ANALYSIS 3

Contents

PART

I

Sequence of Functions

SECTION 1

Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 1 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I \; \forall \epsilon > 0 \; \exists n \in \mathbb{N} \; \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let $x = 0$ then find $\lim_{n\to\infty} f_n(0) =$ some $f(x)$
2. Then let $x \neq 0$ and again find $\lim_{n\to\infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0, 1] \end{cases}.$$

SECTION 2

Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 2 We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; \forall x \in I : \; \sup_{x\in I} |f_n(x) - f(x)| < \epsilon.$$

Remark We can also prove uniform convergence by proving

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

There is also an easy qay to prove uniform convergence of a function by

1. Prove that the sequence of fucntions $f_n(x)$ is pointwise convergent to a function $f(x)$ [1]
2. Define a function $g(x) = |f_n(x) - f(x)|$ and find the maxima of that function at a point x_0 (usually by doing $\mathrm{d}g/\mathrm{d}x = 0$)
3. If $\lim\limits_{n\to\infty} g(x_0) = 0$ then the sequence converges uniformly to $f(x)$

[1] *if the function $f(x)$ is continuous at a point piecewise then the sequence doesn't uniformly converge*

PART

II

Series of Functions

Definition 3 Let $f_n(x)$ be sequence of functions defined on $I \subset \mathbb{R}$, we define the series $S(x)$ to be

$$S(x) = \sum_{n=0}^{\infty} f_n(x).$$

SECTION 3

Convergence of a Numerical Series

In order to prove a series of functions converge we have to prove that it converges for all fixed x.

Theorem 1 Suppose there exists a sequence a_n such that $\forall x, n \ |f_n| \leq a_n$. The Weierstrass test states that if $\sum a_n$ converges then $\sum f_n(x)$ converges uniformly and absolutely

Theorem 2 Let a_n be a sequence of numbers, if $\left|\dfrac{a_{n+1}}{a_n}\right| = l$ then the sequence is a geometric Series

$$\sum_{n=0}^{\infty} a_n \begin{cases} \text{converges} & \text{if } |l| < 1 \\ \text{diverges} & \text{if } |l| \geq 1 \end{cases}.$$

Theorem 3 A harmonic series is defined to be $a_n = \dfrac{1}{n^p}$

$$\sum_{n=0}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}.$$

Theorem 4 Let a_n be a sequence of numbers. The 2 series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} 2^n a_n$ are simultaneously convergent/divergent.

Theorem 5 The sequence $\sum_{n=0}^{\infty} (-1)^n a_n$ is convergent if a_n is decreasing and $\lim_{n \to \infty} a_n = 0$.

Theorem 6 Consider the series $S = \sum_{n=0}^{\infty} a_n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \quad \text{such that} \quad \begin{cases} l < 1 & \text{if } S \text{ converges} \\ l > 1 & \text{if } S \text{ diverges} \\ l = 1 & \text{this test cannot help us} \end{cases}.$$

SECTION 4

Finite Expansion

The general formula for the finite expansion (Taylor-young formula) is

$$f(x) = f(x-a) + \frac{x}{1!} f'(x-a) + \frac{x^2}{2!} f''(x-a) + \cdots + \frac{x^n}{n!} f^{(n)}(x-a) + x^n o(1) \quad x \to a.$$

Some important expansions to keep in mind are

a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

b) $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

c) $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

d) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

e) $\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

f) $\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

g) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

h) $\ln(1-x) = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

i) $(1+x)^\alpha = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n+1} (\alpha - k)}{n!} x^n$

PART III

Power Series

A power series is just a series in the following formula

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$
$$= \sum_{n=0}^{\infty} U_n.$$

SECTION 5

Radius of Convergence

For some values of x a power series can either diverge or converge, to determine the interval of convergence we employ the ratio test

Theorem 7 Let r be the radius of convergence and I be the domain of convergence. If we compute the limit

$$\Gamma = \lim_{n\to\infty} \left| \frac{U_{n+1}}{U_n} \right|.$$

The ratio test states that

$$\begin{cases} \text{if} \quad \Gamma = 0 & \text{then} \quad r = \infty \quad \text{and} \quad I = \mathbb{R} \\ \text{if} \quad \Gamma = \infty & \text{then} \quad r = 0 \quad \text{and} \quad I = \{0\} \end{cases}.$$

In the case that Γ isn't 0 or ∞ we set $\Gamma < 1$ and then we find $|x| < R$, finally we can say that $r = R$ and $I =]-R, R[$. A special case need to be done for the points $-R$ and R to determine if they belong in I.

