Mechanics of Materials
Semester 4

Contents

Chapter 1

Mathematical Concepts

1.1 Tensors

Definition 1.1.1: Einstein Notation

Also known as summation notation, says that if we have a repeated index then we are summing over that index. For example

$$y = c_i \hat{\mathbf{e}}_i.$$

implies that

$$y = \sum_{i=1}^{3} c_i \hat{\mathbf{e}}_i = c_1 \hat{\mathbf{e}}_1 + c_2 \hat{\mathbf{e}}_2 + c_3 \hat{\mathbf{e}}_3.$$

same thing with

$$a_i \cdot b_i = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.$$

Definition 1.1.2

Kronecker delta is defined to be

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

and the permutation symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1),(1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}.$$

And they appear in

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$$
$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \varepsilon_{ijk} \hat{\mathbf{e}}_k$$

Definition 1.1.3: Tensors

In an m-dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components, and obeys certain *transformation rules*

Note:-

Typically $m = 3$ corresponding to the 3D space.

Example 1.1.1

- A rank 0 tensor is a scalar
$$A.$$
- A rank 1 tensor is a vector
$$A\hat{\mathbf{x}} = A_i x_i = A_1 x_1 + A_2 x_2 + A_3 x_3 = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$
- A rank 2 tensor is a matrix
$$A(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = A_{ij} x_i y_j = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Some notable tensors are:

1. Symmetric tensors
$$A_{ij} = A_{ji}.$$
2. Anti-symmetric tensors
$$A_{ij} = -A_{ji}.$$
3. General tensor. It can be represented using a symmetric and an anti symmetric tensor
$$A = A^S + A^A.$$
where
$$A^S = \frac{1}{2}(A + A^T)$$
$$A^A = \frac{1}{2}(A \cdot A^T)$$

The identity tensor is the tensor whose components $I_{ij} = \delta_{ij}$
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The scalar invariants of a tensor

1. $I_1 = \mathrm{tr}(A) = A_{ii} = A_{11} + A_{22} + A_{33}$
2. $I_2 = \frac{1}{2}\left[\mathrm{tr}(A)^2 - tr(A^2)\right] = \frac{1}{2}\left(A_{ii}A_{jj} - A_{ij}A_{ji}\right)$
3. $I_3 = \det(A) = \varepsilon_{iij} T_{i1} T_{j2} T_{k3}$

The characteristic polynomial of a tensor $\det(A - \lambda I)$ can be expressed as
$$\det(A - \lambda I) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3.$$

Definition 1.1.4: Tensor Product

We define the tensor product between 2 tensors X and Y of order 3 to be
$$(X \otimes Y)_{ij} = X_i Y_j.$$
and with a tensor of order 2 T
$$(T \otimes X)_{ij} = T_{ij} X_k.$$

Example 1.1.2 (Tensor Product)

$$\begin{bmatrix} 1 & \alpha \\ \alpha^* & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} & \alpha\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} \\ \alpha^*\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \beta & \alpha & \alpha\beta \\ \beta^* & 1 & \alpha\beta^* & \alpha \\ \alpha^* & \alpha^*\beta & 1 & \beta \\ \alpha^*\beta^* & \alpha^* & \beta^* & 1 \end{bmatrix}$$

https://www.math3ma.com/blog/the-tensor-product-demystified

Note:-

The order of a tensor product $X \otimes Y$ is the sum of the orders of X and Y.

Definition 1.1.5: Contraction

We define the tensor product between 2 tensors X and Y to be

$$X \cdot Y = X_i Y_j.$$

Note:-

From what I understand, a tensor product is the outer product and a contraction is an inner product

$$X \otimes Y = X \times Y^T$$
$$X \cdot Y = X^T \times Y$$

1.2 Tensor Calculus

Definition 1.2.1: Gradient operator

The gradient operator on a scalar tensor is defined to be

$$\nabla f = \frac{\partial f}{\partial x_i}\hat{\mathbf{e}}_i = \frac{\partial f}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial x_3}\hat{\mathbf{e}}_3.$$

in cylindrical coordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z.$$

Definition 1.2.2: Gradient of a vector

The gradient of a vector tensor is

$$\nabla\vec{\mathbf{a}} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}.$$

in cylindrical coordinates

$$\nabla\vec{\mathbf{a}} = \begin{bmatrix} \frac{\partial a_r}{\partial r} & \frac{1}{r}\left(\frac{\partial a_r}{\partial \theta} - a_\theta\right) & \frac{\partial a_r}{\partial z} \\ \frac{\partial a_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta} + a_r\right) & \frac{\partial a_\theta}{\partial z} \\ \frac{\partial a_z}{\partial r} & \frac{1}{r}\frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z} \end{bmatrix}.$$

Note:-

The order of a gradient tensor is 1 order higher than the tensor it operates on.

