Volatility is unobservable. We can only ever estimate and forecast volatility, and this only within the context of an assumed statistical model. So there is no absolute 'true' volatility: what is 'true' depends only on the assumed model. ... Moreover, volatility is only a sufficient statistic for the dispersion of the returns distribution when we make a normality assumption. In other words, volatility does not provide a full description of the risks that are taken by the investment unless we assume the investment returns are normally distributed.

1
Data and methodology

1.1 Data

We worked with daily returns on the Euro Stoxx 50 Price Index[1] denoted in EUR from 31 December, 1986 to 27 April, 2021. It is the leading blue-chip index of the Eurozone, was founded in 1999 and covers 50 of the most liquid and largest (in terms of free-float market capitalization) stocks. For its composition we refer to the factsheet [1].

1.1.1 Descriptives

Table of summary statistics

Equation 1.1 provides the main statistics describing the return series analyzed. Returns are computed with equation (1.1).

$$R_t = 100 \left(\ln \left(I_t \right) - \ln \left(I_{t-1} \right) \right) \tag{1.1}$$

where I_t is the index price at time t and I_{t-1} is the index price at $t-1$.

[1] The same analysis has been performed for the INDEX 1, INDEX 2, INDEX 3 and the INDEX 4 indexes with the same conclusions. The findings of these researches are available upon requests.

The arithmetic mean of the series is 0.017% with a standard deviation of 1.307% and a median of 0.036 which translate to an annualized mean of 4.208% and an annualized standard deviation of 20.748%. The skewness statistic is highly significant and negative at -0.31 and the excess kurtosis is also highly significant and positive at 7.208. These 2 statistics give an overview of the distribution of the returns which has thicker tails than the normal distribution with a higher presence of left tail observations. A formal test such as the Jarque-Bera one with its statistic at 19528.62 and a high statistical significance, confirms the non normality feeling given by the Skewness and Kurtosis.

The right column of table 1.1 displays the same descriptive statistics but for the standardizes residuals obtained from a simple GARCH model as mentioned in table 1.1 in Note 2∗. Again, Skewness statistic at -0.633 with a high statistical significance level and the excess Kurtosis at 5.134 also with a high statistical significance, suggest a non normal distribution of the standardized residuals and the Jarque-Bera statistic at NA, given its high significance, confirms the rejection of the normality assumption.

Descriptive figures

Stylized facts

As can be seen in figure 1.1 the Euro area equity and later, since 1999 the Euro Stoxx 50 went up during the tech ("dot com") bubble reaching an ATH of €5464.43. Then, there was a correction to boom again until the burst of the 2008 financial crisis. After which it decreased significantly. With an ATL at 09 March, 2009 of €1809.98. There is an improvement, but then the European debt crisis, with it's peak in 2010-2012, occurred. From then there was some improvement until the "health crisis", which arrived in Europe, February 2020. This crisis recovered very quickly reaching already values higher then the pre-COVID crisis level.

Table 1.1: Summary statistics of the returns

Statistics	Euro Stoxx 50	Standardized Residuals
Arithmetic Mean	0.0167	-0.0409
Median	0.0357	-0.0193
Maximum	10.4376	5.7126
Minimum	-13.2404	-11.7732
Stdev	1.307	0.9992
Skewness	-0.31	-0.6327
	(0***)	(0***)
Excess Kurtosis	7.2083	5.134
	(0***)	(0***)
Jarque-Bera	19528.6196***	10431.0514***

Notes

[1] This table shows the descriptive statistics of the daily percentage returns of EURO STOXX 50 over the period 1987-01-01 to 2021-04-27 (8954 observations). Including arithmetic mean, median, maximum, minimum, standard deviation, skewness, excess kurtosis and the Jarque-Bera test.

[2] The standardized residual is derived from a maximum likelihood estimation (simple GARCH model) as follows:
$R_t = \alpha_0 + \alpha_1 R_{t-1} + z_t \sigma_t$
$\sigma_t^2 = \beta_0 + \beta_1 \sigma_{t-1}^2 z_{t-1}^2 + \beta_2 \sigma_{t-1}^2$,
Where z is the standard residual (assumed to have a normal distribution).

[3] *, **, *** represent significance levels at the 5

In figure 1.2 the daily log-returns are visualized. A stylized fact that is observable is the volatility clustering. As can be seen: periods of large volatility are mostly followed by large volatility and small volatility by small volatility.

