Differential Geometry

Contents

PART

I

Prerequisites

Section 1

Matrices

Theorem 1 To prove a system of vectors $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_n\}$ is free we prove:

$$\det \begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_1 \\ | & | & | & & | \end{bmatrix} \neq 0.$$

Theorem 2 A transition matrix $P_{B \to B'}$between 2 basis $B = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ and $B' = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ we start by solving the system

$$\begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{\mathbf{v}}_n \\ | \end{bmatrix}.$$

or in other words finding

$$\begin{cases} \vec{\mathbf{v}}_1 & = \alpha_1 \vec{\mathbf{u}}_1 + \beta_1 \vec{\mathbf{u}}_2 + \gamma_1 \vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_2 & = \alpha_2 \vec{\mathbf{u}}_1 + \beta_2 \vec{\mathbf{u}}_2 + \gamma_2 \vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_3 & = \alpha_3 \vec{\mathbf{u}}_1 + \beta_3 \vec{\mathbf{u}}_2 + \gamma_3 \vec{\mathbf{u}}_3 \end{cases}.$$

Finally we say that

$$P_{B \to B'} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

transition matrices are always square and invertible ($\det P \neq 0$)

Remark To find the transition matrix in the inverse direction (from B' to B) we simply do

$$P_{B' \to B} = {P_{B \to B'}}^{-1}.$$

Section 2

Vectors

Definition 1 We define an operation called the scalar product (dot product)

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}}, \vec{\mathbf{v}} &\longmapsto \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = \sum_{n=1}^{n} v_i \cdot u_i. \end{aligned}$$

Definition 2 We define the usual norm on $\mathbb{R}$ to be

$$\begin{aligned} \| \cdot \| : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto \|\vec{\mathbf{u}}\| = \sqrt{\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle}. \end{aligned}$$

Theorem 3 The projection of a vector $\vec{\mathbf{u}}$ on to another vector $\vec{\mathbf{v}}$ is

$$\operatorname{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle}{\|\vec{\mathbf{v}}\|^2} \vec{\mathbf{v}}.$$

Subsection 2.1

GramSchmidt process

The aim of this process is to find a new basis $\Gamma = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n\}$ derived from a basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ such that it is orthonormal or in other words

$$\forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Gamma : \langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle = 0 \quad \text{and} \quad \|\hat{\mathbf{x}}\| = 1.$$

We find it as follows

$$\begin{aligned} \vec{\mathbf{u}}_1 &= \vec{\mathbf{v}}_1 & \hat{\mathbf{e}}_1 &= \frac{\vec{\mathbf{u}}_1}{\|\vec{\mathbf{u}}_1\|} \\ \vec{\mathbf{u}}_2 &= \vec{\mathbf{v}}_2 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_2) & \hat{\mathbf{e}}_2 &= \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|} \\ \vec{\mathbf{u}}_3 &= \vec{\mathbf{v}}_3 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_3) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_3) & \hat{\mathbf{e}}_3 &= \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|} \\ &\vdots \\ \vec{\mathbf{u}}_n &= \vec{\mathbf{v}}_n - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_n) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_n) - \ldots - \operatorname{proj}_{\vec{\mathbf{u}}_{n-1}}(\vec{\mathbf{v}}_n) & \hat{\mathbf{e}}_n &= \frac{\vec{\mathbf{u}}_n}{\|\vec{\mathbf{u}}_n\|} \end{aligned}$$

Conics and Quadrics

PART II

Section 3

Conics

We define a quadric form to be a mapping q

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\vec{\mathbf{u}} \longmapsto q(\vec{\mathbf{u}}) = \begin{bmatrix} \text{---} & {}^T\vec{\mathbf{u}} & \text{---} \end{bmatrix} A \begin{bmatrix} | \\ \vec{\mathbf{u}} \\ | \end{bmatrix}.$$

