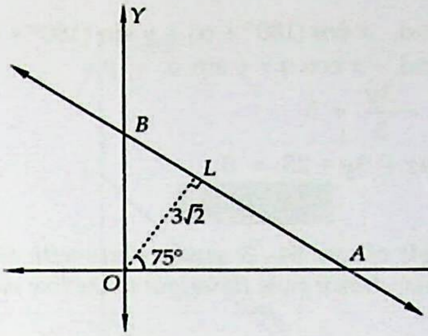

Fig. 23.43

So, the equation of the line is

$$x\cos\alpha + y\sin\alpha = p \text{ or, } x\cos 75° + y\sin 75° = 3\sqrt{2} \quad \text{...(i)}$$

or, $\left(\dfrac{\sqrt{3}-1}{2\sqrt{2}}\right)x + \left(\dfrac{\sqrt{3}+1}{2\sqrt{2}}\right)y = 3\sqrt{2}$ $\left[\because \cos 75° = \dfrac{\sqrt{3}-1}{2\sqrt{2}} \text{ and } \sin 75° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}\right]$

or, $(\sqrt{3}-1)\,x + (\sqrt{3}+1)\,y = 12$, which is the required equation.

EXAMPLE 5 *Find the equation of the straight line upon which the length of the perpendicular from the origin is 5 and the slope of this perpendicular is* $\dfrac{3}{4}$.

SOLUTION Suppose the perpendicular OL drawn from the origin O on the given line makes acute angle α with x-axis. Then, the slope of OL is $\tan\alpha$. But, it is given that the slope of OL is $\dfrac{3}{4}$.

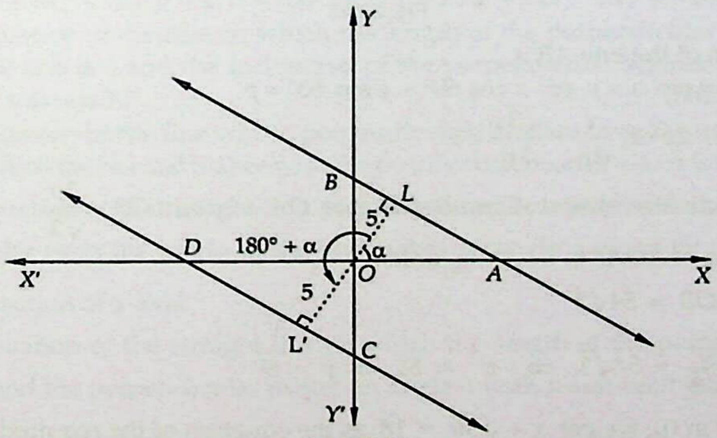

Fig. 23.44

$\therefore \quad \tan\alpha = \dfrac{3}{4}$ [Given]

Since $\tan(180° + \alpha) = \tan\alpha$. So, there are two possible lines AB and CD on which the perpendicular drawn from the origin has slope $\dfrac{3}{4}$.

Now, $\tan\alpha = \dfrac{3}{4} \Rightarrow \sin\alpha = \dfrac{3}{5}$ and $\cos\alpha = \dfrac{4}{5}$

Here, $p = 5$

So, the equations of the required lines are

$$x \cos \alpha + y \sin \alpha = p \text{ and, } x \cos (180° + \alpha) + y \sin (180° + \alpha) = p$$

or, $$x \cos \alpha + y \sin \alpha = p \text{ and, } -x \cos \alpha - y \sin \alpha = p$$

or, $$\frac{4x}{5} + \frac{3y}{5} = 5 \text{ and } \frac{-4x}{5} - \frac{3y}{5} = 5$$

or, $$4x + 3y - 25 = 0 \text{ and } 4x + 3y + 25 = 0$$

LEVEL-2

EXAMPLE 6 *A line forms a triangle of area $54\sqrt{3}$ square units with the coordinate axes. Find the equation of the line if the perpendicular drawn from the origin to the line makes an angle of 60° with the X-axis.*

SOLUTION Let AB be the given line and $OL = p$ be the perpendicular drawn from the origin on the line.

It is given that the perpendicular OL makes 60° angle with x-axis. Therefore, $\alpha = 60°$.

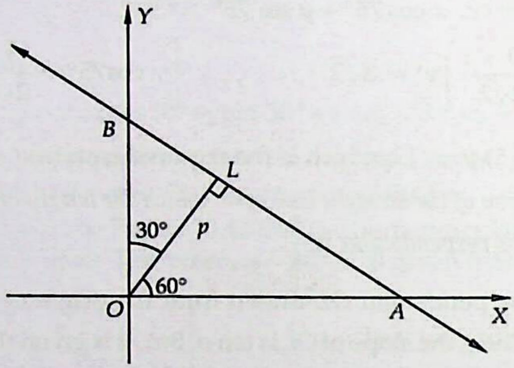

Fig. 23.45

Thus, the equation of the line AB is

$$x \cos \alpha + y \sin \alpha = p \text{ or, } x \cos 60° + y \sin 60° = p$$

or, $$x + \sqrt{3}y = 2p \text{ or, } \frac{x}{2p} + \frac{\sqrt{3}y}{2p} = 1 \qquad \text{...(i)}$$

This, cuts the coordinates axes at A and B such that $OA = 2p$ and $OB = \dfrac{2p}{\sqrt{3}}$.

It is given that area of ΔOAB is $54\sqrt{3}$ sq. units.

$$\therefore \quad \frac{1}{2} \times OA \times OB = 54\sqrt{3}$$

$$\Rightarrow \quad \frac{1}{2} \times 2p \times \frac{2p}{\sqrt{3}} = 54\sqrt{3} \Rightarrow p^2 = 81 \Rightarrow p = 9$$

Substituting $p = 9$ in (i), we get $x + \sqrt{3}y = 18$ as the equation of the required line.

EXAMPLE 7 *A straight canal is $4\frac{1}{2}$ miles from a place and the shortest route from this place to the canal is exactly north-east. A village is 3 miles north and four miles east from the place. Does it lie by the nearest edge of the canal?*

SOLUTION Let the given place be O. Take this as the origin and the east and north directions through O as the x and y axes respectively. Let AB be the nearest edge of the canal. It is given that the canal is $4\frac{1}{2}$ miles from O. This means that the perpendicular distance of AB from O is $4\frac{1}{2}$ miles i.e. $OL \perp AB$ and $OL = 4\frac{1}{2}$. It is also given that OL is exactly north-east. Therefore, $\angle LOA = 45°$.

So, the equation of the edge AB of the canal is

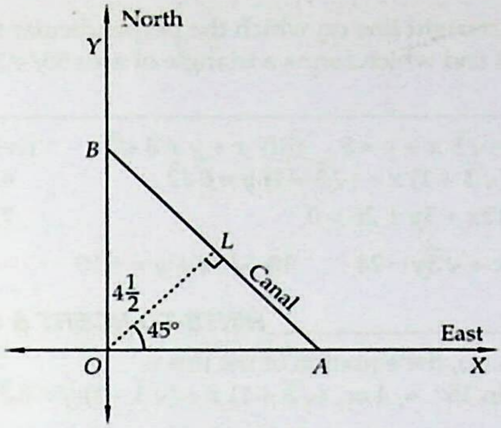

Fig. 23.46

$$x \cos 45° + y \sin 45° = 4\frac{1}{2} \quad \text{or,} \quad \sqrt{2}\,(x + y) = 9 \qquad \text{...(i)}$$

The position of the village is (4, 3). The village will lie on the edge of the canal, if (4, 3) satisfies the equation (i). Clearly, (4, 3) does not satisfy (i). Hence, the village does not lie by the nearer edge of the canal.

EXERCISE 23.7

LEVEL-1

1. Find the equation of a line for which
   (i) $p = 5, \alpha = 60°$ (ii) $p = 4, \alpha = 150°$ (iii) $p = 8, \alpha = 225°$ (iv) $p = 8, \alpha = 300°$
2. Find the equation of the line on which the length of the perpendicular segment from the origin to the line is 4 and the inclination of the perpendicular segment with the positive direction of x-axis is 30°. [NCERT EXEMPLAR]
3. Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with the positive direction of x-axis is 15°. [NCERT]
4. Find the equation of the straight line at a distance of 3 units from the origin such that the perpendicular from the origin to the line makes an angle α given by $\tan\alpha = \frac{5}{12}$ with the positive direction of x-axis.
5. Find the equation of the straight line on which the length of the perpendicular from the origin is 2 and the perpendicular makes an angle α with x-axis such that $\sin\alpha = \frac{1}{3}$.
6. Find the equation of the straight line upon which the length of the perpendicular from the origin is 2 and the slope of this perpendicular is $\frac{5}{12}$.
7. The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of y-axis. Find the equation of the line.
8. Find the value of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$. [NCERT]

LEVEL-2

9. Find the equation of the straight line which makes a triangle of area $96\sqrt{3}$ with the axes and perpendicular from the origin to it makes an angle of 30° with y-axis.

10. Find the equation of a straight line on which the perpendicular from the origin makes an angle of 30° with x-axis and which forms a triangle of area $50/\sqrt{3}$ with the axes.

ANSWERS

1. (i) $x+\sqrt{3}\,y=10$ (ii) $-\sqrt{3}\,x+y=8$ (iii) $x+y+8\sqrt{2}$ (iv) $x-\sqrt{3}\,y=16$
2. $\sqrt{3}x+y=8$
3. $(\sqrt{3}+1)\,x+(\sqrt{3}-1)\,y=8\sqrt{2}$
4. $12\,x+5\,y=39$
5. $2\sqrt{2}\,x+y=6$
6. $12x+5y\pm 26=0$
7. $\sqrt{3}x+y=14$
8. $\theta=\dfrac{7\pi}{6}, p=1$
9. $x+\sqrt{3}y=24$
10. $\sqrt{3}x+y=\pm 10$

HINTS TO NCERT & SELECTED PROBLEMS

3. Here, $\alpha=15°$ and $p=4$. So, the equation of the line is
$$x\cos 15°+y\sin 15° = 4 \text{ or, } (\sqrt{3}+1)\,x+(\sqrt{3}-1)\,y=8\sqrt{2}.$$
8. We have,
$$\sqrt{3}x+y+2 = 0 \Rightarrow -\sqrt{3}x-y = 2 \Rightarrow \left(\frac{-\sqrt{3}}{2}\right)x+\left(-\frac{1}{2}\right)y = 1$$
This is same as $x\cos\theta+y\sin\theta = p$.
$$\therefore \quad \cos\theta = -\frac{\sqrt{3}}{2}, \sin\theta = -\frac{1}{2} \text{ and } p = 1$$
$$\Rightarrow \quad \theta = \frac{7\pi}{6} \text{ and } p = 1$$

22.6.6 DISTANCE FORM OF A LINE

THEOREM *The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of x-axis is* $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r$, *where r is the distance of the point (x, y) on the line from the point (x_1, y_1).*

PROOF Let the given line meets x-axis at A, y-axis at B and passes through the point $Q\,(x_1, y_1)$. Let $P\,(x, y)$ be any point on the line at a distance r from $Q\,(x_1, y_1)$ i.e. $PQ=r$. Draw $PL \perp OX$, $QM \perp OX$ and $QN \perp PL$. Then,

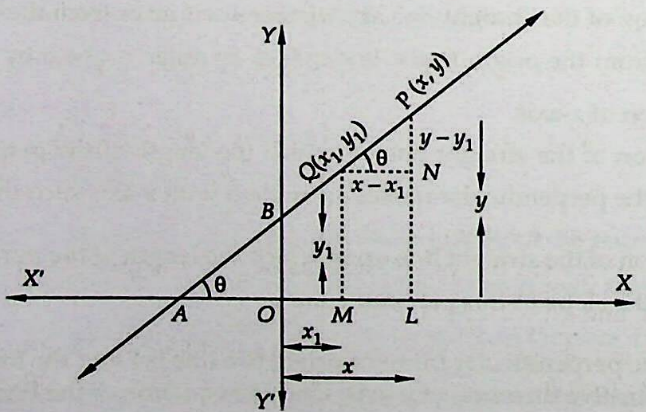

Fig. 23.47

$$QN=ML=OL-OM=x-x_1 \text{ and, } PN=PL-NL=PL-QM=y-y_1.$$
In $\Delta\,PQN$, we have
$$\cos\theta = \frac{QN}{PQ}$$

$$\Rightarrow \quad \cos\theta = \frac{x - x_1}{r} \qquad \text{...(i)}$$

and, $\sin\theta = \frac{PN}{PQ}$

$$\Rightarrow \quad \sin\theta = \frac{y - y_1}{r} \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

This is the required equation of the line in the distance form. **Q.E.D.**

NOTE 1 *The equation of the line is*

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

$$\Rightarrow \quad x - x_1 = r\cos\theta \text{ and } y - y_1 = r\sin\theta \Rightarrow x = x_1 + r\cos\theta \text{ and } y = y_1 + r\sin\theta.$$

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$. If P is on the right side of (x_1, y_1), then r is positive and if P is on the left side of (x_1, y_1), then r is negative. Since different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

NOTE 2 *In the above form one can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point (x_1, y_1) on the line $\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta}$ there are two points viz. $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$ and $(x_1 - r\cos\theta,\ y_1 - r\sin\theta)$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *A straight line is drawn through the point P (2, 3) and is inclined at an angle of 30° with the x-axis. Find the coordinates of two points on it at a distance 4 from P on either side of P.*

SOLUTION Here, $(x_1, y_1) = (2, 3)$, $\theta = 30°$. So, the equation of the line is

$$\frac{x - 2}{\cos 30°} = \frac{y - 3}{\sin 30°}$$

or, $\frac{x - 2}{\frac{\sqrt{3}}{2}} = \frac{y - 3}{\frac{1}{2}}$ or, $x - 2 = \sqrt{3}\,(y - 3)$ or, $x - \sqrt{3}\,y = 2 - 3\sqrt{3}$.

Points on the line at a distance 4 from P (2, 3) are

$$(x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta) \text{ or, } (2 \pm 4\cos 30°,\ 3 \pm 4\sin 30°) \text{ or, } (2 \pm 2\sqrt{3},\ 3 \pm 2)$$

or, $(2 + 2\sqrt{3}, 5)$ and $(2 - 2\sqrt{3}, 1)$.

EXAMPLE 2 *The slope of a straight line through A (3, 2) is 3/4 Find the coordinates of the points on the line that are 5 units away from A.* **[NCERT EXEMPLAR]**

SOLUTION Suppose the given line makes an angle θ with x-axis. It is given that its slope is 3/4.

$$\therefore \quad \tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5} \text{ and } \cos\theta = \frac{4}{5}$$

The equation of the line in distance from is $\frac{x-3}{\cos\theta} = \frac{y-2}{\sin\theta}$ or, $\frac{x-3}{4/5} = \frac{y-2}{3/5}$ and the coordinates of two points P and Q at a distance of 5 units from A are given by

$$\frac{x-3}{4/5}=\frac{y-2}{3/5}=\pm 5$$

Now,

$$\frac{x-3}{4/5}=\frac{y-2}{3/5}=5$$

$\Rightarrow \quad x-3=\frac{4}{5}\times 5$ and, $y-2=\frac{3}{5}\times 5$

$\Rightarrow \quad x=7, y=5$

and, $\quad \frac{x-3}{4/5}=\frac{y-2}{3/5}=-5$

$\Rightarrow \quad x-3=\frac{4}{5}\times -5$ and, $y-2=\frac{3}{5}\times -5$

$\Rightarrow \quad x=-1, y=-1$

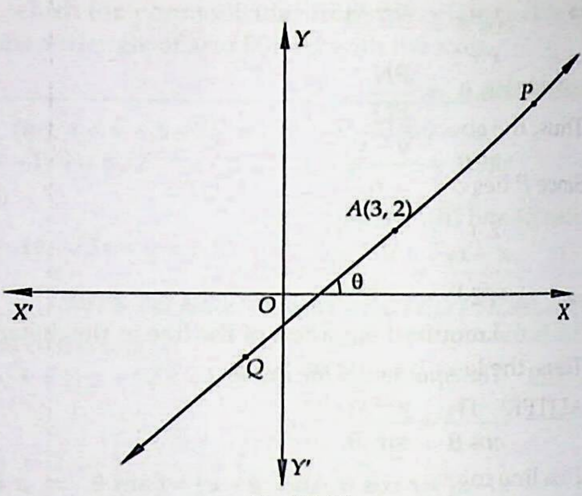

Fig. 23.48

Hence, the coordinates of P and Q are $(7, 5)$ and $(-1, -1)$ respectively.

REMARK *The coordinates of P and Q are* $(3\pm 5\cos\theta, 2\pm 5\sin\theta)$, *where* $\cos\theta=\frac{4}{5}$ *and* $\sin\theta=\frac{3}{5}$.

EXAMPLE 3 *Find the equation of the line through the point A (2, 3) and making an angle of 45° with the x-axis. Also, determine the length of intercept on it between A and the line* $x+y+1=0$.

SOLUTION The equation of a line through A and making an angle of 45° with the x-axis is

$$\frac{x-2}{\cos 45^\circ}=\frac{y-3}{\sin 45^\circ} \Rightarrow \frac{x-2}{\frac{1}{\sqrt{2}}}=\frac{y-3}{\frac{1}{\sqrt{2}}} \Rightarrow x-y+1=0$$

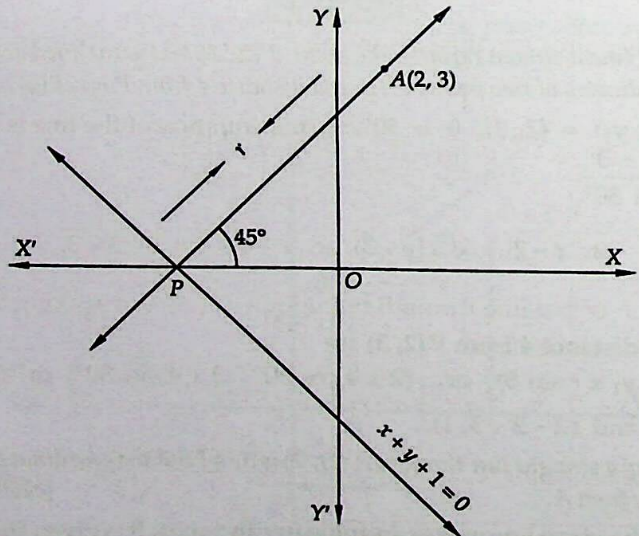

Fig. 23.49

Suppose this line meets the line $x+y+1=0$ at P such that $AP=r$. Then, the coordinates of P are given by

$$\frac{x-2}{\cos 45^\circ}=\frac{y-3}{\sin 45^\circ}=r$$

$\Rightarrow \qquad x = 2 + r\cos 45°,\ y = 3 + r\sin 45°$

$\Rightarrow \qquad x = 2 + \frac{r}{\sqrt{2}},\ y = 3 + \frac{r}{\sqrt{2}}$

Thus, the coordinates of P are $\left(2 + \frac{r}{\sqrt{2}}, 3 + \frac{r}{\sqrt{2}}\right)$.

Since P lies on $x + y + 1 = 0$. Therefore,

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0$$

$\Rightarrow \qquad \sqrt{2}\,r = -6 \Rightarrow r = -3\sqrt{2}$

$\therefore \qquad$ Length $AP = |r| = 3\sqrt{2}$.

Thus, the length of the intercept is $3\sqrt{2}$.

ALITER The equation of the line through A (2, 3) and making an angle of 45° with x-axis is

$$y - 3 = \tan 45°(x - 2) \text{ or, } x - y + 1 = 0.$$

This line intersects $x + y + 1 = 0$ at $P(-1, 0)$.

$\therefore \qquad AP = \sqrt{(2+1)^2 + (3-0)^2} = \sqrt{18} = 3\sqrt{2}$

EXAMPLE 4 *If the straight line through the point P (3, 4) makes an angle $\frac{\pi}{6}$ with x-axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length of PQ.*

SOLUTION The equation of a line passing through P (3, 4) and making an angle $\frac{\pi}{6}$ with x-axis is

$$\frac{x-3}{\cos\frac{\pi}{6}} = \frac{y-4}{\sin\frac{\pi}{6}} = r \text{ or, } \frac{x-3}{\frac{\sqrt{3}}{2}} = \frac{y-4}{\frac{1}{2}} = r$$

where r represents the distance of any point on this line from the given point P (3, 4).