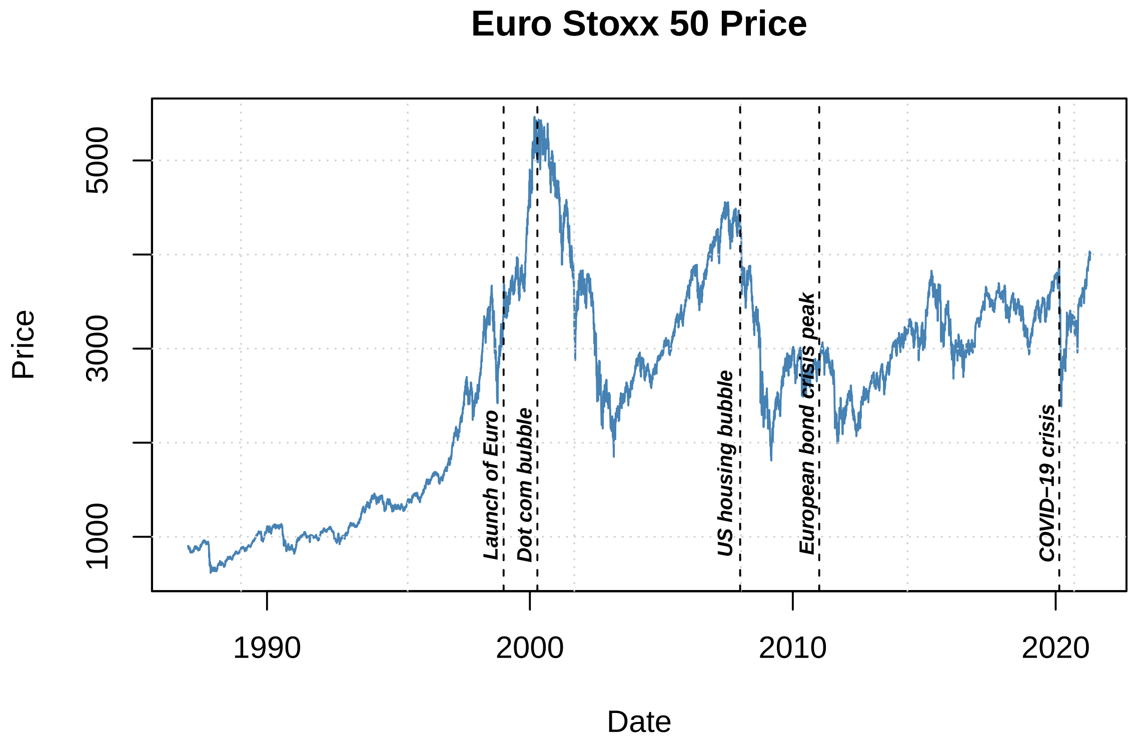

Figure 1.1: Euro Stoxx 50 Price Index prices

In figure 1.4 the density distribution of the log returns are examined. As can be seen, as already mentioned in part **??**, log returns are not really normally distributed. So

