Differential Geometry

Contents

PART

I

Prerequisites

SECTION 1

Matrices

Theorem 1 To prove a system of vectors $\{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \ldots, \vec{\mathbf{u}}_n\}$ is free we prove:

$$\det \begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_1 \\ | & | & | & & | \end{bmatrix} \neq 0.$$

Theorem 2 A transition matrix $P_{B\to B'}$between 2 basis $B = \{\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3\}$ and $B' = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ we start by solving the system

$$\begin{bmatrix} | & | & | & & | \\ \vec{\mathbf{u}}_1 & \vec{\mathbf{u}}_2 & \vec{\mathbf{u}}_3 & \cdots & \vec{\mathbf{u}}_n \\ | & | & | & & | \end{bmatrix} \begin{bmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{\mathbf{v}}_n \\ | \end{bmatrix}.$$

or in other words finding

$$\begin{cases} \vec{\mathbf{v}}_1 & = \alpha_1\vec{\mathbf{u}}_1 + \beta_1\vec{\mathbf{u}}_2 + \gamma_1\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_2 & = \alpha_2\vec{\mathbf{u}}_1 + \beta_2\vec{\mathbf{u}}_2 + \gamma_2\vec{\mathbf{u}}_3 \\ \vec{\mathbf{v}}_3 & = \alpha_3\vec{\mathbf{u}}_1 + \beta_3\vec{\mathbf{u}}_2 + \gamma_3\vec{\mathbf{u}}_3 \end{cases}.$$

Finally we say that

$$P_{B\to B'} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

transition matrices are always square and invertible ($\det P \neq 0$)

Remark To find the transition matrix in the inverse direction (from B' to B) we simply do

$$P_{B'\to B} = {P_{B\to B'}}^{-1}.$$

SECTION 2

Vectors

Definition 1 We define an operation called the scalar product (dot product)

$$\begin{aligned}\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}\\ \vec{\mathbf{u}},\vec{\mathbf{v}} &\longmapsto \langle\vec{\mathbf{u}},\vec{\mathbf{v}}\rangle = \sum_{n=1}^{n} v_i \cdot u_i.\end{aligned}$$

Definition 2 We define the usual norm on $\mathbb{R}$ to be

$$\begin{aligned}\|\cdot\| : \mathbb{R}^n &\longrightarrow \mathbb{R}\\ \vec{\mathbf{u}} &\longmapsto \|\vec{\mathbf{u}}\| = \sqrt{\langle\vec{\mathbf{u}},\vec{\mathbf{u}}\rangle}.\end{aligned}$$

Theorem 3 The projection of a vector $\vec{\mathbf{u}}$ on to another vector $\vec{\mathbf{v}}$ is

$$\operatorname{proj}_{\vec{\mathbf{v}}}(\vec{\mathbf{u}}) = \frac{\langle\vec{\mathbf{u}},\vec{\mathbf{v}}\rangle}{\|\vec{\mathbf{v}}\|^2}\vec{\mathbf{v}}.$$

Subsection 2.1

GramSchmidt process

The aim of this process is to find a new basis $\Gamma = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \ldots, \hat{\mathbf{e}}_n\}$ derived from a basis $B = \{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_n\}$ such that it is orthonormal or in other words

$$\forall \hat{\mathbf{x}}, \hat{\mathbf{y}} \in \Gamma : \langle\hat{\mathbf{x}},\hat{\mathbf{y}}\rangle = 0 \quad \text{and} \quad \|\hat{\mathbf{x}}\| = 1.$$

We find it as follows

$$\begin{aligned}
\vec{\mathbf{u}}_1 &= \vec{\mathbf{v}}_1 & \hat{\mathbf{e}}_1 &= \frac{\vec{\mathbf{u}}_1}{\|\vec{\mathbf{u}}_1\|}\\
\vec{\mathbf{u}}_2 &= \vec{\mathbf{v}}_2 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_2) & \hat{\mathbf{e}}_2 &= \frac{\vec{\mathbf{u}}_2}{\|\vec{\mathbf{u}}_2\|}\\
\vec{\mathbf{u}}_3 &= \vec{\mathbf{v}}_3 - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_3) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_3) & \hat{\mathbf{e}}_3 &= \frac{\vec{\mathbf{u}}_3}{\|\vec{\mathbf{u}}_3\|}\\
&\vdots\\
\vec{\mathbf{u}}_n &= \vec{\mathbf{v}}_n - \operatorname{proj}_{\vec{\mathbf{u}}_1}(\vec{\mathbf{v}}_n) - \operatorname{proj}_{\vec{\mathbf{u}}_2}(\vec{\mathbf{v}}_n) - \ldots - \operatorname{proj}_{\vec{\mathbf{u}}_{n-1}}(\vec{\mathbf{v}}_n) & \hat{\mathbf{e}}_n &= \frac{\vec{\mathbf{u}}_n}{\|\vec{\mathbf{u}}_n\|}
\end{aligned}$$

PART II

Conics and Quadrics

Section 3

Quadratic form

We define a quadric form to be a mapping q

$$\begin{aligned} q : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ \vec{\mathbf{u}} &\longmapsto q(\vec{\mathbf{u}}) = \begin{bmatrix} \text{---} & {}^T\vec{\mathbf{u}} & \text{---} \end{bmatrix} A \begin{bmatrix} | \\ \vec{\mathbf{u}} \\ | \end{bmatrix}. \end{aligned}$$

Where the matrix A is a symmetric matrix.[1]
The conics understudy are

[1] *symmetric matrices ($A = {}^T A$) is always diagonalizable*

$$\begin{aligned}
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 && \text{ellipse (circle if } a = b) \\
&\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 && \text{imaginary ellipse} \\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1 && \text{hyperbola with asymptote } y = \pm \frac{b}{a} x \\
&\left.\begin{aligned} y^2 &= \pm 2px \quad p > 0 \\ x^2 &= \pm 2py \quad p > 0 \end{aligned}\right\} && \text{parabolas} \\
&\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 && \text{union of two straight lines} \\
&\left.\begin{aligned} x &= \text{const} \\ y &= \text{const} \end{aligned}\right\} && \text{straight lines}
\end{aligned}$$