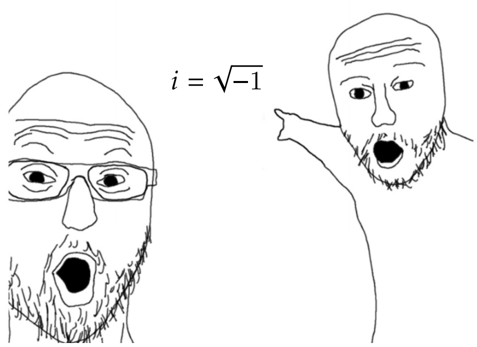
Complex Analysis
Semester 4

Contents

Chapter 1

The Complex Plane

1.1 Algebra of the complex plane



Euler's formulas for sin and cos

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

The n-th roots of unity are the set of complex numbers $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are the complex numbers that satisfy the equation

$$z^n = w.$$

where $w = Re^{i\alpha}$. The solutions equation are

$$\zeta_k = \sqrt[n]{R}e^{i\left(\frac{\alpha+2k\pi}{n}\right)}.$$

1.2 Topology of the complex plane

Theorem 1.2.1

The mapping

$$\begin{aligned} |z| : \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ z = x + yi &\longmapsto |x + yi| = \sqrt{x^2 + y^2}. \end{aligned}$$

defines a norm on $\mathbb{C}$, so the complex plane is a normed space.

Theorem 1.2.2

The mapping

$$\begin{aligned} d(.,.) : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ (z, w) &\longmapsto d(z, w) = |z - w|. \end{aligned}$$

defined a distance on $\mathbb{C}$, so the complex plane is a metric space.

Definition 1.2.1: Neighborhood

We call δ-neighborhood of z_0 an open disk centered at z_0 of radius δ

$$N_\delta(z_0) = \{z \in \mathbb{C} : \ |z - z_0| < \delta\}.$$

We call $N_\delta(z_0) - \{z_0\}$ a deleted δ-neighborhood. ($\{z \in \mathbb{C} : \ 0 < |z - z_0| < \delta\}$)

Definition 1.2.2

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

1. z_0 is called an *interior point* of Ω if
$$\exists \delta > 0, N_\delta(z_0) \subset \Omega.$$
2. z_0 is an *exterior point* of Ω if
$$\exists \delta > 0, N_\delta(z_0) \cap \Omega = \emptyset.$$
3. z_0 is a *boundary point* of Ω if
$$\forall \delta > 0, \ N_\delta(z_0) \cap \Omega \neq \emptyset \quad \text{and} \quad N_\delta(z_0) \cap \underbrace{C_{\mathbb{C}}^{\Omega}}_{\mathbb{C}-\Omega} \neq \emptyset.$$

Definition 1.2.3

The set of all:

1. interior points: $\dot{\Omega}$
2. boundary points: $\partial\Omega$
3. the set $\Omega \cup \partial\Omega$ is called a closure of Ω denoted $\bar{\Omega}$

Definition 1.2.4

We call a set Ω

1. an *open set* if it only contains it's interior points
$$\Omega \cup \partial\Omega = \emptyset \quad \text{and} \quad \Omega = \dot{\Omega}.$$
2. a *closed set* if it contains all it's boundary points
$$\partial\Omega \subset \Omega \quad \text{and} \quad \Omega = \bar{\Omega}.$$

Note:-

Ω is said to be *compact* if it is both *bounded and closed.*

Definition 1.2.5: Limit (accumulation point)

Given a point $z_0 \in \Omega$. z_0 is a limit point if for all $\delta > 0$, $\exists$ infinitely many points $\in N_\delta(z_0)$.

