lines passing through (2, – 1) and making an angle of 45° with the line $6x+5y-8=0$.
4. Find the equations to the straight lines which pass through the point (h, k) and are inclined at angle $\tan^{-1} m$ to the straight line $y=mx+c$.

5. Find the equations to the straight lines passing through the point (2, 3) and inclined at an angle of 45° to the line $3x + y - 5 = 0$.
6. Find the equations to the sides of an isosceles right angled triangle the equation of whose hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point (2, 2).

[NCERT EXEMPLAR]

LEVEL-2

7. The equation of one side of an equilateral triangle is $x - y = 0$ and one vertex is $(2 + \sqrt{3}, 5)$. Prove that a second side is $y + (2 - \sqrt{3})x = 6$ and find the equation of the third side.
8. Find the equations of the two straight lines through (1, 2) forming two sides of a square of which $4x + 7y = 12$ is one diagonal.
9. Find the equations of two straight lines passing through (1, 2) and making an angle of 60° with the line $x + y = 0$. Find also the area of the triangle formed by the three lines.
10. Two sides of an isosceles triangle are given by the equations $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Determine the equation of the third side.
11. Show that the point $(3, -5)$ lies between the parallel lines $2x + 3y - 7 = 0$ and $2x + 3y + 12 = 0$ and find the equation of lines through $(3, -5)$ cutting the above lines at an angle of 45°.
12. The equation of the base of an equilateral triangle is $x + y = 2$ and its vertex is $(2, -1)$. Find the length and equations of its sides. **[NCERT EXEMPLAR]**
13. If two opposite vertices of a square are (1, 2) and (5, 8), find the coordinates of its other two vertices and the equations of its sides.

ANSWERS

1. $y = (\sqrt{3} \pm 2)x$ 2. $x = 0, y + \sqrt{3}x = 0$ 3. $11x - y - 23 = 0$
4. $y = k, (1 - m^2)(y - k) = 2m(x - h)$ 5. $x + 2y - 8 = 0$ and $2x - y - 1 = 0$
6. $7x + y - 16 = 0$ and $x - 7y + 12 = 0$ 7. $y + (2 + \sqrt{3})x = 12 + 4\sqrt{3}$
8. $3x - 11y + 19 = 0$ and $11x + 3y - 17 = 0$
9. $y - 2 = (2 \pm \sqrt{3})(x - 1)$, Area $= \dfrac{3\sqrt{3}}{2}$ sq. units
10. $x - 3y - 31 = 0$ or $3x + y + 7 = 0$ 11. $x - 5y - 28 = 0$ or $5x + y - 10 = 0$
12. $\sqrt{\dfrac{2}{3}}$, $(2 - \sqrt{3})x - y - 5 + 2\sqrt{3} = 0$, $(2 + \sqrt{3})x - y - 5 - 2\sqrt{3} = 0$
13. (6, 3), (0, 7), $x - 5y + 9 = 0$; $5x + y - 7 = 0$; $5x + y - 33 = 0$; $x - 5y + 35 = 0$

HINTS TO SELECTED PROBLEMS

6. The two sides pass through (2, 2) and make an angle of 45° with the line $3x + 4y = 4$.
8. The two sides pass through (1, 2) and make 45° angle with the diagonal having slope $m = -4/7$.
10. Any line through $(1, -10)$ is $y + 10 = m(x - 1)$. Since it makes equal angles, say θ, with the given lines. Therefore,

$$\tan\theta = \frac{m - 7}{1 + 7m} = -\frac{m - (-1)}{1 + m(-1)} \Rightarrow m = -3 \text{ or}, \frac{1}{3}.$$

23.17 FAMILY OF LINES THROUGH THE INTERSECTION OF TWO GIVEN LINES

THEOREM *Prove that the equation of the family of lines passing through the intersection of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is $(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0$, where λ is a parameter.*

PROOF The equations of the lines are

$$a_1 x + b_1 y + c_1 = 0 \quad \text{...(i)}$$

and, $$a_2 x + b_2 y + c_2 = 0 \quad \text{...(ii)}$$

Let (α, β) be the point of intersection of the lines (i) and (ii). Then,

$$a_1 \alpha + b_1 \beta + c_1 = 0 \quad \text{...(iii)}$$

and, $$a_2 \alpha + b_2 \beta + c_2 = 0 \quad \text{...(iv)}$$

Now, consider the equation

$$(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0 \quad \text{...(v)}$$

Clearly, this is a first degree equation in x and y. So it represents a straight line.

We have,

$$(a_1 \alpha + b_1 \beta + c_1) + \lambda (a_2 \alpha + b_2 \beta + c_2) = 0 + \lambda 0 = 0 \quad \text{[Using (iii) and (iv)]}$$

So, (α, β) lies on the line given in equation (v).

Hence, equation (v) represents family lines through the point of intersection of lines (i) and (ii).

Thus, the family of straight lines through the intersection of lines $L_1 = a_1 x + b_1 y + c_1 = 0$ and $L_2 = a_2 x + b_2 y + c_2 = 0$ is

$$(a_1 x + b_1 y + c_1) + \lambda (a_2 x + b_2 y + c_2) = 0$$

i.e. $$L_1 + \lambda L_2 = 0$$

Q.E.D.

