ANALYSIS 3

Contents

Sequence of Functions

Introduction

In previous courses, we analysed the convergence of sequences of numbers (example: $U_n = \left\{ \dfrac{1}{4}, \dfrac{1}{9}, \dfrac{1}{16}, \ldots \right\} = \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2}$) with a series of tests. In this course we will be analysing sequences of *functions* $f_n(x)$.

An example, is $f_n(x) = \dfrac{x}{x+n} = \{f_1, f_2, f_3, \ldots\} = \left\{ \dfrac{x}{x+1}, \dfrac{x}{x+2}, \dfrac{x}{x+3}, \ldots \right\}$.

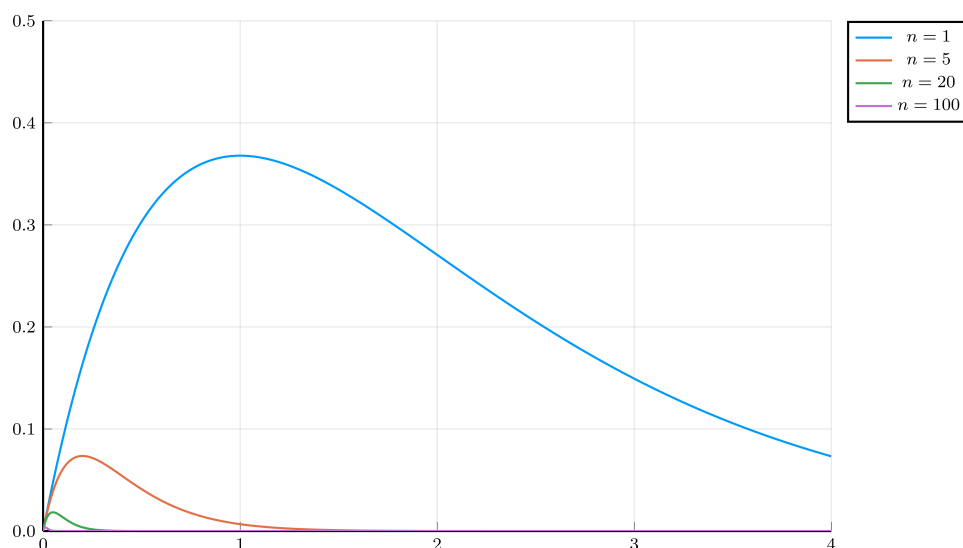There are 2 ways these sequences can converge: pointwise and uniformly

Figure 1. Plot of the sequence $f_n(x) = xe^{-nx}$

Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 1 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R},$

converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I \ \forall \epsilon > 0 \ \exists n \in \mathbb{N} \ \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:
1. Let $x = 0$ then find $\lim\limits_{n \to \infty} f_n(0) =$ some $f(x)$
2. Then let $x \neq 0$ and again find $\lim\limits_{n \to \infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0, 1] \end{cases}.$$

SECTION 3

Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 2 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \ \forall x \in I : \ \sup_{x \in I} |f_n(x) - f(x)| < \epsilon.$$

Remark We can also prove uniform convergence by proving

$$\lim_{n \to \infty} \sup_{x \in I} |f_n(x) - f(x)| = 0.$$

There is also an easy qay to prove uniform convergence of a function by
1. Prove that the sequence of fucntions $f_n(x)$ is pointwise convergent to a function $f(x)$ [1]
2. Define a function $g(x) = |f_n(x) - f(x)|$ and find the maxima of that function at a point x_0 (usually by doing $\mathrm{d}g/\mathrm{d}x = 0$)
3. If $\lim\limits_{n \to \infty} g(x_0) = 0$ then the sequence converges uniformly to $f(x)$

[1] *if the function $f(x)$ is continuous at a point piecewise then the sequence doesn't uniformly converge*

Series of Functions

Definition 3 | Let $f_n(x)$ be sequence of functions defined on $I \subset \mathbb{R}$, we define the series $S(x)$ to be

$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

SECTION 4

Convergence of a Numerical Series

In order to prove a series of functions converge we have to prove that it converges for all fixed x.

Theorem 1 | Suppose there exists a sequence a_n such that $\forall x, n \ |f_n| \leq a_n$. The Weierstrass test states that if $\sum a_n$ converges then $\sum f_n(x)$ converges uniformly and absolutely

Theorem 2 | Let a_n be a sequence of numbers, if $\left| \dfrac{a_{n+1}}{a_n} \right| = l$ then the sequence is a geometric Series

$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

Theorem 3 | A harmonic series is defined to be $a_n = \dfrac{1}{n^p}$

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

Theorem 4 | Let a_n be a sequence of numbers. The 2 series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_n$ are simultaneously convergent/divergent.

Theorem 5 | The sequence $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent if a_n is decreasing and $\lim_{n \to \infty} a_n = 0$.

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Theorem 6 Consider the series $S = \sum\limits_{n=0}^{\infty} a_n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases} .$$

SECTION 5
Finite Expansion

The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!}f'(x-a) + \frac{x^2}{2!}f''(x-a) + \cdots + \frac{x^n}{n!}f^{(n)}(x-a) + x^n o(1) \quad x \to a.$$

Some important expansions to keep in mind are

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n+1}(\alpha - k)}{n!} x^n$$

Power Series

A power series is just a series in the following formula

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$
$$= \sum_{n=0}^{\infty} U_n.$$

SECTION 6

Radius of Convergence

For some values of x a power series can either diverge or converge, to determine the interval of convergence we employ the ratio test

Theorem 7 Let r be the radius of convergence and I be the domain of convergence. If we compute the limit

$$\Gamma = \lim_{n \to \infty} \left| \frac{U_{n+1}}{U_n} \right|.$$

The ratio test states that

$$\begin{cases} \text{if} \quad \Gamma = 0 \quad \text{then} \quad r = \infty \quad \text{and} \quad I = \mathbb{R} \\ \text{if} \quad \Gamma = \infty \quad \text{then} \quad r = 0 \quad \text{and} \quad I = \{0\} \end{cases}.$$

In the case that Γ isn't 0 or ∞ we set $\Gamma < 1$ and then we find $|x| < R$, finally we can say that $r = R$ and $I =]-R, R[$. A special case need to be done for the points $-R$ and R to determine if they belong in I.