1. Plain TeXnology

Theorem T. *All things are not necessarily the same**

2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantities of permutations are considered with relevance to computing efficiencies.

3. The Wide-Awake example Group

We re-think*, re-word, and re-start with a set of attributes, elements or objects, $W = \{$ woozy, vacuous, sleepy, wide-awake $\}$. These elements are used to generate all possible arrangements η which are ordered n-tuples with $1 \le n \le 4$. For example, $\eta = ($ woozy, wide-awake $)$ is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like η.

Let $(\text{Woozy}, \circ, 0, -)$ be the group with the set Woozy, a binary operation $\circ$, a neutral element 0, and for each element $\pi \in$ Woozy there is an inverse element $-\pi \in$ Woozy such that $\pi \circ -\pi = 0$.

For now, here, we call this group's binary operation *composition*. Given two elements $\pi, \eta \in$ Woozy, then $\pi \circ \eta \in$ Woozy and $\eta \circ \pi \in$ Woozy.

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* *TeXbook, texbook.tex, https://www.ctan.org/tex-archive/systems/knuth/dist/tex*

* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

4. Creating the Woozy set

Theorem X. *An ordered set of n elements has $n!$ arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.*

Given a set of objects $W = \{a_1, a_2, ..., a_n\}$. P_n is the set of arrangements given n objects $a_1, ..., a_n \in W$, such as $\{(a_1, a_2, ...a_n), (a_2, a_1, ...), ...\}$. For example, with $W = \{1, 2, 3\}$, we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from $n = 3$ to $n = 4$ as follows. For each element in $P_{n-1} = P_3$, place element a_n in each possible vacuous position to arrive at $P_n = P_4$, that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), ..., (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

5. Accounting for these Arrangements

Adding up all permutations that are so generated we have p_n the number of all elements in P_n

And again, after some re-view, we sense a need to re-word. P_{nn} is the set of permuted n-tuples, and P_n is the, probably bigger, set of all the k-tuples with $k \in \{1, 2, .., n\}$.. In other words, P_n may mean different things, or sets of things. This also applies to quantities that could be denoted like p_{nk}, and p_{nn}, and in case of our big wide-awake bean bag, which we sum up to p_n; probably.

First, we started with $p_n = \sum_{k=1}^{n} k!$ to be the quantity p_n that accounts for all the elements of arrangements in set P_n, with $p_k = k!$ for $1 \leq k \leq n$.

However, on the back of some scrap paper, we jotted down $\{(1), (2), (3), (4)\}$ and saw that $\{(2), (3), (4)\}$ are not included in our sum, and $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$ has 10 2-tuples unaccounted for, etc.)

So, for now, given that $p_{nk} = n(n-1)...(n-k+1)$*, combined with $p_n = \sum_{k=1}^{n} p_{nk}$, we count the number of arrangements of n objects to be $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ or some such like.

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

6. Making concrete Space

We now look at the set W that we enumerated above and apply method 1 to arrange things.

Given W as above, we have
$\text{Woozy}_{41} = \{($ woozy $), ($ vacuous $), ($ sleepy $), ($ wide-awake $)\}$

Then, taking one step at a time and applying method 1, given the set
$\text{Woozy}_{11} = \{($ sleepy $)\}$ together with another element, wide-awake $\in W$, and we get
$\text{Woozy}_{22} = \{($ wide-awake, sleepy $), ($ sleepy, wide-awake $)\}$.

Let's start counting now. We have

$P_{21} = \text{Woozy}_{21} = \{$ (sleepy), (wide-awake) $\}$

$P_{22} = \text{Woozy}_{22} = \{$ (sleepy, wide-awake), (wide-awake, sleepy) $\}$.

To sum up we get
$p_2 = p_{21} + p_{22}$, with
$p_{21} = 2$, the count for the set of two 1-tuples, and
$p_{22} = 2$, the count for set set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,
$p_{21} = \frac{2!}{(2-1)!} = 2$, and $p_{22} = \frac{2!}{(2-2)!} = 2$. and $p_2 = p_{21} + p_{22}$
$p_2 = \sum_{k=1}^{2} \frac{2!}{(2-k)!}$ which has two terms and evaluates to $p_2 = \frac{2}{1!} + \frac{2}{0!}$, and it looks better (or is this just an illusion; however, $1! = 0! = 1$).

Let's take our result from section 5 and adjust.

$p_n = \sum_{k=1}^{n}(n-k+1) * \frac{n!}{(n-k+1)!}$ and since

$(n-k+1)! = (n-k+1) * (n-k) * (n-k-1) * ... * 1$, we may simplify and have

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ which, for our state,

yields the following sum in two terms, (given that $0! = 1$)

$p_2 = \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4$, which agrees with our permutations' making.

And while we are here we set a solid base by calculating the simple case for the set P_1 for which $p_1 = 1$, as we have counted just now.

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ with $n = 1$ yields $p_1 = \frac{1!}{(1-1)!} = 1$ and confirms the basic case.

So, does our formula hold its stepping up. Assuming that $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ is correct, we need to find the terms to make p_{n+1} from that. (to be continued)

7. A few observations to count

$p_{22} = p_{21} = 2$

$p_{33} = p_{32}$

$p_{nk} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$

$1! = 0! = 1$ seems a reasonable cause, since $(n-(n-1))! = 1! = 1$ and $(n-n)! = 0! = 1$.

