Electricity 2

Contents

Electric Current

PART I

Section 1

Current and Current Density

Electric charge in motion constitutes an electric current. In the steady flow of charge in a wire of cross-sectional area A, the total charge passing through this area in unit time is defined to be the electric current I at this place. If a total charge q flows through this area in time t, the current I is given by:

$$I = \frac{q}{t}. \tag{1.1}$$

where I is measured in Ampere. Sometimes it is also convenient to look at the density of the current we're observing over a certain area A called *current density*

$$J = \frac{I}{A}. \tag{1.2}$$

If the area in which the current is running through changes at points 1 and 2, the current between those 2 points remain constant ($I_1 = I_2$) while the current density changes to become

$$\frac{J_1}{J_2} = \frac{A_2}{A_1}.$$

If a positively charged cloud of n particles moves through a space with speed v and charge q, we find that[1]

$$\vec{\mathbf{J}} = nq\vec{\mathbf{v}} = \varrho_\tau \vec{\mathbf{v}}. \tag{1.3}$$

[1] q can be positive or negative

The total current through a *slice* of space mentioned prior it calculated using

$$I = \iint_A \vec{\mathbf{J}}\, \mathrm{d}A\,. \tag{1.4}$$

The volume element here is $\vec{\mathbf{v}}\,\mathrm{d}A\,\mathrm{d}t = \mathrm{d}\tau$

Section 2

Continuity Law

The continuity law for current is

$$\boldsymbol{\nabla} \cdot \left(\vec{\mathbf{J}} + \vec{\mathbf{J}}_c\right) = -\frac{\partial \varrho_\tau}{\partial t}. \tag{2.1}$$

Where
$\vec{\mathbf{J}}$ is the conduction current density.
$\vec{\mathbf{J}}_c = \varrho_\tau \vec{\mathbf{v}}_\tau$ is the convection current density.
$\vec{\mathbf{v}}_\tau$ is the velocity of the volume containing the particles.
There are 2 particular cases for the equation above

1. The volume is not moving ($\vec{\mathbf{v}}_\tau = 0 \implies \vec{\mathbf{J}}_c = 0$). The continuity law of static structures is

$$\boldsymbol{\nabla} \cdot \vec{\mathbf{J}} = -\frac{\partial \varrho_\tau}{\partial t}. \tag{2.2}$$

2. The volume is not moving and we are in a steady state, so the charge density of the particles doesn't change much as they are static so $\partial \varrho_\tau / \partial t = 0$ so

$$\oiint \vec{\mathbf{J}} \, \mathrm{d}A = 0. \tag{2.3}$$

Ohm's and Joule's Law

PART II

Section 3

Electrical Mobility and Conductivity

The velocity of a charge carrier in in an electric field $\vec{\mathbf{E}}$ is

$$\vec{\mathbf{v}} = \mu\vec{\mathbf{E}}. \tag{3.1}$$

where μ is the mobility of the charge carrier.

The current density vector in a conductor with conductivity σ is said to be

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}. \tag{3.2}$$

Ohm's law(doesn't really need an introduction)

$$U = RI. \tag{3.3}$$

where we can find R using

$$R = \frac{\rho l}{A}. \tag{3.4}$$

where A is the surface area of the resistor and l is the length of the conductor[2].

[2] $\sigma = \frac{1}{\rho}$

Section 4

Resistance

The symbol of a resistive conductor is represented by

$$P = UI \tag{4.1}$$
$$W = Pt \tag{4.2}$$

Power loss due to Joule's law

$$P = I^2R = \frac{U^2}{R} \tag{4.3}$$
$$W = I^2Rt \tag{4.4}$$

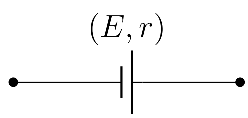
Section 5

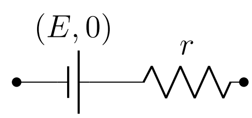
Electric Ciruits

Subsection 5.1

Power Sources/Generators

A DC source is characterized by their electromotive force E and internal resistance r





