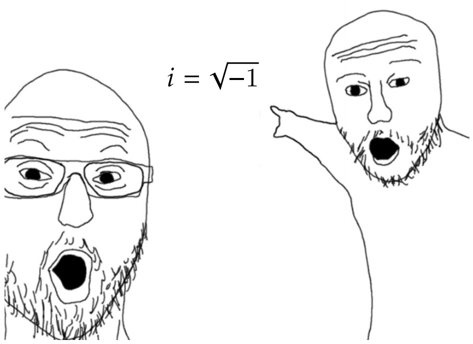
Complex Analysis
Semester 4

Contents

Chapter 1

The Complex Plane

1.1 Algebra of the complex plane



Euler's formulas for sin and cos

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

The n-th roots of unity are the set of complex numbers $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are the complex numbers that satisfy the equation

$$z^n = w.$$

where $w = Re^{i\alpha}$. The solutions equation are

$$\zeta_k = \sqrt[n]{R} e^{i\left(\frac{\alpha + 2k\pi}{n}\right)}.$$

1.2 Topology of the complex plane

Theorem 1.2.1

The mapping

$$|z| : \mathbb{C} \longrightarrow \mathbb{R}^+$$
$$z = x + yi \longmapsto |x + yi| = \sqrt{x^2 + y^2}.$$

defines a norm on $\mathbb{C}$, so the complex plane is a normed space.

Theorem 1.2.2

The mapping

$$d(.,.) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}^+$$
$$(z, w) \longmapsto d(z, w) = |z - w|.$$

defined a distance on $\mathbb{C}$, so the complex plane is a metric space.

Definition 1.2.1: Neighborhood

We call δ-neighborhood of z_0 an open disk centered at z_0 of radius δ

$$N_\delta(z_0) = \{z \in \mathbb{C} : \ |z - z_0| < \delta\}.$$

We call $N_\delta(z_0) - \{z_0\}$ a deleted δ-neighborhood. ($\{z \in \mathbb{C} : \ 0 < |z - z_0| < \delta\}$)

Definition 1.2.2

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

1. z_0 is called an *interior point* of Ω if
$$\exists \delta > 0, N_\delta(z_0) \subset \Omega.$$
2. z_0 is an *exterior point* of Ω if
$$\exists \delta > 0, N_\delta(z_0) \cap \Omega = \emptyset.$$
3. z_0 is a *boundary point* of Ω if
$$\forall \delta > 0, \ N_\delta(z_0) \cap \Omega \neq \emptyset \quad \text{and} \quad N_\delta(z_0) \cap \underbrace{C_{\mathbb{C}}^{\Omega}}_{\mathbb{C}-\Omega} \neq \emptyset.$$

Definition 1.2.3

The set of all:

1. interior points: $\dot{\Omega}$
2. boundary points: $\partial\Omega$
3. the set $\Omega \cup \partial\Omega$ is called a closure of Ω denoted $\bar{\Omega}$

Definition 1.2.4

We call a set Ω

1. an *open set* if it only contains it's interior points

$$\Omega \cup \partial\Omega = \emptyset \quad \text{and} \quad \Omega = \dot{\Omega}.$$

2. a *closed set* if it contains all it's boundary points

$$\partial\Omega \subset \Omega \quad \text{and} \quad \Omega = \bar{\Omega}.$$

Note:-

Ω is said to be *compact* if it is both *bounded and closed.*

Definition 1.2.5: Limit (accumulation point)

Given a point $z_0 \in \Omega$. z_0 is a limit point if for all $\delta > 0$, $\exists$ infinitely many points $\in N_\delta(z_0)$.