REMARK *The equation $L_1 + \lambda L_2 = 0$ represents a line passing through the intersection of the lines $L_1 = 0$ and $L_2 = 0$ which is a fixed point. Hence, $L_1 + \lambda L_2 = 0$ represents a family of straight lines, for different values of λ, which pass through a fixed-point.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the straight line which passes through the point $(2, -3)$ and the point of intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$.*

SOLUTION Any line through the intersection of the lines $x + y + 4 = 0$ and $3x - y - 8 = 0$ has the equation

$$(x + y + 4) + \lambda (3x - y - 8) = 0 \quad \text{...(i)}$$

This will pass through $(2, -3)$, if

$$(2 - 3 + 4) + \lambda (6 + 3 - 8) = 0 \Rightarrow 3 + \lambda = 0 \Rightarrow \lambda = -3.$$

Putting the value of λ in (i), the equation of the required line is $2x - y - 7 = 0$.

ALITER Solving the equations $x + y + 4 = 0$ and $3x - y - 8 = 0$ by cross-multiplication, we get $x = 1, y = -5$. So, the two lines intersect at the point $(1, -5)$. Hence, the required line passes through points $(2, -3)$ and $(1, -5)$ and so its equation is

$$y + 3 = -\frac{5 + 3}{1 - 2}(x - 2) \Rightarrow 2x - y - 7 = 0.$$

EXAMPLE 2 *Find the equation of the straight line which passes through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ and is parallel to (i) x-axis (ii) y-axis (iii) $3x + 4y = 14$.*

SOLUTION The equation of any line through the intersection of the lines $x - y - 1 = 0$ and $2x - 3y + 1 = 0$ is

$$(x - y - 1) + \lambda(2x - 3y + 1) = 0$$

or, $$(2\lambda + 1)x - y(3\lambda + 1) + \lambda - 1 = 0 \qquad \text{...(i)}$$

(i) The line in (i) will be parallel to x-axis if it is of the form $y = \text{constant}$.

$\therefore$ Coefficient of x in (i) $= 0$

$\Rightarrow$ $2\lambda + 1 = 0 \Rightarrow \lambda = -1/2$.

Putting $\lambda = -1/2$ in (i), we get $y = 3$ as is the equation of the required line.

(ii) The line in (i) will be parallel to y-axis if it is of the form $x = \lambda$.

$\therefore$ Coefficient of y in (i) $= 0$

$\Rightarrow$ $3\lambda + 1 = 0 \Rightarrow \lambda = -1/3$.

Putting $\lambda = -1/3$ in (i), we get $x = 4$ as the equation of the required line.

(iii) The line in (i) is parallel to the line $3x + 4y - 14 = 0$. Therefore, their slopes are equal

So, slope of line in equation (i) is same as that of the line $3x + 4y - 14 = 0$.

i.e. $$\frac{2\lambda + 1}{3\lambda + 1} = -\frac{3}{4} \Rightarrow \lambda = -\frac{7}{17}.$$

Putting this value of λ in (i) we get the equation of the required line as $3x + 4y = 24$.

EXAMPLE 3 *Find the equation of the straight line which passes through the point of intersection of the straight lines $x + 2y = 5$ and $3x + 7y = 17$ and is perpendicular to the straight line $3x + 4y = 10$.*

SOLUTION The equation of any line through the intersection of the lines $x + 2y - 5 = 0$ and $3x + 7y - 17 = 0$ is

$$(x + 2y - 5) + \lambda(3x + 7y - 17) = 0$$

or, $$x(3\lambda + 1) + y(7\lambda + 2) - (17\lambda + 5) = 0 \qquad \text{...(i)}$$

This is perpendicular to the line $3x + 4y = 10$.

$\therefore$ Product of their slopes $= -1$.

$$\Rightarrow \quad -\left(\frac{3\lambda + 1}{7\lambda + 2}\right)\left(-\frac{3}{4}\right) = -1 \Rightarrow \lambda = -\frac{11}{37}$$

Putting this value of λ in (i), the equation of the required line is $4x - 3y + 2 = 0$.

EXAMPLE 4 *Obtain the equations of the lines passing through the intersection of lines $4x - 3y - 1 = 0$ and $2x - 5y + 3 = 0$ and equally inclined to the axes.*

SOLUTION The equation of any line through the intersection of the given lines is

$$(4x - 3y - 1) + \lambda(2x - 5y + 3) = 0$$

or, $$x(2\lambda + 4) - y(5\lambda + 3) + 3\lambda - 1 = 0 \qquad \text{...(i)}$$

Let m be the slope of this line. Then, $m = \dfrac{2\lambda + 4}{5\lambda + 3}$.

As the line is equally inclined with the axes.

$$\therefore \quad m = \pm 1 \Rightarrow \frac{2\lambda + 4}{5\lambda + 3} = \pm 1 \Rightarrow \lambda = -1 \text{ or, } 1/3$$

Putting the values of λ in (i), we obtain

$x + y - 2 = 0$ and $x = y$ as the equations of the required lines.