$p_{32} = 2p_{31}$ as $p_{31} = 3$ and $p_{32} = 6$ and it appears that is on the same ground as the previous line of reasoning; the nature of $x!$ being $x(x-1)...3*2*1$, while with increasing k the positive integer sequence is a steady factor. (to be re-worded)

So, $p_{n2} = 2p_{n1}$, $p_{n3} = 3p_{n2}$, and $p_{n4} = 4p_{n3}$, or some such like. (Well, probably it is not a conjecture that will turn out to be true.)

8. The Series of Sequences

$$(\tfrac{4!}{(4-1)!}), \quad (\tfrac{4!}{(4-2)!}), \quad (\tfrac{4!}{(4-3)!}), \quad (\tfrac{4!}{(4-4)!})$$
$$(\tfrac{24}{6}), \quad (\tfrac{24}{2}), \quad (\tfrac{24}{1!}), \quad (\tfrac{24}{0!})$$
$$(P_{41} = 4), (P_{42} = 12), (P_{43} = 24), (P_{44} = 24)$$

{(sequence, series), (Folge, Reihe)}

{1 2 3 4}

{(woozy), (vacuous), (sleepy), (wide-awake)}

{(1), (2), (3), (4)}

{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)}

{ (312), (132), (123), (321), (231), (213), (412), (142), (124), (421), (241), (214) }

We find a 3-tuple, take its inverse, then cycle down. These are order 3 cycles. Then, we find another 3-tuple that is not yet noted, and go back to the first step, until all cycle routes have been followed.

123	321			
231	213			
312	132			
	124	421		
	241	214		
	412	142		
		134	431	
		341	314	
		413	143	
			234	432
			342	324
			234	243

We find four order three cycles and their respective inverse things (we may call, arbitrarily, the first in its row the *id* of its cycle, while the other jumbles in the row follow their leader. Maybe *chief* is a more appropriate designation than *id*.)

3-cycles

123	231	312			
321	213	132			
	124	241	412		
	421	214	142		
		134	341	413	
		431	314	143	
			234	342	423
			432	324	243

{ 1234, 2341, 3412, 4123, 4321, 3214, 2143, 1432, 2134, 1342, 3421, 4213, 4312, 3124, 1243, 2431, 1324, 3241, 2413, 4132, 4231, 2314, 3142, 1423 }

Again, we find three 4-cycles. We reverse to form their inverse. We follow the four permutation instances that are specified by these six order four canonical cycle permutations.

As an additional step, here, in order to give some visual clue as to the uniqueness of each 4-tuple, we have ordered the said permutation cycles to list the canonical cycle representation as the first element of its row. (As a permutation, each element in each of the six rows below will permute the standard permutation, say the 4-tuple 1234, to the identical resulting arrangement.)

How significant is this? We aim to find an understanding of how we may abstract this in the context of group theory. Given the 24 permutations listed, we have only six group elements. Each has four different representations. For example, the group's elements 3412 and 4123 are equal because the permutation $(1234) \circ (3412) = (2341)$ is equal to $(1234) \circ (4123) = (2341)$.

1234	2341	3412	4123		
1432	4321	3214	2143		
	1342	3421	4213	2134	
	1243	2431	4312	3124	
		1423	4231	2314	3142
		1324	3241	2413	4132

4-cycles

9. A Group of Permutations

We fill the vacuous bag with four words to make a start. Let W be the set that we made earlier, namely $\{$ woozy, vacuous, sleepy, wide-awake $\}$.

Next we make four sets of permutations P_{nk} with $k \in \{k|1 \leq k \leq n\}$,

We have P_{41}

(woozy)	
(vacuous)	
	(sleepy)
	(wide-awake)

We have P_{42}

(woozy vacuous)	(vacuous woozy)		
(woozy sleepy)	(sleepy woozy)		
(woozy wide-awake)	(wide-awake woozy)		
	(vacuous sleepy)	(sleepy vacuous)	
	(vacuous wide-awake)	(wide-awake vacuous)	
		(sleepy wide-awake)	(wide-awake sleepy)

We have P_{43}

(woozy vacuous sleepy) (vacuous sleepy woozy) (sleepy woozy vacuous)
(sleepy vacuous woozy) (vacuous woozy sleepy) (woozy sleepy vacuous)

(woozy vacuous wide-awake) (vacuous wide-awake woozy) (wide-awake woozy vacuous)
(wide-awake vacuous woozy) (vacuous woozy wide-awake) (woozy wide-awake vacuous)

(woozy sleepy wide-awake) (sleepy wide-awake woozy) (wide-awake woozy sleepy)
(wide-awake sleepy woozy) (sleepy woozy wide-awake) (woozy wide-awake sleepy)

(vacuous sleepy wide-awake) (sleepy wide-awake vacuous) (wide-awake vacuous sleepy)
(wide-awake sleepy vacuous) (sleepy vacuous wide-awake) (vacuous wide-awake sleepy)

We have P_{44} (now in progress)

(woozy vacuous sleepy wide-awake) (vacuous sleepy wide-awake woozy)
(sleepy wide-awake woozy vacuous) (wide-awake woozy vacuous sleepy)
(woozy wide-awake sleepy vacuous) (wide-awake sleepy vacuous woozy)
(sleepy vacuous woozy wide-awake) (vacuous woozy wide-awake sleepy)