Note:-

If a set is finite then it doesn't have any limit points.
If z_0 is a boundary point of Ω and $\notin \Omega \Longrightarrow z_0$ is a limit point
Ω is a closed set $\Leftrightarrow \Omega \subset \{\text{All limit points}\}$
The set of all limit points of Ω is called the derivative set Ω'

Note:-

A set Ω is bounded if

$$\exists M \in \mathbb{R}_+ \Big/ \forall z \in \Omega \; |z| \leqslant M.$$

Theorem 1.2.3 Bolzano-Weirstrass theorem

Every *bounded infinte* set admits at least one limit point

Paths

A path is a set of complex points Γ where

$$\Gamma = \{z(t) = x(t) + i\, y(t) \; t \in [a, b[\}.$$

A simple path/Jordan arc if it does not cross itself

$$\forall t_1, t_2 \in [a, b[\; t_1 \neq t_2 \Longrightarrow z(t_1) \neq z(t_2).$$

A closed path is a path such that

$$z(a) = z(b).$$

A differentiable path (aka. a contour) is a path of equation $(x(t), y(t))$ such that x and y are of class C^1 on the domain of t.

A piecewise differentiable path is union of several differentiable paths.

Note:-

A set is connected if we can connect 2 points $z_1, z_2 \in \Omega$ using a broken line
A connected set simply connected if we can connect any 2 points within using a straight line (no holes in the set), otherwise it is multiply connected.

Chapter 2

Complex Functions

2.1 Limits and Differentiability

Note:-

When taking limits we can do the 2D limit where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x+iy).$$

then we can take multiple paths to find the limit. However we can't take sufficient paths to prove a limit exists as there could exist one path that causes the limit to not exist, however we can use polar limits to prove that the limit exists. We take $x = r\cos(\theta) - x_0$ and $y = r\sin(\theta) - y_0$

$$\lim_{r\to 0} f\left(r\cos(\theta) - x_0 + i\left(r\sin(\theta) - y_0\right)\right).$$

Theorem 2.1.1 Cauchy-Riemann equations

We define a complex function

$$f(x+iy) = u(x,y) + iv(x,y).$$

If f is differentiable on a point $z_0 = x_0 + iy_0$ then u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note that the converse is not true

To prove that a function f is differentiable at z_0 then we have to prove that u_x, u_y, v_x, and v_y

$$\begin{cases} \text{exist in } \Omega \\ \text{are continous at } (x_0, y_0) \\ \text{satisfy the Cauchy-Riemann equations at } (x_0, y_0) \end{cases}$$

2.1.1 Hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\tanh z = \frac{\sinh z}{\cosh z}$$

Properties

a) $\cosh^2 z - \sinh^2 z = 1$

b) $\cosh^2 z + \sinh^2 z = \cosh 2z$

c) $\cosh z_1 + z_2 = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

d) $\sinh z_1 + z_2 = \sinh z_1 \cdot \cosh z_2 + \sinh z_2 \cdot \cosh z_1$

e) $\cos iz = \cosh z$

f) $\sin iz = i \sinh z$

g) $\cosh iz = \cos z$

h) $\sinh iz = i \sin z$

Theorem 2.1.2

Consider 2 functions u and v, and the 2 curves $u = \alpha$ and $v = \beta$ such that $\alpha, \beta \in \mathbb{R}$. The 2 curves are orthogonal at their intersection points if and only if they satisfy the Cauchy-Riemann conditions.

2.2 Harmonic functions

Definition 2.2.1: Harmonic function

A function $u(x, y)$, of class C^2 and defined on Ω, is said to be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or in other words the Laplacian is equal to 0

$$\Delta u = \nabla^2 u = 0.$$

Theorem 2.2.1

Let a function $f = u + iv$ defined on Ω

$$f \text{ is holomorphic } \Leftrightarrow \begin{cases} u, v \text{ are of class } C^\infty \text{ in } \Omega \\ u, v \text{ satisfy the Cauchy–Riemann equations in } \Omega \\ u, v \text{ are harmonic in } \Omega \end{cases}.$$

