1 Data and methodology

1.1 Data

We worked with daily returns on the Euro Stoxx 50 Price Index[1] denoted in EUR from 01 January, 1987 to 27 April, 2021. It is the leading blue-chip index of the Eurozone, was founded in 1999 and covers 50 of the most liquid and largest (in terms of free-float market capitalization) stocks. For its composition and computation we refer to the factsheet [1]. The Euro Stoxx 50 Price index was chosen while this one has more data available (going back to 1987).

Table 1.1 provides the main statistics describing the return series analyzed. Let daily returns be computed as $R_t = 100\,(\ln P_t - \ln P_{t-1})$,where P_t is the index price at time t and P_{t-1} is the index price at $t-1$.

The arithmetic mean of the series is 0.017% with a standard deviation of 1.307% and a median of 0.036 which translate to an annualized mean of 4.208% and an annualized standard deviation of 20.748%. The skewness statistic is highly significant and negative at -0.31 and the excess kurtosis is also highly significant

[1]The same analysis has been performed for the INDEX 1, INDEX 2, INDEX 3 and the INDEX 4 indexes with the same conclusions. The findings of these researches are available upon requests.

and positive at 7.207. These 2 statistics give an overview of the distribution of the returns which has thicker tails than the normal distribution with a higher presence of left tail observations. A formal test such as the Jarque-Bera one with its statistic at 10429.919 and a high statistical significance, confirms the non normality feeling given by the Skewness and Kurtosis.

Table 1.1: Summary statistics of the returns

Statistics	Euro Stoxx 50	Standardized Residuals
Arithmetic Mean	0.0167	-0.0409
Median	0.0358	-0.0192
Maximum	10.4376	5.7128
Minimum	-13.2404	-11.7738
Stdev	1.307	0.9992
Skewness	-0.31	-0.6326
	(0***)	(0***)
Excess Kurtosis	7.2071	5.1341
	(0***)	(0***)
Jarque-Bera	19520.3072***	10429.9193***

Notes

[1] This table shows the descriptive statistics of the daily percentage returns of EURO STOXX 50 over the period 1987-01-02 to 2021-04-27 (8953 observations). Including arithmetic mean, median, maximum, minimum, standard deviation. The skewness, excess kurtosis with p-value and signicance and the Jarque-Bera test with significance.

[2] The standardized residual is derived from a maximum likelihood estimation (simple GARCH model) as follows:
$R_t = \alpha_0 + \alpha_1 R_{t-1} + z_t \sigma_t$
$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_2 \sigma_{t-1}^2,$
Where z is the standard residual (assumed to have a normal distribution).

[3] *, **, *** represent significance levels at the 5

The right column of table 1.1 displays the same descriptive statistics but for the standardizes residuals obtained from a simple GARCH model as mentioned in table 1.1 in Note 2. Again, Skewness statistic at -0.633 with a high statistical significance level and the excess Kurtosis at 5.134 also with a high statistical significance, suggest a non normal distribution of the standardized residuals and the Jarque-Bera statistic at 19520.307, given its high significance, confirms the rejection of the normality assumption.

As can be seen in figure 1.1 the Euro area equity and later, since 1999 the Euro Stoxx 50 went up during the tech (“dot com”) bubble reaching an ATH of €5464.43. Then, there was a correction to boom again until the burst of the 2008 financial crisis. After which it decreased significantly. With an ATL at 09 March, 2009 of €1809.98. There is an improvement, but then the European debt crisis, with it’s peak in 2010-2012, occurred. From then there was some improvement until the “health crisis”, which arrived in Europe, February 2020. This crisis recovered very quickly reaching already values higher then the pre-COVID crisis level.