(woozy sleepy wide-awake vacuous) (sleepy wide-awake vacuous woozy)
(wide-awake vacuous woozy sleepy) (vacuous woozy sleepy wide-awake)
(woozy vacuous wide-awake sleepy) (sleepy woozy vacuous wide-awake)
(wide-awake sleepy woozy vacuous) (vacuous wide-awake sleepy woozy)

(woozy wide-awake vacuous sleepy) (sleepy woozy wide-awake vacuous)
(vacuous sleepy woozy wide-awake) (wide-awake vacuous sleepy woozy)
(woozy sleepy vacuous wide-awake) (wide-awake woozy sleepy vacuous)
(vacuous wide-awake woozy sleepy) (sleepy vacuous wide-awake woozy)

10. The example 2F 2R cycle and its abstraction

Spinning the cube. Let $\pi \in S$ and $S \subseteq \mathbf{R}$ where $S = \{F, R\}$. We apply the front face-rotation twice, followed by two right-face rotations, that is $\pi = F \circ F \circ R \circ R$.

We just did that on a Rubik's cube app, and counted 6 instances until the initial permutation was reinstantiated. That indicates that π may be of order 6 (that is 24 quarter turns in total.)

After one $FFRR$ we may do 5 more π composites, that is $FFRRFFRRFFRRFFRRFFRR$, to revert the first move.

We could, of course, also have done $-\pi$, that is the inverse permutation of π (with $\pi \circ -\pi = 0$). $-\pi = -R \circ -R \circ -F \circ -F$, which with $-F = f$ and $-R = r$, may be denoted as $rrff$.

Now we will enumerate the three subsets, $F \subseteq \mathbf{R}$, $R \subseteq \mathbf{R}$, and $F \cup R \subseteq \mathbf{R}$. These subsets of permutations contribute to the abstraction of the Rubik's cube's front face F and right face R. The union of these sets $S = R \cup F$ aims to help the enumeration of permutations which we may call (not all that seriously) jumbles, ruffles, permutes, and sometimes but not always rotations or composites. These subsets together with their operation $\circ$ and their neutral element 0 (maybe called ()) form subgroups of the Rubik group $\mathbf{R}$.

The front face is represented by set F, and set R is the right face of the cube. (There is already some ambiguity creeping and ready to interfere. There are nouns like set, permutation, arrangement, and face; then we see verbs such as permute, arrange, or jumble, ruffle, or just rotate by a quarter turn; or some such like.)

Here is the front face $F = \{$ FUL FU FUR FR FDR FD FDL FL FC $\}$, and the right face $R = \{$ RUF RU RUB RB RDB RD RDF RF RC $\}$.

We also see some entanglements, namely $\{$ (FUR RUF), (FDR RDF), (FR RF) $\}$. For example, FUR is the cube of cubes on the front face in set F, while RUF is the same cube (maybe called cuby) on the right face in set R. The elements FUR and RUF are also part of the permutation (here, the action of rotating the face) F, and R respectively.

The rotation of the front face as a permutation cycle is $F =$ (FUL FUR FDR FDL) $\circ$ (FU FR FD FL), and the right face re-arrangement by a clockwise quarter-turn is $R =$ (RUF RUB RDB RDF) $\circ$ (RU RB RD RF), while the entanglements of the front and the right are already lurking and ready to intervene.

Now we look at the FFRR permutation again. This composite jumble has two front face quarter-turns followed by two 90 degree turns of the right face and it gives the permutation $F \circ F \circ R \circ R$, or in other more detailed words ((FUL FDR) (FUR FDL) (FU FD) (FR FL) $\circ$ (RUF RDB) (RUB RDF) (RU RD) (RF RB))

One way to verify this permutation (verb) is to express its effect as a two-line expression, first $F(F \circ F)$, like so

$$F(F \circ F) = \begin{pmatrix} \text{FUL FU FUR FR FDR FD FDL FL} \\ \text{FDR FD FDL FL FUL FU FUR FR} \end{pmatrix},$$

Or in a more verbose way the right face 180 degree turn might look like this:

$$R(R \circ R) = \begin{pmatrix} \text{RFU RU RUB RB RBD RD RDF RF RC} \\ \text{RFU RU RUB RB RBD RD RDF RF RC} \end{pmatrix}$$

$$\circ(\text{RFU RBD}) \circ (\text{RUB RDF}) \circ (\text{RU RD}) \circ (\text{RF RB}) \circ (\text{RC}) =$$

$$\begin{pmatrix} \text{RFU RU RUB RB RBD RD RDF RF RC} \\ \text{RBD RD RFD RF RFU RU RBU RB RC} \end{pmatrix}.$$

(to be corrected, still)

We try to visualise a particular face-to-face entanglement which could be expressed within the context of the respective edge-cubes of the front and the right face.

$$\begin{array}{ccccccc} & \text{FU} & & & & \text{RU} & \\ \text{FL} & & \textbf{FR} & & \textbf{RF} & & \text{RB} \\ & \text{FD} & & & & \text{RD} & \end{array}$$

In terms of the set of permutation cycles the entanglement may read $\{(\text{FR RF}), (\text{RF FR})\}$, and again we note that (FR RF) = (RF FR).

