Mechanics of Materials
Semester 4

Contents

Chapter 1

Mathematical Concepts

1.1 Tensors

Definition 1.1.1: Einstein Notation

Also known as summation notation, says that if we have a repeated index then we are summing over that index. For example

$$y = c_i \hat{\mathbf{e}}_i.$$

implies that

$$y = \sum_{i=1}^{3} c_i \hat{\mathbf{e}}_i = c_1 \hat{\mathbf{e}}_1 + c_2 \hat{\mathbf{e}}_2 + c_3 \hat{\mathbf{e}}_3.$$

same thing with

$$a_i \cdot b_i = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3.$$

Definition 1.1.2

Kronecker delta is defined to be

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

and the permutation symbol

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i = j, \text{ or } j = k, \text{ or } k = i \end{cases}.$$

And they appear in

$$\begin{aligned} \hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j &= \delta_{ij} \\ \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j &= \varepsilon_{ijk} \hat{\mathbf{e}}_k \end{aligned}$$

Definition 1.1.3: Tensors

In an m-dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components, and obeys certain *transformation rules*

Note:-

Typically $m = 3$ corresponding to the 3D space.

Example 1.1.1

- A rank 0 tensor is a scalar
$$A.$$

- A rank 1 tensor is a vector
$$A\hat{\mathbf{x}} = A_i x_i = A_1 x_1 + A_2 x_2 + A_3 x_3 = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

- A rank 2 tensor is a matrix
$$A(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = A_{ij} x_i y_j = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

Some notable tensors are:

1. Symmetric tensors
$$A_{ij} = A_{ji}.$$
2. Anti-symmetric tensors
$$A_{ij} = -A_{ji}.$$
3. General tensor. It can be represented using a symmetric and an anti symmetric tensor
$$A = A^S + A^A.$$
where
$$A^S = \frac{1}{2}(A + A^T)$$
$$A^A = \frac{1}{2}(A \cdot A^T)$$

The identity tensor is the tensor whose components $I_{ij} = \delta_{ij}$
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The scalar invariants of a tensor

1. $I_1 = \mathrm{tr}(A) = A_{ii} = A_{11} + A_{22} + A_{33}$
2. $I_2 = \frac{1}{2}\left[\mathrm{tr}(A)^2 - tr(A^2)\right] = \frac{1}{2}\left(A_{ii}A_{jj} - A_{ij}A_{ji}\right)$
3. $I_3 = \det(A) = \varepsilon_{iij} T_{i1} T_{j2} T_{k3}$

The characteristic polynomial of a tensor $\det(A - \lambda I)$ can be expressed as
$$\det(A - \lambda I) = -\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3.$$

Definition 1.1.4: Tensor Product

We define the tensor product between 2 tensors X and Y of order 3 to be
$$(X \otimes Y)_{ij} = X_i Y_j.$$
and with a tensor of order 2 T
$$(T \otimes X)_{ij} = T_{ij} X_k.$$

Example 1.1.2 (Tensor Product)

$$\begin{bmatrix} 1 & \alpha \\ \alpha^* & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} & \alpha\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} \\ \alpha^*\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & \beta \\ \beta^* & 1 \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \beta & \alpha & \alpha\beta \\ \beta^* & 1 & \alpha\beta^* & \alpha \\ \alpha^* & \alpha^*\beta & 1 & \beta \\ \alpha^*\beta^* & \alpha^* & \beta^* & 1 \end{bmatrix}$$

https://www.math3ma.com/blog/the-tensor-product-demystified

Note:-

The order of a tensor product $X \otimes Y$ is the sum of the orders of X and Y.

Definition 1.1.5: Contraction

We define the tensor product between 2 tensors X and Y to be

$$X \cdot Y = X_i Y_j.$$

Note:-

From what I understand, a tensor product is the outer product and a contraction is an inner product

$$X \otimes Y = X \times Y^T$$
$$X \cdot Y = X^T \times Y$$

1.2 Tensor Calculus

Definition 1.2.1: Gradient operator

The gradient operator on a scalar tensor is defined to be

$$\nabla f = \frac{\partial f}{\partial x_i}\hat{\mathbf{e}}_i = \frac{\partial f}{\partial x_1}\hat{\mathbf{e}}_1 + \frac{\partial f}{\partial x_2}\hat{\mathbf{e}}_2 + \frac{\partial f}{\partial x_3}\hat{\mathbf{e}}_3.$$

in cylindrical coordinates

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z.$$

Definition 1.2.2: Gradient of a vector

The gradient of a vector tensor is

$$\nabla\vec{\mathbf{a}} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}.$$

in cylindrical coordinates

$$\nabla\vec{\mathbf{a}} = \begin{bmatrix} \frac{\partial a_r}{\partial r} & \frac{1}{r}\left(\frac{\partial a_r}{\partial \theta} - a_\theta\right) & \frac{\partial a_r}{\partial z} \\ \frac{\partial a_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta} + a_r\right) & \frac{\partial a_\theta}{\partial z} \\ \frac{\partial a_z}{\partial r} & \frac{1}{r}\frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z} \end{bmatrix}.$$

