Numerical Analysis
Semester 4
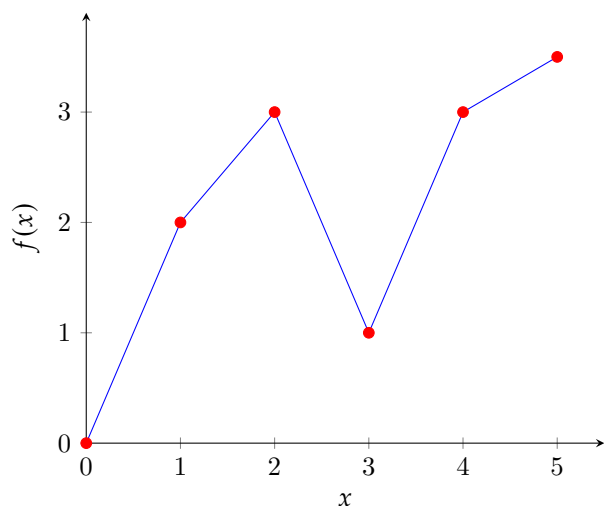
Contents

Chapter 1

Interpolation

1.1 Linear Interpolation

x	$f(x)$
0	0
1	2
2	3
3	1
4	3
5	3.5

Linear interpolation is just drawing lines between the data points.

Definition 1.1.1: Linear Interpolation(lerp) equation

The equation of the lines between data points is

$$y = \frac{y_{i+1} - y_i}{x_{i+1} - x_i}(x - x_i) + y_i.$$

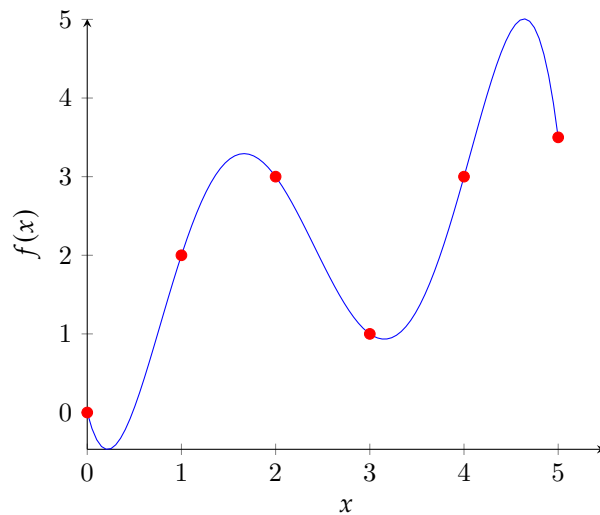**Theorem 1.1.1** Error due to linear interpolation

Let f be a continuous and differentiable on $[a, b]$. We define the error $z(x)$ to be

$$|z(x)| \leqslant \frac{(b-a)^2}{8} \sup_{a \leqslant x \leqslant b} |f''(x)|.$$

1.2 Polynomial Interpolation

1.2.1 Lagrange Polynomials

Really nice video here explaining Lagrange polynomials.

Theorem 1.2.1 Lagrange polynomial equation

Consider a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The Lagrange polynomial for this set of data is

$$L(x) = \sum_{k=0}^{n} y_k \ell_k(x).$$

where

$$\ell_k(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

Case of equidistant points

If the set of x_i are equidistant from each other with a distance of $h = x_{i+1} - x_i$, then we can represent any point as $x_k = x_0 + kh$ where $k \in \mathbb{N}$ and any number $x = x_0 + sh$ where $s \in \mathbb{R}$. We can rewrite the formula as

$$Q(s) = \sum_{k=0}^{n} \ell_k(s) f(x_k).$$

where

$$\ell_k(s) = \prod_{\substack{j=0 \\ j \neq k}}^{n} \frac{s - j}{k - j}.$$

by substitution

$$s = \frac{x - x_0}{h}.$$

Existence

Proof: $P(x)$ belongs to the vectorial space of polynomial of degree of, at most, n. Now, we must fins a basis for this vectorial space. Find the polynomial ℓ_k of degree $\leqslant n$ such that

$$\ell_k(x_i) = \delta_{ki} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}.$$

Then, $\ell_k(x) = \lambda(x - x_0) \ldots (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_n)$ where

$$\lambda = \frac{1}{(x_k - x_0) \ldots (x_k - x_{k-1})(x_k - x_{k+1}) \ldots (x_k - x_n)}.$$