The corner-cubes have a similar entanglement. (re-think, re-word)

$$\begin{array}{ccccccc} \text{FLU} & & \textbf{FUR} & & \textbf{RFU} & & \text{RUB} \\ & \text{Front} & & & & \text{Right} & \\ \text{FDL} & & \textbf{FRD} & & \textbf{RDF} & & \text{RD} \end{array}$$

11. Let's look at the example RUru

The move given in the example* is (RUru) which is said to be a "commutator." It is said to affect seven cubes of which two have no part in the up face. In our abstraction we denote $\gamma \in G$ and $\gamma =$ (RUru), with $G = U \cup R$ and $G \subseteq R$.

Here, we start with the standard permutation (noun) σ with the edge cubes of the up face singled out with the expression $\sigma_U^e =$ (UB UR UF UL).

We re-vise our view of the clockwise quarter turn of the right face's edges, and therefore write $R^e =$ (RU RF RD RB). We also appreciate, now, that the right face's up-edge, RU, is the same as the up face's right edge, UR. In other words, RU = UR. (Sounds trivial, but we find this important to keep in mind, for now.)

So, just do it! That is, let's do the first component of or γ-move. Here it is, looking at two faces with two expressions. $R^2 = R \circ$ (RU RF RD RB), giving $R^2 =$ (RF RU RB RD). Now, at the same time we have $U^R = U \circ$ (UR RF), being aware that UR = RU, and noting the effect that this R-move has on the U-face which we now call U^R, and which is spelled out as (UB RF UF UL).

* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

The second component of γ being the top face quarter turn, U, and we may write $U^{RU} = R \circ U$. The, admittedly verbose expression reads $U_e^{RU} = U^R \circ U$, and then $U_e^R \circ U =$ (UB RF UF UL) $\circ\, U$ and we get $U_e^{RU} =$ (UL UB RF UF).

We also see the entanglement that this re-arrangement implies and we aim to express the effect is has on the R-face. $R_e^R \circ U^{RU}$ which shows the end effect, a ruffle of (RF UB), or some such like (for now unconfirmed.) This makes the R-face to read $R_e^{RU} = R^2 \circ U^{RU}$ and $R_e^{RU} =$ (UB RU RB RD), (still under re-view; in particular our mis-conceptions, the ambiguities, and the mis-representations are creeping all over; however we will whittle things down just to pick up each one of these crawlies and re-permute things to form a set of cohesion; given time.)

Fast forward, pending completion, we have booted the Rubik cube to do the computation γ for us, and just note it down in terms of edge jumbles,

$$\gamma_R^e = (\text{ RF RB RD BU })$$

$$\gamma_U^e = (\text{ UR FR UF UL })$$

Given the standard permutation σ with $\sigma_U^e = (\text{ UB UR UF UL })$, and $\sigma_R^e = (\text{ RU RB RD RF })$, this allows us to calculate the components in terms of permutation cycles

$$\gamma_R^e = (\text{RU RF}) \circ (\text{RB}) \circ (\text{RD}) \circ (\text{RF UL}),$$

and

$$\gamma_U^e = (\text{UB}) \circ (\text{UR FR}) \circ (\text{UF}) \circ (\text{UL}).$$

12. Direction of the permutation which cycles one face

$$\begin{array}{ccc} & 1 & \\ 4 & & 2 \\ & 3 & \end{array} \circ \text{ (clock-wise 1/4 face-turn) } \begin{array}{ccc} & 4 & \\ 3 & & 1 \\ & 2 & \end{array}$$

$$\begin{array}{ccc} & \mathrm{FU} & \\ \mathrm{FL} & & \mathrm{FR} \\ & \mathrm{FD} & \end{array} \circ(\mathrm{FU\ FL\ FD\ FR}) = \begin{array}{ccc} & \mathrm{FL} & \\ \mathrm{FD} & & \mathrm{FU} \\ & \mathrm{FR} & \end{array}$$

$$\begin{array}{ccc} & \mathrm{FU}_1 & \\ \mathrm{FL}_4 & & \mathrm{FR}_2 \\ & \mathrm{FD}_3 & \end{array} \circ(1432) = \begin{array}{ccc} & \mathrm{FL}_4 & \\ \mathrm{FD}_3 & & \mathrm{FU}_1 \\ & \mathrm{FR}_2 & \end{array}$$

$$\begin{array}{ccc} & \text{woozy}_1 & \\ \text{wide-awake}_4 & & \text{vacuous}_2 \\ & \text{sleepy}_3 & \end{array} \circ(1432) = \begin{array}{ccc} & \text{wide-awake}_4 & \\ \text{sleepy}_3 & & \text{woozy}_1 \\ & \text{vacuous}_2 & \end{array}$$

The move F again, with its effect on five faces.

$$\begin{array}{ccc} & \text{woozy}_1 & \\ \text{wide-awake}_4 & & \text{vacuous}_2 \\ & \text{sleepy}_3 & \end{array} \circ (1432) = \begin{array}{ccc} & \text{wide-awake}_4 & \\ \text{sleepy}_3 & & \text{woozy}_1 \\ & \text{vacuous}_2 & \end{array}$$

$$\begin{array}{ccc} & \mathrm{FU}_1 & \\ \mathrm{FL}_4 & & \mathrm{FR}_2 \\ & \mathrm{FD}_3 & \end{array} \circ F = \begin{array}{ccc} & \mathrm{FL}_4 & \\ \mathrm{FD}_3 & & \mathrm{FU}_1 \\ & \mathrm{FR}_2 & \end{array}$$