LEVEL-2

EXAMPLE 5 *If t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$, where λ is an arbitrary constant. Then prove that the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ always passes through a fixed point. Also, find that point.*

SOLUTION Since t_1 and t_2 are roots of the equation $t^2 + \lambda t + 1 = 0$.

$\therefore \quad t_1 + t_2 = -\lambda$ and, $t_1 t_2 = 1$ ...(i)

The equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2) \text{ or, } y(t_2 + t_1) = 2x + 2at_1 t_2 \quad \text{...(ii)}$$

Putting $t_1 + t_2 = -\lambda$ and $t_1 t_2 = 1$ in (ii), we get

$$2x + 2a = -\lambda y \Rightarrow (x + a) + \left(-\frac{\lambda}{2}\right)y = 0$$

This is a line passing through the intersection of the lines $x + a = 0$ and $y = 0$ which is a fixed point having coordinates $(-a, 0)$.

EXAMPLE 6 *Show that the straight lines given by $x(a + 2b) + y(a + 3b) = a + b$ for different values of a and b pass through a fixed point.*

SOLUTION The given equation can be written as

$$a(x + y - 1) + b(2x + 3y - 1) = 0$$

or, $\quad (x + y - 1) + \lambda(2x + 3y - 1) = 0$, where $\lambda = b/a$

This is of the form $L_1 + \lambda L_2 = 0$. So it represents a line passing through the intersection of $x + y - 1 = 0$ and $2x + 3y - 1 = 0$. Solving these two equations, we get the point $(2, -1)$, which is the fixed point.

EXAMPLE 7 *If a, b, c are variables such that $3a + 2b + 4c = 0$, then show that the family of lines given by $ax + by + c = 0$ pass through a fixed point. Also, find that point.*

SOLUTION We have,

$$3a + 2b + 4c = 0 \Rightarrow c = -\frac{3}{4}a - \frac{1}{2}b$$

Substituting this value of c in $ax + by + c = 0$, we get

$$ax + by - \frac{3}{4}a - \frac{1}{2}b = 0$$

$$\Rightarrow \quad a\left(x - \frac{3}{4}\right) + b\left(y - \frac{1}{2}\right) = 0 \Rightarrow \left(x - \frac{3}{4}\right) + \lambda\left(y - \frac{1}{2}\right) = 0, \text{ where } \lambda = \frac{b}{a}.$$

This equation is of the form $L_1 + \lambda L_2 = 0$ which represents a straight line through the intersection of the line $L_1 = 0$ and $L_2 = 0$ i.e. $x - \frac{3}{4} = 0$ and $y - \frac{1}{2} = 0$. Solving these two equations, we get the point $(3/4, 1/2)$, which is a fixed point.

ALITER We have

$$3a + 2b + 4c = 0$$

$$\Rightarrow \quad \left(\frac{3}{4}\right)a + \left(\frac{2}{4}\right)b + c = 0$$

$\Rightarrow \quad (3/4, 1/2)$ lies on the line $ax + by + c = 0$

Hence, the given family of lines pass through the point $(3/4, 1/2)$.

EXAMPLE 8 *Let a, b, c be parameters. Then, the equation $ax + by + c = 0$ will represent a family of straight lines passing through a fixed-point iff there exists a linear relation between a, b and c.*

SOLUTION First, let the equation $ax + by + c = 0$ represent a family of straight lines passing through a fixed-point (α, β) (say) for different values of a, b, c. Then, we have to prove that there is a linear relation between a, b and c. Since, $ax + by + c = 0$ represents a family of lines pasing through (α, β). Therefore,

$a\alpha + b\beta + c = 0$, which is the required linear relation between a, b and c.

Conversely, let there be a linear relation between the parameters a, b, c. Then, we have to prove that the equation $ax + by + c = 0$ represents a family of lines passing through a fixed-point.

Let the linear relation be

$la + mb + nc = 0$, where l, m, n are constants.

$$\Rightarrow \quad \left(\frac{l}{n}\right) a + \left(\frac{m}{n}\right) b + c = 0$$

$\Rightarrow \quad ax + by + c = 0$ passes through the fixed-point $(l/n, m/n)$

EXAMPLE 9 *If the algebraic sum of the perpendiculars from the points* $(2, 0), (0, 2), (1, 1)$ *to a variable line be zero, then prove that the line passes through a fixed-point whose coordinates are* $(1, 1)$.

[NCERT EXEMPLAR]

SOLUTION Let the variable line be $ax + by = 1$. It is given that the algebraic sum of the perpendiculars from the points $(2, 0), (0, 2)$ and $(1, 1)$ to this line is zero. Therefore,

$$\frac{2a + ob - 1}{\sqrt{a^2 + b^2}} + \frac{oa + 2b - 1}{\sqrt{a^2 + b^2}} + \frac{a + b - 1}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \quad 3a + 3b - 3 = 0 \Rightarrow a + b - 1 = 0 \Rightarrow a + b = 1.$$

This is a linear relation between the parameters a and b. So, the equation $ax + by = 1$ represents a family of straight lines passing through a fixed-point. Comparing $ax + by = 1$ and $a + b = 1$, we obtain that the coordinates of the fixed-point are $(1, 1)$.