Where the matrix A is a symmetric matrix.[1]
The conics under study are

[1] *symmetric matrices ($A = {}^T A$) is always diagonalizable*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{ellipse (circle if } a = b\text{)}$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \qquad \text{imaginary ellipse}$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 \qquad \text{hyperbola with asymptote } y = \pm\frac{b}{a}x$$
$$\left.\begin{aligned} y^2 &= \pm 2px \quad p > 0 \\ x^2 &= \pm 2py \quad p > 0 \end{aligned}\right\} \qquad \text{parabolas}$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \qquad \text{union of two straight lines}$$
$$\left.\begin{aligned} x &= \text{const} \\ y &= \text{const} \end{aligned}\right\} \qquad \text{straight lines}$$

Subsection 3.1

Identification of the conics

Let the general equation of all conics be:

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

- $\boxed{\text{if } b = 0}$: then we simply group together the terms x^2 and x as well as y^2 and y followed by completing the square to get an equation of a conic.
- $\boxed{\text{if } b \neq 0}$: in this case we have to introduce a new system of reference which eliminates the existence of xy
  We do this by first defining a quadratic form $q(x, y) = ax^2 + 2bxy +$

cy^2 using a matrix

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

which we diagonalize in to an or tho normal age-basis which we project our equation in to in order to get rid of the xy term

Example Find the nature of the conic

$$\Gamma : 5x^2 - 4xy + 8y^2 + \frac{20}{\sqrt{5}}x - \frac{80}{\sqrt{5}}y + 4 = 0.$$

Let $q(x,y) = 5x^2 - 4xy + 8y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =^T \vec{\mathbf{u}} A \vec{\mathbf{u}}$. We find that the matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$ with eigenvalues $\vec{\mathbf{u}}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{u}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, the age vectors are already orthogonal so we just find $\vec{\mathbf{e}}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{e}}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, finally

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad D = \begin{pmatrix} 4 & \\ & 9 \end{pmatrix}.$$

We define $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to be any vector with basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$x = \frac{2}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(2\alpha - \beta)$$
$$y = \frac{1}{\sqrt{5}}\alpha + \frac{2}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(\alpha + 2\beta)$$

now we substitute x and y with α and β into Γ and we manipulate the expression until we get

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

$\therefore \Gamma$ is an ellipse.

Subsection 3.2

Tangent to a conic at point B

Theorem 4 The normal to vector to a conic Γ

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

at a point $B \in \Gamma$ is defined to be

$$\nabla f(B) = \begin{pmatrix} \left.\dfrac{\partial f}{\partial x}\right|_{(x_B, y_B)} \\ \left.\dfrac{\partial f}{\partial y}\right|_{(x_B, y_B)} \end{pmatrix}.$$

where $f(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$

The equation of a tangent to a conic at a point B is

$$a(x - x_B) + b(y - y_B) = 0.$$

where a and b are respectively the x and y components of the normal vector at B

Section 4

Quadrics

Definition 3 A quadric is any surface in 3D space with an equation of the form:

$$\underbrace{ax^2 + by^2 + cz^2 + 2dyz + 2exy + 2fxy}_{q(x,y,z)\text{:quadratic form of 3 variables}} + \underbrace{gx + hy + iz}_{\text{linear part}} + \underbrace{j}_{\text{constant}} = 0.$$

The quadrics under study are[2]

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	Ellipsoid
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	Hyperboliod of one sheet
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$	Hyperboliod of 2 sheets
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$	Asymptote cone
$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2pz$	Hyperbolic paraboloid
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = z^2$	Elliptic cone

If a one of variables is missing in the equation then the surface is said to be "(Conic name)-ic Cylinder". For example "Hyperbolic cylinder",

[2] *if $a = b$ the surface is a surface of revolution of axis (Oz)*

"Circular cylinder", and "Elliptical cylinder"

PART III

Parametric Curves

A vector function/parametric curve is a function of the form

$$\begin{aligned} \vec{\mathbf{F}} : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \vec{\mathbf{F}}(t) = (x(t), y(t)). \end{aligned}$$

With a domain of definition $\mathbb{D}_{\vec{\mathbf{F}}} = \mathbb{D}_x \cap \mathbb{D}_y$

Remark The length of a curve when $t \in [a, b]$ is

$$\int_a^b \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\, \mathrm{d}t\,.$$

Section 5

Symmetry

Consider the domain of definition to be $\mathbb{R}$.
If a function is is even($f(-x) = f(x)$) or odd($f(-x) = -f(x)$) the domain of study $\mathbb{D}_S$ is only $[0, +\infty[$, and it is symmetric with respect to some axis.(refer to the table)

y \ x	Even	Odd
Even	None	y-axis
Odd	x-axis	Center O

Table 1. Axis of symmetry of $\vec{\mathbf{F}}(t)$ depending on the nature of x and y.