Note:-

If a set is finite then it doesn't have any limit points.
If z_0 is a boundary point of Ω and $\notin \Omega \Longrightarrow z_0$ is a limit point
Ω is a closed set $\Leftrightarrow \Omega \subset \{\text{All limit points}\}$
The set of all limit points of Ω is called the derivative set Ω'

Note:-

A set Ω is bounded if
$$\exists M \in \mathbb{R}_+ \Big/ \forall z \in \Omega\ |z| \leqslant M.$$

Theorem 1.2.3 Bolzano-Weirstrass theorem

Every *bounded infinte* set admits at least one limit point

Paths

A path is a set of complex points Γ where
$$\Gamma = \{z(t) = x(t) + i\,y(t)\ t \in [a, b[\}\,.$$

A simple path/Jordan arc if it does not cross itself
$$\forall t_1, t_2 \in [a, b[\ t_1 \neq t_2 \Longrightarrow z(t_1) \neq z(t_2).$$

A closed path is a path such that
$$z(a) = z(b).$$

A differentiable path (aka. a contour) is a path of equation $(x(t), y(t))$ such that x and y are of class C^1 on the domain of t.

A piecewise differentiable path is union of several differentiable paths.

Note:-

A set is connected if we can connect 2 points $z_1, z_2 \in \Omega$ using a broken line
A connected set simply connected if we can connect any 2 points within using a straight line (no holes in the set), otherwise it is multiply connected.

Chapter 2

Complex Functions

2.1 Limits and Differentiability

Note:-

When taking limits we can do the 2D limit where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x+iy).$$

then we can take multiple paths to find the limit. However we can't take sufficient paths to prove a limit exists as there could exist one path that causes the limit to not exist, however we can use polar limits to prove that the limit exists. We take $x = r\cos(\theta) - x_0$ and $y = r\sin(\theta) - y_0$

$$\lim_{r\to 0} f\left(r\cos(\theta) - x_0 + i\left(r\sin(\theta) - y_0\right)\right).$$

Theorem 2.1.1 Cauchy-Riemann equations

We define a complex function

$$f(x+iy) = u(x,y) + iv(x,y).$$

If f is differentiable on a point $z_0 = x_0 + iy_0$ then u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that the converse is not true

To prove that a function f is differentiable at z_0 then we have to prove that u_x, u_y, v_x, and v_y

$$\begin{cases} \text{exist in } \Omega \\ \text{are continous at } (x_0, y_0) \\ \text{satisfy the Cauchy-Riemann equations at } (x_0, y_0) \end{cases}$$

2.1.1 Hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\tanh z = \frac{\sinh z}{\cosh z}$$

Properties

a) $\cosh^2 z - \sinh^2 z = 1$

b) $\cosh^2 z + \sinh^2 z = \cosh 2z$

c) $\cosh z_1 + z_2 = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

d) $\sinh z_1 + z_2 = \sinh z_1 \cdot \cosh z_2 + \sinh z_2 \cdot \cosh z_1$

e) $\cos iz = \cosh z$

f) $\sin iz = i \sinh z$

g) $\cosh iz = \cos z$

h) $\sinh iz = i \sin z$

Theorem 2.1.2

Consider 2 functions u and v, and the 2 curves $u = \alpha$ and $v = \beta$ such that $\alpha, \beta \in \mathbb{R}$. The 2 curves are orthogonal at their intersection points if and only if they satisfy the Cauchy-Riemann conditions.

2.2 Harmonic functions

Definition 2.2.1: Harmonic function

A function $u(x, y)$, of class C^2 and defined on Ω, is said to be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or in other words the Laplacian is equal to 0

$$\Delta u = \nabla^2 u = 0.$$

Theorem 2.2.1

Let a function $f = u + iv$ defined on Ω

$$f \text{ is holomorphic } \Leftrightarrow \begin{cases} u, v \text{ are of class } C^\infty \text{ in } \Omega \\ u, v \text{ satisfy the Cauchy–Riemann equations in } \Omega \\ u, v \text{ are harmonic in } \Omega \end{cases}.$$

Chapter 3

Integrals

Definition 3.0.1: Complex Integral

Let Ω be an open subset of $\mathbb{C}$ and Γ a piecewise differentiable path from z_1 to z_2. We define the integral of f along the path to be 2 different line integrals:

$$\int f(z)\,\mathrm{d}z = \int_\Gamma (u+iv)(\mathrm{d}x + i\,\mathrm{d}y) = \int_\Gamma (u\,\mathrm{d}x - v\,\mathrm{d}y) + i\int_\Gamma (v\,\mathrm{d}x + u\,\mathrm{d}y).$$