Note:-

The order of a gradient tensor is 1 order higher than the tensor it operates on.

Definition 1.2.3: Divergence

The divergence is defined to be

$$\nabla \cdot \vec{\mathbf{a}} = \mathrm{tr}(\nabla\vec{\mathbf{a}}).$$

unlike the gradient, it reduces the order of the tensor.

Definition 1.2.4: Laplacian

The Laplacian is is the composition of a divergence and a gradient. It keeps the same order of the tensor

$$\Delta f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}.$$

Definition 1.2.5: Rotation

Rotation mostly applies to vector tensors and retains the same order as it

$$(\nabla \times \vec{\mathbf{a}})_i = \varepsilon_{ijk}\frac{\partial a_k}{\partial x_j}.$$

$$\nabla \times \vec{\mathbf{a}} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\hat{\mathbf{e}}_1 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\hat{\mathbf{e}}_2 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\hat{\mathbf{e}}_3.$$

in cylindrical coordinates

$$\nabla \times \vec{\mathbf{a}} = \left(\frac{1}{r}\frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial a_r}{\partial r} - \frac{\partial a_z}{\partial x_r}\right)\hat{\mathbf{e}}_\theta + \left(\frac{\partial a_\theta}{\partial r} - \frac{1}{r}\frac{\partial a_r}{\partial \theta} + \frac{a_\theta}{r}\right)\hat{\mathbf{e}}_z.$$

Note:-

$$\begin{aligned} \nabla \times (\nabla\vec{\mathbf{a}}) &= 0 \\ \nabla \cdot (\nabla \times \vec{\mathbf{a}}) &= 0 \\ \nabla(ab) &= a(\nabla b) + b(\nabla a) \end{aligned}$$

Theorem 1.2.1 Ostrogradsky's theorem

Denote $\iiint_D \dots \mathrm{d}V$ as a volume integral and $\iint_S \dots \hat{\mathbf{n}}\mathrm{d}S$ as a surface integral.

$$\iiint_D \nabla f \mathrm{d}V = \iint_S f\hat{\mathbf{n}}\mathrm{d}S$$
$$\iiint_D \nabla \cdot \vec{\mathbf{U}}\mathrm{d}V = \iint_S \vec{\mathbf{U}}\hat{\mathbf{n}}\mathrm{d}S$$
$$\iiint_D \nabla \cdot T\mathrm{d}V = \iint_S T\hat{\mathbf{n}}\mathrm{d}S$$

Chapter 2

Deformation

We consider a body under some deformation, at time $t = 0$, a point P on that body can be described as

$$\vec{\mathbf{X}} = X_k \hat{\mathbf{e}}_k.$$

after some time t the object has deformed and the position of the point P is now $\vec{\mathbf{x}}$. The relation between it's initial position and it's new position is

$$\vec{\mathbf{x}} = \vec{\mathbf{\Phi}}(\vec{\mathbf{X}}, t).$$

where $\vec{\mathbf{\Phi}}$ is a bijective transformation($\forall \vec{\mathbf{\Phi}}, \exists \vec{\mathbf{\Phi}}^{-1}$). The vector $\vec{\mathbf{x}}$ is a function of the initial position and time. The displacement vector is

$$\vec{\mathbf{u}}\left(\vec{\mathbf{X}}, t\right) = \vec{\mathbf{x}} - \vec{\mathbf{X}}.$$

velocity vector

$$\vec{\mathbf{v}}\left(\vec{\mathbf{X}}, t\right) = \frac{\partial \vec{\mathbf{x}}}{\partial t}.$$

and acceleration vector

$$\vec{\mathbf{a}} = \frac{\partial \vec{\mathbf{v}}}{\partial t}.$$

We consider a point P on a body and 2 points on the same body Q_1 and Q_2 described with respect to the point P. The differentials of Q_1 and Q_2 are

