9.2 Mon.

$\mathbb{N}$: nature numbers $1, 2, 3, \cdots$ $\mathbb{Q}$: rational #s $\frac{p}{q}, p, q \in \mathbb{Z}$ $q \neq 0$

$\mathbb{Z}$: intergers $\cdots -2, -1, 0, 1, 2 \cdots$

real number are the union of algebraic and transcendental numbers

$\mathbb{R}$: real #s (irrationals) one of them must be uncountale

algebraic # : a root of polynomials (with coeffs in $\mathbb{Z}$) $\sqrt{2}, \sqrt[3]{7} \cdots$

transcendental #: π, e, $e + \pi$? $e\pi$? one of them is.

Def. Two sets A, B are said to be equiv. if $\exists$ 1-1 onto map from A to B & in that case, we say A and B have the same cardinal #s

Def. A set A is said to be countable if either A is finite or A is equiv. to $\mathbb{N}$

Otherwise, A is uncountable.

Def. The set of all subsets of A is denoted by 2^A

Lemma. $2^{\mathbb{N}}$ is uncountable

Pf. Suppose $2^{\mathbb{N}}$ is countable, i.e.

$2^{\mathbb{N}} = \{A_1, A_2, A_3, \cdots\}$

理发师为不给自己理发的人理发

理发师理发

理发师自己.

Define $B = \{k \in \mathbb{N} \mid k \notin A_k\}$, then $\exists n \in \mathbb{N}$, s.t. $B = A_n$

不给自己理发的人

(i) $n \in A_n$ 给自己理发

(ii) $n \notin A_n$ 不给自己理发

Th'm $\mathbb{R}$ is uncountable

step I: $\mathbb{R} \sim (0,1)$

$f(x) = \frac{1}{\pi} \tan^{-1} x + \frac{1}{2}$ $x \in \mathbb{R}$

Step II: $2^{\mathbb{N}} \sim (0,1)$

Define $f: 2^{\mathbb{N}} \to (0,1)$

$A \in 2^{\mathbb{N}} \Rightarrow A \subseteq \mathbb{N}$

$f(A)=0.a_1a_2a_3\cdots$ where $a_i=\begin{cases}2 & \text{if } i\in A\\ 4 & \text{if } i\notin A\end{cases}$

Define $g(a)=\{j\in\mathbb{N} \mid a_j=0\}\in 2^{\mathbb{N}}$ e.g. $0.1=\frac{1}{2}$ $g(0.1)=\{2,3,4,\cdots\}$

For $a\in(0,1)$, define: $a=0.a_1a_2\cdots$ where

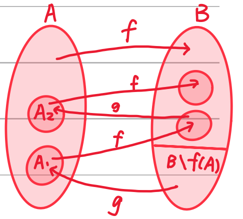
$a_1=\begin{cases}0 & \text{if } a\in(0,\frac{1}{2})\\ 1 & \text{if } a\in[\frac{1}{2},1)\end{cases}$ 用二进制表示的数

After $a_1,\cdots a_{j-1}$ are chosen define $a_j=$ the largest interger in $\{0,1\}$ s.t.

$\frac{a_1}{2}+\frac{a_2}{2}+\cdots\frac{a_j}{2}\le a$

Schroder - Bernstein theorem

If $f:A\to B$ & $g:B\to A$ are both 1-1 then $A\sim B$



$A_1, A_2, \cdots$ are disjoint

Pf. if one of f,g is onto, we've done

Otherwise (i.e. f,g neither is onto)

$A_1=g(B\setminus f(A))$ $A_2=g(f(A_1))$ $\cdots$ $A_{n+1}=g(f(A_n))$

$S=\bigcup_{n=1}^{\infty}A_n$ Define $F(a)=\begin{cases}f(a) & a\in A\setminus S\\ g^{-1}(a) & a\in S\end{cases}$ $F:A\to B$

claim: F is 1-1 & onto

Onto: Let $b\in B$:

(i) $g(b)\in S\Rightarrow F(g(b))=g^{-1}(g(b))=b$

(ii) $g(b)\notin S\Rightarrow b\in f(A)\Rightarrow\exists a\in A$ s.t. $f(a)=b$ 希望能证出 $f(a)=F(a)=b$ 那么 $\exists a\in A$. s.t. $F(a)=b$

[if $b\notin f(A)\Rightarrow g(b)\in S$]

Need to show: $a\notin S$ ($\Rightarrow F(a)=f(a)=b$) $S=A_1\cup A_2\cup\cdots=A_1\cup(gf(A_1)\cup gf(A_2)\cdots)$ $=A_1\cup gf(A_1\cup A_2\cdots)=A_1\cup gf(S)$

if $a\in S\Rightarrow g(b)=g(f(a))\in gf(S)\subseteq S$ contradict!

1-1: Suppose F is not 1-1 $\Rightarrow\exists a_1\in A\setminus S$ & $a_2\in S$

s.t. $F(a_1)=F(a_2)$ 因为 f. g^{-1} 均为 1-1 i.e. $f(a_1)=g^{-1}(a_2)$

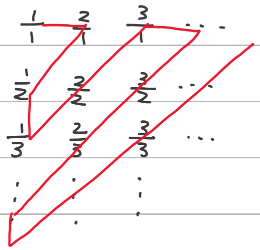
$\Rightarrow gf(a_1)=a_2\in S=A_1\cup gf(A_1)\cup gf^2(A_1)\cup\cdots$

$gf(a_1)\in A_1=g(B\setminus f(A))\Rightarrow f(a_1)\in B\setminus f(A)$

$gf(a_1)\in A_2=gf(A_1)\Rightarrow a_1\in A_1\subseteq S$ ※ $\cdots$ By induction

$a_1\notin A_1, A_2, \cdots$

Th'm $\mathbb{Q}$ is countable



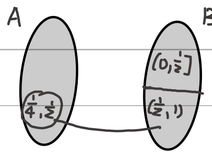
为什么 $\frac{1}{1}$ $\frac{2}{2}$ $\frac{3}{3}$ 不一样?

Example

To construct a 1-1 onto map between $A=(0,\frac{1}{2}]$ & $B=(0,1)$

$f(a)=a$ $\quad g(b)=\frac{1}{2}b$

$S=\bigcup_{i=1}^{\infty}\left(\frac{1}{4i},\frac{1}{2i}\right)$ $\quad F(a)=\begin{cases} a & a\in A\setminus S \\ 2a & a\in S \end{cases}$



9.4 Wed

可能 converge 到 $\mathbb{R}\setminus\mathbb{Q}$

$\{1.4, 1.41, 1.414, \cdots\} \to \sqrt{2}$ $\quad \mathbb{Q}$ is not complet

$\mathbb{R}$: real #s "Any cauchy seq. of real #s conv." ⇒ "complete"

complete the space ⇒ add all the number of cauchy seq. conv. to to the space put all the numbers like $\sqrt{2}$ to $\mathbb{Q}$ to complete it.

Def. (1) A seq. of rational #s. $\{q_1, q_2, \cdots q_n, \cdots\}$ is a Cauchy seq. if $\forall \varepsilon>0, \exists N$ s.t. $|q_m-q_n|<\varepsilon$ for all $m,n\geq N$

(2) Two Cauchy seq. $\{p_1,p_2,\cdots p_n,\cdots\}$ & $\{q_1,q_2,\cdots q_k\cdots\}$ of rational #s are equiv. if $\forall \varepsilon>0, \exists N>0$, s.t. $|p_k-q_k|<\varepsilon$ $\forall k\geq N$ & we write $\{p_1,\cdots\}\sim\{q_1,q_2,\cdots\}$

Rk. Verify "$\sim$" is an equiv. ralation defined on the set of all Cauchy seqs. of rational #s.

(3) real number is defined to be an equiv. class of Cauchy seqs of rational #s.

Def. Two real #s x, y, we have

$x \sim \{x_1, \cdots, x_n, \cdots\}$, $x_j \in \mathbb{Q}$

$y \sim \{y_1, \cdots y_n, \cdots\}$, $y_j \in \mathbb{Q}$

$x+y \sim \{x_1+y_1, \cdots x_n+y_n, \cdots\}$ $\quad$ $xy \sim \{x_1y_1, \cdots x_ny_n, \cdots\}$

$|x| \sim \{|x_1|, \cdots |x_n|, \cdots\}$

Prop. The set of all real #s is complete i.e. $\forall$ Cauchy seq. of real numbers must conv. in $\mathbb{R}$

(pf.)

algebraic #s

Def. A # $x \in \mathbb{R}$ is algebraic if it satisfies the equ.

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_n, \cdots a_1, a_0 \in \mathbb{Z}$ & NOT all 0. And we say that x is of degree n if $a_n \neq 0$ & x is NOT a root of any poly. of lower degree

$\mathbb{Q}$ 都是 algebraic #s

Def. A real # x is transcendental if it is NOT algebraic

Prop. The set of all algebraic #s is countable

Step I: let $\mathcal{P}_n$ = the set of all poly. with integel coffs of degree n. i.e. $\mathcal{P}_n = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_n, \cdots a_1, a_0 \in \mathbb{Z}, a_n \neq 0\}$

$\mathcal{P}_n \sim \{(a_n, \cdots, a_1, a_0) \mid a_j \in \mathbb{Z}\} = \mathbb{Z}^{n+1}$ is countable

Step II. let $\mathcal{R}_n$ = the set of all real roots of poly. in $\mathcal{P}_n$. Then

$\mathcal{R}_n \hookrightarrow \mathcal{P}_n^n$ & $\mathcal{R}_n$ is countable

Step III. $\bigcup_{n=1}^{\infty} \mathcal{R}_n$ is countable

Corollary The set of all transcendental #s is uncountable

Q: How fast can we approx. a real # by rationals?

$\sim$ Diophantine approx.

order越高 逼近越快

Def. A real number $r \in \mathbb{R}$ is said to be approximable by rationals to order n if $\exists$ constant $K (= K(r))$ s.t. the ineq.

$G_n(\alpha) = \{\frac{p}{q} \in \mathbb{Q} \mid 0 < |\alpha - \frac{p}{q}| < \frac{1}{q^n}\}$

the set of n-good approximation

α is n-approximable when $|G_n(\alpha)| = \infty$

$$|r - \frac{p}{q}| \leq \frac{K}{q^n}$$

has infinitely many solutions $\frac{p}{q} \in \mathbb{Q}$ with $q > 0$ when p & q are relatively prime in $\mathbb{Z}$

Lemma A rational # $\frac{p}{q}$ is approximable by rationals to order 1 & no higher than 1

$\forall q, \exists p_1, p_2$. s.t. $|\frac{p}{q} - r| < \frac{2}{q}$

$|p_1 - p_2| < 4$, take q prime. one of p_1, p_2 must be relatively prime with q

Pf. $\left|\frac{p}{q}-\frac{p_k}{q_k}\right| = \left|\frac{p_k q - p q_k}{q_k q}\right| \geq \frac{1}{q_k q} = \frac{\frac{1}{q}}{q_k} \xrightarrow{} \frac{p}{q} \neq \frac{p_k}{q_k}$

$\Rightarrow \frac{p}{q}$ cannot be app. by rationals to order higher than 1

Observe. $\frac{\frac{1}{q}}{q_k} \leq \left|\frac{p_k}{q_k}-\frac{p}{q}\right| \leq \frac{k}{q_k^n}$ $(n>1)$ contradicts.

Th'm An algebraic # ξ of degree n is NOT approximable by rationals to any order higher than n.

(pf.) let ξ be a root of $f(x)=a_n x^n+\cdots+a_1x+a_0$

$\exists \lambda>0$, s.t. $f(x)\neq 0$ in $[\xi-\lambda,\xi+\lambda]$, $x\neq\xi$

$\forall \frac{p}{q}\in[\xi-\lambda,\xi+\lambda], \frac{p}{q}\neq\xi$ of $\left|f(\frac{p}{q})\right| = \left|a_n\frac{p^n}{q^n}+\cdots+a_1\frac{p}{q}+a_0\right|$

$= \left|\frac{a_np^n+\cdots+a_1pq^{n-1}a_0q^n}{q^n}\right| \geq \frac{1}{q^n}$

$\left|f(\frac{p}{q})\right| = \left|f(\frac{p}{q})-f(\xi)\right| = \left|\frac{p}{q}-\xi\right|\left|f'(\eta)\right| \leq M\left|\frac{p}{q}-\xi\right|$

$\frac{\frac{1}{M}}{q^n} \leq \left|\frac{p}{q}-\xi\right|$ $\forall \frac{p}{q}\neq\xi$ in $[\xi-\lambda,\xi+\lambda]$

$\frac{\lambda}{q^n} < \lambda \leq \left|\frac{p}{q}-\xi\right|$ $\forall \frac{p}{q}\notin[\xi-\lambda,\xi+\lambda]$

take $c=\min\left\{\frac{\frac{1}{M}}{q^n},\frac{\lambda}{q^n}\right\}$

if ξ can be approxed by degree higher than n.

? $\frac{c}{q^n}$ $\quad \frac{\lambda}{q^n} < \left|\xi-\frac{p}{q}\right| < \frac{c}{q^k}$ $\frac{\tilde{c}}{q^k}$? $k>n$ contradict!

Example $\xi=\sum_{n=1}^{\infty}\frac{1}{10^{n!}}$ can be approximable by rationals to order ∞

$\left|\sum_{n=1}^{\infty}\frac{1}{10^{n!}}-\frac{p}{q}\right| < \frac{c}{q^N}$ for $\forall N>0$ take $\frac{p}{q}=\sum_{n=1}^{k}\frac{1}{10^{n!}}$, $k\geq N$

Pf. Suppose L is approximable by rationals of order n & not higher, we claim: $\exists k$ st. $\left|\xi-\frac{p}{q}\right| < \frac{k}{q^{n+1}}$ has ∞ many solutions

Let $\xi_k = \frac{1}{10^{1!}}+\frac{1}{10^{2!}}+\cdots+\frac{1}{10^{k!}} = \frac{\eta_k}{10^{k!}}$ $\eta_k \rightarrow$ 最后一位是 1 so $\gcd(\eta_k, 10^{k!})=1$

$|L-\xi_k| = \frac{1}{10^{(k+1)!}}+\frac{1}{10^{(k+2)!}}+\cdots = \frac{1}{10^{(k+1)!}}\left[1+\frac{1}{10^{k+2}}+\frac{1}{10^{(k+2)(k+3)}}+\cdots\right]$

$\leq \frac{2}{(10^{k!})^{k+1}}$

$\left|L-\frac{p}{q}\right| < \frac{2}{q^{k+1}}$ $\quad \left|L-\frac{p}{q}\right| < \frac{k}{q^{n+1}}$ $(\frac{p}{q}=\xi_k, k\geq n+1)$ $\rightarrow\!\!\!\!\times$

Limits & Continuity

Def. (1) Let: $f: D \to R$ when $D \subseteq R$ is an open set & $c \in D$. f is said to be continuous at c if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(c)| < \varepsilon \quad \forall\ |x - c| < \delta$$

(2) if f is continuous at every pt. in D, we say that f is continuous on D

Example $D = (0,1)$

$$f(x) = \begin{cases} 0, & x \notin Q \\ \frac{1}{q}, & \text{if } x = \frac{p}{q} \ (p, q \text{ are prime}) \end{cases}$$

Observation The set of all points in $(0,1) \setminus Q$

Question $\exists ?\ g$ s.t. the set of all pts where g is continuous is $Q \cap (0,1)$

Example Let $f(x) = \begin{cases} \frac{x-4}{\sqrt{x}-2} & x > 0 \ \& \ x \neq 4 \\ 4 & \text{if } x = 4 \end{cases}$ show that f is cont. at $x = 4$

(Pf.) $\forall \varepsilon > 0$ given, to find $\delta > 0$, s.t.

$$\left|\frac{x-4}{\sqrt{x}-2} - 4\right| < \varepsilon \quad \forall\ |x-4| < \delta$$

$$|\sqrt{x} - 2| < \varepsilon \qquad 4 - \delta < x < 4 + \delta$$

$$2 - \varepsilon < \sqrt{x} < \varepsilon + 2$$

$\forall \varepsilon > 0$. set $\varepsilon' = \min\{2, \varepsilon\}$

$2 - \varepsilon' < \sqrt{x} < 2 + \varepsilon' \Rightarrow 4 - 4\varepsilon' + \varepsilon'^2 < x < 4 + 4\varepsilon' + \varepsilon'^2$

take $\delta = 2\varepsilon' \Rightarrow \forall \varepsilon > 0$ given, set $\delta = 2\min\{\varepsilon, 2\}$

11) (Bolzano-Weierstnass)

Every bdd seq. in $\mathbb{R}$ has a conv. subseq. (Completeness)

12) (Heine-Borel) Every open cover of a bdd closed subset of $\mathbb{R}$ has a finite subcover

13) ((Cantor's Nested Interval Lemma) $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}] = \phi$

Let $I_n = [a_n, b_n]$, with $a_n \uparrow$ $b_n \downarrow$ & $(b_n - a_n) \searrow 0$ as $n \to \infty$

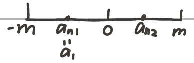
then $\exists c \in \mathbb{R}$ s.t. $\bigcap_{n=1}^{\infty} [a_n, b_n] = \{c\}$

(pf) (1) Let $\{a_n\}$ be a bdd seq. in $\mathbb{R}$, say $\{a_n\} \subseteq [-M, M]$

Pick $\{a_{n_k}\}$

$a_{n_1} = a_1$ $\{a_n\} \subseteq [M, 0] \cup [0, M]$

Suppose $[0, M] \cap \{a_n\}$ is ∞ Pick a_{n_2} to be an element in $[0, M] \cap \{a_n\}$ with $n_2 > n_1$



Suppose $\{a_n\} \cap [0, \frac{M}{2}]$ is infinite

Pick $n_3 > n_2$ s.t. $a_{n_3} \in \{a_n\} \cap [0, \frac{M}{2}]$

$\{a_{n_k}\}$ Observe : $\{a_{n_{k_1}}, a_{n_{k_2}}, \dots\} \subseteq$ an interval of length $\frac{2M}{2^{k-1}} \to 0$

$\Rightarrow \{a_{n_k}\}$ is cauchy $\Rightarrow \{a_{n_k}\}$ conv.

