1 Empirical Findings

1.1 Density of the returns

```
## Loading required package: readxl
## Loading required package: xts
## Loading required package: zoo
##
## Attaching package: 'zoo'
## The following objects are masked from 'package:base':
##
##     as.Date, as.Date.numeric
## Loading required package: PerformanceAnalytics
##
## Attaching package: 'PerformanceAnalytics'
## The following object is masked from 'package:graphics':
##
##     legend
## Loading required package: kableExtra
## Loading required package: fitdistrplus
```

```
## Loading required package: MASS
## Loading required package: survival
## Loading required package: fGarch
## Loading required package: timeDate
##
## Attaching package: 'timeDate'
## The following objects are masked from 'package:PerformanceAnalytics':
##
##     kurtosis, skewness
## Loading required package: timeSeries
##
## Attaching package: 'timeSeries'
## The following object is masked from 'package:zoo':
##
##     time<-
## Loading required package: fBasics
## Loading required package: tree
## Loading required package: sgt
## Loading required package: optimx
## Loading required package: numDeriv
## Loading required package: devtools
## Loading required package: usethis
## Registered S3 method overwritten by 'cli':
##   method     from
##   print.tree tree
## Loading required package: stringr
## Loading required package: rugarch
## Loading required package: parallel
```

```
##
## Attaching package: 'rugarch'
## The following object is masked from 'package:fitdistrplus':
##
##     fitdist
## The following object is masked from 'package:stats':
##
##     sigma
## Loading required package: plyr
## Loading required package: dplyr
##
## Attaching package: 'dplyr'
## The following objects are masked from 'package:plyr':
##
##     arrange, count, desc, failwith, id, mutate, rename, summarise,
##     summarize
## The following objects are masked from 'package:timeSeries':
##
##     filter, lag
## The following object is masked from 'package:MASS':
##
##     select
## The following object is masked from 'package:kableExtra':
##
##     group_rows
## The following objects are masked from 'package:xts':
##
##     first, last
```

```
## The following objects are masked from 'package:stats':
##
##     filter, lag
## The following objects are masked from 'package:base':
##
##     intersect, setdiff, setequal, union
## Skipping install of 'xtsExtra' from a github remote, the SHA1 (7d0b2b52) has not
##   Use `force = TRUE` to force installation
```

1.1.1 MLE distribution parameters

In table 1.1 we can see the estimated parameters of the unconditional distribution functions. Note that the student-t and skewed student-t distribution are usually noted with degrees of freedom as parameters. For consistency, we have parameterized them using limiting cases of the SGT-distribution. Note that to read the degrees of freedom for the two distributions, it is simply 2η. They are presented for the Skewed Generalized T-distribution (SGT) and limiting cases thereof previously discussed. Additionally, maximum likelihood score and the Akaike Information Criterion (AIC) is reported to compare goodness of fit of the different distributions but also taking into account simplicity of the models. We find that the SGT-distribution has the highest maximum likelihood score of all. All other distributions have relatively similar likelihood scores, though slightly lower and are therefore not the optimal distributions. However, when considering AIC it is a tie between SGT and SGED. This provides some indication that we have a valid case to test the suitability of different SGED-GARCH VaR models as an alternative for the SGT-GARCH VaR models. While sacrificing some goodness of fit, the SGED distribution has the advantage of requiring one parameter less, which could possibly result in less errors due to misspecification and easier implementation. For the SGT parameters the standard deviation and skewness are both significant at the 1% level. For the SGED parameters, the standard deviation and the skewness are both significant at respectively the 1% and 5% level. Both distributions are right-skewed. For both distributions the shape parameters are significant at the 1% level, though the q parameter was not estimated as it is by design set to infinity due to the SGED being a limiting case of SGT.[1]

Additionally, for every distribution fitted with MLE, plots are generated to compare the theoretical distribution with the observed returns. We see that except for the normal distribution which is quite off, the theoretical distributions are close to the actual data, except that they are too peaked. This problem is the least present for the SGT distribution.

[1] To check whether the relative ranking of distributions still holds in different periods, we have calculated the maximum likelihood score and AIC for three smaller periods: The period up to the dotcom collapse (1987-2001), up to the GFC (2002-2009) and up to the present Covid-crash (2009-2021). There is no qualitative difference in relative ranking with these subsamples. Results are reported in the appendix.

