

Statistics
Semester 4

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Chapter 1

Revision of Probability

I'm simply gonna list rules.

$$\mathbb{E}(X) = \mu = \sum_{i \in \Omega} X_i \Pr(X_i)$$

$$\mathbb{E}(g(X)) = \sum_{i \in \Omega} g(X_i) \Pr(X_i)$$

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$$

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y) \quad \text{if both variables are independent}$$

$$\text{Var}(X) = \sigma^2 = \mathbb{E}(X^2) - \mu^2$$

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{cov}(X, Y)$$

where

$$\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y).$$

1.1 Discrete Distributions

1. Uniform discrete law

$$X(\Omega) = \{1, 2, 3, \dots, n\}$$

$$\Pr(X = k) = \frac{1}{n} \quad \forall k = 1, 2, 3, \dots, n$$

$$\begin{cases} \mathbb{E}(X) = \frac{n+1}{2} \\ \text{Var}(X) = \frac{n^2-1}{12} \end{cases}$$

2. Bernoulli law of parameters p ($0 < p < 1$)

$$X \sim B(p)$$

$$X(\Omega) = \{0, 1\}$$

$$\Pr(X = 1) = p \quad \Pr(X = 0) = 1 - p$$

$$\begin{cases} \mathbb{E}(X) = p \\ \text{Var}(X) = p(1 - p) \end{cases}$$

3. Binomial law of parameters n and p

$$\begin{aligned} X &\sim \text{Bin}(n, p) \\ X(\Omega) &= \{1, 2, \dots, n\} \\ \Pr(X = k) &= C_n^k p^k q^{n-k} \quad \forall k \in \{0, 1, 2, \dots, n\} \\ \begin{cases} \mathbb{E}(X) = np \\ \text{Var}(X) = np(1-p) \end{cases} \end{aligned}$$

4. Hypergeometric law

$$\begin{aligned} X &\sim \mathcal{H}(N, n, p) \\ X(\Omega) &= [\max\{0, n - N + M\}, \min\{M, n\}] \\ \Pr(X = k) &= \frac{C_M^k \cdot C_{N-M}^{n-k}}{C_N^n} \quad \forall k \in X(\Omega) \\ \begin{cases} \mathbb{E}(X) = np \\ \text{Var}(X) = np(1-p) \left(\frac{N-n}{N-1}\right) \end{cases} \end{aligned}$$

5. Geometric law

$$\begin{aligned} X &\sim G(p) \\ X(\Omega) &= \mathbb{N}^* \\ \Pr(X = k) &= p(1-p)^{k-1} \quad \forall k \in \mathbb{N}^* \\ \begin{cases} \mathbb{E}(X) = \frac{1}{p} \\ \text{Var}(X) = \frac{1-p}{p^2} \end{cases} \end{aligned}$$

6. Poisson's law of parameter λ ($\lambda \in \mathbb{R}_+^*$)

$$\begin{aligned} X &\sim \mathcal{P}(\lambda) \\ X(\Omega) &= \mathbb{N} \\ \Pr(X = k) &= e^{-\lambda} \frac{\lambda^k}{k!} \quad \forall k \in \mathbb{N} \\ \begin{cases} \mathbb{E}(X) = \lambda \\ \text{Var}(X) = \lambda \end{cases} \end{aligned}$$

1.2 Continuous Distributions

1. Uniform law

$$\begin{aligned} f(x) &= \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{else} \end{cases} \\ \begin{cases} \mathbb{E}(x) = \frac{a+b}{2} \\ \text{Var}(x) = \frac{(b-a)^2}{12} \end{cases} \end{aligned}$$

2. Continuous law

$$\begin{aligned} x &\sim \xi(\lambda) \\ f(x) &= \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{else} \end{cases} \\ \begin{cases} \mathbb{E}(x) = \frac{1}{\lambda} \\ \text{Var}(x) = \frac{1}{\lambda^2} \end{cases} \end{aligned}$$

3. Normal law

$$x \sim \mathcal{N}(\mu, \sigma)$$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\begin{cases} \mathbb{E}(x) = \mu \\ \text{Var}(x) = \sigma^2 \end{cases}$$

For $\mathcal{N}(0, 1)$

$$\Phi(z) = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

$$\pi(z) = \Phi(z) - 0.5 = \int_0^z \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx.$$