13) $a_n \uparrow$ & bdd above by b_1 B-W $\Rightarrow a_n \to c$

$b_n \downarrow$ & bdd below by a_1 B-W $\Rightarrow b_n \to d$

$0 \le d - c \le b_n - a_n \to 0$

$\therefore d = c$

9.11 Wedn.

Def. Let $E \subseteq \mathbb{R}$, an arbitrary subset of $\mathbb{R}$. An open cover $\{U_i\}_{i\in I}$ of E is a collection of opensets whose union contains E; i.e. $E \subseteq \bigcup_{i\in I} U_i$

A finite subcover of $\{U_i\}_{i\in I}$ is a finite sub-collection whose union still covers E

Example: $E=(0,1]$, $U_n=(\frac{1}{n}, 1+\frac{1}{n})$, $\{U_n \mid n\in\mathbb{N}\}$ is an open cover for E

$(0,1] \subseteq \bigcup_{n=1}^{\infty} (\frac{1}{n}, 1+\frac{1}{n})$, but $\{U_n \mid n\in\mathbb{N}\}$ does not have any finite subcover

(Pf). $E=[0,M]$ & $\{U_i \mid i\in I\}$ is an open cover

Suppose for contradiction that $\{U_i\}$ does not have any finite subcover

Write $E=E_1 \cup \tilde{E}_1 = [0,\frac{M}{2}] \cup [\frac{M}{2}, M]$

$\Rightarrow$ One of E_1 or $\tilde{E}_1$ cannot have a finite subcover, say, $E_1=[0,\frac{M}{2}]$ does not have a finite subcover.

write $E_1=[0,\frac{M}{2}]=[0,\frac{M}{4}]\cup[\frac{M}{4},\frac{M}{2}]=E_2\cup\tilde{E}_2$

Then, one of $E_2, \tilde{E}_2$, say E_2, does not have any finite subcover. Write $E_2=[0,\frac{M}{4}]=[0,\frac{M}{8}]\cup[\frac{M}{8},\frac{M}{4}]$
$=E_3\cup\tilde{E}_3$

& repeat the above arguement

We obtain $E_1, E_2, \cdots E_k \cdots$ with the property that none of them has a finite subcover

& $|E_k|=\frac{M}{2^k}\to 0$ as $k\to\infty$. Then Nested Interval Lemma $\Rightarrow \bigcap_{k=1}^{\infty} E_k=\{a\}$

Then $a\in U_1 \Rightarrow \exists\ \delta>0$, st $(a-\delta, a+\delta)\subseteq U_1$

可能是多个 interval 拼起来

Now the general case: E is a bdd closed set. Say $E\subseteq[-k,k]$ Set $V=(-k-1,k+1)\setminus E$

$\Rightarrow \{V, U_i\}_{i\in I}$ is an open cover for $[-k,k]$ Our previous proof now garentees that $\exists$ a finite subcover (S) for $[-k,k]$. Now if $V\notin S$ then we're done. If $V\in S$, then $S\setminus\{V\}$ covers E

Corollary Suppose that $[a,b]$ (E (bdd closed set)) is covered by an open cover $\mathcal{F}$ (a family of open intervals)

Then $\exists\, \rho > 0$ s.t. every interval of of the form $(c-\rho, c+\rho)$, where $c \in [a,b]$ (E) is contained in one of the intervals in $\mathcal{F}$ (ρ : Lebesgue #)

Extreme-Value Th'm

Suppose that f is continuous on $[a,b]$ $-\infty < a < b < \infty$. Then both $\sup_{a \le x \le b} f(x)$ & $\inf_{a \le x \le b} f(x)$ are assumed ($\exists\, c$ s.t $f(c) = M$)

pf. Let $\alpha = \sup_{a \le x \le b} f(x) \Rightarrow \exists\, f(x_n) \to \alpha$ because the range of f is bdd

Since $\{x_n\}$ is bdd, Bolzano-Weierstrass $\Rightarrow$ $\{x_n\}$ has a conv. subseq. $x_{n_k} \to c \in [a,b]$

Continuity of f $\Rightarrow f(x_{n_k}) \to f(c) \Rightarrow f(c) = \alpha$

IVT

suppose that f is continuous on $[a,b]$ & $f(a) \ne f(b)$, $\exists\, c \in (a,b)$ s.t. $f(c) = \alpha$

(pf.) W.L.O.G. assume $f(a) < \alpha < f(b)$

(i) if $\alpha = f(\frac{a+b}{2})$, then we're done

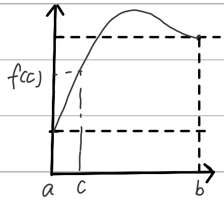
(ii) 〃 $\alpha < f(\frac{a+b}{2})$, then seq $a_1 = a$ $b_1 = \frac{a+b}{2}$

i.e. $f(a_1) < \alpha < f(b_1)$

repeat : if $a = f(\frac{a_1+b_1}{2})$ then we're done

if $a < f(\frac{a_1+b_1}{2})$, then $a_2 = a_1$ $b_2 = \frac{a_1+b_1}{2}$

if $a > f(\frac{a_1+b_1}{2})$, then $a_2 = \frac{a_1+b_1}{2}$, $b_2 = b_1$



Nested Interval Lemma $\Rightarrow \bigcap_{k=1}^{\infty} [a_k, b_k] = \{c\}$

$f(c) \leftarrow f(a_k) < \alpha < f(b_k) \to f(c)$ $\therefore f(c) = \alpha$

Th'm A continuous function f on $[a,b]$ $-\infty < a < b < \infty$ must be uniformly continuous on $[a,b]$

Def. ① f is cont. at $c \in E$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t $|f(x) - f(c)| < \varepsilon \ \forall |x - c| < \delta$

② f is unif. cont. on E if $\forall \varepsilon > 0, \exists \delta > 0$ st $|f(x) - f(c)| < \varepsilon \ \forall |x - c| < \delta$ $\delta = \delta(\varepsilon)$

(Pf) $\forall \varepsilon > 0, x \in [a,b], \exists \delta > 0$ s.t. $|f(y) - f(x)| < \varepsilon$ if $|y - x| < \delta$ $\delta(\varepsilon, x)$

$S = \{(x - \delta(\varepsilon, x), x + \delta(\varepsilon, x)) | x \in [a,b]\}$ is an open cover for $[a,b]$ Heine-Borel $\Rightarrow$

$\exists$ a finite sub-cover & then $\exists$ a Lebesgue # $\rho > 0$

s.t $\forall c \in [a,b]$ $(c - \rho, c + \rho) \subseteq (x - \delta(\varepsilon, x), x + \delta(\varepsilon, x))$ for some $x \in [a,b]$

$|f(y) - f(c)| \leq |f(y) - f(x)| + |f(x) - f(c)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ $\forall |y - c| < \rho$ $(y \in (c - \rho, c + \rho))$

9.14 Sat.

metric space

Def. Let S be a set & $d: S \times S \to \mathbb{R}$ be a fcn. We say that (S, d) is a metric space if the following hold:

(i) $d(x,y) \geq 0 \ \forall x, y \in S$ & $d(x,y) = 0 \Leftrightarrow x = y$

(ii) $d(x,y) = d(y,x) \ \forall x, y \in S$

(iii) $d(x,z) \leq d(x,y) + d(y,z) \ \forall x, y, z \in S$ (Triangle ineq.)

(d: a metric / a distance fcn)

Example (1) On $\mathbb{R}$, $d(x,y) = |x - y|$ $\begin{cases} x = (x_1, x_2) \\ y = (y_1, y_2) \end{cases}$

(2) On $\mathbb{R}^2$, $d(x,y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$d_1(x,y) = |x_1 - x_2| + |y_1 - y_2|$

$\check{d}(x,y) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

All $d, d_1, \check{d}$ are metrics

equivlant property of point set topology are the same

Def. Suppose d_1, d_2 are two metrics on S. We say that d_1, d_2 are equivalent if $\exists$ positive constants c, C st. $\forall x, y \in S$ $c\, d_1(x,y) \leq d_2(x,y) \leq C\, d_1(x,y)$

Rk $d, d_1, \tilde{d}$ in Example (2) above are equiv.

Pf. ① $\tilde{d}(x,y) \le d(x,y) \le 2\tilde{d}(x,y)$

② $\frac{1}{\sqrt{2}} d_1(x,y) \le d(x,y) \le d_1(x,y)$

③ $\tilde{d}(x,y) \le d_1(x,y) \le 2\tilde{d}(x,y)$

Generalization $(\mathbb{R}^n, d)$ $(\mathbb{R}^n, d_1)$ & $(\mathbb{R}^n, \tilde{d})$ are equiv.

(3) Let $\mathscr{C}([a,b])$ be the set of all cont. fcns on $[a,b]$ & define

$d(f,g) = \left(\int_a^b |f(x) - g(x)|^2 dx\right)^{\frac{1}{2}}$

$d_1(f,g) = \int_a^b |f(x) - g(x)| dx$

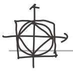
$\tilde{d}(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$ $\qquad d_p(f,g) = \left(\int_a^b |f(x) - g(x)|^p dx\right)^{\frac{1}{p}}$ $1 \le p < \infty$

(4) S: any set $x, y \in S$ counter examples

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \ne y \end{cases}$$

$\Rightarrow (S,d)$ is a metric space

Exercise: What is the shape of unit $B = \{x \in S \mid d(x,0) \le 1\}$ for the metric spaces $(\mathbb{R}^2, d)$ $(\mathbb{R}^2, d_1)$ & $(\mathbb{R}^2, \tilde{d})$



Def: (1) In a metric space (S,d), a seq of points $\{p_n\}$ is said to converge to a point $p \in S$

if $d(p_n, p) \to 0$ as $n \to \infty$

(2) $B(p_0, r) = \{p \in S \mid d(p_0, p) < r\}$ the open ball centered at p_0 with radius r

$\overline{B(p_0, r)} = \{$ $\le r\}$ Closed

(3) Let $A \subseteq S$ be a subset & $P_0 \in A$ we say that "P_0" is an isolated points of A if $\exists r>0$ s.t $B(P_0,r) \cap A = \{P_0\}$

every point in example 4 is isolated points

(4) Let $A \subseteq S$, $P_0 \in S$, we say that P_0 is limit point of A if every open ball $B(p_0,r)$ contains a point $p \in A$ s.t. $p \neq P_0$

in example 4. $S=\mathbb{R}$ $A=[0,1)$ $\forall x \in A$ is not limit point

(5) $A \subseteq S$ is closed if it contains all of its limit points

(6) $A \subseteq S$ is open if $\forall p \in A$ $\exists r>0$ s.t. $B(p,r) \subseteq A$

Prop (1) P_0 is a limit point of $A \Leftrightarrow \exists$ a seq. $\{p_n\} \subseteq A$ s.t $P_n \neq P_0$ & $P_n \to P_0$ as $n \to \infty$

(2) A is closed $\Leftrightarrow S \setminus A$ is open

suppose $S\setminus A$ is not open $\exists x_0 \in S\setminus A$, $\forall B(x_0,\delta)$, $\exists p \in B(x_0,\delta)$ $p \notin S\setminus A$
take $P_n \in B(x_0,\frac{1}{n})$ $d(P_n,x_0)<\frac{1}{n}$ $\exists \{P_n\} \to x_0$. x_0 is a limit point of A
$x_0 \in A$ $\rightarrow\leftarrow$

(3) S & the empty set $\emptyset$ are both open & closed

(4) The union of any family of open set is open

(5) The intersection of any family of closed sets is closed

(6) The intersection of any finite family of open sets is open

(7) The union of any finite family of closed sets is closed

examples ① $(0,1) = \bigcup_{n=3}^{\infty} [\frac{1}{n}, 1-\frac{1}{n}]$

② $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, 1+\frac{1}{n}) = [0,1]$

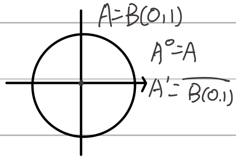
Def ① if $B(p,r) \subseteq A (\subseteq S)$ then we say that p is an interior pt. of A & we set

A° = the set of all interior pts of A

② A' = the set of all limit pts of A, the derived set of A

③ $\overline{A} = A \cup A'$ the closure of A

④ $\partial A = \overline{A} \setminus A^\circ$ the boundary of A



Prop

① A', $\bar{A}$ & ∂A are closed

② A° is open

prop (i) Any open set in $\mathbb{R}$ is the union of a countable family of disjoint open intervals

(ii) An open set in $\mathbb{R}^n$ is the union of a countable family of closed cells whose interiors are disjoint

Def An open cell in $\mathbb{R}^n$ is $\{(x_1, x_2, \cdots, x_n) \mid a_i < x_i < b_i, \ i=1,\cdots n\}$

A closed cell in $\mathbb{R}^n$ is $\{(x_1, x_2, \cdots x_n) \mid a_i \le x_i \le b_i, \ i=1,\cdots n\}$

9.18 Wed. https://math.stackexchange.com/questions/674982/difference-between-closed-bounded-and-compact-sets

Def. ① A subset A of a metric space S is compact if every seq. in A has a conv. subseq. in A

Definition. A set A in a metric space S is **compact** if and only if each sequence of points $\{p_n\}$ in A contains a subsequence, say $\{q_n\}$, which converges to a point p_0 in A.

② A is bdd if $A \subseteq B(p, r)$ for some $p \in S$ & $r > 0$

Example On $\mathbb{R}^1$ $[0,2]$ is compact

$\{\frac{1}{n}\} \to 0$ $(0,1)$ is not compact because $0 \notin A$

Th'm (1) A compact set in a metric space is bdd & closed

(2) A bdd closed subset of $\mathbb{R}^n$ is compact

Example $\mathcal{C}([0,1])$ = the set of all cont. fcns on $[0,1]$ equipped with the metric

$d(f,g) = \max_{0 \le x \le 1} |f(x) - g(x)|$ compactness in function space $\Leftrightarrow$ ① $\mathcal{C}(A)$ where A is compact ② all fcns are unif bdd & equi-cont.

$A = \overline{B(\underset{\text{0 fcn}}{0}, \underset{\text{radius}}{1})} = \{f \in \mathcal{C}([0,1]) \mid \max_{0 \le x \le 1} |f(x)| \le 1\}$

bdd, closed (It contains all of its limit point)

Let f be a limit pt. of A; i.e $\exists \{f_n\} \subseteq A$

s.t. $f_n \to f$ (i.e. $d(f_n, f) \to 0$) $\max_{0 \le x \le 1} |f_n(x) - f(x)| \to 0$

i.e. $f_n \to f$ uniformly (So f is cont).

Date: /

Theorem 4.2.18

If $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of functions converging uniformly to a function $f(x)$ on $0 < |x - x_0| < \eta$, and if for each n the limit

$$\lim_{x \to x_0} f_n(x) = a_n$$

exists, then both the limits

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{x \to x_0} f(x)$$

exist and they are equal, i.e.

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x).$$

Theorem 4.2.19.

If $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of continuous functions on E converging uniformly to a function $f(x)$ on E, then $f(x)$ is also continuous on E.

Theorem 4.2.20

If $\{f_n\}$ is a sequence of continuous functions on the interval $[a, b]$ converging uniformly to a function f, then

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

For any $x_0 \in [a, b]$, the sequence $\int_{x_0}^x f_n(t)dt$ uniformly converges to $\int_{x_0}^x f(t)dt$ on $[a, b]$.

Theorem 4.2.21

Assume $\{f_n(x)\}_{n=1}^{\infty}$ converges to $f(x)$ on $[a, b]$,

each $f_n(x)$ is continuously differentiable and

$\{f_n'(x)\}_{n=1}^{\infty}$ uniformly converges on $[a, b]$,

then f is continuously differentiable on $[a, b]$,

$$f'(x) = \frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x), \quad \forall x \in [a, b],$$

and $\{f_n(x)\}_{n=1}^{\infty}$ also converges uniformly on $[a, b]$.

4.2.19

W.T.S $\forall \varepsilon > 0, \exists \delta_1 > 0$. s.t $|f(x) - f(c)| < \varepsilon$ whenever $|x-c| < \delta_1$

f_n is cont $\exists \delta_2$ $|f_n(x) - f_n(c)| < \varepsilon$ whenever $|x - c| < \delta_2$

$\{f_n\}$ is unif. conv. $\forall \varepsilon > 0$. $\exists N(\varepsilon) > 0$. s.t. $|f_n(x) - f(x)| < \varepsilon$ $|f_n(c) - f(c)| < \varepsilon$ whenever $n > N$

$|f(x) - f(c)| = |f_n(x) - f(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| < 3\varepsilon$ take $\delta = \min\{\delta_1, \delta_2\}$

" A space X is compact if and only if every collection of closed subsets of X satisfying the finite intersection property has non-empty intersection."

($\Rightarrow$)

Proof. $\Rightarrow$ Let X be compact. Let $\mathcal{C}$ be a collection of closed subsets of X. We show that if $\mathcal{C}$ has the finite intersection property, then it has non-empty intersection. Suppose that $\bigcap \mathcal{C} = \phi$. Then $\mathcal{U} = \{X - C : C \in \mathcal{C}\}$ is an open cover of X. By the compactness of X, $\mathcal{U}$ has a finite subcover $\{X - C_0, X - C_1, ..., X - C_n\}$. it follows that $\bigcap_{i \le n} C_i = \phi$.

($\Leftarrow$)

The argument itself is straightforward. Suppose that X is not compact; then it has an open cover $\mathscr{U}$ with no finite subcover. For each $U \in \mathscr{U}$ let $F_U = X \setminus U$, and let $\mathscr{F} = \{F_U : U \in \mathscr{U}\}$; clearly $\mathscr{F}$ is a family of closed sets. Let $\mathscr{F}_0$ be any finite subset of $\mathscr{F}$. There is a finite $\mathscr{U}_0 \subseteq \mathscr{U}$ such that $\mathscr{F}_0 = \{F_U : U \in \mathscr{U}_0\}$. Then

$$\bigcap \mathscr{F}_0 = \bigcap_{U\in\mathscr{U}_0} F_U = \bigcap_{U\in\mathscr{U}_0}(X \setminus U) = X \setminus \bigcup \mathscr{U}_0 .$$

$\mathscr{U}$ has no finite subcover, so $\bigcup \mathscr{U}_0 \ne X$, and therefore

$$\bigcap \mathscr{F}_0 = X \setminus \bigcup \mathscr{U}_0 \ne \varnothing .$$

Thus, $\mathscr{F}$ is centred: every finite subset of $\mathscr{F}$ has non-empty intersection. But

$$\bigcap \mathscr{F} = \bigcap_{U\in\mathscr{U}}(X \setminus U) = X \setminus \bigcup \mathscr{U} = \varnothing ,$$

since $\mathscr{U}$ is a cover of X, so $\mathscr{F}$ is a centred family of closed sets in X whose intersection is empty.

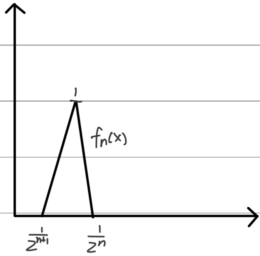
Compactness is a powerful property of spaces, and is used in many ways in many different areas of mathematics. One is via appeal to local-to-global principles; one establishes local control on some function or other quantity, and then uses compactness to boost the local control to global control. Another is to locate maxima or minima of a function, which is particularly useful in calculus of variations. A third is to partially recover the notion of a limit when dealing with non-convergent sequences, by accepting the need to pass to a subsequence of the original sequence. (Note however that different subsequences may converge to different limits; compactness guarantess the existence of a limit point, but not uniqueness.) Compactness of one object also tends to beget compactness of other objects; for instance, the image of a compact set under a continuous map is still compact, and the product of finitely many or even infinitely many compact sets continues to be compact (this is known as *Tychonoff's theorem*).

