Analysis 3

Contents

PART

I

Sequence of Functions

1 Introduction

In previous courses, we analysed the convergence of sequences of numbers (example: $U_n = \left\{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\right\} = \sum_{n=1}^{\infty} \frac{1}{n^2}$) with a series of tests. In this course we will be analysing sequences of *functions* $f_n(x)$.
An example, is $f_n(x) = \frac{x}{x+n} = \{f_1, f_2, f_3, \ldots\} = \left\{\frac{x}{x+1}, \frac{x}{x+2}, \frac{x}{x+3}, \ldots\right\}$.

There are 2 ways these sequences can converge: pointwise and uniformly

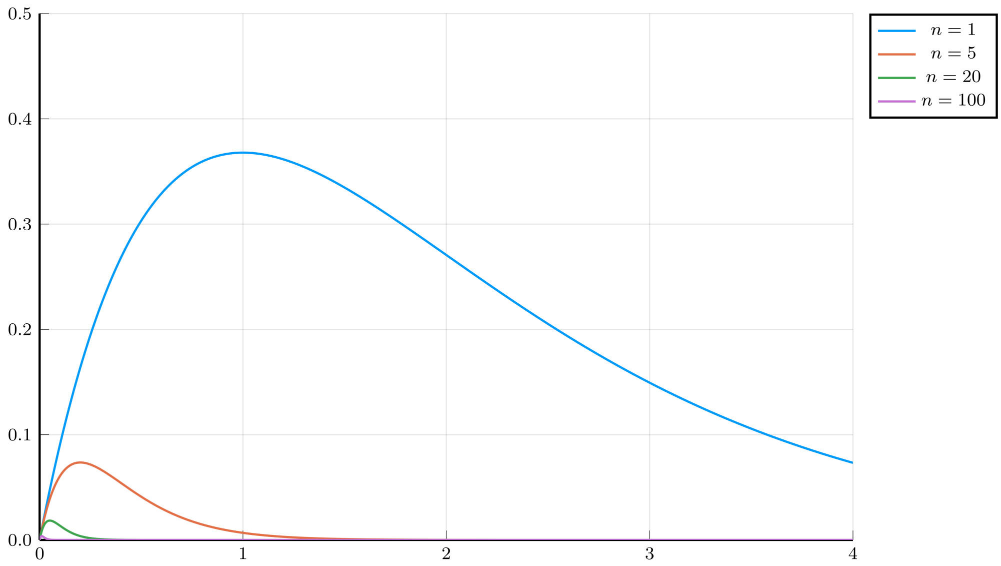

Figure 1. Plot of the sequence $f_n(x) = xe^{-nx}$

2 Pointwise convergence

This is a very natural way of proving convergence since all you have to do is fix f_n to a point x then the sequence just becomes an ordinary sequence of numbers, and if they all converge to a number we can define a limit function f and say that they converge to f pointwisely.

Definition 2.1. We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges pointwise to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall x \in I\ \forall \epsilon > 0\ \exists n \in \mathbb{N}\ \forall n \geq \mathbb{N} : \quad |f_n(x) - f(x)| < \epsilon.$$

You either prove convergence using the definition or by doing:

1. Let $x = 0$ then find $\lim_{n\to\infty} f_n(0) = \text{some } f(x)$
2. Then let $x \neq 0$ and again find $\lim_{n\to\infty} f_n(x) = f(x)$
3. If neither of the results are unbounded $\pm\infty$ then we say $f_n(x)$ is convergent to some $f(x)$

Remark. if the result of step 1 is $g(x)$ and step 2 results in $h(x)$ where $g(x) \neq h(x)$ then we define the limit function:

$$f(x) = \begin{cases} g(x) & x = 0 \\ h(x) & x \in]0,1] \end{cases}.$$

3 Uniform convergence

The idea of uniform convergence is that the sequence always approaches it's limit function as the value of n increases.

