1 Data and methodology

1.1 Data

We worked with daily returns on the EURO STOXX 50 Index denoted in EUR from 31 December, 1986 to 27 April, 2021. It is the leading blue-chip index of the Eurozone, was founded in 1999 and covers 50 of the most liquid and largest (in terms of free-float market capitalization) stocks. Its composition is reviewed annually in September, from each of the 19 EURO STOXX Supersector indices the biggest stocks are selected until the coverage is at 60% of the free-float market cap of each of the EURO STOXX Supersector index then all the current EURO STOXX 50 stocks are used in the selection list from which the largest 40 in terms of free-float market cap are selected and the remaining 10 stocks are chosen among those ranked between 41 and 60 [1].

The calculation of the index is made with the @ref(eq:Laspeyres.formula), that measures the changes in price of the index for fixed weights.

$$\text{Index}_{\ t} = \frac{\sum_{i=1}^{n}\left(p_{it} \cdot s_{it} \cdot ff_{it} \cdot cf_{it} \cdot x_{it}\right)}{D_t} = \frac{M_t}{D_t}(\#eq: Laspeyres.formula) \quad (1.1)$$

where: t = Time the index is computed n = Number of companies in the index p_{it} = Price of company (i) at time (t) s_{it} = Number of shares of company (i) at time (t) ff_{it} = Free float factor of company (i) at time (t) cf_{it} = Weighting cap factor of company (i) at time (t) x_{it} = Exchange rate from local currency into index currency for company (i) at time (t) M_t = Free-float market capitalization of the index at time (t) D_t = Divisor of the index at time (t)

Changes in weights caused by corporate actions are proportionally distributed across the components of the index and the index Divisor is computed with the @ref(eq:Price.weighted)

$$D_{t+1} = D_t \cdot \frac{\sum_{i=1}^{n} (p_{it} \cdot s_{it} \cdot ff_{it} \cdot cf_{it} \cdot x_{it}) \pm \Delta MC_{t+1}}{\sum_{i=1}^{n} (p_{it} \cdot s_{it} \cdot ff_{it} \cdot cf_{it} \cdot x_{it})} (\#eq: Price.weighted) \tag{1.2}$$

where: ΔMC_{t+1} = Difference between the closing market capitalization of the index and the adjusted closing market capitalization of the index

(Optional)

The same analysis has been performed for the INDEX 1, INDEX 2, INDEX 3 and the INDEX 4 indexes with not qualitatively different conclusions(....hopefully....). The findings of these researches are available upon requests.

1.1.1 Descriptives

Table of summary statistics

Here comes a table and description of the stats

Table 1.1: Summary statistics of the returns

Statistics	Eurostoxx.50	Standardized.Residuals
Arithmetic Mean	0.0167	-0.0409
Median	0.0357	-0.0193
Maximum	10.4376	5.7126
Minimum	-13.2404	-11.7732
Stdev	1.307	0.9992
Skewness	-0.31	-0.6327
	(0***)	(0***)
Excess Kurtosis	7.2083	5.134
	(0***)	(0***)
Jarque-Bera	19528.6196***	10431.0514***

Note: This table shows the descriptive statistics of the returns of over the period 1987-01-01 to 2021-04-27. Including minimum, median, arithmetic average, maximum, standard deviation, skewness and excess kurtosis.

Descriptive figures

Stylized facts As can be seen in figure 1.1 the Euro area equity and later, since 1999 the EuroStoxx 50 went up during the tech (“dot com”) bubble reaching an ATH of €5464.43. Then, there was a correction to boom again until the burst of the 2008 financial crisis. After which it decreased significantly. With an ATL at 09 March, 2009 of €1809.98. There is an improvement, but then the European debt crisis of 2011 occurred. With then some improvement again, with the decline beginning of 2020. To then recover very quickly to already values higher then the pre-COVID crisis level.

