

DIFFERENTIAL GEOMETRY

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Prerequisites

SECTION 1

Matrices

Theorem 1 To prove a system of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is free we prove:

$$\det \begin{bmatrix} \left| \begin{array}{c} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \\ \vdots \\ \vec{u}_n \end{array} \right| \end{bmatrix} \neq 0.$$

Theorem 2 A transition matrix $P_{B \rightarrow B'}$ between 2 basis $B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ and $B' = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ we start by solving the system

$$\begin{bmatrix} \left| \begin{array}{c} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \\ \vdots \\ \vec{u}_n \end{array} \right| \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \begin{bmatrix} \left| \begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \\ \vdots \\ \vec{v}_n \end{array} \right| \end{bmatrix}.$$

or in other words finding

$$\begin{cases} \vec{v}_1 = \alpha_1 \vec{u}_1 + \beta_1 \vec{u}_2 + \gamma_1 \vec{u}_3 \\ \vec{v}_2 = \alpha_2 \vec{u}_1 + \beta_2 \vec{u}_2 + \gamma_2 \vec{u}_3 \\ \vec{v}_3 = \alpha_3 \vec{u}_1 + \beta_3 \vec{u}_2 + \gamma_3 \vec{u}_3 \end{cases}.$$

Finally we say that

$$P_{B \rightarrow B'} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

transition matrices are always square and invertible ($\det P \neq 0$)

Remark To find the transition matrix in the inverse direction (from B' to B) we simply do

$$P_{B' \rightarrow B} = P_{B \rightarrow B'}^{-1}.$$

SECTION 2

Vectors

Definition 1 We define an operation called the scalar product (dot product)

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{u}, \vec{v} \longmapsto \langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^n v_i \cdot u_i.$$

Definition 2 We define the usual norm on \mathbb{R} to be

$$\| \cdot \| : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{u} \longmapsto \|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}.$$

Theorem 3 The projection of a vector \vec{u} on to another vector \vec{v} is

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}.$$

SUBSECTION 2.1

GramSchmidt process

The aim of this process is to find a new basis $\Gamma = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ derived from a basis $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ such that it is orthonormal or in other words

$$\forall \hat{x}, \hat{y} \in \Gamma : \langle \hat{x}, \hat{y} \rangle = 0 \quad \text{and} \quad \|\hat{x}\| = 1.$$

We find it as follows

$$\begin{aligned} \vec{u}_1 &= \vec{v}_1 & \hat{e}_1 &= \frac{\vec{u}_1}{\|\vec{u}_1\|} \\ \vec{u}_2 &= \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2) & \hat{e}_2 &= \frac{\vec{u}_2}{\|\vec{u}_2\|} \\ \vec{u}_3 &= \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3) & \hat{e}_3 &= \frac{\vec{u}_3}{\|\vec{u}_3\|} \\ &\vdots & & \\ \vec{u}_n &= \vec{v}_n - \text{proj}_{\vec{u}_1}(\vec{v}_n) - \text{proj}_{\vec{u}_2}(\vec{v}_n) - \dots - \text{proj}_{\vec{u}_{n-1}}(\vec{v}_n) & \hat{e}_n &= \frac{\vec{u}_n}{\|\vec{u}_n\|} \end{aligned}$$

Conics and Quadrics

SECTION 3

Conics

We define a quadric form to be a mapping q

$$q : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$\vec{u} \longmapsto q(\vec{u}) = \begin{bmatrix} \text{---} & {}^t\vec{u} & \text{---} \end{bmatrix} A \begin{bmatrix} | \\ | \\ | \end{bmatrix} \vec{u}.$$

Where the matrix A is a symmetric matrix.¹

The conics under study are

¹symmetric matrices ($A = {}^tA$) is always diagonalizable

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse (circle if $a = b$)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	imaginary ellipse
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$	hyperbola with asymptote $y = \pm \frac{b}{a}x$
$\left. \begin{array}{l} y^2 = \pm 2px \quad p > 0 \\ x^2 = \pm 2py \quad p > 0 \end{array} \right\}$	parabolas
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	union of two straight lines
$\left. \begin{array}{l} x = \text{const} \\ y = \text{const} \end{array} \right\}$	straight lines

SUBSECTION 3.1

Identification of the conics

Let the general equation of all conics be:

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

- if $b = 0$: then we simply group together the terms x^2 and x as well as y^2 and y followed by completing the square to get an equation of a conic.
- if $b \neq 0$: in this case we have to introduce a new system of reference which eliminates the existence of xy
We do this by first defining a quadratic form $q(x, y) = ax^2 + 2bxy +$

cy^2 using a matrix

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

which we diagonalize in to an or tho normal age-basis which we project our equation in to in order to get rid of the xy term

Example Find the nature of the conic

$$\Gamma : 5x^2 - 4xy + 8y^2 + \frac{20}{\sqrt{5}}x - \frac{80}{\sqrt{5}}y + 4 = 0.$$

Let $q(x, y) = 5x^2 - 4xy + 8y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = {}^t \vec{u} A \vec{u}$. We find that the matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$ with eigenvalues $\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, the age vectors are already orthogonal so we just find $\vec{e}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{e}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, finally

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad D = \begin{pmatrix} 4 & \\ & 9 \end{pmatrix}.$$

We define $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ to be any vector with basis $\{\vec{e}_1, \vec{e}_2\}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$\begin{aligned} x &= \frac{2}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(2\alpha - \beta) \\ y &= \frac{1}{\sqrt{5}}\alpha + \frac{2}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(\alpha + 2\beta) \end{aligned}$$

now we substitute x and y with α and β into Γ and we manipulate the expression until we get

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

$\therefore \Gamma$ is an ellipse.

