$$\Rightarrow \quad 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

$$\Rightarrow \quad 1 \geq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \geq \lim_{\theta \to 0} \cos\theta \text{ or, } \lim_{\theta \to 0} \cos\theta \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$\Rightarrow \quad 1 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$\Rightarrow \quad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

(iii) We have,

$$\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \frac{1}{\cos\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \lim_{\theta \to 0} \frac{1}{\cos\theta} = 1 \times 1 = 1$$

(iv) We have,

$$\lim_{\theta \to a} \frac{\sin(\theta - a)}{\theta - a} = \lim_{h \to 0} \frac{\sin(a + h - a)}{(a + h - a)} \quad \left[\text{Using: } \lim_{\theta \to a} \frac{\sin(\theta - a)}{\theta - a} = \lim_{\theta \to a^+} \frac{\sin(\theta - a)}{\theta - a}\right]$$

$$= \lim_{h \to 0} \frac{\sin h}{h} = 1$$

(v) We have,

$$\lim_{\theta \to a} \frac{\tan(\theta - a)}{\theta - a} = \lim_{h \to 0} \frac{\tan(a + h - a)}{a + h - a} \quad \left[\text{Using: } \lim_{\theta \to a} \frac{\tan(\theta - a)}{\theta - a} = \lim_{\theta \to a^+} \frac{\tan(\theta - a)}{\theta - a}\right]$$

$$= \lim_{h \to 0} \frac{\tan h}{h} = 1$$

### 29.7.1 EVALUATION OF TRIGONOMETRIC LIMITS WHEN THE VARIABLE TENDS TO ZERO

Following examples will illustrate the procedure.

## ILLUSTRATIVE EXAMPLES

### LEVEL-1

**EXAMPLE 1** *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\sin 3x}{x}$ (ii) $\lim_{x \to 0} \frac{\sin 5x}{2x}$ (iii) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$ **[NCERT]**

(iv) $\lim_{x \to 0} \frac{\sin^2 ax}{\sin^2 bx}$ (v) $\lim_{x \to 0} \frac{\sin^2 3x}{x^2}$

**SOLUTION** (i) We have,

$$\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \left(3 \times \frac{\sin 3x}{3x}\right) = 3 \lim_{x \to 0} \frac{\sin 3x}{3x} = 3(1) = 3 \quad \left[\because \lim_{x \to 0} \frac{\sin 3x}{3x} = 1\right]$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 5x}{2x} = \lim_{x \to 0} \left(\frac{5}{2} \times \frac{\sin 5x}{5x}\right) = \frac{5}{2} \lim_{x \to 0} \frac{\sin 5x}{5x} = \frac{5}{2}(1) = \frac{5}{2} \quad \left[\because \lim_{x \to 0} \frac{\sin 5x}{5x} = 1\right]$$

(iii) We have,

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax}{\left(\frac{\sin bx}{bx}\right)bx}=\frac{a}{b}\,\frac{\lim_{x\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{x\to 0}\left(\frac{\sin bx}{bx}\right)}=\frac{a}{b}\frac{(1)}{(1)}=\frac{a}{b}$$

(iv) We have,

$$\lim_{x\to 0}\frac{\sin^2 ax}{\sin^2 bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)(ax)\left(\frac{\sin ax}{ax}\right)ax}{\left(\frac{\sin bx}{bx}\right)(bx)\left(\frac{\sin bx}{bx}\right)bx}$$

$$=\frac{a^2}{b^2}\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\left(\frac{\sin ax}{ax}\right)}{\left(\frac{\sin bx}{bx}\right)\left(\frac{\sin bx}{bx}\right)}=\frac{a^2}{b^2}\times\frac{1}{1}=\frac{a^2}{b^2}$$

(v) We have,

$$\lim_{x\to 0}\frac{\sin^2 3x}{x^2}=\lim_{x\to 0}\frac{\sin 3x}{x}\times\frac{\sin 3x}{x}$$

$$=\lim_{x\to 0}3\left(\frac{\sin 3x}{3x}\right)\times 3\left(\frac{\sin 3x}{3x}\right)$$

$$=3\lim_{x\to 0}\frac{\sin 3x}{3x}\times 3\lim_{x\to 0}\frac{\sin 3x}{3x}=(3)(3)=9$$

**EXAMPLE 2** *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ **[NCERT EXEMPLAR]** (ii) $\lim_{x\to 0}\frac{1-\cos 2x}{x}$

(iii) $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (iv) $\lim_{x\to 0}\frac{1-\cos 2mx}{1-\cos 2nx}$ **[NCERT EXEMPLAR]**

(v) $\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}$ **[NCERT EXEMPLAR]**

SOLUTION (i) We have,

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}$$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\times\frac{\sin x}{x}\right)=2\lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{\sin x}{x}=2(1)(1)=2.$$

(ii) We have,

$$\lim_{x\to 0}\frac{1-\cos 2x}{x}=\lim_{x\to 0}\frac{2\sin^2 x}{x}$$

$$=2\lim_{x\to 0}\left(\frac{\sin x}{x}\times\sin x\right)=2\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\sin x\right)=2(1)(0)=0$$

(iii) We have,

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \lim_{x \to 0} \frac{2\sin^2 x/2}{x^2}$$

$$= 2 \lim_{x \to 0} \left( \frac{\sin x/2}{x} \times \frac{\sin x/2}{x} \right)$$

$$= 2 \lim_{x \to 0} \left( \frac{\sin x/2}{x/2} \times \frac{1}{2} \times \frac{1}{2} \frac{\sin x/2}{x/2} \right)$$

$$= \frac{2}{4} \left( \lim_{x \to 0} \frac{\sin x/2}{x/2} \right) \times \left( \lim_{x \to 0} \frac{\sin x/2}{x/2} \right) = \frac{2}{4}(1)(1) = \frac{1}{2}$$

(iv) We have,

$$\lim_{x \to 0} \frac{1-\cos 2mx}{1-\cos 2nx}$$

$$= \lim_{x \to 0} \frac{2\sin^2 mx}{2\sin^2 nx} = \lim_{x \to 0} \frac{\sin^2 mx}{\sin^2 nx}$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \times \frac{\frac{\sin mx}{mx} \times mx}{\frac{\sin nx}{nx} \times nx} \right\} = \frac{m^2}{n^2} \left\{ \lim_{x \to 0} \frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}} \right\} \times \left\{ \lim_{x \to 0} \frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}} \right\}$$

$$= \frac{m^2}{n^2} \left\{ \frac{\lim\limits_{x \to 0} \frac{\sin mx}{mx}}{\lim\limits_{x \to 0} \frac{\sin nx}{nx}} \right\} \times \left\{ \frac{\lim\limits_{x \to 0} \frac{\sin mx}{mx}}{\lim\limits_{x \to 0} \frac{\sin nx}{nx}} \right\} = \frac{m^2}{n^2} \left(\frac{1}{1}\right)\left(\frac{1}{1}\right) = \frac{m^2}{n^2}$$

(v) We have,

$$\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx}$$

$$= \lim_{x \to 0} \frac{2\sin^2\left(\frac{mx}{2}\right)}{2\sin^2\left(\frac{nx}{2}\right)} = \lim_{x \to 0} \left( \frac{\sin \frac{mx}{2}}{\sin \frac{nx}{2}} \right)^2$$

$$= \lim_{x \to 0} \left\{ \frac{\left( \frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right) \frac{mx}{2}}{\left( \frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right) \cdot \frac{nx}{2}} \right\}^2 = \left\{ \frac{m}{n} \frac{\lim\limits_{x \to 0} \left( \frac{\sin \frac{mx}{2}}{\frac{mx}{2}} \right)}{\lim\limits_{x \to 0} \left( \frac{\sin \frac{nx}{2}}{\frac{nx}{2}} \right)} \right\}^2 = \left( \frac{m \times 1}{n \times 1} \right)^2 = \frac{m^2}{n^2}$$

**EXAMPLE 3** *Evaluate the following limits:*

(i) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$ (ii) $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{\sin^3 x}$ **[NCERT EXEMPLAR]**

**SOLUTION** (i) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$$

$$= \lim_{x \to 0} \left( \frac{\sin x - \sin x \cos x}{x^3 \cos x} \right)$$

$$= \lim_{x \to 0} \left\{ \frac{\sin x (1 - \cos x)}{x^3 \cos x} \right\} = \lim_{x \to 0} \left\{ \frac{\sin x}{x} \times \frac{1 - \cos x}{x^2} \times \frac{1}{\cos x} \right\}$$

$$= \left\{ \lim_{x \to 0} \frac{\sin x}{x} \right\} \times \left\{ \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \times 4} \right\} \times \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \left\{ \lim_{x \to 0} \frac{\sin x}{x} \right\} \times \frac{1}{2} \times \left\{ \lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \right\} \times \lim_{x \to 0} \frac{1}{\cos x} = 1 \times \frac{1}{2} (1)^2 \times \frac{1}{1} = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\sin x (1 - \cos x)}{\cos x \sin^3 x}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x \sin^2 x} = \lim_{x \to 0} \frac{1 - \cos x}{\cos x (1 - \cos^2 x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\cos x (1 + \cos x)(1 - \cos x)} = \lim_{x \to 0} \frac{1}{\cos x (1 + \cos x)} = \frac{1}{2}$$

**EXAMPLE 4** *Evaluate:* $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$

**SOLUTION** We have,

$$\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3} \qquad \left( \frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3} = \lim_{x \to 0} \frac{\sin 2x (1 - \cos 2x)}{x^3 \cos 2x} = \lim_{x \to 0} \frac{\sin 2x \times 2 \sin^2 x}{x^3 \cos 2x}$$

$$= 2 \lim_{x \to 0} \frac{\tan 2x \sin^2 x}{x^3} = 4 \lim_{x \to 0} \left( \frac{\tan 2x}{2x} \right) \left( \frac{\sin x}{x} \right)^2 = 4 (1) (1)^2 = 4.$$

**EXAMPLE 5** *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2}$ (ii) $\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x}$ **[NCERT EXEMPLAR]**

(iii) $\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$

**SOLUTION** (i) We have,

$$\lim_{x \to 0} \frac{\cos Ax - \cos Bx}{x^2} \qquad \left( \text{form } \frac{0}{0} \right)$$

$$= \lim_{x \to 0} \frac{2 \sin\left(\frac{A+B}{2}\right)x \ \sin\left(\frac{B-A}{2}\right)x}{x^2} \qquad \left[\because \ \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}\right]$$

$$= 2 \lim_{x \to 0} \left\{ \frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x} \times \left(\frac{A+B}{2}\right) \frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x} \times \left(\frac{B-A}{2}\right) \right\} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= 2\left(\frac{B+A}{2}\right)\left(\frac{B-A}{2}\right)\left\{ \lim_{x \to 0} \frac{\sin\left(\frac{A+B}{2}\right)x}{\left(\frac{A+B}{2}\right)x} \right\} \times \left\{ \lim_{x \to 0} \frac{\sin\left(\frac{B-A}{2}\right)x}{\left(\frac{B-A}{2}\right)x} \right\}$$

$$= \left(\frac{B^2 - A^2}{2}\right)(1)(1) = \frac{B^2 - A^2}{2}$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 3x}{2x + \sin 3x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{\sin 2x}{x} + \frac{\sin 3x}{x}}{2 + \frac{\sin 3x}{x}} \qquad [\text{Dividing } N^r \text{ and } D^r \text{ by } x]$$

$$= \lim_{x \to 0} \frac{2\left(\frac{\sin 2x}{2x}\right) + 3\left(\frac{\sin 3x}{3x}\right)}{2 + 3\left(\frac{\sin 3x}{3x}\right)} = \frac{2 \lim_{x \to 0} \left(\frac{\sin 2x}{2x}\right) + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)}{2 + 3 \lim_{x \to 0} \left(\frac{\sin 3x}{3x}\right)}$$

$$= \frac{2 \times 1 + 3 \times 1}{2 + 3 \times 1} = \frac{5}{5} = 1$$

(iii) We have,

$$\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x} = \lim_{x \to 0} \frac{2 \sin 4x \cos 2x}{2 \sin x \cos 4x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left\{ \frac{\frac{\sin 4x}{4x} \times 4x \times \cos 2x}{\frac{\sin x}{x} \times x \times \cos 4x} \right\} = 4 \ \frac{\lim_{x \to 0} \frac{\sin 4x}{4x} \times \lim_{x \to 0} \cos 2x}{\lim_{x \to 0} \frac{\sin x}{x} \times \lim_{x \to 0} \cos 4x} = 4\left(\frac{1 \times 1}{1 \times 1}\right) = 4$$

**EXAMPLE 6** *Evaluate the following limits:*

(i) $\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y}$ **[NCERT EXEMPLAR]** (ii) $\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x}$

**SOLUTION** (i) We have,

$$\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x \sec x}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} \frac{x(\sec(x+y) - \sec x) + y \sec(x+y)}{y} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{y \to 0} x \left\{ \frac{\sec(x+y) - \sec x}{y} \right\} + \lim_{y \to 0} \frac{y \sec(x+y)}{y}$$

$$= \lim_{y \to 0} x \left\{ \frac{\cos x - \cos(x+y)}{y \cos x \cos(x+y)} \right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \left\{ \frac{\cos x - \cos(x+y)}{y} \times \frac{x}{\cos x \cos(x+y)} \right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \left\{ \frac{2 \sin\left(x + \frac{y}{2}\right) \sin\left(\frac{y}{2}\right)}{2\left(\frac{y}{2}\right)} \times \frac{x}{\cos x \cos(x+y)} \right\} + \lim_{y \to 0} \sec(x+y)$$

$$= \lim_{y \to 0} \sin\left(x + \frac{y}{2}\right) \times \lim_{y \to 0} \frac{\sin\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)} \times \lim_{y \to 0} \frac{x}{\cos x \cos(x+y)} + \lim_{y \to 0} \sec(x+y)$$

$$= \sin x \times 1 \times \frac{x}{\cos^2 x} + \sec x = x \tan x \sec x + \sec x$$