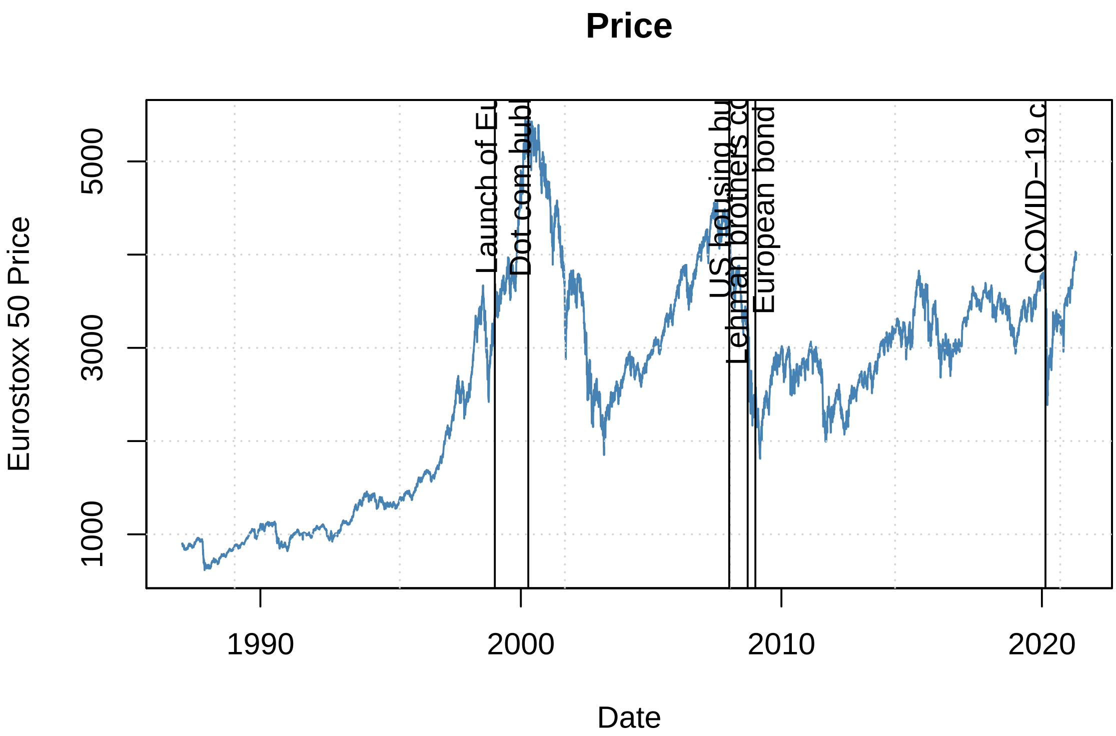

Figure 1.1: Eurostoxx 50 Price Index prices

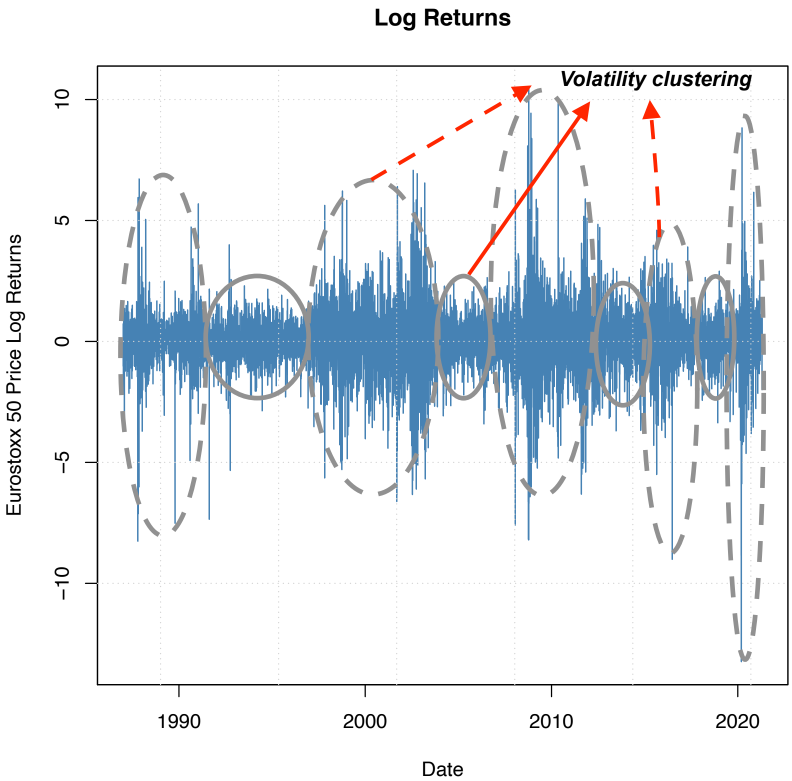

Figure 1.2: Eurostoxx 50 Price Index log returns

In figure 1.2 the daily log-returns are visualized. There is already

