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T-Spherical Fuzzy Frank Aggregation Operators and Their Application to Decision Making With Unknown Weight Information

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ABSTRACT The current study presents a novel multi-criteria decision making (MCDM) approach to address decision analysis with T-spherical fuzzy data, whose weights of criteria are fully unknown. To serve the purpose, we design some generalized operational laws, namely Frank operational laws for T-spherical fuzzy numbers (T-SFNs) using Frank t-norm and t-conorm. Then, based on the proposed operations, a range of T-spherical fuzzy aggregation operators is developed to aggregate T-spherical fuzzy information efficiently. Also, discuss their particular cases and desirable properties are well proved. Next, we propound the T-spherical fuzzy entropy measure and its capability to fulfil the required properties. Then it is further used for criteria weight determination in the proposed aggregation based MCDM approach. In addition, a descriptive example is provided for viewing the applicability of the established approach. Lastly, the superiority and validity of this approach are highlighted by parameter analysis and comparative analysis.

INDEX TERMS Frank t-norm and t-conorm, frank aggregation operators, MCDM, T-spherical fuzzy set.

I. INTRODUCTION

Decision-making is a frequent and daily action in human existence to select the best alternative based on a set of criteria. Because of its great capacity to represent information uncertainty, decision-making has been broadly explored and fruitfully applied to management, economics, and other disciplines in the last few years. Owing to the uncertainty of decision data, using fuzzy set theory to solve decision making problems has grown popular in recent years. The notion of fuzzy set (FS) was pioneered by Zadeh [1], and, since then, it becomes a hot topic for researchers. Till date, many theoretical developments on FS have been put forward to cope with uncertain situations. However, in some cases, the concept of FS is ineffective. For instance, when a person is presented with knowledge in the form of truth and falsity grades, the FS theory is incapable of dealing with it. To address such difficulties, Atanassove [2] created the theory of intuitionistic FS (IFS) by inserting the term of falsehood degree in the disadvantage of FS. Compared to current FS, IFS is far more helpful due to its constraint that the total of truth and falsity degrees is between [0, 1]. IFS is a comprehensive and strong approach for dealing with complex and untrustworthy data in decision making situations. Many researchers have applied IFS theory in many domains [3]–[6].

But, if a person provides such values, the total of which exceeds the unit interval, the IFS is unable to handle it. Yager [7] explored Pythagorean FS (PyFS) for dealing with such problems by altering the IFS rule that the sum of the squares of the truth and falsity degrees are confined to [0, 1]. Compared to the current IFS, PyFS is a more powerful approach for dealing with complex and untrustworthy information in decision making problems. Many researchers have applied PyFS theory in a variety of disciplines [8]–[10]. Furthermore, Yager [11] changed the PyFS condition to study the theory of q-rung orthopair FS (qROFS) with a requirement that the total of the q-powers of the truth and falsity degrees cannot be greater than the unit interval. Because of its structure, the q-ROFS has been widely used and garnered increased interest from many researchers. Numerous authors have used the q-ROFS theory to the disadvantages of various disciplines [12]–[14].

Although q-ROFS can cope with partial and unclear assessment information, it cannot deal with conflicting information

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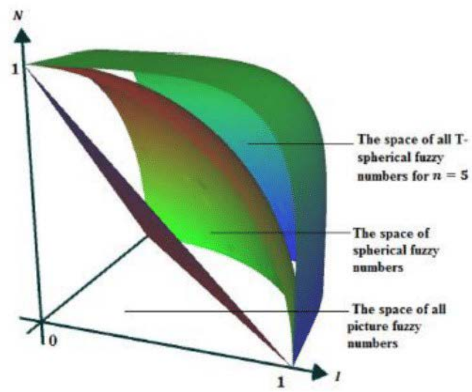

FIGURE 1. Comparison of spaces of TSFNs with PFNs and SFNs.

well in real-world circumstances. For instance, in Son's work [15], voting results for the election of village director may be split into three categories: "vote for," "neutral voting," and "vote against." "Neutral voting" implies "no vote." that the voting paper is white and rejects both agree and disapprove for the candidate, yet it still accepts the vote. This case occurred in reality, but q-ROFS was incapable of dealing with it. Cuong *et al.* [16] developed Picture FS (PFS) to address these issues, which includes three types of information: yes, neutral, and no. It is capable of dealing with inconsistencies in data. Many notable contributions have been made to the study of PFS until now; for instance, Wei [17] proposed various unique aggregation operators for PFS and addressed their applicability in decision making issues. Ashraf *et al.* [18] identified flaws in pre-existing operational rules and developed innovative enhanced aggregation operators to deal with uncertainty in complicated real world decision making problems in picture fuzzy setting. Khan *et al.* [19] defined the new extension, namely generalized Picture fuzzy soft set (PFSS) and addressed its potential applications in group decision making problems. But, there were still some difficulties; when a person provides such numbers, the total of which exceeds the unit interval, the PFS is unable to handle it. Mahmood *et al.* [20] initiated spherical FS (SFS) for dealing with such difficulties by changing the PFS rule such that the total of the squares of the truth, abstinence, and falsehood degrees is confined to [0, 1]. Compared to existing PFS, SFS is a more robust approach for dealing with complex and untrustworthy information in decision making problems. Furthermore, Mahmood *et al.* [20] changed the SFS condition to examine the theory of T-spherical FS (T-SFS) with a requirement that the total of the t-powers of the truth, abstinence, and falsity degrees is not greater than the unit interval. Because of its generalized structure, the T-SFS has been widely used and drawn more interest from many researchers [21]–[23]. The geometrical analysis of the spaces of PFS, SFS, and T-SFS spaces is depicted in Fig. 1.

As aggregation operators play a key role in decision-making issues, numerous scholars have made significant contributions to introducing aggregation operators for spherical fuzzy environments. Ashraf and Abdullah [24] presented some families of aggregation operators based on Archimedean t-norm and t-conorm with spherical fuzzy information. Chinram *et al.* [25] worked on the uncertainty to probe the best power plant location in Pakistan using SFS and Yager aggregation operators. But, with the passage of time, it is noticed that these operators [24], [25] cannot model DMs opinions when we obtain information in the form of a triplet like $\langle 0.6, 0.7, 0.6\rangle$ where the sum of squares of truth, abstinence, and falsehood degrees exceeds 1, i.e., $0.6^2 + 0.7^2 + 0.6^2 \nleq 1$. To tackle such issues, Garg *et al.* [26] studied some geometric aggregation operators based on their developed T-spherical fuzzy operational rules. Quek and his coworkers [27] investigated some generalized T-spherical fuzzy weighted aggregation operators and applied them to a problem related to the degree of pollution. Liu *et al.* [28] initiated the concept of normal T-SFNs and their relevant theory. They further put forward normal T-spherical fuzzy Maclaurin symmetric mean operator and explored a novel MCDM approach. Guleria and Bajaj [29] several averaging and geometric aggregation operators for T-spherical fuzzy soft numbers. Based on their proposed operators, they provided an MCDM technique for handling complex decision making problems. Zeng *et al.* [30] highlighted the drawbacks of the existing intuitionistic fuzzy and T-spherical fuzzy Einstein aggregation operators and studied some novel T-spherical fuzzy Einstein aggregation operators along with their desired properties. However, their proposed operations still have several weak points, which are pointed out in Section VII-D. Recently, Ullah *et al.* [31] combined the concept of T-SFNs and Hamacher aggregation operators.

From the above analysis, it is evident that the existing aggregation operators concentrated on the algebraic, Einstein, and Hamacher norms under T-SFSs to formulate the combination process. Algebraic, Einstein, and Hamacher product and sum are not only fundamental T-SFS operations that characterize the union and intersection of two T-SFSs. A generalized norm may be utilized to build a general union and intersection under T-spherical fuzzy information; that is, instances of deferent-norms families can be used to execute the corresponding intersections and unions under T-spherical fuzzy environment. Frank product and sum are suitable substitutes of the algebraic, Einstein and Hamacher product for an intersection and union and can deliver smooth estimates of the algebraic product and sum. But it seems that in the literature, there is no research on aggregation operators utilizing these operations on T-SFSs.

In view of the motivations mentioned above, the main novelties of the current study are delineated as follows:

1) To define some generalized operation rules for T-SFNs by using Frank t-norm and Frank t-conorm, which can provide more choices for the DMs.
2) To originate T-spherical fuzzy Frank arithmetic and geometric aggregation operators, including T-spherical fuzzy Frank weighted average (T-SFFWA) operator, T-spherical fuzzy Frank ordered weighted averaging

(T-SFFOWA) operator, T-spherical fuzzy Frank hybird averaging (T-SFFHA) operator, T-spherical fuzzy Frank weighted geometric (T-SFFWG) operator, T-spherical fuzzy Frank ordered weighted geometric (T-SFFOWG) operator, T-spherical fuzzy Frank hybrid geometric (T-SFFHG) operator. Further, some basic properties like idempotency, monotonicity, boundedness, homogeneity and some limiting cases of these operators are also investigated.

3) To propose the entropy measure for T-spherical fuzzy information, which can help to obtain the unknown weights information of the criteria.
4) To develop an MCDM model based on the proposed T-SFFHA operator and T-SFFHG to handle the T-spherical fuzzy decision problems with unknown weight information.
5) A practical case concerning the investment problem is considered to manifest the implementation of the suggested approach. Moreover, the impacts of the parameters on the ranking results of alternatives are analyzed in depth.

The setup of this paper is listed as follows: Some basic knowledge of T-SFNs and the notion of Frank t-norm and t-conorm are briefly reviewed in Section II. Section III presents the Frank operations of T-SFNs and some of their characteristics. Section IV, presented as the cornerstone of this work, develops T-SFFHWA and T-SFFHWG operators together with the associated proof of their required properties. T-spherical fuzzy entropy measure is added in Section V along with the detailed proof of its characteristics. Section VI constructs an MCDM method based on proposed operators. Section VII gives a case study concerning company selection for investment to illustrate the implementation of the proposed approach. Detail sensitivity analysis and comparative study are also performed in this section. The last section presents some concluding remarks summarizing the article.

II. SOME BASIC CONCEPTS

The notion of T-SFS is propounded by Mahmood *et al.* [20] as a synthesis of SFS to offer a broader range of preferences for DMs and enable them to express their hesitation about an alternative. Some basic definitions of T-SFS and terms relevant to planned work are delineated as follows.

Definition 1 [20]: Let Y be a given nonempty set. A T-spherical fuzzy set (SFS) $\mathcal{S}$ on Y is given by

$$\mathcal{S} = \{(y, \sigma(y), \vartheta(y), \varrho(y)) \,|y \in Y\}, \tag{1}$$

where $\sigma(y)$, $\vartheta(y)$, $\varrho(y) \in [0, 1]$ denote the membership, neutral and non-membership grades of $y \in Y$ to the set $\mathcal{S}$, respectively, with the restriction that $0 \leq \sigma^t(y) + \vartheta^t(y) + \varrho^t(y) \leq 1$. The degree of refusal is $\pi(y) = \sqrt[t]{1 - \sigma^t(y) - \vartheta^t(y) - \varrho^t(y)}$. For convince, $(\sigma(y), \vartheta(y), \varrho(y))$ is called a T-spherical fuzzy number (T-SFN), labelled by $\mathcal{S} = (\sigma, \vartheta, \varrho)$.

Remark 1:

- The Definition 1 reduced to SFS if we set $t = 2$.
- The Definition 1 reduced to PFS if we set $t = 1$.
- The Definition 1 reduced to q-OFS if we set $\vartheta = 0$.
- The Definition 1 reduced to PyFS if we set $t = 2$ and $\vartheta = 0$.
- The Definition 1 reduced to IFS if we set $t = 1$ and $\vartheta = 0$.

Definition 2: [32]: Let $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ and $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ be two T-SFNs and $\eta > 0$, then

1) $\mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\begin{array}{c}\sqrt[t]{\sigma_1^t + \sigma_2^t - \sigma_1^t\sigma_2^t}, \vartheta_1\vartheta_2, \\ \varrho_1\varrho_2\end{array}\right);$
2) $\mathcal{S}_1 \otimes \mathcal{S}_2 = \left(\begin{array}{c}\sigma_1\sigma_2, \sqrt[t]{\vartheta_1^t + \vartheta_2^t - \vartheta_1^t\vartheta_2^t}, \\ \sqrt[t]{\varrho_1^t + \varrho_2^t - \varrho_1^t\varrho_2^t}\end{array}\right);$
3) $\mathcal{S}_1^\eta = \left(\begin{array}{c}\sigma_1^\eta, \sqrt[t]{1 - (1 - \vartheta_1^t)^\eta}, \\ \sqrt[t]{1 - (1 - \varrho_1^t)^\eta}\end{array}\right);$
4) $\eta\mathcal{S}_1 = \left(\sqrt[t]{1 - (1 - \sigma_1^t)^\eta}, \vartheta_1^\eta, \varrho_1^\eta\right);$
5) $\mathcal{S}_1^c = (\varrho_1, \vartheta_1, \sigma_1)$.

Definition 3 [20], [33]: $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ and $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ be any two T-SFNs, let $S(\mathcal{S}_1) = \sigma_1^t - \vartheta_1^t - \varrho_1^t + \left(\frac{\exp^{\sigma_1^t - \vartheta_1^t - \varrho_1^t}}{\exp^{\sigma_1^t - \vartheta_1^t - \varrho_1^t} + 1} - \frac{1}{2}\right)\pi^t$ and $S(\mathcal{S}_2) = \sigma_2^t - \vartheta_2^t - \varrho_2^t + \left(\frac{\exp^{\sigma_2^t - \vartheta_2^t - \varrho_2^t}}{\exp^{\sigma_2^t - \vartheta_2^t - \varrho_2^t} + 1} - \frac{1}{2}\right)\pi^t$ be the score values of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and let $A(\mathcal{S}_1) = \sigma_1^t + \vartheta_1^t + \varrho_1^t$ and $A(\mathcal{S}_2) = \sigma_2^t + \vartheta_2^t + \varrho_2^t$ be the accuracy values of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then,

1) If $S(\mathcal{S}_1) < S(\mathcal{S}_1)$, then $\mathcal{S}_1 < \mathcal{S}_2$;
2) If $S(\mathcal{S}_1) = S(\mathcal{S}_1)$, then
   a. If $A(\mathcal{S}_1) < A(\mathcal{S}_1)$, then $\mathcal{S}_1 < \mathcal{S}_2$;
   b. If $S(\mathcal{S}_1) = S(\mathcal{S}_1)$, then $\mathcal{S}_1 = \mathcal{S}_2$.

