Differential Geometry

Contents

PART

I

Prerequisites

Section 1

Matrices

Theorem 1 To prove a system of vectors $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_n\}$ is free we prove:

$$\det \begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_1 \\ | & | & | & & | \end{bmatrix} \neq 0.$$

Theorem 2 A transition matrix $P_{B\to B'}$between 2 basis $B = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ and $B' = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ we start by solving the system

$$\begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{\mathbf{v}}_n \\ | \end{bmatrix}.$$

or in other words finding

$$\begin{cases} \vec{\mathbf{v}}_1 & = \alpha_1\vec{\mathbf{u}}_1 + \beta_1\vec{\mathbf{u}}_2 + \gamma_1\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_2 & = \alpha_2\vec{\mathbf{u}}_1 + \beta_2\vec{\mathbf{u}}_2 + \gamma_2\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_3 & = \alpha_3\vec{\mathbf{u}}_1 + \beta_3\vec{\mathbf{u}}_2 + \gamma_3\vec{\mathbf{u}}_3 \end{cases}.$$

Finally we say that

$$P_{B\to B'} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

transition matrices are always square and invertible ($\det P \neq 0$)

Remark To find the transition matrix in the inverse direction (from B' to B) we simply do

$$P_{B'\to B} = {P_{B\to B'}}^{-1}.$$

Section 2

Vectors

Definition 1 We define an operation called the scalar product (dot product)

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}}, \vec{\mathbf{v}} &\longmapsto \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = \sum_{n=1}^{n} v_i \cdot u_i. \end{aligned}$$

Definition 2 We define the usual norm on $\mathbb{R}$ to be

$$\begin{aligned} \| \cdot \| : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto \|\vec{\mathbf{u}}\| = \sqrt{\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle}. \end{aligned}$$

Theorem 3 The projection of a vector $\vec{\mathbf{u}}$ on to another vector $\vec{\mathbf{v}}$ is

$$\operatorname{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle}{\|\vec{\mathbf{v}}\|^2} \vec{\mathbf{v}}.$$

Subsection 2.1

GramSchmidt process

The aim of this process is to find a new basis $\Gamma = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n\}$ derived from a basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ such that it is orthonormal or in other words

$$\forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Gamma : \langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle = 0 \quad \text{and} \quad \|\hat{\mathbf{x}}\| = 1.$$

We find it as follows

$$\begin{aligned}
\vec{\mathbf{u}}_1 &= \vec{\mathbf{v}}_1 & \hat{\mathbf{e}}_1 &= \frac{\vec{\mathbf{u}}_1}{\|\vec{\mathbf{u}}_1\|} \\
\vec{\mathbf{u}}_2 &= \vec{\mathbf{v}}_2 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_2) & \hat{\mathbf{e}}_2 &= \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|} \\
\vec{\mathbf{u}}_3 &= \vec{\mathbf{v}}_3 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_3) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_3) & \hat{\mathbf{e}}_3 &= \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|} \\
&\vdots \\
\vec{\mathbf{u}}_n &= \vec{\mathbf{v}}_n - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_n) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_n) - \ldots - \operatorname{proj}_{\vec{\mathbf{u}}_{n-1}}(\vec{\mathbf{v}}_n) & \hat{\mathbf{e}}_n &= \frac{\vec{\mathbf{u}}_n}{\|\vec{\mathbf{u}}_n\|}
\end{aligned}$$

PART II

Conics and Quadrics

Section 3

Conics

We define a quadric form to be a mapping q

$$\begin{aligned} q : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto q(\vec{\mathbf{u}}) = \begin{bmatrix} \rule{1em}{0.4pt} & {}^T\vec{\mathbf{u}} & \rule{1em}{0.4pt} \end{bmatrix} A \begin{bmatrix} | \\ \vec{\mathbf{u}} \\ | \end{bmatrix}. \end{aligned}$$