EXAMPLE 10 *A ray of light is sent along the line* $x - 2y - 3 = 0$ *upon reaching the line* $3x - 2y - 5 = 0$, *the ray is reflected from it. Find the equation of the line containing the reflected ray.*

SOLUTION The point of intersection of the lines $x - 2y - 3 = 0$ and $3x - 2y - 5 = 0$ is $B(1, -1)$. BP is the normal at P. Clearly, BP passes through $B(1, -1)$ and is perpendicular to $3x - 2y - 5 = 0$. So, equation of BP is

$$y + 1 = -(2/3)(x - 1) \text{ or, } 2x + 3y + 1 = 0$$

Since the reflected ray passes through the intersection of $x - 2y - 3 = 0$ and the normal $2x + 3y + 1 = 0$. Therefore, equation of the reflected ray is

$$x - 2y - 3 + \lambda(2x + 3y + 1) = 0 \quad \text{...(i)}$$

or, $$x(1 + 2\lambda) + y(3\lambda - 2) + (\lambda - 3) = 0 \quad \text{...(ii)}$$

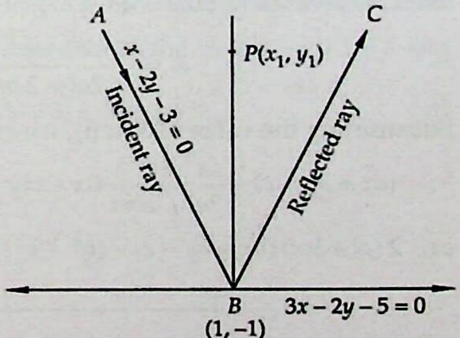

Fig. 23.96

Let $P(x_1, y_1)$ be an arbitrary point on the normal at P. Then, P is equidistant from the incident ray and the reflected ray.

$$\therefore \quad \left|\frac{x_1 - 2y_1 - 3}{\sqrt{1 + 4}}\right| = \left|\frac{(x_1 - 2y_1 - 3) + \lambda(2x_1 + 3y_1 + 1)}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}}\right|$$

$$\Rightarrow \quad \left|\frac{x_1 - 2y_1 - 3}{\sqrt{5}}\right| = \left|\frac{(x_1 - 2y_1 - 3) + \lambda \times 0}{\sqrt{(1 + 2\lambda)^2 + (3\lambda - 2)^2}}\right| \quad \left[\begin{array}{l} \because (x_1, y_1) \text{ lies on } 2x + 3y + 1 = 0 \\ \therefore 2x_1 + 3y_1 + 1 = 0 \end{array}\right]$$

$$\Rightarrow \quad \left|\frac{1}{\sqrt{5}}\right| = \left|\frac{1}{\sqrt{(2\lambda+1)^2+(3\lambda-2)^2}}\right|$$

$$\Rightarrow \quad 5 = (2\lambda+1)^2+(3\lambda-2)^2$$

$$\Rightarrow \quad 13\lambda^2 - 8\lambda = 0 \Rightarrow \lambda = 0 \text{ or, } \lambda = \frac{8}{13}$$

Since $\lambda = 0$ is not possible. So, $\lambda = \frac{8}{13}$ Putting the value of λ in (ii), we get $29x - 2y - 31 = 0$ as the equation of the line containing the reflected ray.

EXAMPLE 11 *Lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv lx + my + n = 0$ intersect at a point P and make an angle θ with each other. Find the equation of the line L different from L_2 which passes through P and makes the same angle with L_1.*

SOLUTION Since the required line L passes through the intersection of $L_1 = 0$ and $L_2 = 0$. So, equation of the required line L is

$$L_1 + \lambda L_2 = 0 \text{ i.e. } (ax + by + c) + \lambda(lx + my + n) = 0 \qquad \text{...(i)}$$

where λ is a parameter.

Since L_1 is the angle bisector of $L = 0$ and $L_2 = 0$. Therefore any point $A(x_1, y_1)$ on L_1 is equidistant from $L = 0$ and $L_2 = 0$.

$$\Rightarrow \quad \frac{|l x_1 + m y_1 + n|}{\sqrt{l^2+m^2}} = \frac{|(a x_1 + b y_1 + c) + \lambda(l x_1 + m y_1 + n)|}{\sqrt{(a+\lambda l)^2 + (b+\lambda m)^2}} \qquad \text{...(ii)}$$

But, $A(x_1, y_1)$ lies on L_1. So, it must satisfy the equation of L_1 i.e. $ax + by + c = 0$.

$$\therefore \quad ax_1 + by_1 + c = 0$$

Substituting $a x_1 + b y_1 + c = 0$ in (ii), we get

$$\frac{|l x_1 + m y_1 + n|}{\sqrt{l^2+m^2}} = \frac{|0 + \lambda(l x_1 + m y_1 + n)|}{\sqrt{(a+\lambda l)^2 + (b+\lambda m)^2}}$$

$$\Rightarrow \quad \lambda^2(l^2+m^2) = (a+\lambda l)^2 + (b+\lambda m)^2$$

$$\Rightarrow \quad \lambda = -\frac{a^2+b^2}{2al + 2bm}$$

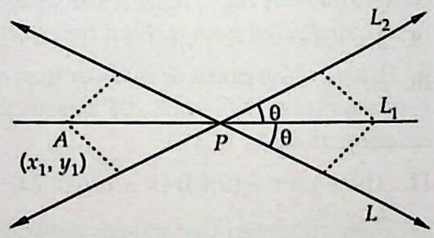

Fig. 23.97

Substituting the value of λ in (i), we get

$$(ax + by + c) - \frac{(a^2+b^2)}{2al + 2bm}(lx + my + n) = 0$$

or, $2(al + bm)(ax + by + c) - (a^2 + b^2)(lx + my + n) = 0$ as the equation of the required line L.