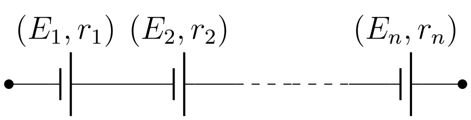
The voltage supplied by the source is

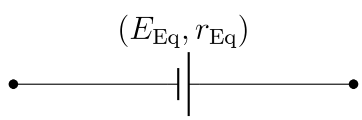
$$V_{\text{source}} = E - rI.$$

and it's efficiency is

$$\eta = 1 - \frac{rI}{E}.$$

5.1.1 In Series

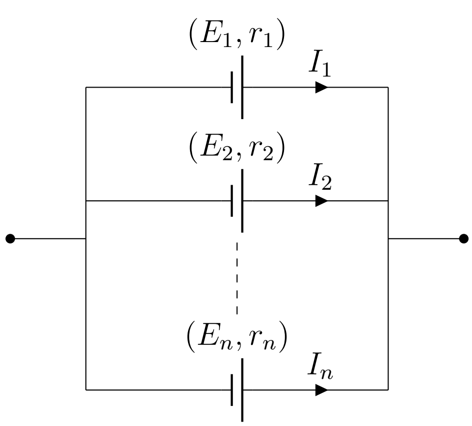


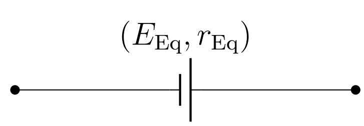


where

$$E_{\text{Eq}} = \sum_{i=1}^{n} E_i$$
$$r_{\text{Eq}} = \sum_{i=1}^{n} r_i$$

5.1.2 In Parallel





In case of *identical* sources:

$$I = \sum_{i=1}^{n} I_i$$
$$\frac{1}{r_{\text{Eq}}} = \sum_{i=1}^{n} \frac{1}{r_i}$$

and

$$V_{\text{source}} = E - r_{\text{Eq}} I.$$

Subsection 5.2

Loads

An electrical load is an electrical component or portion of a circuit that consumes electric power. Electric loads are represented by a counter electromotive force e and internal resistance r' (with the exception of resistors)

All formulas for generators are the same as loads with the exception of the efficiency

$$\eta = 1 - \frac{r' I}{U}.$$

PART III

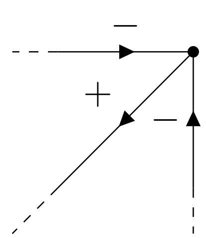
Circuits Analysis Techniques

Terminology:

Node A point where two or more circuit elements join.
Essential node A node where three or more circuit elements join.
Branch A path that connects two nodes
Loop A path whose last node is the same as the starting node
Mesh A loop that does not enclose any other loops

- **KVL**: the algebraic sum of all the voltages around any closed path in a circuit equals zero.
- **KCL**: the algebraic sum of all the currents at any node in a circuit equals zero.

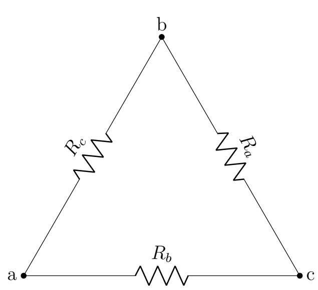
To use Kirchhoff's current law, an algebraic sign corresponding to a reference direction must be assigned to every current at the node. Assigning a positive sign to a current leaving a node requires assigning a negative sign to a current entering a node.

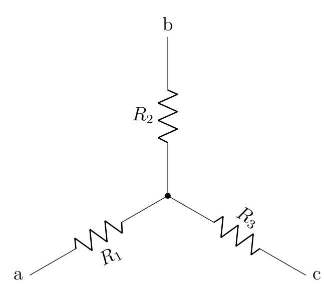


Section 6

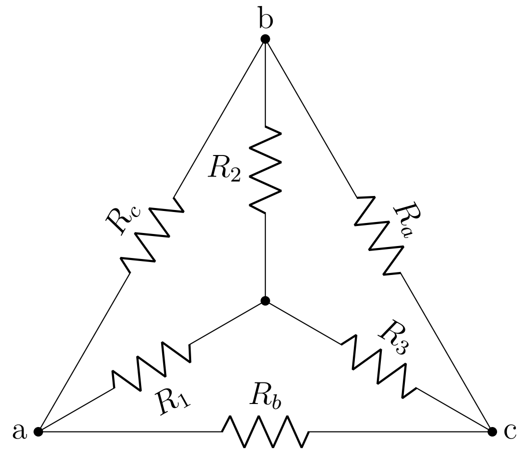
Wye-Delta and Delta-Wye Transformations

This transformation of a set of resistors configured in the shape of the letter Δ to a configuration of a shape of the letter Y.





