1. Plain TEXnology

Theorem T. *All things are not necessarily the same**

2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantites of permutations are considered with relevance to computing efficiencies.

3. The Wide-Awake example Group

We re-think*, re-word, and re-start with a set of attributes, elements or objects, $W = \{$ woozy, vacuous, sleepy, wide-awake $\}$. These elements are used to generate all possible arrangements η which are orderd n-tuples with $1 \leq n \leq 4$. For example, $\eta =$ (woozy, wide-awake) is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like η.

Let $(\text{Woozy}, \circ, 0, -)$ be the group with the set Woozy, a binary operation $\circ$, a neutral elment 0, and for each element $\pi \in$ Woozy there is an inverse element $-\pi \in$ Woozy such that $\pi \circ -\pi = 0$.

For now, here, we call this group's binary operation *composition*. Given two elements $\pi, \eta \in$ Woozy, then $\pi \circ \eta \in$ Woozy and $\eta \circ \pi \in$ Woozy.

* *TEXbook, texbook.tex, https://www.ctan.org/tex-archive/systems/knuth/dist/tex*
* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

4. Creating the Woozy set

Theorem X. *An ordered set of n elements has $n!$ arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.*

Given a set of objects $W = \{a_1, a_2, ..., a_n\}$. P_n is the set of arrangements given n objects $a_1, ..., a_n \in W$, such as $\{(a_1, a_2, ...a_n), (a_2, a_1, ...), ...\}$. For example, with $W = \{1, 2, 3\}$, we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from $n = 3$ to $n = 4$ as follows. For each element in $P_{n-1} = P_3$, place element a_n in each possible vacuous position to arrive at $P_n = P_4$, that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), ..., (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

5. Accounting for these Arrangements

Adding up all permutations that are so generated we have p_n the number of all elements in P_n

And again, after some re-view, we sense a need to re-word. P_{nn} is the set of permuted n-tuples, and P_n is the, probably bigger, set of all the k-tuples with $k \in \{1, 2, .., n\}$.. In other words, P_n may mean different things, or sets of things. This also applies to quantities that could be denoted like p_{nk}, and p_{nn}, and in case of our big wide-awake bean bag, which we sum up to p_n; probably.

First, we started with $p_n = \sum_{k=1}^{n} k!$ to be the quantity p_n that accounts for all the elements of arrangements in set P_n, with $p_k = k!$ for $1 \le k \le n$.

However, on the back of some scrap paper, we jotted down $\{(1), (2), (3), (4)\}$ and saw that $\{(2), (3), (4)\}$ are not included in our sum, and $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$ has 10 2-tuples unaccounted for, etc.)

So, for now, given that $p_{nk} = n(n-1)...(n-k+1)$*, combined with $p_n = \sum_{k=1}^{n} p_{nk}$, we count the number of arrangements of n objects to be $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ or some such like.

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

6. Making concrete Space

We now look at the set W that we enumerated above and apply method 1 to arrange things.

Given W as above, we have
$\text{Woozy}_{41} = \{(\text{ woozy }), (\text{ vacuus }), (\text{ sleepy }), (\text{ wide-awake })\}$

Then, taking one step at a time and applying method 1, given the set
$\text{Woozy}_{11} = \{(\text{ sleepy })\}$ together with another element, wide-awake $\in W$, and we get
$\text{Woozy}_{22} = \{(\text{ wide-awake, sleepy }), (\text{ sleepy, wide-awake })\}$.

Let's start counting now. We have

$P_{21} = \text{Woozy}_{21} = \{ \ (\text{ sleepy }), (\text{ wide-awake }) \ \}$

$P_{22} = \text{Woozy}_{22} = \{ \ (\text{ sleepy, wide-awake }), (\text{ wide-awake, sleepy }) \ \}$.

To sum up we get
$p_2 = p_{21} + p_{22}$, with
$p_{21} = 2$, the count for the set of two 1-tuples, and
$p_{22} = 2$, the count for set set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,
$p_{21} = \frac{2!}{(2-1)!} = 2$, and $p_{22} = \frac{2!}{(2-2)!} = 2$. and $p_2 = p_{21} + p_{22}$
$p_2 = \sum_{k=1}^{2} \frac{2!}{(2-k)!}$ which has two terms and evaluates to $p_2 = \frac{2}{1!} + \frac{2}{0!}$, and it looks better (or is this just an illusion; however, $1! = 0! = 1$).

Let's take our result from section 5 and adjust.

$p_n = \sum_{k=1}^{n} (n-k+1) * \frac{n!}{(n-k+1)!}$ and since

$(n-k+1)! = (n-k+1) * (n-k) * (n-k-1) * ... * 1$, we may simplify and have

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ which, for our state,

yields the following sum in two terms, (given that $0! = 1$)

$p_2 = \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4$, which agrees with our permutations' making.