As an important tool in information fusion, T-spherical fuzzy aggregation operator has received much attention, Mahmood *et al.* [34] propound the T-spherical fuzzy weighted averaging (T-SFWA) operator and the T-spherical fuzzy weighted geometric (T-SFWG) operator as follows:

Definition 4 [34]: Let $\mathcal{S}_i\,(i = 1, 2, \ldots, n)$ be a collection of SFNs, then the T-spherical fuzzy weighted averaging (T-SFWA) operator is a mapping $\mathcal{S}^n \longrightarrow \mathcal{S}$ such that

$$\begin{aligned} &T - SFWA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \\ &= w_1\mathcal{S}_1 \oplus w_2\mathcal{S}_2 \oplus \ldots \oplus w_n\mathcal{S}_n \\ &= \left(\left(1 - \prod_{k=1}^{n}\left(1 - \sigma_k^t\right)^{w_k}\right)^{1/t}, \prod_{k=1}^{n}(\vartheta_k)^{w_k}, \prod_{k=1}^{n}(\varrho_k)^{w_k}\right), \end{aligned} \tag{2}$$

where $w = \{w_1, w_2, \ldots, w_n\}^T$ is the weight vector of $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n)$ such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 5 [34]: Let $\mathcal{S}_i\,(i = 1, 2, \ldots, n)$ be a collection of SFNs, then the T-spherical fuzzy weighted geometric

(T-SFWG) operator is a mapping $\mathcal{S}^n \longrightarrow \mathcal{S}$ such that

$$
\begin{aligned}
&T-SFWG\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \\
&\quad = w_1\mathcal{S}_1 \otimes w_2\mathcal{S}_2 \otimes \ldots \otimes w_n\mathcal{S}_n \\
&\quad = \left(\begin{array}{c} \prod_{k=1}^{n}\left(\sigma_k\right)^{w_k}, \left(1-\prod_{k=1}^{n}\left(1-\vartheta_k^t\right)^{w_k}\right)^{1/t} \\ , \left(1-\prod_{k=1}^{n}\left(1-\varrho_k^t\right)^{w_k}\right)^{1/t} \end{array}\right), \qquad (3)
\end{aligned}
$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n)$ such that $0 \leq w_i \leq 1$ and $\sum_{i=1}^{n} w_i = 1$.

Triangular norms have been intensively investigated, initiating from Zadeh presented max and min operation as a pair of triangular norm and triangular conorm. We can make reference to several triangular norms and corresponding triangular conorms, like product t-norm and probabilistic sum t-conorm [35], Einstein t-norm and t-conorm [36], Lukasiewicz t-norm and t-conorm [37], Hamacher t-norm and t-conorm [38] etc., are vehicles for operations on FSs. Frank operations include Frank's product and Frank's sum, which are examples of triangular norms and triangular conorms, respectively.

Frank t-norm T_F and Frank t-conorm S_F are defined in the following way.

$$
T_F\left(y_1, y_2\right) = \log_\tau\left(1 + \frac{\left(\tau^{y_1}-1\right)\left(\tau^{y_2}-1\right)}{\tau-1}\right) \quad \forall \; \left(y_1, y_2\right) \in [0,1]^2, \qquad (4)
$$

$$
S_F\left(y_1, y_2\right) = 1-\log_\tau\left(1 + \frac{\left(\tau^{1-y_1}-1\right)\left(\tau^{1-y_2}-1\right)}{\tau-1}\right) \quad \forall \; \left(y_1, y_2\right) \in [0,1]^2. \qquad (5)
$$

It is pointed out that the Frank t-norm and Frank t-conorm have the following properties [39].

$$
T_F\left(y_1, y_2\right) + S_F\left(y_1, y_2\right) = y_1 + y_2, \qquad (6)
$$

$$
\eta\left(\mathcal{S}_1 \oplus \mathcal{S}_2\right) = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)\left(\tau^{1-\sigma_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)\left(\tau^{\vartheta_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)\left(\tau^{\varrho_2^t}-1\right)}{\tau-1}\right)}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}
\end{array}\right)
$$

$$
= \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}, \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}
\end{array}\right). \qquad (8)
$$

$$\frac{\partial T_F(y_1,y_2)}{\partial y_1}+\frac{\partial S_F(y_1,y_2)}{\partial y_1}=1. \tag{7}$$

Based on limit theory, one can easily verify the following desirable results [39].

1). If $\tau \longrightarrow 1$, then $T_F(y_1,y_2) \longrightarrow y_1+y_2-y_1y_2$, $S_F(y_1,y_2) \longrightarrow y_1y_2$, the Frank t-norm and Frank t-conorm are reduced to probabilistic product and probabilistic sum.

2). If $\tau \longrightarrow \infty$, then $T_F(y_1,y_2) \longrightarrow \min(y_1+y_2,1)$, $S_F(y_1,y_2) \longrightarrow \max(0,y_1+y_2-1)$, the Frank t-norm and Frank t-conorm are reduced to the Lukasiewicz product and Lukasiewicz sum, respectively.

III. FRANK OPERATIONS OF T-SPHERCAL FUZZY SET

This section is dedicated to present Frank operations of T-SFNs and study some interesting properties of these operations.

Definition 6: Let $\mathcal{S}_1=(\sigma_1,\vartheta_1,\varrho_1)$ and $\mathcal{S}_2=(\sigma_2,\vartheta_2,\varrho_2)$ be two T-SFNs and $\eta>0$, then

$$\begin{aligned}
\eta\mathcal{S}_1\oplus\eta\mathcal{S}_2 &= \left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}\end{array}\right)\oplus\left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}\end{array}\right)\\
&= \left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{\eta-1}}\right)}-1\right)}{\tau-1}\right)}\end{array}\right)\\
&= \left(\begin{array}{c}\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^\eta\left(\tau^{1-\sigma_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)},\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^\eta\left(\tau^{\vartheta_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)},\\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^\eta\left(\tau^{\varrho_2^t}-1\right)^\eta}{(\tau-1)^{2\eta-1}}\right)}\end{array}\right).
\end{aligned} \tag{9}$$

$$1)\ \mathcal{S}_1 \oplus \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)\left(\tau^{1-\sigma_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)\left(\tau^{\vartheta_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)\left(\tau^{\varrho_2^t}-1\right)}{\tau-1}\right)} \end{array} \right); \quad 2)\ \mathcal{S}_1 \otimes \mathcal{S}_2 = \left(\begin{array}{c} \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)\left(\tau^{\sigma_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)\left(\tau^{1-\vartheta_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_1^t}-1\right)\left(\tau^{1-\varrho_2^t}-1\right)}{\tau-1}\right)} \end{array} \right);$$

$$\begin{aligned}
\eta_1\mathcal{S} \oplus \eta_2\mathcal{S} &= \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)} \end{array} \right) \oplus \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)} \end{array} \right) \\
&= \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}-1\right)\left(\tau^{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)}{\tau-1}\right)} \end{array} \right) \\
&= \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}, \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)} \end{array} \right). \qquad (10)
\end{aligned}$$

And

$$(\eta_1+\eta_2)\,\mathcal{S} = \left(\begin{array}{c} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}, \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1+\eta_2}}{(\tau-1)^{\eta_1+\eta_2-1}}\right)} \end{array} \right). \qquad (11)$$

3) $\mathcal{S}_1^{\eta} = \left(\begin{array}{c} \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)} \end{array} \right)$;

4) $\eta\mathcal{S}_1 = \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\eta}}{(\tau-1)^{\eta-1}}\right)} \end{array} \right)$;

5) $\mathcal{S}_1^c = (\varrho_1, \vartheta_1, \sigma_1)$.

Remark 2:

- The operational laws provided in Definition 6 reduced to SFNs if we set $t = 2$.
- The operational laws provided in Definition 6 reduced to PFNS if we set $t = 1$.
- The operational laws provided in Definition 6 reduced to q-OFNS if we set $\vartheta = 0$.
- The operational laws provided in Definition 6 reduced to PyFS if we set $t = 2$ and $\vartheta = 0$.
- The operational laws provided in Definition 6 reduced to IFS if we set $t = 1$ and $\vartheta = 0$.

Based on the operational laws given in Definition 6, we examine the following results.

Theorem 1: Let $\mathcal{S}_\ell = (\sigma_\ell, \vartheta_\ell, \varrho_\ell)\ (\ell = 1, 2)$ and $\mathcal{S} = (\sigma, \vartheta, \varrho)$ be three T-SFNs, and $\eta, \eta_1, \eta_2 > 0$, then

1) $\mathcal{S}_1 \oplus \mathcal{S}_2 = \mathcal{S}_2 \oplus \mathcal{S}_1$;
2) $\mathcal{S}_1 \otimes \mathcal{S}_2 = \mathcal{S}_2 \otimes \mathcal{S}_1$;
3) $\eta(\mathcal{S}_1 \oplus \mathcal{S}_2) = \eta\mathcal{S}_1 \oplus \eta\mathcal{S}_2$;
4) $(\mathcal{S}_1 \otimes \mathcal{S}_2)^{\eta} = \mathcal{S}_1^{\eta} \otimes \mathcal{S}_2^{\eta}$;
5) $\eta_1\mathcal{S} \oplus \eta_2\mathcal{S} = (\eta_1 + \eta_2)\mathcal{S}$;
6) $\mathcal{S}^{\eta_1} \otimes \mathcal{S}^{\eta_2} = \mathcal{S}^{\eta_1+\eta_2}$;
7) $(\eta_1\eta_2)\mathcal{S} = \eta_1(\eta_2\mathcal{S})$.

Proof: We prove only parts 1, 3, 5 and 7 and similarly for others.

1. It is obvious.
3. $\mathcal{S}_1 \oplus \mathcal{S}_2$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)\left(\tau^{1-\sigma_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)\left(\tau^{\vartheta_2^t}-1\right)}{\tau-1}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)\left(\tau^{\varrho_2^t}-1\right)}{\tau-1}\right)} \end{array} \right),$$

by the Frank operational law (4) in Definition 6, it follows (8), as shown at the bottom of page 4. Now, we get (9), as shown at the bottom of page 5.

$$\eta_1(\eta_2\mathcal{S}) = \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)}, \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}-1\right)^{\eta_1}}{(\tau-1)^{\eta_1-1}}\right)} \end{array} \right)$$

$$= \left(\begin{array}{c} \sqrt[t]{1-\log_{\tau}\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)}, \\ \sqrt[t]{\log_{\tau}\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_1\eta_2}}{(\tau-1)^{\eta_1\eta_2-1}}\right)} \end{array} \right) = (\eta_1\eta_2)\mathcal{S}$$

$$
\begin{aligned}
&T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2\right)=\varpi_1\mathcal{S}_1\oplus\varpi_2\mathcal{S}_2\\
&=\left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)}
\end{array}\right)\\
&=\left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}
\end{array}\right)
=\left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{\log_\tau\left(1+\frac{\left(1+\frac{\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}
\end{array}\right)\\
&=\left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\left(\left(\tau^{1-\sigma_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\sigma_2^t}-1\right)^{\varpi_2}\right)\right)},\\
\sqrt[t]{\log_\tau\left(1+\left(\left(\tau^{\vartheta_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{\vartheta_2^t}-1\right)^{\varpi_2}\right)\right)},\\
\sqrt[t]{\log_\tau\left(1+\left(\left(\tau^{\varrho_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{\varrho_2^t}-1\right)^{\varpi_2}\right)\right)}
\end{array}\right).
\end{aligned}
$$

$$
\begin{aligned}
&T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{k+1}\right) \\
&\quad = T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{k}\right) \oplus \varpi_{k+1}\mathcal{S}_{k+1} \\
&\quad = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}
\end{array}\right)
\oplus
\left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\left(\tau^{1-\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\left(\tau^{\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\left(\tau^{\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}\right)}
\end{array}\right) \\
&\quad = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\dfrac{\left(\tau^{1-\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\dfrac{\left(\tau^{\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\dfrac{\left(\tau^{\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)^{-1}}}}{\tau-1}\right)}
\end{array}\right) \\
&\quad = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{\sum_{j=1}^{k}(\tau-1)^{\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)^{-1}}}\right)}
\end{array}\right) \\
&\quad = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)-1}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)-1}}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\dfrac{\Pi_{j=1}^{k}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{\sum_{j=1}^{k+1}(\tau-1)^{\varpi(j)-1}}\right)}
\end{array}\right) \\
&\quad = \left(\begin{array}{c}
\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)}, \\
\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)}
\end{array}\right)
\end{aligned}
$$

5. From Eqs. (8) and (9), we get $\eta(\mathcal{S}_1 \oplus \mathcal{S}_2) = \eta\mathcal{S}_1 \oplus \eta\mathcal{S}_2$ (9) and (10), as shown at the bottom of page 6.
   Thus, from Eqs. (10) and (11), as shown at the bottom of page 6, we get the required result.
7. Since

$$\eta_2\mathcal{S} = \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\sigma^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\vartheta^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\varrho^t}-1\right)^{\eta_2}}{(\tau-1)^{\eta_2-1}}\right)} \end{pmatrix}.$$

   From this, we can further write equation, as shown at the bottom of page 7.

■

Theorem 2: Let $\mathcal{S}_1 = (\sigma_1, \vartheta_1, \varrho_1)$ and $\mathcal{S}_2 = (\sigma_2, \vartheta_2, \varrho_2)$ be two T-SFNs, then

1) $(\mathcal{S}_1 \oplus \mathcal{S}_2)^c = \mathcal{S}_1^c \otimes \mathcal{S}_2^c$;
2) $(\mathcal{S}_1 \otimes \mathcal{S}_2)^c = \mathcal{S}_1^c \oplus \mathcal{S}_2^c$.

Proof: The proof is trivial, therefore it is omitted here. ■

IV. AGGREGATION OPERATORS ABOUT T-SFNs BASED FRANK OPERATIONS

Based on the proposed Frank operation laws, in what follows, we propose a series of weighted aggregation operators for T-SFNs.

A. T-SPHERICAL FUZZY FRANK AVERAGING OPERATORS

Definition 7: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank weighted averaging operator (T-SFFWA) is:

$$T-SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \oplus_{j=1}^{n}\left(\varpi_j\mathcal{S}_j\right), \tag{12}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n}\varpi_j = 1$. Especially, if $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the T-SFFWA operator reduces to the T-spherical fuzzy Frank averaging (T-SFFA) operator of dimension n, which is given as follows:

$$T-SFFA\,(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \frac{1}{n}\oplus_{j=1}^{n}\left(\mathcal{S}_j\right). \tag{13}$$

Theorem 3: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFWA operator is still a T-SFN, and

$$T-SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)} \end{pmatrix}. \tag{14}$$

Proof: We prove it by mathematical induction on n.

For $n = 2$, we have equation, as shown in page 8.

Thus, result holds for $n = 2$.

If Eq. (14) holds for $n = k$, then for $n = k + 1$, we have equation, as shown in the previous page.

Thus, results holds for $n = k+1$ and hence, by the principle of mathematical induction, result given in Eq. (14) holds for all positive integer n. ■

Example 1: Let $\mathcal{S}_1 = (0.4, 0.3, 0.5)$, $\mathcal{S}_2 = (0.7, 0.3, 0.4)$, $\mathcal{S}_3 = (0.6, 0.7, 0.8)$ be three T-SFNs, and $\varpi = (0.4, 0.3, 0.3)^T$ be the weight vector of $\mathcal{S}_j\,(j = 1, 2, 3)$. Suppose $\tau = 2$, then by Definition 7 and Theorem 3, we can get (t=4): see equation, as shown at the bottom of the page.

