

## 1. Plain T<sub>E</sub>Xnology

**Theorem T.** *All things are not necessarily the same\**

## 2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantities of permutations are considered with relevance to computing efficiencies.

## 3. The Wide-Awake example Group

We re-think\*, re-word, and re-start with a set of attributes, elements or objects,  $W = \{ \text{woozy, vacuous, sleepy, wide-awake} \}$ . These elements are used to generate all possible arrangements  $\eta$  which are ordered  $n$ -tuples with  $1 \leq n \leq 4$ . For example,  $\eta = ( \text{woozy, wide-awake} )$  is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like  $\eta$ .

Let  $(\text{Woozy}, \circ, 0, -)$  be the group with the set Woozy, a binary operation  $\circ$ , a neutral element 0, and for each element  $\pi \in \text{Woozy}$  there is an inverse element  $-\pi \in \text{Woozy}$  such that  $\pi \circ -\pi = 0$ .

For now, here, we call this group's binary operation *composition*. Given two elements  $\pi, \eta \in \text{Woozy}$ , then  $\pi \circ \eta \in \text{Woozy}$  and  $\eta \circ \pi \in \text{Woozy}$ .

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\* *T<sub>E</sub>Xbook*, *texbook.tex*, <https://www.ctan.org/tex-archive/systems/knuth/dist/tex>

\* The Mathematics of the Rubiks Cube, <http://web.mit.edu/sp.268/www/rubik.pdf>

#### 4. Creating the Woozy set

**Theorem X.** *An ordered set of  $n$  elements has  $n!$  arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.\*

Given a set of objects  $W = \{a_1, a_2, \dots, a_n\}$ .  $P_n$  is the set of arrangements given  $n$  objects  $a_1, \dots, a_n \in W$ , such as  $\{(a_1, a_2, \dots, a_n), (a_2, a_1, \dots), \dots\}$ . For example, with  $W = \{1, 2, 3\}$ , we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from  $n = 3$  to  $n = 4$  as follows. For each element in  $P_{n-1} = P_3$ , place element  $a_n$  in each possible vacuous position to arrive at  $P_n = P_4$ , that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), \dots, (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

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\* TAoCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

## 5. Accounting for these Arrangements

Adding up all permutations that are so generated we have  $p_n$  the number of all elements in  $P_n$

And again, after some re-view, we sense a need to re-word.  $P_{nn}$  is the set of permuted  $n$ -tuples, and  $P_n$  is the, probably bigger, set of all the  $k$ -tuples with  $k \in \{1, 2, \dots, n\}$ . In other words,  $P_n$  may mean different things, or sets of things. This also applies to quantities that could be denoted like  $p_{nk}$ , and  $p_{nn}$ , and in case of our big wide-awake bean bag, which we sum up to  $p_n$ ; probably.

First, we started with  $p_n = \sum_{k=1}^n k!$  to be the quantity  $p_n$  that accounts for all the elements of arrangements in set  $P_n$ , with  $p_k = k!$  for  $1 \leq k \leq n$ .

However, on the back of some scrap paper, we jotted down  $\{(1), (2), (3), (4)\}$  and saw that  $\{(2), (3), (4)\}$  are not included in our sum, and  $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$  has 10 2-tuples unaccounted for, etc.)

So, for now, given that  $p_{nk} = n(n-1)\dots(n-k+1)^*$ , combined with  $p_n = \sum_{k=1}^n p_{nk}$ , we count the number of arrangements of  $n$  objects to be  $p_n = \sum_{k=1}^n \frac{n!}{(n-k)!}$  or some such like.

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\* TAOCP chapter 1.2.5, <https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html>

## 6. Making concrete Space

We now look at the set  $W$  that we enumerated above and apply method 1 to arrange things.

Given  $W$  as above, we have  
 $Woozy_{41} = \{ ( \text{woozy} ), ( \text{vacuus} ), ( \text{sleepy} ), ( \text{wide-awake} ) \}$

Then, taking one step at a time and applying method 1, given the set  
 $Woozy_{11} = \{ ( \text{sleepy} ) \}$  together with another element, wide-awake  $\in W$ , and we get  
 $Woozy_{22} = \{ ( \text{wide-awake, sleepy} ), ( \text{sleepy, wide-awake} ) \}.$

Let's start counting now. We have

$P_{21} = Woozy_{21} = \{ ( \text{sleepy} ) \}$ , and another  $P_{21} = \{ ( \text{wide-awake} ) \}$

$P_{22} = Woozy_{22} = \{ ( \text{sleepy, wide-awake} ), ( \text{wide-awake, sleepy} ) \}.$

To sum up we get

$p_2 = (p_{21} + p_{21}) + p_{22}$ , with

$p_{21} = 1$ , the count for each set of one 1-tuple, and

$p_{22} = 2$ , the count for the one set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,

$p_{21} = \frac{2!}{(2-1+1)!} = 1$ , and  $p_{22} = \frac{2!}{(2-2+1)!} = 2$ . and  $p_2 = 2 * p_{21} + p_{22}$

$p_2 = \sum_{k=1}^2 \frac{2!}{(2-k+1)!}$  which has two terms and evaluates to  $p_2 = \frac{2}{2} + \frac{2}{1}$ , and it looks like there is something wrong here!