And while we are here we set a solid base by calculating the simple case for the set P_1 for which $p_1 = 1$, as we have counted just now.

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ with $n = 1$ yields $p_1 = \frac{1!}{(1-1)!} = 1$ and confirms the basic case.

So, does our formula hold its stepping up. Assuming that $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ is correct, we need to find the terms to make p_{n+1} from that. (to be continued)

7. A few observations to count

$p_{22} = p_{21} = 2$

$p_{33} = p_{32}$

$p_{nk} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$

$1! = 0! = 1$ seems a reasonable cause, since $(n-(n-1))! = 1! = 1$ and $(n-n)! = 0! = 1$.

$p_{32} = 2p_{31}$ as $p_{31} = 3$ and $p_{32} = 6$ and it appears that is on the same ground as the previous line of reasoning; the nature of $x!$ being $x(x-1)...3*2*1$, while with increasing k the positive integer sequence is a steady factor. (to be re-worded)

So, $p_{n2} = 2p_{n1}$, $p_{n3} = 3p_{n2}$, and $p_{n4} = 4p_{n3}$, or some such like. (Well, probably it is not a conjecture that will turn out to be true.)

8. The Series of Sequences

$$
\begin{array}{llll}
(\frac{4!}{(4-1)!}), & (\frac{4!}{(4-2)!}), & (\frac{4!}{(4-3)!}), & (\frac{4!}{(4-4)!}) \\
(\frac{24}{6}), & (\frac{24}{2}), & (\frac{24}{1!}), & (\frac{24}{0!}) \\
(P_{41} = 4), & (P_{42} = 12), & (P_{43} = 24), & (P_{44} = 24)
\end{array}
$$

{(sequence, series), (Folge, Reihe)}

{1 2 3 4}

{(woozy), (vacuous), (sleepy), (wide-awake)}

{(1), (2), (3), (4)}

{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)}

{ (312), (132), (123), (321), (231), (213), (412), (142), (124), (421), (241), (214) }

We find a 3-tuple, take its inverse, then cycle down. These are order 3 cycles. Then, we find another 3-tuple that is not yet noted, and go back to the first step, until all cycle routes have been followed.

123	321			
231	213			
312	132			
	124	421		
	241	214		
	412	142		
		134	431	
		341	314	
		413	143	
			234	432
			342	324
			234	243

We find four order three cycles and their respective inverse things (we may call, arbitrarily, the first in its row the *id* of its cycle, while the other jumbles in the row follow their leader. Maybe *chief* is a more appropriate designation than *id*.)

3-cycles

123	231	312			
321	213	132			
	124	241	412		
	421	214	142		
		134	341	413	
		431	314	143	
			234	342	423
			432	324	243

{ 1234, 2341, 3412, 4123, 4321, 3214, 2143, 1432, 2134, 1342, 3421, 4213, 4312, 3124, 1243, 2431, 1324, 3241, 2413, 4132, 4231, 2314, 3142, 1423 }

Again, we find three 4-cycles. We reverse to form their inverse. We follow the four permutation instances that are specified by these six order four canonical cycle permutations.

As an additional step, here, in order to give some visual clue as to the uniqueness of each 4-tuple, we have ordered the said permutation cyles to list the canonical cycle representation as the first element of its row. (As a permutation, each element in each of the six rows below will permute the standard permutation, say the 4-tuple 1234, to the identical resulting arrangement.)

How significant is this? We aim to find an understanding of how we may abstract this in the context of group theory. Given the 24 permutations listed, we have only six group elements. Each has four different representations. For example, the group's elements 3412 and 4123 are equal because the permutation $(1234) \circ (3412) = (2341)$ is equal to $(1234) \circ (4123) = (2341)$.

1234	2341	3412	4123		
1432	4321	3214	2143		
	1342	3421	4213	2134	
	1243	2431	4312	3124	
		1423	4231	2314	3142
		1324	3241	2413	4132

4-cycles

9. A Group of Permutations

We fill the vacuous bag with four words to make a start. Let W be the set that we made earlier, namely { woozy, vacuous, sleepy, wide-awake }.