Table 1.1: Maximum likelihood estimates of unconditional distribution functions

dist	α	β	ξ	κ	η	*LLH*	AIC
SGT	0.014	1.441	-0.02	1.233	4.959	-8850.419	17710.84
	(0.025)	(0.031)***	(0.016)	(0.085)***	(1.368)***		
SGED	0.015	1.42	0.008	0.898	Inf	-8859.217	17710.84
	(0.003)***	(0.015)***	(0.002)***	(0.018)***			
GED	0	1.418	0	0.899	Inf	-8859.537	17725.07
	(0.002)	(0.023)***		(0.022)***			
ST	0.012	1.635	-0.043	2	2.817	-8873.843	17755.69
	(0.02)	(0.076)***	(0.016)***		(0.132)***		
T	0.045	1.64	0	2	1.403	-8877.165	17760.33
	(0.015)***	(0.078)***			(0.133)***		
Normal	0.01	1.439	0	2	Inf	-9477.660	18959.32
	(0.02)	(0.014)***					

Notes
Table contains parameter estimates for SGT-distribution and some of its limiting cases. The underlying data is the daily return series of the Euro Stoxx 50 for the period between December 31. 1986 and April 27. 2021. Standard errors are reported between brackets. *LLH* is the maximum log-likelihood value. *, ** and *** point out significance at 10

1.2 Constant higher moments

Table 1.2 presents the maximum likelihood estimates for 8 symmetric and asymmetric GARCH models based on the ST distribution with constant skewness and kurtosis parameters (t values are presented in parenthesis). The parameters in the conditional mean equations (α_0) are all statistically significant with t values from 0.201 to 4.323. The AR(1) coefficient, α_1, has parameters going from -0.049 to -0.032 with t values ranging from -3.802 to -2.569 not suggesting a high significance and indicating slight negative autocorrelation. The GARCH parameters in the conditional variance equations (β_0) are generally statistically significant with t values ranging from 1.512 to 11.432. The results of β_1 and β_2 show the presence of significant time-variation in the conditional volatility of the Euro Stoxx 50 Price Index daily returns, in fact, the sum of β_1 and β_2 for the GARCH parameters is close to one (from 0.827 to 1), suggesting the presence of persistence in the volatility of the returns. The parameter ξ is highly significant for all the 8 models tested with values ranging from 0.885 to 0.918 confirming the presence of Skewness in the returns. The shape parameter η, which, in our case, measures the number of degrees of freedom, determining the tail behavior, is significant for all the models and ranges between 6.306 and 8.127. The parameter γ, which is present only for EGARCH and GJRGARCH is significant and with values around 0.14. The absolute value function in family GARCH models (NAGARCH, TGARCH and AVGARCH) is subject to the $shift$ and the rot parameters whose values are always positive and statistically significant. According to the log likelihood values (LLH), displayed in table 1.2, the model with the highest value is AVGARCH while, excluding the non-standard GARCH models from the analysis, the model that performs best is EGARCH.

Table 1.2: Maximum likelihood estimates of the ST-GARCH models with constant skewness and kurtosis parameters

	SGARCH	IGARCH	EGARCH	GJRGARCH	EWMA	NAGARCH	TGARCH	AVGARCH
α_0	0.052	0.052	0.014	0.019	0.055	0.004	0.013	0.003
	(4.323)	(3.907)	(1.262)	(1.574)	(3.798)	(0.312)	(1.009)	(0.201)
α_1	-0.048	-0.048	-0.033	-0.043	-0.049	-0.034	-0.038	-0.032
	(-3.802)	(-3.572)	(-2.688)	(-3.489)	(-3.396)	(-2.717)	(-3.336)	(-2.569)
β_0	0.017	0.014	0.004	0.021	0	0.024	0.023	0.024
	(4.194)	(4.049)	(1.512)	(5.112)		(7.609)	(5.202)	(11.432)
β_1	0.095	0.1	-0.155	0	0.072	0.06	0.078	0.068
	(8.44)	(9.03)	(-14.319)	(0)	(9.211)	(4.61)	(9.322)	(16.631)
β_2	0.899	0.9	0.982	0.902	0.928	0.787	0.922	0.899
	(84.002)		(2399.932)	(68.804)		(36.109)	(103.357)	(7288.415)
ξ	0.918	0.918	0.89	0.895	0.915	0.885	0.891	0.885
	(54.745)	(54.754)	(52.93)	(54.569)	(58.951)	(53.5)	(53.159)	(52.625)
η	6.602	6.306	7.9	7.736	7.248	8.124	8.067	8.127
	(10.158)	(11.559)	(9.074)	(9.17)	(12.708)	(8.826)	(8.941)	(8.892)
γ			0.107	0.177				
			(10.04)	(8.9)				
$shift$						1.567		0.393
						(4.726)		(28.665)
rot							1	1
							(14.43)	(8.566)
LLH	-8303.694	-8304.437	-8158.11	-8186.06	-8328.667	-8143.563	-8154.785	-8143.141