Remark The power series $f(x)$ is continous and will always uniformly converge in the interval of convergence I

Theorem 8 If $\sum a_n x^n$ and $\sum b_n x^n$ be 2 power series with radii R_1 and R_2. For the power series $\sum (a_n + b_n) x^n$ the radius of convergence R

$$R = \min\{R_1, R_2\}.$$

Theorem 9 If $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is a power series with radius R then $S'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^n$ as well as $\int_{x_0}^{x} S(t)\, dt$ both a radius of R

The general term a_k of a power series $S(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is equal to

$$a_k = \frac{S^{(k)}(x_0)}{k!}.$$

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\
y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} \\
y'' &= 2a_2 + 6a_3 x + \cdots + n(n-1) a_n x^{n-2} + n(n+1) a_{n+1} x^{n-1}
\end{aligned}$$

PART

IV

Integrals Depending on a Parameter

We define an integratable function

$$\begin{aligned} f : I \times U &\longrightarrow \mathbb{R} \times \mathbb{R} \\ t, x &\longmapsto f(t, x). \end{aligned}$$

where t is a parameter.
Let

$$F(x) = \int_a^b f(t, x)\,\mathrm{d}t\,.$$

Theorem 10 If f is of class C^p then F is also of class C^n and

$$\frac{\partial^p F}{\partial x^p} = \int_a^b \frac{\partial^p f}{\partial x^p}\,\mathrm{d}t\,.$$

Remark F is differentiable if $\dfrac{\partial f}{\partial x}$ is convergent.

Section 6

Improper Integral

Theorem 11 If $\int_a^b |f(t)|\,\mathrm{d}t$ is convergent then $\int_a^b f(t)\,\mathrm{d}t$ is also convergent and $\left|\int_a^b f(t)\,\mathrm{d}t\right| \leq \int_a^b |f(t)|\,\mathrm{d}t$

Theorem 12 The Weierstrass test states that $\int_a^b f(t, x)\,\mathrm{d}t$ is convergent if there exists a function $g(t)$ such that $|f(t, x)| \leq g(t)$ and $\int_a^b g(t)\,\mathrm{d}t$

PART

V

Fourier Series

Definition 4 A Fourier series is a series of functions of general term

$$u_n(x) = a_0 + a_n \cos(nx) + b_n \sin(nx).$$

SECTION 7

Trigonometric Coefficients

The Fourier coefficients of a function defined on an interval $F \subset \mathbb{R}$ of period $T = \dfrac{2\pi}{\omega} \implies \omega = \dfrac{2\pi}{T}$ are

1. $\sin n\,\pi = 0$
2. $\sin n\,\pi/2 = (-1)^n$
3. $\cos n\,\pi = (-1)^n$
4. $\cos n\,\pi/2 = 0$

$$\begin{aligned} a_0 &= \frac{1}{T}\int_F f(x)\ \mathrm{d}x \\ a_n &= \frac{2}{T}\int_F f(x)\cos\omega n x\ \mathrm{d}x \\ b_n &= \frac{2}{T}\int_F f(x)\sin\omega n x\ \mathrm{d}x \end{aligned}$$

SUBSECTION 7.1

Even Functions

If a function has a domain $F = [-\ell;\ell]$ and $f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients $(T = |F|)$ become

$$\begin{aligned} a_0 &= \frac{1}{\ell}\int_0^\ell f(x)\ \mathrm{d}x \\ a_n &= \frac{2}{\ell}\int_0^\ell f(x)\cos nx\ \mathrm{d}x \\ b_n &= 0 \end{aligned}$$

SUBSECTION 7.2

Odd Functions

If a function has a domain $F = [-\ell;\ell]$ and $-f(x) = f(-x)\ \forall x \in \mathbb{R}$ the Fourier coefficients $(T = |F|)$ become

$$\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{2}{\ell}\int_0^\ell f(x)\sin nx\ \mathrm{d}x \end{aligned}$$

PART

VI

Laplace Transforms

The Laplace transform of a function is defined as

$$F(p) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty e^{-pt} f(t)\,\mathrm{d}t\,.$$

It only exists if the integral above converges.

