1 Data and methodology

1.1 Data

We worked with daily returns on the Euro Stoxx 50 Return Index[1] retrieved from Datastream denoted in EUR from 02 January, 2001 to 19 May, 2021. The choice of daily data is motivated as follows. The primary interest in this paper is (c)VaR models for banks internal trading desks. Their positions are usually short-term, making risk management at the daily level the most appropriate. As such, in reference to the literature review, regulators require VaR forecast for one day in advance. All following analysis could be applied on monthly returns as well. To get some intuition on how returns differ at a wider aggregation, table X in the appendix shows summary statistics for monthly data instead of daily. The Euro Stoxx 50 is the leading blue-chip index of the Eurozone, was founded in 1999 and covers 50 of the most liquid and largest (in terms of free-float market capitalization) stocks. For its composition and computation we refer to the factsheet [1]. Given that in the 20th century computing return series was time consuming, the Eurostoxx 50 Return index is x shorter than the Euro Stoxx 50 Price index (going back to 2001). As a robustness check, we ran all subsequent analysis for the longer price index

[1]The same analysis has been performed for the INDEX 1, INDEX 2, INDEX 3 and the INDEX 4 indexes with the same conclusions. The findings of these researches are available upon requests.

as well. This did not yield a qualitative difference in terms of most efficient model(s).

Table 1.1 provides the main statistics describing the return series analyzed. Let daily returns be computed as $R_t = 100\,(\ln P_t - \ln P_{t-1})$,where P_t is the index price at time t and P_{t-1} is the index price at $t-1$.

The arithmetic mean of the series is 0.01% with a standard deviation of 1.44% and a median of 0.02 which translate to an annualized mean of 2.57% and an annualized standard deviation of 22.85%. The skewness statistic is highly significant and negative at -0.22 and the excess kurtosis is also highly significant and positive at 6.54. These 2 statistics give an overview of the distribution of the returns which has thicker tails than the normal distribution with a higher presence of left tail observations. A formal test such as the Jarque-Bera one with its statistic at 739.89 and a high statistical significance, confirms the non normality feeling given by the Skewness and Kurtosis.
The right column of table 1.1 displays the same descriptive statistics but for the standardizes residuals obtained from a simple GARCH model as mentioned in table 1.1 in Note 2. Again, Skewness statistic at -0.328 with a high statistical significance level and the excess Kurtosis at 1.706 also with a high statistical significance, suggest a non normal distribution of the standardized residuals and the Jarque-Bera statistic at 9511.18, given its high significance, confirms the rejection of the normality assumption.

As can be seen in figure 1.1 the Euro area equity and later, since 1999 the Euro Stoxx 50, went up during the tech ("dot com") bubble reaching an ATH of €1734.84. Then, there was a correction to boom again until the burst of the 2008 financial crisis. After which it decreased significantly. With an ATL at 12 March, 2003 of €405.23. There is an improvement, but then the European debt crisis, with its peak in 2010-2012, occurred. From then there was some improvement until the "health

Table 1.1: Summary statistics of the returns

Statistics	Euro Stoxx 50	Standardized Residuals
Arithmetic Mean	0.0102	-0.0512
Median	0.0152	-0.0314
Maximum	10.4372	5.8561
Minimum	-13.2164	-6.4432
Stdev	1.4391	0.9986
Skewness	-0.2191	-0.3285
	(0***)	(0***)
Excess Kurtosis	6.5382	1.7055
	(0***)	(0***)
Jarque-Bera	9511.1796***	739.8884***

Notes

[1] This table shows the descriptive statistics of the daily percentage returns of Euro Stoxx 50 over the period 2001-01-03 to 2021-05-19 (5316 observations). Including arithmetic mean, median, maximum, minimum, standard deviation. The skewness, excess kurtosis with p-value and signicance and the Jarque-Bera test with significance.

[2] The standardized residual is derived from a maximum likelihood estimation (simple GARCH model) as follows:
$R_t = \alpha_0 + \alpha_1 R_{t-1} + z_t \sigma_t$
$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_2 \sigma_{t-1}^2$,
Where z is the standard residual (assumed to have a normal distribution).

[3] *, **, *** represent significance levels at the 5

crisis", which arrived in Europe, February 2020. This crisis recovered very quickly reaching already values higher then the pre-COVID crisis level.

In figure 1.2 the daily log-returns are visualized. A stylized fact that is observable is the volatility clustering. As can be seen: periods of large volatility are mostly followed by large volatility and small volatility by small volatility.