If a curve $x(t+T) = x(t)$ and $y(t+T) = y(t)$ then the curve is T-periodic. Then the domain if study $\mathbb{D}_S = [0, T] \cap \mathbb{D}_{\vec{\mathbf{F}}}$ or $= \left[-\frac{T}{2}, \frac{T}{2}\right] \cap \mathbb{D}_{\vec{\mathbf{F}}}$.

Remark The tangent line of a curve at $t = t_0$ is

$$-y'(t_0)(x - x(t_0)) + x'(t_0)(y - y(t_0)) = 0.$$

and the normal is

$$x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0.$$

Section 6

Infinite Branches

- If $\lim\limits_{t\to t_0} x(t) = \pm\infty$ and $\lim\limits_{t\to t_0} y(t) = y_0$ then the line $y = y_0$ is a horizontal asymptote.
- If $\lim\limits_{t\to t_0} x(t) = x_0$ and $\lim\limits_{t\to t_0} y(t) = \pm\infty$ then the line $x = x_0$ is a vertical asymptote.

- If $\lim_{t\to t_0} x(t) = \pm\infty \quad$ and $\quad \lim_{t\to t_0} y(t) = \pm\infty$ then we study $\dfrac{y(t)}{x(t)}$
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = \pm\infty$ then the curve admits a parabolic directed by (Oy).
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = 0$ then the curve admits a parabolic directed by (Ox).
  - If $\lim_{t\to t_0} \dfrac{y(t)}{x(t)} = a \in \mathbb{R}^*$ then we study $y(t) - ax(t)$
    * If $\lim_{t\to t_0} y(t) - ax(t) = b \in \mathbb{R}$ then the curve admits an oblique asymptote $y = ax + b$
    * If $\lim_{t\to t_0} y(t) - ax(t) = \pm\infty$ then the curve admits an asymptotic direction $y = ax$

Section 7

Particular Points

A point is said to be stationary if $\vec{\mathbf{F}}'(t) = 0$, regular if $\vec{\mathbf{F}}'(t) = 0$, and biregular if $\det\left(\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t)\right) \neq 0$.

The first non zero vector in the set $\{\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t), \vec{\mathbf{F}}'''(t), \dots, \vec{\mathbf{F}}^{(k)}(t)\}$ is $\vec{\mathbf{F}}^{(p)}$ is used to define the tangent vector to the curve

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{F}}^{(p)}(t)}{\left\|\vec{\mathbf{F}}^{(p)}(t)\right\|}.$$

$$(T) : y = \frac{y^{(p)}(t)}{x^{(p)}(t)}(x - x(t)) + y(t).$$

Remark

- $\vec{\mathbf{F}}'(t_0) = 0 \implies t = t_0$ is a stationary point (reflection point of 1/2 kind).
- $\vec{\mathbf{F}}'(t_0) \neq 0 \implies t = t_0$ is an inflection point or normal shape point.
- $\det\left(\vec{\mathbf{F}}'(t_0), \vec{\mathbf{F}}''(t_0)\right) = 0 \implies t = t_0$ is a reflection or inflection point (not biregular).