Section 8

Transforms of some functions

Subsection 8.1

Unit step function

Also known as Heaviside's unit step function, it is defined as

$$u(t) = \begin{cases} 1 & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}.$$

$$\mathcal{L}\left\{u(t)\right\} = \frac{1}{p} \quad \text{for} \quad \mathrm{Re}(p) > 0.$$

Subsection 8.2

Dirac Delta Function

The Dirac Delta function

$$\delta(t) = \begin{cases} \infty & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0 \end{cases}.$$

$$\mathcal{L}\left\{\delta(t)\right\} = 1.$$

Subsection 8.3

Usual Elementary functions

a) $\mathcal{L}\left\{1\right\} = \dfrac{1}{p}$

b) $\mathcal{L}\left\{t\right\} = \dfrac{1}{p^2}$

c) $\mathcal{L}\left\{t^n\right\} = \dfrac{n!}{p^{n+1}}$

d) $\mathcal{L}\left\{\sin\omega t\right\} = \dfrac{\omega}{p^2 + \omega^2}$

e) $\mathcal{L}\{\cos \omega t\} = \dfrac{p}{p^2+\omega^2}$

f) $\mathcal{L}\{\sinh \omega t\} = \dfrac{\omega}{p^2-\omega^2}$

g) $\mathcal{L}\{\cosh \omega t\} = \dfrac{p}{p^2-\omega^2}$

h) $\mathcal{L}\left\{e^{at}\right\} = \dfrac{1}{p-a}$

Section 9

Properties of the Transform

1. Linearity:
$$\mathcal{L}\{\lambda f + \mu g\} = \lambda \mathcal{L}\{f\} + \mu \mathcal{L}\{g\}.$$
2. Homothety:
$$\mathcal{L}\{f(kt)\} = \frac{1}{k} F(\frac{p}{k}).$$
3. Derivation:
$$\begin{aligned}\mathcal{L}\{f'(t)\} &= p\mathcal{L}\{f(t)\} - f(0^+)\\ \mathcal{L}\{f''(t)\} &= p^2\mathcal{L}\{f(t)\} - pf(0^+) - f'(0^+)\\ \mathcal{L}\left\{f^{(n)}(t)\right\} &= p^n\mathcal{L}\{f(t)\} - \sum_{k=1}^{n} p^{n-k} f^{(k-1)}(0^+)\end{aligned}$$
4. Integration:
$$\mathcal{L}\left\{\int_0^t f(u)\,\mathrm{d}u\right\} = \frac{F(p)}{p}.$$
5. Initial value theorem:
$$f(0^+) = \lim_{p\to\infty} p\mathcal{L}\{f(t)\}.$$
6. Final value theorem:
$$f(\infty) = \lim_{p\to 0} p\mathcal{L}\{f(t)\}.$$

Remark

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}p}F(p)\\ \mathcal{L}\left\{t^2 f(t)\right\} &= \frac{\mathrm{d}^2}{\mathrm{d}p^2}F(p)\\ \mathcal{L}\{t^n f(t)\} &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}p^n}F(p)\end{aligned}$$

Remark Convolution over a domain $I \subset \mathbb{R}$ is defined as

$$f(t) * g(t) = \int_I f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_I f(t-\tau)g(\tau)\,\mathrm{d}\tau.$$

and it's trasform is

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(p) \cdot G(p).$$

Section 10

Translation

In the time domain:

$$\mathcal{L}\left\{f(t-a)\right\} = e^{-ap}F(p).$$

In the p-domain:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(p+a).$$

Definition of a system of DEs

PART VII

Suppose we have a vector of functions $\vec{\mathbf{x}} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$, a differential of $\vec{\mathbf{x}}$ is

