Digital Signal Processing
Semester 7

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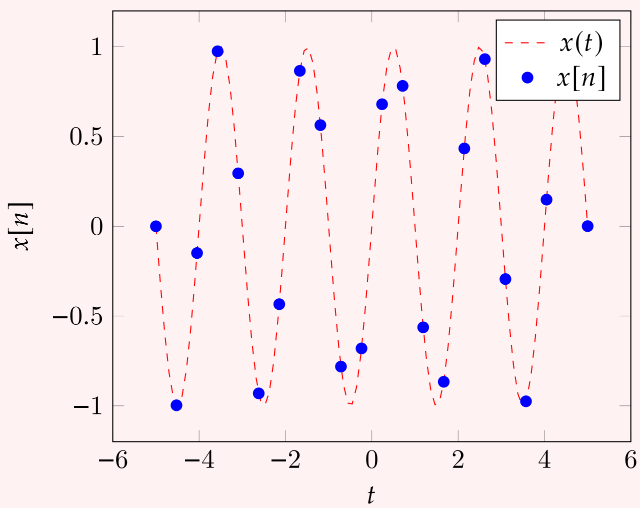
Contents

Chapter 1

General Definitions for Digital Signals

Definition 1.0.1: Digital Signal

A digital signal is a sequence of numbers $x[n]$ where $n \in \mathbb{Z}$. The signal is said to be discrete in time. The signal can be represented as a function of time $x(t)$ where $t = nT_s$ and T_s is the sampling period $x[n] = x(nT_s)$.



Power and Energy

The energy of a signal is defined as

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2.$$

And the power as

$$P_x = \lim_{N\to\infty} \frac{\sum_{n=-N}^{N} |x[n]|^2}{2N+1}.$$

Periodic Signal

A digital periodic signal is a signal $x[n]$ with a period $N \in \mathbb{Z}$ such that $\forall n \in \mathbb{Z} x(n+N) = x[n]$.

Example 1.0.1

For a periodic signal $\cos(\pi n_0 n)$ the condition for periodicity becomes

$$\cos(\pi n_0 n) = \cos(\pi n_0 (n+N)).$$

$$\begin{aligned} \pi n_0 N &= 2k\pi \\ N &= \frac{2k}{n_0} \end{aligned}$$

We choose the smallest possible $k \in \mathbb{Z}$ to obtain an $N \in \mathbb{Z}$.

Even and Odd

$$\begin{aligned} x[n] &= x[-n] \quad \text{Even} \\ x[n] &= -x[-n] \quad \text{Odd} \end{aligned}$$

$$x[n] = \underbrace{\frac{x[n]+x[-n]}{2}}_{\text{Even}} + \underbrace{\frac{x[n]-x[-n]}{2}}_{\text{Odd}}.$$

Usually Used Signals

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \quad ; \quad u[n] = \begin{cases} 1 & n \geqslant 0 \\ 0 & n < 0 \end{cases} \quad ; \quad \text{sgn}[n] = \begin{cases} 1 & n \geqslant 0 \\ -1 & n < 0 \end{cases} \quad ; \quad x[n] = a^n \quad a \in \mathbb{C}$$

1.1 Linear Time Invariant Systems

For a LTI system, an input $x[n]$ will produce an output $y[n]$ such that the relation can be expressed as

1. Analytically: E.g. $y[n] = x[n] - 2x[n-1]$.
2. As a function of the impulse response $h[n]$: $y[n] = f(x[n], x[n-1], \ldots, h[n], h[n-1], \ldots)$.