Definition 3.1. We say that a sequence of functions f_n where $f_n : I \to \mathbb{R}, I \subset \mathbb{R}$, converges uniformly to function $f : I \to \mathbb{R}$ on the interval I if:

$$\forall \epsilon > 0 \, \exists N \in \mathbb{N} \, \forall n \geq N \, \forall x \in I : \sup_{x\in I} |f_n(x) - f(x)| < \epsilon.$$

Remark. We can also prove uniform convergence by proving

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0.$$

Let f_n be a sequence of functions defined on $E \subset \mathbb{R}\}$. We will say that f_n converges pointwise to the function f if for every $\varepsilon > 0$ and for every $x \in E, \exists N \in \mathbb{N}$ such that for al $n > N$ $|f_n(x) - f(x)| < \varepsilon$ we write $\lim_{n\to\infty} f_n(x) = f(x)$.[1]

[1] *The integer N depends on ε and x ;$N(\varepsilon, x)$*

Example. Let $f_x(x) = \dfrac{x}{x+n}$;$x \in [0,1] = E$, study the pointwise convergence of f_n.

$$\lim_{n\to\infty} f_n(x) = 0.$$

We conclude that f_n converges pointwise to $f(x) = 0 \quad \forall x \in [0,1]$

Example. let $f_n(x) = \dfrac{nx}{1+nx}$ where $x \in [0,1] = E$. Study the pointwise convergence of f_n.

- $x = 0$,$x \to +\infty \implies nx \to \infty$. Undetermined form.
  $f_n(0) = 0 \implies \lim_{n\to\infty} f_n(0) = 0$
- $x \neq 0$ (x is fixed)
  $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{nx}{1+nx} = 1$

Then f_n converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \in]0,1] \end{cases}.$$

3.1 Second method

using the definition of the pointwise convergence.

$$\forall x \in E, \forall \varepsilon > 0\ \exists N \in \mathbb{N} / \forall n > N\ |f_n(x) - f(x)| < \varepsilon.$$

we must find N first for $x = 0$ $|f_n(0) - f(0)| = 0 - 0 = 0 < \varepsilon$ then the choice of N is arbitrary.
for $x \neq 0$

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx}\right| = \left|\frac{1}{nx+1}\right| = \frac{1}{nx+1} < \varepsilon.$$

We choose N such that $N \geq \dfrac{1-\varepsilon}{\varepsilon x}$

4 Uniform convergence

Definition 4.1. Let f_n be a sequence of functions defined on $E \subset \mathbb{R}$. We say that f_n converges uniformly to the limit function f if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} / \forall n > N \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon.$$

$f(x)$ is the limit function $x \in E$.[2]

[2] N *depends on only* $\varepsilon, N(\varepsilon)$

Remark.
- Pointwise convergence means that at every point the sequence of function has its own speed of convergence (that can be very fast at some points and very slow at others)
- Uniform convergence means there is an overall speed of convergence

Example. $f_n(x) = \dfrac{1}{n}x^2$.
$\lim\limits_{n\to\infty} f_n(x) = 0$ the sequence $\dfrac{1}{n}x^2$ CV pointwise to $f(x) = 0$ (no overall speed of CV for all points)

Example. $g_n(x) = \dfrac{\sin(nx)}{n}$; $x \neq 0$
$\lim\limits_{n\to\infty} g_n(x) = 0$ the sequence CV uniform to $g(x) = 0$ (there is an overall speed of CV for all points)

Remark. To prove the uniform convergence of a sequence of functions f_n defined on E to the limit function, we may prove the pointwise convergence by proving the integer N is independent of x or

$\lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0$

Example. Let $f_n(x) = \dfrac{x}{x+n}$; $x \in [0,1]$ and $n \in \mathbb{N}$.
Study the uniform convergence of $f_n(x)$ to $f(x) = 0$ by showing that $N = N(\varepsilon)$.

$\lim_{n\to\infty} f_n(x) = 0$

$$|f_n(x) - f(x)| = \frac{x}{x+n} < \varepsilon \implies n > x\left(\frac{1}{\varepsilon} - 1\right).$$

$x \in [0,1] \implies n > \dfrac{1}{\varepsilon}$, $N > \dfrac{1}{\varepsilon} \implies N = N(\varepsilon)$ then f_n converges uniformly to $f(x) = 0$

Example. Let $f_n(x) = \dfrac{nx}{1+nx}$ defined on $[1,2]$. Show the uniform convergence of f_n to the limit function using sup.