1.3 Convergence

Theorem 1.3.1 Chebyshev's inequality

Let X be a random variable of expectation $\mathbb{E}(X)$ and variance $\text{Var}(X)$. Then $\forall \varepsilon$

$$\Pr(|X - \mathbb{E}(X)| \geq \varepsilon) \leq \frac{\text{Var}(X)}{\varepsilon^2}.$$

it can also be stated as

$$\Pr(|X - \mathbb{E}(X)| < \varepsilon) \geq 1 - \frac{\text{Var}(X)}{\varepsilon^2}.$$

We say a sequence of random variables X_n converges to a ($X_n \xrightarrow{\Pr} a$) if $\forall \varepsilon$

$$\lim_{n \rightarrow +\infty} \Pr(|X_n - a| > \varepsilon) = 0.$$

or

$$\lim_{n \rightarrow +\infty} \Pr(|X_n - a| \leq \varepsilon) = 1.$$

Theorem 1.3.2 Weak law of large numbers

Consider a random variable (X_n) of mean μ and variance σ^2 . Consider the random variable $\tilde{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$. It can be shown that \tilde{X}_n converges to μ meaning $\forall \varepsilon$

$$\lim_{n \rightarrow +\infty} \Pr(|\tilde{X}_n - \mu| > \varepsilon) = 0.$$

1.4 Approximations

Theorem 1.4.1 Binomial by a Poisson

$$\text{Bin}(n, p) \sim \mathcal{P}(np) \quad \text{if} \quad \begin{cases} n \geq 30 \\ p \leq 0.1 \\ np < 15 \end{cases}.$$

Theorem 1.4.2 Hypergeometric by a Binomial

$$\mathcal{H}(N, n, p) \sim \text{Bin}(n, p) \quad \text{if } n \leq 0.05N.$$

Theorem 1.4.3 De Moivre–Laplace theorem

$$\text{Bin}(n, p) \sim \mathcal{N}\left(np, \sqrt{np(1-p)}\right) \quad \text{if } \begin{cases} n \geq 30 \\ np \geq 5 \\ n(1-p) \geq 5 \end{cases}.$$

In this case the event $X = k$ can be replaced by $k - 0.5 < X < l + 0.5$

Theorem 1.4.4 Central limit theorem

Let (X_n) be a sequence of independent random variables following the same law of expectation μ and of standard deviation σ . Let $S_n = \sum_{i=1}^n X_i$ and $S_n^* = \frac{S_n - n\mu}{\sigma\sqrt{n}}$. It can be shown that S_n^* converges in law to $\mathcal{N}(0, 1)$.

$$\begin{aligned} \mathbb{E}(S_n) &= n\mu \\ \text{Var}(S_n) &= n\sigma^2 \end{aligned}$$

1.5 Further laws

Theorem 1.5.1 Chi square law

Let X_1, X_2, \dots, X_n be n independent random variables following the standard normal law $\mathcal{N}(0, 1)$. Let $Y = X_1^2 + X_2^2 + \dots + X_n^2$. We say that Y follows a chi-square law with n degrees of freedom. $Y \sim \chi_n^2$.

$$\begin{aligned} \mathbb{E}(Y) &= n \\ \text{Var}(Y) &= 2n \end{aligned}$$

It can be shown that the density function of Y is

$$f(x) = \begin{cases} \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} x^{\frac{n}{2}-1} e^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{else} \end{cases}.$$

where Γ is the gamma function

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt \quad \forall x > 0.$$

Theorem 1.5.2 Student law (t-distribution)

Let X, Z be two independent random variables such that $X \sim \mathcal{N}(0, 1)$ and $Z \sim \chi_n^2$. Hence the random variable

$$T = \frac{X}{\sqrt{\frac{Z}{n}}}.$$

is said to be following a student law. $T \sim \mathcal{T}_n$

$$f(t) = \frac{1}{\sqrt{n\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}.$$

Chapter 2

Estimators

Let θ be a certain characteristic of a population P of N individuals, for example letting θ be the expectation of a certain random variable X defined over the population. We take a sample of size $n < N$ of the population to estimate the value of θ .

