CHAPTER 14

MEAN VALUE THEOREMS

14.1 ROLLE'S THEOREM

STATEMENT *Let f be a real valued function defined on the closed interval [a, b] such that*

(i) it is continuous on the closed interval [a, b] (ii) it is differentiable on the open interval (a, b) and, (iii) $f(a) = f(b)$.

Then, there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

GEOMETRICAL INTERPRETATION OF ROLLE'S THEOREM *Let $f(x)$ be a real valued function defined on [a, b] such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ and it is possible to draw a unique tangent at every point on the curve $y = f(x)$ between points A and B. Also, the ordinates at the end points of the interval [a, b] are equal. Then, there exists at least one point $(c, f(c))$ lying between A and B on the curve $y = f(x)$ where tangent is parallel to x-axis.*

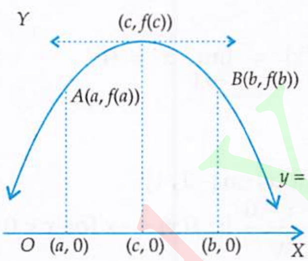

Fig. 14.1

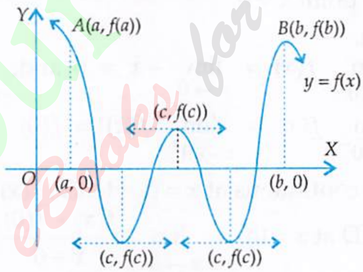

Fig. 14.2

ALGEBRAIC INTERPRETATION OF ROLLE'S THEOREM Let $f(x)$ be a polynomial with a and b as its roots. Since a polynomial function is everywhere continuous and differentiable and a and b are roots of $f(x)$, therefore $f(a) = f(b) = 0$. So, $f(x)$ satisfies conditions of Rolle's theorem. Consequently, there exists $c \in (a, b)$ such that $f'(c) = 0$ i.e. $f'(x) = 0$ at $x = c$. In other words $x = c$ is a root of $f'(x)$. Thus, algebraically Rolle's theorem can be interpreted as follows:

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

REMARK *On Rolle's theorem generally two types of problems are formulated.*

(a) *To check the applicability of Rolle's theorem to a given function on a given interval,*

(b) *To verify Rolle's theorem for a given function on a given interval. In both types of problems we first check whether $f(x)$ satisfies conditions of Rolle's theorem or not. The following results are very helpful in doing so.*

  (i) *A polynomial function is everywhere continuous and differentiable.*

  (ii) *The exponential function, sine and cosine functions are every where continuous and differentiable.*

  (iii) *Logarithmic function is continuous and differentiable in its domain.*

  (iv) *tan x is not continuous at $x = \pm \pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots$*

  (v) *$|x|$ is not differentiable at $x = 0$.*

  (vi) *If $f'(x)$ tends to $\pm\infty$ as $x \to k$, then $f(x)$ is not differentiable at $x = k$.*

For example, if $f(x)=(2x-1)^{1/2}$, *then* $f'(x)=\dfrac{1}{\sqrt{2x-1}}$ *is such that* $\lim\limits_{x\to(1/2)^+} f'(x)=\infty$.

So, $f(x)$ *is not differentiable at* $x=1/2$.

(vii) *The sum, difference, product and quotient of continuous (differentiable) functions is continuous (differentiable).*

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I TO CHECK THE APPLICABILITY OF ROLLE'S THEOREM

EXAMPLE 1 *Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals:*

(i) $f(x)=|x|$ *on* $[-1,1]$ (ii) $f(x)=3+(x-2)^{2/3}$ *on* $[1,3]$ (iii) $f(x)=\tan x$ *on* $[0,\pi]$

SOLUTION (i) We have,

$$f(x)=|x|=\begin{cases}-x, & \text{when } -1\le x<0\\ x, & \text{when } 0\le x\le 1\end{cases}$$

Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous and differentiable for all $x<0$ as well as for all $x>0$ except possibly at $x=0$. So, consider the point $x=0$.

We find that,

$$\lim_{x\to0^-} f(x)=\lim_{x\to0} -x=0 \text{ and, } \lim_{x\to0^+} f(x)=\lim_{x\to0} x=0$$

$$\therefore \quad \lim_{x\to0^-} f(x)=\lim_{x\to0^+} f(x)=f(0)$$

Thus, $f(x)$ is continuous at $x=0$. Hence, $f(x)$ is continuous on $[-1,1]$.

Now, $$(\text{LHD at } x=0)=\lim_{x\to0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{-x-0}{x} \quad [\because f(x)=-x \text{ for } x<0 \text{ and } f(0)=0]$$

$$=\lim_{x\to0}\frac{-x}{x}=\lim_{x\to0} -1=-1$$

and, $$(\text{RHD at } x=0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{x-0}{x} \quad [\because f(x)=x \text{ for } x>0]$$

$$=\lim_{x\to0}\frac{x}{x}=\lim_{x\to0} 1=1$$

Clearly, $(\text{LHD at } x=0)\neq(\text{RHD at } x=0)$. This means that $f(x)$ is not differentiable at $x=0\in(-1,1)$. Thus, the condition of derivability at each point of $(-1,1)$ is not satisfied. Hence, Rolle's theorem is not applicable to $f(x)=|x|$ on $[-1,1]$.

(ii) Given that $f(x)=3+(x-2)^{2/3}$, $x\in[1,3]$. We find that $f'(x)=\frac{2}{3}(x-2)^{-1/3}$.

Clearly, $\lim\limits_{x\to2^+} f'(x)=\infty$. So, $f(x)$ is not differentiable at $x=2\in(1,3)$. Hence, Rolle's theorem is not applicable to $f(x)=3+(x-2)^{2/3}$ on the interval $[1,3]$.

(iii) We have, $f(x)=\tan x$, $x\in[0,\pi]$. We find that $x=\frac{\pi}{2}\in[0,\pi]$ such that $f(x)$ is not continuous at $x=\frac{\pi}{2}$. So, the condition of continuity at each point of $[0,\pi]$ is not satisfied. Hence, Rolle's theorem is not applicable to $f(x)=\tan x$ on the interval $[0,\pi]$.

EXAMPLE 2 *Discuss the applicability of Rolle's theorem on the function* $f(x) = \begin{cases} x^2+1, \text{ when } 0 \le x \le 1 \\ 3-x, \text{ when } 1 < x \le 2 \end{cases}$

[NCERT EXEMPLAR]

SOLUTION Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous and differentiable at all points except possibly at $x = 1$. Let us now check the differentiability of $f(x)$ at $x = 1$. We find that

$$(\text{LHD at } x=1) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1} \frac{(x^2+1)-(1+1)}{x-1} \quad [\because f(x) = x^2+1 \text{ for } 0 \le x \le 1]$$

$$= \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 2$$

and, $$(\text{RHD at } x=1) = \lim_{x \to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1} \frac{(3-x)-(1+1)}{x-1} = \lim_{x \to 1} \frac{-(x-1)}{x-1} = -1$$

Clearly(LHD at $x=1$) $\neq$ (RHD at $x=1$). So, $f(x)$ is not differentiable at $x = 1$. Thus, the condition of differentiability at each point of the given interval is not satisfied. Hence, Rolle's theorem is not applicable to the given function on the interval [0, 2].