EXERCISE 23.19

LEVEL-1

1. Find the equation of a straight line through the point of intersection of the lines $4x - 3y = 0$ and $2x - 5y + 3 = 0$ and parallel to $4x + 5y + 6 = 0$.
2. Find the equation of a straight line passing through the point of intersection of $x + 2y + 3 = 0$ and $3x + 4y + 7 = 0$ and perpendicular to the straight line $x - y + 9 = 0$.
3. Find the equation of the line passing through the point of intersection of $2x - 7y + 11 = 0$ and $x + 3y - 8 = 0$ and is parallel to (i) x-axis (ii) y-axis.
4. Find the equation of the straight line passing through the point of intersection of $2x + 3y + 1 = 0$ and $3x - 5y - 5 = 0$ and equally inclined to the axes.

5. Find the equation of the straight line drawn through the point of intersection of the lines $x + y = 4$ and $2x - 3y = 1$ and perpendicular to the line cutting off intercepts 5, 6 on the axes.

LEVEL-2

6. Prove that the family of lines represented by $x(1+\lambda) + y(2-\lambda) + 5 = 0$, λ being arbitrary, pass through a fixed point. Also, find the fixed point.

7. Show that the straight lines given by $(2+k)x + (1+k)y = 5 + 7k$ for different values of k pass through a fixed point. Also, find that point.

8. Find the equation of the straight line passing through the point of intersection of $2x + y - 1 = 0$ and $x + 3y - 2 = 0$ and making with the coordinate axes a triangle of area 3/8 sq. units.

9. Find the equation of the straight line which passes through the point of intersection of the lines $3x - y = 5$ and $x + 3y = 1$ and makes equal and positive intercepts on the axes.

10. Find the equations of the lines through the point of intersection of the lines $x - 3y + 1 = 0$ and $2x + 5y - 9 = 0$ and whose distance from the origin is $\sqrt{5}$.

11. Find the equations of the lines through the point of intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ whose distance from the point $(3, 2)$ is 7/5. [NCERT EXEMPLAR]

ANSWERS

1. $28x + 35y - 48 = 0$ 2. $x + y + 2 = 0$ 3. $13y = 27, 13x = 23$
4. $19x + 19y + 3 = 0$ or, $19x - 19y - 23 = 0$ 5. $25x - 30y - 23 = 0$
6. $(-5/3, -5/3)$ 7. $(-2, 9)$
8. $3x + 4y - 3 = 0$ or, $12x + y - 3 = 0$ 9. $5x + 5y = 7$
10. $2x + y - 5 = 0$ 11. $3x - 4y + 6 = 0,\ 4x - 3y + 1 = 0$

HINTS TO SELECTED PROBLEMS

11. The equation of the family of lines through the intersection of the lines $x - y + 1 = 0$ and $2x - 3y + 5 = 0$ is

$$(x - y + 1) + \lambda(2x - 3y + 5) = 0 \text{ or, } x(2\lambda + 1) + y(-3\lambda - 1) + 5\lambda + 1 = 0 \quad \text{...(i)}$$

This is at a distance of $\frac{7}{5}$ units from the point $(3, 2)$.

$$\therefore \quad \frac{|3(2\lambda+1) + 2(-3\lambda-1) + 5\lambda + 1|}{\sqrt{(2\lambda+1)^2 + (-3\lambda-1)^2}} = \frac{7}{5}$$

$$\Rightarrow \quad \frac{|5\lambda + 2|}{\sqrt{13\lambda^2 + 10\lambda + 2}} = \frac{7}{5}$$

$$\Rightarrow \quad 25(5\lambda+2)^2 = 49(13\lambda^2 + 10\lambda + 2) \Rightarrow 6\lambda^2 - 5\lambda - 1 = 0 \Rightarrow \lambda = 1, -\frac{1}{6}$$

Substituting the values of λ in (i), we obtain

$3x - 4y + 6 = 0$ and $4x - 3y + 1 = 0$ as the required equations of the line.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. Write an equation representing a pair of lines through the point (a, b) and parallel to the coordinate axes.