Theorem 3.0.1 Parametrization of the path

If the path Γ is parametrized by $\gamma(t) = x(t) + iy(t)$ where x, y are of class c^1 on $[a, b]$ then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t\,.$$

Theorem 3.0.2 ML-rule

In a path of Γ of length L, we can approximate the value of an integral along that path

$$\left|\int_\Gamma f(z)\,\mathrm{d}z\right| \leqslant M \cdot L.$$

where

$$M = \sup_{z\in\Gamma} |f(z)| \quad \text{and} \quad L = \text{Length of the path } \Gamma = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t\,.$$

Theorem 3.0.3 Cauchy's theorem

Let Γ be a simple closed curve. Let f be a holomorphic function on Γ and inside Γ, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0.$$

Note:-

Green-Riemann theorem states that

$$\oint_{\partial\Omega} (P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y) = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y\,.$$

Note:-

$$\int_{\Gamma^-} f(z)\,\mathrm{d}z = -\int_{\Gamma} f(z)\,\mathrm{d}z\,.$$

A consequence of Cauchy's theorem is that if a closed path C contains a discontinuity then the path of integration doesn't matter as long as the new path also contains the exact same discontinuity.

Theorem 3.0.4

Let Ω be a simply closed region. Let f be a holomorphic function on Ω, z_1 and z_2 be 2 point $\in \Omega$. Then the integral of $f(z)$ is independent of the path taken from z_1 to z_2

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z\,.$$

Theorem 3.0.5 Liouville's theorem

- f is holomorphic in $\mathbb{C}$
- f is bounded in $\mathbb{C}$

$$\exists M \in \mathbb{R}_+, \forall z \in \mathbb{C}, |f(z) \leqslant M|.$$

then f is constant in $\mathbb{C}$

Theorem 3.0.6 Mean value theorem

Let γ_r be a circle of center a and radius $r > 0$. If f is a holomorphic on and in γ_r then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right)\mathrm{d}\theta\,.$$

Theorem 3.0.7 Cauchy's integral formula

Let Γ is a simple closed curve and the function $f(z)$ is holomorphic on Γ and its interior. Then:

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{z-a}\,\mathrm{d}z\,.$$

and the general form of the formula is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{(z-a)^{n+1}}\,\mathrm{d}z\,.$$

Theorem 3.0.8 Tangent half-angle substitution

We can transform the integral of the form

$$\int f(\sin x, \cos x)\,\mathrm{d}x = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\mathrm{d}x\,.$$

by letterfonts

$$t = \tan\frac{x}{2}.$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

3.1 Primitives

Definition 3.1.1: Primitives

Let f be a complex function, defined in an open set $\Omega \subset \mathbb{C}$.
We call a primitive of f on Ω, any function F such that F is holomorphic in Ω and $\forall z \in \Omega \; F'(z) = f(z)$

$$F(z) = \int f(z)\,\mathrm{d}z\,.$$

Note:-

If f admits a primitive on the open set Ω then f is holomorphic in Ω

Let the path γ goes from z_1 to z_2 in Ω then

$$\int_\gamma f(z)\,\mathrm{d}z = F(z_2) - F(z_2).$$

Note:-

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \;\Rightarrow f \text{ is holomorphic in } \Omega.$$

Chapter 4

Multivalued Functions

4.1 Complex Logarithm

We define the complex logarithm to be

$$\log(z) = \ln|z| + i\arg(z).$$

where

$$\arg\left(re^{i\theta}\right) = \theta + 2k\pi \quad k \in \mathbb{Z}.$$

We define the principle valued logarithm to be

$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z).$$

where

$$\mathrm{Arg}\left(re^{i\theta}\right) = \text{principle value of } \arg\left(re^{i\theta}\right) \quad \theta \in]-\pi,\pi].$$