The $(n + 1)$ polynomials $\ell_k(x)$ for a system of generators in the vectorial space of polynomials of degree at most n.

$$\lambda_0 \ell_0(x) + \lambda_1 \ell_1(x) + \cdots + \lambda_k \ell_k(x) + \cdots + \lambda_n \ell_n(x) = 0.$$

for $x = x_k$

$$\lambda_0 \ell_0(x_k) + \lambda_1 \ell_1(x_k) + \cdots + \lambda_k \ell_k(x_k) + \cdots + \lambda_n \ell_n(x_k) = 0$$
$$0 + 0 + \cdots + \lambda_k 1 + \cdots + 0 = 0 \lambda_k \qquad\qquad = 0.$$

$\therefore$ the set of ℓ_k for a basis in the vector space $\Rightarrow$ there has to exist a polynomial passing through the given set of points.

$\square$

Uniqueness

Proof: Let P and Q be 2 Lagrange polynomials of degrees $\leqslant n \big/ P(x_i) = Q(x_i) = f(x_i) \quad \forall i = 0, 1, \ldots, n$.

Let

$$\left.\begin{array}{l} R = P - Q \text{ of degree } \leqslant n \\ R = 0 \ (n + 1) \text{ times} \end{array}\right\} R \equiv 0 \Rightarrow P = Q \ \forall x.$$

$\square$

1.2.2   Newton Polynomial

> **Definition 1.2.1: Newton Polynomial equation**
>
> Consider a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. The Newton polynomial for this set of data is
>
> $$p_n(x) = \underbrace{a_0}_{A_0} + \underbrace{a_1(x - x_0)}_{A_1} + \underbrace{a_2(x - x_0)(x - x_1)}_{A_2} + \cdots + \underbrace{a_n \prod_{i=0}^{n-1}(x - x_i)}_{A_n}.$$
>
> where
>
> $$a_i = f[x_0, x_1, \ldots, x_i].$$
>
> Here $f[\ldots]$ is the divided difference of the inputted data.

> **Definition 1.2.2: Backwards formula**
>
> $$P_n(x) = f_n + A_1 + A_2 + \cdots + A_n.$$
>
> where
>
> $$A_i = f[x_n, x_{n-1}, \ldots, x_{n-i}] \prod_{j=n-i+1}^{n}(x - x_j).$$

The divided difference has 2 formulas, the recurrence formula

$$f[x_0, x_1, \ldots, x_{n+1}] = \frac{f[x_1, x_2, \ldots, x_{n+1}] - f[x_0, x_1, \ldots, x_n]}{x_{n+1} - x_0}.$$

and a general formula

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=1}^{n} \frac{y_i}{\prod_{\substack{k=0 \\ k \neq i}}^{n}(x_i - x_k)}.$$

Now forget you ever saw those cause there is an easier method to finding the divided difference.

Divided Difference Table

$$
\begin{array}{ll}
x_0 & y_0 \\
& \frac{y_1-y_0}{x_1-x_0} = f[x_0,x_1] \\
x_1 & y_1 & & \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} \\
& \frac{y_2-y_1}{x_2-x_1} = f[x_1,x_2] & & & \cdots \\
x_2 & y_2 & & \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} & & \cdots \\
& \frac{y_3-y_2}{x_3-x_2} = f[x_2,x_3] & & & \cdots \\
x_3 & y_3 & & \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2} \\
& \frac{y_4-y_3}{x_4-x_3} = f[x_3,x_4] \\
x_4 & y_4
\end{array}
$$

After we have constructed the table we can find the divided difference we want by looking at the top diagonal

$$
\begin{array}{ll}
x_0 & y_0 \\
& f[x_0,x_1] \\
x_1 & y_1 & & f[x_0,x_1,x_2] \\
& \cdots & & & f[x_0,x_1,x_2,x_3] \\
x_2 & y_2 & & \cdots & & f[x_0,x_1,x_2,x_3,x_4] \\
& \cdots & & & \cdots \\
x_3 & y_3 & & \cdots \\
& \cdots \\
x_4 & y_4
\end{array}
$$

Case of equidistant points

Bla bla bla the formula becomes

$$
P(t) = a_0 + a_1(t-0) + a_2(t-0)(t-1) + \cdots + a_n \prod_{i=0}^{n-1}(t-i).
$$

where in this case

$$
a_k = \frac{\nabla^k[y](x_k)}{k!}.
$$

and

$$
x = x_0 + th.
$$

where $\nabla^k[y]$ is the discrete difference.

