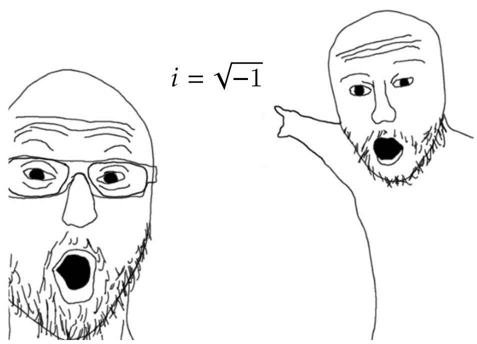
# Complex Analysis
Semester 4

# Contents

# Chapter 1

# The Complex Plane

## 1.1 Algebra of the complex plane




Euler's formulas for sin and cos

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

The $n$-th roots of unity are the set of complex numbers $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are the complex numbers that satisfy the equation

$$z^n = w.$$

where $w = Re^{i\alpha}$. The solutions equation are

$$\zeta_k = \sqrt[n]{R}e^{i\left(\frac{\alpha+2k\pi}{n}\right)}.$$

## 1.2 Topology of the complex plane

**Theorem 1.2.1**

The mapping

$$|z| : \mathbb{C} \longrightarrow \mathbb{R}^+$$
$$z = x + yi \longmapsto |x + yi| = \sqrt{x^2 + y^2}.$$

defines a norm on $\mathbb{C}$, so the complex plane is a normed space.

**Theorem 1.2.2**
The mapping

$$d(.,.) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}^+$$
$$(z, w) \longmapsto d(z, w) = |z - w|.$$

defined a distance on $\mathbb{C}$, so the complex plane is a metric space.

**Definition 1.2.1: Neighborhood**

We call $\delta$-neighborhood of $z_0$ an open disk centered at $z_0$ of radius $\delta$

$$N_\delta(z_0) = \{z \in \mathbb{C} : \ |z - z_0| < \delta\}.$$

We call $N_\delta(z_0) - \{z_0\}$ a deleted $\delta$-neighborhood. ($\{z \in \mathbb{C} : \ 0 < |z - z_0| < \delta\}$)

**Definition 1.2.2**

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

1. $z_0$ is called an *interior point* of $\Omega$ if
$$\exists \delta > 0, N_\delta(z_0) \subset \Omega.$$
2. $z_0$ is an *exterior point* of $\Omega$ if
$$\exists \delta > 0, N_\delta(z_0) \cap \Omega = \emptyset.$$
3. $z_0$ is a *boundary point* of $\Omega$ if
$$\forall \delta > 0, \ N_\delta(z_0) \cap \Omega \neq \emptyset \quad \text{and} \quad N_\delta(z_0) \cap \underbrace{C_{\mathbb{C}}^{\Omega}}_{\mathbb{C}-\Omega} \neq \emptyset.$$

**Definition 1.2.3**

The set of all:

1. interior points: $\dot{\Omega}$
2. boundary points: $\partial\Omega$
3. the set $\Omega \cup \partial\Omega$ is called a closure of $\Omega$ denoted $\bar{\Omega}$

**Note:-**

$$\dot{\Omega} \subset \Omega \subset \bar{\Omega}.$$

and

$$\Omega \text{ is open} \Leftrightarrow \begin{cases} \Omega \cap \partial\Omega = \emptyset \\ \Omega = \dot{\Omega} \end{cases} .$$

If $\Omega$ is not open from all sides then it is not open. Same thing with closed.
If $\Omega$ is not open then it is not connected.

$\Omega$ is said to be *compact* if it is both *bounded and closed*.

**Theorem 1.2.3** Bolzano-Weirstrass theorem

Every *bounded infinte* set admits at least one limit point

**Paths**

A path is a set of complex points $\Gamma$ where

$$\Gamma = \{z(t) = x(t) + i\, y(t)\ t \in [a, b[\} .$$

A simple path/Jordan arc if it does not cross itself

$$\forall t_1, t_2 \in [a, b[\ t_1 \neq t_2 \Longrightarrow z(t_1) \neq z(t_2).$$

A closed path is a path such that

$$z(a) = z(b).$$

# Chapter 2

# Complex Functions

## 2.1 Limits and Differentiability

**Note:-**

When taking limits we can do the 2D limit where $x = \text{Re}(z)$ and $y = \text{Im}(z)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x+iy).$$

then we can take multiple paths to find the limit. However we can't take suffecient paths to prove a limit exists as there could exist one path that causes the limit to not exist, however we can use polar limits to prove that the limit exists. We take $x = r\cos(\theta) - x_0$ and $y = r\sin(\theta) - y_0$

$$\lim_{(r,\theta)\to(0,0)} f\left(r\cos(\theta) - x_0 + i\left(r\sin(\theta) - y_0\right)\right).$$

**Theorem 2.1.1** Cauchy-Riemann equations

We define a complex function

$$f(x+iy) = u(x,y) + iv(x,y).$$

If $f$ is differentiable on a point $z_0 = x_0 + iy_0$ then $u$ and $v$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$$
$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