<table>
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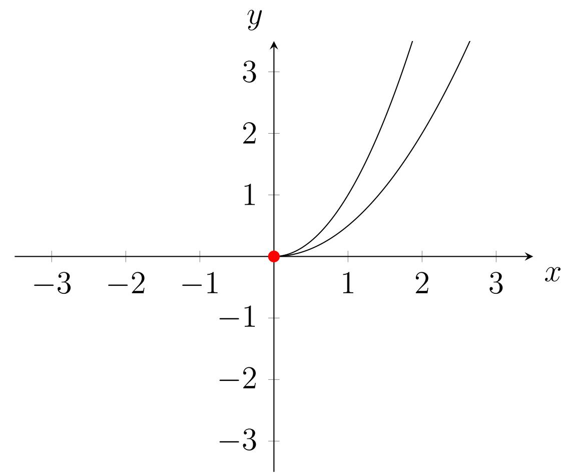
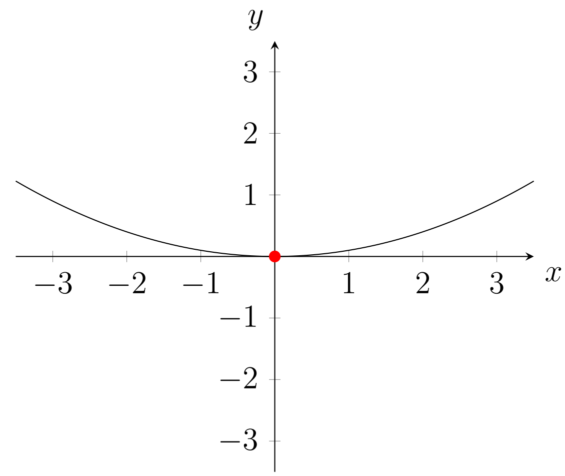
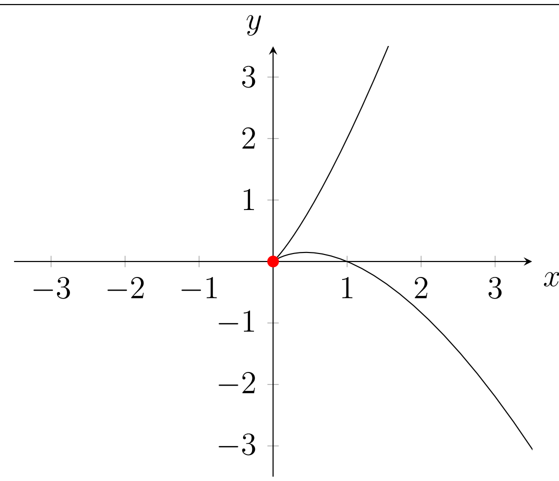
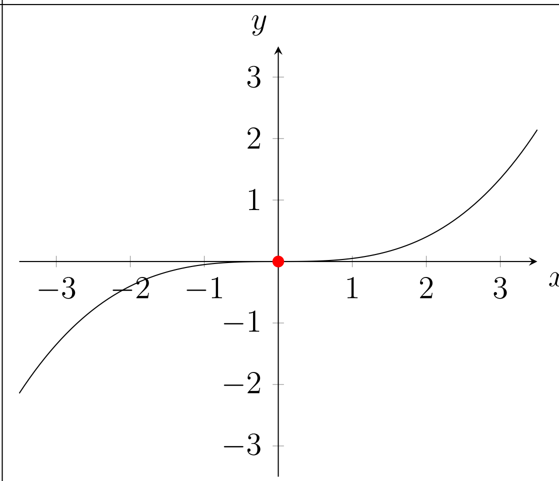
<tr><td>Even</td><td>

Reflection point of the second kind</td><td>

Normal shape point</td></tr>
<tr><td>Odd</td><td>

Reflection point of the first kind</td><td>

Inflection point</td></tr>
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Parametric Curves and Surfaces in 3D

PART IV

Section 8

3D Curves

A parametric is defined using a vector function

$$\vec{\mathbf{F}}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

The rules of limits, continuity, and differentiability are all hold if they also hold on the individual components $(x(t), y(t), z(t))$

$$\mathrm{d}\vec{\mathbf{F}}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\mathbf{i}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\mathbf{j}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\mathbf{k}}.$$

Remark If $\left\|\vec{\mathbf{F}}(t)\right\| = \text{cnst}$ then $\vec{\mathbf{F}}(t) \perp \dfrac{\mathrm{d}\vec{\mathbf{F}}(t)}{\mathrm{d}t}$ for all t

Subsection 8.1

Tangent and Normal Vectors

The tangent (T_0) to a curve at a point $t = t_0$ is directed by

$$\frac{\mathrm{d}\vec{\mathbf{F}}(t)}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \\ \mathrm{d}z/\mathrm{d}t \end{pmatrix}.$$

The normal plane of a curve is the plane who is perpendicular to the tangent plane. The normal vector of this plane is the directing vector the to the tangent plane.