Next we make four sets of permutaions P_{nk} with $k \in \{k|1 \leq k \leq n\}$,

We have P_{41}

(woozy)	
(vacuous)	
	(sleepy)
	(wide-awake)

We have P_{42}

(woozy vacuous)	(vacuous woozy)		
(woozy sleepy)	(sleepy woozy)		
(woozy wide-awake)	(wide-awake woozy)		
	(vacuous sleepy)	(sleepy vacuous)	
	(vacuous wide-awake)	(wide-awake vacuous)	
		(sleepy wide-awake)	(wide-awake sleepy)

We have P_{43}

(woozy vacuous sleepy) (vacuous sleepy woozy) (sleepy woozy vacuous)
(sleepy vacuous woozy) (vacuous woozy sleepy) (woozy sleepy vacuous)

(woozy vacuous wide-awake) (vacuous wide-awake woozy) (wide-awake woozy vacuous)
(wide-awake vacuous woozy) (vacuous woozy wide-awake) (woozy wide-awake vacuous)

(woozy sleepy wide-awake) (sleepy wide-awake woozy) (wide-awake woozy sleepy)
(wide-awake sleepy woozy) (sleepy woozy wide-awake) (woozy wide-awake sleepy)

(vacuous sleepy wide-awake) (sleepy wide-awake vacuous) (wide-awake vacuous sleepy)
(wide-awake sleepy vacuous) (sleepy vacuous wide-awake) (vacuous wide-awake sleepy)

We have P_{44} (now in progress)

(woozy vacuous sleepy wide-awake) (vacuous sleepy wide-awake woozy)
(sleepy wide-awake woozy vacuous) (wide-awake woozy vacuos sleepy)
(woozy wide-awake sleepy vacuous) (wide-awake sleepy vacuous woozy)
(sleepy vacuous woozy wide-awake) (vacuous woozy wide-awake sleepy)

(woozy sleepy wide-awake vacuous) (sleepy wide-awake vacuous woozy)
(wide-awake vacuous woozy sleepy) (vacuous woozy sleepy wide-awake)
(woozy vacuous wide-awake sleepy) (sleepy woozy vacuous wide-awake)
(wide-awake sleepy woozy vacuous) (vacuous wide-awake sleepy woozy)

(woozy wide-awake vacuous sleepy) (sleepy woozy wide-awake vacuous)
(vacuous sleepy woozy wide-awake) (wide-awake vacuous sleepy woozy)
(woozy sleepy vacuous wide-awake) (wide-awake woozy sleepy vacuous)
(vacuous wide-awake woozy sleepy) (sleepy vacuous wide-awake woozy)

10. The example 2F 2R cycle and its abstraction

Spinning the cube. Let $\pi \in S$ and $S \subseteq \mathbf{R}$ where $S = \{F, R\}$. We apply the front face-rotation twice, followed by two right-face rotations, that is $\pi = F \circ F \circ R \circ R$.

We just did that on a Rubik's cube app, and counted 6 instances until the initial permutation was reinstantiated. That indicates that π may be of order 6 (that is 24 quarter turns in total.)

After one $FFRR$ we may do 5 more π composites, that is $FFRRFFRRFFRRFFRRFFRR$, to revert the first move.

We could, of course, also have done $-\pi$, that is the inverse permutation of π (with $\pi \circ -\pi = 0$). $-\pi = -R \circ -R \circ -F \circ -F$, which with $-F = f$ and $-R = r$, may be denoted as $rrff$.

Now we will enumerate the three subsets, $F \subseteq \mathbf{R}$, $R \subseteq \mathbf{R}$, and $F \cup R \subseteq \mathbf{R}$. These subsets of permuations contribute to the abstraction of the Rubik's cube's front face F and right face R. The union of these sets $S = R \cup F$ aims to help the enumertion of permutations which we may call (not all that seriously) jumbles, ruffles, permutes, and sometimes but not always rotations or composites. These subsets together with their operation $\circ$ and their neutral element 0 (maybe called ()) form subroups of the Rubik group $\mathbf{R}$.

The front face is represented by set F, and set R is the right face of the cube. (There is already some ambiguity creeping and ready to interfere. There are nouns like set, permutation, arrangement, and face; then we see verbs such as permute, arrange, or jumble, ruffle, or just rotate by a quarter turn; or some such like.)

Here is the front face $F = \{$ FUL FU FUR FR FDR FD FDL FL FC $\}$, and the right face $R = \{$ RUF RU RUB RB RDB RD RDF RF RC $\}$.

We also see some entanglements, namely $\{$ (FUR RUF), (FDR RDF), (FR RF) $\}$. For example, FUR is the cube of cubes on the front face in set F, while RUF is the same cube (maybe called cuby) on the right face in set R. The elements FUR and RUF are also part of the permutation (here, the action of rotating the face) F, and R respectively.

The rotation of the front face as a permutation cycle is $F =$ (FUL FUR FDR FDL) $\circ$ (FU FR FD FL), and the right face re-arrangement by a clockwise quarter-turn is $R =$ (RUF RUB RDB RDF) $\circ$ (RU RB RD RF), while the entaglements of the front and the right are already lurking and ready to intervene.