Let Y_n be a function of the random variables X_1, X_2, \dots, X_n . Y_n is called an estimator of θ if

$$\lim_{n \rightarrow +\infty} \mathbb{E}(Y_n) = \theta.$$

a consistent estimator if

$$\lim_{n \rightarrow +\infty} \text{Var}(Y_n) = 0.$$

and an unbiased estimator if

$$\mathbb{E}(Y_n) = \theta \quad \forall n \in \mathbb{N}^*.$$

the value y_n of Y_n computed from any observed sample is called point estimation of θ

2.1 Point estimation of the mean

Let X be a random variable defined over the population P of the expected value μ and standard deviation σ . Consider a sample (X_1, X_2, \dots, X_n) of size n , randomly selected from P such that X_i are independent and follow the same law.

Consider the statistic $\bar{X}_n = \frac{X_1 + X_2 + \dots + X_n}{n}$, it is a random variable whose distribution is called the sample distribution of the mean.

$$\begin{aligned} \mathbb{E}(\bar{X}_n) &= \mu \\ \text{Var}(\bar{X}_n) &= \frac{\sigma^2}{n} \end{aligned}$$

Since $\text{Var}(\bar{X}_n) \xrightarrow{n \rightarrow +\infty} 0$ then \bar{X}_n is a consistent unbiased estimator of the mean μ .

Note:-

The standard deviation of \bar{X}_n is called standard error of the mean

$$\sigma(\bar{X}_n) = \frac{\sigma}{\sqrt{n}}.$$

Due to the central limit theorem, as the sample size gets larger and larger \bar{X}_n approaches a normal distribution $\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$.

2.2 Point estimator of the variance

2.2.1 Suppose μ is unknown

Consider the random variable S^2 (estimator of σ^2)

$$S^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

The expectation of S^2 can be proved to be

$$\mathbb{E}(S^2) = \frac{n-1}{n} \sigma^2.$$

Since $\mathbb{E}(S^2) \xrightarrow{n \rightarrow +\infty} \sigma^2$ then S^2 is a biased estimator of σ^2 .

Consider the random variable S'^2

$$S'^2 = \frac{n}{n-1} S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

Since $\mathbb{E}(S'^2) = \sigma^2$ then S'^2 is an unbiased estimator of σ^2 .

Hence σ can be estimated by

$$S' = \sqrt{\frac{n}{n-1}} S.$$

and

$$\sigma(\bar{X}_n) = \frac{S}{\sqrt{n-1}}.$$

where

σ^2 variance of the population.

S^2 variance of the sample.

$\sigma^2(\bar{X}_n)$ variance of the distribution of the sample mean.

S'^2 corrected variance of the sample.

Note:-

It is better to estimate σ^2 using S'^2 than S^2 since S^2 is a biased estimator. However, if n (sample size) is big enough ($\frac{n}{n-1} \approx 1$), then σ^2 can be estimated by S^2

2.2.2 Suppose μ is known

Consider the random variable Z^2 (not the variance of the sample)

$$Z^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2.$$

Since $\mathbb{E}(Z^2) = \sigma^2$ then Z^2 is an unbiased estimator of σ^2 thus the value $z^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2$ is a point estimation of the variance σ^2 of the population.

Note:-

If $n > 0.05N$ and if the sample is selected without replacement then the value of the variance changes to become

$$\text{Var}(\bar{X}_n) = \left(\frac{N-n}{N-1} \right) \frac{\sigma^2}{n}.$$

and the standard error becomes

$$\sigma(\bar{X}_n) = \frac{\sigma}{\sqrt{n}} \sqrt{\frac{N-n}{N-1}}.$$

If the variance of the population is not known then we can use S^2 or Z^2 to estimate $\text{Var}(\bar{X}_n)$

$$\text{Var}(\bar{X}_n) = \left(\frac{N-n}{N-1} \right) \frac{S^2}{n-1}.$$

and the standard error with

$$\sigma(\bar{X}_n) = \frac{S}{\sqrt{n-1}} \sqrt{\frac{N-n}{N-1}}.$$

2.3 Point estimation of a proportion (percentage)

Consider a population P of individuals with a proportion p if individuals having a certain characteristic θ . Let (a_1, a_2, \dots, a_n) be a sample randomly selected P . We define for each individual a_i the Bernoulli random variable X_i as follows

$$\begin{cases} X_i = 1 & \text{if } a_i \text{ has the characteristic } \theta \text{ with probability } p \\ X_i = 0 & \text{else} \end{cases}.$$

Let $Y_n = \frac{X_1 + X_2 + \dots + X_n}{n} = \frac{1}{n} \sum_{i=1}^n X_i$. Y_n is the random variable giving the proportion of individuals of the sample that have the characteristic θ .