Type II ON VERIFICATION OF ROLLE'S THEOREM FOR A GIVEN FUNCTION DEFINED ON A GIVEN INTERVAL

EXAMPLE 3 *Verify Rolle's theorem for the function* $f(x) = x^2 - 5x + 6$ *on the interval* [2, 3].

[CBSE 2002C]

SOLUTION A polynomial function is everywhere differentiable and hence continuous. Therefore, $f(x)$ is continuous on $[2, 3]$ and differentiable on $(2, 3)$. We find that

$$f(2) = 2^2 - 5 \times 2 + 6 = 0 \text{ and } f(3) = 3^2 - 5 \times 3 + 6 = 0$$

So, $f(2 = f(3)$. Thus, all the conditions of Rolle's theorem are satisfied. Now, we have to show that there exists some $c \in (2, 3)$ such that $f'(c) = 0$. For this we proceed as follows.

We have,

$$f(x) = x^2 - 5x + 6 \Rightarrow f'(x) = 2x - 5$$

$$\therefore \quad f'(x) = 0 \Rightarrow 2x - 5 = 0 \Rightarrow x = 2.5$$

Thus, $c = 2.5 \in (2, 3)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 4 *Verify Rolle's theorem for the function* $f(x) = x(x-3)^2$, $0 \le x \le 3$.

SOLUTION We have, $f(x) = x^3 - 6x^2 + 9x$. We know that a polynomial function is everywhere differentiable and hence continuous also. So, $f(x)$ being a polynomial function is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Also, $f(0) = f(3) = 0$. Thus, all the conditions of Rolle's theorem are satisfied. Now we have to show that there exists $c \in (0, 3)$ such that $f'(c) = 0$.

We have,

$$f(x) = x^3 - 6x^2 + 9x \Rightarrow f'(x) = 3x^2 - 12x + 9$$

$$\therefore \quad f'(x) = 0 \Rightarrow 3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = 1, 3$$

Clearly, $c = 1 \in (0, 3)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 5 *Verify Rolle's theorem for the function* $f(x) = x^3 - 6x^2 + 11x - 6$ *on the interval* [1,3].

SOLUTION Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on [1, 3] and differentiable on (1, 3). Further, we find that:

$$f(1) = 1^3 - 6 \times 1^2 + 11 \times 1 - 6 = 0 \text{ and } f(3) = 3^3 - 6 \times 3^2 + 11 \times 3 - 6 = 0$$

$\therefore \quad f(1) = f(3)$

Thus, all the three conditions of Rolle's theorem are satisfied. Now we have to show that there exists $c \in (1, 3)$ such that $f'(c) = 0$.

We have,

$$f(x) = x^3 - 6x^2 + 11x - 6 \Rightarrow f'(x) = 3x^2 - 12x + 11$$

$$\therefore \quad f'(x) = 0 \Rightarrow 3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{12 \pm \sqrt{144 - 132}}{6} \Rightarrow x = 2 \pm \frac{1}{\sqrt{3}}$$

Clearly, both the values of x lie in the interval $(1, 3)$. Thus, $c = 2 \pm \frac{1}{\sqrt{3}} \in (2, 3)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

EXAMPLE 6 *Verify Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$ on the interval $[a, b]$, where m, n are positive integers.*

SOLUTION We have, $f(x) = (x-a)^m (x-b)^n$, where $m, n \in N$. On expanding $(x-a)^m$ and $(x-b)^n$ by binomial theorem and then taking the product, we find that $f(x)$ is a polynomial of degree $(m+n)$. Since a polynomial function is everywhere differentiable and so continuous also. Therefore, $f(x)$ is continuous on $[a, b]$ and is derivable on (a, b). Also, $f(a) = f(b) = 0$. Thus, all the three conditions of Rolle's theorem are satisfied. Now, we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$.

We have,

$$f(x) = (x-a)^m (x-b)^n$$

$$\Rightarrow \quad f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$$

$$\Rightarrow \quad f'(x) = (x-a)^{m-1}(x-b)^{n-1}\{m(x-b) + n(x-a)\} = (x-a)^{m-1}(x-b)^{n-1}\{x(m+n) - (mb+na)\}$$

$$\therefore \quad f'(x) = 0$$

$$\Rightarrow \quad (x-a)^{m-1}(x-b)^{n-1}\{x(m+n) - (mb+na)\} = 0$$

$$\Rightarrow \quad (x-a) = 0 \text{ or, } (x-b) = 0 \text{ or, } x(m+n) - (mb+na) = 0 \Rightarrow x = a \text{ or, } x = b \text{ or, } x = \frac{mb+na}{m+n}$$

Clearly, $x = \frac{mb+na}{m+n}$ divides (a, b) into the ratio $m : n$. Therefore, $\frac{mb+na}{m+n} \in (a, b)$.

Thus, $c = \frac{mb+na}{m+n} \in (a, b)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 7 *Verify Rolle's theorem for the function $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$.*

SOLUTION Clearly, $f(x)$ is defined for all $x \in [-2, 2]$ and has a unique value for each $x \in [-2, 2]$. So, at each point of $[-2, 2]$, the limit of $f(x)$ is equal to the value of the function. Therefore, $f(x)$ is continuous on $[-2, 2]$.

Now, $\quad f(x) = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{4-x^2}}$

Clearly, $f'(x) = \frac{-x}{\sqrt{4-x^2}}$ exists for all $x \in (-2, 2)$. So, $f(x)$ is differentiable on $(-2, 2)$.

Also, $\quad f(-2) = f(2) = 0$. Thus, all the three conditions of Rolle's theorem are satisfied.

Now we have to show that there exists $c \in (-2, 2)$ such that $f'(c) = 0$.

We have,

$$f(x) = \sqrt{4-x^2} \Rightarrow f'(x) = \frac{-x}{\sqrt{4-x^2}}$$

$$\therefore \quad f'(x) = 0 \Rightarrow \frac{-x}{\sqrt{4-x^2}} = 0 \Rightarrow x = 0$$

Clearly, $c = 0 \in (-2, 2)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 8 *Verify Rolle's theorem for the function* $f(x) = \log\left\{\frac{x^2+ab}{x(a+b)}\right\}$ *on* $[a, b]$, *where* $0 < a < b$.