Thus $f \in B(0,1) \Rightarrow B(0,1)$ is closed

Define $g_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2^{n+1}} \text{ or } x \geq \frac{1}{2^n} \\ (x - \frac{1}{2^{n+1}}) 2^{n+2} & x \in (\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}) \\ -(x - \frac{1}{2^n}) 2^{n+2} & x \in (\frac{3}{2^{n+2}}, \frac{1}{2^n}) \end{cases}$

$\Rightarrow |g_n(x)| \leq 1$

$\therefore \{g_n\} \subseteq B(0,1)$



Q: Does $\{g_n\}$ have a cont. subseq.?

If $n \neq m \Rightarrow \max_{0 \leq x \leq 1} |g_n(x) - g_m(x)| = 1$

i.e. $d(g_n, g_m) = 1 \quad \forall m \neq n$

$\therefore \{g_n\}$ cannot have any conv. subseq.

Given a topological space X, we can equip the space of bounded real or complex-valued functions over X with the uniform norm topology, with the uniform metric defined by

$$d(f,g) = \|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|.$$

Then uniform convergence simply means convergence in the uniform norm topology:

$$\lim_{n \to \infty} \|f_n - f\|_\infty = 0.$$

(Pf) (1) (Bddness) If A is NOT Bdd $\Rightarrow$

$B(a;n)$ $n = 1, 2, 3, \cdots$ does not contain A, i.e. $\exists\, a_n \in A \setminus B(a,n)$

Consider the seq. $\{a_n\} \subseteq A$ it has no conv. Subseq ($\because d(a, a_{n_k}) \geq n_k$)

assume not closed $\exists$ a limit point $P_0 \notin A$

$\exists \{P_n\} \to P_0$ $\forall$ subsequence of $\{P_n\} \to P_0$.

(Closedness: Immediate) A is compact so $P_0 \in A$ contradicti.

(2) $S = \mathbb{R}^N$

$N = 1$ $A \subseteq \mathbb{R}^1$ is bdd and closed

Let $\{a_n\} \subseteq A$ be a seq.

B-W th'm $\Rightarrow \exists$ conv. subseq. in A

$N = 2$ $\{a_n = (x_n, y_n)\}$ bdd seq. in $\mathbb{R}^2 \Rightarrow \{x_n\}$ bdd in $\mathbb{R}^1$ $\therefore \exists$ conv. subseq. $\{x_{n_k}\}$

Consider $\{y_{n_k}\}$ a bdd seq. in $\mathbb{R}^1$

$\therefore$ it contains a conv. subseq. $\{y_{n_{k_l}}\}$ $\quad a_{n_{k_l}} = (x_{n_{k_l}}, y_{n_{k_l}})$

Def A seq. $\{P_n\}$ is a metric space (S, d)

is a Cauchy seq. if $\forall \varepsilon > 0, \exists N > 0$. s.t. $d(P_n, P_m) < \varepsilon \quad \forall n, m \geq N$

Th'm Let $S_1 \supseteq S_2 \cdots \supseteq S_n \supseteq \cdots$ be a seq. of non-empty closed subsets of a compact set A in a metric space (S,d) Then $\bigcap_{m=1}^{\infty} S_m$ is non-empty

pf). Pick a pt (arbitrary) $P_n \in S_n$ $\{P_n\} \in A \Rightarrow \{P_n\}$ has a conv. subseq. $\{P_{n_k}\}$

i.e. $P_{n_k} \to P_0 \in A$

$\forall m$, $\{P_{n_k} ; n_k \geq m\} \subseteq S_m \Rightarrow P_0 \in S_m$ ($\because S_m$ is closed)

$\therefore P_0 \in \bigcap_{m=1}^{\infty} S_m$

Def $A \subseteq S$ is said to be totally bdd if $\forall \delta > 0$, $\exists$ finite # of points $p_1, \cdots p_n$ in A s.t. $A \subseteq \bigcup_{k=1}^{\infty} B(P_k, \delta)$

Rk A is totally bdd $\Rightarrow$ A is bdd

> Prove that a metric space is totally bounded if and only if every sequence has a Cauchy subsequence.

Bolzano-Weierstrass is stating boundness w.r.t. the usual metric in $\mathbb{R}^n$ is the same as total boundness

Lemma if A is compact in S, then A is totally bdd

(Pf.) Pick $P_1 \in A$ $B(P, \delta) \supseteq A$ we're done

Otherwise, pick $P_2 \in A \setminus B(P_1, \delta)$ if $A \subseteq B(P_2, \delta) \cup B(P_1, \delta)$, we're done

If this process stops after finite # of iterations we're done

Otherwise, we have a seq. $P_1, P_2, \cdots$ in A with the property that $d(p_i, p_j) \geq \delta$ $\forall i \neq j$

$\therefore \{P_n\}$ cannot have a conv. subseq. Contradict!

Theorem 6.21. ***In any metric space a closed subset of a compact set is compact.***

B A

its a closed subset. So every $\{P_n\}$ in B must have a conv. subseq. Other wise $\{P_n\}$ also in A, then A is not compact. Suppose B is not compact $\{P_{n_k}\} \to P_0 \notin B$ B is a limit point by def. the B is not closed $\to$ ✗

Th'm (Heine-Borel)

Let A be a compact set in a metric space (S,d) & $\mathcal{F}$ be a family of open sets which covers A. Then $\mathcal{F}$ has a finite subcover (i.e. a finite subfamily that covers A)

Pf is important!

(Pf.) Suppose: $\mathcal{F}$ does not have a finite subcover

Pick $\delta = \frac{1}{2}$ $\exists P_{11}, P_{12}, \cdots P_{1k} \in A$ & $A \subseteq \bigcup_{i=1}^{k_1} B(P_{1i}, \frac{1}{2})$

One of $A \cap \overline{B(P_{1i}, \frac{1}{2})}$ cannot have a finite subcover

say, $A \cap \overline{B(P_1, \frac{1}{2})}$ which is compact

So, F is an open cover for $A \cap \overline{B(P_1, \frac{1}{2})}$

Pick $\delta = \frac{1}{2^2}$, $\exists P_{21}, P_{22}, \cdots P_{2k}$ s.t.

$A \cap \overline{B(P_1, \frac{1}{2})} \subseteq \bigcup_{i=1}^{k_2} B(P_{2i}, \frac{1}{2^2})$

Likewise one of $A\cap\overline{B(P_1,\frac{1}{2})}\cap\overline{B(P_{2_i},\frac{1}{2^2})}$

doesn't have a finite subcover, denoted

by $A\cap\overline{B(P_1,\frac{1}{2})}\cap\overline{B(P_2,\frac{1}{2^2})}$

Repeating the process, we have: $P_1, P_2, \cdots P_n, \cdots$

s.t. $A\cap\overline{B(P_1,\frac{1}{2})}\cap\cdots\cap\overline{B(P_n,\frac{1}{2^n})}$ doesn't have a finite subcover

Observe $d(P_n, P_{n+1})\le\frac{1}{2^n}\Rightarrow\{P_n\}$ is Cauchy

$\forall m>n$ $d(P_n,P_m)\le\frac{1}{2^n}+\frac{1}{2^{n+1}}+\cdots+\frac{1}{2^m}\le\frac{1}{2^{n-1}})$

$\Rightarrow P_n\to P_0\in A$

$\mho$ is one element from the open cover

Since F is an open cover $\exists\mho\in F$ s.t. $P_0\in\mho$ $\therefore$ $\exists\,\varepsilon>0$. s.t. $B(P_0,\varepsilon)\subseteq\mho$ & then $\overline{B(P_n,\frac{1}{2^n})}\subseteq B(P_0,\varepsilon)\subseteq\mho$

$\forall n$ large $A\cap\overline{B(P_1,\frac{1}{2})}\cap\cdots\cap\overline{B(P_n,\frac{1}{2^n})}\subseteq\mho$

Lebesgue Lemma: Let $A\subseteq S$ be cpt & F be an open cover of A Then $\exists\,\rho>0$ s.t $\forall p\in A$

$B(P,\rho)\subseteq$ One of the open sets in F

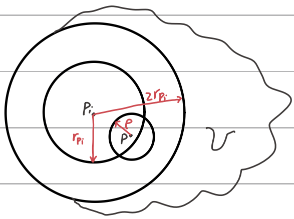
(Pf). $\forall p\in A$ $\exists\,\mho\in F$ s.t. $p\in\mho$

$\therefore\exists r_p>0.$ s.t. $B(P,2r_p)\subseteq\mho$

$\tilde{F}=\{B(P,r_p)\mid P\in A\}$ is an open cover

H-B thm $\Rightarrow\exists$ a finite subcover $\{B(P_1,r_{P_1}),\cdots,B(P_n,r_{P_n})\}$

set $\rho=\min\{r_{P_1},\cdots r_{P_n}\}>0$



Claim: $\forall p\in A$, $B(P,\rho)\subseteq B(P_i,2r_{P_i})$ for some i $\quad d(P,P_i)\le\rho+r_{P_i}\le 2r_{P_i}$

Def. Let $A\subseteq S$ & $f: A\to R$

We say that $f(p)\to l$ as $p\to P_0$ in A

if (i) P_0 is a limit pt. of A &

(ii) $\forall\varepsilon>0.\exists\,\delta>0$ s.t. $|f(p)-l|<\varepsilon$ for all $p\in A$ with $0<d(P,P_0)<\delta$ ($\lim_{\substack{p\to P_0\\ p\in A}} f(p)=l$)

Def. We say that f is cont. at P_0 w.r.t. A if $P_0\in A$ & $\lim_{\substack{p\to P_0\\ P\in A}} f(p)=f(P_0)$

We say that f is cont. on A if it is cont. w.r.t. A at every pt. in A

Extreme-Value Th'm : Suppose $f: A \to \mathbb{R}$ is cont. & A is cpt. Then $\sup_A f$ & $\inf_A f$ (both) are assumed

Unif. Cont. Suppose that A is cpt. & $f: A \to \mathbb{R}$ is cont. w.r.t. A. Then f is unif. cont. on A

3 Let $f: (S_1, d_1) \to (S_2, d_2)$ we say f is cont. at $P_0 \in S_1$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $d_2(f(P), f(P_0)) < \varepsilon$

$\forall P \in S_1$ with $d_1(P, P_0) < \delta$

Th'm Let $f: (S_1, d_1) \to (S_2, d_2)$. Then f is cont $\Leftrightarrow$ for every open set U in S_2, $f^{-1}(U)$ is open in S_1 pre-image of U

$f^{-1}(A) = \{ x \in S_1 \mid f(x) \in A \}$

(Pf) ($\Rightarrow$) U is open in S_2, need to show $f^{-1}(U)$ (i.e. $f(P) \in U$) is open $\forall P \in f^{-1}(U)$ $\exists B(P, \delta) \subseteq f^{-1}(U)$

$\exists B(f(P), \varepsilon) \subseteq U$ for some $\varepsilon > 0$

$\exists \delta > 0$, s.t. $d_2(f(Q), f(P)) < \varepsilon$ $\forall d_1(Q, P) < \delta$ (by the continuity of f)

i.e. $f(B(P, \delta)) \subseteq B(f(P), \varepsilon) \subseteq U$

i.e. $B(P, \delta) \subseteq f^{-1}(U)$

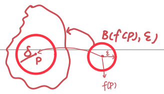
($\Leftarrow$) $\forall \varepsilon > 0$. $\exists \delta > 0$, s.t. $d_2(f(P), f(Q)) < \varepsilon$ $\forall P, Q \in S_1$ with $d_1(P, Q) < \delta$

Given $\varepsilon > 0$. $B(f(P), \varepsilon)$ is open in S_2 U

$\Rightarrow f^{-1}(B(f(P), \varepsilon))$ is open in S_1

i.e. $B(P, \delta) \subseteq f^{-1}(B(f(P), \varepsilon))$ because $P \in f^{-1}(B(f(P), \varepsilon))$

$f(B(P, \delta)) \subseteq B(f(P), \varepsilon)$



if $d_1(P, Q) < \delta$ then $d_2(f(Q), f(P)) < \varepsilon$

Let (S_1, d_1) and (S_2, d_2) be metric spaces and suppose that f is a mapping on S_1 into S_2. Show that f is continuous if and only if for every closed set A in S_2, the set $f^{-1}(A)$ is closed in S_1.

(Pf). ($\Rightarrow$) $\forall V \in S_2$ V is a closed set, then $S_2 - V$ is an open set. $f^{-1}(S_2 - V) = S_1 - f^{-1}(V)$ is an open set

So $f^{-1}(V)$ is closed

($\Leftarrow$) $\forall V \in S_2$ is an open set, $S_2 - V$ is closed set $\Rightarrow$ $f^{-1}(S_2 - V) = S_1 - f^{-1}(V)$ is closed, so

$f^{-1}(V)$ is open. Then f is cont.

Boundness: consider $b(x,y) = \min\{d(x,y), 1\}$ b & d produce exactly the same open sets in X

not arbitrary metric space have cauchy subseq

> Let (X, d) be a metric space. The following are equivalent
>
> - (X, d) is complete.
> - (Nested Set Property) Any decreasing sequence of closed sets $F_1 \supseteq F_2 \supseteq \cdots$ in X with $\operatorname{diam}(F_i) \to 0$ has $\bigcap_{i=1}^{\infty} F_i \neq \emptyset$ (in fact the intersection contains exactly one point).
> - (Bolzano–Weierstraß metric space formulation) Every infinite, totally bounded subset of X has a limit point in X.

9.25 Wed.

Example 1. $f: (-1, 1) \to R$ by $f(x) = x^2$ $[0,2]$ closed in R, $f^{-1}([0,2]) = \underset{\text{closed}}{(-1,1)}$ $f^{-1}((-1,1)) = \underset{\text{open}}{(-1,1)}$

$S_1 = (-1,1)$ the whole space (closed & open)

Cont. fcn. doesn't map open/closed set to open/closed set

$f(x) = x^2$ $f((-\frac{1}{2}, \frac{1}{2})) = [0, \frac{1}{4})$

$f(x) = \arctan x$ $f(\underset{\text{closed}}{(-\infty, \infty)}) = (-\frac{\pi}{2}, \frac{\pi}{2})$

Theorem 6.42. *Let A be a subset of a metric space (S_1, d_1), and suppose f is a continuous function on A into a metric space (S_2, d_2).*

(a) *If A is compact, then $f(A)$ is compact.*
(b) *If A is connected, then $f(A)$ is connected.*
(c) *If A is compact, then f is uniformly continuous on A.*
(d) *If A is compact and f is one-to-one, then f^{-1} is continuous.*
(e) If A is open, then $f^{-1}(A)$ is open
(f) If A is closed, then $f^{-1}(A)$ is closed

Def. Let A be a subset of a metric space (S,d). Consider (A,d) as a metric (sub)space. A set $C \subseteq A$ is open in A if C is open when considered as a set in (A,d) (Similiarly as "closed")

Example $A=\{0,1,2,3,\cdots\} \subseteq R$

是在原集合上取点, 取一个 ball $C=\{2,4\}$ closed in $\mathbb{R}$, not open in $\mathbb{R}$ $B_R(2,\varepsilon) \subseteq \{2,4\}$ X

closed in A, open in A $B_A(2,\varepsilon)=\{2\} \subseteq C$

Prop. Suppose that (A,d) is a metric subspace of (S,d) & $C \subseteq A$. Then

(i) C is open in $A \Leftrightarrow \exists$ a set G open in S s.t. $C=A \cap G$

(ii) C is closed in $A \Leftrightarrow \exists$ a set F closed in S s.t. $C=A \cap F$

(iii) C is connected in $A \Leftrightarrow C$ is connected in S ("connectedness" is intrinsic)

Theorem 6.42. *Let A be a subset of a metric space (S_1, d_1), and suppose f is a continuous function on A into a metric space (S_2, d_2).*

(a) *If A is compact, then $f(A)$ is compact.*
(b) *If A is connected, then $f(A)$ is connected.*
(c) *If A is compact, then f is uniformly continuous on A.*
(d) *If A is compact and f is one-to-one, then f^{-1} is continuous.*

Let $(A,d_1) \subseteq (S_1,d_1)$ & $f:(S_1,d_1) \to (S_2,d_2)$ be cont.

Then (i) A is cpt $\Rightarrow$ so in $f(A)$ 反过来不对 e.g. $f:(-1,1) \to R$ $f(x)=1$ $f^{-1}(1)=(-1,1)$

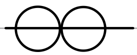
(ii) A is connected $\Rightarrow$ so is $f(A)$

Connectedness

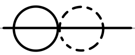
(a) **A set S in a metric space is connected $\Leftrightarrow$ S is not the union of two nonempty disjoint subsets, each of which is open in S.**
(b) **A set S in a metric space is connected $\Leftrightarrow$ the only subsets of S which are both open and closed in S are S itself and the empty set, $\varnothing$.**

Example

$A=B(0,1) \cup B(2,1)$



$C=\overline{B(0,1)} \cup \overline{B(2,1)}$



$D=\overline{B(0,1)} \cup B(2,1)$

Def. a set $A \subseteq S$ is connected if A cannot be represented as the union of two non-empty, disjoint sets neither contains a limit point of the other

A is not connected $\Leftrightarrow A=\Sigma_1 \cup \Sigma_2$ $\overline{\Sigma_1} \cap \Sigma_2 = \Sigma_1 \cap \overline{\Sigma_2} = \phi$

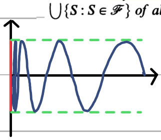
path-connected $\Rightarrow$ connected

7. Prove that if A is a connected set in a metric space and $A \subset B \subset \bar{A}$, then B is connected.

Theorem 6.33. *Let $\mathscr{F}$ be any family of connected subsets of a metric space X such that any two members of $\mathscr{F}$ have a common point. Then the union $A = \bigcup\{S : S \in \mathscr{F}\}$ of all the sets in $\mathscr{F}$ is connected.*

Example

$A=\{(x,y) \in \mathbb{R}^2 \mid y=\sin\frac{1}{x}, x>0\} \cup \{(0,y) \mid -1 \le y \le 1\}$



Connected but not path-connected

Intermediate-Value Th'm

Let $f: A \to \mathbb{R}$ be continuous where A is a connected set in (S,d). Then the range of f, $R(f)$ is either a pt. in $\mathbb{R}$ or an interval in $\mathbb{R}$

(pf) if f is a constant fcn $\Rightarrow R(f)$ is a pt.

Otherwise $\exists y_1, y_2 \in R(f)$ s.t. $y_1 < y_2$

Claim: $(y_1, y_2) \subseteq R(f)$

Suppose not. i.e. suppose $\exists c \in (y_1, y_2) \setminus R(f)$

$f^{-1}(-\infty, c) = \{x \in A \mid f(x) < c\}$ open $\neq \phi$

$f^{-1}(c, \infty) = \{x \in A \mid f(x) > c\}$ open $\neq \phi$

$A = f^{-1}(-\infty, c) \cup f^{-1}(c, \infty)$ $f^{-1}(-\infty, c) \cap f^{-1}(c, +\infty) = \phi$

$\Rightarrow A$ is not connected $\times$

9.30 Mon.

complete + totally bdd = compactness

Def. A metric space (S,d) is complete if every Cauchy seq. conv.

