Analysis 3

Contents

PART I

Sequence of Functions

SECTION 1

Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 1 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I \, \forall \epsilon > 0 \, \exists n \in \mathbb{N} \, \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let $x = 0$ then find $\lim\limits_{n\to\infty} f_n(0) =$ some $f(x)$
2. Then let $x \neq 0$ and again find $\lim\limits_{n\to\infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0, 1] \end{cases}.$$

SECTION 2

Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 2 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \forall x \in I : \sup_{x\in I} |f_n(x) - f(x)| < \epsilon.$$

Remark We can also prove uniform convergence by proving

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

There is also an easy qay to prove uniform convergence of a function by

1. Prove that the sequence of fucntions $f_n(x)$ is pointwise convergent to a function $f(x)$ [1]
2. Define a function $g(x) = |f_n(x) - f(x)|$ and find the maxima of that function at a point x_0 (usually by doing $\mathrm{d}g/\mathrm{d}x = 0$)
3. If $\lim_{n\to\infty} g(x_0) = 0$ then the sequence converges uniformly to $f(x)$

[1] *if the function $f(x)$ is continuous at a point piecewise then the sequence doesn't uniformly converge*

PART

II

Series of Functions

Definition 3 Let $f_n(x)$ be sequence of functions defined on $I \subset \mathbb{R}$, we define the series $S(x)$ to be
$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

Section 3

Convergence of a Numerical Series

In order to prove a series of functions converge we have to prove that it converges for all fixed x.

Theorem 1 Suppose there exists a sequence a_n such that $\forall x, n \ |f_n| \leq a_n$. The Weierstrass test states that if $\sum a_n$ converges then $\sum f_n(x)$ converges uniformly and absolutely

Theorem 2 Let a_n be a sequence of numbers, if $\left|\dfrac{a_{n+1}}{a_n}\right| = l$ then the sequence is a geometric Series
$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

Theorem 3 A harmonic series is defined to be $a_n = \dfrac{1}{n^p}$
$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

Theorem 4 Let a_n be a sequence of numbers. The 2 series $\sum\limits_{n=0}^{\infty} a_n$ and $\sum\limits_{n=0}^{\infty} 2^n a_n$ are simultaneously convergent/divergent.

Theorem 5 The sequence $\sum\limits_{n=0}^{\infty} (-1)^n a_n$ is convergent if a_n is decreasing and $\lim\limits_{n \to \infty} a_n = 0$.

Theorem 6 Consider the series $S = \sum_{n=0}^{\infty} a_n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases}.$$

Section 4

Finite Expansion

The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!} f'(x-a) + \frac{x^2}{2!} f''(x-a) + \cdots + \frac{x^n}{n!} f^{(n)}(x-a) + x^n o(1) \quad x \to a.$$

Some important expansions to keep in mind are

a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

b) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

c) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

d) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

e) $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

f) $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

g) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

h) $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

i) $(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n+1} (\alpha - k)}{n!} x^n$

PART

III

Power Series

A power series is just a series in the following formula

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x - x_0)^n \\ &= \sum_{n=0}^{\infty} U_n. \end{aligned}$$

SECTION 5

Radius of Convergence

For some values of x a power series can either diverge or converge, to determine the interval of convergence we employ the ratio test

Theorem 7 Let r be the radius of convergence and I be the domain of convergence. If we compute the limit

$$\Gamma = \lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right|.$$

The ratio test states that

$$\begin{cases} \text{if} \quad \Gamma = 0 \quad \text{then} \quad r = \infty \quad \text{and} \quad I = \mathbb{R} \\ \text{if} \quad \Gamma = \infty \quad \text{then} \quad r = 0 \quad \text{and} \quad I = \{0\} \end{cases}.$$

In the case that Γ isn't 0 or ∞ we set $\Gamma < 1$ and then we find $|x| < R$, finally we can say that $r = R$ and $I =]-R, R[$. A special case need to be done for the points $-R$ and R to determine if they belong in I.