SOLUTION We have,

$$f(x) = \log\left\{\frac{x^2+ab}{x(a+b)}\right\} = \log(x^2+ab) - \log x - \log(a+b).$$

Logarithmic function is differentiable and hence continuous in its domain. Therefore, $f(x)$ is continuous on $[a, b]$ and differential on (a, b). We find that

$$f(a) = \log\left(\frac{a^2+ab}{a(a+b)}\right) = \log 1 = 0 \text{ and, } f(b) = \log\left(\frac{b^2+ab}{b\,(a+b)}\right) = \log 1 = 0.$$

So, $f(a) = f(b)$. Thus, all the three conditions of Rolle's theorem are satisfied.

Now, we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$.

We have,

$$f(x) = \log(x^2+ab) - \log x - \log(a+b) \Rightarrow f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x} = \frac{x^2-ab}{x\,(x^2+ab)}$$

$$\therefore \quad f'(x) = 0 \Rightarrow \frac{x^2-ab}{x\,(x^2+ab)} = 0 \Rightarrow x^2 = ab \Rightarrow x = \sqrt{ab}$$

Clearly, $a < \sqrt{ab} < b$. Therefore, $c = \sqrt{ab} \in (a, b)$ is such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 9 *Verify Rolle's theorem for each of the following functions on indicated intervals :*

(i) $f(x) = \sin^2 x$ on $0 \le x \le \pi$ (ii) $f(x) = \sin x + \cos x - 1$ on $[0, \pi/2]$

(iii) $f(x) = \sin x - \sin 2x$ on $[0, \pi]$

SOLUTION (i) Since sine function is everywhere continuous and differentiable and the product of two continuous (differentiable) functions is continuous (differentiable). Therefore, $f(x) = \sin^2 x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. We also find that

$$f(0) = \sin^2 0 = 0 \text{ and } f(\pi) = \sin^2 \pi = 0. \text{ Therefore, } f(0) = f(\pi)$$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem. Now, we have to show that there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

We have,

$$f(x) = \sin^2 x \Rightarrow f'(x) = 2\sin x\cos x = \sin 2x$$

$$\therefore \quad f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = \pi \Rightarrow x = \pi/2.$$

Since $c = \pi/2 \in (0, \pi)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

(ii) Since $\sin x$ and $\cos x$ are everywhere continuous and differentiable. Therefore, $f(x) = \sin x + \cos x - 1$ is continuous on $[0, \pi/2]$ and differentiable on $(0, \pi/2)$.

Also, $f(0) = \sin 0 + \cos 0 - 1 = 0$ and $f(\pi/2) = \sin \pi/2 + \cos \pi/2 - 1 = 1 - 1 = 0$. Therefore, $f(0) = f(\pi/2)$. Thus, $f(x)$ satisfies conditions of Rolle's theorem on $[0, \pi/2]$. Therefore, there exists $c \in (0, \pi/2)$ such that $f'(c) = 0$.

Now, $f(x) = \sin x + \cos x - 1 \Rightarrow f'(x) = \cos x - \sin x$

$\therefore \quad f'(x) = 0 \Rightarrow \cos x - \sin x = 0 \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$

Clearly, $c = \pi/4 \in (0, \pi/2)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

(iii) Since sine function is everywhere continuous and differentiable, therefore so are $\sin x$ and $\sin 2x$. Consequently, $f(x) = \sin x - \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. We find that $f(0) = \sin 0 - \sin 0 = 0$ and $f(\pi) = \sin \pi - \sin 2\pi = 0$. Therefore, $f(0) = f(\pi)$.

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[0, \pi]$. Consequently there exists $c \in (0, \pi)$ such that $f'(c) = 0$.

Now, $f(x) = \sin x - \sin 2x \Rightarrow f'(x) = \cos x - 2\cos 2x$

$\therefore \quad f'(x) = 0 \Rightarrow \cos x - 2\cos 2x = 0 \Rightarrow \cos x - 2(2\cos^2 x - 1) = 0 \Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$$\Rightarrow \quad \cos x = \frac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or, } -0.5931 \Rightarrow x = \cos^{-1}(0.8431) \text{ or, } \cos^{-1}(-0.5931)$$

$$\Rightarrow \quad x = \cos^{-1}(0.8431) \text{ or, } 180^\circ - \cos^{-1}(0.5931) \qquad [\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$$

$$\Rightarrow \quad x = 32^\circ\, 32' \text{ or, } x = 126^\circ\, 23'$$

Thus, $c = 32^\circ\, 32'; 126^\circ\, 23' \in (0, \pi)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

EXAMPLE 10 *Verify Rolle's theorem for each of the following functions on the indicated intervals:*

(i) $f(x) = x(x+3)\, e^{-x/2}$ on $[-3, 0]$. **[NCERT EXEMPLAR]**

(ii) $f(x) = e^x (\sin x - \cos x)$ on $[\pi/4, 5\pi/4]$.

SOLUTION (i) Since a polynomial function and an exponential function are everywhere continuous and differentiable. Therefore, $f(x)$, being product of these two, is continuous on $[-3, 0]$ and differentiable on $(-3, 0)$. We find that $f(-3) = -3(-3+3)\, e^{3/2} = 0$ and $f(0) = 0$. Therefore, $f(-3) = f(0)$. Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[-3, 0]$. Consequently, there exists $c \in (-3, 0)$ such that $f'(c) = 0$.

Now, $f(x) = x(x+3)\, e^{-x/2}$

$$\Rightarrow \quad f'(x) = (2x+3)\, e^{-x/2} + (x^2 + 3x)\left(-\frac{1}{2}\right) e^{-x/2} = e^{-x/2}\left\{\frac{-x^2 + x + 6}{2}\right\}$$

$$\therefore \quad f'(x) = 0 \Rightarrow e^{-x/2}\left\{\frac{-x^2 + x + 6}{2}\right\} = 0 \Rightarrow -x^2 + x + 6 = 0 \Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow \quad (x-3)(x+2) = 0 \Rightarrow x = -2, 3$$

Thus, $c = -2 \in (-3, 0)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

(ii) An exponential function and sine and cosine functions are everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $[\pi/4, 5\pi/4]$ and differentiable on $(\pi/4, 5\pi/4)$.

$$\text{Also,} \quad f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

$$\text{and,} \quad f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{5\pi/4}\left(-\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = 0$$

$\therefore \quad f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$

Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem on $[\pi/4, 5\pi/4]$. Consequently, there exists $c \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$.

Now, $\quad f(x) = e^x (\sin x - \cos x)$

$\Rightarrow \quad f'(x) = e^x (\sin x - \cos x) + e^x (\cos x + \sin x) = 2e^x \sin x$

$\therefore \quad f'(x) = 0 \Rightarrow 2e^x \sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = \pi \qquad [\because e^x \neq 0]$

Thus, $c = \pi \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$. Hence, Rolle's theorem is verified.

BASED ON LOWER ORDER THINKING SKILLS (LOTS)

Type III MISCELLANEOUS EXAMPLES

EXAMPLE 11 *It is given that for the function $f(x) = x^3 - 6x^2 + ax + b$ on $[1, 3]$, Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b, if $f(1) = f(3) = 0$.*

SOLUTION We are given that $f(1) = f(3) = 0$.