Now, we look at some TeXnicalities* in order to show these ruffles. We want to utilise the \halign control sequence to layout our cube-faces. We also want to use TeX definitions to ease the writing of this TeX-text. We aim for some yield of visual clues as to the Rubik group model for these permutations. (re-word)

Here is the basic idea. We draw, using TeX text, the four edge cubes: Up U_1, Right R_2, Down D_3, and Left L_4.

$$\begin{array}{ccc} & U_1 & \\ L_4 & & R_2 \\ & D_3 & \end{array}$$

Then with the help of TeX and its \halign control sequence, we abstract the drawing of this particular face.

$$\begin{array}{ccc} & U_1 & \\ L_4 & & R_2 \\ & D_3 & \end{array}$$

* TeXbook, texbook.tex, https://www.ctan.org/tex-archive/systems/knuth/dist/tex

Now we parameterise the four edge cubes that we see on this face.

$$\begin{array}{ccc} & U_1 & \\ L_4 & & R_2 \\ & D_3 & \end{array}$$

This allows us, now, to facilitate these abstraction and apply things to our favourite set.

$$\begin{array}{ccc} & \text{woozy}_1 & \\ \text{wide-awake}_4 & & \text{vacuous}_2 \\ & \text{sleepy}_3 & \end{array}$$

And we may even show a plain and clear view of these face rotations, in our own perception, by using the set $\{1, 2, 3, 4\}$ as the Group's set.

$$\begin{array}{ccc} & 1_1 & \\ 4_4 & & 2_2 \\ & 3_3 & \end{array}$$

Let's see a representation of our adjoining faces. These are the four faces that adjoin this one. This one is face $(1, 2, 3, 4)$. We say that the face in front of us is face 0 and substitution yields $(01, 02, 03, 04)$.

$$\begin{array}{ccc} & 01_1 & \\ 05_4 & & 03_2 \\ & 04_3 & \end{array}$$

Our edge cube 01 is adjoined to face 1. (language) and face one sees its edge cube 10 to be adjoined to us. Yes! we are face 0.

$$\begin{array}{ccc} & 12_1 & \\ 15_4 & & 13_2 \\ & 10_3 & \end{array}$$

And here we take a break to reflect, and to replenish our writing finger non-fossil, no-no-nothing fuel.

Another approach to get these faces to show themselves side by side.

$$\begin{array}{ccc} & 01_1 & \\ 05_4 & & 03_2 \\ & 04_3 & \end{array}$$

Now, then! we backup a few steps and just move straight ahead.

$$\begin{array}{ccc} & B_1 & \\ L_4 & & R_2 \\ & F_3 & \end{array}$$

$$\begin{array}{ccc} & U_1 & \\ L_4 & & R_2 \\ & D_3 & \end{array} \qquad \begin{array}{ccc} & U_1 & \\ F_4 & & B_2 \\ & D_3 & \end{array} \qquad \begin{array}{ccc} & U_1 & \\ R_4 & & L_2 \\ & D_3 & \end{array} \qquad \begin{array}{ccc} & U_1 & \\ B_4 & & F_2 \\ & D_3 & \end{array}$$

$$\begin{array}{ccc} & F_1 & \\ L_4 & & R_2 \\ & B_3 & \end{array}$$

We show the edges as number pairs where the first digit of each two-digit number gives the cube's face visible on this cube of cubes' face. The second such digit gives the adjoining face of the standard permutation σ, where the faces are: 0 to represent the front face, 1 its back, 2 to the left, 3 on the right, 4 faces up, and 5 looks down.

We break down σ and show just its edges, with
$\sigma^e_F = (02, 05, 03, 04)^e_F$, $\sigma^e_B = (12, 14, 13, 15)^e_B$,
$\sigma^e_U = (21, 25, 20, 23)^e_U$, $\sigma^e_D = (30, 35, 31, 32)^e_D$,
$\sigma^e_L = (42, 40, 43, 41)^e_L$, and $\sigma^e_R = (52, 51, 53, 50)^e_R$. *(this is unchecked and unfinished)*

$$
\begin{array}{cccccccccccc}
 & 21_1 & & & & & & & & & & \\
23_4 & & 25_2 & & & & & & & & & \\
 & 20_3 & & & & & & & & & & \\
 & 02_1 & & & 52_1 & & & 12_1 & & & 42_1 & \\
04_4 & & 05_2 & 50_4 & & 51_2 & 15_4 & & 14_2 & 41_4 & & 40_2 \\
 & 03_3 & & & 53_3 & & & 13_3 & & & 43_3 & \\
 & 30_1 & & & & & & & & & & \\
32_4 & & 35_2 & & & & & & & & & \\
 & 31_3 & & & & & & & & & &
\end{array}
$$

How can we see that this resembles some kind of reality? First, since this is the standard permutation we have all the edge cubes show their face in one colour, the colour of the front face being 0, for example.

Second, one edge joins two faces and therefore, for example the front face 0 and the right face 5 have a shared edge-cube. This shared edge appears on the front-face on the right *front*$_2$, here denoted as 05_2. On the adjoining face, the right one in our example, this shared edge is written as 50_4 being on the left, *right*$_4$, of the right Rubik's cube face.