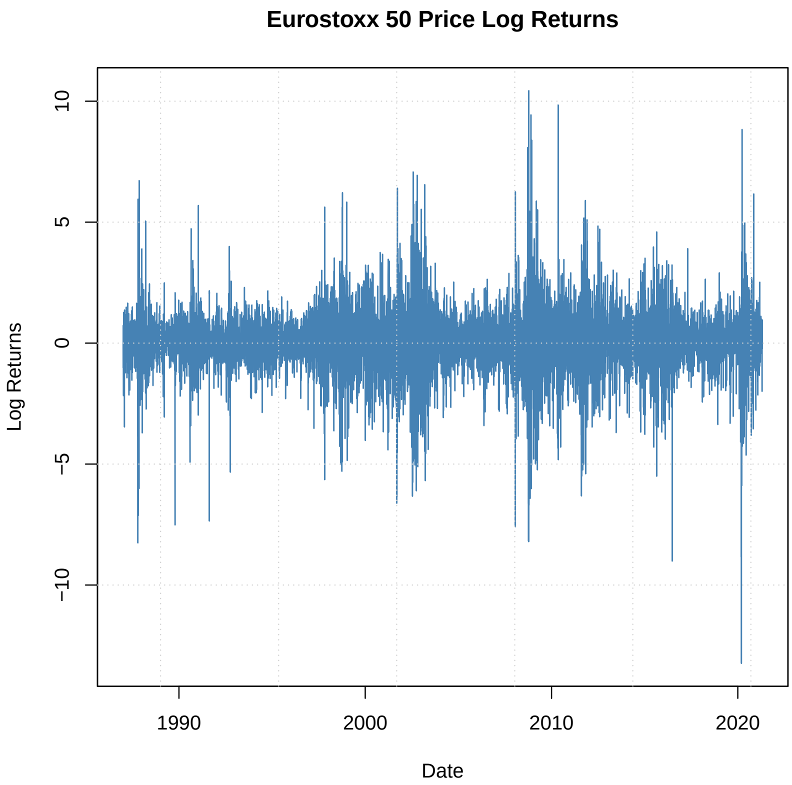

Figure 1.2: Euro Stoxx 50 Price Index log returns

ACF plots: to do...

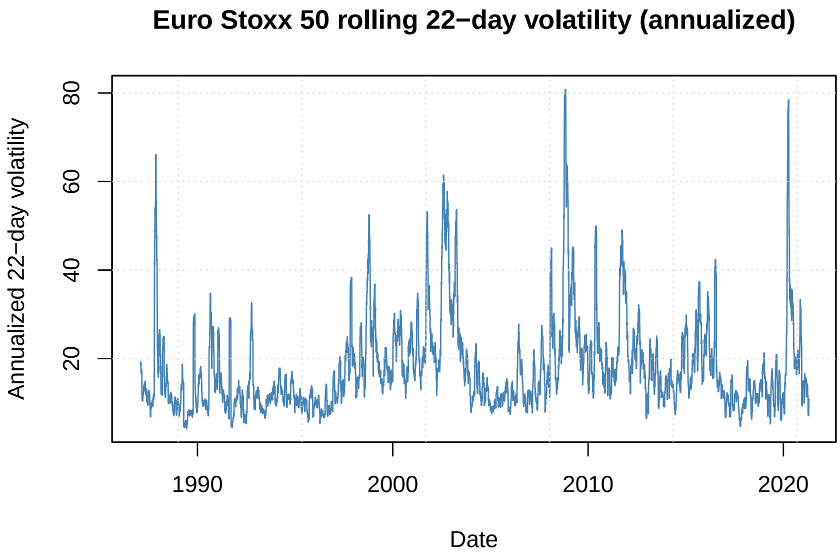

Figure 1.3: Euro Stoxx 50 rolling volatility (22 days, calculated over 252 days)

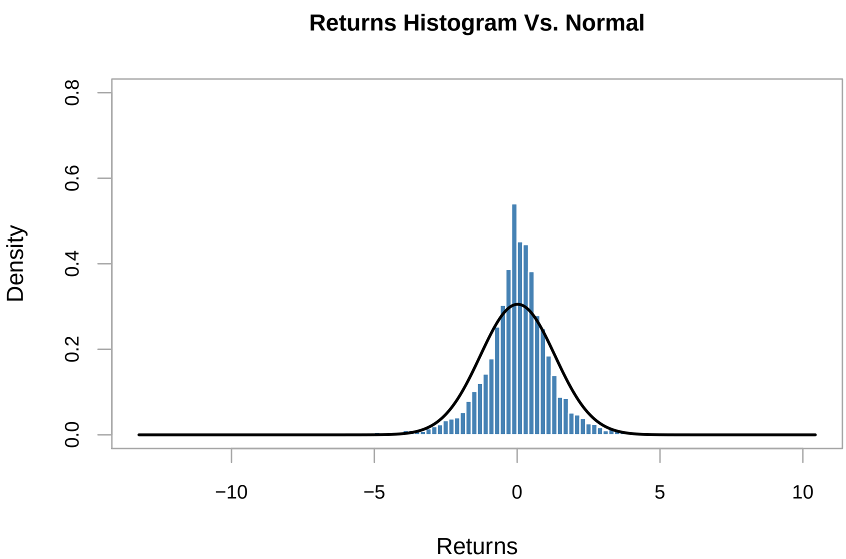

Figure 1.4: Density vs. Normal Euro Stoxx 50 log returns)