The coordinates of any point Q on this line are $\left(3 + \frac{\sqrt{3}}{2}r,\ 4 + \frac{r}{2}\right)$. If Q lies on $12x + 5y + 10 = 0$, then

$$12\left(3 + \frac{\sqrt{3}}{2}r\right) + 5\left(4 + \frac{r}{2}\right) + 10 = 0 \Rightarrow r = \frac{-132}{12\sqrt{3} + 5}$$

Hence, length $PQ = \frac{132}{12\sqrt{3} + 5}$.

ALITER The equation of the line through the point $P(3, 4)$ and making an angle of $\frac{\pi}{6}$ with x-axis is

$$y - 4 = \tan\frac{\pi}{6}(x - 3) \text{ or, } x - \sqrt{3}y + 4\sqrt{3} - 3 = 0$$

This intersects the line $12x + 5y + 10 = 0$ at $Q\left(\frac{15 - 30\sqrt{3}}{5 + 12\sqrt{3}}, \frac{48\sqrt{3} - 46}{5 + 12\sqrt{3}}\right)$.

$$\therefore \qquad PQ = \sqrt{\left(\frac{15 - 30\sqrt{3}}{5 + 12\sqrt{3}} - 3\right)^2 + \left(\frac{48\sqrt{3} - 46}{5 + 12\sqrt{3}} - 4\right)^2} = \frac{132}{5 + 12\sqrt{3}}$$

EXAMPLE 5 *The line joining two points A (2, 0), B (3, 1) is rotated about A in anti-clockwise direction through an angle of 15°. Find the equation of the line in the new position. If point B goes to point C in the new position, what will be the coordinates of C?*

SOLUTION The slope m of the line AB is given by $m = \frac{1-0}{3-2} = 1$. So, AB makes an angle of 45° with x-axis. Now, AB is rotated through 15° in anticlockwise direction and so it makes an angle of 60° with x-axis in its new position AC. Clearly, AC passes through A (2, 0) and makes an angle of 60° with x-axis. Therefore, the equation of AC in distance form is

$$\frac{x-2}{\cos 60°} = \frac{y-0}{\sin 60°} \quad \text{or,} \quad \frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}}$$

Clearly, $AB = \sqrt{(3-2)^2 + (1-0)^2} = \sqrt{2}$.

The point C is at a distance $\sqrt{2}$ from A. So, the coordinates of C are given by

$$\frac{x-2}{\frac{1}{2}} = \frac{y-0}{\frac{\sqrt{3}}{2}} = \sqrt{2}$$

$$\Rightarrow \quad x = 2 + \frac{1}{2}\sqrt{2} = 2 + \frac{1}{\sqrt{2}} \text{ and } y = \frac{\sqrt{3}}{2} \times \sqrt{2} = \frac{\sqrt{6}}{2}$$

Hence, the coordinates of C are $\left(2 + \frac{1}{\sqrt{2}}, \frac{\sqrt{6}}{2}\right)$.

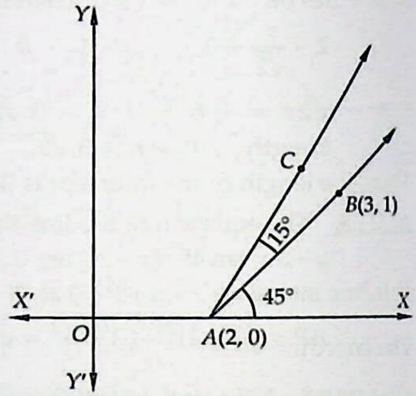

Fig. 23.50

EXAMPLE 6 *Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.* **[NCERT]**

SOLUTION The equation in distance form of the line passing through P (4, 1) and making an angle of 135° with the positive x-axis is

$$\frac{x-4}{\cos 135°} = \frac{y-1}{\sin 135°}$$

Suppose it cuts $4x - y = 0$ at Q such that $PQ = r$. Then, the coordinates of Q are given by

$$\frac{x-4}{\cos 135°} = \frac{y-1}{\sin 135°} = r$$

$$\Rightarrow \quad \frac{x-4}{-1/\sqrt{2}} = \frac{y-1}{1/\sqrt{2}} = r$$

$$\Rightarrow \quad x = 4 - \frac{r}{\sqrt{2}}, y = 1 + \frac{r}{\sqrt{2}}$$

So, the coordinates of Q are $\left(4 - \frac{r}{\sqrt{2}}, 1 + \frac{r}{\sqrt{2}}\right)$.

Clearly, Q lies on $4x - y = 0$.

$$\therefore \quad 16 - \frac{4r}{\sqrt{2}} - 1 - \frac{r}{\sqrt{2}} = 0 \Rightarrow \frac{5r}{\sqrt{2}} = 15 \Rightarrow r = 3\sqrt{2}$$

Hence, required distance is $3\sqrt{2}$ units.

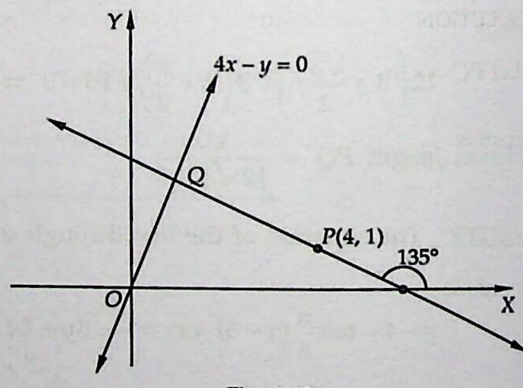

Fig. 23.51

EXAMPLE 7 *Find the distance of the point* (2, 3) *from the line* $2x - 3y + 9 = 0$ *measured along a line* $x - y + 1 = 0$.

SOLUTION The slope of the line $x - y + 1 = 0$ is 1. So it makes an angle of 45° with x-axis.

The equation of a line, in distance form, passing through P (2, 3) and making an angle of 45° with x-axis is

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°}$$

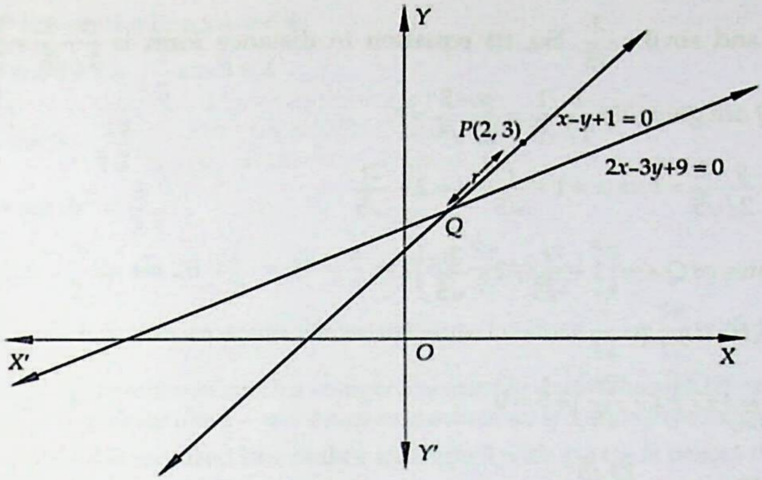

Fig. 23.52

The coordinates of any point Q on this line are given by

$$\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r \qquad \left[\text{Using}: \frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r\right]$$

So, the coordinates of Q are $(2 + r\cos 45°,\ 3 + r\sin 45°)$ or, $\left(2+\frac{r}{\sqrt{2}},\ 3+\frac{r}{\sqrt{2}}\right)$

If this point lies on the line $2x - 3y + 9 = 0$, then

$$4 + r\sqrt{2} - 9 - \frac{3r}{\sqrt{2}} + 9 = 0 \Rightarrow r = 4\sqrt{2} \Rightarrow PQ = 4\sqrt{2}$$

Hence, the required distance is $4\sqrt{2}$ units.

EXAMPLE 8 *Find the distance of the line* $4x + 7y + 5 = 0$ *from the point* $(1, 2)$ *along the line* $2x - y = 0$. **[NCERT]**

SOLUTION Clearly, line $2x - y = 0$ passes through $P(1, 2)$ and intersects $4x + 7y + 5 = 0$ at Q. Let $PQ = r$. If $2x - y = 0$ makes an angle θ with x-axis. Then, $\tan\theta = 2$ and hence $\sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$. Thus, the line $2x - y = 0$ passes through $P(1, 2)$ and makes an angle θ with x-axis such

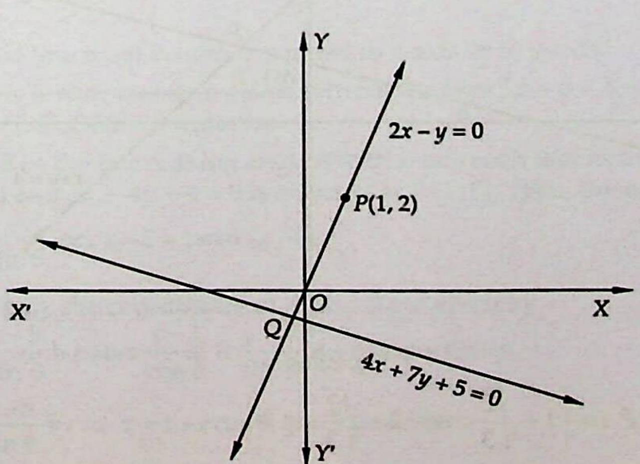

Fig. 23.53

that $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$. So, its equation in distance form is $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}}$ and the coordinates of Q are given by $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r$.

Now, $\frac{x-1}{1/\sqrt{5}} = \frac{y-2}{2/\sqrt{5}} = r \Rightarrow x = 1 + \frac{r}{\sqrt{5}}, y = 2 + \frac{2r}{\sqrt{5}}$

So, the coordinates of Q are $\left(1+\frac{r}{\sqrt{5}}, 2+\frac{2r}{\sqrt{5}}\right)$.

As Q lies on $4x+7y+5=0$.

$\therefore \quad 4\left(1+\frac{r}{\sqrt{5}}\right)+7\left(2+\frac{2r}{\sqrt{5}}\right)+5=0$

$\Rightarrow \quad \frac{18r}{\sqrt{5}} = -23 \Rightarrow r = -\frac{23\sqrt{5}}{18}$

Hence, $PQ = |r| = \frac{23\sqrt{5}}{18}$.

LEVEL-2

EXAMPLE 9 *In what direction a line be drawn through the point* (1, 2) *that its point of intersection with the line* $x+y=4$ *is at a distance* $\frac{\sqrt{6}}{3}$ *from the given point.* **[NCERT EXEMPLAR]**

SOLUTION Let the line drawn through A (1, 2) makes an angle θ with the positive direction of x-axis and intersects the line $x+y=4$ at P such that $AP = \frac{\sqrt{6}}{3}$. Then, the coordinates of P are given by

$$\frac{x-1}{\cos\theta} = \frac{y-2}{\sin\theta} = \frac{\sqrt{6}}{3} \Rightarrow x = 1+\sqrt{\frac{2}{3}}\cos\theta, y = 2+\sqrt{\frac{2}{3}}\sin\theta$$

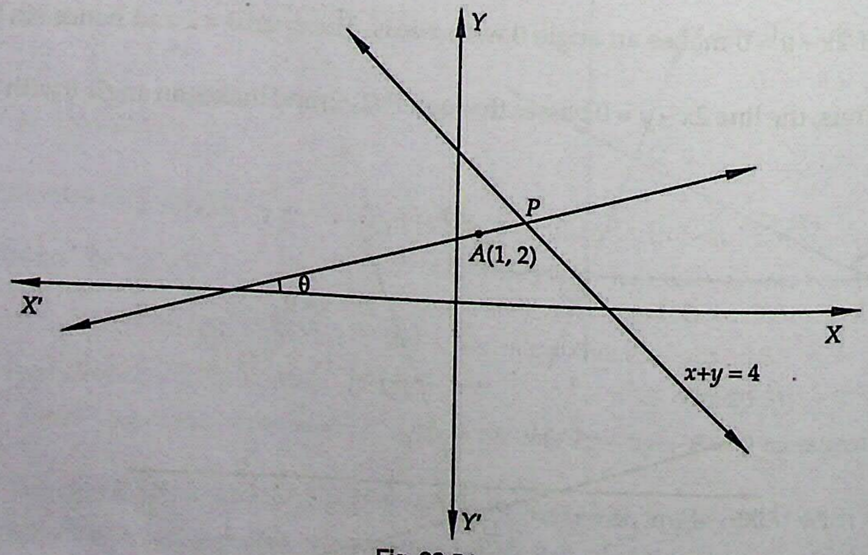

Fig. 23.54

So, the coordinates of P are $\left(1+\sqrt{\frac{2}{3}}\cos\theta, 2+\sqrt{\frac{2}{3}}\sin\theta\right)$.

Clearly, point P lies on the line $x + y = 4$.

$\therefore \quad 1+\sqrt{\frac{2}{3}}\cos\theta+2+\sqrt{\frac{2}{3}}\sin\theta=4$

$\Rightarrow \quad \cos\theta+\sin\theta=\sqrt{\frac{3}{2}}$

$\Rightarrow \quad (\cos\theta+\sin\theta)^2=\frac{3}{2}$

$\Rightarrow \quad 1+\sin 2\theta=\frac{3}{2} \Rightarrow \sin 2\theta=\frac{1}{2} \Rightarrow 2\theta=\frac{\pi}{6}$ or, $2\theta=\frac{5\pi}{6} \Rightarrow \frac{\pi}{12}, \frac{5\pi}{12}$

Hence, the line drawn makes an angle whose measure is either $\frac{\pi}{12}$ or $\frac{5\pi}{12}$ with the x-axis.

EXAMPLE 10 *Find the direction in which a straight line must be drawn through the point $(-1, 2)$ so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from the point.* **[NCERT]**

SOLUTION Suppose the required line makes an angle θ with x-axis. It passes through the point $P(-1, 2)$. So, its equation is

$$\frac{x-(-1)}{\cos\theta}=\frac{y-2}{\sin\theta} \text{ or, } \frac{x+1}{\cos\theta}=\frac{y-2}{\sin\theta}$$

The coordinates of a point Q on this line at a distance of 3 units from $P(-1, 2)$ are given by

$$\frac{x+1}{\cos\theta}=\frac{y-2}{\sin\theta}=3$$

$\Rightarrow \quad x=-1+3\cos\theta,\ y=2+3\sin\theta$

So, the coordinates of Q are

$$(-1+3\cos\theta, 2+3\sin\theta)$$

If point Q lies on $x + y = 4$, then

$$-1+3\cos\theta+2+3\sin\theta=4$$

$\Rightarrow \quad 3\cos\theta+3\sin\theta=3$

$\Rightarrow \quad \cos\theta+\sin\theta=1$

$\Rightarrow \quad (\cos\theta+\sin\theta)^2=1$

$\Rightarrow \quad 1+\sin 2\theta=1$

$\Rightarrow \quad \sin 2\theta=0$

$\Rightarrow \quad 2\theta=0$ or $2\theta=\pi$

$\Rightarrow \quad \theta=0$ or $\theta=\frac{\pi}{2}$

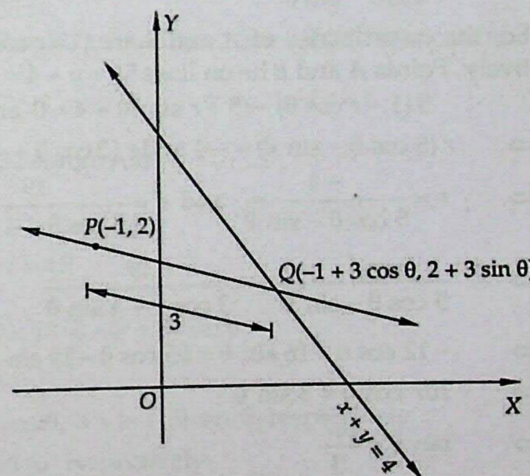

Fig. 23.55

Hence, the required line must be either parallel to x-axis or to y-axis.

EXAMPLE 11 *A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point $(1, 5)$. Obtain its equation.* **[NCERT]**

SOLUTION Let AB be the line making angle θ with x-axis such that its intercept AB between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at $P(1, 5)$. Then, the equation of the line is

$$\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta} \text{ or, } y-5=\tan\theta\,(x-1) \quad \ldots\text{(i)}$$

Let $AP = BP = r$. Then, the coordinates of A and B are given by

$$\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r \text{ and } \frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=-r \text{ respectively.}$$

Now, $\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r \Rightarrow x=1+r\cos\theta,\ y=5+r\sin\theta$

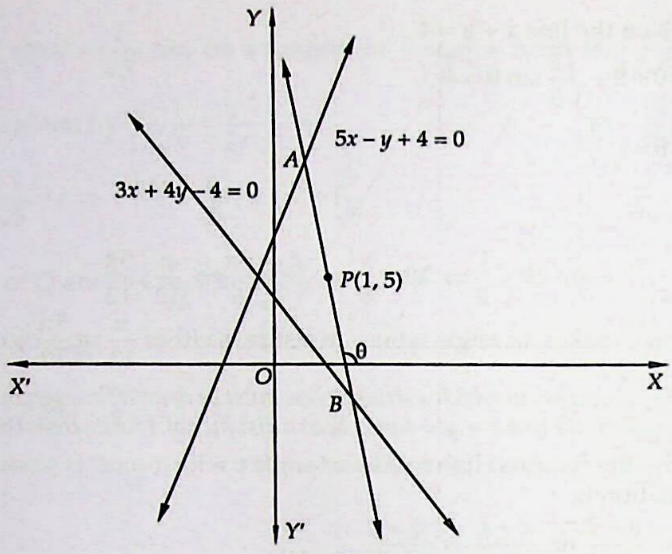

Fig. 23.56

l, $\dfrac{x-1}{\cos\theta}=\dfrac{y-5}{\sin\theta}=-r \Rightarrow x=1-r\cos\theta,\ y=5-r\sin\theta$

the co-ordinates of A and B are $(1+r\cos\theta, 5+r\sin\theta)$ and $(1-r\cos\theta, 5-r\sin\theta)$ respec-ely. Points A and B lie on lines $5x-y+4=0$ and $3x+4y-4=0$ respectively.

. $5(1+r\cos\theta)-(5+r\sin\theta)+4=0$ and, $3(1-r\cos\theta)+4(5-r\sin\theta)-4=0$

$\Rightarrow \quad r(5\cos\theta-\sin\theta)=-4$ and $r(3\cos\theta+4\sin\theta)=19$

$\Rightarrow \quad r=\dfrac{-4}{5\cos\theta-\sin\theta}$ and $r=\dfrac{19}{3\cos\theta+4\sin\theta}$

$\Rightarrow \quad \dfrac{-4}{5\cos\theta-\sin\theta}=\dfrac{19}{3\cos\theta+4\sin\theta}$

$\Rightarrow \quad -12\cos\theta-16\sin\theta=95\cos\theta-19\sin\theta$

$\Rightarrow \quad 107\cos\theta=3\sin\theta$

$\Rightarrow \quad \tan\theta=\dfrac{107}{3}$

Putting the value of $\tan\theta$ in (i), we obtain

$$y-5=\tan\theta(x-1) \text{ or, } y-5=\frac{107}{3}(x-1)$$

or, $107x-3y-92=0$ as the required equation of the line.

