9.2 Mom.

First order equation

$$F(t, y, y') = 0$$

$$y' = \underset{\sim}{f(t,y)}$$ linear function

Linear Eq.

$$y' + a(t)\, y = b(t)$$

(1) $a \equiv 0$ : $y' = b(t) \Rightarrow y = \int b(t)dt$

(2) $b \equiv 0$ : $y' + a(t) y = 0$ ( $1^{st}$ order homogeneous DE)

$$y' = -a(t) y$$

$$\frac{y'}{y} = -a(t)$$

$$\ln|y| = -\int a(t)dt + C$$

$$|y| = e^{c} e^{-\int a(t)dt} \Rightarrow y = \tilde{c}\, e^{-\int a(t)dt}$$ if $y(t_0) = 0 \Rightarrow y(t) \equiv 0$

(3) General case

$$y = c\, e^{-\int a(t)dt}$$ $c$ is a function

$$y = c(t)\, e^{-\int a(t)dt}$$ variation of parameter

$$y' = c'(t) e^{-\int a(t)dt} + c(t) e^{-\int a(t)dt}(-a(t))$$

$$= c'(t) e^{-\int a(t)dt} + y(-a(t))$$

$$y' + a(t) y = c'(t) e^{-\int a(t)dt} = b(t)$$

want: $c'(t)\, e^{-\int a(t)dt} = b(t)$

$$c'(t) = b(t)\, e^{\int a(t)dt}$$

$$c(t) = \int b(t)\, e^{\int a(t)dt}\, dt$$

Example $y' - 2ty = t$

Step 1: $u' - 2tu = 0$

$$(u\, e^{\int -2t\, dt})' = 0$$

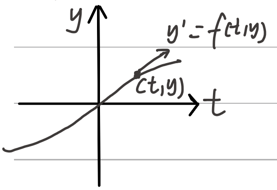
$$y' + a(t) y = b(t)$$

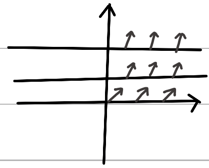
$$(y e^{\int a(t)dt})' = b(t) e^{\int a(t)dt}$$

$$y e^{\int a(t)dt} = \int b(t) e^{\int a(t)dt}\, dt$$

## Geometric meaning

$y' = f(t,y)$ Example $y' = 1+y^2$



9.4 Wed.

$y' = f(t)g(y)$ separable equ.

$\int \frac{dy}{g(y)} = \int f(t)dt \Rightarrow G(y) = F(t)$ Solve y from

Example $\begin{cases} y' = 1+y^2 \\ \eta(0) = 0 \end{cases}$ : $\underbrace{\int \frac{dy}{1+y^2}}_{= \tan^{-1} y} = \int dt = t+c \Rightarrow \begin{cases} y = \tan(t+c) \\ c=0 \end{cases}$

malthus $p'(t) = cp(t)$

$\frac{p'}{p} = c_1$

$\ln|p| = c_1 t + c_2$

$p(t) = e^{c_1 t} e^{c_2} = ke^{ct}$

1837 Verhulst

$p'(t) = r(1 - \frac{p(t)}{K})p(t)$ r: growth rate

$p(0) = p_0$ k: carrying capacity

$p' = rp(1 - \frac{p}{k})$ logistic equ

$p' = p(a - bp)$

$\int \frac{dp}{p(a-bp)} = \int dt = t+c$

$\int (\frac{1/a}{p} + \frac{b/a}{a-bp})dp = \frac{1}{a}\ln|p| - \frac{1}{a}\ln|a-bp| = \frac{1}{a}\ln\frac{p}{|a-bp|} = t+c$

$\frac{p}{|a-bp|} = e^{at} \cdot \tilde{C}$

$\frac{p_0}{|a-bp_0|} = \tilde{C}$

(i) $P_0 < \frac{b}{a}$ : $\frac{P}{a-bP} = \tilde{c}e^{at} \Rightarrow P = \frac{a}{b+\frac{1}{\tilde{c}}e^{-at}}$ $P\uparrow$ $P \to \frac{a}{b}$ but never reach

Example A commercial fishery is assumed to have carrying capacity 10,000 kg of certain kind of fish. Suppose the annual growth rate of the total fish (kg) is governed by the equ.

$\frac{dP}{dt} = P(1-\frac{P}{10000})$ & initially $P(0) = 2000$ kg

① What is the fish population after 1 year?

Suppose that after waiting for a certain period of time the owner decides to harvest 2400 kg of fish annually at a constant rate. Then

② What is the diff equ. governing the fish population now?

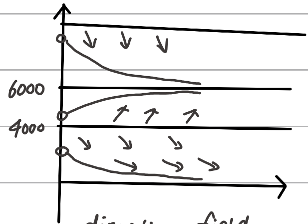
③ What is the minimal waiting period you'd recommend to the owner

① $P(1) = \frac{1}{\frac{1}{10000} + \frac{1-1/5}{2000}e^{-t}} = 4047$ kg

② $P' = P(1-\frac{P}{10000}) - 2400$

$= -\frac{1}{10000}[(P-6000)(P-4000)]$

minimal waiting period: about 1 year



direction field

## Orthogonal Trajectories

Example Given a family of curves $y = cx^2, c \in \mathbb{R}$

find the orthogonal trajectories

$$\begin{cases} y' = 2cx \to c = \frac{y}{x^2} \\ y' = -\frac{1}{2cx} = -\frac{x}{2y} \end{cases}$$

$\frac{x^2}{2} + y^2 = k$

9.9 Mon

Newton's Cooling law

$T(t)$ : the temp of a body

$A$: the temp of the ambient space

$T'(t) = k(A-T)$ , $k>0$

$\frac{dT}{A-T} = k\,dt \Rightarrow \ln|A-T| = -kt + C \Rightarrow |A-T| = Ce^{-kt}$

$A - T(t) = Ce^{-kt}$ $(t=0 \Rightarrow A - T(0) = C)$

$A - T(t) = (A - T(0))e^{-kt}$

Example

$T_Y(t) = A + (T_Y(0) - A)e^{-kt}$

$T_F(t) = A + (T_c - A)e^{-kt}$

$T_{Y(10)} = A + \left(\frac{T_c + rT_m}{1+r} - A\right)e^{-10k}$

$\tilde{T}_{F(10)} = \frac{T_{F(10)} + rT_m}{1+r} = \frac{A + (T_c - A)e^{-10k} + rT_m}{1+r}$

$T_c$ : coffee's temp (served) $T_m$ : cream's temp

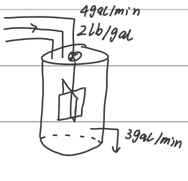
$A$: room temp $r$: proportion of cream and coffee $T_{Y(0)} = \frac{T_c + rT_m}{1+r}$

$\tilde{T}_{F(10)} - T_{Y(10)} = \frac{1}{1+r}\left[A + rT_m + (T_c - A)^{-10k} - (1+r)A - \{T_c + rT_m - (1+r)A\}e^{-10k}\right]$

$= \frac{1}{1+r}\left[rT_m - rA - \{rT_m - rA\}e^{-10k}\right] = (T_m - A)\underbrace{\frac{r}{1+r}}_{>0}\underbrace{[1 - e^{-10k}]}_{>0}$

A mixture problem

A 120 gal tank initially contains 90 lb of salt dissolved in 90 gal of water. Brine containing 2lb/gal of salt flows into the tank at the rate of 4gal/min and the mixture of the well-stilled solution in the tank flows out at the rate of 3gal/min. How much salt the tank contains when it is full




$y(t)$ = the ammount of salt in the tank

$y(0) = 90\,lb$

$y' = 2lb/gal \cdot 4gal/min - \frac{y(t)}{90+t}\cdot 3$

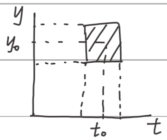
$(e^{\int \frac{3}{90+t}dt}\, y)' = 8\, e^{\int \frac{3}{90+t}dt} = 8\cdot e^{3\ln(90+t)}$

$y = 2(90+t) + \frac{C}{(90+t)^3}$ $\qquad$ $y(0) = 180 + \frac{C}{90^3} \Rightarrow C = -90^4$

$= 2(90+t) - \frac{90^4}{(90+t)^3}$

$y(30) = 240 - \frac{3^5 \cdot 10}{4^3}$

$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ $\quad$ Local: $|t-t_0|$ small



assume: $f, \frac{\partial f}{\partial y}$ are continuous on a neighborhood of $(t_0, y_0)$ & $M = \max\limits_{\substack{|t-t_0|\le a \\ |y-y_0|\le b}} |f(t,y)|$ $\quad L = \max\limits_{\substack{|t-t_0|\le a \\ |y-y_0|\le b}} |f_y(t,y)|$

$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds$

uniqueness assume $y_1(t), y_2(t)$ both are sols.

$|y_1(t) - y_2(t)| = \left|\int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s))\,ds\right| \le \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_2(s))|\,ds = \int_{t_0}^{t} |f_y(s, \tilde{y})| \cdot |y_1(s) - y_2(s)|\,ds$

$\le L \int_{t_0}^{t} |y_1(s) - y_2(s)|\,ds$

Set $w(t) = |y_1(t) - y_2(t)| \ge 0$ $\qquad$ $w(t) \le L \underbrace{\int_{t_0}^{t} w(s)\,ds}_{z(t)}$

i.e. $z'(t) \le L z(t)$

$[e^{-Lt} z(t)]' \le 0$

$e^{-Lt} z(t) \le e^{-Lt_0} z(t_0) = 0 \Rightarrow z(t) = 0$

example $\begin{cases} y' = \frac{3}{2} y^{\frac{1}{3}} \\ y(0) = 0 \end{cases} \Rightarrow \begin{array}{l} y \equiv 0 \\ y = t^{\frac{3}{2}} \end{array}$ $\quad$ $f(t,y)$在 original 不连续

9.11 Wedn.

Existence Th'm

Suppose that $f$ & $\frac{\partial f}{\partial y}$ are cont. on $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$

Then the IVP (initial-value problem)

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \quad (DE)$$

$\alpha(a,b,f)$ $\begin{cases} ① \alpha \le a \\ ② M\alpha \le b \end{cases} \Rightarrow \alpha = \min\{a, \frac{b}{M}\}$

has a unique solution for $t \in (t_0 - \alpha, t_0 + \alpha)$ for some $\alpha > 0$ small

Set $M = \max_{(t,y)\in R} |f(t,y)|$ $\quad L = \max_{(t,y)\in R} |\frac{\partial f}{\partial y}(t,y)|$

$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds$ (IE) $t > t_0$

Rk: (DE) & (IE) are equiv.

$y_0(t) \equiv y_0$ like Taylor expansion. e.g. $\begin{cases} y' = 1 + y^2 \\ y(0) = 0 \end{cases}$

$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))ds$

$y_1(t) = 0$

$y_2(t) = t + \frac{t^3}{3}$

$y_3(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{1}{63}t^7$

$\tan(t) = t + \frac{t^3}{3} + \frac{2t^5}{15} + \frac{17t^7}{315} + \frac{62}{2835}t^9$

$y_2(t) = y_0 + \int_{t_0}^{t} f(s, y_1(s))ds$

$y_3(t) = y_0 + \int_{t_0}^{t} f(s, y_2(s))ds$

$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s))ds$

... (Picard iteration)

$y_\infty(t) = y_0 + \int_{t_0}^{t} f(s, y_\infty(s))ds$

$|y_{n+1}(t) - y_n(t)| \le \int_{t_0}^{t} |f(s, y_n(s)) - f(s, y_{n-1}(s))| ds = \int_{t_0}^{t} |\frac{\partial f}{\partial y}(s,*)| |y_n(s) - y_{n-1}(s)| ds$

$(s, y_n(s)) \in R$ we need
$(s, y_n(s)) \in R$ garentee $= \int_{t_0}^{t} |\frac{\partial f}{\partial y}(s,*)| |y_n(s) - y_{n-1}(s)| ds \le L \int_{t_0}^{t} |y_n(s) - y_{n-1}(s)| ds \le L^n M \frac{(t-t_0)^{n+1}}{(n+1)!}$

$|y_1(t) - y_0(t)| \le \int_{t_0}^{t} |f(s, y_0(s))| ds \le \underbrace{M(t-t_0)}_{\le M\alpha < b} \le b$ choose $\alpha$

$|y_2(t) - y_1(t)| \le \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_0(s))| ds \le L \int_{t_0}^{t} |y_1(s) - y_0(s)| ds \le L \cdot M \frac{(t-t_0)^2}{2}$

$|y_3(t) - y_2(t)| \le \int_{t_0}^{t} |f(s, y_2(s)) - f(s, y_1(s))| ds \le L \int_{t_0}^{t} |y_2(s) - y_1(s)| ds \le L^2 M \int_{t_0}^{t} \frac{(s-s_0)^2}{2} ds = L^2 M \frac{(t-t_0)^3}{6}$

make sure $y_1(t)$, $y_2(t)$, $y_3(t)$ lie in $[y_0 - b, y_0 + b]$

$y_{n+1}(t) = y_0(t) + (y_1(t) - y_0(t)) + (y_2(t) - y_1(t)) + \cdots + (y_{n+1}(t) - y_n(t))$

$\left( |y_{n+1}(t) - y_0(t)| \le \sum_{k=0}^{n} |y_{k+1}(t) - y_k(t)| \le \sum_{k=0}^{n} L^k M \frac{\alpha^{k+1}}{(k+1)!} = \frac{M}{L} \sum_{k=0}^{n} \frac{(\alpha L)^{k+1}}{(k+1)!} \right)$

$|y_n(t) - y_m(t)| \le |y_n(t) - y_{n-1}(t)| + \cdots + |y_{m+1}(t) - y_m(t)| \le \sum_{k=m}^{n-1} L^k M \frac{\alpha^{k+1}}{(k+1)!} \le \sum_{k=m}^{\infty} L^k M \frac{\alpha^{k+1}}{(k+1)!}$

$= (\sum_{k=0}^{\infty} - \sum_{k=0}^{m-1})(L^k M \frac{\alpha^{k+1}}{(k+1)!}) = \frac{M}{L}[e^{\alpha L} - \sum_{k=0}^{m-1} \frac{(\alpha L)^{k+1}}{(k+1)!}]$ $\therefore y_n(t)$ is cauchy

Need to verify $y_n(t) \in R$ $\forall$ $t - t_0 < \alpha$

Know $n=1$

Induction $|y_{n+1}(t)-y_0| \le \int_{t_0}^{t} |f(s,y_n(s))|\,ds \le \int_{t_0}^{t} M\,ds$ by induction hypothesis $\forall\, t_0<t<t_0+\alpha$ $y_n(t)\in R$

$= M(t-t_0) \le M\alpha \le b$

W.T.S. $y$ is cont. $\{y_n(t)\}$ is unif. cont.