$$\begin{aligned} \mathrm{d}\vec{\mathbf{X}}_1 &= \vec{\mathbf{X}}_{Q_1} - \vec{\mathbf{X}}_P \\ \mathrm{d}\vec{\mathbf{X}}_2 &= \vec{\mathbf{X}}_{Q_2} - \vec{\mathbf{X}}_P \end{aligned}$$

and after the deformation

$$\begin{aligned} \mathrm{d}\vec{\mathbf{x}}_1 &= \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P + \mathrm{d}\vec{\mathbf{X}}_1, t\right) - \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P, t\right) \\ \mathrm{d}\vec{\mathbf{x}}_2 &= \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P + \mathrm{d}\vec{\mathbf{X}}_2, t\right) - \vec{\mathbf{\Phi}}\left(\vec{\mathbf{X}}_P, t\right) \end{aligned}$$

we define a differential tensor of the transformation

$$\mathbf{F}\left(\vec{\mathbf{X}}, t\right) = \frac{\partial \vec{\mathbf{\Phi}}}{\partial \vec{\mathbf{X}}}.$$

aka the Jacobian matrix

$$\mathbf{F} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix}.$$

The differential can be written as

$$\mathrm{d}\vec{\mathbf{x}}_1 = \mathbf{F}\left(\vec{\mathbf{X}}_P, t\right)\cdot \mathrm{d}\vec{\mathbf{X}}_1$$
$$\mathrm{d}\vec{\mathbf{x}}_2 = \mathbf{F}\left(\vec{\mathbf{X}}_P, t\right)\cdot \mathrm{d}\vec{\mathbf{X}}_2$$

The Jacobian is also useful for a change of reference when integrating

$$\int_v a(\vec{\mathbf{x}})\mathrm{d}v = \int_V a\left(\vec{\mathbf{x}}\left(\vec{\mathbf{X}}, t\right)\right)\det(\mathbf{F})\mathrm{d}V.$$

The relation between vectors before and after deformation

$$\mathrm{d}\vec{\mathbf{x}}_1 \cdot \mathrm{d}\vec{\mathbf{x}}_2 = \mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C} \cdot \mathrm{d}\vec{\mathbf{X}}_2.$$

where $\mathbf{C}$ is the Cauchy–Green deformation tensor

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}.$$

The elongation after deformation in a given direction

$$\delta(\mathrm{d}\vec{\mathbf{X}}) = \frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}\vec{\mathbf{X}}} - 1 = \frac{\mathrm{d}\sqrt{\mathrm{d}\vec{\mathbf{X}}\cdot \mathbf{C}\cdot \mathrm{d}\vec{\mathbf{X}}}}{\mathrm{d}\vec{\mathbf{X}}} - 1.$$

We define

$$\lambda = \frac{\mathrm{d}\sqrt{\mathrm{d}\vec{\mathbf{X}}\cdot \mathbf{C}\cdot \mathrm{d}\vec{\mathbf{X}}}}{\mathrm{d}\vec{\mathbf{X}}} = \delta + 1.$$

$$\delta\begin{cases} > 0 & \text{elongation in the direction of } \mathrm{d}\vec{\mathbf{x}} \\ < 0 & \text{contraction in the direction of } \mathrm{d}\vec{\mathbf{x}} \end{cases}.$$

Consider 2 orthogonal vectors, X_1 and X_2. The new angle formed $\alpha = \frac{\pi}{2} - \gamma$ is calculated using the formula

$$\sin(\gamma) = \frac{\mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C}\cdot \mathrm{d}\vec{\mathbf{X}}_2}{\sqrt{\mathrm{d}\vec{\mathbf{X}}_1 \cdot \mathbf{C}\cdot \mathrm{d}\vec{\mathbf{X}}_1}\cdot\sqrt{\mathrm{d}\vec{\mathbf{X}}_2 \cdot \mathbf{C}\cdot \mathrm{d}\vec{\mathbf{X}}_2}}.$$

We define the Green-Lagrangian strain tensor to be

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}).$$

The diagonal elements of $\mathbf{E}$ represent scaling of basis vectors of the body, while non-diagonal elements represent the change in angle (rotation).

