Analysis 3

Contents

PART

I

Sequence of Functions

SECTION 1

Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 1 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I\ \forall \epsilon > 0\ \exists n \in \mathbb{N}\ \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let $x = 0$ then find $\lim\limits_{n\to\infty} f_n(0) =$ some $f(x)$
2. Then let $x \neq 0$ and again find $\lim\limits_{n\to\infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0, 1] \end{cases}.$$

SECTION 2

Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 2 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N\ \forall x \in I : \sup_{x\in I} |f_n(x) - f(x)| < \epsilon.$$

Remark We can also prove uniform convergence by proving

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

There is also an easy qay to prove uniform convergence of a function by

1. Prove that the sequence of fucntions $f_n(x)$ is pointwise convergent to a function $f(x)$ [1]
2. Define a function $g(x) = |f_n(x) - f(x)|$ and find the maxima of that function at a point x_0 (usually by doing $\mathrm{d}g/\mathrm{d}x = 0$)
3. If $\lim_{n\to\infty} g(x_0) = 0$ then the sequence converges uniformly to $f(x)$

[1] *if the function $f(x)$ is continuous at a point piecewise then the sequence doesn't uniformly converge*

PART

II

Series of Functions

Definition 3 Let $f_n(x)$ be sequence of functions defined on $I \subset \mathbb{R}$, we define the series $S(x)$ to be

$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

Section 3

Convergence of a Numerical Series

In order to prove a series of functions converge we have to prove that it converges for all fixed x.

Theorem 1 Suppose there exists a sequence a_n such that $\forall x, n \ |f_n| \leq a_n$. The Weierstrass test states that if $\sum a_n$ converges then $\sum f_n(x)$ converges uniformly and absolutely

Theorem 2 Let a_n be a sequence of numbers, if $\left|\frac{a_{n+1}}{a_n}\right| = l$ then the sequence is a geometric Series

$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

Theorem 3 A harmonic series is defined to be $a_n = \frac{1}{n^p}$

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

Theorem 4 Let a_n be a sequence of numbers. The 2 series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_n$ are simultaneously convergent/divergent.

Theorem 5 The sequence $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent if a_n is decreasing and $\lim_{n \to \infty} a_n = 0$.

Theorem 6 Consider the series $S = \sum_{n=0}^{\infty} a_n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases}.$$

Section 4

Finite Expansion

The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!}f'(x-a) + \frac{x^2}{2!}f''(x-a) + \cdots + \frac{x^n}{n!}f^{(n)}(x-a) + x^n o(1) \quad x \to a.$$

Some important expansions to keep in mind are

a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

b) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

c) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

d) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

e) $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

f) $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

g) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

h) $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

i) $(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n+1}(\alpha - k)}{n!} x^n$

PART III

Power Series

A power series is just a series in the following formula

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= \sum_{n=0}^{\infty} U_n. \end{aligned}$$

SECTION 5

Radius of Convergence

For some values of x a power series can either diverge or converge, to determine the interval of convergence we employ the ratio test

Theorem 7 Let r be the radius of convergence and I be the domain of convergence. If we compute the limit

$$\Gamma = \lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right|.$$

The ratio test states that

$$\begin{cases} \text{if} \quad \Gamma = 0 \quad \text{then} \quad r = \infty \quad \text{and} \quad I = \mathbb{R} \\ \text{if} \quad \Gamma = \infty \quad \text{then} \quad r = 0 \quad \text{and} \quad I = \{0\} \end{cases}.$$

In the case that Γ isn't 0 or ∞ we set $\Gamma < 1$ and then we find $|x| < R$, finally we can say that $r = R$ and $I =]-R, R[$. A special case need to be done for the points $-R$ and R to determine if they belong in I.

Remark The power series $f(x)$ is continous and will always uniformly converge in the interval of convergence I

Theorem 8 If $\sum a_n x^n$ and $\sum b_n x^n$ be 2 power series with radii R_1 and R_2. For the power series $\sum (a_n + b_n) x^n$ the radius of convergence R

$$R = \min\{R_1, R_2\}.$$

Theorem 9 If $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is a power series with radius R then $S'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^n$ as well as $\int_{x_0}^{x} S(t)\, dt$ both a radius of R

The general term a_k of a power series $S(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is equal to

$$a_k = \frac{S^{(k)}(x_0)}{k!}.$$

$$\begin{aligned} y &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \\ y' &= a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} \\ y'' &= 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2} + n(n+1)a_{n+1}x^{n-1} \end{aligned}$$