$|y(x)-y(c)| \le |y(x)-y_n(x)| + |y_n(x)-y_n(c)| + |y_n(c)-y(c)| < 3\varepsilon$

9.14 Sat.

$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \xrightarrow[\text{bring in the form}]{\text{want it}} \frac{d}{dt}\phi(t,y)=0$

$\phi_t + \phi_y \frac{dy}{dt} = 0$ To find $\phi$ s.t. $\phi_t = M$ & $\phi_y = N$

$\phi_{ty} = M_y$ $\phi_{yt} = N_t$ $\Rightarrow M_y = N_t$

Example $\frac{y^2}{2} + 2ye^t + (y+e^t)y' = 0$

$\begin{cases} \phi_t = e^t(\frac{y^2}{2}+2ye^t) \Rightarrow \phi = \frac{e^t y^2}{2} + ye^{2t} + h(y) \\ \phi_y = e^t(y+e^t) \Rightarrow \phi = \frac{y^2}{2}e^t + e^{2t}y + k(t) \end{cases}$ $\frac{d}{dt}\phi(t,y)=0$

$M_y = y+2e^t$ $N_t = e^t$ $y^2\frac{e^t}{2} + e^{2t}y + c = 0 \Rightarrow y = \frac{-e^{2t} \pm \sqrt{e^{4t}-2ce^t}}{e^t}$

$\mu(M+Ny'=0) \Rightarrow \mu M + \mu N y' = 0$ $(\mu M)_y = (\mu N)_t$

$\mu_y M + \mu M_y = \mu_t N + \mu N_t$

(i) if $\mu$ is a function of $t$:

$\Rightarrow \mu M_y = \mu_t N + \mu N_t$ $\mu_t = \frac{\mu(M_y - N_t)}{N}$

(ii) If $\mu$ is a fcn of $y$ only

$\mu_t = \mu \frac{y+2e^t-e^t}{y+e^t} = \mu \Rightarrow \mu = e^t$

$(\frac{y^2}{2}e^t + 2ye^{2t}) + (y\,e^t + e^{2t})y' = 0$

Th'm Let $R$ be a rectangle in $t$-$y$ plane. Then $\exists\, \phi(t,y)$ s.t. $\phi_t = M$ & $\phi_y = N$

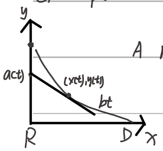
$\Leftrightarrow N_t = M_y$

$\Leftarrow$ Set $\phi(t,y) = \int M(t,y)\,dt + \int [N(t,y) - \int M_y(t,y)\,dt]\,dy$

$\phi_t = M(t,y) + \int [N_t - M_y]\,dy = M$

$\phi_y = \int M_y(t,y)\,dt + N(t,y) - \int M_y(t,y)\,dt = N$

# Example




A rabbit starts at the origin & runs up the y-axis with speed a towards its burrow at (0,R). At the same time, a dog starts at (C,D) & pursues the the rabbit with speed b. What is the path of the dog?

$$y'(t) = \frac{at-y(t)}{0-x(t)} \Rightarrow xy' = y-at$$

$$\frac{d}{dx}(xy') = \frac{d}{dx}(y-at) \Rightarrow y' + xy'' = y' - a\frac{dt}{dx}$$

$$bt = \int_x^c \sqrt{1+y'^2(z)}\,dz \Rightarrow b\frac{dt}{dx} = -\sqrt{1+y'^2(x)}$$

$$xy'' = r\sqrt{1+y'^2(x)} \qquad u=y'$$

$$xu' = r\sqrt{1+u^2(x)}$$

$$u+\sqrt{1+u^2(x)} = \tilde{C}x^r$$

$$y' = u = \frac{1}{2}\left(\tilde{C}x^r - \frac{1}{\tilde{C}x^r}\right) \qquad \begin{cases} y'(0)=0 \\ y(c)=0 \end{cases}$$

9.18 Wedn.

## Air Resistance

A bolt is shot straight upward with initial velocity $V_0 = 49m/sec$ at the ground level from a crossbow. Compute the maximum hieght & $T_{up}$. $T_{down}$ if

(i) air resistance is neglected

(ii) air resistance is taken into consideration

(i) $ma = mv' = F = -mg$

$v' = -g$

$v(t) = v_0 - \int_0^t g = v_0 - gt = 49 - 9.8t$

$v(T_{up}) = 0 = 49 - 9.8T_{up} \qquad T_{up} = 5 \text{ sec}$

$h'(t) = v(t)$

$h(t) = h(0) + \int_0^t v(s)ds = (49s - \frac{9.8}{2}s^2)\Big|_0^t = 49t - 4.9t^2$

$h(5) = 122.5\,m$

Slow $p=1$
large $p=2$

(ii) $mv' = ma = F = -mg + F_r \quad F_r = kv^p \quad 1 \le p \le 2$

$v' = -g - \frac{k}{m} v \ (\rho = \frac{k}{m})$

$(e^{\rho t} v)' = -e^{\rho t} g$

$v(t) = e^{-\rho t}(V_0 + \frac{g}{\rho}) - \frac{g}{\rho}$

$h(t) = \int_0^t [e^{-\rho s}(V_0 + \frac{g}{\rho}) - \frac{g}{\rho}] ds = \frac{1}{\rho}(V_0 + \frac{g}{\rho})(1 - e^{-\rho t}) - \frac{g}{\rho} t$

$v(T_{up}) = 0 = e^{-\rho T_{up}}(49 + \frac{9.8}{0.04}) - \frac{9.8}{0.04}$

$T_{up} = 4.55$ sec

$h(T_{up}) = 107.8\,m$ max height

$T_{down} = T_{total} - T_{up}$

$h(T_{total}) = 0$

$\frac{1}{\rho}(V_0 + \frac{g}{\rho})(1 - e^{-\rho T}) = \frac{g}{\rho} T \qquad (49 + \frac{9.8}{0.04})(1 - e^{-0.04T}) = 9.8T$

# Chapter 2 $\quad 2^{nd}$ order linear equation

$y'' = f(t, y, y')$

$y'' + p(t)y' + q(t)y = r(t)$

$\underbrace{y'' + p(t)y' + q(t)y}_{L(y) \equiv} = 0$

def $\ker(L) \equiv \{y \mid L[y] = 0\}$ $\quad L$: a linear operator

Prop. $\ker(L)$ is a vector space of dimension 2

pf. ① vector space

Th'm (Existence & Uniqueness)

Linear equation 所以可以在整个区间有解

If $p(t), q(t)$ are cont. on $(a,b)$ then the IVP has a unique sol on $[a,b]$

$$\begin{cases} y''+p(t)y'+q(t)y=0 \\ y(t_0)=\alpha \\ y'(t_0)=\beta \end{cases}$$

$y_1$ : a sol. $\begin{cases} y''+p(t)y'+q(t)y=0 \\ y(t_0)=1,\ y'(t_0)=0 \end{cases}$

$W[y_1,y_2](0)=-1$

$y_2$ : a sol. $\begin{cases} y''+p(t)y'+q(t)y=0 \\ y(t_0)=0,\ y'(t_0)=1 \end{cases}$

$\tilde{y}=\alpha y_1+\beta y_2 \Rightarrow \begin{cases} \tilde{y}(t_0)=\alpha \\ \tilde{y}'(t_0)=\beta \end{cases}$ dimension is at most 2

$\alpha y_1+\beta y_2=0$ ← 0 fcn $\quad \begin{cases} (\alpha y_1+\beta y_2)(t_0)=0 \Rightarrow \alpha=0 \\ (\alpha y_1+\beta y_2)'(t_0)=0 \Rightarrow \beta=0 \end{cases}$

Wronskian if $y_1$ $y_2$ are 2 sols. then

$$W[y_1,y_2](t)=\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

Prop. $W'+p(t)W=0$ There fore $W(t)=W(t_0)\,e^{-\int_{t_0}^{t} p(s)ds}$

($\Rightarrow$ $W(t)$ is either $\equiv 0$ or never 0)

(Pf) $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$

$W'=y_1'y_2'+y_1y_2''-y_2'y_1'-y_2y_1''$

$=y_1(-py_2'-qy_2)-y_2(-py_1'-qy_1)$

$=-py_1y_2'-qy_1y_2+y_2py_1'+qy_2y_1$

$=p(-y_1y_2'+y_2y_1')=-pW$

$W'+pW=0 \Rightarrow e^{\int_{t_0}^{t} p(s)ds}\,w(t)=W(t_0)$

prop Two sols $y_1$, $y_2$ are l.dep. $\Leftrightarrow$ $W[y_1,y_2](t)=0$ for all $t$

(Pf) ($\Rightarrow$) Suppose $y_1$, $y_2$ are l.dep.

i.e. $\exists c$ s.t. $y_1(t)=c\,y_2(t)$ $\forall t$

$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} cy_2 & y_2 \\ cy_2' & y_2' \end{vmatrix} = cy_2 y_2' - cy_2' y_2 = 0$

($\Leftarrow$) $W[y_1, y_2](t) = 0$

i.e. $y_1 y_2'(t) - y_2 y_1'(t) = 0$

case 1. $y_1\ y_2\ (t)$ is never $0$

$\frac{y_2'(t)}{y_2(t)} = \frac{y_1'(t)}{y_1(t)}$

$\ln|y_2(t)| = \ln|y_1(t)| + C$

$|y_2(t)| = \tilde{C}\,|y_1(t)| \Rightarrow y_2(t) = \bar{c}\, y_1(t)$

case 2 $y_1 y_2(t) = 0$ for some $t_0$

$\Rightarrow y = \frac{y_1'(t_0)}{y_2'(t_0)} y_2$

Subcase 1: $y_1(t_0) = 0 = y_2(t_0)$

$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 0 \\ y'(t_0) = y_1'(t_0) \end{cases}$ $\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = 0 \\ y'(t_0) = \frac{y_1'(t_0)}{y_2'(t_0)} y_2'(t_0) \end{cases}$ $\Rightarrow y_1 = \frac{y_1'(t_0)}{y_2'(t_0)} y_2$

$\exists!$ thm

Subcase 2: $y_1(t_0) = 0$ $y_2(t_0) \neq 0$

$y_1 y_2' = y_2 y_1'$ $\forall t$ $y_1(t_0) = 0$ $y_2(t_0) \neq 0$ $\Rightarrow \frac{y_1(t_0)}{y_2(t_0)} y_2'(t_0) = y_1'(t_0)$

set $y_3(t) = \frac{y_1(t_0)}{y_2(t_0)} y_2(t)$

$\begin{cases} y_3'' + p(t)y_3' + q(t)y_3 = 0 \\ y_3(t_0) = y_1(t_0) \\ y_3'(t_0) = \frac{y_1(t_0)}{y_2(t_0)} y_2'(t_0) \end{cases}$ $\Rightarrow y_3(t) = y_1(t)$ $\forall t$

# Constant coeffs. ($p(t) \equiv p$ $q(t) \equiv q$)

Example $y'' - 3y' + 2y = 0$ $m^2 - 3m + 2 = 0$ characteristic eqn.

$\frac{d^2}{dt^2} y - 3\frac{d}{dt} y + 2y = 0 \xrightarrow{D = \frac{d}{dt}} D^2 y - 3Dy + 2y = 0 \Rightarrow (D-2)\underbrace{(D-1)\,y}_{=z} = 0$

$(D-2)z = 0 \Rightarrow z' - 2z = 0 \Rightarrow (e^{-2t} z)' = 0 \Rightarrow z = ce^{2t}$

$y' - y = ce^{2t}$

$(e^{-t}y)' = c_1 e^{t} \Rightarrow y = c_1 e^{2t} + c_2 e^{t}$ $\{e^{2t}, e^{t}\}$ l. indep.

$L[y] = ay'' + by' + cy = 0$

characteristic eqn $am^2 + bm + c = 0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ $m_1, m_2$

$e^{m_1 t}$ $e^{m_2 t}$ $b^2-4ac>0$

Case 1 : $b^2-4ac>0$

2 real roots $m_1, m_2 \Rightarrow y = C_1 e^{m_1 t} + C_2 e^{m_2 t}$

Case 2 $b^2-4ac<0$

(1) $e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$ $W[e^{\alpha t}\cos\beta t, e^{\alpha t}\sin\beta t] = \begin{vmatrix} e^{\alpha t}\cos\beta t & e^{\alpha t}\sin\beta t \\ & \end{vmatrix}$

(2) $e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$

$y_1 = (1)+(2) = e^{\alpha t}\cos\beta t$ $y_2 = (1)-(2) = e^{\alpha t}\sin\beta t$

Case 3 $b^2-4ac=0$

Then a double root $r$ $(=r_1=r_2)$

$y = C_1 t e^{rt} + C_2 e^{rt}$

Example ① $4y'' + 4y' + 5y = 0$

$b^2-4ac = 16 - 4\cdot4\cdot5 = -64 < 0$

$\alpha \pm i\beta = \frac{-4 \pm \sqrt{-64}}{8} = -\frac{1}{2} \pm i$

$y = C_1 e^{-\frac{1}{2}t}\cos t + C_2 e^{-\frac{1}{2}}\sin t$

② $y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$ $(r+2)^2 = 0$ $r = -2, -2$

$y = ce^{-2t}$ Variation parameter

$y = C(t)e^{-2t}$ $y' = C'(t)e^{-2t} - 2C(t)e^{-2t}$ $y'' = C''(t)e^{-2t} - 2C'(t)e^{-2t} - 2C'(t)e^{-2t} + 4C(t)e^{-2t}$

$y'' + 4y' + 4y = e^{-2t}[C'' - 4C' + 4C + 4C' - 8C + 4C] = e^{-2t}C''$

need $C''=0$ i.e. $C' = \text{const.}$ i.e. $C = C_1 t + C_2$

$y=(C_1t+C_2)e^{-2t}=C_1te^{-2t}+C_2e^{-2t}$

Example $(1-t^2)y''+2ty'-2y=0 \quad -1<t<1$

There is a sol. $y=t$ $\quad y=C(t)\,t$ variation para.

