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Quantum mechanics and violations of the sure-thing principle: The use of probability interference and other concepts

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ABSTRACT

The use of quantum mechanical concepts in social science is a fairly new phenomenon. This paper uses one of quantum mechanics' most basic concepts, probability interference, to explain the violation of an important decision theory principle (the 'sure-thing principle'). We also attempt to introduce other quantum mechanical concepts in relation to the sure-thing principle violation.

1. Introduction

Since about ten years ago, some research has appeared on using quantum-mechanical principles (or concepts) in resolutely non-quantum settings such as social science. Some of the work to date has exhibited a varied range of applications. The interested researcher who is not informed about those developments may, rightly so, question the rationale for using quantum mechanical concepts in settings other than quantum mechanics. We believe we can appease such concerns. As an example, research on the interface of quantum mechanics and cognitive systems, with applications in particular, to the fields of psychology and the foundational levels of probability, shows great promise. The work by Busemeyer and Wang (2007) and Busemeyer, Wang, and Townsend (2006) has used one of the most basic concepts of quantum mechanics, probability interference, to aid in explaining violations of the sure-thing principle. This principle, as we will discuss in Section 3, is actively used in some of the expected utility models.

The objective of this paper is to continue the research endeavour of investigating how we can use simple quantum-mechanical principles in a social science setting, in particular cognitive systems. We will continue using the concept of probability interference to explain the violations of the sure-thing principle but in a different way from the Busemeyer and Wang (2007) and Busemeyer et al. (2006) papers. We then also proceed to show how the wave function, another very basic concept in quantum mechanics, can be fruitfully used to explaining arbitrage. Following the work of Bossaerts, Ghirardato, Guarnaschelli, and Zame (2007) we use the connection they developed between arbitrage and the Ellsberg (1961) paradox (this paradox is a school example of where the sure-thing principle goes wrong), to show how the wave function, as an information function, can be used to model variations in the existence of arbitrage.

As we have indicated above, the applications of quantum-mechanical concepts, in settings other than quantum mechanics (as physical theory), have found their way in various areas of research. We mention:

- economics (Danilov & Lambert-Mogiliansky, 2006; Franco, 2007, 2008; La Mura, 2003, 2005; Lambert Mogiliansky, Zamir, & Zwirn, 2003)
- finance (Accardi & Boukas, 2006; Baaquie, 2005; Choustova, 2006, 2007; Haven, 2005, 2007, 2008a, 2008b; Segal & Segal, 1998)
- others areas of social science (Khrennikov, 1999, 2002, 2004)
- other areas (Aerts & D'Hooghe, 1996; Aerts, Aerts, Broekaert, & Gabora, 2000; Aerts et al., 2003; Bruza, Lawless, van Rijsbergen, & Sofge, 2007).

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In the next section of the paper we introduce the so-called 'double slit experiment'. This is the experiment 'par excellence', which explains the probability interference concept. In section three of the paper, we highlight the violation of the sure-thing principle and the related Ellsberg paradox. We attempt to explain how the violation of the sure-thing principle can be explained with the concept of probability interference. We provide for a rigorous discussion of the concept of 'context'. In section four we consider again the Ellsberg paradox but now in its relation to arbitrage and ambiguity. We use the quantum mechanical wave function to explain arbitrage and ambiguity.

2. Explaining probability interference with the 'double slit' experiment

For a novice to the field of quantum mechanics, the concept of probability interference is not such an easy concept to grasp from first principles. A possible reason to explain this hindrance in understanding interference, could be related to the fact one uses the idea of superposition. The so-called 'double slit experiment' will aid us in understanding the notion of superposition. See also Khrennikov and Haven (2007).

Consider an 'electron gun' which produces a beam of electrons. Consider an electron detector which counts the number of electrons landing on a given area. Assume that the landing area, where the electrons arrive, is divided by two slits which are each of equal width. Assume there exists some space between the slits. Let us call slit 1, s_1 and slit 2, s_2. The double slit experiment covers the three possible cases:

(1) slit s_1 is open and slit s_2 is closed
(2) slit s_1 is closed and slit s_2 is open
(3) slit s_1 is open and slit s_2 is open.

Let us step away from the quantum-mechanical environment for the moment, and let us consider instead of having an electron gun, we have a gun firing small metal pellets. Assume each pellet has a diameter which is small relative to the slit openings. We note that the slit openings would certainly be much larger than what is suggested by slits s_1 and s_2 in the case where electrons are used.

Clearly, in case 1, a lot of pellets will pile up behind slit s_1 but few will pile up behind slit s_2. We assume some piling behind s_2 will occur because pellets will scatter off at the edges of slit s_2. The same argument can be used in the second case. In the third case, when both slits are open we would expect, intuitively, that pellets would pile up in roughly equal amounts behind both slits.

What may be much less intuitive is to explain what happens under the same three cases (assuming slit lengths of s_1 and s_2 are now much smaller) when we consider electrons. We follow Morrison (1990). With electrons, once both slits are open, initially spots form onto the landing area, behind the slits, in a random fashion. Strangely enough, after some time, the electrons start forming an interference pattern. Morrison (1990) remarks, that at first, the electrons behave like particles (say pellets) but with time moving forward, the 'pellets' start behaving like waves! What is even more bizarre, is that if the electron gun were to fire off a single electron, there would still be interference! In effect, this would indicate that the electron interferes with itself. This is a clearly counter-intuitive result.

The modelling of the electron firing-slit phenomenon can be done in the following way. We still follow, adapting notation slightly, Morrison (1990). We remark that Busemeyer et al. (2006) also provide for a very intuitive background on the discussion which is now following.

We use the notations $p_{s_1}(x,t)$ and $p_{s_2}(x,t)$ to denote, respectively, the probability that the electron, at time t, arrives at position x, when slit s_1 (slit s_2) is open. How can we express the probability for the case when both slits s_1 and s_2 are open? In the 'pellet' case, we could use the probability rule for mutually exclusive events:

$$p_{s_1s_2}(x,t) = p_{s_1}(x,t) + p_{s_2}(x,t). \tag{1}$$

As one could expect, the interference phenomenon, which is observed when electrons rather than pellets are used, is not translated in this formula.

As we have indicated at the beginning of this section, the difficulty with modelling the electron phenomenon, as is well explained in Morrison (1990), is that we need to introduce, mathematically, the idea of superposition so as to translate the phenomenon of interference. Probability distributions cannot be superposed *but* one could superpose, so called probability waves (also called probability amplitudes), $\psi_{s_i}(x,t)$, for when slit s_1 $(i = 1)$ is open and slit s_2 $(i = 2)$ is open. A crucial relationship is the following:

$$|\psi(x,t)|^2 \propto \text{ probability density function.} \tag{2}$$

$|\psi(x,t)|^2$ is obtained by executing a multiplication of the probability amplitude with its, so called complex conjugate. The complex conjugate of ψ is often denoted as ψ^*. In the basic quantum mechanics literature one then often encounters the notation: $|\psi(x,t)|^2 = \psi^*\psi$. We provide below for an example of this multiplication.

Using (2), we can write the probability that the electron, at time t, arrives in position x when slit s_1 is open as:

$$p_{s_1}(x,t) \propto \left|\psi_{s_1}(x,t)\right|^2. \tag{3}$$

We can write the same for $p_{s_2}(x,t)$.