Where the matrix A is a symmetric matrix.[1]
The conics under study are

[1] *symmetric matrices ($A = {}^T A$) is always diagonalizable*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{ellipse (circle if } a = b\text{)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \qquad \text{imaginary ellipse}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \qquad \text{hyperbola with asymptote } y = \pm\frac{b}{a}x$$

$$\left.\begin{aligned} y^2 &= \pm 2px \quad p > 0 \\ x^2 &= \pm 2py \quad p > 0 \end{aligned}\right\} \qquad \text{parabolas}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad \text{union of two straight lines}$$

$$\left.\begin{aligned} x &= \text{const} \\ y &= \text{const} \end{aligned}\right\} \qquad \text{straight lines}$$

SUBSECTION 3.1

Identification of the conics

Let the general equation of all conics be:

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

- $\boxed{\text{if } b = 0}$: then we simply group together the terms x^2 and x as well as y^2 and y followed by completing the square to get an equation of a conic.
- $\boxed{\text{if } b \neq 0}$: in this case we have to introduce a new system of reference which eliminates the existence of xy
  We do this by first defining a quadratic form $q(x, y) = ax^2 + 2bxy + cy^2$ using a matrix

  $$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  which we diagonalize in to an or tho normal age-basis which we project our equation in to in order to get rid of the xy term

Example Find the nature of the conic

$$\Gamma : 5x^2 - 4xy + 8y^2 + \frac{20}{\sqrt{5}}x - \frac{80}{\sqrt{5}}y + 4 = 0.$$

Let $q(x,y) = 5x^2 - 4xy + 8y^2 = \left(\begin{smallmatrix} x & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 & -2 \\ -2 & 8 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) =^T \vec{\mathbf{u}}A\vec{\mathbf{u}}$. We find that the matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$ with eigenvalues $\vec{\mathbf{u}}_1 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $\vec{\mathbf{u}}_2 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, the age vectors are already orthogonal so we just find $\vec{\mathbf{e}}_1 = \dfrac{1}{\sqrt{5}}\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $\vec{\mathbf{e}}_2 = \dfrac{1}{\sqrt{5}}\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, finally

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad D = \begin{pmatrix} 4 & \\ & 9 \end{pmatrix}.$$

We define $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ to be any vector with basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P\begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{aligned} x &= \frac{2}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(2\alpha - \beta) \\ y &= \frac{1}{\sqrt{5}}\alpha + \frac{2}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(\alpha + 2\beta) \end{aligned}$$

now we substitute x and y with α and β into Γ and we manipulate the expression until we get

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

$\therefore \Gamma$ is an ellipse.

Subsection 3.2

Tangent to a conic at point B

Theorem 4 The normal to vector to a conic Γ

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

at a point $B \in \Gamma$ is defined to be

$$\nabla f(B) = \begin{pmatrix} \left.\dfrac{\partial f}{\partial x}\right|_{(x_B, y_B)} \\ \left.\dfrac{\partial f}{\partial y}\right|_{(x_B, y_B)} \end{pmatrix}.$$

where $f(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$

The equation of a tangent to a conic at a point B is

$$a(x - x_B) + b(y - y_B) = 0.$$

where a and b are respectively the x and y components of the normal vector at B

SECTION 4

Quadrics

Definition 3 A quadric is any surface in 3D space with an equation of the form:

$$\underbrace{ax^2 + by^2 + cz^2 + 2dyz + 2exy + 2fxy}_{q(x,y,z)\text{:quadratic form of 3 variables}} + \underbrace{gx + hy + iz}_{\text{linear part}} + \underbrace{j}_{\text{constant}} = 0.$$

The quadrics under study are[2]

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 && \text{Ellipsoid}\\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 && \text{Hyperboliod of one sheet}\\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 && \text{Hyperboliod of 2 sheets}\\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 && \text{Asymptote cone}\\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2pz && \text{Hyperbolic paraboloid}\\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2 && \text{Elliptic cone}
\end{aligned}$$

[2] *if $a = b$ the surface is a surface of revolution of axis (Oz)*

If a one of variables is missing in the equation then the surface is said to be "(Conic name)-ic Cylinder". For example "Hyperbolic cylinder", "Circular cylinder", and "Elliptical cylinder"