Now we look at the FFRR permutation again. This composite jumble has two front face quarter-turns followed by two 90 degree turns of the right face and it gives the permutation $F \circ F \circ R \circ R$, or in other more detailed words ((FUL FDR) (FUR FDL) (FU FD) (FR FL) $\circ$ (RUF RDB) (RUB RDF) (RU RD) (RF RB))

One way to verify this permutation (verb) is to express its effect as a two-line expression, first $F(F \circ F)$, like so

$$F(F \circ F) = \begin{pmatrix} \text{FUL FU FUR FR FDR FD FDL FL} \\ \text{FDR FD FDL FL FUL FU FUR FR} \end{pmatrix},$$

Or in a more verbose way the right face 180 degree turn might look like this:

$$R(R \circ R) = \begin{pmatrix} \text{RFU RU RUB RB RBD RD RDF RF RC} \\ \text{RFU RU RUB RB RBD RD RDF RF RC} \end{pmatrix}$$

$$\circ(\text{RFU RBD}) \circ (\text{RUB RDF}) \circ (\text{RU RD}) \circ (\text{RF RB}) \circ (\text{RC}) =$$

$$\begin{pmatrix} \text{RFU RU RUB RB RBD RD RDF RF RC} \\ \text{RBD RD RFD RF RFU RU RBU RB RC} \end{pmatrix}.$$

(to be corrected, still)

We try to visualise a particular face-to-face entanglement which could be expressed within the context of the respective edge-cubes of the front and the right face.

	FU		RU	
FL	**FR**	**RF**		RB
	FD		RD	

In terms of the set of permutation cycles the entaglement may read $\{(\mathrm{FR\ RF}), (\mathrm{RF\ FR})\}$, and again we note that $(\mathrm{FR\ RF}) = (\mathrm{RF\ FR})$.

The corner-cubes have a similar entanglement. (re-thing, re-word)

FLU	**FUR**	**RFU**	RUB
Front		Right	
FDL	**FRD**	**RDF**	RD

11. Let's look at the example RUru

The move given in the example* is (RUru) which is said to be a "commutator." It is said to affect seven cubes of which two have no part in the up face. In our abstraction we denote $\gamma \in G$ and $\gamma =$ (RUru), with $G = U \cup R$ and $G \subseteq R$.

Here, we start with the standard permutation (noun) σ with the edge cubes of the up face singled out with the expression $\sigma_U^e =$ (UB UR UF UL).

We re-vise our view of the clockwise quarter turn of the right face's edges, and therefore write $R^e =$ (RU RF RD RB). We also appreciate, now, that the right face's up-edge, RU, is the same as the up face's right edge, UR. In other words, RU = UR. (Sounds trivial, but we find this important to keep in mind, for now.)

So, just do it! That is, let's do the first component of or γ-move. Here it is, looking at two faces with two expressions. $R^2 = R \circ$ (RU RF RD RB), giving $R^2 =$ (RF RU RB RD). Now, at the same time we have $U^R = U \circ$ (UR RF), being aware that UR = RU, and noting the effect that this R-move has on the U-face which we now call U^R, and which is spelled out as (UB RF UF UL).

* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

The second component of γ being the top face quarter turn, U, and we may write $U^{RU} = R \circ U$. The, admittedly verbose expression reads $U_e^{RU} = U^R \circ U$, and then $U_e^R \circ U =$ (UB RF UF UL) $\circ\, U$ and we get $U_e^{RU} =$ (UL UB RF UF).

We also see the entanglement that this re-arrangement implies and we aim to express the effect is has on the R-face. $R_e^R \circ U^{RU}$ which shows the end effect, a ruffle of (RF UB), or some such like (for now unconfirmed.) This makes the R-face to read $R_e^{RU} = R^2 \circ U^{RU}$ and $R_e^{RU} =$ (UB RU RB RD), (still under re-view; in particular our mis-conceptions, the ambiguities, and the mis-representation are creeping all over; however we will wittle things down just to pick up each one of these crawlies and re-permute things to form a set of cohesion; given time.)

Fast forward, pending completion, we have booted the Rubik cube to do the computation γ for us, and just note it down in terms of edge jumbles,

$$\gamma_R^e = (\text{ RF RB RD BU })$$

$$\gamma_U^e = (\text{ UR FR UF UL })$$

Given the standard permutaion σ with $\sigma_U^e = (\text{ UB UR UF UL })$, and $\sigma_R^e = (\text{ RU RB RD RF })$, this allows us to calculate the components in terms of permutation cycles

$$\gamma_R^e = (\text{RU RF}) \circ (\text{RB}) \circ (\text{RD}) \circ (\text{RF UL}),$$

and

$$\gamma_U^e = (\text{UB}) \circ (\text{UR FR}) \circ (\text{UF}) \circ (\text{UL}).$$