(ii) We have,

$$\lim_{x \to 0} \frac{\sec 4x - \sec 2x}{\sec 3x - \sec x} = \lim_{x \to 0} \left( \frac{\frac{\cos 2x - \cos 4x}{\cos 2x \cos 4x}}{\frac{\cos x - \cos 3x}{\cos x \cos 3x}} \right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 0} \left( \frac{\cos 2x - \cos 4x}{\cos x - \cos 3x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right) = \lim_{x \to 0} \left( \frac{2 \sin 3x \sin x}{2 \sin 2x \sin x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right)$$

$$= \lim_{x \to 0} \left( \frac{\frac{\sin 3x}{3x} \times 3x}{\frac{\sin 2x}{2x} \times 2x} \times \frac{\cos x \cos 3x}{\cos 2x \cos 4x} \right) = \frac{3}{2} \frac{\lim_{x \to 0} \frac{\sin 3x}{3x}}{\lim_{x \to 0} \frac{\sin 2x}{2x}} \times \lim_{x \to 0} \frac{\cos x \cos 3x}{\cos 2x \cos 4x} = \frac{3}{2}\left(\frac{1}{1} \times \frac{1}{1}\right) = \frac{3}{2}$$

**EXAMPLE 7** *Evaluate:* $\lim_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$

**SOLUTION** We have,

$$\lim_{x \to 0} \frac{\cot 2x - \operatorname{cosec} 2x}{x}$$

$$= \lim_{x \to 0} \frac{\frac{\cos 2x}{\sin 2x} - \frac{1}{\sin 2x}}{x} = \lim_{x \to 0} \frac{\cos 2x - 1}{x \sin 2x}$$

$$= \lim_{x \to 0} \frac{-(1 - \cos 2x)}{x \sin 2x} = -\lim_{x \to 0} \frac{2 \sin^2 x}{x(2 \sin x \cos x)} = -\lim_{x \to 0} \frac{\tan x}{x} = -1$$

**EXAMPLE 8** *Evaluate:* $\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$ **[NCERT EXEMPLAR]**

**SOLUTION** We have,

$$\lim_{x \to 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x} = \lim_{x \to 0} \frac{(\sin x - \sin 3x) + (\sin 5x - \sin 3x)}{x}$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos 2x + 2 \sin x \cos 4x}{x} = \lim_{x \to 0} \frac{2 \sin x (\cos 4x - \cos 2x)}{x}$$

$$= 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \times \lim_{x \to 0} (\cos 4x - \cos 2x) = 2 \times 1 \times 0 = 0$$

ALITER $\lim\limits_{x \to 0} \dfrac{\sin x - 2 \sin 3x + \sin 5x}{x}$

$$= \lim_{x \to 0} \left\{\left(\frac{\sin x}{x}\right) - 2\left(\frac{\sin 3x}{x}\right) + \left(\frac{\sin 5x}{x}\right)\right\}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - 2 \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin 5x}{x}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} - 2 \times 3 \left(\lim_{x \to 0} \frac{\sin 3x}{3x}\right) + 5\left(\lim_{x \to 0} \frac{\sin 5x}{5x}\right) = 1 - 2 \times 3 \times 1 + 5 \times 1 = 0$$

## LEVEL-2

EXAMPLE 9 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{\tan x + 4 \tan 2x - 3 \tan 3x}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{\tan x + 4\left(\dfrac{2 \tan x}{1 - \tan^2 x}\right) - 3\left(\dfrac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}\right)}{x^2 \tan x}$$

$$= \lim_{x \to 0} \frac{1 + \dfrac{8}{1 - \tan^2 x} - 3\left(\dfrac{3 - \tan^2 x}{1 - 3 \tan^2 x}\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{(1 - \tan^2 x)(1 - 3 \tan^2 x) + 8(1 - 3 \tan^2 x) - 3(3 - \tan^2 x)(1 - \tan^2 x)}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{1 - 4 \tan^2 x + 3 \tan^4 x + 8 - 24 \tan^2 x - 9 + 12 \tan^2 x - 3 \tan^4 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= \lim_{x \to 0} \frac{-16 \tan^2 x}{x^2 (1 - \tan^2 x)(1 - 3 \tan^2 x)}$$

$$= -16 \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 \times \frac{1}{1 - \tan^2 x} \times \frac{1}{1 - 3 \tan^2 x} = -16$$

EXAMPLE 10 *Evaluate:* $\lim\limits_{x \to 0} \dfrac{1 - \cos x \sqrt{\cos 2x}}{x^2}$

SOLUTION We have,

$$\lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} = \lim_{x \to 0} \frac{1 - \cos x \sqrt{\cos 2x}}{x^2} \times \frac{1 + \cos x \sqrt{\cos 2x}}{1 + \cos x \sqrt{\cos 2x}}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x \cos 2x}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x (2 \cos^2 x - 1)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{1 - 2 \cos^4 x + \cos^2 x}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{(1 - \cos^2 x)\ (1 + 2 \cos^2 x)}{x^2 (1 + \cos x \sqrt{\cos 2x})}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \lim_{x \to 0} \frac{1 + 2 \cos^2 x}{\left\{1 + \cos x \sqrt{\cos 2x}\right\}} = 1 \times \left(\frac{1+2}{1+1}\right) = \frac{3}{2}$$

**EXAMPLE 11** *Evaluate:* $\lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4}\right)$

**SOLUTION** We have,

$$\lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2} - \cos \frac{x^2}{4} + \cos \frac{x^2}{2} \cos \frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} \frac{8}{x^8} \left\{\left(1 - \cos \frac{x^2}{2}\right) - \cos \frac{x^2}{4} \left(1 - \cos \frac{x^2}{2}\right)\right\}$$

$$= \lim_{x \to 0} \frac{8}{x^8} \left(1 - \cos \frac{x^2}{2}\right) \left(1 - \cos \frac{x^2}{4}\right)$$

$$= \lim_{x \to 0} 8 \left\{\frac{1 - \cos \frac{x^2}{2}}{x^4}\right\} \left\{\frac{1 - \cos \frac{x^2}{4}}{x^4}\right\}$$

$$= \lim_{x \to 0} 8 \times \frac{2 \sin^2 \frac{x^2}{4}}{x^4} \times \frac{2 \sin^2 \frac{x^2}{8}}{x^4}$$

$$= \lim_{x \to 0} 32 \times \left\{\frac{\sin \frac{x^2}{4}}{x^2}\right\}^2 \times \left\{\frac{\sin \frac{x^2}{8}}{x^2}\right\}^2$$

$$= 32 \lim_{x \to 0} \left\{\frac{\sin \frac{x^2}{4}}{4\left(\frac{x^2}{4}\right)}\right\}^2 \times \left\{\frac{\sin \frac{x^2}{8}}{8\left(\frac{x^2}{8}\right)}\right\}^2 = 32 \times \left(\frac{1}{4}\right)^2 \times \left(\frac{1}{8}\right)^2 = 32 \times \frac{1}{16} \times \frac{1}{64} = \frac{1}{32}$$

**EXAMPLE 12** *Evaluate:* $\lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$

SOLUTION Clearly,

$$\cos x \cos 2x \cos 3x = \frac{1}{2}\{2\cos x \cos 2x \cos 3x\} = \frac{1}{2}\{(2\cos x \cos 2x)\cos 3x\}$$

$$= \frac{1}{2}\left\{(\cos 3x + \cos x)\cos 3x\right\} = \frac{1}{2}\left\{\cos^2 3x + \cos 3x \cos x\right\}$$

$$= \frac{1}{4}\left\{2\cos^2 3x + 2\cos 3x \cos x\right\} = \frac{1}{4}\left\{1 + \cos 6x + \cos 4x + \cos 2x\right\}$$

$$\therefore \quad \lim_{x \to 0} \frac{1 - \cos x \cos 2x \cos 3x}{\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{1 - \frac{1}{4}(1 + \cos 6x + \cos 4x + \cos 2x)}{\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{4 - 1 - \cos 6x - \cos 4x - \cos 2x}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{(1 - \cos 6x) + (1 - \cos 4x) + (1 - \cos 2x)}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 3x + 2\sin^2 2x + 2\sin^2 x}{4\sin^2 2x}$$

$$= \lim_{x \to 0} \frac{\frac{\sin^2 3x}{x^2} + \frac{\sin^2 2x}{x^2} + \frac{\sin^2 x}{x^2}}{2\left(\frac{\sin^2 2x}{x^2}\right)}$$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin 3x}{x}\right)^2 + \left(\frac{\sin 2x}{x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2\left(\frac{\sin 2x}{x}\right)^2}$$

$$= \lim_{x \to 0} \frac{9 \times \left(\frac{\sin 3x}{3x}\right)^2 + 4 \times \left(\frac{\sin 2x}{2x}\right)^2 + \left(\frac{\sin x}{x}\right)^2}{2 \times 4\left(\frac{\sin 2x}{2x}\right)^2}$$

$$= \frac{9 \times 1 + 4 \times 1 + 1}{8} = \frac{14}{8} = \frac{7}{4}$$

## EXERCISE 29.7

### LEVEL-1

*Evaluate the following limits:*

1. $\lim_{x \to 0} \frac{\sin 3x}{5x}$
2. $\lim_{x \to 0} \frac{\sin x^0}{x}$
3. $\lim_{x \to 0} \frac{x^2}{\sin x^2}$
4. $\lim_{x \to 0} \frac{\sin x \cos x}{3x}$

5. $\lim\limits_{x \to 0} \frac{3 \sin x - 4 \sin^3 x}{x}$

6. $\lim\limits_{x \to 0} \frac{\tan 8x}{\sin 2x}$

7. $\lim\limits_{x \to 0} \frac{\tan mx}{\tan nx}$

8. $\lim\limits_{x \to 0} \frac{\sin 5x}{\tan 3x}$

9. $\lim\limits_{x \to 0} \frac{\sin x^\circ}{x^\circ}$

10. $\lim\limits_{x \to 0} \frac{7x \cos x - 3 \sin x}{4x + \tan x}$

11. $\lim\limits_{x \to 0} \frac{\cos ax - \cos bx}{\cos cx - \cos dx}$

12. $\lim\limits_{x \to 0} \frac{\tan^2 3x}{x^2}$

13. $\lim\limits_{x \to 0} \frac{1 - \cos mx}{x^2}$

14. $\lim\limits_{x \to 0} \frac{3 \sin 2x + 2x}{3x + 2 \tan 3x}$

15. $\lim\limits_{x \to 0} \frac{\cos 3x - \cos 7x}{x^2}$

16. $\lim\limits_{\theta \to 0} \frac{\sin 3\theta}{\tan 2\theta}$

17. $\lim\limits_{x \to 0} \frac{\sin x^2 (1 - \cos x^2)}{x^6}$

18. $\lim\limits_{x \to 0} \frac{\sin^2 4x^2}{x^4}$

19. $\lim\limits_{x \to 0} \frac{x \cos x + 2 \sin x}{x^2 + \tan x}$

20. $\lim\limits_{x \to 0} \frac{2x - \sin x}{\tan x + x}$

21. $\lim\limits_{x \to 0} \frac{5x \cos x + 3 \sin x}{3x^2 + \tan x}$

22. $\lim\limits_{x \to 0} \frac{\sin 3x - \sin x}{\sin x}$

23. $\lim\limits_{x \to 0} \frac{\sin 5x - \sin 3x}{\sin x}$

24. $\lim\limits_{x \to 0} \frac{\cos 3x - \cos 5x}{x^2}$

25. $\lim\limits_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin^2 x}$

26. $\lim\limits_{x \to 0} \frac{\sin (2 + x) - \sin (2 - x)}{x}$

27. $\lim\limits_{h \to 0} \frac{(a + h)^2 \sin (a + h) - a^2 \sin a}{h}$

28. $\lim\limits_{x \to 0} \frac{\tan x - \sin x}{\sin 3x - 3 \sin x}$

29. $\lim\limits_{x \to 0} \frac{\sec 5x - \sec 3x}{\sec 3x - \sec x}$

30. $\lim\limits_{x \to 0} \frac{1 - \cos 2x}{\cos 2x - \cos 8x}$

31. $\lim\limits_{x \to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$

32. $\lim\limits_{x \to 0} \frac{\sin (a + x) + \sin (a - x) - 2 \sin a}{x \sin x}$

33. $\lim\limits_{x \to 0} \frac{x^2 - \tan 2x}{\tan x}$

34. $\lim\limits_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$ **[NCERT EXEMPLAR]**

35. $\lim\limits_{x \to 0} \frac{x \tan x}{1 - \cos x}$

36. $\lim\limits_{x \to 0} \frac{x^2 + 1 - \cos x}{x \sin x}$

37. $\lim\limits_{x \to 0} \frac{\sin 2x (\cos 3x - \cos x)}{x^3}$

38. $\lim\limits_{x \to 0} \frac{2 \sin x^\circ - \sin 2x^\circ}{x^3}$

39. $\lim\limits_{x \to 0} \frac{x^3 \cot x}{1 - \cos x}$

40. $\lim\limits_{x \to 0} \frac{x \tan x}{1 - \cos 2x}$

41. $\lim\limits_{x \to 0} \frac{\sin (3 + x) - \sin (3 - x)}{x}$

42. $\lim\limits_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

43. $\lim_{x\to 0} \dfrac{3\sin^2 x - 2\sin x^2}{3x^2}$

44. $\lim_{x\to 0} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$

45. $\lim_{x\to 0} \dfrac{1-\cos 4x}{x^2}$

46. $\lim_{x\to 0} \dfrac{x\cos x+\sin x}{x^2+\tan x}$

47. $\lim_{x\to 0} \dfrac{1-\cos 2x}{3\tan^2 x}$

48. $\lim_{\theta\to 0} \dfrac{1-\cos 4\theta}{1-\cos 6\theta}$

49. $\lim_{x\to 0} \dfrac{ax+x\cos x}{b\sin x}$ [NCERT]

50. $\lim_{\theta\to 0} \dfrac{\sin 4\theta}{\tan 3\theta}$

51. $\lim_{x\to 0} \dfrac{2\sin x-\sin 2x}{x^3}$

52. $\lim_{x\to 0} \dfrac{1-\cos 5x}{1-\cos 6x}$

53. $\lim_{x\to 0} \dfrac{\operatorname{cosec} x-\cot x}{x}$

54. $\lim_{x\to 0} \dfrac{\sin 3x+7x}{4x+\sin 2x}$

55. $\lim_{x\to 0} \dfrac{5x+4\sin 3x}{4\sin 2x+7x}$

56. $\lim_{x\to 0} \dfrac{3\sin x-\sin 3x}{x^3}$

57. $\lim_{x\to 0} \dfrac{\tan 2x-\sin 2x}{x^3}$

58. $\lim_{x\to 0} \dfrac{\sin ax+bx}{ax+\sin bx}$ [NCERT]