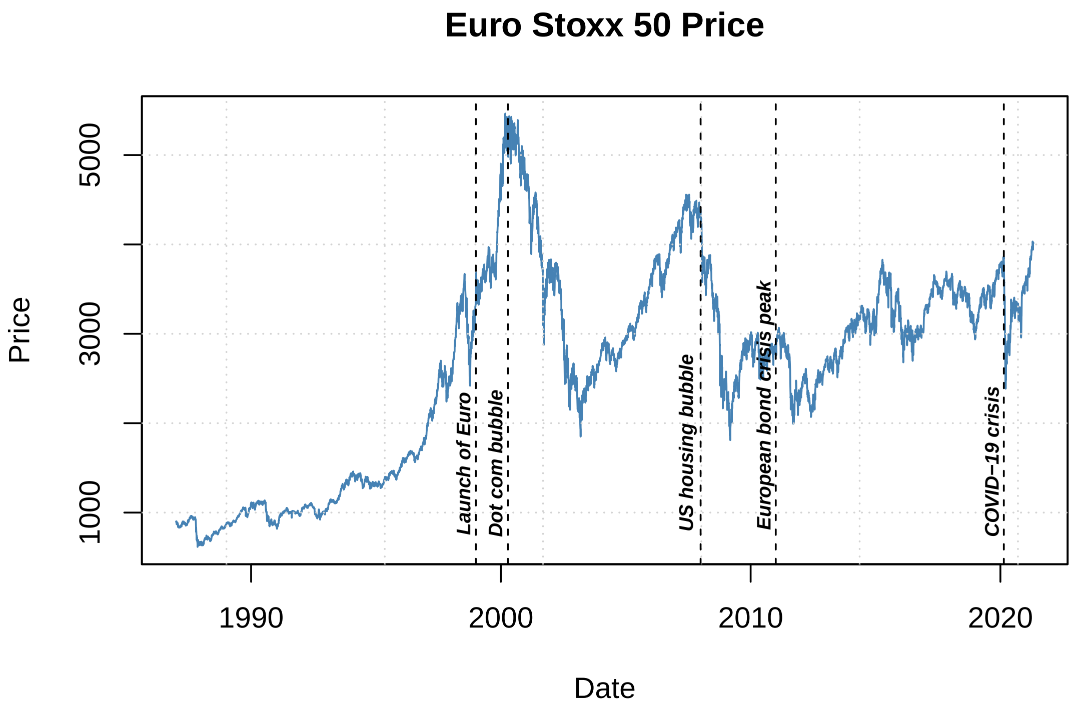

Figure 1.1: Euro Stoxx 50 Price Index prices

In figure 1.2 the daily log-returns are visualized. A stylized fact that is observable is the volatility clustering. As can be seen: periods of large volatility are mostly followed by large volatility and small volatility by small volatility.

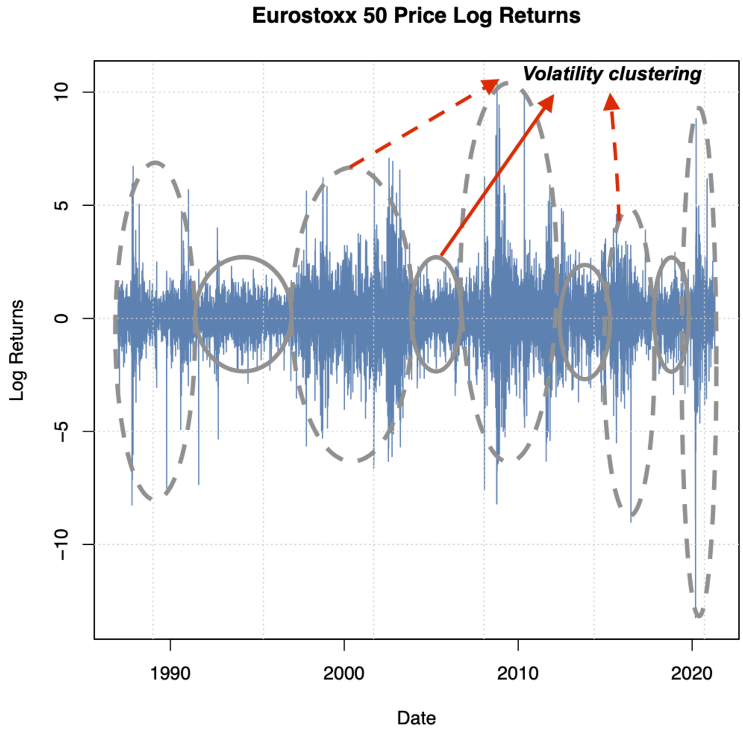

Figure 1.2: Euro Stoxx 50 Price Index log returns

In figure 1.3 you can see a proxy for risk, the rolling volatility over one month (22 trading days) calculated using a rolling window of 252 days. As in figure 1.2, you can see again the pattern of volatility clustering arise.