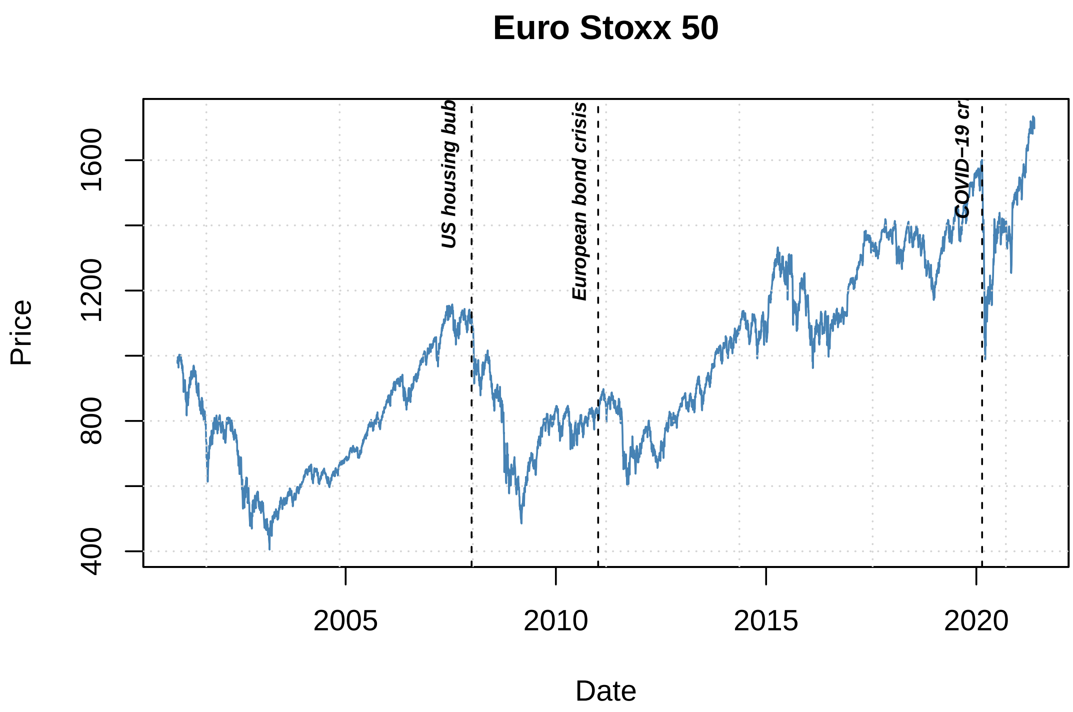

Figure 1.1: Euro Stoxx 50 prices

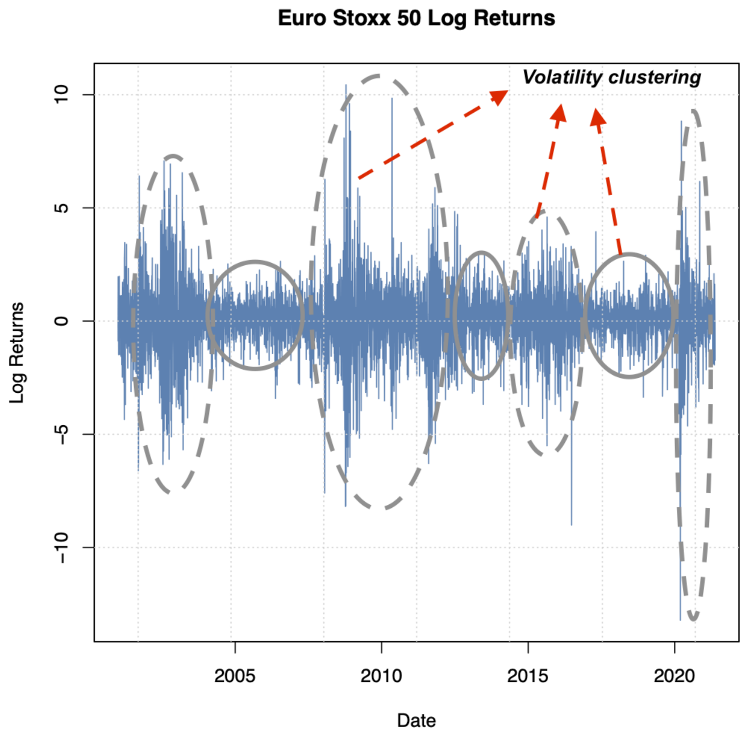

Figure 1.2: Euro Stoxx 50 log returns

In figure 1.3 the density distribution of the log returns are examined. As can be seen, as already mentioned in part **??**, log returns are not really normally distributed.