## LEVEL-2

59. $\lim_{x\to 0} (\operatorname{cosec} x-\cot x)$

60. $\lim_{x\to 0} \dfrac{\{\sin(\alpha+\beta)x+\sin(\alpha-\beta)x+\sin 2\alpha x\}}{\cos^2\beta x-\cos^2\alpha x}$

61. $\lim_{x\to 0} \dfrac{\cos ax-\cos bx}{\cos cx-1}$

62. $\lim_{h\to 0} \dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

63. If $\lim_{x\to 0} kx\operatorname{cosec} x=\lim_{x\to 0} x\operatorname{cosec} kx$, find $k$.

## ANSWERS

1. $\frac{3}{5}$ 2. $\frac{\pi}{180}$ 3. 1 4. $\frac{1}{3}$ 5. 3 6. 4 7. $\frac{m}{n}$ 8. $\frac{5}{3}$

9. 1 10. $\frac{4}{5}$ 11. $\frac{a^2-b^2}{c^2-d^2}$ 12. 9 13. $\frac{m^2}{2}$ 14. $\frac{8}{9}$ 15. 20

16. $\frac{3}{2}$ 17. $\frac{1}{2}$ 18. 16 19. 3 20. $\frac{1}{2}$ 21. 8 22. 2 23. 2

24. 8 25. $\frac{1}{3}$ 26. $2\cos 2$ 27. $2a\sin a+a^2\cos a$ 28. $\frac{-1}{8}$

29. 2 30. $\frac{1}{15}$ 31. 3 32. $-\sin a$ 33. $-2$ 34. $\frac{1}{4\sqrt{2}}$ 35. 2

36. $\frac{3}{2}$ 37. $-8$ 38. $\left(\frac{\pi}{180}\right)^3$ 39. 2 40. $\frac{1}{2}$ 41. $2\cos 3$ 42. 4 43. $\frac{1}{3}$

44. 1 45. 8 46. 2 47. $\frac{2}{3}$ 48. $\frac{4}{9}$ 49. $\frac{a+1}{b}$ 50. $\frac{4}{3}$ 51. 1

52. $\frac{25}{36}$ 53. $\frac{1}{2}$ 54. $\frac{5}{3}$ 55. $\frac{17}{15}$ 56. 4 57. 4 58. 1 59. 0

60. $\frac{2\alpha}{\alpha^2-\beta^2}$ 61. $\frac{a^2-b^2}{c^2}$ 62. $a^2\cos a+2a\sin a$ 63. $k=\pm 1$

## HINTS TO NCERT & SELECTED PROBLEMS

42. $\lim\limits_{x \to 0} \dfrac{\cos 2x - 1}{\cos x - 1} = \lim\limits_{x \to 0} \dfrac{(2\cos^2 x - 1) - 1}{\cos x - 1} = \lim\limits_{x \to 0} \dfrac{2(\cos^2 x - 1)}{\cos x - 1}$

$= \lim\limits_{x \to 0} \dfrac{2(\cos x - 1)(\cos x + 1)}{\cos x - 1} = \lim\limits_{x \to 0} 2(\cos x + 1) = 4$

49. $\lim\limits_{x \to 0} \dfrac{ax + x\cos x}{b\sin x} = \lim\limits_{x \to 0} \dfrac{a + \cos x}{b\left(\dfrac{\sin x}{x}\right)} = \dfrac{a+1}{b \times 1} = \dfrac{a+1}{b}$

58. $\lim\limits_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx} = \lim\limits_{x \to 0} \dfrac{\dfrac{\sin ax}{x} + b}{a + \dfrac{\sin bx}{x}}$ [Dividing $N^r$ and $D^r$ by $x$]

$= \lim\limits_{x \to 0} \dfrac{a\left(\dfrac{\sin ax}{ax}\right) + b}{a + b\left(\dfrac{\sin bx}{bx}\right)} = \dfrac{a \times 1 + b}{a + b \times 1} = \dfrac{a+b}{a+b} = 1$

### 29.7.2 EVALUATION OF TRIGONOMETRIC LIMITS WHEN VARIABLE TENDS TO A NON-ZERO QUANTITY

So far we have been discussing trigonometric limits when $x \to 0$. Now we will discuss evaluation of trigonometric limits when $x$ tends to a non-zero real number. As we have already assumed that $\lim\limits_{x \to a} f(x)$ always exists.

$\therefore \quad \lim\limits_{x \to a} f(x) = \lim\limits_{x \to a^+} f(x)$ $\quad [\because \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) \Leftrightarrow \lim\limits_{x \to a} f(x)]$

$= \lim\limits_{h \to 0} f(a+h)$

Thus, we have the following algorithm to evaluate the said type of limits.

**ALGORITHM**

STEP I *Obtain the problem. Suppose $x \to a$, where $a$ is a non-zero real number.*

STEP II *Replace $x \to a$ by $h \to 0$ and $x$ by $(a+h)$.*

STEP III *Solve the problem by using formulae discussed in the previous section.*

Following examples will illustrate the above procedure.

REMARK *In order to evaluate $\lim\limits_{x \to a} f(x)$, where $a$ is a non-zero real number, we may use the following results:*

(i) $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$ (ii) $\lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$

## ILLUSTRATIVE EXAMPLES

**LEVEL-1**

**EXAMPLE 1** *Evaluate:*

(i) $\lim\limits_{x \to \pi} \dfrac{\sin x}{\pi - x}$ (ii) $\lim\limits_{x \to \pi} \dfrac{\sin 3x - 3\sin x}{(\pi - x)^3}$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos x}{\pi - 2x}$ (iv) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cos 3x + 3\cos x}{\left(\dfrac{\pi}{2} - x\right)^3}$

SOLUTION (i) $\lim_{x \to \pi} \frac{\sin x}{(\pi - x)} = \lim_{h \to 0} \frac{\sin(\pi + h)}{(\pi - (\pi + h))} = \lim_{h \to 0} \frac{-\sin h}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1.$

ALITER $\lim_{x \to \pi} \frac{\sin x}{\pi - x} = \lim_{x \to \pi} \frac{\sin(\pi - x)}{(\pi - x)} = 1.$

(ii) $\lim_{x \to \pi} \frac{\sin 3x - 3\sin x}{(\pi - x)^3} = \lim_{x \to \pi} \frac{3\sin x - 4\sin^3 x - 3\sin x}{(\pi - x)^3}$

$$= -4 \lim_{x \to \pi} \frac{\sin^3 x}{(\pi - x)^3}$$

$$= -4 \lim_{h \to 0} \frac{\sin^3(\pi + h)}{\{\pi - (\pi + h)\}^3}$$

$$= -4 \lim_{h \to 0} \frac{(-\sin h)^3}{(-h)^3} = -4 \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^3 = -4 \times (1)^3 = -4$$

ALITER $\lim_{x \to \pi} \frac{\sin 3x - 3\sin x}{(\pi - x)^3} = \lim_{x \to \pi} \frac{3\sin x - 4\sin^3 x - 3\sin x}{(\pi - x)^3}$

$$= -4 \lim_{x \to \pi} \frac{\sin^3(\pi - x)}{(\pi - x)^3} = -4 \lim_{x \to \pi} \left\{\frac{\sin(\pi - x)}{\pi - x}\right\}^3 = -4 \times (1)^3 = -4$$

(iii) $\lim_{x \to \pi/2} \frac{\cos x}{\pi - 2x} = \lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2} + h\right)}{\pi - 2\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-\sin h}{-2h} = \frac{1}{2} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{2} \times 1 = \frac{1}{2}$

ALITER $\lim_{x \to \pi/2} \frac{\cos x}{\pi - 2x} = \lim_{x \to \pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = \frac{1}{2} \lim_{x \to \pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = \frac{1}{2} \times 1 = \frac{1}{2}$

(iv) $\lim_{x \to \pi/2} \frac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim_{x \to \pi/2} \frac{4\cos^3 x - 3\cos x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3}$

$$= 4 \lim_{x \to \frac{\pi}{2}} \frac{\cos^3 x}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim_{h \to 0} \frac{\cos^3\left(\frac{\pi}{2} + h\right)}{\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\}}$$

$$= 4 \lim_{h \to 0} \frac{(-\sin h)^3}{(-h)^3} = 4 \lim_{h \to 0} \left(\frac{\sin h}{h}\right)^3 = 4 \times (1)^3 = 4$$

ALITER $\lim_{x \to \pi/2} \frac{\cos 3x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = \lim_{x \to \pi/2} \frac{4\cos^3 x - 3\cos x + 3\cos x}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim_{x \to \pi/2} \frac{\cos^3 x}{\left(\frac{\pi}{2} - x\right)^3}$.

$$= 4 \lim_{x \to \pi/2} \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)^3} = 4 \lim_{x \to \pi/2} \left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)}\right\}^3 = 4 \times (1)^3 = 4$$

**EXAMPLE 2** *Evaluate:*

(i) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\frac{\pi}{2} - x}$ (ii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\tan 2x}{x - \frac{\pi}{2}}$

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right) \tan x$ [NCERT]

**SOLUTION** (i) $$\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right)}{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-\tan h}{-h} = \lim_{h \to 0} \frac{\tan h}{h} = 1$$

<u>ALITER</u> $$\lim_{x \to \frac{\pi}{2}} \frac{\cot x}{\left(\frac{\pi}{2} - x\right)} = \lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = 1$$

(ii) $$\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{h \to 0} \frac{\tan 2\left(\frac{\pi}{2} + h\right)}{\left(\frac{\pi}{2} + h\right) - \frac{\pi}{2}} = \lim_{h \to 0} \frac{\tan(\pi + 2h)}{h} = -\lim_{h \to 0} \frac{\tan 2h}{h}$$

$$= -2 \lim_{h \to 0} \frac{\tan 2h}{2h} = -2 \times 1 = -2$$

<u>ALITER</u> $$\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} - \frac{\tan(\pi - 2x)}{x - \frac{\pi}{2}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{-\left(\frac{\pi}{2} - x\right)} = -2 \lim_{x \to \frac{\pi}{2}} \frac{\tan 2\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = -2 \times 1 = -2$$

(iii) $$\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x \qquad (0 \times \infty \text{ form})$$

$$= \lim_{h \to 0} \left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\} \tan\left(\frac{\pi}{2} + h\right) = \lim_{h \to 0} -h \times -\cot h = \lim_{h \to 0} \frac{h}{\tan h} = 1$$

<u>ALITER</u> $$\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \tan x \qquad (0 \times \infty \text{ form})$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\cot x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{\tan\left(\frac{\pi}{2} - x\right)} = 1.$$

**EXAMPLE 3** *Evaluate the following limits:*

(i) $\lim\limits_{n \to \infty} 2^n \sin \dfrac{a}{2^n}$ (ii) $\lim\limits_{x \to \infty} x \tan \dfrac{1}{x}$

(iii) $\lim\limits_{x \to \infty} 2^{x-1} \tan\left(\dfrac{a}{2^x}\right)$

**SOLUTION** (i) $\lim\limits_{n \to \infty} 2^n \sin \dfrac{a}{2n}$ $(\infty \times 0 \text{ form})$

$$= \lim_{n \to \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{1}{2^n}\right)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n \to \infty} \left\{ \frac{\sin\left(\frac{a}{2^n}\right)}{\left(\frac{a}{2^n}\right)} \right\} \times a = 1 \times a = a$$

(ii) $\lim\limits_{x \to \infty} x \tan \dfrac{1}{x}$ $(\infty \times 0 \text{ form})$

$$= \lim_{x \to \infty} \left\{ \frac{\tan\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \right\} = 1$$

(iii) $$\lim_{x \to \infty} 2^{x-1} \tan\left(\frac{a}{2^x}\right) = \frac{a}{2} \lim_{x \to \infty} \frac{\tan\left(\frac{a}{2x}\right)}{\left(\frac{a}{2x}\right)} = \frac{a}{2} \times 1 = \frac{a}{2}$$

**EXAMPLE 4** *Evaluate the following elements:*

(i) $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\sin x - \cos x}{x - \frac{\pi}{4}}$ **[NCERT EXEMPLAR]**

(ii) $\lim\limits_{x \to \pi/6} \dfrac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$ **[NCERT EXEMPLAR]**

(iii) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a}$

(iv) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$ **[NCERT EXEMPLAR]**

**SOLUTION** (i) We have,

$$\lim_{\pi \to \pi/2} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{4} + h\right) - \cos\left(\frac{\pi}{4} + h\right)}{\frac{\pi}{4} + h - \frac{\pi}{4}} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin \frac{\pi}{4} \cos h + \cos \frac{\pi}{4} \sin h - \cos \frac{\pi}{4} \cos h + \sin \frac{\pi}{4} \sin h}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2\left(\frac{1}{\sqrt{2}} \sin h\right)}{h} = \frac{2}{\sqrt{2}} \lim_{h \to 0} \frac{\sin h}{h} = \sqrt{2}\,(1) = \sqrt{2}$$

**ALITER** $$\lim_{\pi \to \pi/4} \frac{\sin x - \cos x}{x - \frac{\pi}{4}} = \lim_{x \to \pi/4} \frac{\sqrt{2}\left\{\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x\right\}}{x - \frac{\pi}{4}}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4}}{x - \frac{\pi}{4}}$$

$$= \sqrt{2} \lim_{x \to \pi/4} \frac{\sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = \sqrt{2} \times 1 = \sqrt{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\left(\sin x \cos \frac{\pi}{6} - \cos x \sin \frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = 2 \lim_{x \to \pi/6} \frac{\sin\left(x - \frac{\pi}{6}\right)}{\left(x - \frac{\pi}{6}\right)} = 2.$$