EXAMPLE 12 *Find the equation of the line passing through the point (2, 3) and making an intercept of length 3 units between the lines $y+2x=2$ and $y+2x=5$.*

SOLUTION The equations of the given lines are

$2x+y=2$ ...(i) and, $2x+y=5$ ...(ii)

We observe that the lines given by equations (i) and (ii) are parallel. Suppose a line passing through $A(2, 3)$ and intercepting length $BC=3$ between lines (i) and (ii) makes an angle θ with x-axis. The equation of this line in distance form is

$$\frac{x-2}{\cos\theta}=\frac{y-3}{\sin\theta} \quad \text{...(iii)}$$

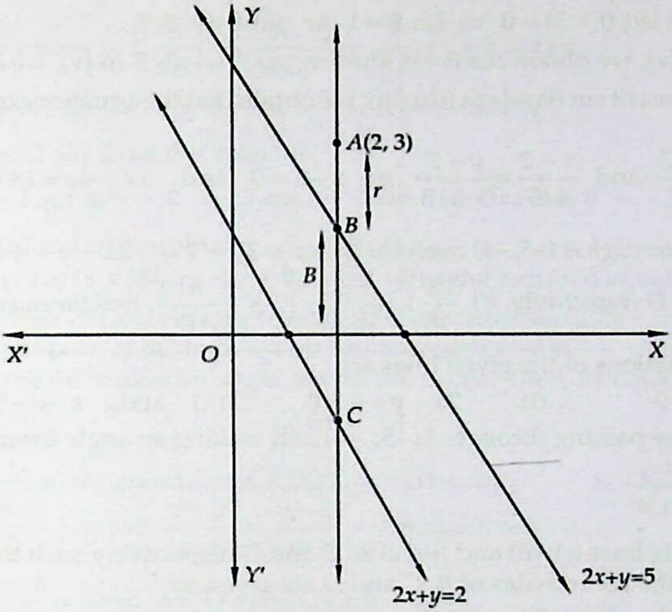

Fig. 23.57

Let $AB = r$. Then, $AC = AB + BC = (r+3)$. Clearly, B and C are points on line (iii) at distances r and $r+3$ respectively from A.

So, the coordinates of B and C are given by

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \text{ and } \frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r+3 \text{ respectively.}$$

Now,

$$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r \Rightarrow x = 2 + r\cos\theta,\ y = 3 + r\sin\theta$$

and, $$\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r+3 \Rightarrow x = 2 + (r+3)\cos\theta,\ y = 3 + (r+3)\sin\theta$$

So, the coordinates of B and C are

$$(2 + r\cos\theta, 3 + r\sin\theta) \text{ and } (2 + (r+3)\cos\theta, 3 + (r+3)\sin\theta) \text{ respectively.}$$

We observe that points B and C lie on lines (i) and (ii) respectively.

$\therefore$ $2(2 + r\cos\theta) + (3 + r\sin\theta) = 2$ and, $2\{2 + (r+3)\cos\theta\} + \{3 + (r+3)\sin\theta\} = 5$

$\Rightarrow$ $r(2\cos\theta + \sin\theta) = -5$ and, $(r+3)(2\cos\theta + \sin\theta) = -2$

$\Rightarrow$ $r = \dfrac{-5}{2\cos\theta + \sin\theta}$ and $r + 3 = \dfrac{-2}{2\cos\theta + \sin\theta}$

$\Rightarrow$ $\dfrac{-5}{2\cos\theta + \sin\theta} + 3 = \dfrac{-2}{2\cos\theta + \sin\theta}$ [On eliminating r]

$\Rightarrow$ $3 = \dfrac{3}{2\cos\theta + \sin\theta}$ ...(iv)

$\Rightarrow$ $2\cos\theta + \sin\theta = 1$

$\Rightarrow$ $2\cos\theta = 1 - \sin\theta$ ...(v)

$\Rightarrow$ $4\cos^2\theta = 1 + \sin^2\theta - 2\sin\theta$

$\Rightarrow$ $4(1 - \sin^2\theta) = 1 + \sin^2\theta - 2\sin\theta$

$\Rightarrow$ $5\sin^2\theta - 2\sin\theta - 3 = 0$

$\Rightarrow \quad (\sin\theta - 1)(5\sin\theta + 3) = 0 \Rightarrow \sin\theta = 1$ or $\sin\theta = -3/5$

Putting $\sin\theta = 1$ in (v), we obtain $\cos\theta = 0$. Putting $\sin\theta = -3/5$ in (v), we obtain $\cos\theta = 4/5$.
Substituting the values of $\sin\theta$ and $\cos\theta$ in (iii), we obtain that the equations of the required lines are

$$\frac{x-2}{0} = \frac{y-3}{1} \text{ and } \frac{x-2}{4/5} = \frac{y-3}{-3/5} \text{ or, } x-2=0 \text{ and } 3x+4y=18.$$

EXAMPLE 13 *A line through $A(-5,-4)$ meets the lines $x+3y+2=0$, $2x+y+4=0$ and $x-y-5=0$ at the points B, C and D respectively, if* $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$ *find the equation of the line.*

SOLUTION The equations of the given lines are

$$x+3y+2=0 \quad \ldots\text{(i)}, \quad 2x+y+4=0 \quad \ldots\text{(ii)} \quad \text{and,} \quad x-y-5=0 \quad \ldots\text{(iii)}$$

The equation of a line passing through $A(-5,-4)$ and making an angle θ with x-axis is

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} \quad \ldots\text{(iv)}$$

Suppose this line cuts lines (i), (ii) and (iii) at B, C and D respectively such that $AB = r_1$, $AC = r_2$ and $AD = r_3$. Then, the coordinates of B, C and D are given by

$$\frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_1, \ \frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_2 \text{ and } \frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = r_3 \text{ respectively.}$$

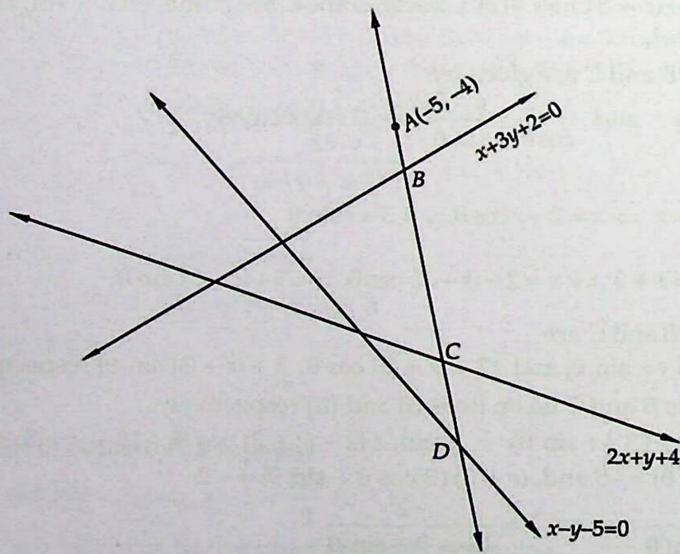

Fig. 23.58

The coordinates of B, C and D are $(-5+r_1\cos\theta, -4+r_1\sin\theta)$, $(-5+r_2\cos\theta, -4+r_2\sin\theta)$, and $(-5+r_3\cos\theta, -4+r_3\sin\theta)$ respectively.

Points B, C and D lie on lines (i), (ii) and (iii) respectively.

$\therefore \quad (-5+r_1\cos\theta) + 3(-4+r_1\sin\theta) + 2 = 0$, $2(-5+r_2\cos\theta) + (-4+r_2\sin\theta) + 4 = 0$

and $\quad (-5+r_3\cos\theta) - (-4+r_3\sin\theta) - 5 = 0$

$$\Rightarrow \quad r_1 = \frac{15}{\cos\theta + 3\sin\theta}, \ r_2 = \frac{10}{2\cos\theta + \sin\theta} \text{ and } r_3 = \frac{6}{\cos\theta - \sin\theta}$$

$$\Rightarrow \quad \frac{15}{AB} = \cos\theta + 3\sin\theta, \ \frac{10}{AC} = 2\cos\theta + \sin\theta \text{ and } \frac{6}{AD} = \cos\theta - \sin\theta$$

Substituting these values in $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, we obtain

$$(\cos\theta + 3\sin\theta)^2 + (2\cos\theta + \sin\theta)^2 = (\cos\theta - \sin\theta)^2$$

$$\Rightarrow \quad 4\cos^2\theta + 12\sin\theta\cos\theta + 9\sin^2\theta = 0$$

$$\Rightarrow \quad (2\cos\theta + 3\sin\theta)^2 = 0 \Rightarrow 2\cos\theta + 3\sin\theta = 0 \Rightarrow \tan\theta = -2/3$$

Putting $\tan\theta = -2/3$ in (iv), we obtain

$y + 4 = -(2/3)(x+5)$ or, $2x + 3y + 22 = 0$ as the required equation of the line.

EXAMPLE 14 *The sides AB and AC of a triangle ABC are respectively* $2x + 3y = 29$ *and* $x + 2y = 16$ *respectively. If the mid-point of BC is (5, 6) then find the equation of BC.*

SOLUTION Suppose BC makes an angle θ with the x-axis. Then, its equation is

$$\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} \quad \text{...(i)}$$

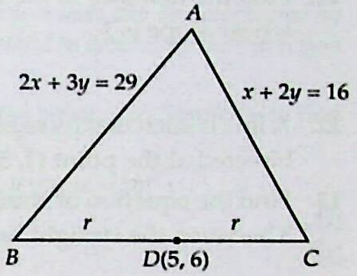

Fig. 23.59

Let $BD = CD = r$. Then, the coordinates B and C are given by

$$\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = -r \text{ and, } \frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = r \text{ respectively.}$$

Now, $\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = -r \Rightarrow x = 5 - r\cos\theta,\ y = 6 - r\sin\theta$

and, $\frac{x-5}{\cos\theta} = \frac{y-6}{\sin\theta} = r \Rightarrow x = 5 + r\cos\theta,\ y = 6 + r\sin\theta$

Thus, the coordinates of B and C are $B(5 - r\cos\theta, 6 - r\sin\theta)$ and $C(5 + r\cos\theta, 6 + r\sin\theta)$ respectively which lie on lines $2x + 3y = 29$ and $x + 2y = 16$ respectively.

$$\therefore \quad 2(5 - r\cos\theta) + 3(6 - r\sin\theta) = 29 \text{ and, } (5 + r\cos\theta) + 2(6 + r\sin\theta) = 16$$

$$\Rightarrow \quad r = \frac{-1}{2\cos\theta + 3\sin\theta} \text{ and } r = \frac{-1}{\cos\theta + 2\sin\theta}$$

$$\Rightarrow \quad \frac{-1}{2\cos\theta + 3\sin\theta} = \frac{-1}{\cos\theta + 2\sin\theta} \quad \text{[On eliminating } r\text{]}$$

$$\Rightarrow \quad 2\cos\theta + 3\sin\theta = \cos\theta + 2\sin\theta \Rightarrow \sin\theta = -\cos\theta \Rightarrow \tan\theta = -1 \Rightarrow \theta = 3\pi/4$$

Putting $\theta = \frac{3\pi}{4}$ in (i), we obtain

$$\frac{x-5}{\cos 3\pi/4} = \frac{y-6}{\sin 3\pi/4} \text{ or, } x + y - 11 = 0 \text{ as the required equation of the line.}$$

EXERCISE 23.8

LEVEL-1

1. A line passes through a point $A(1, 2)$ and makes an angle of 60° with the x-axis and intersects the line $x + y = 6$ at the point P. Find AP.
2. If the straight line through the point $P(3, 4)$ makes an angle $\pi/6$ with the x-axis and meets the line $12x + 5y + 10 = 0$ at Q, find the length PQ.
3. A straight line drawn through the point $A(2, 1)$ making an angle $\pi/4$ with positive x-axis intersects another line $x + 2y + 1 = 0$ in the point B. Find length AB.
4. A line a drawn through $A(4, -1)$ parallel to the line $3x - 4y + 1 = 0$. Find the coordinates of the two points on this line which are at a distance of 5 units from A.
5. The straight line through $P(x_1, y_1)$ inclined at an angle θ with the x-axis meets the line $ax + by + c = 0$ in Q. Find the length of PQ.

6. Find the distance of the point (2, 3) from the line $2x - 3y + 9 = 0$ measured along a line making an angle of 45° with the x-axis.
7. Find the distance of the point (3, 5) from the line $2x + 3y = 14$ measured parallel to a line having slope 1/2.
8. Find the distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to a line having slope 3/4.
9. Find the distance of the point (3, 5) from the line $2x + 3y = 14$ measured parallel to the line $x - 2y = 1$.
10. Find the distance of the point (2, 5) from the line $3x + y + 4 = 0$ measured parallel to the line $3x - 4y + 8 = 0$.
11. Find the distance of the line $2x + y = 3$ from the point $(-1, -3)$ in the direction of the line whose slope is 1.

LEVEL-2

12. A line is such that its segment between the straight lines $5x - y - 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.
13. Find the equation of straight line passing through $(-2, -7)$ and having an intercept of length 3 between the straight lines $4x + 3y = 12$ and $4x + 3y = 3$.

ANSWERS

1. $3(\sqrt{3} - 1)$
2. $\dfrac{132}{5 + 12\sqrt{3}}$
3. $\dfrac{5\sqrt{2}}{3}$
4. $(8, 2), (0, -4)$
5. $\left|\dfrac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}\right|$
6. $4\sqrt{2}$
7. $\sqrt{5}$
8. 5 units
9. $\sqrt{5}$
10. 5
11. $\dfrac{8\sqrt{2}}{3}$
12. $83x - 35y + 92 = 0$
13. $x + 2 = 0, 7x + 24y + 182 = 0$

23.7 TRANSFORMATION OF GENERAL EQUATION IN DIFFERENT STANDARD FORMS

The general equation of a straight line is $Ax + By + C = 0$ which can be transformed to various standard forms as discussed below.

(i) *Transformation of $Ax + By + C = 0$ in the slope intercept form ($y = mx + c$):*
The equation of the line is

$$Ax + By + C = 0 \Rightarrow By = -Ax - C \Rightarrow y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

This is of the form $y = mx + c$, where $m = -\dfrac{A}{B}$ and, $c = -\dfrac{C}{B}$.

Thus, for the straight line $Ax + By + C = 0$, we have

$$m = \text{Slope} = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} \quad \text{and, Intercept on } y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

NOTE *To determine the slope of a line by the formula $m = -\dfrac{\text{Coefficient of } x}{\text{Coefficient of } y}$, we must first transfer all terms in the equation on one side.*

(ii) *Transformation of $Ax + By + C = 0$ in intercept form* $\left(\frac{x}{a} + \frac{y}{b} = 1\right)$:

The equation of the line is

$$Ax + By + C = 0 \Rightarrow Ax + By = -C \Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1 \Rightarrow \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1$$

This is of the form $\frac{x}{a} + \frac{y}{b} = 1$.

Thus, for the straight line $Ax + By + C = 0$, we have

Intercept on x-axis $= -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}$, Intercept on y-axis $= -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$

<u>NOTE</u> *As discussed above the intercepts made by a line with the coordinate axes can be determined by reducing its equation to intercept form. We may also use the following method to determine the intercepts on the coordinate axes:*

For intercept on x-axis : Put $y = 0$ in the equation of the line and find the value of x. Similarly to find y-intercept, put $x = 0$ in the equation of the line and find the value of y.

(iii) *Transformation of $Ax + By + C = 0$ in the normal form* ($x \cos \alpha + y \sin \alpha = p$):

We have, $$Ax + By + C = 0 \quad \text{...(i)}$$

Let $$x \cos \alpha + y \sin \alpha - p = 0 \quad \text{...(ii)}$$

be the normal form of $Ax + By + C = 0$.

Then, (i) and (ii) represent the same straight line.

$$\therefore \quad \frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p}$$

$$\Rightarrow \quad \cos \alpha = -\frac{Ap}{C} \text{ and, } \sin \alpha = -\frac{Bp}{C} \quad \text{...(iii)}$$

$$\Rightarrow \quad \cos^2 \alpha + \sin^2 \alpha = \frac{A^2p^2}{C^2} + \frac{B^2p^2}{C^2}$$

$$\Rightarrow \quad 1 = \frac{p^2}{C^2}(A^2 + B^2) \Rightarrow p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

But, p denotes the length of the perpendicular from the origin to the line and is always positive.

$$\therefore \quad p = \frac{|C|}{\sqrt{A^2 + B^2}}$$

Putting the value of p in (iii), we get

$$\cos \alpha = -\frac{A}{\sqrt{A^2 + B^2}} \text{ and, } \sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}$$

So, the equation (ii) takes the form

$$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y - \frac{C}{\sqrt{A^2 + B^2}} = 0$$

or, $$-\frac{A}{\sqrt{A^2 + B^2}}x - \frac{B}{\sqrt{A^2 + B^2}}y = \frac{C}{\sqrt{A^2 + B^2}}$$

This is the required normal form of the line $Ax + By + C = 0$.

In order to transform the general equation of a line to the normal form, we use the following steps :

STEP I *Shift the constant term on the RHS and make it positive*

STEP II *Divide both sides by* $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$

The equation so obtained is in the normal form.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Transform the equation of the line* $\sqrt{3}\,x + y - 8 = 0$ *to (i) slope intercept form and find its slope and y-intercept (ii) intercept form and find intercepts on the coordinate axes (iii) normal form and find the inclination of the perpendicular segment from the origin on the line with the axis and its length.*

SOLUTION (i) We have,

$\sqrt{3}x + y - 8 = 0 \Rightarrow y = -\sqrt{3}x + 8$, which is the slope intercept form of the given line.

$\therefore$ Slope $= -\sqrt{3}$, and y-intercept $= 8$

(ii) We have,

$\sqrt{3}\,x + y - 8 = 0 \Rightarrow \dfrac{x}{8/\sqrt{3}} + \dfrac{y}{8} = 1$, which is the intercept form of the given line.

So, x-intercept $= \dfrac{8}{\sqrt{3}}$ and, y-intercept $= 8$

(iii) We have, $\sqrt{3}\,x + y - 8 = 0$ or, $\sqrt{3}\,x + y = 8$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$\Rightarrow \quad \frac{\sqrt{3}}{\sqrt{(\sqrt{3})^2 + 1^2}}\,x + \frac{1}{\sqrt{(\sqrt{3})^2 + 1^2}}\,y = \frac{8}{\sqrt{(\sqrt{3})^2 + 1^2}}$$

$\Rightarrow \quad \dfrac{\sqrt{3}}{2}x + \dfrac{1}{2}y = 4$, which is the normal form of the given line.

Comparing this equation with $x\cos\alpha + y\sin\alpha = p$, we obtain

$$\cos\alpha = \frac{\sqrt{3}}{2},\ \sin\alpha = \frac{1}{2} \text{ and } p = 4.$$

As $\sin\alpha$ and $\cos\alpha$ both are positive, therefore α is in first quadrant and is equal to $\pi/6$. Hence, for the given line, we have $\alpha = \pi/6$ and $p = 4$.

EXAMPLE 2 *Reduce the lines* $3x - 4y + 4 = 0$ *and* $4x - 3y + 12 = 0$ *to the normal form and hence determine which line is nearer to the origin.*

SOLUTION The equation of the first line is

$$3x - 4y + 4 = 0$$

$$\Rightarrow \quad -3x + 4y = 4$$

Dividing throughout by $\sqrt{(-3)^2 + (4)^2}$, we obtain

$$-\frac{3x}{\sqrt{(-3)^2 + 4^2}} + \frac{4y}{\sqrt{(-3)^2 + 4^2}} = \frac{4}{\sqrt{(-3)^2 + 4^2}}$$

or, $$-\frac{3}{5}x + \frac{4}{5}y = \frac{4}{5}$$

This is the normal form of $3x - 4y + 4 = 0$ from which we find that the length of the perpendicular from the origin to it is given by $p_1 = 4/5$.

The equation of the second line is

$$4x - 3y + 12 = 0 \text{ or, } -4x + 3y = 12$$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$-\frac{4x}{\sqrt{(-4)^2+3^2}} + \frac{3y}{\sqrt{(-4)^2+3^2}} = \frac{12}{\sqrt{(-4)^2+3^2}}$$

or, $\quad -\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$

This is the normal form of $4x - 3y + 12 = 0$ from which we find that the length of the perpendicular from the origin is given by $p_2 = \frac{12}{5}$.