Note:-

A system is said to be relaxed if the output is zero for zero input. (i.e. the only energy source is the input)

An input $x[n]$ can be decomposed into a sum of shifted impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

If a system is linear, then the output of the system to the input $x[n]$ can be expressed as a sum of the outputs to the shifted impulses

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \underbrace{x[n] * h[n]}_{\text{Convolution}}.$$

Note:-

The convolution of two signals $x[n]$ and $h[n]$ is commutative

$$x[n] * h[n] = h[n] * x[n].$$

If the system is causal, then the impulse response $h[n]$ is zero for $n < 0$. The output of the system can be expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k].$$

If both the input and the impulse response are causal, then the output can be expressed as

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k].$$

1.1.1 Stability

A system is said to be stable if the output is bounded for a bounded input.

$$|x[n]| \leqslant \alpha \quad \forall n \implies |y[n]| \leqslant \beta.$$

$$|y[n]| \leqslant \sum_{k=-\infty}^{\infty} \alpha |h[n-k]|.$$

So a system is stable if it satisfies the sufficient condition

$$\sum_{n=-\infty}^{\infty} |h[n]| \leqslant \gamma.$$

Definition 1.1.1: Difference Equation

A difference equation is a relation between the input $x[n]$, the output $y[n]$ and the impulse response $h[n]$ of a system. The relation can be expressed as

$$y[n] = -\sum_{k=0}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

where a_k and b_k are constants $\in \mathbb{C}$.

- If $N = 0$, the system is said to be a moving average system.
- If $M = 0$, the system is said to be an autoregressive system.

1.2 Response of LTI Systems to an input $x[n]$

There are three methods to find the output of an LTI system to an input $x[n]$.

1. Integration (point by point)
2. Convolution (using the impulse response)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k].$$

3. Finding the analytical expression for the output

1.2.1 Analytical Expression

The analytical expression of $y[n]$ is the sum of the terms

$$y[n] = y_{\text{ZI}}[n] + y_{\text{ZS}}[n].$$

Where $y_{\text{ZI}}[n]$ is the zero input response (the response of the system to the input $x[n] = 0$) and $y_{\text{ZS}}[n]$ is the zero state response (the response of the system to the input $x[n]$ when the system is relaxed).

Zero Input Response

It is the solution to the equation

$$y_{\text{ZI}}[n] = -\sum_{k=1}^{N} a_k y[n-k].$$

If the zero input response is non-zero, then the zero state response is 0.

Zero State Response

It is the solution to the equation

$$y_{\text{ZS}}[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

The N initial conditions become 0 if the system is relaxed.

The zero state response is comprised of the homogenous response and the forced response.

$$y_{\text{ZS}}[n] = y_h[n] + y_f[n].$$

In other words, consider the following difference equation

$$y[n] = \underbrace{0.2y[n-1] - 0.1y[n-2]}_{y_{\text{ZI}}[n]} + \underbrace{x[n] - x[n-1]}_{y_{\text{ZS}}[n]}.$$

Characteristic Equation

The characteristic equation of a difference equation is the equation obtained by setting the output to zero and substituting $y[n] = \lambda^n$.

$$\lambda^n = \sum_{k=1}^{N} a_k \lambda^{n-k}.$$

The roots of the characteristic equation are the poles of the system, and the system can be expressed as the sum of the roots

$$y_h[n] = \sum_{i=1}^{N} C_i {\lambda_i}^n \quad \text{where} \quad C_i \text{ is a } i\text{-th order polynomial } = \sum_{j=0}^{i} c_j n^j.$$

The constants c_i can be found by substituting the initial conditions.
To find the forced response $y_f[n]$ we substitute it (along with $x[n]$) with $ku[n]$ where k is a constant. Then we find the value of k that satisfies the difference equation.
Finally, y_{ZS} is the sum of the homogenous and forced responses (when the system is relaxed).

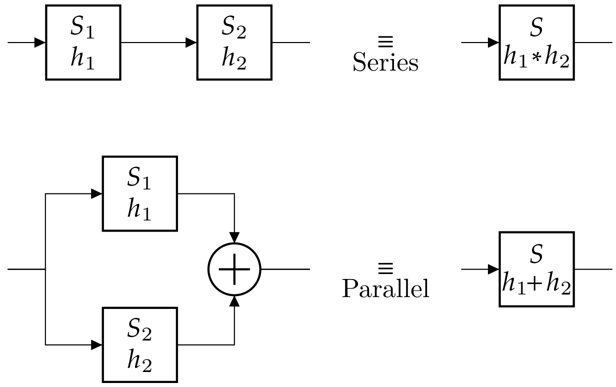
$$y_{\text{ZS}}[n] = \sum_{i=1}^{N} C_i {\lambda_i}^n + ku[n] \quad \text{where} \quad n \geqslant 0.$$

And y_{ZI} is the solution to the difference equation with $x[n] = 0$.

$$y_{\text{ZI}}[n] = \sum_{i=1}^{N} C_i {\lambda_i}^n.$$

Note:-
The terms C_i are different in y_{ZI} and y_{ZS}, but the terms λ_i are the same.