Definition 1.2.3: Divergence

The divergence is defined to be

$$\nabla \cdot \vec{\mathbf{a}} = \operatorname{tr}(\nabla\vec{\mathbf{a}}).$$

unlike the gradient, it reduces the order of the tensor.

Definition 1.2.4: Laplacian

The Laplacian is is the composition of a divergence and a gradient. It keeps the same order of the tensor

$$\Delta f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}.$$

Definition 1.2.5: Rotation

Rotation mostly applies to vector tensors and retains the same order as it

$$(\nabla \times \vec{\mathbf{a}})_i = \varepsilon_{ijk}\frac{\partial a_k}{\partial x_j}.$$

$$\nabla \times \vec{\mathbf{a}} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\hat{\mathbf{e}}_3.$$

in cylindrical coordinates

$$\nabla \times \vec{\mathbf{a}} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial a_r}{\partial r} - \frac{\partial a_z}{\partial x_r}\right)\hat{\mathbf{e}}_\theta + \left(\frac{\partial a_\theta}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta} + \frac{a_\theta}{r}\right)\hat{\mathbf{e}}_z.$$

Note:-

$$\begin{aligned} \nabla \times (\nabla\vec{\mathbf{a}}) &= 0 \\ \nabla \cdot (\nabla \times \vec{\mathbf{a}}) &= 0 \\ \nabla(ab) &= a(\nabla b) + b(\nabla a) \end{aligned}$$

Theorem 1.2.1 Ostrogradsky's theorem

Denote $\iiint_D \ldots \mathrm{d}V$ as a volume integral and $\iint_S \ldots \hat{\mathbf{n}}\mathrm{d}S$ as a surface integral.

$$\iiint_D \nabla f \mathrm{d}V = \iint_S f \hat{\mathbf{n}} \mathrm{d}S$$
$$\iiint_D \nabla \cdot \vec{\mathbf{U}} \mathrm{d}V = \iint_S \vec{\mathbf{U}} \hat{\mathbf{n}} \mathrm{d}S$$
$$\iiint_D \nabla \cdot T \mathrm{d}V = \iint_S T \hat{\mathbf{n}} \mathrm{d}S$$

Chapter 2

Deformation

We consider a body under some deformation, at time $t = 0$, a point P on that body can be described as

$$\vec{\mathbf{X}} = X_k \hat{\mathbf{e}}_k.$$

after some time t the object has deformed and the position of the point P is now $\vec{\mathbf{x}}$. The relation between it's initial position and it's new position is

$$\vec{\mathbf{x}} = \vec{\mathbf{\Phi}}(\vec{\mathbf{X}}, t).$$

where $\vec{\mathbf{\Phi}}$ is a bijective transformation($\forall \vec{\mathbf{\Phi}}, \exists \vec{\mathbf{\Phi}}^{-1}$). The vector $\vec{\mathbf{x}}$ is a function of the initial position and time. The displacement vector is

$$\vec{\mathbf{u}}\left(\vec{\mathbf{X}}, t\right) = \vec{\mathbf{x}} - \vec{\mathbf{X}}.$$

velocity vector

$$\vec{\mathbf{v}}\left(\vec{\mathbf{X}}, t\right) = \frac{\partial \vec{\mathbf{x}}}{\partial t}.$$

and acceleration vector

$$\vec{\mathbf{a}} = \frac{\partial \vec{\mathbf{v}}}{\partial t}.$$

We consider a point P on a body and 2 points on the same body Q_1 and Q_2 described with respect to the point P. The differentials of Q_1 and Q_2 are

$$\begin{aligned} \mathrm{d}\vec{\mathbf{X}}_1 &= \vec{\mathbf{X}}_{Q_1} - \vec{\mathbf{X}}_P \\ \mathrm{d}\vec{\mathbf{X}}_2 &= \vec{\mathbf{X}}_{Q_2} - \vec{\mathbf{X}}_P \end{aligned}$$

and after the deformation

$$\begin{aligned} \mathrm{d}\vec{\mathbf{x}}_1 &= \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P + \mathrm{d}\vec{\mathbf{X}}_1, t\right) - \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P, t\right) \\ \mathrm{d}\vec{\mathbf{x}}_2 &= \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P + \mathrm{d}\vec{\mathbf{X}}_2, t\right) - \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P, t\right) \end{aligned}$$

we define a differential tensor of the transformation

$$\mathbf{F}\left(\vec{\mathbf{X}}, t\right) = \frac{\partial \vec{\mathbf{\Phi}}}{\partial \vec{\mathbf{X}}}.$$

aka the Jacobian matrix

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}.$$

The differential can be written as

$$\mathrm{d}\vec{\mathbf{x}}_1 = \mathbf{F}\left(\vec{\mathbf{X}}_P, t\right) \cdot \mathrm{d}\vec{\mathbf{X}}_1$$
$$\mathrm{d}\vec{\mathbf{x}}_2 = \mathbf{F}\left(\vec{\mathbf{X}}_P, t\right) \cdot \mathrm{d}\vec{\mathbf{X}}_2$$