$$\Pr(X_i = 1) = \frac{\text{number of individuals of the population having } \theta}{\text{total number of individuals}} = p$$

$$\Pr(X_i = 0) = 1 - p$$

Thus $X_1 + X_2 + \dots + X_n \sim \text{Bin}(n, p)$

$$\mathbb{E}(X_1 + X_2 + \dots + X_n) = np$$

$$\text{Var}(X_1 + X_2 + \dots + X_n) = np(1 - p)$$

$$\mathbb{E}(Y_n) = p$$

$$\text{Var}(Y_n) = \frac{p(1 - p)}{n}$$

Hence Y_n is a consistent unbiased estimator of p . Therefore any observed value y_n of Y_n is a point estimator of P , meaning p is estimated by the proportion of the sample.

2.4 Confidence interval

2.4.1 Confidence interval for the mean

1. **Suppose that $n \geq 30$, the population is normally distributed, and σ is known**

Let X be a random variable defined over a population P of mean $\mathbb{E}(X) = \mu$ and of variance $\text{Var}(X) = \sigma^2$.

Here we consider that $\bar{X}_n \sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$. Hence $\frac{\bar{X}_n - \mu}{\sigma_{\bar{X}_n}} \sim \mathcal{N}(0, 1)$.

Given the probability γ (level of confidence), we can find t such that

$$\Pr\left(-t \leq \frac{\bar{X}_n - \mu}{\sigma_{\bar{X}_n}} \leq t\right) = \gamma$$

$$\Pr(\bar{X}_n - t\sigma_{\bar{X}_n} \leq \mu \leq \bar{X}_n + t\sigma_{\bar{X}_n}) = \gamma$$

where $\pi(t) = \frac{\gamma}{2}$. Knowing γ we get t . Therefore a $\gamma\%$ confidence interval for the mean μ is given by

$$\text{IC}_\gamma(\mu) = [\bar{x}_n - t\sigma_{\bar{X}_n}, \bar{x}_n + t\sigma_{\bar{X}_n}].$$

where

$$\sigma_{\bar{X}_n} = \begin{cases} \frac{\sigma}{\sqrt{n}} & \text{if } \sigma \text{ is known} \\ \frac{S}{\sqrt{n-1}} & \text{if } \sigma \text{ is unknown (estimated by } S' = \sqrt{\frac{n}{n-1}}S) \end{cases}.$$

2. Suppose that $n < 30$, the population is normally distributed, and σ is unknown:

Using the table of student distributed knowing γ , we determine t such that

$$\Pr\left(\bar{X}_n - t\frac{S}{\sqrt{n-1}} \leq \mu \leq \bar{X}_n + t\frac{S}{\sqrt{n-1}}\right) = \gamma.$$

hence the confidence interval for the mean μ is

$$\text{IC}_\gamma(\mu) = \left[\bar{X}_n - t\frac{S}{\sqrt{n-1}}, \bar{X}_n + t\frac{S}{\sqrt{n-1}}\right].$$

Theorem 2.4.1

- (a) \bar{X}_n and S^2 are two independent random variance.
- (b) The random variable $n\frac{S^2}{\sigma^2}$ follows a chi-square law with $n - 1$ degrees of freedom.

Theorem 2.4.2

The random variable

$$\tilde{T} = \frac{\bar{X}_n - \mu}{\frac{S'}{\sqrt{n}}} = \frac{\bar{X}_n - \mu}{\frac{S}{\sqrt{n-1}}}.$$

follows a student law (t-distribution) with $n - 1$ degrees of freedom

3. Suppose that $n < 30$, the population is not normally distributed:

In this case we cannot use the normal distributed nor the student distribution. However we can use Chebyshev's inequality.

$$\Pr(|\bar{X}_n - \mu| \leq \varepsilon) \geq 1 - \frac{\sigma_{\bar{X}_n}^2}{\varepsilon^2}.$$

Take $\varepsilon = t\sigma_{\bar{X}_n}$

$$\Pr(\bar{X}_n - t\sigma_{\bar{X}_n} \leq \mu \leq \bar{X}_n + t\sigma_{\bar{X}_n}) \geq 1 - \frac{1}{t^2}.$$

Then we set $1 - \frac{1}{t^2}$ equal to γ solve for t and find the interval as follows

$$\text{IC}_\gamma = [\bar{x}_n - t\sigma_{\bar{X}_n}, \bar{x}_n + t\sigma_{\bar{X}_n}].$$