Clearly, $p_2 > p_1$. Therefore, the line $3x - 4y + 4 = 0$ is nearer to the origin.

EXAMPLE 3 *Find the values of k for which the line $(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0$ is*

(i) *parallel to the x-axis.* (ii) *parallel to the y-axis.* (iii) *passing through the origin.*

[NCERT]

SOLUTION Let m be the slope of the line

$$(k-3)x - (4-k^2)y + k^2 - 7k + 6 = 0 \qquad \text{...(i)}$$

Then,

$$m = -\frac{(k-3)}{-(4-k^2)} = \frac{k-3}{4-k^2}$$

(i) If the line is parallel to x-axis, then

$$\text{Slope} = 0 \Rightarrow \frac{k-3}{4-k^2} = 0 \Rightarrow k-3 = 0 \Rightarrow k = 3$$

(ii) If the line is parallel to y-axis, then

$$\frac{1}{m} = 0 \Rightarrow \frac{4-k^2}{k-3} = 0 \Rightarrow 4-k^2 = 0 \Rightarrow k = \pm 2$$

(iii) If the line passes through the origin, then (0, 0) must satisfy the equation (i).

$\therefore \quad (k-3)\times 0 - (4-k^2)\times 0 + k^2 - 7k + 6 = 0 \Rightarrow (k-1)(k-6) = 0 \Rightarrow k = 1, 6.$

LEVEL-2

EXAMPLE 4 *Find the equation of a line with slope 2 and the length of the perpendicular from the origin equal to $\sqrt{5}$.*

SOLUTION Let the y-intercept of the required line be c. Then, its equation is

$$y = 2x + c \qquad \text{...(i)}$$

$$\Rightarrow \quad -2x + y = c$$

Dividing throughout by $\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$, we obtain

$$-\frac{2}{\sqrt{(-2)^2+1^2}}x + \frac{y}{\sqrt{(-2)^2+1^2}} = \frac{c}{\sqrt{(-2)^2+1^2}} \text{ or, } -\frac{2}{\sqrt{5}}x + \frac{y}{\sqrt{5}} = \frac{c}{\sqrt{5}}$$

This is the normal form of line (i). Therefore, RHS represents the length of the perpendicular from the origin. But, the length of the perpendicular from the origin is given to be $\sqrt{5}$.

$$\therefore \quad \left|\frac{c}{\sqrt{5}}\right| = \sqrt{5} \Rightarrow |c| = 5 \Rightarrow c = \pm 5$$

Putting $c = \pm 5$ in (i), we obtain the equations of the required lines as $y = 2x \pm 5$.

EXAMPLE 5 *Prove that the slope of a line is invariant under the translation of the axes.*

SOLUTION Let the equation of a straight line referred to a system of coordinate axes be

$$ax + by + c = 0 \qquad \text{...(i)}$$

The slope of this line is $m = -\frac{a}{b}$.

Now, let the origin be shifted to the point (h, k) under some translation of the axes. Then, any point (X, Y) with respect to the new system of coordinate axes is given by the relation

$$x = X + h \text{ and } y = Y + k$$

where (x, y) are the coordinates of the point in the old system of coordinate axes.

The equation of line (i) in the new system of axes is given by

$$a(X + h) + b(Y + k) + c = 0 \text{ or, } aX + bY + (ah + bk + c) = 0 \quad \text{...(ii)}$$

Let m' be the slope of this line. Then, $m' = -\frac{a}{b}$. Clearly, $m = m'$.

Hence, the slope of a straight line is invariant under the translation of coordinate axes.

EXAMPLE 6 *The line $2x - y = 5$ turns about the point on it, whose ordinate and abscissae are equal, through an angle of 45° in the anti-clockwise direction. Find the equation of the line in the new position.*

SOLUTION If the line $2x - y = 5$ makes an angle θ with x-axis. Then, $\tan\theta = 2$.

Let $P(\alpha, \alpha)$ be a point on the line $2x - y = 5$. Then,

$$2\alpha - \alpha = 5 \Rightarrow \alpha = 5$$

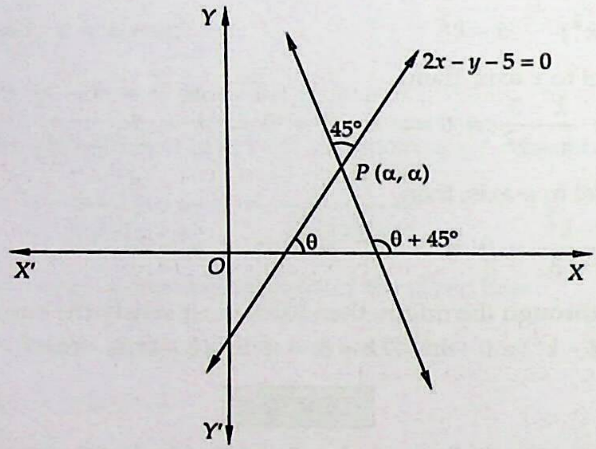

Fig. 23.60

So, the coordinates of P are $(5, 5)$.

If the line $2x - y - 5 = 0$ is rotated about point P through 45° in anti-clockwise direction, then the line in its new position makes angle $\theta + 45°$ with x-axis. Let m' be the slope of the line in its new position. Then,

$$m' = \tan(\theta + 45°) = \frac{\tan\theta + \tan 45°}{1 - \tan\theta \tan 45°} = \frac{2 + 1}{1 - 2 \times 1} = -3$$

Thus, the line in its new position passes through $P(5, 5)$ and has slope $m' = -3$.

So, its equation is $y - 5 = m'(x - 5)$ or, $y - 5 = -3(x - 5)$ or, $3x + y - 20 = 0$

EXAMPLE 7 *Find the coordinates of one vertex of an equilateral triangle with centroid at the origin and the opposite side $x + y - 2 = 0$.* **[NCERT EXEMPLAR]**

SOLUTION Let ABC be an equilateral triangle having $x + y - 2 = 0$ as the equation of side BC and opposite vertex A. Let the coordinates of its vertices be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. It is given that the centroid of ΔABC is at the origin.

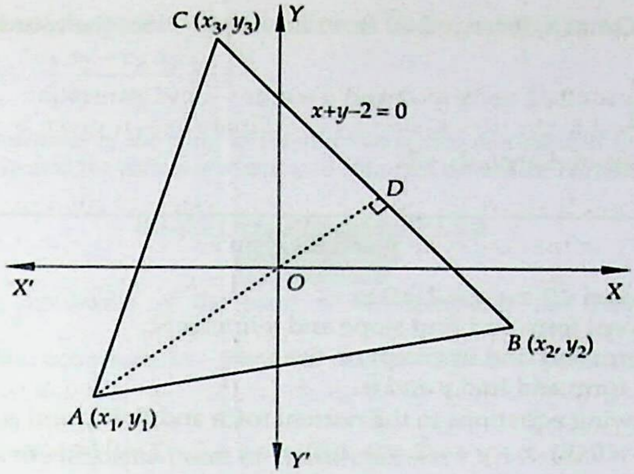

Fig. 23.61

$\therefore \quad \dfrac{x_1+x_2+x_3}{3}=0$ and $\dfrac{y_1+y_2+y_3}{3}=0$

$\Rightarrow \quad x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0$

$\Rightarrow \quad x_2+x_3=-x_1$ and $y_2+y_3=-y_1$ ...(i)

It is given that ΔABC is an equilateral triangle. Therefore, median $AD \perp BC$ and so $OA \perp BC$.

$\therefore \quad$ Slope of $OA \times$ Slope of $BC = -1$

$\Rightarrow \quad \dfrac{y_1-0}{x_1-0} \times -1 = 1 \qquad \left[\text{Equation of } BC \text{ is } x+y-2=0 \therefore \text{ Slope of } BC = -\dfrac{1}{1} = -1\right]$

$\Rightarrow \quad y_1 = x_1$ ...(ii)

Clearly, D is the mid-point of BC. So, the coordinates of D are

$\left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right) = \left(-\dfrac{x_1}{2}, -\dfrac{y_1}{2}\right)$ [Using (i)]

Point D lies on BC whose equation is $x+y-2=0$.

$\therefore \quad -\dfrac{x_1}{2} - \dfrac{y_1}{2} - 2 = 0 \Rightarrow x_1 + y_1 + 4 = 0$ ...(iii)

Solving (ii) and (iii), we obtain $x_1 = -2$, $y_1 = -2$.

Hence, the coordinates of A are $(-2, -2)$.

ALITER It is given that ABC is an equilateral triangle with centroid at the origin O. Therefore, $OA \perp BC$ and so

Slope of $OA \times$ Slope of $BC = -1$

$\Rightarrow \quad$ Slope of $OA \times -1 = -1$

$\Rightarrow \quad$ Slope of $OA = 1$

Thus, OA makes an angle of 45° with x-axis.

So, the equation OA in distance form is

$$\frac{x-0}{\cos 45°} = \frac{y-0}{\sin 45°} \text{ or, } \frac{x-0}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}}$$

The equation of BC in normal form is

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} = \sqrt{2}$$

$\therefore \quad OD = \sqrt{2}$

Since O is the centroid of ΔABC. Therefore, $OA = 2\,(OD) = 2\sqrt{2}$.

Thus, A is a point on OA at a distance $2\sqrt{2}$ from the origin O. So, the coordinates of A are given by

$$\frac{x-0}{1/\sqrt{2}} = \frac{y-0}{1/\sqrt{2}} = -2\sqrt{2} \Rightarrow x = -2 \text{ and } y = -2$$

Hence, the co-ordinates of A are $(-2, -2)$.

EXERCISE 23.9

LEVEL-1

1. Reduce the equation $\sqrt{3}\,x + y + 2 = 0$ to:
   (i) slope-intercept form and find slope and y-intercept;
   (ii) intercept form and find intercept on the axes;
   (iii) the normal form and find p and α.
2. Reduce the following equations to the normal form and find p and α in each case:
   (i) $x + \sqrt{3}\,y - 4 = 0$ (ii) $x + y + \sqrt{2} = 0$ (iii) $x - y + 2\sqrt{2} = 0$ (iv) $x - 3 = 0$
   (v) $y - 2 = 0$.
3. Put the equation $\frac{x}{a} + \frac{y}{b} = 1$ to the slope intercept form and find its slope and y-intercept.
4. Reduce the lines $3x - 4y + 4 = 0$ and $2x + 4y - 5 = 0$ to the normal form and hence find which line is nearer to the origin.
5. Show that the origin is equidistant from the lines $4x + 3y + 10 = 0$; $5x - 12y + 26 = 0$ and $7x + 24y = 50$.
6. Find the values of θ and p, if the equation $x\cos\theta + y\sin\theta = p$ is the normal form of the line $\sqrt{3}x + y + 2 = 0$. [NCERT]
7. Reduce the equation $3x - 2y + 6 = 0$ to the intercept form and find the x and y intercepts.

LEVEL-2

8. The perpendicular distance of a line from the origin is 5 units and its slope is -1. Find the equation of the line.

ANSWERS

1. (i) Slope $= -\sqrt{3}$, y-intercept $= -2$ (ii) x-intercept $= -\frac{2}{\sqrt{3}}$, y-intercept $= -2$
   (iii) $p = 1, \alpha = 210°$
2. (i) $p = 2, \alpha = \frac{\pi}{3}$ (ii) $p = 1, \alpha = 225°$ (iii) $p = 2, \alpha = 135°$ (iv) $p = 3, \alpha = 0$
   (v) $p = 2, \alpha = \frac{\pi}{2}$
3. Slope $= -\frac{b}{a}$, y-intercept $= b$
4. $3x - 4y + 4 = 0$
6. $\alpha = 210°, p = 1$
7. x-intercept $= -2$, y-intercept $= 3$
8. $x + y - 5\sqrt{2} = 0$

23.8 POINT OF INTERSECTION OF TWO LINES

Let the equations of two lines be

$$a_1 x + b_1 y + c_1 = 0 \quad \text{...(i)}$$

and, $$a_2 x + b_2 y + c_2 = 0 \quad \text{...(ii)}$$

Suppose these two lines intersect at a point $P(x_1, y_1)$. Then, (x_1, y_1) satisfies each of the given equations.

$\therefore$ $a_1 x_1 + b_1 y_1 + c_1 = 0$ and, $a_2 x_1 + b_2 y_1 + c_2 = 0$

Solving these two equations by cross-multiplication, we get

$$\frac{x_1}{b_1 c_2 - b_2 c_1} = \frac{y_1}{c_1 a_2 - c_2 a_1} = \frac{1}{a_1 b_2 - a_2 b_1} \Rightarrow x_1 = \frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, y_1 = \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}$$

Hence, the coordinates of the point of intersection of lines (i) and (ii) are:

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right)$$

NOTE *To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y so obtained determine the coordinates of the point of intersection.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the coordinates of the point of intersection of the lines* $2x - y + 3 = 0$ *and* $x + 2y - 4 = 0$.

SOLUTION Solving the equations $2x - y + 3 = 0$ and $x + 2y - 4 = 0$, simultaneously we obtain

$$\frac{x}{4-6} = \frac{y}{3+8} = \frac{1}{4+1} \Rightarrow \frac{x}{-2} = \frac{y}{11} = \frac{1}{5} \Rightarrow x = -\frac{2}{5}, \ y = \frac{11}{5}$$

Hence, $(-2/5, 11/5$ is the required point of intersection.

EXAMPLE 2 *Find the area of the triangle formed by the lines* $y = x$, $y = 2x$ *and* $y = 3x + 4$.

SOLUTION The given equations are

$$y = x \quad ...(i) \qquad y = 2x \quad ...(ii) \qquad \text{and} \quad y = 3x + 4 \quad ...(iii)$$

Suppose the equations (i), (ii) and (iii) represent the sides AB, BC and CA respectively of a triangle ABC.

Solving (i) and (ii), we get: $x = 0$ and $y = 0$. Thus, AB and BC intersect at $B(0, 0)$.

Solving (ii) and (iii), we obtain: $x = -4$, $y = -8$. Thus, BC and CA intersect at $C(-4, -8)$.

Solving (iii) and (i), we get: $x = -2$ and $y = -2$. So, CA and AB intersect at $A(-2, -2)$.

Thus, the coordinates of the vertices of the triangle ABC are : $A(-2, -2)$, $B(0, 0)$ and $C(-4, -8)$.

$$\therefore \qquad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} -2 & -2 & 1 \\ 0 & 0 & 1 \\ -4 & -8 & 1 \end{vmatrix} = 4 \text{ sq. units.}$$

EXAMPLE 3 *Find the equations of the medians of a triangle formed by the lines* $x + y - 6 = 0$, $x - 3y - 2 = 0$ *and* $5x - 3y + 2 = 0$.

SOLUTION The given equations are :

$$x + y - 6 = 0 \quad ...(i) \qquad x - 3y - 2 = 0 \quad ...(ii) \qquad \text{and} \qquad 5x - 3y + 2 = 0 \quad ...(iii)$$

Suppose equations (i), (ii) and (iii) represent the sides, AB, BC and CA respectively of triangle ABC.

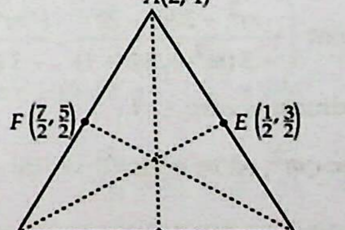

Fig. 23.62

Solving (i) and (ii), we get: $x = 5$ and $y = 1$. Thus, AB and BC intersect at $B(5, 1)$.

Solving (ii) and (iii), we get: $x = -1$ and $y = -1$. Thus, BC and CA intersect at $C(-1, -1)$.

Solving (i) and (iii), we get: $x = 2$ and $y = 4$. Thus, AB and CA intersect at $A(2, 4)$.

Thus, the coordinates of the vertices A, B and C of triangle ABC are $(2, 4)$, $(5, 1)$ and $(-1, -1)$ respectively. Let D, E and F be the mid-points of sides BC, CA and AB respectively. Then, the coordinates of D, E and F are

$$D\left(\frac{5-1}{2}, \frac{1-1}{2}\right)=(2, 0); E\left(\frac{2-1}{2}, \frac{4-1}{2}\right)=\left(\frac{1}{2}, \frac{3}{2}\right) \text{and } F\left(\frac{2+5}{2}, \frac{4+1}{2}\right)=\left(\frac{7}{2}, \frac{5}{2}\right)$$

respectively.

The median AD passes through $A(2, 4)$ and $D(2, 0)$. So, its equation is

$$y-4 = \frac{0-4}{2-2}(x-2)$$

$$\Rightarrow \quad x-2 = \frac{2-2}{0-4}(y-4) \Rightarrow x-2 = 0 \Rightarrow x = 2$$

The median BE passes through points $B(5, 1)$ and $E(1/2, 3/2)$. So, its equation is

$$y-1 = \frac{\frac{3}{2}-1}{\frac{1}{2}-5}(x-5)$$

$$\Rightarrow \quad y-1 = -\frac{1}{9}(x-5) \Rightarrow x+9y-14=0$$

The median CF passes through points $C(-1, -1)$ and $F(7/2, 5/2)$. So, its equation is

$$y+1 = \frac{\frac{5}{2}+1}{\frac{7}{2}+1}(x+1) \Rightarrow y+1 = \frac{7}{9}(x+1) \Rightarrow 7x-9y-2=0$$

Hence, the equations of the medians of the triangle are $x=2$, $x+9y-14=0$ and $7x-9y-2=0$.

EXAMPLE 4 *Find the value of m for which the lines $mx+(2m+3)y+m+6=0$ and $(2m+1)x+(m-1)y+m-9=0$ intersect at a point on y-axis.*

SOLUTION The equations of the lines are

$$mx+(2m+3)y+m+6=0 \quad \text{...(i)}$$

$$(2m+1)x+(m-1)y+m-9=0 \quad \text{...(ii)}$$

Solving these two equations by cross-multiplication, we obtain

$$\frac{x}{(2m+3)(m-9)-(m-1)(m+6)} = \frac{y}{(2m+1)(m+6)-m(m-9)} = \frac{1}{m(m-1)-(2m+1)(2m+3)}$$

$$\Rightarrow \quad \frac{x}{m^2-20m-21} = \frac{y}{m^2+22m+6} = \frac{1}{-3(m^2+3m+1)}$$

$$\Rightarrow \quad x = \frac{m^2-20m-21}{-3(m^2+3m+1)} \text{ and } y = \frac{m^2+22m+6}{-3(m^2+3m+1)}$$

So, given lines intersect at the point $\left(\frac{m^2-20m-21}{-3(m^2+3m+1)}, \frac{m^2+22m+6}{-3(m^2+3m+1)}\right)$.

If it lies on y-axis, then its x-coordinate is zero.

$$\therefore \quad \frac{m^2-20m-21}{-3(m^2+3m+1)} = 0 \Rightarrow m^2-20m-21=0 \Rightarrow (m-21)(m+1)=0 \Rightarrow m=-1, 21$$

EXAMPLE 5 *Find the area of the triangle formed by the lines $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$.* **[NCERT]**

SOLUTION Let $y=m_1x+c_1$, $y=m_2x+c_2$ and $x=0$ be the sides AB, BC and CA respectively of a triangle ABC. Solving $y=m_1x+c_1$ and $y=m_2x+c_2$ as linear equations in x, y, we get

$$x = \frac{c_2-c_1}{m_1-m_2}, \quad y = \frac{m_1c_2-m_2c_1}{m_1-m_2}$$

So, the coordinates of B are $\left(\frac{c_2-c_1}{m_1-m_2}, \frac{m_1c_2-m_2c_1}{m_1-m_2}\right)$.