Notes

This table shows the maximum likelihood estimates of various ST-GARCH models. The daily returns used on the Euro Stoxx 50 Price index cover the period from 03 January, 2001 to 19 May, 2021 (5316 observations). The mean process is modeled as follows: $R_t = \alpha_0 + \alpha_1 \times R_{t-1} + \varepsilon_t$ Where, in the 8 GARCH models estimated, γ is the asymmetry in volatility, ξ, κ and η are constant and t statistics are displayed in parenthesis. LLH is

As you can see in table 1.3 the AIC for the skewed student's t-distribution (ST) is the best from all the models. As also shown in appendix part **??**. The best in all distributions of the GARCH models seems to be the NAGARCH model, but we do not want to overfit our models because of an in-sample estimation. That is why a parsimonous model like the EGARCH (which has the highest maximum likelihood for the standard GARCH models that are considered), but also the model AVGARCH will be examined using the ST distribution while it has the second-best (lowest) AIC.

Table 1.3: Model selection according to AIC

	SGARCH	IGARCH	EWMA	EGARCH	GJRGARCH	NAGARCH	TGARCH	AVGARCH
N	3.174	3.176	3.198	3.114	3.124	3.107	3.111	3.107
T	3.130	3.130	3.140	3.079	3.089	3.074	3.077	3.074
ST	3.127	3.127	3.135	3.072	3.083	3.067	3.071	3.067
GED	3.128	3.128	3.139	3.080	3.089	3.075	3.079	3.076
SGED	3.125	3.126	3.136	3.075	3.084	3.069	3.073	3.069

Notes

This table shows the AIC value for the respective model. With on the rows the distributions.

1.2.1 Value-at-risk

As already mentioned 2 candidate models seem to be very appropriate. This includes the EGARCH and the NAGARCH So to check if they perform well out-of-sample we conduct a backtest by using a rolling forecasting technique. A simple graph is shown in figure 1.1 for the EGARCH-ST model. It seems that the VaR model for $\alpha = 0.05$ underestimates the downside events, while the VaR model for $\alpha = 0.01$ captures more of the downside events.

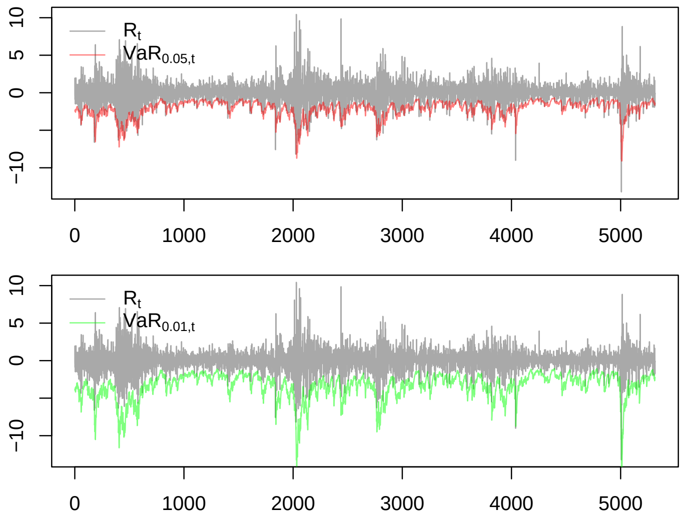


Figure 1.1: Value-at-Risk (in-sample) for the EGARCH-ST model

Let us examine this further using a rolling window approach whilst forecasting 1-day ahead results with re-estimating parameters every year.

Figure 1.2 shows that choosing an appropriate forecast period is important (with here the Eurobond crisis, the Brexit and Covid-crisis), so in order to avoid a look-ahead bias this rolling window approach was used instead of a static forecast method.