PART

IV

Integrals Depending on a Parameter

We define an integratable function

$$\begin{aligned} f : I \times U &\longrightarrow \mathbb{R} \times \mathbb{R} \\ t, x &\longmapsto f(t, x). \end{aligned}$$

where t is a parameter.
Let

$$F(x) = \int_a^b f(t, x) \, \mathrm{d}t \, .$$

Theorem 10 If f is of class C^p then F is also of class C^n and

$$\frac{\partial^p F}{\partial x^p} = \int_a^b \frac{\partial^p f}{\partial x^p} \, \mathrm{d}t \, .$$

Remark F is differentiable if $\dfrac{\partial f}{\partial x}$ is convergent.

Section 6

Improper Integral

Theorem 11 If $\int_a^b |f(t)| \, \mathrm{d}t$ is convergent then $\int_a^b f(t) \, \mathrm{d}t$ is also convergent and $\left| \int_a^b f(t) \, \mathrm{d}t \right| \leq \int_a^b |f(t)| \, \mathrm{d}t$

Theorem 12 The Weierstrass test states that $\int_a^b f(t, x) \, \mathrm{d}t$ is convergent if there exists a function $g(t)$ such that $|f(t, x)| \leq g(t)$ and $\int_a^b g(t) \, \mathrm{d}t$

PART

V

Fourier Series

Definition 4 A Fourier series is a series of functions of general term

$$u_n(x) = a_0 + a_n \cos(nx) + b_n \sin(nx).$$

Section 7

Trigonometric Coefficients

The Fourier coefficients of a function defined on an interval $F \subset \mathbb{R}$ of period $T = \dfrac{2\pi}{\omega} \implies \omega = \dfrac{2\pi}{T}$ are

1. $\sin n\,\pi = 0$
2. $\sin n\,\pi/2 = (-1)^n$
3. $\cos n\,\pi = (-1)^n$
4. $\cos n\,\pi/2 = 0$

$$a_0 = \frac{1}{T}\int_F f(x)\ \mathrm{d}x$$
$$a_n = \frac{2}{T}\int_F f(x)\cos\omega n x\ \mathrm{d}x$$
$$b_n = \frac{2}{T}\int_F f(x)\sin\omega n x\ \mathrm{d}x$$

Subsection 7.1

Even Functions

If a function has a domain $F = [-\ell;\ell]$ and $f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients ($T = |F|$) become

$$a_0 = \frac{1}{\ell}\int_0^\ell f(x)\ \mathrm{d}x$$
$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos nx\ \mathrm{d}x$$
$$b_n = 0$$

Subsection 7.2

Odd Functions

If a function has a domain $F = [-\ell;\ell]$ and $-f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients ($T = |F|$) become

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin nx\ \mathrm{d}x$$

PART

VI

Laplace Transforms

The Laplace transform of a function is defined as

$$F(p) = \mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} e^{-pt} f(t)\,\mathrm{d}t\,.$$

It only exists if the integral above converges.

Section 8

Transforms of some functions

Subsection 8.1

Unit step function

Also known as Heaviside's unit step function, it is defined as

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}.$$

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{p} \quad \text{for} \quad \operatorname{Re}(p) > 0.$$

Subsection 8.2

Dirac Delta Function

The Dirac Delta function

$$\delta(t) = \begin{cases} \infty & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}.$$

$$\mathcal{L}\left\{\delta(t)\right\} = 1.$$

Subsection 8.3

Usual Elementary functions

a) $\mathcal{L}\left\{1\right\} = \dfrac{1}{p}$

b) $\mathcal{L}\left\{t\right\} = \dfrac{1}{p^2}$

c) $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{p^{n+1}}$

d) $\mathcal{L}\left\{\sin \omega t\right\} = \dfrac{\omega}{p^2 + \omega^2}$

e) $\mathcal{L}\{\cos\omega t\} = \dfrac{p}{p^2+\omega^2}$ f) $\mathcal{L}\{\sinh\omega t\} = \dfrac{\omega}{p^2-\omega^2}$

g) $\mathcal{L}\{\cosh\omega t\} = \dfrac{p}{p^2-\omega^2}$ h) $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{p-a}$

