

1. Plain T_EXnology

Theorem T. *All things are not necessarily the same**

2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantities of permutations are considered with relevance to computing efficiencies.

3. The Wide-Awake example Group

We re-think*, re-word, and re-start with a set of attributes, elements or objects, $W = \{ \text{woozy, vacuous, sleepy, wide-awake} \}$. These elements are used to generate all possible arrangements η which are ordered n -tuples with $1 \leq n \leq 4$. For example, $\eta = (\text{woozy, wide-awake})$ is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like η .

Let $(\text{Woozy}, \circ, 0, -)$ be the group with the set Woozy, a binary operation \circ , a neutral element 0, and for each element $\pi \in \text{Woozy}$ there is an inverse element $-\pi \in \text{Woozy}$ such that $\pi \circ -\pi = 0$.

For now, here, we call this group's binary operation *composition*. Given two elements $\pi, \eta \in \text{Woozy}$, then $\pi \circ \eta \in \text{Woozy}$ and $\eta \circ \pi \in \text{Woozy}$.

* *T_EXbook*, *texbook.tex*, <https://www.ctan.org/tex-archive/systems/knuth/dist/tex>

* The Mathematics of the Rubiks Cube, <http://web.mit.edu/sp.268/www/rubik.pdf>

4. Creating the Woozy set

Theorem X. *An ordered set of n elements has $n!$ arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.*

Given a set of objects $W = \{a_1, a_2, \dots, a_n\}$. P_n is the set of arrangements given n objects $a_1, \dots, a_n \in W$, such as $\{(a_1, a_2, \dots, a_n), (a_2, a_1, \dots), \dots\}$. For example, with $W = \{1, 2, 3\}$, we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from $n = 3$ to $n = 4$ as follows. For each element in $P_{n-1} = P_3$, place element a_n in each possible vacuous position to arrive at $P_n = P_4$, that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), \dots, (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

* TAoCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

5. Accounting for these Arrangements

Adding up all permutations that are so generated we have p_n the number of all elements in P_n

And again, after some re-view, we sense a need to re-word. P_{nn} is the set of permuted n -tuples, and P_n is the, probably bigger, set of all the k -tuples with $k \in \{1, 2, \dots, n\}$. In other words, P_n may mean different things, or sets of things. This also applies to quantities that could be denoted like p_{nk} , and p_{nn} , and in case of our big wide-awake bean bag, which we sum up to p_n ; probably.

First, we started with $p_n = \sum_{k=1}^n k!$ to be the quantity p_n that accounts for all the elements of arrangements in set P_n , with $p_k = k!$ for $1 \leq k \leq n$.

However, on the back of some scrap paper, we jotted down $\{(1), (2), (3), (4)\}$ and saw that $\{(2), (3), (4)\}$ are not included in our sum, and $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$ has 10 2-tuples unaccounted for, etc.)

So, for now, given that $p_{nk} = n(n-1)\dots(n-k+1)^*$, combined with $p_n = \sum_{k=1}^n p_{nk}$, we count the number of arrangements of n objects to be $p_n = \sum_{k=1}^n \frac{n!}{(n-k)!}$ or some such like.

* TAOCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

6. Making concrete Space

We now look at the set W that we enumerated above and apply method 1 to arrange things.

Given W as above, we have
 $Woozy_{41} = \{ (\text{woozy}), (\text{vacuus}), (\text{sleepy}), (\text{wide-awake}) \}$

Then, taking one step at a time and applying method 1, given the set
 $Woozy_{11} = \{ (\text{sleepy}) \}$ together with another element, wide-awake $\in W$, and we get
 $Woozy_{22} = \{ (\text{wide-awake, sleepy}), (\text{sleepy, wide-awake}) \}$.

Let's start counting now. We have

$P_{21} = Woozy_{21} = \{ (\text{sleepy}) \}$, and another $P_{21} = \{ (\text{wide-awake}) \}$

$P_{22} = Woozy_{22} = \{ (\text{sleepy, wide-awake}), (\text{wide-awake, sleepy}) \}$.

