Differential Geometry

Contents

PART

I

Prerequisites

SECTION 1

Matrices

Theorem 1 To prove a system of vectors $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \dots, \vec{\mathbf{u}}_n\}$ is free we prove:

$$\det \begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_1 \\ | & | & | & & | \end{bmatrix} \neq 0.$$

Theorem 2 A transition matrix $P_{B\to B'}$between 2 basis $B = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ and $B' = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ we start by solving the system

$$\begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{\mathbf{v}}_n \\ | \end{bmatrix}.$$

or in other words finding

$$\begin{cases} \vec{\mathbf{v}}_1 & = \alpha_1\vec{\mathbf{u}}_1 + \beta_1\vec{\mathbf{u}}_2 + \gamma_1\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_2 & = \alpha_2\vec{\mathbf{u}}_1 + \beta_2\vec{\mathbf{u}}_2 + \gamma_2\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_3 & = \alpha_3\vec{\mathbf{u}}_1 + \beta_3\vec{\mathbf{u}}_2 + \gamma_3\vec{\mathbf{u}}_3 \end{cases}.$$

Finally we say that

$$P_{B\to B'} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

transition matrices are always square and invertible ($\det P \neq 0$)

Remark To find the transition matrix in the inverse direction (from B' to B) we simply do

$$P_{B'\to B} = {P_{B\to B'}}^{-1}.$$

SECTION 2

Vectors

Definition 1 We define an operation called the scalar product (dot product)

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}}, \vec{\mathbf{v}} &\longmapsto \langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle = \sum_{n=1}^{n} v_i \cdot u_i. \end{aligned}$$

Definition 2 We define the usual norm on $\mathbb{R}$ to be

$$\begin{aligned} \| \cdot \| : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto \|\vec{\mathbf{u}}\| = \sqrt{\langle \vec{\mathbf{u}}, \vec{\mathbf{u}} \rangle}. \end{aligned}$$

Theorem 3 The projection of a vector $\vec{\mathbf{u}}$ on to another vector $\vec{\mathbf{v}}$ is

$$\mathrm{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\langle \vec{\mathbf{u}}, \vec{\mathbf{v}} \rangle}{\|\vec{\mathbf{v}}\|^2} \vec{\mathbf{v}}.$$

Subsection 2.1

GramSchmidt process

The aim of this process is to find a new basis $\Gamma = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n\}$ derived from a basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ such that it is orthonormal or in other words

$$\forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Gamma : \langle \hat{\mathbf{x}}, \hat{\mathbf{y}} \rangle = 0 \quad \text{and} \quad \|\hat{\mathbf{x}}\| = 1.$$

We find it as follows

$$\begin{aligned} \vec{\mathbf{u}}_1 &= \vec{\mathbf{v}}_1 & \hat{\mathbf{e}}_1 &= \frac{\vec{\mathbf{u}}_1}{\|\vec{\mathbf{u}}_1\|} \\ \vec{\mathbf{u}}_2 &= \vec{\mathbf{v}}_2 - \mathrm{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_2) & \hat{\mathbf{e}}_2 &= \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|} \\ \vec{\mathbf{u}}_3 &= \vec{\mathbf{v}}_3 - \mathrm{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_3) - \mathrm{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_3) & \hat{\mathbf{e}}_3 &= \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|} \\ &\vdots \\ \vec{\mathbf{u}}_n &= \vec{\mathbf{v}}_n - \mathrm{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_n) - \mathrm{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_n) - \ldots - \mathrm{proj}_{\vec{\mathbf{u}}_{n-1}}(\vec{\mathbf{v}}_n) & \hat{\mathbf{e}}_n &= \frac{\vec{\mathbf{u}}_n}{\|\vec{\mathbf{u}}_n\|} \end{aligned}$$

PART II

Conics and Quadrics

Section 3

Conics

We define a quadric form to be a mapping q

$$\begin{aligned} q : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto q(\vec{\mathbf{u}}) = \begin{bmatrix} \rule{1em}{0.4pt} & {}^T\vec{\mathbf{u}} & \rule{1em}{0.4pt} \end{bmatrix} A \begin{bmatrix} | \\ \vec{\mathbf{u}} \\ | \end{bmatrix}. \end{aligned}$$