(iii) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{\sin (a + h) - \sin a}{a + h - a}$$

$$= \lim_{h \to 0} \frac{\sin a \cos h + \cos a \sin h - \sin a}{h} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \left(\frac{1 - \cos h}{h}\right)$$

$$= \lim_{h \to 0} \cos a \frac{\sin h}{h} - \sin a \lim_{h \to 0} \left(\frac{1 - \cos h}{h}\right)$$

$$= \cos a \left(\lim_{h \to 0} \frac{\sin h}{h}\right) - \sin a \left(\lim_{h \to 0} \frac{2 \sin^2 \frac{h}{2}}{h}\right)$$

$$= \cos a \lim_{h \to 0} \frac{\sin h}{h} - 2 \sin a \lim_{h \to 0} \left(\frac{\frac{\sin h/2}{h}}{h/2}\right)^2 \times \frac{h}{4} = \cos a \times 1 - 2 \sin a\,(1)^2 \times 0 = \cos a$$

**ALITER** $$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = \lim_{x \to a} \frac{2 \sin\left(\frac{x - a}{2}\right) \cos\left(\frac{x + a}{2}\right)}{2\left(\frac{x - a}{2}\right)}$$

$$= \lim_{x \to a} \left\{\frac{\sin\left(\frac{x - a}{2}\right)}{\frac{x - a}{2}}\right\} \times \cos\left(\frac{x + a}{2}\right) = \cos a$$

(iv) We have,

$$\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$$

$$= \lim_{x \to a} \frac{\sin x - \sin a}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right) \cos\left(\frac{x+a}{2}\right)}{x - a} (\sqrt{x} + \sqrt{a})$$

$$= \lim_{x \to a} \frac{2 \sin\left(\frac{x-a}{2}\right)}{2\left(\frac{x-a}{2}\right)} \times \cos\left(\frac{x+a}{2}\right) \times (\sqrt{x} + \sqrt{a}) = 1 \times \cos a \times (\sqrt{a} + \sqrt{a}) = 2\sqrt{a} \cos a$$

**EXAMPLE 5** *Evaluate the following limits:*

(i) $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$ (ii) $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

SOLUTION (i) We have,

$$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left(\frac{\pi}{2} + h\right)}{\left\{\pi - 2\left(\frac{\pi}{2} + h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2} = \lim_{h \to 0} \frac{2 \sin^2 h}{4h^2} = \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin h}{h}\right)\left(\frac{\sin h}{h}\right)$$

$$= \frac{1}{2}\left(\lim_{h \to 0} \frac{\sin h}{h}\right)\left(\lim_{h \to 0} \frac{\sin h}{h}\right) = \frac{1}{2}$$

<u>ALITER</u> $$\lim_{x \to \pi/2} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{1 - \cos(\pi - 2x)}{(\pi - 2x)^2} = \lim_{x \to \pi/2} \frac{2 \sin^2\left(\frac{\pi}{2} - x\right)}{4\left(\frac{\pi}{2} - x\right)^2}$$

$$= \frac{1}{2} \lim_{x \to \pi/2} \left\{\frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)}\right\}^2 = \frac{1}{2} \times (1)^2 = \frac{1}{2}$$

(ii) We have,

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2} = \lim_{h \to 0} \frac{1 - \sin\left(\frac{\pi}{2} + h\right)}{\left\{\frac{\pi}{2} - \left(\frac{\pi}{2} + h\right)\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{1 - \cos h}{h^2} = \lim_{h \to 0} 2 \frac{\sin^2 \frac{h}{2}}{h^2} = \frac{2}{4} \lim_{h \to 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)\left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right) = \frac{1}{2} (1) (1) = \frac{1}{2}$$

**ALITER** $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^2} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{4\left(\frac{\pi}{4}-\frac{x}{2}\right)^2}$

$$= \frac{2}{4} \lim_{x \to \frac{\pi}{2}} \left\{ \frac{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\left(\frac{\pi}{4}-\frac{x}{2}\right)} \right\}^2 = \frac{2}{4} \times (1)^2 = \frac{1}{2}$$

**EXAMPLE 6** *Evaluate the following limits:*

(i) $\lim\limits_{x \to a} \dfrac{\cos x - \cos a}{\cot x - \cot a}$ (ii) $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ **[NCERT EXEMPLAR]**

(iii) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x - \cos x}{(\pi - 2x)^3}$

**SOLUTION** (i) We have,

$$\lim_{x \to a} \frac{\cos x - \cos a}{\cot x - \cot a} = \lim_{x \to a} \frac{\cos x - \cos a}{\cos x \sin a - \sin x \cos a} \cdot \sin x \sin a \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to a} \frac{-2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)}{-2\sin\left(\frac{x-a}{2}\right)\cos\left(\frac{x-a}{2}\right)} \sin x \sin a$$

$$= \lim_{x \to a} \frac{\sin\left(\frac{x+a}{2}\right)}{\cos\left(\frac{x-a}{2}\right)} \sin x \sin a = \frac{\sin a}{1} \times \sin a \sin a = \sin^3 a$$

(ii) We have,

$$\lim_{x \to \pi/2} (\sec x - \tan x) \qquad (\text{form } \infty - \infty)$$

$$= \lim_{x \to \pi/2} \left(\frac{1-\sin x}{\cos x}\right) \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \left\{ \frac{1 - \sin(\pi/2 + h)}{\cos\left(\frac{\pi}{2}+h\right)} \right\}$$

$$= \lim_{h \to 0} \frac{1-\cos h}{-\sin h} = \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{-2\sin h/2 \cos h/2} = -\lim_{h \to 0} \tan h/2 = -\tan 0 = 0$$

**ALITER** $\lim\limits_{x \to \pi/2} (\sec x - \tan x) = \lim\limits_{x \to \pi/2} \dfrac{1-\sin x}{\cos x} = \lim\limits_{x \to \pi/2} \dfrac{1-\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)}$

$$= \lim_{x \to \pi/2} \frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\pi}{2}\right)} = \lim_{x \to \pi/2} \tan\left(\frac{\pi}{4}-\frac{x}{2}\right) = 0$$

(iii) We have,

$$\lim_{x \to \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$$

$$= \lim_{h \to 0} \frac{\cot\left(\frac{\pi}{2} + h\right) - \cos\left(\frac{\pi}{2} + h\right)}{\left\{\pi - 2\left(\frac{\pi}{2} + h\right)\right\}^3} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{-\tan h + \sin h}{-8h^3}$$

$$= \lim_{h \to 0} \frac{-\sin h (1 - \cos h)}{\cos h (-8h^3)}$$

$$= \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \frac{1 - \cos h}{h^2} = \frac{1}{8} \lim_{h \to 0} \frac{\tan h}{h} \times \lim_{h \to 0} \frac{2\sin^2 \frac{h}{2}}{h^2}$$

$$= \frac{1}{8} \times \frac{2}{4} \lim_{h \to 0} \frac{\sin^2 \frac{h}{2}}{\left(\frac{h}{2}\right)^2} = \frac{1}{16}$$

**LEVEL-2**

**EXAMPLE 7** *Evaluate:* $\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

SOLUTION We have,

$$\lim_{x \to \frac{\pi}{6}} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2} = \lim_{h \to 0} \frac{2 - \sqrt{3}\cos\left(\frac{\pi}{6} + h\right) - \sin\left(\frac{\pi}{6} + h\right)}{\left\{6\left(\frac{\pi}{6} + h\right) - \pi\right\}^2} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{h \to 0} \frac{2 - \sqrt{3}\left(\cos\frac{\pi}{6}\cos h - \sin\frac{\pi}{6}\sin h\right) - \left(\sin\frac{\pi}{6}\cos h + \cos\frac{\pi}{6}\sin h\right)}{36h^2}$$

$$= \lim_{h \to 0} \frac{2 - \frac{3}{2}\cos h + \frac{\sqrt{3}}{2}\sin h - \frac{1}{2}\cos h - \frac{\sqrt{3}}{2}\sin h}{36h^2} = \lim_{h \to 0} \frac{2(1 - \cos h)}{36h^2}$$

$$= \frac{1}{18} \lim_{h \to 0} \frac{2\sin^2\left(\frac{h}{2}\right)}{h^2} = \frac{1}{9} \lim_{h \to 0} \left(\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right)^2 \times \frac{1}{4} = \frac{1}{9} \times (1)^2 \times \frac{1}{4} = \frac{1}{36}$$

ALITER $\lim_{x \to \pi/6} \frac{2 - \sqrt{3}\cos x - \sin x}{(6x - \pi)^2}$

$$= \lim_{x \to \pi/6} \frac{2 - 2\left\{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right\}}{(6x - \pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 - 2\left\{\cos x \cos\frac{\pi}{6} + \sin x \sin\frac{\pi}{6}\right\}}{(6x - \pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 - 2\cos\left(x - \frac{\pi}{6}\right)}{(6x - \pi)^2}$$

$$= \lim_{x \to \pi/6} \frac{2 \times 2\sin^2\left(\frac{x}{2} - \frac{\pi}{12}\right)}{144\left(\frac{x}{2} - \frac{\pi}{12}\right)^2} = \frac{4}{144} \lim_{x \to \pi/6} \left\{\frac{\sin\left(\frac{x}{2} - \frac{\pi}{12}\right)}{\left(\frac{x}{2} - \frac{\pi}{12}\right)}\right\}^2 = \frac{1}{36} \times (1)^2 = \frac{1}{36}$$

**EXAMPLE 8** *Prove that:* $\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = -4$ **[NCERT EXEMPLAR]**

**SOLUTION** We have,

$$\lim_{x \to \pi/4} \frac{\tan^3 x - \tan x}{\cos(x + \pi/4)} = \lim_{x \to \pi/4} \frac{\tan x(\tan x - 1)(\tan x + 1)}{\cos(x + \pi/4)}$$

$$= \lim_{x \to \pi/4} \frac{\tan x(\sin x - \cos x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\lim_{x \to \pi/4} \frac{\tan x(\cos x - \sin x)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x\right)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \lim_{x \to \pi/4} \frac{\tan x \cos(x + \pi/4)(\tan x + 1)}{\cos x \cos(x + \pi/4)}$$

$$= -\sqrt{2} \times \lim_{x \to \pi/4} \frac{\tan x(\tan x + 1)}{\cos x} = -\sqrt{2} \times 2\sqrt{2} = -4.$$

**EXAMPLE 9** *Evaluate :* $\lim_{n \to \infty} \left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$

**SOLUTION** We have,

$$\lim_{n \to \infty} \left(\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3} \ldots \cos\frac{x}{2^n}\right)$$

$$= \lim_{n \to \infty} \frac{\sin\left(2^n \times \frac{x}{2^n}\right)}{2^n \sin\left(\frac{x}{2^n}\right)} \quad \left[\because \cos A \cos 2A \ldots \cos 2^{n-1} A = \frac{\sin 2^n A}{2^n \sin A}\right]$$

$$= \lim_{n \to \infty} \frac{\sin x}{x \left\{ \dfrac{\sin\left(\dfrac{x}{2^n}\right)}{\left(\dfrac{x}{2^n}\right)} \right\}} = \frac{\sin x}{x} \qquad \left[ \because \lim_{n \to \infty} \frac{\sin\left(\dfrac{x}{2^n}\right)}{\left(\dfrac{x}{2^n}\right)} = 1 \right]$$

**EXAMPLE 10** *Evaluate :* $\lim\limits_{x \to \pi/4} \dfrac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$

**SOLUTION** We have,

$$\lim_{x \to \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x} = \lim_{x \to \pi/4} \frac{2^{5/2} - \left\{(\cos x + \sin x)^2\right\}^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{2^{5/2} - (1 + \sin 2x)^{5/2}}{2 - (1 + \sin 2x)}$$

$$= \lim_{x \to \pi/4} \frac{(1 + \sin 2x)^{5/2} - 2^{5/2}}{(1 + \sin 2x) - 2}$$

$$= \lim_{u \to 2} \frac{u^{5/2} - 2^{5/2}}{u - 2}, \text{ where } u = 1 + \sin 2x$$

$$= \frac{5}{2} \times (2)^{5/2 - 1} = \frac{5}{2} \times 2^{3/2} = 5\sqrt{2}$$

**EXAMPLE 11** *If $\alpha, \beta$ are the roots of $ax^2 + bx + c$, then evaluate* $\lim\limits_{x \to \beta} \dfrac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2}$.

**SOLUTION** It is given that $\alpha$ and $\beta$ are the roots of the given equation $ax^2 + bx + c = 0$.

$\therefore \quad ax^2 + bx + c = a(x - \alpha)(x - \beta)$

Now,

$$\lim_{x \to \beta} \frac{1 - \cos(ax^2 + bx + c)}{(x - \beta)^2} = \lim_{x \to \beta} \frac{1 - \cos\{a(x - \alpha)(x - \beta)\}}{(x - \beta)^2}$$

$$= \lim_{x \to \beta} \frac{2\sin^2\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}}{(x - \beta)^2}$$

$$= 2 \lim_{x \to \beta} \left[ \frac{\sin\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}}{\left\{\dfrac{a(x - \alpha)(x - \beta)}{2}\right\}} \right]^2 \times \frac{a^2(x - \alpha)^2}{4}$$

$$= 2 \times \frac{a^2(\beta - \alpha)^2}{4} = \frac{a^2(\beta - \alpha)^2}{2}$$

## EXERCISE 29.8

### LEVEL-1

*Evaluate the following limits:*

1. $\lim_{x \to \pi/2} \left(\frac{\pi}{2} - x\right) \tan x$

2. $\lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\cos x}$

3. $\lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{1 - \sin x}$

4. $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2}\,(\pi/3 - x)}$

5. $\lim_{x \to a} \frac{\cos x - \cos a}{x - a}$

6. $\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{x - \frac{\pi}{4}}$

7. $\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\left(\frac{\pi}{2} - x\right)^2}$

8. $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$

9. $\lim_{x \to a} \frac{a \sin x - x \sin a}{ax^2 - xa^2}$

10. $\lim_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

11. $\lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2 - \sin x} - 1}{\left(\frac{\pi}{2} - x\right)^2}$

12. $\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2} - \cos x - \sin x}{\left(\frac{\pi}{4} - x\right)^2}$

13. $\lim_{x \to \frac{\pi}{8}} \frac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