To sum up we get

$p_2 = (p_{21} + p_{21}) + p_{22}$, with

$p_{21} = 1$, the count for each set of one 1-tuple, and

$p_{22} = 2$, the count for the one set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,

$p_{21} = \frac{2!}{(2-1+1)!} = 1$, and $p_{22} = \frac{2!}{(2-2+1)!} = 2$. and $p_2 = 2 * p_{21} + p_{22}$

$p_2 = \sum_{k=1}^2 \frac{2!}{(2-k+1)!}$ which has two terms and evaluates to $p_2 = \frac{2}{2} + \frac{2}{1}$, and it looks like there is something wrong here!

Let's take our result from section 5 and adjust.

$$\begin{aligned}
 p_n &= \sum_{k=1}^n (n-k+1) * \frac{n!}{(n-k+1)!} \text{ and since} \\
 (n-k+1) &= (n-k+1) * (n-k) * (n-k-1) * \dots * 1 \text{ we can simplify and have} \\
 p_n &= \sum_{k=1}^n \frac{n!}{(n-k)!} \text{ which, for our state,} \\
 &\text{yields the following sum in two terms, (given that } 0! = 1) \\
 p_2 &= \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4, \text{ which agrees with our permutations' making.}
 \end{aligned}$$

And while we are here we set a solid base by calculating the simple case for the set P_1 for which $p_1 = 1$, as we have counted just now.

$$p_n = \sum_{k=1}^n \frac{n!}{(n-k)!} \text{ with } n = 1 \text{ yields } p_1 = \frac{1!}{(1-1)!} = 1 \text{ and confirms the basic case.}$$

So, does our formula hold its stepping up. Assuming that $p_n = \sum_{k=1}^n \frac{n!}{(n-k)!}$ is correct, we need to find the terms to make p_{n+1} from that. (to be continued)

7. A few observations to count

$$p_{22} = p_{21} = 2$$

$$p_{33} = p_{32}$$

$$p_{nk} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

$1! = 0! = 1$ seems a reasonable cause, since $(n - (n-1))! = 1! = 1$ and $(n - n)! = 0! = 1$.

$p_{32} = 2p_{31}$ as $p_{31} = 3$ and $p_{32} = 6$ and it appears that is on the same ground as the previous line of reasoning; the nature of $x!$ being $x(x-1)\dots 3 * 2 * 1$, while with increasing k the positive integer sequence is a steady factor. (to be re-worded)

So, $p_{n2} = 2p_{n1}$, $p_{n3} = 3p_{n2}$,

$p_{n4} = 4p_{n3}$, or some such like, if you get my drift.

(well, maybe not)

8. The Series of Sequences

$$\left(\frac{4!}{(4-1)!} \right), \left(\frac{4!}{(4-2)!} \right), \left(\frac{4!}{(4-3)!} \right), \left(\frac{4!}{(4-4)!} \right)$$

$$\left(\frac{24}{6} \right), \left(\frac{24}{2} \right), \left(\frac{24}{1!} \right), \left(\frac{24}{0!} \right)$$

$$\{ (\text{sequence, series}), (\text{Folge, Reihe}) \}$$

$$\{ 1 \ 2 \ 3 \ 4 \}$$

$$\{ (\text{woozy}), (\text{vacuous}), (\text{sleepy}), (\text{wide-awake}) \}$$

$$\{ (1), (2), (3), (4) \}$$

$$\{ (12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43) \}$$

$$\{ (312), (132), (123), (321), (231), (213), (412), (142), (124), (421), (241), (214) \}$$

$$\{ 123 \ 321 \ 231 \ 213 \ 312 \ 132 \ 124 \ 421 \ 241 \ 214 \ 412 \ 142 \ 134 \ 431 \ 341 \ 314 \ 413 \ 143 \ 234 \ 432 \ 342 \ 324 \ 234 \ 243 \}$$

$$\{ 1234, 2341, 3412, 4123, 4321, 3214, 2143, 1432, 2134, 1342, 3421, 4213, 4312, 3124, 1243, 2431, 1324, 3241, 2413, 4132, 4231, 2314, 3142, 1423 \}$$