Probability amplitudes are waves and therefore a superposition is intuitive. One can thus write, that the superposition, denoted by $\psi_{s_1s_2}(x,t)$, is:

$$\psi_{s_1s_2}(x,t) = \psi_{s_1}(x,t) + \psi_{s_2}(x,t). \tag{4}$$

The key relationship to finally generate the probability interference term is as follows. Consider:

$$p_{s_1s_2}(x,t) \propto \left|\psi_{s_1}(x,t) + \psi_{s_2}(x,t)\right|^2. \tag{5}$$

To develop this argument, we need some background information. We recall that a complex number z can be denoted as $z = x + iy$, where x is the real part and y is the imaginary part. z can also be expressed as $z = re^{i\theta}$ where $r = \sqrt{x^2+y^2}$. Note that r is also denoted as $|z|$ (amplitude). The angle θ, also called the phase, is equal to $\tan^{-1}\frac{y}{x}$, with $x = r\cos\theta$ and $y = r\sin\theta$.

The wave function can be written in terms of a phase and amplitude, as follows:

$$\psi_{s_1}(x,t) = \left|\psi_{s_1}(x,t)\right| e^{iS_{s_1}(x,t)}, \tag{6}$$

where $S_{s_1}(x,t)$ is the phase of the wave function (when slit s_1 is open). The amplitude is $\left|\psi_{s_1}(x,t)\right|$.

We hinted above to the notion of complex conjugate. We have an example of such conjugate. The complex conjugate, $\psi^*_{s_1}(x,t)$, is defined as: $\psi^*_{s_1}(x,t) = \left|\psi_{s_1}(x,t)\right| e^{-iS_{s_1}(x,t)}$.

Substituting (6) into (5) (and similarly for $\psi_{s_2}(x,t)$), we obtain:

$$p_{s_1s_2}(x,t) = \left|\psi_{s_1}(x,t)\right|^2 + \left|\psi_{s_2}(x,t)\right|^2 + 2\left|\psi_{s_1}(x,t)\right|\left|\psi_{s_2}(x,t)\right|\cos(S_{s_1} - S_{s_2}). \tag{7}$$

This equation includes the probability interference term, which is the third term on the right-hand side of the above equation: $2\left|\psi_{s_1}(x,t)\right|\left|\psi_{s_2}(x,t)\right|\cos(S_{s_1} - S_{s_2})$. If this term is non-zero, the total probability formula can be either sub- (or super-) additive. See also Khrennikov (2007b) for a more in-depth discussion.

3. How can probability interference explain the violation of the sure-thing principle?

3.1. What is the sure-thing principle? What is the Ellsberg paradox?

The sure-thing principle is a key principle in the so called Savage (1954) expected utility theory. This theory allows for probability to be entirely subjective. In that respect, Savage's theory is quite different from the 'expected utility workhorse' which is mostly used in basic micro-economic theory: the von Neumann and Morgenstern (1947) expected utility. This approach to expected utility only allows for objective probability.

The sure-thing principle is easy to exemplify via the following experimental setting. Assume there are four gambles, G_1, G_2, G_3 and G_4, which are divided into two sets. Set 1 consists of G_1 and G_2, while set 2 consists of G_3 and G_4. Experiment participants are instructed to indicate whether they prefer, in set 1, G_1 over G_2 (or the opposite). Similarly for the set 2 gambles. We will, as is described in Khrennikov and Haven (2007), assume the gamble has payoffs in 3 states of nature: n_1, n_2 and n_3. We make the simple assumption that the payoff of the in-set (that is *within* sets 1 and 2) gambles in state n_3 are identical.

Given the context we have described just now, the sure-thing principle becomes straightforward. It simply says that the preference of gambles in set 1 and set 2 will be unaffected by the (in-set) identical outcomes of the gambles in state n_3.

Busemeyer and Wang (2007) and Busemeyer et al. (2006) indicate, that in the experiment Tversky and Shafir (1992) ran, the authors found that in many instances players were violating this principle!

A very famous paradox which implies a violation of the sure-thing principle is the Ellsberg paradox. The most common form of presenting the paradox is as follows. Assume there exists an urn consisting of 30 red balls and 60 other balls (blue and green). The exact proportion of green and blue balls is thus unknown. Four gambles are considered and they are again divided into two sets. Experiment participants are asked to express a preference between set 1 gambles: 1 and 2 and between set 2 gambles: 3 and 4. The gamble's payoffs are as follows.

(1) Gamble 1 ($G1$): you receive 1 unit of currency (uoc) if you draw a red ball
(2) Gamble 2 ($G2$): you receive 1 unit of currency (uoc) if you draw a blue ball
(3) Gamble 3 ($G3$): you receive 1 unit of currency (uoc) if you draw a red or green ball
(4) Gamble 4 ($G4$): you receive 1 unit of currency (uoc) if you draw a blue or green ball.

It is intuitive to observe that most of the experiment participants will prefer $G1$ over $G2$, $G1 \succ G2$. One knows there is a 1/3 chance of winning $G1$, while it is unsure what the odds are in the second gamble, $G2$. Using similar reasoning it is expected that experiment participants will express $G4 \succ G3$.

This paradox violates the sure-thing principle. In Khrennikov and Haven (2007) we go to a great length in explaining this paradox. In this paper, we provide for a much more condensed explanation.[1] Assume the preference $G1 \succ G2$ is expressed when the green ball does not occur. If the green ball were to occur then clearly we would express indifference over both gambles; i.e. $G1 \sim G2$. Therefore, we can say that $G1 \succ G2$ if the green ball does not occur, while we have a weak preference, $G1 \succeq G2$ if the green ball occurs. The sure-thing principle indicates that $G1$ should be preferred whether the green ball appears or not. If the green ball does not occur then, $G3$ is like $G1$ and $G4$ is like $G2$ and the sure principle indicates that $G3$ should be preferred.

[1] Thanks to Jerome Busemeyer for pointing out this explanation.

3.2. Findings of the Tversky and Shafir (1992) experiment

As we have already indicated above, the violation of the sure-thing principle is conclusively tested in the Tversky and Shafir (1992) experiment. The work by Busemeyer and Wang (2007) and Busemeyer et al. (2006) expends a lot of effort on explaining how this violation can be explained with probability interference. In this paper we continue using the same Tversky and Shafir (1992) experimental outcome but we will attempt to explain the result in a different way as compared to the approach Busemeyer and Wang (2007) and Busemeyer et al. (2006) follow. We still continue using the probability interference argument though.[2]

Busemeyer and Wang (2007) and Busemeyer et al. (2006) describe the experiment as follows. Participants are informed they have an equal chance to win \$ 200 or lose \$ 100. Although experiment participants must play the first gamble, they have the option to play it for a second time. Their decision to play the gamble for a second time is conditioned upon three possible scenarios:

- they are informed they have won the first gamble
- they are informed they have lost the first gamble
- they are not informed of the outcome of the first gamble.

It is intuitive to make the statement, Busemeyer and Wang (2007) and Busemeyer et al. (2006) make, that if participants want to play the second gamble, irrespective of whether the information they received was that they had won or lost the first gamble, they should still want to play the second gamble if they have no information as to whether they had won or lost in the first gamble. This intuitive conclusion is just another way of translating the sure-thing principle. But the Tversky and Shafir (1992) study found that:

- 69% of experiment participants opt for a second gamble knowing they won in the first gamble
- 59% of experiment participants opt for a second gamble knowing they lost in the first gamble.

However, 36% of the experiment participants will opt for a second gamble if they have no information on whether they won or lost in the first gamble. Furthermore, 64% of the participants will reject playing for a second gamble if they have no information on whether they won or lost in the first gamble.