14. $\lim_{x \to a} \frac{\cos x - \cos a}{\sqrt{x} - \sqrt{a}}$

15. $\lim_{x \to \pi} \frac{\sqrt{5 + \cos x} - 2}{(\pi - x)^2}$

16. $\lim_{x \to a} \frac{\cos \sqrt{x} - \cos \sqrt{a}}{x - a}$

17. $\lim_{x \to a} \frac{\sin \sqrt{x} - \sin \sqrt{a}}{x - a}$

18. $\lim_{x \to 1} \frac{1 - x^2}{\sin 2\pi x}$

19. $\lim_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}}$, where $f(x) = \sin 2x$

20. $\lim_{x \to 1} \frac{1 + \cos \pi x}{(1 - x)^2}$

21. $\lim_{x \to 1} \frac{1 - x^2}{\sin \pi x}$

22. $\lim_{x \to \frac{\pi}{4}} \frac{1 - \sin 2x}{1 + \cos 4x}$

23. $\lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$

24. $\lim_{n \to \infty} n \sin\left(\frac{\pi}{4n}\right) \cos\left(\frac{\pi}{4n}\right)$

25. $\lim_{n \to \infty} 2^{n-1} \sin\left(\frac{a}{2^n}\right)$

26. $\lim_{n \to \infty} \frac{\sin\left(\frac{a}{2^n}\right)}{\sin\left(\frac{b}{2^n}\right)}$

27. $\lim_{x \to -1} \frac{x^2 - x - 2}{(x^2 + x) + \sin(x + 1)}$

28. $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 2x + \sin(x - 2)}$

29. $\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$

30. $\lim_{x \to \frac{\pi}{4}} \frac{1-\tan x}{1-\sqrt{2} \sin x}$

31. $\lim_{x \to \pi} \frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}$

32. $\lim_{x \to \pi/4} \frac{\sqrt{\cos x}-\sqrt{\sin x}}{x-\pi/4}$

33. $\lim_{x \to 1} \frac{1-\frac{1}{x}}{\sin \pi (x-1)}$

34. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x-3}{\operatorname{cosec} x-2}$ [NCERT EXEMPLAR]

**LEVEL-2**

35. $\lim_{x \to \frac{\pi}{4}} \frac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}$

36. $\lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right) \sin x - 2\cos x}{\left(\frac{\pi}{2}-x\right)+\cot x}$

37. $\lim_{x \to \frac{\pi}{4}} \frac{\cos x-\sin x}{\left(\frac{\pi}{4}-x\right)(\cos x+\sin x)}$

38. $\lim_{x \to \pi} \frac{1-\sin\frac{x}{2}}{\cos\frac{x}{2}\left(\cos\frac{x}{4}-\sin\frac{x}{4}\right)}$ [NCERT EXEMPLAR]

***ANSWERS***

1. 1 2. 2 3. 2 4. 3 5. $-\sin a$ 6. $-2$ 7. $\frac{1}{2}$ 8. $\frac{4}{3}$

9. $\frac{a\cos a-\sin a}{a^2}$ 10. $\frac{1}{4\sqrt{2}}$ 11. $\frac{1}{4}$ 12. $\frac{1}{\sqrt{2}}$ 13. $\frac{1}{16}$ 14. $-2\sqrt{a}\sin a$

15. $\frac{1}{8}$ 16. $-\frac{1}{2\sqrt{a}}\sin\sqrt{a}$ 17. $\frac{1}{2\sqrt{a}}\cos\sqrt{a}$ 18. $\frac{-1}{\pi}$ 19. 0 20. $\frac{\pi^2}{2}$

21. $\frac{2}{\pi}$ 22. $\frac{1}{4}$ 23. $\frac{1}{2}$ 24. $\frac{\pi}{4}$ 25. $\frac{a}{2}$ 26. $\frac{a}{b}$ 27. $\infty$ 28. 1

29. $\frac{2}{\pi}$ 30. 2 31. $\frac{1}{4}$ 32. $-\frac{1}{2^{1/4}}$ 33. $\frac{1}{\pi}$ 34. 4 35. $\frac{1}{16\sqrt{2}}$

36. $-\frac{1}{2}$ 37. 1 38. $\sqrt{2}$

### 29.7.3 EVALUATION OF TRIGONOMETRIC LIMITS BY FACTORISATION

Sometimes trigonometric limits can also be evaluated by factorisation method as illustrated in the following examples.

## ILLUSTRATIVE EXAMPLES

**LEVEL-1**

**EXAMPLE 1** *Evaluate the following limits:*

(i) $\lim_{x \to \pi/4} \frac{\sec^2 x-2}{\tan x-1}$

(ii) $\lim_{x \to \pi} \frac{1+\sec^3 x}{\tan^2 x}$

(iii) $\lim_{x \to \pi/2} \frac{1-\sin^3 x}{\cos^2 x}$

(iv) $\lim_{x \to \pi} \frac{1+\cos^3 x}{\sin^2 x}$

(v) $\lim_{x \to \pi/2} \frac{\sqrt{2}-\sqrt{1+\sin x}}{\sqrt{2}\cos^2 x}$

(vi) $\lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

SOLUTION (i) We have,

$$\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} (\tan x + 1) = 2.$$

(ii) We have,

$$\lim_{x \to \pi} \frac{1 + \sec^3 x}{\tan^2 x} = \lim_{x \to \pi} \frac{(1 + \sec^3 x)}{(\sec^2 x - 1)} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to \pi} \frac{(\sec x + 1)(\sec^2 x - \sec x + 1)}{(\sec x + 1)(\sec x - 1)}$$

$$= \lim_{x \to \pi} \frac{\sec^2 x - \sec x + 1}{\sec x - 1} = \frac{1 + 1 + 1}{-2} = -\frac{3}{2}$$

(iii) We have,

$$\lim_{x \to \pi/2} \frac{1 - \sin^3 x}{\cos^2 x}$$

$$= \lim_{x \to \pi/2} \frac{(1 - \sin x)(1 + \sin x + \sin^2 x)}{(1 - \sin x)(1 + \sin x)} = \lim_{x \to \pi/2} \frac{1 + \sin x + \sin^2 x}{1 + \sin x} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}$$

(iv) We have,

$$\lim_{x \to \pi} \frac{1 + \cos^3 x}{\sin^2 x}$$

$$= \lim_{x \to \pi} \frac{(1 + \cos x)(1 - \cos x + \cos^2 x)}{(1 - \cos x)(1 + \cos x)} = \lim_{x \to \pi} \frac{1 - \cos x + \cos^2 x}{1 - \cos x} = \frac{1 + 1 + 1}{1 + 1} = \frac{3}{2}$$

(v) We have,

$$\lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2} \cos^2 x} = \lim_{x \to \frac{\pi}{2}} \frac{\sqrt{2} - \sqrt{1 + \sin x}}{\sqrt{2} \cos^2 x} \times \frac{\sqrt{2} + \sqrt{1 + \sin x}}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2 - (1 + \sin x)}{\sqrt{2}(1 - \sin^2 x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)}{\sqrt{2}(1 - \sin x)(1 + \sin x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\sqrt{2}(1 + \sin x)} \times \frac{1}{\sqrt{2} + \sqrt{1 + \sin x}}$$

$$= \frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{8}$$

(vi) We have,

$$\lim_{x \to 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\sqrt{2} - \sqrt{2} \cos \frac{x}{2}}{(1 - \cos x)(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sqrt{2}\left(1 - \cos \frac{x}{2}\right)}{2 \sin^2 \frac{x}{2} (1 + \cos x)} = \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1 - \cos \frac{x}{2}}{\left(1 - \cos \frac{x}{2}\right)\left(1 + \cos \frac{x}{2}\right)(1 + \cos x)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{1}{\left(1 + \cos \frac{x}{2}\right)(1 + \cos x)} = \frac{1}{\sqrt{2}} \frac{1}{(1+1)(1+1)} = \frac{1}{4\sqrt{2}}$$

**EXAMPLE 2** *Evaluate:* $\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

SOLUTION We have,

$$\lim_{x \to \pi/6} \frac{2 \sin^2 x + \sin x - 1}{2\sin^2 x - 3 \sin x + 1} = \lim_{x \to \pi/6} \frac{(2 \sin x - 1)(\sin x + 1)}{(2 \sin x - 1)(\sin x - 1)} = \lim_{x \to \pi/6} \frac{\sin x + 1}{\sin x - 1} = \frac{\frac{1}{2} + 1}{\frac{1}{2} - 1} = -3$$

## EXERCISE 29.9

### LEVEL-1

*Evaluate the following limits:*

1. $\lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x}$
2. $\lim_{x \to \frac{\pi}{4}} \frac{\operatorname{cosec}^2 x - 2}{\cot x - 1}$
3. $\lim_{x \to \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$
4. $\lim_{x \to \frac{\pi}{4}} \frac{2 - \operatorname{cosec}^2 x}{1 - \cot x}$
5. $\lim_{x \to \pi} \frac{\sqrt{2 + \cos x} - 1}{(\pi - x)^2}$
6. $\lim_{x \to \frac{3\pi}{2}} \frac{1 + \operatorname{cosec}^3 x}{\cot^2 x}$

### ANSWERS

1. $\frac{1}{2}$  2. 2  3. 4  4. 2  5. $-\frac{3}{2}$  6. $\frac{1}{4}$

## 29.8 EVALUATION OF EXPONENTIAL AND LOGARITHMIC LIMITS

Sometimes, following expansions are useful in evaluating limits. Students are advised to learn these expansions.

1. $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \ldots$
2. $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \ldots$
3. $\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \ldots$
4. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$
5. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$

6. $a^x = 1 + x(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \ldots$

7. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$

8. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$

9. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

**THEOREM** Prove that: (i) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$. (ii) $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$.

PROOF (i) Using expansion of $a^x$, we obtain

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \frac{1 + x\log_e a + \frac{x^2}{2!}(\log_e a)^2 + \ldots - 1}{x}$$

$$\Rightarrow \lim_{x \to 0} \frac{a^x - 1}{x} = \lim_{x \to 0} \left\{\log_e a + \frac{x}{2!}(\log_e a)^2 + \ldots\right\} = \log_e a.$$

(ii) Using expansion of $\log(1 + x)$, we obtain

$$\lim_{x \to 0} \frac{\log(1+x)}{x} = \lim_{x \to 0} \frac{x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots}{x} = \lim_{x \to 0} 1 - \frac{x}{2} + \frac{x^2}{3} \ldots = 1.$$

**COROLLARY** Putting $a = e$ in (i) we get

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1.$$

## ILLUSTRATIVE EXAMPLES

### LEVEL-1

**EXAMPLE 1** *Evaluate the following limits:*

(i) $\lim_{x \to 0} \frac{a^x - b^x}{x}$ (ii) $\lim_{x \to 0} \frac{a^x - b^x}{\sin x}$

(iii) $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1}$ (iv) $\lim_{x \to 0} \frac{3^x - 2^x}{\tan x}$

(v) $\lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x}$ (vi) $\lim_{x \to 1} \frac{a^{x-1} - 1}{\sin \pi x}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{(a^x - 1) - (b^x - 1)}{x} = \lim_{x \to 0} \frac{a^x - 1}{x} - \lim_{x \to 0} \frac{b^x - 1}{x} = \log a - \log b = \log\left(\frac{a}{b}\right).$$

ALITER $$\lim_{x \to 0} \frac{a^x - b^x}{x} = \lim_{x \to 0} \frac{b^x\left\{\left(\frac{a}{b}\right)^x - 1\right\}}{x} = b^0 \times \log\left(\frac{a}{b}\right) = \log\left(\frac{a}{b}\right)$$

(ii) We have,

$$\lim_{x\to 0} \frac{a^x - b^x}{\sin x} = \lim_{x\to 0} \left\{\left(\frac{a^x-1}{\sin x}\right) - \left(\frac{b^x-1}{\sin x}\right)\right\}$$

$$= \lim_{x\to 0} \left(\frac{a^x-1}{x}\times\frac{x}{\sin x}\right) - \lim_{x\to 0}\left(\frac{b^x-1}{x}\times\frac{x}{\sin x}\right)$$

$$= \lim_{x\to 0} \frac{a^x-1}{x}\times \lim_{x\to 0}\frac{x}{\sin x} - \lim_{x\to 0}\frac{b^x-1}{x}\times\lim_{x\to 0}\frac{x}{\sin x}$$

$$= (\log a)\times 1 - (\log b)\times 1 = \log\left(\frac{a}{b}\right).$$

(iii) We have,

$$\lim_{x\to 0}\frac{2^x-1}{\sqrt{1+x}-1}$$

$$= \lim_{x-0}\frac{2^x-1}{\sqrt{1+x}-1}\times\frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)} = \lim_{x-0}\frac{2^x-1}{x}\times\left\{\sqrt{1+x}+1\right\}$$

$$= \lim_{x\to 0}\frac{2^x-1}{x}\times\lim_{x\to 0}(\sqrt{1+x}+1) = (\log 2)\,2 = 2\log 2.$$

(iv) We have,

$$\lim_{x\to 0}\frac{3^x-2^x}{\tan x} = \lim_{x\to 0}\frac{(3^x-1)-(2^x-1)}{x}\times\frac{x}{\tan x}$$

$$= \left\{\lim_{x\to 0}\frac{3^x-1}{x} - \lim_{x\to 0}\frac{2^x-1}{x}\right\}\times\lim_{x\to 0}\frac{x}{\tan x} = (\log 3 - \log 2)\times 1 = \log\left(\frac{3}{2}\right)$$

(v) Let $y = \sin x$. Then, $y \to 0$ as $x \to 0$.

$$\therefore \quad \lim_{x\to 0}\frac{a^{\sin x}-1}{\sin x} = \lim_{y\to 0}\frac{a^y-1}{y} = \log a$$

(vi) We have,

$$\lim_{x\to 1}\frac{a^{x-1}-1}{\sin \pi x} = \lim_{h\to 0}\frac{a^{1+h-1}-1}{\sin\pi(1+h)} = \lim_{h\to 0}\frac{a^h-1}{-\sin\pi h}$$

$$= \frac{-1}{\pi}\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)\left(\frac{\pi h}{\sin\pi h}\right) = -\frac{1}{\pi}\left(\lim_{h\to 0}\frac{a^h-1}{h}\right)\left(\lim_{h\to 0}\frac{\pi h}{\sin\pi h}\right) = -\frac{1}{\pi}\log a$$

