

# ANALYSIS 3

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# Sequence of Functions

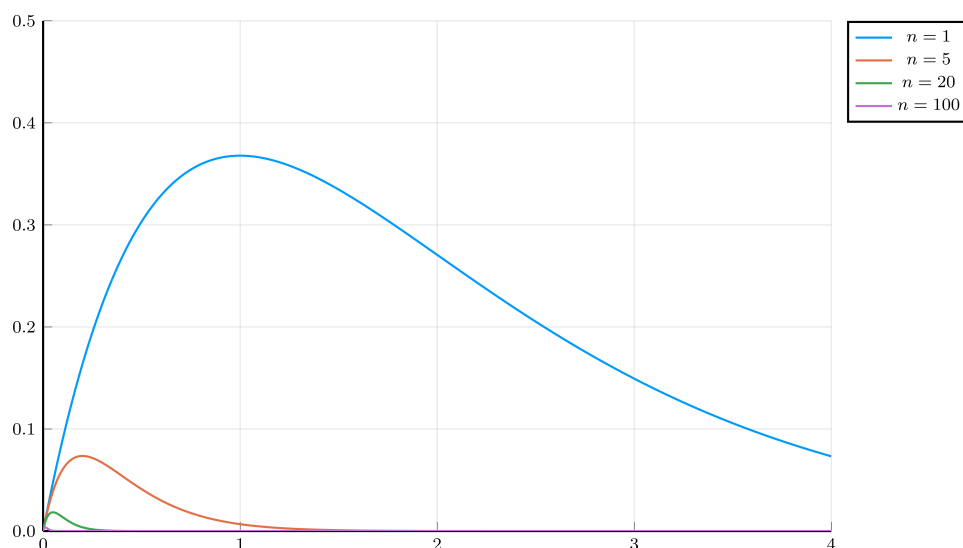
## SECTION 1

### Introduction

In previous courses, we analysed the convergence of sequences of numbers (example:  $U_n = \left\{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\right\} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ ) with a series of tests. In this course we will be analysing sequences of *functions*  $f_n(x)$ .

An example, is  $f_n(x) = \frac{x}{x+n} = \{f_1, f_2, f_3, \dots\} = \left\{\frac{x}{x+1}, \frac{x}{x+2}, \frac{x}{x+3}, \dots\right\}$ .

There are 2 ways these sequences can converge: pointwise and uniformly



**Figure 1.** Plot of the sequence  $f_n(x) = x e^{-nx}$

## SECTION 2

### Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix  $f_n$  to a point  $x$  then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function  $f$  and say that they converge to  $f$  pointwisely.

**Definition 1** We say that a sequence of functions  $f_n$  where  $f_n : I \rightarrow \mathbb{R}, I \subset \mathbb{R}$ ,

converges pointwise to function  $f : I \rightarrow \mathbb{R}$  on the interval  $I$  if:

$$\forall x \in I \forall \epsilon > 0 \exists n \in \mathbb{N} \forall n \geq N : |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let  $x = 0$  then find  $\lim_{n \rightarrow \infty} f_n(0) = \text{some } f(x)$
2. Then let  $x \neq 0$  and again find  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$
3. If neither of the results are unbounded  $\pm\infty$  then we say  $f_n(x)$  is convergent to some  $f(x)$

*Remark* if the result of step 1 is  $g(x)$  and step 2 results in  $h(x)$  where  $g(x) \neq h(x)$  then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in ]0, 1] \end{cases}.$$

## SECTION 3

# Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of  $n$  increases.

**Definition 2** We say that a sequence of functions  $f_n$  where  $f_n : I \rightarrow \mathbb{R}, I \subset \mathbb{R}$ , converges uniformly to function  $f : I \rightarrow \mathbb{R}$  on the interval  $I$  if:

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall x \in I : \sup_{x \in I} |f_n(x) - f(x)| < \epsilon.$$

*Remark* We can also prove uniform convergence by proving

$$\lim_{n \rightarrow \infty} \sup_{x \in I} |f_n(x) - f(x)| = 0.$$

There is also an easy way to prove uniform convergence of a function by

1. Prove that the sequence of functions  $f_n(x)$  is pointwise convergent to a function  $f(x)$  <sup>1</sup> if the function  $f(x)$  is continuous
2. Define a function  $g(x) = |f_n(x) - f(x)|$  and find the maxima of that function at a point  $x_0$  (usually by doing  $dg/dx = 0$ ) <sup>if the sequence doesn't uniformly converge</sup>
3. If  $\lim_{n \rightarrow \infty} g(x_0) = 0$  then the sequence converges uniformly to  $f(x)$

# Series of Functions

**Definition 3** Let  $f_n(x)$  be sequence of functions defined on  $I \subset \mathbb{R}$ , we define the series  $S(x)$  to be

$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

## SECTION 4

### Convergence of a Numerical Series

In order to prove a series of functions converge we have to prove that it converges for all fixed  $x$ .

**Theorem 1** Suppose there exists a sequence  $a_n$  such that  $\forall x, n \ |f_n| \leq a_n$ . The Weierstrass test states that if  $\sum a_n$  converges then  $\sum f_n(x)$  converges uniformly and absolutely

**Theorem 2** Let  $a_n$  be a sequence of numbers, if  $\left| \frac{a_{n+1}}{a_n} \right| = l$  then the sequence is a geometric Series

$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

**Theorem 3** A harmonic series is defined to be  $a_n = \frac{1}{n^p}$

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

**Theorem 4** Let  $a_n$  be a sequence of numbers. The 2 series  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} 2^n a_n$  are simultaneously convergent/divergent.

**Theorem 5** The sequence  $\sum_{n=0}^{\infty} (-1)^n a_n$  is convergent if  $a_n$  is decreasing and  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Theorem 6**

Consider the series  $S = \sum_{n=0}^{\infty} a_n$

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases}.$$

## SECTION 5

## Finite Expansion

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The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!} f'(x-a) + \frac{x^2}{2!} f''(x-a) + \cdots + \frac{x^n}{n!} f^{(n)}(x-a) + x^n o(1) \quad x \rightarrow a.$$

Some important expansions to keep in mind are

$$\begin{array}{l} e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \\ \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \end{array} \left| \begin{array}{l} \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \\ \ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n} \end{array} \right| \begin{array}{l} \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \\ \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \\ (1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n-1} (\alpha - k)}{n!} x^n \end{array}$$