Section 9

Properties of the Transform

1. Linearity:
$$\mathcal{L}\{\lambda f + \mu g\} = \lambda\mathcal{L}\{f\} + \mu\mathcal{L}\{g\}.$$
2. Homothety:
$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F(\frac{p}{k}).$$
3. Derivation:
$$\begin{aligned}\mathcal{L}\{f'(t)\} &= p\mathcal{L}\{f(t)\} - f(0^+)\\ \mathcal{L}\{f''(t)\} &= p^2\mathcal{L}\{f(t)\} - pf(0^+) - f'(0^+)\\ \mathcal{L}\left\{f^{(n)}(t)\right\} &= p^n\mathcal{L}\{f(t)\} - \sum_{k=1}^{n} p^{n-k}f^{(k-1)}(0^+)\end{aligned}$$
4. Integration:
$$\mathcal{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = \frac{F(p)}{p}.$$
5. Initial value theorem:
$$f(0^+) = \lim_{p\to\infty} p\mathcal{L}\{f(t)\}.$$
6. Final value theorem:
$$f(\infty) = \lim_{p\to 0} p\mathcal{L}\{f(t)\}.$$

Remark

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}p}F(p)\\ \mathcal{L}\left\{t^2 f(t)\right\} &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}F(p)\\ \mathcal{L}\{t^n f(t)\} &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)\end{aligned}$$

Remark Convolution over a domain $I \subset \mathbb{R}$ is defined as

$$f(t) * g(t) = \int_I f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_I f(t-\tau)g(\tau)\,\mathrm{d}\tau.$$

and it's trasform is

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(p) \cdot G(p).$$

Section 10

Translation

In the time domain:

$$\mathcal{L}\left\{f(t-a)\right\} = e^{-ap}F(p).$$

In the p-domain:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(p+a).$$

PART VII

Systems of Differential Equations

Consider a system of first order differential equations (S)

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}.$$

in matrix form the equation can be written as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} + \vec{\mathbf{b}}.$$

and the initial condition can be written as

$$\vec{\mathbf{x}}_0(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Section 11

Solving The System of DEs

There 3 main ways of solving systems of DEs:

1. Laplace Transform
2. Change of Basis
3. Solving Matrix Formula

Remark Let A be a diagonalizable matrix

$$A = PDP^{-1}.$$

so we define that

$$e^{At} = Pe^{Dt}P^{-1}.$$

or in other words

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{pmatrix} P^{-1}.$$

Subsection 11.1

Change of Basis

We consider a new system of DEs to be

$$\frac{\mathrm{d}\vec{\mathbf{y}}}{\mathrm{d}t} = P^{-1}AP\vec{\mathbf{y}} + P^{-1}\vec{\mathbf{b}}.$$

which simplifies to

$$\frac{\mathrm{d}\vec{\mathbf{y}}}{\mathrm{d}t} = D\vec{\mathbf{y}} + \vec{\mathbf{B}}.$$

in this new system we can solve for $\vec{\mathbf{y}}$

$$\begin{cases} \dfrac{\mathrm{d}y_1}{\mathrm{d}t} = \lambda_1 y_1 + B_1 \\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t} = \lambda_2 y_2 + B_2 \\ \vdots \\ \dfrac{\mathrm{d}y_n}{\mathrm{d}t} = \lambda_n y_n + B_n \end{cases}.$$

Remark The solution to a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y + \beta.$$

is

$$y = c_1 e^{\alpha t} - \frac{\beta}{\alpha}.$$

after we find the solution to the new system, we can simply obtain the solution to the original system by

$$\vec{\mathbf{x}} = P\vec{\mathbf{y}}.$$

and by substituting t_0 in $\vec{\mathbf{x}}$ we can solve for the constant terms $(c_1, c_2, \ldots, c_n)$ using $\vec{\mathbf{x}}_0$.

Subsection 11.2

Solving Matrix Formula

The formula for a system of first order equations is

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p.$$

Where

$$\begin{aligned}
\vec{\mathbf{x}}_h &= V(t,t_0)\vec{\mathbf{x}}_0 \\
\vec{\mathbf{x}}_p &= \int_{t_0}^{t} V(t,u)\vec{\mathbf{b}}(u)\,\mathrm{d}u
\end{aligned}$$

where

$$V(t,t_0) = X(t)X^{-1}(t_0).$$

if $t = 0$ then the formula becomes

$$\vec{\mathbf{x}} = e^{At}\vec{\mathbf{x}}_0 + \int_0^t e^{A(t-u)}\vec{\mathbf{b}}(u)\,\mathrm{d}u\,.$$