2. Write the coordinates of the orthocentre of the triangle formed by the lines $x^2 - y^2 = 0$ and $x + 6y = 18$.
3. If the centroid of a triangle formed by the points (0, 0), ($\cos\theta$, $\sin\theta$) and ($\sin\theta$, $-\cos\theta$ lies on the line $y = 2x$, then write the value of $\tan\theta$.
4. Write the value of $\theta \in \left(0, \frac{\pi}{2}\right)$ for which area of the triangle formed by points $O(0, 0)$, $A(a\cos\theta, b\sin\theta)$ and $B(a\cos\theta, -b\sin\theta)$ is maximum.
5. Write the distance between the lines $4x + 3y - 11 = 0$ and $8x + 6y - 15 = 0$.
6. Write the coordinates of the orthocentre of the triangle formed by the lines $xy = 0$ and $x + y = 1$.
7. If the lines $x + ay + a = 0$, $bx + y + b = 0$ and $cx + cy + 1 = 0$ are concurrent, then write the value of $2abc - ab - bc - ca$.
8. Write the area of the triangle formed by the coordinate axes and the line $(\sec\theta - \tan\theta)\, x + (\sec\theta + \tan\theta)\, y = 2$.
9. If the diagonals of the quadrilateral formed by the lines $l_1 x + m_1 y + n_1 = 0$, $l_2 x + m_2 y + n_2 = 0$, $l_1 x + m_1 y + n_1' = 0$ and $l_2 x + m_2 y + n_2' = 0$ are perpendicular, then write the value of $l_1^2 - l_2^2 + m_1^2 - m_2^2$.
10. Write the coordinates of the image of the point (3, 8) in the line $x + 3y - 7 = 0$.
11. Write the integral values of m for which the x-coordinate of the point of intersection of the lines $y = mx + 1$ and $3x + 4y = 9$ is an integer.
12. If $a \neq b \neq c$, write the condition for which the equations $(b - c)\, x + (c - a)\, y + (a - b) = 0$ and $(b^3 - c^3)\, x + (c^3 - a^3)\, y + (a^3 - b^3) = 0$ represent the same line.
13. If a, b, c are in G.P. write the area of the triangle formed by the line $ax + by + c = 0$ with the coordinates axes.
14. Write the area of the figure formed by the lines $a|x| + b|y| + c = 0$.
15. Write the locus of a point the sum of whose distances from the coordinates axes is unity.
16. If a, b, c are in A.P., then the line $ax + by + c = 0$ passes through a fixed point. Write the coordinates of that point.
17. Write the equation of the line passing through the point $(1, -2)$ and cutting off equal intercepts from the axes.
18. Find the locus of the mid-points of the portion of the line $x\sin\theta + y\cos\theta = p$ intercepted between the axes.

ANSWER

1. $(x - a)(y - b) = 0$	2. (0, 0)	3. -3	4. $\pi/4$
5. 7/10 units	6. (0, 0)	7. -1	8. 2 sq. units
9. 0	10. $(-1, -4)$	11. $-1, -2$	12. $a + b + c = 0$
13. $\frac{1}{2}$ Sq. units	14. $\frac{2c^2}{\lvert ab\rvert}$ Sq. units	15. A square	16. $(1, -2)$
17. $x + y + 1 = 0$	18. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$		

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. L is a variable line such that the algebraic sum of the distances of the points $(1, 1)$, $(2, 0)$ and $(0, 2)$ from the line is equal to zero. The line L will always pass through
(a) $(1, 1)$ (b) $(2, 1)$ (c) $(1, 2)$ (d) none of these

2. The acute angle between the medians drawn from the acute angles of a right angled isosceles triangle is
(a) $\cos^{-1}\left(\frac{2}{3}\right)$ (b) $\cos^{-1}\left(\frac{3}{4}\right)$ (c) $\cos^{-1}\left(\frac{4}{5}\right)$ (d) $\cos^{-1}\left(\frac{5}{6}\right)$

3. The distance between the orthocentre and circumcentre of the triangle with vertices $(1, 2)$. $(2, 1)$ and $\left(\frac{3+\sqrt{3}}{2}, \frac{3+\sqrt{3}}{2}\right)$ is
(a) 0 (b) $\sqrt{2}$ (c) $3+\sqrt{3}$ (d) none of these

4. The equation of the straight line which passes through the point $(-4, 3)$ such that the portion of the line between the axes is divided internally by the point in the ratio $5:3$ is
(a) $9x - 20y + 96 = 0$ (b) $9x + 20y = 24$
(c) $20x + 9y + 53 = 0$ (d) none of these

5. The point which divides the join of $(1, 2)$ and $(3, 4)$ externally in the ratio $1:1$
(a) lies in the III quadrant (b) lies in the II quadrant
(c) lies in the I quadrant (d) cannot be found

6. A line passes through the point $(2, 2)$ and is perpendicular to the line $3x + y = 3$. Its y-intercept is
(a) $1/3$ (b) $2/3$ (c) 1 (d) $4/3$

7. If the lines $ax + 12y + 1 = 0$, $bx + 13y + 1 = 0$ and $cx + 14y + 1 = 0$ are concurrent, then a, b, c are in
(a) H.P. (b) G.P. (c) A.P. (d) none of these

8. The number of real values of λ for which the lines $x - 2y + 3 = 0$, $\lambda x + 3y + 1 = 0$ and $4x - \lambda y + 2 = 0$ are concurrent is
(a) 0 (b) 1 (c) 2 (d) Infinite

