Analysis 3

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PART

I

Sequence of Functions

1 Introduction

In previous courses, we analysed the convergence of sequences of numbers (example: $U_n = \left\{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots\right\} = \sum_{n=1}^{\infty} \frac{1}{n^2}$) with a series of tests. In this course we will be analysing sequences of *functions* $f_n(x)$.
An example, is $f_n(x) = \frac{x}{x+n} = \{f_1, f_2, f_3, \dots\} = \left\{\frac{x}{x+1}, \frac{x}{x+2}, \frac{x}{x+3}, \dots\right\}$.

There are 2 ways these sequences can converge: pointwise and uniformly

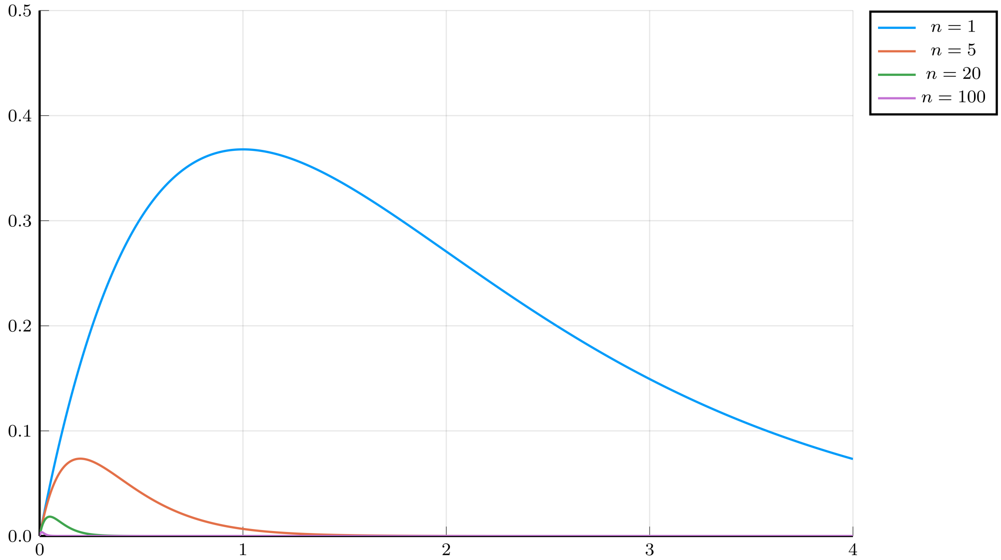

Figure 1. Plot of the sequence $f_n(x) = xe^{-nx}$

2 Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 2.1. We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I \; \forall \epsilon > 0 \; \exists n \in \mathbb{N} \; \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let $x = 0$ then find $\lim_{n\to\infty} f_n(0) = \text{some } f(x)$
2. Then let $x \neq 0$ and again find $\lim_{n\to\infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark. if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0,1] \end{cases}.$$

3 Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 3.1. We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \forall x \in I : \sup_{x\in I} |f_n(x) - f(x)| < \epsilon.$$

Remark. We can also prove uniform convergence by proving

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

There is also an easy qay to prove uniform convergence of a function by

1. Prove that the sequence of fucntions $f_n(x)$ is pointwise convergent to a function $f(x)$ [1]
2. Define a function $g(x) = |f_n(x) - f(x)|$ and find the maxima of that function at a point x_0 (usually by doing $\mathrm{d}g/\mathrm{d}x = 0$)
3. If $\lim_{n\to\infty} g(x_0) = 0$ then the sequence converges uniformly to $f(x)$

[1] *if the function $f(x)$ is continuous at a point piecewise the the sequence doesn't uniformly*

Series of Functions

PART

II

Definition 3.2. Let $f_n(x)$ be sequence of functions defined on $I \subset \mathbb{R}$, we define the series $S(x)$ to be

$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

4 Reminder: Convergence of a Series

In order to prove a series of functions converge we have to prove that it converges for all fixed x.

Theorem 4.1. Suppose there exists a sequence a_n such that $\forall x, n\ |f_n| \leq a_n$. The Weierstrass test states that if $\sum a_n$ converges then $\sum f_n(x)$ converges uniformly and absolutely

Theorem 4.2. Let a_n be a sequence of numbers, if $\left|\frac{a_{n+1}}{a_n}\right| = l$ then the sequence is a geometric Series

$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

Theorem 4.3. A harmonic series is defined to be $a_n = \frac{1}{n^p}$

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

Theorem 4.4. Let a_n be a sequence of numbers. The 2 series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_n$ are simultaneously convergent/divergent.

Theorem 4.5. The sequence $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent if a_n is decreasing and $\lim_{n \to \infty} a_n = 0$.

Theorem 4.6. Consider the series $S = \sum_{n=0}^{\infty} a_n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases}.$$

5 Finite Expansion

The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!}f'(x-a) + \frac{x^2}{2!}f''(x-a) + \cdots + \frac{x^n}{n!}f^{(n)}(x-a) + x^n o(1) \quad x \to a.$$

Some important expansions to keep in mind are

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$
$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$	$\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^n}{n}$	$(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n+1}(\alpha-k)}{n!} x^n$