

Complex Analysis

Semester 4

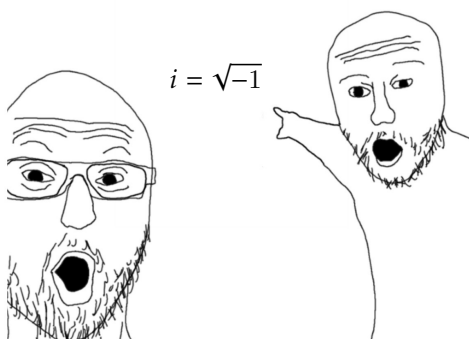
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Chapter 1

The Complex Plane

1.1 Algebra of the complex plane



Euler's formulas for sin and cos

$$\begin{aligned}\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ \tan(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}\end{aligned}$$

The n -th roots of unity are the set of complex numbers $(\zeta_1, \zeta_2, \dots, \zeta_n)$ are the complex numbers that satisfy the equation

$$z^n = w.$$

where $w = Re^{i\alpha}$. The solutions equation are

$$\zeta_k = \sqrt[n]{R}e^{i(\frac{\alpha+2k\pi}{n})}.$$

1.2 Topology of the complex plane

Theorem 1.2.1

The mapping

$$|z| : \mathbb{C} \longrightarrow \mathbb{R}^+$$

$$z = x + yi \longmapsto |x + yi| = \sqrt{x^2 + y^2}.$$

defines a norm on \mathbb{C} , so the complex plane is a normed space.

Theorem 1.2.2

The mapping

$$d(.,.) : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{R}^+$$

$$(z, w) \longmapsto d(z, w) = |z - w|.$$

defined a distance on \mathbb{C} , so the complex plane is a metric space.

Definition 1.2.1: Neighborhood

We call δ -neighborhood of z_0 an open disk centered at z_0 of radius δ

$$N_\delta(z_0) = \{z \in \mathbb{C} : |z - z_0| < \delta\}.$$

We call $N_\delta(z_0) - \{z_0\}$ a deleted δ -neighborhood. ($\{z \in \mathbb{C} : 0 < |z - z_0| < \delta\}$)

Definition 1.2.2

Let $z_0 \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$.

1. z_0 is called an *interior point* of Ω if

$$\exists \delta > 0, N_\delta(z_0) \subset \Omega.$$

2. z_0 is an *exterior point* of Ω if

$$\exists \delta > 0, N_\delta(z_0) \cap \Omega = \emptyset.$$

3. z_0 is a *boundary point* of Ω if

$$\forall \delta > 0, N_\delta(z_0) \cap \Omega \neq \emptyset \quad \text{and} \quad N_\delta(z_0) \cap \underbrace{\mathbb{C}^\Omega_{\mathbb{C}}}_{\mathbb{C}-\Omega} \neq \emptyset.$$

Definition 1.2.3

The set of all:

1. interior points: $\dot{\Omega}$
2. boundary points: $\partial\Omega$
3. the set $\Omega \cup \partial\Omega$ is called a closure of Ω denoted $\bar{\Omega}$

Note:-

$$\dot{\Omega} \subset \Omega \subset \bar{\Omega}.$$

and

$$\Omega \text{ is open} \Leftrightarrow \begin{cases} \Omega \cap \partial\Omega = \emptyset \\ \Omega = \dot{\Omega} \end{cases}.$$

If Ω is not open from all sides then it is not open. Same thing with closed.

If Ω is not open then it is not connected.

Ω is said to be *compact* if it is both *bounded and closed*.

Theorem 1.2.3 Bolzano-Weirstrass theorem

Every *bounded infinte* set admits at least one limit point

Paths

A path is a set of complex points Γ where

$$\Gamma = \{z(t) = x(t) + i y(t) \mid t \in [a, b]\}.$$

A simple path/Jordan arc if it does not cross itself

$$\forall t_1, t_2 \in [a, b[\quad t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2).$$

A closed path is a path such that

$$z(a) = z(b).$$

Chapter 2

Complex Functions

2.1 Limits and Differentiability

Note:-

When taking limits we can do the 2D limit where $x = \text{Re}(z)$ and $y = \text{Im}(z)$

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x + iy).$$

then we can take multiple paths to find the limit. However we can't take sufficient paths to prove a limit exists as there could exist one path that causes the limit to not exist, however we can use polar limits to prove that the limit exists. We take $x = r \cos(\theta) - x_0$ and $y = r \sin(\theta) - y_0$

$$\lim_{(r,\theta) \rightarrow (0,0)} f(r \cos(\theta) - x_0 + i(r \sin(\theta) - y_0)).$$

Theorem 2.1.1 Cauchy-Riemann equations

We define a complex function

$$f(x + iy) = u(x, y) + iv(x, y).$$

If f is differentiable on a point $z_0 = x_0 + iy_0$ then u and v satisfy the Cauchy-Riemann equations:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

Note that the converse is not true

To prove that a function f is differentiable at z_0 then we have to prove that u and v

$$\begin{cases} \text{exist in } \Omega \\ \text{are continuous at } (x_0, y_0) \\ \text{satisfy the Cauchy-Riemann equations at } (x_0, y_0) \end{cases}$$