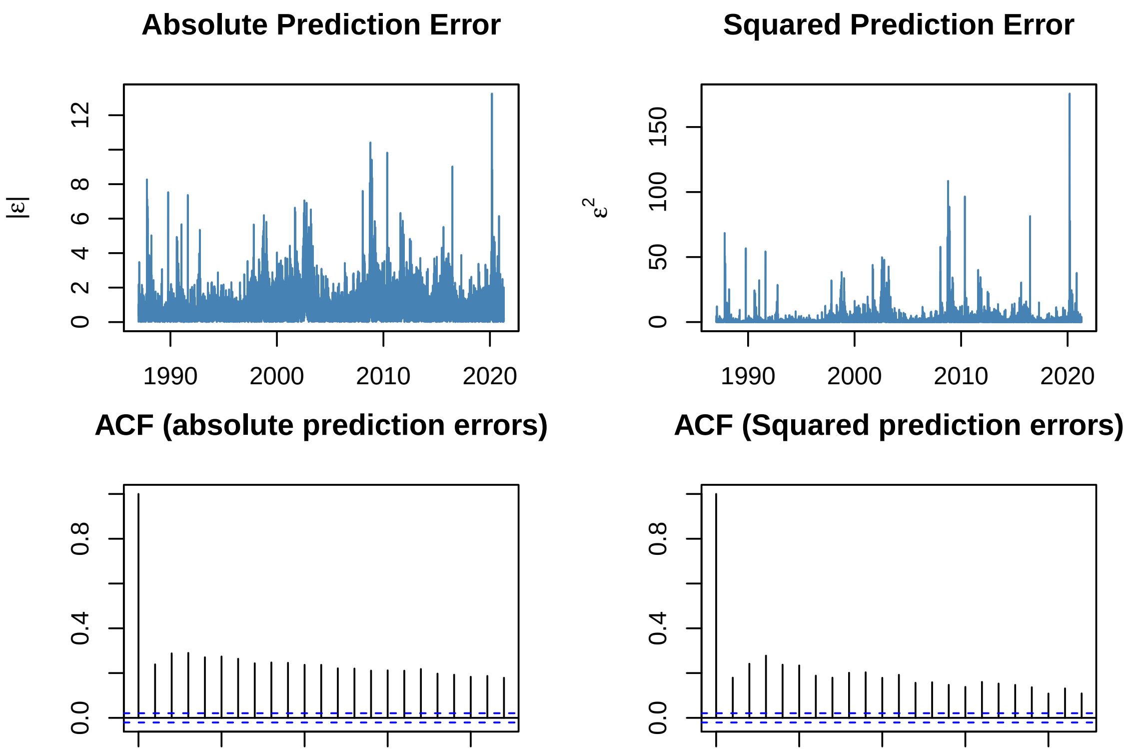

Figure 1.5: Absolute prediction errors

1.2 Methodology

1.2.1 Garch models

As already mentioned in part **??**, GARCH models GARCH, EGARCH, IGARCH, GJRGARCH, NGARCH, TGARCH and NAGARCH (or TSGARCH) will be estimated. Additionally the distributions will be examined as well, including the normal, student-t distribution, skewed student-t distribution, generalized error distribution, skewed generalized error distribution and the skewed generalized t distribution. They will be estimated using maximum likelihood. As already mentioned, fortunately, Ghalanos [2] has made it easy for us to implement this methodology in the R language (v.3.6.1) with the package "rugarch" v.1.4-4 (*R univariate garch*), which gives us a bit more time to focus on the results and the interpretation.

Maximum likelihood estimation is a method to find the distribution parameters that best fit the observed data, through maximization of the likelihood function, or the computationally more efficient log-likelihood function (by taking the natural logarithm). It is assumed that the return data is i.i.d. and that there is some underlying parametrized density function f with one or more parameters that generate the data, defined as a vector θ (equation (1.3)). These functions are based on the joint probability distribution of the observed data (equation (1.5)). Subsequently, the (log)likelihood function is maximized using an optimization algorithm (equation (1.7)).

$$y_1, y_2, ..., y_N \sim i.i.d \tag{1.2}$$

$$y_i \sim f(y|\theta) \tag{1.3}$$

$$L(\theta) = \prod_{i=1}^{N} f(y_i|\theta) \tag{1.4}$$

$$\log(L(\theta)) = \sum_{i=1}^{N} \log f(y_i|\theta) \tag{1.5}$$

$$\theta^* = arg \max_{\theta}[L] \tag{1.6}$$

$$\theta^* = arg \max_{\theta}[\log(L)] \tag{1.7}$$

1.2.2 ACD models

Following Ghalanos [3], arguments of ACD models are specified as in Hansen [4]. The density function $f(y|\alpha)$ has parameters $\alpha_t = (\mu_t, \sigma_t, \nu_t)$, with equation (1.8), the conditional mean equation. Equation (1.9) as the conditional variance. And $\nu_t = \nu(\theta, x_t)$ the remaining parameters of the distribution like the skewness and kurtosis (shape) parameters.