Example 1. $\mathbb{R}$ is complete 2. $\mathbb{Q}$ is not complete

As it happens $\mathbb{R}^n$ with its usual metric is complete. And a further theorem tells is that a subset of a complete space is itself complete if and only if it's closed in the larger space. This is the reason that being closed in $\mathbb{R}^n$ is such a special property.

3. $\mathscr{C}[a,b]$ = the set of all cont. fcns from $[a,b]$ to $\mathbb{R}$, equipped with the metric $d(f,g) = \sup_{a \le x \le b} |f(x) - g(x)|$

is a complete metric space

4. $(0,1)$ is incomplete

5. $[0,1]$ is complete

5. Let $I = \{x : 0 \leqslant x \leqslant 1\}$ be the unit interval in $\mathbb{R}^1$ and let every $x \in I$ be represented in a ternary expansion:

$$x = 0 \cdot a_1 a_2 a_3, \ldots,$$

where each a_i has the value 0, 1, or 2. Let A be the subset of I such that every point of A has only zeros or twos in its expansion.

(a) Show that A is an uncountable set.

(b) Show that A is nowhere dense in I.

6. Any closed subset of a complete metric space is complete

Def. Let A be a subset of a metric space (S,d). We say that

$\forall$ ball contains pts in A

(1) A is dense in S if $\bar{A} = S \Leftrightarrow \forall \varepsilon > 0, P \in S, \exists q \in A$ s.t. $q \in B(P, \varepsilon) \Leftrightarrow \forall p \in S, \exists \{q_n\} \subseteq A$, s.t. $\{q_n\} \to p$

(2) A is nowhere dense in S if $\bar{A}$ contains no balls of S (i.e. $\mathring{\bar{A}} = \emptyset$)

Rk (i) A is nowhere dense $\Leftrightarrow \bar{A}$ is nowhere dense

For example, let S be $\mathbb{R}^1$, let A be all the rational numbers, and let B be all the irrational numbers. Since $\bar{A} = \mathbb{R}^1$ we see that $\bar{A} \supset B$ and so A is dense in B. Note that A and B are disjoint.

(ii) "Dense" "nowhere dense" are NOT complementary e.g. $(0,1)$ in $\mathbb{R}$ neither dense nor nowhere dense

Lemma (i) A is nowhere dense in S

(ii) Every open ball B of S contains an open ball B_1 s.t. $B_1 \cap A = \emptyset$

(iii) Every open ball B of S contains an open ball B_1 s.t. $\overline{B_1} \cap \overline{A} = \emptyset$

are equiv.

(pf) (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)

(pf) (i) $\Rightarrow$ (iii)

Let B be an open ball in S

A : nowheredense $\Rightarrow B \setminus \overline{A} \neq \emptyset$ because $\overline{A}$ contains no ball in S

$\Rightarrow \exists$ an open ball $\tilde{B}_1 \subseteq B \setminus \overline{A}$ 开集挖掉一个闭集，剩下开集. $P_0 \in B_0 \setminus \overline{A}$ $\exists B(P_0, \varepsilon) \subseteq B_0 \setminus \overline{A}$

Suppose $\tilde{B}_1$ has radius r_1 centered at p_1 we let B_1 be the ball of radius $r_1/2$ centered at p_1

then $\overline{B_1} \cap \overline{A} = \emptyset$

(ii) $\Rightarrow$ (i) Suppose that A is NOT nowhere dense.

Then $\overline{A}$ contains an open ball B_0 (ii) guarantees that B_0 contains an open ball B_1

s.t. $B_1 \cap A = \emptyset$, this in turn implies that $B_1 \cap \overline{A} = \emptyset$ ($\because B$ is open$\cdots$)

$\Rightarrow B_1 \not\subseteq \overline{A}$ Contradiction

Def. (1) $A \subseteq S$ is said to be of 1^{st} Category if A is a countable union of nowhere dense Subset e.g. $\mathbb{Q}$

(2) $A \subseteq S$ is said to be of 2^{nd} Category if it is NOT of the 1^{st} Category

Baire Category Thm https://math.stackexchange.com/questions/165696/your-favourite-application-of-the-baire-category-theorem

(1) A complete metric space S is of the 2^{nd} Category

Suppose the open set $A \Rightarrow$ is of 1^{st} category, then $S \setminus A$ is dense $\Leftrightarrow \forall$ ball contains a point in $S \setminus A$. However, $P \in A$ $\exists B(P, \varepsilon) \subseteq A$, $B(P, \varepsilon)$ doesn't contain any pts in $S \setminus A$ ※

(2) A nonempty open set in a complete metric space is of the 2^{nd} Catogory

(3) if A is of 1^{st} Catogory in a complete metric space S. Then $S \setminus A$ is dense in S.

(pf). $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is nowhere dense, $\forall n$ Let $B_0 = B(P_0, r_0)$ be an arbitrary open ball in S

As A_1 is nowhere dense, Lemma (iii) $\Rightarrow B_1 = B(P_1, r_1) \subseteq B$ & $\overline{B_1} \cap \overline{A_1} = \emptyset$ W.L.O.G we assume $r_1 \leq r_0/2$

and $\overline{B_1} \subseteq B_0$ Now, repeating the argument for B_1, we have,

$\exists\, B_2 = B(P_2, r_2)$ s.t. $\overline{B_2} \cap \overline{A_2} = \phi$ & $\overline{B_2} \subseteq B_1$ with $r_2 \le r_1/2$

Continue this way we obtain $B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots B_n \supseteq \cdots$ $\underset{B(P_n, r_n)}{\shortparallel}$

$r_{n+1} \le r_n/2$ & $\overline{B_{n+1}} \subseteq B_n$ with $\overline{B_n} \cap \overline{A_n} = \phi$

$d(P_n, P_m) \le r_n \le \frac{r_0}{2^n}$ $\forall m > n \Rightarrow \{P_n\}$ is cauchy

$P_n \Rightarrow q \in S$ (S is complete)

Since $P_m \in \overline{B_n}$ $\forall m \ge n \Rightarrow q \in \overline{B_n}$ $\forall_n \Rightarrow q \notin A_n \Rightarrow q \notin A$

i.e. $q \in S \setminus A$ & $q \in B_0$ ($\because q \in \overline{B_1} \in B_0$)

$\forall$ ball contains a point in $S\setminus A$ so $S\setminus A$ is dense

E_n is nowhere dense $\Leftrightarrow E_n$ is closed, $S \setminus E_n$ is dense

If A is of first category then $A^\circ = \phi$ (In complete metric space) (If not $p \in A^\circ$, $B(P, \varepsilon) \subseteq A^\circ$ A contains a open ball)

Every countable intersection of dense open sets is dense

(pf. $\bigcap_{n=1}^{\infty} A_n$ A_n is dense open set, so $S \setminus A_n$ is closed $\Rightarrow E_n = S - A_n$ is nowhere dense

$\bigcup_{n=1}^{\infty} E_n$ is of first category $\Rightarrow S - \bigcup_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} (S - E_n)$ is dense)

A topological space X is called a **Baire space** if it satisfies any of the following equivalent conditions:[1] [7] [8]

1. Every countable intersection of dense open sets is dense.
2. Every countable union of closed sets with empty interior has empty interior.
3. Every meagre set has empty interior.
4. Every nonempty open set is nonmeagre.[note 1]
5. Every comeagre set is dense.
6. Whenever a countable union of closed sets has an interior point, at least one of the closed sets has an interior point.

Proposition: Assuming BCT holds, every non-empty complete metric space is a Baire space.

Proof: Suppose by way of contradiction that X is a non-empty complete metric space, but isn't a Baire space. Then by Lemma 2, there is some countable collection $\{A_n\}_{n=1}^{\infty}$ of closed nowhere-dense subsets of X whose union doesn't have empty interior--meaning there is some non-empty open U such that

$$U \subseteq \bigcup_{n=1}^{\infty} A_n.$$

Now, in particular, we can take a non-empty open set V such that $\overline{V} \subseteq U$. (Why?) Then

$$\overline{V} \subseteq \bigcup_{n=1}^{\infty} A_n,$$

so

$$\overline{V} = \overline{V} \cap \left(\bigcup_{n=1}^{\infty} A_n \right) = \bigcup_{n=1}^{\infty} (\overline{V} \cap A_n). \qquad (\#)$$

But $\overline{V}$ is a non-empty complete metric space by Lemma 3, so by BCT, $\overline{V}$ is not a countable union of closed nowhere-dense subsets of $\overline{V}$. But each $\overline{V} \cap A_n$ is closed and nowhere-dense in $\overline{V}$ (why?), so $(\#)$ gives us the desired contradiction. □

Def (1) The oscillation of a fcn $f: E \to \mathbb{R}$ is defined as $w(f, E) = \sup_{x,y \in E} |f(x) - f(y)|$

(2) The oscillation of a fcn f at a single point x_0 is defined as $w(f, x_0) = \lim_{\delta \to 0} w(f, B(x_0, \delta))$

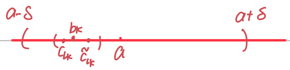
Rk f is cont. at $x_0 \Leftrightarrow w(f, x_0) = 0$

Prop. Suppose that $f: \mathbb{R} \to \mathbb{R}$ is cont. on a dense subset of $\mathbb{R}$. Then the set of all discontinuous points of f must be of 1^{st} category

(Pf). Denote the set of all discontinuous pts of f by D. Then $D = \bigcup_{n=1}^{\infty} D_n$ where

$D_n = \{x \in \mathbb{R} \mid w(f, x) \geq \frac{1}{n}\}$ 条件的转化

Observe: D_n is closed



claim: D_n is nowhere dense in $\mathbb{R}$

Suppose for contradiction, for some n fixed D_n is NOT nowhere dense; i.e. $\overline{D_n}$ contains an open interval I

Since f is cont. on a dense subset of $\mathbb{R}$ we conclude that $\exists\, a \in I$ s.t. $w(f, a) = 0$

$a \in \overline{D_n} \Rightarrow a$ is a limit pt

& $\exists$ a seq. $\{b_k\} \subseteq D_n$ & $b_k \to a$ $\quad$ $\exists$ a seq. $\{x_k\} \subseteq \overline{D_n}$, & $x_k \to a$ $\quad$ a is in an open ball so a is a limit point of $\overline{D_n}$, then a is a limit point of D_n, so $\exists$ seq $\{b_n\} \to a$ $b_n \in B_n$

as f is cont. at a, $\exists\, \delta > 0$ s.t.

$|f(x) - f(a)| < \frac{1}{4n} \quad \forall |x-a| < \delta$

Since $b_k \to a$, $b_k \in (a-\delta, a+\delta)$ $\forall k$ large

Since $b_k \in D_n$, $w(f, b_k) \geq \frac{1}{n}$

$\Rightarrow \exists\ C_{k_l}, \tilde{C}_{k_l} \to b_k$ as $l \to \infty$ s.t.

$|f(C_{k_l}) - f(\tilde{C}_{k_l})| \geq \frac{1}{2n}$

$|f(C_{k_l}) - f(a)| < \frac{1}{4n}$ $\forall l$ large

$|f(C_{k_l}) - f(\tilde{C}_{k_l})| \leq |f(C_{k_l}) - f(a)| + |f(a) - f(\tilde{C}_{k_l})| \approx \frac{1}{2n}$

Theorem 15.4. *Let S_1, S_2 be metric spaces and suppose that S_2 is complete. Let A be a subset of S_1 and $f: A \to S_2$ be a mapping which is uniformly continuous on A. Then there is a unique mapping $f^*: \bar{A} \to S_2$ which is uniformly continuous on $\bar{A}$ and such that $f^*(p) = f(p)$ for all $p \in A$.*

Def. Let $A \subseteq$ a metric space (S,d). We define $\varphi(p) := d(p,A) = \inf\{d(p,a) \mid a \in A\}$

$d(p,A)$ is cont. but may not be differentiable

$\sqrt{x^2+y^2} - 1$

$|\varphi(p) - \varphi(q)| < d(p,q)$ unif cont.

prop1. $d(p,A)$ is cont. & $|d(p,A) - d(q,A)| \leq d(p,q)$ $\forall p,q \in S$

(pf) $\forall \varepsilon > 0, \exists a \in A$ st. $d(p,a) < d(p,A) + \varepsilon$

$\forall q \in S$, we have $d(q,A) \leq d(q,a) \leq d(q,p) + d(p,a) < d(q,p) + d(p,A) + \varepsilon$

$\Rightarrow d(q,A) - d(p,A) < d(p,q) + \varepsilon$ $\quad d(p,A) - d(q,A) < d(q,p) + \varepsilon$

Prop2. Let A, B be two closed sets in (S,d). If $A \cap B = \phi$, then $\exists$ a cont. fcn $\psi: S \to [0,1]$

s.t. $\psi \equiv 0$ on A & $\psi \equiv 1$ on B

pf. $\psi(p) = \dfrac{d(p,A)}{d(p,A) + d(p,B)}$ verify the positivity $d(p,A) + d(p,B)$

(Rk $d(p,A) = 0 \Leftrightarrow p \in A$ Since A is closed)

Tietze Extension th'm

Let A be a $\boxed{\text{closed}}$ set in a metric space (S,d) & $f: A \to \mathbb{R}$ be a bdd cont. fcn. Then $\exists$ a cont.

fcn $g: S \to \mathbb{R}$ s.t. $g \equiv f$ on A & $|g| \leq \sup_A |f|$ on S

(pf) assume that $\sup_A |f| \leq M$ & $M > 0$

Set $f_1 \equiv f$ on A & let $A_1 = \{p \in A \mid f_1(p) \leq -\frac{1}{3}M\}$ $\quad B_1 = \{p \in A \mid f_1(p) \geq \frac{1}{3}M\}$

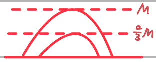
$\Rightarrow A_1 \cap B_1 = \phi$ & both are closed

By prop2, $\exists \psi \in C(S, \mathbb{R})$ s.t. $\psi_1 \equiv -\frac{1}{3}M$ on A_1 & $\equiv \frac{1}{3}M$ on B_1 $\quad |\psi_1| \leq \frac{1}{3}M$ on S

$\psi_1(p) = \frac{2}{3}M(\psi(p) - \frac{1}{2})$

(on A)

Set $f_2 = f_1 - \varphi_1 \Rightarrow |f_2| \le \frac{2}{3}M$



Repeating this argument

$\varphi_2 \equiv -\frac{1}{3}(\frac{2}{3}M)$ on $A_2 = \{P \in A \mid f_2(P) \le -\frac{1}{3}(\frac{2}{3}M)\}$

$\varphi_2 \equiv \frac{1}{3}(\frac{2}{3}M)$ on $B_2 = \{P \in A \mid f_2(P) \ge \frac{1}{3}(\frac{2}{3}M)\}$

& $|\varphi_2| \le \frac{1}{3}(\frac{2}{3}M)$

Finally

$\varphi_n \equiv -\frac{1}{3}(\frac{2}{3})^{n-1}M$ on $A_n = \{P \in A \mid f_n(P) \le -\frac{1}{3}(\frac{2}{3})^{n-1}M\}$

$\frac{1}{3}(\frac{2}{3})^{n-1}M$ on $B_n = \{P \in A \mid f_n(P) \ge \frac{1}{3}(\frac{2}{3})^{n-1}M\}$

& $f_{n+1} = f_n - \varphi_n \Rightarrow |f_{n+1}| \le (\frac{2}{3})^n M$

Let $g_n = \varphi_1 + \varphi_2 + \cdots + \varphi_n \in C(S, \mathbb{R})$

Claim: $\{g_n\}$ conv. unif. on S

$|g_m - g_n| = |\varphi_{n+1} + \cdots + \varphi_m| \le \frac{1}{3}(\frac{2}{3})^n[1 + \frac{2}{3} + (\frac{2}{3})^2 + \cdots]M = (\frac{2}{3})^n M$ $(m > n)$

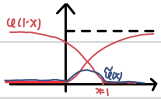
Let $g = \lim g_n \in C(S, \mathbb{R})$

$g_n = f_1 - f_2 + f_2 - f_3 + \cdots + f_n - f_{n+1} = f_1 - f_{n+1} \longrightarrow f_1$ as $n \to \infty$

10.14 Mon.

Approximating Cont. fcns.

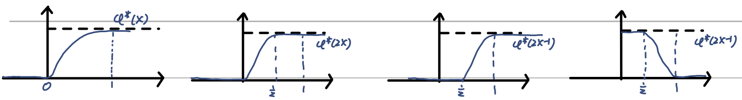
$\varphi(x) = \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$



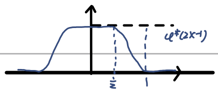
$\varphi \in C^\infty(\mathbb{R})$

$\tilde{\varphi}(x) = \varphi(x)\varphi(1-x)$ is C^∞ & $\tilde{\varphi} \equiv 0$ on $\mathbb{R} \setminus (0,1)$

set $\varphi^*(x) = \dfrac{\int_0^x \tilde{\varphi}(t)dt}{\int_0^1 \tilde{\varphi}(t)dt} = \begin{cases} 0 & x \le 0 \\ 1 & x \ge 1 \end{cases}$



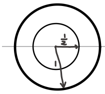
$\psi(x) = \begin{cases} 1 - \varphi^*(2x-1) & x \ge 0 \\ \text{even} & x < 0 \end{cases}$



Definitions. Let $f: I \to \mathbb{R}^1$ be infinitely differentiable at a point $a \in I$ and suppose that the series (9.15) has a positive radius of convergence. Then f is said to be **analytic at** a. A function f is **analytic** on a domain if and only if it is analytic at each point of its domain.