3.3. A rigorous development of the concept of 'context'

As was mentioned above, we have as objective in this paper to explain the sure-thing principle with probability interference, but in a different way from the Busemeyer and Wang (2007) and Busemeyer et al. (2006) paper. To achieve this purpose we will need to employ a very important concept, known under the name of 'context'. In the sequel of this paper, we will denote *context*, as C. The notion of context C, in quantum mechanics, can be explained as a complex of experimental physical conditions (Khrennikov (2004)). One way to begin to understand the concept of 'context' is as follows. Probabilistic conditioning in classical probability is always performed with respect to an *event*. In what follows below, we will define conditional probabilities with respect to a context. We can consider contexts in other areas besides quantum mechanics. As an example, we can define the concept of 'context' in psychology or in any other social science setting.

As the title of this sub-section suggests, we present here a rigorous development on the concept of 'context'. We note that the next sub-section will provide for an application of some of

[2] Note that there exist two main types of interference terms: the so called 'trigonometric' interference term (which we deal with in this section) and the 'hyperbolic' interference term (see Section 3.5).

the concepts we developed here, in the Tversky and Shafir (1992) experimental setting. The treatment presented in this sub-section can also be found in Khrennikov (2007a and 2007c).

For any model M we need to determine the set of contexts, $\mathcal{C}$ (please see below) for M. For *some* models (not for all models!) one can construct a set theoretic representation of contexts. As an example, the set Ω is said to contain all the possible parameters (physical, mental etc...) of M.

A probabilistic model, $\mathcal{P}$, has two ingredients: (i) a set of contexts, denoted $\mathcal{C}$, and (ii) a set of observables, denoted $\mathcal{O}$. Each observable, $a \in \mathcal{O}$ can be measured under $C \in \mathcal{C}$.[3] We denote the set of possible values of the observable $a \in \mathcal{O}$ by X_a. We only consider discrete (dichotomous) observables.

Consider the following two axioms (see f.i Khrennikov (2007c))

Axiom 3.1. For any observable $a \in \mathcal{O}$ and its value $y \in X_a$, there is defined a context, say C_y (corresponding to the y-selection (see below)): if we perform a measurement of the observable 'a' under the complex of (physical) conditions C_y, then we obtain the value $a = y$ with probability 1. We assume that the set of contexts $\mathcal{C}$ contains C_y selection contexts.[4] for all observables $a \in \mathcal{O}$ and $y \in X_a$

Example 3.1. Let a be an observable with a dichotomous outcome. We could say the observable corresponds to a question (which is part of an experiment for instance) with a binary outcome '0' or '1'. The so-called C_1 selection context would then for instance group together experiment participants who have indicated '1' as their answer (similarly for the C_0 selection context).

Axiom 1 ensures that the y selection contexts are well defined.

Axiom 3.2. Contextual (conditional) probabilities $p_C^a(y) \equiv P(a = y \mid C)$ are defined for any context $C \in \mathcal{C}$ and any observable $a \in \mathcal{O}$.

Hence, for any context $C \in \mathcal{C}$ and any observable $a \in \mathcal{O}$, there exists a probability to observe the fixed value $a = y$ under C. Transition probabilities play an important role here: $p^{b|a}(x \mid y) \equiv P(b = x \mid C_y)$ with $a, b \in \mathcal{O}$. Note also that $y \in X_a$ and $x \in X_b$ and C_y is the $[a = y]$-selection context. By Axiom 3.2, for any context $C \in \mathcal{C}$, the set of probabilities: $\{P(a = y \mid C) : a \in \mathcal{O}\}$ is well defined. The corresponding collection of data $D(\mathcal{O}, C)$ consists of contextual probabilities: $P(a = y \mid C), P(b = x \mid C), P(b = x \mid C_y), P(a = y \mid C_x) \dots$ where $a, b \in \mathcal{O}$. There also exists a family of probabilistic data $D(\mathcal{O}, C)$ for all contexts $C \in \mathcal{C}$ which we denote by $\mathcal{D}(\mathcal{O}, \mathcal{C})$.

The formulation of the two above axioms leads to the following definition: the Växjö model (see f.i. Khrennikov (2007c)).

Definition 3.1. An observational contextual statistical model of reality is a triple $M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C}))$, where $\mathcal{C}$ is a set of contexts and $\mathcal{O}$ is a set of observables which satisfy Axioms 3.1 and 3.2, and $\mathcal{D}(\mathcal{O}, \mathcal{C})$ is probabilistic data about contexts $\mathcal{C}$ obtained with the aid of observables belonging to $\mathcal{O}$.

Inside model M, observables belonging to the set $\mathcal{O}$ give the only possible references about a context $C \in \mathcal{C}$.

We also provide for a second definition which deals with the concept of supplementarity.

Definition 3.2. Let $a, b \in \mathcal{O}$. The observable b is said to be supplementary to the observable a if $p^{b|a}(x \mid y) \equiv P(b = x \mid C_y) \neq 0$ for all $y \in X_a$ and $x \in X_b$.

[3] C can denote for instance a finance, economics or any other context.

[4] Please note that the next subsection uses similar notation in the setting of the Tversky and Shafir (1992) experiment.

The supplementarity in Definition 3.2. can be explained as follows. Following Khrennikov (2005) the physical observables $a, b \in \mathcal{O}$ produce supplementary statistical information for some context C. This means that the contextual probability distribution (say of b) cannot be reconstructed on the basis of the probability distribution of observable a (violating thus the classical total probability formula). It is the supplementarity of the observables which creates interference of the so called contextual probabilities. Of course, the notion of supplementarity has a very close association to the notion of Bohr's complementarity. The crucial difference is that the latter implies mutual exclusivity of observations of a and b. In probabilistic terms this means that the quantum formalism provides for conditional probabilities, but not for a joint probability distribution. We do not impose such a constraint. Our supplementary observables may have (or may not have) the well defined joint probability distribution. We just do not take care of this. The supplementarity allows to consider quantum-like probabilities, for both ensembles of macroscopic physical systems and cognitive systems.

In "ordinary quantum mechanics" the principle of complementarity is formulated by using the Hilbert space formalism: quantum states are represented by normalized vectors and observables by self-adjoint operators. The complementarity of observables, say a and b, is equivalent to the non-commutativity of corresponding operators, $\widehat{a}, \widehat{b} : [\widehat{a}, \widehat{b}] = \widehat{ab} - \widehat{ba} \neq 0$.

In the appendix (please see Appendix A), we explain the connection between the conventional Hilbert space formulation and our contextual probabilities.

As explained in the appendix of Khrennikov (2007a), it is important to mention that in general, transition probabilities can exhibit (cognitive) context dependency: $p_C^{b|a}(x \mid y)$ does not necessarily equal $p^{b|a}(x \mid y)$. In other words the probability of a result from b after observing a result from a also depends on the original context C. Can the dependency of $p_C^{b|a}(x \mid y)$ on C be reducible so that $p_C^{b|a}(x \mid y) = p^{b|a}(x \mid y)$ and no longer depends on C? We can give an affirmative answer. Those probabilities could be obtained as follows. Assume first that there exist cognitive systems which interact with a context C. We can build up an ensemble S_C of cognitive systems representing the context C. Elements of the ensemble S_C will interact with the selection-context C_y (see Axiom 3.1 above). As an example consider a group of graduate mathematics students, who are required to sit an exam. They need to answer a question 'a', which is part of the exam. We can say each student, ω, belongs to the ensemble S_C where C will spell out the mental and social conditions under which graduate mathematics students will have been trained (example: high levels of concentration; staying late in the office; etc...). Assume question 'a' is a mathematics exam question, which relates to study material students were instructed *not* to know. If student ω is so surprised that he/she forgets about his 'C-training', then the transition probabilities do not depend on C: $p^{b|a}(x \mid y)$. Assume the mathematics graduate students are required to answer a 'b' question, which follows the 'a' question. It is not unreasonable to assume the student ω, having been subjected to such an intensity of surprise when faced with the 'a' question, will continue to forget about his C-training when he faces question 'b'. Hence, it is entirely possible that the number of mathematics graduate students who still use the original C-context (e.g. training) to reply to the b-question (following the a-question) will be very small compared to the total number of graduate mathematics students belonging to S_C.