Theorem 4: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFWA

$$T-SFFWA\,(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\prod_{j=1}^{3}\left(2^{1-\sigma_j^4}-1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\varrho_j^4}-1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$\times \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\left(2^{1-.4^4}-1\right)^{.4}\left(2^{1-.7^4}-1\right)^{.3}\left(2^{1-.6^4}-1\right)^{.3}\right)} \\ \sqrt[4]{\log_2\left(1+\left(2^{.3^4}-1\right)^{.4}\left(2^{.3^4}-1\right)^{.3}\left(2^{.7^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1+\left(2^{.5^4}-1\right)^{.4}\left(2^{.4^4}-1\right)^{.3}\left(2^{.8^4}-1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.5940, 0.3887, 0.5418)$$

operator approaches the following limit

$$\lim_{\tau\longrightarrow 1} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\sigma_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\varrho_j^t\right)^{\varpi_j}} \end{pmatrix}. \tag{15}$$

Proof: As $\tau\longrightarrow 1$, then

$$\begin{pmatrix} \Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}, \\ \Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}, \\ \Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j} \end{pmatrix} \longrightarrow (0,0,0)$$

by log property and the rule of infinitesimal changes, we have

$$\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

$$\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right) = \frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau} \longrightarrow \frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}$$

Based on Taylor's expansion formula, we have

$$\tau^{1-\sigma_j^t} = 1+\left(1-\sigma_j^t\right)\ln\tau+\frac{\left((1-\sigma_j^t)\ln\tau\right)^2}{2!}+\ldots$$

$$\tau^{\vartheta_j^t} = 1+\left(\vartheta_j^t\right)\ln\tau+\frac{\left((\vartheta_j^t)\ln\tau\right)^2}{2!}+\ldots$$

$$\tau^{\varrho_j^t} = 1+\left(\varrho_j^t\right)\ln\tau+\frac{\left((\varrho_j^t)\ln\tau\right)^2}{2!}+\ldots$$

Also, since $\tau > 1$, then $\ln\tau > 0$, $\tau^{1-\sigma_j^t} = 1+\left(1-\sigma_j^t\right)\ln\tau+O(\ln\tau)$, $\tau^{\vartheta_j^t} = 1+\left(\vartheta_j^t\right)\ln\tau+O(\ln\tau)$, $\tau^{\varrho_j^t} = 1+\left(\varrho_j^t\right)\ln\tau+O(\ln\tau)$.

It follows that $\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \left(\left(1-\sigma_j^t\right)\ln\tau\right)^{\varpi_j}$

$$\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right)\Pi_{j=1}^{n}(\ln\tau)^{\varpi_j}$$

$$\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right)\ln(\tau)^{\sum_{j=1}^{n}\varpi_j}$$

$$\frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(1-\sigma_j^t\right).$$

Analogously, we can get $\frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(\vartheta_j^t\right)$ and $\frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^{n}\left(\varrho_j^t\right)$.

Then, we have

$$\lim_{\tau\longrightarrow 1} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right)$$

$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}} \end{pmatrix}$$

$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix} \sqrt[t]{1-\frac{\Pi_{j=1}^{n}\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}}, \\ \sqrt[t]{\frac{\Pi_{j=1}^{n}\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}}, \\ \sqrt[t]{\frac{\Pi_{j=1}^{n}\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\sigma_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\varrho_j^t\right)^{\varpi_j}} \end{pmatrix}$$

which completes the proof. ■

Theorem 5: Let $\mathcal{S}_j = \left(\sigma_j,\vartheta_j,\varrho_j\right)$ $(j=1,2,\ldots,n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau\longrightarrow\infty$, the T-SFFWA operator approaches the following limit

$$\lim_{\tau\longrightarrow\infty} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{\left(\sum_{j=1}^{n}\varpi_j\left(\sigma_j^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(\vartheta_j^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n}\varpi_j\left(\varrho_j^t\right)\right)} \end{pmatrix}. \tag{16}$$

Proof: According to Theorem 3, we have

$$\lim_{\tau\longrightarrow\infty} T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right) = \begin{pmatrix} \lim_{\tau\longrightarrow\infty}\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}, \\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}, \\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)} \end{pmatrix}$$

Using limit rules, logarithmic transform and L'Hospital's rule, it follows equation, shown at the bottom of the page, which completes the proof of Theorem 5. ∎

Theorem 6 (Idempotency): Let $\mathcal{T}_j = \left(\sigma_j,\vartheta_j,\varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, if $\mathcal{S}_j = \mathcal{S}_0\ \forall\ j$, then

$$T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right) = \mathcal{S}_0. \tag{17}$$

Proof: Since for all j $\mathcal{S}_j = \mathcal{S}_0 = (\sigma_0,\vartheta_0,\varrho_0)$, and $\sum_{j=1}^n \varpi_j = 1$ so by Theorem 3, we have

$$T-SFFWA\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_0^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_0^t}-1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_0^t}-1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \left(\sqrt[t]{1-\log_\tau \tau^{1-\sigma_0^t}}, \sqrt[t]{\log_\tau \tau^{\vartheta_0^t}}, \sqrt[t]{\log_\tau \tau^{\varrho_0^t}}\right)$$

$$= (\sigma_0,\vartheta_0,\varrho_0) = \mathcal{S}_0.$$

Thus, proof is completed. ∎

Theorem 7 (Monotonicity): Let $\mathcal{S}_j = \left(\sigma_j,\vartheta_j,\varrho_j\right)$ $(j = 1, 2, \ldots, n)$ and $\dot{\mathcal{S}}_j = \left(\dot{\sigma}_j,\dot{\vartheta}_j,\dot{\varrho}_j\right)$ $(j = 1, 2, \ldots, n)$ be two

$$\begin{pmatrix} \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{\lim_{\tau\longrightarrow\infty}\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{\lim_{\tau\longrightarrow\infty}\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\frac{\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(1-\sigma_j^t\right)\frac{\tau^{-\sigma_j^t}}{\tau^{1-\sigma_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\frac{\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\vartheta_j^t-1}}{\tau^{\vartheta_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\frac{\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(\varrho_j^t\right)\frac{\tau^{\varrho_j^t-1}}{\tau^{\varrho_j^t}-1}\right)}{\frac{1}{\tau}}}, \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(1-\sigma_j^t\right)\frac{\tau^{1-\sigma_j^t}}{\tau^{1-\sigma_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\vartheta_j^t}}{\tau^{\vartheta_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n \varpi_j\left(\varrho_j^t\right)\frac{\tau^{\varrho_j^t}}{\tau^{\varrho_j^t}-1}\right)} \end{pmatrix}$$

$$= \left(\sqrt[t]{1-\left(\sum_{j=1}^n \varpi_j\left(1-\sigma_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n \varpi_j\left(\vartheta_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n \varpi_j\left(\varrho_j^t\right)\right)}\right)$$

$$= \left(\sqrt[t]{\left(\sum_{j=1}^n \varpi_j\left(\sigma_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n \varpi_j\left(\vartheta_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n \varpi_j\left(\varrho_j^t\right)\right)}\right)$$

families of T-SFNs such that $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j \ \forall \ j$, then

$$T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \geq T-SFFWA\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n\right). \quad (18)$$

Proof: According to Definition 3, when $\sigma_j \geq \dot{\sigma}_j, \vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j \ \forall \ j$, then

$$\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_j^t}-1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\dot{\sigma}_j^t}-1\right)^{\varpi_j}\right)},$$

$$\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_j^t}-1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\dot{\vartheta}_j^t}-1\right)^{\varpi_j}\right)} \text{ and}$$

$$\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_j^t}-1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\dot{\varrho}_j^t}-1\right)^{\varpi_j}\right)}$$

Thus, $S\left(T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right)\right) \geq S(T-SFFWA\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n\right))$

Hence, $T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \geq T-SFFWA\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n\right)$. ■

Theorem 8 (Boundedness): Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and let $\mathcal{S}^- = \left(\min_{1\leq j\leq n}\sigma_j, \ \max_{1\leq j\leq n}\vartheta_j, \max_{1\leq j\leq n}\varrho_j\right)$, $\mathcal{S}^+ = (\max_{1\leq j\leq n}\sigma_j, \min_{1\leq j\leq n}\vartheta_j, \min_{1\leq j\leq n}\varrho_j)$, then

$$\mathcal{S}^- \leq T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \leq \mathcal{S}^+. \quad (19)$$

Proof: Since for all j, $\min_{1\leq j\leq n}\sigma_j \leq \sigma_j \leq \max_{1\leq j\leq n}\sigma_j$, $\min_{1\leq j\leq n}\vartheta_j \leq \vartheta_j \leq \max_{1\leq j\leq n}\vartheta_j$ and $\min_{1\leq j\leq n}\varrho_j \leq \varrho_j \leq \max_{1\leq j\leq n}\varrho_j$, thereby on the basis of idempotency and monotonicity, we get

$$\mathcal{S}^- \leq T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \leq \mathcal{S}^+.$$

■

Theorem 9 (Shift-Invariance): Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs and $\dot{\mathcal{S}} = \left(\dot{\sigma}, \dot{\vartheta}, \dot{\varrho}\right)$ be any other T-SFNs, then

$$T-SFFWA\left(\mathcal{S}_1 \oplus \dot{\mathcal{S}}, \mathcal{S}_2 \oplus \dot{\mathcal{S}}, \ldots, \mathcal{S}_n \oplus \dot{\mathcal{S}}\right) = T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \oplus \dot{\mathcal{S}}. \quad (20)$$

Theorem 10 (Homogeneity): Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs and $\eta > 0$ be any real number, then

$$T-SFFWA\left(\eta\mathcal{S}_1, \eta\mathcal{S}_2, \ldots, \eta\mathcal{S}_n\right) = \eta T-SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right). \quad (21)$$

The proof of the above two theorems can be easily derived from the proposed Frank operational laws of T-SFNs; thus, it is omitted here due to the space limitations.

Definition 8: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank ordered weighted averaging (T-SFFOWA) operator is:

$$T-SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \oplus_{j=1}^n\left(w_j\mathcal{S}_{\delta(j)}\right), \quad (22)$$

where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ is the position weights of $\mathcal{S}_j$ $(j = 1, 2, .., n)$ such that $w_j > 0$ and $\sum_{j=1}^n w_j = 1$. $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\mathcal{S}_{\delta(j-1)} \geq \mathcal{S}_{\delta(j)}$ for $j = 2, 3.., n$.

Theorem 11: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFOWA operator is still a T-SFN, and

$$T-SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\sigma_{\delta(j)}^t}-1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\vartheta_{\delta(j)}^t}-1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\varrho_{\delta(j)}^t}-1\right)^{w_j}\right)} \end{pmatrix}. \quad (23)$$

Proof: The proof of this result is similar to that of Theorem 3, and so we omit here. ■

Example 2: Let $\mathcal{S}_1 = (0.3, 0.3, 0.5)$, $\mathcal{S}_2 = (0.7, 0.4, 0.5)$, $\mathcal{S}_3 = (0.6, 0.7, 0.8)$ be three T-SFNs, then according to Definition 3 we can get (t=4):

$$S\left(\mathcal{S}_1\right) = -0.0769, \ S\left(\mathcal{S}_2\right) = 0.1775, \ S\left(\mathcal{S}_3\right) = -0.5482$$

Since $S\left(\mathcal{S}_2\right) > S\left(\mathcal{S}_1\right) > S\left(\mathcal{S}_3\right)$, we have $\mathcal{S}_{\delta(1)} = (0.7, 0.4, 0.5)$, $\mathcal{S}_{\delta(2)} = (0.3, 0.3, 0.5)$, $\mathcal{S}_{\delta(3)} =$, $(0.6, 0.7, 0.8)$ and $w = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the T-SFFOWA operator. Suppose $\tau = 2$, then by Definition 8 and Theorem 11, we can get equation, as shown at the bottom of the next page.

Theorem 12: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFOWA operator approaches the following limit

$$\lim_{\tau\longrightarrow 1} T-SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1-\Pi_{j=1}^n\left(1-\sigma_{\delta(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\Pi_{j=1}^n\left(\vartheta_{\delta(j)}^t\right)^{w_j}}, \\ \sqrt[t]{\Pi_{j=1}^n\left(\varrho_{\delta(j)}^t\right)^{w_j}} \end{pmatrix}. \quad (24)$$

Theorem 13: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFOWA operator approaches the following limit

$$\lim_{\tau\longrightarrow\infty} T-SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{\left(\sum_{j=1}^n w_j\left(\sigma_{\delta(j)}^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n w_j\left(\vartheta_{\delta(j)}^t\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^n w_j\left(\varrho_{\delta(j)}^t\right)\right)} \end{pmatrix}. \quad (25)$$

As similar to those of T-SFFWA operator, the T-SFFOWA operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFOWA operator has the following desirable results.

Theorem 14: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then we have the following:

i). If $w = (1, 0, \ldots, 0)^T$ then $T - SFFOWA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \max\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$.

ii). If $w = (0, 0, \ldots, 1)^T$ then $T - SFFOWA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \min\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$.

iii). If $w_j = 1$ and $\varpi_i = 0\,(i \neq j)$ then $T - SFFOWA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \mathcal{S}_{\delta(j)}$ where $\mathcal{S}_{\delta(j)}$ is the jth largest of $\mathcal{S}_j$, $(j = 1, 2, \ldots, n)$.

Based on the definition of T-SFFWA and T-SFFOWA operators, we can see that the T-SFFWA operator can weights only the SFNs while T-SFFOWA operator weights only the ordered position of SFNs. In real-world practical situations, we should both consider the two aspects at the same time. Therefore, to overcome this limitation, we define the hybrid averaging operator [40] based on Frank t-norm and t-conorm, which weight both the given T-SFNs and their ordered positions.

Definition 9: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank hybrid averaging (T-SFFHA) operator is:

$$T - SFFHA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \oplus_{j=1}^{n} \left(w_j \hat{\mathcal{S}}_{\delta(j)}\right), \tag{26}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector associated with T-SFFHA such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is the weight vector of $\mathcal{S}_j$ $(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. $\hat{\mathcal{S}}_{\delta(j)}$ is the jth largest of the weighted T-SFNs $\hat{\mathcal{S}}_j \left(\hat{\mathcal{S}}_j = (n\varpi_j)\,\mathcal{S}_j, j = 1, 2, .., n\right)$ and n is the balancing coefficient.

Theorem 15: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFHA operator is still a T-SFN, and

$$T - SFFHA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{1-\hat{\sigma}_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\hat{\vartheta}_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\hat{\varrho}_{\delta(j)}^t} - 1\right)^{w_j}\right)} \end{pmatrix}, \tag{27}$$

Proof: The proof of this result is similar to that of Theorem 3, and so we omit here. ■

Example 3: Let $\mathcal{S}_1 = (0.4, 0.5, 0.2)$, $\mathcal{S}_2 = (0.6, 0.7, 0.8)$ $\mathcal{S}_3 = (0.3, 0.6, 0.5)$, be three T-SFNs (t=4), and $\varpi = (0.4, 0.4, 0.2)^T$ is the weight vector of $\mathcal{S}_j$ $(j = 1, 2, 3)$. Suppose $\tau = 2$, then according to Definition 6, we can get the weighted T-SFNs:

$$\hat{\mathcal{S}}_1 = 3 \times 0.4 \times \mathcal{S}_1 = \begin{pmatrix} \sqrt[4]{1 - \log_2\left(1 + \frac{\left(2^{1-0.4^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.5^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.2^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)} \end{pmatrix} = (0.4185, 0.4289, 0.1423);$$

$$\hat{\mathcal{S}}_2 = 3 \times 0.4 \times \mathcal{S}_2 = \begin{pmatrix} \sqrt[4]{1 - \log_2\left(1 + \frac{\left(2^{1-0.6^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.7^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{\log_2\left(1 + \frac{\left(2^{0.8^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)} \end{pmatrix}$$

$$T - SFFOWA\left(\mathcal{S}_{\delta(1)}, \mathcal{S}_{\delta(2)}, \mathcal{S}_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{1 - \log_2\left(1 + \prod_{j=1}^{3}\left(2^{1-\sigma_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2\left(1 + \prod_{j=1}^{3}\left(2^{\vartheta_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2\left(1 + \prod_{j=1}^{3}\left(2^{\varrho_{\delta(j)}^4} - 1\right)^{w_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{1 - \log_2\left(1 + \left(2^{1-.7^4} - 1\right)^{.3}\left(2^{1-.3^4} - 1\right)^{.4}\left(2^{1-.6^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1 + \left(2^{.4^4} - 1\right)^{.3}\left(2^{.3^4} - 1\right)^{.4}\left(2^{.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1 + \left(2^{.5^4} - 1\right)^{.3}\left(2^{.5^4} - 1\right)^{.4}\left(2^{.8^4} - 1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.5861, 0.4236, 0.5786).$$

$$= (0.6264, 0.6464, 0.7617);$$

$$\hat{\mathcal{S}}_3 = 3 \times 0.2 \times \mathcal{S}_3$$

$$= \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.3^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.6^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} \end{pmatrix}$$

$$= (0.2641, 0.7478, 0.6743).$$

According to Definitions 3, we can get the score of $\hat{\mathcal{S}}_j\ (j = 1, 2, 3)$:

$$S\left(\hat{\mathcal{S}}_1\right) = -0.0044, \quad S\left(\hat{\mathcal{S}}_2\right) = -0.3868,$$
$$S\left(\hat{\mathcal{S}}_3\right) = -0.5745.$$

Since $S\left(\hat{\mathcal{S}}_1\right) > S\left(\hat{\mathcal{S}}_2\right) > S\left(\hat{\mathcal{S}}_3\right)$, we have $\hat{\mathcal{S}}_{\delta(1)} = (0.4185, 0.4289, 0.1423)$, $\hat{\mathcal{S}}_{\delta(2)} = (0.6264, 0.6464, 0.7617)$, $\hat{\mathcal{S}}_{\delta(3)} = (0.2641, 0.7478, 0.6743)$. Suppose $w = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the T-SFFHA operator. Then by Definition 9 and Theorem 15, we can get equation, as shown at the bottom of the page.