(i) $\psi(x)=\psi(x_1,\cdots,x_n)=\psi(x_1)\cdots\psi(x_n)$

(ii) $\psi_n(x)=\psi(|x|)=\psi(r)$ $\quad r\ge 0$

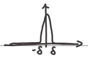


$\equiv 0$ $\quad$ set $\rho(x)=\dfrac{\psi_n(x)}{\int_{\mathbb{R}^n}\psi_n(x)}$ $\quad \left(\int_{\mathbb{R}^n}\rho(x)dx=1\right)$

mollifier

Def. A (any) nonnegative fcn $\rho\in C^\infty(\mathbb{R}^n)$ s.t. $\rho\equiv 0$ outside $B(0,1)$ & $\int_{\mathbb{R}^n}\rho=1$ is called a mollifier

Rk For any $\delta>0$, $\rho_\delta(x)=\frac{1}{\delta^n}\rho(\frac{x}{\delta})$ 缩小δ倍

$\delta\to 0$  $\int_{\mathbb{R}^n}\rho_\delta=1$

当δ很小时 $\int_{B(x,\delta)}\rho_\delta(x-\xi)f(\xi)d\xi$

$f(\xi)\cong f(x)$, $f(x)\underbrace{\int_{B(x,\delta)}\rho_\delta(x-\xi)d\xi}_{\cong 1}\cong f(x)$

Thm Let $f:\mathbb{R}^n\to\mathbb{R}$ be a cont. fcn. & ρ be a mollifier on $\mathbb{R}^n$. Then for any $\delta>0$, $f_\delta(x)=\int_{\mathbb{R}^n}\rho_\delta(x-\xi)f(\xi)d\xi$

is C^∞ & $f_\delta\to f$ unif. on every compact subset in $\mathbb{R}^n$ as $\delta\to 0$

(pf) $f_\delta\in C^\infty(\mathbb{R}^n)$

claim: $f_\delta\to f$ on $B(0,R)$ unif ($\forall R$)

i.e. $\forall\varepsilon>0$ $\exists\delta_0>0$ s.t. $|f_\delta(x)-f(x)|<\varepsilon$ $\quad\forall x\in B(0,R)$ & $\delta<\delta_0$

Since f is unif cont. on $\overline{B(0,R+1)}$

$\exists 1>\delta_0>0$ s.t. $|f(x)-f(y)|<\varepsilon$ $\forall$ $|x-y|<\delta_0$ in $\overline{B(0,R+1)}$

take $\delta<\delta_0$

$$|f_\delta(x)-f(x)|=\left|\int_{\mathbb{R}^n}\rho_\delta(x-\xi)f(\xi)d\xi-f(x)\underbrace{\int_{\mathbb{R}^n}\rho_\delta(x-\xi)d\xi}_{=1}\right|\le\int_{\mathbb{R}^n}\rho_\delta(x-\xi)|f(\xi)-f(x)|d\xi=\int_{B(x,\delta)}|\rho_\delta(x-\xi)|\cdot|\underbrace{f(\xi)-f(x)}_{<\varepsilon}|d\xi<\varepsilon\int_{B(x,\delta)}|\rho(x-\xi)|d\xi=\varepsilon$$

$\forall x\in B(0,R)$

Corollary 1. Let $f\in C(A,\mathbb{R})$ where $A\subseteq\mathbb{R}^n$ is compact. Then $\exists$ a seq. $\{f_k\}\subseteq C^\infty(\mathbb{R}^n,\mathbb{R})$ s.t. $f_k\to f$ unif. on A

Corollary 2. Suppose $f\in C^k(A^\circ,\mathbb{R})$ where A° is the interior of A. Then f_δ & $D^\alpha f_\delta$ ($|\alpha|\le k$) conv. unif. to f & $D^\alpha f$

resp. on every compact subset of A° $\forall B(0,R)$

在 compact set 上 $f\in C^k$ 被 $\{f_n\}\in C^\infty$ 各阶逼近

Stone – Weierstrass Approximation

Set-up S: A (any) compact metric sp.

$C(S)$: all cont. fcn s $S\to\mathbb{R}$ $\quad (d(f,g)=\max_{x\in S}|f(x)-g(x)|$ sup norm

Def (i) $\mathcal{L} \subseteq C(S)$ is said to seperate points in S. if for each pair of pts. $p,q \in S$, $\exists f \in \mathcal{L}$ st. $f(p) \neq f(q)$

a can equal to b

(ii) $\mathcal{L} \subseteq C(S)$ is said to separate pts. in S & $\mathbb{R}$ if for every pair of pts. $p,q \in S$ & every pair of # $a,b \in \mathbb{R}$

$\exists f \in \mathcal{L}$ s.t $f(p)=a$ & $f(q)=b$

Strone Approx Th'm

Suppose that $\mathcal{L}$ has the following properties

(i) $\mathcal{L}$ seperates pts in compact S & $\mathbb{R}$

(ii) $f,g \in \mathcal{L} \Rightarrow \max\{f,g\}$ & $\min\{f,g\} \in \mathcal{L}$ Then

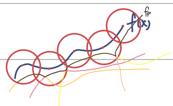
by metric $d(f,g) = \max_{p \in S} |f(p)-g(p)|$

All fcns. in $C(S)$ can be approximated uniformly by fcn. in $\mathcal{L}$

(pf.) $\forall p,q \in S, a,b \in \mathbb{R}, \exists f \in \mathcal{L}$ s.t. $f(p)=a$ $f(q)=b$

Now, let $f \in C(S)$ with $f(p)=a$, $f(q)=b$

$\exists g_{pq} \in \mathcal{L}$, s.t. $g_{pq}(p)=a$ & $g_{pq}(q)=b$ $\{g_{pq}\}$ 只有 p q 两点有限制



$\forall \varepsilon > 0$, Since g_{pq} & f are cont.

neighborhood

$\exists$ neighborhood $N(q)$ s.t. $\begin{cases} g_{pq}(s) > g_{pq}(q) - \frac{\varepsilon}{2} & \forall s \in N(q) \\ f(q) > f(s) - \frac{\varepsilon}{2} & \forall s \in N(q) \end{cases}$ $|g_{pq}(s) - g_{pq}(q)| < \frac{\varepsilon}{2}$

i.e. $g_{pq}(s) > g_{pq}(q) - \frac{\varepsilon}{2} = f(q) - \frac{\varepsilon}{2} > f(s) - \varepsilon$ $\forall s \in N(q)$ 这时 p,q are fixed

(we do this for every $q \in S$) $\{N(q) \mid q \in S\}$ is an open cover for S. S compact $\Rightarrow \exists q_1, \cdots q_n$, s.t.

$S \subseteq N(q_1) \cup \cdots \cup N(q_n)$ 这时 p fixed, vary q.

$\forall s \in S$ $\exists k$ s.t. $f(s) - \varepsilon < g_{pq_k}(s)$ $g_{pq_k}(p) = a$ $g_{pq_k}(q_k) = f(q_k)$

$g_p(p) = f(p)$ g_{pq_k} may not equal to $f(q_k)$ since $f(p_k) < g_{pq_i}(p_k)$ for some i

Set $\underbrace{g_p(s) = \max_{1 \le k \le n} \{g_{pq_k}(s)\}}_{\in \mathcal{L}} \Rightarrow g_p \in \mathcal{L}$ & $f(s) - \varepsilon < g_p(s)$ $\forall s \in S$

By same argument $\exists N(p)$ s.t $g_p(p) = a$ since $g_{pq_k}(p) = a$ for all q_k

$g_p(s) < f(s) + \varepsilon$ $\forall s \in N(p)$

Similarly, $\{N(p)\}$ Covers S & therefore, $\exists p_1, \cdots, p_m$ s.t. $S \subseteq N(p_1) \cup \cdots \cup N(p_m)$

& $\forall s \in S$ $\exists l$ s.t. $f(s) > g_{p_l}(s) - \varepsilon$

Set $g(s) = \min_{1 \le l \le m} \{g_{p_l}(s)\} \Rightarrow f(s) > g(s) - \varepsilon$

$f(s) + \varepsilon > g(s) > f(s) - \varepsilon$

Def. A set $A \subseteq \mathscr{C}(S)$ forms an algebra of fcns in $\mathscr{C}(S)$ if

(i) $f, g \in A \Rightarrow \alpha f + \beta g \in A \quad \forall \alpha, \beta \in \mathbb{R}$

(ii) $f, g \in A \Rightarrow f \cdot g \in A$

Stone-Weierstrass Approx Th'm

Corollary (Weierstrass approximation theorem). *Let S be a closed bounded set in $\mathbb{R}^N$. Then any continuous function on S can be approximated uniformly on S by a polynomial in the coordinates $x_1, x_2, \ldots, x_N$.*

Let S be a compact metric space & $A \subseteq \mathscr{C}(S)$ is an algebra which seperate pts in S, contains the constant fcn 1. Then any $f \in \mathscr{C}(S)$ can be approximated unif. by fcn. in A (A is dense in S)

P_n (poly) $\to h$ as $n \to \infty$ i.e. $\|P_n - h\| \to 0$ as $n \to \infty$

(pf.) let $\mathscr{L} \subseteq \mathscr{C}(S)$ be the subset which can be approximated by fcn. in A

Claim: (1) $\mathscr{L}$ Seperates pts. in S & $\mathbb{R}$ (A seperates S & $\mathbb{R}$)

(2) $f, g \in \mathscr{L} \Rightarrow \max\{f, g\}, \min\{f, g\} \in \mathscr{L}$

For (1), let $p \ne q$ in S & $a, b \in \mathbb{R}$, $\exists f \in A$ st. $f(p) \ne f(q)$

Solve $\begin{cases} \alpha f(p) + \beta \cdot 1 = a \\ \alpha f(q) + \beta \cdot 1 = b \end{cases}$ $\quad \begin{vmatrix} f(p) & 1 \\ f(q) & 1 \end{vmatrix} \ne 0$

$h = \alpha f + \beta \cdot 1$

$h \in \mathscr{L}$

$h(p) = a$

$h(q) = b$

$\boxed{h = \alpha \cdot f + \beta \cdot 1} \in A$

$f \in A$

$\max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$

$\max\{f, g\} = \frac{1}{2}(f + g - |f - g|)$

Suffices to show: $|f|$ can be approx. by poly. in f ($\because A$ is algebra)

Lemma 1: On $[-1, 1]$ $\quad 1 + \frac{1}{2} \sum_{n=0}^{N} \frac{(-1)^n (n+1)(2n)!}{2^{2n} [(n+1)!]^2} x^{n+1} \to \sqrt{x+1}$ unif as $N \to \infty$

Lemma 2: The fcn $|x|$ can be approx unif. on $[-M, M]$ by poly.

Pf set $u = x^2 - 1$ $\quad |x| = \sqrt{1+u}$ can be approximated by poly. in u (by lemma 1

if A is a dense subset of $C([0,1])$ (wrt uniform convergence, of course) and, for $f \in C([0,1])$, $\int_0^1 f(x) g(x) dx = 0 \quad \forall g \in A$ then $f = 0$.

Ok, two more hints: i) If A is the dense subset approximate f by functions $g_k \in A$ converging uniformly. Uniform convergence implies

$$0 = \int f g_k \to \int f^2$$

① A seperates S, $A \subseteq \mathscr{L} \Rightarrow \mathscr{L}$ Seperates S & $\mathbb{R}$

② $\forall f, g \in A$, $\max(f, g)$, $\min(f, g) \in A$ (and we get poly. $\subseteq A$)

$\forall f, g \in \mathscr{L}$ $f_n \to f$, $g_n \to g$ unif. $f_n, g_n \in A$, so $\max(f, g), \min(f, g) \in \mathscr{L}$ (Since f, g can be approximated by fcns in A)

$\mathscr{C}(S)$ can be approx by $\mathscr{L}$, $\mathscr{L}$ can be approx by A, so $\mathscr{C}(S)$ can be approx by A

Separating sets can be used to formulate a version of the Stone–Weierstrass theorem for real-valued functions on a compact Hausdorff space X, with the topology of uniform convergence. It states that any subalgebra of this space of functions is dense if and only if it separates points. This is the version of the theorem originally proved by Marshall H. Stone.[1]

Weierstrass approximation theorem — Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - p\| < \varepsilon$.

As a consequence of the Weierstrass approximation theorem, one can show that the space $\mathrm{C}[a, b]$ is separable: the polynomial functions are dense, and each polynomial function can be uniformly approximated by one with rational coefficients; there are only countably many polynomials with rational coefficients. Since $\mathrm{C}[a, b]$ is metrizable and separable it follows that $\mathrm{C}[a, b]$ has cardinality at most $2^{\aleph_0}$. (Remark: This cardinality result also follows from the fact that a continuous function on the reals is uniquely determined by its restriction to the rationals.)

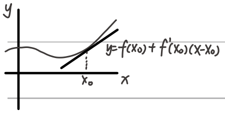
Stone starts with an arbitrary compact Hausdorff space X and considers the algebra $\mathrm{C}(X, \mathbf{R})$ of real-valued continuous functions on X, with the topology of uniform convergence. He wants to find subalgebras of $\mathrm{C}(X, \mathbf{R})$ which are dense. It turns out that the crucial property that a subalgebra must satisfy is that it *separates points*: a set A of functions defined on X is said to separate points if, for every two different points x and y in X there exists a function p in A with $p(x) \neq p(y)$. Now we may state:

Stone–Weierstrass theorem (real numbers) — Suppose X is a compact Hausdorff space and A is a subalgebra of $\mathrm{C}(X, \mathbf{R})$ which contains a non-zero constant function. Then A is dense in $\mathrm{C}(X, \mathbf{R})$ if and only if it separates points.

Mon 10.28

Def Let $f:\mathbb{R}\to\mathbb{R}$ if $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ exists, then we say that f is differentiable at x_0 & denote

$f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$ as its derivative at x_0



Physically, $f'(x_0)$ is the rate of change at x_0

Rolle's Thm

Suppose $f\in C[a,b]$ & it is differentiable in (a,b) if $f(a)=f(b)=0$ then $\exists\, x_0\in(a,b)$ s.t $f'(x_0)=0$

(Pf). Since $[a,b]$ is compact & $f\in C[a,b]$, we conclude that f assume its max & min at some

pts $x_0, \bar{x}_0$ in $[a,b]$

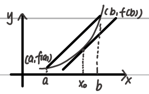
If $f(x_0)=f(\bar{x}_0)=0 \Rightarrow f\equiv 0$ on $[a,b] \Rightarrow f'\equiv 0$ on $[a,b]$

Otherwise, either $f(x_0)>0$ or $f(\bar{x}_0)<0$

W.L.O.G. assume $f(x_0)>0$ $\quad f'(x_0)=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=0$ $\quad f(x_0+h)-f(x_0)<0,\ h>0$; $f(x_0+h)-f(x_0)>0,\ h<0$ $\quad$ the number is 0

mean value theorem

Suppose $f\in C[a,b]$ & is differentiable in (a,b) Then $\exists\, x_0\in(a,b)$ s.t. $f'(x_0)=\frac{f(b)-f(a)}{a-b}$



Set $F(x)=f(x)-\frac{f(b)-f(a)}{b-a}(x-a)-f(a)$ $\qquad F(a)=0=F(b) \Rightarrow F'(x_0)=0$

$F'(x_0)=f'(x_0)-\frac{f(b)-f(a)}{b-a}=0$

Generalised mean value theorem

Suppose $f,g\in C[a,b]$ & f',g' exist in (a,b) with $g'(x)\neq 0$ $\forall x\in(a,b)$. Then $g(b)\neq g(a)$ & $\exists\, x_0\in(a,b)$ s.t. $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(x_0)}{g'(x_0)}$

(Pf) $\phi(x)=(f(x)-f(a))-\frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a))$

$\phi(a)=\phi(b)=0$ $\quad \phi'(x_0)=f'(x_0)-\frac{f(b)-f(a)}{g(b)-g(a)}g'(x_0)=0$

If $g(b)=g(a)$ M-V-T $\Rightarrow \exists\, x$ s.t. $g'(x_1)=0$

L'Hospital's Rule (%)

(1) Suppose $f,g\in C[a,b]$ with f' g' exist in (a,b) & $g'(x)\neq 0$ $\forall x\in(a,b)$ if $f(a)=g(a)=0$ & $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$

Then $\lim_{x\to a^+}\frac{f(x)}{g(x)}=L$

(pf) $\left|\frac{f(x)}{g(x)}-L\right| = \left|\frac{f(x)-f(x_0)}{g(x)-g(x_0)}-L\right| = \left|\frac{f'(x_0)}{g'(x_0)}-L\right|$ for some $x\in(a,x)$

as $x\to a^+ \Rightarrow x_0\to a^+ \Rightarrow \left|\frac{f'(x_0)}{g'(x_0)}-L\right|\to 0$

(2) Suppose $f,g\in C[a,\infty)$ with f' g' exist in (a,∞) & $g'(x)\neq 0$ $\forall x\in(a,\infty)$ if $\lim_{x\to\infty} f(x)=0=\lim_{x\to\infty} g(x)$

& $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=L$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=L$

(pf) set $\tilde{f}(z)=f(\frac{1}{z})$ $z\in(0,\frac{1}{a})$ & $\tilde{f}(0)=0$

$\tilde{g}(z)=g(\frac{1}{z})$ $z\in(0,\frac{1}{a})$ & $\tilde{g}(0)=0$

$\Rightarrow \tilde{f},\tilde{g}\in C[0,\frac{1}{a})$

$$\left.\begin{aligned}\tilde{f}'(z)&=f'(\tfrac{1}{z})\tfrac{-1}{z^2}\\ \tilde{g}'(z)&=g'(\tfrac{1}{z})\tfrac{-1}{z^2}\end{aligned}\right\}\quad \frac{\tilde{f}'(z)}{\tilde{g}'(z)}=\frac{f'(\frac{1}{z})}{g'(\frac{1}{z})}\to L \text{ as } z\to\infty \qquad \frac{\tilde{f}(z)}{\tilde{g}(z)}=\frac{f(\frac{1}{z})}{g(\frac{1}{z})}\to L$$

(3) L'Hospital's Rule (∞/∞)

Suppose f', g' exists in (a,b) & $\lim_{x\to a^+} f(x)=\lim_{x\to a^+} g(x)=\infty$. If $g'(x)\neq 0$ in (a,b) & $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$ then $\lim_{x\to a^+}\frac{f(x)}{g(x)}=L$

$\forall\varepsilon>0, \exists\delta>0$ s.t. $\left|\frac{f'(x)}{g'(x)}-L\right|<\varepsilon/2$ $\forall x\in(a,a+\delta)$

Fix $c\in(a,a+\delta)$ & consider $x\in(a,c)$

$\left|\frac{f(x)}{g(x)}-L\right| \le \left|\frac{f(x)}{g(x)}-\frac{f(x)-f(c)}{g(x)-g(c)}\right| + \left|\frac{f(x)-f(c)}{g(x)-g(c)}-L\right|$

$\left|\frac{f(x)-f(c)}{g(x)-g(c)}-L\right| = \left|\frac{f'(x_0)}{g'(x_0)}-L\right| < \frac{\varepsilon}{2}$ by Gen'd M-V thm & $x_0\in(x,c)$

$$\left|\frac{f(x)}{g(x)}-\frac{f(x)-f(c)}{g(x)-g(c)}\right| = \left|\frac{g(c)}{g(x)}-\frac{g(c)}{g(x)}\cdot\frac{f(x)-f(c)}{g(x)-g(c)}\right| \le \left|\frac{g(c)}{g(x)}\right| + \left|\frac{g(c)}{g(x)}\cdot\overbrace{\frac{f(x)-f(c)}{g(x)-g(c)}}^{\frac{f'(x_1)}{g'(x_1)}\ x_1\in(x,c)}\right| = \left|\frac{f(c)}{g(x)}\right| + \left|\frac{g(c)}{g(x)}\cdot\frac{f'(x_1)}{g'(x_1)}\right| \le \left|\frac{f(c)}{g(x)}\right| + \left|\frac{g(c)}{g(x)}\right|(|L|+\varepsilon/2)\to 0 \text{ as } x\to a^+$$

$-f(x)g(c)+f(c)g(c)-f(c)g(c)+g(x)f(c)$

$\Rightarrow \exists\tilde{\delta}>0$ s.t. $\left|\frac{f(c)}{g(x)}\right|+\left|\frac{g(c)}{g(x)}\right|\cdot(|L|+\varepsilon/2)\le\varepsilon/2$ $\forall x\in(a,a+\tilde{\delta})$

$\forall x\in(a,a+\min\{\delta,\tilde{\delta}\})$, we have $\left|\frac{f(x)}{g(x)}-L\right|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