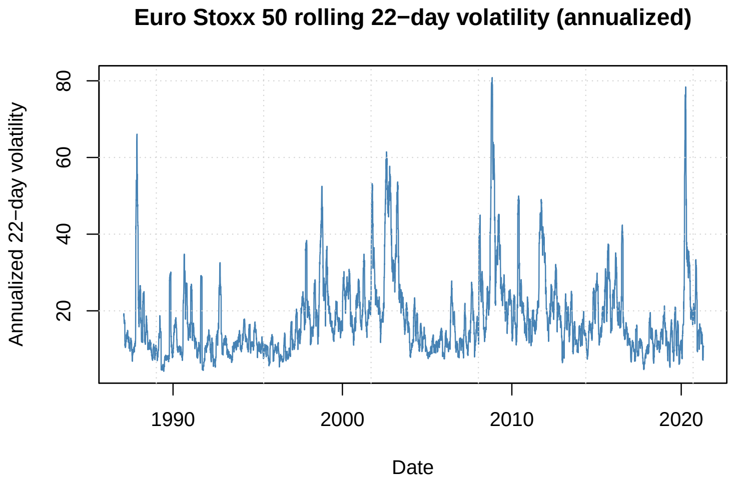

Figure 1.3: Euro Stoxx 50 rolling volatility (22 days, calculated over 252 days)

In figure 1.4 the density distribution of the log returns are examined. As can be seen, as already mentioned in part **??**, log returns are not really normally distributed.

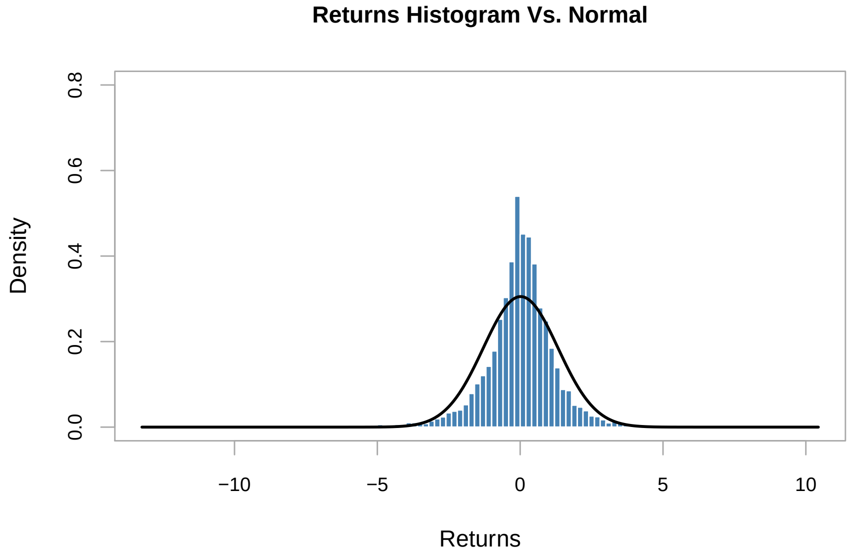

Figure 1.4: Density vs. Normal Euro Stoxx 50 log returns)

In figure 1.5 the prediction errors (in absolute values and squared) are visualized in autocorrelation function plots. It is common practice to check this, while in GARCH models the variance is for a large extent driven by the square of the prediction errors. The first component[2] α_0 is set equal to the sample average. As can be seen there is presence of large positive autocorrelation. This reflects, again, the presence of volatility clusters.