$y'=C'(t)t+C(t) \quad y''=C''(t)t+2C'(t)$

$(1-t^2)y''+2ty'-2y=(1-t^2)C''t+2C'=0$

$C''+\frac{2}{t(1-t^2)}C'=0$

$\left(e^{\int\frac{2}{t(1-t^2)}dt}\,C'\right)'=0$

$\frac{t^2}{1-t^2}C'=\text{cost.} \Rightarrow C'=\frac{1-t^2}{t^2}=\frac{1}{t^2}-1 \qquad C=-\frac{1}{t}-t+\text{cost.}$

$y=C(t)t=\left(-\frac{1}{t}-t\right)\cdot t=-1-t^2$

$y=C_1t+C_2(1+t^2)$

$y''+p(t)y'+q(t)y=r(t) \quad (*)$

Suppose ① $y''+p(t)y'+q(t)y=0$ has 2 l. indep. sols. $y_1(t)$ $y_2(t)$

② $(*)$ has a particular sol. $y_p(t)$

Then every sol. of $(*)$ takes the form

$C_1y_1(t)+C_2y_2(t)+y_p(t)$

Suppose $y(t)$ is a sol of $(*)$ Then $y-y_p$

$$\begin{cases} y''+p(t)y'+q(t)y=r(t) \\ y_p''+p(t)y_p'+q(t)y_p=r(t) \end{cases} \Rightarrow \underbrace{(y-y_p)''}_{z''}+p(t)\underbrace{(y'-y_p')}_{z'}+q(t)\underbrace{(y-y_p)}_{z}=0$$

$\Rightarrow \exists C_1, C_2\in\mathbb{R}$ s.t. $\underset{y-y_p}{z}=C_1y_1(t)+C_2y_2(t)$

$y=C_1y_1+C_2y_2+y_p$

$y_p=C_1(t)y_1+C_2(t)y_2$

$y_p=u_1y_1+u_2y_2$

$y_p'=\underbrace{(u_1'y_1+u_2'y_2)}_{\text{set this to zero}}+(u_1y_1'+u_2y_2')$

$y_p'' + p y_p' + q y_p$

$= u_1(\overbrace{y_1'' + p y_1' + q y_1}^{0}) + u_2(\overbrace{y_2'' + p y_2' + q y_2}^{0}) + u_1' y_1' + u_2' y_2'$

$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = r \end{cases} \quad \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ r \end{pmatrix}$

$u_1' = \dfrac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{-r y_2}{y_1 y_2' - y_2 y_1'} \qquad u_2' = \dfrac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{r y_1}{y_1 y_2' - y_2 y_1'}$

Example $\quad y'' + y' + y = t^2$

$y'' + y' + y = 0 \qquad r^2 + r + 1 = 0 \qquad r = \frac{-1 \pm \sqrt{3}}{2} \qquad y_1 = e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t), \quad y_2 = e^{-\frac{1}{2}}\sin(\frac{\sqrt{3}}{2}t)$

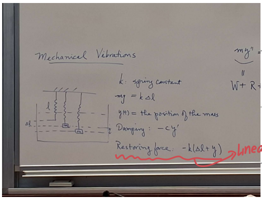
$W = y_1 y_2' - y_1' y_2 = \begin{vmatrix} e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t & e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t \\ -\frac{1}{2}e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{\sqrt{3}}{2}e^{-\frac{1}{2}}\sin\frac{\sqrt{3}}{2}t & -\frac{1}{2}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t + e^{-\frac{1}{2}}\cos\frac{\sqrt{3}}{2}t \cdot \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{\sqrt{3}}{2}e^{-t}$

$u_1' = \dfrac{-t^2 e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}e^{-t}} = \frac{-2}{\sqrt{3}} t^2 e^{\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$u_2' = \dfrac{t^2 e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}e^{-t}} = \frac{2}{\sqrt{3}} t^2 e^{\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$

9.30 Mon

mechanical Vibrations




linear

$my'' = -cy' - k(\Delta l + y) + mg + F$

$my'' = -cy' - ky + F$

$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F}{m}$

(a) Free Vibrations (undamped $c=0$ $F=0$)

$y'' + \frac{k}{m}y = 0 \quad \omega_0^2 = \frac{k}{m} \qquad y'' + \omega_0^2 y = 0 \quad (\pm i\omega_0)$

$y(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t) = \sqrt{a^2+b^2}\cos(\omega_0 t - \delta)$

(b) Damped free vibrations

$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0$

$r^2 + \frac{c}{m}r + \omega_0^2 = 0$

$r = \dfrac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\omega_0^2}}{2} = \dfrac{-c \pm \sqrt{c^2 - 4m^2\omega_0^2}}{2m}$

(i) $c^2 - 4w_0^2 m^2 > 0$ 2 real roots $r_1 < r_2 < 0$

$$y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} = e^{r_2 t}(C_1 e^{(r_1 - r_2)t} + C_2) \qquad \text{Overdamped}$$

(ii) $c^2 - 4w_0^2 m^2 = 0$

$$y(t) = C_1 e^{rt} + C_2 t e^{rt} = e^{rt}(C_1 + C_2 t) \qquad \text{Critically damped}$$

(iii) $c^2 - 4w_0^2 m^2 < 0$

$$y(t) = e^{-\frac{c}{2m}t}\left(a\cos\sqrt{w_0^2 - \frac{c^2}{4m^2}}\,t + b\sin\sqrt{w_0^2 - \frac{c^2}{4m^2}}\,t\right)$$

$$= e^{-\frac{c}{2m}t} \cdot R \cdot \cos(\mu t - \delta) \qquad \text{undamped}$$

$$y'' + \frac{c}{m}y' + w_0^2 y = F_0\cos(wt)$$

$$w_0^2 y_p(t) = w_0^2 a\cos(wt) + b w_0^2 \sin(wt)$$

$$c' y_p' = -c' a w \sin(wt) + c b w \cos(wt) \qquad c' = \frac{c}{m}$$

$$y_p'' = -a w^2 \cos(wt) - b w^2 \sin(wt)$$

$$\cos(wt)(-aw^2 + bc'w + aw_0^2) + \sin(wt)(-bw^2 - c'aw + bw_0^2) = F_0 \cos wt$$

$$\begin{cases} a(w_0^2 - w^2) + c'wb = F_0 \\ -c'wa + (w_0^2 - w^2)b = 0 \end{cases} \quad a = \frac{\begin{vmatrix} F_0 & c'w \\ 0 & w_0^2 - w^2 \end{vmatrix}}{\begin{vmatrix} w_0^2 - w^2 & c'w \\ -c'w & w_0^2 - w \end{vmatrix}} = \frac{F_0(w_0^2 - w^2)}{(w_0^2 - w^2)^2 + (c'w)^2} \qquad b = \frac{c'wF_0}{(w_0^2 - w^2)^2 + (c'w)^2}$$

$$y_p = \frac{F_0}{(w_0^2 - w^2)^2 + (c'w)^2}\left[(w_0^2 - w^2)\cos(wt) + c'w\sin(wt)\right] = \frac{F_0}{(w_0^2 - w^2) + (c'w)^2}\left[(w_0^2 - w^2)\cos wt + c'w\sin wt\right] = \frac{F_0}{\sqrt{(w_0^2 - w^2)^2 + (c'w)^2}}\cos(wt - \delta)$$

Q What is the behavior when $w \to w_0$ & $c \to 0$

$$y(t) = C_1\cos(w_0 t) + C_2\sin(w_0 t)$$

$$y_p(t) = \frac{F_0}{w_0^2 - w^2}\cos(wt)$$

$$y(t) = \frac{F_0}{w_0^2 - w^2}(\cos(wt) - \cos(w_0 t)) = \frac{F_0}{2w_0(-1)}\, t\sin(wt)$$

10.9 Wedn

Example $y'' + 4y = \cos 2t$

$r^2 + 4 = 0 \quad r = \pm 2i$

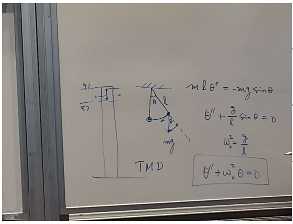
$y_1 = e^{2it} \quad y_2 = e^{-2it}$

$y_p = u_1 y_1 + u_2 y_2 \Rightarrow$ (V. of P.)

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}}{W[y_1, y_2]} = \frac{1}{4i} \Rightarrow u_1 = \frac{1}{4i}t \qquad y_p = \frac{1}{4i}\underset{\uparrow}{te^{2it}} + \frac{1}{16}e^{2it}$$

$$\frac{1}{4i}t(\cos 2t + i\sin 2t)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}}{W[y_1, y_2]} = \frac{e^{4it}}{-4i} \Rightarrow u_2 = \frac{1}{16}e^{4it}$$




$\theta = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

$\frac{2\pi}{\sqrt{g}}\sqrt{l} = 7 \Rightarrow l = \left(\frac{7}{2\pi}\right)^2 \cdot 9.8 = 12.2m$

Series solutions

Examples $y'' - 2ty' - 2y = 0$

$y(t) = a_0 + a_1 t + \cdots = \sum_{n=0}^{\infty} a_n t^n$

$y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$

$y''(t) = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n$

$y'' - 2ty' - 2y = \sum_{n=0}^{\infty} t^n \left[(n+2)(n+1)a_{n+2} - 2na_n - 2a_n\right] = 0$

$a_{n+2} = \frac{2(n+1)}{(n+2)(n+1)} a_n = \frac{2}{n+2} a_n$

$a_{2k+2} = \frac{2}{2k+2} \cdot \frac{2}{2k} \cdot \frac{2}{2k-2} \cdots a_0$

$a_0 = 1, a_1 = 0 \quad a_{2k} = \frac{2}{2k} \cdot \frac{2}{2(k-1)} \cdots 1 (a_0) = \frac{1}{k!}$ $\quad y(t) = \sum_{k=0}^{\infty} a_{2k} t^{2k} = \sum_{k=0}^{\infty} \frac{1}{k!} t^{2k} = e^{t^2}$

$a_0 = 1, a_1 = 0 \quad a_{2k+1} = \cdots$

Th'm Consider the eqn $P(t)y'' + Q(t)y' + R(t)y = 0$

if $\frac{Q(t)}{P(t)}$ & $\frac{Q(t)}{q(t)}$ have Taylor Series expansions

conv. on the interval $|t-t_0| < \rho$ then every sol. of the equ. must have a Taylor series expansion

which conv. at least on the interval $|t-t_0| < \rho$

Expand $\frac{1}{1+t^2}$ into its Taylor series

10.14 Mon

$P(t)y'' + Q(t)y' + R(t)y = 0$

Example $(1+t^2)y'' + 3ty' + y = 0$

$y=\sum_{k=0}^{\infty} a_n t^n \quad y'=\sum_{k=0}^{\infty} n a_n t^{n-1} \quad y''=\sum_{k=0}^{\infty} n(n-1)a_n t^{n-2}=\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m$

$(1+t^2)y''+3ty'+y=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+n(n-1)a_n+3na_n+a_n\right]t^n$

Need: $(n+2)(n+1)a_{n+2}+(n+1)^2a_n=0 \qquad a_{n+2}=-\frac{n+1}{n+2}a_n$

$a_0=0 \quad a_1=1: \quad a_1=a_3=a_5=a_7\cdots=0$

$n=2k \qquad a_{2k+2}=(-1)^{k+1}\frac{2k+1}{2k+2}\cdot\frac{2k-1}{2k}\cdots\frac{1}{2}a_0=(-1)^{k+1}\frac{(2k+1)\cdots 1}{2^{k+1}(k+1)!}$

$y=1-\frac{1}{2}k^2+\cdots+(-1)^k\cdot\frac{(2k-1)\cdots 3\cdot 1}{2^k k!}t^{2k}+\cdots$

$$\left|\frac{\frac{(-1)^{k+1}(2k+1)\cdots 3\cdot 1}{2^{k+1}(k+1)!}t^{2k+2}}{\frac{(-1)(2k-1)\cdots 3\cdot 1}{2^k\ k!}t^{2k}}\right|<1$$

Euler's equ. $\quad t^2y''+\alpha ty'+\beta y=0,\ t=0$ 0 is a singular point

$y''+\frac{\alpha}{t}y'+\frac{\beta}{t^2}y=0$

$t=0$ is a regular singular point.