Example Evaluate $\lim_{x\to 0}\frac{1+x-e^x}{2x^2}$

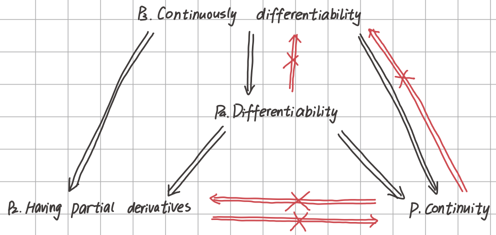
$e^x=1+x+\frac{x^2}{2}+\cdots$ $\quad 1+x-e^x=-\frac{x^2}{2}-\cdots$

$\frac{1+x-e^x}{2x^2}=-\frac{1}{4}+O(x)\to-\frac{1}{4}$

$\lim_{x\to 0}\frac{1+x-e^x}{2x^2} = \lim_{x\to 0}\frac{1-e^x}{4x} = \lim_{x\to 0}-\frac{e^x}{4}=-\frac{1}{4}$

exist

Inverse fcns. 4.2



A fcn is differentiable at (x_0, y_0) if

① $f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + a\Delta x + b\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \quad \varepsilon_1, \varepsilon_2 \to 0 \quad$ as $\Delta x, \Delta y \to 0$

② $f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + a\Delta x + b\Delta y + o(\sqrt{\Delta x^2+\Delta y^2}) \quad$ as $\Delta x, \Delta y \to 0$

③ $\dfrac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} \to 0$

(i) $P_1 \not\Rightarrow P_2$: $f(x,y) = \sqrt{x^2+y^2}$

$P_2 \not\Rightarrow P_1$: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

(ii) $P_1 \not\Rightarrow P_3 \quad f(x,y) = \sqrt{x^2+y^2}$

$P_3 \Rightarrow P_1$

(Pf). $|f(x_0+h, y_0+k) - f(x,y)|$

$\leq |f(x_0+h, y_0+k) - f(x_0+h, y_0)| + |f(x_0+h, y_0) - f(x_0, y_0)|$

$= |f_y(x_0+h, y_0+\theta_1 k)| \cdot k + |f_x(x_0+\theta_2 h, y_0)| \cdot h$

$\leq |f_x(x_0, y_0)| h + |f_y(x_0, y_0)| \cdot k + \varepsilon_1 h + \varepsilon_2 k$

$\varepsilon_1 = f_x(x_0+\theta_1 h, y_0+k) - f_x(x_0, y_0)$

$\varepsilon_2 = f_y(x_0, y_0+\theta_2 k) - f_y(x_0, y_0)$

(iii) $P_1 \not\Rightarrow P_4 \quad f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

$P_4 \Rightarrow P_1$

(V) $P_2 \not\Rightarrow P_4 \quad f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

(Vi) $P_4 \not\Rightarrow P_3$

$f(x,y) = \begin{cases} (x^2+y^2) \sin \dfrac{1}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2 = 0 \end{cases}$

$f: \mathbb{R}^N \to \mathbb{R}$

partial derivative

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x_1, x_2, \cdots, x_i+h, \cdots x_N) - f(x_1, x_2, \cdots, x_i, \cdots x_N)}{h}$$

$x=(x_1, x_2, \cdots, x_N)$ (Fix $x_1, \cdots x_N$ variables, except x_i, & view f as a fcn from $\mathbb{R} \to \mathbb{R}$ where x_i is the variable)

Example $f: \mathbb{R}^2 \to \mathbb{R}$ s.t. f has all partial derivatives in a nbhd of 0 but f is discontinuous at $0=(0,0)$

10.30 Wedn.

partial derivatives

$f: \mathbb{R}^N \to \mathbb{R}$ $\quad x=(x_1, \cdots, x_N)$

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h\to 0} \frac{f(x_1, \cdots, x_i+h, \cdots, x_N) - f(x_1, \cdots, x_N)}{h}$$

fundamental Lemma

Suppose that $\frac{\partial f}{\partial x_i}$, $i=1, \cdots, N$ all cont. on a $B(a,r)$ centered at a, then f is cont. at a

In fact, $f(a+h) = f(a) + \sum_{i=1}^{N} \left[\frac{\partial f}{\partial x_i}(a) + \varepsilon_i(h)\right] h_i$ $\quad \begin{cases} a=(a_1, \cdots, a_N) \\ h=(h_1, \cdots, h_N) \\ \& \ \varepsilon_i(h) \text{ is cont at } h=0 \text{ with } \varepsilon_i(0)=0 \end{cases}$

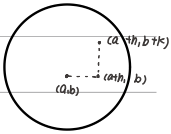
Rk $N=1$ if $f: \mathbb{R} \to \mathbb{R}$ is diff. at a, then f is cont. at a.

$N=2$ if $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$ exist at a, then f is cont. at a?

Example

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & \text{otherwise} \end{cases} \qquad x=ky \quad f(ky,y) = \frac{ky^2}{k^2y^2+y^2} = \frac{k}{k^2+1}$$

(Pf). $N=2$ $\quad f(a+h, b+k) - f(a,b) = \frac{\partial f}{\partial y}(a+h, \xi)\, k + \frac{\partial f}{\partial x}(\eta, b)\, h$



$$\begin{cases} \varepsilon_1(h,k) = \frac{\partial f}{\partial x}(\eta, b) - \frac{\partial f}{\partial x}(a,b) \\ \varepsilon_2(h,k) = \frac{\partial f}{\partial y}(a+h, \xi) - \frac{\partial f}{\partial y}(a,b) \end{cases}$$

$\int_0^h f_x(a+s, b)\, ds$ let $s=th$

$\varepsilon_1(h,k) = f(a+h, b) - f(a,b) = \left(\int_0^1 f_x(a+th, b)\, dt\right) h$ ✡

Th'm Suppose $f: \mathbb{R}^N \to \mathbb{R}$ with $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial x_i \partial x_j}$ all being cont. at point a $\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a)$ $\quad \forall i,j = 1, \cdots, N$

✡ $N=2$ Define $\Delta_2 f = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a,b) = \begin{cases} u(h) - u(0) = u'(s_1) h = \frac{\partial f}{\partial y \partial x}(a+s_1, b+t_2)\, kh \\ \psi(k) - \psi(0) = \psi'(k_1) k = \frac{\partial f}{\partial x \partial y}(a+s_2, b+t_1)\, hk \end{cases}$

Set $\varphi(s) = f(a+s, b+k) - f(a+s, b)$ $\quad s \in (0,h)$ $\quad \varphi'(s) = \frac{\partial f}{\partial x}(a+s, b+k) - \frac{\partial f}{\partial x}(a+s, b) = \frac{\partial^2 f}{\partial y \partial x}(a+s, b+t_2)\, k$ $\quad t_2 \in (0,k)$

$\psi(t) = f(a+h, b+t) - f(a, b+t)$ $\quad t \in (0,k)$ $\quad \psi'(t) = \frac{\partial f}{\partial y}(a+h, b+t) - \frac{\partial f}{\partial y}(a, b+t) = \frac{\partial^2 f}{\partial x \partial y}(a+s_2, b+t)\, h$ $\quad s_2 \in (0,h)$

Taylor's th'm $(N=1)$

Let $f: R \to R$ Suppose on $(a-r, a+r)$ f & its first n derivatives are cont. and $f^{(n+1)}$ exists.

Then $f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n$

Proof. We give a proof for $a = 0$ and $x > 0$. Let

$$R(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k, \quad g(x) = x^{n+1}.$$

Then $R(x)$ has up to n^{th} derivatives on $[0, x]$ and has $(n+1)^{\text{th}}$ derivatives in $(0, x)$, and

$$R(0) = R'(0) = \cdots = R^{(n)}(0) = 0, \quad R^{(n+1)}(\xi) = f^{(n+1)}(\xi),$$
$$g(0) = g'(0) = \cdots g^{(n)}(0) = 0, \quad g^{(n+1)}(\xi) = (n+1)!.$$

Repeating using the Cauchy's Mean Value Theorem we have

$$\frac{R(x)}{g(x)} = \frac{R(x) - R(0)}{g(x) - g(0)} = \frac{R'(c_1)}{g'(c_1)} = \frac{R'(c_1) - R'(0)}{g'(c_1) - g'(0)} = \frac{R''(c_2)}{g''(c_2)} = \cdots$$
$$= \frac{R^{(n+1)}(\xi)}{g^{(n+1)}(\xi)} = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $0 < \xi < c_n < c_{n-1} < \cdots < c_1 < x$. Hence

$$R(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}.$$

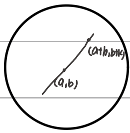
Th'm Let $f: R^N \to R$. Suppose on $B(a,r)$ f & all of its partial derivatives of order $\leq n$ are cont. and its

$(n+1)$ partial derivatives exists.

Then $(N=2)$ $\quad f: R^2 \to R$ $\quad B$ is a ball Centered at (a,b) with radius r where f all of its partial

derivatives of order $\leq n$ are cont. and its $(n+1)$ partial derivatives exists.

Then $f(a+h, b+k) = f(a,b) + f_x(a,b)h + f_y(a,b)k + \frac{1}{2!}(f_{xx}(a,b)h^2 + 2f_{xy}(a,b)hk + f_{yy}(a,b)k^2] + \cdots + \cdots + R_n$



$\phi(t) = f(a+th, b+tk)$

$\phi(t) = \phi(0) + \phi'(0)t + \frac{\phi''(0)}{2!}t^2 + \cdots + \frac{\phi^{(n)}(0)}{n!}t^n + R_n$

$\phi'(t) = \frac{\partial f}{\partial x}(a+th, b+tk)h + \frac{\partial f}{\partial y}(a+th, b+tk)k = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}) f$

$\phi''(t) = \frac{\partial^2 f}{\partial x^2}(a+th, b+tk)h^2 + \frac{\partial^2 f}{\partial y \partial x}(a+th, b+tk)hk + \frac{\partial^2 f}{\partial x \partial y}(a+th, b+tk)kh + \frac{\partial^2 f}{\partial y^2}(a+th, b+tk)k^2 = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f$

$\phi'''(t) = \frac{d}{dt}[(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2] f = [(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2] \frac{d}{dt} f = [(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2] (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}) f = (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^3 f$

remember $N=2$

11.4 Mon

Def. $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$, $\alpha_i \geq 0$, integer

- order of α $\quad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$
- $\alpha ! = \alpha_1 ! \alpha_2 ! \cdots \alpha_N !$
- $\alpha + \beta = (\alpha_1 + \beta_1, \cdots, \alpha_N + \beta_N)$
- $x \in \mathbb{R}^N$: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$
- polynomial in $\mathbb{R}^N$: $\quad f(x) = \sum_{|\alpha| \leq n} C_\alpha x^\alpha$
- $P(\xi_1, \cdots, \xi_N) = \sum_{|\alpha| \leq n} C_\alpha \xi^\alpha \Rightarrow \quad P(D_1, \cdots, D_N) = \sum_{|\alpha| \leq n} C_\alpha D^\alpha \quad \frac{\partial}{\partial x_j} = D_j$

Binomial Th'm $\quad x, y \in \mathbb{R}, n \in \mathbb{N} \Rightarrow (x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

multi-nomial th'm

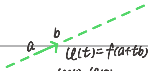
$$(x_1 + \cdots + x_N)^n = \sum_{\alpha_1 + \cdots + \alpha_N = n} \frac{n!}{\alpha_1 ! \cdots \alpha_N !} x_1^{\alpha_1} \cdots x_N^{\alpha_N} = \sum_{|\alpha| = n} \frac{n!}{\alpha !} x^\alpha$$

(pf). Binomial : induction on n

multi-nomial : induction on N

Def : Let $f : \mathbb{R}^N \to \mathbb{R}$, $a, b \in \mathbb{R}^N$ $|b| = 1$. Then direction derivative of f in the direction b at point a is

$$(D_b f)(a) = \lim_{t \to 0} \frac{f(a+tb) - f(a)}{t} \qquad d_b f \quad \nabla f(a) \cdot b$$



$\varphi(t) = f(a+tb)$

$\frac{\varphi(t) - \varphi(0)}{t} = \varphi'(0)$

Lemma 7.4. *Suppose that $f : \mathbb{R}^N \to \mathbb{R}^1$ and all its partial derivatives up to and including order n are continuous in a ball $B(a, r)$. Then*

$$(d_b)^n f(a) = \sum_{|\alpha| = n} \frac{n!}{\alpha!} b^\alpha D^\alpha f(a). \tag{7.26}$$

Lemma $\quad [(D_b)^n f](a) = \sum_{|\alpha| = n} \frac{n!}{\alpha !} b^\alpha D^\alpha f(a)$

Taylor expansion of $f(x, y)$ about a point (a, b) is

$$f(x,y) = f(a,b) + (x-a)\frac{\partial f}{\partial x}|_{(a,b)} + (y-b)\frac{\partial f}{\partial y}|_{(a,b)} + \frac{(x-a)^2}{2}\frac{\partial^2 f}{\partial x^2}|_{(a,b)} + \frac{(y-b)^2}{2}\frac{\partial^2 f}{\partial y^2}|_{(a,b)} + (x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}|_{(a,b)}$$

for the third order terms you have

$$\frac{(x-a)^3}{3!}\partial_x^{(3)} f|_{(a,b)} + \frac{3(x-a)^2(y-b)}{3!}\partial_{xxy} f|_{(a,b)} + \frac{3(x-a)(y-b)^2}{3!}\partial_{yyx} f|_{(a,b)} + \frac{(y-b)^3}{3!}\partial_y^{(3)} f|_{(a,b)}$$

Taylor's $\quad (a, b) \in \mathbb{R}^2$

$$f(a+h, b+k) = f(a+b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)(a,b) + \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)(a,b) + \frac{1}{3!}\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f\right](a,b) + \cdots + \frac{1}{n!}\left[\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\right](a,b) + R(x)$$

Taylor's Let $f : \mathbb{R}^n \to \mathbb{R}$. Suppose on $B(a,r)$ f & all of its partial derivatives of order $\leq n$ are cont.

& its $(n+1)$ derivatives exist

Then $\forall x \in B(a,r)$, $\exists \xi \in \overline{ax}$ (line segment connecting a & x)

$f(x) = \sum_{|\alpha|\le n} \frac{1}{\alpha!} [(D^\alpha f)(a)] (x-a)^\alpha + \sum_{|\alpha|=n+1} \frac{1}{\alpha!} (D^\alpha f)(\xi)(x-a)^\alpha$

(pf) $\phi(t) = (a+tb)$ $b = \frac{x-a}{|x-a|}$ Apply Taylor's Th'm in $\mathbb{R}$

$(x-a)^\alpha = (b|x-a|)^\alpha = b^\alpha |x-a|^{|\alpha|} = b^\alpha t^{|\alpha|}$

Def: Suppose $f: \mathbb{R}^n \to \mathbb{R}$ has partial derivatives at point $a \in \mathbb{R}^N$. We say that a is critical pt. of f if $\frac{\partial f}{\partial x_i}(a) = 0 \quad \forall i = 1, \cdots, N$

Th'm (2^{nd} Derivative Test) Suppose $f: \mathbb{R}^N \to \mathbb{R}$ & its partial derivatives up to & including 2 are cont. in $B(a,r)$ where a is a critical pt. of f. Let

Let $Q(h) = \sum_{|\alpha|=2} \frac{1}{\alpha!} (D^\alpha f)(a) h^\alpha = \frac{1}{2!} \sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(a) h_i h_j$

$$A = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}$$

$\vec{h}^T A h$

$$D^2 f(x_0, y_0) = \begin{pmatrix} f_{xx}(x_0, y_0) & f_{yx}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix}.$$

THEOREM 5.4.8 (Second derivative test for local extreme values). *Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (x_0, y_0) and that*

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$

Then:

(i) f has a local maximum at (x_0, y_0) if

$$f_{xx} < 0, \quad f_{xx} f_{yy} - f_{xy}^2 > 0 \quad \text{at} \quad (x_0, y_0).$$

(The matrix $D^2 f(x_0, y_0)$ is negative definite.)

(ii) f has a local minimum at (x_0, y_0) if

$$f_{xx} > 0, \quad f_{xx} f_{yy} - f_{xy}^2 > 0 \quad \text{at} \quad (x_0, y_0).$$

(The matrix $D^2 f(x_0, y_0)$ is positive definite.)

(iii) f has a saddle point at (x_0, y_0) if

$$f_{xx} f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$

(The matrix $D^2 f(x_0, y_0)$ is indefinite.)

(iv) the test is inconclusive at (x_0, y_0) if

$$f_{xx} f_{yy} - f_{xy}^2 = 0 \quad \text{at} \ (x_0, y_0).$$

(The matrix $D^2 f(x_0, y_0)$ is degenerate.)

In mathematics, **Sylvester's criterion** is a necessary and sufficient criterion to determine whether a Hermitian matrix is positive-definite.

Sylvester's criterion states that a *n* × *n* Hermitian matrix *M* is positive-definite if and only if all the following matrices have a positive determinant:

- the upper left 1-by-1 corner of *M*,
- the upper left 2-by-2 corner of *M*,
- the upper left 3-by-3 corner of *M*,
- $\vdots$
- *M* itself.

In other words, all of the *leading* principal minors must be positive. By using appropriate permutations of rows and columns of *M*, it can also be shown that the positivity of *any* nested sequence of *n* principal minors of *M* is equivalent to *M* being positive-definite.[1]

where $h=(h_1, h_2, \cdots, h_N)$ Then

(i) $Q(h)>0 \ \forall h\neq 0 \Rightarrow f$ has a strict local min at a positive definite

(ii) $Q(h)<0 \ \forall h\neq 0 \Rightarrow f$ has a strict local max at a negative definite

(iii) $Q(h)$ changes sign $\Rightarrow f$ has a saddle point.

$A=\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right)_{i,j=1}^{N}$ is positive definite

$Q(h)=h^T A h$

$\Leftrightarrow Q(h)>0 \quad \forall h\neq 0$

$h^T\begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n\end{pmatrix}h=\lambda_1 h_1^2+\cdots+\lambda_N h_N^2 \geq \lambda_1 h_1^2+\cdots+\lambda_1 h_N^2=\lambda_1|h|^2 \Rightarrow Q(h)\geq\lambda|h|^2$ for some $\lambda>0$

$\geq \min\{\lambda_1,\cdots,\lambda_n\}|h|^2$

$f(x)=f(a)+(x_1-a_1)\frac{\partial f}{\partial x_1}(a)+(x_2-a_2)\frac{\partial f}{\partial x_2}(a)+\frac{1}{2!}\left[\frac{\partial^2 f}{\partial x_1^2}(a)(x_1-a_1)^2+2\frac{\partial^2 f}{\partial x_1\partial x_2}(a)(x_1-a_1)(x_2-a_2)+\frac{\partial^2 f}{\partial x_2^2}(a)(x_2-a_2)^2\right]+o(|x-a|^2)$

$f(x)=f(a)+Q(x-a)+o(|x-a|^2)$ For $|x-a|$ small

$\geq f(a)+\lambda|x-a|^2+o(|x-a|^2)$

$f(x)-f(a)\geq(\lambda+\underbrace{o(1)}_{|o(1)|\leq\frac{\lambda}{2}})|x-a|^2\geq\frac{\lambda}{2}|x-a|^2>0 \qquad (x\neq a)$

Def. Let $A\subseteq\mathbb{R}^N$ be an open set & $f:A\to\mathbb{R}$ f is said to be differentiable at a pt. $a\in A$ if $\exists$ a fcn

approx a fcn by l. fcn.