Figure 1.2: Selected period to start forecast from

If we look at the results of the rolling window, we can for example compare as in figure 1.3 the EGARCH-ST (with skewed student's t-distribution) with the EGARCH-N (with normal distribution). The EGARCH-N seems to capture the extreme events a bit less compared EGARCH-ST. But let us formally test this.

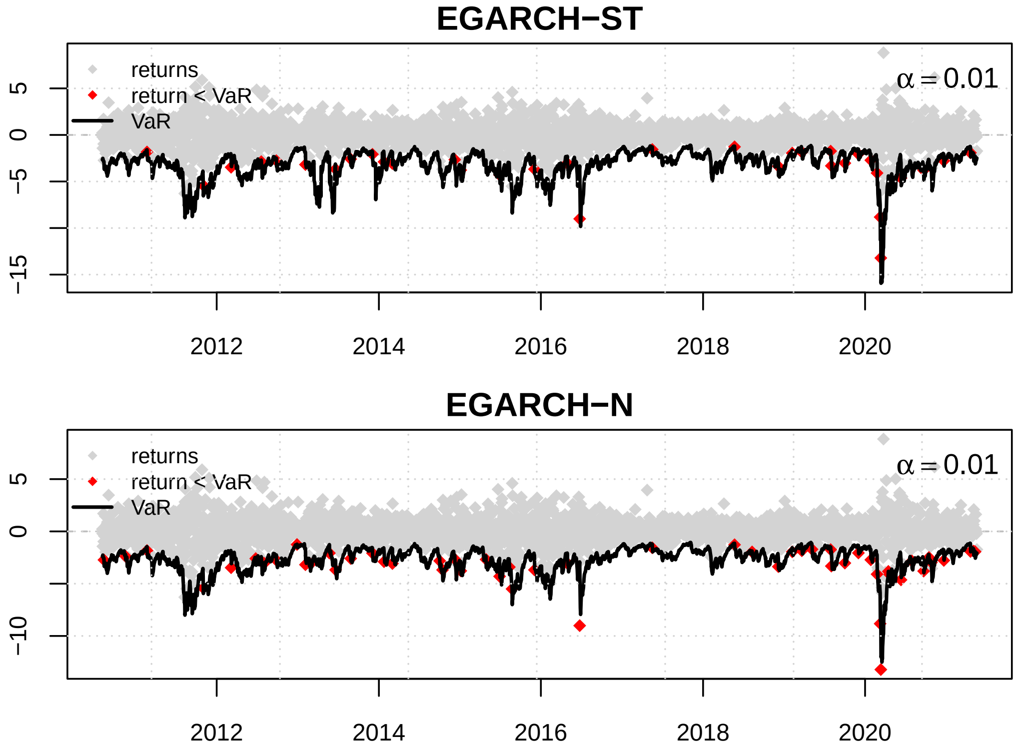


Figure 1.3: Comparison between VaR-EGARCH-ST and VaR-NAGARCH-N

Table 1.4 cointains multiple statistics related to the performance of different combinations of GARCH models and distribution concerning VaR and ES. More specifically, it includes the ratios of actual to expected exceedances for both VaR and ES, the unconditional coverage test by Kupiec [1], the conditional coverage by Christoffersen, Hahn, and Inoue [2] and the dynamic quantile test of Engle and Manganelli [3]. The ES ratio is derived from the ES test statistic by **McNeil2000**

r

Table 1.4: VaR and ES test statistics

	EGARCH	GJRGARCH	TGARCH	NAGARCH	AVGARCH
Panel A: SGED					
AE VaR	1.21	1.24	1.21	1.21	1.21
AE ES	1.21***	1.25**	1.21**	1.21*	1.21**
UC	1.15	1.56	1.15	1.15	1.15
CC	1.98	2.44	1.98	1.98	1.98
DQ	7.76	13.29	12.58	5.22	5.85
Panel B: GED					
AE VaR	1.46	1.46	1.49	1.46	1.38
AE ES	1.46***	1.46**	1.5***	1.46**	1.39***
UC	5.18**	5.18**	5.97**	5.18**	3.76*
CC	6.4**	6.4**	7.24**	6.4**	4.86*
DQ	20.23***	16.56**	19.58**	11.69	10.51
Panel C: ST					
AE ES	1.21	1.24	1.28	1.17	1.24
AE ES	1.21	1.25	1.29	1.18	1.25
UC	1.15	1.56	2.03	0.8	1.56
CC	1.98	2.44	2.96	1.58	2.44
DQ	15.07*	13	13.94*	4.55	6.84
Panel D: T					
AE VaR	1.53	1.56	1.56	1.49	1.6
AE ES	1.54	1.57	1.57*	1.5	1.61
UC	6.8***	7.68***	7.68***	5.97**	8.61***
CC	8.14**	9.08**	9.08**	7.24**	10.07***
DQ	0.01	18.99**	22.76***	17.38**	25.56***
Panel E: N					
AE VaR	1.95	1.88	1.95	1.74	1.78
AE ES	1.96***	1.89***	1.96***	1.75***	1.79***
UC	20.22***	17.58***	20.22***	12.76***	13.9***
CC	22.41***	19.61***	22.41***	14.5***	15.71***
DQ	36.5***	32.87***	39.47***	27.1***	37.61***