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \dfrac{nx}{1+nx} = 1$ (x is fixed). f_n converges pointwise to $f(x) = 0$ on $[1,2]$.
$\lim_{n\to\infty} \sup |f_n(x) - f(x)|$
Let $g(x) = |f_n(x) - f(x)| = \left|\dfrac{nx}{1+nx} - 1\right| = \left|-\dfrac{1}{1+nx}\right| = \dfrac{1}{1+nx}$
$g'(x) = -\dfrac{n}{(1+nx)^2} < 0\ \forall x \in [1,2]$ TABLE OF VAR HERE
$\lim_{n\to\infty} \sup |f_n(x) - f(x)| = \lim_{n\to\infty} \sup g(x) = \lim_{n\to\infty} \dfrac{1}{1+n} = 0.$
$\therefore f_n$ converges uniformly to $f(x) = 1$

Example. Let $f_n(x) = ne^{-nx}$; $x \in [0, +\infty[$.

1. Study the pointwise convergence of f_n
2. Study the pointwise convergence of f_n on $[0, +\infty[$
3. Study the uniform convergence of f_n on $[1, +\infty[$

1. $x = 0$; $\lim_{n\to\infty} f_n(x) = 1$
   $x \neq 0$ (x is fixed)
   $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx} = \lim_{n\to\infty} \dfrac{nx}{e^{nx}} = 0\ \forall x \in [0,1]$
2. $|f_n(x) - f(x)| = g(x) = nxe^{-nx}$.
   $g'(x) = ne^{-nx}(1 - nx)$
   $g'(x) = 0 \implies x = \dfrac{1}{n}$
   $\sup |f_n(x) - f(x)| = \dfrac{1}{e} \neq 0$ then f_n doesn't converge uniformly to $f(x) = 0$ for $x \in [0, +\infty[$
3. $x \in [1, +\infty[\ \dfrac{1}{n} < 1$
   $g(x)$ decreases $\implies \sup |f_n(x) - f(x)| = g(1) = ne^{-n}$
   $\lim_{n\to\infty} \sup |f_n(x) - f(x)| = \lim_{n\to\infty} \dfrac{n}{e^n} = 0$ then f_n converges uni-

formly to $f(x) = 0$ for $x \in [1, +\infty[$

5 Sequence of continuous functions

Theorem 5.1. If a sequence of continuous functions on $E \subset \mathbb{R}$ converges uniformly to the limit function f then f is continuous.

Remark.
- If a sequence of continuous functions f_n converges pointwise to a discontinuous function f, then the convergence is not uniform.
- The continuity of the limit function $f(x)$ on E is a necessary but not sufficient condition for the uniform sequence of functions f_n.

In the previous year, you took numberical sequences, for example $U_n = \frac{1}{n+1} \implies \{\frac{1}{2}, \frac{1}{3}, \ldots\}$, but this year we will be studying sequences of *functions* so we have to study 2 variables, the usual n and x

In this case, there 2 types of convergence pointwise and uniform. In pointwise, we let x be a fixed variable and instead of studying the entire continuoum of values of n we study only one value. so

Definition 5.1. If f_n is a sequence of functions defined on $x \in E \subset \mathbb{R}$, we say that f_n converges pointwise to the functions f if for $\forall \epsilon > 0$ and $\forall x \in E, \exists N \in \mathbb{N}$ such that for all $n > N$ $|f_n(x) - f(x)| < \epsilon$we write $\lim_{n\to\infty} f_n(x) = f(x)$.

Example. $f_n(x) = \frac{x}{x+n}$ $x \in [0,1] = E \subset \mathbb{R}$
Study the pointwise convergence of f_n.
We let x be fixed and take the limit $\lim_{n\to\infty} f_n(x) = 0$. We conclude that f_n converges

Example. $f_n(x) = \frac{nx}{1+nx}$ $x \in [0,1] = E$.
Study the pointwise convergence of f_n.

- $x = 0$ $f_n(0) = \frac{0}{1+0} = 0$
- x fixed $x \in]0,1]$ $\lim_{n\to\infty} \frac{nx}{1+nx} = \lim_{n\to\infty} \frac{x}{x} = 1$ (by Hopital)

IF:

1. $\lim_{n\to a} f(x) = 0$ and $\lim_{n\to a}$ type 0^0
2. $\lim_{n\to a} f(x) = \infty$ and $\lim_{n\to a} g(x) = 0$ type ∞^0
3. $\lim_{n\to a}$ and $\lim_{n\to a} g(x) = \infty$ type 1^∞

Example. $f_n(x) = \frac{nx}{1+n}$ using the definition