$L(x)=f(a)+\sum_{k=1}^{N}C_k(x_k-a_k)$ s.t. $\lim_{\substack{x\to a\\ \& x\neq a}}\frac{|f(x)-L(x)|}{\|x-a\|}=0$

And then L is the derivative of f at a (or total derivative)

Prop1 Let f be differentiable at a. Then

(1) the derivative $L(x)$ at a is unique

(2) f is cont. at a

(3) All partial derivatives of f at a exist

Prop2. if all partial derivatives of f at a are cont. at a, then f is differentiable at a

pf of prop. 1

(i) Suppose both $L(x)=f(a)+\sum_1^N C_k(x_k-a_k)$ & $\overline{L}(x)=f(a)+\sum_1^N\overline{C_k}(x_k-a_k)$

$|L(x)-\overline{L}(x)|\leq|L(x)-f(x)|+|f(x)-\overline{L}(x)|$

$\parallel$

$|\sum_{k=1}^{N}(C_k-\overline{C_k})(x_k-a_k)|$ pick $x=a+he_j$ $x_j=a+h$, $x_k=a_k$ $\forall k\neq j$

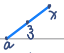
$|C_j-\bar{C}_j|\cdot h \le |L(x)-f(x)|+|f(x)-\bar{L}(x)| \Leftrightarrow |C_j-\bar{C}_j| \le \frac{|f(x)-L(x)|}{\|x-a\|}+\frac{|f(x)-\bar{L}(x)|}{\|x-a\|} \to 0$ as $x\to a$

(ii) $|f(x)-f(a)| \le \underbrace{|f(x)-L(x)|}_{\substack{\to 0 \\ \text{as } x\to a}} + \underbrace{|L(x)-f(a)|}_{\substack{\| \\ \left|\sum_{k=1}^{N} C_k(x_k-a_k)\right| \to 0}}$

(iii) Pick $x=a+he_k$

$$\frac{f(a+he_k)-f(a)}{h} = \underbrace{\frac{f(x)-L(x)}{h}}_{\to 0} + \underbrace{\frac{L(x)-f(a)}{h}}_{\substack{\| \\ \sum_{j=1}^{N} C_j(x_j-a_j) = \frac{C_k h}{h} = C_k}} = C_k \Rightarrow \frac{\partial f}{\partial x_k}(a) = C_k$$

Prof of 2 Set $L(x) = f(a)+\sum_{k=1}^{N}\frac{\partial f}{\partial x_k}(a)(x_k-a_k)$



$f(x)-f(a)=\nabla f(\xi)(x-a)$

$=\frac{\partial f}{\partial x_1}(\xi)(x_1-a_1)+\cdots+\frac{\partial f}{\partial x_N}(\xi)(x_N-a_N)$

$$f(x)-L(x)=f(a)+\sum_{k=1}^{N}\frac{\partial f}{\partial x_k}(\xi)(x_k-a_k)-L(x)$$

$$\le \sum_{k=1}^{N}\left|\frac{\partial f}{\partial x_k}(\xi)-\frac{\partial f}{\partial x_k}(a)\right|\,|(x_k-a_k)|$$

$$\frac{|f(x)-L(x)|}{\|x-a\|} \le \sum_{k=1}^{N}\underbrace{\left|\frac{\partial f}{\partial x_k}(\xi)-\frac{\partial f}{\partial x_k}(a)\right|}_{\to 0 \text{ as } x\to a}\cdot\underbrace{\frac{|(x_k-a_k)|}{\|x-a\|}}_{\le 1}$$

Definitions. Suppose that f has all first partial derivatives at a point a in $\mathbb{R}^N$. The **gradient of** f is the element in $\mathbb{R}^N$ whose components are

$$(f_{,1}(a), f_{,2}(a), \ldots, f_{,N}(a)).$$

We denote the gradient of f by ∇f or grad f.

Suppose that A is a subset of $\mathbb{R}^N$ and $f: A \to \mathbb{R}^1$ is differentiable on A. Let $h=(h_1, h_2, \ldots, h_N)$ be an element of $\mathbb{R}^N$. We define the **total differential** df as that function from $A \times \mathbb{R}^N \to \mathbb{R}^1$, given by the formula

$$df(x, h) = \sum_{k=1}^{N} f_{,k}(x)h_k. \tag{7.38}$$

Using the inner or dot product notation for elements in $\mathbb{R}^N$, we may also write

$$df(x, h) = \nabla f(x)\cdot h.$$

Lemma 14.3. *Let G be an open set in $\mathbb{R}^{m+n}$ and $F: G \to \mathbb{R}^n$ a vector function with continuous first partial derivatives. Suppose that the straight line segment L joining $(\bar{x}, \bar{y})$ and $(\bar{\bar{x}}, \bar{y})$ is in G and that there are two positive constants M_1, M_2 such that*

$$|\nabla_x F| \leqslant M_1 \quad \text{and} \quad |\nabla_y F| \leqslant M_2$$

for all points (x, y) on the segment L. Then

$$|F(\bar{\bar{x}}, \bar{\bar{y}}) - F(\bar{x}, \bar{y})| \leqslant M_1\cdot|\bar{\bar{x}}-\bar{x}| + M_2\cdot|\bar{\bar{y}}-\bar{y}|.$$

Theorem 7.13 (Chain rule). *Suppose that each of the functions $g^1, g^2, \ldots, g^N$ is a mapping from $\mathbb{R}^M$ into $\mathbb{R}^1$ and that each g^i is differentiable at a point $b=(b_1, b_2, \ldots, b_M)$. Suppose that $f: \mathbb{R}^N \to \mathbb{R}^1$ is differentiable at the point $a = (g^1(b), g^2(b), \ldots, g^N(b))$. Form the function*

$$H(x) = f[g^1(x), g^2(x), \ldots, g^N(x)].$$

Then H is differentiable at b and

$$dH(b, h) = \sum_{i=1}^{N} f_{,i}[g(b)]\, dg^i(b, h).$$

Corollary (Inverse function theorem). *Suppose that f is defined on an open set A in $\mathbb{R}^1$ with values in $\mathbb{R}^1$. Also assume that f' is continuous on A and that $f(x_0) = y_0$, $f'(x_0) \neq 0$. Then there is an interval I containing y_0 such that the inverse function of f, denoted f^{-1}, exists on I and has a continuous derivative there. Furthermore, the derivative $(f^{-1})'$ is given by the formula*

$$(f^{-1}(y))' = \frac{1}{f'(x)}, \tag{14.4}$$

where $y = f(x)$.

Theorem 14.4 (Inverse function theorem). *Let G be an open set in $\mathbb{R}^m$ containing the point $\bar{x}$. Suppose that $f\colon G \to \mathbb{R}^m$ is a function of class C^1 and that*

$$\bar{y} = f(\bar{x}), \qquad \det \nabla_x f(\bar{x}) \neq 0.$$

Then there are positive numbers h and k such that the ball $B_m(\bar{x}, k)$ is in G and for each $y \in B_m(\bar{y}, h)$ there is a unique point $x \in B_m(\bar{x}, k)$ with $f(x) = y$. If g is defined to be the inverse function determined by the ordered pairs (y, x) with the domain of g consisting of $B_m(\bar{y}, h)$ and range of g in $B_m(\bar{x}, k)$, then g is a function of class C^1. Furthermore, $f[g(y)] = y$ for $y \in B_m(\bar{y}, h)$.

Remarks. The Inverse function theorem for functions of one variable (Corollary to Theorem 14.1) has the property that the function is one-to-one over the entire domain in which the derivative does not vanish. In Theorem 14.4, the condition $\det \nabla_x f \neq 0$ does not guarantee that the inverse (vector) function will be one-to-one over its domain. To see this consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f^1 = x_1^2 - x_2^2, \qquad f^2 = 2x_1x_2, \tag{14.18}$$

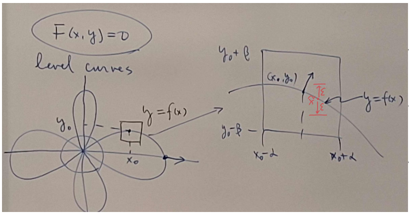
Implicit fcn th'm

Let $F(x,y)$ be a cont on a domain D in $\mathbb{R}^2$.

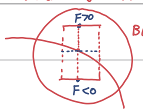
Assume (i) $(x_0,y_0) \in D$ & $F(x_0,y_0)=0$

(ii) $\frac{\partial F}{\partial y}$ is cont. in D & $\frac{\partial F}{\partial y}(x_0,y_0) \neq 0$ gradient is not horizontal

Then $\exists\ \alpha, \beta > 0$ s.t the eqn $F(x,y)=0$ in the open rectangle $(x_0-\alpha, x_0+\alpha) \times (y_0-\beta, y_0+\beta)$ uniquely determines a cont. fcn. $y=f(x)$, s.t. $F(x,f(x))=0$ $\forall\ x \in (x_0-\alpha, x_0+\alpha)$



W.L.O.G $\frac{\partial F}{\partial y}(x_0,y_0) > 0$ in $B((x_0,y_0), 2\beta)$



$B((x_0,y_0), 2\beta)$

$F(x_0, y_0+\beta) > 0$ $\therefore \exists\ \alpha$ small s.t. $F(x, y_0+\beta) > 0$

$F(x_0, y_0-\beta) < 0$ $F(x, y_0-\beta) < 0$

IVT $\Rightarrow \forall\ x \in (x_0-\alpha, x_0+\alpha)\ \exists!\ y \in (y_0-\beta, y_0+\beta)$ s.t $F(x,y)=0$

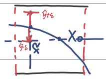
Set $y=f(x) \Rightarrow F(x, f(x))=0$ $\forall\ x \in [x_0-\alpha, x_0+\alpha]$

Let $\tilde{x} \in (x_0-\alpha, x_0+\alpha)$. Then $\exists\, \tilde{y} \in (y_0-\beta, y_0+\beta)$

s.t. $F(\tilde{x},\tilde{y})=0, \hat{y}=f(\tilde{x})$

$\forall \varepsilon>0$. W.L.O.G. (Pick ε smaller if necessary)

assume $(\hat{y}-\varepsilon, \hat{y}+\varepsilon) \subseteq (y_0-\beta, y_0+\beta)$



$F(\tilde{x}, \hat{y}+\varepsilon)>0$

$F(\tilde{x}, \hat{y}-\varepsilon)<0$

$F(\tilde{x}, \hat{y}+\varepsilon)>0$ $F(\tilde{x}, \hat{y}-\varepsilon)<0$ Choose $\delta>0$ small s.t. $(\tilde{x}-\delta, \tilde{x}+\delta) \subseteq (x_0-\alpha, x_0+\alpha)$ & $F(x, \hat{y}-\varepsilon)<0<F(x, \hat{y}+\varepsilon)$ $\forall x \in (\tilde{x}-\delta, \tilde{x}+\delta)$

$\Rightarrow \forall\, x \in (\tilde{x}-\delta, \tilde{x}+\delta)$ $\exists!\, y \in (\hat{y}-\varepsilon, \hat{y}+\varepsilon)$ s.t. $F(x,y)=0$

By uniqueness, $y=f(x)$ $(y \in (\hat{y}-\varepsilon, \hat{y}+\varepsilon))$ $|y-\hat{y}|<\varepsilon$ whenever $|x-\tilde{x}|<\delta$

Furthermore, if $\frac{\partial F}{\partial x}$ is cont. in D, then f is cont. diff. & $f'(x) = \frac{\frac{\partial F}{\partial x}(x, f(x))}{\frac{\partial F}{\partial y}(x, f(x))}$

(Pf.) $F(x, f(x))=0$ $x \in (x_0-\alpha, x_0+\alpha) \Rightarrow \frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x))\, f'(x)=0$

And if F has partial derivatives up to order k, then f is differentiable up to order k

11.11 Mon.

Implicit fcn th'm $(\mathbb{R}^N \to \mathbb{R}^1)$

assume $F: D(\subseteq \mathbb{R}^{N+1}) \to \mathbb{R}$

& $(\bar{x}, \bar{y}) = (x_1, \cdots, x_N, \bar{y}) \in D$

and (i) F is cont. in D

(ii) $\frac{\partial F}{\partial y}$ is cont. in D

(iii) $F(\bar{x}, \bar{y})=0$ & $\frac{\partial F}{\partial y}(\bar{x}, \bar{y}) \neq 0$

Then $\exists\, \alpha, \beta>0$ & a cont. fcn.

$f: B(\bar{x}, \alpha) \to (\bar{y}-\beta, \bar{y}+\beta)$ s.t. $f(\bar{x})=\bar{y}$ and $F(x,y)=0 \Leftrightarrow y=f(x)$ $\forall (x,y) \in B(\bar{x}, \alpha) \times (\bar{y}-\beta, \bar{y}+\beta)$

moreover, if $\frac{\partial F}{\partial x_j}$ is cont. in D, $\forall j=1, \cdots, N$, then f is differentiable in $B(\bar{x}, \alpha)$ with $\frac{\partial f}{\partial x_j}(x) = -\frac{F_{x_j}(x, f(x))}{F_y(x, f(x))}$

$\forall j=1, \cdots, N$. Likewise, if $F \in \mathcal{C}^k(D) \Rightarrow f \in \mathcal{C}^k(B(\bar{x}, \alpha))$

Implicit fcn Th'm $(\mathbb{R}^2 \to \mathbb{R}^2)$

assume $F(x,y,u,v)$, $G(x,y,u,v): D\ (\subseteq \mathbb{R}^{2+2} \to \mathbb{R}^2)$ & $(x_0,y_0,u_0,v_0) \in D$, and

(i) F & G: continuous in D

(ii) F & G have cont. partial derivatives in D

(iii) $F(x_0,y_0,u_0,v_0) = 0 = G(x_0,y_0,u_0,v_0)$ & $J = \left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_{(x_0,y_0,u_0,v_0)} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}_{(x_0,y_0,u_0,v_0)} \neq 0$

Then $\exists\, \alpha, \beta > 0$, & $f, g: B((x_0,y_0),\alpha) \longrightarrow B((u_0,v_0),\beta)$

both cont. s.t $u_0 = f(x_0,y_0)$ $v_0 = g(x_0,y_0)$ &

$$\begin{cases} u = f(x,y) \\ v = g(x,y) \end{cases} \Leftrightarrow \begin{cases} F(x,y,u,v) = 0 \\ G(x,y,u,v) = 0 \end{cases} \quad \forall (x,y,u,v) \in B((x_0,y_0),\alpha) \times B((u_0,v_0),\beta)$$

Moreover, we have $\dfrac{\partial f}{\partial x} = -\dfrac{1}{J}\,\dfrac{\partial(F,G)}{\partial(x,v)}$ $\quad \dfrac{\partial f}{\partial y} = -\dfrac{1}{J}\cdot\dfrac{\partial(F,G)}{\partial(y,v)}$

$\dfrac{\partial g}{\partial x} = -\dfrac{1}{J}\cdot\dfrac{\partial(F,G)}{\partial(u,x)}$ $\quad \dfrac{\partial g}{\partial y} = -\dfrac{1}{J}\,\dfrac{\partial(F,G)}{\partial(u,y)}$

$\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0$ every column / row $\neq 0$

(pf) $J \neq 0 \Rightarrow$ W.L.O.G assume $G_v(x_0,y_0,u_0,v_0) \neq 0$ $\quad |\quad G: \mathbb{R}^4 \to \mathbb{R}\ (x,y,u,v)$

The previous Implicit fcn thm $\Rightarrow \exists\, \varphi(x,y,u) = v \Leftrightarrow G(x,y,u,v) = 0$

consider $F(x,y,u,\varphi(x,y,u)) = 0$

Set $\Phi(x,y,u) = F(x,y,u,\varphi(x,y,u))$ $\quad \dfrac{\partial \Phi}{\partial u} = F_u + F_v \varphi_u = F_u - F_v \dfrac{G_u}{G_v} = \dfrac{F_u G_v - F_v G_u}{G_v} \neq 0$ at (x_0,y_0,u_0)

Previous Implicit function theorem $\Rightarrow \exists\, u = f(x,y)$ s.t $\Phi(x,y,f(x,y)) = 0$

Set $g(x,y) = \varphi(x,y,f(x,y)) \Rightarrow \begin{cases} F(x,y,f(x,y),g(x,y)) = 0 \\ G(x,y,f(x,y),g(x,y)) = 0 \end{cases}$ $\xRightarrow{\text{cramer's}}$ $f_x = \dfrac{\begin{vmatrix} -F_x & F_v \\ -G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \dfrac{-F_x G_v + G_x F_v}{F_u G_v - F_v G_u} = -\dfrac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$

$$\begin{cases} F_x + F_u f_x + F_v g_x = 0 \\ G_x + G_u f_x + G_v g_x = 0 \end{cases} \qquad \begin{cases} F_y + F_u f_y + F_v g_y = 0 \\ G_y + G_u f_y + G_v g_y = 0 \end{cases}$$

Definition. A mapping $f: U \to V$, where U, V are open subsets of $\mathbb{R}^m$, is a C^k-**diffeomorphism**, k is a non-negative integer, if

(i) $f \in C^k(U,V)$;

(ii) f is one-to-one and onto.

(iii) $f^{-1} \in C^k(V,U)$.

A C^0-diffeomorphism is called a **homeomorphism**.