$\therefore \quad 1^3 - 6 \times 1 + a + b = 3^3 - 6 \times 3^2 + 3a + b = 0 \Rightarrow a + b = 5$ and $3a + b = 27$

Solving these two equations for a and b, we get: $a = 11$ and $b = -6$.

We now verify whether for these values of a and b, $f'(c)$ is zero or not.
We have,

$$f(x) = x^3 - 6x^2 + ax + b \Rightarrow f(x) = x^3 - 6x^2 + 11x - 6 \qquad [\because a = 11, b = -6]$$

$$\Rightarrow \quad f'(x) = 3x^2 - 12x + 11$$

$$\therefore \quad f'(c) = 3c^2 - 12c + 11 = 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 - 12\left(2 + \frac{1}{\sqrt{3}}\right) + 11 = 12 + \frac{12}{\sqrt{3}} + 1 - 24 - \frac{12}{\sqrt{3}} + 11 = 0$$

Hence, $a = 11$ and $b = -6$.

EXAMPLE 12 *It is given that for the function f given by $f(x) = x^3 + bx^2 + ax$, $x \in [1, 3]$. Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. Find the values of a and b.*

SOLUTION It is given that the Rolle's theorem holds for $f(x)$ defined on $[1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$.

$$\therefore \quad f(1) = f(3) \text{ and } f'(c) = 0$$

$$\Rightarrow \quad 1 + b + a = 27 + 9b + 3a \text{ and } 3c^2 + 2bc + a = 0$$

$$\Rightarrow \quad 2a + 8b + 26 = 0 \text{ and } 3\left(2 + \frac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \text{ and } a + 4b + 13 + \frac{2}{\sqrt{3}}(b + 6) = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \text{ and } 0 + \frac{2}{\sqrt{3}}(b + 6) = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \text{ and } b = -6 \Rightarrow a = 11 \text{ and } b = -6$$

EXAMPLE 13 *Find the point on the curve $y = \cos x - 1$, $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ at which the tangent is parallel to the x-axis.*

SOLUTION Let $f(x) = \cos x - 1$. Clearly, $f(x)$ is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. Also, $f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - 1 = -1 = f\left(\frac{3\pi}{2}\right)$.

Thus, all the conditions of Rolle's theorem are satisfied. Consequently, there exists at least one point $c \in (\pi/2, 3\pi/2)$ for which $f'(c) = 0$. But,

$$f'(c)=0 \Rightarrow -\sin c=0 \Rightarrow c=\pi \Rightarrow f(c)=\cos\pi-1=-2$$

By geometrical interpretation of Rolle's theorem $(c, f(c))$ is the point on $y=\cos x-1$ where tangent is parallel to x-axis. Hence, $(\pi, -2)$ is the required point on the given curve.

EXERCISE 14.1

BASIC

1. Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals:
   (i) $f(x)=3+(x-2)^{2/3}$ on $[1, 3]$
   (ii) $f(x)=[x]$ for $-1 \le x \le 1$, where $[x]$ denotes the greatest integer not exceeding x
   (iii) $f(x)=\sin\dfrac{1}{x}$ for $-1 \le x \le 1$ (iv) $f(x)=2x^2-5x+3$ on $[1, 3]$
   (v) $f(x)=x^{2/3}$ on $[-1, 1]$ (vi) $f(x)=\begin{cases}-4x+5, & 0 \le x \le 1\\ 2x-3, & 1 < x \le 2\end{cases}$
2. Verify Rolle's theorem for each of the following functions on the indicated intervals:
   (i) $f(x)=x^2-8x+12$ on $[2, 6]$ (ii) $f(x)=x^2-4x+3$ on $[1, 3]$ **[NCERT, CBSE 2007]**
   (iii) $f(x)=(x-1)(x-2)^2$ on $[1, 2]$ (iv) $f(x)=x(x-1)^2$ on $[0, 1]$ **[NCERT EXEMPLAR]**
   (v) $f(x)=(x^2-1)(x-2)$ on $[-1, 2]$ (vi) $f(x)=x(x-4)^2$ on $[0, 4]$
   (vii) $f(x)=x(x-2)^2$ on $[0, 2]$ (viii) $f(x)=x^2+5x+6$ on $[-3, -2]$
3. Verify Rolle's theorem for each of the following functions on the indicated intervals:
   (i) $f(x)=\cos 2(x-\pi/4)$ on $[0, \pi/2]$ (ii) $f(x)=\sin 2x$ on $[0, \pi/2]$
   (iii) $f(x)=\cos 2x$ on $[-\pi/4, \pi/4]$ (iv) $f(x)=e^x \sin x$ on $[0, \pi]$
   (v) $f(x)=e^x \cos x$ on $[-\pi/2, \pi/2]$ (vi) $f(x)=\cos 2x$ on $[0, \pi]$
   (vii) $f(x)=\dfrac{\sin x}{e^x}$ on $0 \le x \le \pi$ (viii) $f(x)=\sin 3x$ on $[0, \pi]$
   (ix) $f(x)=e^{1-x^2}$ on $[-1, 1]$ (x) $f(x)=\log(x^2+2)-\log 3$ on $[-1, 1]$
   [NCERT EXEMPLAR]
   (xi) $f(x)=\sin x+\cos x$ on $[0, \pi/2]$ (xii) $f(x)=2\sin x+\sin 2x$ on $[0, \pi]$
   (xiii) $f(x)=\dfrac{x}{2}-\sin\dfrac{\pi x}{6}$ on $[-1, 0]$ (xiv) $f(x)=\dfrac{6x}{\pi}-4\sin^2 x$ on $\left[0, \dfrac{\pi}{6}\right]$
   (xv) $f(x)=4^{\sin x}$ on $[0, \pi]$ (xvi) $f(x)=x^2-5x+4$ on $[1, 4]$ **[CBSE 2007]**
   (xvii) $f(x) = \sin^4 x+\cos^4 x$ on $\left[0, \dfrac{\pi}{2}\right]$ **[NCERT EXEMPLAR]**
   (xviii) $f(x) = \sin x-\sin 2x$ on $[0, \pi]$

BASED ON LOTS

7. Using Rolle's theorem, find points on the curve $y=16-x^2$, $x \in [-1, 1]$, where tangent is parallel to x-axis. **[NCERT, CBSE 2000]**
8. At what points on the following curves, is the tangent parallel to x-axis?
   (i) $y = x^2$ on $[-2, 2]$ (ii) $y = e^{1-x^2}$ on $[-1, 1]$
   (iii) $y = 12(x+1)(x-2)$ on $[-1, 2]$.
9. If $f:[-5, 5] \to R$ is differentiable and if $f'(x)$ doesn't vanish anywhere, then prove that $f(-5) \ne f(5)$. **[NCERT]**
10. Examine if Rolle's theorem is applicable to any one of the following functions:
    (i) $f(x)=[x]$ for $x \in [5, 9]$ (ii) $f(x)=[x]$ for $x \in [-2, 2]$ **[NCERT]**
    Can you say something about the converse of Rolle's Theorem from these functions?