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}.$$

Suppose that $\vec{\mathbf{x}}$ verifies the equation

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = f\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}, \cdots, \frac{\mathrm{d}^n x}{\mathrm{d}t^n}, t\right).$$

We can transform the above equation to a vector system by taking

$$\begin{aligned} x &= x_1 \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= x_2 \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= x_3 \\ &\vdots \\ \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} &= x_n \end{aligned}$$

PART VIII

Homogeneous Systems of Linear DEs

A linear Differential equation is an equation of the form:

$$a_n(t)x^{(n)} + \cdots + a_1(t)x' + a_0(t)x = f(t).$$

The equation becomes homogeneous when $f(t) = 0$.

A system of linear DEs would be in the form:[2]

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n \end{cases}.$$

[2] *We assume that all a_{ij} are continuous on the interval of study*

In vector form the system is expressed as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \vec{\mathbf{x}} \quad \text{where} \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{given} \quad \vec{\mathbf{x}}_0(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Remark An interesting fact about homogeneous linear DEs is that if the initial condition is ever zero ($\vec{\mathbf{x}}(t_0) = \vec{\mathbf{0}}$) then $\vec{\mathbf{x}}(t) = \vec{\mathbf{0}}$ is a solution to that DE and in fact the only solution due to the uniqueness theorem.

SECTION 11

Fundamental Solutions

For any given system of homogeneous linear DEs there exists a set of n functions such they for a linearly independent basis for a general solution of said DEs, in other words for a given DE there exists a set of vector functions $(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\zeta}}_2, \ldots, \vec{\boldsymbol{\zeta}}_n)$ such that

$$\vec{\mathbf{x}}(t) = c_1\vec{\boldsymbol{\zeta}}_1(t) + c_2\vec{\boldsymbol{\zeta}}_2(t) + \cdots + c_n\vec{\boldsymbol{\zeta}}_n(t) \quad \text{where} \quad c_{1,2,\ldots,n} \in \mathbb{R}.$$

We define the fundamental matrix of the system

$$X = \begin{pmatrix} \vec{\boldsymbol{\zeta}}_1 & \vec{\boldsymbol{\zeta}}_2 & \dots & \vec{\boldsymbol{\zeta}}_n \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & & \ddots & \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nn} \end{pmatrix}.$$

The system can be written in terms of X as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX.$$

Remark The fundamental solutions are linearly independent $\implies \det(X) \neq 0$

Subsection 11.1

Wronskian of vector functions

Consider the vector functions:

$$\vec{\boldsymbol{\phi}}_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix} \quad \cdots \quad \vec{\boldsymbol{\phi}}_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}.$$

The Wronskian is defined to the determinant:

$$W(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \cdots, \vec{\boldsymbol{\phi}}_n) = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & \ddots & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}.$$

If the Wronskian $= 0$ then the functions $(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \dots, \vec{\boldsymbol{\phi}}_n)$ are said to be linearly independent.
When dealing with DEs the concept of a Wronskian can be applied to *non-vector functions* as follows

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & \ddots & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

Section 12

n-the order Homogeneous Linear DE

We define the notation $\mathrm{D}^n x = \dfrac{\mathrm{d}^n x}{\mathrm{d}t^n}$.

An n-the order linear DE is any equation of the form:

$$\mathrm{D}^n x + a_1(t)\mathrm{D}^{n-1} x + \cdots + a_{n-1}(t)\mathrm{D}\, x + a_n(t)x = 0.$$

We can then write the equation in vector form

$$\begin{aligned} x &= x_1 \\ \mathrm{D}\, x &= x_2 \\ \mathrm{D}^2 x &= x_3 \\ &\vdots \\ \mathrm{D}^{n-1} x &= x_n \\ \mathrm{D}^n x &= -a_n x_1 - a_{n-1}x_2 - \cdots - a_1 x_n \end{aligned}$$

and we take

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{D}\, x \\ \vdots \\ \mathrm{D}^{n-1} x \end{pmatrix}.$$

then we can write the system as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{pmatrix}.$$