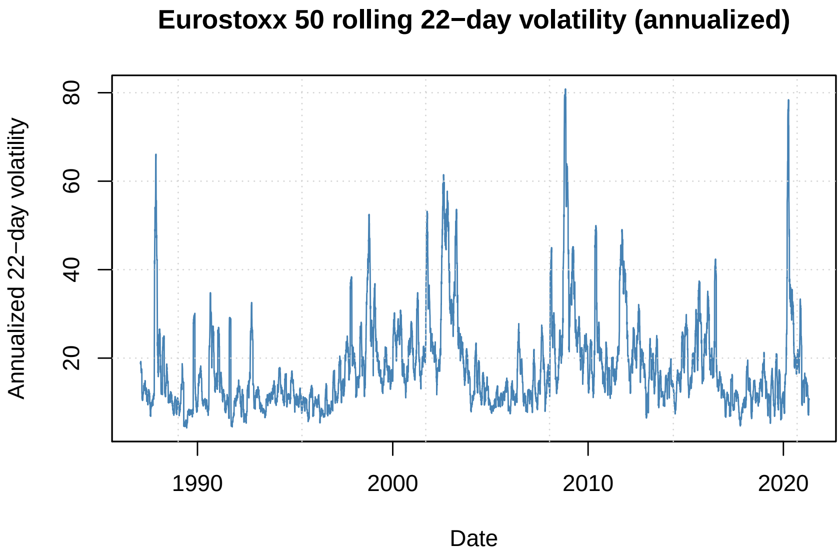

Figure 1.3: Eurostoxx 50 rolling volatility (22 days, calculated over 252 days)

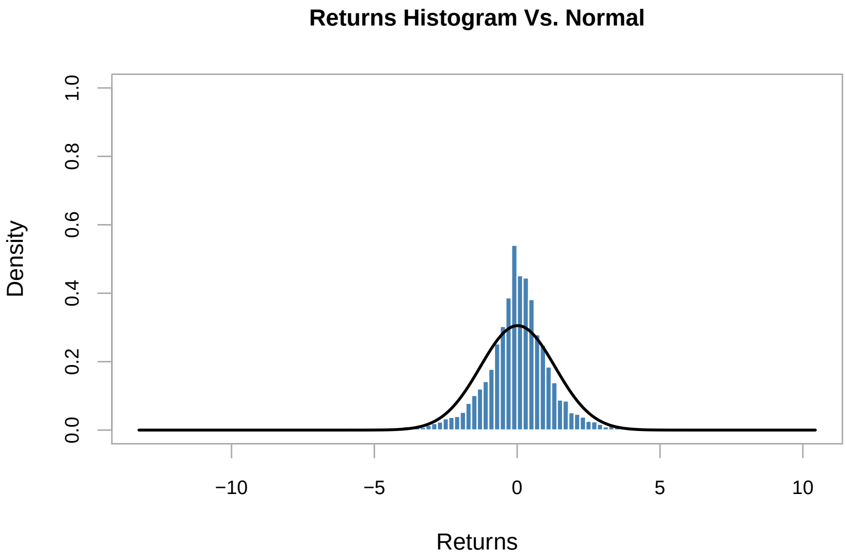

Figure 1.4: Density vs. Normal Eurostoxx 50 log returns)