To find the equivalent resistances we apply



$$R_1 = \frac{R_b R_c}{R_a + R_b + R_c} \qquad R_a = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_1}$$
$$R_2 = \frac{R_a R_c}{R_a + R_b + R_c} \qquad R_b = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_2}$$
$$R_3 = \frac{R_a R_b}{R_a + R_b + R_c} \qquad R_c = \frac{R_1R_2 + R_2R_3 + R_3R_1}{R_3}$$

Section 7

Divider Circuits

Subsection 7.1

Voltage Divider

At timesespecially in electronic circuitsdeveloping more than one voltage level from a single voltage supply is necessary. One way of doing this is by using a voltage-divider circuit.

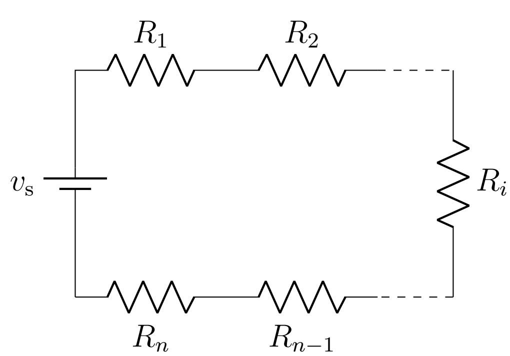


Figure 3. An n resistor voltage divider

$$v_i = v_s \frac{R_i}{\sum_{j=1}^{n} R_j}.$$

Subsection 7.2

Current Divider

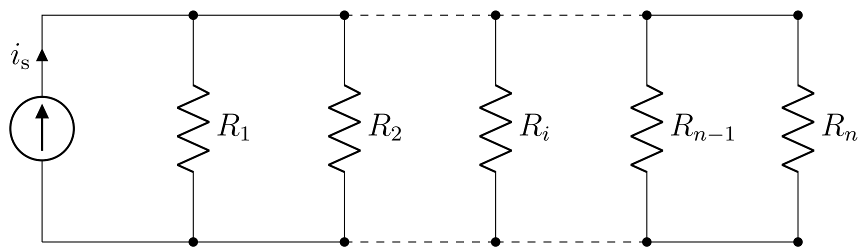


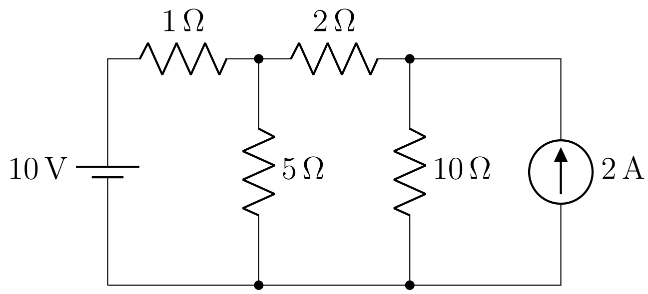
Figure 3. An n resistor current divider

$$i_i = i_s \frac{R_{Eq}}{R_i}.$$

Section 8

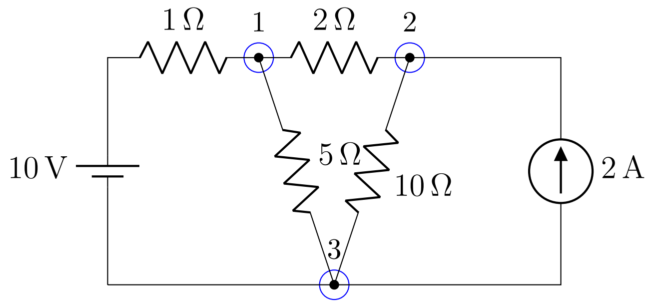
Node Voltage Method

We introduce the node-voltage method by using the essential nodes of the circuit.
To better understand node voltage method, we will apply directly on a circuit. Consider the following circuit:

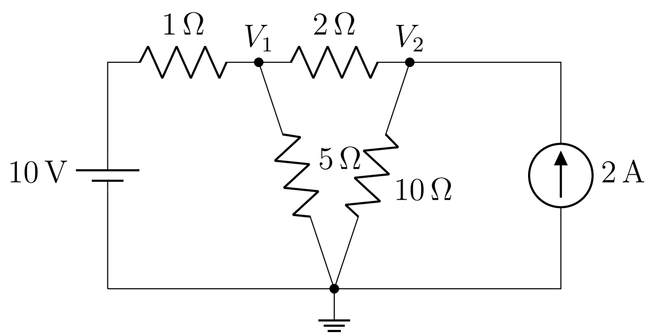


To find the voltages across the resistors are

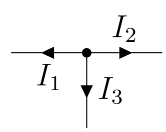
1. Assign the essential nodes (nodes with 3 or more branches):



2. Choose a reference node: select one of the essential nodes to be a reference. Usually we choose the lowest node. The assign voltages to the other nodes.

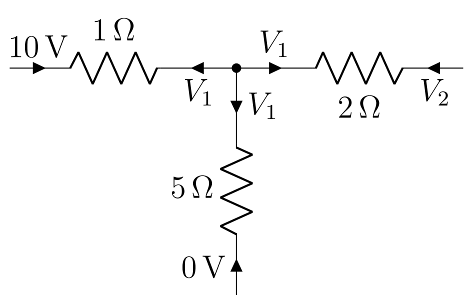


3. Apply KCL at each node to determine the voltage. For example let's look at node 1:
   Assume that there are currents leaving the node from all directions and that carry the voltage V_1 with them where by KCL



$$I_1 + I_2 + I_3 = 0.$$

   We let these currents carry our voltage V_1 with them, now we find the values of the currents. So the equation for node 1 is



$$\frac{V_1 - 10}{1} + \frac{V_2}{5} + \frac{V_1 - V_2}{2} = 0.$$

   Similarly for node 2

$$\frac{V_2 - V_1}{2} + \frac{V_2}{10} - 2 = 0.$$

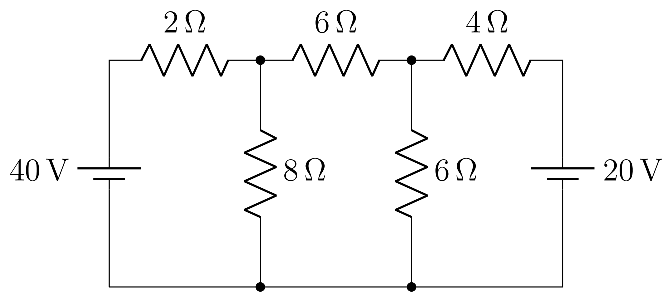
   By simply solving the equations we get

$$V_1 = 9.09\,\mathrm{V}$$
$$V_2 = 10.91\,\mathrm{V}$$

Section 9

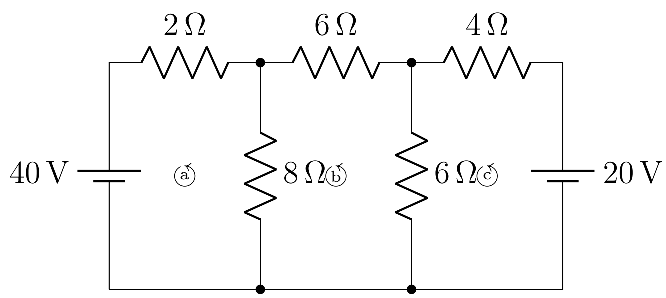
Mesh Current Method

We introduce the mesh current method by using the meshes of the circuit. To better understand mesh current method, we will apply directly on a circuit. Consider the following circuit:



Steps:

1. Assign the meshes (loop with no other loop inside): We have 3 meshes: a, b, and c
2. Define a current running in each mesh to be flowing in the counter clock wise direction.
3. Apply KVL in each mesh to find the currents.