Subsection 12.1

DE from a Set of Fundamental Solutions

Given a set of fundamental solutions $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, due to the uniqueness theorem those solutions only satisfy one DE. To find that DE we simply compute

$$W(x, \zeta_1, \zeta_2, \ldots, \zeta_n) = \begin{vmatrix} x & \zeta_1 & \cdots & \zeta_n \\ \mathrm{D}\, x & \mathrm{D}\, \zeta_1 & \cdots & \mathrm{D}\, \zeta_n \\ \vdots & & \ddots & \\ \mathrm{D}^n x & \mathrm{D}^n \zeta_1 & \cdots & \mathrm{D}^n \zeta_n \end{vmatrix} = 0.$$

Example | Given the fundamental set of solutions $(e^{\omega t}, e^{-\omega t})$, find the second order homogeneous equation for that set of solutions:

$$
\begin{aligned}
&W(x, e^{\omega t}, e^{-\omega t}) = 0 \\
\implies &\begin{vmatrix} x & e^{\omega t} & e^{-\omega t} \\ x' & \omega e^{\omega t} & -\omega e^{-\omega t} \\ x'' & \omega^2 e^{\omega t} & \omega^2 e^{-\omega t} \end{vmatrix} = 0 \\
\implies &x'' - \omega^2 x = 0.
\end{aligned}
$$

Non-Homogeneous Systems of Linear DEs

PART IX

A system of Non-homogeneous DE is a system of the form

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}.$$

or in matrix form

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} + \vec{\mathbf{b}}.$$

Section 13

General Solution Formula

The general formula for a non-homogeneous DE is given by the general solution and the particular solution

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_g(t) + \vec{\mathbf{x}}_p(t).$$

Where

$\vec{\mathbf{x}}_g(t)$ is the solution to the DE if $\vec{\mathbf{b}}(t) = 0$

$$\begin{aligned} \vec{\mathbf{x}}_g(t) &= X(t)\vec{\mathbf{k}} \\ \vec{\mathbf{x}}_p(t) &= X(t)\int_{t_0}^{t} X^{-1}(u)\vec{\mathbf{b}}(u)\,\mathrm{d}u \end{aligned}$$

Where X is the solution to

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}}.$$

Let us denote

$$X(t)X^{-1}(t_0) = V(t, t_0).$$

We can prove that

$$X(t) = V(t, t_0).$$

Remark 1. $V(t_1, t_2)V(t_2, t_3) = V(t_1, t_3)$

2. $V(t_1, t_1) = I$

Thus we can write the solution to be

$$\vec{\mathbf{x}}(t) = V(t, t_0)\vec{\mathbf{x}}(t_0) + \int_{t_0}^{t} V(t, u)\vec{\mathbf{b}}(u)\,\mathrm{d}u\,.$$

The solution of the equation of the form

$$\sum_{k=0}^{n} a_k(t)\mathrm{D}^{n-k}\, x \ = b(t).$$

along with its homogeneous form ($b(t) = 0$) with initial conditions $x(t_0) = c_1, \mathrm{D}\, x \ = c_2, \ldots, \mathrm{D}^{n-1}\, x\ (t_0) = c_n$ have a solution given by the formula

$$x(t) = u_1(t, t_0)x(t_0) + u_2(t, t_0)\mathrm{D}\, x(t_0) \ + \cdots + u_n\mathrm{D}^{n-1}\, x(t_0)\ .$$

Where $(u_1, u_2, \ldots, u_n)$ are the fundamental solutions to the homogeneous equation verifying $\mathrm{D}^{j-1}\, u_i(t_0, t_0) \ = \delta_{ij}$.

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Section 14

Method of Variation of Constant

In this method we look for solutions of the equation of the form

$$x = z_1 x_1 + z_2 x_2 + \cdots + z_n x_n.$$

Where z_i are a function of t. We are looking for a solution verifying $\sum_{i=1}^{n} \mathrm{D}\, z_i\ \mathrm{D}^j\, x_i \ = 0$ for $j = 0, 1, \ldots, n-2$

At this point the course become extremely useless and redundant so I'll just cut to the chase here.