Third, we spot an error, as we experience just how to go about error spotting. The left face (4) has in its *down* position (*left*$_3$) a cube 43_3; and this feels correct.

But which edge is shared by the down cube (3) and the left cube (4)? Answer: there should be a cube $34_?$ on the *down* face. We see that 30_1 joins with *front* (0) correctly, and we deduce that 34_4 must replace 32_4.

Again, with one correction to read 34_4.

$$
\begin{array}{cccccccccccc}
 & 21_1 & & & & & & & & & & \\
23_4 & & 25_2 & & & & & & & & & \\
 & 20_3 & & & & & & & & & & \\
 & 02_1 & & & 52_1 & & & 12_1 & & & 42_1 & \\
04_4 & & 05_2 & 50_4 & & 51_2 & 15_4 & & 14_2 & 41_4 & & 40_2 \\
 & 03_3 & & & 53_3 & & & 13_3 & & & 43_3 & \\
 & 30_1 & & & & & & & & & & \\
34_4 & & 35_2 & & & & & & & & & \\
 & 31_3 & & & & & & & & & &
\end{array}
$$

Thanks, yes we see another error and we substitute 24_4 for the erroneous 23_4. This error hints at another tell-tale sign. Each face should have exactly two even and two odd edges, as a consequence of how our faces are numbered. *(This applies to the standard permutation σ^e and of course, after a jumble we might see flipped edges.)*

$$
\begin{array}{cccccccccccc}
 & 21_1 & & & & & & & & & & \\
24_4 & & 25_2 & & & & & & & & & \\
 & 20_3 & & & & & & & & & & \\
 & 02_1 & & & 52_1 & & & 12_1 & & & 42_1 & \\
04_4 & & 05_2 & 50_4 & & 51_2 & 15_4 & & 14_2 & 41_4 & & 40_2 \\
 & 03_3 & & & 53_3 & & & 13_3 & & & 43_3 & \\
 & 30_1 & & & & & & & & & & \\
34_4 & & 35_2 & & & & & & & & & \\
 & 31_3 & & & & & & & & & &
\end{array}
$$

13. The concrete move RUru

We start with the representation of the standard permutation σ^e in the context of edge cubelets.

$$\begin{array}{ccc} & 21_1 & \\ 24_4 & & 25_2 \\ & 20_3 & \end{array}$$

$$\begin{array}{ccc} & 02_1 & \\ 04_4 & & 05_2 \\ & 03_3 & \end{array} \quad \begin{array}{ccc} & 52_1 & \\ 50_4 & & 51_2 \\ & 53_3 & \end{array} \quad \begin{array}{ccc} & 12_1 & \\ 15_4 & & 14_2 \\ & 13_3 & \end{array} \quad \begin{array}{ccc} & 42_1 & \\ 41_4 & & 40_2 \\ & 43_3 & \end{array}$$

$$\begin{array}{ccc} & 30_1 & \\ 34_4 & & 35_2 \\ & 31_3 & \end{array}$$

And we re-write in a somewhat narrative style (as we perceive it so to be.)

$\sigma_F^e = (02, 05, 03, 04)_F^e$, $\sigma_B^e = (12, 14, 13, 15)_B^e$,

$\sigma_U^e = (21, 25, 20, 24)_U^e$, $\sigma_D^e = (30, 35, 31, 34)_D^e$,

$\sigma_L^e = (42, 40, 43, 41)_L^e$, and $\sigma_R^e = (52, 51, 53, 50)_R^e$.

Now we apply the move R, which spells out to be $(1432)^e_R$, if you know what this means. We perceive the meaning to be $R_1 \to R_4$, $R_2 \to R_1$, $R_3 \to R_2$, and $R_4 \to R_3$, and we think this to be a clockwise 1/4 turn around the x axis, that is the axis from left to right. *(We admit that the concept of direction such as left and right is a woozy one; but it's not vacuous.)*

$$
\begin{array}{cccccccccccc}
 & 21_1 & & & & & & & & & & \\
24_4 & & 05_2 & & & & & & & & & \\
 & 20_3 & & & & & & & & & & \\
 & 02_1 & & & 50_1 & & & 12_1 & & & 42_1 & \\
04_4 & & 35_2 & 53_4 & & 52_2 & 25_4 & & 14_2 & 41_4 & & 40_2 \\
 & 03_3 & & & 51_3 & & & 13_3 & & & 43_3 & \\
 & 30_1 & & & & & & & & & & \\
34_4 & & 15_2 & & & & & & & & & \\
 & 31_3 & & & & & & & & & &
\end{array}
$$

We note that with the move of the right facelets we also see changes on the adjoining faces, namely F_2 sees $05 \to 35$, B_4 moves $15 \to 25$, U_2 substitutes $25 \to 05$, and finally D_2 transposes its edge $35 \to 15$. (to verify)

Whittling things down, we see things scattered around us, such as

{(set, coset), (Definitionsmenge, Wertemenge)},
{ injective, surjective, bijective }.