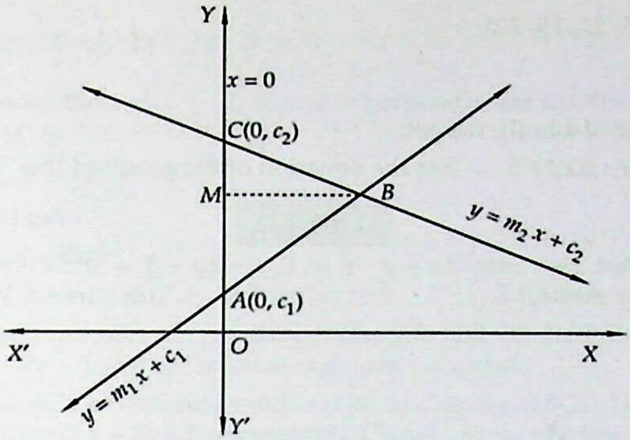

Fig. 23.63

Solving $y = m_2 x + c_2$ and $x = 0$, we get: $x = 0, y = c_2$. So, coordinates of C are $(0, c_2)$.
Similarly, by solving $x = 0$ and $y = m_1 x + c_1$, we get the coordinates of A as $(0, c_1)$.

$$\therefore \quad \text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 0 & c_1 & 1 \\ 0 & c_2 & 1 \\ \dfrac{c_2 - c_1}{m_1 - m_2} & \dfrac{m_1 c_2 - m_2 c_1}{m_1 - m_2} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{c_2 - c_1}{m_1 - m_2}\right)(c_1 - c_2) = \frac{1}{2}\frac{(c_1 - c_2)^2}{m_1 - m_2} \text{ in magnitude.}$$

ALITER Given lines are

$$y = m_1 x + c_1 \qquad \text{... (i)}$$
$$y = m_2 x + c_2 \qquad \text{... (ii)}$$
$$x = 0 \qquad \text{... (iii)}$$

Lines (i) and (ii) intersect line (iii) at $A\,(0, c_1)$ and $C\,(0, c_2)$ respectively.

Solving (i) and (ii), we obtain the coordinates of B as $\left(\dfrac{c_2 - c_1}{m_1 - m_2}, \dfrac{m_1 c_2 - m_2 c_1}{m_1 - m_2}\right)$.

$$\therefore \qquad BM = x\text{-coordinate of } B = \frac{c_2 - c_1}{m_1 - m_2}$$

Clearly, $AC = |c_2 - c_1|$

$$\therefore \qquad \text{Area of } \Delta ABC = \frac{1}{2}\text{ Base} \times \text{Height} = \frac{1}{2}(AC \times BM)$$

$$= \frac{1}{2}|c_2 - c_1| \times \left|\frac{c_2 - c_1}{m_1 - m_2}\right| = \frac{1}{2}\frac{(c_2 - c_1)^2}{|m_1 - m_2|}$$

EXAMPLE 6 *Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.*

SOLUTION On solving the equations $x - 7y + 5 = 0$ and $3x + y = 0$, we get:

$$x = -\frac{5}{22} \text{ and } y = \frac{15}{22}.$$

So, the given lines intersect at the point whose coordinates are : $(-5/22, 15/22)$.

We know that, the equation of a line parallel to y-axis is of the form x=constant. So, let the equation of the required line be

$$x = \lambda \qquad \text{...(i)}$$

It passes through $(-5/22, 15/22)$.

$$\therefore \qquad \frac{-5}{22} = \lambda$$

Substituting the value of λ in (i), we get:

$x = -5/22$ or, $22x + 5 = 0$ as the equation of the required line.

LEVEL-2

EXAMPLE 7 *Show that the lines* $4x + y - 9 = 0, x - 2y + 3 = 0, 5x - y - 6 = 0$ *make equal intercepts on any line of gradient 2.*

SOLUTION The equation of any line of gradient 2 is

$$y = 2x + c \qquad \text{...(i)}$$

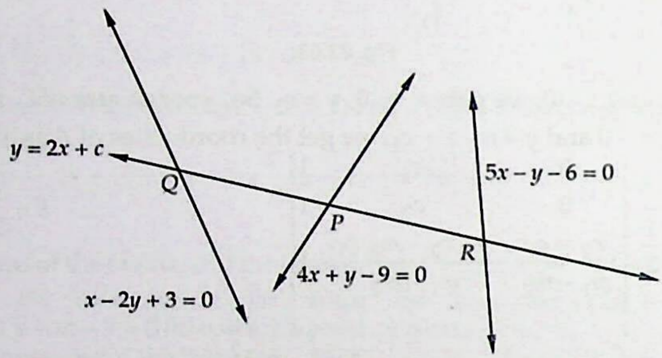

Fig. 23.64

The equations of given lines are

$$4x + y - 9 = 0 \qquad \text{...(ii)}$$
$$x - 2y + 3 = 0 \qquad \text{...(iii)}$$
$$5x - y - 6 = 0 \qquad \text{...(iv)}$$

Solving (i) with (ii), (iii) and (iv) respectively, we obtain the coordinates of P, Q and R as

$$P\left(\frac{3}{2} - \frac{c}{6}, 3 + \frac{2c}{3}\right), Q\left(1 - \frac{2c}{3}, 2 - \frac{c}{3}\right) \text{and } R\left(2 + \frac{c}{3}, 4 + \frac{5c}{3}\right)$$

Clearly, P is the mid-point of QR. Therefore $PQ = PR$.

Hence, lines (ii), (iii) and (iv) make equal intercepts on any line of gradient 2.

EXAMPLE 8 *Two vertices of a triangle are* $(3, -1)$ *and* $(-2, 3)$ *and its orthocentre is at the origin. Find the coordinates of the third vertex.*

SOLUTION Clearly, A is the intersection of sides AB and AC of ΔABC. Side AB passes through $B(3, -1)$ and is perpendicular to OC whose slope is $-3/2$. So, equation of side AB is

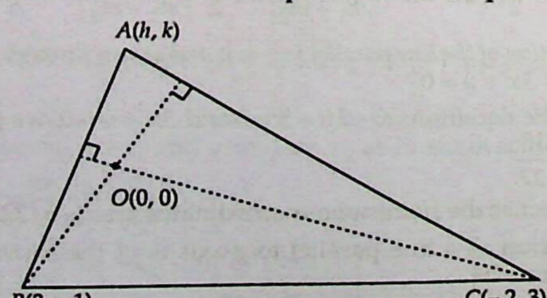

Fig. 23.65

$$y+1 = \frac{2}{3}(x-3) \text{ or, } 2x-3y-9 = 0 \quad ...(i)$$

Similarly, side AC passes through $C(-2, 3)$ and is perpendicular to OB whose slope is $-1/3$. So, equation of side AC is

$$y-3 = 3(x+2) \text{ or, } 3x-y+9=0 \quad ...(ii)$$

Solving (i) and (ii), we get

$$x = -36/7, y = -45/7$$

Hence, coordinates of A are $(-36/7, -45/7)$

EXAMPLE 9 *Two consecutive sides of a parallelogram are $4x+5y=0$ and $7x+2y=0$. If the equation of one diagonal is $11x+7y=9$, find the equation of the other diagonal.*

SOLUTION Let AB and AD be consecutive sides of parallelogram $ABCD$. Let the equations of AB and AD be $4x+5y=0$ and $7x+2y=0$ respectively. Clearly, these two lines intersect at $A(0, 0)$.

Solving $11x+7y = 9$ and $4x+5y = 0$, we get: $x = 5/3$ and $y = -4/3$

So, the coordinates of B are $(5/3, -4/3)$.

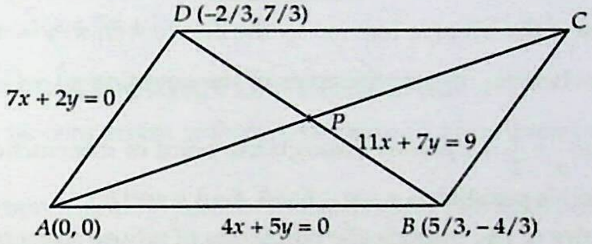

Fig. 23.66

Similarly, by solving $11x+7y = 9$ and $7x+2y = 0$, we obtain that the coordinates of D are $(-2/3, 7/3)$.

We know that the diagonals of a parallelogram bisect each other. So, P is the mid-point of BD and hence its coordinates are $\left(\frac{\frac{5}{3}-\frac{2}{3}}{2}, \frac{-\frac{4}{3}+\frac{7}{3}}{2}\right)$ or, $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Clearly, AC passes through $A(0, 0)$ and $C(1/2, 1/2)$.

Hence, equation of AC is $y-0 = \frac{\frac{1}{2}-0}{\frac{1}{2}-0}(x-0)$ or, $y = x$.

EXERCISE 23.10

LEVEL-1

1. Find the point of intersection of the following pairs of lines :
   (i) $2x-y+3=0$ and $x+y-5=0$ (ii) $bx+ay=ab$ and $ax+by=ab$.
   (iii) $y = m_1 x + \frac{a}{m_1}$ and $y = m_2 x + \frac{a}{m_2}$.
2. Find the coordinates of the vertices of a triangle, the equations of whose sides are:
   (i) $x+y-4=0$, $2x-y+3=0$ and $x-3y+2=0$
   (ii) $y(t_1+t_2) = 2x+2at_1t_2$, $y(t_2+t_3) = 2x+2at_2t_3$ and, $y(t_3+t_1) = 2x+2at_1t_3$.

3. Find the area of the triangle formed by the lines
   (i) $y = m_1 x + c_1, y = m_2 x + c_2$ and $x = 0$
   (ii) $y = 0, x = 2$ and $x + 2y = 3$.
   (iii) $x + y - 6 = 0, x - 3y - 2 = 0$ and $5x - 3y + 2 = 0$
4. Find the equations of the medians of a triangle, the equations of whose sides are:
$$3x + 2y + 6 = 0, 2x - 5y + 4 = 0 \text{ and } x - 3y - 6 = 0$$
5. Prove that the lines $y = \sqrt{3}x + 1, y = 4$ and $y = -\sqrt{3}x + 2$ form an equilateral triangle.
6. Classify the following pairs of lines as coincident, parallel or intersecting:
   (i) $2x + y - 1 = 0$ and $3x + 2y + 5 = 0$ (ii) $x - y = 0$ and $3x - 3y + 5 = 0$
   (iii) $3x + 2y - 4 = 0$ and $6x + 4y - 8 = 0$.
7. Find the equation of the line joining the point (3, 5) to the point of intersection of the lines $4x + y - 1 = 0$ and $7x - 3y - 35 = 0$.
8. Find the equation of the line passing through the point of intersection of the lines $4x - 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes. [NCERT]

LEVEL-2

9. Show that the area of the triangle formed by the lines $y = m_1 x, y = m_2 x$ and $y = c$ is equal to $\frac{c^2}{4}(\sqrt{33} + \sqrt{11})$, where m_1, m_2 are the roots of the equation $x^2 + (\sqrt{3} + 2)x + \sqrt{3} - 1 = 0$.
10. If the straight line $\frac{x}{a} + \frac{y}{b} = 1$ passes through the point of intersection of the lines $x + y = 3$ and $2x - 3y = 1$ and is parallel to $x - y - 6 = 0$, find a and b.
11. Find the orthocentre of the triangle the equations of whose sides are $x + y = 1, 2x + 3y = 6$ and $4x - y + 4 = 0$.
12. Three sides AB, BC and CA of a triangle ABC are $5x - 3y + 2 = 0, x - 3y - 2 = 0$ and $x + y - 6 = 0$ respectively. Find the equation of the altitude through the vertex A.
13. Find the coordinates of the orthocentre of the triangle whose vertices are $(-1, 3), (2, -1)$ and $(0, 0)$.
14. Find the coordinates of the incentre and centroid of the triangle whose sides have the equations $3x - 4y = 0, 12y + 5x = 0$ and $y - 15 = 0$.
15. Prove that the lines $\sqrt{3}x + y = 0, \sqrt{3}y + x = 0, \sqrt{3}x + y = 1$ and $\sqrt{3}y + x = 1$ form a rhombus.
16. Find the equation of the line passing through the intersection of the lines $2x + y = 5$ and $x + 3y + 8 = 0$ and parallel to the line $3x + 4y = 7$.
17. Find the equation of the straight line passing through the point of intersection of the lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$.
[NCERT EXEMPLAR]

ANSWERS

1. (i) $\left(\frac{2}{3}, \frac{13}{3}\right)$ (ii) $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$ (iii) $\left(\frac{a}{m_1 m_2}, a\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right)$
2. (i) $\left(\frac{1}{3}, \frac{11}{3}\right), \left(-\frac{7}{5}, \frac{1}{5}\right), \left(\frac{5}{2}, \frac{3}{2}\right)$ (ii) $(at_1^2, 2at_1), (at_2^2, 2at_2), (at_3^2, 2at_3)$
3. (i) $\frac{(c_1 - c_2)^2}{2(m_1 - m_2)}$ (ii) 0 (iii) 12. sq. units
4. $41x - 112y - 70 = 0$, $16x - 59y - 120 = 0$ and $25x - 53y + 50 = 0$

6. (i) intersecting (ii) parallel (iii) coincident
7. $12x - y - 31 = 0$ 8. $x + y + 13 = 0$ 10. $a = 1, b = -1$
11. $\left(\frac{3}{7}, \frac{22}{7}\right)$ 12. $3x + y - 10 = 0$ 13. $(-4, -3)$ 14. $(-1, 8), \left(-\frac{16}{3}, 10\right)$
16. $3x + 4y + 3 = 0$ 17. $5x + 3y + 8 = 0$

HINTS TO NCERT & SELECTED PROBLEM

8. Given lines intersect at $(-8, -5)$. The equation of a line making equal intercepts on the coordinates axes is $\frac{x}{a} + \frac{y}{a} = 1$ or, $x + y = a$. It passes through $(-8, -5)$.
$\therefore \quad -8 - 5 + a = 0 \Rightarrow a = 13$
Hence, the equation of the line is $x + y = 13$.
17. The lines $5x - 6y - 1 = 0$ and $3x + 2y + 5 = 0$ intersect at the point $(-1, -1)$. The slope of the line $3x - 5y + 11 = 0$ is $3/5$. So, the slope of a line perpendicular to it is $-5/3$. Hence, the equation of the required line is
$y + 1 = -5/3\,(x + 1)$ or, $5x + 3y + 8 = 0$.

23.9 CONDITION OF CONCURRENCY OF THREE LINES

Three lines are said to be concurrent if they pass through a common point i.e. they meet at a point.

Thus, if three lines are concurrent the point of intersection of two lines lies on the third line. Let

$$a_1 x_1 + b_1 y + c_1 = 0 \quad ...(i) \qquad a_2 x + b_2 y + c_2 = 0 \quad ...(ii) \qquad a_3 x + b_3 y + c_3 = 0 \quad ...(iii)$$

be three concurrent lines. Then the point of intersection of (i) and (ii) must lie on the third.

The coordinates of the point of intersection of (i) and (ii) are:

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right) \qquad \text{[See section 22.8]}$$

This point must lies on line (iii).

$$\therefore \quad a_3\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}\right) + b_3\left(\frac{c_1 a_2 - c_2 a_1}{a_1 b_2 - a_2 b_1}\right) + c_3 = 0$$

$$\Rightarrow \quad a_3 (b_1 c_2 - b_2 c_1) + b_3 (c_1 a_2 - c_2 a_1) + c_3 (a_1 b_2 - a_2 b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

ANOTHER CONDITION OF CONCURRENCY OF THREE LINES

Three lines

$$L_1 \equiv a_1 x + b_1 y + c_1 = 0; \quad L_2 \equiv a_2 x + b_2 y + c_2 = 0; \quad L_3 \equiv a_3 x + b_3 y + c_3 = 0$$

are concurrent iff there exist constants $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

i.e. $$\lambda_1 (a_1 x + b_1 y + c_1) + \lambda_2 (a_2 x + b_2 y + c_2) + \lambda_3 (a_3 x + b_3 y + c_3) = 0.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Prove that the lines* $3x + y - 14 = 0, x - 2y = 0$ *and* $3x - 8y + 4 = 0$ *are concurrent.*

SOLUTION Given lines are $3x + y - 14 = 0,\ x - 2y + 0 = 0$ and $3x - 8y + 4 = 0$.

We have, $\begin{vmatrix} 3 & 1 & -14 \\ 1 & -2 & 0 \\ 3 & -8 & 4 \end{vmatrix} = 3(-8+0) - 1(4-0) - 14(-8+6) = -24 - 4 + 28 = 0.$

So, the given lines are concurrent.

EXAMPLE 2 *Show that the lines* $x - y - 6 = 0, 4x - 3y - 20 = 0$ *and* $6x + 5y + 8 = 0$ *are concurrent. Also, find their common point of intersection.*

SOLUTION The given lines are

$$x - y - 6 = 0 \quad ...(i) \qquad 4x - 3y - 20 = 0 \quad ...(ii) \qquad 6x + 5y + 8 = 0 \quad ...(iii)$$

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{x}{20-18} = \frac{y}{-24+20} = \frac{1}{-3+4} \Rightarrow x = 2 \text{ and } y = -4.$$

Thus, the first two lines intersect at the point $(2, -4)$. Putting $x = 2$ and $y = -4$ in (iii), we get

$$6 \times 2 + 5 \times -4 + 8 = 0$$

Thus, the point $(2, -4)$ lies on line (iii).

Hence, the given lines are concurrent and their common point of intersection is $(2, -4)$.

EXAMPLE 3 *Find the value of* λ, *if the lines* $3x - 4y - 13 = 0, 8x - 11y - 33 = 0$ *and* $2x - 3y + \lambda = 0$ *are concurrent.*

SOLUTION The given lines are concurrent, if

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 3(-11\lambda - 99) + 4(8\lambda + 66) - 13(-24 + 22) = 0 \Rightarrow -\lambda - 7 = 0 \Rightarrow \lambda = -7$$

ALITER The given equations are

$$3x - 4y - 13 = 0 \quad ...(i) \qquad 8x - 11y - 33 = 0 \quad ...(ii) \text{ and, } 2x - 3y + \lambda = 0 \quad ...(iii)$$

Solving equations (i) and (ii), we get $x = 11$ and $y = 5$. Thus, $(11, 5)$ is the point of intersection of lines (i) and (ii). The given lines will be concurrent if they pass through the common point i.e. the point of intersection of any two lies on the third. Therefore, the point $(11, 5)$ must lie on the line (iii).

$$\therefore \quad 2 \times 11 - 3 \times 5 + \lambda = 0 \Rightarrow \lambda = -7.$$

EXAMPLE 4 *If the lines* $a_1 x + b_1 y + 1 = 0, a_2 x + b_2 y + 1 = 0$ *and* $a_3 x + b_3 y + 1 = 0$ *are concurrent, show that the points* $(a_1, b_1), (a_2, b_2)$ *and* (a_3, b_3) *are collinear.*

SOLUTION The given lines are

$$a_1 x + b_1 y + 1 = 0 \quad ...(i) \qquad a_2 x + b_2 y + 1 = 0 \quad ...(ii) \quad \text{and} \quad a_3 x + b_3 y + 1 = 0 \quad ...(iii).$$

If these lines are concurrent, we must have

$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0$, which is the condition of collinearity of three points $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3).

Hence, if the given lines are concurrent, the given points are collinear.

LEVEL-2

EXAMPLE 5 *If the lines* $ax + y + 1 = 0$, $x + by + 1 = 0$ *and* $x + y + c = 0$ *are concurrent* $(a \neq b \neq c \neq 1)$, *prove that* $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$.

SOLUTION The equations of the given lines are

$$ax + y + 1 = 0$$
$$x + by + 1 = 0$$
$$x + y + c = 0$$

If these lines are concurrent, then

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \quad a\begin{vmatrix} b & 1 \\ 1 & c \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & c \end{vmatrix} + 1\begin{vmatrix} 1 & b \\ 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad a(bc-1) - (c-1) + (1-b) = 0 \Rightarrow abc - a - c + 1 + 1 - b = 0 \Rightarrow abc = a + b + c - 2 \qquad \text{...(i)}$$

Now, $$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = \frac{(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b)}{(1-a)(1-b)(1-c)}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + ab + bc + ca - abc}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{1 - (a+b+c) + (ab + bc + ca) - (a+b+c-2)} \qquad \text{[Using (i)]}$$

$$= \frac{3 - 2(a+b+c) + ab + bc + ca}{3 - 2(a+b+c) + ab + bc + ca} = 1$$

EXAMPLE 6 *Show that the following lines are concurrent:*

$$L_1 = (a-b)x + (b-c)y + (c-a) = 0$$
$$L_2 = (b-c)x + (c-a)y + (a-b) = 0$$

and, $$L_3 = (c-a)x + (a-b)y + (b-c) = 0.$$

SOLUTION Clearly,

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0, \text{ where } \lambda_1 = \lambda_2 = \lambda_3 = 1$$

Hence, the given lines are concurrent.

