1. Plain TEXnology

Theorem T. *All things are not necessarily the same**

2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantites of permutations are considered with relevance to computing efficiencies.

3. The Wide-Awake example Group

We re-think*, re-word, and re-start with a set of attributes, elements or objects, W = { woozy, vacuous, sleepy, wide-awake }. These elements are used to generate all possible arrangements η which are orderd n-tuples with $1 \le n \le 4$. For example, η = (woozy, wide-awake) is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like η.

Let $(\mathrm{Woozy}, \circ, 0, -)$ be the group with the set Woozy, a binary operation $\circ$, a neutral elment 0, and for each element $\pi \in \mathrm{Woozy}$ there is an inverse element $-\pi \in \mathrm{Woozy}$ such that $\pi \circ -\pi = 0$.

For now, here, we call this group's binary operation *composition*. Given two elements $\pi, \eta \in \mathrm{Woozy}$, then $\pi \circ \eta \in \mathrm{Woozy}$ and $\eta \circ \pi \in \mathrm{Woozy}$.

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* *TEXbook, texbook.tex, https://www.ctan.org/tex-archive/systems/knuth/dist/tex*
* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

4. Creating the Woozy set

Theorem X. *An ordered set of n elements has $n!$ arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.*

Given a set of objects $W = \{a_1, a_2, ..., a_n\}$. P_n is the set of arrangements given n objects $a_1, ..., a_n \in W$, such as $\{(a_1, a_2, ...a_n), (a_2, a_1, ...), ...\}$. For example, with $W = \{1, 2, 3\}$, we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from $n = 3$ to $n = 4$ as follows. For each element in $P_{n-1} = P_3$, place element a_n in each possible vacuous position to arrive at $P_n = P_4$, that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), ..., (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

5. Accounting for these Arrangements

Adding up all permutations that are so generated we have p_n the number of all elements in P_n

And again, after some re-view, we sense a need to re-word. P_{nn} is the set of permuted n-tuples, and P_n is the, probably bigger, set of all the k-tuples with $k \in \{1, 2, .., n\}$.. In other words, P_n may mean different things, or sets of things. This also applies to quantities that could be denoted like p_{nk}, and p_{nn}, and in case of our big wide-awake bean bag, which we sum up to p_n; probably.

First, we started with $p_n = \sum_{k=1}^{n} k!$ to be the quantity p_n that accounts for all the elements of arrangements in set P_n, with $p_k = k!$ for $1 \leq k \leq n$.

However, on the back of some scrap paper, we jotted down $\{(1), (2), (3), (4)\}$ and saw that $\{(2), (3), (4)\}$ are not included in our sum, and $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$ has 10 2-tuples unaccounted for, etc.)

So, for now, given that $p_{nk} = n(n-1)...(n-k+1)$*, combined with $p_n = \sum_{k=1}^{n} p_{nk}$, we count the number of arrangements of n objects to be $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ or some such like.

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

6. Making concrete Space

We now look at the set W that we enumerated above and apply method 1 to arrange things.

Given W as above, we have
$\text{Woozy}_{41} = \{($ woozy $), ($ vacuus $), ($ sleepy $), ($ wide-awake $)\}$

Then, taking one step at a time and applying method 1, given the set
$\text{Woozy}_{11} = \{($ sleepy $)\}$ together with another element, wide-awake $\in W$, and we get
$\text{Woozy}_{22} = \{($ wide-awake, sleepy $), ($ sleepy, wide-awake $)\}$.

Let's start counting now. We have

$P_{21} = \text{Woozy}_{21} = \{$ (sleepy), (wide-awake) $\}$

$P_{22} = \text{Woozy}_{22} = \{$ (sleepy, wide-awake), (wide-awake, sleepy) $\}$.

To sum up we get
$p_2 = p_{21} + p_{22}$, with
$p_{21} = 2$, the count for the set of two 1-tuples, and
$p_{22} = 2$, the count for set set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,
$p_{21} = \frac{2!}{(2-1)!} = 2$, and $p_{22} = \frac{2!}{(2-2)!} = 2$. and $p_2 = p_{21} + p_{22}$
$p_2 = \sum_{k=1}^{2} \frac{2!}{(2-k)!}$ which has two terms and evaluates to $p_2 = \frac{2}{1!} + \frac{2}{0!}$, and it looks better (or is this just an illusion; however, $1! = 0! = 1$).

Let's take our result from section 5 and adjust.

$p_n = \sum_{k=1}^{n} (n-k+1) * \frac{n!}{(n-k+1)!}$ and since

$(n-k+1) = (n-k+1) * (n-k) * (n-k-1) * \ldots * 1$ we can simplify and have

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ which, for our state,

yields the following sum in two terms, (given that $0! = 1$)

$p_2 = \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4$, which agrees with our permutations' making.