11. It is given that the Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + cx$, $x \in [1, 2]$ at the point $x = \frac{4}{3}$. Find the values of b and c.

ANSWERS

1. (i) Not applicable (ii) Not applicable (iii) Not applicable
   (iv) Not applicable (v) Not applicable (vi) Not applicable.

7. $(0, 16)$ 8. (i) $(0,0)$ (ii) $(0, e)$ (iii) $(1/2, -27)$ 11. $b = -5, c = 8$

14.2 LAGRANGE'S MEAN VALUE THEOREM

STATEMENT *Let $f(x)$ be a function defined on $[a, b]$ such that it is continuous on $[a, b]$ and differentiable on (a, b). Then, there exists a real number $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.*

PROOF Consider a function $\phi(x) = f(x) + Ax$, where A is a constant to be chosen in such a way that $\phi(a) = \phi(b)$.

Now,

$$\phi(a) = \phi(b) \Rightarrow f(a) + Aa = f(b) + Ab \Rightarrow f(b) - f(a) = -A(b - a) \Rightarrow A = -\left\{\frac{f(b) - f(a)}{b - a}\right\} \quad ...(i)$$

(i) Since $f(x)$ is continuous on $[a, b]$ and Ax, being a polynomial function, is everywhere continuous. Therefore, $\phi(x)$, being the sum of two continuous functions $f(x)$ and Ax, is continuous on $[a, b]$.

(ii) As $f(x)$ is differentiable on (a, b) and Ax, being a polynomial function, is everywhere differentiable. So, $\phi(x)$, being the sum of two differentiable functions $f(x)$ and Ax, is differentiable on (a, b).

Also, $\phi(a) = \phi(b)$. Thus, all the three conditions of Rolle's theorem are satisfied by $\phi(x)$ on $[a, b]$. So, there must exist some $c \in (a, b)$ such that $\phi'(c) = 0$.

Now, $\phi(x) = f(x) + Ax \Rightarrow \phi'(x) = f'(x) + A \Rightarrow \phi'(c) = f'(c) + A$

$$\therefore \quad \phi'(c) = 0 \Rightarrow f'(c) + A = 0 \Rightarrow f'(c) = -A \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{[Using (i)]}$$

Q.E.D.

GEOMETRICAL INTERPRETATION *Let $f(x)$ be a function defined on $[a, b]$, and let APB be the curve represented by $y = f(x)$. Then, coordinates of A and B are $(a, f(a))$ and $(b, f(b))$ respectively. Suppose the chord AB makes an angle ψ with the axis of x. Then, from the triangle ARB, we have*

$$\tan\psi = \frac{BR}{AR} \Rightarrow \tan\psi = \frac{f(b) - f(a)}{b - a}$$

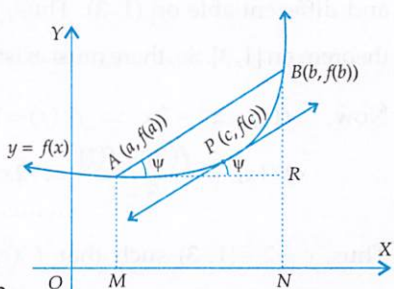

Fig.14.3

By Lagrange's Mean Value theorem, we obtain

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\Rightarrow \tan\psi = f'(c)$

$\Rightarrow$ Slope of the chord AB = Slope of the tangent at $(c, f(c))$

Thus, we arrive at the following geometrical interpretation of Lagrange's mean value theorem:

Let $f(x)$ be a function defined on $[a, b]$ such that the curve $y = f(x)$ is a continuous curve between points $A(a, f(a))$ and $B(b, f(b))$ and at every point on the curve, except at the end points, it is possible to draw a unique tangent. Then there exists a point on the curve such that the tangent there at is parallel to the chord joining the end points of the curve.

ILLUSTRATIVE EXAMPLES

BASED ON BASIC CONCEPTS (BASIC)

Type I VERIFICATION OF LAGRANGE'S MEAN VALUE THEOREM

EXAMPLE 1 *Verify Lagrange's mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ on the interval* $[3, 5]$.

SOLUTION We have, $f(x) = (x-3)(x-6)(x-9) = x^3 - 18x^2 + 99x - 162$.

Since a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $[3, 5]$ and differentiable on $(3, 5)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. So, there must exist at least one real number $c \in (3, 5)$ such that

$$f'(c) = \frac{f(5)-f(3)}{5-3}$$

Now, $f(x) = x^3 - 18x^2 + 99x - 162$

$\Rightarrow \quad f'(x) = 3x^2 - 36x + 99,\ f(5) = (5-3)(5-6)(5-9) = 8$ and $f(3) = 0$

$\therefore \quad f'(x) = \dfrac{f(5)-f(3)}{5-3}$

$\Rightarrow \quad 3x^2 - 36x + 99 = \dfrac{8-0}{5-3} \Rightarrow 3x^2 - 36x + 99 = 4$

$\Rightarrow \quad 3x^2 - 36x + 95 = 0 \Rightarrow x = \dfrac{36 \pm \sqrt{1296-1140}}{6} = \dfrac{36 \pm 12.49}{6} = 8.8, 4.8$

Clearly, $c = 4.8 \in (3, 5)$ such that $f'(c) = \dfrac{f(5)-f(3)}{5-3}$. Hence, Lagrange's mean value theorem is verified.

EXAMPLE 2 *Verify Lagrange's mean value theorem for the following functions on the indicated intervals. Also, find a point c in the indicated interval:*

(i) $f(x) = x(x-2)$ *on* $[1, 3]$ (ii) $f(x) = x(x-1)(x-2)$ *on* $[0, 1/2]$

SOLUTION (i) We have, $f(x) = x(x-2) = x^2 - 2x$. We know that a polynomial function is everywhere continuous and differentiable. So, $f(x)$ being a polynomial, is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem on $[1, 3]$. So, there must exist at least one real number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3)-f(1)}{3-1}$.

Now, $f(x) = x^2 - 2x \Rightarrow f'(x) = 2x - 2,\ f(3) = 9 - 6 = 3$ and $f(1) = 1 - 2 = -1$.

$\therefore \quad f'(x) = \dfrac{f(3)-f(1)}{3-1} \Rightarrow 2x - 2 = \dfrac{3-(-1)}{3-1} \Rightarrow 2x - 2 = 2 \Rightarrow x = 2$

Thus, $c = 2 \in (1, 3)$ such that $f'(c) = \dfrac{f(3)-f(1)}{3-1}$. Hence, Lagrange's mean value theorem is verified for $f(x)$ on $[1, 3]$.