$y=t^r \quad y'=rt^{r-1} \quad y''=r(r-1)t^{r-2}$

$0=r(r-1)+\alpha r+\beta=r^2+(\alpha-1)r+\beta=0 \qquad r=\frac{1-\alpha\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

$P(t)y''+Q(t)y'+R(t)y=0$

$y''+p(t)y'+q(t)y=0$

$p(t)=\frac{p_0}{t}+p_1+p_2t+\cdots$

$q(t)=\frac{q_0}{t^2}+\frac{q_1}{t}+q_2+q_3t+\cdots$

$y=t^r(a_0+a_1t+\cdots)$

$t^2y''+t^2p(t)y'+t^2q(t)y=0$

$y=t^r(a_0+a_1t+\cdots)=\sum_{n=0}^{\infty}a_nt^{n+r}$

$y'=\sum_{n=0}^{\infty}(n+r)a_nt^{n+r-1}$

$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nt^{n+r}$

$n=0 \quad t^r: \quad r(r-1)a_0+ra_0p_0+a_0q_0=0$

$n=1 \quad t^{r+1}: \quad (1+r)ra_1+p_0(1+r)a_1+p_1ra_0+q_0a_1+q_1a_0=0$

Bessel's eqn. of order $\frac{1}{2}$

$t^2y'' + ty' + (t^2-\frac{1}{4})y = 0$

$y = \sum_{n=0}^{\infty} a_n t^{n+r}$ $a_0 \neq 0$

$ty' = \sum_{n=0}^{\infty} (n+r) a_n t^{n+r}$

$t^2y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r}$

$(t^2-\frac{1}{4})y = \sum_{m=2}^{\infty} a_{m-2} t^{m+r} - \frac{1}{4}\sum_{n=0}^{\infty} a_n t^{n+r}$

$t^r$: $r(r-1)a_0 + ra_0 - \frac{1}{4}a_0 = 0$ $r^2 - \frac{1}{4} = 0$ $r = \pm\frac{1}{2}$

$t^{r+1}$: $(1+r)ra_1 + (1+r)a_1 - \frac{1}{4}a_1$ $a_1[r^2+2r+\frac{3}{4}]$

$r=\frac{1}{2} \Rightarrow a_1 = 0$ $r = -\frac{1}{2} \Rightarrow a_1 \in \mathbb{R}$ $a_1$ can be any number

$r = \frac{1}{2}$ $a_1 = 0$ $a_0 = 1$ $n \geq 2$, $t^{n+r}$ $(n+r)(n+r-1)a_n + (n+r)a_n + a_{n-2} - \frac{1}{4}a_n = 0$

$(n+r)^2 a_n - \frac{1}{4}a_n + a_{n-2} = 0 \Rightarrow a_n = \frac{-1}{n(n+1)} a_{n-2}$ $a_1 = a_3 = \cdots = 0$

$y = t^{\frac{1}{2}}[1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 + \cdots + \frac{(-1)^k}{(2k+1)!}t^{2k} + \cdots] = \frac{1}{\sqrt{t}}\sin t$

10.16 wedn.

Example $y'' + 3y' + 2y = e^t$

$$\begin{cases} x_1 = y & x_1' = x_2 \\ x_2 = y' & x_2' = -3x_2 - 2x_1 + e^t \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -3x_2 - 2x_1 + e^t \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

$$\mathbb{X}' = \underset{2\times 2}{A}\mathbb{X} + e^t$$

example $y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y(t) = g(t)$

$x_1 = y$
$x_2 = y'$
$\vdots$
$x_n = y^{(n-1)}$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & \cdots & & & \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ g(t) \end{pmatrix}$$

$\mathbb{X}'(t) = A\mathbb{X}(t) + \vec{b}(t)$

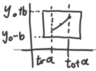
$$A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ & \cdots & & \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix} \qquad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

(Existence / Uniqueness)

Th'm The IVP $\begin{cases} \mathbb{X}'(t) = A\mathbb{X}(t) + \vec{g}(t) \\ \mathbb{X}(t_0) = \mathbb{X}_0 \end{cases}$

has a unique sol. (in the entire "interval")

(pf.) The same as that of $1^{st}$ order equ. (Picard iteration)




$\begin{vmatrix} a(t) & b(t) \\ c(t) & d(t) \end{vmatrix}' = (ad-bc)' = \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix} = \begin{vmatrix} a' & b \\ c' & d \end{vmatrix} + \begin{vmatrix} a & b' \\ c & d' \end{vmatrix}$

$\mathbb{X}' = A\mathbb{X}$

$\begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix}' = \begin{vmatrix} c_{11}' & c_{12}' & \cdots & c_{1n}' \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21}' & c_{22}' & \cdots & c_{2n}' \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{vmatrix} + \cdots +$

$\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \rightarrow i^{th}$ $\quad \vec{\mathbb{X}}_i(t_0) = \vec{e}_i$, $i = 1, \cdots n$

Wronskian

$W[\vec{\mathbb{X}}_1, \cdots \vec{\mathbb{X}}_n](t) = \begin{vmatrix} | & | & & | \\ \vec{\mathbb{X}}_1(t) & \vec{\mathbb{X}}_2(t) & \cdots & \vec{\mathbb{X}}_n(t) \\ | & | & & | \end{vmatrix}$

$W'(t) = -(a_{11}(t) + a_{22}(t) + \cdots + a_{nn}(t))\, W$

$W' + (trA)\, W = 0 \qquad W(t) = e^{-\int trA} \cdot W(t_0)$

$y' = ay \qquad y = e^{at} y_0 = e^{At}\, \mathbb{X}(0)$

10.28 Mon

Th'm ($\vec{g} \equiv 0$) Homogeneous

The set of all sols. to (*) (with $\vec{g} \equiv 0$) is a vector space of

dim $n$

wronskiam

$W[\vec{\mathbb{X}}_1, \cdots \vec{\mathbb{X}}_n](t) = \begin{vmatrix} | & & | \\ \vec{\mathbb{X}}_1(t) & \cdots & \vec{\mathbb{X}}_n(t) \\ | & & | \end{vmatrix}$

$A(t)$ is a constant matrix $\qquad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$

$\vec{\mathbb{X}}' = A\vec{\mathbb{X}} \Rightarrow \vec{\mathbb{X}} = e^{At}\vec{c} \qquad e^{A} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$

suppose you can diagonalize $A$ i.e. $\exists Q$ s.t. $QAQ^{-1} = D$ is diagonal

$e^{A} = \begin{pmatrix} 1 + a_1 + \frac{a_1^k}{2} + \cdots + \frac{a_1^k}{k!} + \cdots & & & \\ & 1 + a_2 + \frac{a_2^2}{2} + \cdots + \frac{a_2^k}{k!} + \cdots & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} = \begin{pmatrix} e^{a_1} & & 0 \\ & e^{a_2} & \\ 0 & & \ddots \\ & & & e^{a_n} \end{pmatrix}$

$A^k = QD^kQ^{-1}$

$e^A = QIQ^{-1} + QDQ^{-1} + \frac{QD^2Q^{-1}}{2!} + \cdots + \frac{QD^kQ^{-1}}{k!} + \cdots = Qe^{D}Q$

Example $\quad X' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} X$

$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (\lambda - 5)(\lambda + 1) = 0$

$\lambda_1 = -1 \qquad v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Q = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

$\lambda_2 = 5 \qquad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$X(t) = e^{At} X(0)$

$\begin{cases} Av = (-1)v \\ Aw = (5)w \end{cases}$

$X_1(t) = e^{At} v = (I + A + \frac{A^2}{2!} + \cdots)\vec{v} = \vec{v} + \lambda t \vec{v} + \frac{\lambda^2 t^2}{2!}\vec{v} + \cdots = e^{\lambda t}\vec{v}$

$\frac{d}{dt}(e^{\lambda t}\vec{v}) = \lambda_1 e^{\lambda t}\vec{v}$

$A(e^{\lambda t}\vec{v}) = e^{\lambda t} A\vec{v} = \lambda_1 e^{\lambda_1 t}\vec{v}$

$\frac{d}{dt}(e^{\lambda t}\vec{v}) = A(e^{\lambda t}\vec{v})$

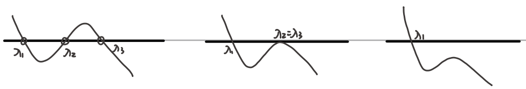
General sol. $\quad e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\vec{X}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ -c_1 + 2c_2 \end{pmatrix} \qquad \begin{cases} c_1 = 1/3 \\ c_2 = 2/3 \end{cases}$

$X' = AX, \quad A: 3\times 3$

$0 = |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = -\lambda^3 +$




3 possibilities

(i) 3 real distinct roots

(ii) 3 real roots not all distinct

(iii) 1 real root & 2 complex conjugate roots

Case (iii) 3 roots $\lambda \in \mathbb{R}$, $\alpha \pm i\beta$

$\alpha + i\beta$ $\quad \vec{w_1} + i\vec{w_2}$ (Pf) $A\vec{w_1} = (\alpha + i\beta)\vec{w_1} = \alpha\vec{w_1} + i\beta\vec{w_1}$

$A(\vec{w_1} + i\vec{w_2}) = (\alpha + i\beta)(\vec{w_1} + i\vec{w_2}) = (\alpha\vec{w_1} - \beta\vec{w_2}) + i(\beta\vec{w_1} + \alpha\vec{w_2})$

$A(\vec{w_1} - i\vec{w_2}) = (\alpha - i\beta)(\vec{w_1} - i\vec{w_2})$

$e^{(\alpha+i\beta)t}(\vec{w_1} + i\vec{w_2}) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{w_1} + i\vec{w_2}) = e^{\alpha t}(\cos\beta t\,\vec{w_1} - \sin\beta t\,\vec{w_2}) + ie^{\alpha t}(\sin\beta t\,\vec{w_1} + \cos\beta t\,\vec{w_2})$

$e^{(\alpha-i\beta)t}(\vec{w_1} - i\vec{w_2}) = e^{\alpha t}(\cos\beta t\,\vec{w_1} - \sin\beta t\,\vec{w_2}) - ie^{\alpha t}(\sin\beta t\,\vec{w_1} + \cos\beta t\,\vec{w_2})$

Claim: (1) Both $R(t)$ $I(t)$ are sols.

(2) $R(t)$ $I(t)$ are l.indep.

(1) $(R(t) + iI(t))' = A(R(t) + iI(t))$

$R'(t) = AR(t)$ $\quad iI'(t) = AI(t)$

(2) Lemma $\{\vec{w_1}, \vec{w_2}\}$ l.indep.

suppose $\vec{w_2} = c\vec{w_1}$ for some $c \in \mathbb{R}$

$\Rightarrow \vec{w_1} + i\vec{w_2} = (1 + ic)\vec{w_1}$

$A(\vec{w_1} + i\vec{w_2}) = A(1 + ic)\vec{w_1} = (1 + ic)A\vec{w_1}$

$\quad \| $

$(\alpha + i\beta)(\vec{w_1} + i\vec{w_2})$

$A\vec{w_1} = \frac{\alpha + i\beta}{1 + ic}(\vec{w_1} + i\vec{w_2}) = \frac{(1 - ic)(\alpha + i\beta)}{1 + c^2}(\vec{w_1} + i\vec{w_2}) = RHS$

$Im(RHS) = \frac{1}{1+c^2}[(\alpha + c\beta)\vec{w_2} + (\beta - c\alpha)\vec{w_1}] = \frac{1}{1+c^2}[\alpha c + c^2\beta + \beta - c\alpha]\vec{w_1} = \frac{1}{1+c^2}(1 + c^2)\beta\vec{w_1} = \beta\vec{w_1}$ $\times$

Pf of (2) Suppose NOT i.e. Suppose $C_1R(t) + C_2I(t) = 0$

$C_1e^{\alpha t}[\cos\beta t\,\vec{w_1} - \sin\beta t\,\vec{w_2}] + C_2e^{\alpha t}[\cos\beta t\,\vec{w_2} + \sin\beta t\,\vec{w_1}] = 0$

$\vec{w_1}[C_1\cos\beta t + C_2\sin\beta t] + \vec{w_2}[C_2\cos\beta t - C_1\sin\beta t] = 0$

$\begin{cases} C_1\cos\beta t + C_2\sin\beta t = 0 \\ -C_1\sin\beta t + C_2\cos\beta t = 0 \end{cases} \Rightarrow (\cos^2\beta t + \sin^2\beta t)C_2 = 0 \Rightarrow C_2 = 0$

(ii) Example $\mathbb{X}' = \underbrace{\begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}}_{A}\mathbb{X}$

$$0=|A-\lambda I| = \begin{vmatrix} 3-\lambda & -1 & 0 \\ 1 & 1-\lambda & 0 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda\end{vmatrix} = -(\lambda-2)^3 \qquad \lambda=2,2,2$$

$$0=(A-2I)\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}1&-1&0\\1&-1&0\\-1&0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} = \begin{pmatrix}v_1-v_2\\v_1-v_2\\-v_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \vec v = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\vec v = (A-2I)\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} = \begin{pmatrix}w_1-w_2\\w_1-w_2\\-w_1\end{pmatrix} \qquad w_1=-1=w_2 \qquad W=\begin{pmatrix}-1\\-1\\0\end{pmatrix}$$

$$\vec w = (A-2I)\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} = \begin{pmatrix}u_1-u_2\\u_1-u_2\\-u_1\end{pmatrix} \qquad \vec u = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

first sol $e^{2t}\vec v$ $\qquad (Ae^{2t}\vec v = e^{2t}A\vec v = 2e^{2t}\vec v = (e^{2t}\vec v)')$

$(A-2I)(A-2I)\vec w = (A-2I)\vec v = 0$

$(A-2I)\vec w = \vec v$

$$e^{At}\vec w = e^{2It}e^{(A-2I)t}\vec w = e^{2t}[e^{(A-2I)t}\vec w] = e^{2t}[I+(A-2I)t+\frac{(A-2I)^2}{2}t^2+\cdots]\vec w = e^{2t}[\vec w+t\vec v] = e^{2t}\begin{pmatrix}-1\\-1\\t\end{pmatrix}$$

$I+2It+\frac{(2It)^2}{2!}+\cdots+\frac{(2t)^n I}{n!}+\cdots = Ie^{2t}$

$$e^{At}\vec u = e^{2tI}[e^{(A-2I)t}\vec u] = e^{2t}[\vec u+(A-2I)t\vec u+\frac{(A-2I)^2}{2}t^2\vec u+\frac{(A-2I)^3}{3!}t^3\vec u+\cdots] = e^{2t}[\vec u+t\vec w+\frac{t^2}{2}\vec v] = e^{2t}\begin{pmatrix}-t\\1-t\\\frac{t^2}{2}\end{pmatrix}$$

$$\left[c_1\begin{pmatrix}0\\0\\1\end{pmatrix}+c_2\begin{pmatrix}-1\\-1\\t\end{pmatrix}+c_3\begin{pmatrix}-t\\1-t\\t^2/2\end{pmatrix}\right]$$