Log is defined on $\mathbb{C}^*$ however it is only continous on $\mathbb{C} - \mathbb{R}_-$

$$\begin{aligned}
\Omega_1 &= \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \\
\Omega_2 &= \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \\
\Omega_3 &= \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}
\end{aligned}$$

$$\mathbb{C} - \mathbb{R}_- = \Omega_1 \cup \Omega_2 \cup \Omega_3.$$

$$\mathrm{Arg}(x+iy) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & \text{if } z \in \Omega_1 \\ \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } z \in \Omega_2 \\ \arccos\left(-\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } z \in \Omega_3 \end{cases}.$$

$$\begin{aligned}
\mathrm{Log}(z_1 \cdot z_2) &= \mathrm{Log}(z_1) + \mathrm{Log}(z_2) + 2k\pi i \\
\mathrm{Log}\left(\frac{z_1}{z_2}\right) &= \mathrm{Log}(z_1) - \mathrm{Log}(z_2) + 2k\pi i \\
\overline{\mathrm{Log}(z)} &= \mathrm{Log}(\bar{z})
\end{aligned}$$

4.1.1 Generalisation of the Complex Logarithm

We define

$$\mathrm{Log}_\alpha(z) = \ln|z| + i\,\mathrm{Arg}(z) \quad \mathrm{Arg}(z) \in]\alpha, \alpha + 2\pi].$$

It is continuous on $\mathbb{C} - \Delta$ where

$$\Delta = \left\{ re^{i\alpha} : r \geqslant 0 \right\}.$$

4.2 n-th root

We call the square root of $z = re^{i\theta}$ any complex number $w \in \mathbb{C}$ satisfying

$$w^2 = z.$$

$$w = \sqrt{r} e^{i\left(\frac{\theta}{2} + k\pi\right)}.$$

And the n-th root becomes

$$w = \sqrt[n]{r} e^{i\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)}.$$

4.3 Power Functions

A power function is any function $f(z) = z^\alpha$. For $\alpha \in \mathbb{Z}$ the function is single valued, while for values $\alpha \notin \mathbb{Z}$ the function is multivalued. The principle determination of f

$$z^\alpha = e^{\alpha\,\mathrm{Log}(z)}.$$

4.4 Inverse Trig Functions

It can be shown that

$$\mathrm{Arcsin}(z) = \frac{1}{i} \log\left(iz + \sqrt{1 - z^2}\right).$$

is holomorphic on $\mathbb{C} -]-\infty, -1] \cup [1, +\infty[$ and

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arcsin}(z) = \frac{1}{\sqrt{1 - z^2}}.$$

$$\mathrm{Arccos}(z) = \frac{1}{i} \mathrm{Log}\left(z + \sqrt{z^2 - 1}\right) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arccos}(z) = -\frac{1}{\sqrt{1 - z^2}}$$

$$\mathrm{Arctan}(z) = \frac{i}{2} \mathrm{Log}\left(\frac{i + z}{i - z}\right) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arctan}(z) = \frac{1}{1 + z^2}$$

$$\mathrm{Arcsinh}(z) = \mathrm{Log}\left(z + \sqrt{z^2 + 1}\right) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arcsinh}(z) = \frac{1}{\sqrt{1 + z^2}}$$

$$\mathrm{Arccosh}(z) = \mathrm{Log}\left(z + \sqrt{z^2 - 1}\right) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arccosh}(z) = \frac{1}{\sqrt{z^2 - 1}}$$

$$\mathrm{Arctanh}(z) = \frac{1}{2} \mathrm{Log}\left(\frac{1 + z}{1 - z}\right) \qquad \frac{\mathrm{d}}{\mathrm{d}z} \mathrm{Arctanh}(z) = \frac{1}{1 - z^2}$$

Chapter 5

Series

Definition 5.0.1: Power Series

We define a power series to be a function of the from

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Note:-

A power series is always convergent at $z = z_0$ and it's value is a_0.

Theorem 5.0.1

Consider a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

1. If this power series converges at $z_1 \neq z_0$ then it converges absolutely for all z such that

$$|z - z_0| < |z_1 - z_0|.$$

2. If this power series diverges at $z_2 \neq z_0$ then it diverges for all z such that

$$|z - z_0| > |z_2 - z_0|.$$

5.0.1 Radius of Convergence

Consider a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. We conider the cases for convergence.

1. **Case 1**: The power series cvonverges at z_0 only. Then the radius of convergence is 0.
2. **Case 2**: The power series converges for all z in a disc D of radius R centered at z_0. Then the radius of convergence is R. The power series may or many no converge in ∂D.
3. **Case 3**: The power series converges for all z. Then the radius of convergence is ∞.