And while we are here we set a solid base by calculating the simple case for the set P_1 for which $p_1 = 1$, as we have counted just now.

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ with $n = 1$ yields $p_1 = \frac{1!}{(1-1)!} = 1$ and confirms the basic case.

So, does our formula hold its stepping up. Assuming that $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ is correct, we need to find the terms to make p_{n+1} from that. (to be continued)

7. A few observations to count

$p_{22} = p_{21} = 2$

$p_{33} = p_{32}$

$p_{nk} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$

$1! = 0! = 1$ seems a reasonable cause, since $(n-(n-1))! = 1! = 1$ and $(n-n)! = 0! = 1$.

$p_{32} = 2p_{31}$ as $p_{31} = 3$ and $p_{32} = 6$ and it appears that is on the same ground as the previous line of reasoning; the nature of $x!$ being $x(x-1)...3 * 2 * 1$, while with increasing k the positive integer sequence is a steady factor. (to be re-worded)

So, $p_{n2} = 2p_{n1}$, $p_{n3} = 3p_{n2}$, and $p_{n4} = 4p_{n3}$, or some such like. (Well, probably it is not a conjecture that will turn out to be true.)

8. The Series of Sequences

$$(\tfrac{4!}{(4-1)!}), \quad (\tfrac{4!}{(4-2)!}), \quad (\tfrac{4!}{(4-3)!}), \quad (\tfrac{4!}{(4-4)!})$$
$$(\tfrac{24}{6}), \quad (\tfrac{24}{2}), \quad (\tfrac{24}{1!}), \quad (\tfrac{24}{0!})$$
$$(P_{41} = 4), (P_{42} = 12), (P_{43} = 24), (P_{44} = 24)$$

{(sequence, series), (Folge, Reihe)}

{1 2 3 4}

{(woozy), (vacuous), (sleepy), (wide-awake)}

{(1), (2), (3), (4)}

{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)}

{ (312), (132), (123), (321), (231), (213), (412), (142), (124), (421), (241), (214) }

We find a 3-tuple, take its inverse, then cycle down. These are order 3 cycles. Then, we find another 3-tuple that is not yet noted, and go back to the first step, until all cycle routes have been followed.

123	321			
231	213			
312	132			
	124	421		
	241	214		
	412	142		
		134	431	
		341	314	
		413	143	
			234	432
			342	324
			234	243

We find four order three cycles and their respective inverse things (we may call, arbitrarily, the first in its row the *id* of its cycle, while the other jumbles in the row follow their leader. Maybe *chief* is a more appropriate designation than *id*.)

3-cycles

123	231	312			
321	213	132			
	124	241	412		
	421	214	142		
		134	341	413	
		431	314	143	
			234	342	423
			432	324	243

{ 1234, 2341, 3412, 4123, 4321, 3214, 2143, 1432, 2134, 1342, 3421, 4213, 4312, 3124, 1243, 2431, 1324, 3241, 2413, 4132, 4231, 2314, 3142, 1423 }

Again, we find three 4-cycles. We reverse to form their inverse. We follow the four permutation instances that are specified by these six order four canonical cycle permutations.

As an additional step, here, in order to give some visual clue as to the uniqueness of each 4-tuple, we have ordered the said permutation cyles to list the canonical cycle representation as the first element of its row. (As a permutation, each element in each of the six rows below will permute the standard permutation, say the 4-tuple 1234, to the identical resulting arrangement.)

How significant is this? We aim to find an understanding of how we may abstract this in the context of group theory. Given the 24 permutations listed, we have only six group elements. Each has four different representations. For example, the group's elements 3412 and 4123 are equal because the permutation $(1234) \circ (3412) = (2341)$ is equal to $(1234) \circ (4123) = (2341)$.

1234	2341	3412	4123		
1432	4321	3214	2143		
	1342	3421	4213	2134	
	1243	2431	4312	3124	
		1423	4231	2314	3142
		1324	3241	2413	4132

4-cycles

9. A Group of Permutations

We fill the vacuous bag with four words to make a start. Let W be the set that we made earlier, namely $\{$ woozy, vacuous, sleepy, wide-awake $\}$.

Next we make four sets of permutaions P_{nk} with $k \in \{k|1 \leq k \leq n\}$,

We have P_{41}

woozy	
vacuous	
	sleepy
	wide-awake

We have P_{44}

woozy	b	c	d	
vacuous	b	c	d	
	sleepy	b	c	d
	wide-awake	b	c	d