Theorem 16: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)\ (j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFHA operator approaches the following limit

$$\lim_{\tau \longrightarrow 1} T - SFFHA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{1-\Pi_{j=1}^{n}\left(1-\hat{\sigma}_{\delta(j)}^{t}\right)^{w_j}}, \\ \sqrt[t]{\Pi_{j=1}^{n}\left(\hat{\vartheta}_{\delta(j)}^{t}\right)^{w_j}}, \sqrt[t]{\Pi_{j=1}^{n}\left(\hat{\varrho}_{\delta(j)}^{t}\right)^{w_j}} \end{pmatrix}. \quad (28)$$

Theorem 17: Let $\mathcal{S}_j = \left(\sigma_j, \vartheta_j, \varrho_j\right)\ (j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFHA operator approaches the following limit

$$\lim_{\tau \longrightarrow \infty} T - SFFHA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) = \begin{pmatrix} \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\hat{\sigma}_{\delta(j)}^{t}\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n} w_j\left(\hat{\vartheta}_{\delta(j)}^{t}\right)\right)}, \\ \sqrt[t]{1-\left(\sum_{j=1}^{n} w_j\left(\hat{\varrho}_{\delta(j)}^{t}\right)\right)} \end{pmatrix}. \quad (29)$$

As similar to those of T-SFFWA operator, the T-SFFHA operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFHA operator has the following special cases.

Corollary 18: T-SFFWA operator is a special case of the T-SFFHA operator.

Proof: Let $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &T - SFFHA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \\ &= w_1\hat{\mathcal{S}}_{\delta(1)} \oplus w_2\hat{\mathcal{S}}_{\delta(2)} \oplus \ldots \oplus w_n\hat{\mathcal{S}}_{\delta(n)} \\ &= \frac{1}{n}\left(\hat{\mathcal{S}}_{\delta(1)} \oplus \hat{\mathcal{S}}_{\delta(2)} \oplus \ldots \oplus \hat{\mathcal{S}}_{\delta(n)}\right) \\ &= \varpi_1\mathcal{S}_1 \oplus \varpi_2\mathcal{S}_2 \oplus \ldots \oplus \varpi_n\mathcal{S}_n \\ &= T - SFFWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right). \end{aligned}$$

■

Corollary 19: T-SFFOWA operator is a special case of the T-SFFHA operator.

Proof: Let $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &T - SFFHA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \\ &= w_1\hat{\mathcal{S}}_{\delta(1)} \oplus w_2\hat{\mathcal{S}}_{\delta(2)} \oplus \ldots \oplus w_n\hat{\mathcal{S}}_{\delta(n)} \\ &= w_1\mathcal{S}_{\delta(1)} \oplus w_2\mathcal{S}_{\delta(2)} \oplus \ldots \oplus w_n\mathcal{S}_{\delta(n)} \\ &= T - SFFOWA\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right). \end{aligned}$$

■

$$T - SFFHA\left(\hat{\mathcal{S}}_{\delta(1)}, \hat{\mathcal{S}}_{\delta(2)}, \hat{\mathcal{S}}_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\prod_{j=1}^{3}\left(2^{1-\hat{\sigma}_{\delta(j)}^4}-1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\hat{\vartheta}_{\delta(j)}^4}-1\right)^{w_j}\right)}, \\ \sqrt[4]{\log_2\left(1+\prod_{j=1}^{3}\left(2^{\hat{\varrho}_{\delta(j)}^4}-1\right)^{w_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{1-\log_2\left(1+\left(2^{1-0.4185^4}-1\right)^{.3}\left(2^{1-0.6264^4}-1\right)^{.4}\left(2^{1-0.2641^4}-1\right)^{.3}\right)} \\ \sqrt[4]{\log_2\left(1+\left(2^{0.4289^4}-1\right)^{.3}\left(2^{0.6464^4}-1\right)^{.4}\left(2^{0.7478^4}-1\right)^{.3}\right)}, \\ \sqrt[4]{\log_2\left(1+\left(2^{0.1423^4}-1\right)^{.3}\left(2^{0.7617^4}-1\right)^{.4}\left(2^{0.6743^4}-1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.5216, 0.5995, 0.4501).$$

$$
\begin{aligned}
T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2\right) &= \mathcal{S}_1^{\varpi_1}\otimes\mathcal{S}_2^{\varpi_2}\\
&=\left(\begin{array}{c}
\sqrt[t]{\log_\tau\left(1+\dfrac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}\right)\right)}-1\right)\left(\tau^{1-\left(1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}\right)\right)}-1\right)}{\tau-1}\right)}
\end{array}\right)\\
&=\left(\begin{array}{c}
\sqrt[t]{\log_\tau\left(1+\dfrac{\left(1+\frac{\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\left(1+\frac{\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\dfrac{\left(1+\frac{\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}}{(\tau-1)^{\varpi_1-1}}-1\right)\left(1+\frac{\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}}{(\tau-1)^{\varpi_2-1}}-1\right)}{\tau-1}\right)}
\end{array}\right)\\
&=\left(\begin{array}{c}
\sqrt[t]{\log_\tau\left(1+\left(\left(\tau^{\sigma_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{\sigma_2^t}-1\right)^{\varpi_2}\right)\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\left(\left(\tau^{1-\vartheta_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\vartheta_2^t}-1\right)^{\varpi_2}\right)\right)},\\
\sqrt[t]{1-\log_\tau\left(1+\left(\left(\tau^{1-\varrho_1^t}-1\right)^{\varpi_1}\right)\left(\left(\tau^{1-\varrho_2^t}-1\right)^{\varpi_2}\right)\right)}
\end{array}\right).
\end{aligned}
$$

$$T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_{k+1}\right)=T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_k\right)\otimes\mathcal{S}_{k+1}^{\varpi_{k+1}}$$

$$=\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{pmatrix}\otimes\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\frac{\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}}{\tau-1}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}}{\tau-1}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\frac{\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\varpi(k+1)-1}}}{\tau-1}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k}\varpi(j)-1}(\tau-1)(\tau-1)^{\varpi(k+1)-1}}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)-1}}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\frac{\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}}{(\tau-1)^{\sum_{j=1}^{k+1}\varpi(j)-1}}\right)}\end{pmatrix}$$

$$=\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\left(\tau^{\sigma_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\vartheta_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^{k}\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\left(\tau^{1-\varrho_{(k+1)}^t}-1\right)^{\varpi(k+1)}\right)}\end{pmatrix}.$$

B. T-SPHERICAL FUZZY FRANK GEOMETRIC OPERATORS

This section is devoted to provide a series of T-spherical fuzzy Frank geometric aggregation operators based on proposed Frank operations. In geometric aggregation operators, we will further discuss the T-SFFWG, T-SFFOWG and T-SFFHWG and also the basic definitions, remarks and results, corollary for these operators which are based on the Frank t-norm and t-conorm.

Definition 10: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank weighted geometric operator (T-SFFWG) is:

$$T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \otimes_{j=1}^{n} (\mathcal{S}_j)^{\varpi_j}, \quad (30)$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is the weight vector of $\mathcal{S}_j$ $(j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. Especially, if $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then the T-SFFWG operator reduces to the T-spherical fuzzy Frank geometric (T-SFFG) operator of dimension n, which is given as follows:

$$T - SFFA(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \otimes_{j=1}^{n} (\mathcal{S}_j)^{\frac{1}{n}}. \quad (31)$$

Theorem 20: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFWG operator is still a T-SFN, and

$$T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right)} \end{pmatrix}. \quad (32)$$

Proof: We prove it by mathematical induction on n.

For $n = 2$, we have equation, as shown at the page 16.

Thus, result holds for $n = 2$.

If Eq. (32) holds for $n = k$, then for $n = k + 1$, we have equation, as shown at the previous page.

Thus, results holds for $n = k+1$ and hence, by the principle of mathematical induction, result given in Eq. (32) holds for all positive integer n. ■

Example 4 (Continued From Example 1): According to Definition 10 and Theorem 20, we have equation, as shown at bottom of the page.

Theorem 21: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFWG operator approaches the following limit

$$\lim_{\tau \longrightarrow 1} T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\Pi_{j=1}^{n} \left(\sigma_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \vartheta_j^t\right)^{\varpi_j}}, \\ \sqrt[t]{1 - \Pi_{j=1}^{n} \left(1 - \varrho_j^t\right)^{\varpi_j}} \end{pmatrix}. \quad (33)$$

Proof: As $\tau \longrightarrow 1$, then $\left(\Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}, \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}, \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right) \longrightarrow (0, 0, 0)$ by log property and the rule of infinitesimal changes, we have

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln \left(1 + \Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}\right)}{\ln \tau} \longrightarrow \frac{\Pi_{j=1}^{n} \left(\tau^{\sigma_j^t} - 1\right)^{\varpi_j}}{\ln \tau}$$

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}\right)}{\ln \tau} \longrightarrow \frac{\Pi_{j=1}^{n} \left(\tau^{1-\vartheta_j^t} - 1\right)^{\varpi_j}}{\ln \tau}$$

$$\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right) = \frac{\ln \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}\right)}{\ln \tau} \longrightarrow \frac{\Pi_{j=1}^{n} \left(\tau^{1-\varrho_j^t} - 1\right)^{\varpi_j}}{\ln \tau}$$

$$T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3) = \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\sigma_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\vartheta_j^4} - 1\right)^{\varpi_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\varrho_j^4} - 1\right)^{\varpi_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \left(2^{.4^4} - 1\right)^{.4} \left(2^{.7^4} - 1\right)^{.3} \left(2^{.6^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.3^4} - 1\right)^{.4} \left(2^{1-.3^4} - 1\right)^{.3} \left(2^{1-.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-.5^4} - 1\right)^{.4} \left(2^{1-.4^4} - 1\right)^{.3} \left(2^{1-.8^4} - 1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.5361, 0.5358, 0.6417).$$

Based on Taylor's expansion formula, we have

$$\tau^{\vartheta_j^t} = 1 + \left(\sigma_j^t\right)\ln\tau + \frac{\left(\left(\sigma_j^t\right)\ln\tau\right)^2}{2!} + \dots$$
$$\tau^{1-\sigma_j^t} = 1 + \left(1-\vartheta_j^t\right)\ln\tau + \frac{\left(\left(1-\vartheta_j^t\right)\ln\tau\right)^2}{2!} + \dots$$
$$\tau^{1-\sigma_j^t} = 1 + \left(1-\varrho_j^t\right)\ln\tau + \frac{\left(\left(1-\varrho_j^t\right)\ln\tau\right)^2}{2!} + \dots$$

Also, since $\tau > 1$, then $\ln\tau > 0$, $\tau^{\sigma_j^t} = 1 + \left(\sigma_j^t\right)\ln\tau + O(\ln\tau)$, $\tau^{1-\vartheta_j^t} = 1 + \left(1-\vartheta_j^t\right)\ln\tau + O(\ln\tau)$, $\tau^{1-\varrho_j^t} = 1 + \left(1-\varrho_j^t\right)\ln\tau + O(\ln\tau)$.

It follows that $\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \left(\left(\sigma_j^t\right)\ln\tau\right)^{\varpi_j}$

$$\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^n\left(\sigma_j^t\right)\Pi_{j=1}^n(\ln\tau)^{\varpi_j}$$
$$\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j} \longrightarrow \Pi_{j=1}^n\left(\sigma_j^t\right)\ln(\tau)^{\sum_{j=1}^n\varpi_j}$$
$$\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^n\left(\sigma_j^t\right).$$

Analogously, we can get $\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^n\left(1-\vartheta_j^t\right)$ and $\frac{\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{\ln\tau} \longrightarrow \Pi_{j=1}^n\left(1-\varrho_j^t\right)$.

Then, we have

$$\lim_{\tau\longrightarrow 1} T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_n\right)$$
$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{pmatrix}$$
$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix}\sqrt[t]{\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}},\\ \sqrt[t]{1-\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}},\\ \sqrt[t]{1-\frac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}\end{pmatrix}$$
$$= \lim_{\tau\longrightarrow 1}\begin{pmatrix}\sqrt[t]{\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{\ln\tau}},\\ \sqrt[t]{1-\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}},\\ \sqrt[t]{1-\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{\ln\tau}}\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt[t]{\Pi_{j=1}^n\left(\sigma_j^t\right)^{\varpi_j}},\\ \sqrt[t]{1-\Pi_{j=1}^n\left(1-\vartheta_j^t\right)^{\varpi_j}},\\ \sqrt[t]{1-\Pi_{j=1}^n\left(1-\varrho_j^t\right)^{\varpi_j}}\end{pmatrix}$$

which completes the proof. ■

Theorem 22: Let $\mathcal{S}_j = \left(\sigma_j,\vartheta_j,\varrho_j\right)$ $(j = 1, 2, \dots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFWG operator approaches the following limit

$$\lim_{\tau\longrightarrow\infty} T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_n\right) = \begin{pmatrix}\sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(\sigma_j^t\right)\right)},\\ \sqrt[t]{\left(\sum_{j=1}^n\varpi_j\left(\vartheta_j^t\right)\right)},\\ \sqrt[t]{\left(\sum_{j=1}^n\varpi_j\left(\varrho_j^t\right)\right)}\end{pmatrix}. \tag{34}$$

Proof: According to Theorem 20, we have

$$\lim_{\tau\longrightarrow\infty} T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_n\right)$$
$$= \begin{pmatrix}\lim_{\tau\longrightarrow\infty}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)},\\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)},\\ \lim_{\tau\longrightarrow\infty}\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}\end{pmatrix}.$$

Using limit rules, logarithmic transform and L'Hospital's rule, it follows equation, as shown at the bottom of the next page, which completes the proof of Theorem 22. ■

Theorem 23 (Idempotency): Let $\mathcal{T}_j = \left(\sigma_j,\vartheta_j,\varrho_j\right)$ $(j = 1, 2, \dots, n)$ be a family of T-SFNs, if $\mathcal{S}_j = \mathcal{S}_0 \forall j$, then

$$T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_n\right) = \mathcal{S}_0. \tag{35}$$

Proof: Since for all j $\mathcal{S}_j = \mathcal{S}_0 = (\sigma_0,\vartheta_0,\varrho_0)$, and $\sum_{j=1}^n\varpi_j = 1$ so by Theorem 20, we have

$$T-SFFWG\left(\mathcal{S}_1,\mathcal{S}_2,\dots,\mathcal{S}_n\right)$$
$$= \begin{pmatrix}\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_0^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_0^t}-1\right)^{\varpi_j}\right)},\\ \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_0^t}-1\right)^{\varpi_j}\right)}\end{pmatrix}$$
$$= \begin{pmatrix}\sqrt[t]{\log_\tau\tau^{\sigma_0^t}}, \sqrt[t]{1-\log_\tau\tau^{1-\vartheta_0^t}},\\ \sqrt[t]{1-\log_\tau\tau^{1-\varrho_0^t}}\end{pmatrix}$$
$$= (\sigma_0,\vartheta_0,\varrho_0) = \mathcal{S}_0.$$

Thus, proof is completed. ■

Theorem 24 (Monotonicity): Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ and $\dot{\mathcal{S}}_j = (\dot{\sigma}_j, \dot{\vartheta}_j, \dot{\varrho}_j)$ $(j = 1, 2, \ldots, n)$ be two families of T-SFNs such that $\sigma_j \geq \dot{\sigma}_j$, $\vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j \ \forall\ j$, then

$$T-SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \geq T-SFFWG\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n\right). \quad (36)$$

Proof: According to Definition 3, when $\sigma_j \geq \dot{\sigma}_j$, $\vartheta_j \leq \dot{\vartheta}_j$ and $\varrho_j \leq \dot{\varrho}_j \ \forall\ j$, then

$$\sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)} \leq \sqrt[t]{\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{\dot{\sigma}_j^t}-1\right)^{\varpi_j}\right)},$$

$$\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\dot{\vartheta}_j^t}-1\right)^{\varpi_j}\right)} \text{ and}$$

$$\sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)} \geq \sqrt[t]{1-\log_\tau\left(1+\Pi_{j=1}^n\left(\tau^{1-\dot{\varrho}_j^t}-1\right)^{\varpi_j}\right)}.$$

Thus, $S\left(T-SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right)\right) \geq S(T-SFFWG\,(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n))$.