$$\mu_t = \mu(\theta, x_t) = E(y_t \mid x_t) \tag{1.8}$$

$$\sigma_t^2 = \sigma^2(\theta, x_t) = E\left(\left(y_t - \mu_t^2\right) \mid x_t\right) \tag{1.9}$$

To further explain the difference between GARCH and ACD. The scaled innovations are given by equation (1.10). The conditional density is given by equation (1.11) and related to the density function $f(y|\alpha)$ as in equation (1.2.2).

$$z_t(\theta) = \frac{y_t - \mu(\theta, x_t)}{\sigma(\theta, x_t)} \tag{1.10}$$

$$g(z \mid \eta_t) = \frac{d}{dz} P(z_t < z \mid \eta_t) \tag{1.11}$$

$$f\left(y_t \mid \mu_t, \sigma_t^2, \eta_t\right) = \frac{1}{\sigma_t} g\left(z_t \mid \eta_t\right) \tag{1.12}$$

Again Ghalanos [3] makes it easier to implement the somewhat complex ACD models using the R language with package “racd”.

1.2.3 Analysis Tests VaR and cVaR

Unconditional coverage test of Kupiec [5]

A number of tests are computed to see if the value-at-risk estimations capture the actual losses well. A first one is the unconditional coverage test by Kupiec [5]. The unconditional coverage or proportion of failures method tests if the actual value-at-risk exceedances are consistent with the expected exceedances (a chosen percentile, e.g. 1% percentile) of the VaR model. Following Kupiec [5] and Ghalanos [6], the number of exceedences follow a binomial distribution (with thus probability equal to the significance level or expected proportion) under the null hypothesis of a correct VaR model. The test is conducted as a likelihood ratio test with statistic like in equation (1.13), with p the probability of an exceedence for a confidence level, N the sample size and X the number of exceedences. The null hypothesis states that the test statistic LR^{uc} is χ^2-distributed with one degree of freedom or that the probability of failure $\hat{p}$ is equal to the chosen percentile α.

$$LR^{uc} = -2 \ln \left(\frac{(1-p)^{N-X} p^X}{\left(1 - \frac{X}{N}\right)^{N-X} \left(\frac{X}{N}\right)^X} \right) \tag{1.13}$$

Conditional coverage test of Christoffersen, Hahn, and Inoue [7]

Christoffersen, Hahn, and Inoue [7] proposed the conditional coverage test. It is tests for unconditional covrage and serial independence. The serial independence is important while the LR^{uc} can give a false picture while at any point in time it classifies inaccurate VaR estimates as "acceptably accurate" [8]. For a certain VaR estimate an indicator variable, $I_t(\alpha)$, is computed as equation (1.14).

$$I_t(\alpha) = \begin{cases} 1 & \text{if exceedence occurs} \\ 0 & \text{if no exceedence occurs} \end{cases} . \tag{1.14}$$

It involves a likelihood ratio test's null hypothesis is that the statistic is χ^2-distributed with two degrees of freedom or that the probability of violation $\hat{p}$ (unconditional coverage) as well as the conditional coverage (independence) is equal to the chosen percentile α.

Dynamic quantile test

Engle and Manganelli [9] with the aim to provide completeness to the conditional coverage test of Christoffersen, Hahn, and Inoue [7] developed the Dynamic quantile test. It consists in testing some restriction in a . . . (work-in-progress).

References

[1] *Calculation guide STOXX ®.* Tech. rep. 2020.

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[6] Alexios Ghalanos. *Introduction to the rugarch package. (Version 1.4-3).* 2020. URL: `http://cran.r-project.org/web/packages/rugarch/`.

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[8] Turan G. Bali and Panayiotis Theodossiou. "A conditional-SGT-VaR approach with alternative GARCH models". In: *Annals of Operations Research* 151.1 (Feb. 22, 2007), pp. 241–267. URL: `http://link.springer.com/10.1007/s10479-006-0118-4`.

[9] Robert F. Engle and S. Manganelli. *CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles.* Tech. rep. San Diego: UC San Diego, 1999. URL: `http://www.jstor.org/stable/1392044`.