Notes
Table contains the ratio of actual to expected exceedances for VaR and Expected Shortfall, ratio, the unconditional and conditional coverage test statistic and the dynamic quantile test statistic. *, ** and *** point out significance at 10

1.3 Time-varying higher moments

2
Robustness checks

In order to check if the models are specified correctly, some specification checks have to be performed. The specification checks have to be done on the standardized residuals of the estimated GARCH model given by the following equation:

$$\hat{Z}_t = \frac{\hat{\varepsilon}_t}{\hat{\sigma}_t} = \frac{R_t - \hat{\mu}}{\hat{\sigma}_t}$$

.

2.2 Panel A displays a first Ljung-box test on the absolute value of the standardized residuals of the GARCH models.

The description of table

Table 2.1: ARCH LM Test

	SGARCH	EGARCH	AVGARCH	NAGARCH	GJRGARCH	TGARCH	IGARCH	EWMA
Norm	32.322*	26.461	23.081	23.474	34.475**	26.991	33.857*	42.773***
T	34.687**	25.958	22.063	22.218	37.875**	27.634	34.912**	40.719***
ST	34.605**	26.138	22.129	22.36	38.174**	27.718	34.756**	39.559**
GED	33.431*	25.973	22.433	22.598	36.379**	27.023	34.228**	41.433***
SGED	33.393*	26.173	22.319	22.698	36.859**	27.167	34.071**	40.155***

Notes

[1] This table shows the ARCH LM statistics value for the respective model

[2] DESCRIBE PVALUES ***, **, *

Table 2.2: Diagnostic Tests for Heteroscedasticity

	SGARCH	EGARCH	AVGARCH	NAGARCH	GJRGARCH	TGARCH	IGARCH	EWMA
Panel A: Ljung Box Test on the standardized squared values of the residuals								
Norm	31.157*	25.321	22.263	23.03	32.186*	26.208	32.727*	44.066***
T	33.907**	24.81	21.321	21.862	34.34**	26.658	34.183**	41.765***
ST	33.961**	25.024	21.412	22.051	34.607**	26.811	34.187**	40.605***
GED	32.493*	24.826	21.63	22.191	33.365*	26.142	33.361*	42.627***
SGED	32.569*	25.065	21.57	22.341	33.747*	26.351	33.342*	41.333***
Panel B: ARCH LM Test on the standardized squared values of the residuals								
Norm	32.322*	26.461	23.081	23.474	34.475**	26.991	33.857*	42.773***
T	34.687**	25.958	22.063	22.218	37.875**	27.634	34.912**	40.719***
ST	34.605**	26.138	22.129	22.36	38.174**	27.718	34.756**	39.559**
GED	33.431*	25.973	22.433	22.598	36.379**	27.023	34.228**	41.433***
SGED	33.393*	26.173	22.319	22.698	36.859**	27.167	34.071**	40.155***

Notes

Table displays the Ljung box statistics and the ARCH LM Test for the standardized squared residuals of the models analyzed. The underlying data is the daily return series of the Euro Stoxx 50 for the period between 2001-01-03 and 2021-05-19.

*, ** and *** point out respectively significance at 10

The null hypothesis of the test in both panels are described as follows:

H_0: $Corr(Z_t^2,Z_{t-1}^2)$=$Corr(Z_t^2,Z_{t-2}^2)$= ... =$Corr(Z_t^2,Z_{t-22}^2) = 0$

Residual heteroscedasticity

Ljung-Box test on the squared or absolute standardized residuals.