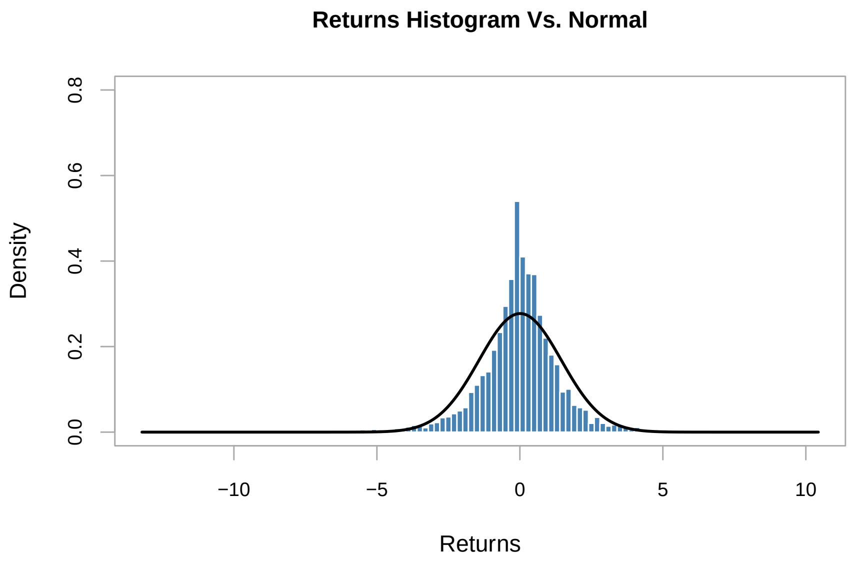

Figure 1.3: Density vs. Normal Euro Stoxx 50 log returns)

In figure 1.4 the prediction errors (in absolute values and squared) are visualized in autocorrelation function plots. It is common practice to check this, while in GARCH models the variance is for a large extent driven by the square of the prediction errors. The first component[2] α_0 is set equal to the sample average. As can be seen there is presence of large positive autocorrelation. This reflects, again, the presence of volatility clusters.