PART III

Parametric Curves

A vector function/parametric curve is a function of the form

$$\begin{aligned}
\vec{\mathbf{F}} : \mathbb{R} &\longrightarrow \mathbb{R}^2\\
t &\longmapsto \vec{\mathbf{F}}(t) = (x(t), y(t)).
\end{aligned}$$

With a domain of definition $\mathbb{D}_{\vec{\mathbf{F}}} = \mathbb{D}_x \cap \mathbb{D}_y$

Remark The length of a curve when $t \in [a, b]$ is

$$\int_a^b \sqrt{y'^2(t) + x'^2(t)}\,\mathrm{d}t\,.$$

Section 5

Symmetry

Consider the domain of definition to be $\mathbb{R}$.
If a function is is even($f(-x) = f(x)$) or odd($f(-x) = -f(x)$) the domain of study $\mathbb{D}_S$ is only $[0, +\infty[$, and it is symmetric with respect to some axis.(refer to the table)

$y \backslash x$	Even	Odd
Even	None	y-axis
Odd	x-axis	Center O

Table 1. Axis of symmetry of $\vec{\mathbf{F}}(t)$ depending on the nature of x and y.

If a curve $x(t+T) = x(t)$ and $y(t+T) = y(t)$ then the curve is T-periodic.
Then the domain if study $\mathbb{D}_S = [0, T] \cap \mathbb{D}_{\vec{\mathbf{F}}}$ or $= \left[-\frac{T}{2}, \frac{T}{2}\right] \cap \mathbb{D}_{\vec{\mathbf{F}}}$.

Remark The tangent line of a curve at $t = t_0$ is

$$-y'(t_0)(x - x(t_0)) + x'(t_0)(y - y(t_0)) = 0.$$

and the normal is

$$x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0.$$

Section 6

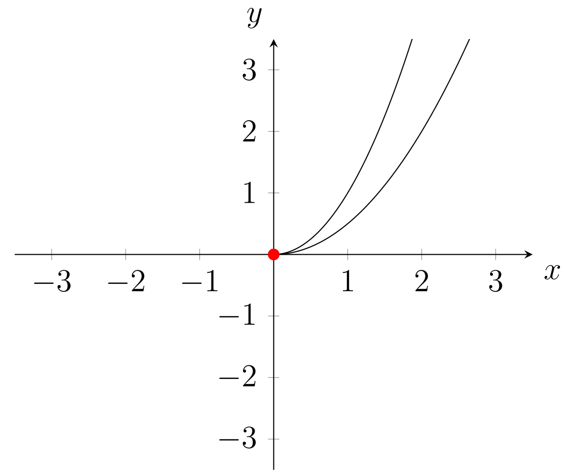
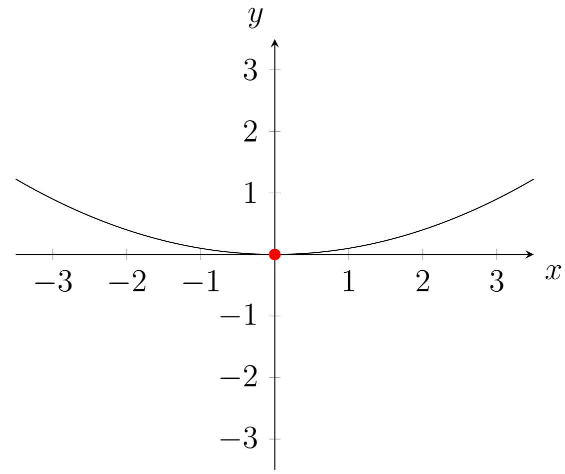
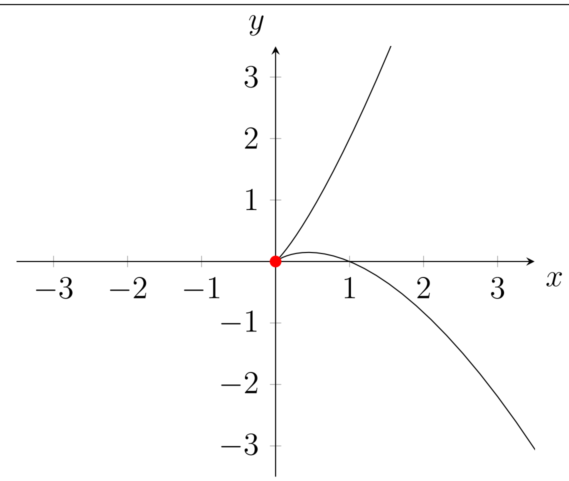
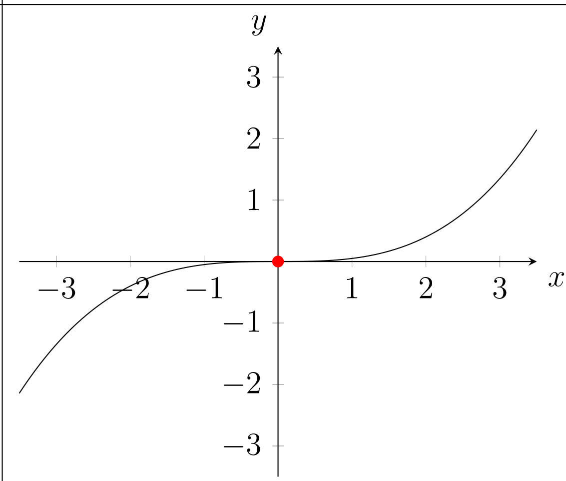
Particular Points