$$
\nabla[y](x_i) = y(x_i + h) - y(x_i).
$$

and the backwards formula is

$$
P(t) = f_n + A_1 + A_2 + \cdots + A_n.
$$

where

$$
A_i = \frac{\bar{\nabla}^i f_n}{i!} \prod_{j=n-i+1}^{n}(t-j).
$$

> **Definition 1.2.3: Discrete Difference**
>
> Forward discrete difference:
>
> $$\nabla[y](x_i) = y(x_i + h) - y(x_i)$$
> $$\nabla^2[y](x_i) = \nabla[y](x_i + h) - \nabla[y](x_i)$$
> $$= y(x_i + 2h) - 2y(x_i + h) + y(x_i)$$
> $$\nabla^k[y](x_i) = \nabla\left(\nabla^{k-1}[y](x_i)\right)$$
>
> Backwards discrete difference:
>
> $$\bar{\nabla}[y](x_i) = y(x_i) - y(x_i - h)$$
> $$\bar{\nabla}^k[y](x_i) = \bar{\nabla}\left(\bar{\nabla}^{k-1}[y](x_i)\right)$$

$$
\begin{array}{llllll}
x_0 & y_0 & & & & \\
 & & \nabla[y](x_i) & & & \\
x_1 & y_1 & & \nabla^2[y](x_i) & & \\
 & & \ldots & & \nabla^3[y](x_i) & \\
x_2 & y_2 & & \ldots & & \nabla^4[y](x_i) \\
 & & \ldots & & \ldots & \\
x_3 & y_3 & & \ldots & & \\
 & & \ldots & & & \\
x_4 & y_4 & & & &
\end{array}
$$

1.2.3 Error due to polynomial interpolation

Let $f(x)$ be of class C^{n+1} $\quad \forall x \in [a,b]$ and let the polynomial $P(x)$ interpolate it.

The error function is bounded by

$$|\text{Error}| = |f(x) - P(x)| \leqslant \frac{\left|\prod_{i=0}^{n}(x - x_i)\right|}{(n+1)!} \sup_{x \in [a,b]} \left|f^{(n+1)}(x)\right|.$$

1.2.4 Hermite Interpolation

> **Definition 1.2.4: Hermite interpolation formula**
>
> Consider $(n+1)$ sets of point (x_i, y_i, y_i') representing $f(x)$ $(y_i = f(x_i)$ and $y_i' = f'(x_i))$, the hermite polynomial $P(x)$ interpolates $f(x)$ such that $P'(x) = f'(x)$.
>
> $$P(x) = \sum_{i=0}^{n} h_i(x)y_i + \sum_{i=0}^{n} k_i(x)y_i'.$$
>
> where
>
> $$h_i(x) = \left(1 - 2(x - x_i)\ell_i'(x_i)\right)\ell_i^2(x)$$
> $$k_i(x) = (x - x_i)\ell_i^2(x)$$
> $$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Existence

Proof:

$$P(x) = \sum_{i=0}^{n} h_i(x)y_i + \sum_{i=0}^{n} k_i(x)y_i'.$$

where

$$h_i(x) = \left(1 - 2(x - x_i)\ell_i'(x_i)\right)\ell_i^2(x)$$
$$k_i(x) = (x - x_i)\ell_i^2(x)$$
$$\ell_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Let $i \neq j$.

$$k_i(x_j) = (x_j - x_i)\ell_i^2(x_j) = 0$$
$$k_i(x_i) = (x_i - x_i)\ell_i^2(x_i) = 0$$

and

$$h_i(x_j) = (1 - 2(x_j - x_i)\ell_i'(x_i))\ell_i^2(x_j) = 0$$
$$h_i(x_j) = (1 - 2(x_i - x_i)\ell_i'(x_i))\ell_i^2(x_i) = 1$$

We conclude that $P(x_i) = f(x_i)$

Now we have to prove that $P'(x_i) = f'(x_i)$

$$h_i'(x) = -2\ell_i'(x_i)\ell_i^2(x) + 2(1 - 2(x - x_i)\ell_i'(x_i))\ell_i(x)\ell_i'(x)$$
$$k_i'(x) = \ell_i^2(x) + 2(x - x_i)\ell_i(x)\ell_i'(x)$$

$$h_i'(x_j) = -2\ell_i'(x_i)\ell_i^2(x_j) + 2(1 - 2(x_j - x_i)\ell_i'(x_i))\ell_i(x_j)\ell_i'(x_j) = 0$$
$$h_i'(x_i) = -2\ell_i'(x_i)\ell_i^2(x_i) + 2(1 - 2(x_i - x_i)\ell_i'(x_i))\ell_i(x_i)\ell_i'(x_i) = 0$$

$$k_i'(x_j) = \ell_i^2(x - j) + 2(x_j - x_i)\ell_i(x_j)\ell_i'(x_j) = 0$$
$$k_i'(x_j) = \ell_i^2(x - i) + 2(x_i - x_i)\ell_i(x_i)\ell_i'(x_i) = 1$$