Let's take our result from section 5 and adjust.

$$\begin{aligned}
 p_n &= \sum_{k=1}^n (n-k+1) * \frac{n!}{(n-k+1)!} \text{ and since} \\
 (n-k+1) &= (n-k+1) * (n-k) * (n-k-1) * \dots * 1 \text{ we can simplify and have} \\
 p_n &= \sum_{k=1}^n \frac{n!}{(n-k)!} \text{ which, for our state,} \\
 &\text{yields the following sum in two terms, (given that } 0! = 1) \\
 p_2 &= \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4, \text{ which agrees with our permutations' making.}
 \end{aligned}$$

And while we are here we set a solid base by calculating the simple case for the set  $P_1$  for which  $p_1 = 1$ , as we have counted just now.

$$p_n = \sum_{k=1}^n \frac{n!}{(n-k)!} \text{ with } n = 1 \text{ yields } p_1 = \frac{1!}{(1-1)!} = 1 \text{ and confirms the basic case.}$$

So, does our formula hold its stepping up. Assuming that  $p_n = \sum_{k=1}^n \frac{n!}{(n-k)!}$  is correct, we need to find the terms to make  $p_{n+1}$  from that. (to be continued)

## 7. A few observations to count

$$p_{22} = p_{21} = 2$$

$$p_{33} = p_{32}$$

$$p_{nk} = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

$1! = 0! = 1$  seems a reasonable cause, since  $(n - (n-1))! = 1! = 1$  and  $(n - n)! = 0! = 1$ .

$p_{32} = 2p_{31}$  as  $p_{31} = 3$  and  $p_{32} = 6$  and it appears that is on the same ground as the previous line of reasoning; the nature of  $x!$  being  $x(x-1)\dots 3 * 2 * 1$ , while with increasing  $k$  the positive integer sequence is a steady factor. (to be re-worded)

So,  $p_{n2} = 2p_{n1}$ ,  $p_{n3} = 3p_{n2}$ ,

$p_{n4} = 4p_{n3}$ , or some such like, if you get my drift.

(well, maybe not)

## 8. The Series of Sequences

$$\left( \frac{4!}{(4-1)!} \right), \left( \frac{4!}{(4-2)!} \right), \left( \frac{4!}{(4-3)!} \right), \left( \frac{4!}{(4-4)!} \right)$$

$$\left( \frac{24}{6} \right), \left( \frac{24}{2} \right), \left( \frac{24}{1!} \right), \left( \frac{24}{0!} \right)$$

$$\{(\text{sequence, series}), (\text{Folge, Reihe})\}$$

$$\{1\ 2\ 3\ 4\}$$

$$\{(\text{woozy}), (\text{vacuous}), (\text{sleepy}), (\text{wide-awake})\}$$

$$\{(\ 1), (\ 2), (\ 3), (\ 4)\}$$

$$\{(\ 12), (\ 21), (\ 13), (\ 31), (\ 14), (\ 41), (\ 23), (\ 32), (\ 24), (\ 42), (\ 34), (\ 43)\}$$

$$\{(\ 312), (\ 132), (\ 123), (\ 321), (\ 231), (\ 213), (\ 412), (\ 142), (\ 124), (\ 421), (\ 241), (\ 214)\}$$

We find a 3-tuple, take its inverse, then cycle down. These are order 3 cycles. Then, we find another 3-tuple that is not yet noted, and go back to the first step, until all cycle routes have been followed.

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123 321
231 213
312 132
    124 421
    241 214
    412 142
        134 431
        341 314
        413 143
            234 432
            342 324
            234 243

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We find four order three cycles and their respective inverse things (we may call, arbitrarily, the first in its row the *id* of its cycle, while the other jumbles in the row follow their leader. Maybe *chief* is a more appropriate designation than *id*.)

123	231	312	
321	213	132	
124	241	412	
421	214	142	
134	341	413	
431	314	143	
234	342	423	
432	324	243	

$\{ 1234, 2341, 3412, 4123, 4321, 3214, 2143, 1432, 2134, 1342, 3421, 4213, 4312, 3124, 1243, 2431, 1324, 3241, 2413, 4132, 4231, 2314, 3142, 1423 \}$