As we have underlined in the appendix of Khrennikov (2007a), it is important to remark that the situation described just now corresponds to what one would find in conventional quantum mechanics! In the case of non-commutative observables, $p^{b|a}(x \mid y)$

will not depend on the original context C (the context preceding the $a = y$ selection).[5] The $a = y$ selection, in quantum mechanics, will destroy the memory on the preceding (physical) context C.

What we have provided for here, is what one could call a 'contextual interpretation' of the so called von Neumann projection postulate (von Neumann, 1955). It needs to be underlined, that we do *not* know the general situation for cognitive systems. One cannot exclude the possibility that the von Neumann projection postulate is violated by cognitive systems. What would such violation imply? In such a case we would not be able to construct the conventional quantum representation of contexts by complex probability amplitudes!

Let us consider the following postulate.

Postulate 3.1. *The "von Neumann postulate for a mental observable" says that for any pair a, b of supplementary mental observables the transition probability $p^{b|a}(x \mid y)$ is completely determined by the preceding preparation — context C_y corresponding to the $[a = y]$-selection.*

We remark that by Axiom 3.1, $p^{b|b}(x \mid x) = 1$.

A weaker form of the above postulate could also be formulated.

Postulate 3.2. *The "Weak von Neumann postulate for mental observable" says there exist supplementary mental observables a, b such that the transition probability $p^{b|a}(x \mid y)$ is completely determined by the preceding preparation — context C_y corresponding to the $[a = y]$-selection.*

Our formulation of a contextual analogue of von Neumann's postulate is not based (at least from the very beginning) on the Hilbert space formalism. We do not consider the state vector and its collapse. Instead we operate solely with contextual (conditional) probabilities. In the appendix we explain the connection between the original von Neumann postulate and our probabilistic one. We also discuss the inter-relation between the von Neumann postulate, in the form of Postulate 3.2, and the classical Markov property.

3.4. More on the use of the concept of 'context' in the Tversky and Shafir (1992) experiment

How can the methodology developed in the above sub-section be applied to the setting of the Tversky and Shafir (1992) experiment? We can define, as in Khrennikov (2008), the following (please see also Section 3.2. for the description of the Tversky and Shafir (1992) experiment).

- let the roulette which generates a loss outcome, $a = -$, or a win outcome, $a = +$, be denoted as A
- the context C indicates the case where the lottery outcome of the first game is unknown
- the context C_+^A indicates the case where the lottery outcome of the first game is a known win ($a = +$)
- the context C_-^A indicates the case where the lottery outcome of the first game is a known loss ($a = -$)
- the experiment participant is denoted by the letter B.
- B is allowed to have actions $b = +$ or $b = -$ which indicate respectively that B decides or does not decide to play the second game
- the contexts in which B takes his actions $b = +$ or $b = -$, are either C, C_+^A or C_-^A
- contextual probabilities can be defined as follows:

(1) $P(b = + \mid C)$ denotes the probability of B deciding to play the second game given the lottery outcome of the first game is unknown
(2) $P(b = - \mid C)$ denotes the probability of B not deciding to play the second game given the lottery outcome of the first game is unknown
(3) $P(b = \pm \mid C_-^A)$ denotes the probability of B deciding to play the second game ($b = +$) (or not to play the second game ($b = -$)) given the lottery outcome of the first game is a known loss
(4) $P(b = \pm \mid C_+^A)$ denotes the probability of B deciding to play the second game ($b = +$) (or not to play the second game ($b = -$)) given the lottery outcome of the first game is a known win.

Using this notation, the Tversky and Shafir (1992) experimental findings can be written as:

- $P(b = + \mid C) = 0.36$ and the supplement probability (see Definition 3.2 above) probability: $P(b = - \mid C) = 0.64$
- $P(b = + \mid C_-^A) = 0.59$ and the supplement probability: $P(b = - \mid C_-^A) = 0.41$
- $P(b = + \mid C_+^A) = 0.69$ and the supplement probability: $P(b = - \mid C_+^A) = 0.31$.

Those probabilities can be put into a matrix of transition probabilities. We say that such a square matrix is stochastic if each of the rows of this matrix consists of nonnegative real numbers, which sum to 1. This is a common property of matrices of transition probabilities.

Let us now consider the concept of a *doubly-stochastic matrix*. In a doubly-stochastic matrix all entries are nonnegative and all rows and *all columns* sum to 1. In stochastic matrix terms, using our terminology, we obtain:

$$P = \begin{pmatrix} P(b = + \mid C_+^A) & P(b = - \mid C_+^A) \\ P(b = + \mid C_-^A) & P(b = - \mid C_-^A) \end{pmatrix}. \tag{8}$$

Using the results of the Tversky and Shafir (1992) experiment, we get:

$$P = \begin{pmatrix} 0.69 & 0.31 \\ 0.59 & 0.41 \end{pmatrix}. \tag{9}$$

The (classical) law of total probability,[6] using the notations we have introduced above, can be written as follows.

$$P(b = \pm) = P(a = +)P(b = \pm \mid C_+^A) + P(a = -)P(b = \pm \mid C_-^A). \tag{10}$$

Looking back to the end of Section 3.2., we recall that 36% of experiment participants will opt for a second gamble if they have no information on whether they won or lost in the first gamble. If $P(a = +) = P(a = -) = \frac{1}{2}$, we then find that: $P(a = +)P(b = + \mid C_+^A) + P(a = -)P(b = + \mid C_-^A) = \frac{1}{2}(0.69) + \frac{1}{2}(0.59) = 0.64$, which is clearly not equal to the 36%. Remark that the 64% corresponds exactly to the percentage of participants who reject playing for a second gamble when there was no information about the outcome of the first gamble. Similarly, the percentage of experiment participants who will not gamble for a second time if they have no information as to the outcome of the first gamble, is again, not the average of the probabilities of not gambling a second time given the experiments participants have been informed of the

[5] This is valid for the case where observables have non-degenerate spectra.

[6] This law can be found everywhere. For instance Wikipedia has an article, entitled "Law of total probability".

previous outcome: $P(a=+)P(b=-\mid C_+^A)+P(a=-)P(b=-\mid C_-^A)=\frac{1}{2}(0.31)+\frac{1}{2}(0.41)=0.36$.

The transition probability matrix in (9) is not doubly-stochastic. In Khrennikov (2008), we discuss the reasons why this is the case. We repeat the arguments here. In conventional quantum mechanics (in a two dimensional Hilbert space), the matrices of transition probabilities are always doubly-stochastic (von Neumann, 1955). The non-double-stochasticity of the transition probability matrices can also be observed in other experimental contexts such as the prisoner's dilemma (Khrennikov, 2008) and the so called 'Bari' experiment (Conte et al., 2006). See also Khrennikov and Haven (2006). We also note that double-stochasticity has a close link with Definition 3.2, above. See Khrennikov (2007c).

Why do we have, in the context of the Tversky and Shafir (1992) experiment, what we could call a "non-doubly-stochasticity paradox" ? There may be two possible reasons:

- the statistics resulting from the experiment are neither classical (i.e., they do not conform to the Kolmogorov measure-theoretic model) nor are they quantum (i.e. no self-adjoint operators can describe those statistics)
- could it be that the observables corresponding to real and possible actions (in the Tversky and Shafir (1992) experiment: $b=+$ or $b=-$ are examples of such actions) are not complete? From a quantum mechanics point of view, this would mean that the observables should be represented not in the two-dimensional (mental qubit) Hilbert space, but in Hilbert space of a higher dimension.[7]

The second reason immediately implies the query how to find the real dimension of the quantum state space? As is explained in Khrennikov (2008), if this dimension is not determined by values of complementary observables a and b, then we should be able to find an answer to the question: "Which are those additional mental observables which could complete the model?"

