

## 1. Plain T<sub>E</sub>Xnology

**Theorem T.** *All things are not necessarily the same\**

## 2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantities of permutations are considered with relevance to computing efficiencies.

## 3. The Wide-Awake example Group

We re-think\*, re-word, and re-start with a set of attributes, elements or objects,  $W = \{ \text{woozy, vacuous, sleepy, wide-awake} \}$ . These elements are used to generate all possible arrangements  $\eta$  which are ordered  $n$ -tuples with  $1 \leq n \leq 4$ . For example,  $\eta = ( \text{woozy, wide-awake} )$  is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like  $\eta$ .

Let  $(\text{Woozy}, \circ, 0, -)$  be the group with the set Woozy, a binary operation  $\circ$ , a neutral element 0, and for each element  $\pi \in \text{Woozy}$  there is an inverse element  $-\pi \in \text{Woozy}$  such that  $\pi \circ -\pi = 0$ .

For now, here, we call this group's binary operation *composition*. Given two elements  $\pi, \eta \in \text{Woozy}$ , then  $\pi \circ \eta \in \text{Woozy}$  and  $\eta \circ \pi \in \text{Woozy}$ .

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\* *T<sub>E</sub>Xbook*, *texbook.tex*, <https://www.ctan.org/tex-archive/systems/knuth/dist/tex>

\* The Mathematics of the Rubiks Cube, <http://web.mit.edu/sp.268/www/rubik.pdf>

#### 4. Creating the Woozy set

**Theorem X.** *An ordered set of  $n$  elements has  $n!$  arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.\*

Given a set of objects  $W = \{a_1, a_2, \dots, a_n\}$ .  $P_n$  is the set of arrangements given  $n$  objects  $a_1, \dots, a_n \in W$ , such as  $\{(a_1, a_2, \dots, a_n), (a_2, a_1, \dots), \dots\}$ . For example, with  $W = \{1, 2, 3\}$ , we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from  $n = 3$  to  $n = 4$  as follows. For each element in  $P_{n-1} = P_3$ , place element  $a_n$  in each possible vacuous position to arrive at  $P_n = P_4$ , that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), \dots, (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

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\* TAOCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

## 5. Accounting for these Arrangements

Adding up all permutations that are so generated we have  $p_n$  the number of all elements in  $P_n$

And again, after some re-view, we sense a need to re-word.  $P_{nn}$  is the set of permuted  $n$ -tuples, and  $P_n$  is the, probably bigger, set of all the  $k$ -tuples with  $k \in \{1, 2, \dots, n\}$ . In other words,  $P_n$  may mean different things, or sets of things. This also applies to quantities that could be denoted like  $p_{nk}$ , and  $p_{nn}$ , and in case of our big wide-awake bean bag, which we sum up to  $p_n$ ; probably.

First, we started with  $p_n = \sum_{k=1}^n k!$  to be the quantity  $p_n$  that accounts for all the elements of arrangements in set  $P_n$ , with  $p_k = k!$  for  $1 \leq k \leq n$ .

However, on the back of some scrap paper, we jotted down  $\{(1), (2), (3), (4)\}$  and saw that  $\{(2), (3), (4)\}$  are not included in our sum, and  $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$  has 10 2-tuples unaccounted for, etc.)

So, for now, given that  $p_{nk} = n(n-1)\dots(n-k+1)^*$ , combined with  $p_n = \sum_{k=1}^n p_{nk}$ , we count the number of arrangements of  $n$  objects to be  $p_n = \sum_{k=1}^n \frac{n!}{(n-k)!}$  or some such like.

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\* TAOCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

## 6. Making concrete Space

We now look at the set  $W$  that we enumerated above and apply method 1 to arrange things.

Given  $W$  as above, we have

$$\text{Woozy}_{41} = \{(\text{ woozy }), (\text{ vacuus }), (\text{ sleepy }), (\text{ wide-awake })\}$$

Then, taking one step at a time and applying method 1, given the set

$\text{Woozy}_{11} = \{(\text{ sleepy })\}$  together with another element,  $\text{wide-awake} \in W$ , and we get

$$\text{Woozy}_{22} = \{(\text{ wide-awake, sleepy }), (\text{ sleepy, wide-awake })\}.$$