EXAMPLE 7 *Show that the altitudes of a triangle are concurrent.*

SOLUTION Let ABC be a triangle such that the coordinates of its vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let AD, BE and CF be the altitudes drawn from the vertices A, B, C respectively to the opposite sides BC, CA and, AB respectively. Then,

Slope of $BC = \frac{y_2 - y_3}{x_2 - x_3}$, Slope of $CA = \frac{y_3 - y_1}{x_3 - x_1}$ and, Slope of $AB = \frac{y_2 - y_1}{x_2 - x_1}$

Since $AD \perp BC$, $BE \perp CA$ and $CF \perp AB$. Therefore,

Slope of $AD = -\frac{x_2 - x_3}{y_2 - y_3}$, Slope of $BE = -\frac{x_3 - x_1}{y_3 - y_1}$ and, Slope of $CF = -\frac{x_2 - x_1}{y_2 - y_1}$.

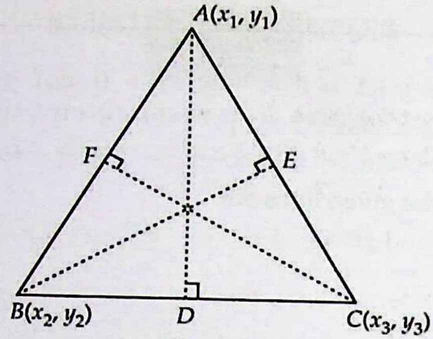

Fig. 23.67

The equation of altitude AD is

$$y - y_1 = -\frac{x_2 - x_3}{y_2 - y_3}(x - x_1)$$

i.e. $L_1 \equiv x(x_2 - x_3) + y(y_2 - y_3) - x_1(x_2 - x_3) - y_1(y_2 - y_3) = 0$ ...(i)

Similarly, equations of altitudes BE and CF are

$$L_2 \equiv x(x_3 - x_1) + y(y_3 - y_1) - x_2(x_3 - x_1) - y_2(y_3 - y_1) = 0 \quad \text{...(ii)}$$

and, $L_3 \equiv x(x_1 - x_2) + y(y_1 - y_2) - x_3(x_1 - x_2) - y_3(y_1 - y_2) = 0$ ...(iii)

Clearly, $1 . L_1 + 1 . L_2 + 1 . L_3 = 0$.

Hence, the altitudes AD, BE and CF are concurrent.

EXAMPLE 8 *Prove analytically that the medians of a triangle are concurrent.*

SOLUTION Let ABC be a triangle the coordinates of whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$. Let D, E and F be the mid-points of sides BC, CA and AB respectively. The coordinates of D, E and F are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$, $E\left(\frac{x_3 + x_1}{2}, \frac{y_3 + y_1}{2}\right)$ and $F\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ respectively.

Equation of median AD is

$$y - y_1 = \frac{y_1 - \frac{y_2 + y_3}{2}}{x_1 - \frac{x_2 + x_3}{2}}(x - x_1)$$

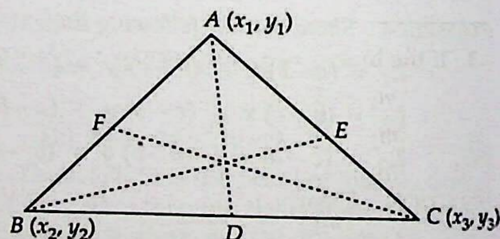

Fig. 23.68

or, $y - y_1 = \frac{2y_1 - y_2 - y_3}{2x_1 - x_2 - x_3}(x - x_1)$

or, $(2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y - x_1(2y_1 - y_2 - y_3) + y_1(2x_1 - x_2 - x_3) = 0$

or, $L_1 \equiv (2y_1 - y_2 - y_3)x - (2x_1 - x_2 - x_3)y + x_1(y_2 + y_3) - y_1(x_2 + x_3) = 0$ ...(i)

Similarly, equations of medians BE and CF are respectively

$$L_2 \equiv (2y_2 - y_1 - y_3)x - (2x_2 - x_1 - x_3)y + x_2(y_1 + y_3) - y_2(x_1 + x_3) = 0 \quad \text{...(ii)}$$

$$L_3 \equiv (2y_3 - y_1 - y_2)x - (2x_3 - x_1 - x_2)y + x_3(y_1 + y_2) - y_3(x_1 + x_2) = 0 \quad \text{...(iii)}$$

We observe that

$$1 . L_1 + 1 . L_2 + 1 . L_3 = 0 \quad \text{(identically)}$$

Hence, medians AD, BE and CF are concurrent.

EXERCISE 23.11

LEVEL-1

1. Prove that the following sets of three lines are concurrent:
   (i) $15x - 18y + 1 = 0, 12x + 10y - 3 = 0$ and $6x + 66y - 11 = 0$
   (ii) $3x - 5y - 11 = 0, 5x + 3y - 7 = 0$ and $x + 2y = 0$
   (iii) $\frac{x}{a} + \frac{y}{b} = 1, \frac{x}{b} + \frac{y}{a} = 1$ and $y = x$.
2. For what value of λ are the three lines $2x - 5y + 3 = 0, 5x - 9y + \lambda = 0$ and $x - 2y + 1 = 0$ concurrent?
3. Find the conditions that the straight lines $y = m_1 x + c_1, y = m_2 x + c_2$ and $y = m_3 x + c_3$ may meet in a point. **[NCERT]**
4. If the lines $p_1 x + q_1 y = 1, p_2 x + q_2 y = 1$ and $p_3 x + q_3 y = 1$ be concurrent, show that the points $(p_1, q_1), (p_2, q_2)$ and (p_3, q_3) are collinear.

LEVEL-2

5. Show that the straight lines $L_1 = (b + c)x + ay + 1 = 0$, $L_2 = (c + a)x + by + 1 = 0$ and $L_3 = (a + b)x + cy + 1 = 0$ are concurrent.
6. If the three lines $ax + a^2 y + 1 = 0, bx + b^2 y + 1 = 0$ and $cx + c^2 y + 1 = 0$ are concurrent, show that at least two of three constants a, b, c are equal.
7. If a, b, c are in A.P., prove that the straight lines $ax + 2y + 1 = 0, bx + 3y + 1 = 0$ and $cx + 4y + 1 = 0$ are concurrent.
8. Show that the perpendicular bisectors of the sides of a triangle are concurrent.

ANSWERS

2. $\lambda = 4$ 3. $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$

HINTS TO NCERT & SELECTED PROBLEM

3. If the lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ and $y = m_3 x + c_3$ are concurrent, then

$$\begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} m_1 & c_1 & 1 \\ m_2 & c_2 & 1 \\ m_3 & c_3 & 1 \end{vmatrix} = 0 \qquad \text{[Interchanging second and third column]}$$

$$\Rightarrow \quad m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$$

23.10 LINES PARALLEL AND PERPENDICULAR TO A GIVEN LINE

LINE PARALLEL TO A GIVEN LINE

THEOREM 1 *Prove that the equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.*

PROOF Let m be the slope of the line $ax + by + c = 0$. Then,

$$m = -\frac{a}{b} \qquad \left[\text{Using}: m = -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}\right]$$

The required line is parallel to the given line. So, the slope of the required line is also m. Let c_1 be the y-intercept of the required line. Then, its equation is

$$y = mx + c_1$$

$$\Rightarrow \quad y = -\frac{a}{b}x + c_1$$

$$\Rightarrow \quad ax + by - bc_1 = 0$$

$$\Rightarrow \quad ax + by + \lambda = 0, \quad \text{where } \lambda = -bc_1 = \text{constant.}$$

Q.E.D.

NOTE *To write a line parallel to a given line we keep the expression containing x and y same and simply replace the given constant by an unknown constant λ. The value of λ can be determined by some given condition.*

LINE PERPENDICULAR TO A GIVEN LINE

THEOREM 2 *Prove that the equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant.*

PROOF Let m_1 be the slope of the given line and m_2 be the slope of a line perpendicular to the given line. Then, $m_1 = -\frac{a}{b}$. As the lines are perpendicular.

$$\therefore \quad m_1 m_2 = -1 \Rightarrow m_2 = -\frac{1}{m_1} = \frac{b}{a}$$

Let c_2 be the y-intercept of the required line. Then, its equation is

$$y = m_2 x + c_2$$

$$\Rightarrow \quad y = \frac{b}{a}x + c_2$$

$$\Rightarrow \quad bx - ay + ac_2 = 0$$

$$\Rightarrow \quad bx - ay + \lambda = 0, \quad \text{where } \lambda = ac_2 = \text{constant.}$$

Q.E.D.

To write a line perpendicular to a given line we may use the following algorithm.

ALGORITHM

STEP I *Interchange x and y.*

STEP II *If the coefficients of x and y in the given equation are of the same sign make them of opposite signs and if the coefficients are of opposite signs make them of the same sign.*

STEP III *Replace the given constant by a new constant λ which is determined by a given condition.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the equation of the line which is parallel to $3x - 2y + 5 = 0$ and passes through the point $(5, -6)$.*

SOLUTION The equation of any line parallel to the line $3x - 2y + 5 = 0$ is

$$3x - 2y + \lambda = 0 \qquad \text{...(i)}$$

This passes through $(5, -6)$.

$$\therefore \quad 3 \times 5 - 2 \times -6 + \lambda = 0 \Rightarrow \lambda = -27.$$

Putting $\lambda = -27$ in (i), we obtain $3x - 2y - 27 = 0$ as the required equation.

ALITER The slope of the given line is 3/2. Therefore, the slope of the required line is also 3/2. Since the required line passes through (5, –6), so its equation is

$$y + 6 = \frac{3}{2}(x - 5) \quad \text{or,} \quad 3x - 2y - 27 = 0 \qquad [\text{Using: } y - y_1 = m(x - x_1)]$$

EXAMPLE 2 *Find the equation of the straight line that passes through the point (3, 4) and perpendicular to the line* $3x + 2y + 5 = 0$.

SOLUTION The equation of a line perpendicular to $3x + 2y + 5 = 0$ is

$$2x - 3y + \lambda = 0 \quad \text{...(i)}$$

This passes through the point (3, 4).

$\therefore \quad 3 \times 2 - 3 \times 4 + \lambda = 0 \Rightarrow \lambda = 6$

Putting $\lambda = 6$ in (i), we obtain $2x - 3y + 6 = 0$ as the required equation.

ALITER The slope of the given line is $-3/2$. Since the required line is perpendicular to the given line. So, the slope of the required line is $2/3$. As it passes through (3, 4). So, its equation is

$$y - 4 = \frac{2}{3}(x - 3) \text{ or, } 2x - 3y + 6 = 0 \quad [\text{Using}: y - y_1 = m(x - x_1)].$$

EXAMPLE 3 *Find the equation of the line perpendicular to* $x - 7y + 5 = 0$ *and having x- intercept 3.*

SOLUTION The equation of a line perpendicular to $x - 7y + 5 = 0$ is

$$7x + y + \lambda = 0 \quad \text{...(i)}$$

Its x-intercept is 3. This means that the line cuts x-axis at a distance of 3 units from the origin. Consequently, it passes through the point (3, 0) on x-axis.

$\therefore \quad 21 + 0 + \lambda = 0 \Rightarrow \lambda = -21$

Putting $\lambda = -21$ in (i), we obtain $7x + y - 21 = 0$ as the equation of the required line.

EXAMPLE 4 *Find the coordinates of the foot of the perpendicular drawn from the point* $(1, -2)$ *on the line* $y = 2x + 1$.

SOLUTION Let M be the foot of the perpendicular drawn from $P(1, -2)$ on the line $y = 2x + 1$. Then, M is the point of intersection of $y = 2x + 1$ and a line passing through $P(1, -2)$ and perpendicular to $y = 2x + 1$. The equation of a line perpendicular to $y = 2x + 1$ or, $2x - y + 1 = 0$ is

$$x + 2y + \lambda = 0 \quad \text{...(i)}$$

This passes through $P(1, -2)$.

$\therefore \quad 1 - 4 + \lambda = 0 \Rightarrow \lambda = 3$

Putting $\lambda = 3$ in (i), we get

$$x + 2y + 3 = 0$$

Point M is the point of intersection of the lines

$$2x - y + 1 = 0 \text{ and } x + 2y + 3 = 0.$$

Solving these equations by cross-multiplication, we get

$$\frac{x}{-5} = \frac{y}{-5} = \frac{1}{5} \Rightarrow x = -1, \ y = -1.$$

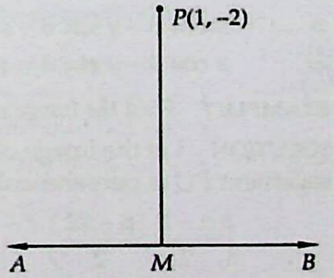

Fig. 23.69

Hence, the coordinates of the foot of the perpendicular are $(-1, -1)$.

EXAMPLE 5 *Find the equation of a straight line parallel to* $2x + 3y + 11 = 0$ *and which is such that the sum of its intercepts on the axes is 15.*

SOLUTION The equation of a line parallel to $2x + 3y + 11 = 0$ is

$$2x + 3y + \lambda = 0, \ \lambda \text{ is a constant} \quad \text{...(i)}$$

To find x-intercept of this line, we put $y = 0$ in its equation. Putting $y = 0$ in (i), we get

$\Rightarrow \quad 2x + \lambda = 0 \Rightarrow x = -\lambda/2$

So, x-intercept $= -\lambda/2$.

To find y-intercept of this line, we put $x = 0$ in its equation. Putting $x = 0$ in (i), we get

$$3y + \lambda = 0 \Rightarrow y = -\lambda/3$$

So, y-intercept $= -\lambda/3$.

It is given that the sum of the intercepts of the line (i) on the coordinate axes is 15.

$$\therefore \quad \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15 \Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

Putting $\lambda = -18$ in (i), we get: $2x + 3y - 18 = 0$.

Hence, the equation of the required line is $2x + 3y - 18 = 0$.

EXAMPLE 6 *Show that the equation of a line passing through* $(a\cos^3\theta, a\sin^3\theta)$ *and perpendicular to the line* $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ *is* $x\cos\theta - y\sin\theta = a\cos 2\theta$.

SOLUTION The equation of a line perpendicular to the line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$ is

$$x\,\mathrm{cosec}\,\theta - y\sec\theta + \lambda = 0, \ \lambda \text{ is a constant} \quad \text{...(i)}$$

This line passes through $(a\cos^3\theta, a\sin^3\theta)$.

$$\therefore \quad a\cos^3\theta\,\mathrm{cosec}\,\theta - a\sin^3\theta\sec\theta + \lambda = 0$$

$$\Rightarrow \quad \lambda = a(\sin^3\theta\sec\theta - \cos^3\theta\,\mathrm{cosec}\,\theta)$$

Putting the value of λ in (i), we get

$$x\,\mathrm{cosec}\,\theta - y\sec\theta + a(\sin^3\theta\sec\theta - \cos^3\theta\,\mathrm{cosec}\,\theta) = 0$$

$$\Rightarrow \quad \frac{x}{\sin\theta} - \frac{y}{\cos\theta} + a\left(\frac{\sin^3\theta}{\cos\theta} - \frac{\cos^3\theta}{\sin\theta}\right) = 0$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta + a(\sin^4\theta - \cos^4\theta) = 0$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta + a(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta) = 0$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta - a(\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta - a\cos 2\theta = 0$$

$$\Rightarrow \quad x\cos\theta - y\sin\theta = a\cos 2\theta.$$

EXAMPLE 7 *Find the image of the point* $(-8, 12)$ *with respect to the line mirror* $4x + 7y + 13 = 0$.

SOLUTION Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$. Then, the line segement PQ is perpendicularly bisected at R. So, the coordinates of R are

$$\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right).$$

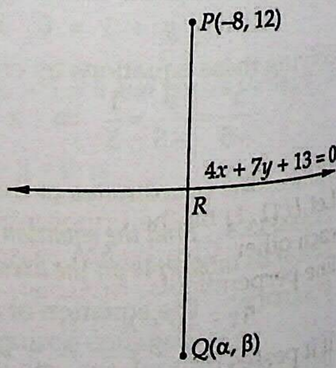

Fig. 23.70

As it lies on $4x + 7y + 13 = 0$.

$$\therefore \quad 2\alpha - 16 + \frac{7\beta + 84}{2} + 13 = 0 \Rightarrow 4\alpha + 7\beta + 78 = 0 \quad \text{...(i)}$$

The line segment PQ is perpendicular to AB.

$$\therefore \quad (\text{Slope of } AB) \times (\text{Slope of } PQ) = -1$$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad \text{...(ii)}$$

Solving (i) and (ii), we get: $\alpha = -16, \beta = -2$.

Hence, the image of $(-8, 12)$ in the line mirror $4x + 7y + 13 = 0$ is $(-16, -2)$.

LEVEL-2

EXAMPLE 8 *A person stranding at a junction (crossing) of two straight paths represented by the equations* $2x-3y-4=0$ *and* $3x-4y-5=0$, *wants to reach the path whose equation is* $6x-7y+8=0$ *in the least time. Find the equation of the path that he should follow.* **[NCERT]**

SOLUTION The lines $2x-3y-4=0$ and $3x-4y-5=0$ intersect at $(-1,-2)$. In order to reach the path, represented by the equation $6x-7y+8=0$, in the least time, the person should move along the line passing through A and perpendicular to $6x-7y+8=0$.

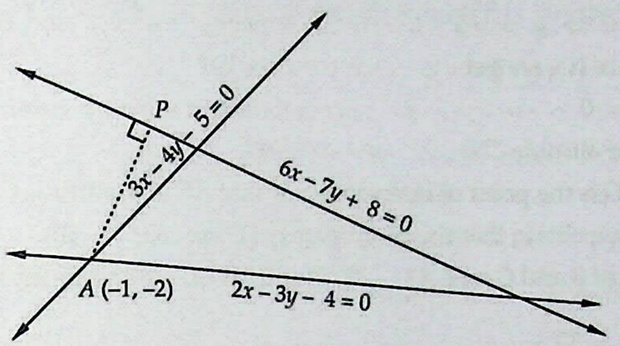

Fig. 23.71

Clearly, slope of the line $6x-7y+8=0$ is $\frac{6}{7}$. Therefore, slope of a line perpendicular to it is $-\frac{7}{6}$.

Hence, the equation of the required path is

$$y+2 = -\frac{7}{6}(x+1) \text{ or, } 7x+6y+19 = 0$$

EXAMPLE 9 *The equations of two sides of a triangle are* $3x-2y+6 = 0$ *and* $4x+5y-20 = 0$ *and the orthocentre is* $(1,1)$. *Find the equation of the third side.* **[NCERT]**

SOLUTION Let the equations of sides AB and AC of triangle ABC be respectively

$$3x-2y+6 = 0 \quad \text{...(i)}$$

and, $$4x+5y-20 = 0 \quad \text{...(ii)}$$

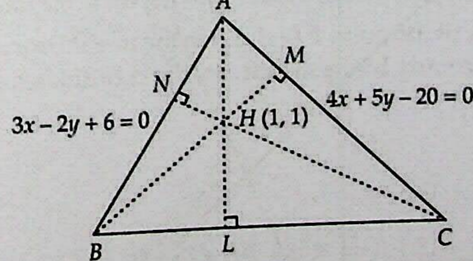

Fig. 23.72

Let $H(1,1)$ be the orthocentre of triangle ABC where the altitudes AL, BM and CN intersect each other. Clearly, BM passes through $H(1,1)$ and is perpendicular to AC. The equation of a line perpendicular to AC is

$$5x-4y+\lambda = 0 \quad \text{...(iii)}$$

If it passes through the orthocentre $H(1,1)$, then

$$5-4+\lambda = 0 \Rightarrow \lambda = -1$$

Substituting $\lambda = -1$ in (iii), we get

$$5x-4y-1 = 0 \quad \text{...(iv)}$$

This is the equation of altitude BM.