As can be seen

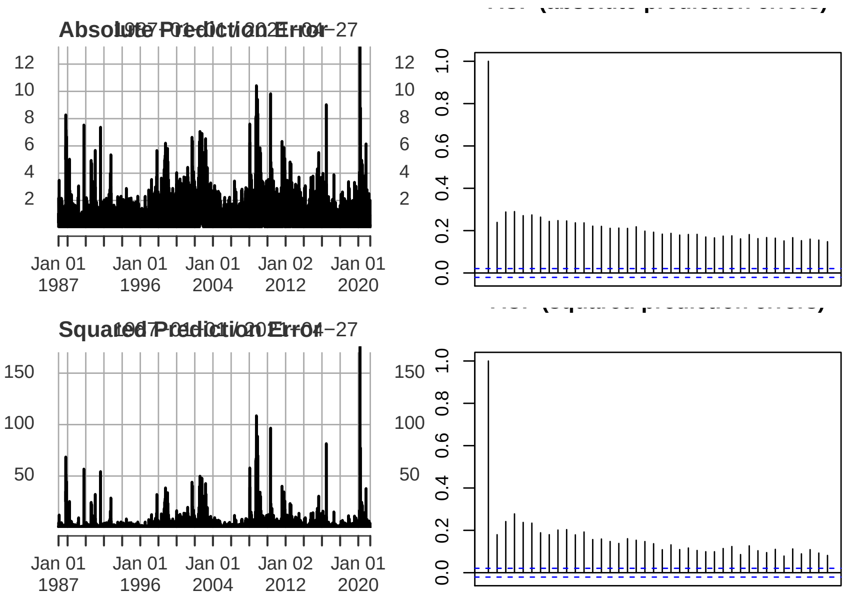

Figure 1.5: Absolute prediction errors

heteroscedasticity

1.2 Methodology

1.2.1 Garch models

As already mentioned in . . . , GARCH models GARCH, EGARCH, IGARCH, GJR-GARCH, NGARCH, TGARCH and NAGARCH (or TSGARCH) will be estimated. Additionally the distributions will be examined as well, including the normal, student-t distribution, skewed student-t distribution, generalized error distribution, skewed generalized error distribution and the skewed generalized t distribution.

They will be estimated using maximum likelihood. As already mentioned, fortunately, Alexios Ghalanos [2] has made it easy for us to implement this methodology in the R language (v.3.6.1) with the package "rugarch" v.1.4-4 (*R univariate garch*), which gives us a bit more time to focus on the results and the interpretation.

Maximum likelihood estimation is a method to find the distribution parameters that best fit the observed data, through maximization of the likelihood function, or the computationally more efficient log-likelihood function (by taking the natural logarithm). It is assumed that the return data is i.i.d. and that there is some underlying parametrized density function f with one or more parameters that generate the data, defined as a vector θ ((1.4)). These functions are based on the joint probability distribution of the observed data (equation (1.6)). Subsequently, the (log)likelihood function is maximized using an optimization algorithm (equation (1.8)).

$$y_1, y_2, ..., y_N \sim i.i.d \tag{1.3}$$

$$y_i \sim f(y|\theta) \tag{1.4}$$

$$L(\theta) = \prod_{i=1}^{N} f(y_i|\theta) \tag{1.5}$$

$$\log(L(\theta)) = \sum_{i=1}^{N} \log f(y_i|\theta) \tag{1.6}$$

$$\theta^* = arg\max_{\theta}[L] \tag{1.7}$$

$$\theta^* = arg\max_{\theta}[\log(L)] \tag{1.8}$$

1.2.2 ACD models

Following Ghalanos [3], arguments of ACD models are specified as in Hansen [4]. The density function $f(y|\alpha)$ has parameters $\alpha_t = (\mu_t, \sigma_t, \nu_t)$, with equation (1.9), the conditional mean equation. Equation (1.10) as the conditional variance. And $\nu_t = \nu(\theta, x_t)$ the remaining parameters of the distribution like the skewness and kurtosis (shape) parameters.

$$\mu_t = \mu(\theta, x_t) = E(y_t \mid x_t) \tag{1.9}$$

$$\sigma_t^2 = \sigma^2(\theta, x_t) = E\left(\left(y_t - \mu_t^2\right) \mid x_t\right) \tag{1.10}$$

To further explain the difference between GARCH and ACD. The scaled innovations are given by equation (1.11). The conditional density is given by equation (1.12) and related to the density function $f(y|\alpha)$ as in equation (1.2.2).

$$z_t(\theta) = \frac{y_t - \mu(\theta, x_t)}{\sigma(\theta, x_t)} \tag{1.11}$$

$$g(z \mid \eta_t) = \frac{d}{dz} P(z_t < z \mid \eta_t) \tag{1.12}$$

$$f\left(y_t \mid \mu_t, \sigma_t^2, \eta_t\right) = \frac{1}{\sigma_t} g(z_t \mid \eta_t) \tag{1.13}$$

Again Ghalanos [3] makes it easier to implement the somewhat complex ACD models using the R language with package "racd".