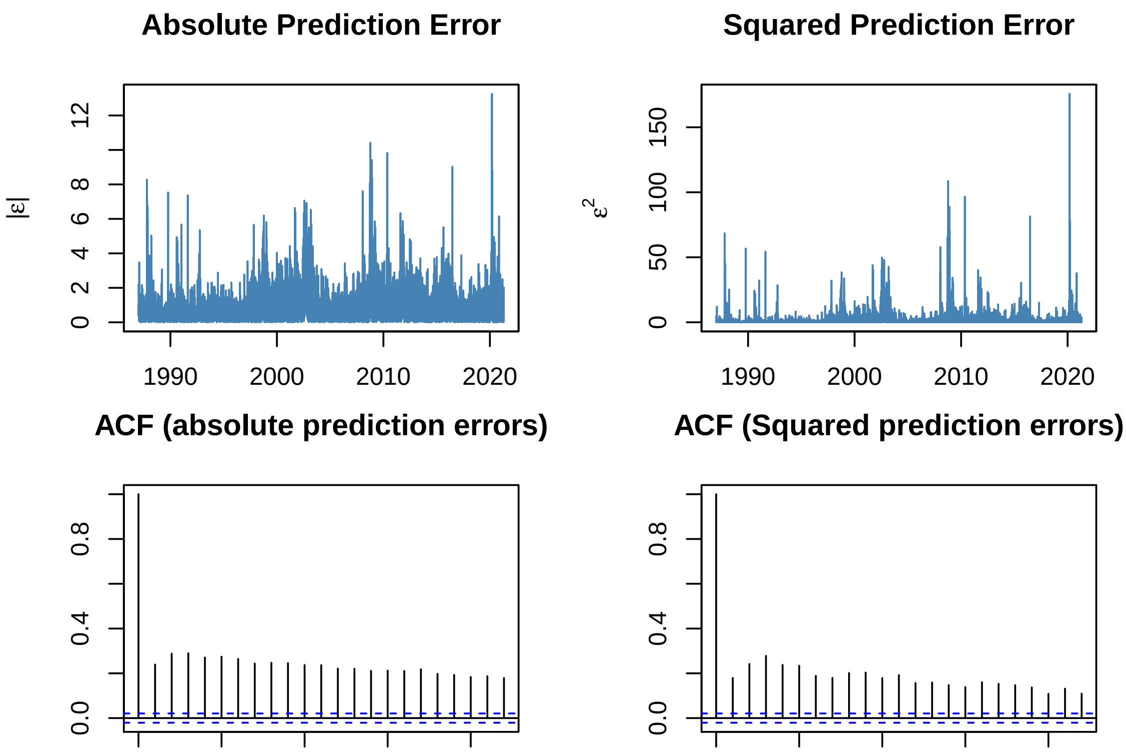

Figure 1.5: Absolute prediction errors

[2] α_0 is most of the time referred to as the μ in the conditional mean equation. Here we have followed Bali, Mo, and Tang [2].

1.2 Methodology

1.2.1 Garch models

As already mentioned in part **??**, the following models: symmetric GARCH, EGARCH, IGARCH, GJRGARCH, NGARCH, TGARCH and NAGARCH (or TSGARCH) will be estimated. Additionally the distributions will be examined as well, including the normal, student-t distribution, skewed student-t distribution, generalized error distribution, skewed generalized error distribution and the skewed generalized t distribution. They will be estimated using maximum likelihood[3].

Maximum likelihood estimation is a method to find the distribution parameters that best fit the observed data, through maximization of the likelihood function, or the computationally more efficient log-likelihood function (by taking the natural logarithm). It is assumed that the return data is i.i.d. and that there is some underlying parametrized density function f with one or more parameters that generate the data, defined as a vector θ in equation (1.2). These functions are based on the joint probability distribution of the observed data as in equation (1.3). Subsequently, the (log)likelihood function is maximized using an optimization algorithm shown inequation (1.4).

$$y_1, y_2, ..., y_N \sim i.i.d \tag{1.1}$$

$$y_i \sim f(y|\theta) \tag{1.2}$$

$$L(\theta) = \prod_{i=1}^{N} f(y_i|\theta) \tag{1.3}$$

$$\log(L(\theta)) = \sum_{i=1}^{N} \log f(y_i|\theta)$$

[3]As already mentioned, fortunately, Ghalanos [3] has made it easy for us to implement this methodology in the R language[4] [4] with the package "rugarch" v.1.4-4 (*R univariate garch*), which gives us a bit more time to focus on the results and the interpretation.

$$\theta^* = arg \max_{\theta}[L] \tag{1.4}$$

$$\theta^* = arg \max_{\theta}[\log(L)]$$

1.2.2 ACD models

Following Ghalanos [5], arguments of ACD models are specified as in Hansen [6]. The density function $f(y|\alpha)$ has parameters $\alpha_t = (\mu_t, \sigma_t, \nu_t)$, with equation (1.5), the conditional mean equation. Equation (1.6) as the conditional variance. And $\nu_t = \nu(\theta, x_t)$ the remaining parameters of the distribution like the skewness and kurtosis (shape) parameters.

$$\mu_t = \mu\left(\theta, x_t\right) = E\left(y_t \mid x_t\right) \tag{1.5}$$

$$\sigma_t^2 = \sigma^2\left(\theta, x_t\right) = E\left(\left(y_t - \mu_t^2\right) \mid x_t\right) \tag{1.6}$$

To further explain the difference between GARCH and ACD. The scaled innovations are given by equation (1.7). The conditional density is given by equation (1.8) and related to the density function $f(y|\alpha)$ as in equation (1.2.2).

$$z_t(\theta) = \frac{y_t - \mu\left(\theta, x_t\right)}{\sigma\left(\theta, x_t\right)} \tag{1.7}$$

$$g\left(z \mid \eta_t\right) = \frac{d}{dz} P\left(z_t < z \mid \eta_t\right) \tag{1.8}$$

$$f\left(y_t \mid \mu_t, \sigma_t^2, \eta_t\right) = \frac{1}{\sigma_t} g\left(z_t \mid \eta_t\right) \tag{1.9}$$

Again Ghalanos [5] makes it easier to implement the somewhat complex ACD models using the R language with package "racd".