The vertex B of ΔABC is the intersection point of side AB and altitude BM. Solving their equations given by (i) and (iii), we get $x = -13$ and $y = -33/2$.

So, coordinates of B are $(-13, -33/2)$.

The altitude CN is perpendicular to AB. So, let its equation be

$$2x + 3y + \mu = 0 \quad \text{...(v)}$$

If it passes through the orthocentre $H(1, 1)$, then

$$2 + 3 + \mu = 0 \Rightarrow \mu = -5$$

Substituting $\mu = -5$ in (v), we get

$$2x + 3y - 5 = 0 \quad \text{...(vi)}$$

This is the equation of altitude CN.

The vertex C of ΔABC is the point of intersection of side AC and altitude CN.

Solving (ii) and (vi), we obtain that the coordinates of C are $(35/2, -10)$.

Thus, the coordinates of B and C are $(-13, -33/2)$ and $(35/2, -10)$ respectively. Hence, equation of side BC is

$$y + \frac{33}{2} = \frac{-10 + \frac{33}{2}}{\frac{35}{2} + 13}(x + 13) \text{ or, } \frac{2y + 33}{2} = \frac{13}{61}(x + 13) \text{ or, } 26x - 122y - 1675 = 0$$

EXAMPLE 10 *A straight line L is perpendicular to the line* $5x - y = 1$. *The area of the triangle formed by the line L and the coordinate axes is 5 square units. Find the equation of the line L.*

SOLUTION The equation of a line L perpendicular to the line $5x - y = 1$ is

$$x + 5y + \lambda = 0 \quad \text{...(i)}$$

This line meets x-axis at $y = 0$. Putting $y = 0$, we get $x = -\lambda$. So, the line L meets x-axis at $A(-\lambda, 0)$.

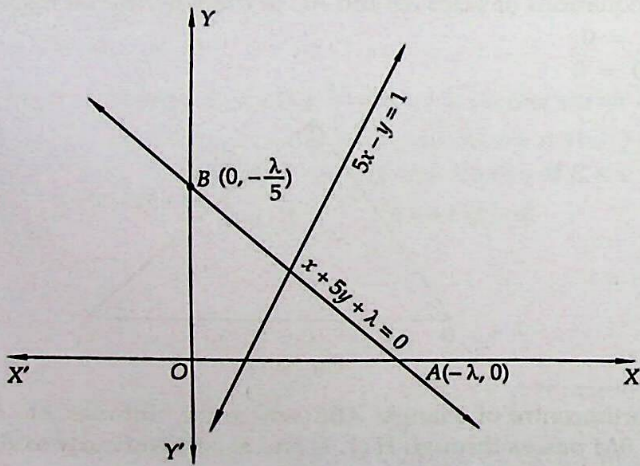

Fig. 23.73

The line L meets y-axis at $x = 0$. Putting $x = 0$ in the equation $x + 5y + \lambda = 0$, we obtain $y = -\lambda/5$.

So, the line L meets y-axis at $B(0, -\lambda/5)$.

Thus, the line L meets the coordinate axes at A and B such that $OA = -\lambda$ and $OB = -\lambda/5$.

It is given that

Area of $\Delta OAB = 5$

$$\Rightarrow \quad \frac{1}{2}(OA)(OB) = 5$$

$$\Rightarrow \quad \frac{1}{2}(-\lambda)\left(-\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}$$

Substituting the value of λ in (i), we obtain $x + 5y \pm 5\sqrt{2} = 0$ as the equations of the required line L.

EXAMPLE 11 *Let $P(x_1, y_1)$ be a point and let $ax + by + c = 0$ be a line. If $L(h, k)$ is the foot of perpendicular drawn from P on this line and $Q(\alpha, \beta)$ is the image of P in the given line, then prove that*

(i) $\dfrac{h - x_1}{a} = \dfrac{k - y_1}{b} = -\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$ (ii) $\dfrac{\alpha - x_1}{a} = \dfrac{\beta - y_1}{b} = -2\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$

SOLUTION Suppose PQ makes an angle θ with x-axis. Since PQ is perpendicular to $ax + by + c = 0$.

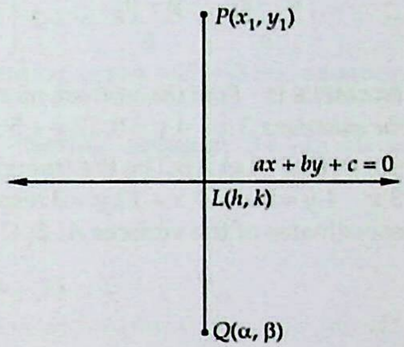

Fig. 23.74

$\therefore$ Slope of PQ × (Slope of $ax + by + c = 0$) $= -1$

$$\Rightarrow \quad \tan\theta \times -\frac{a}{b} = -1$$

$$\Rightarrow \quad \tan\theta = \frac{b}{a}$$

$$\Rightarrow \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}} \text{ and } \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

Since PQ passes through $P(x_1, y_1)$ and makes an angle θ with x-axis. Therefore, equation of PQ (in distance form) is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta}$.

Let $PL = LQ = r$. Then, coordinates of L and Q are given by

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \text{ and } \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = 2r \text{ respectively.}$$

$$\therefore \quad \frac{h - x_1}{\cos\theta} = \frac{k - y_1}{\sin\theta} = r \qquad \text{...(i)}$$

$$\text{and,} \quad \frac{\alpha - x_1}{\cos\theta} = \frac{\beta - y_1}{\sin\theta} = 2r \qquad \text{...(ii)}$$

Now, $\dfrac{h - x_1}{\cos\theta} = \dfrac{k - y_1}{\sin\theta} = r \Rightarrow h = x_1 + r\cos\theta,\ k = y_1 + r\sin\theta$

So, the coordinates of L are $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$.

Point L lies on $ax + by + c = 0$.

$$\therefore \quad a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \qquad \left[\because \cos\theta = \frac{a}{\sqrt{a^2 + b^2}},\ \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}\right]$$

$$3y + \lambda = 0 \Rightarrow y = -\lambda/3$$

So, y-intercept $= -\lambda/3$.

It is given that the sum of the intercepts of the line (i) on the coordinate axes is 15.

$$\therefore \quad \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15 \Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

Putting $\lambda = -18$ in (i), we get: $2x + 3y - 18 = 0$.

Hence, the equation of the required line is $2x + 3y - 18 = 0$.

EXAMPLE 6 *Show that the equation of a line passing through* $(a \cos^3 \theta, a \sin^3 \theta)$ *and perpendicular to the line* $x \sec \theta + y \operatorname{cosec} \theta = a$ *is* $x \cos \theta - y \sin \theta = a \cos 2\theta$.

SOLUTION The equation of a line perpendicular to the line $x \sec \theta + y \operatorname{cosec} \theta = a$ is

$$x \operatorname{cosec} \theta - y \sec \theta + \lambda = 0, \ \lambda \text{ is a constant} \quad \text{...(i)}$$

This line passes through $(a \cos^3 \theta, a \sin^3 \theta)$.

$$\therefore \quad a \cos^3 \theta \operatorname{cosec} \theta - a \sin^3 \theta \sec \theta + \lambda = 0$$

$$\lambda = a (\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta)$$

ing the value of λ in (i), we get

$$x \operatorname{cosec} \theta - y \sec \theta + a (\sin^3 \theta \sec \theta - \cos^3 \theta \operatorname{cosec} \theta) = 0$$

$$\frac{x}{\sin \theta} - \frac{y}{\cos \theta} + a\left(\frac{\sin^3 \theta}{\cos \theta} - \frac{\cos^3 \theta}{\sin \theta}\right) = 0$$

$$x \cos \theta - y \sin \theta + a (\sin^4 \theta - \cos^4 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta + a (\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a (\cos^2 \theta - \sin^2 \theta) = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta - a \cos 2\theta = 0$$

$$\Rightarrow \quad x \cos \theta - y \sin \theta = a \cos 2\theta.$$

EXAMPLE 7 *Find the image of the point* $(-8, 12)$ *with respect to the line mirror* $4x + 7y + 13 = 0$.

SOLUTION Let the image of the point $P(-8, 12)$ in the line mirror AB be $Q(\alpha, \beta)$. Then, the line segement PQ is perpendicularly bisected at R. So, the coordinates of R are

$$\left(\frac{\alpha - 8}{2}, \frac{\beta + 12}{2}\right).$$

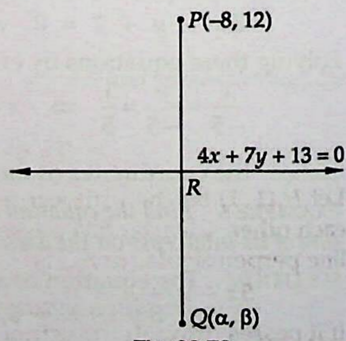

Fig. 23.70

As it lies on $4x + 7y + 13 = 0$.

$$\therefore \quad 2\alpha - 16 + \frac{7\beta + 84}{2} + 13 = 0 \Rightarrow 4\alpha + 7\beta + 78 = 0 \quad \text{...(i)}$$

The line segment PQ is perpendicular to AB.

$$\therefore \quad (\text{Slope of } AB) \times (\text{Slope of } PQ) = -1$$

$$\Rightarrow \quad -\frac{4}{7} \times \frac{\beta - 12}{\alpha + 8} = -1$$

$$\Rightarrow \quad 7\alpha - 4\beta + 104 = 0 \quad \text{...(ii)}$$

Solving (i) and (ii), we get: $\alpha = -16, \beta = -2$.

Hence, the image of $(-8, 12)$ in the line mirror $4x + 7y + 13 = 0$ is $(-16, -2)$.

LEVEL-2

EXAMPLE 8 *A person stranding at a junction (crossing) of two straight paths represented by the equations* $2x-3y-4=0$ *and* $3x-4y-5=0$, *wants to reach the path whose equation is* $6x-7y+8=0$ *in the least time. Find the equation of the path that he should follow.* **[NCERT]**

SOLUTION The lines $2x-3y-4=0$ and $3x-4y-5=0$ intersect at $(-1,-2)$. In order to reach the path, represented by the equation $6x-7y+8=0$, in the least time, the person should move along the line passing through A and perpendicular to $6x-7y+8=0$.

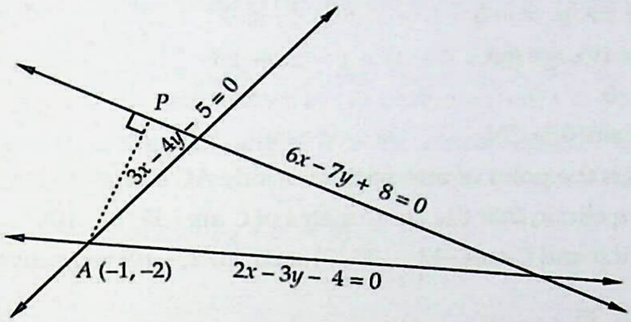

Fig. 23.71

Clearly, slope of the line $6x-7y+8=0$ is $\frac{6}{7}$. Therefore, slope of a line perpendicular to it is $-\frac{7}{6}$.

Hence, the equation of the required path is

$$y+2=-\frac{7}{6}(x+1) \text{ or, } 7x+6y+19=0$$

EXAMPLE 9 *The equations of two sides of a triangle are* $3x-2y+6=0$ *and* $4x+5y-20=0$ *and the orthocentre is* $(1,1)$. *Find the equation of the third side.* **[NCERT]**

SOLUTION Let the equations of sides AB and AC of triangle ABC be respectively

$$3x-2y+6=0 \quad \text{...(i)}$$

and, $$4x+5y-20=0 \quad \text{...(ii)}$$

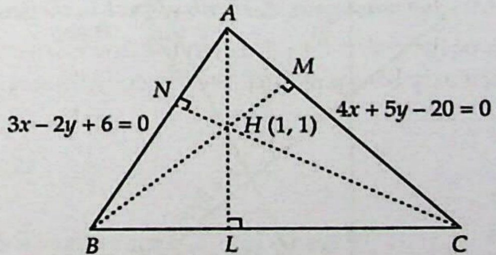

Fig. 23.72

Let $H(1,1)$ be the orthocentre of triangle ABC where the altitudes AL, BM and CN intersect each other. Clearly, BM passes through $H(1,1)$ and is perpendicular to AC. The equation of a line perpendicular to AC is

$$5x-4y+\lambda=0 \quad \text{...(iii)}$$

If it passes through the orthocentre $H(1,1)$, then

$$5-4+\lambda=0 \Rightarrow \lambda=-1$$

Substituting $\lambda=-1$ in (iii), we get

$$5x-4y-1=0 \quad \text{...(iv)}$$

This is the equation of altitude BM.

The vertex B of ΔABC is the intersection point of side AB and altitude BM. Solving their equations given by (i) and (iii), we get $x = -13$ and $y = -33/2$.

So, coordinates of B are $(-13, -33/2)$.

The altitude CN is perpendicular to AB. So, let its equation be

$$2x + 3y + \mu = 0 \qquad \text{...(v)}$$

If it passes through the orthocentre $H(1, 1)$, then

$$2 + 3 + \mu = 0 \Rightarrow \mu = -5$$

Substituting $\mu = -5$ in (v), we get

$$2x + 3y - 5 = 0 \qquad \text{...(vi)}$$

This is the equation of altitude CN.

The vertex C of ΔABC is the point of intersection of side AC and altitude CN.

Solving (ii) and (vi), we obtain that the coordinates of C are $(35/2, -10)$.

Thus, the coordinates of B and C are $(-13, -33/2)$ and $(35/2, -10)$ respectively. Hence, equation of side BC is

$$y + \frac{33}{2} = \frac{-10 + \frac{33}{2}}{\frac{35}{2} + 13}(x + 13) \text{ or, } \frac{2y + 33}{2} = \frac{13}{61}(x + 13) \text{ or, } 26x - 122y - 1675 = 0$$

EXAMPLE 10 *A straight line L is perpendicular to the line $5x - y = 1$. The area of the triangle formed by the line L and the coordinate axes is 5 square units. Find the equation of the line L.*

SOLUTION The equation of a line L perpendicular to the line $5x - y = 1$ is

$$x + 5y + \lambda = 0 \qquad \text{...(i)}$$

This line meets x-axis at $y = 0$. Putting $y = 0$, we get $x = -\lambda$. So, the line L meets x-axis at $A(-\lambda, 0)$.

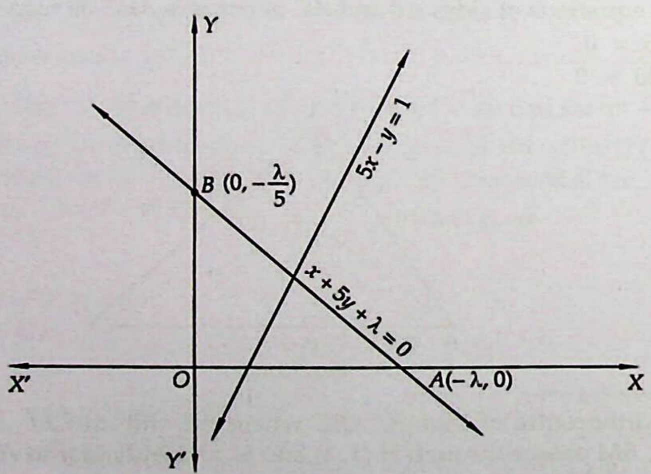

Fig. 23.73

The line L meets y-axis at $x = 0$. Putting $x = 0$ in the equation $x + 5y + \lambda = 0$, we obtain $y = -\lambda/5$.

So, the line L meets y-axis at $B(0, -\lambda/5)$.

Thus, the line L meets the coordinate axes at A and B such that $OA = -\lambda$ and $OB = -\lambda/5$.

It is given that

$$\text{Area of } \Delta OAB = 5$$

$$\Rightarrow \quad \frac{1}{2}(OA)(OB) = 5$$

$$\Rightarrow \quad \frac{1}{2}(-\lambda)\left(-\frac{\lambda}{5}\right) = 5 \Rightarrow \lambda^2 = 50 \Rightarrow \lambda = \pm 5\sqrt{2}$$

Substituting the value of λ in (i), we obtain $x + 5y \pm 5\sqrt{2} = 0$ as the equations of the required line L.

EXAMPLE 11 *Let $P(x_1, y_1)$ be a point and let $ax + by + c = 0$ be a line. If $L(h, k)$ is the foot of perpendicular drawn from P on this line and $Q(\alpha, \beta)$ is the image of P in the given line, then prove that*

(i) $\dfrac{h - x_1}{a} = \dfrac{k - y_1}{b} = -\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$ (ii) $\dfrac{\alpha - x_1}{a} = \dfrac{\beta - y_1}{b} = -2\left(\dfrac{ax_1 + by_1 + c}{a^2 + b^2}\right)$

SOLUTION Suppose PQ makes an angle θ with x-axis. Since PQ is perpendicular to $ax + by + c = 0$.

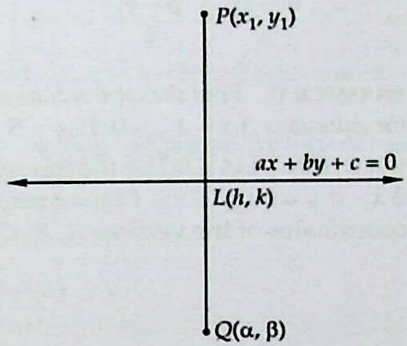

Fig. 23.74

$\therefore$ Slope of $PQ \times$ (Slope of $ax + by + c = 0$) $= -1$

$$\Rightarrow \quad \tan\theta \times -\frac{a}{b} = -1$$

$$\Rightarrow \quad \tan\theta = \frac{b}{a}$$

$$\Rightarrow \quad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}} \text{ and } \cos\theta = \frac{a}{\sqrt{a^2 + b^2}}$$

Since PQ passes through $P(x_1, y_1)$ and makes an angle θ with x-axis. Therefore, equation of PQ (in distance form) is $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta}$.

Let $PL = LQ = r$. Then, coordinates of L and Q are given by

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r \text{ and } \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = 2r \text{ respectively.}$$

$$\therefore \quad \frac{h - x_1}{\cos\theta} = \frac{k - y_1}{\sin\theta} = r \qquad \text{...(i)}$$

$$\text{and,} \quad \frac{\alpha - x_1}{\cos\theta} = \frac{\beta - y_1}{\sin\theta} = 2r \qquad \text{...(ii)}$$

Now, $\dfrac{h - x_1}{\cos\theta} = \dfrac{k - y_1}{\sin\theta} = r \Rightarrow h = x_1 + r\cos\theta,\ k = y_1 + r\sin\theta$

So, the coordinates of L are $(x_1 + r\cos\theta,\ y_1 + r\sin\theta)$.

Point L lies on $ax + by + c = 0$.

$$\therefore \quad a(x_1 + r\cos\theta) + b(y_1 + r\sin\theta) + c = 0$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{a\cos\theta + b\sin\theta}$$

$$\Rightarrow \quad r = -\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \qquad \left[\because \cos\theta = \frac{a}{\sqrt{a^2 + b^2}},\ \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}\right]$$

(i) Substituting the values of $\cos\theta$, $\sin\theta$ and r in (i), we get

$$\frac{h-x_1}{\frac{a}{\sqrt{a^2+b^2}}}=\frac{k-y_1}{\frac{b}{\sqrt{a^2+b^2}}}=-\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad \frac{h-x_1}{a}=\frac{k-y_1}{b}=-\frac{ax_1+by_1+c}{a^2+b^2}$$

(ii) Substituting the values of $\cos\theta$, $\sin\theta$ and r in (ii), we get

$$\frac{\alpha-x_1}{\frac{a}{\sqrt{a^2+b^2}}}=\frac{\beta-y_1}{\frac{b}{\sqrt{a^2+b^2}}}=-2\left(\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right)$$

$$\Rightarrow \quad \frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=-2\left(\frac{ax_1+by_1+c}{a^2+b^2}\right)$$

EXAMPLE 12 *Find the centroid, incentre circum-centre and orthocentre of the triangle whose sides have the equations* $3x-4y=0$, $12y+5x=0$ *and* $y-15=0$

SOLUTION Let ABC be the triangle whose sides BC, CA and AB have the equations $y-15=0$, $3x-4y=0$ and $5x+12y=0$ respectively. Solving these equations pair wise we can obtain the coordinates of the vertices A, B, C as $A(0, 0)$, $B(-36, 15)$ and $C(20, 15)$ respectively.