1.2.3 Control Tests

Unconditional coverage test of Kupiec [5]

A number of tests are computed to see if the value-at-risk estimations capture the actual losses well. A first one is the unconditional coverage test by Kupiec [5]. The unconditional coverage or proportion of failures method tests if the actual value-at-risk exceedances are consistent with the expected exceedances (a chosen percentile, e.g. 1% percentile) of the VaR model. Following Kupiec [5] and Ghalanos [6], the number of exceedence follow a binomial distribution (with thus probability equal to the significance level or expected proportion) under the null hypothesis of a correct VaR model. The test is conducted as a likelihood ratio test with statistic like in equation (1.14), with p the probability of an exceedence for a confidence level, N the sample size and X the number of exceedence. The null hypothesis states that the test statistic LR^{uc} is χ^2-distributed with one degree of freedom or that the probability of failure $\hat{p}$ is equal to the chosen percentile α.

$$LR^{uc} = -2\ln\left(\frac{(1-p)^{N-X}p^X}{\left(1-\frac{X}{N}\right)^{N-X}\left(\frac{X}{N}\right)^X}\right) \tag{1.14}$$

Conditional coverage test of Christoffersen, Hahn, and Inoue [7]

Christoffersen, Hahn, and Inoue [7] proposed the conditional coverage test. It is tests for unconditional covrage and serial independence. The serial independence is important while the LR^{uc} can give a false picture while at any point in time it classifies inaccurate VaR estimates as “acceptably accurate” [8]. For a certain VaR estimate an indicator variable, $I_t(\alpha)$, is computed as equation (1.15).

$$I_t(\alpha) = \begin{cases} 1 & \text{if exceedence occurs} \\ 0 & \text{if no exceedence occurs} \end{cases}. \tag{1.15}$$

It involves a likelihood ratio test's null hypothesis is that the statistic is χ^2-distributed with two degrees of freedom or that the probability of violation $\hat{p}$

(unconditional coverage) as well as the conditional coverage (independence) is equal to the chosen percentile α.

Dynamic quantile test

Engle and Manganelli [9] with the aim to provide completeness to the conditional coverage test of Christoffersen, Hahn, and Inoue [7] developed the Dynamic quantile test. It consists in testing some restriction in a

References

[1] *CALCULATION GUIDE STOXX ®*. Tech. rep. 2020.

[2] Alexios Ghalanos. *rugarch: Univariate GARCH models.* R package version 1.4-4. 2020.

[3] Alexios Ghalanos. *racd: Autoregressive Conditional Density Models.* http://www.unstarched.net, https://bitbucket.org/alexiosg/. 2016.

[4] Bruce E. Hansen. “Autoregressive Conditional Density Estimation”. In: *International Economic Review* 35.3 (1994), pp. 705–730.

[5] P.H. Kupiec. “Techniques for Verifying the Accuracy of Risk Measurement Models”. In: *Journal of Derivatives* 3.2 (1995), pp. 73–84.

[6] Alexios Ghalanos. *Introduction to the rugarch package. (Version 1.4-3)*. Tech. rep. 2020. URL: `http://cran.r-project.org/web/packages/`.

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[8] Turan G. Bali and Panayiotis Theodossiou. “A conditional-SGT-VaR approach with alternative GARCH models”. In: *Annals of Operations Research* 151.1 (Feb. 22, 2007), pp. 241–267. URL: `http://link.springer.com/10.1007/s10479-006-0118-4`.

[9] Robert F. Engle and S. Manganelli. *CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles.* Tech. rep. San Diego: UC San Diego, 1999. URL: `http://www.jstor.org/stable/1392044`.