SUBSECTION 3.2

Tangent to a conic at point B

Theorem 4

The normal to vector to a conic Γ

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

at a point $B \in \Gamma$ is defined to be

$$\nabla f(B) = \begin{pmatrix} \frac{\partial f}{\partial x} \Big|_{(x_B, y_B)} \\ \frac{\partial f}{\partial y} \Big|_{(x_B, y_B)} \end{pmatrix}.$$

where $f(x, y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$

The equation of a tangent to a conic at a point B is

$$a(x - x_B) + b(y - y_B) = 0.$$

where a and b are respectively the x and y components of the normal vector at B

SECTION 4

Quadrics

Definition 3

A quadric is any surface in 3D space with an equation of the form:

$$\underbrace{ax^2 + by^2 + cz^2 + 2dyz + 2exy + 2fxy}_{q(x,y,z): \text{quadratic form of 3 variables}} + \underbrace{gx + hy + iz}_{\text{linear part}} + \underbrace{j}_{\text{constant}} = 0.$$

The quadrics under study are²

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboliod of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboliod of 2 sheets
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Asymptote cone
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2pz$	Hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$	Elliptic cone

²if $a = b$ the surface is a surface of revolution of axis (Oz)

If a one of variables is missing in the equation then the surface is said to be "(Conic name)-ic Cylinder". For example "Hyperbolic cylinder",

"Circular cylinder", and "Elliptical cylinder"

Parametric Curves

A vector function/parametric curve is a function of the form

$$\begin{aligned}\vec{\mathbf{F}} : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto \vec{\mathbf{F}}(t) = (x(t), y(t)).\end{aligned}$$

With a domain of definition $\mathbb{D}_{\vec{\mathbf{F}}} = \mathbb{D}_x \cap \mathbb{D}_y$

Remark The length of a curve when $t \in [a, b]$ is

$$\int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2} dt.$$

SECTION 5

Symmetry

Consider the domain of definition to be \mathbb{R} .

If a function is even ($f(-x) = f(x)$) or odd ($f(-x) = -f(x)$) the domain of study \mathbb{D}_S is only $[0, +\infty[$, and it is symmetric with respect to some axis. (refer to the table)

If a curve $x(t+T) = x(t)$ and $y(t+T) = y(t)$ then the curve is T -periodic.

Then the domain of study $\mathbb{D}_S = [0, T] \cap \mathbb{D}_{\vec{\mathbf{F}}}$ or $= \left[-\frac{T}{2}, \frac{T}{2}\right] \cap \mathbb{D}_{\vec{\mathbf{F}}}$.

Remark The tangent line of a curve at $t = t_0$ is

$$-y'(t_0)(x - x(t_0)) + x'(t_0)(y - y(t_0)) = 0.$$

and the normal is

$$x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) = 0.$$

SECTION 6

Particular Points

A point is said to be stationary if $\vec{\mathbf{F}}'(t) = 0$, regular if $\vec{\mathbf{F}}'(t) \neq 0$, and biregular if $\det(\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t)) \neq 0$.