$\gamma=\{\vec v, \vec w, \vec u\} \qquad \beta=\{\vec e_1, \vec e_2, \vec e_3\}$

$$[A]_\gamma = \begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}$$

$$[A]_\beta = Q[A]_\gamma Q^{-1} \Rightarrow [A]_\gamma = [I]_\beta^\gamma [A]_\beta [I]_\gamma^\beta$$

$$e^A = e^{Q[A]_\gamma Q^{-1}} = Qe^{[A]_\gamma}Q^{-1}$$

11.4 Mon

Def $\mathbb{X}(t)$ is called a fundamentally matrix sol. to $\mathbb{X}'=A\mathbb{X}$

if the column vectors of $\mathbb{X}(t)$ form a set of $n$ L.indep. sol. of (*)

Example $\qquad \mathbb{X}' = \begin{pmatrix}3&-1&0\\1&1&0\\-1&0&2\end{pmatrix}\mathbb{X}$

eigenvalues $\quad 2, 2, 2$

eigenvector $\vec v = \begin{pmatrix}0\\0\\1\end{pmatrix}$ $\qquad$ general eigen vector $\vec w = \begin{pmatrix}-1\\-1\\0\end{pmatrix}$ $\quad \vec u = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$$e^{At} = \begin{pmatrix} | & | & | \\ e^{At}e_1 & e^{At}e_2 & e^{At}e_3 \\ | & | & | \end{pmatrix} \qquad \mathbb{X}(t) = \begin{pmatrix}0 & -e^{2t} & -te^{2t} \\ 0 & -e^{2t} & e^{2t}-te^{2t} \\ e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t}\end{pmatrix}$$

$$\mathbb{X}(t)\vec c = \begin{pmatrix} | & & | \\ \vec{\mathbb{X}}_1(t) & \cdots & \vec{\mathbb{X}}_n(t) \\ | & & | \end{pmatrix}\begin{pmatrix}c_1\\c_2\\\vdots\\c_n\end{pmatrix} = \begin{pmatrix} | \\ c_1\vec{\mathbb{X}}_1(t)+\cdots+c_n\vec{\mathbb{X}}_n(t) \\ | \end{pmatrix}$$

$\Phi(t)\cdot B$ is a fundamental matrix sol.

$\Phi(t)\Phi^{-1}(0)\big|_{t=0} = I = e^{At}\big|_{t=0} \Rightarrow e^{At} = \Phi(t)\Phi^{-1}(0)$

$\Phi(0) = \begin{pmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Phi^{-1}(0) = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$

$e^{At} = \begin{pmatrix} 0 & -1 & -t \\ 0 & -1 & 1-t \\ 1 & t & \frac{t^2}{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} e^{2t} = e^{2t}\begin{pmatrix} 1+t & -t & 0 \\ t & 1-t & 0 \\ -t-\frac{t^2}{2} & \frac{t^2}{2} & 1 \end{pmatrix}$

Nonhomogeneous Linear system

$\vec{X}' = A\vec{X} + \vec{f}(t)$ $\quad A$: $n\times n$ constant matrix

① solve $\vec{X}' = A\vec{X}$ $\quad \Phi(t)\vec{c}$, $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

② Variation of Parameters

$\Phi(t)\vec{u}(t) = \Phi(t)\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{pmatrix}$

$\vec{X}(t) = \Phi(t)\vec{u}(t)$ $\quad \vec{X}'(t) = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) \xrightarrow{\Phi'(t)=A\Phi(t)} A\Phi(t)\vec{u}(t) + \Phi(t)\vec{u}'(t) = A\vec{X} + \Phi(t)\vec{u}'(t)$

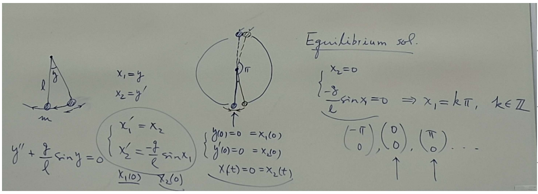
$\Phi(t)\vec{u}'(t) = \vec{f}(t) \Rightarrow \vec{u}'(t) = \Phi^{-1}(t)\vec{f}(t) \Rightarrow \vec{u}(t) = \vec{u}(t_0) + \int_{t_0}^{t}\Phi^{-1}(s)\vec{f}(s)\,ds$

$\vec{X}(t) = \Phi(t)\vec{u}(t_0) + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\vec{f}(s)\,ds$

$\Phi(t) = e^{At}$ $\quad \vec{X}(t) = e^{At}\vec{u}(t_0) + \int_{t_0}^{t} e^{At}e^{-As}\vec{f}(s)\,ds = e^{At}\vec{u}(t_0) + \int_{t_0}^{t} e^{A(t-s)}\vec{f}(s)\,ds$

$e^{A+B} = e^Ae^B \Leftrightarrow A, B$ commute $\qquad \begin{matrix} e^A = I + A + \frac{A^2}{2!} + \cdots \\ e^B = I + B + \frac{B^2}{2!} + \cdots \end{matrix} \Big\}\ e^{A+B} = I + (A+B) + \frac{(A+B)^2}{2!} + \cdots + \frac{(A+B)^n}{n!} + \cdots$

11.6 Wean.




Autonomous system

(*) $\vec{X}' = f(\vec{X})$ $\qquad \vec{X}' = f(t, \vec{X})$

Def. a sol. $\vec{X} = \vec{\phi}(t)$ of (*) is said to be stable if $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. for any sol. $\vec{\psi}(t)$ of (*) with $\|\vec{\psi}(t) - \vec{\phi}(t)\| < \varepsilon$ for all $t > 0$ ($t$: forward). A sol which is not stable is said to be unstable

Continuous dependence on initial values :

$\forall T>0$, & $\forall \varepsilon>0$, $\exists \delta>0$ s.t. for any sol. $\vec{\psi}(t)$ of $(*)$ with $\|\vec{\psi}(0)-\vec{\phi}(0)\|<\delta \Rightarrow \|\vec{\psi}(t)-\vec{\phi}(t)\|<\varepsilon$ $\forall 0<t<T$

($\vec{\phi}(t)$: any sol of $(*)$) $\quad \sup \|\vec{\psi}(t)-\vec{\phi}(t)\| < \varepsilon$

Linear system $\vec{\mathbb{X}}'=A\vec{\mathbb{X}}$ $(+)$ $n\times n$ matrix

proposition Every sol. of $(+)$ is stable $\Leftrightarrow$ The zero sol. of $(+)$ $\vec{\phi}(t)\equiv 0$ is stable

(pf). $\forall\varepsilon>0$, $\exists\delta>0$, s.t. if $\vec{\psi}(t)$ is a sol. of $(+)$ with $\|\vec{\psi}(0)\|<\delta \Rightarrow \|\vec{\psi}(t)\|<\varepsilon$ $\forall t>0$

Let $\vec{\xi}(t)$ be any sol. of $(*)$ & $\varepsilon>0$ be given, and $\vec{\eta}(t)$ be another sol. of $(*)$ with $\|\vec{\eta}(0)-\vec{\xi}(0)\|<\delta$

Then $\|\vec{\eta}(t)-\vec{\xi}(t)\|<\varepsilon$ $\forall t>0$, Since if $\vec{\psi}(t)=(\vec{\eta}-\vec{\xi})(t)$ then $\vec{\psi}(t)$ is a sol. of $(+)$ & $\|\vec{\psi}(0)\|<\delta$

so $\|\vec{\psi}(t)\|<\varepsilon$ $\forall t>0$

To determine the stability properties of the zero sol of $(+)$, we have 3 cases:

(1) at least one eigenvalue of $A$ is positive or has a positive real parts $\Rightarrow$ 0 Sol is unstable (actually Sol $\to\infty$)

(2) All eigenvalues of $A$ are negative or have a negative real parts $\Rightarrow$ 0 Sol is stable

(3) All eigenvalues of $A$ are nonpositive or have nonegative real parts $\Rightarrow$ need to be careful & need more info.

Prop 1 In case (1) all sol are unstable

(pf.) Only have to show: 0 sol is unstable

(i) Suppose $\lambda>0$ is an eigenvalue of $A$ with $\vec{v}$ as its eigenvector (corresponding)

Set $\vec{\psi}(t)=ce^{\lambda t}\vec{v}$, $c>0$, to be chosen

$\|\vec{\psi}(0)\|=\|c\vec{v}\|=|c|\,\|\vec{v}\|<\delta$ for any $\delta$ if $c<\frac{\delta}{\|\vec{v}\|}$

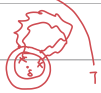
$\|\vec{\psi}(t)\|=\|c\vec{v}e^{\lambda t}\|=e^{\lambda t}|c|\,\|\vec{v}\|\to\infty$ as $t\to\infty$

(ii) Suppose $\alpha+i\beta$ is an eigenvalue of $A$ with $\alpha>0$. Let $\vec{v}_1+i\vec{v}_2$ be a corresp. eigenvector.

$e^{(\alpha+i\beta)t}(\vec{v}_1+i\vec{v}_2)=e^{\alpha t}(\cos\beta t+i\sin\beta t)(\vec{v}_1+i\vec{v}_2)=e^{\alpha t}\left([\cos\beta t\cdot\vec{v}_1-\sin\beta t\,\vec{v}_2]+i\,[\cos\beta t\cdot\vec{v}_2+\sin\beta t\cdot\vec{v}_1]\right)$

$e^{\alpha t}[(\cos\beta t)\vec{v}_1-(\sin\beta t)\vec{v}_2]$ take $t_k=\frac{2k\pi}{\beta}$ $e^{\frac{2k\pi\alpha}{\beta}}\vec{v}_1\to\infty$

upper bound



$e^{\alpha t}[\cos\beta t\cdot\vec{v}_1-\sin\beta t\,\vec{v}_2]\to\infty$ $\vec{v}_1,\vec{v}_2$ are L. indep. $\Rightarrow \cos\beta t\cdot\vec{v}_1-\sin\beta t\cdot\vec{v}_2$

$\cos\beta t\cdot\vec{v}_1-\sin\beta t\,\vec{v}_2=0$ $\quad$ T

$t\in[0,2\pi]$ $\tilde{c}=\inf\{\|\cos\beta t\cdot\vec{v}_1-\sin\beta t\cdot\vec{v}_2\| \mid t\in(0,2\pi)\}>0$ $\Leftrightarrow \cos\beta t=\sin\beta t=0$ ※ $\quad \delta T<\varepsilon$

$e^{\alpha t}[\cos\beta t\cdot\vec{v}_1-\sin\beta t\,\vec{v}_2]\geq\tilde{c}\,e^{\alpha t}$

Prop. 2. In case (2), all sols are stable

(pf). $\lambda_1, \lambda_2, \cdots, \lambda_k$ are negative eigenvalues $\lambda_{k+1}, \cdots, \lambda_n \in \mathbb{C}$ with negative real parts

$e^{\lambda_j t}\vec{v_j}$

generalized eigenvectors

$e^{\lambda_j t}\,[C_1\vec{v_j} + C_2 t\vec{w_j} + \cdots + C_{m_j} t^{m_j-1}\vec{v_j}] \to 0$ $\quad \psi(t) = e^{At}\psi(0)$

$\vec{X}' = A\vec{X}$ $\quad \lambda_1, \cdots, \lambda_k, \lambda_{k+1}, \cdots, \lambda_n$ eigenvalues

$\lambda_1 = i\beta_1, \cdots, \lambda_k = i\beta_k$, $\beta_j \in \mathbb{R}$, $j = 1, \cdots, k$ $\quad \lambda_{k+1}, \cdots, \lambda_k$ : have negative real parts

Prop 3: Suppose that $\lambda_1, \cdots, \lambda_k$ have multiplicities

有足够的 eigenvector

$m_1, \cdots, m_k$, resp. Then all sols of (*) are stable $\Leftrightarrow$ A has $m_j$ l. indep. eigenvectors for $\lambda_j = i\beta_j$, $j = 1, \cdots, k$

Example 1 $\quad \mathbb{X}' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbb{X}$

$0 = |A - \lambda I| = \lambda^2 \Rightarrow \lambda = 0, 0$

$0 = (A - 0I)\vec{v} \Rightarrow \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\vec{v} = (A - \lambda I)\vec{w} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ 0 \end{pmatrix}$ $\quad \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$e^{At}\vec{v} = \vec{v}$ $\quad e^{At}\vec{w} = \vec{w} + t\vec{v}$

$\vec{w} + t\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix} \to \infty$ unbdd as $t \to \infty$

Example 2 $\quad \mathbb{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbb{X}$

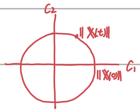
$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$ $\quad \therefore \lambda = i, -i$

$V_2 = iV_1$ $\quad \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$(\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \left[\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + i\left[\cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

general sol. $\quad C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix} = \begin{pmatrix} C_1\cos t + C_2\sin t \\ -C_1\sin t + C_2\cos t \end{pmatrix} = \mathbb{X}(t)$

$\mathbb{X}(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$




$\|\mathbb{X}(t)\|^2 = (C_1\cos t + C_2\sin t)^2 + (-C_1\sin t + C_2\cos t)^2 = C_1 + C_2 = \|\mathbb{X}(0)\|^2$

$\forall \varepsilon > 0, \exists \delta > 0$, s.t. if $\|\mathbb{X}(0) - 0\| < \delta$ then $\|\mathbb{X}(t) - 0\| < \varepsilon$ $\forall t > 0$ $\quad \therefore$ 0 is stable $\quad$ doesn't go away

渐近

Def The sol $\phi(t)$ is said to be asymptotically stable if $\exists \delta > 0$ s.t $\|\phi(t) - \psi(t)\| \to 0$ as $t \to \infty$ $\forall$ sols

$\Psi(t)$ with $\|\Psi(0)-\vec{\phi}(0)\|<\delta$

Consider an autonomous nonlinear dynamical system

$$\dot{x}=f(x(t)), \quad x(0)=x_0,$$

where $x(t)\in\mathcal{D}\subseteq\mathbb{R}^n$ denotes the system state vector, $\mathcal{D}$ an open set containing the origin, and $f:\mathcal{D}\to\mathbb{R}^n$ is a continuous vector field on $\mathcal{D}$. Suppose $f$ has an equilibrium at $x_e$ so that $f(x_e)=0$ then

1. This equilibrium is said to be **Lyapunov stable** if for every $\epsilon>0$ there exists a $\delta>0$ such that if $\|x(0)-x_e\|<\delta$ then for every $t\geq 0$ we have $\|x(t)-x_e\|<\epsilon$.
2. The equilibrium of the above system is said to be **asymptotically stable** if it is Lyapunov stable and there exists $\delta>0$ such that if $\|x(0)-x_e\|<\delta$ then $\lim_{t\to\infty}\|x(t)-x_e\|=0$.