A point is said to be stationary if $\vec{\mathbf{F}}'(t) = 0$, regular if $\vec{\mathbf{F}}'(t) = 0$, and biregular if $\det\left(\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t)\right) \neq 0$.

The first non zero vector in the set $\{\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t), \vec{\mathbf{F}}'''(t), \ldots, \vec{\mathbf{F}}^{(k)}(t)\}$ is $\vec{\mathbf{F}}^{(p)}$ is used to define the tangent vector to the curve

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{F}}^{(p)}(t)}{\left\|\vec{\mathbf{F}}^{(p)}(t)\right\|}.$$

$$(T) : y = \frac{y^{(p)}(t)}{x^{(p)}(t)}(x - x(t)) + y(t).$$

q \ p	Even	Odd
Even	 Reflection point of the second kind	 Normal shape point
Odd	 Reflection point of the first kind	 Inflection point

Remark

- $\vec{\mathbf{F}}'(t_0) = 0 \implies t = t_0$ is a stationary point (reflection point of 1/2 kind).
- $\vec{\mathbf{F}}'(t_0) \neq 0 \implies t = t_0$ is an inflection point or normal shape point.
- $\det\left(\vec{\mathbf{F}}'(t_0), \vec{\mathbf{F}}''(t_0)\right) = 0 \implies t = t_0$ is a reflection or inflection point (not biregular).

Section 7

Infinite Branches

- If $\lim_{t \to t_0} x(t) = \pm\infty$ and $\lim_{t \to t_0} y(t) = y_0$ then the line $y = y_0$ is a horizontal asymptote.

- If $\lim_{t\to t_0} x(t) = x_0$ and $\lim_{t\to t_0} y(t) = \pm\infty$ then the line $x = x_0$ is a vertical asymptote.
- If $\lim_{t\to t_0} x(t) = \pm\infty$ and $\lim_{t\to t_0} y(t) = \pm\infty$ then we study $\dfrac{y(t)}{x(t)}$
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = \pm\infty$ then the curve admits a parabolic directed by (Oy).
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = 0$ then the curve admits a parabolic directed by (Ox).
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = a \in \mathbb{R}^*$ then we study $y(t) - ax(t)$
    * If $\lim_{t\to t_0} y(t) - ax(t) = b \in \mathbb{R}$ then the curve admits an oblique asymptote $y = ax + b$
    * If $\lim_{t\to t_0} y(t) - ax(t) = \pm\infty$ then the curve admits an asymptotic direction $y = ax$