(ii) We have, $f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$. Since $f(x)$ is a polynomial function and a polynomial function is everywhere continuous and differentiable. Therefore, $f(x)$ is continuous on $[0, 1/2]$ and differentiable on $(0, 1/2)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. So, there must exist at least one real number $c \in (0, 1/2)$ such that $f'(c) = \dfrac{f(1/2) - f(0)}{1/2 - 0}$.

Now, $f(x) = x^3 - 3x^2 + 2x \Rightarrow f'(x) = 3x^2 - 6x + 2$, $f(0) = 0$ and $f\left(\dfrac{1}{2}\right) = \dfrac{1}{8} - \dfrac{3}{4} + 1 = \dfrac{3}{8}$.

$$\therefore \quad f'(x) = \frac{f(1/2) - f(0)}{(1/2) - 0}$$

$$\Rightarrow \quad 3x^2 - 6x + 2 = \frac{(3/8) - 0}{(1/2) - 0}$$

$$\Rightarrow \quad 3x^2 - 6x + 2 = \frac{3}{4} \Rightarrow 12x^2 - 24x + 5 = 0 \Rightarrow x = \frac{24 \pm \sqrt{336}}{24} = 1 \pm \frac{\sqrt{21}}{6}$$

Clearly, $c = 1 - \dfrac{\sqrt{21}}{6} \in \left(0, \dfrac{1}{2}\right)$ such that $f'(c) = \dfrac{f(1/2) - f(0)}{(1/2) - 0}$. Hence, Lagrange's mean value theorem is verified.

EXAMPLE 3 *Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x-2}$ defined on the interval $[2, 3]$, where the tangent is parallel to the chord joining the end points of the curve.*

SOLUTION Let $f(x) = \sqrt{x-2}$. Since for each $x \in [2, 3]$, the function $f(x)$ attains a unique definite value. So, $f(x)$ is continuous on $[2, 3]$. Also, $f'(x) = \dfrac{1}{2\sqrt{x-2}}$ exists for all $x \in (2, 3)$. So, $f(x)$ is differentiable on $(2, 3)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (2, 3)$ such that $f'(c) = \dfrac{f(3) - f(2)}{3 - 2}$.

Now, $f(x) = \sqrt{x-2} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{x-2}}$, $f(3) = 1$ and $f(2) = 0$

$$\therefore \quad f'(x) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x-2}} = \frac{1-0}{3-2} \Rightarrow \frac{1}{2\sqrt{x-2}} = 1 \Rightarrow 4(x-2) = 1 \Rightarrow x - 2 = \frac{1}{4} \Rightarrow x = \frac{9}{4}$$

Clearly, $c = \dfrac{9}{4} \in (2, 3)$ such that $f'(c) = \dfrac{f(3) - f(2)}{3 - 2}$.

We find that $f(c) = \sqrt{\dfrac{9}{4} - 2} = \dfrac{1}{2}$. Thus, $(c, f(c))$ i.e. $(9/4, 1/2)$ is a point on the curve $y = \sqrt{x-2}$ such that the tangent at it is parallel to the chord joining the end points of the curve.

EXAMPLE 4 *Verify Lagrange's mean value theorem for the following functions on the indicated intervals.*

(i) $f(x) = x - 2\sin x$ on $[-\pi, \pi]$ (ii) $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$

(iii) $f(x) = \log_e x$ on $[1, 2]$ (iv) $f(x) = \begin{cases} 2 + x^3 & \text{if } x \le 1 \\ 3x & \text{if } x > 1 \end{cases}$ on $[-1, 2]$

SOLUTION (i) Since x and $\sin x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. So, there must exist at least one $c \in (-\pi, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}.$$

Now, $f(x) = x - 2 \sin x$

$\Rightarrow$ $f'(x) = 1 - 2 \cos x$, $f(\pi) = \pi - 2 \sin \pi = \pi$ and $f(-\pi) = -\pi - 2 \sin(-\pi) = -\pi$

$\therefore$ $$f'(x) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$$

$\Rightarrow$ $$1 - 2 \cos x = \frac{\pi - (-\pi)}{\pi - (-\pi)} \Rightarrow 1 - 2 \cos x = 1 \Rightarrow \cos x = 0 \Rightarrow x = \pm \pi/2$$

Thus, $c = \pm (\pi/2) \in (-\pi, \pi)$ such that $f'(c) = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)}$. Hence Lagrange's mean value theorem is verified.

(ii) Since $\sin x$ and $\sin 2x$ are everywhere continuous and differentiable, therefore $f(x)$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Thus, $f(x)$ satisfies both the conditions of Lagrange's mean value theorem. Consequently, there exists at least one $c \in (0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

Now,

$f(x) = 2 \sin x + \sin 2x \Rightarrow f'(x) = 2 \cos x + 2 \cos 2x$, $f(0) = 0$ and $f(\pi) = 2 \sin \pi + \sin 2\pi = 0$

$\therefore$ $$f'(x) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$\Rightarrow$ $$2 \cos x + 2 \cos 2x = \frac{0 - 0}{\pi - 0} \Rightarrow 2 \cos x + 2 \cos 2x = 0 \Rightarrow \cos x + \cos 2x = 0$$

$\Rightarrow$ $\cos 2x = -\cos x \Rightarrow \cos 2x = \cos(\pi - x) \Rightarrow 2x = \pi - x \Rightarrow 3x = \pi \Rightarrow x = \pi/3$

Thus, $c = \frac{\pi}{3} \in (0, \pi)$ such that $f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$. Hence, Lagrange's mean value theorem is verified.

(iii) The logarithmic function $f(x) = \log_e x$ is differentiable and so continuous for all $x > 0$. So, $f(x)$ is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Thus, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (1, 2)$ such that $f'(c) = \frac{f(2) - f(1)}{2 - 1}$.

Now, $f(x) = \log_e x \Rightarrow f'(x) = \frac{1}{x}$, $f(2) = \log_e 2$ and $f(1) = \log_e 1 = 0$

$\therefore$ $$f'(x) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow \frac{1}{x} = \frac{\log_e 2 - 0}{2 - 1} \Rightarrow \frac{1}{x} = \log_e 2 \Rightarrow x = \frac{1}{\log_e 2} = \log_2 e$$

Now, $2 < e < 4. \Rightarrow \log_2 2 < \log_2 e < \log_2 4 \Rightarrow 1 < \log_2 e < 2.$

Thus, $c = \log_2 e \in (1, 2)$ such that $f'(c) = \frac{f(2) - f(1)}{2 - 1}$. Hence, Lagrange's mean value theorem is verified.

(iv) We observe that $2 + x^3$ and $3x$ are polynomial functions. Therefore, $f(x)$ is continuous and differentiable for all values of x except possibly at $x = 1$.