2.1.1 Hyperbolic functions

$$\begin{aligned}\cosh z &= \frac{e^z + e^{-z}}{2} \\ \sinh z &= \frac{e^z - e^{-z}}{2} \\ \tanh z &= \frac{\sinh z}{\cosh z}\end{aligned}$$

Properties

- | | |
|--|--|
| a) $\cosh^2 z - \sinh^2 z = 1$ | b) $\cosh^2 z + \sinh^2 z = \cosh 2z$ |
| c) $\cosh z_1 + z_2 = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$ | d) $\sinh z_1 + z_2 = \sinh z_1 \cdot \cosh z_2 + \sinh z_2 \cdot \cosh z_1$ |
| e) $\cos iz = \cosh z$ | f) $\sin iz = i \sinh z$ |
| g) $\cosh iz = \cos z$ | h) $\sinh iz = i \sin z$ |

2.2 Harmonic functions

Definition 2.2.1: Harmonic function

A function $u(x, y)$, of class C^2 and defined on Ω , is said to be harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or in other words the Laplacian is equal to 0

$$\Delta u = \nabla^2 u = 0.$$

Theorem 2.2.1

Let a function $f = u + iv$ defined on Ω

$$f \text{ is holomorphic} \Leftrightarrow \begin{cases} u, v \text{ are of class } C^\infty \text{ in } \Omega \\ u, v \text{ satisfy the Cauchy-Riemann equations in } \Omega \\ u, v \text{ are harmonic in } \Omega \end{cases}.$$

Chapter 3

Integrals

Definition 3.0.1: Complex Integral

Let Ω be an open subset of \mathbb{C} and Γ a piecewise differentiable path from z_1 to z_2 . We define the integral of f along the path to be 2 different line integrals:

$$\int_{\Gamma} f(z)dz = \int_{\Gamma} (u + iv)(dx + idy) = \int_{\Gamma} (u dx - v dy) + i \int_{\Gamma} (v dx + u dy).$$

Theorem 3.0.1 Parametrization of the path

If the path Γ is parametrized by $\gamma(t) = x(t) + iy(t)$ where x, y are of class c^1 on $[a, b]$ then

$$\int_{\Gamma} f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)dt.$$

Theorem 3.0.2 ML-rule

In a path of Γ of length L , we can approximate the value of an integral along that path

$$\left| \int_{\Gamma} f(z)dz \right| \leq M \cdot L.$$

where

$$M = \sup_{z \in \Gamma} |f(z)| \quad \text{and} \quad L = \text{Length of the path } \Gamma = \int_a^b \sqrt{x'(t)^2 + y'(t)^2} dt.$$

Theorem 3.0.3 Cauchy's theorem

Let Γ be a simple closed curve. Let f be a holomorphic function on Γ and inside Γ , then

$$\oint_{\Gamma} f(z)dz = 0.$$

Note:-

Green-Riemann theorem states that

$$\oint_{\partial\Omega} (P(x, y)dx + Q(x, y)dy) = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Note:-

$$\int_{\Gamma^-} f(z)dz = - \int_{\Gamma} f(z)dz.$$

A consequence of Cauchy's theorem is that if a closed path C contains a discontinuity then the path of integration doesn't matter as long as the new path also contains the exact same discontinuity.

Theorem 3.0.4

Let Ω be a simply closed region. Let f be a holomorphic function on Ω , z_1 and z_2 be 2 point $\in \Omega$. Then the integral of $f(z)$ is independent of the path taken from z_1 to z_2

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Theorem 3.0.5 Liouville's theorem

- f is holomorphic in \mathbb{C}
- f is bounded in \mathbb{C}

$$\exists M \in \mathbb{R}_+, \forall z \in \mathbb{C}, |f(z)| \leq M.$$

then f is constant in \mathbb{C}

Theorem 3.0.6 Mean value theorem

Let γ_r be a circle of center a and radius $r > 0$. If f is a holomorphic on and in γ_r then

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

Theorem 3.0.7 Cauchy's integral formula

Let Γ is a simple closed curve and the function $f(z)$ is holomorphic on Γ and its interior. Then:

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma^+} \frac{f(z)}{z-a} dz.$$

and the general form of the formula is

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_{\Gamma^+} \frac{f(z)}{(z-a)^{n+1}} dz.$$

3.1 Primitives

Definition 3.1.1: Primitives

Let f be a complex function, defined in an open set $\Omega \subset \mathbb{C}$.

We call a primitive of f on Ω , any function F such that F is holomorphic in Ω and $\forall z \in \Omega F'(z) = f(z)$

$$F(z) = \int f(z)dz.$$

Note:-

If f admits a primitive on the open set Ω then f is holomorphic in Ω

Let the path γ goes from z_1 to z_2 in Ω then

$$\int_{\gamma} f(z)dz = F(z_2) - F(z_1).$$

Note:-

$$\oint_{\gamma} f(z)dz = 0 \Rightarrow f \text{ is holomorphic in } \Omega.$$