1. Plain TEXnology

Theorem T. *All things are not necessarily the same**

2. Permutations

TAoCP in chapter 1.2.5 gives two methods to generate all permutations of a given ordered set. Quantites of permutations are considered with relevance to computing efficiencies.

3. The Wide-Awake example Group

We re-think*, re-word, and re-start with a set of attributes, elements or objects, $W = \{$ woozy, vacuous, sleepy, wide-awake $\}$. These elements are used to generate all possible arrangements η which are orderd n-tuples with $1 \le n \le 4$. For example, $\eta = ($ woozy, wide-awake $)$ is a 2-tuple. Now the set Woozy is the set of all permutations that jumble such elements like η.

Let $(\text{Woozy}, \circ, 0, -)$ be the group with the set Woozy, a binary operation $\circ$, a neutral elment 0, and for each element $\pi \in$ Woozy there is an inverse element $-\pi \in$ Woozy such that $\pi \circ -\pi = 0$.

For now, here, we call this group's binary operation *composition*. Given two elements $\pi, \eta \in$ Woozy, then $\pi \circ \eta \in$ Woozy and $\eta \circ \pi \in$ Woozy.

* *TEXbook, texbook.tex, https://www.ctan.org/tex-archive/systems/knuth/dist/tex*
* The Mathematics of the Rubiks Cube, http://web.mit.edu/sp.268/www/rubik.pdf

4. Creating the Woozy set

Theorem X. *An ordered set of n elements has $n!$ arrangements.*

This had a little consideration. Here, we convey our understanding of the Permutations and Factorials section.*

Given a set of objects $W = \{a_1, a_2, ..., a_n\}$. P_n is the set of arrangements given n objects $a_1, ..., a_n \in W$, such as $\{(a_1, a_2, ...a_n), (a_2, a_1, ...), ...\}$. For example, with $W = \{1, 2, 3\}$, we have

$$P_3 = \{(123), (231), (312), (132), (321), (213)\}.$$

Method 1, now, moves from $n = 3$ to $n = 4$ as follows. For each element in $P_{n-1} = P_3$, place element a_n in each possible vacuous position to arrive at $P_n = P_4$, that is

$$P_4 = \{(a_n a_1 a_2 a_3), (a_1 a_n a_2 a_3), (a_1 a_2 a_n a_3), (a_1 a_2 a_3 a_n), ..., (a_n a_2 a_1 a_3), (a_2 a_n a_1 a_3), (a_2 a_1 a_n a_3), (a_2 a_1 a_3 a_n)\}$$

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

5. Accounting for these Arrangements

Adding up all permutations that are so generated we have p_n the number of all elements in P_n

And again, after some re-view, we sense a need to re-word. P_{nn} is the set of permuted n-tuples, and P_n is the, probably bigger, set of all the k-tuples with $k \in \{1, 2, .., n\}$.. In other words, P_n may mean different things, or sets of things. This also applies to quantities that could be denoted like p_{nk}, and p_{nn}, and in case of our big wide-awake bean bag, which we sum up to p_n; probably.

First, we started with $p_n = \sum_{k=1}^{n} k!$ to be the quantity p_n that accounts for all the elements of arrangements in set P_n, with $p_k = k!$ for $1 \leq k \leq n$.

However, on the back of some scrap paper, we jotted down $\{(1), (2), (3), (4)\}$ and saw that $\{(2), (3), (4)\}$ are not included in our sum, and $\{(12), (21), (13), (31), (14), (41), (23), (32), (24), (42), (34), (43)\}$ has 10 2-tuples unaccounted for, etc.)

So, for now, given that $p_{nk} = n(n-1)...(n-k+1)$*, combined with $p_n = \sum_{k=1}^{n} p_{nk}$, we count the number of arrangements of n objects to be $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ or some such like.

* TAoCP chapter 1.2.5, https://www-cs-faculty.stanford.edu/%7Eknuth/taocp.html

6. Making concrete Space

We now look at the set W that we enumerated above and apply method 1 to arrange things.

Given W as above, we have
$\text{Woozy}_{41} = \{($ woozy $), ($ vacuus $), ($ sleepy $), ($ wide-awake $)\}$

Then, taking one step at a time and applying method 1, given the set
$\text{Woozy}_{11} = \{($ sleepy $)\}$ together with another element, wide-awake $\in W$, and we get
$\text{Woozy}_{22} = \{($ wide-awake, sleepy $), ($ sleepy, wide-awake $)\}$.

Let's start counting now. We have

$P_{21} = \text{Woozy}_{21} = \{$ (sleepy) $\}$, and another $P_{21} = \{$ (wide-awake) $\}$

$P_{22} = \text{Woozy}_{22} = \{$ (sleepy, wide-awake), (wide-awake, sleepy) $\}$.

To sum up we get
$p_2 = (p_{21} + p_{21}) + p_{22}$, with
$p_{21} = 1$, the count for each set of one 1-tuple, and
$p_{22} = 2$, the count for the one set of two 2-tuples that we have created so far.

Compare things with the calculations that we made earlier,
$p_{21} = \frac{2!}{(2-1+1)!} = 1$, and $p_{22} = \frac{2!}{(2-2+1)!} = 2$. and $p_2 = 2 * p_{21} + p_{22}$
$p_2 = \sum_{k=1}^{2} \frac{2!}{(2-k+1)!}$ which has two terms and evaluates to $p_2 = \frac{2}{2} + \frac{2}{1}$, and it looks like there is something wrong here!

Let's take our result from section 5 and adjust.

$p_n = \sum_{k=1}^{n} (n-k+1) * \frac{n!}{(n-k+1)!}$ and since

$(n-k+1) = (n-k+1) * (n-k) * (n-k-1) * ... * 1$ we can simplify and have

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ which, for our state,

yields the following sum in two terms, (given that $0! = 1$)

$p_2 = \frac{2}{1} + \frac{2}{1} = 2 + 2 = 4$, which agrees with our permutations' making.

And while we are here we set a solid base by calculating the simple case for the set P_1 for which $p_1 = 1$, as we have counted just now.

$p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ with $n = 1$ yields $p_1 = \frac{1!}{(1-1)!} = 1$ and confirms the basic case.

So, does our formula hold its stepping up. Assuming that $p_n = \sum_{k=1}^{n} \frac{n!}{(n-k)!}$ is correct, we need to find the terms to make p_{n+1} from that. (to be continued)

7. A few observations to count

$p_{22} = p_{21} = 2$
$p_{33} = p_{32}$
$p_{nk} = n(n-1)...(n-k+1) = \frac{n!}{(n-k)!}$
$1! = 0! = 1$ seems a reasonable cause, since $(n-(n-1))! = 1! = 1$ and $(n-n)! = 0! = 1$.

$p_{32} = 2p_{31}$ as $p_{31} = 3$ and $p_{32} = 6$ and it appears that is on the same ground as the previous line of reasoning; the nature of $x!$ being $x(x-1)...3*2*1$, while with increasing k the positive integer sequence is a steady factor. (to be re-worded)

So, $p_{n2} = 2p_{n1}$, $p_{n3} = 3p_{n2}$,
$p_{n4} = 4p_{n3}$, or some such like, if you get my drift.
(well, maybe not)