We define $\mathrm{d}L$ to the magnitude of $\mathrm{d}\vec{\mathbf{X}}$ and $\hat{\mathbf{N}}$ to be its direction vector.

$$\mathrm{d}\vec{\mathbf{X}} = \mathrm{d}L\hat{\mathbf{N}}.$$

similarly for $\mathrm{d}\vec{\mathbf{x}}$

$$\mathrm{d}\vec{\mathbf{x}} = \mathrm{d}l\hat{\mathbf{n}}.$$

it follows that

$$\frac{1}{2}\left(\frac{\mathrm{d}l^2 - \mathrm{d}L^2}{\mathrm{d}L^2}\right) = \hat{\mathbf{N}}\cdot\mathbf{E}\cdot\hat{\mathbf{N}}.$$

and the angle between the 2 transformed vectors becomes $\alpha = \frac{\pi}{2} - \gamma$

$$\frac{1}{2}\sin(\gamma)\frac{\mathrm{d}l_1}{\mathrm{d}L_1}\frac{\mathrm{d}l_2}{\mathrm{d}L_2} = \hat{\mathbf{N}}_1\cdot\mathbf{C}\cdot\hat{\mathbf{N}}_2.$$

We can decompose the gradient tensor $\mathbf{F}$ in to 2 other tensors where $\mathbf{R}$ is an orthogonal matrix ($\mathbf{R}^T = \mathbf{R}^{-1}$) and $\mathbf{U}$ is a symmetric matrix

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U}.$$

$$\mathbf{C} = \mathbf{U} \cdot \mathbf{U}.$$

and

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}).$$

in a small displacement

$$\mathbf{E} = \frac{1}{2}\left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}} + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T \cdot \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right).$$

We can ignore the quadratic terms to obtain the strain tensor for small displacement $\boldsymbol{\varepsilon}$

$$\boldsymbol{\varepsilon} \approx \frac{1}{2}\left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}} + \left(\frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}\right)^T\right).$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right).$$

Using the above definition we can explicitly define the matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix}.$$

We can also prove that γ between the 2 base vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ is

$$\frac{\gamma}{2} = \varepsilon_{12}.$$

Plane strain in a displacement plane $\vec{\mathbf{u}} = (u, v)$ of a body that has a unit thickness $\mathrm{d}x$ and $\mathrm{d}y$. The Jacobian becomes

$$\mathbf{F} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\mathrm{d}x' = \mathbf{F}\begin{bmatrix} \mathrm{d}x \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x}\mathrm{d}x + \mathrm{d}x \\ \frac{\partial v}{\partial x}\mathrm{d}x \end{bmatrix}$$

$$\mathrm{d}y' = \mathbf{F}\begin{bmatrix} 0 \\ \mathrm{d}y \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial y}\mathrm{d}y \\ \frac{\partial v}{\partial y}\mathrm{d}y + \mathrm{d}y \end{bmatrix}$$

The strain tensor becomes

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} \end{bmatrix}.$$

where

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

We can scale it up to 3D ($\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$ and $\vec{\mathbf{u}} = (u, v, w)$)

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_{yy} & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_{zz} \end{bmatrix}.$$

Consider a small displacement, to study the change in volume we need to look at the Jacobian. To do that we define the tensor

$$\mathbf{H} = \frac{\partial \vec{\mathbf{u}}}{\partial \vec{\mathbf{X}}}.$$

such that

$$\mathbf{F} = \mathbf{H} + \mathbf{I}.$$

The Jacobian becomes

$$J = \det \mathbf{F} = 1 + \operatorname{tr} \mathbf{H} = 1 + \operatorname{tr} \boldsymbol{\varepsilon}.$$

For a tensor to be considered a stress tensor it has to satisfy the 6 compatibility equations

$$\frac{\partial^2 \varepsilon_{11}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial X_1^2} = 2\frac{\partial^2 \varepsilon_{12}}{\partial X_1 \partial X_2} \qquad \frac{\partial^2 \varepsilon_{11}}{\partial X_2 \partial X_3} + \frac{\partial}{\partial X_1}\left(\frac{\partial \varepsilon_{23}}{\partial X_1} - \frac{\partial \varepsilon_{31}}{\partial X_2} - \frac{\partial \varepsilon_{12}}{\partial X_3}\right) = 0$$
$$\frac{\partial^2 \varepsilon_{22}}{\partial X_2^2} + \frac{\partial^2 \varepsilon_{33}}{\partial X_2^2} = 2\frac{\partial^2 \varepsilon_{23}}{\partial X_2 \partial X_3} \qquad \frac{\partial^2 \varepsilon_{22}}{\partial X_3 \partial X_1} + \frac{\partial}{\partial X_2}\left(\frac{\partial \varepsilon_{31}}{\partial X_2} - \frac{\partial \varepsilon_{12}}{\partial X_3} - \frac{\partial \varepsilon_{23}}{\partial X_1}\right) = 0$$
$$\frac{\partial^2 \varepsilon_{33}}{\partial X_1^2} + \frac{\partial^2 \varepsilon_{11}}{\partial X_3^2} = 2\frac{\partial^2 \varepsilon_{31}}{\partial X_3 \partial X_1} \qquad \frac{\partial^2 \varepsilon_{33}}{\partial X_1 \partial X_2} + \frac{\partial}{\partial X_3}\left(\frac{\partial \varepsilon_{12}}{\partial X_3} - \frac{\partial \varepsilon_{23}}{\partial X_1} - \frac{\partial \varepsilon_{31}}{\partial X_2}\right) = 0$$