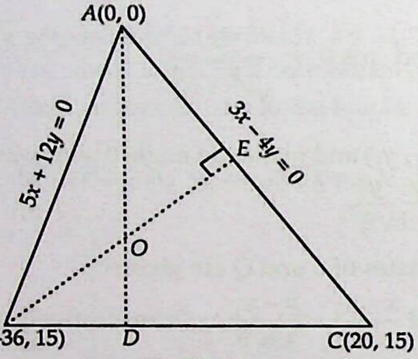

Fig. 23.75

Centroid: The coordinates of the centroid are

$$\left(\frac{0-36+20}{3},\frac{0+15+15}{3}\right)=\left(-\frac{16}{3},10\right) \qquad \left[\text{Using: }\left(\frac{x_1+x_2+x_3}{3},\frac{y_1+y_2+y_3}{3}\right)\right]$$

In centre: We have,

$$a=BC=\sqrt{(-36-20)^2+(15-15)^2}=56,\ b=CA=\sqrt{20^2+15^2}=25,$$

and, $\quad c=AB=\sqrt{(-36-0)^2+(15-0)^2}=39.$

Using $\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$ the coordinates of the incentre are

$$\left(\frac{56\times 0+25\times -36+39\times 20}{56+25+39},\frac{56\times 0+25\times 15+39\times 15}{56+25+39}\right)=(-1,8)$$

Circum-centre: Let (x, y) be the coordinates of the circum-centre O (say). Then, $OA = OB = OC$.

Now, $OA = OB$

$\Rightarrow \quad OA^2 = OB^2$

$\Rightarrow \quad x^2 + y^2 = (x+36)^2 + (y-15)^2$

$\Rightarrow \quad 72x - 30y + 1521 = 0$ ...(i)

and, $OB = OC$

$\Rightarrow \quad OB^2 = OC^2$

$\Rightarrow \quad (x+36)^2 + (y-15)^2 = (x-20)^2 + (y-15)^2$

$\Rightarrow \quad 112x + 896 = 0 \Rightarrow x = -8$ ...(ii)

Solving (i) and (ii), we get : $x = -8$ and, $y = 63/2$.

So, the coordinates of circumcentre are $(-8, 63/2)$.

Orthocentre: AD is a line passing through $A\ (0, 0)$ and perpendicular to $y - 15 = 0$. So, equation of AD is $x = 0$.

The equation of any line perpendicular to side AC having equation $3x - 4y = 0$ is $4x + 3y + \lambda = 0$. If it passes through $B(-36, 15)$, then

$$-144 + 45 + \lambda = 0 \Rightarrow \lambda = 99.$$

So, the equation of BE is $4x + 3y + 99 = 0$.

Solving the equations of AD and BE, we obtain $x = 0$ and $y = -33$.

Hence, the coordinates of the orthocentre are $(0, -33)$.

EXAMPLE 13 *Find the circumcentre of the triangle whose sides are* $3x - y + 3 = 0$, $3x + 4y + 3 = 0$ *and* $x + 3y + 11 = 0$.

SOLUTION Let ABC be the triangle whose sides AB, BC and CA have the equations $3x - y + 3 = 0$, $3x + 4y + 3 = 0$ and $x + 3y + 11 = 0$ respectively.

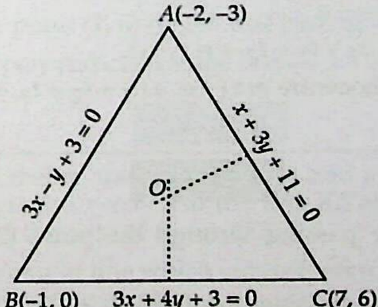

Fig. 23.76

The circumcentre of ΔABC is the point of concurrence of its perpendicular bisectors. So, let us first find the perpendicular bisector of sides BC and AC.

Solving the equations of AB and AC; BC and AB; AC and BC in pairs we obtain that the coordinates of the vertices A, B and C are $(-2, -3)$, $(-1, 0)$ and $(7, -6)$ respectively.

The equation of a line perpendicular to BC is $4x - 3y + \lambda = 0$.

This will pass through $(3, -3)$, the mid point of BC, if

$$12 + 9 + \lambda = 0 \Rightarrow \lambda = -21.$$

Putting $\lambda = -21$ in $4x - 3y + \lambda = 0$, we get

$$4x - 3y - 21 = 0. \quad \text{...(i)}$$

as the equation of the perpendicular bisector of BC.

The equation of a line perpendicular to AC is

$$3x - y + \lambda_1 = 0.$$

This will pass through $(5/2, -9/2)$ i.e. the mid point of AC, if

$$\frac{15}{2} + \frac{9}{2} + \lambda_1 = 0 \Rightarrow \lambda_1 = -12.$$

Putting $\lambda_1 = -12$ in $3x - y + \lambda_1 = 0$, we get

$$3x - y - 12 = 0. \quad \text{...(ii)}$$

as the perpendicular bisector of AC.

Solving (i) and (ii), we get: $x = 3, y = -3$.

Hence, the coordinates of the circumcentre of ΔABC are $(3, -3)$.

EXAMPLE 14 *Find the orthocentre of the triangle whose vertices are* $(at_1 t_2, a(t_1 + t_2))$, $(at_2 t_3, a(t_2 + t_3))$ *and* $(at_1 t_3, a(t_1 + t_3))$.

SOLUTION Let ABC be a triangle whose vertices are $A(at_1 t_2, a(t_1 + t_2))$, $B(at_2 t_3, a(t_2 + t_3))$ and $C(at_1 t_3, a(t_1 + t_3))$. The orthocentre of ΔABC is the point of concurrence of its altitudes. So, let us find their equations.

Clearly,

$$\text{Slope of } BC = \frac{a(t_2 + t_3) - a(t_1 + t_3)}{at_2 t_3 - at_1 t_3} = \frac{1}{t_3} \text{ and, Slope of } AC = \frac{a(t_1 + t_3) - a(t_1 + t_2)}{at_1 t_3 - at_1 t_2} = \frac{1}{t_1}.$$

The equation of the line through A perpendicular to BC i.e. the altitude through vertex A is

$$y - a(t_1 + t_2) = -t_3(x - at_1 t_2) \quad \text{...(i)}$$

The equation of the line through B perpendicular to AC i.e. the altitude through vertex B is

$$y - a(t_2 + t_3) = -t_1(x - at_2 t_3) \quad \text{...(ii)}$$

The point of intersection of (i) and (ii) is the orthocentre.

Subtracting (ii) from (i), we get $x = -a$.

Putting $x = -a$ in (i), we get $y = a(t_1 + t_2 + t_3 + t_1 t_2 t_3)$.

Hence, the coordinates of the orthocentre are $(-a, a(t_1 + t_2 + t_3 + t_1 t_2 t_3))$.

EXERCISE 23.12

LEVEL-1

1. Find the equation of a line passing through the point (2, 3) and parallel to the line $3x - 4y + 5 = 0$.
2. Find the equation of a line passing through (3, − 2) and perpendicular to the line $x - 3y + 5 = 0$
3. Find the equation of the perpendicular bisector of the line joining the points (1, 3) and (3, 1).
4. Find the equations of the altitudes of a ΔABC whose vertices are $A(1, 4)$, $B(-3, 2)$ and $C(-5, -3)$.
5. Find the equation of a line which is perpendicular to the line $\sqrt{3}x - y + 5 = 0$ and which cuts off an intercept of 4 units with the negative direction of y-axis.
6. If the image of the point (2, 1) with respect to a line mirror is (5, 2), find the equation of the mirror.
7. Find the equation of the straight line through the point (α, β) and perpendicular to the line $lx + my + n = 0$.

8. Find the equation of the straight line perpendicular to $2x - 3y = 5$ and cutting off an intercept 1 on the positive direction of the x-axis.
9. Find the equation of the straight line perpendicular to $5x - 2y = 8$ and which passes through the mid-point of the line segment joining (2, 3) and (4, 5).
10. Find the equation of the straight line which has y-intercept equal to 4/3 and is perpendicular to $3x - 4y + 11 = 0$.
11. Find the equation of the right bisector of the line segment joining the points (a, b) and (a_1, b_1).
12. Find the image of the point (2, 1) with respect to the line mirror $x + y - 5 = 0$.
13. If the image of the point (2, 1) with respect to the line mirror be (5, 2), find the equation of the mirror.
14. Find the equation to the straight line parallel to $3x - 4y + 6 = 0$ and passing through the middle point of the join of points (2, 3) and (4, – 1).
15. Prove that the lines $2x - 3y + 1 = 0$, $x + y = 3$, $2x - 3y = 2$ and $x + y = 4$ form a parallelogram.
16. Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point where it meets the y-axis. [NCERT]
17. The perpendicular from the origin to the line $y = mx + c$ meets it at the point $(-1, 2)$. Find the values of m and c. [NCERT]
18. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).
19. The line through $(h, 3)$ and (4, 1) intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.
20. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror. [NCERT]
21. Find the coordinates of the foot of the perpendicular from the point $(-1, 3)$ to the line $3x - 4y - 16 = 0$. [NCERT]
22. Find the projection of the point (1, 0) on the line joining the points $(-1, 2)$ and (5, 4).
23. Find the equation of a line perpendicular to the line $\sqrt{3}\,x - y + 5 = 0$ and at a distance of 3 units from the origin.

LEVEL-2

24. The line $2x + 3y = 12$ meets the x-axis at A and y-axis at B. The line through (5, 5) perpendicular to AB meets the x-axis and the line AB at C and E respectively. If O is the origin of coordinates, find the area of figure $OCEB$.
25. Find the equation of the straight line which cuts off intercepts on x-axis twice that on y-axis and is at a unit distance from the origin.
26. The equations of perpendicular bisectors of the sides AB and AC of a triangle ABC are $x - y + 5 = 0$ and $x + 2y = 0$ respectively. If the point A is $(1, -2)$, find the equation of the line BC.

ANSWERS

1. $3x - 4y + 6 = 0$ 2. $3x + y - 7 = 0$ 3. $y = x$
4. $2x + 5y - 22 = 0, 6x + 7y + 4 = 0, 2x + y + 13 = 0$
5. $x + \sqrt{3}\,y + 4\sqrt{3} = 0$ 6. $3x + y - 12 = 0$
7. $m(x - \alpha) = l(y - \beta)$ 8. $3x + 2y - 3 = 0$ 9. $2x + 5y - 26 = 0$ 10. $4x + 3y - 4 = 0$
11. $2x(a_1 - a) + 2y(b_1 - b) + (a^2 + b^2) - (a_1^2 + b_1^2) = 0$ 12. (4, 3)

13. $3x + y = 12$ 14. $3x - 4y = 5$ 16. $2x - 3y + 18 = 0$ 17. $m = 1/2, c = 5/2$
18. $2x + y - 5 = 0$ 19. $22/9$ 20. $(-1, -4)$ 21. $\left(\frac{68}{25}, -\frac{49}{25}\right)$
22. $\left(\frac{1}{5}, \frac{12}{5}\right)$ 23. $x + \sqrt{3}\, y \pm 6 = 0$ 24. $\frac{23}{3}$ sq. units 25. $x + 2y \pm \sqrt{5} = 0$
26. $14x + 23y - 40 = 0$ 27. $5x + 3y + 8 = 0$

HINTS TO NCERT & SELECTED PROBLEMS

16. The line $\frac{x}{4} + \frac{y}{6} = 1$ cuts y-axis at $(0, 6)$ and has slope is $-\frac{3}{2}$.

Hence, equation of the required line is $y - 6 = \frac{2}{3}(x - 0)$ or, $2x - 3y + 18 = 0$

17. Clearly, $(-1, 2)$ lies on $y = mx + c$.

$\therefore \quad 2 = -m + c$ ...(i)

The line joining the origin to $(-1, 2)$ is perpendicular to $y = mx + c$.

$\therefore \quad \frac{2-0}{-1-0} \times m = -1 \Rightarrow m = \frac{1}{2}$

Putting $m = \frac{1}{2}$ in (i), we get $c = \frac{5}{2}$.

20. Let the image of the point $P(3, 8)$ in the line mirror $x + 3y = 7$ be $Q(\alpha\ \beta)$. Then, PQ is perpendicularly bisected at M. The coordinates of M are $\left(\frac{\alpha+3}{2}, \frac{\beta+8}{2}\right)$. Since M lies on $x + 3y = 7$.

$\therefore \quad \frac{\alpha+3}{2} + 3\left(\frac{\beta+8}{2}\right) = 7$

$\Rightarrow \quad \alpha + 3\beta + 13 = 0$ ...(i)

Line segment PQ is perpendicular to $x + 3y = 7$.

$\therefore \quad \frac{\beta-8}{\alpha-3} \times -\frac{1}{3} = -1$

$\Rightarrow \quad 3\alpha - 9 = \beta - 8$

$\Rightarrow \quad 3\alpha - \beta - 1 = 0$ ...(ii)

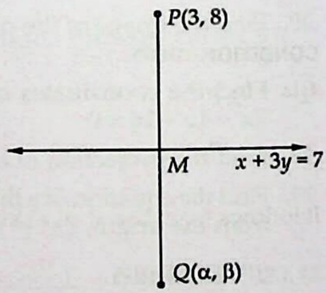

Fig. 23.77

Solving (i) and (ii), we get $\alpha = -1$ and $\beta = -4$.

Hence, the coordinates of Q are $(-1, -4)$.

21. Let $M(\alpha, \beta)$ be the foot of perpendicular from $P(-1, 3)$ on the line $3x - 4y - 16 = 0$. Then,

$\frac{\beta-3}{\alpha+1} \times \frac{3}{4} = -1$

$\Rightarrow \quad 4\alpha + 3\beta - 5 = 0$ ...(i)

Point $M(\alpha, \beta)$ lies on $3x - 4y - 16 = 0$.

$\therefore \quad 3\alpha - 4\beta - 16 = 0$ ...(ii)

Solving (i) and (ii), we get

$\alpha = \frac{68}{25}$ and $\beta = -\frac{49}{25}$.

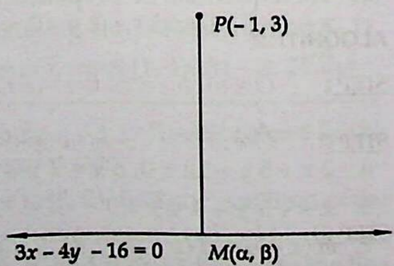

Fig. 23.78

Hence, the coordinates of M are $(68/25, -49/25)$.

23.11 ANGLE BETWEEN TWO STRAIGHT LINES WHEN THEIR EQUATIONS ARE GIVEN

THEOREM *Prove that the acute angle θ between the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ is given by*

$$\tan\theta = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|$$

PROOF Let m_1 and m_2 be the slopes of the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$. Then,

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2}$$

Now, $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$\Rightarrow \quad \tan\theta = \left|\frac{-\frac{a_1}{b_1} + \frac{a_2}{b_2}}{1 + \left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}\right| \Rightarrow \tan\theta = \left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right| \Rightarrow \theta = \tan^{-1}\left|\frac{a_2 b_1 - a_1 b_2}{a_1 a_2 + b_1 b_2}\right|$$

Q.E.D.

CONDITION FOR THE LINES TO BE PARALLEL *If the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are parallel, then*

$$m_1 = m_2 \Rightarrow -\frac{a_1}{b_1} = -\frac{a_2}{b_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

CONDITION FOR THE LINES TO BE PERPENDICULAR *If the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are perpendicular, then*

$$m_1 m_2 = -1 \Rightarrow -\frac{a_1}{b_1} \times -\frac{a_2}{b_2} = -1 \Rightarrow a_1 a_2 + b_1 b_2 = 0$$

It follows from the above discussion that the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ are:

(i) Coincident, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (ii) Parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$.

(iii) Intersecting, if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (iv) Perpendicular, if $a_1 a_2 + b_1 b_2 = 0$

To find the acute angle between two lines when their slopes are given, we may use the following algorithm.

ALGORITHM

STEP I *Obtain the equations of the lines.*

STEP II *Obtain the slopes m_1 and m_2 of two lines by using the formula: Slope $= -\frac{\text{Coeff. of } x}{\text{Coeff. of } y}$.*

STEP III *Use the formula :* $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ *to find the acute angle θ between the lines.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Find the angles between the pairs of straight lines*

(i) $x-\sqrt{3}y-5 = 0$ and $\sqrt{3}\,x+y-7=0$ (ii) $y=(2-\sqrt{3})\,x+5$ and $y=(2+\sqrt{3})\,x-7$.

SOLUTION (i) The equations of two straight lines are:

$$x-\sqrt{3}y-5=0 \quad \text{...(i)} \qquad \text{and} \qquad \sqrt{3}\,x+y-7=0 \quad \text{...(ii)}$$

Let m_1 and m_2 be the slopes of these two lines. Then,

$$m_1 = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \text{ and } m_2 = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

We observe that $m_1 m_2 = -1$. Thus, the two lines are at right angle.

(ii) Let m_1 and m_2 be the slopes of the straight lines $y=(2-\sqrt{3})\,x+5$ and $y=(2+\sqrt{3})\,x-7$ respectively. Then, $m_1 = 2-\sqrt{3}$ and $m_2 = 2+\sqrt{3}$.

Let θ be the angle between the lines. Then,

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1 m_2}\right| = \left|\frac{(2-\sqrt{3})-(2+\sqrt{3})}{1+(2-\sqrt{3})(2+\sqrt{3})}\right| = \left|-\frac{2\sqrt{3}}{1+4-3}\right| = \sqrt{3}$$

$$\Rightarrow \qquad \theta = \pi/3$$

Thus, the acute angle between the lines is of 60°.

EXAMPLE 2 *Find the tangent of the angle between the lines whose intercepts on the axes are respectively $a, -b$ and $b, -a$.*

SOLUTION The line which cuts off intercepts a and $-b$ on the coordinate axes passes through points $A\ (a, 0)$ and $B\ (0, -b)$.

$$\therefore \qquad \text{Slope of line } AB = m_1 = \frac{-b-0}{0-a} = \frac{b}{a}$$

The line which cuts off intercepts b and $-a$ on the coordinate axes passes through points $C\ (b, 0)$ and $D\ (0, -a)$.

$$\therefore \qquad \text{Slope of line } CD = m_2 = \frac{-a-0}{0-b} = \frac{a}{b}$$

Let θ be the angle between AB and CD. Then,

$$\tan\theta = \pm\frac{m_1-m_2}{1+m_1 m_2} = \pm\frac{\frac{b}{a}-\frac{a}{b}}{1+\frac{b}{a}\cdot\frac{a}{b}} = \pm\frac{b^2-a^2}{2ab}$$

EXAMPLE 3 *Find the obtuse angle between the lines $x-2y+3=0$ and $3x+y-1=0$.*

SOLUTION Let m_1 and m_2 be the slopes of the straight lines $x-2y+3=0$ and $3x+y-1=0$. Then,

$$m_1 = -\frac{1}{-2} = \frac{1}{2} \text{ and } m_2 = -\frac{3}{1} = -3.$$

Let θ be the angle between the given lines. Then,

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1 m_2}\right| = \left|\frac{\frac{1}{2}+3}{1-\frac{3}{2}}\right| = 7 \Rightarrow \theta = \tan^{-1}(7).$$

Thus, the acute angle between the lines is $\tan^{-1}(7)$ and the obtuse angle is $\pi-\tan^{-1}(7)$.