<u>ALITER</u> $\displaystyle\lim_{x\to 1}\frac{a^{x-1}-1}{\sin\pi x} = \lim_{x\to 1}\frac{a^{x-1}-1}{x-1}\times\frac{x-1}{\sin(\pi-\pi x)}$

$$= \lim_{x\to 1}\frac{a^{x-1}-1}{x-1}\times\frac{\pi(x-1)}{-\pi\sin\pi(x-1)} = \log_e a\times\frac{1}{-\pi} = -\frac{1}{\pi}\log_e a$$

**EXAMPLE 2** *Evaluate the following limits:*

(i) $\displaystyle\lim_{x\to 0}\frac{10^x-2^x-5^x+1}{x\tan x}$ (ii) $\displaystyle\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$

(iii) $\displaystyle\lim_{x\to 0}\frac{2^{3x}-3^x}{\sin 3x}$ (iv) $\displaystyle\lim_{x\to 0}\frac{3^{2x}-2^{3x}}{x}$

(v) $\lim_{x \to 0} \frac{3^{2x}-1}{2^{3x}-1}$ (vi) $\lim_{x \to 0} \frac{\sin 3x}{3^x - 1}$

SOLUTION (i) We have,

$$\lim_{x \to 0} \frac{10^x - 2^x - 5^x + 1}{x \tan x}$$

$$= \lim_{x \to 0} \frac{5^x \cdot 2^x - 2^x - 5^x + 1}{x \tan x}$$

$$= \lim_{x \to 0} \frac{5^x (2^x - 1) - (2^x - 1)}{x \tan x} = \lim_{x \to 0} \frac{(5^x - 1)(2^x - 1)}{x \tan x} = \lim_{x \to 0} \frac{5^x - 1}{x} \times \frac{2^x - 1}{x} \times \frac{x}{\tan x}$$

$$= \lim_{x \to 0} \frac{5^x - 1}{x} \times \lim_{x \to 0} \frac{2^x - 1}{x} \times \lim_{x \to 0} \frac{x}{\tan x} = (\log 5)(\log 2)(1) = (\log 5)(\log 2)$$

(ii) We have,

$$\lim_{x \to 0} \frac{3^x + 3^{-x} - 2}{x^2}$$

$$= \lim_{x \to 0} \frac{3^{2x} - 2 \times 3^x + 1}{3^x \times x^2} = \lim_{x \to 0} \left(\frac{3^x - 1}{x}\right)^2 \times \frac{1}{3^x} = (\log 3)^2 \times \left(\frac{1}{3}\right)^0 = (\log 3)^2$$

(iii) We have,

$$\lim_{x \to 0} \frac{2^{3x} - 3^x}{\sin 3x}$$

$$= \lim_{x \to 0} \left\{\frac{(2^{3x} - 1)}{\sin 3x} - \frac{(3^x - 1)}{\sin 3x}\right\}$$

$$= \lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x} \times \frac{3x}{\sin 3x}\right) - \left(\lim_{x \to 0} \frac{3^x - 1}{3x} \times \frac{3x}{\sin 3x}\right)$$

$$= (\log 2) \times 1 - \frac{1}{3}(\log 3) = \log 2 - \frac{1}{3} \log 3.$$

(iv) We have,

$$\lim_{x \to 0} \frac{3^{2x} - 2^{3x}}{x}$$

$$= \lim_{x \to 0} \left\{\left(\frac{3^{2x} - 1}{x}\right) - \left(\frac{2^{3x} - 1}{x}\right)\right\}$$

$$= \lim_{x \to 0} \left(\frac{3^{2x} - 1}{2x} \times 2\right) - \lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x} \times 3\right) = 2 \log 3 - 3 \log 2 = \log 9 - \log 8 = \log\left(\frac{9}{8}\right)$$

(v) We have,

$$\lim_{x \to 0} \frac{3^{2x} - 1}{2^{3x} - 1}$$

$$= \lim_{x \to 0} \frac{\left(\frac{3^{2x} - 1}{2x}\right) \times 2x}{\left(\frac{2^{3x} - 1}{3x}\right) \times 3x} = \frac{2}{3} \times \frac{\lim_{x \to 0} \left(\frac{3^{2x} - 1}{2x}\right)}{\lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x}\right)} = \frac{2}{3}\left(\frac{\log 3}{\log 2}\right) = \frac{\log 3^2}{\log 2^3} = \frac{\log 9}{\log 8}.$$

(vi) We have,

$$\lim_{x\to 0}\frac{\sin 3x}{3^x-1}=\lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{3x}{3^x-1}=3\lim_{x\to 0}\frac{\sin 3x}{3x}\times\lim_{x\to 0}\frac{x}{3^x-1}=\frac{3}{\log 3}$$

**EXAMPLE 3** *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{e^{-x}-1}{x}$ (ii) $\lim_{x\to 0}\frac{e^x-e^{-x}}{x}$

(iii) $\lim_{x\to 1}\frac{x-1}{\log_e x}$ (iv) $\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$

**SOLUTION** (i) Putting $-x=y$, we obtain

$$\lim_{x\to 0}\frac{e^{-x}-1}{x}=\lim_{y\to 0}\frac{e^y-1}{-y}=-\lim_{y\to 0}\frac{e^y-1}{y}=-1$$

(ii) We have,

$$\lim_{x\to 0}\frac{e^x-e^{-x}}{x}=\lim_{x\to 0}\left\{\left(\frac{e^x-1}{x}\right)-\left(\frac{e^{-x}-1}{x}\right)\right\}=\left(\lim_{x\to 0}\frac{e^x-1}{x}\right)+\left(\lim_{x\to 0}\frac{e^{-x}-1}{-x}\right)=1+1=2$$

(iii) We have,

$$\lim_{x\to 1}\frac{x-1}{\log_e x}=\lim_{h\to 0}\frac{1+h-1}{\log_e(1+h)}=\lim_{h\to 0}\frac{h}{\log_e(1+h)}=\frac{1}{\lim_{h\to 0}\frac{\log(1+h)}{h}}=\frac{1}{1}=1$$

<u>ALITER</u> $$\lim_{x\to 1}\frac{x-1}{\log_e x}=\lim_{x\to 1}\frac{x-1}{\log_e\{1+(x-1)\}}=\lim_{x\to 1}\frac{1}{\frac{\log_e\{1+(x-1)\}}{x-1}}=\frac{1}{1}=1$$

(iv) We have,

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{x^2}$$

$$=\lim_{x\to 0}\frac{e^{2x}+1-2e^x}{x^2\,e^x}=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\times e^x=\lim_{x\to 0}\left(\frac{e^x-1}{x}\right)^2\times\lim_{x\to 0}e^x=(1^2)\times e^0=1.$$

**EXAMPLE 4** *Evaluate the following limits:*

(i) $\lim_{x\to 0}\frac{\log(5+x)-\log(5-x)}{x}$ (ii) $\lim_{x\to 5}\frac{\log x-\log 5}{x-5}$

**SOLUTION** (i) We have,

$$\lim_{x\to 0}\frac{\log(5+x)-\log(5-x)}{x}$$

$$=\lim_{x\to 0}\frac{\log\left\{5\left(1+\frac{x}{5}\right)\right\}-\log\left\{5\left(1-\frac{x}{5}\right)\right\}}{x}$$

$$=\lim_{x\to 0}\frac{\left\{\log 5+\log\left(1+\frac{x}{5}\right)\right\}-\left\{\log 5+\log\left(1+\frac{x}{5}\right)\right\}}{x}$$

$$=\lim_{x\to 0}\frac{\log\left(1+\frac{x}{5}\right)-\log\left(1-\frac{x}{5}\right)}{x}$$

$$= \lim_{x \to 0} \frac{1}{5} \times \frac{\log\left(1+\frac{x}{5}\right)}{x/5} - \lim_{x \to 0} \frac{\log\left(1-\frac{x}{5}\right)}{-x/5} \times \frac{1}{(-5)} = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

(ii) $$\lim_{x \to 5} \frac{\log x - \log 5}{x-5} = \lim_{h \to 0} \frac{\log(5+h) - \log 5}{h} = \lim_{h \to 0} \frac{\log\left(\frac{5+h}{5}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{5}\right)}{\frac{h}{5}} \times \frac{1}{5} = \frac{1}{5}$$

<u>ALITER</u> $$\lim_{x \to 5} \frac{\log x - \log 5}{x-5} = \lim_{x \to 5} \frac{\log\left(\frac{x}{5}\right)}{x-5} = \lim_{x \to 5} \frac{\log\left(1+\frac{x}{5}-1\right)}{x-5}$$

$$= \lim_{x \to 5} \frac{\log\left(1+\frac{x-5}{5}\right)}{\left(\frac{x-5}{5}\right)} \times \frac{1}{5} = 1 \times \frac{1}{5} = \frac{1}{5}$$

**LEVEL-2**

**EXAMPLE 5** *Prove that:* $\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}} = -\frac{4}{3}$.

**SOLUTION** We have,

$$\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{3-x} - 3^{x/2}}$$

$$= \lim_{x \to 2} \frac{(3^x)^2 - 12(3^x) + 27}{3^3 - (3^{x/2})^3}$$

$$= \lim_{x \to 2} \frac{(3^x - 3)\ (3^x - 9)}{(3 - 3^{x/2})\ (9 + 3 \times 3^{x/2} + 3^x)}$$

$$= - \lim_{x \to 2} \frac{(3^x - 3)(3^{x/2} - 3)(3^{x/2} + 3)}{(3^{x/2} - 3)(3^x + 3 \times 3^{x/2} + 9)}$$

$$= - \lim_{x \to 2} \frac{(3^x - 3)\ (3^{x/2} + 3)}{(3^x + 3 \times 3^{x/2} + 9)} = -\frac{(9-3)(3+3)}{(9+9+9)} = -\frac{4}{3}$$

**EXAMPLE 6** *Evaluate:* $\lim_{x \to e} \frac{\log x - 1}{x - e}$

**SOLUTION** We have,

$$\lim_{x \to e} \frac{\log x - 1}{x-e} = \lim_{h \to 0} \frac{\log(e+h) - 1}{e+h-e} = \lim_{h \to 0} \frac{\log(e+h) - \log e}{h} \qquad [\because \log e = 1]$$

$$= \lim_{h \to 0} \frac{\log\left(1+\frac{h}{e}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{e}\right)}{\frac{h}{e}} \times \frac{1}{e}$$

$$= 1 \times \frac{1}{e} = \frac{1}{e} \qquad \left[\because \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

**ALITER** $\lim\limits_{x \to e} \dfrac{\log x - 1}{x - e} = \lim\limits_{x \to e} \dfrac{\log x - \log e}{x - e} = \lim\limits_{x \to e} \dfrac{\log\left(\dfrac{x}{e}\right)}{x - e}$

$$= \lim_{x \to e} \frac{\log\left(1 + \dfrac{x}{e} - 1\right)}{x - e} = \lim_{x \to e} \frac{\log\left(1 + \dfrac{x - e}{e}\right)}{\left(\dfrac{x - e}{e}\right) e} = \frac{1}{e} \times 1 = \frac{1}{e}$$

## EXERCISE 29.10

### LEVEL-1

*Evaluate the following limits:*

1. $\lim\limits_{x \to 0} \dfrac{5^x - 1}{\sqrt{4 + x} - 2}$
2. $\lim\limits_{x \to 0} \dfrac{\log(1 + x)}{3^x - 1}$
3. $\lim\limits_{x \to 0} \dfrac{a^x + a^{-x} - 2}{x^2}$
4. $\lim\limits_{x \to 0} \dfrac{a^{mx} - 1}{b^{nx} - 1}$, $n \neq 0$
5. $\lim\limits_{x \to 0} \dfrac{a^x + b^x - 2}{x}$
6. $\lim\limits_{x \to 0} \dfrac{9^x - 2.6^x + 4^x}{x^2}$
7. $\lim\limits_{x \to 0} \dfrac{8^x - 4^x - 2^x + 1}{x^2}$
8. $\lim\limits_{x \to 0} \dfrac{a^{mx} - b^{nx}}{x}$
9. $\lim\limits_{x \to 0} \dfrac{a^x + b^x + c^x - 3}{x}$
10. $\lim\limits_{x \to 2} \dfrac{x - 2}{\log_a(x - 1)}$
11. $\lim\limits_{x \to 0} \dfrac{5^x + 3^x + 2^x - 3}{x}$
12. $\lim\limits_{x \to \infty} (a^{1/x} - 1)\, x$
13. $\lim\limits_{x \to 0} \dfrac{a^{mx} - b^{nx}}{\sin kx}$
14. $\lim\limits_{x \to 0} \dfrac{a^x + b^x - c^x - d^x}{x}$
15. $\lim\limits_{x \to 0} \dfrac{e^x - 1 + \sin x}{x}$
16. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{e^x - 1}$
17. $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$
18. $\lim\limits_{x \to 0} \dfrac{e^{2x} - e^x}{\sin 2x}$
19. $\lim\limits_{x \to a} \dfrac{\log x - \log a}{x - a}$
20. $\lim\limits_{x \to 0} \dfrac{\log(a + x) - \log(a - x)}{x}$
21. $\lim\limits_{x \to 0} \dfrac{\log(2 + x) + \log 0.5}{x}$
22. $\lim\limits_{x \to 0} \dfrac{\log(a + x) - \log a}{x}$
23. $\lim\limits_{x \to 0} \dfrac{\log(3 + x) - \log(3 - x)}{x}$
24. $\lim\limits_{x \to 0} \dfrac{8^x - 2^x}{x}$
25. $\lim\limits_{x \to 0} \dfrac{x(2^x - 1)}{1 - \cos x}$
26. $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - 1}{\log(1 + x)}$
27. $\lim\limits_{x \to 0} \dfrac{\log|1 + x^3|}{\sin^3 x}$
28. $\lim\limits_{x \to \pi/2} \dfrac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x}$
29. $\lim\limits_{x \to 0} \dfrac{e^x - 1}{\sqrt{1 - \cos x}}$
30. $\lim\limits_{x \to 5} \dfrac{e^x - e^5}{x - 5}$