Stability for Non Linear System

$\vec{X}'=A\vec{X}+\vec{g}(\vec{X})$ $A$: $n\times n$ matrix $\vec{g}(\vec{X})=o(\|\vec{X}\|)$ near $\vec{X}=0$ ($\because \vec{X}=0$ is a sol. — Steady state)

Th'm (1) if at least one eigenvalues of $A$ has positive real points, then $\vec{X}=0$ is unstable

Strictly local → (2) if all eigenvalues of $A$ have negative real parts, then $\vec{X}=0$ is asymptotically stable

(3) The stability of $\vec{X}=0$ cannot be determined

Example $\vec{X}'=\underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{A}\vec{X}\pm\underbrace{\begin{pmatrix}-x_1(x_1^2+x_2^2)\\-x_2(x_1^2+x_2^2)\end{pmatrix}}_{g(x)}$

$\frac{1}{2}\cdot\frac{d}{dt}\|X(t)\|^2=\frac{1}{2}(X_1^2(t)+X_2^2(t))'=X_1X_1'+X_2X_2'=X_1(X_2\pm[-X_1(X_1^2+X_2^2)])+X_2[-X_1\pm[-X_2(X_1^2+X_2^2)]]=\mp(X_1^2+X_2^2)^2=\mp\|X\|^4$

"+": $\frac{d}{dt}\|X\|^2<0\Rightarrow\|X\|^2\searrow$ "−" $\varphi(t)=\|X\|^2\Rightarrow\varphi'(t)=2\varphi^2\Rightarrow\varphi\uparrow\infty$ as $t\to\infty$ $\therefore$ 0 is unstable

"−": $\frac{d}{dt}\|X\|^2>0\Rightarrow\|X\|\nearrow$ "+" $\varphi'(t)=-2\varphi^2\Rightarrow\varphi(t)\searrow 0$ as $t\to\infty$ $\therefore$ 0 is asymp. stable

11.13 Wedn.

(Pf of (2)) $\vec{X}'=A\vec{X}+\vec{g}(\vec{X}(t))$ $(*)$

Since all eigenvalues of $A$ have negative real parts, $\exists\ \alpha, k$, both positive, s.t.

(1) $\|e^{At}\vec{X}(0)\|\leq ke^{-\alpha t}\|\vec{X}(0)\|$ $\forall t>0$ $\forall\vec{c}$

(2) A sol. for $(*)$ can be written as $\vec{X}(t)=e^{At}\vec{X}(0)+\int_0^t e^{A(t-s)}\vec{g}(\vec{X}(s))\,ds$

(3) $\|\vec{g}(x)\|\leq\varepsilon\|\vec{X}\|$ $\forall\|\vec{x}\|<\delta$

Strategy: In order to apply (3), we need to apply (3), we need to find $\eta>0$, s.t.

if $\|\vec{x}(0)\|<\eta$ then $\|\vec{X}(t)\|<\delta$ $\forall t>0$

$\vec{X}(t)=e^{At}\vec{X}(0)+\int_0^t e^{A(t-s)}\vec{g}(\vec{X}(s))\,ds$

$\|\vec{X}(t)\|\leq\|e^{At}\vec{X}(0)\|+\int_0^t\|e^{A(t-s)}\vec{g}(\vec{X}(s))\|\,ds$

$\leq ke^{-\alpha t}\|\vec{X}(0)\|+\int_0^t ke^{-\alpha(t-s)}\|\vec{g}(\vec{X}(s))\|\,ds$

(3) $\leq ke^{-\alpha t}\|\vec{X}(0)\|+\varepsilon ke^{-\alpha t}\int_0^t e^{\alpha s}\|\vec{X}(s)\|\,ds$

$$\underbrace{e^{\alpha t}\|\vec{x}(t)\|}_{z(t)} \le k\|\vec{x}(0)\| + \varepsilon k \int_0^t e^{\alpha s}\|\vec{x}(s)\|ds$$

$$z(t) \le k\|x(0)\| + \varepsilon k \underbrace{\int_0^t z(s)ds}_{u(t)}$$

$$u'(t) \le k\|x(0)\| + \varepsilon k\, u(t)$$

$$(e^{-\varepsilon k t}u)' \le e^{-\varepsilon k t} k \|\vec{x}(0)\|$$

$$e^{-\varepsilon k t}u \le k\|\vec{x}(0)\| \int_0^t e^{-\varepsilon k s} ds = k\|\vec{x}(0)\| \cdot \frac{1}{\varepsilon k}(1 - e^{-\varepsilon k t})$$

$$u(t) \le \frac{1}{\varepsilon}\|x(0)\|(e^{\varepsilon k t} - 1)$$

$$\|e^{\alpha t}x(t)\| = z(t) \le k\|x(0)\| + k\|x(0)\|(e^{\varepsilon k t} - 1)$$

$$\|x(t)\| \le K\|x(0)\|e^{(\varepsilon k - \alpha)t} \quad \forall t \ge 0$$

choose $\varepsilon = \frac{\alpha}{2k}$

(4) $\|x(t)\| \le k\|x(0)\|e^{-\frac{\alpha}{2}t} \quad \forall t \ge 0$

choose $\eta = \delta/k$, Then for $\|x(0)\| < \eta$

$$\|x(t)\| \le k\eta e^{-\frac{\alpha}{2}t} = \delta e^{\frac{\alpha}{2}t} < \delta \quad \forall t \ge 0$$

(4) $\Rightarrow$ $\|x(t)\| \to 0$ as $t \to \infty$

Reverse: (pf) $\alpha, k$ : determined by $A$

Set $\varepsilon = \frac{\alpha}{2k}$, which determines $\delta$. Now, let $\eta = \frac{\delta}{k}$

For $\|x(0)\| < \eta$ claim $\|x(t)\| < \delta \ \forall t \ge 0$

Suppose that $\exists \tau > 0$ s.t $\|x(\tau)\| = \delta$ & $\|x(t)\| < \delta \ \forall t < \tau$

Our previous argument applies to $x(t)$ for all $t < \tau$

i.e. $\|x(t)\| \le k\|x(0)\|e^{-\frac{\alpha}{2}t} \quad \forall t < \tau$

$$\|x(\tau)\| \le k\|x(0)\|e^{-\frac{\alpha}{2}t} < k\eta = k \cdot \frac{\delta}{k}$$

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

at $\vec{x}_0$, $F(\vec{x}_0) = 0$ (i.e. $\vec{x}(t) \equiv \vec{x}_0$ is a sol) An equilibrium

$$\vec{f}(x) = \vec{f}(x_0) + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} (\vec{x} - \vec{x}_0) + o(\|\vec{x} - \vec{x}_0\|)$$

$\vec{f} = (f_1, f_2, \cdots f_n)$

$(\vec{x} - \vec{x}_0)' = A(\vec{x} - \vec{x}_0) + g(\vec{x} - \vec{x}_0)$

$\vec{y}' = A\vec{y} + g(\vec{y})$

Example $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1-xy \\ x-y^3 \end{pmatrix}$ determine all equilibria & their stability

$\begin{cases} 1-xy=0 \\ x-y^3=0 \end{cases} \Rightarrow \begin{cases} xy=1 \\ x=y^3 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \end{cases} \quad \begin{cases} x=-1 \\ y=-1 \end{cases}$

At $(1,1)$ $\frac{\partial(f,g)}{\partial(x,y)} = \begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}\Big|_{(1,1)} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$ $f(\lambda) = (\lambda+2)^2$ Stable

At $(-1,-1)$ $\begin{pmatrix} -y & -x \\ 1 & -3y^2 \end{pmatrix}\Big|_{(-1,-1)} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$ $f(\lambda) = \lambda^2+2\lambda-4$ unstable

Phase plane

$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ $f(\cancel{t}, x, y)$ → no "t" here autonomous

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g(x,y)}{f(x,y)}$

Example

$\begin{cases} x' = y^2 \\ y' = x^2 \end{cases}$ $\begin{cases} x(0) = x \\ y(0) = y \end{cases}$

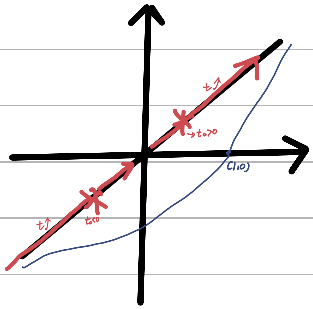
$\frac{dy}{dx} = \frac{x^2}{y^2} \Rightarrow y^3 - x^3 = C \Rightarrow (y-x)(y^2+xy+x^2) = C$

(i) $C=0 \Rightarrow y=x$

$x_0 = y_0 = 0 \Rightarrow x(t) = 0 = y(t)$

$x_0 = y_0 > 0$

$x_0 = y_0 < 0$




(ii) $x_0 = 1$ $y_0 = 0$ $C = -1$

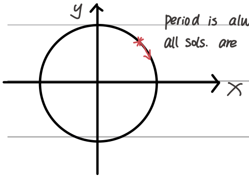
$y^3 = x^3 - 1 = x^3(1 - \frac{1}{x^3}) \Rightarrow y = x(1 - \frac{1}{x^3})^{\frac{1}{3}} = x(1 - \frac{1}{3} \cdot \frac{1}{x^3})$

Example

(1) $\begin{cases} x' = y \\ y' = -x \end{cases}$

$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow x^2 + y^2 = C$

$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad y = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

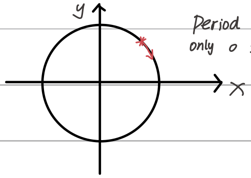



period is always $2\pi$
all sols. are stable

(2) $\begin{cases} x' = y\,(1+x^2+y^2) \\ y' = -x\,(1+x^2+y^2) \end{cases}$ $\quad x^2+y^2 = x_0^2 + y_0^2 = r_0^2$ $\quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1+r_0^2 \\ -(1+r_0^2) & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

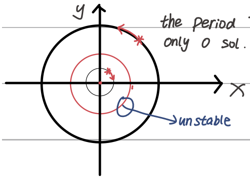
$\frac{dy}{dx} = -\frac{x}{y}$

$y = C_1 \begin{pmatrix} \cos(1+r_0^2)t \\ -\sin(1+r_0^2)t \end{pmatrix} + C_2 \begin{pmatrix} \sin(1+r_0^2)t \\ \cos(1+r_0^2)t \end{pmatrix}$




Period is $\frac{2\pi}{1+r_0^2}$
only 0 sol. is stable

(3) $\begin{cases} x' = y\,(1-x^2-y^2) \\ y' = -x\,(1-x^2-y^2) \end{cases}$ $\quad$ equilibrium pts $(0,0)$, $x^2+y^2=1$




the period $\frac{2\pi}{|1-r_0^2|}$
only 0 sol. is stable

Def An orbit is the sol. curve $\{(x(t), y(t)) \mid -\infty < t < \infty\}$ on the phase plane

$$(*) \quad \mathbb{X}' = \vec{f}(\mathbb{X}) = \begin{pmatrix} f_1(x_1, \cdots, x_n) \\ f_2(x_1, \cdots, x_n) \\ \vdots \\ f_n(x_1, \cdots, x_n) \end{pmatrix}$$

Prop 1 (Existence / Unique)

Suppose that $f_1, \cdots f_n$ are cont. differentiable. Then for every initial value $\mathbb{X}_0$, there is a unique orbit through $\mathbb{X}_0$

Cor. No two orbit intersect each other

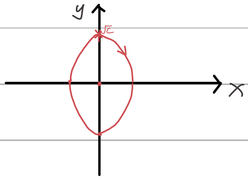
Prop 2. Let $\mathbb{X} = \vec{\phi}(t)$ be a sol of $(*)$ if $\vec{\phi}(t_0+T) = \vec{\phi}(t_0)$ for some $t_0$ & $T>0$, then $\vec{\phi}(t+T) = \vec{\phi}(t)$ $\forall t$ (i.e. $\vec{\phi}(t)$ is a periodic sol.)