Subsection 8.2

Osculating Plane

The osculating plane to a curve Γ is the plane (π) directed by the 2 vectors $\vec{\mathbf{F}}^{(p)}(t_0)$ and $\vec{\mathbf{F}}^{(q)}(t_0)$ where

$$\begin{cases} \vec{\mathbf{F}}^{(p)}(t) & \text{first non-zero derivative vector} \\ \vec{\mathbf{F}}^{(q)}(t) & \text{first non-zero derivative vector which isn't collinear to } \vec{\mathbf{F}}^{(p)}(t) \text{ where } q > p \end{cases}.$$

The plane has an equation at a point $M_0(x_0, y_0, z_0)$

$$\det\left(\overrightarrow{M_0\vec{\mathbf{F}}}(t), \vec{\mathbf{F}}^{(p)}(t), \vec{\mathbf{F}}^{(q)}(t)\right) = \begin{vmatrix} x - x_0 & a & d \\ y - y_0 & b & e \\ z - z_0 & c & f \end{vmatrix} = \alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0) = 0.$$

where

$$\vec{\mathbf{F}}^{(p)}(t) \times \vec{\mathbf{F}}^{(q)}(t) = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

Subsection 8.3

Infinite Branches

A curve Γ is said to have an infinite branch at t_0 if

$$\lim_{t \to t_0} \left\| \vec{\mathbf{F}}(t) \right\| = +\infty.$$

If

$$\lim_{t \to t_0} = \frac{\vec{\mathbf{F}}(t)}{\left\| \vec{\mathbf{F}}(t) \right\|} = \vec{\mathbf{n}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Then the curve admits a parabolic branch at t_0 directed by $\vec{\mathbf{n}}$

Now let (Δ_t) a straight line passing through a point on the curve $M(t)$ and directed by $\vec{\mathbf{n}}$. Let $m(t)$ be the point where (Δ_t) intersects with the (xy) plane

- if $\lim_{t \to t_0} \left\| \overrightarrow{Om}(t) \right\| = +\infty$ then the curve admits a parabolic branch directed by $\vec{\mathbf{n}}$.
- if $\lim_{t \to t_0} m(t) = A$ then the curve admits an asymptote passing through A and directed by $\vec{\mathbf{n}}$

Section 9

Parametric Surfaces

We call a vector function of 3 variables, a mapping of the form

$$\vec{\mathbf{F}}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}.$$

Partial derivatives of the vector function

$$\frac{\partial \vec{\mathbf{F}}}{\partial u} = \vec{\mathbf{F}}_u(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}.$$

$$\frac{\partial \vec{\mathbf{F}}}{\partial v} = \vec{\mathbf{F}}_v(u,v) = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}.$$

The tangent vector at a point is

$$\vec{\mathbf{T}} = \vec{\mathbf{F}}_u \cdot u'(t) + \vec{\mathbf{F}}_u \cdot v'(t).$$

The normal vector at a point is

$$\vec{\mathbf{n}} = \vec{\mathbf{F}}_u \times \vec{\mathbf{F}}_v.$$

The curve has a tangent plane at a point $M_0(x_0, y_0, z_0)$ given by

$$\det\left(\overrightarrow{M_0\vec{\mathbf{F}}}(u,v), \vec{\mathbf{F}}_u, \vec{\mathbf{F}}_v\right) = \begin{vmatrix} x - x_0 & a & d \\ y - y_0 & b & e \\ z - z_0 & c & f \end{vmatrix} = \alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0) = 0.$$

where

$$\vec{\mathbf{F}}_u \times \vec{\mathbf{F}}_v = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