The Tversky and Shafir (1992) experimental outcome indicates, beyond any doubt, that the classical total probability law can be violated. However, we must be careful in arguing for an alternative probability rule! The alternative probability law which includes the probability interference term, is a good candidate. However, this alternative is not unique. It is important to stress that non-classical statistical data can be explained by avenues, other than the conventional quantum model.[8]

3.5. The law of total probability: Contextual probabilities and the interference term

In Section 3.4. we provided for the definition of contextual probabilities in the setting of the Tversky and Shafir (1992) experiment.

Recall that (10) was formulated with contextual probabilities which are referring to different contexts. In Khrennikov (2008) we posed the fundamental question: "can those different contexts be amalgamated?" Not always! Hence, we must immediately question the general relevance of using (10) in a contextual probabilistic setting. However, even though with this caveat, it needs to be stressed that the total probability rule for contextual probabilities does not contradict the Bayesian approach.

In this subsection we would like to re-write (10) in such a way that it incorporates the interference term required by the Tversky and Shafir (1992) experimental results. We follow Khrennikov (2008). The difference between the left-hand side and right-hand side in (10) is a measure of incompatibility between contexts C and $C_A^{\pm}$. Let us denote this measure of incompatibility by the symbol $\delta_{\pm}$. In words, the measure of incompatibility can be interpreted as a 'measure of impossibility' to combine these contexts (C and $C_A^{\pm}$) in a single space of elementary contexts.

Eq. (10) can now be augmented, with the trigonometric interference term, in the following way.

$$P(b=\pm)=P(a=+)P_{\pm,+}+P(a=-)P_{\pm,-}+2\cos\theta_{\pm}\sqrt{\Pi}, \quad (11)$$

where $P_{\pm,+}\equiv P(b=\pm\mid C_+^A)$ and similarly for $P_{\pm,-}$. We remark that the two first product terms on the right-hand side of (11) are exactly the same terms as they appear on the right-hand side of (10). We note that: $\Pi=P(a=+\mid C)P(a=-\mid C)P_{\pm,+}P_{\pm,-}$ (see also our definition of Π below).

Using Dirac (1930), quantum systems theory prescribes that the coefficient (or measure) of incompatibility has the form $2\cos\theta$ (where the angle θ is called 'the phase'[9]) multiplied by the normalization factor, which is equal to the square root of the product Π of all probabilities on the right-hand side of (10). Thus, we write:

$$\delta_{\pm}=2\cos\theta\,\sqrt{\Pi}. \quad (12)$$

The term in (12) is also called, in conventional quantum mechanics, interference. Henceforth, we define the normalized coefficient of incompatibility of contexts, as:

$$\lambda=\frac{\delta_{\pm}}{2\sqrt{\Pi}}. \quad (13)$$

We can now consider two types of interference[10]:

- trigonometric interference (conventional quantum mechanics): we can write $\lambda=\cos\theta$, where $\theta=\arccos\lambda$ and of course there is an upper bound of one
- hyperbolic interference (Khrennikov, 2005): $\lambda=\pm\cosh\theta$, where $\theta=\operatorname{arccosh}|\lambda|$.

Let us recall the discussion we had at the end of Section 3.4, about the non-double-stochasticity. In case hyperbolic interference is to be employed, the 'higher dimension' argument can not be used.

In the setting of the Tversky and Shafir (1992) experiment, we obtain the following values for $\delta_{\pm}$ and $\lambda_{\pm}$:

- $\delta_+=-0.28$, and hence $\lambda_+=-0.44$.
- The probabilistic phase is then $\theta_+=2.03$.

Since, in the general case $\delta_++\delta_-=0$ (Khrennikov, 2004), we obtain:

- $\delta_-=0.28$, and hence $\lambda_-=0.79$.
- The probabilistic phase is then $\theta_-=0.66$.

In the special case, for a doubly-stochastic matrix of transition probabilities (please see Section 3.4) this law can be derived in the conventional quantum formalism.

For our purposes (please see Khrennikov (2008)), there exists an elementary formula from the algebra of complex numbers:

$$k=k_1+k_2+2\sqrt{k_1k_2}\cos\theta=|\sqrt{k_1}+e^{i\theta}\sqrt{k_2}|^2, \quad (14)$$

7 This possibility was pointed out by Jerome Busemeyer and Ariane Lambert-Mogiliansky during the recent workshop "Applying quantum principles to psychological phenomena" (International Center for Mathematical Modeling in Physics and Cognitive Sciences, University of Växjö, Sweden; 17–18 September, 2007).

8 This point of view came to being from private communications between Andrei Khrennikov, Luigi Accardi and Diederik Aerts.

9 Please see also under Eq. (5) — Section 2, where we discuss the phase.

10 λ can be interpreted as a measure of context interference.

for real numbers $k_1, k_2 > 0, \theta \in [0, 2\pi]$. Thus:

$$k = |\psi|^2, \tag{15}$$

where $\psi = \sqrt{k_1} + e^{i\theta}\sqrt{k_2}$. We note that ψ is a wave function.

Let us compare this formula and the interference law of total probability (11). We set $k = P(b = \pm)$, $k_1 = P(a = +)P_{\pm,+}$, $k_2 = P(a = -)P_{\pm,-}$. We introduce the complex probability amplitudes:

$$\psi(\pm) = \sqrt{P(a = +)P_{\pm,+}} + e^{i\theta_\pm}\sqrt{P(a = -)P_{\pm,-}}. \tag{16}$$

The wave function defined above could be called the 'mental wave function' (it is defined on the set $\{+, -\}$ and takes complex values) representing the context C via observables a and b.

Our formalism mimics the conventional quantum formalism presented in Section 2. In particular, Born's rule which is postulated in quantum mechanics arises also in our formalism:

$$P(b = \pm) = |\psi(\pm)|^2. \tag{17}$$

Unlike quantum physics, this rule is not postulated. We designed the mental wave in such a way that this rule holds true. However, it does not play such a fundamental role as in quantum physics. In our approach the basic rule is the formula of total probability including the interference term of (11). It is important that one searches for the invariance of the theta term in (11) and discover how experimental contexts influence this parameter.

We could suggest that the brain applies such an algorithm to contextual probabilistic data. This would then mean that the brain is assumed to be able to construct the complex probability amplitude, which we have coined as the 'mental wave' function. Remark that the brain would thus work with such amplitudes and not with original probabilities!