Cor every closed orbit is periodic

(pf) $\vec{X}(t) = \vec{X}(t_0) + \int_{t_0}^{t} \vec{f}(\vec{X}(s))\,ds$ Picard iteration

Example $z'' + z + z^3 = 0$

$\begin{cases} x = z \\ y = z' = x' \end{cases}$ $\begin{cases} x' = y \\ y' = -x - x^3 \end{cases}$




$\frac{dy}{dx} = \frac{-x-x^3}{y} \Rightarrow \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^4}{4} = C$

$y = \pm\sqrt{C - x^2 - \frac{x^4}{2}}$

11.20 Wedn

$\mathbb{X}' = A\mathbb{X}$ $A: 2\times 2$

$0 = |A - \lambda I|$

(I) A has real eigenvalues only

(i) $\lambda_2 < \lambda_1 < 0$ (ii) $0 < \lambda_1 < \lambda_2$ (iii) $\lambda_2 < 0 < \lambda_1$ (iv) $\lambda_1 = \lambda_2\ (=\lambda) < 0$

(V) $\lambda_1 = \lambda_2\ (=\lambda) > 0$

(a) $A = \lambda I$ $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

(b) $A \neq \lambda I$ $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

(II) A has 2 complex conjugate eigenvalues $\alpha \pm i\beta$

(i) $\alpha < 0$ (ii) $\alpha > 0$ (iii) $\alpha = 0$

(I) (i) $\lambda_2 < \lambda_1 < 0$

$\vec{V}_2$ $\vec{V}_1$

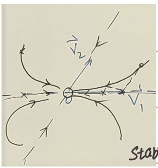
$\mathbb{X}(t) = C_1 e^{\lambda_1 t}\vec{V}_1 + C_2 e^{\lambda_2 t}\vec{V}_2$ $C_1, C_2 \in \mathbb{R}$

$= e^{\lambda_1 t}[C_1\vec{V}_1 + \underbrace{C_2 e^{(\lambda_2-\lambda_1)t}}_{\to 0}\vec{V}]$

(ii) $0 < \lambda_1 < \lambda_2$ $\vec{V}_1$ $\vec{V}_2$

$\mathbb{X}(t) = C_1 e^{\lambda_1 t}\vec{V}_1 + C_2 e^{\lambda_2 t}\vec{V}_2$

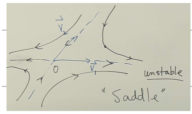
$= e^{\lambda_2 t}[C_1 e^{\overbrace{(\lambda_1-\lambda_2)}^{<0} t}\vec{V}_1 + C_2\vec{V}_2]$




Stable

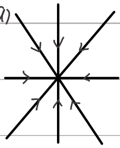
(iii) $\lambda_2 < 0 < \lambda_1$

$\mathbb{X}(t) = C_1 e^{\lambda_1 t}\vec{V}_1 + C_2 e^{\lambda_2 t}\vec{V}_2$




(iv)

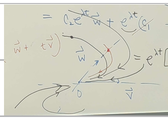
(a)



(b) $A \neq \lambda I$ $\quad \vec{V}$ $\quad \vec{W}$ gen'd eigenvector

$\mathbb{X}(t) = C_1 e^{\lambda t}\vec{V} + C_2 e^{\lambda t}(\vec{W} + t\vec{V}) = C_2 e^{\lambda t}\vec{W} + e^{\lambda t}(C_1 + C_2 t)\vec{V}$

$= e^{\lambda t}[C_2\vec{W} + C_1\vec{V} + C_2 t\vec{V}]$

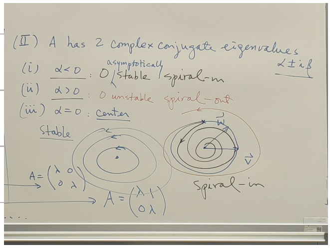


(II) $\mathbb{X}(t) = e^{(\alpha+i\beta)t}(\vec{V} + i\vec{W}) = e^{\alpha t}[(\cos\beta t\cdot\vec{V} - \sin\beta t\cdot\vec{W}) + i(\cos\beta t\,\vec{W} + \sin\beta t\,\vec{V})]$

$\mathbb{X}(t) = e^{\alpha t}[C_1(\cos\beta t\cdot\vec{V} - \sin\beta t\cdot\vec{W}) + C_2(\sin\beta t\cdot\vec{V} + \cos\beta t\cdot\vec{W})]$

$= e^{\alpha t}[(C_1\cos\beta t + C_2\sin\beta t)\vec{V} + (-C_1\sin\beta t + C_2\cos\beta t)\vec{W}]$

$= e^{\alpha t}\underbrace{\sqrt{C_1^2+C_2^2}[\sin(\beta t + \theta_1)\vec{V} - \sin(\beta t - \theta_2)\vec{W}]}$




$\vec{\mathbb{X}} = \vec{f}(\mathbb{X})$ $\quad \vec{f}(\vec{\mathbb{X}}_0) = 0$

Analyze sol near $\mathbb{X}_0$

$\vec{\mathbb{X}}' = D\vec{f}(\mathbb{X}_0)(\vec{\mathbb{X}} - \vec{\mathbb{X}}_0)$

$(\vec{\mathbb{X}} - \vec{\mathbb{X}}_0)'$ $\quad \vec{\mathbb{Y}} = \vec{\mathbb{X}} - \vec{\mathbb{X}}_0$

$\vec{\mathbb{Y}}' = Df(\vec{\mathbb{X}}_0)\vec{\mathbb{Y}}$ $\quad \vec{\mathbb{Y}} = 0$

Th'm Let $\lambda_1, \lambda_2$ be the 2 eigenvalues of $D\vec{f}(\vec{\mathbb{X}}_0)$. Then the "local" behavior of sol s to (*) are as follows

(1) $\lambda_2 < \lambda_1 < 0$ : Stable node

(2) $\lambda_2 = \lambda_1 < 0$ : Stable node or stable focus (spiral-in)

(3) $\lambda_2 < 0 < \lambda_1$ : Unstable saddle

(4) $0 < \lambda_2 = \lambda_1$ unstable node or unstable focus (Spiral-out)

(5) $0 < \lambda_1 < \lambda_2$ : unstable node

(6) $\lambda_1, \lambda_2 = \alpha \pm i\beta : \alpha = 0$ center, stable or unstable focus (spiral-in or spiral-out)

(7) $\lambda_1, \lambda_2 = \alpha \pm i\beta : \alpha < 0$ stable focus (spiral in)

(8) $\lambda_1, \lambda_2 = \alpha \pm i\beta ; \alpha > 0$ stable focus (spiral-out)

**Theorem 3.** *Let each of the functions $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ have continuous partial derivatives with respect to $x_1, \ldots, x_n$. Then, the initial-value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}^0$ has one, and only one solution $\mathbf{x} = \mathbf{x}(t)$, for every $\mathbf{x}^0$ in $\mathsf{R}^n$.*

**Lemma 1.** *If $\mathbf{x} = \boldsymbol{\phi}(t)$ is a solution of (1), then $\mathbf{x} = \boldsymbol{\phi}(t+c)$ is again a solution of (1).* Autonomous system

**Property 1.** (Existence and uniqueness of orbits.) Let each of the functions $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ have continuous partial derivatives with respect to $x_1, \ldots, x_n$. Then, there exists one, and only one, orbit through every point $\mathbf{x}^0$ in $\mathsf{R}^n$. In particular, if the orbits of two solutions $\mathbf{x} = \boldsymbol{\phi}(t)$ and $\mathbf{x} = \boldsymbol{\psi}(t)$ of (1) have one point in common, then they must be identical.

**Property 2.** Let $\mathbf{x} = \boldsymbol{\phi}(t)$ be a solution of (1). If $\boldsymbol{\phi}(t_0 + T) = \boldsymbol{\phi}(t_0)$ for some $t_0$ and $T > 0$, then $\boldsymbol{\phi}(t+T)$ is identically equal to $\boldsymbol{\phi}(t)$. In other words, if a solution $\mathbf{x}(t)$ of (1) returns to its starting value after a time $T > 0$, then it must be periodic, with period $T$ (i.e. it must repeat itself over every time interval of length $T$.)

11.25 Mon.

$\mathbb{X} = f'(\mathbb{X}) = \begin{pmatrix} f_1(\mathbb{X}) \\ f_2(\mathbb{X}) \\ \vdots \\ f_n(\mathbb{X}) \end{pmatrix} \quad \mathbb{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Prop 1: Existence / Uniqueness

Prop 2: Closed orbit : periodic sol.

**Lemma 2.** *Suppose that a solution* $\mathbf{x}(t)$ *of* (1) *approaches a vector* $\boldsymbol{\xi}$ *as* $t$ *approaches infinity. Then,* $\boldsymbol{\xi}$ *is an equilibrium point of* (1).

Prop 3: Suppose a sol. $\mathbb{X}(t) \to \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$ as $t \to \infty$ Then $\vec{f}(\xi) = 0$

(pf) $\mathbb{X}(t) \to \xi$ as $t \to \infty \Rightarrow \mathbb{X}(t+h) \to \xi$ as $t \to \infty$ i.e. $x_1(t+h) - x_1(t) \to \xi_1 - \xi_1 = 0$

$\nearrow x_1'(t+\tilde{h}) \to 0 \quad x_1(t+\tilde{h}) \to \xi_1 \Rightarrow x_1'(t+\tilde{h}) \to f_1(\xi) \to 0$

LHS $= h x_1'(t+\tilde{h}) \quad 0 < \tilde{h} < h \quad \to 0 \quad \therefore f_1(\xi) = 0$

Likewise $\vec{f}(\xi) = 0$

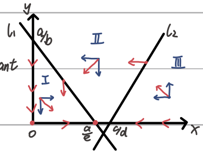
Example $\begin{cases} x' = ax - bxy - ex^2 = \\ y' = -cy + dxy - fy^2 = \end{cases} \begin{pmatrix} x(a - by - ex) \\ y(-c + dx - fy) \end{pmatrix}$ a.b.c.d.e.f all positive constants

x: prey
y: predater

$\frac{a}{e} < \frac{c}{d}$ Analyze all sols. with $x(0) > 0$ $y(0) > 0$ can't leave the first quardrant

$\begin{cases} 0 = x(a - by - ex) \Rightarrow x = 0 \quad \text{or} \quad a = by + ex \quad l_1 \\ 0 = y(-c + dx - fy) \Rightarrow y = 0 \quad \text{or} \quad c = dx - fy \quad l_2 \end{cases}$

$(0,0)$, $(0, \cancel{\tfrac{c}{f}})$, $(\frac{a}{e}, 0)$,



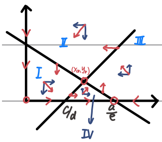

(1) All sols in III must enter II

(2) Sols in II has 2 possibilities (i) enter I (ii) Conv. to $(\frac{a}{e}, 0)$

(3) Sols. in I must conv. to $(\frac{a}{e}, 0)$

In conclusion, all sols $\to (\frac{a}{e}, 0)$ as $t \to \infty$

Example 2 $\frac{a}{e} > \frac{c}{d}$



1) Sol. in III have 2 possibilities (i) enter II (ii) $\to (X_*, Y_*)$

2) Sol. in II have 2 possibilities (i) enter I (ii) $\to (X_*, Y_*)$

3) Sol. in I have 2 possibilities (i) enter IV (ii) $\to (X_*, Y_*)$

4) Sol. in IV have 2 possibilities (i) enter III (ii) $\to (X_*, Y_*)$

In summary: (i) $\to (X_*, Y_*)$ (ii) III $\to$ II $\to$ I $\to$ IV $\to$ III $\cdots$

At $(X_*, Y_*)$ linearize

$\mathbb{X}' = A\mathbb{X}$ + higher order term

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x_0, y_0)} = \begin{pmatrix} a - bY_* - 2eX_* & -bX_* \\ dY_* & -c + dX_* - 2fY_* \end{pmatrix} = \begin{pmatrix} -eX_* & -bX_* \\ dY_* & -fY_* \end{pmatrix} \qquad |A - \lambda I| = 0 \Rightarrow \lambda = \frac{-(eX_* + fY_*) \pm \sqrt{\Delta}}{2}$$

$Re(\lambda)$ is always $< 0$ & thus $(X_*, Y_*)$ is stable

Poincare-Bendixson

Suppose that a sol. of the system

$$* \begin{cases} x'(t) = f(x,y) \\ y'(t) = g(x,y) \end{cases}$$ remains in a bdd region of the $(x,y)$-plane, which contains no equilibrium pts of $(*)$

Then $(x(t), y(t))$ must either be a periodic sol. or it spiral into a single closed curve

which is the orbit of a periodic sol.

Example $\begin{cases} x'(t) = -y + x(1-x^2-y^2) \\ y'(t) = x + y(1-x^2-y^2) \end{cases}$

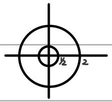
Solve all equilibrium pts $\begin{cases} -y + x(1-x^2-y^2) = 0 \\ x + y(1-x^2-y^2) = 0 \end{cases}$

$(x^2+y^2)(1-x^2-y^2) = 0$

$x^2 + y^2 = 0 \Rightarrow (x,y) = (0,0) \leftarrow$ only equilibrium pt.

Since the term $x^2+y^2$ appears prominently in both equations, it suggests itself to introduce polar coordinates $r, \theta$, where $x = r\cos\theta, y = r\sin\theta$, and to rewrite (3) in terms of $r$ and $\theta$. To this end, we compute

$$\frac{d}{dt}r^2 = 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$= 2(x^2+y^2) - 2(x^2+y^2)^2 = 2r^2(1-r^2).$$
$$\frac{d\theta}{dt} = \frac{d}{dt}\arctan\frac{y}{x} = \frac{1}{x^2}\frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{1+(y/x)^2} = \frac{x^2+y^2}{x^2+y^2} = 1.$$




$\frac{1}{2} \cdot \frac{d}{dt}(x^2+y^2) = \frac{1}{2}(r^2)' = xx' + yy'$ polar coordinate

11.27 Wedn.

(*) $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ Poincare - Bendixson

Suppose that $\Omega$ is a bdd closed region & $\Omega$ contains no equilibrium pt of (*). If (*) has one sol. lies entirely in $\Omega$ for all time $t$ large (i.e. $\exists\ t_0$ s.t. $(x(t), y(t)) \in \Omega\ \ \forall t \geq t_0$), then either the sol. $(x(t), y(t))$ is periodic or it spirals into a periodic sol (i.e. a closed orbit) as $t \to \infty$

all forward time, don't care $t \to -\infty$

Rk. In practice, we show all sols. start inside $\Omega$ stay in $\Omega$ for all forward time. To achieve this, we'll find $\Omega$ s.t. the vector field $(f, g)$ is pointing inside

Example: Prove that the equ

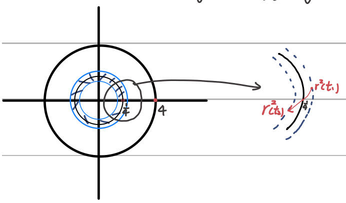
$$x'' + (x^2 + 2(x')^2 - 1)x' + x = 0 \quad \text{has a (nontrival) periodic sol.}$$