Section 12

Fundamental Solutions

For any given system of homogeneous linear DEs there exists a set of n functions such they for a linearly independent basis for a general solution of said DEs, in other words for a given DE there exists a set of vector functions $(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\zeta}}_2, \dots, \vec{\boldsymbol{\zeta}}_n)$ such that

$$\vec{\mathbf{x}}(t) = c_1\vec{\boldsymbol{\zeta}}_1(t) + c_2\vec{\boldsymbol{\zeta}}_2(t) + \dots + c_n\vec{\boldsymbol{\zeta}}_n(t) \quad \text{where} \quad c_{1,2,\dots,n} \in \mathbb{R}.$$

We define the fundamental matrix of the system

$$X = \begin{pmatrix} \vec{\boldsymbol{\zeta}}_1 & \vec{\boldsymbol{\zeta}}_2 & \dots & \vec{\boldsymbol{\zeta}}_n \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & & \ddots & \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nn} \end{pmatrix}.$$

The system can be written in terms of X as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX.$$

Remark The fundamental solutions are linearly independent $\implies \det(X) \neq 0$

Subsection 12.1

Wronskian of vector functions

Consider the vector functions:

$$\vec{\boldsymbol{\phi}}_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix} \quad \cdots \quad \vec{\boldsymbol{\phi}}_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}.$$

The Wronskian is defined to the determinant:

$$W(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \cdots, \vec{\boldsymbol{\phi}}_n) = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & \ddots & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}.$$

If the Wronskian $= 0$ then the functions $(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \ldots, \vec{\boldsymbol{\phi}}_n)$ are said to be linearly independent.
When dealing with DEs the concept of a Wronskian can be applied to *non-vector functions* as follows

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & \ddots & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

Section 13

Solving n-th order Homogeneous Linear DE

We define the notation $\mathrm{D}^n x = \dfrac{\mathrm{d}^n x}{\mathrm{d}t^n}$.
An n-the order linear DE is any equation of the form:

$$\mathrm{D}^n x + a_1(t)\mathrm{D}^{n-1} x + \cdots + a_{n-1}(t)\mathrm{D}\, x + a_n(t)x = 0.$$

We can then write the equation in vector form

$$\begin{aligned} x &= x_1 \\ \mathrm{D}\, x &= x_2 \\ \mathrm{D}^2 x &= x_3 \\ &\vdots \\ \mathrm{D}^{n-1} x &= x_n \\ \mathrm{D}^n x &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n \end{aligned}$$

and we take

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{D}\, x \\ \vdots \\ \mathrm{D}^{n-1} x \end{pmatrix}.$$

then we can write the system as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{pmatrix}.$$

Subsection 13.1

DE from a Set of Fundamental Solutions

Given a set of fundamental solutions $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, due to the uniqueness theorem those solutions only satisfy one DE. To find that DE we simply compute

$$W(x, \zeta_1, \zeta_2, \ldots, \zeta_n) = \begin{vmatrix} x & \zeta_1 & \cdots & \zeta_n \\ \mathrm{D}\,x & \mathrm{D}\,\zeta_1 & \cdots & \mathrm{D}\,\zeta_n \\ \vdots & & \ddots & \\ \mathrm{D}^n\,x & \mathrm{D}^n\,\zeta_1 & \cdots & \mathrm{D}^n\,\zeta_n \end{vmatrix} = 0.$$

Example Given the fundamental set of solutions $(e^{\omega t}, e^{-\omega t})$, find the second order homogeneous equation for that set of solutions:

$$\begin{aligned} & W(x, e^{\omega t}, e^{-\omega t}) = 0 \\ \implies & \begin{vmatrix} x & e^{\omega t} & e^{-\omega t} \\ x' & \omega e^{\omega t} & -\omega e^{-\omega t} \\ x'' & \omega^2 e^{\omega t} & \omega^2 e^{-\omega t} \end{vmatrix} = 0 \\ \implies & x'' - \omega^2 x = 0. \end{aligned}$$

Subsection 13.2

Method of Variation of constants

Consider a non-homogeneous linear DE

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) x^{(i)}(t) = b(t).$$

and let $(x_1, x_2, \ldots, x_n)$ be a solution to the corresponding homogeneous differential equation. Then the particular solution to the equation is given by

$$x_p = \sum_{i=0}^{n} z_i(t) x_i(t).$$

such that $z_i(t)$ satisfies the condition

$$\sum_{i=1}^{n} z_i'(t) x^{(j)}(t) = 0 \quad \text{for} \quad j = 0, 1, \ldots, n-2.$$

We substitute x_p in to the original DE, and along with the previous condition, we obtain a linear system of equations dependent on z_i'. Then we simply find z_i' and integrate to get back z_i then finally obtain a general solution to the non-homogeneous DE.