Section 15

PART X

Systems of Differential Equations

Consider a system of first order differential equations (S)

$$\begin{cases} \dfrac{\mathrm{d}x_1}{\mathrm{d}t} = a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + b_1(t) \\ \dfrac{\mathrm{d}x_2}{\mathrm{d}t} = a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + b_2(t) \\ \vdots \\ \dfrac{\mathrm{d}x_n}{\mathrm{d}t} = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + b_n(t) \end{cases}.$$

in matrix form the equation can be written as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} + \vec{\mathbf{b}}.$$

and the initial condition can be written as

$$\vec{\mathbf{x}}_0(t_0) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

SECTION 16

Solving The System of DEs

There 3 main ways of solving systems of DEs:

1. Laplace Transform
2. Change of Basis
3. Solving Matrix Formula

Remark Let A be a diagonalizable matrix

$$A = PDP^{-1}.$$

so we define that

$$e^{At} = Pe^{Dt}P^{-1}.$$

or in other words

$$e^{At} = P \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{\lambda_n t} \end{pmatrix} P^{-1}.$$

SUBSECTION 16.1

Change of Basis

We consider a new system of DEs to be

$$\frac{\mathrm{d}\vec{\mathbf{y}}}{\mathrm{d}t} = P^{-1}AP\vec{\mathbf{y}} + P^{-1}\vec{\mathbf{b}}.$$

which simplifies to

$$\frac{\mathrm{d}\vec{\mathbf{y}}}{\mathrm{d}t} = D\vec{\mathbf{y}} + \vec{\mathbf{B}}.$$

in this new system we can solve for $\vec{\mathbf{y}}$

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}t} = \lambda_1 y_1 + B_1 \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} = \lambda_2 y_2 + B_2 \\ \vdots \\ \frac{\mathrm{d}y_n}{\mathrm{d}t} = \lambda_n y_n + B_n \end{cases}.$$

Remark The solution to a differential equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \alpha y + \beta.$$

is

$$y = c_1 e^{\alpha t} - \frac{\beta}{\alpha}.$$

after we find the solution to the new system, we can simply obtain the solution to the original system by

$$\vec{\mathbf{x}} = P\vec{\mathbf{y}}.$$

and by substituting t_0 in $\vec{\mathbf{x}}$ we can solve for the constant terms $(c_1, c_2, \ldots, c_n)$ using $\vec{\mathbf{x}}_0$.

SUBSECTION 16.2

Solving Matrix Formula

The formula for a system of first order equations is

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p.$$

Where

$$\vec{\mathbf{x}}_h = V(t,t_0)\vec{\mathbf{x}}_0$$
$$\vec{\mathbf{x}}_p = \int_{t_0}^{t} V(t,u)\vec{\mathbf{b}}(u)\,\mathrm{d}u$$

where

$$V(t,t_0) = X(t)X^{-1}(t_0).$$

if $t = 0$ then the formula becomes

$$\vec{\mathbf{x}} = e^{At}\vec{\mathbf{x}}_0 + \int_0^t e^{A(t-u)}\vec{\mathbf{b}}(u)\,\mathrm{d}u\,.$$

Section 17

Fundamental Solutions

For any given system of homogeneous linear DEs there exists a set of n functions such they for a linearly independent basis for a general solution of said DEs, in other words for a given DE there exists a set of vector functions $(\vec{\boldsymbol{\zeta}}_1, \vec{\boldsymbol{\zeta}}_2, \ldots, \vec{\boldsymbol{\zeta}}_n)$ such that

$$\vec{\mathbf{x}}(t) = c_1\vec{\boldsymbol{\zeta}}_1(t) + c_2\vec{\boldsymbol{\zeta}}_2(t) + \cdots + c_n\vec{\boldsymbol{\zeta}}_n(t) \quad \text{where} \quad c_{1,2,\ldots,n} \in \mathbb{R}.$$