DESCRIPTION OF GMM TEST HERE!!!

Table 2.3: GMM Tests

	SGARCH	EGARCH	AVGARCH	NAGARCH	GJRGARCH	TGARCH	IGARCH	EWMA
Panel A: SGED								
Mean	-0.036**	-0.002	0.007	0.006	-0.006	-0.001	-0.036**	-0.039***
Variance	0.015**	0.009	0.007	0.007	0.01	0.008	-0.016**	0.121***
Skewness	-0.46**	-0.392	-0.374	-0.381	-0.406	-0.374	-0.443**	-0.566***
Excess Kurtosis	1.974**	1.782	1.809	1.803	1.693	1.696	1.755**	3.821***
Panel B: GED								
Mean	-0.051***	-0.024*	-0.015	-0.017	-0.026*	-0.022	-0.05***	-0.051***
Variance	0.004***	0.004*	0.004	0.004	0.003*	0.003	-0.024***	0.12***
Skewness	-0.49***	-0.452*	-0.438	-0.444	-0.459*	-0.435	-0.476***	-0.602***
Excess Kurtosis	1.858***	1.763*	1.815	1.809	1.662*	1.689	1.661***	3.824***
Panel C: ST								
Mean	-0.042***	-0.009	0.002	0.001	-0.012	-0.007	-0.042***	-0.049***
Variance	0.006***	0.005	0.003	0.003	0.004	0.005	-0.016***	0.124***
Skewness	-0.473***	-0.414	-0.388	-0.396	-0.421	-0.393	-0.46***	-0.596***
Excess Kurtosis	1.944***	1.79	1.809	1.808	1.671	1.702	1.777***	3.855***
Panel D: T								
Mean	-0.059***	-0.03**	-0.019	-0.02	-0.032**	-0.027*	-0.059***	-0.062***
Variance	-0.007***	0.001**	0.003	0.002	-0.001**	0.002*	-0.024***	0.124***
Skewness	-0.511***	-0.473**	-0.452	-0.458	-0.476**	-0.451*	-0.501***	-0.638***
Excess Kurtosis	1.812***	1.783**	1.848	1.84	1.658**	1.707*	1.674***	3.856***
Panel E: N								
Mean	-0.051***	-0.007	0.004	0.002	-0.01	-0.004	-0.051***	-0.047***
Variance	0.001***	0.001	0	0	0.001	0	-0.035***	0.118***
Skewness	-0.482***	-0.391	-0.371	-0.378	-0.405	-0.372	-0.465***	-0.583***
Excess Kurtosis	1.775***	1.684	1.736	1.721	1.605	1.616	1.516***	3.807***

Notes

Table displays the GMM test statistics for the standardized residuals. The underlying data is the daily return series of the Euro Stoxx 50 for the period between 2001-01-03 and 2021-05-19.

The null hypothesis of the tests for each variable is described as follows: FORMULA $H_0 = ...$ HERE!!!