The formula for z_1 and z_2 are given for 2nd order DEs of the form

$$x'' + P(t)x' + Q(t)x = g(x).$$

$$z_1 = -\int \frac{x_2 g}{W(x_1, x_2)}\,\mathrm{d}t$$
$$z_2 = -\int \frac{x_1 g}{W(x_1, x_2)}\,\mathrm{d}t$$

Example Solve:

$$x'' + x = \sec(t) = \frac{1}{\cos(x)}.$$

We know the solution to the homogeneous equation is $x_h = c_1\cos(t) + c_2\sin(t)$ (refer to the next chapter for a method for solving the homogeneous equation)
We substitute the constants with our parameters

$$x = z_1\cos(t) + z_2\sin(t).$$

Our condition for the parameters becomes

$$z_1'\cos(t) + z_2'\sin(t) = 0.$$

and substituting x in the original DE we obtain

$$-z_1'\sin(t) + z_2'\cos(t) = \sec(t).$$

Solving the system

$$\begin{cases} z_1' = -\tan(t) & \Rightarrow z_1 = \ln|\cos(t)| + c_1 \\ z_2' = 1 & \Rightarrow z_2 = t + c_2 \end{cases}.$$

Finally

$$x = (\ln|\cos(t)| + c_1)\cos(t) + (t + c_2)\sin(t).$$

Not sure why we're solving systems of DEs before we solve DEs in general, besides half the course is basically redundant

Linear Differential Equations

PART VIII

Let the notation denote

$$\mathrm{D}^n\, x = \frac{\mathrm{d}^n x}{\mathrm{d}t^n}.$$

Section 14

Differential Operators

Subsection 14.1

Definition

Let's call the set of differential operators $\mathbb{L}$, a differential operator $A \in \mathbb{L}$ is defined to be

$$A = a_0 \mathrm{D}^0 + a_1 \mathrm{D}^1 + a_2 \mathrm{D}^2 + \cdots + a_n \mathrm{D}^n .$$

The order of a linear operator is the highest power of D denoted

$$\gamma(A) = \max(\text{Power of D}).$$

Differential operators form a ring $(\mathbb{L}, +, \cdot)$.

They behave just like normal polynomials with the exception of $A \cdot B \neq B \cdot A$ which also has the exception of being equal if $a_k = \text{cnst}$ and $b_k = \text{cnst}$

Subsection 14.2

Differential Operators as DEs

We can use differential operators to write DEs in the form

$$Ax = b(t).$$

Let $A, B \in \mathbb{L}$ and $\gamma(B) \leq \gamma(A)$ we can prove that $\exists Q, R \in \mathbb{L}\ A = QB + R$ (Q: quotient, R: remainder)

Algebra 1 long division, yum

Remark When dealing with differential operators, treat D as a variable. Thus differential operators can be though of as polynomials of D.

If the operators A and B each correspond to a *homogeneous* DE then we can compute $\gcd(A, B)$

- $\gcd(A, B) = 1$ then the differential equations do not have a common solution between them.
- $\gcd(A, B) \neq 1$ then the differential equations have a common solution, which is the solution to the DE corresponding to $R = \gcd(A, B)$

14.2.1 Case of known k solutions

If we have a homogeneous DE, whose corresponding operator is A of order n, that we know k solutions of we can construct an operator B of order k whose solutions is said k solutions. Then the remaining $n-k$ solutions are the solutions are the solutions for $\gcd(A, B)$. We can construct B using

$$W(x, x_1, x_2, \ldots, x_k) = 0.$$

Subsection 14.3

Linear Equations with Constant Coefficients

The characteristic polynomial of a homogeneous DE is the polynomial where D is substituted with r, it is denoted as $P(r)$ and its solutions can help find the solutions to the DE.
If $P(r)$ has a solution in the reals ($\mathbb{R}$) then the solution of the associated DE is

$$P(r) = (r-\alpha)^{} = 0 \Rightarrow x = C_{-1}e^{\alpha t}.$$

On the other hand if the solutions to $P(r)$ were complex then the solutions become

is the order of the root of the polynomial

$$P(r) = ((r-\omega)(r-\bar{\omega}))^{} = 0 \Rightarrow x = e^{\mathrm{Re}(\omega)t}\left(U_{-1}\cos(\mathrm{Im}(\omega)t) + i\,V_{-1}\sin(\mathrm{Im}(\omega)t)\right).$$

where C_n, U_n, and V_n are polynomials with constant coefficients $(c_1, c_2, \ldots, c_n)$ and degree n. $\left(\sum_{i=0}^{n} c_i t^i\right)$

Remark The principle of super position states that for a linear differential equation of the form

$$Ax = \underbrace{b(t)}_{b_1(t)+b_2(t)}.$$

the particular solutions of the equations with b_1 and b_2, being x_1 and x_2 respectively, add up to form the particular solution of the original equation
Generally, we find the solutions to the homogeneous equations then use variation of parameters/constants to find the particular solution.