We define the fundamental matrix of the system

$$X = \begin{pmatrix} \vec{\boldsymbol{\zeta}}_1 & \vec{\boldsymbol{\zeta}}_2 & \ldots & \vec{\boldsymbol{\zeta}}_n \end{pmatrix} = \begin{pmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1n} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2n} \\ \vdots & & \ddots & \\ \zeta_{n1} & \zeta_{n2} & \cdots & \zeta_{nn} \end{pmatrix}.$$

The system can be written in terms of X as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = AX.$$

Remark The fundamental solutions are linearly independent $\implies \det(X) \neq 0$

Subsection 17.1

Wronskian of vector functions

Consider the vector functions:

$$\vec{\boldsymbol{\phi}}_1(t) = \begin{pmatrix} \phi_{11}(t) \\ \phi_{21}(t) \\ \vdots \\ \phi_{n1}(t) \end{pmatrix} \quad \cdots \quad \vec{\boldsymbol{\phi}}_n(t) = \begin{pmatrix} \phi_{1n}(t) \\ \phi_{2n}(t) \\ \vdots \\ \phi_{nn}(t) \end{pmatrix}.$$

The Wronskian is defined to the determinant:

$$W(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \cdots, \vec{\boldsymbol{\phi}}_n) = \begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & \ddots & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix}.$$

If the Wronskian $= 0$ then the functions $(\vec{\boldsymbol{\phi}}_1, \vec{\boldsymbol{\phi}}_2, \ldots, \vec{\boldsymbol{\phi}}_n)$ are said to be linearly independent.
When dealing with DEs the concept of a Wronskian can be applied to *non-vector functions* as follows

$$W(\phi_1, \phi_2, \cdots, \phi_n) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & \ddots & \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix}.$$

Section 18

Solving n-th order Homogeneous Linear DE

We define the notation $\mathrm{D}^n x = \dfrac{\mathrm{d}^n x}{\mathrm{d}t^n}$.
An n-the order linear DE is any equation of the form:

$$\mathrm{D}^n x + a_1(t)\mathrm{D}^{n-1} x + \cdots + a_{n-1}(t)\mathrm{D}\, x + a_n(t)x = 0.$$

We can then write the equation in vector form

$$\begin{aligned} x &= x_1 \\ \mathrm{D}\, x &= x_2 \\ \mathrm{D}^2 x &= x_3 \\ &\vdots \\ \mathrm{D}^{n-1} x &= x_n \\ \mathrm{D}^n x &= -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n \end{aligned}$$

and we take

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x \\ \mathrm{D}\, x \\ \vdots \\ \mathrm{D}^{n-1} x \end{pmatrix}.$$

then we can write the system as

$$\frac{\mathrm{d}\vec{\mathbf{x}}}{\mathrm{d}t} = A\vec{\mathbf{x}} \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{pmatrix}.$$

Subsection 18.1

DE from a Set of Fundamental Solutions

Given a set of fundamental solutions $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, due to the uniqueness theorem those solutions only satisfy one DE. To find that DE we simply compute

$$W(x, \zeta_1, \zeta_2, \ldots, \zeta_n) = \begin{vmatrix} x & \zeta_1 & \cdots & \zeta_n \\ \mathrm{D}\, x & \mathrm{D}\, \zeta_1 & \cdots & \mathrm{D}\, \zeta_n \\ \vdots & & \ddots & \\ \mathrm{D}^n\, x & \mathrm{D}^n\, \zeta_1 & \cdots & \mathrm{D}^n\, \zeta_n \end{vmatrix} = 0.$$

Example Given the fundamental set of solutions $(e^{\omega t}, e^{-\omega t})$, find the second order homogeneous equation for that set of solutions:

$$\begin{aligned} & W(x, e^{\omega t}, e^{-\omega t}) = 0 \\ \implies & \begin{vmatrix} x & e^{\omega t} & e^{-\omega t} \\ x' & \omega e^{\omega t} & -\omega e^{-\omega t} \\ x'' & \omega^2 e^{\omega t} & \omega^2 e^{-\omega t} \end{vmatrix} = 0 \\ \implies & x'' - \omega^2 x = 0. \end{aligned}$$

Subsection 18.2

Method of Variation of constants