- Mesh a: $-40 + 2I_a + 8(I_a - I_b) = 0$
- Mesh b: $8(I_b - I_a) + 6I_b + 6(I_b - I_c) = 0$
- Mesh c:$6(I_c - I_b) + 4I_c + 20 = 0$

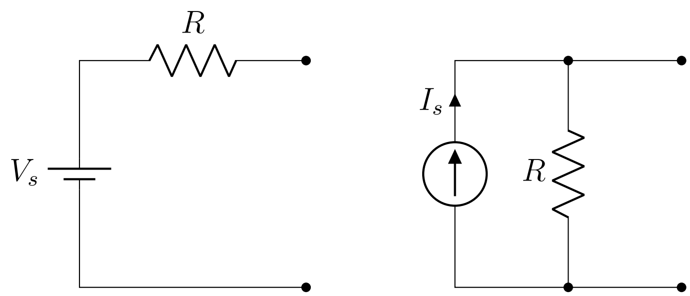
By solving the equations we get that

$$\begin{aligned} I_a &= 5.6\,\mathrm{A} \\ I_b &= 2\,\mathrm{A} \\ I_c &= -0.8\,\mathrm{A} \end{aligned}.$$

Section 10

Source Transformations

$$V_s = RI_s.$$



Section 11

Superposition

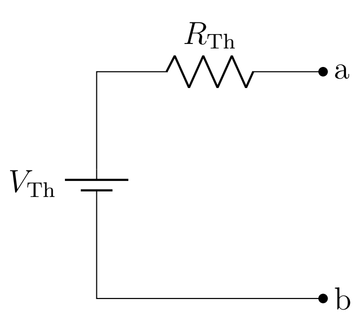
The principle of superposition allows us to find the currents within a circuit by analyzing 2 different circuits and adding the algebraic of the calculated currents.
The first circuit is obtained by *replacing volatge sources with a short circuit*, then finding $i_1, i_2, \ldots, i_n$.
The second circuit is obtained by *replacing current sources with an open circuit*, then finding $i'_1, i'_2, \ldots, i'_n$.

Then finally taking their sum.

Section 12

Thevenin and Norton Equivalents

Thevenin equivalent circuit is an independent voltage source V_{Th} in series with a resistor R_{Th}, which replaces an interconnection of sources and resistors
This series combination of V_{Th} and R_{Th} is equivalent to the original circuit in the sense that, if we connect the same load across the terminals a,b of each circuit, we get the same voltage and current at the terminals of the load.
A Thevenin equivalent circuit looks like



First, we note that if the load resistance is infinitely large, we have an open-circuit condition. The open-circuit voltage at the terminals a,b is

V_{Th}. By hypothesis, this must be the same as the open-circuit voltage at the terminals a,b in the original circuit. Therefore, to calculate the Thevenin voltage V_{Th},we simply calculate the open-circuit voltage in the original circuit.
Finding R_{Th}:
Method 1:
We replace the terminals of ab with short circuit and find the current across it. then

$$I_{\mathrm{sc}} = \frac{V_{\mathrm{Th}}}{I_{\mathrm{Th}}}.$$

Method 2:
First deactivate all sources[3] then find the resistance seen looking through ab.

[3] *replace volatge sources with short circuits and current ones with open circuits*

Section 13

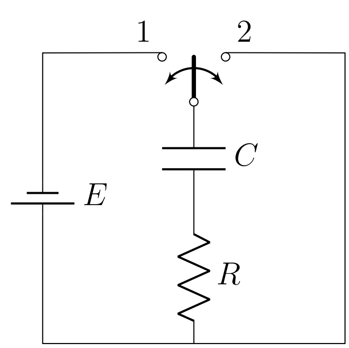
Capacitors

Remark

$$u_c = \frac{q}{C}.$$

$$i = C\frac{\mathrm{d}u_c}{\mathrm{d}t}.$$

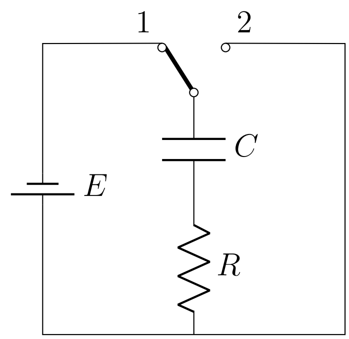
Consider the following circuit



Subsection 13.1

Charging

At $t = 0$,the switch is in position 1, the capacitor is fully discharged $q_{t_0} = 0$.



$$\begin{aligned} E &= v_R + v_c \\ &= Ri + \frac{q}{C} \\ \frac{E}{R} &= \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC} \end{aligned}$$

General solution to the DE

$$q = Ke^{-\frac{t}{RC}} + EC.$$

$$i = \frac{\mathrm{d}q}{\mathrm{d}t} = \frac{E}{R}e^{-\frac{t}{RC}}.$$

$\tau = RC$ is the time constant of the RC circuit.

