Mechanics 2

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Definition 1 A double product between 3 vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$ is defined as

$$\vec{\mathbf{u}} \wedge (\vec{\mathbf{v}} \wedge \vec{\mathbf{w}}) = (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}} - (\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}}.$$

Definition 2 The mixed product of 3 vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}} \in \mathbb{R}^n$ is defined as

$$\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \wedge \vec{\mathbf{w}}) = \det \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}.$$

SUBSECTION 1.2

Antisymmetric mapping

Definition 3 In Euclidean space $\mathbb{R}^n$ a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is symmetric if

$$\forall \vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n : \vec{\mathbf{u}} \cdot f(\vec{\mathbf{v}}) = \vec{\mathbf{v}} \cdot f(\vec{\mathbf{u}}).$$

and antisymmetric if

$$\forall \vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n : \vec{\mathbf{u}} \cdot f(\vec{\mathbf{v}}) = -\vec{\mathbf{v}} \cdot f(\vec{\mathbf{u}}).$$

Remark The (anti)symmetric mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is linear

$$\forall \vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n : f(\alpha\vec{\mathbf{u}} + \beta\vec{\mathbf{v}}) = (-)(\alpha f(\vec{\mathbf{u}}) + \beta f(\vec{\mathbf{v}})).$$

Theorem 1 The (anti)symmetric mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, there exists a unique vector $R \in \mathbb{R}^n$ which is called the *characteristic* vector of f such that

$$\forall \vec{\mathbf{u}} \in \mathbb{R}^n \ f(\vec{\mathbf{u}}) = \vec{\mathbf{R}} \wedge \vec{\mathbf{u}}.$$

SUBSECTION 1.3

Torsors

A torsor is a mathematical element used in mechanics to define certain forces.

A torsor consists of a resultant vector $\vec{\mathbf{R}}$ and a moment vector $\vec{\mathbf{M}}_O$ about a point O[1], these elements are called *elements of reduction* of the torsor. We represent a torsor like so

$$[\mathrm{T}(A)] = \begin{bmatrix} \vec{\mathbf{R}} & \vec{\mathbf{M}}_O \end{bmatrix}.$$

[1] *to change the moment vector to be about a point A* $\vec{\mathbf{M}}_A = \vec{\mathbf{M}}_O + \vec{\mathbf{R}} \wedge \vec{\mathbf{AB}}$

Definition 4 The scalar invariant of a torsor [T], denoted $I_{[\mathrm{T}]}$, is the scalar product of the torsor with the moment at a point A.[2]

$$I_{[\mathrm{T}]} = \vec{\mathbf{R}} \cdot \vec{\mathbf{M}}_A.$$

[2] *the invariant is independent of the point the moment is chosen about*

Consider 2 torsors $[\mathrm{T}_1(A)] = \begin{bmatrix} \vec{\mathbf{R}}_1 & \vec{\mathbf{M}}_{1A} \end{bmatrix}$ and $[\mathrm{T}_2(A)] = \begin{bmatrix} \vec{\mathbf{R}}_2 & \vec{\mathbf{M}}_{2A} \end{bmatrix}$. The properties of a torsor are:

- **Equality**: $[\mathrm{T}_1(A)] = [\mathrm{T}_2(A)] \Leftrightarrow \vec{\mathbf{R}}_1 = \vec{\mathbf{R}}_2$ and $\vec{\mathbf{M}}_{1A} = \vec{\mathbf{M}}_{2A}$
- **Automoment**: $[\mathrm{T}_1(A)] \cdot [\mathrm{T}_1(A)] = 2I_{[\mathrm{T}]}$
- **Mutiplication by a scalar**: $\forall \lambda \in \mathbb{R} : [\mathrm{T}_1(A)] = \lambda[\mathrm{T}_1(A)] \Leftrightarrow \vec{\mathbf{R}}_1 = \lambda\vec{\mathbf{R}}_2$ and $\vec{\mathbf{M}}_{1A} = \lambda\vec{\mathbf{M}}_{1A}$
- **Comoment**:[3] $[\mathrm{T}_1(A)] \cdot [\mathrm{T}_2(A)] = \vec{\mathbf{R}}_1 \cdot \vec{\mathbf{M}}_{2A} + \vec{\mathbf{R}}_2 + \vec{\mathbf{M}}_{1A}$

[3] *independent of the point A*

1.3.1 Central Axis of a Torsor

Definition 5 The central point A of a torsor is a point where the moment $\vec{\mathbf{M}}_A$ is collinear with the resultant $\vec{\mathbf{R}}$:

$$\vec{\mathbf{M}}_A \wedge \vec{\mathbf{R}} = 0.$$

and

$$\vec{\mathbf{M}}_A = k\vec{\mathbf{R}} \quad k \in \mathbb{R}.$$

Definition 6 The central axis (Δ) of a torsor is the straight line of the set of central points

$$\Delta = \left\{ A \;/\; \vec{\mathbf{M}}_A \wedge \vec{\mathbf{R}} = 0 \right\}.$$

The formula(parametric) for the central axis Δ is

$$(\Delta) = \frac{\vec{\mathbf{R}} \wedge \vec{\mathbf{M}}_O}{\|\vec{\mathbf{R}}\|^2} + t\vec{\mathbf{R}} \quad t \in \mathbb{R}.$$

Remark Let $A \in \Delta$,

1. If $\vec{\mathbf{M}}_A = 0$ or $\vec{\mathbf{M}}_A \wedge \vec{\mathbf{R}} = 0$ then Δ is a straight line passing through A with a direction vector $\vec{\mathbf{R}}$.
2. If $\vec{\mathbf{M}}_A \neq 0$ then Δ is a straight line passing through B and a direction vector $\vec{\mathbf{R}}$ such that $\vec{AB} = \dfrac{\vec{\mathbf{R}} \wedge \vec{\mathbf{M}}_A}{\|\vec{\mathbf{R}}\|^2}$

1.3.2 Torsor with Scalar Invariant $= 0$

If [T] is a torsor with $I_{[\mathrm{T}]} = 0$, then the torsor is $\begin{cases} \text{null} \\ \text{a slider} \\ \text{a couple} \end{cases}$.

- **Null**: $\vec{\mathbf{R}} = \vec{\mathbf{M}}_A = 0$
- **Slider**: $I_{[\mathrm{T}]} = 0$, $\vec{\mathbf{R}} \neq 0$,and $\forall C \in \Delta\ \vec{\mathbf{M}}_C = 0$
- **Couple**: $I_{[\mathrm{T}]} = 0, \vec{\mathbf{R}} = 0$, and $\vec{\mathbf{M}}_A \neq 0$
  A couple torsor is a uniform field $\vec{\mathbf{M}}_A = \vec{\mathbf{M}}_B$ and $\nexists\Delta$