We observe the effects of this rotation, R^e as it produces five permutation cycles of order four, namely

$R^e_R = (50, 53, 51, 52)^e_R$
$R^e_F = (05, 35, 15, 25)^e_F$
$R^e_U = (15, 25, 05, 35)^e_U$
$R^e_B = (25, 05, 35, 15)^e_B$
$R^e_D = (35, 15, 25, 05)^e_R$

From a different view we see how the two-faced facelet 05 will cycle through its positions in five different faces.

$05^e_F = F^e_2$
$05^e_U = U^e_2$
$05^e_B = B^e_4$
$05^e_D = D^e_2$
$50^e_R = (R^1_e, R^4_e, R^3_e, R^2_e)$.

(We also appreciate, from our position, how the abstractions may stay just abstract until we actually follow things through, and maybe even consider their views in their own surroundings.)

In summary, we have
$05^e = (F^e_2, D^e_2, B^e_4, U^e_2)^e$ and
$50^e_R = (R^1_e, R^4_e, R^3_e, R^2_e)$.

(Just now we made a correction, here, $B^e_2 \rightarrow B^e_4$; some rationale, that may give a tell-tale sign of why this is the way it must be. B and F are opposing faces; We are looking at a R permuter. The edge that is shared between the front in position F^e_2 must be, once rotated by the orthagonal face R, in the opposite position on the back face B^e_4.)

14. The group of three slices

So called slices are sandwiched between opposing faces. The group, so generated from the set of moves { 2R2l, 2F2b, 2U2d } is said to have 12 elements*.

Maybe we need a reminder of the group abstraction, and maybe we should convince ourselves that this is a group, and maybe we should account for all the elements, the completeness, the associativities of the compositions, the identity element, and the existence of inverse elements.

First, we count the basic generating elements: 1. $FFbb$, 2. $RRll$, 3. $UUdd$.
Next, we compose with two basics: 4. $FFbb \circ RRll$, 5. $FFbb \circ UUdd$, 6. $RRll \circ UUdd$
Last come compositions from three basic generating elements:
7. $(FFbb \circ RRll) \circ UUdd$, 8. $(RRll \circ FFbb) \circ UUdd$, 9. $(FFbb \circ UUdd) \circ RRll$,
10.$(UUdd \circ FFbb) \circ RRll$, 11.$(RRll \circ UUdd) \circ FFbb$, 12.$(UUdd \circ RRll) \circ FFbb$.

(This is the first attempt of enumerating this group and the result is unverified.)

We have, just now, taken some pictures of this group and explored what we want to take forward. We will look at a subgroup of the three slices subgroup of the Rubik group. We will just consider the centre-cubelets, their permutation cycles, including their orientation. We will also try to find the different aspects, or view points that will help us to understand what happens with this group, how we denote things, and how this simple example is possibly abstracted.

We already have seen different view points of the cube, such as the view from the outside, and the view from a particular edge-cubelet. In the case of the S^+ group of three slices with only centre-cubelets we at-

* https://books.google.co.uk/books?id=%5fn1vr0%5fRbXoC&lpg=PA786&pg=PA786

tempt to describe the twelve elements, as given above, from a view of a corner cube, FUR that is not a part of the underlying set of cubelets (only centre facelets, now), of our observations.

We are observing, sitting on the FUR corner, as the elementary permuters effect the rotational moves, in 180 degree quantities. Sitting where we are sitting, we see the slices rotate (while in effect we, that is the FUR= $025 = 250 = 520$ face that we sit on, will rotate in pairs with our opposite face.)

Let's spell out, from an outside view, the move S_F^+, with $S_F^+ = (FFbb)$.

We start with the representation of the standard permutation σ^+ in the context of centre cubelets and the observer's, our cubelet FUR. Yes, we add the orientation of the centre facelet, its spin, as a sub-prefix, one of $\{0, 1, 2, 3\}$ for 0 degrees, 90 degrees, 180 degrees, or 270 degrees, or $0, 1/4, 1/2$, and $3/4$ facelet turns.

σ^+_{025}, at the start is

1 2
$_0 2^+$
4 $_3 250$
1 $_2 025$ $_1 502$ 2 1 2 1 2
$_0 0^+$ $_0 5^+$ $_0 1^+$ $_0 4^+$
4 3 4 3 4 3 4 3
1 2
$_0 3^+$
4 3

$\sigma^{+}_{025} \circ F^{+}$, and the centre facelet of our front face spins by a 1/4-turn, and we, the observer, spin likewise

1 2
$_{0}2^{+}$
4 3
1 2 1 $_{2}250$ 1 2 1 2
$_{1}0^{+}$ $_{0}5^{+}$ $_{0}1^{+}$ $_{0}4^{+}$
4 $_{3}025$ 4 3 4 3 4 3
1 $_{2}502$
$_{0}3^{+}$
4 3

And again we apply F^{+},

1 2
$_{0}2^{+}$
4 3
1 2 1 2 1 2 1 2
$_{2}0^{+}$ $_{0}5^{+}$ $_{0}1^{+}$ $_{0}4^{+}$
$_{4}025$ 3 4 3 4 3 4 $_{3}250$
$_{1}502$ 2
$_{0}3^{+}$
4 3

(The observer feels woozy—are we lost?)