$\therefore P'(x_i) = f'(x_i)$ □

Uniqueness

Proof: Suppose that there exists 2 polynomials P and Q of degree $n \leqslant 2n + 1$ such that $P(x_i) = Q(x_i) = f(x_i)$ and $P'(x_i) = Q'(x_i) = f'(x_i) \ \forall i = 0, 1, \ldots, n$.

Let $R(x) = P(x) - Q(x)$.

$R = 0 \ (n + 1)$ times $\Rightarrow$ according to Rolle's theorem $\exists n$ points $\neq x_i \Big/ R' = 0$

$R' = 0 \ n$ times as a consequence of Rolle's theorem then

$$\left. \begin{array}{l} R'(x) = 0 \ (2n + 1) \text{ times} \\ R'(x) \text{ is of degree } 2n \end{array} \right\} R'(x) = 0 \ \forall x.$$

$$R'(x) = 0 \Rightarrow R(x) = \text{cnst} \quad \text{and} \quad R(x_i) = P(x_i) - Q(x_i) = 0 \Rightarrow \text{cnst} = 0.$$

$$R(x) = P(x) - Q(x) = 0 \ \forall x.$$

$\therefore P(x) = Q(x)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Chapter 2

Finding $f(x) = 0$

We will assume that every function is defined in the interval $I = [a, b]$ and that every $x_0 \in I$

2.1 Bisection Method

Suppose that f is a continuous and monotone function on the domain $I = [a, b]$ such that $f(a)f(b) < 0 \Rightarrow \exists r \in\ ]a, b[: \ f(r) = 0$.

At each step in the algorithm, in an iteration we let $c = (a + b)/2$, then we check the value of $f(c)$, if it is 0 then we are done.
However when it is not, then we define a new interval such that

$$I = \begin{cases} [a, c] & \text{if } f(c)f(a) < 0 \\ [c, b] & \text{if } f(c)f(b) < 0 \end{cases}.$$

We repeat this step until the length of the interval reaches a certain number (for example $|b - a| < 10^{-5}$), at this point we stop and the best guess for the root would be $(a + b)/2$

Error of the Bisection Method

After n iterations, the error of the approximated root would be

$$\text{Error} \leqslant \frac{|b - a|}{2^{n+1}}.$$

2.2 Lagrange Method

Suppose that f is a continuous and monotone function on the domain $I = [a, b]$ such that $f(a)f(b) < 0 \Rightarrow \exists r \in\ ]a, b[: \ f(r) = 0$.

The starting value of x_0 depends on the value of f

$$x_0 = \begin{cases} a & \text{if } f(a)f''(a) < 0 \\ b & \text{if } f(b)f''(b) < 0 \end{cases}.$$

then we can find a new guess x depending on the value of x_0

- if $x_0 = a$

$$x_1 = x_0 - \frac{b - x_0}{f(b) - f(x_0)} f(x_0).$$

- if $x_0 = b$

$$x_1 = x_0 - \frac{a - x_0}{f(a) - f(x_0)} f(x_0).$$

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Error from Lagrange Method

For the first iteration
$$\text{Error} \leqslant \sup_{x \in [a,b]} |f''(x)| \frac{(b-a)^2}{8}.$$

For the second iteration
$$M_2 = \sup_{x \in [a,b]} |f''(x)|.$$

- if $x_0 = a$
$$\text{Error} \leqslant \frac{M_2}{8} \left| \frac{(b-x_0)^3}{f(b)-f(x_0)} \right|.$$

- if $x_0 = b$
$$\text{Error} \leqslant \frac{M_2}{8} \left| \frac{(a-x_0)^3}{f(a)-f(x_0)} \right|.$$

2.3 Newton Method

Suppose that f is a continuous and monotone function on the domain $I = [a,b]$ such that $f(a)f(b) < 0 \Rightarrow \exists r \in \ ]a,b[: \ f(r) = 0$.