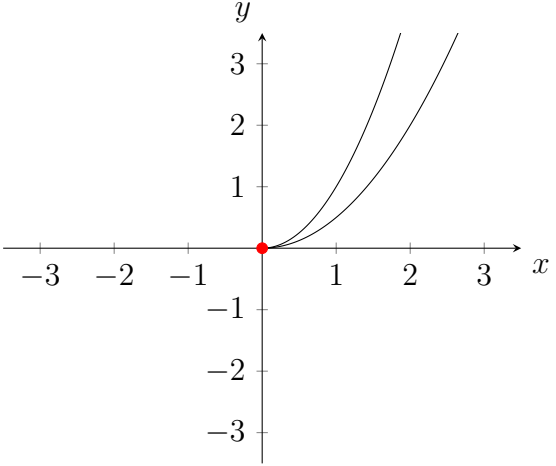
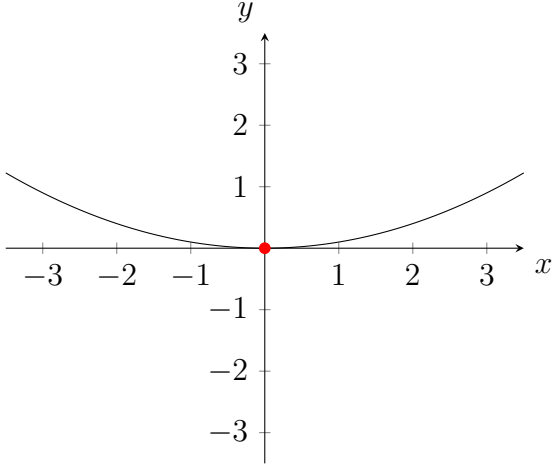
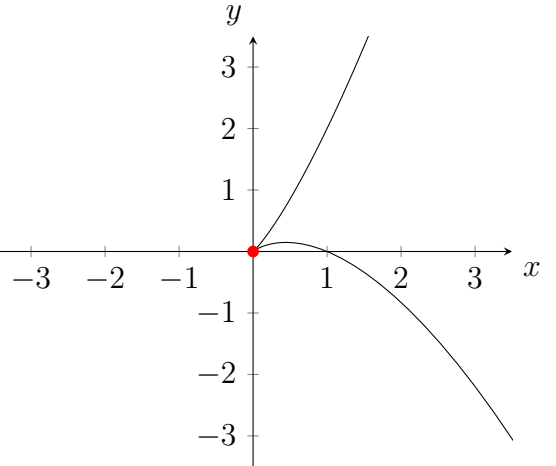
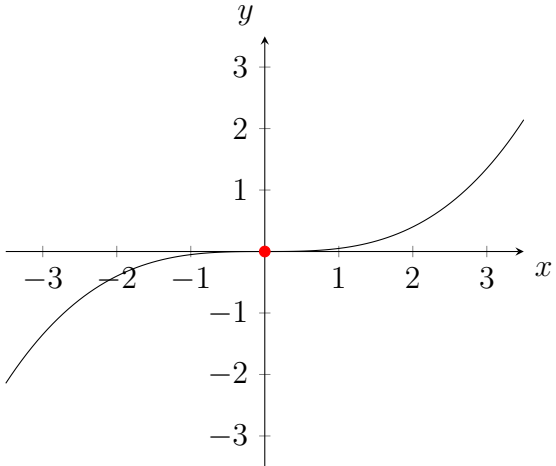
$\begin{matrix} x \\ y \end{matrix}$	Even	Odd
Even	None	y -axis
Odd	x -axis	Center O

Table 1. Axis of symmetry of $\vec{\mathbf{F}}(t)$ depending on the nature of x and y .

The first non zero vector in the set $\{\vec{\mathbf{F}}'(t), \vec{\mathbf{F}}''(t), \vec{\mathbf{F}}'''(t), \dots, \vec{\mathbf{F}}^{(k)}(t)\}$ is $\vec{\mathbf{F}}^{(p)}$ is used to define the tangent vector to the curve

$$\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{F}}^{(p)}(t)}{\|\vec{\mathbf{F}}^{(p)}(t)\|}.$$

$$(T) : y = \frac{y^{(p)}(t)}{x^{(p)}(t)}(x - x(t)) + y(t).$$

$\begin{array}{c} p \\ \backslash \\ q \end{array}$	Even	Odd
Even		
Odd		

Remark

- $\vec{\mathbf{F}}'(t_0) = 0 \implies t = t_0$ is a stationary point (reflection point of 1/2 kind).
- $\vec{\mathbf{F}}'(t_0) \neq 0 \implies t = t_0$ is an inflection point or normal shape point.
- $\det(\vec{\mathbf{F}}'(t_0), \vec{\mathbf{F}}''(t_0)) = 0 \implies t = t_0$ is a reflection or inflection point (not biregular).

SECTION 7

Infinite Branches

- If $\lim_{t \rightarrow t_0} x(t) = \pm\infty$ and $\lim_{t \rightarrow t_0} y(t) = y_0$ then the line $y = y_0$ is a horizontal asymptote.

- If $\lim_{t \rightarrow t_0} x(t) = x_0$ and $\lim_{t \rightarrow t_0} y(t) = \pm\infty$ then the line $x = x_0$ is a vertical asymptote.
- If $\lim_{t \rightarrow t_0} x(t) = \pm\infty$ and $\lim_{t \rightarrow t_0} y(t) = \pm\infty$ then we study $\frac{y(t)}{x(t)}$
 - If $\lim_{t \rightarrow t_0} \frac{y(t)}{x(t)} = \pm\infty$ then the curve admits a parabolic directed by (Oy) .
 - If $\lim_{t \rightarrow t_0} \frac{y(t)}{x(t)} = 0$ then the curve admits a parabolic directed by (Ox) .
 - If $\lim_{t \rightarrow t_0} \frac{y(t)}{x(t)} = a \in \mathbb{R}^*$ then we study $y(t) - ax(t)$
 - * If $\lim_{t \rightarrow t_0} y(t) - ax(t) = b \in \mathbb{R}$ then the curve admits an oblique asymptote $y = ax + b$
 - * If $\lim_{t \rightarrow t_0} y(t) - ax(t) = \pm\infty$ then the curve admits an asymptotic direction $y = ax$