The incompatible contexts, in the case of the Tversky and Shafir (1992) experiment outcome, can then be represented as:

$$\psi(+) \approx 0.59 + e^{2.03i}\,0.54; \quad \psi(-) \approx 0.39 + e^{0.66i}\,0.45. \tag{18}$$

To make a stronger connection between Eq. (4) of Section 2 and the above equation, let us consider the following. Define $\psi_{+,+} = \sqrt{P(a = +)P_{+,+}} = \sqrt{0.5(0.69)}$; $\psi_{-,+} = \sqrt{P(a = -)P_{-,+}} = \sqrt{0.5(0.59)}e^{2.03i}$ and $\psi_+ = \psi_{+,+} + \psi_{-,+}$. Then, $\left|\psi_{+,+}\right|^2 = P(a = +)P(b = + \mid a = +) = (0.5)(0.69)$ and $\left|\psi_{-,+}\right|^2 = P(a = -)P(b = + \mid a = -) = (0.5)(0.59)$ *but* $|\psi_+|^2 = \left|\psi_{+,+} + \psi_{-,+}\right|^2 = P(b = +) = 0.36$. Similarly, $\psi_{+,-} = \sqrt{P(a = +)P_{+,-}} = \sqrt{0.5(0.31)}$; $\psi_{-,-} = \sqrt{P(a = -)P_{-,-}} = \sqrt{0.5(0.41)}e^{0.66i}$ and $\psi_- = \psi_{+,-} + \psi_{-,-}$. Then $\left|\psi_{+,-}\right|^2 = P(a = +)P(b = - \mid a = +) = (0.5)(0.31)$ and $\left|\psi_{-,-}\right|^2 = P(a = -)P(b = - \mid a = -) = (0.5)(0.41)$ *but* $|\psi_-|^2 = \left|\psi_{+,-} + \psi_{-,-}\right|^2 = P(b = -) = 0.64$. In this model, we can represent the initial state as: $\psi_0 = \begin{bmatrix}\sqrt{0.5}\\ \sqrt{0.5}\end{bmatrix}$ and we can represent the operator as a matrix: $V = \begin{bmatrix}\sqrt{0.69} & \sqrt{0.59}e^{2.03i}\\ \sqrt{0.31} & \sqrt{0.41}e^{0.66i}\end{bmatrix}$ and then the final state is $\psi = V\psi_0$. Note that V is not a transition matrix and it is not a unitary matrix either.

4. The Ellsberg paradox, arbitrage and the information wave function

In this section of the paper we intend to utilise the wave function concept in the relationship the Ellsberg paradox has with financial arbitrage. Important work has already been performed in this area. The approach by La Mura (2005) provides for a quantum-like approach which is able to take into account the Ellsberg's paradox. The formalism proposed in that paper uses only real numbers, and the parameters of the matrices involved are suitably chosen. We also need to mention other important work by Franco (2008) who provides for a description, within the quantum formalism, of the two-color Ellsberg's paradox (the simpler version of the paradox) which focuses attention on the subjective probabilities. Finally, Bordley (2005) argues for the use of probability amplitudes in the context of another important paradox: Allais paradox.

4.1. The Ellsberg paradox and arbitrage

In Khrennikov and Haven (2007) we describe the connection between the Ellsberg paradox and arbitrage. Bossaerts et al. (2007), interpret the red, green and blue balls as financial securities.[11] It is assumed in Bossaerts et al. (2007) that each security pays the same fixed amount (1 unit of currency) according to the draw of the color. One could claim, as in Bossaerts et al. (2007), that the 'blue' and 'green' securities are 'ambiguous', in the sense that the distribution of their payoffs, in the Ellsberg paradox, is unknown.

If we denote the probability of drawing a red ball, as p_r, then $p_r = 1/3$. It is quite easy to observe that $G1 \succ G2$ implies that $p_r > p_b$, assuming the units of currency one can gain will coincide with the level of utility (or satisfaction) of this gain. Similarly, the choice $G4 \succ G3$ will imply the *opposite*: $p_b > p_r$. Hence, if the price of the 'red' security, which we denote as q_r, was a function of the 'information' content of prize winning,[12] then the price of the 'red' security should exceed the price of the 'blue' security (in gambles 1 and 2): $q_r > q_b$. However, in gambles 3 and 4, the opposite effect will occur: $q_r < q_b$.

If we let gambles 1 and 2 to refer to one market, say market X. Moreover, if we let gambles 3 and 4 refer to another market, say market Y, then one can simultaneously buy a red security in market Y and sell an identically same red security in market X. Similarly, we can simultaneously buy the blue security in the X market and sell an identically same blue security in the Y market. Hence, as Bossaerts et al. (2007) remark, the Ellsberg paradox in effect admits an arbitrage opportunity.

What is an arbitrage opportunity? It probably is best to define this concept by saying what it is not. Higham (2004) defines the absence of arbitrage in the following way: "(t)here is never an opportunity to make a risk-free profit that gives a greater return than that provided by the interest from a bank deposit". We assume the bank deposit interest rate is equal to the risk-free rate of return, which is the return given to assets which cannot default (i.e. you assume the bank cannot go bankrupt). This rate of return is very different from the risky rate of return which typically characterises risky assets such as stocks. Assume the risky rate of return were to exist on the (non-defaultable) bank deposit. Then, the difference between the risky rate of return and the risk-free rate of return would be the arbitrage return. The difference in return has *not* occurred because there is a change in the risk position of the asset (it still is a non-defaultable bank deposit).

If we were to apply this reasoning to the case of the Ellsberg paradox 'security price' gamble, then the act of simultaneously buying a 'blue' security at a lower price in market X and selling a same 'blue' security in a different market Y, at a higher price, would constitute an arbitrage opportunity. The difference in price (and hence the ensuing price return) has *not* occurred because there is a change in the risk position of the 'blue' security. In effect the 'blue' security is identically the same security whether it is bought and

11 A share of a stock, or a bond are all examples of such securities. Please see also below for the formulation of the non-arbitrage theorem.

12 We remark that the distribution of payoffs for the red security is known. We can not say the same about the blue and green securities.

sold in market X or market Y. The same reasoning can be applied to the 'red' security.

We can thus claim that the Ellsberg paradox, when extended into the security setting we described above, will allow for, what Bossaerts et al. (2007) proposed, an arbitrage opportunity. In Section 3, we have attempted to explain the paradox via the probability interference argument. Hence, it is reasonable to claim that the existence of arbitrage could be explained by making recourse to quantum physical concepts. The question then becomes: which concepts?

4.2. The discrete state space non-arbitrage theorem

We need two concepts to explain arbitrage: (i) the use of the discrete state space non-arbitrage theorem and (ii) the quantum mechanical wave function as an information function (see the next subsection below). The discrete state space non-arbitrage theorem, which we will consider in simple form[13] in this subsection, forms the basis of a lot of financial asset pricing theory. Harrison and Kreps (1979) is the key reference in this regard and Duffie (1996) discusses the proof of the theorem. Etheridge (2002) also provides for an excellent treatment of the subject. There exists a more high powered (continuous state space) version of the theorem (see Kabanov and Stricker (2005)). The theorem below (discrete state space) details the conditions under which no-arbitrage is guaranteed. We follow Etheridge (2002). See also Haven (2008b) where this theorem is also discussed.

Theorem 4.1. *Assume there are N tradable assets (some assets may be risky and some not) and their prices, at time t_0 are given by $\overrightarrow{p_0} = (p_0^1, p_0^2, \ldots, p_0^N)$. Assume there exists a K (where K indicates the K states of the world) dimensional state price vector $\overrightarrow{\Phi} = (\Phi_1, \Phi_2, \ldots, \Phi_K)$ which is strictly positive in all coordinates. Consider the following model:* $\begin{pmatrix} p_0^1 \\ p_0^2 \\ \cdot \\ \cdot \\ p_0^N \end{pmatrix} = \Phi_1 \begin{pmatrix} D_{11} \\ D_{21} \\ \cdot \\ \cdot \\ D_{N1} \end{pmatrix} + \Phi_2 \begin{pmatrix} D_{12} \\ D_{22} \\ \cdot \\ \cdot \\ D_{N2} \end{pmatrix} + \cdots \Phi_K \begin{pmatrix} D_{1K} \\ D_{2K} \\ \cdot \\ \cdot \\ D_{NK} \end{pmatrix}$*, where each N dimensional vector $\overrightarrow{D_1}, \ldots, \overrightarrow{D_K}$ is the security price vector at time t_1, if the market is, respectively, in state $1, \ldots, K$. For the market model described here there is no arbitrage if and only if there is a state price vector.*

Etheridge (2002) defines the vector $\overrightarrow{\Phi_{prob}} = \left(\frac{\Phi_1}{\Phi_0}, \frac{\Phi_2}{\Phi_0}, \ldots, \frac{\Phi_K}{\Phi_0}\right)$, where each coordinate is a probability and $\Phi_0 = \exp(-rT)$ is the discount rate (continuously discounted) at the risk-free rate of return, r, and T is time.