Continuity at $x = 1$: We find that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2 + x^3) = 2 + 1^3 = 3, \quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1} 3x = 3 \times 1 = 3 \text{ and } f(1) = 2 + 1^3 = 3.$$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$. So, $f(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$: We find that

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{2 + x^3 - 3}{x - 1} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1} x^2 + x + 1 = 1^2 + 1 + 1 = 3$$

and, $$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{3x - 3}{x - 1} = \lim_{x \to 1} \frac{3(x-1)}{(x-1)} = 3$$

Clearly, (LHD at $x = 1$) = (RHD at $x = 1$). So, $f(x)$ is differentiable at $x = 1$.

Thus, $f(x)$ is continuous and differentiable on $[-1, 2]$. So, both the conditions of Lagrange's mean value theorem are satisfied. Consequently, there must exist some $c \in (-1, 2)$ such that

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

Now,

$$f(x) = \begin{cases} 2 + x^3, & x \le 1 \\ 3x, & x > 1 \end{cases} \Rightarrow f'(x) = \begin{cases} 3x^2, & x \le 1 \\ 3, & x > 1 \end{cases} \quad f(-1) = 2 + (-1)^3 = 1 \text{ and } f(2) = 3(2) = 6$$

$$\therefore \quad f'(x) = \frac{f(2) - f(-1)}{2 - (-1)} \Rightarrow f'(x) = \frac{6 - 1}{2 - (-1)} = \frac{5}{3}$$

Since $f'(x) = 3$ for $x \ge 1$, the value of x must be less than 1.

$\therefore \quad f'(x) = 5/3$

$\Rightarrow \quad 3x^2 = 5/3$ [$\because$ $x < 1$ and for $x < 1$, $f'(x) = 3x^2$]

$\Rightarrow \quad x^2 = 5/9 \Rightarrow x = \pm\sqrt{5}/3$

Clearly, $c = \pm \frac{\sqrt{5}}{3} \in (-1, 2)$ such that $f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$. Hence, Lagrange's mean value theorem is verified.

BASED ON HIGHER ORDER THINKING SKILLS (HOTS)

Type I ON PROVING INEQUALITIES BY USING LAGRANGE'S MEAN VALUE THEOREM

EXAMPLE 5 *Using Lagrange's mean value theorem, show that* $\sin x < x$ *for* $x > 0$.

SOLUTION Consider the function $f(x) = x - \sin x$ defined on the interval $[0, x]$, where $x > 0$. Clearly, $f(x)$ is everywhere continuous and differentiable. So, it is continuous on $[0, x]$ and differentiable on $(0, x)$. Consequently, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \quad \text{[By Lagrange's mean value theorem]}$$

$$\Rightarrow \quad 1 - \cos c = \frac{x - \sin x}{x}$$

$$\Rightarrow \quad \frac{x - \sin x}{x} > 0 \quad [\because 1 - \cos c > 0]$$

$$\Rightarrow \quad x - \sin x > 0 \Rightarrow x > \sin x \Rightarrow \sin x < x \text{ for all } x. \quad [\because x > 0]$$

EXAMPLE 6 *Using mean value theorem, prove that* $\tan x > x$ *for all* x $(0, \pi/2)$.

SOLUTION Let x be any point in the interval $(0, \pi/2)$. Consider the function f given by

$$f(x) = \tan x - x, \text{ where } x \in [0, x] \subset (0, \pi/2)$$

Clearly, $f(x)$ is continuous on $[0, x]$ and differentiable on $(0, x)$. So, there exists $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \quad \sec^2 c - 1 = \frac{(\tan x - x) - 0}{x}$$

$$\Rightarrow \quad \frac{\tan x - x}{x} > 0 \qquad \left[\because \sec^2 c > 1 \text{ for all } c \in (0, x) \subset \left(0, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad \tan x - x > 0 \qquad [\because x > 0]$$

$$\Rightarrow \quad \tan x > x \quad \text{for all } x \in (0, 2).$$

EXAMPLE 7 *Using Lagrange's mean value theorem, prove that* $\frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a}$, *where* $0 < a < b$.

SOLUTION Consider the function f given by $f(x) = \log_e x$, $x \in [a, b]$, $0 < a < b$.

Clearly, it is continuous on $[a, b]$ and differentiable on (a, b). So, by Lagrange's mean value theorem there exist $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{1}{c} = \frac{\log b - \log a}{b - a} \qquad \left[\because f(x) = \log x \Rightarrow f'(x) = \frac{1}{x}\right]$$

Now, $\quad c \in (a, b)$

$$\Rightarrow \quad a < c < b \Rightarrow \frac{1}{b} < \frac{1}{c} < \frac{1}{a} \Rightarrow \frac{1}{b} < \frac{\log b - \log a}{b - a} < \frac{1}{a} \Rightarrow \frac{b-a}{b} < \log\left(\frac{b}{a}\right) < \frac{b-a}{a} \qquad [\because b - a > 0]$$

Type II MISCELLANEOUS APPLICATIONS OF LAGRANGE'S MEAN VALUE THEOREM

EXAMPLE 8 *Let f and g be differentiable on $[0, 1]$ such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$ and $g(1) = 2$. Show that there exists $c \in (0, 1)$ such that $f'(c) = 2g'(c)$.*

SOLUTION Consider the function ϕ given by

$$\phi(x) = \{g(1) - g(0)\} f(x) - \{f(1) - f(0)\} g(x) \quad \text{for all } x \in [0, 1]$$

or, $$\phi(x) = 2f(x) - 4g(x) \quad \text{for all } x \in [0, 1].$$

Since $f(x)$ and $g(x)$ are differentiable on $[0, 1]$. Therefore, $\phi(x)$ is differentiable on $[0, 1]$. As $\phi(x)$ is differentiable on $[0, 1]$. So, it is also continuous on $[0, 1]$. Consequently, by Lagrange's mean value theorem there exists $c \in (0, 1)$ such that

$$\phi'(c) = \frac{\phi(1) - \phi(0)}{1 - 0}$$

$$\Rightarrow \quad 2f'(c) - 4g'(c) = [2f(1) - 4g(1)] - [2f(0) - 4g(0)]$$

$$\Rightarrow \quad 2f'(c) - 4g'(c) = 4 - 4 \Rightarrow 2f'(c) = 4g'(c) \Rightarrow f'(c) = 2g'(c)$$

EXAMPLE 9 *Let f be a twice differentiable function such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there exists at least one value γ between a and b for which $f''(\gamma) < 0$.*

SOLUTION Let us consider the function f on $[a, b]$.$\Rightarrow$ $x > \sin x \Rightarrow \sin x < x$ for all x.It is given that f is twice differentiable. Therefore, f' and f both exist and are continuous on $[a, b]$.