9. The equations of the sides AB, BC and CA of ΔABC are $y - x = 2$, $x + 2y = 1$ and $3x + y + 5 = 0$ respectively. The equation of the altitude through B is
(a) $x - 3y + 1 = 0$ (b) $x - 3y + 4 = 0$ (c) $3x - y + 2 = 0$ (d) none of these

10. If p_1 and p_2 are the lengths of the perpendiculars from the origin upon the lines $x \sec\theta + y \operatorname{cosec}\theta = a$ and $x\cos\theta - y\sin\theta = a\cos 2\theta$ respectively, then
(a) $4p_1^2 + p_2^2 = a^2$ (b) $p_1^2 + 4p_2^2 = a^2$
(c) $p_1^2 + p_2^2 = a^2$ (d) none of these

11. Area of the triangle formed by the points $((a+3)(a+4), a+3)$, $((a+2)(a+3), (a+2))$ and $((a+1)(a+2), (a+1))$ is
(a) $25a^2$ (b) $5a^2$ (c) $24a^2$ (d) none of these

12. If $a + b + c = 0$, then the family of lines $3ax + by + 2c = 0$ pass through fixed point
(a) $(2, 2/3)$ (b) $(2/3, 2)$ (c) $(-2, 2/3)$ (d) none of these

13. The line segment joining the points $(-3, -4)$ and $(1, -2)$ is divided by y-axis in the ratio
(a) $1:3$ (b) $2:3$ (c) $3:1$ (d) $3:2$

14. The area of a triangle with vertices at $(-4, -1)$, $(1, 2)$ and $(4, -3)$ is
(a) 17 (b) 16 (c) 15 (d) none of these

15. The line segment joining the points $(1, 2)$ and $(-2, 1)$ is divided by the line $3x + 4y = 7$ in the ratio
(a) $3:4$ (b) $4:3$ (c) $9:4$ (d) $4:9$

16. If the point $(5, 2)$ bisects the intercept of a line between the axes, then its equation is
(a) $5x + 2y = 20$ (b) $2x + 5y = 20$ (c) $5x - 2y = 20$ (d) $2x - 5y = 20$

17. $A(6, 3)$, $B(-3, 5)$, $C(4, -2)$ and $D(x, 3x)$ are four points. If $\Delta DBC : \Delta ABC = 1:2$, then x is equal to
(a) $11/8$ (b) $8/11$ (c) 3 (d) none of these

18. If p be the length of the perpendicular from the origin on the line $x/a + y/b = 1$, then
(a) $p^2 = a^2 + b^2$ (b) $p^2 = \frac{1}{a^2} + \frac{1}{b^2}$ (c) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$ (d) none of these

19. The equation of the line passing through $(1, 5)$ and perpendicular to the line $3x - 5y + 7 = 0$ is
(a) $5x + 3y - 20 = 0$ (b) $3x - 5y + 7 = 0$ (c) $3x - 5y + 6 = 0$ (d) $5x + 3y + 7 = 0$

20. The figure formed by the lines $ax \pm by \pm c = 0$ is
(a) a rectangle (b) a square (c) a rhombus (d) none of these

21. Two vertices of a triangle are $(-2, -1)$ and $(3, 2)$ and third vertex lies on the line $x + y = 5$. If the area of the triangle is 4 square units, then the third vertex is
(a) $(0, 5)$ or, $(4, 1)$ (b) $(5, 0)$ or, $(1, 4)$ (c) $(5, 0)$ or, $(4, 1)$ (d) $(0, 5)$ or, $(1, 4)$

22. The inclination of the straight line passing through the point $(-3, 6)$ and the mid-point of the line joining the point $(4, -5)$ and $(-2, 9)$ is
(a) $\pi/4$ (b) $\pi/6$ (c) $\pi/3$ (d) $3\pi/4$

23. Distance between the lines $5x + 3y - 7 = 0$ and $15x + 9y + 14 = 0$ is
(a) $\frac{35}{\sqrt{34}}$ (b) $\frac{1}{3\sqrt{34}}$ (c) $\frac{35}{3\sqrt{34}}$ (d) $\frac{35}{2\sqrt{34}}$

24. The angle between the lines $2x - y + 3 = 0$ and $x + 2y + 3 = 0$ is
(a) 90° (b) 60° (c) 45° (d) 30°

25. The value of λ for which the lines $3x + 4y = 5$, $5x + 4y = 4$ and $\lambda x + 4y = 6$ meet at a point is
(a) 2 (b) 1 (c) 4 (d) 3

26. Three vertices of a parallelogram taken in order are $(-1, -6)$, $(2, -5)$ and $(7, 2)$. The fourth vertex is
(a) $(1, 4)$ (b) $(4, 1)$ (c) $(1, 1)$ (d) $(4, 4)$

27. The centroid of a triangle is (2, 7) and two of its vertices are (4, 8) and (−2, 6). The third vertex is

(a) (0, 0) (b) (4, 7) (c) (7, 4) (d) (7, 7)

28. If the lines $x + q = 0$, $y - 2 = 0$ and $3x + 2y + 5 = 0$ are concurrent, then the value of q will be

(a) 1 (b) 2 (c) 3 (d) 5

29. The medians AD and BE of a triangle with vertices $A(0, b)$, $B(0, 0)$ and $C(a, 0)$ are perpendicular to each other, if