$GMM \ \ test$

Table 2.4: Jarque-Bera Test on standardized residuals

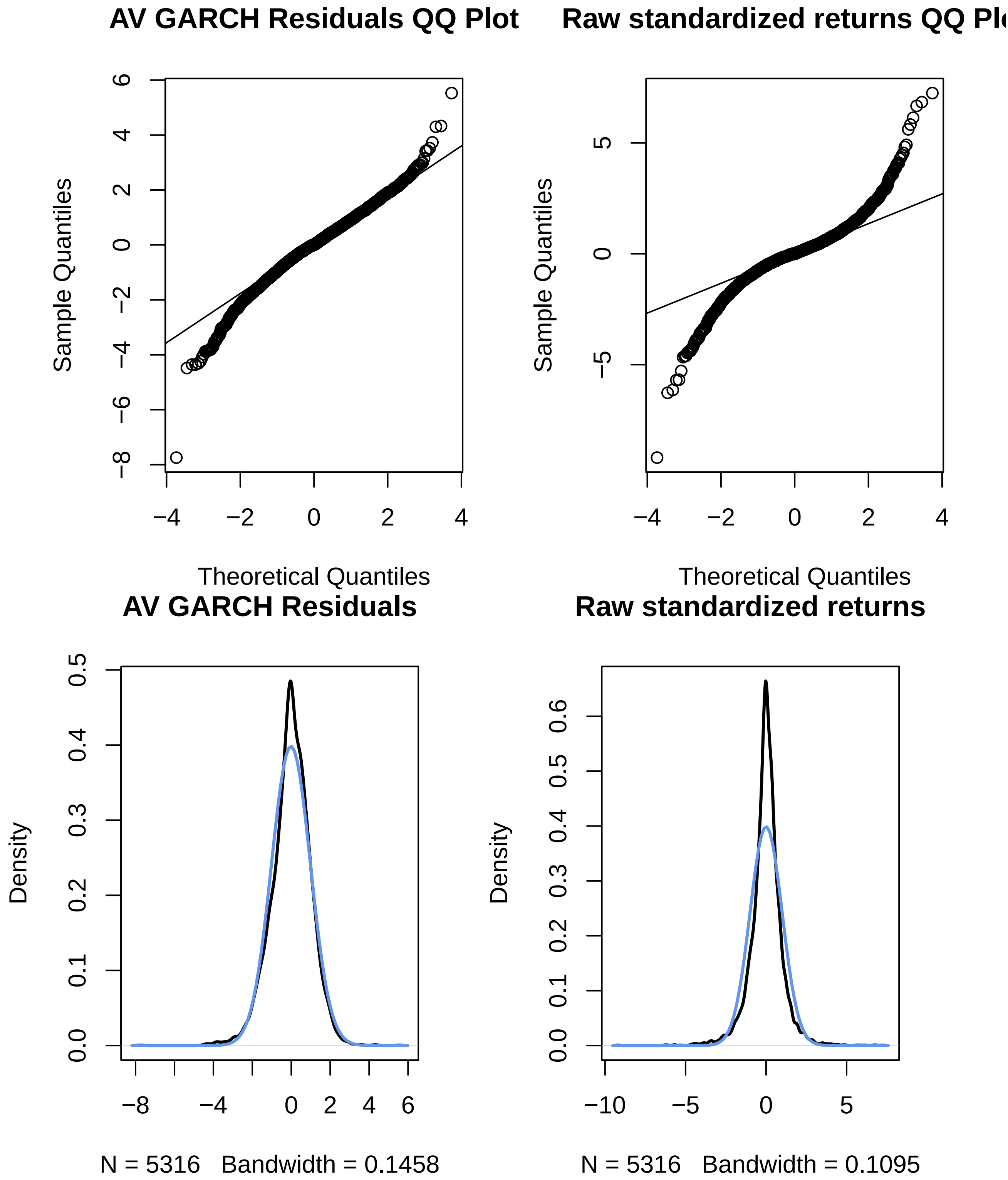
	SGARCH	EGARCH	AVGARCH	NAGARCH	GJRGARCH	TGARCH	IGARCH	EWMA
Norm	739.888***	735.992***	791.977***	784.475***	676.533***	679.569***	804.423***	1382.222***
T	822.929***	789.877***	839.862***	845.41***	703.17***	718.176***	851.465***	1350.929***
ST	845.468***	787.192***	833.066***	842.468***	700.469***	712.976***	884.557***	1360.366***
GED	785.374***	759.794***	810.002***	811.834***	685.775***	697.098***	841.477***	1367.132***
SGED	803.315***	758.538***	810.37***	811.343***	684.155***	691.427***	869.762***	1372.361***

Notes

Table displays the Jarque-Bera statistics value for the residuals of the models analyzed. The underlying data is the daily return series of the Euro Stoxx 50 for the period between 2001-01-03 and 2021-05-19.

*, ** and *** point out respectively significance at 10

The null hypothesis is described as follows: WRITE NULL HYPOTHESIS HERE



zero-mean unit-variance not skewed no excess kurtosis no serial correlation in the squares no serial correlation in the cubes no serial correlation in the squares

Other diagnostics? robustness analysis?

References

[1] P.H. Kupiec. “Techniques for Verifying the Accuracy of Risk Measurement Models”. In: *Journal of Derivatives* 3.2 (1995), pp. 73–84.

[2] Peter Christoffersen, Jinyong Hahn, and Atsushi Inoue. “Testing and comparing Value-at-Risk measures”. In: *Journal of Empirical Finance* 8.3 (July 2001), pp. 325–342.

[3] Robert F Engle and Simone Manganelli. “CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles”. In: *Journal of Business and Economic Statistics* 22.4 (Oct. 1, 2004), pp. 367–381. URL: https://www.tandfonline.com/doi/full/10.1198/073500104000000370.