Theorem 5.0.2

Consider a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with radius of convergence R. Then the series converges uniformly on any closed disc γ_r of radius $r < R$ centered at z_0.

$$\gamma_r = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

5.1 Taylor Series

Definition 5.1.1: Taylor Series

Consider a function f holomorphic on a disc D of radius R centered at z_0. Then the Taylor series of f centered at z_0 is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \quad \forall n \in \mathbb{N}.$$

Note:-

The Taylor series of $f(z)$ at $z = 0$ is called the Maclaurin series of $f(z)$.

Theorem 5.1.1

The radius of convergence R of the Taylor series of f at z_0 is given by $R = |z_T - z_0|$ where z_T is the closest singularity of f to z_0.

Note:-

1. If f is holomorphic on $\mathbb{C}$ then z_T is located at infinity hence $R = \infty$.
2. f is holomorphic on an open set Ω if an only if the Taylor series of f converges to f on Ω.

Note:-

The Taylor expension of some common functions are

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad R = 1$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \qquad R = 1$$

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \qquad R = 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad R = \infty$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \qquad R = \infty$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} \qquad R = \infty$$

$$\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad R = \infty$$

$$\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad R = \infty$$

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} (\alpha - k) \frac{z^n}{n!} \qquad R \text{ depends on } \alpha$$

Note:-

z_0 is a zero of order n of f if $f(z_0) = f'(z_0) = \cdots = f^{(n-1)}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$.

Corollary 5.1.1

Consider a function $h = f \cdot g$ where f and g are holomorphic around z_0. If z_0 is a zero of order n of f and z_0 is a zero of order m of g then z_0 is a zero of order $n + m$ of h.

Corollary 5.1.2

Consider a function f holomorphic in an open Ω containing z_0, such that $f \neq 0$ in Ω. If z_0 is a zero of f then z_0 is an isolated zero.

Corollary 5.1.3

Consider a function f holomorphic in an open set Ω. Let K be compact set in Ω such that $f \neq 0$ on K. Then f admits a finite number

Definition 5.1.2: Poles

Consider 2 functions f and g holomorphic in a neighborhood of z_0. Let $h = \frac{f}{g}$. If $g(z_0) \neq z_0$ then z_0 is a regular point. However if $g(z_0) = 0$ then we have several cases of study. Let l be the order of the root of f at z_0 and u be the order of the root of g at z_0.

- If $l < u$ then z_0 is a pole of order $u - l$ of h.
- If $l \geqslant u$ then z_0 is a removable singularity of h.

Definition 5.1.3: Memomorphic functions

A function f is meromorphic in an open set Ω if $\exists A \subset \Omega$

1. A admits no accumulation points
2. f is holomorphic in $\Omega - A$.
3. Each point of A is a pole of f.

5.2 Laurent Series

Definition 5.2.1: Laurent Series

Consider a function f holomorphic in a disk γ_r of center z_0 not including z_0 and radii $r < R$. Then the Laurent series of f centered at z_0 is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \quad \forall n \in \mathbb{Z}.$$

The negative terms of the Laurent series are called the principal part of the Laurent series, while the positive terms are called the regular part of the Laurent series.
The domain of convergence of the Laurent series is the punctured disk of center z_0 and radius $R = |z_T - z_0|$ where z_T is the closest singularity of f to z_0.

Definition 5.2.2: Analytic Extensions

Let γ_r be a punctured disk of center z_0 and radius r. Let f be a holomorphic function in γ_r(not necessarily holomorphic at z_0). Then the following statements are equivalent

1. f is bounded in the deleted neighborhood of z_0.
2. The Laurent expansion of f at $z - 0$ is a Taylor expansion.
3. f admits an analytic extension at z_0.

Note:-

If f is holomorphic at z_0 then it is bounded in a neighborhood of z_0. So the Laurent expansion of f at z_0 coincides with its Taylor expansion at z_0.

Let f be a function expandable into a Laurent series at z_0, and non-holomorphic at z_0. This is equivalent to say z_0 is an isolated singular point of f. We distinguish then three situations for this singular point:

1. If the Laurent series of f at z_0 has only positive terms, then z_0 is a removable singularity of f.
2. If the Laurent series of f at z_0 has only a finite number k of negative terms, then z_0 is a pole of f of order k.
3. If the Laurent series of f at z_0 has an infinite number of negative terms, then z_0 is either an essential singularity of f or a pole of f.