Hence, $T-SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \geq T-SFFWG\left(\dot{\mathcal{S}}_1, \dot{\mathcal{S}}_2, \ldots, \dot{\mathcal{S}}_n\right)$. ■

Theorem 25 (Boundedness): Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and let $\mathcal{S}^- = \left(\min_{1\leq j\leq n}\sigma_j,\ \max_{1\leq j\leq n}\vartheta_j, \max_{1\leq j\leq n}\varrho_j\right)$, $\mathcal{S}^+ = \left(\max_{1\leq j\leq n}\sigma_j,\ \min_{1\leq j\leq n}\vartheta_j,\ \min_{1\leq j\leq n}\varrho_j\right)$, then

$$\mathcal{S}^- \leq T-SFFWG\left(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\right) \leq \mathcal{S}^+. \quad (37)$$

$$\begin{aligned}
&\begin{pmatrix} \sqrt[t]{\lim_{\tau\longrightarrow\infty}\dfrac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\dfrac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\dfrac{\ln\left(1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}\right)}{\ln\tau}} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\dfrac{\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(\vartheta_j^t\right)\frac{\tau^{\sigma_j^t-1}}{\tau^{\sigma_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\dfrac{\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(1-\vartheta_j^t\right)\frac{\tau^{-\vartheta_j^t}}{\tau^{1-\vartheta_j^t}-1}\right)}{\frac{1}{\tau}}}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\dfrac{\frac{\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(1-\varrho_j^t\right)\frac{\tau^{-\varrho_j^t}}{\tau^{1-\varrho_j^t}-1}\right)}{\frac{1}{\tau}}}, \end{pmatrix} \\
&= \begin{pmatrix} \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{\sigma_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(\sigma_j^t\right)\frac{\tau^{\sigma_j^t}}{\tau^{\sigma_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\vartheta_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(1-\vartheta_j^t\right)\frac{\tau^{1-\vartheta_j^t}}{\tau^{1-\vartheta_j^t}-1}\right)}, \\ \sqrt[t]{1-\lim_{\tau\longrightarrow\infty}\frac{\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}{1+\Pi_{j=1}^n\left(\tau^{1-\varrho_j^t}-1\right)^{\varpi_j}}\left(\sum_{j=1}^n\varpi_j\left(1-\varrho_j^t\right)\frac{\tau^{1-\varrho_j^t}}{\tau^{1-\varrho_j^t}-1}\right)} \end{pmatrix} \\
&= \left(\sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(\sigma_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(1-\vartheta_j^t\right)\right)}, \sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(1-\varrho_j^t\right)\right)}\right) \\
&= \left(\sqrt[t]{1-\left(\sum_{j=1}^n\varpi_j\left(\sigma_j^t\right)\right)}, \sqrt[t]{\left(\sum_{j=1}^n\varpi_j\left(\vartheta_j^t\right)\right)}, \sqrt[t]{\left(\sum_{j=1}^n\varpi_j\left(\varrho_j^t\right)\right)}\right)
\end{aligned}$$

Proof: Since for all j, $\min_{1\le j\le n}\sigma_j \le \sigma_j \le \max_{1\le j\le n}\sigma_j$, $\min_{1\le j\le n}\vartheta_j \le \vartheta_j \le \max_{1\le j\le n}\vartheta_j$ and $\min_{1\le j\le n}\varrho_j \le \varrho_j \le \max_{1\le j\le n}\varrho_j$, thereby on the basis of idempotency and monotonicity, we get

$\mathcal{S}^- \le T-SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \le \mathcal{S}^+$. ■

Theorem 26 (Shift-Invariance): Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs and $\dot{\mathcal{S}} = (\dot{\sigma}, \dot{\vartheta}, \dot{\varrho})$ be any other T-SFNs, then

$$T-SFFWG(\mathcal{S}_1 \otimes \dot{\mathcal{S}}, \mathcal{S}_2 \otimes \dot{\mathcal{S}}, \ldots, \mathcal{S}_n \otimes \dot{\mathcal{S}}) = T-SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \otimes \dot{\mathcal{S}}. \quad (38)$$

Theorem 27 (Homogeneity): Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs and $\eta > 0$ be any real number, then

$$T-SFFWG(\eta\mathcal{S}_1, \eta\mathcal{S}_2, \ldots, \eta\mathcal{S}_n) = \eta T-SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n). \quad (39)$$

The proof of the above two theorems can be easily derived from the proposed Frank operational laws of T-SFNs; thus, it is omitted here due to the space limitations.

Definition 11: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank ordered weighted geometric (T-SFFOWG) operator is:

$$T-SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \otimes_{j=1}^{n}\left(\mathcal{S}_{\delta(j)}^{w_j}\right), \quad (40)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the position weights of $\mathcal{S}_j\,(j = 1, 2, .., n)$ such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\mathcal{S}_{\delta(j-1)} \ge \mathcal{S}_{\delta(j)}$ for $j = 2, 3.., n$.

Theorem 28: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFOWG operator is still a T-SFN, and

$$T-SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{\sigma_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{1-\vartheta_{\delta(j)}^t} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau\left(1 + \Pi_{j=1}^{n}\left(\tau^{1-\varrho_{\delta(j)}^t} - 1\right)^{w_j}\right)} \end{pmatrix}. \quad (41)$$

Proof: The proof of this result is similar to that of Theorem 20, and so we omit here. ■

Example 5 (Continued From Example 2): According to Definition 11 and Theorem 28, we can get equation, as shown at the bottom of the page.

Theorem 29: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFOWG operator approaches the following limit

$$\lim_{\tau\longrightarrow 1} T-SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\Pi_{j=1}^{n}\left(\sigma_{\delta(j)}^t\right)^{w_j}}, \sqrt[t]{1 - \Pi_{j=1}^{n}\left(1 - \vartheta_{\delta(j)}^t\right)^{w_j}}, \\ \sqrt[t]{1 - \Pi_{j=1}^{n}\left(1 - \varrho_{\delta(j)}^t\right)^{w_j}} \end{pmatrix}. \quad (42)$$

Theorem 30: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)$ $(j = 1, 2, \ldots, n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFOWG operator approaches the following limit

$$\lim_{\tau\longrightarrow \infty} T-SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1 - \left(\sum_{j=1}^{n} w_j\left(\sigma_{\delta(j)}^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\vartheta_{\delta(j)}^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^{n} w_j\left(\varrho_{\delta(j)}^t\right)\right)} \end{pmatrix}. \quad (43)$$

As similar to those of T-SFFWG operator, the T-SFFOWG operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFOWG operator has the following desirable results.

$$T-SFFOWG\left(\mathcal{S}_{\delta(1)}, \mathcal{S}_{\delta(2)}, \mathcal{S}_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{\log_2\left(1 + \prod_{j=1}^{3}\left(2^{\sigma_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2\left(1 + \prod_{j=1}^{3}\left(2^{1-\vartheta_{\delta(j)}^4} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2\left(1 + \prod_{j=1}^{3}\left(2^{1-\varrho_{\delta(j)}^4} - 1\right)^{w_j}\right)} \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt[4]{\log_2\left(1 + \left(2^{.7^4} - 1\right)^{.3}\left(2^{.3^4} - 1\right)^{.4}\left(2^{.6^4} - 1\right)^{.3}\right)} \\ \sqrt[4]{1 - \log_2\left(1 + \left(2^{1-.4^4} - 1\right)^{.3}\left(2^{1-.3^4} - 1\right)^{.4}\left(2^{1-.7^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2\left(1 + \left(2^{1-.5^4} - 1\right)^{.3}\left(2^{1-.5^4} - 1\right)^{.4}\left(2^{1-.8^4} - 1\right)^{.3}\right)} \end{pmatrix}$$
$$= (0.4788, 0.5438, 0.6509).$$

Theorem 31: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\ (j = 1, 2, \ldots, n)$ be a family of T-SFNs, then we have the following:

i). If $w = (1, 0, \ldots, 0)^T$ then $T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \max\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$.

ii). If $w = (0, 0, \ldots, 1)^T$ then $T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \min\{\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n\}$.

iii). If $w_j = 1$ and $\varpi_i = 0\ (i \neq j)$ then $T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \mathcal{S}_{\delta(j)}$ where $\mathcal{S}_{\delta(j)}$ is the jth largest of $\mathcal{S}_j$, $(j = 1, 2, \ldots, n)$.

Based on the definition of T-SFFWG and T-SFFOWG operators, we can see that the T-SFFWG operator can weights only the SFNs while T-SFFOWG operator weights only the ordered position of SFNs. In real-world practical situations, we should both consider the two aspects at the same time. Therefore, to overcome this limitation, we define the hybrid geometric operator [40] based on Frank t-norm and t-conorm, which weight both the given T-SFNs and their ordered positions.

Definition 12: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\ (j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the T-spherical fuzzy Frank hybrid geometric (T-SFFHG) operator is:

$$T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \otimes_{j=1}^{n} \left(\hat{\mathcal{S}}_{\delta(j)}\right)^{w_j}, \quad (44)$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector associated with T-SFFHG such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$, $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)^T$ is the weight vector of $\mathcal{S}_j\ (j = 1, 2, .., n)$ such that $\varpi_j > 0$ and $\sum_{j=1}^{n} \varpi_j = 1$. $\hat{\mathcal{S}}_{\delta(j)}$ is the jth largest of the weighted T-SFNs $\hat{\mathcal{S}}_j \left(\hat{\mathcal{S}}_j = (\mathcal{S}_j)^{n\varpi_j}, j = 1, 2, .., n\right)$ and n is the balancing coefficient.

Theorem 32: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\ (j = 1, 2, \ldots, n)$ be a family of T-SFNs, then the result obtained by using the T-SFFHG operator is still a T-SFN, and

$$T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{\log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{\hat{\sigma}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\hat{\vartheta}^t_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[t]{1 - \log_\tau \left(1 + \Pi_{j=1}^{n} \left(\tau^{1-\hat{\varrho}^t_{\delta(j)}} - 1\right)^{w_j}\right)} \end{pmatrix}, \quad (45)$$

Proof: The proof of this result is similar to that of Theorem 20, and so we omit here. ■

Example 6 (Continued From Example 3): According to Definition 6, we can get the weighted T-SFNs:

$$\hat{\mathcal{S}}_1 = \mathcal{S}_1^{3\times 0.4} = \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \frac{\left(2^{0.4^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \frac{\left(2^{1-0.5^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \frac{\left(2^{1-0.2^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)} \end{pmatrix} = (0.3275, 0.5227, 0.2093);$$

$$\hat{\mathcal{S}}_2 = \mathcal{S}_2^{3\times 0.4} = \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \frac{\left(2^{0.6^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \frac{\left(2^{1-0.7^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \frac{\left(2^{1-0.8^4} - 1\right)^{3\times 0.4}}{(2-1)^{3\times 0.4 - 1}}\right)} \end{pmatrix} = (0.5353, 0.7290, 0.8292);$$

$$T - SFFHG\left(\hat{\mathcal{S}}_{\delta(1)}, \hat{\mathcal{S}}_{\delta(2)}, \hat{\mathcal{S}}_{\delta(3)}\right) = \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \prod_{j=1}^{3} \left(2^{\hat{\sigma}^4_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\hat{\vartheta}^4_{\delta(j)}} - 1\right)^{w_j}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \prod_{j=1}^{3} \left(2^{1-\hat{\varrho}^4_{\delta(j)}} - 1\right)^{w_j}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt[4]{\log_2 \left(1 + \left(2^{0.5012^4} - 1\right)^{.3} \left(2^{0.3275^4} - 1\right)^{.4} \left(2^{0.5353^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-0.5307^4} - 1\right)^{.3} \left(2^{1-0.5227^4} - 1\right)^{.4} \left(2^{1-0.7290^4} - 1\right)^{.3}\right)}, \\ \sqrt[4]{1 - \log_2 \left(1 + \left(2^{1-0.4410^4} - 1\right)^{.3} \left(2^{1-0.2093^4} - 1\right)^{.4} \left(2^{1-0.8292^4} - 1\right)^{.3}\right)} \end{pmatrix}$$

$$= (0.4317, 0.6143, 0.6486).$$

$$\hat{\mathcal{S}}_3 = \mathcal{S}_3^{3\times 0.2} = \begin{pmatrix} \sqrt[4]{\log_2\left(1+\frac{\left(2^{0.3^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.6^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)}, \\ \sqrt[4]{1-\log_2\left(1+\frac{\left(2^{1-0.5^4}-1\right)^{3\times 0.2}}{(2-1)^{3\times 0.4-1}}\right)} \end{pmatrix} = (0.5012, 0.5307, 0.4410).$$

According to Definition 3, we can get the score of $\hat{\mathcal{S}}_j\ (j=1,2,3)$:

$$S\left(\hat{\mathcal{S}}_1\right) = -0.0799,\ S\left(\hat{\mathcal{S}}_2\right) = -0.6995,\ S\left(\hat{\mathcal{S}}_3\right) = -0.0651.$$

Since $S\left(\hat{\mathcal{S}}_3\right) > S\left(\hat{\mathcal{S}}_1\right) > S\left(\hat{\mathcal{S}}_2\right)$, we have $\hat{\mathcal{S}}_{\delta(1)} = (0.5012, 0.5307, 0.4410)$, $\hat{\mathcal{S}}_{\delta(2)} = (0.3275, 0.5227, 0.2093)$, $\hat{\mathcal{S}}_{\delta(3)} = (0.5353, 0.7290, 0.8292)$. Suppose $w = (0.3, 0.4, 0.3)^T$ is the weight vector associated with the T-SFFHG operator. Then by Definition 12 and Theorem 32, we can get:

Theorem 33: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\ (j=1,2,\ldots,n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow 1$, the T-SFFHG operator approaches the following limit $\lim_{\tau\longrightarrow 1} T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) =$

$$\begin{pmatrix} \sqrt[t]{\Pi_{j=1}^n \left(\hat{\sigma}_{\delta(j)}^t\right)^{w_j}}, \sqrt[t]{1-\Pi_{j=1}^n\left(1-\hat{\vartheta}_{\delta(j)}^t\right)^{w_j}}, \\ \sqrt[t]{1-\Pi_{j=1}^n\left(1-\hat{\varrho}_{\delta(j)}^t\right)^{w_j}} \end{pmatrix}. \tag{46}$$

Theorem 34: Let $\mathcal{S}_j = (\sigma_j, \vartheta_j, \varrho_j)\ (j=1,2,\ldots,n)$ be a family of T-SFNs, and $\tau > 1$. As $\tau \longrightarrow \infty$, the T-SFFHG operator approaches the following limit

$$\lim_{\tau\longrightarrow\infty} T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) = \begin{pmatrix} \sqrt[t]{1-\left(\sum_{j=1}^n w_j\left(\hat{\sigma}_{\delta(j)}^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^n w_j\left(\hat{\vartheta}_{\delta(j)}^t\right)\right)}, \\ \sqrt[t]{\left(\sum_{j=1}^n w_j\left(\hat{\varrho}_{\delta(j)}^t\right)\right)} \end{pmatrix}. \tag{47}$$

As similar to those of T-SFFWG operator, the T-SFFHG operator also follows the boundedness, idempotency and monotonicity, shift-invariance, homogeneity properties. Besides the aforementioned properties, the T-SFFHG operator has the following special cases.

