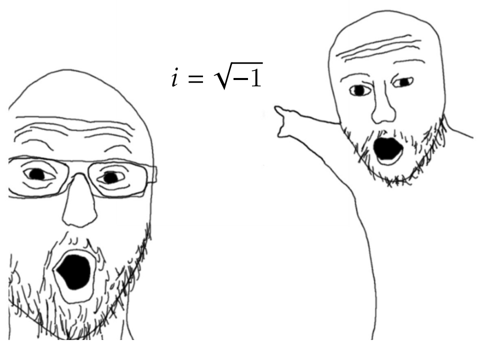
Complex Analysis
Semester 4

Contents

Chapter 1

The Complex Plane

1.1 Algebra of the complex plane



Euler's formulas for sin and cos

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\tan(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

The n-th roots of unity are the set of complex numbers $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ are the complex numbers that satisfy the equation

$$z^n = w.$$

where $w = Re^{i\alpha}$. The solutions equation are

$$\zeta_k = \sqrt[n]{R} e^{i\left(\frac{\alpha+2k\pi}{n}\right)}.$$

1.2 Topology of the complex plane

Theorem 1.2.1

The mapping

$$\begin{aligned} |z| : \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ z = x + yi &\longmapsto |x + yi| = \sqrt{x^2 + y^2}. \end{aligned}$$

defines a norm on $\mathbb{C}$, so the complex plane is a normed space.

Theorem 1.2.2

The mapping

$$\begin{aligned} d(.,.) : \mathbb{C} \times \mathbb{C} &\longrightarrow \mathbb{R}^+ \\ (z, w) &\longmapsto d(z, w) = |z - w|. \end{aligned}$$

defined a distance on $\mathbb{C}$, so the complex plane is a metric space.

Definition 1.2.1: Neighborhood

We call δ-neighborhood of z_0 an open disk centered at z_0 of radius δ

$$N_\delta(z_0) = \{z \in \mathbb{C} : \ |z - z_0| < \delta\}.$$

We call $N_\delta(z_0) - \{z_0\}$ a deleted δ-neighborhood. ($\{z \in \mathbb{C} : \ 0 < |z - z_0| < \delta\}$)

Definition 1.2.2

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

1. z_0 is called an *interior point* of Ω if

$$\exists \delta > 0, N_\delta(z_0) \subset \Omega.$$

2. z_0 is an *exterior point* of Ω if

$$\exists \delta > 0, N_\delta(z_0) \cap \Omega = \emptyset.$$

3. z_0 is a *boundary point* of Ω if

$$\forall \delta > 0, \ N_\delta(z_0) \cap \Omega \neq \emptyset \quad \text{and} \quad N_\delta(z_0) \cap \underbrace{C_{\mathbb{C}}^{\Omega}}_{\mathbb{C}-\Omega} \neq \emptyset.$$

Definition 1.2.3

The set of all:

1. interior points: $\dot{\Omega}$
2. boundary points: $\partial\Omega$
3. the set $\Omega \cup \partial\Omega$ is called a closure of Ω denoted $\bar{\Omega}$

Note:-

$$\dot{\Omega} \subset \Omega \subset \bar{\Omega}.$$

and

$$\Omega \text{ is open} \Leftrightarrow \begin{cases} \Omega \cap \partial\Omega = \emptyset \\ \Omega = \dot{\Omega} \end{cases} .$$

If Ω is not open from all sides then it is not open. Same thing with closed.
If Ω is not open then it is not connected.

Ω is said to be *compact* if it is both *bounded and closed*.

Theorem 1.2.3 Bolzano-Weirstrass theorem
Every *bounded infinte* set admits at least one limit point

Paths

A path is a set of complex points Γ where

$$\Gamma = \{z(t) = x(t) + i\, y(t)\ t \in [a, b[\} .$$

A simple path/Jordan arc if it does not cross itself

$$\forall t_1, t_2 \in [a, b[\ t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2).$$

A closed path is a path such that

$$z(a) = z(b).$$

Chapter 2

Complex Functions

2.1 Limits and Differentiability

Note:-

When taking limits we can do the 2D limit where $x = \text{Re}(z)$ and $y = \text{Im}(z)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x+iy).$$

then we can take multiple paths to find the limit. However we can't take suffecient paths to prove a limit exists as there could exist one path that causes the limit to not exist, however we can use polar limits to prove that the limit exists. We take $x = r\cos(\theta) - x_0$ and $y = r\sin(\theta) - y_0$

$$\lim_{(r,\theta)\to(0,0)} f\left(r\cos(\theta) - x_0 + i\left(r\sin(\theta) - y_0\right)\right).$$

Theorem 2.1.1 Cauchy-Riemann equations

We define a complex function

$$f(x+iy) = u(x,y) + iv(x,y).$$

If f is differentiable on a point $z_0 = x_0 + iy_0$ then u and v satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0)$$
$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0).$$

Note that the converse is not true

To prove that a function f is differentiable at z_0 then we have to prove that u and v

$$\begin{cases} \text{exist in } \Omega \\ \text{are continous at } (x_0,y_0) \\ \text{satisfy the Cauchy-Riemann equations at } (x_0,y_0) \end{cases}$$

2.1.1 Hyperbolic functions

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\tanh z = \frac{\sinh z}{\cosh z}$$

Properties

a) $\cosh^2 z - \sinh^2 z = 1$

b) $\cosh^2 z + \sinh^2 z = \cosh 2z$

c) $\cosh z_1 + z_2 = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$

d) $\sinh z_1 + z_2 = \sinh z_1 \cdot \cosh z_2 + \sinh z_2 \cdot \cosh z_1$

e) $\cos iz = \cosh z$

f) $\sin iz = i \sinh z$

g) $\cosh iz = \cos z$

h) $\sinh iz = i \sin z$

2.2 Harmonic functions

Definition 2.2.1: Harmonic function

A function $u(x, y)$, of class C^2 and defined on Ω, is said to be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or in other words the Laplacian is equal to 0

$$\Delta u = \nabla^2 u = 0.$$

Theorem 2.2.1

Let a function $f = u + iv$ defined on Ω

$$f \text{ is holomorphic } \Leftrightarrow \begin{cases} u, v \text{ are of class } C^\infty \text{ in } \Omega \\ u, v \text{ satisfy the Cauchy–Riemann equations in } \Omega \\ u, v \text{ are harmonic in } \Omega \end{cases}.$$