The Jacobian is also useful for a change of reference when integrating

$$\int_v a(\vec{\mathbf{x}})\mathrm{d}v = \int_V a\left(\vec{\mathbf{x}}\left(\vec{\mathbf{X}}, t\right)\right) \det(\mathbf{F})\mathrm{d}V.$$

The relation between vectors before and after deformation

$$\mathrm{d}\vec{\mathbf{x}}_1 \cdot \mathrm{d}\vec{\mathbf{x}}_2 = \mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}_2.$$

where $\mathbf{C}$ is the Cauchy–Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}.$$

The elongation after deformation in a given direction

$$\delta(\mathrm{d}\vec{\mathbf{X}}) = \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}\vec{\mathbf{X}}} - 1 = \frac{\mathrm{d}\sqrt{\mathrm{d}\vec{\mathbf{X}} \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}}}{\mathrm{d}\vec{\mathbf{X}}} - 1.$$

We define

$$\lambda = \frac{\mathrm{d}\sqrt{\mathrm{d}\vec{\mathbf{X}} \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}}}{\mathrm{d}\vec{\mathbf{X}}} = \delta + 1.$$

$$\delta \begin{cases} > 0 & \text{elongation in the direction of } \mathrm{d}\vec{\mathbf{x}} \\ < 0 & \text{contraction in the direction of } \mathrm{d}\vec{\mathbf{x}} \end{cases}.$$

Consider 2 orthogonal vectors, X_1 and X_2. The new angle formed $\alpha = \frac{\pi}{2} - \gamma$ is calculated using the formula

$$\sin(\gamma) = \frac{\mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}_2}{\sqrt{\mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}_1} \cdot \sqrt{\mathrm{d}\vec{\mathbf{X}}_2 \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}_2}}.$$

We define the Green-Lagrangian strain tensor to be

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}).$$

The diagonal elements of $\mathbf{E}$ represent scaling of basis vectors of the body, while non-diagonal elements represent the change in angle (rotation).

We define $\mathrm{d}L$ to the magnitude of $\mathrm{d}\vec{\mathbf{X}}$ and $\hat{\mathbf{N}}$ to be its direction vector.

$$\mathrm{d}\vec{\mathbf{X}} = \mathrm{d}L\hat{\mathbf{N}}.$$

similarly for $\mathrm{d}\vec{\mathbf{x}}$

$$\mathrm{d}\vec{\mathbf{x}} = \mathrm{d}l\hat{\mathbf{n}}.$$

it follows that

$$\frac{1}{2}\left(\frac{\mathrm{d}l^2 - \mathrm{d}L^2}{\mathrm{d}L^2}\right) = \hat{\mathbf{N}} \cdot \mathbf{E} \cdot \hat{\mathbf{N}}.$$

and the angle between the 2 transformed vectors becomes $\alpha = \frac{\pi}{2} - \gamma$

$$\frac{1}{2}\sin(\gamma)\frac{\mathrm{d}l_1}{\mathrm{d}L_1}\frac{\mathrm{d}l_2}{\mathrm{d}L_2} = \hat{\mathbf{N}}_1 \cdot \mathbf{C} \cdot \hat{\mathbf{N}}_2.$$

We can decompose the gradient tensor $\mathbf{F}$ in to 2 other tensors where $\mathbf{R}$ is an orthogonal matrix ($\mathbf{R}^T = \mathbf{R}^{-1}$) and $\mathbf{U}$ is a symmetric matrix

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}.$$

$$\mathbf{C} = \mathbf{U} \cdot \mathbf{U}.$$

and

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}).$$

in a small displacement

$$\mathbf{E} = \frac{1}{2}\left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}} + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T \cdot \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right).$$

We can ignore the quadratic terms to obtain the strain tensor for small displacement $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} \approx \frac{1}{2}\left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}} + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T\right).$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right).$$

Using the above definition we can explicitly define the matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix}.$$

We can also prove that γ between the 2 base vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is

$$\frac{\gamma}{2} = \varepsilon_{12}.$$