Example 4.1. Assume, that a bond price at time t_0 is normalised to 1. Assume this is price p_0^1 (in Theorem 4.1 above), $p_0^1 = 1$. Assume the payoff of the bond in any of the K states of the world is the same. Hence, we can write that: $D_{11} = D_{12} = \cdots D_{1K}$. If the states of the world are evocative of the possible states of the economy, then the bond pays the same no matter what the states of the economy are. Hence, it is a risk-free bond. Assume, the bond payoff in any of the states of nature, $1, 2, 3, \ldots, K$, is each time: $1 + r$, where r is the risk free rate of return (please see Section 4.1 for a discussion of this type of return). We can thus write: $p_0^1 = 1 = \Phi_1(1 + r) + \Phi_2(1 + r) \ldots \Phi_K(1 + r)$. Given the positivity of $\overrightarrow{\Phi}$ and the fact that the risk-free rate cannot be negative, we can set: $p_1 \equiv (1 + r)\Phi_1; p_2 \equiv (1 + r)\Phi_2; \ldots p_K \equiv (1 + r)\Phi_K$; where $p_1, p_2, \ldots, p_K$ are so-called risk neutral probabilities. See also Neftci (2000) for more examples.

[13] We mean in a discrete state form (as opposed to a continuous state form).

4.3. The wave function as an 'information wave function'

We would like to keep in mind our objective on how risk-neutral probabilities can be connected to probabilities which are generated via an (information) wave function. It is appropriate to discuss this issue because of the connection arbitrage has with the Ellsberg paradox (and the sure-thing principle). Please see Section 4.1 above. Before we can really tackle this issue, we need to define what we mean with an 'information wave function'.

Bohm and Hiley (1993) and Bohm (1952) did propose that the wave function, when interpreted as a so called 'pilot wave' steers a particle (please see below (under Definition 4.1.) for more information on this). Following the work of Bohm and Hiley (1993), Khrennikov (1999) was the first to show how a quantum-mechanical wave function can be used to model information in a macroscopic setting. Please see Khrennikov (1999, 2002, 2004) but see also Choustova (2006, 2007) and Haven, (2005, 2008a, 2007).

Hence, in heuristic terms, if the wave function symbolises information, and if the particle's position symbolises an asset price, then the 'pilot wave' model holds the promise to providing for a set of analytical tools which can aid in understanding how information can steer asset prices.

We can define, as for instance in Haven (2008a), the information wave function as follows.

Definition 4.1. We define the information wave function, denoted $\psi(q, t)$, as $\psi(q, t) : \mathbb{R} \to \mathbb{C}$ such that $\psi(q, t) \equiv R(q, t) \exp(iA(q, t))$; where $R(q, t)$ is some function of the position variable q and time t; $A(q, t)$ is some other function of the position variable q and time t. Moreover, i is a complex number.

As is further remarked in Haven (2008a), (i) $R(q, t)$ is the absolute value (amplitude of $\psi(q, t)$) of the complex number $\psi(q, t) = a$ and ii) $A(q, t)$ is the argument (phase of $\psi(q, t)$) of the complex number a.

We want to cite here too (as in Haven (2008a)) the work by Holland (1993). Holland (1993) remarks that physics Nobel prize winner, Prince Louis de Broglie attributed two roles to the wave function, $\psi(q, t)$ (p. 16): "not only does it determine the likely location of a particle it also influences the location by exerting a force on the orbit." From this statement we can intuitively understand the term 'pilot wave theory'. Furthermore, we can see appear the particle-trajectory interpretation of quantum mechanics.

4.4. Macroscopic significance of the Bohmian quantum potential

The term 'pilot wave theory' can probably be best understood by defining the so called 'quantum potential' of Bohmian mechanics. The interested reader is referred to Holland (1993) for an excellent treatment on (i) how this type of potential (naturally) emerges in Bohmian mechanics and (ii) how this potential compares to the real potential. The quantum potential, Q, is defined as follows. See Choustova (2006, 2007) and Khrennikov (1999, 2004), who were the first to interpret Q in a macroscopic context.

Definition 4.2. $Q(q, t) \equiv -\frac{1}{R(q,t)} \frac{\partial^2 R(q,t)}{\partial q^2}$.

In the papers by Choustova (2006) and Khrennikov (1999, 2004), Q is named the 'financial mental potential'. The 'financial mental force' (see Choustova (2006) and Khrennikov (1999, 2004)) is defined as follows.

Definition 4.3. $F(q, t) \equiv -\frac{\partial Q(q,t)}{\partial q}$.

Consider the following example. See also Haven (2008a,b).

Example 4.2. Let $q = p$ be the price of some asset. Let $R(p) \equiv p^3$. Then $Q(p) = -\frac{6}{p^2}$ and $F(p) = \frac{12}{p^3}$. This indicates that with the price p increasing, the effect of $F(p)$ becomes smaller and smaller. So the price should stop increasing.

As we remarked in Haven (2008a), $F(p)$ can thus be interpreted as a pricing rule. We hope the background we presented here can support the claim of using the wave function, $\psi(q, t)$ as an information wave function.

4.5. Connecting risk neutral probabilities with the information wave function

Consider, as in Choustova (2006), the space $L^2(\mathbb{R})$ of square integrable functions $\psi : \mathbb{R} \to \mathbb{C}$. Assume that ψ belongs to the space $L^2(\mathbb{R}, dM)$ with respect to some measure M on $\mathbb{R}$ such that: $\|\psi\|^2 = \int_{\mathbb{R}} |\psi(q)|^2 \, \mathrm{d}M(q) < \infty$.
Consider the following proposition drawn from Haven (2008b).

Proposition 4.1. *Let there exist an N-dimensional asset price vector $\overrightarrow{p_0}$ and a K-dimensional state price vector $\overrightarrow{\Phi}$. Let there exist a K-dimensional probability vector $\overrightarrow{\Phi_{prob}} = \left(\frac{\Phi_1}{\Phi_0}, \frac{\Phi_2}{\Phi_0}, \frac{\Phi_3}{\Phi_0}, \ldots, \frac{\Phi_K}{\Phi_0}\right)$. Let $N = K$ and let $\Phi_0 = \exp(-rT)$ be fixed. Let there be an information wave function $\psi(q)$ and a measure M on $\mathbb{R}$. Let each of the probabilities in $\overrightarrow{\Phi_{prob}}$ be drawn from $\|\psi\|^2 = \int_{\mathbb{R}} |\psi(q)|^2 \, dM(q)$ for each of a respective set of lower and upper bound values of the integral. Let the state prices which are in $\overrightarrow{\Phi_{prob}}$ guarantee no arbitrage. Consider now an information wave function $\psi^*(q)$ which has a different functional form from $\psi(q)$ in the following way. For the same measure M on $\mathbb{R}$ and for the same respective set of lower and upper bound values of the integral we used for $\int_{\mathbb{R}} |\psi(q)|^2 \, dM(q)$, we write $\int_{\mathbb{R}} |\psi^*(q)|^2 \, dM(q)$ such that the functions $|\psi^*(q)|^2$ and $|\psi(q)|^2$ can be allowed to overlap on different intervals of their domain but under the constraint that at least one probability drawn from $\|\psi^*\|^2 = \int_{\mathbb{R}} |\psi^*(q)|^2 \, dM(q)$ must be different from the corresponding probabilities drawn from $\int_{\mathbb{R}} |\psi(q)|^2 \, dM(q)$. Under those conditions will the change in the information wave function from $\psi(q)$ to $\psi^*(q)$ trigger arbitrage.*

From the above, it seems reasonable that we can use the information wave function in the basic notion of probability value.
What have we shown in this section? We have been using a very basic quantum mechanical tool to model a very basic economics phenomenon (arbitrage) which happens to be linked to a basic paradox in psychology (Ellsberg's paradox).