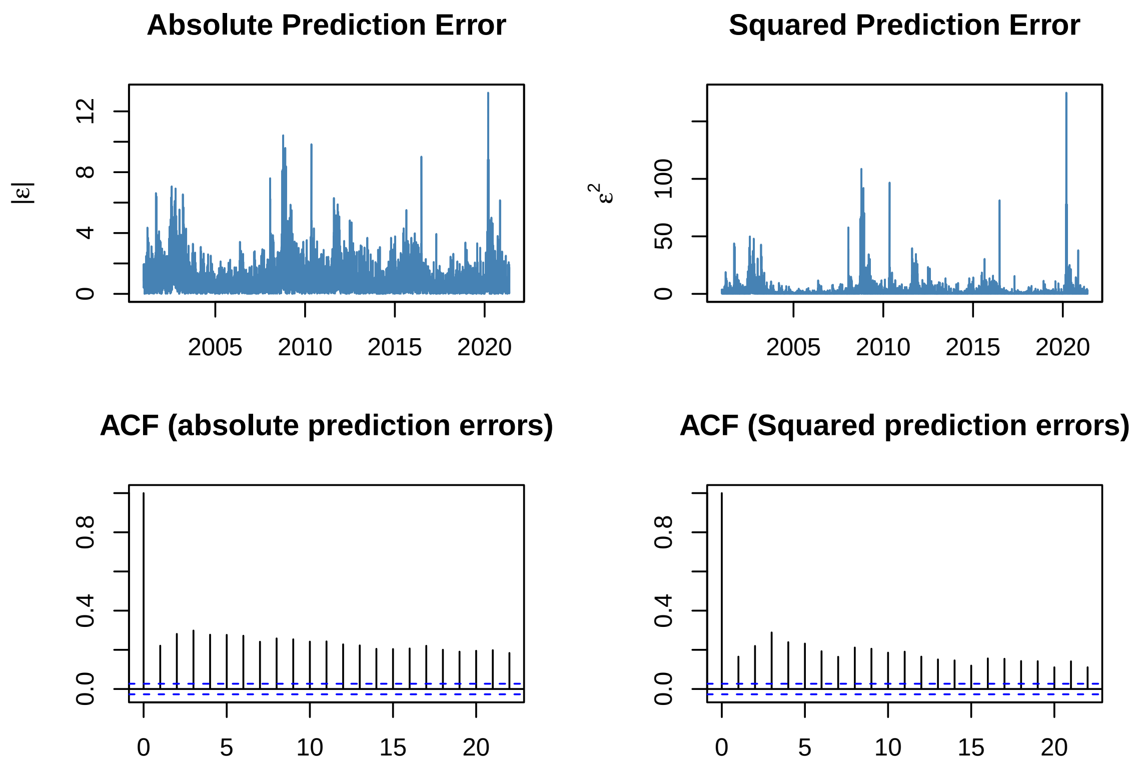

Figure 1.4: Absolute prediction errors

[2] α_0 is most of the time referred to as the μ in the conditional mean equation. Here we have followed Bali, Mo, and Tang [2].

1.2 Methodology

1.2.1 Garch models

As already mentioned in part **??**, the following models: symmetric GARCH, EGARCH, IGARCH, GJRGARCH, NGARCH, TGARCH and NAGARCH (or TSGARCH) will be estimated. Additionally the distributions will be examined as well, including the normal, student-t distribution, skewed student-t distribution, generalized error distribution, skewed generalized error distribution and the skewed generalized t distribution. They will be estimated using maximum likelihood[3].

Maximum likelihood estimation is a method to find the distribution parameters that best fit the observed data, through maximization of the likelihood function, or the computationally more efficient log-likelihood function (by taking the natural logarithm). It is assumed that the return data is i.i.d. and that there is some underlying parametrized density function f with one or more parameters that generate the data, defined as a vector θ in equation (1.2). These functions are based on the joint probability distribution of the observed data as in equation (1.3). Subsequently, the (log)likelihood function is maximized using an optimization algorithm shown inequation (1.4).

$$y_1, y_2, ..., y_N \sim i.i.d \tag{1.1}$$

$$y_i \sim f(y|\theta) \tag{1.2}$$

$$L(\theta) = \prod_{i=1}^{N} f(y_i|\theta) \tag{1.3}$$

$$\log(L(\theta)) = \sum_{i=1}^{N} \log f(y_i|\theta)$$

[3]As already mentioned, fortunately, Ghalanos [3] has made it easy for us to implement this methodology in the R language[^data-meth-4] [4] with the package "rugarch" v.1.4-4 (*R univariate garch*), which gives us a bit more time to focus on the results and the interpretation.

$$\theta^* = arg \max_{\theta}[L] \tag{1.4}$$

$$\theta^* = arg \max_{\theta}[\log(L)] \tag{1.5}$$

After estimation of the GARCH models in-sample, out-sample analysis is done by performing a rolling window approach. With assumptions: a window of 2500 observations and re-estimation every year.

|**Note for profs: choices**:

– n.start = 2500 ==> sample = 2500 observations to forecast the next,

– refit.every = 252 (trading days in a year) ==> so recompute the parameters every year,

– solver = hybrid using cluster = 10 to run on 10 cores to speed up the process of estimation of the roll object (took 5-10 minutes per backtest with some solvers, now with parallel package. . .)

– forecast.length = 1 ==> length of total forecast for which out of sample data from dataset will be used for testing |

1.2.2 ACD models

Following Ghalanos [5], arguments of ACD models are specified as in Hansen [6]. The skewness and kurtosis (shape) parameters which are constant in GARCH models (or time-invariant), are here time-varying following a piecewise linear dynamic. In equation (1.6) the parameters of the GARCH-ACD model are specified.

$$\begin{aligned}
&y_t = \alpha_0 + \alpha_1 \times \varepsilon_t, \\
&\varepsilon_t = \sigma_t \times z_t, \\
&z_t \sim \Delta\left(0, 1, \rho_t, \zeta_t\right), \\
&\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2, \\
&\rho_t = \chi_0 + \chi_1 z_{t-1} I_{z_{t-1}<x} + \chi_2 z_{t-1} I_{z_{t-1}\geqslant x} + \xi_1 \bar{\rho}_{t-1}, \\
&\zeta_t = \kappa_0 + \kappa_1 \left|z_{t-1}\right| I_{z_{t-1}<x} + \kappa_2 \left|z_{t-1}\right| I_{z_{t-1}\geqslant x} + \psi_1 \bar{\zeta}_{t-1},
\end{aligned} \tag{1.6}$$

where y_t, z_t and σ_t are familiar from GARCH models. ρ_t and $zeta_t$ are respectively the time-varying skewness and shape parameter (with shape parameter meaning here the tail-tickness) and the standardized residuals z_t follows a distribution Δ that has a skewness and shape parameter. ρ_t and $zeta_t$ are following a piecewise linear dynamic with I the indicator variable (taking one if the underlying expression is true, 0 otherwise). x is a treshold value set to 0.