Chapter 6

Residues

Definition 6.0.1: Residue

Let f be a function analytic in a punctured open disk V centred at z_0 (f may not be analytic at z_0). Then f is expandable into a Laurent series at z_0 and this Laurent series is given by:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z-z_0)^{n+1}} \, \mathrm{d}z \quad \forall n \in \mathbb{Z}.$$

The coefficient a_{-1} is called the residue of f at z_0 and is denoted by $\mathrm{Res}(f, z_0)$.

$$\oint_{\gamma_r} f(z) \, \mathrm{d}z = 2\pi i a_{-1}.$$

Theorem 6.0.1 Residue Theorem

Let Γ be a simple closed curve and f be a function analytic in an open set containing Γ except for a finite number of singularities $z_1, \ldots, z_n$ inside Γ. Then

$$\oint_{\Gamma} f(z) \, \mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

6.1 Calculating Residues

1. **Case of a simple pole**: Suppose that z_0 is a simple pole of the function f. The Laurent series of f at z_0 is

$$f(z) = \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

   and

$$a_{-1} = \mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z).$$

2. **Case of a pole of order** k: Suppose that z_0 is a pole of order k of the function f. The Laurent series of f at z_0 is

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots .$$

Thus

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}} \left[(z - z_0)^k f(z) \right].$$

3. **Case of an essential singularity or a pole of high order**: In this case, to calculate the residue of f at z_0, it is sufficient to expand f into a Laurent series at z_0 and examine the coefficient of the term $(z - z_0)^{-1}$

Proposition 6.1.1

In the particular case where $f(z) = \frac{P(z)}{Q(z)}$ with $P(z_0) \neq 0$ and z_0 being a simple zero of $Q(z)$, then

$$\operatorname{Res}(f, z_0) = \frac{P(z_0)}{Q'(z_0)}.$$

6.2 Applications of the Residue Theorem

6.2.1 Jordan's lemmas

Lenma 6.2.1

Let f be a continuous function in $\Omega = \{z \in \mathbb{C} : 0 < |z| < R, 0 \leqslant \theta_1 \leqslant \operatorname{Arg}(z) \leqslant \theta_2 \leqslant 2\pi\}$, where $R > 0$ and θ_1 and θ_2 are both fixed.
Consider an arc of circle γ_r of center 0 and radius r contained in Ω. Then if

$$\lim_{z \to 0} z f(z) = 0 \quad \text{then} \quad \lim_{r \to 0} \int_{\gamma_r} f(z)\, \mathrm{d}z = 0.$$

Lenma 6.2.2

Let f be a continuous function in $\Omega = \{z \in \mathbb{C} : |z| > R, 0 \leqslant \theta_1 \leqslant \operatorname{Arg}(z) \leqslant \theta_2 \leqslant 2\pi\}$, where $R > 0$ and θ_1 and θ_2 are both fixed. Consider an arc of circle γ_r of center 0 and radius r contained in Ω. Then if

$$\lim_{|z| \to \infty} z f(z) = 0 \quad \text{then} \quad \lim_{r \to \infty} \int_{\gamma_r} f(z)\, \mathrm{d}z = 0.$$

Lenma 6.2.3

Let f be a continuous function in $\Omega = \{z \in \mathbb{C} : |z| > R, 0 \leqslant \theta_1 \leqslant \operatorname{Arg}(z) \leqslant \theta_2 \leqslant \pi\}$, where $R > 0$ and θ_1 and θ_2 are both fixed. Consider an arc of circle γ_r of center 0 and radius r contained in Ω, and let m be a positive constant. Then if

$$\lim_{|z| \to \infty} f(z) = 0 \quad \text{then} \quad \lim_{r \to \infty} \int_{\gamma_r} f(z) e^{imz}\, \mathrm{d}z = 0.$$