Note:-

- if σ is known then $\sigma_{\bar{X}_n} = \frac{\sigma}{\sqrt{n}}$
- if σ is unknown then we replace $\sigma_{\bar{X}_n}$ by its point estimator $\frac{S'}{\sqrt{n}} = \frac{S}{\sqrt{n-1}}$

2.4.2 Confidence interval for a proportion (percentage)

same setup as last time. If we assume this time that $\text{Bin}(n, p) \approx \mathcal{N}(np, \sqrt{np(1-p)})$ if $(n \geq 30, np, n(1-p) \geq 5)$ then we can say $Y_n \sim \mathcal{N}\left(p, \sqrt{\frac{p(1-p)}{n}}\right)$. Knowing γ we can determine t such that

$$\Pr\left(-t \leq \frac{Y_n - p}{\sigma_{Y_n}} \leq t\right) = \gamma.$$

The confidence interval becomes

$$[y_n - t\sigma_{Y_n}, y_n + t\sigma_{Y_n}].$$

where $\sigma_{Y_n} = \sqrt{\frac{p(1-p)}{n}}$ estimated by

$$\sqrt{\frac{n}{n-1}} \sqrt{\frac{y_n(1-y_n)}{n}} = \sqrt{\frac{y_n(1-y_n)}{n-1}}.$$

Therefore the confidence interval becomes

$$\text{IC}_\gamma(p) = \left[y_n - t \sqrt{\frac{y_n(1-y_n)}{n-1}}, y_n + t \sqrt{\frac{y_n(1-y_n)}{n-1}} \right].$$

Note:-

If $n \geq 100$ then $\frac{n}{n-1} \approx 1$, then the confidence interval s

$$[y_n - t \sqrt{\frac{y_n(1-y_n)}{n}}, y_n + t \sqrt{\frac{y_n(1-y_n)}{n}}].$$

Note:-

If the sample is selected without replace and if $n > 0.05N$ then we shall put a correcting factor $\frac{N-n}{N-1}$ to $\sigma_{Y_n} = \sqrt{\frac{p(1-p)}{n}}$, thus the confidence interval for proportion p becomes

$$\left[y_n - t \sqrt{\frac{N-n}{N-1}} \sqrt{\frac{y_n(1-y_n)}{n-1}}, y_n + t \sqrt{\frac{N-n}{N-1}} \sqrt{\frac{y_n(1-y_n)}{n-1}} \right].$$

2.4.3 Confidence interval for the variance

Assume $X \sim \mathcal{N}(\mu, \sigma)$ and X_1, X_2, \dots, X_n n independent random variables and identically distributed as X . We set the variables

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$S^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

$$S'^2 = \frac{n}{n-1} S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

$$Z^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2$$

we have

$$\begin{aligned}\bar{X}_n &\sim \mathcal{N}\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \\ n \frac{S^2}{\sigma^2} &\sim \chi_{n-1}^2 \\ n \frac{Z^2}{\sigma^2} &\sim \chi_n^2\end{aligned}$$

1. Suppose μ is unknown

Since $n \frac{S^2}{\sigma^2} \sim \chi_{n-1}^2$, then we determine the values $v_{\alpha/2}$ and $v_{1-\alpha/2}$ from the chi-square table such that

$$\Pr\left(v_{\alpha/2} \leq \frac{nS^2}{\sigma^2} \leq v_{1-\alpha/2}\right) = \gamma = 1 - \alpha.$$

therefore a confidence interval of level γ (risk α) is given by

$$IC_{\gamma}(\sigma^2) = \left[\frac{nS^2}{v_{1-\alpha/2}}, \frac{nS^2}{v_{\alpha/2}} \right] = \left[\frac{(n-1)S'^2}{v_{1-\alpha/2}}, \frac{(n-1)S'^2}{v_{\alpha/2}} \right].$$

2. Suppose μ is known

From the chi-square table, we determine the values of the quantities $v_{\alpha/2}$ and $v_{1-\alpha/2}$ for the law χ_n^2 such that

$$\Pr\left(v_{\alpha/2} \leq \frac{nZ^2}{\sigma^2} \leq v_{1-\alpha/2}\right) = \gamma.$$

Therefore the confidence interval of level γ is given by

$$IC_{\gamma}(\sigma^2) = \left[\frac{nz^2}{v_{1-\alpha/2}}, \frac{nz^2}{v_{\alpha/2}} \right].$$