At $t = 0$, $q = 0$, $i = \dfrac{E}{R}$, the capacitor acts as a short circuit.
At $t \to \infty$, $q = EC$, $i = 0$, the capacitor acts as a open circuit.

The energy dissipated by the resistor is

$$W = \int_0^\infty Ri^2 \,\mathrm{d}t = \cdots = \frac{1}{2}E^2C.$$

The energy stored in the capacitor is

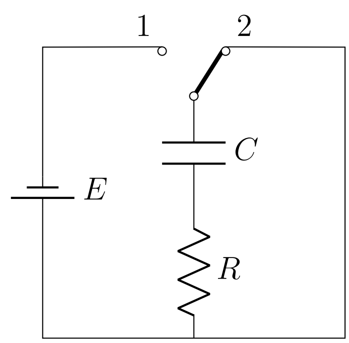
$$W = \frac{1}{2}E^2C.$$

The total energy delivered by the source

$$W_{\text{Total}} = E^2C.$$

Subsection 13.2

Discharging



$$\begin{aligned} v_c - v_R &= 0 \\ \frac{q}{C} - Ri &= 0 \\ \frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{RC} &= 0 \end{aligned}$$

The solution of the DE is

$$q = ECe^{-\frac{t}{RC}}.$$

$$i = -\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{E}{R}e^{-\frac{t}{RC}}.$$

For $t \to \infty: \ q = 0, \ i = 0.$ The capacitor is fully charged after $\approx 5\tau$.

PART

IV

Magnetostatics

SECTION 14

Biot-Savart Law

The Lorentz force law states that

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right).$$

and the magnitude is

$$\left\|\vec{\mathbf{F}}\right\| = |q| \, \|\vec{\mathbf{v}}\| \left\|\vec{\mathbf{B}}\right\| \sin\theta.$$

The magnetic field of a steady line current is given by the Biot-Savart law:

$$\vec{\mathbf{B}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_C \frac{I \, \mathrm{d}\vec{\ell} \times \hat{\mathbf{r}}}{\|\vec{\mathbf{r}}\|^2} = \frac{\mu_0}{4\pi}\int_C \frac{I \, \mathrm{d}\vec{\ell} \times \vec{\mathbf{r}}}{\|\vec{\mathbf{r}}\|^3}.$$

The integration is along the current path, in the direction of the flow; $\mathrm{d}\ell$ is an element of length along the wire, and $\vec{\mathbf{r}}$ is the vector from the source to the point $\mathbf{r}$. The constant μ_0 is called the permeability of free space

$$\mu_0 = 4\pi \times 10^{-7}.$$

The flux Φ across a surface S is

$$\Phi = \oiint_S \vec{\mathbf{B}} \, \mathrm{d}\vec{\mathbf{S}} = \iiint_V \boldsymbol{\nabla} \cdot \vec{\mathbf{B}} \, \mathrm{d}v = 0.$$

The induction vector derives from a vector potential

$$\vec{\mathbf{B}} = \boldsymbol{\nabla} \times \vec{\mathbf{A}} \quad \text{where} \quad \vec{\mathbf{A}} = \frac{\mu_0 I}{2\pi}\int \frac{1}{r} \, \mathrm{d}\vec{\ell}.$$

SECTION 15

Ampère's circuital law

Ampère's law is defined *in a closed loop* to be

$$\oint \vec{\mathbf{B}} \, \mathrm{d}\vec{\ell} = \mu_0 I_{\text{enc}}.$$

Recall Coulomb's law and Gauss's law from the previous semester and notice the pattern

$$\begin{cases} \text{Electrostatics:} & \text{Coulomb} \rightarrow \text{Gauss} \\ \text{Magnetostatics:} & \text{Biot-Savart} \rightarrow \text{Ampère} \end{cases}.$$