Corollary 35: T-SFFWG operator is a special case of the T-SFFHG operator.

Proof: Let $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \\ &= \hat{\mathcal{S}}_{\delta(1)}^{w_1} \otimes \hat{\mathcal{S}}_{\delta(2)}^{w_2} \otimes \ldots \otimes \hat{\mathcal{S}}_{\delta(n)}^{w_n} \\ &= \left(\hat{\mathcal{S}}_{\delta(1)} \otimes \hat{\mathcal{S}}_{\delta(2)} \otimes \ldots \otimes \hat{\mathcal{S}}_{\delta(n)}\right)^{\frac{1}{n}} \\ &= \mathcal{S}_1^{\varpi_1} \otimes \mathcal{S}_2^{\varpi_2} \otimes \ldots \otimes \mathcal{S}_n^{\varpi_n} \\ &= T - SFFWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n). \end{aligned}$$

■

Corollary 36: T-SFFOWG operator is a special case of the T-SFFHG operator.

Proof: Let $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$, then

$$\begin{aligned} &T - SFFHG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n) \\ &= \hat{\mathcal{S}}_{\delta(1)}^{w_1} \otimes \hat{\mathcal{S}}_{\delta(2)}^{w_2} \otimes \ldots \otimes \hat{\mathcal{S}}_{\delta(n)}^{w_n} \\ &= \mathcal{S}_{\delta(1)}^{w_1} \otimes \mathcal{S}_{\delta(2)}^{w_2} \otimes \ldots \otimes \mathcal{S}_{\delta(n)}^{w_n} \\ &= T - SFFOWG(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n). \end{aligned}$$

■

V. ENTROPY MEASURE FOR T-SPHERICAL FUZZY SET

In this part, the entropy measure for T-spherical fuzzy set is given in detail and the required proof in terms of satisfying given properties is shared.

Definition 13: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two T-SFSs on Y. A real-valued function E :T-SFS $\longrightarrow [0,1]$ is an entropy measure for SFSs if it is provided with the following properties:

p_1. $E(\mathcal{S}_1) = 0$ iff $\mathcal{S}_1$ is a crisp set;

p_2. $E(\mathcal{S}_1) = 1$ iff $\sigma_1(y) = \varrho_1(y)$ and $\vartheta_1(y) = \sqrt[t]{0.25}$ for all $y \in Y$;

p_3. $E(\mathcal{S}_1) = E\left(\mathcal{S}_1^c\right)$;

p_4. $E(\mathcal{S}_1) \leq E(\mathcal{S}_2)$ if $\vartheta_2^t(y) \leq \vartheta_1^t(y)$ and $\sigma_1^t(y) \leq \sigma_2^t(y) \leq \varrho_2^t(y) \leq \varrho_1^t(y)$ or $\varrho_1^t(y) \leq \varrho_2^t(y) \leq \sigma_2^t(y) \leq \sigma_1^t(y)$ for all $y \in Y$.

Theorem 37: Let $\mathcal{S}$ be a T-SFS on Y. The mapping

$$E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^n\left(1-\frac{4}{5}\left[\left|\sigma^t(y_i)-\varrho^t(y_i)\right|+\left|\vartheta^t(y_i)-0.25\right|\right]\right), \tag{48}$$

is an entropy measure for T-SFS.

Proof:

p_1. Let $\mathcal{S}$ be a crisp set. Then, we have $\sigma^t(y_i) = 1$, $\varrho^t(y_i) = 0$, $\vartheta^t(y_i) = 0$ or $\sigma^t(y_i) = 0$, $\varrho^t(y_i) = 1$, $\vartheta^t(y_i) = 0\ \forall y_i \in Y$. If $\sigma^t(y_i) = 1$, $\varrho^t(y_i) = 0$, $\vartheta^t(y_i) = 0$, then

$$\begin{aligned} E(\mathcal{S}) &= \frac{1}{n}\sum_{i=1}^n\left(1-\frac{4}{5}\left[\left|\sigma^t(y_i)-\varrho^t(y_i)\right|\right.\right. \\ &\quad \left.\left.+\left|\vartheta^t(y_i)-0.25\right|\right]\right) = \frac{1}{n}\sum_{i=1}^n\left(1-\frac{4}{5}\left[|1-0|\right.\right. \\ &\quad \left.\left.+|0-0.25|\right]\right) = \frac{1}{n}\sum_{i=1}^n\left(1-\frac{4}{5}[1.25]\right) = 0. \end{aligned}$$

Analogously, when $\sigma^t(y_i) = 0$, $\varrho^t(y_i) = 1$, $\vartheta^t(y_i) = 0\ \forall y_i \in Y$, we can easily show that $E(\mathcal{S}) = 0$. Conversely, suppose that $E(\mathcal{S}) = 0$. Thus,

$$\frac{4}{5}\left[\left|\sigma^t(y_i)-\varrho^t(y_i)\right|+\left|\vartheta^t(y_i)-0.25\right|\right] = 1, \tag{49}$$

Eq. (49) $\Rightarrow$ $\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right| = 1.25$.

There are four possibilities: The first one is $\left(\sigma^t(y_i) - \varrho^t(y_i)\right) + \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow \sigma^t(y_i) - \varrho^t(y_i) + \vartheta^t(y_i) = 1.5 \Rightarrow \sigma^t(y_i) + \vartheta^t(y_i) = \varrho^t(y_i) + 1.5$.

The second is $\left(\sigma^t(y_i) - \varrho^t(y_i)\right) - \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow \sigma^t(y_i) - \varrho^t(y_i) - \vartheta^t(y_i) = 1 \Rightarrow \sigma^t(y_i) = \vartheta^t(y_i) + \varrho^t(y_i)1$.

The third is $-\left(\sigma^t(y_i) - \varrho^t(y_i)\right) + \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow -\sigma^t(y_i) + \varrho^t(y_i) + \vartheta^t(y_i) = 1.5 \Rightarrow \varrho^t(y_i) + \vartheta^t(y_i) = \sigma^t(y_i) + 1.5$.

The last one is $-\left(\sigma^t(y_i) - \varrho^t(y_i)\right) - \left(\vartheta^t(y_i) - 0.25\right) = 1.25 \Rightarrow -\sigma^t(y_i) + \varrho^t(y_i) - \vartheta^t(y_i) = 1 \Rightarrow \varrho^t(y_i) = \sigma^t(y_i) + \vartheta^t(y_i) + 1$.

In all these possibilities, the inequality $0 \le \sigma^t(y_i) + \vartheta^t(y_i) + \varrho^t(y_i) \le 1$ is not satisfied for all $y_i \in Y$. So Eq. (49) can hold when $\sigma^t(y_i) = 1, \varrho^t(y_i) = 0, \vartheta^t(y_i) = 0$ or $\sigma^t(y_i) = 0, \varrho^t(y_i) = 1, \vartheta^t(y_i) = 0$ $\forall y_i \in Y$. Consequently, it is proved that $\mathcal{S}$ is a crisp set.

p_2. Let $\sigma^t(y_i) = \varrho^t(y_i)\ \vartheta^t(y_i) = 0.25\ \forall y_i \in Y$. Then $E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}[0 + 0]\right) = 1$. Conversely, suppose that $E(\mathcal{S}) = 1$. Then $\frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right] = 0$ $\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right| = 0$. Hence, $\sigma(y_i) = \varrho(y_i)$ and $\vartheta(y_i) = \sqrt[t]{0.25}\ \forall\ y_i \in Y$.

p_3. $E(\mathcal{S}) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma^t(y_i) - \varrho^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\varrho^t(y_i) - \sigma^t(y_i)\right| + \left|\vartheta^t(y_i) - 0.25\right|\right]\right) = E(\mathcal{S}^c)$.

p_4. Since $\vartheta_2^t(y) \le \vartheta_1^t(y)$ and $\sigma_1^t(y) \le \sigma_2^t(y) \le \varrho_2^t(y) \le \varrho_1^t(y)$, we have $\left|\sigma_2^t(y_i) - \varrho_2^t(y_i)\right| \le \left|\sigma_1^t(y_i) - \varrho_1^t(y_i)\right|$ and $\left|\vartheta_2^t(y_i) - 0.25\right| \le \left|\vartheta_1^t(y_i) - 0.25\right|$. Therefore $E(\mathcal{S}_1) = \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma_1^t(y_i) - \varrho_1^t(y_i)\right| + \left|\vartheta_1^t(y_i) - 0.25\right|\right]\right) \le \frac{1}{n}\sum_{i=1}^{n}\left(1 - \frac{4}{5}\left[\left|\sigma_2^t(y_i) - \varrho_2^t(y_i)\right| + \left|\vartheta_2^t(y_i) - 0.25\right|\right]\right) = E(\mathcal{S}_2)$.

Analogously, if $\vartheta_2^t(y) \le \vartheta_1^t(y)$ and $\varrho_1^t(y) \le \varrho_2^t(y) \le \sigma_2^t(y) \le \sigma_1^t(y)$ for all $y \in Y$, then $E(\mathcal{S}_1) \le E(\mathcal{S}_2)$.

■

VI. PROPOSED METHOD OF SFS FOR MCDM PROBLEMS

This section focuses on presenting an MCDM method based on the proposed Frank aggregation operators to handle decision making problems with T-spherical fuzzy information.

A. DECISION-MAKING METHOD

Let $o = \{o_1, o_2, \ldots, o_m\}$ be a set of alternatives, and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_n\}$ be the set of criteria, whose weight vector is unknown. The characteristics of each alternative $o_i\ (i = 1, 2, .., m)$ with respect to each criteria is characterized in terms of T-SFNs $\mathcal{S}_{ij} = \left(\sigma_{ij} + \vartheta_{ij} + \varrho_{ij}\right);\ 0 \le \sigma_{ij}^t + \vartheta_{ij}^t + \varrho_{ij}^t \le 1$ Then, in the following, we construct an MCDM method based on the proposed operator to cope with the decision-making problems with T-spherical fuzzy information, which mainly involves the following steps.

Step 1: Formation of decision matrix:

Collect the T-spherical fuzzy information from experts about the finite set of alternatives observing the criteria in the form of matrix $M = \left[\mathcal{S}_{ij}\right]$. Also, decide the least value of t for which every triplet of the provided information lies in the frame of T-SFNs.

Step 2: Normalization:

Transform the decision matrix $M = \left[\mathcal{S}_{ij}\right]$ into the normalized form $\widetilde{M} = \left[\widetilde{\mathcal{S}}_{ij}\right]$ by the below formula:

$$\widetilde{\mathcal{S}}_{ij} = \begin{cases} \mathcal{S}_{ij}, & \text{if for benefit criteria} \\ \left(\mathcal{S}_{ij}\right)^c, & \text{for cost criteria.} \end{cases}$$

where $\left(\mathcal{S}_{ij}\right)^c$ is the complement of $\mathcal{S}_{ij}$.

Step 3: Criteria weight determination:

The stated entropy measure for T-SFSs is used to derive the weights of criteria. First, Eq. (48) is applied for each jth criteria to obtain the entropy measure that represents the information dispersion in handled criteria, and then all the criteria's entropies are used to derive their weights $\left(\varpi_j\right)$.

$$E_j = \frac{1}{n}\sum_{i=1}^{m}\left(1 - \frac{4}{5}\left[\left|\sigma_{ij}^t - \varrho_{ij}^t\right| + \left|\vartheta_{ij}^t - 0.25\right|\right]\right) \tag{50}$$

Divergence representing the intrinsic information of jth criteria is computed through $div_j = 1 - E_j$. The objective criteria weights can be computed as follows [41]:

$$\varpi_j = \frac{div_j}{\sum_{j=1}^{n} div_j}. \tag{51}$$

Step 4: Aggregation:

Aggregate the T-SFNs $\mathcal{S}_{ij}\ (j = 1, 2, \ldots, n)$ for each alternative $o_i\ (i = 1, 2, \ldots, n)$ into the overall preference value $\mathcal{S}_i$ either by using the proposed T-SFFHWA or T-SFFHWG operators. Mathematically,

$$\mathcal{S}_i = T - SFFHWA_{\varpi,w}\left(\mathcal{S}_{i1}, \mathcal{S}_{i2}, \ldots, \mathcal{S}_{im}\right), \tag{52}$$

$$\mathcal{S}_i = T - SFFHWG_{\varpi,w}\left(\mathcal{S}_{i1}, \mathcal{S}_{i2}, \ldots, \mathcal{S}_{im}\right), \tag{53}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_n)$ is the weight vector of criteria derived in Step 3. And $w = (w_1, w_2, \ldots, w_n)$ is the weight vector associated the aggregation operator such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$.

Step 5: Score values:

Compute the score values of $S(\mathcal{S}_i)$ $(i = 1, 2, \ldots, m)$ of the overall values $\mathcal{S}_i (i = 1, 2, \ldots, m)$.

Step 6: Ranking:

Rank the alternatives o_i $(i = 1, 2, \ldots, m)$ according to the score values $S(\mathcal{S}_i)$ and select the best one.

VII. ILLUSTRATIVE EXAMPLE

This section first uses a numerical example to demonstrate the working procedure of the suggested MCDM method, then performs a series of experiments to examine the influence of various specific operators and parameter values on the aggregation results.

A. EXAMPLE

The presented MCDM method is demonstrated using a numerical example to determine the best industry for investment from five possible industries (adopted from Ref. [12]).

To make full use of idle capital, the company's board of directors decided to invest in a new industry. Following the preliminary investigation, four industries were selected as potential investment targets. The four alternative industries are manufacturing industry (o_1), real estate development industry (o_2), education and training industry (o_3) and medical industry (o_4). To determine the best choice for investment, the directors have set up a panel of experts. These experts were called to rate the four alternatives industries based on four criteria namely, level of capital gain (κ_1), market potential (κ_2), growth potential (κ_3) and political stability (κ_4), whose weights are unknown. To offer sufficient freedom in the evaluation of the values of the five criteria of each alternative industry, the experts were allowed to use T-SFNs. The evaluation information of experts is detailed in the following matrix.