1.2.3 Analysis Tests VaR and cVaR

Unconditional coverage test of Kupiec [7]

A number of tests are computed to see if the value-at-risk estimations capture the actual losses well. A first one is the unconditional coverage test by Kupiec [7]. The unconditional coverage or proportion of failures method tests if the actual value-at-risk exceedances are consistent with the expected exceedances (a chosen percentile, e.g. 1% percentile) of the VaR model. Following Kupiec [7] and Ghalanos [8], the number of exceedences follow a binomial distribution (with thus probability equal to the significance level or expected proportion) under the null hypothesis of a correct VaR model. The test is conducted as a likelihood ratio test with statistic like in equation (1.10), with p the probability of an exceedence for a confidence level, N the sample size and X the number of exceedences. The null hypothesis states that the test statistic LR^{uc} is χ^2-distributed with one degree of freedom or that the probability of failure $\hat{p}$ is equal to the chosen percentile α.

$$LR^{uc} = -2 \ln \left(\frac{(1-p)^{N-X} p^X}{\left(1 - \frac{X}{N}\right)^{N-X} \left(\frac{X}{N}\right)^X} \right) \tag{1.10}$$

Conditional coverage test of Christoffersen, Hahn, and Inoue [9]

Christoffersen, Hahn, and Inoue [9] proposed the conditional coverage test. It is tests for unconditional covrage and serial independence. The serial independence is important while the LR^{uc} can give a false picture while at any point in time it classifies inaccurate VaR estimates as "acceptably accurate" [10]. For a certain VaR estimate an indicator variable, $I_t(\alpha)$, is computed as equation (1.11).

$$I_t(\alpha) = \begin{cases} 1 & \text{if exceedence occurs} \\ 0 & \text{if no exceedence occurs} \end{cases}. \tag{1.11}$$

It involves a likelihood ratio test's null hypothesis is that the statistic is χ^2-distributed with two degrees of freedom or that the probability of violation $\hat{p}$ (unconditional coverage) as well as the conditional coverage (independence) is equal to the chosen percentile α. While it tests both unconditional coverage as independence of violations, only this test has been performed and the unconditional coverage test is not reported.

Dynamic quantile test

engle2004 provides an alternative test to specify if a VaR model is appropriately specified by proposing the dynamic quantile test. This test specifies the occurence of an exceedance (here hit) as in (1.12), with $I(.)$ a function that indicates when there is a hit, based on the actual return being lower than the predicted VaR. θ is the confidence level. They test jointly H_0 that the expected value of hit is zero and that it is uncorrelated with any variables known at the beginning of the period (B), notably the current VaR estimate and hits in previous periods, specified as lagged hits. This is done by regressing hit on these variables as in (1.13). $X\delta$ corresponds to the matrix notation. Under H_0, this regression should have no explanatory power. As a final step, a χ^2-distributed test statistic is constructed as in (1.14).

$$Hit_t = I\left(R_t < -\,\mathrm{VaR}_t(\theta)\right) - \theta, \tag{1.12}$$

$$\begin{gathered} Hit_t = \delta_0 + \delta_1 Hit_{t-1} + \ldots + \delta_p Hit_{t-p} + \delta_{p+1} VaR_t + \\ \delta_{p+2} I_{year1,t} + \ldots + \delta_{p+2+n} I_{yearn,t} + u_t \end{gathered} \tag{1.13}$$

$$Hit_t = X\delta + u_t \quad u_t = \begin{cases} -\theta & \text{prob } (1-\theta) \\ (1-\theta) & \text{prob } \theta \end{cases}$$

$$\frac{\hat{\delta}'_{OLS} X' X \hat{\delta}^a_{OLS}}{\theta(1-\theta)} \sim \chi^2(p+n+2) \tag{1.14}$$

References

[1] *Calculation guide STOXX ®*. Tech. rep. 2020.

[2] Turan G. Bali, Hengyong Mo, and Yi Tang. "The role of autoregressive conditional skewness and kurtosis in the estimation of conditional VaR". In: *Journal of Banking and Finance* 32.2 (Feb. 2008). Publisher: North-Holland, pp. 269–282.

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