Remark The power series $f(x)$ is continous and will always uniformly converge in the interval of convergence I

Theorem 8 If $\sum a_n x^n$ and $\sum b_n x^n$ be 2 power series with radii R_1 and R_2. For the power series $\sum (a_n + b_n) x^n$ the radius of convergence R

$$R = \min\{R_1, R_2\}.$$

Theorem 9 If $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is a power series with radius R then $S'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^n$ as well as $\int_{x_0}^{x} S(t)\, dt$ both a radius of R

The general term a_k of a power series $S(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is equal to

$$a_k = \frac{S^{(k)}(x_0)}{k!}.$$

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} \\
y'' &= 2a_2 + 6a_3 x + \cdots + n(n-1) a_n x^{n-2} + n(n+1) a_{n+1} x^{n-1}
\end{aligned}$$

PART IV

Integrals Depending on a Parameter

We define an integratable function

$$\begin{aligned} f : I \times U &\longrightarrow \mathbb{R} \times \mathbb{R} \\ t, x &\longmapsto f(t, x). \end{aligned}$$

where t is a parameter.
Let

$$F(x) = \int_a^b f(t, x) \, \mathrm{d}t \,.$$

Theorem 10 If f is of class C^p then F is also of class C^n and

$$\frac{\partial^p F}{\partial x^p} = \int_a^b \frac{\partial^p f}{\partial x^p} \, \mathrm{d}t \,.$$

Remark F is differentiable if $\dfrac{\partial f}{\partial x}$ is convergent.

Section 6

Improper Integral

Theorem 11 If $\int_a^b |f(t)| \, \mathrm{d}t$ is convergent then $\int_a^b f(t) \, \mathrm{d}t$ is also convergent and $\left| \int_a^b f(t) \, \mathrm{d}t \right| \leq \int_a^b |f(t)| \, \mathrm{d}t$

Theorem 12 The Weierstrass test states that $\int_a^b f(t, x) \, \mathrm{d}t$ is convergent if there exists a function $g(t)$ such that $|f(t, x)| \leq g(t)$ and $\int_a^b g(t) \, \mathrm{d}t$

PART

V

Fourier Series

Definition 4 A Fourier series is a series of functions of general term

$$u_n(x) = a_0 + a_n \cos(nx) + b_n \sin(nx).$$

Section 7

Trigonometric Coefficients

The Fourier coefficients of a function defined on an interval $F \subset \mathbb{R}$ of period $T = \frac{2\pi}{\omega} \implies \omega = \frac{2\pi}{T}$ are

$$a_0 = \frac{1}{T}\int_F f(x)\ \mathrm{d}x$$
$$a_n = \frac{2}{T}\int_F f(x)\cos \omega n x\ \mathrm{d}x$$
$$b_n = \frac{2}{T}\int_F f(x)\sin \omega n x\ \mathrm{d}x$$

1. $\sin n\,\pi = 0$
2. $\sin n\,\pi/2 = (-1)^n$
3. $\cos n\,\pi = (-1)^n$
4. $\cos n\,\pi/2 = 0$

Subsection 7.1

Even Functions

If a function has a domain $F = [-\ell;\ell]$ and $f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients $(T = |F|)$ become

$$a_0 = \frac{1}{\ell}\int_0^\ell f(x)\ \mathrm{d}x$$
$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\cos nx\ \mathrm{d}x$$
$$b_n = 0$$

Subsection 7.2

Odd Functions

If a function has a domain $F = [-\ell;\ell]$ and $-f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients $(T = |F|)$ become

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{2}{\ell}\int_0^\ell f(x)\sin nx\ \mathrm{d}x$$

PART VI

Laplace Transforms

The Laplace transform of a function is defined as

$$F(p) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-pt} f(t)\,\mathrm{d}t\,.$$

It only exists if the integral above converges.