Where the matrix A is a symmetric matrix.[1]
The conics under study are

[1] *symmetric matrices ($A = {}^T A$) is always diagonalizable*

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 && \text{ellipse (circle if } a = b\text{)} \\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 && \text{imaginary ellipse} \\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 && \text{hyperbola with asymptote } y = \pm\frac{b}{a}x \\
&\left.\begin{array}{l} y^2 = \pm 2px \quad p > 0 \\ x^2 = \pm 2py \quad p > 0 \end{array}\right\} && \text{parabolas} \\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 && \text{union of two straight lines} \\
&\left.\begin{array}{l} x = \text{const} \\ y = \text{const} \end{array}\right\} && \text{straight lines}
\end{aligned}$$

Subsection 3.1

Identification of the conics

Let the general equation of all conics be:

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

- $\boxed{\text{if } b = 0}$: then we simply group together the terms x^2 and x as well as y^2 and y followed by completing the square to get an equation of a conic.
- $\boxed{\text{if } b \neq 0}$: in this case we have to introduce a new system of reference which eliminates the existence of xy
  We do this by first defining a quadratic form $q(x, y) = ax^2 + 2bxy + cy^2$ using a matrix

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  which we diagonalize in to an or tho normal age-basis which we project our equation in to in order to get rid of the xy term

Example Find the nature of the conic

$$\Gamma : 5x^2 - 4xy + 8y^2 + \frac{20}{\sqrt{5}}x - \frac{80}{\sqrt{5}}y + 4 = 0.$$

Let $q(x, y) = 5x^2 - 4xy + 8y^2 = \left(\begin{smallmatrix} x & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 5 & -2 \\ -2 & 8 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) =^T \vec{\mathbf{u}} A \vec{\mathbf{u}}$. We find that the matrix A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 9$ with eigenvalues $\vec{\mathbf{u}}_1 = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $\vec{\mathbf{u}}_2 = \left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, the age vectors are already orthogonal so we just find $\vec{\mathbf{e}}_1 = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$ and $\vec{\mathbf{e}}_2 = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right)$, finally

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad D = \begin{pmatrix} 4 & \\ & 9 \end{pmatrix}.$$

We define $\left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right)$ to be any vector with basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

$$x = \frac{2}{\sqrt{5}}\alpha - \frac{1}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(2\alpha - \beta)$$
$$y = \frac{1}{\sqrt{5}}\alpha + \frac{2}{\sqrt{5}}\beta = \frac{1}{\sqrt{5}}(\alpha + 2\beta)$$

now we substitute x and y with α and β into Γ and we manipulate the expression until we get

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} = 1.$$

$\therefore \Gamma$ is an ellipse.

SUBSECTION 3.2

Tangent to a conic at point B

Theorem 4 The normal to vector to a conic Γ

$$\Gamma : ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

at a point $B \in \Gamma$ is defined to be

$$\nabla f(B) = \begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{(x_B, y_B)} \\ \left.\frac{\partial f}{\partial y}\right|_{(x_B, y_B)} \end{pmatrix}.$$

where $f(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$

The equation of a tangent to a conic at a point B is

$$a(x - x_B) + b(y - y_B) = 0.$$

where a and b are respectively the x and y components of the normal vector at B

SECTION 4

Quadrics

Definition 3 A quadric is any surface in 3D space with an equation of the form:

$$\underbrace{ax^2 + by^2 + cz^2 + 2dyz + 2exy + 2fxy}_{q(x,y,z)\text{:quadratic form of 3 variables}} + \underbrace{gx + hy + iz}_{\text{linear part}} + \underbrace{j}_{\text{constant}} = 0.$$

The quadrics under study are[2]

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboliod of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboliod of 2 sheets
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	Asymptote cone
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2pz$	Hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z^2$	Elliptic cone

If a one of variables is missing in the equation then the surface is said to be "(Conic name)-ic Cylinder". For example "Hyperbolic cylinder", "Circular cylinder", and "Elliptical cylinder"

[2] *if $a = b$ the surface is a surface of revolution of axis (Oz)*