Chapter 3

Integrals

Definition 3.0.1: Complex Integral

Let Ω be an open subset of $\mathbb{C}$ and Γ a piecewise differentiable path from z_1 to z_2. We define the integral of f along the path to be 2 different line integrals:

$$\int f(z)\mathrm{d}z = \int_\Gamma (u + iv)(\mathrm{d}x + i\mathrm{d}y) = \int_\Gamma (u\mathrm{d}x - v\mathrm{d}y) + i\int_\Gamma (v\mathrm{d}x + u\mathrm{d}y).$$

Theorem 3.0.1 Parametrization of the path

If the path Γ is parametrized by $\gamma(t) = x(t) + iy(t)$ where x, y are of class c^1 on $[a, b]$ then

$$\int_\Gamma f(z)\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\mathrm{d}t.$$

Theorem 3.0.2 ML-rule

In a path of Γ of length L, we can approximate the value of an integral along that path

$$\left|\int_\Gamma f(z)\mathrm{d}z\right| \leqslant M \cdot L.$$

where

$$M = \sup_{z\in\Gamma} |f(z)| \quad \text{and} \quad L = \text{Length of the path } \Gamma = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t.$$

Theorem 3.0.3 Cauchy's theorem

Let Γ be a simple closed curve. Let f be a holomorphic function on Γ and inside Γ, then

$$\oint_\Gamma f(z)\mathrm{d}z = 0.$$

Note:-

Green-Riemann theorem states that

$$\oint_{\partial\Omega} (P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y) = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Note:-

$$\int_{\Gamma^-} f(z)\mathrm{d}z = -\int_{\Gamma} f(z)\mathrm{d}z.$$

A consequence of Cauchy's theorem is that if a closed path C contains a discontinuity then the path of integration doesn't matter as long as the new path also contains the exact same discontinuity.

Theorem 3.0.4

Let Ω be a simply closed region. Let f be a holomorphic function on Ω, z_1 and z_2 be 2 point $\in \Omega$. Then the integral of $f(z)$ is independent of the path taken from z_1 to z_2

$$\int_{\gamma_1} f(z)\mathrm{d}z = \int_{\gamma_2} f(z)\mathrm{d}z.$$

Theorem 3.0.5 Liouville's theorem

- f is holomorphic in $\mathbb{C}$
- f is bounded in $\mathbb{C}$

$$\exists M \in \mathbb{R}_+, \forall z \in \mathbb{C}, |f(z) \leqslant M|.$$

then f is constant in $\mathbb{C}$

Theorem 3.0.6 Mean value theorem

Let γ_r be a circle of center a and radius $r > 0$. If f is a holomorphic on and in γ_r then

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right)\mathrm{d}\theta.$$

Theorem 3.0.7 Cauchy's integral formula

Let Γ is a simple closed curve and the function $f(z)$ is holomorphic on Γ and its interior. Then:

$$f(a) = \frac{1}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{z-a}\mathrm{d}z.$$

and the general form of the formula is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\oint_{\Gamma^+} \frac{f(z)}{(z-a)^{n+1}}\mathrm{d}z.$$

Theorem 3.0.8 Tangent half-angle substitution

We can transform the integral of the form

$$\int f(\sin x, \cos x)\mathrm{d}x = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\mathrm{d}x.$$

by letterfonts

$$t = \tan\frac{x}{2}.$$

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t$$

3.1 Primitives

Definition 3.1.1: Primitives

Let f be a complex function, defined in an open set $\Omega \subset \mathbb{C}$.
We call a primitive of f on Ω, any function F such that F is holomorphic in Ω and $\forall z \in \Omega\ F'(z) = f(z)$

$$F(z) = \int f(z)\mathrm{d}z.$$

Note:-

If f admits a primitive on the open set Ω then f is holomorphic in Ω

Let the path γ goes from z_1 to z_2 in Ω then

$$\int_\gamma f(z)\mathrm{d}z = F(z_2) - F(z_2).$$

Note:-

$$\oint_\gamma f(z)\mathrm{d}z = 0 \ \Rightarrow f \text{ is holomorphic in } \Omega.$$