B. THE PROCESS OF SOLVING

In the following, we utilize T-SFFHA operator and T-SFFHG operator in the provided MCDM approach with T-spherical fuzzy information.

1) USING T-SFFHA OPERATOR

Step 1. The provided decision matrix is listed in Table 1. On account of this, we determine the value of 't' for which the given data lie in T-spherical fuzzy information.

As $0.6 + 0.7 + 0.6 = 1.9 > 1$,

for $t = 2$, $0.6^2 + 0.7^2 + 0.6^2 = 1.21 > 1$,

for $t = 3$, $0.6^3 + 0.7^3 + 0.6^3 = 0.775 < 1$. Analogously, for $t = 3$, all the given data lie in the T-spherical fuzzy information.

Step 2. Consider all the criteria κ_j $(j = 1, 2, 3, 4)$ are the benefit criteria, thus, the criteria values of alternatives o_i $(i = 1, 2, 3, 4)$ do not need normalization.

Step 3. In the light of Eq. (48), the criteria weight is calculated as given below.

$$\varpi = (0.3576, 0.2337, 0.2344, 0.1743),$$
$$n\varpi = (1.4304, 0.9348, 0.9376, 0.6972).$$

Step 4. Using the T-spherical fuzzy information listed in Table 1, the values of $\hat{\mathcal{S}}_{ij} = (n\varpi_j)\,\mathcal{S}_{ij}$, (suppose n=4, $\tau = 2$),

TABLE 1. T-spherical fuzzy decision matrix.

	κ_1	κ_2	κ_3	κ_4
o_1	$(0.7, 0.3, 0.4)$	$(0.3, 0.3, 0.5)$	$(0.6, 0.7, 0.6)$	$(0.3, 0.4, 0.2)$
o_2	$(0.5, 0.5, 0.4)$	$(0.6, 0.3, 0.4)$	$(0.3, 0.2, 0.8)$	$(0.7, 0.2, 0.4)$
o_3	$(0.6, 0.9, 0.2)$	$(0.7, 0.3, 0.3)$	$(0.5, 0.4, 0.3)$	$(0.4, 0.7, 0.5)$
o_4	$(0.8, 0.2, 0.2)$	$(0.5, 0.6, 0.2)$	$(0.4, 0.1, 0.3)$	$(0.5, 0.7, 0.4)$

are worked out as given below.

$$
\begin{aligned}
\hat{\mathcal{S}}_{11} &= (0.7713, 0.1702, 0.2580),\\
\hat{\mathcal{S}}_{12} &= (0.2934, 0.3267, 0.5256),\\
\hat{\mathcal{S}}_{13} &= (0.5883, 0.7172, 0.6214),\\
\hat{\mathcal{S}}_{14} &= (0.2663, 0.5406, 0.3366),\\
\hat{\mathcal{S}}_{21} &= (0.5595, 0.3575, 0.2580),\\
\hat{\mathcal{S}}_{22} &= (0.5878, 0.3267, 0.4271),\\
\hat{\mathcal{S}}_{23} &= (0.2937, 0.2227, 0.8120),\\
\hat{\mathcal{S}}_{24} &= (0.6303, 0.3366, 0.5406),\\
\hat{\mathcal{S}}_{31} &= (0.6675, 0.8578, 0.0950),\\
\hat{\mathcal{S}}_{32} &= (0.6867, 0.3267, 0.3267),\\
\hat{\mathcal{S}}_{33} &= (0.4899, 0.4259, 0.3255),\\
\hat{\mathcal{S}}_{34} &= (0.3555, 0.7857, 0.6280),\\
\hat{\mathcal{S}}_{41} &= (0.8675, 0.0950, 0.0950),\\
\hat{\mathcal{S}}_{42} &= (0.4894, 0.6224, 0.2238),\\
\hat{\mathcal{S}}_{43} &= (0.3917, 0.1163, 0.3255),\\
\hat{\mathcal{S}}_{44} &= (0.4455, 0.7857, 0.5406).
\end{aligned}
$$

Based on the score function, we have

$$
\begin{aligned}
\hat{\mathcal{S}}_{\delta(11)} &= \hat{\mathcal{S}}_{11} = (0.7713, 0.1702, 0.2580),\\
\hat{\mathcal{S}}_{\delta(12)} &= \hat{\mathcal{S}}_{12} = (0.2934, 0.3267, 0.5256),\\
\hat{\mathcal{S}}_{\delta(13)} &= \hat{\mathcal{S}}_{14} = (0.2663, 0.5406, 0.3366),\\
\hat{\mathcal{S}}_{\delta(14)} &= \hat{\mathcal{S}}_{13} = (0.5883, 0.7172, 0.6214),\\
\hat{\mathcal{S}}_{\delta(21)} &= \hat{\mathcal{S}}_{21} = (0.5595, 0.3575, 0.2580),\\
\hat{\mathcal{S}}_{\delta(22)} &= \hat{\mathcal{S}}_{22} = (0.5878, 0.3267, 0.4271),\\
\hat{\mathcal{S}}_{\delta(23)} &= \hat{\mathcal{S}}_{24} = (0.6303, 0.3366, 0.5406),\\
\hat{\mathcal{S}}_{\delta(24)} &= \hat{\mathcal{S}}_{23} = (0.2937, 0.2227, 0.8120),\\
\hat{\mathcal{S}}_{\delta(31)} &= \hat{\mathcal{S}}_{32} = (0.6867, 0.3267, 0.3267),\\
\hat{\mathcal{S}}_{\delta(32)} &= \hat{\mathcal{S}}_{33} = (0.4899, 0.4259, 0.3255),\\
\hat{\mathcal{S}}_{\delta(33)} &= \hat{\mathcal{S}}_{31} = (0.6675, 0.8578, 0.0950),\\
\hat{\mathcal{S}}_{\delta(34)} &= \hat{\mathcal{S}}_{34} = (0.3555, 0.7857, 0.6280),\\
\hat{\mathcal{S}}_{\delta(41)} &= \hat{\mathcal{S}}_{41} = (0.8675, 0.0950, 0.0950),\\
\hat{\mathcal{S}}_{\delta(42)} &= \hat{\mathcal{S}}_{43} = (0.3917, 0.1163, 0.3255),\\
\hat{\mathcal{S}}_{\delta(43)} &= \hat{\mathcal{S}}_{42} = (0.4894, 0.6224, 0.2238),\\
\hat{\mathcal{S}}_{\delta(44)} &= \hat{\mathcal{S}}_{44} = (0.4455, 0.7857, 0.5406).
\end{aligned}
$$

Now applying the T-SFFHA operator Eq. (26), having associated weight vector $w = (0.3, 0.2, 0.3, 0.2)^T$ to get the overall preference values $\mathcal{S}_i$ of the alternative

$o_i \, (i = 1, 2, \ldots, n)$:

$$\mathcal{S}_1 = (0.5922, 0.3691, 0.3857),$$
$$\mathcal{S}_2 = (0.5604, 0.3138, 0.4526),$$
$$\mathcal{S}_3 = (0.6082, 0.5579, 0.2583),$$
$$\mathcal{S}_4 = (0.6747, 0.2700, 0.2233).$$

Step 5: By Definition 3, we calculate the score values $S(\mathcal{S}_i) \, (i = 1, 2, 3, 4)$ of the overall preference values $\mathcal{S}_i \, (i = 1, 2, 3, 4)$ as follows:

$$S(\mathcal{S}_1) = 0.1171, \quad S(\mathcal{S}_2) = 0.0615,$$
$$S(\mathcal{S}_3) = 0.0391, \quad S(\mathcal{S}_4) = 0.3218.$$

Step 6: Based on the above score values the ranking order of alternatives is: $o_4 \succ o_1 \succ o_2 \succ o_3$, where the symbol "$\succ$" means "superior to". Thus, the most desirable company is o_4.

2) USING T-SFFHG OPERATOR

Step 4. Using the T-spherical fuzzy information listed in Table 1, and $\hat{\mathcal{S}}_{ij} = (\mathcal{S}_{ij})^{n\varpi_j}$, (suppose n=4, $\tau = 2$), the results are worked out as given below.

$$\hat{\mathcal{S}}_{11} = (0.5893, 0.3376, 0.4492),$$
$$\hat{\mathcal{S}}_{12} = (0.3267, 0.2934, 0.4894),$$
$$\hat{\mathcal{S}}_{13} = (0.6214, 0.6873, 0.5883),$$
$$\hat{\mathcal{S}}_{14} = (0.4447, 0.3555, 0.1774),$$
$$\hat{\mathcal{S}}_{21} = (0.3575, 0.5595, 0.4492),$$
$$\hat{\mathcal{S}}_{22} = (0.6224, 0.2934, 0.3913),$$
$$\hat{\mathcal{S}}_{23} = (0.3255, 0.1958, 0.7873),$$
$$\hat{\mathcal{S}}_{24} = (0.7857, 0.1774, 0.3555),$$
$$\hat{\mathcal{S}}_{31} = (0.4681, 0.9496, 0.2253),$$
$$\hat{\mathcal{S}}_{32} = (0.7179, 0.2934, 0.2934),$$
$$\hat{\mathcal{S}}_{33} = (0.5244, 0.3917, 0.2937),$$
$$\hat{\mathcal{S}}_{34} = (0.5406, 0.6303, 0.4455),$$
$$\hat{\mathcal{S}}_{41} = (0.7199, 0.2253, 0.2253),$$
$$\hat{\mathcal{S}}_{42} = (0.5256, 0.5878, 0.1956),$$
$$\hat{\mathcal{S}}_{43} = (0.4259, 0.0979, 0.2937),$$
$$\hat{\mathcal{S}}_{44} = (0.6280, 0.6303, 0.3555).$$

Based on the score function, we have

$$\hat{\mathcal{S}}_{\delta(11)} = \hat{\mathcal{S}}_{11} = (0.5893, 0.3376, 0.4492),$$
$$\hat{\mathcal{S}}_{\delta(12)} = \hat{\mathcal{S}}_{14} = (0.4447, 0.3555, 0.1774),$$
$$\hat{\mathcal{S}}_{\delta(13)} = \hat{\mathcal{S}}_{12} = (0.3267, 0.2934, 0.4894),$$
$$\hat{\mathcal{S}}_{\delta(14)} = \hat{\mathcal{S}}_{13} = (0.6214, 0.6873, 0.5883),$$
$$\hat{\mathcal{S}}_{\delta(21)} = \hat{\mathcal{S}}_{24} = (0.7857, 0.1774, 0.3555),$$
$$\hat{\mathcal{S}}_{\delta(22)} = \hat{\mathcal{S}}_{22} = (0.6224, 0.2934, 0.3913),$$
$$\hat{\mathcal{S}}_{\delta(23)} = \hat{\mathcal{S}}_{21} = (0.3575, 0.5595, 0.4492),$$
$$\hat{\mathcal{S}}_{\delta(24)} = \hat{\mathcal{S}}_{23} = (0.3255, 0.1958, 0.7873),$$
$$\hat{\mathcal{S}}_{\delta(31)} = \hat{\mathcal{S}}_{32} = (0.7179, 0.2934, 0.2934),$$
$$\hat{\mathcal{S}}_{\delta(32)} = \hat{\mathcal{S}}_{33} = (0.5244, 0.3917, 0.2937),$$
$$\hat{\mathcal{S}}_{\delta(33)} = \hat{\mathcal{S}}_{34} = (0.5406, 0.6303, 0.4455),$$
$$\hat{\mathcal{S}}_{\delta(34)} = \hat{\mathcal{S}}_{31} = (0.4681, 0.9496, 0.2253),$$
$$\hat{\mathcal{S}}_{\delta(41)} = \hat{\mathcal{S}}_{41} = (0.7199, 0.2253, 0.2253),$$
$$\hat{\mathcal{S}}_{\delta(42)} = \hat{\mathcal{S}}_{43} = (0.4259, 0.0979, 0.2937),$$
$$\hat{\mathcal{S}}_{\delta(43)} = \hat{\mathcal{S}}_{44} = (0.6280, 0.6303, 0.3555),$$
$$\hat{\mathcal{S}}_{\delta(44)} = \hat{\mathcal{S}}_{42} = (0.5256, 0.5878, 0.1956).$$

Applying the T-SFFHG operator Eq. (44), having associated weight vector $w = (0.3, 0.2, 0.3, 0.2)^T$ to get the overall preference values $\mathcal{S}_i$ of the alternative $o_i \, (i = 1, 2, \ldots, n)$:

$$\mathcal{S}_1 = (0.4734, 0.46298, 0.4730),$$
$$\mathcal{S}_2 = (0.5018, 0.3979, 0.5499),$$
$$\mathcal{S}_3 = (0.5699, 0.7152, 0.3471),$$
$$\mathcal{S}_4 = (0.5863, 0.4996, 0.2867).$$

Step 5: By definition 3, we calculate the score values $S(\mathcal{S}_i) \, (i = 1, 2, 3, 4)$ of the overall preference values $\mathcal{S}_i \, (i = 1, 2, 3, 4)$ as follows:

$$S(\mathcal{S}_1) = -0.1160, \quad S(\mathcal{S}_2) = -0.1195,$$
$$S(\mathcal{S}_3) = -0.2451, \quad S(\mathcal{S}_4) = 0.0619.$$

Step 6: Based on the above score values the ranking order of alternatives is: $o_4 \succ o_1 \succ o_2 \succ o_3$, where the symbol "$\succ$" means "superior to". Thus, the most desirable company is o_4.

C. PARAMETER ANALYSIS

The two parameters (τ, t) associated with the established model certainly have an effect on the final outcomes; therefore, we attempt to examine their effects.

1) THE INFLUENCE OF THE PARAMETER τ

To represent the impact of different values of parameter τ, we employ several values of τ in our suggested methodology to rank the alternatives. The ranking results are depicted in Table 2 and Figs. 2 and 3.

As shown in Table 2, the score values with various parameter τ alter, but the ranking order of the alternatives are changeless, indicating that the suggested approach has the feature of isotonicity, allowing DMs to select the adequate value based on their preferences. By examining Figs. 2 and 3, we can see that the score values generated by the T-SFFHWG operator rise as the parameter grow for the same option, but the score values produced by T-SFFHWA operator get lower as the parameter τ increases within the interval $(1, 5]$. Further from Table 2, one can notice that the aggregated score values of alternatives o_2 and o_3 obtained by T-SFFHWA operator are quite close for smaller values of τ, but as the value of τ increases, their gap of difference also becomes enlarge. This implies that the T-SFFHWA operator with the larger value of τ has a strong sense of differentiation.

TABLE 2. Ranking of alternatives with different values of τ.