Chapter 3

Stress

External forces can be classified as body forces and surface forces. Body forces are forces that act on the entire body, such as gravity. Surface forces are forces that act on the surface of the body, such as pressure.

A body is at equilibrium if the external forces resisted by internal forces.

3.1 Stress Vectors

$$\vec{\mathbf{T}} = \lim_{\mathrm{d}s \to 0} \frac{\mathrm{d}\vec{\mathbf{F}}}{\mathrm{d}s}.$$

The stress vector is a vector that acts on a surface. It is defined as the force per unit area acting on a surface $\mathrm{d}s$. It can be split in to 2 vectors

$$\vec{\mathbf{T}} = \vec{\mathbf{T}}_n + \vec{\mathbf{T}}_t.$$

where $\vec{\mathbf{T}}_n$ is the normal stress vector and $\vec{\mathbf{T}}_t$ is the shear stress vector.

To know the stress vector of a plane we can look at T in 3 perpendicular directions. The stress vector is then

$$\sigma_{ij} = \vec{\mathbf{T}}_i \cdot \hat{\mathbf{e}}_j.$$

where the index i is the direction of the normal vector and j is the direction of the stress vector.

$$\vec{\mathbf{T}} = \begin{pmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \end{pmatrix}.$$

To find the stress at every plane we can use the stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}.$$

Note:-

The stress tensor is symmetric.

The stress at a plane P with normal vector $\hat{\mathbf{n}}$ is

$$\vec{\mathbf{T}} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}.$$

$$\vec{\mathbf{T}}_n = \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}$$
$$\vec{\mathbf{T}}_t = \vec{\mathbf{T}} - \vec{\mathbf{T}}_n$$

3.2 Priciple Stresses

The principle stresses are the maximum and minimum stresses at a point. They are the eigenvalues of the stress tensor. The eigenvectors are the directions of the principle stresses.

$$\boldsymbol{\sigma}_p = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

3.3 Plane stress

Plane stress is when the stress in the z direction is 0. This means that $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $\vec{\mathbf{T}}$ be the stress vector acting on a plane with normal vector $\hat{\mathbf{n}}$, such that $\hat{\mathbf{n}}$ is making an angle θ with the horizontal.

$$\vec{\mathbf{T}} = \begin{pmatrix} T_1 \\ T_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sigma_{11}\cos\theta + \sigma_{12}\sin\theta \\ \sigma_{12}\cos\theta + \sigma_{22}\sin\theta \\ 0 \end{pmatrix}.$$

$$\begin{aligned} \sigma_n &= \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta \\ \tau_{nt} &= -\frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta \end{aligned}$$

3.3.1 Mohr's Circle in Plane Stress

Mohr's circle is a graphical method of finding the principle stresses. It's a circle with centr $(\sigma_{11} + \sigma_{22})/2$ and radius $\sqrt{\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$. The sign convention in Mohr circle space is that τ_{nt} is in the negative y direction and σ_n is in the positive x direction, and an angle of θ in real space is an angle of 2θ in Mohr circle space.
The principle stresses are the x intercepts of the circle.

$$\sigma_{1,2} = \frac{\sigma_{11} + \sigma_{22}}{2} \pm \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{xy}^2}.$$

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \tau_{xy}^2}.$$

The principle angles $\theta_{1,2}$ are

$$\tan 2\theta_{1,2} = \frac{2\tau_{xy}}{\sigma_{11} - \sigma_{22}}.$$

3.4 Mohr's Circle in 3D

Mohr's circle in case of 3D stress consists of 3 circles. We calculate the 3 principle stresses $\sigma_{1,2,3}$ and the 3 principle directions of those stresses $n_{1,2,3}$.