31. $\lim_{x \to 0} \frac{e^{x+2} - e^2}{x}$

32. $\lim_{x \to \pi/2} \frac{e^{\cos x} - 1}{\cos x}$

33. $\lim_{x \to 0} \frac{e^{3+x} - \sin x - e^3}{x}$

34. $\lim_{x \to 0} \frac{e^x - x - 1}{2}$

35. $\lim_{x \to 0} \frac{e^{3x} - e^{2x}}{x}$

36. $\lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x}$

37. $\lim_{x \to 0} \frac{e^{bx} - e^{ax}}{x}$ where $0 < a < b$

38. $\lim_{x \to 0} \frac{e^{\tan x} - 1}{x}$

39. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

40. $\lim_{x \to 0} \frac{3^{2+x} - 9}{x}$

41. $\lim_{x \to 0} \frac{a^x - a^{-x}}{x}$

42. $\lim_{x \to 0} \frac{x(e^x - 1)}{1 - \cos x}$

43. $\lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)}$

## ANSWERS

| | | | |
|---|---|---|---|
| 1. $4 \log 5$ | 2. $\frac{1}{\log_e 3}$ | 3. $(\log_e a)^2$ | 4. $\frac{m}{n} \frac{\log a}{\log b}$ |
| 5. $\log (ab)$ | 6. $\left(\log \frac{3}{2}\right)^2$ | 7. $(\log 4)(\log 2)$ | 8. $\log\left(\frac{a^m}{b^n}\right)$ |
| 9. $\log (abc)$ | 10. $\log a$ | 11. $\log 30$ | 12. $\log a$ |
| 13. $\frac{1}{k} \log\left(\frac{a^m}{b^n}\right)$ | 14. $\log\left(\frac{ab}{cd}\right)$ | 15. 2 | 16. 2 |
| 17. 1 | 18. $\frac{1}{2}$ | 19. $\frac{1}{a}$ | 20. $\frac{2}{a}$ |
| 21. $\frac{1}{2}$ | 22. $\frac{1}{a}$ | 23. $\frac{2}{3}$ | 24. $\log 4$ |
| 25. $\log 4$ | 26. $\frac{1}{2}$ | 27. 1 | 28. $\log a$ |
| 29. Does not exist | 30. $e^5$ | 31. $e^2$ | 32. 1 |
| 33. $e^3 - 1$ | 34. 0 | 35. 1 | 36. 1 |
| 37. $(b - a)$ | 38. 1 | 39. 1 | 40. $9 . \log_e 3$ |
| 41. $2 \log_e 2$ | 42. 2 | 43. $\frac{2}{\pi} \log_e 2$ | |

## HINTS TO SELECTED PROBLEMS

10. $$\lim_{x \to 2} \frac{x-2}{\log_a (x-1)} = \lim_{x \to 2} \frac{(x-2)}{\log_e (x-1) \times \log_a e}$$

$$= \lim_{x \to 2} \frac{1}{\frac{\log_e \{1 + (x-2)\}}{x-2}} \times \log_e a = \frac{1}{1} \times \log_e a = \log_e a$$

**29.** We have,

$$\lim_{x \to 0} \frac{e^x - 1}{\sqrt{1 - \cos x}} = \lim_{x \to 0} \frac{e^x - 1}{\sqrt{2}\left|\sin \frac{x}{2}\right|}$$

Now, $$\lim_{x \to 0^-} \frac{e^x - 1}{\sqrt{2}\left|\sin \frac{x}{2}\right|} = -\frac{1}{\sqrt{2}} \lim_{x \to 0} \frac{e^x - 1}{\sin \frac{x}{2}} = -\frac{2}{\sqrt{2}} \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right) \times \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}}\right) = -\sqrt{2}$$

and, $$\lim_{x \to 0^+} \frac{e^x - 1}{\sqrt{2}\left|\sin \frac{x}{2}\right|} = \frac{2}{\sqrt{2}} \lim_{x \to 0^+} \left(\frac{e^x - 1}{x}\right)\left(\frac{\frac{x}{2}}{\sin \frac{x}{2}}\right) = \sqrt{2}.$$

$$\therefore \quad \lim_{x \to 0^-} \frac{e^x - 1}{\sqrt{2}\left|\sin \frac{x}{2}\right|} \neq \lim_{x \to 0^+} \frac{e^x - 1}{\sqrt{2}\left(\sin \frac{x}{2}\right)}$$

**33.** $$\lim_{x \to 0} \left\{\frac{e^{3+x} - e^3}{x} - \frac{\sin x}{x}\right\} = \lim_{x \to 0} \left\{e^3\left(\frac{e^x - 1}{x}\right) - \frac{\sin x}{x}\right\} = e^3 \times 1 - 1 = e^3 - 1$$

**39.** $$\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} = \lim_{x \to 0} e^{\sin x}\left(\frac{e^{x - \sin x} - 1}{x - \sin x}\right) = e^0 \times 1 = 1$$

**42.** $$\lim_{x \to 0} x\left(\frac{e^x - 1}{1 - \cos x}\right) = \lim_{x \to 0} x^2\left(\frac{\frac{e^x - 1}{x}}{2\sin^2 \frac{x}{2}}\right) = 2\lim_{x \to 0}\left(\frac{e^x - 1}{x}\right)\left(\frac{\frac{x}{2}}{\sin \frac{x}{2}}\right)^2 = 2 \times 1 \times 1^2 = 2.$$

**43.** $$\lim_{x \to \pi/2} \frac{2^{-\cos x} - 1}{x\left(x - \frac{\pi}{2}\right)} = \lim_{x \to \pi/2} \frac{2^{\sin(x - \pi/2)} - 1}{x - \frac{\pi}{2}} \times \frac{1}{x}$$

$$= \lim_{x \to \pi/2} \frac{2^{\sin\left(x - \frac{\pi}{2}\right)} - 1}{\sin\left(x - \frac{\pi}{2}\right)} \times \frac{\sin\left(x - \frac{\pi}{2}\right)}{x - \frac{\pi}{2}} \times \frac{1}{x} = \log_e 2 \times 1 \times \frac{2}{\pi} = \frac{2}{\pi} \log_e 2$$

## 29.9 EVALUATION OF LIMITS OF THE FORM $1^\infty$

To evaluate exponential limits of the form $1^\infty$, we use the following result which is stated and proved as a theorem.

**THEOREM** *If* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ *such that* $\lim_{x \to a} \frac{f(x)}{g(x)}$ *exists, then*

$$\lim_{x \to a} \left\{1 + f(x)\right\}^{\frac{1}{g(x)}} = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

<u>PROOF</u> Let $A = \lim_{x \to a} \left\{1 + f(x)\right\}^{\frac{1}{g(x)}}$. Then,

$$\log_e A = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{g(x)}$$

$$\Rightarrow \quad \log_e A = \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} \times \frac{f(x)}{g(x)}$$

$$\Rightarrow \quad \log_e A = \lim_{x \to a} \frac{f(x)}{g(x)} \qquad \left[\because \lim_{x \to a} \frac{\log\{1 + f(x)\}}{f(x)} = 1\right]$$

$$\Rightarrow \quad A = e^{\lim_{x \to a} \frac{f(x)}{g(x)}}$$

Q.E.D

**REMARK** *The above result can also be restated in the following form:*

*If* $\lim_{x \to a} f(x) = 1$ *and* $\lim_{x \to a} g(x) = \infty$ *such that* $\lim_{x \to a} \{f(x) - 1\} g(x)$ *exists, then*

$$\lim_{x \to a} \{f(x)\}^{g(x)} = e^{\lim_{x \to a} \{f(x) - 1\} g(x)}$$

**PARTICULAR CASES**

(i) $\lim_{x \to 0} (1 + x)^{1/x} = e$ (ii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(iii) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^{\lambda}$ (iv) $\lim_{x \to \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^{\lambda}$

**Following examples will illustrate the applications of the above results.**

## ILLUSTRATIVE EXAMPLES

### LEVEL-1

**EXAMPLE 1** *Evaluate the following limits:*

(i) $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$ (ii) $\lim_{x \to 0} (1 + \sin x)^{2 \cot x}$

(iii) $\lim_{x \to 1} (\log_3 3x)^{\log_x 3}$ (iv) $\lim_{x \to 0} (\cos x)^{\cot x}$

**SOLUTION** (i) We have,

$$\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x = e^{\lim_{x \to \infty} \frac{2}{x} \times x} = e^2$$

(ii) We have,

$$\lim_{x \to 0} (1 + \sin x)^{2 \cot x} = e^{\lim_{x \to 0} \sin x \times 2 \cot x} = e^{\lim_{x \to 0} 2 \cos x} = e^2$$

(iii) We have,

$$\lim_{x \to 1} (\log_3 3x)^{\log_x 3} = \lim_{x \to 1} (\log_3 3 + \log_3 x)^{\log_x 3}$$

$$= \lim_{x \to 1} (1 + \log_3 x)^{\frac{1}{\log_3 x}} = e^{\lim_{x \to 1} \log_3 x \times \frac{1}{\log_3 x}} = e^1 = e$$

(iv) We have,

$$\lim_{x \to 0} (\cos x)^{\cot x}$$

$$= \lim_{x \to 0} \{1 + \cos x - 1\}^{\cot x}$$

$$= \lim_{x \to 0} \{1 - (1 - \cos x)\}^{\cot x}$$

$$= \lim_{x \to 0} \left\{1 - 2\sin^2\left(\frac{x}{2}\right)\right\}^{\cot x}$$

$$= e^{\lim\limits_{x \to 0} -2\sin^2(x/2) \times \cot x}$$

$$= e^{\lim\limits_{x \to 0} \frac{-2\sin^2(x/2)\ \cos x}{2\sin(x/2)\cos x/2}} = e^{\lim\limits_{x \to 0} -\tan(x/2) \times \cos x} = e^0 = 1$$

**EXAMPLE 2** *Evaluate the following limits:*

(i) $\lim\limits_{x \to 0} \left(\dfrac{a^x + b^x + c^x}{3}\right)^{1/x}$ (ii) $\lim\limits_{x \to a} \left(2 - \dfrac{a}{x}\right)^{\tan \pi x/2a}$

**SOLUTION** (i) We have,

$$\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$$

$$\lim_{x \to 0} \left\{1 + \frac{a^x + b^x + c^x - 3}{3}\right\}^{1/x} = \lim_{x \to 0} \left\{1 + \frac{(a^x - 1) + (b^x - 1) + (c^x - 1)}{3}\right\}^{1/x}$$

$$= e^{\lim\limits_{x \to 0} \frac{a^x - 1}{3x} + \frac{b^x - 1}{3x} + \frac{c^x - 1}{3x}} = e^{\frac{1}{3}\left\{\lim\limits_{x \to 0} \frac{a^x - 1}{x} + \lim\limits_{x \to 0} \frac{b^x - 1}{x} + \lim\limits_{x \to 0} \frac{c^x - 1}{x}\right\}}$$

$$= e^{\frac{1}{3}\{\log a + \log b + \log c\}} = e^{\log (abc)^{1/3}} = (abc)^{1/3}$$

(ii) We have,

$$\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \pi x/2a}$$

$$= \lim_{x \to a} \left\{1 + \left(1 - \frac{a}{x}\right)\right\}^{\tan \pi x/2a}$$

$$= e^{\lim\limits_{x \to a} \left(1 - \frac{a}{x}\right) \tan \frac{\pi x}{2a}} = e^{\lim\limits_{x \to a} \left(\frac{x - a}{x}\right) \tan \frac{\pi x}{2a}} = e^{l}, \text{ where } l = \lim_{x \to a} \left(\frac{x - a}{x}\right) \tan \frac{\pi x}{2a}$$

Now, $l = \lim\limits_{x \to a} \left(\dfrac{x - a}{x}\right) \tan \dfrac{\pi x}{2a} = \lim\limits_{x \to a} \left(\dfrac{x - a}{x}\right) \cot \left(\dfrac{\pi}{2} - \dfrac{\pi x}{2a}\right)$

$$\Rightarrow \quad l = \lim_{x \to a} \left(\frac{x - a}{x}\right) \cot \frac{\pi}{2a}(a - x)$$

$$\Rightarrow \quad l = \lim_{x \to a} \left(\frac{x - a}{x}\right) \times \frac{1}{\tan \frac{\pi}{2a}(a - x)}$$

$$\Rightarrow \quad l = -\lim_{x \to a} \frac{(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{1}{x} = -\lim_{x \to a} \frac{\frac{\pi}{2a}(a-x)}{\tan \frac{\pi}{2a}(a-x)} \times \frac{2a}{\pi x} = -\frac{2}{\pi}$$

Hence, $\lim_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan \frac{\pi x}{2a}} = e^{-2/\pi}$.