Applying Lagrange's Mean value Theorem to f on the intervals $[a, c]$ and $[c, b]$ respectively, we get

$$\frac{f(c) - f(a)}{c - a} = f'(\alpha), \text{ where } \alpha \in (a, c) \text{ and, } \frac{f(b) - f(c)}{b - c} = f'(\beta), \text{ where } \beta \in (c, b)$$

$$\Rightarrow \quad \frac{f(c)}{c-a} = f'(\alpha) \text{ and } \frac{-f(c)}{b-c} = f'(\beta) \qquad [\because f(a) = f(b) = 0]$$

Clearly, $a < \alpha < \beta < b$. It is given that $f'(x)$ is continuous on $[a, b]$ and $[\alpha, \beta] \subset [a, b]$. Therefore, $f'(x)$ is continuous on $[\alpha, \beta]$ and differentiable on (α, β). Applying Lagrange's Mean Value Theorem to $f'(x)$ and $[\alpha, \beta]$.

$$\frac{f'(\beta) - f'(\alpha)}{\beta - \alpha} = f''(\gamma), \text{ where } \alpha < \gamma < \beta$$

$$\Rightarrow \quad f''(\gamma) = \frac{1}{\beta - \alpha}\left\{-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}\right\} \Rightarrow f''(\gamma) = -\frac{f(c)}{\beta - \alpha} \times \frac{(b-a)}{(b-c)(c-a)} < 0$$

EXERCISE 14.2

BASIC

1. Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each case find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem:
   (i) $f(x) = x^2 - 1$ on $[2, 3]$
   (ii) $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$
   (iii) $f(x) = x(x-1)$ on $[1, 2]$
   (iv) $f(x) = x^2 - 3x + 2$ on $[-1, 2]$
   (v) $f(x) = 2x^2 - 3x + 1$ on $[1, 3]$
   (vi) $f(x) = x^2 - 2x + 4$ on $[1, 5]$
   (vii) $f(x) = 2x - x^2$ on $[0, 1]$
   (viii) $f(x) = (x-1)(x-2)(x-3)$ on $[0, 4]$
   (ix) $f(x) = \sqrt{25 - x^2}$ on $[-3, 4]$
   (x) $f(x) = \tan^{-1} x$ on $[0, 1]$
   (xi) $f(x) = x + \frac{1}{x}$ on $[1, 3]$ **[CBSE 2000]**
   (xii) $f(x) = x(x+4)^2$ on $[0, 4]$
   (xiii) $f(x) = \sqrt{x^2 - 4}$ on $[2, 4]$ **[CBSE 2002]**
   (xiv) $f(x) = x^2 + x - 1$ on $[0, 4]$ **[CBSE 2002]**
   (xv) $f(x) = \sin x - \sin 2x - x$ on $[0, \pi]$
   (xvi) $f(x) = x^3 - 5x^2 - 3x$ on $[1, 3]$ **[NCERT]**
2. Discuss the applicability of Lagrange's mean value theorem for the function $f(x) = |x|$ on $[-1, 1]$.
3. Show that the lagrange's mean value theorem is not applicable to the function $f(x) = \frac{1}{x}$ on $[-1, 1]$.
4. Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function $f(x) = \frac{1}{4x-1}, 1 \le x \le 4$.

BASED ON LOTS

5. Find a point on the parabola $y = (x-4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).
6. Find a point on the curve $y = x^2 + x$, where the tangent is parallel to the chord joining (0, 0) and (1, 2).
7. Find a point on the parabola $y = (x-3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).
8. Find the points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining (1, – 2) and (2, 2).
9. Find a point on the curve $y = x^3 + 1$ where the tangent is parallel to the chord joining (1, 2) and (3, 28).

BASED ON HOTS

10. Let C be a curve defined parametrically as $x = a\cos^3\theta,\ y = a\sin^3\theta,\ 0 \le \theta \le \frac{\pi}{2}$. Determine a point P on C, where the tangent to C is parallel to the chord joining the points $(a, 0)$ and $(0, a)$. **[CBSE 2014]**

11. Using Lagrange's mean value theorem, prove that

$$(b-a)\sec^2 a < \tan b - \tan a < (b-a)\sec^2 b, \text{ where } 0 < a < b < \frac{\pi}{2}.$$

ANSWERS

1. (i) $c = 5/2$ (ii) $c = 1/3$ (iii) $c = 3/2$ (iv) $c = 1/2$ (v) $c = 2$
(vi) $c = 3$ (vii) $c = 1/2$ (viii) $c = 2 \pm \frac{2}{\sqrt{3}}$ (ix) $c = \pm\frac{1}{\sqrt{2}}$ (x) $c = \sqrt{\frac{4}{\pi} - 1}$
(xi) $c = \sqrt{3}$ (xii) $c = \frac{-8 + 4\sqrt{13}}{3}$ (xiii) $c = \sqrt{6}$ (xiv) $c = 2$
(xv) $c = \cos^{-1}\left(\frac{1 \pm \sqrt{33}}{8}\right)$ (xvi) $c = \frac{7}{3}$

2. Not applicable 5. $(9/2, 1/4)$ 6. $(1/2, 3/4)$ 7. $(7/2, 1/4)$

8. $\left(\pm\sqrt{\frac{7}{3}}, \mp\frac{2}{3}\sqrt{\frac{7}{3}}\right)$ 9. $\left(\sqrt{\frac{13}{3}}, \left(\frac{13}{3}\right)^{3/2} + 1\right)$ 11. $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$

FILL IN THE BLANKS TYPE QUESTIONS (FBQs)

1. A function $f(x) = 1 + \frac{1}{x}$ is defined on the closed interval [1, 3]. A point in the interval, where the function satisfies the mean value theorem, is ....

2. For the function $f(x) = 8x^2 - 7x + 5,\ x \in [-6, 6]$, the value of c for the Lagrange's mean value theorem is ....

3. If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3] satisfies Roll's theorem for $c = 2 + \frac{1}{\sqrt{3}}$, then $a =$ ...., $b =$ ....

4. It is given that for the function $f(x) = x^3 - 6x^2 + ax + b$ on [1, 3], Rolle's theorem holds with $c = 2 + \frac{1}{\sqrt{3}}$. If $f(1) = f(3) = 0$, then $a =$ ...., $b =$ ....

5. For the function $f(x) = \log_e x,\ x \in [1, 2]$, the value of c for the Lagrange's mean value theorem is ....

6. The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is ....

ANSWERS

1. $\sqrt{3}$ 2. 0 3. $a = 11, b \in R$ 4. $a = 11, b = -6$ 5. $\log_2 e$ 6. 1

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. If $f(x) = Ax^2 + Bx + C$ is such that $f(a) = f(b)$, then write the value of c in Rolle's theorem.
2. State Rolle's theorem.
3. State Lagrange's mean value theorem.
4. If the value of c prescribed in Rolle's theorem for the function $f(x) = 2x(x-3)^n$ on the interval $[0, 2\sqrt{3}]$ is $\frac{3}{4}$, write the value of n (a positive integer).
5. Find the value of c prescribed by Lagrange's mean value theorem for the function $f(x) = \sqrt{x^2 - 4}$ defined on [2, 3].

ANSWERS

1. $\frac{a+b}{2}$ 4. 3 5. $\sqrt{5}$