1.3 Multiple LTI Systems



Types of systems

Denote by $\mathcal{S}$ a system with impulse response $h[n]$. Then

- $\mathcal{S}$ is said to be *causal* if $h[n] = 0$ for $n < 0$.
- $\mathcal{S}$ is said to be *stable* if $|x[n]| \leqslant \alpha \implies |\mathcal{S}\{x[n]\}| \leqslant \beta$.
- $\mathcal{S}$ is said to be *invertible* if there exists a system $\mathcal{S}^{-1}$ such that $\mathcal{S}^{-1}\mathcal{S} = \delta[n]$.
- $\mathcal{S}$ is said to be *linear* if $\mathcal{S}\{a_1x_1[n] + a_2x_2[n]\} = a_1\mathcal{S}\{x_1[n]\} + a_2\mathcal{S}\{x_2[n]\}$.
- $\mathcal{S}$ is said to be *time invariant* if $\mathcal{S}\{x[n]\} = y[n] \implies \mathcal{S}\{x[n-k]\} = y[n-k]$.

Chapter 2

Frequency Analysis of Signals

2.1 Fourier Transform

Definition 2.1.1: Discrete Fourier Transform

The Fourier Transform of a signal $x[n]$ is defined as

$$X(f) = \mathcal{F}_t\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.$$

The Fourier Transform of a signal $x[n]$ exists if the signal is absolutely summable. I.e.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty.$$

Properties

- Linearity: $\mathcal{F}\{a_1x_1[n] + a_2x_2[n]\} = a_1X_1(f) + a_2X_2(f)$.
- Time Shifting: $\mathcal{F}\{x[n-k]\} = e^{-j\omega k}X(f)$.
- Inversion: $\mathcal{F}\{x[-n]\} = X(-f)$.
- Frequency Shifting: $\mathcal{F}\left\{e^{j\omega_0 n}x[n]\right\} = X(f-f_0)$.
- Convolution: $\mathcal{F}\{x[n] * h[n]\} = X(f) \cdot H(f)$.
- Multiplication: $\mathcal{F}\{x[n] \cdot h[n]\} = X(f) * H(f)$.
- Derivative: $\mathcal{F}\{nx[n]\} = \dfrac{j}{2\pi}\dfrac{\mathrm{d}X(f)}{\mathrm{d}f}$.

Power Spectral Density

The power spectral density of a signal $x[n]$ is defined as

$$S_x(f) = |X(f)|^2 = \frac{\mathrm{d}E}{\mathrm{d}f}.$$

2.2 System Response

Let $H(f)$ be the Transfer Function of a system.

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\mathcal{F}\{y[n]\}}{\mathcal{F}\{x[n]\}}.$$

Take for example the difference equation

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$
$$Y(f) = -\sum_{k=1}^{N} a_k e^{-j\omega k} Y(f) + \sum_{k=0}^{M} b_k e^{-j\omega k} X(f)$$
$$\frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

2.2.1 Frequency Response

The frequency response of a system to an input $x[n]$ is defined as

$$|y[n]| = |x[n]| \cdot |H(f_0)| \quad \text{and} \quad \arg\{y[n]\} = \arg\{x[n]\} + \arg\{H(f_0)\}.$$

Where f_0 is the frequency of the input signal.

For example, if $x[n] = 3\sin\left(\frac{\pi}{5}n + \frac{2\pi}{3}\right)$ and $H(f)\big|_{s=\frac{1}{10}} = 2e^{j\frac{\pi}{3}}$, then the output will be

$$y[n] = 6\sin\left(\frac{\pi}{5}n + \frac{2\pi}{3} + \frac{\pi}{3}\right).$$

Note:-

The frequency of a constant signal is 0.