(a) $a = \frac{b}{2}$ (b) $b = \frac{a}{2}$ (c) $ab = 1$ (d) $a = \pm\sqrt{2}\, b$

30. The equation of the line with slope $-3/2$ and which is concurrent with the lines $4x + 3y - 7 = 0$ and $8x + 5y - 1 = 0$ is

(a) $3x + 2y - 63 = 0$ (b) $3x + 2y - 2 = 0$

(c) $2y - 3x - 2 = 0$ (d) none of these

31. The vertices of a triangle are (6, 0), (0, 6) and (6, 6). The distance between its circumcentre and centroid is

(a) $2\sqrt{2}$ (b) 2 (c) $\sqrt{2}$ (d) 1

32. A point equidistant from the line $4x + 3y + 10 = 0$, $5x - 12y + 26 = 0$ and $7x + 24y - 50 = 0$ is

(a) (1, −1) (b) (1, 1) (c) (0, 0) (d) (0, 1)

33. The ratio in which the line $3x + 4y + 2 = 0$ divides the distance between the lines $3x + 4y + 5 = 0$ and $3x + 4y - 5 = 0$ is

(a) 1 : 2 (b) 3 : 7 (c) 2 : 3 (d) 2 : 5

34. The coordinates of the foot of the perpendicular from the point (2, 3) on the line $x + y - 11 = 0$ are

(a) (−6, 5) (b) (5, 6) (c) (−5, 6) (d) (6, 5)

35. The reflection of the point (4, −13) about the line $5x + y + 6 = 0$ is

(a) (−1, −14) (b) (3, 4) (c) (0, 0) (d) (1, 2)

ANSWERS

1. (a)	2. (c)	3. (a)	4. (a)	5. (d)	6. (d)	7. (c)	8. (a)
9. (b)	10. (a)	11. (d)	12. (b)	13. (c)	14. (a)	15. (d)	16. (b)
17. (a)	18. (c)	19. (a)	20. (c)	21. (b)	22. (d)	23. (c)	24. (a)
25. (b)	26. (b)	27. (b)	28. (c)	29. (d)	30. (b)	31. (c)	32. (c)
33. (b)	34. (b)	35. (a)					

SUMMARY

1. Every first degree equation in x, y represents a straight line.
2. The trigonometrical tangent of the angle that a non-vertical line makes with the positive direction of the x-axis in anticlockwise sense is called the slope or gradient of the line.
3. The slope m of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{m_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

4. Slope of a horizontal line is zero and slope of a vertical ine is undefined.
5. An acute angle θ between the lines having slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$$

6. Two lines are parallel if and only if their slopes are equal.
7. Two lines are perpendicular if and only if the product of their slopes is -1.
8. Three points P, Q and R are collinear if and only if

   Slope of PQ = Slope of QR
9. If a straight line cuts x-axis at A and the y-axis at B, then OA and OB are known as the intercepts of the line on x-axis and y-axis respectively.
10. The equation of a line parallel to x-axis at a distance a from it is $y = a$ or $y = -a$ according as it is above or below x-axis.
11. The equation of a line parallel to y-axis at a distance b from it is $x = b$ or $x = -b$ according as it is on the right or on left side of y-axis.
12. The equation of x-axis is $y = 0$.
13. The equation of y-axis is $x = 0$.
14. The equation of a line with slope m and making an intercept c on y-axis is $y = mx + c$.
15. The equation of a line with slope m and passing through the origin is $y = mx$.
16. The equation of the line which passes through the point (x_1, y_1) and has slope m is

$$y - y_1 = m(x - x_1)$$

17. The equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

18. The equation of the line making intercepts a and b on x and y-axis respectively is $\frac{x}{a} + \frac{y}{b} = 1$.
19. The equation of the straight line upon which the length of the perpendicular from the origin is p and the angle between this perpendicular and positive x-axis is α is given by $x \cos\alpha + y \sin\alpha = p$.
20. The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is

$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$, where r is the distance of the point (x, y) on the line from the point (x_1, y_1).

   The coordinates of any point on the line at a distance r from the point (x_1, y_1) are

   $(x_1 \pm r\cos\theta, y_1 \pm r\sin\theta)$
21. The slope of the line $ax + by + c = 0$ is

$$-\frac{a}{b} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

22. Three lines $L_1 \equiv a_1 x + b_1 y + c_1 = 0$, $L_2 \equiv a_2 x + b_2 y + c_2 = 0$ and, $L_3 \equiv a_3 x + b_3 y + c_3 = 0$ are concurrent, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Also, these lines are concurrent iff there exist scalars $\lambda_1, \lambda_2, \lambda_3$ such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

23. The equation of a line parallel to the line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.
24. The equation of a line perpendicular to the line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.
25. The perpendicular distance (d) of a line $ax + by + c = 0$ from a point (x_1, y_1) is given by

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

26. The distance (d) between the parallel lines $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$ is given by $d = \dfrac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.
27. The equations of the lines passing through (x_1, y_1) and making an angle α with the line $y = mx + c$ are given by

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \pm m \tan \alpha} (x - x_1).$$