Section 8

Transforms of some functions

Subsection 8.1

Unit step function

Also known as Heaviside's unit step function, it is defined as

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}.$$

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{p} \quad \text{for} \quad \mathrm{Re}(p) > 0.$$

Subsection 8.2

Dirac Delta Function

The Dirac Delta function

$$\delta(t) = \begin{cases} \infty & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}.$$

$$\mathcal{L}\left\{\delta(t)\right\} = 1.$$

Subsection 8.3

Usual Elementary functions

a) $\mathcal{L}\left\{1\right\} = \dfrac{1}{p}$

b) $\mathcal{L}\left\{t\right\} = \dfrac{1}{p^2}$

c) $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{p^{n+1}}$

d) $\mathcal{L}\left\{\sin \omega t\right\} = \dfrac{\omega}{p^2 + \omega^2}$

e) $\mathcal{L}\{\cos\omega t\} = \dfrac{p}{p^2+\omega^2}$ f) $\mathcal{L}\{\sinh\omega t\} = \dfrac{\omega}{p^2-\omega^2}$

g) $\mathcal{L}\{\cosh\omega t\} = \dfrac{p}{p^2-\omega^2}$ h) $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{p-a}$

Section 9

Properties of the Transform

1. Linearity:
$$\mathcal{L}\{\lambda f + \mu g\} = \lambda\mathcal{L}\{f\} + \mu\mathcal{L}\{g\}.$$
2. Homothety:
$$\mathcal{L}\{f(kt)\} = \frac{1}{k}F(\frac{p}{k}).$$
3. Derivation:
$$\begin{aligned}\mathcal{L}\{f'(t)\} &= p\mathcal{L}\{f(t)\} - f(0^+)\\ \mathcal{L}\{f''(t)\} &= p^2\mathcal{L}\{f(t)\} - pf(0^+) - f'(0^+)\\ \mathcal{L}\left\{f^{(n)}(t)\right\} &= p^n\mathcal{L}\{f(t)\} - \sum_{k=1}^{n} p^{n-k}f^{(k-1)}(0^+)\end{aligned}$$
4. Integration:
$$\mathcal{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = \frac{F(p)}{p}.$$
5. Initial value theorem:
$$f(0^+) = \lim_{p\to\infty} p\mathcal{L}\{f(t)\}.$$
6. Final value theorem:
$$f(\infty) = \lim_{p\to 0} p\mathcal{L}\{f(t)\}.$$

Remark
$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}p}F(p)\\ \mathcal{L}\left\{t^2f(t)\right\} &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}F(p)\\ \mathcal{L}\{t^nf(t)\} &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)\end{aligned}$$

Remark Convolution over a domain $I \subset \mathbb{R}$ is defined as
$$f(t) * g(t) = \int_I f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_I f(t-\tau)g(\tau)\,\mathrm{d}\tau.$$

and it’s trasform is

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(p) \cdot G(p).$$

Section 10

Translation

In the time domain:

$$\mathcal{L}\left\{f(t-a)\right\} = e^{-ap} F(p).$$

In the p-domain:

$$\mathcal{L}\left\{e^{at} f(t)\right\} = F(p+a).$$

Definition of a system of DEs

PART VII

Suppose we have a vector of functions $\vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ we say that $\vec{\mathbf{x}}$ verifies a DE of order n if

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = \mathbf{f}(\vec{\mathbf{x}}, t).$$

ie

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = f_1(x_1, x_2, \ldots, x_n, t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = f_2(x_1, x_2, \ldots, x_n, t) \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = f_n(x_1, x_2, \ldots, x_n, t) \end{cases} .$$

Suppose that $\vec{\mathbf{x}}$ verifies the equation

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \cdots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}, t\right).$$

We can transform the above equation to a vector system by taking

$$\begin{aligned} x &= x_1 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= x_2 \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= x_3 \\ &\vdots \\ \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} &= x_n \end{aligned}$$

Theorem 13 A mapping $\mathbf{g}$ satisfies the LipShitz condition if

$$\exists k > 0 / \forall \vec{\mathbf{u}}, \vec{\mathbf{v}} \in D \qquad \|\mathbf{g}(\vec{\mathbf{u}}) - \mathbf{g}(\vec{\mathbf{v}})\| \leq k \|\vec{\mathbf{u}} - \vec{\mathbf{v}}\| .$$

Theorem 14 A mapping $\mathbf{f}$ of the system

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = \mathbf{f}(\vec{\mathbf{x}}, t).$$

admits a unique solution given an initial condition if *it satisfies the Lipshitz condition.*