τ	T-SFFHA		T-SFFHG	
	Score values	Ranking	Score values	Ranking
$\tau \longrightarrow 1$	0.1219,0.0656,0.0525,0.3291	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1222,-0.1286,-0.2659,0.0523	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 2$	0.1171,0.0615,0.0391,0.3218	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1160,-0.1195,-0.2451,0.0619	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 5$	0.1119,0.0568,0.0210,0.3138	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0797,-0.1101,-0.2208,0.0732	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 10$	0.1160,0.0582,0.0195,0.3205	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.1046,-0.1049,-0.2056,0.0806	$o_4 \succ o_1 \succ o_2 \succ o_3$
$\tau = 50$	0.1255,0.0634,0.0156,0.3136	$o_4 \succ o_1 \succ o_2 \succ o_3$	-0.0956,-0.0971,-0.1793,0.0951	$o_4 \succ o_1 \succ o_2 \succ o_3$

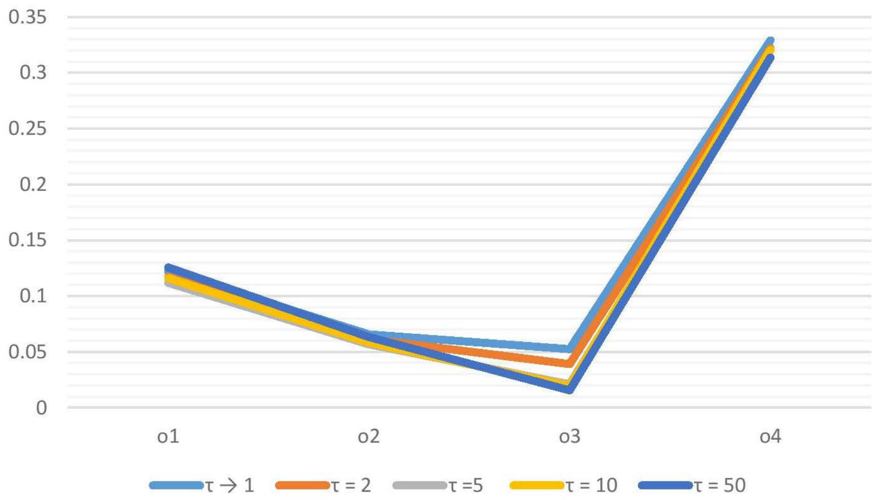

FIGURE 2. Ranking of alternatives by T-SFFHA operator for different values of τ.

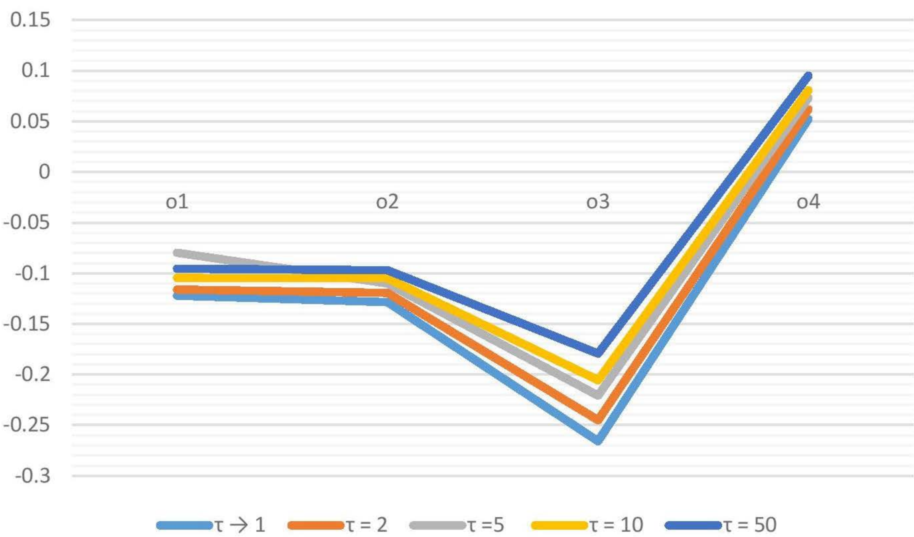

FIGURE 3. Ranking of alternatives by T-SFFHG operator for different values of τ.

2) THE INFLUENCE OF THE PARAMETER t

To examine the impact of various values of the parameter t on the ranking order of alternatives, we adapt different values of t in Step 4 of the suggested MCDM approach. The derived results are shown in Tables 3, 4 and Figures 4 and 5 (setting $\tau = 2$).

From Table 3, we can observe that the score values obtained by T-SFFHA operator become smaller and smaller as the value of t increases while those obtained by T-SFFHG operator rise as the value of t increases. However, the rating results are exactly the same; the best alternative is all o_4. The parameter t can be viewed as the ‘‘DM's attitude.'' The operator is T-SFFHA suitable for modelling pessimistic DMs, whereas the T-SFFHA operator is considered to be useful in reflecting optimistic DMs. Suppose we use T-SFFHA operator as an aggregation tool for the decision process. In that

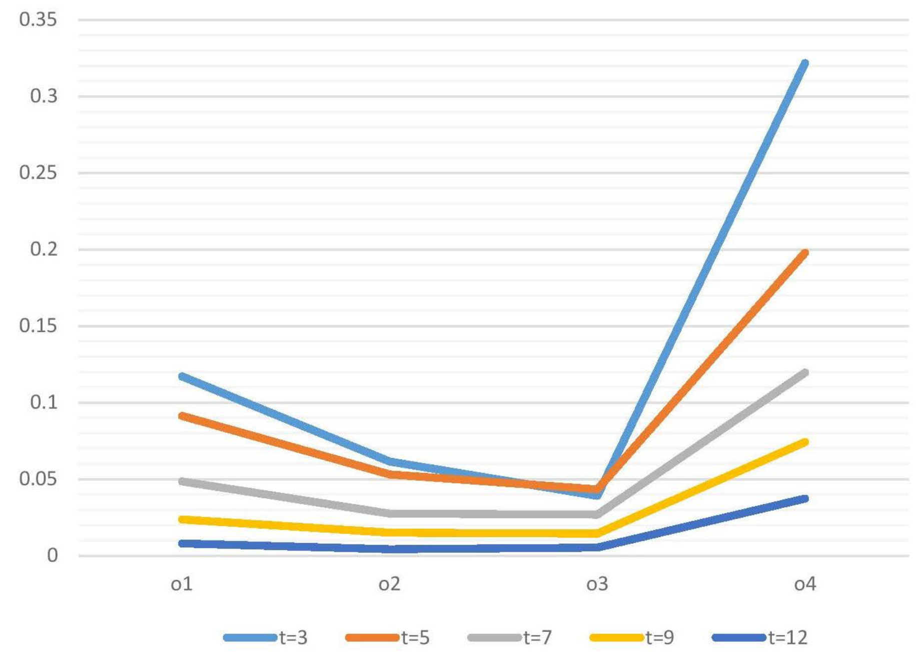

FIGURE 4. Ranking of alternatives by T-SFFHA operator for different values of t.

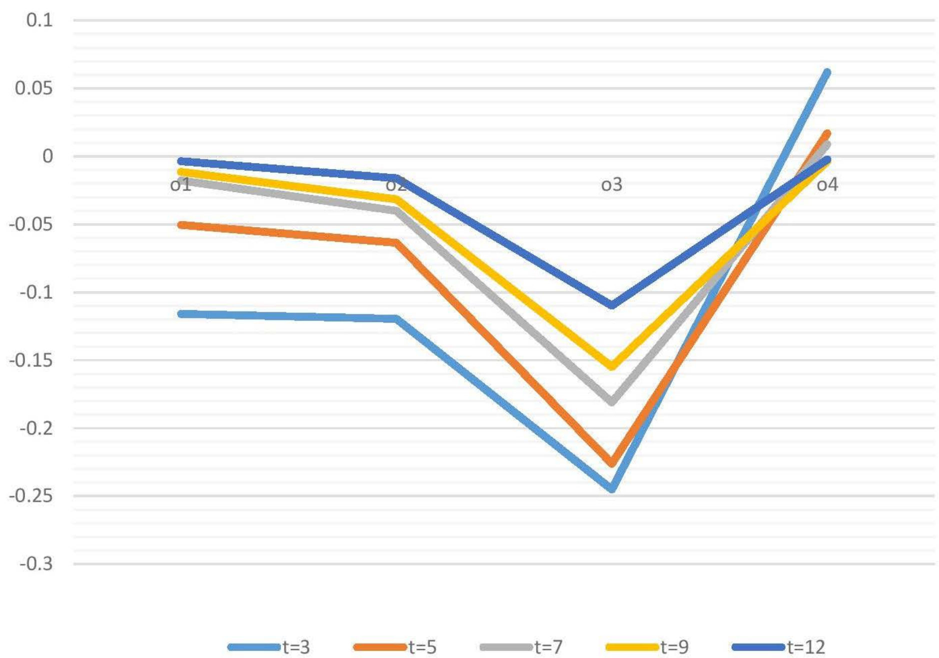

FIGURE 5. Ranking of alternatives by T-SFFHG operator for different values of t.

case, the higher value of t indicates that DMs have a more pessimistic attitude and vice versa in the case of the T-SFHG operator. So, different DMs can choose the value of t based on their attitude.

D. THE ANALYSIS OF COMPARISON

In order to verify the effectiveness of the developed method, in what follows, we compare our established method with three previous MCDM methods based on T-spherical fuzzy Einstein hybrid interactive averaging (T-SFEHIA) [30], T-spherical fuzzy Einstein hybrid interactive geometric (T-SFEHIG) [30], T-spherical fuzzy Hamacher averaging (TSFHHA) [31], T-spherical fuzzy Hamacher geometric (TSFHHG) [31], spherical fuzzy number weighted averaging operator (SFNWA) [24], spherical fuzzy number weighted geometric operator (SFNWG) [24] operators.

TABLE 3. Ranking results for different values of t using T-SFFHA.

Parameter	score values	Ranking
$t=3$	0.1171,0.0615,0.0391,0.3218	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=5$	0.0914,0.0530,0.0433,0.1979	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=7$	0.0484,0.0275,0.0268,0.1196	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=9$	0.0237,0.0151,0.0144,0.0742	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=12$	0.0079,0.0042,0.0052,0.0373	$o_4 \succ o_1 \succ o_2 \succ o_3$

TABLE 4. Ranking results for different values of t using T-SFFHG.

Parameter	score values	Ranking
$t=3$	-0.1160,-0.1195,-0.2451,0.0619	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=5$	-0.0505,-0.06351,-0.2261,0.0169	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=7$	-0.0179,-0.0400,-0.1810,0.0089	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=9$	-0.0114,-0.0315,-0.1549,-0.0035	$o_4 \succ o_1 \succ o_2 \succ o_3$
$t=12$	-0.0037,-0.0162,-0.1098,-0.0023	$o_4 \succ o_1 \succ o_2 \succ o_3$

TABLE 5. Ranking results for different existing aggregation operators.

Parameter	score values	Ranking
T-SFEHIA operator [30]	-0.0207,-0.2146,-0.9255,0.0230	$o_4 \succ o_1 \succ o_2 \succ o_3$
T-SFEHIG operator [30]	0.0296,-0.0676,-0.1309,0.1639	$o_4 \succ o_1 \succ o_2 \succ o_3$
TSFHHA operator [31]	0.1525,0.1048,0.1091,0.3257	$o_4 \succ o_1 \succ o_3 \succ o_2$
TSFHHG operator [31]	-0.1520,-0.1529,-0.2851,-0.0041	$o_4 \succ o_1 \succ o_2 \succ o_3$
SFNWA operator [24]	Not applicable	
SFNWG operator [24]	Not applicable	

The comparison results obtained by these existing methods and our proposed methods are listed in Table 5.

Based on Table 5, we can see that the ranking results obtained by applying T-SFEHIA and T-SFEHIG operators [30] are consistent with the result obtained by the proposed operators. But these operators are based on Einstein operational rules [30] which are not authentic. For instance, if we take $\mathcal{S} = (0.6, 0.9, 0.2)$, then clearly $0.6^3 + 0.9^3 + 0.2^3 = 0.953 < 1$. Now accordingly to the operational rules of [30], if we take scalar multiple of $\mathcal{S}$ with scalar $\eta = 1.4304$, we get $\eta\mathcal{S} = (0.6724, 0.9540, 0.1463)$. From this, we have $0.6724^3 + 0.9540^3 + 0.1463^3 = 1.1754 \nless 1$. Analogously, if we take scalar power with η, we get $\mathcal{S}^\eta = (0.5135, 0.9540, 0.2253)$. Again we have $0.5135^3 + 0.9540^3 + 0.2253^3 = 1.015 \nless 1$. This seems unsuitable because the operating results are not valid for $t = 3$, and may cause unfruitful results in the decision process. In addition, the proposed operators include a parameter, which can adjust the aggregate value based on the real decision needs and capture many existing hesitant fuzzy aggregation operators. Therefore, the benefit is that the proposed operators come with their higher generality and flexibility.

In the second comparison TSFHHA and TSFHHG operators [31] are adopted to address the problem presented in subsection VII-A. From Table 5, it can be observed that likewise T-SFEHIA and T-SFEHIG operators, the ranking result obtained by utilizing TSFHHG operator is consistent with our proposed operators' results. However, in the case that TSFHHA operator is applied we have o_4, o_1, o_3, o_2. The alternatives o_2 and o_3 have interchanged their positions. Thus the ranking of alternatives is slightly different from that derived by the proposed and other existing operators. However, the best alternative remains the same in all cases. This verifies the validity of the developed operators.

Finally, we compare our developed approach with the method from Ashraf and Abdullah [24] for the considered problem, where the method from [24] is structured on the Archimedean t-norm and t-conorm based spherical fuzzy weighted averaging and geometric operators. Obviously, these operators have a general representation by Archimedean t-norm and t-conorm. But, Ashraf and Abdullah's method also has some failure cases because its scope of application is very narrow. It can only handle the MCDM problems expressed by PFNs or SFNs. Therefore it cannot be employed to solve the aforementioned MCDM problem. On the contrary, the proposed method has the parameter t, which can express the spherical fuzzy information more flexible. So, in the real application environment, the proposed method is more applicable.

In comparison to the aforementioned aggregation operators, the proposed work has the following key advantages during implementation.

1). Unlike Ullah *et al.* [31], the proposed approach remains unchanged on the same output ranking for different operators.

2). The developed method includes two parameters, which can adjust the aggregate value based on real-world decision needs and capture various spherical fuzzy aggregation operators. As a result, the suggested operators have a higher degree of generality and flexibility.

3). Unlike the previous methods [24], [30], [31], the stated method employ the proposed entropy measure for the criteria weight determination, and the derived weights are then used in the decision process. Thus the proposed method is more useful in the situation where the criteria weights are completely unknown.

4). T-spherical fuzzy Frank operators can tackle the problems considered in the existing literature [24], [42], [43], but the existing operators of IFSs, PyFSs, PFSs and q-ROFSs cannot address the problems described in T-spherical fuzzy environment.

VIII. CONCLUSION

In the procedure of decision making, owing to the increasing complexity and uncertainty of real life scenarios, the decision information is more suitably expressed in terms of T-SFNs. Though several information fusion techniques have been explored to aggregate spherical fuzzy information, all these techniques are limited to algebraic, Einstein or Hamacher t-norm and t-conorm and some of them have certain limitations as pointed out in Section VII-D. Motivated by these defects and beneficial characteristics of Frank t-norm and t-conorm, we explored novel generalized operational rules of T-SFNs to build T-spherical fuzzy aggregation operators that comply with the principles of Frank t-norm and t-conorm. Keeping in mind the significance of ordered position and argument itself, the notions of T-SFFHA and T-SFFHG are provided. Some desirable properties and special cases of these operators are also studied comprehensibly. Besides, T-spherical fuzzy entropy measure is proposed along with detailed proof of its characteristics. Then, based on the proposed operators and entropy measure, an MCDM method is established to handle the complex decision making problems. The presented method has a good improvement in terms of criteria weights, that is, it utilizes the proposed entropy measure to find the criteria weights and can address the completely unknown weight information problems accurately. A practical case is provided to elaborate on the implication of the suggested method for selecting the best investment company. Further, we examined the impact of the parameters τ and t in the decision procedure and reported the stability stage of sorting results. Finally, a comparative analysis is conducted with some existing approaches to highlight the feasibility and supremacy of the presented work.

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