Note that the converse is not true

To prove that a function $f$ is differentiable at $z_0$ then we have to prove that $u$ and $v$

$$\begin{cases} \text{exist in } \Omega \\ \text{are continous at } (x_0,y_0) \\ \text{satisfy the Cauchy-Riemann equations at } (x_0,y_0) \end{cases}$$

### 2.1.1 Hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\tanh z = \frac{\sinh z}{\cosh z}$$

Properties

a) $\cosh^2 z - \sinh^2 z = 1$

b) $\cosh^2 z + \sinh^2 z = \cosh 2z$

c) $\cosh z_1 + z_2 = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

d) $\sinh z_1 + z_2 = \sinh z_1 \cdot \cosh z_2 + \sinh z_2 \cdot \cosh z_1$

e) $\cos iz = \cosh z$

f) $\sin iz = i \sinh z$

g) $\cosh iz = \cos z$

h) $\sinh iz = i \sin z$

## 2.2 Harmonic functions

**Definition 2.2.1: Harmonic function**

A function $u(x, y)$, of class $C^2$ and defined on $\Omega$, is said to be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or in other words the Laplacian is equal to 0

$$\Delta u = \nabla^2 u = 0.$$

**Theorem 2.2.1**

Let a function $f = u + iv$ defined on $\Omega$

$$f \text{ is holomorphic } \Leftrightarrow \begin{cases} u, v \text{ are of class } C^\infty \text{ in } \Omega \\ u, v \text{ satisfy the Cauchy–Riemann equations in } \Omega \\ u, v \text{ are harmonic in } \Omega \end{cases}.$$

# Chapter 3

# Integrals

**Definition 3.0.1: Complex Integral**

Let $\Omega$ be an open subset of $\mathbb{C}$ and $\Gamma$ a piecewise differentiable path from $z_1$ to $z_2$. We define the integral of $f$ along the path to be 2 different line integrals:

$$\int f(z)\,\mathrm{d}z = \int_\Gamma (u+iv)(\,\mathrm{d}x + i\,\mathrm{d}y) = \int_\Gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_\Gamma (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

**Theorem 3.0.1** Parametrization of the path

If the path $\Gamma$ is parametrized by $\gamma(t) = x(t) + iy(t)$ where $x, y$ are of class $c^1$ on $[a, b]$ then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,\mathrm{d}t.$$

**Theorem 3.0.2** $ML$-rule

In a path of $\Gamma$ of length $L$, we can approximate the value of an integral along that path

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leqslant M\cdot L.$$

where

$$M = \sup_{z\in\Gamma}|f(z)| \quad \text{and} \quad L = \text{Length of the path } \Gamma = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t.$$

**Theorem 3.0.3** Cauchy's theorem

Let $\Gamma$ be a simple closed curve. Let $f$ be a holomorphic function on $\Gamma$ and inside $\Gamma$, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

**Note:-**

Green-Riemann theorem states that

$$\oint_{\partial\Omega} (P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y) = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

**Note:-**

$$\int_{\Gamma^-} f(z)\,\mathrm{d}z = -\int_{\Gamma} f(z)\,\mathrm{d}z.$$

A consequence of Cauchy's theorem is that if a closed path C contains a discontinuity then the path of integration doesn't matter as long as the new path also contains the exact same discontinuity.

**Theorem 3.0.4**

Let $\Omega$ be a simply closed region. Let $f$ be a holomorphic function on $\Omega$, $z_1$ and $z_2$ be 2 point $\in \Omega$. Then the integral of $f(z)$ is independent of the path taken from $z_1$ to $z_2$

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

**Theorem 3.0.5** Liouville's theorem

- $f$ is holomorphic in $\mathbb{C}$
- $f$ is bounded in $\mathbb{C}$

$$\exists M \in \mathbb{R}_+, \forall z \in \mathbb{C}, |f(z) \leqslant M|.$$

then $f$ is constant in $\mathbb{C}$

**Theorem 3.0.6** Mean value theorem

Let $\gamma_r$ be a circle of center $a$ and radius $r > 0$. If $f$ is a holomorphic on and in $\gamma_r$ then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right)\mathrm{d}\theta.$$

**Theorem 3.0.7** Cauchy's integral formula

Let $\Gamma$ is a simple closed curve and the function $f(z)$ is holomorphic on $\Gamma$ and its interior. Then:

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

and the general form of the formula is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z.$$

## 3.1 Primitives

**Definition 3.1.1: Primitives**

Let $f$ be a complex function, defined in an open set $\Omega \subset \mathbb{C}$.
We call a primitive of $f$ on $\Omega$, any function $F$ such that $F$ is holomorphic in $\Omega$ and $\forall z \in \Omega\ F'(z) = f(z)$

$$F(z) = \int f(z)\,\mathrm{d}z.$$

**Note:-**

If $f$ admits a primitive on the open set $\Omega$ then $f$ is holomorphic in $\Omega$

Let the path $\gamma$ goes from $z_1$ to $z_2$ in $\Omega$ then

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_2).$$

**Note:-**

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \implies f \text{ is holomorphic in } \Omega.$$