4.6. Ambiguity and the information wave function

Let us re-visit once more the Ellsberg paradox. We had 30 red balls and 60 blue and green balls. We do not know the proportions of blue or green balls. Clearly, one could make the assumption there is an equal amount of green and blue balls. Ambiguity is now indeed completely ruled out. As we have remarked in Khrennikov and Haven (2007), using this rule would indicate one is using the principle of insufficient reason (Kreps, 1988, p. 146) — which says: "if (one) has no reason to suspect that one outcome is more likely than another, then by reasons of symmetry the outcomes are equally likely and hence equally likely probabilities can be ascribed to them".
Given the assumption of equal amounts of green and blue balls, there is indifference between gambles 1 and 2, and between gambles 3 and 4. If the price of the red security, q_r, is again a function of the information content of prize winning, then (in gambles 1 and 2): $q_r = q_b$. Furthermore, in gambles 3 and 4: $q_r = q_b$. Clearly, there is no possibility for arbitrage.

We can use Proposition 4.1. Let each of the probabilities in $\overrightarrow{\Phi_{prob}}$ be drawn from $\|\psi\|^2 = \int_{\mathbb{R}} |\psi(q)|^2 \, dM(q)$ for each of a respective set of lower and upper bound values of the integral. Let the state prices which are in $\overrightarrow{\Phi_{prob}}$ guarantee no arbitrage. In the context of the Ellsberg paradox (without ambiguity), we can given specific meaning to this particular information wave function. This information wave function contains now highly precise information: "we have an equal chance of obtaining a blue or green ball (i.e. security)". Any information change, which is reflected into a functional form change from $\psi(q)$ to some $\psi^*(q)$, so as to echo the existence of ambiguity, will yield arbitrage.

Example 4.3. Assume that the measure M(q) is a uniform measure and that $\psi(q)$ is constant then the prices q_i will have a uniform distribution.

Thus, in summary, we have two information wave functions which have highly specific information. One information wave function contains the information of 'no ambiguity', while the other wave function contains the information of 'existence of ambiguity'.

5. Conclusion

In this paper we have tried to show how probability interference and the use of contextual and wave function based probabilities, can be used in a social science setting. Those concepts can help us to better understand issues related to cognition and decision making. We strongly believe that more work is needed in this area.
Here are some of the issues which need more treatment in the future.

- Why do we have, in the context of the Tversky and Shafir (1992) experiment, what we could call a "non-doubly-stochasticity paradox"? We have mentioned two possible reasons. Are there more possible reasons?
- It is highly important we can incorporate variables, in the functional form of the wave function, which echo certain experimental factors.

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Appendix. Comparing von Neumann's postulate for quantum physical and mental observables, complementarity and supplementarity

A.1. Born's rule

We recall that a self-adjoint operator $\widehat{a}$ has a purely discrete non-degenerate spectrum if (a) all its eigenvalues are distinct: $\widehat{a}e_j = y_j e_j, y_j \neq y_i, i \neq j$; (b) the system of eigenvectors $\{e_j\}$ is a basis (and it is orthogonal, since $\widehat{a}$ is self-adjoint; the latter also implies that all y_j are real). Thus, each state ψ can be represented as a superposition of eigenstates:

$$\psi = \sum_j c_j e_j, \tag{A.1}$$

where $c_j = (\psi, e_j), \sum_j |c_j|^2 = 1$. Here $(\cdot, \cdot)$ denotes the scalar product in the state space. By Born's rule (which provides the probabilistic interpretation of quantum formalism) we have for the probability to obtain the result $a = y_j$ for measurements of an ensemble of systems prepared in the same state ψ:

$$P_\psi(a = y_j) = |(\psi, e_j)|^2. \tag{A.2}$$

A.2. Von Neumann's Postulate

We now remind the von Neumann's postulate in "ordinary quantum mechanics".

Projection Postulate. *Let a be a physical observable represented by a self-adjoint operator $\widehat{a}$ having a purely discrete non-degenerate spectrum. Any measurement of the observable a on the quantum state ψ induces a transition from the state ψ into one of the eigenvectors e^a_k of the operator $\widehat{a}$.*

von Neumann (1955), p. 216 writes: "Under the above assumption on $\widehat{a}$, a measurement of a has the consequence of changing each state ψ into one of the states $e^a_1, e^a_2, \ldots,$ which are connected with respective results of measurement $y_1, y_2, \ldots$ The probabilities of these changes are therefore equal to the measurement probabilities for $y_1, y_2, \ldots,$".

Thus, by the conventional interpretation of quantum mechanics (the Copenhagen interpretation) each measurement of the observable a induces the "collapse" of superposition (A.1) into one of the eigenvectors. However, such an interpretation of von Neumann's postulate has never been commonly accepted! Even orthodox adherents of Copenhagen's interpretations have been permanently expressing doubts in the validity of this postulate. Nevertheless, von Neumann's postulate implies special probabilistic predictions which are very natural and which are commonly accepted. A probabilistic message of this postulate was encoded in "probabilities of these changes are therefore equal to the measurement probabilities".

Let us consider another observable, say b, represented by a self-adjoint operator $\widehat{b}$. For simplicity, we assume again that it has a purely discrete non-degenerate spectrum: $\widehat{b}e^b_j = x_j e^b_j$. Suppose that one performs conditional measurements: first of a and then of b. It is assumed that a huge ensemble of quantum systems prepared in some state ψ is available. How can one find the conditional ("transition") probability $P_\psi(b = x_j \mid a = y_i)$? Suppose that the result of the first measurement is $a = y_i$. Thus, the state ψ has been collapsed into the state e^a_i. Now the b-measurement is performed. Suppose the result $b = x_j$ is obtained. To find the probability of this result, one need just apply Born's rule to the measurement of b for the state e^a_i. The answer is: $P_\psi(b = x_j \mid a = y_i) = |(e^b_j, e^a_i)|^2$. It is crucial for our considerations that it *does not depend on the initial state ψ, but only on the preceding measurement.* The latter can be interpreted as preparation: the selection of all systems giving the result $a = y_j$. In particular, one can omit the index ψ, and operate with probabilities $p^{b|a}(b = x_j \mid a = y_i) \equiv P(b = x_j \mid a = y_i)$. This is the idea which has been encoded in Postulate 3.2.

A.3. The Markovian property and quantum-like non-Kolmogorovness

In such a probabilistic formulation von Neumann's postulate reminds us of the ordinary Markov property. The crucial difference is that the latter is formulated in the classical probabilistic framework: all probabilities are based on a single Kolmogorov probability space. In the quantum framework as well as in our (more general) contextual quantum-like framework, conditional and more general contextual probabilities need not be related to the same Kolmogorov space.

A.4. Complementarity

Let a and b be two incompatible quantum observables. They are represented by non-commuting operators $\widehat{a}, \widehat{b}$. Let us consider the case of observables with purely discrete non-degenerate spectra. For purposes of our paper, it suffices to restrict considerations to the two-dimensional case. Non-commutativity of operators immediately implies that $p^{b|a}(b = x_j \mid a = y_i) = |(e^b_j, e^a_i)|^2 \neq 0$.

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