The starting value of x_0 depends on the value of f

$$x_0 = \begin{cases} a & \text{if } f(a)f''(a) > 0 \\ b & \text{if } f(b)f''(b) > 0 \end{cases}.$$

Then the new guess for the root would be
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

2.3.1 Improved Newton Method

To improve the method we first let $\eta = b - a$, and we define condition
$$\frac{\eta M_2}{2|f'(x_0)|} < 1.$$

if the condition is not satisfied we need to choose another interval $[a_1, b_1] \subset I$ where $f(a_1)f(b_1) < 0$

Error due to Newton Method

For one iteration
$$\text{Error} =\leqslant \frac{\eta^2 M_2}{2|f'(x_0)|} \quad \text{where} \quad M_2 = \sup_{x \in [x_0 - \eta, x_0 + \eta]} |f''(x)|.$$

2.4 Fixed Point Iteration Method

If a function can be converted to the form $x = g(x)$ along with the sequence $x_{n+1} = g(x_n)$ with initial guess x_0, then it is called a fixed point scheme.
The scheme converges if

- $\forall x \in [a,b] : \ g(x) \in [a,b]$

- g is strictly contracting meaning that $\exists \varepsilon \in \mathbb{R} \ 0 \leqslant \varepsilon < 1$

$$\forall x, y \in [a,b], \ |g(x) - g(y)| \leqslant \varepsilon |x - y|.$$

then $\forall x_0$ the sequence converges to $l \in [a,b]$

$$\sup_{x\in[a,b]} |g'(x)| = L < 1 \Rightarrow g(x) \text{ is strictly contracting.}$$

Let l be the solution to $g(l) = l$

- If $|g'(l)| < 1$ then there exists an interval I containing l for which the sequence converges to l

- If $|g'(l)| > 1$ then the sequence diverges

2.5 Order of Convergence

Order of convergence (Rate of convergence) tells us how the error decreases between 2 iterations. The order of convergence p of a sequence is defined to be

$$\lim_{n\to+\infty} \left| \frac{x_{n+1} - l}{(x_n - l)} \right| \in \mathbb{R}_+^*.$$

The order of convergence of

- Lagrange Method

$$g'(l) = \frac{(b - l)^2}{2f(b)} f''(c).$$

  If $f''(c) \neq 0$ then $g'(l) \neq 0$ then the order is 1.

- Newton method, if $g'(l) = 0$ then the order is at least 2.

We stop the iteration method when

- First case $g'(x) < 0$, then we stop iteration when

$$|x_{n+1} - r| < \varepsilon.$$

- Second case $g'(x) > 0$, then we stop iteration when

$$|f(x_n)| < \eta.$$

  where

$$\eta = \varepsilon \inf |f'(x)|.$$

2.6 Polynomial Shenanigans

2.6.1 Roots of $x^3 + px + q = 0$

Let $y_1(x) = x^3 + px$ and $y_2(x) = -q$

- $p \geqslant 0 \Rightarrow \exists 1$ root

- $p < 0$ then we have 3 separate cases

$$27q^2 + 4p^3 \begin{cases} = 0 & \text{we have 2 separate real roots (one double and one single)} \\ > 0 & \text{we have one real root} \\ < 0 & \text{we have 3 separate real roots} \end{cases}.$$

2.6.2 Roots of $x^3 + ax^2 + bx + c = 0$

If we replace x with $X + h$ where $h = -\frac{a}{3}$, we can get the cubic in the form

$$X^3 + PX + Q = 0.$$

where

$$P = -\frac{a^2}{3} + b$$

$$Q = \frac{2a^3}{27} - \frac{ab}{3} + c$$

2.6.3 Roots of $x^4 + ax^3 + bx^2 + cx + d = 0$

If we replace x with $X + h$ where $h = -\frac{a}{4}$, we can get the quartic in the form

$$X^4 + PX^2 + QX + R = 0.$$

where

$$P = -\frac{3a^2}{8} + b$$

$$Q = \frac{a^3}{8} - \frac{ab}{2} + c$$

$$R = -\frac{3a^4}{256} - \frac{ac}{4} + d$$

Let the circle C be the circle of radius $\left(-\frac{Q}{2}, \frac{1-P}{2}\right)$ and of radius $\sqrt{\left(\frac{P-1}{2}\right)^2 + \frac{Q^2}{4} - R}$.

The roots of the polynomial $X^4 + PX^2 + QX + R = 0$ are the intersection of the circle C and the parabola $Y = X^2$