$$n_1{}^2 = \frac{\tau_n{}^2 + (\sigma_n - \sigma_2)(\sigma_n - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)}$$
$$n_2{}^2 = \frac{\tau_n{}^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_3)}{(\sigma_2 - \sigma_1)(\sigma_2 - \sigma_3)}$$
$$n_3{}^2 = \frac{\tau_n{}^2 + (\sigma_n - \sigma_1)(\sigma_n - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)}$$

Thus we obtain the 3 equations of the 3 circles

$$C_1 = \left(\frac{\sigma_1 + \sigma_2}{2}, 0\right) \qquad R_1 = \frac{\sigma_1 - \sigma_2}{2}$$
$$C_2 = \left(\frac{\sigma_2 + \sigma_3}{2}, 0\right) \qquad R_2 = \frac{\sigma_2 - \sigma_3}{2}$$
$$C_3 = \left(\frac{\sigma_3 + \sigma_1}{2}, 0\right) \qquad R_3 = \frac{\sigma_1 - \sigma_3}{2}$$

3.5 Equation of Equilibrium

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0.$$

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{f} = 0.$$

Chapter 4

Elasticity

4.1 Tensile Test

Nominal stress

$$\sigma = \frac{F}{S_0}.$$

Nominal strain

$$\varepsilon = \frac{\Delta L}{L_0}.$$

Hooke's Law (in case $\sigma < R_e$ where R_e is the yield strength)

$$\sigma = E\varepsilon.$$

Where E is the Young's Modulus.

We define the Poisson's ratio in case of an isotropic material as

$$\vartheta_x = -\frac{\varepsilon_x}{\varepsilon_z}$$
$$\vartheta_y = -\frac{\varepsilon_y}{\varepsilon_z}$$

$$\sigma_z = E\varepsilon_z.$$

$$\vartheta_x = \vartheta_y = \vartheta.$$

4.2 Pure Shear Test

$$\tau = \frac{F}{A}.$$

$$2\varepsilon_{xy} = \tau_{xy} = \tan\theta.$$

If the material is still elastic, the

$$\tau_{xy} = G\varepsilon_{xy}.$$

Where $G = \mu$ is the shear modulus.

$$G = \mu = \frac{E}{2(1+\vartheta)}.$$

4.3 Generalized Hooke's Law

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \vartheta(\sigma_2 + \sigma_3)\right)$$
$$\varepsilon_2 = \frac{1}{E}\left(\sigma_2 - \vartheta(\sigma_1 + \sigma_3)\right)$$
$$\varepsilon_3 = \frac{1}{E}\left(\sigma_3 - \vartheta(\sigma_1 + \sigma_2)\right)$$

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$
$$\varepsilon_{ij} = \frac{1}{E}\left((1+\vartheta)\sigma_{ij} - \vartheta\sigma_{kk}\delta_{ij}\right)$$

where λ is the Lame's constant.

$$\lambda = \frac{E\vartheta}{(1+\vartheta)(1-2\vartheta)}.$$

4.4 Deformation Approach

1. Postulate the strain field.
2. Verify the boundary conditions.
3. Verify the Lamé-Navier equations.
$$(\lambda + 2\mu)\nabla(\nabla \cdot \mathbf{u}) + +\nabla \cdot \vec{\mathbf{f}} = 0.$$
4. Find the strain components first and then the stress components.
5. Check boundary conditions in stress.

4.5 Stress Approach

1. Postulate the stress filed.
2. Verify the conditions of equilibrium.
3. Verify the boundary conditions in terms of stress.
4. Verify the equations of Beltrami-Michell.

$$\nabla^2\sigma_i j + \frac{1}{1+\vartheta}\frac{\partial^2\sigma_{kk}}{\partial x_i \partial x_j} + \frac{\vartheta}{1+\vartheta}\frac{\mathrm{d}f_l}{\mathrm{d}x_l} + \frac{\partial f_j}{\partial x_i} + \frac{\partial f_i}{\partial x_j} = 0.$$

5. Verify the boundary conditions in displacement.

4.6 Stress functions

We call Φ a stress function if it verfies

$$\frac{\partial^4\Phi}{\partial x_1^4} + 2\frac{\partial^4\Phi}{\partial x_1{}^2\partial x_2{}^2} + \frac{\partial^4\Phi}{\partial x_2^4} = 0.$$

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2}$$
$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2}$$
$$\sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2}$$