PART

V

Polar Curves

A polar curve defined to be

$$\vec{\mathbf{F}}(\theta) = \begin{pmatrix} \rho(\theta)\cos(\theta) \\ \rho(\theta)\sin(\theta) \end{pmatrix}.$$

It can be defined by

$$\rho(\theta).$$

Section 10

Periodicity

$\rho(\theta + T) = \rho(\theta)$	ρ is T-periodic
$\rho(-\theta) = \rho(\theta)$	(Ox) is an axis of symmetry
$\rho(-\theta) = -\rho(\theta)$	(Oy) is an axis of symmetry
$\rho(\pi - \theta) = \rho(\theta)$	(Oy) is an axis of symmetry
$\rho(\pi - \theta) = -\rho(\theta)$	(Ox) is an axis of symmetry
$\rho(\pi + \theta) = \rho(\theta)$	O is center of symmetry
$\rho(\pi + \theta) = -\rho(\theta)$	ρ is π periodic
$\rho(2\pi + \theta) = \rho(\theta)$	ρ is 2π periodic
$\rho(\theta_0 - \theta) = \rho(\theta)$	then $\theta = \dfrac{\theta_0}{2}$ is an axis of symmetry
$\rho(\theta_0 - \theta) = -\rho(\theta)$	then $\theta = \dfrac{\theta_0}{2} + \dfrac{\pi}{2}$ is an axis of symmetry

Section 11

Study of Tangent Points

We define the angle ν at a point of θ_0

$$\tan(\nu) = \lim_{\theta \to \theta_0} \frac{\rho(\theta)}{\rho'(\theta)}.$$

The slope of the tangent at a point θ_0 to be

$$\tan(\varphi) = \tan(\theta_0 + \nu).$$

Section 12

Infinite Branches

- if $\rho(\theta)\sin(\theta-\theta_0) \underset{\theta\to\theta_0}{\longrightarrow} A$ then the line $y = A$ is an oblique asymptote relative to the orthonormal system $(O, \hat{\mathbf{u}}, \hat{\mathbf{v}})$ where $(\hat{\mathbf{i}}, \hat{\mathbf{u}}) = \theta_0$.
- if $\rho(\theta)\sin(\theta-\theta_0) \underset{\theta\to\theta_0}{\longrightarrow} \pm\infty$ then the curve admits a parabolic branch of direction $\theta = \theta_0$

Cartesian equation of an asymptote in the usual system

$$-\sin(\theta_0)x + \cos(\theta_0)y = A.$$

The equation in polar form

$$\rho = \frac{A}{\sin(\theta-\theta_0)}.$$

Subsection 12.1

When $\theta \to \pm\infty$

- if $\rho(\theta) \to 0$ then the curve admits O as a point asymptote (limit point).
- if $\rho(\theta) \to \pm\infty$ then the curve admits a spiral asymptote.
- if $\rho(\theta) \to R$ then the curve admits a circle asymptote of radius R.

Remark The arc length of a polar curve

$$L = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta = \int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + \left(\frac{\mathrm{d}\rho}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta\,.$$

The area under a polar curve is

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2}\rho^2\,\mathrm{d}\theta\,.$$

PART

VI

Envelopes and Evolutes

Section 13

Envelopes

We define $\mathcal{C}_\lambda$ be a family curve with associated equation

$$f(x, y, \lambda) = 0.$$

We can form a system of equations to find $x(t)$ and $y(t)$

$$\begin{cases} f(x, y, \lambda) = 0 \\ f_\lambda(x, y, \lambda) = 0 \end{cases}.$$

The normal to an envelope is

$$(x - x(t))x'(t) + (y - y(t))y'(t) = 0.$$

Section 14

Evolutes

Given a curve defined by

$$\vec{\mathbf{F}}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

the parametric form of the evolute of the curve is

$$\alpha(t) = x - y' \frac{x'^2 + y'^2}{\det \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}$$

$$\beta(t) = y + x' \frac{x'^2 + y'^2}{\det \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}}$$

such that the evolute is

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}.$$