Now an inverse B face turn, in our group this is denoted b^+, and note the anti-clockwise spin of centre-facelet from ${}_0 1^+ \rightarrow {}_3 1^+$

1 2
${}_0 2^+$
4 3
1 2 1 2 1 2 1 2
${}_2 0^+$ ${}_0 5^+$ ${}_3 1^+$ ${}_0 4^+$
${}_4 025$ 3 4 3 4 3 4 ${}_3 250$
${}_1 502$ 2
${}_0 3^+$
4 3
(The observer is down, left in front—resting vacuously.)

And another b^+ 1/4 anti-clock-wise turn with its respective spin of facelet ${}_3 1^+ \rightarrow {}_2 1^+$

1 2
${}_0 2^+$
4 3
1 2 1 2 1 2 1 2
${}_2 0^+$ ${}_0 5^+$ ${}_2 1^+$ ${}_0 4^+$
${}_4 025$ 3 4 3 4 3 4 ${}_3 250$
${}_1 502$ 2
${}_0 3^+$
4 3

(The observer feels like having been taken for a ride—wooziness prevails)

15. Taking a more visual approach to these cubelets

A visual approach to the mathematics of the Rubik cube has been started.*The *Persistence of Vision Ray Tracer* allows to specify objects, such as the cubelets, and the cube in a programming language with an appropriate vocabulary. A fundamental connection between the mathematics of the cube and of its visual model is achieved with the help of the application *POV-ray*.

This is a very basic use of the *POV-ray language*. We define a square polygon for each of our six faces. We have 0, 1 for front and back; 2, 3 are up and down; and 4 and 5 face left and right. In other words these faces are F, B, U, D,L, and R.

Then the six facelets are cubed together and a particle, a cublete is constructed. With these particles, or cubelets, we assemble a $3x3x3$ cube that represents and allows us to model the Rubik cube's permutation groups.

We have tried two approaches to modify the rotation of one of the Rubik cube's faces. First, the obvious route is to define a graphic representation, in POV-ray language, of the front face, the back face, together with the middle slice. Given this model we can apply the F-permutation by simply rotating the F face of the cube.

A different approach, which may be more practical, is to rotate each respective cubelet of the front face to achieve the same desired result, which is to rotate the front face of the Rubik cube.

In terms of possible permutations, this consideration shows that it is obvious that we have many more possible arrangements; for example, we may now have more than one edge cube of the same colours. This is, of course, not possible with the real Rubik cube.

It also opens up more views onto the mathematical models of the cube. For example, we could devise a metric of the distance between different cube arrangements. (re-think, re-word, re-write)

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* https://github.com/the-number/R/blob/explore/0001/gnubik/explore/a/p/6/in-pictures.org

16. Persistence of Vision and each cubelets' orientation

We start with the face. The cube has six faces. The cube is made up of $3x3x3$ smaller cubes, little cubes, cubelets. In the approach we take at the moment, each little cube, each cubelet, has also six faces. (However, the real Rubik cube is not like that: there edge cubelets have two small faces, facelets we say, corners show three, and each of the six faces' centre cublets has just one face, its centre cubelet's facelet)

We start with a set $A = \{0, 1, 2, 3, 4, 5\}$ as we continue to abstract things within the concepts of the set and the group.

We have 3 subsets $B_0, B_2, B_4 \subset A$, with $B_0 = \{0, 1\}$, $B_2 = \{2, 3\}$, and $B_4 = \{4, 5\}$. These subsetes model our opposing faces and facelets such as the front is opposite the back, up is opposite down, and left opposes right.

We have another three subsets $C_0 = \{3, 4, 5, 6\} = C_1$, $C_2 = \{0, 1, 4, 5\} = C_3$, and $C_4 = \{0, 1, 2, 3\} = C_5$. These are ajacent faces and facelets.

There is a relation between these sets, $B_n = A \backslash C_n$ and $C_n = A \backslash B_n$. Also we have $B_n \cup C_n = A$. We also have $B_k = B_{k+1}$ for $k \in \{0, 2, 4\}$, and furthermore this holds for the sets of adjacent faces where $C_k = C_{k+1}$ for $k \in \{0, 2, 4\}$.

Now we can enumerate the set of adjacent faces. Given $k \in \{0, 2, 4\}$, we can let $b \in B_k$ and $c \in C_k$, and then enumerte the adjacent pairs (b, c) that are the elements of our group $(A, \circ)$. (*It is unclear, at this stage of considerations, that we have a group here.)*

We enumerate $B_0 \circ C_0$, with $B = \{0, 1\}$ and $C = \{2, 3, 4, 5\}$ to have $\{02, 03, 04, 05\} \cup \{12, 13, 14, 15\}$. Similarly we have $B_2 \circ C_2 \cup B_3 \circ C_3$ to be $\{20, 21, 24, 25, 30, 31, 34, 35\}$, and finally $\{40, 41, 42, 43, 50, 51, 52, 53\}$.

We may also consider how to measure the distance between such arrangements. We assign a metric of 2 to (b_k, b_{k+1}), that is the permutation between opposite faces. Conversely, and for example, with $k = 0$ we have a corresponding permuter element of (c_2, c_4, c_3, c_5) with a 1 unit distance between one arrangement and its successor.

The opposing cycles are of order 2, while the adjacent cycles are of order 4. Considering the parity of such permutation cycles and given that $(c_2, c_4, c_3, c_5) = (c_2, c_4) \circ (c_2, c_3) \circ (c_2, c_5)$ the parity of the adjacent cycles turns out to be $3 \pmod 2$, "odd" in other words.