Section 15

Particular Forms of $b(t)$

- $Ax = ke^{\beta t}$: Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the associated $P(r)$ and $_1, _2, \ldots, _n$ be the respective order of those roots.
  - If $\beta \neq \alpha_i$, then we look for a particular solution of the form $x_p = ce^{\beta t}$ where c is a constant to be determined.
  - If $\beta = \alpha_i$, then we look for a particular solution of the form $x_p = ct^{_i}e^{\beta t}$ where c is a constant to be determined.

- $a_0x'' + a_1x' + a_2x = k$: We look for a particular solution of the form $x_p = c$
  - If $a_2 = 0$ we look for a particular solution of the form $x_p = ct$
  - If $a_1 = a_2 = 0 \implies x_p = ct^2$

- $a_0x'' + a_1x' + a_2x = I_n$ (Polynomial of degree n): We look for a particular solution of the form $x_p = Q_n$ (another polynomial of degree n)
  - If $a_2 = 0$ we look for a particular solution of the form $x_p = Q_{n+1}$
  - If $a_1 = a_2 = 0 \implies x_p = Q_{n+2}$

- $a_0x'' + a_1x' + a_2x = I_ne^{\beta t}$:
  - If $P(\beta) \neq 0$ we look for a particular solution of the form $x_p = Q_ne^{\beta t}$
  - If $P(\beta) = 0$ and the order of $\beta = 1$ we look for a particular solution of the form $x_p = Q_{n+1}e^{\beta t}$
  - If $P(\beta) = 0$ and the order of $\beta = 2$ we look for a particular solution of the form $x_p = Q_{n+2}e^{\beta t}$

- $a_0x'' + a_1x' + a_2x = k\sin(\beta t) + h\cos(\beta t)$:
  - If $P(\pm i\beta) \neq 0$ we look for $x_p = c_1\sin(\beta t) + c_2\cos(\beta t)$
  - If $P(\pm i\beta) = 0$ we look for $x_p = t\,[c_1\sin(\beta t) + h\cos(\beta t)]$

- $a_0x'' + a_1x' + a_2x = e^{\lambda t}\,[k\sin(\beta t) + h\cos(\beta t)]$:
  - If $P(\lambda \pm i\beta) \neq 0$ we look for $x_p = e^{\lambda t}\,[c_1\sin(\beta t) + h\cos(\beta t)]$
  - If $P(\lambda \pm i\beta) = 0$ we look for $x_p = te^{\lambda t}\,[k\sin(\beta t) + h\cos(\beta t)]$

- $a_0x'' + a_1x' + a_2x = I_n\sin(\beta t) + Q_n\cos(\beta t)$:
  - If $P(\pm i\beta) \neq 0$ we look for $x_p = R_n\sin(\beta t) + S_n\cos(\beta t)$
  - If $P(\pm i\beta) = 0$ we look for $x_p = R_{n+1}\sin(\beta t) + S_{n+1}\cos(\beta t)$

Section 16

Euler's Equations

Euler's DE is a DE of the form $a_i \in \mathbb{R}$

$$a_0t^n\mathrm{D}^n\,x\, + a_1t^{n-1}\mathrm{D}^{n-1}\,x\, + \cdots + a_nx = 0.$$

- **Method One**:
  We assume that $x = t^r$, by substituting x in the equation we get a new equation of the form $I(r)t^r = 0$ where $I(r)$ is a polynomial of degree n. The solutions to $I(r)$ are q real solutions $r_1, r_2, \ldots, r_n$, thus $x = c_1t^{r_1} + c_2t^{r_2} + \cdots + c_nt^{r_n}$.

- **Method Two**:
  We let $t = e^u$ and $\Delta = \dfrac{\mathrm{d}}{\mathrm{d}u}$. We can prove that

  $$\mathrm{D}^p\, x \; = e^{-pu}\Delta(\Delta - 1)\cdots(\Delta - (p+1)).$$

  After doing this substitution, the equation is transformed in to linear equation with constant coefficients.