Lenma 6.2.4

Consider an open disk D centered at z_0 and a function f holomorphic in $D - z_0$ such that z_0 is a simple pole of f. Let γ_r be an *arc* of a circle of center z_0 and radius r contained in D and of angle α. Then

$$\lim_{r \to 0} \int_{\gamma_r} f(z)\, \mathrm{d}z = \alpha i \operatorname{Res}(f, z_0).$$

6.2.2 Evaluation of real integrals

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, \mathrm{d}\theta$

This type of integral can be evaluated by the substitution $z = e^{i\theta}$, $\mathrm{d}z = ie^{i\theta}\,\mathrm{d}\theta$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, and $\sin\theta = \frac{1}{2i}(z - \frac{1}{z})$. Hence

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, \mathrm{d}\theta = \int_\gamma \frac{1}{iz} f\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \mathrm{d}z\,.$$

Then we can simply evaluate the integral using the residue theorem, Cauchy's integral formula, or the Cauchy's theorem.

Integrals of the form $\int_{-\infty}^{\infty} f(x)\, \mathrm{d}x$

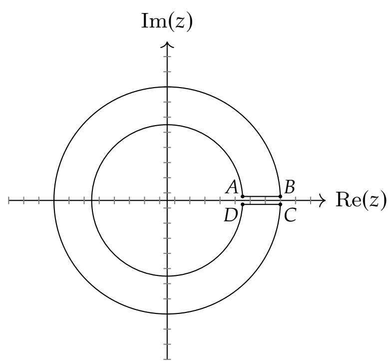
Where $f(x) = \frac{P(x)}{Q(x)}$ with $\deg Q(x) \geqslant \deg P(x) + 2$ and $Q(x)$ has no real roots, then we can evaluate the integral by considering the path Γ consisting of the real axis and a semicircle of radius R in the positive imaginary plane, where R is chosen to be big enough such that Γ encloses all the singularities of f

Integrals of the form $\int_{-\infty}^{\infty} f(x)e^{imx}\, \mathrm{d}x$, $\int_{-\infty}^{\infty} f(x)\cos(mx)\, \mathrm{d}x$, and $\int_{-\infty}^{\infty} f(x)\sin(mx)\, \mathrm{d}x$

Where $f(x) = \frac{P(x)}{Q(x)}$ with $\deg Q(x) \geqslant \deg P(x) + 1$, $Q(x)$ has no real roots, and m is a real positive number, then we can evaluate the integral by considering the same semicircle path as before.

Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{x^\alpha}\, \mathrm{d}x$

Where $f(x) = \frac{P(x)}{Q(x)}$ with $\deg Q(x) \geqslant \deg P(x) + 1$, $Q(x)$ has no real roots, and $\alpha \in]0, 1[$, then we integrate the function $g(z) = \frac{f(z)}{z^\alpha}$ where $z^\alpha = e^{\alpha \operatorname{Log}(z)}$ along the closed path $\Gamma = ABCDA$ shown below were $A(r, 0^+)$, $B(R, 0^+)$, $C(r, 0^-)$, and $D(R, 0^-)$



6.3 Special Functions

Definition 6.3.1: Gamma funcction

The *Gamma function* is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, \mathrm{d}t\,.$$

Some properties of the Gamma function are

1. $\Gamma(z + 1) = z\Gamma(z)$
2. $\Gamma(n + 1) = n!$

3. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$

4. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Definition 6.3.2: Beta function

The *Beta function* is defined as

$$B(x,y) = \int_0^1 (1-t)^{x-1} t^{y-1} \, \mathrm{d}y \, .$$

Some properties of the Beta function are

1. $B(x,y) = B(y,x)$

2. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

3. $B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta \, \mathrm{d}\theta$

4. $B(p,1-p) = \frac{\pi}{\sin(\pi p)}$

Definition 6.3.3: Gauss error functions

The *Gauss error functions* are defined as

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, \mathrm{d}t \quad \text{and} \quad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z).$$

Some properties of the Gauss error functions are

1. $\mathrm{erf}(-z) = -\mathrm{erf}(z)$

2. $\lim_{z\to\infty} \mathrm{erf}(z) = 1$

3. $\lim_{z\to-\infty} \mathrm{erf}(z) = -1$

4. $\mathrm{erf}(z) + \mathrm{erfc}(z) = 1$

Theorem 6.3.1 Stirling's approximation

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n .$$