**LEVEL-2**

**EXAMPLE 3** *Evaluate the following limits:*

(i) $\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x$ (ii) $\lim_{x \to \infty} \left(\frac{x^2+4x+3}{x^2+2x-5}\right)^x$

**SOLUTION** (i) We have,

$$\lim_{x \to \infty} \left(\frac{x+5}{x-1}\right)^x = \lim_{x \to \infty} \left(1 + \frac{6}{x-1}\right)^x = e^{\lim_{x \to \infty} \frac{6x}{x-1}} = e^6$$

(ii) We have

$$\lim_{x \to \infty} \left(\frac{x^2+4x-3}{x^2+2x-5}\right)^x = \lim_{x \to \infty} \left\{1 + \frac{2x+8}{x^2+2x-5}\right\}^x = e^{\lim_{x \to \infty} \frac{2x^2+8x}{x^2+2x-5}} = e^2$$

**EXAMPLE 4** *Evaluate the following limits:*

(i) $\lim_{x \to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$ (ii) $\lim_{x \to 0} (\cos 2x)^{1/x^2}$

**SOLUTION** (i) We have,

$$\lim_{x \to 0} \left\{\tan\left(\frac{\pi}{4} + x\right)\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{\frac{1+\tan x}{1-\tan x}\right\}^{1/x}$$

$$= \lim_{x \to 0} \left\{1 + \frac{2\tan x}{1-\tan x}\right\}^{1/x} = e^{\lim_{x \to 0} \frac{2\tan x}{1-\tan x} \times \frac{1}{x}} = e^{\lim_{x \to 0} \frac{2}{1-\tan x} \times \frac{\tan x}{x}} = e^2$$

(ii) We have,

$$\lim_{x \to 0} (\cos 2x)^{1/x^2} = \lim_{x \to 0} \left\{1 + (\cos 2x - 1)\right\}^{1/x^2} = e^{\lim_{x \to 0} \frac{\cos 2x - 1}{x^2}} = e^{-\lim_{x \to 0} \frac{2\sin^2 x}{x^2}} = e^{-2}$$

## EXERCISE 29.11

**LEVEL-1**

*Evaluate the following limits:*

1. $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$
2. $\lim_{x \to 0^+} \left\{1 + \tan^2 \sqrt{x}\right\}^{1/2x}$
3. $\lim_{x \to 0} (\cos x)^{1/\sin x}$
4. $\lim_{x \to 0} (\cos x + \sin x)^{1/x}$

5. $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x}$

6. $\lim_{x \to \infty} \left\{ \frac{x^2 + 2x + 3}{2x^2 + x + 5} \right\}^{\frac{3x-2}{3x+2}}$

7. $\lim_{x \to 1} \left\{ \frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 3} \right\}^{\frac{1 - \cos(x-1)}{(x-1)^2}}$

8. $\lim_{x \to 0} \left\{ \frac{e^x + e^{-x} - 2}{x^2} \right\}^{1/x^2}$

9. $\lim_{x \to a} \left\{ \frac{\sin x}{\sin a} \right\}^{\frac{1}{x-a}}$

10. $\lim_{x \to \infty} \left\{ \frac{3x^2 + 1}{4x^2 - 1} \right\}^{\frac{x^3}{1+x}}$

### ANSWERS

1. $e^x$ 2. $\sqrt{e}$ 3. 1 4. $e$ 5. $e^{ab}$ 6. $\frac{1}{2}$ 7. $\sqrt{\frac{5}{6}}$ 8. $e^{1/12}$ 9. $e^{\cot a}$ 10. 0

### VERY SHORT ANSWER QUESTIONS (VSAQs)

*Answer each of the following questions in one word or one sentence or as per exact requirement of the question:*

1. Write the value of $\lim_{x \to 0} \frac{\sqrt{1 - \cos 2x}}{x}$.
2. Write the value of $\lim_{x \to 0^-} [x]$.
3. Write the value of $\lim_{x \to 0^+} [x]$.
4. Write the value of $\lim_{x \to 1^-} x - [x]$.
5. Write the value of $\lim_{x \to 0^-} \frac{\sin [x]}{[x]}$.
6. Write the value of $\lim_{x \to \pi} \frac{\sin x}{x - \pi}$.
7. Write the value of $\lim_{x \to \infty} \frac{\sin x}{x}$.
8. Write the value of $\lim_{x \to 2} \frac{|x-2|}{x-2}$.
9. Write the value of $\lim_{x \to 0} \frac{\sin x^\circ}{x}$.
10. Write the value of $\lim_{x \to 0^-} \frac{\sin x}{\sqrt{x}}$.
11. Write the value of $\lim_{x \to 0} \frac{\sin x}{\sqrt{1+x} - 1}$.
12. Write the value of $\lim_{x \to -\infty} (3x + \sqrt{9x^2 - x})$.
13. Write the value of $\lim_{n \to \infty} \frac{n! + (n+1)!}{(n+1)! + (n+2)!}$.
14. Write the value of $\lim_{x \to \pi/2} \frac{2x - \pi}{\cos x}$.
15. Write the value of $\lim_{n \to \infty} \frac{1 + 2 + 3 + \ldots + n}{n^2}$.

### ANSWERS

1. Does not exit 2. $-1$ 3. 1 4. 1 5. sin 1 6. $-1$ 7. 0 8. Does not exist 9. $\frac{\pi}{180}$ 10. Does not exist 11. 2 12. $\frac{1}{6}$ 13. 0 14. $-2$ 15. $\frac{1}{2}$

### MULTIPLE CHOICE QUESTIONS (MCQs)

*Mark the correct alternative in each of the following:*

1. $\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$ is equal to

(a) 1 (b) 1/2 (c) 1/3 (d) 0

2. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$ is equal to

(a) 0 (b) 1 (c) 1/2 (d) 2

3. If $f(x) = x \sin(1/x),\ x \neq 0$, then $\lim_{x \to 0} f(x) =$

(a) 1 (b) 0 (c) $-1$ (d) does not exist

4. $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x}$ is

(a) 0 (b) 1 (c) 2 (d) 4

5. $\lim_{x \to 0} \dfrac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} =$

(a) 10/3 (b) 3/10 (c) 6/5 (d) 5/6

6. $\lim_{x \to 0} \dfrac{x}{\tan x}$ is

(a) 0 (b) 1 (c) 4 (d) not defined

7. $\lim_{n \to \infty} \left\{ \dfrac{1}{1 - n^2} + \dfrac{2}{1 - n^2} + \ldots + \dfrac{n}{1 - n^2} \right\}$ is equal to

(a) 0 (b) $-1/2$ (c) 1/2 (d) none of these

8. $\lim_{x \to \infty} \dfrac{\sin x}{x}$ equals

(a) 0 (b) $\infty$ (c) 1 (d) does not exist

9. $\lim_{x \to 0} \dfrac{\sin x^0}{x}$ is equal to

(a) 1 (b) $\pi$ (c) $x$ (d) $\pi / 180$

10. $\lim_{x \to 3} \dfrac{x - 3}{|x - 3|}$, is equal to

(a) 1 (b) $-1$ (c) 0 (d) does not exist

11. $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$ is equal to

(a) $na^n$ (b) $na^{n-1}$ (c) $na$ (d) 1

12. $\lim_{x \to \pi/4} \dfrac{\sqrt{2} \cos x - 1}{\cot x - 1}$ is equal to

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{2\sqrt{2}}$ (d) 1

13. $\lim_{x \to \infty} \dfrac{\sqrt{x^2 - 1}}{2x + 1}$ is equal to

(a) 1 (b) 0 (c) $-1$ (d) 1/2

14. $\lim_{h \to 0} 2 \left\{ \dfrac{\sqrt{3} \sin(\pi/6 + h) - \cos(\pi/6 + h)}{\sqrt{3}\, h (\sqrt{3} \cos h - \sin h)} \right\}$ is equal to

(a) 2/3 (b) 4/3 (c) $-2\sqrt{3}$ (d) $-4/3$

15. $\lim_{h\to 0}\left\{\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right\}=$

(a) $-1/12$ (b) $-4/3$ (c) $-16/3$ (d) $-1/48$

16. $\lim_{n\to\infty}\left\{\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\ldots+\frac{1}{(2n+1)(2n+3)}\right\}$ is equal to

(a) 0 (b) 1/2 (c) 1/9 (d) 2

17. $\lim_{x\to 1}\frac{\sin\pi x}{x-1}$ is equal to

(a) $-\pi$ (b) $\pi$ (c) $-\frac{1}{\pi}$ (d) $\frac{1}{\pi}$

18. If $\lim_{x\to 1}\frac{x+x^2+x^3+\ldots+x^n-n}{x-1}=5050$, then $n$ equals

(a) 10 (b) 100 (c) 150 (d) none of these

19. The value of $\lim_{x\to\infty}\frac{\sqrt{1+x^4}+(1+x^2)}{x^2}$ is

(a) $-1$ (b) 1 (c) 2 (d) none of these

20. $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1

21. $\lim_{x\to\pi/3}\frac{\sin\left(\frac{\pi}{3}-x\right)}{2\cos x-1}$ is equal to

(a) $\sqrt{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

22. $\lim_{x\to 3}\frac{\sum_{r=1}^{n}x^r-\sum_{r=1}^{n}3^r}{x-3}$ is equal to

(a) $\frac{(2n-1)\times 3^n}{4}$ (b) $\frac{(2n-1)\times 3^n+1}{4}$ (c) $(2n-1)\,3^n+1$ (d) $\frac{(2n-1)\times 3^n-1}{4}$

23. $\lim_{n\to\infty}\frac{1-2+3-4+5-6+\ldots.+(2n-1)-2n}{\sqrt{n^2+1}+\sqrt{n^2-1}}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) $-1$

24. If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$, then $\lim_{x\to 0} f(x)$ equals

(a) 1 (b) 0 (c) $-1$ (d) none of these.

25. $\lim_{n \to \infty} \dfrac{n!}{(n+1)!+n!}$ is equal to

(a) $\frac{1}{2}$ (b) 0 (c) 2 (d) 1

26. $\lim_{x \to \pi/4} \dfrac{4\sqrt{2}-(\cos x+\sin x)^5}{1-\sin 2x}$ is equal to

(a) $5\sqrt{2}$ (b) $3\sqrt{2}$ (c) $\sqrt{2}$ (d) none of these

27. $\lim_{x \to 2} \dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$ is equal to

(a) $\dfrac{1}{8\sqrt{3}}$ (b) $\dfrac{1}{\sqrt{3}}$ (c) $8\sqrt{3}$ (d) $\sqrt{3}$

28. $\lim_{x \to \infty} a^x \sin\left(\dfrac{b}{a^x}\right)$, $a, b > 1$ is equal to

(a) $b$ (b) $a$ (c) $a \log_e b$ (d) $b \log_e a$

29. $\lim_{x \to 0} \dfrac{8}{x^8}\left\{1-\cos\dfrac{x^2}{2}-\cos\dfrac{x^2}{4}+\cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4}\right\}$ is equal to

(a) 1/16 (b) −1/16 (c) 1/32 (d) −1/32

30. If $\alpha$ is a repeated root of $ax^2+bx+c=0$, then $\lim_{x \to \alpha} \dfrac{\tan(ax^2+bx+c)}{(x-\alpha)^2}$ is

(a) $a$ (b) $b$ (c) $c$ (d) 0

31. The value of $\lim_{x \to 0} \dfrac{\sqrt{a^2-ax+x^2}-\sqrt{a^2+ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$ is

(a) $a$ (b) $\sqrt{a}$ (c) $-a$ (d) $-\sqrt{a}$

32. The value of $\lim_{x \to 0} \dfrac{1-\cos x+2\sin x-\sin^3 x-x^2+3x^4}{\tan^3 x-6\sin^2 x+x-5x^3}$ is

(a) 1 (b) 2 (c) −1 (d) −2

33. $\lim_{\theta \to \pi/2} \dfrac{1-\sin\theta}{(\pi/2-\theta)\cos\theta}$ is equal to

(a) 1 (b) −1 (c) 1/2 (d) −1/2

34. The value of $\lim_{x \to \pi/2} (\sec x-\tan x)$ is

(a) 2 (b) −1 (c) 1 (d) 0

35. The value of $\lim_{x \to \infty} \dfrac{n!}{(n+1)!-n!}$ is

(a) 0 (b) 1 (c) −1 (d) none of these

36. The value of $\lim_{n \to \infty} \dfrac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}$ is

(a) 0 (b) −1 (c) 1 (d) none of these

37. The value of $\lim_{x \to \infty} \dfrac{(x+1)^{10}+(x+2)^{10}+\ldots+(x+100)^{10}}{x^{10}+10^{10}}$ is

(a) 10 (b) 100 (c) $10^{10}$ (d) none of these

38. The value of $\lim\limits_{n \to \infty} \left\{ \dfrac{1+2+3+\ldots+n}{n+2} - \dfrac{n}{2} \right\}$ is

(a) 1/2 (b) 1 (c) −1 (d) −1/2

39. $\lim\limits_{x \to 1} [x-1]$, where $[\cdot]$ is the greatest integer function, is equal to

(a) 1 (b) 2 (c) 0 (d) does not exist

40. $\lim\limits_{x \to \infty} \dfrac{|x|}{x}$ is equal to

(a) 1 (b) −1 (c) 0 (d) does not exist

41. $\lim\limits_{x \to 0} \dfrac{|\sin x|}{x}$ is

(a) 1 (b) −1 (c) 0 (d) does not exist

42. If $f(x) = \begin{cases} \dfrac{\sin [x]}{[x]} & , [x] \neq 0 \\ 0 & , [x] = 0 \end{cases}$, where $[\cdot]$ denotes the greatest integer function, then $\lim\limits_{x \to 0} f(x)$ is equal to

(a) 1 (b) 0 (c) −1 (d) none of these

## ANSWERS

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. (c) | 2. (d) | 3. (b) | 4. (a) | 5. (a) | 6. (b) | 7. (b) | 8. (a) |
| 9. (d) | 10. (d) | 11. (b) | 12. (b) | 13. (d) | 14. (b) | 15. (d) | 16. (b) |
| 17. (a) | 18. (b) | 19. (b) | 20. (a) | 21. (c) | 22. (b) | 23. (b) | 24. (b) |
| 25. (a) | 26. (a) | 27. (a) | 28. (a) | 29. (c) | 30. (a) | 31. (d) | 32. (b) |
| 33. (c) | 34. (d) | 35. (a) | 36. (c) | 37. (b) | 38. (d) | 39. (d) | 40. (d) |
| 41. (d) | 42. (d) | | | | | | |

## SUMMARY

1. $\lim\limits_{x \to a} f(x)$ exists $\Leftrightarrow \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$

For a function $f(x)$ and a real number a, $\lim\limits_{x \to a} f(x)$ and $f(a)$ may not be same. In fact:

(i) $\lim\limits_{x \to a} f(x)$ exists but $f(a)$ (the value of $f(x)$ at $x = a$) may not exists

(ii) The value $f(a)$ exists but $\lim\limits_{x \to a} f(x)$ does not exist

(iii) $\lim\limits_{x \to a} f(x)$ and $f(a)$ both exist but are unequal

(iv) $\lim\limits_{x \to a} f(x)$ and $f(a)$ both exist and are equal.

3. Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If $l$ and $m$ both exist, then

(i) $\lim\limits_{x \to a} k f(x) = k \lim\limits_{x \to a} f(x)$

(ii) $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l + m$

(iii) $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = lm$

(iv) $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$ (v) $\lim\limits_{x \to a} \{f(a)\}^{g(x)} = l^m$

**4. Following are some standard limits:**

(i) $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ (ii) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

(iii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$ (iv) $\lim\limits_{x \to a} \dfrac{\sin(x-a)}{x-a} = 1$

(v) $\lim\limits_{x \to a} \dfrac{\tan(x-a)}{x-a} = 1$ (vi) $\lim\limits_{x \to 0} \dfrac{\log(1+x)}{x} = 1$

(vii) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a,\ a \neq 0, a > 1$ (viii) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$