$\begin{cases} x' = y \\ y' = -(x^2+2y^2-1)y - x \end{cases}$

Equilibrium pts: $y=0 \Rightarrow x=0$ $(0,0)$

$\frac{d}{dt}\left[\frac{1}{2}(x^2+y^2)\right] = xx' + yy' = xy - (x^2+2y^2-1)y^2 - xy = y^2(1-x^2-2y^2)$

$(\frac{1}{2}r^2)' = y^2(1-x^2-2y^2) \begin{cases} \geq y^2(1-2x^2-2y^2) = y^2(1-2(\frac{1}{4})^2) \geq 0 & \text{on } r=\frac{1}{4} \\ \leq y^2(1-x^2-y^2) = y^2(1-4^2) \leq 0 & \text{on } r=4 \end{cases}$




$r^2(t_1) > r^2(t_2)$ $(t_1 < t_2)$

$\exists\ t_3$ s.t. $(r^2)'(t_3) < 0$

Limit cycles

Def. A closed trajectory (or orbit) $C$ is called a limit cycle of (*) if

itself is already a sol.

if orbits of (*) spirals into it or away from it

[Sec 4.8 #5 P436] Find all limit cycles of

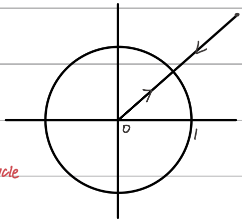
$\begin{cases} x' = x - x^3 - xy^2 = x(1-x^2-y^2) \\ y' = y - y^3 - yx^2 = y(1-y^2-x^2) \end{cases}$

equilibrium pts

$0 = x(1-x^2-y^2) \Rightarrow x=0$ or $x^2+y^2=1$

$0 = y(1-x^2-y^2) \Rightarrow y=0$ or $x^2+y^2=1$

$(0,0)$ $x^2+y^2=1$ the system has no limit cycle




[#7 Sec 4.8]

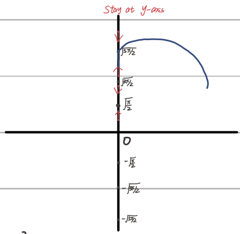
$$\begin{cases} x' = xy + x\cos(x^2+y^2) \\ y' = -x^2 + y\cos(x^2+y^2) \end{cases}$$

equilibrium pt

$$\begin{cases} 0 = x(y+\cos(x^2+y^2)) \Rightarrow x=0 \text{ or } y = -\cos(x^2+y^2) \\ 0 = -x^2 + y\cos(x^2+y^2) \end{cases}$$

$(0,0)$ $(0, \pm\sqrt{(n+\frac{1}{2})\pi})$ $n \in N^*$

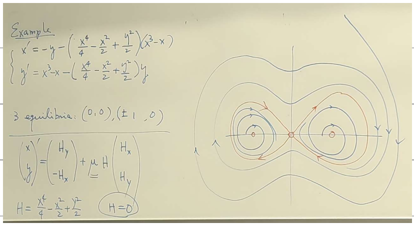
$\frac{d}{dt}(\frac{1}{2}r^2) = xx' + yy' = x^2y + x^2\cos r^2 + (-x^2y) + y^2\cos r^2 = r^2\cos r^2$




Example

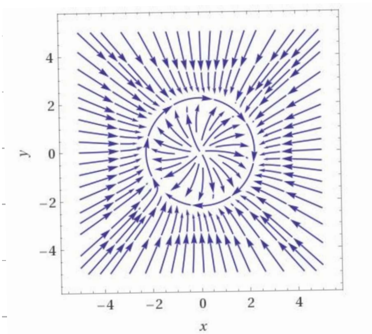
$$\begin{cases} x' = -y - (\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2})(x^3-x) \\ y' = x^3 - x - (\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2})y \end{cases}$$

3 equilibrium : $(0,0)$ $(\pm 1, 0)$




https://math.stackexchange.com/questions/2985533/how-to-determine-the-existence-of-limit-cycle

> Given the dynamical system :
>
> $$x_1' = x_2 + 2\mu x_1(5 - x_1^2 - x_2^2)$$
> $$x_2' = -x_1 + 2\mu x_2(5 - x_1^2 - x_2^2)$$
>
> where $(x_1, x_2) \in \mathbb{R}^2$ and $\mu > 0$ a constant. Applying polar coordinates, determine the *omega limit set* $\omega(x_0)$ for any given vector $(x_1, x_2)$.




**Discussion :**

I used the polar coordinates substitution :

$$x_1 = r\cos\theta$$
$$x_2 = r\sin\theta$$

$$rr' = x_1x_1' + x_2x_2'$$
$$\theta' = \frac{x_1x_2' - x_2x_1'}{r^2}$$

$$r' = 2\mu r(5 - r^2)$$
$$\theta' = -1$$

$$\begin{cases} x = \sqrt{5}\cos(-t) \\ y = \sqrt{5}\sin(-t) \end{cases}$$ is a sol.

(a) According to Green's theorem in the plane, if $C$ is a closed curve which is sufficiently "smooth," and if $f$ and $g$ are continuous and have continuous first partial derivatives, then

$$\oint_C [f(x,y)\,dy - g(x,y)\,dx] = \iint_R [f_x(x,y) + g_y(x,y)]\,dx\,dy$$

where $R$ is the region enclosed by $C$. Assume that $x(t)$, $y(t)$ is a periodic solution of $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$, and let $C$ be the orbit of this solution. Show that for this curve, the line integral above is zero.

(b) Suppose that $f_x + g_y$ has the same sign throughout a simply connected region $D$ in the $x-y$ plane. Show that the system of equations $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ can have no periodic solution which is entirely in $D$.

Dec. 2nd Mon

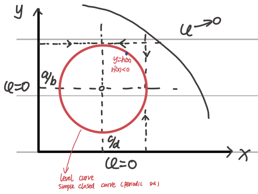
Prey-predator system

$$\begin{cases} x' = x(a-by) \\ y' = y(-c+dx) \end{cases} \Rightarrow \frac{dy}{dx} = \frac{y(-c+dx)}{x(a-by)}$$

$$\frac{a-by}{y}\,dy = \frac{-c+dx}{x}\,dx$$

$$y^a \cdot e^{-by} = x^{-c} \cdot e^{dx}\,\tilde{c} \Rightarrow \frac{y^a}{e^{by}} \cdot \frac{x^c}{e^{dx}} = \tilde{c}$$

Let $u(x,y) = \frac{x^c}{e^{dx}} \cdot \frac{y^a}{e^{by}}$ $(= f(x)\,g(y))$ $\qquad Du(x,y) = \langle f'(x)g(y),\ f(x)\cdot g'(y)\rangle = 0$




$f'(x) = x^{c-1} e^{-dx}(c - xd) = 0 \Rightarrow x = \frac{c}{d}$ $\qquad x < \frac{c}{d} \Rightarrow f' > 0 \qquad x > \frac{c}{d} \Rightarrow f' < 0$

$g'(x) = y^{a-1} e^{-by}(a-by) = 0$ $\qquad y < \frac{a}{b} \Rightarrow g' > 0 \qquad y > \frac{a}{b} \Rightarrow g' < 0$

Sol. $(x(t), y(t))$ must lie on a level curve of $u$

$u(x,y) = C \qquad 0 < C < \max u(x,y) = u(\frac{c}{d}, \frac{a}{b})$

Prop. For any periodic sol. $(x(t), y(t))$ of $(*)$ we have the averages of $x(t)$, $y(t)$ satisfy

$$\bar{x} \equiv \left(\frac{1}{T}\int_0^T x(t)\,dt\right) = \frac{c}{d} \qquad \bar{y} \left(\equiv \frac{1}{T}\int_0^T y(t)\,dt\right) = \frac{a}{b}$$

(pf.) $\frac{x'}{x} = a - by$ integrate from 0 to T

$$\ln x(t)\Big|_0^T = aT - b\int_0^T y(t)\,dt \Rightarrow \ln x(T) - \ln x(0) = 0 \Rightarrow \frac{a}{b} = \frac{1}{T}\int_0^T y(t)\,dt \equiv \bar{y}$$

$\bar{x}$ can be obtained by integrating the 2nd equ.

$$\begin{cases} x' = x(a-by) \\ y' = y(-c+dx) \end{cases} \xrightarrow{\text{Reducing fishing}} \begin{cases} x' = [(a+\varepsilon)-by]x \\ y' = [-(c-\varepsilon)+dx]y \end{cases} \Rightarrow \begin{cases} \bar{x} = \frac{c-\varepsilon}{d} \\ \bar{y} = \frac{a+\varepsilon}{b} \end{cases}$$

die out 的容易程度

Increasing fishing : reducing $a \to a-\varepsilon$ prey 更容易 die out

reducing $c \to c+\varepsilon$ predator 更容易 die out

## Lotka-Volterra Competition System

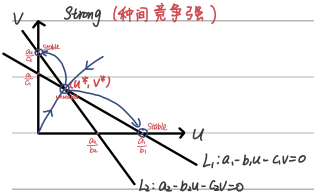
$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v) \\ v_t = v(a_2 - b_2 u - c_2 v) \end{cases}$$

$u, v$: competing species

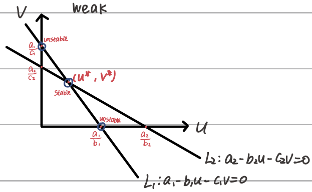
(i) $\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$ (strong) $b_2, c_1$ large    $b_1, c_2$ : intra specific comp. 同种

(ii) $\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$ (weak)    $b_2, c_1$ : inter specific comp. 异种

Strong (种间竞争强)




weak




Prop. The equilibrium $(u_*, v_*)$ is

(1) (locally) unstable in the strong competition case

(2) globally asymptotically stable in the weak competition case

12.4 Wedn.

$$\begin{cases} u_t = u(a_1 - b_1 u - c_1 v) \\ v_t = v(a_2 - b_2 u - c_2 v) \end{cases}$$

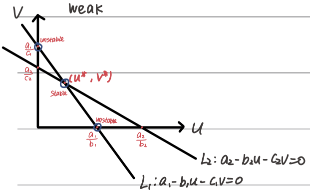
competition system

Logistic equ.

$$y' = ry\left(1 - \frac{y}{K}\right) = y\left(r - \frac{r}{K}y\right)$$

$r$ ↑ growth rate

$K$ ↑ carrying capacity



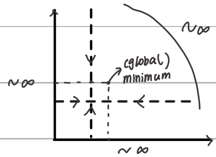

$$\begin{cases} f(u,v) = u(a_1 - b_1 u - c_1 v) \\ g(u,v) = v(a_2 - b_2 u - c_2 v) \end{cases}$$

$$X' = AX \qquad A = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix}_{(u^*, v^*)} = \begin{pmatrix} a_1 - 2b_1 u_* - c_1 v_* & -c_1 u_* \\ -b_2 v_* & a_2 - b_2 u_* - 2c_2 v_* \end{pmatrix} = \begin{pmatrix} -b_1 u_* & -c_1 u_* \\ -b_2 u_* & -c_2 u_* \end{pmatrix}$$

$$0 = \begin{vmatrix} -b_1 u_* - \lambda & -c_1 u_* \\ -b_2 v_* & -c_2 v_* - \lambda \end{vmatrix} = \lambda^2 + (b_1 u_* + c_2 v_*)\lambda + \underbrace{(b_1 c_2 - b_2 c_1)}_{>0} u_* v_*$$

$$\lambda = \frac{-(b_1 u_* + c_2 v_*) \pm \sqrt{(b_1 u_* + c_2 v_*)^2 - 4(b_1 c_2 - b_2 c_1) u_* v_*}}{2}$$

global asymptotic stability




$$E(\overset{\text{sols.}}{u(t)}, v(t)) = b_2\left[u(t) - u_* - v_* \ln \frac{u(t)}{u_*}\right] + c_1\left[v(t) - v_* - v_* \ln \frac{v(t)}{v_*}\right]$$

$\frac{d}{dt} E(u(t), v(t)) \le 0$ & "=" holds $\Leftrightarrow \begin{cases} u \equiv u_* \\ v \equiv v_* \end{cases}$

$\frac{\partial}{\partial u} E(u,v) = b_2\left(1 - \frac{u_*}{u}\right) \quad \begin{cases} < 0 & \text{if } u < u_* \\ > 0 & \text{if } u > u_* \end{cases}$

$\frac{\partial}{\partial u} E(u,v) = c_1\left(1 - \frac{v_*}{v}\right)$

$\frac{d}{dt} E(u(t), v(t)) < 0$ as long as $(u(t), v(t)) \neq (u_*, v_*)$

Lyapunov functional

$$\frac{d}{dt} E(u,v) = b_2\left(u_t - u_* \frac{u_t}{u}\right) + c_1\left(v_t - v_* \frac{v_t}{v}\right) = b_2\left[(u - u_*)(\overset{b_1 u_* + c_1 v_*}{\overset{\shortparallel}{a_1}} - b_1 u - c_1 v)\right] + c_1\left[(v - v_*)(\overset{b_2 u_* + c_2 v_*}{\overset{\shortparallel}{a_2}} - b_2 u - c_2 v)\right]$$

$$= b_2 (u - u_*)[b_1(u_* - u) + c_1(v_* - v)] + c_1(v - v_*)[b_2(u_* - u) + c_2(v_* - v)]$$

$$= -b_1 b_2\left[(u - u_*)^2 + \frac{2c_1}{b_1}(u - u_*)(v - v_*) + \frac{c_1}{b_1} \cdot \frac{c_2}{b_2}(v - v_*)^2\right]$$

$\tilde{u} = u - u_*$ $\tilde{v} = v - v_*$

$$= -b_1 b_2\left[\underbrace{\tilde{u}^2 + 2\frac{c_1}{b_1}\tilde{u}\tilde{v} + \frac{c_1^2}{b_1^2}\tilde{v}^2}_{(\tilde{u} + \frac{c_1}{b_1}\tilde{v})^2} + \frac{c_1}{b_1}\underbrace{\left(\frac{c_2}{b_2} - \frac{c_1}{b_1}\right)}_{>0}\tilde{v}^2\right]$$

Stability (for all time forward)

asymptotic stability

orbital stability (orbit start close remain close) limit cycles

Lyapunov stability