Again Ghalanos [5] makes it easier to implement the somewhat complex ACD models using the R language with package "racd".

1.2.3 Analysis Tests VaR and cVaR

Unconditional coverage test of Kupiec [7]

A number of tests are computed to see if the value-at-risk estimations capture the actual losses well. A first one is the unconditional coverage test by Kupiec [7]. The unconditional coverage or proportion of failures method tests if the actual value-at-risk exceedances are consistent with the expected exceedances (a chosen percentile, e.g. 1% percentile) of the VaR model. Following Kupiec [7] and Ghalanos [8], the number of exceedences follow a binomial distribution (with thus probability equal to the significance level or expected proportion) under the null hypothesis of a correct VaR model. The test is conducted as a likelihood ratio test with statistic like in equation (1.7), with p the probability of an exceedence for a confidence level, N the sample size and X the number of exceedences. The null hypothesis states that the test statistic LR^{uc} is χ^2-distributed with one degree of freedom or that the probability of failure $\hat{p}$ is equal to the chosen percentile α.

$$LR^{uc} = -2\ln\left(\frac{(1-p)^{N-X}p^X}{\left(1-\frac{X}{N}\right)^{N-X}\left(\frac{X}{N}\right)^X}\right) \tag{1.7}$$

Conditional coverage test of Christoffersen, Hahn, and Inoue [9]

Christoffersen, Hahn, and Inoue [9] proposed the conditional coverage test. It is tests for unconditional covrage and serial independence. The serial independence is important while the LR^{uc} can give a false picture while at any point in time it classifies inaccurate VaR estimates as “acceptably accurate” [10]. For a certain VaR estimate an indicator variable, $I_t(\alpha)$, is computed as equation (1.8).

$$I_t(\alpha) = \begin{cases} 1 & \text{if exceedence occurs} \\ 0 & \text{if no exceedence occurs} \end{cases} . \tag{1.8}$$

It involves a likelihood ratio test's null hypothesis is that the statistic is χ^2-distributed with two degrees of freedom or that the probability of violation $\hat{p}$ (unconditional coverage) as well as the conditional coverage (independence) is equal to the chosen percentile α. While it tests both unconditional coverage as independence of violations, only this test has been performed and the unconditional coverage test is not reported.

Dynamic quantile test

Engle and Manganelli [11] provides an alternative test to specify if a VaR model is appropriately specified by proposing the dynamic quantile test. This test specifies the occurence of an exceedance (here hit) as in (1.9), with $I(.)$ a function that indicates when there is a hit, based on the actual return being lower than the predicted VaR. θ is the confidence level. They test jointly H_0 that the expected value of hit is zero and that it is uncorrelated with any variables known at the beginning of the period (B), notably the current VaR estimate and hits in previous periods, specified as lagged hits. This is done by regressing hit on these variables as in (1.10). $X\delta$ corresponds to the matrix notation. Under H_0, this regression should have no explanatory power. As a final step, a χ^2-distributed test statistic with m degrees of freedom equal to the parameters to be estimated (constant, number of hits and VaR estimate) is constructed as in (1.11).

$$Hit_t = I\left(R_t < -\mathrm{VaR}_t(\theta)\right) - \theta, \tag{1.9}$$

$$\begin{gathered} Hit_t = \delta_0 + \delta_1 Hit_{t-1} + \ldots + \delta_p Hit_{t-p} + \delta_{p+1} VaR_t + \\ \delta_{p+2} I_{year1,t} + \ldots + \delta_{p+2+n} I_{yearn,t} + u_t \end{gathered} \tag{1.10}$$

$$Hit_t = X\delta + u_t \quad u_t = \begin{cases} -\theta & \text{prob } (1-\theta) \\ (1-\theta) & \text{prob } \theta \end{cases}$$

$$\frac{\hat{\delta}'_{OLS} X' X \hat{\delta}^a_{OLS}}{\theta(1-\theta)} \sim \chi^2(m) \tag{1.11}$$

ES Test

The Expected Shortfall test by McNeil and Frey [12] tests whether the excess conditional shortfall has a mean of zero. Under the alternative hypothesis, this mean is greater than zero. This test uses a one sided t-statistic and bootstrapped p-values, as the distribution of the excess conditional shortfall is not assumed to be normal.

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