Theorem 5.5.5 (Inverse function theorem). *If a mapping $f: G \to \mathbb{R}^m$ of a domain $G \subset \mathbb{R}^m$ is such that*

(i) $f \in C^k(G,\mathbb{R}^m)$, $\quad k \geqslant 1$;

(ii) $y_0 = f(x_0)$ *at* $x_0 \in G$;

(iii) $D_x f(x_0)$ *is invertible;*

then there exists a neighborhood $U(x_0) \subset G$ of x_0 and a neighborhood $V(y_0)$ of y_0 such that

$$f: U(x_0) \to V(y_0)$$

is a C^k-diffeomorphism. Moreover, if $x \in U(x_0)$ and $y = f(x) \in V(y_0)$, then

$$(D_y f^{-1})(y) = (D_x f(x))^{-1}.$$

Example

$F(x,y,u,v) = -x^2 y + u^2 + v^2 = 0$

$G(x,y,u,v) = -xy - u + v + 1 = 0$

$F(2,1,1,2) = 0 = G(2,1,1,2)$

$\dfrac{\partial(F,G)}{\partial(x,y)} = \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix} \neq 0$ $\quad \dfrac{\partial(F,G)}{\partial(x,u)} = \begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix} \neq 0$ $\quad \dfrac{\partial(F,G)}{\partial(x,v)} = 0$ $\quad \dfrac{\partial(F,G)}{\partial(y,u)} \neq 0$ $\quad \dfrac{\partial(F,G)}{\partial(y,v)} \neq 0$ $\quad \dfrac{\partial(F,G)}{\partial(u,v)} \neq 0$

11.13 Wedn

Set up

- $f: I \to R$ bdd $I = [a,b]$

- A partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$

mash $|\mathcal{P}| = \max_{1 \le j \le m} (x_j - x_{j-1})$ $\Delta x_j = x_j - x_{j-1}$

- Lower Riemann sum $L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i$

- Upper Riemann Sum $U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \Delta x_i$

$m_i = \inf_{[x_{i-1}, x_i]} f$ $M_i = \sup_{[x_{i-1}, x_i]} f$

(b) *If Δ' is a refinement of Δ, then*

$$S_-(f, \Delta) \leqslant S_-(f, \Delta') \leqslant S^+(f, \Delta') \leqslant S^+(f, \Delta).$$

Lower Riemann Integral $\underline{\int_a^b} f(x)dx = \sup_{\mathcal{P}} L(\mathcal{P}, f)$ $\mathcal{P}_1 \subseteq \mathcal{P}_2$

(c) *If Δ_1 and Δ_2 are any two subdivisions of I, then*

$$S_-(f, \Delta_1) \leqslant S^+(f, \Delta_2).$$

Upper Riemann Integral $\overline{\int_a^b} f(x)dx = \inf_{\mathcal{P}} U(\mathcal{P}, f)$

Theorem 5.5. *Suppose that f is bounded on an interval $I = \{x: a \leqslant x \leqslant b\}$. Then f is integrable $\Leftrightarrow$ for every $\varepsilon > 0$ there is a subdivision Δ of I such that*

$$S^+(f, \Delta) - S_-(f, \Delta) < \varepsilon. \quad (5.9)$$

Def. we say that f is (Riemann) integrable

if $\underline{\int_a^b} f = \overline{\int_a^b} f$ & then we denote the common value by $\int_a^b f(x)dx$

Example

$$D(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 1 & x \in \mathbb{Q} \cap [0,1] \end{cases}$$

Question ① Let $\{f_n\}$ be a seq. of (Riemann) integral

$\mathbb{Q} \cap [0,1] = \{a_1, a_2, a_3, \cdots\}$

$f_n(x) = \begin{cases} 1 & \text{if } x = a_1, \cdots, a_n \\ 0 & \text{otherwise} \end{cases}$ $f_n(x) \to D$ as $n \to \infty$

② f is (Riemann) integrable $\Leftrightarrow$ f is cont. almost everywhere (Lebesgue thm)

Example

12) $R(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0,1], (p,q) = 1 \\ 0 & \text{otherwise} \end{cases}$

integrable

13) $f(x) = \begin{cases} \sin\frac{1}{x} & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$

14) $\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right]$

$= \lim_{n \to \infty} \frac{1}{n}\left[\frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \cdots + \frac{1}{1+1}\right]$

$= \int_0^1 \frac{1}{1+x} dx = \ln(1+x)\Big|_0^1 = \ln 2$

Prop (1) Any cont. fcn on $[a,b]$ is integrable

(2) Any monotone fcn on $[a,b]$ is integrable

(3) If f is cont. except for finitely many points on $[a,b]$, then f is integrable

(pf) (1) cont $\Rightarrow$ unif cont

Let f be cont on $[a,b]$ Then f is unif. cont. i.e. $\forall \varepsilon>0, \exists \delta>0$, s.t. $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$ if $|x-y|<\delta$

$\forall \varepsilon>0, \exists \delta>0$ s.t $L(P,f) \geq U(P,f)-\varepsilon$

Take any $P=\{a=x_0<x_1<\cdots<x_n=b\}$ with $|P|<\delta \Rightarrow \max\limits_{[x_{j-1},x_j]} f - \min\limits_{[x_{j-1},x_j]} f < \frac{\varepsilon}{b-a}$

$U(P,f)-L(P,f)=\sum_{j=1}^{n}(M_j-m_j)\Delta x_j \leq \frac{\varepsilon}{b-a}\sum_{j=1}^{n}\Delta x_j = \varepsilon$

Def we say that a seq. $\{f_n\}$ conv. unif. to f on $[a,b]$ if $\forall \varepsilon>0, \exists N>0$, s.t $|f_n(x)-f(x)|<\varepsilon$ $\forall x\in[a,b]$

Prop. Let $\{f_n\}$ be a seq. of integrable fcn on $[a,b]$ if f_n conv. unif to f, then f is also integrable

(pf) Step I $\exists M>0$ s.t. $|f_n(x)|\leq M$ $\forall n$ & $\forall x$ (i.e. $\{f_n\}$ is unif bdd

Since $f_n\to f$ unif pick $\varepsilon=1$ in the Def.

$\Rightarrow \exists N$ s.t $|f_n(x)-f(x)|\leq 1$ $\forall n\geq N$ & $\forall x\in[a,b]$

$\Rightarrow \forall m,n\geq 0$ $|f_n(x)-f_m(x)|\leq|f_n(x)-f(x)|+|f(x)-f_m(x)| \Rightarrow |f_n(x)|\leq 2+|f_N(x)|$ $\forall x\in[a,b]$ $\forall n\geq N$

$M=\max\{\sup\limits_{a\leq x\leq b}|f_1(x)|,\cdots,\sup\limits_{a\leq x\leq b}|f_N(x)|\}+2 \Rightarrow |f_n(x)|\leq M$ $\forall n$ & $\forall x\in[a,b]$

(In particular, $|f(x)|\leq M$ $\forall x\in[a,b]$)

Step II. f is integrable

Set $\varepsilon_n=\sup\limits_{a\leq x\leq b}|f_n(x)-f(x)|$ Then $\varepsilon_n\to 0$ as $n\to\infty$

$f_n(x)-\varepsilon_n\leq f(x)\leq f_n(x)+\varepsilon_n$

$\int_a^b f_n(x)-\varepsilon_n(b-a)=\underline{\int_a^b}(f_n-\varepsilon_n)\leq\underline{\int_a^b}f(x)dx\leq\overline{\int_a^b}f(x)dx\leq\overline{\int_a^b}(f_n(x)+\varepsilon)=\int_a^b f_n(x)dx+\varepsilon_n(b-a)$

$$\overline{\int_a^b}f(x)-\underline{\int_a^b}f(x)\leq 2\varepsilon_n(b-a)$$

(2) $R(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in [0,1],\ (p,q)=1 \\ 0 & \text{otherwise} \end{cases}$

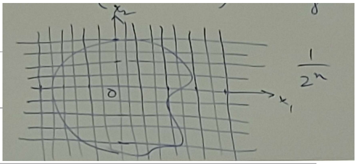
11.18 Mon

Def. (i) $\{ x=(x_1, \cdots, x_N) \mid a_i < x_i < b_i,\ i=1,\cdots,N \}$ $a=(a_1,\cdots a_n)$ $b=(b_1,\cdots,b_n) \in \mathbb{R}^N$

is called an open sell in $\mathbb{R}^N$

(ii) $\{ x=(x_1,\cdots,x_N) \mid a_i \le x_i \le b_i,\ i=1,\cdots,N \}$ is called a closed cell in $\mathbb{R}^N$

(iii) if $b_1-a_1 = b_2-a_2 = \cdots = b_N-a_N$, then the cell is called a hypercube



Set up On $\mathbb{R}^N$ divide into n^{th} grid $\frac{k_i-1}{2^n} \le x_i \le \frac{k_i}{2^n}$, $i=1,\cdots,N$

where k_i $(i=1,\cdots,N)$ are integers

For a bdd set $S \subseteq \mathbb{R}^N$, we have inner, outer & boundary cubes

Set $V_n^-(S) = \frac{1}{2^{nN}} \cdot (\#\text{ of inner cubes})$

$V_n^+(S) = V_n^-(S) + \frac{1}{2^{nN}} \cdot (\#\text{ of boundary cubes})$

Prop (i) $V_n^-(S) \le V_{n+1}^-(S) \le V_{n+1}^+(S) \le V_n^+(S)$

(ii) $V_n^-(S) \uparrow V^-(S)$ inner volumn of S

(iii) $V_n^+(S) \downarrow V^+(S)$ outer volumn of S

(iv) $V^-(S) \le V^+(S)$

Def (1) If $V^-(S) = V^+(S)$, then S has a volumn $V(S)$

(2) A set $S \subseteq \mathbb{R}^N$ is called a figure if it has a volumn

Prop. (1) S_1, S_2 are figures in $\mathbb{R}^N \Rightarrow$ So as $S_1 \cup S_2$ $S_1 \cap S_2$ & $S_1 \setminus S_2$

(2) The boundary of any figures has volumn 0

Let F be a figure in $\mathbb{R}^N$. A subdivision of F is a finite collection of figures $\Delta = \{F_1, \cdots, F_k\}$

s.t. no two of them have common interior pts & $\bigcup_{j=1}^{k} F_j = F$ The mesh of Δ, denoted by $\|\Delta\|$,

$\sup\{d(x,y) \mid x,y \in F_j\}$

is $\|\Delta\| = \max_{1\le j\le k} \text{diam}(F_j)$

Def A figure $F \subseteq \mathbb{R}^N$ is regular if $\forall \varepsilon > 0$, $\exists$ a subdivision $\Delta = \{F_1, \cdots, F_k\}$ with $\|\Delta\| < \varepsilon$ & $V(F_j) > 0$

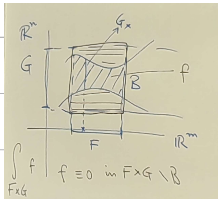
$j = 1, \cdots, k$

We are now ready to define Upper Riemann Sums, Lower Riemann Sums & then upper Riemann integrals, Lower Riemann integrals & then Riemann integrals & Riemann integrable fcns on $\mathbb{R}^N$

Prop. Let F be a figure in $\mathbb{R}^m$

Prop Let $B = \{(x,y) \in \mathbb{R}^{m+n} \mid x \in F, y \in G_x\}$ where F is a figure in $\mathbb{R}^m$ & G_x is a figure in $\mathbb{R}^n$ for each $x \in F$. Suppose $f: B \to \mathbb{R}$ is integrable over B & suppose for each $x \in F$, $f(x, \cdot)$ is integrable over G_x. Then $\phi(x) = \int_{G(x)} f(x,y)\,dy$ is integrable over F & $\int_B f = \int_F [\int_G f(x,y)\,dy]\,dx$

Th'm F: figure in $\mathbb{R}^m$, G: figure in $\mathbb{R}^n$ & f bdd on $F \times G$. Then



maximal over rect. first take a line maximal over boundary

i) $\overline{\int}_{F \times G} f \underbrace{dxdy}_{dv} \geq \overline{\int}_F [\overline{\int}_G f(x,y)\,dy]\,dx]$ iii) $\underline{\int}_{F \times G} f \underbrace{dxdy}_{dv} \geq \underline{\int}_F [\underline{\int}_G f(x,y)\,dy]\,dx]$

ii) $\overline{\int}_{F \times G} f \underbrace{dxdy}_{dv} \geq \overline{\int}_G [\overline{\int}_F f(x,y)\,dx]\,dy]$ iiii) $\underline{\int}_{F \times G} f \underbrace{dxdy}_{dv} \geq \underline{\int}_G [\underline{\int}_F f(x,y)\,dx]\,dy]$

THEOREM 6.1.13 (Fubini's Theorem on rectangles). *Assume $f(x,y)$ is integrable over the closed rectangular region R: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$.*

(a) If for each $x \in [a,b]$ the integral $\int_c^d f(x,y)dy$ exists, then the iterated integral $\int_a^b \int_c^d f(x,y)dydx$ exists, and

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx.$$

(b) If for each $y \in [c,d]$ the integral $\int_a^b f(x,y)dx$ exists, then the iterated integral $\int_c^d \int_a^b f(x,y)dxdy$ exists, and

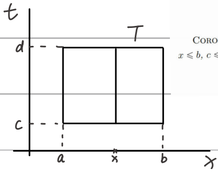
$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy.$$

COROLLARY 6.1.14. *If $f(x,y)$ is continuous on the closed rectangular region R: $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, then*

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx.$$

11.20 Wedn.

Fcns defined by integrals



Set-up: $T = [a,b] \times [c,d]$

$f: T \to \mathbb{R}$ & $\phi(x) = \int_c^d f(x,t)\,dt$

Th'm (Leibniz)

Suppose f & $\frac{\partial f}{\partial x}$ are cont on T

Then $\frac{d}{dx} \int_c^d f(x,t)\,dt = \int_c^d \frac{\partial f}{\partial x}(x,t)\,dt \qquad a < x < b$

(Pf) $\phi(x+h) - \phi(x) = \int_c^d (f(x+h,t) - f(x,t))\,dt = \int_c^d [\underbrace{\int_x^{x+h} \frac{\partial f}{\partial x}(z,t)\,dz}_{f(x+h,t) - f(x,t)}]\,dt$

$\frac{\phi(x+h) - \phi(x)}{h} - \int_c^d \frac{\partial f}{\partial x}(x,t)\,dt = \int_c^d [\frac{1}{h}\int_x^{x+h} \frac{\partial f}{\partial x}(z,t)\,dz - \underbrace{\frac{\partial f}{\partial x}(x,t)}_{\frac{1}{h}\int_x^{x+h} \frac{\partial f}{\partial x}(x,t)dt}]\,dt = \int_c^d \frac{1}{h}\int_x^{x+h} \underbrace{[\frac{\partial f}{\partial x}(z,t) - \frac{\partial f}{\partial x}(x,t)]}_{= o(1) \text{ as } h \to 0}\,dz\,dt$

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\left|\frac{\partial f}{\partial x}(z,t) - \frac{\partial f}{\partial x}(x,t)\right| < \varepsilon \quad \forall |z - x| < \delta$

For $h<\delta$ $\left|\frac{\phi(x+h)-\phi(x)}{h} - \int_c^d \frac{\partial f}{\partial x}(x,t)\,dt\right| \le \int_c^d \frac{1}{h}\int_x^{x+h} \left|\frac{\partial f}{\partial x}(z,t) - \frac{\partial f}{\partial x}(x,t)\right| dz\,dt \le \int_c^d \frac{1}{h}\,\varepsilon\int_x^{x+h} dz\,dt = \int_c^d \varepsilon\,dt = \varepsilon(d-c)$

i.e. as $h\to 0$ $\frac{\phi(x+h)-\phi(x)}{h} \longrightarrow \int_c^d \frac{\partial f}{\partial x}(x,t)\,dt$

$F(x,y,z) = \int_y^z f(x,t)\,dt$

$\frac{\partial f}{\partial x} = \int_y^z \frac{\partial f}{\partial x}(x,t)\,dt$

$\frac{\partial f}{\partial y} = -f(x,y)$

$\frac{\partial f}{\partial z} = f(x,z)$

Gen'd Leibniz Rule

Suppose that (i) f & $\frac{\partial f}{\partial x}$ are cont. on T

(ii) $h_0, h_1 : [a,b] \to [c,d]$ are cont. differentiable (C')

Then $\phi(x) = \int_{h_0(x)}^{h_1(x)} f(x,t)\,dt$ is cont. differentiable

& $\phi'(x) = f(x,h_1(x))h_1'(x) - f(x,h_0(x))h_0'(x) + \int_{h_0(x)}^{h_1(x)} \frac{\partial f}{\partial x}(x,t)\,dt$

(Pf) $\phi(t) = F(x, h_0(x), h_1(x))$, by chain rule

Example $f(x,t) = \begin{cases} \frac{1}{t}\sin(xt) & t\neq 0 \\ x & t=0 \end{cases}$ $\phi(t) = \int_0^{\frac{\pi}{2}} f(x,t)\,dt$

N.T.S. f. $\frac{\partial f}{\partial x}$ are cont.

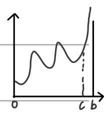
as for f when $t\to 0$ $\lim_{t\to 0}\frac{\sin(xt)}{t} = \lim_{x\to 0}\frac{x\cos(xt)}{1} = x$ so f is cont at $t=0$

as for $\frac{\partial f}{\partial x}$ when $t=0$ $\frac{\partial f}{\partial x}(x,0) = \lim_{h\to 0}\frac{f(x,h)-f(x,0)}{h} = \lim_{h\to 0}\frac{(h+x)-x}{h} = 1$ when $t\neq 0$ $\frac{\partial f}{\partial x}(x,t) = \cos(xt)$ cont.

$\therefore \frac{\partial f}{\partial x}$ is cont. on R

$\Rightarrow \phi(x) = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial x}(x,t)\,dt = \int_0^{\frac{\pi}{2}} \cos(xt)\,dt = \frac{1}{x}\sin(xt)\Big|_0^{\frac{\pi}{2}} = \frac{1}{x}\sin(\frac{\pi}{2}x)$

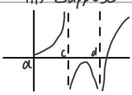
Improper Integrals



Def. (i) On $[a,b)$, f is integrable on $[a,c]$ for every $c<b$ if $\lim_{c\to b}\int_a^c f(x)\,dx$ exists, then we say the integral $\int_a^b f(x)\,dx = \lim_{c\to b}\int_a^c f(x)\,dx$ converges

Otherwise, we say the integral $\int_a^b f(x)\,dx$ diverges

(ii) Suppose $[a,b]$ f is bdd except in a neighborhood of $d\in(a,b)$. Then we say that $\int_a^b f(x)\,dx$



converges if $\lim_{c \to d_-} \int_a^c f(x)dx$ & $\lim_{c \to d^+} \int_c^d f(x)\,dx$ exist

Th'm (Comparison test)

Suppose that $f \in C[a,b)$ & $|f(x)| \le g(x)$ $\forall x \in [a,b)$ if $\int_a^b g(x)dx$ conv. $\Rightarrow \int_a^b f(x)dx$ conv.

(pf.) $G(x) = \int_a^x g(t)\,dt$ $\uparrow$ $(\le \int_a^b g(x)dx)$

Case 1. $f \ge 0$ Set $F(x) = \int_a^x f(t)dt \uparrow \le G(x)$

Case 2 General f.

$$\left.\begin{aligned} f_+(x) &= \frac{|f(x)| + f(x)}{2} \\ f_-(x) &= \frac{|f(x)| - f(x)}{2} \end{aligned}\right. \quad = \max\{f(x), 0\}$$

Both $f_+, f_- \ge 0$ & $f_+(x) + f_-(x) = |f(x)| \le g(x)$

$\therefore \int_a^b f_+$ & $\int_a^b f_-$ conv. $\Rightarrow \int_a^b f(x) = \int_a^b f_+(x) - \int_a^b f_-(x)$ also conv.

Def let $f: [a, \infty) \to \mathbb{R}$ Suppose

$\int_a^c f(x)dx$ exists $\forall c > a$ if $\lim_{c \to \infty} \int_a^c f(x)dx$ exists, then we say the integral $\int_a^\infty f(x)dx$ conv.

& $\int_a^\infty f(x)dx = \lim_{a \to \infty} \int_a^c f(x)dx$

Leibniz Rule for fcns defined by improper integrals

(1) $T = [a,b] \times [c,d]$

(2) $f: T \to \mathbb{R}$ is cont. with cont. $\frac{\partial f}{\partial x}$

(3) $\phi(x) = \int_c^d f(x,t)\,dt$ exists (as an improper integral)

(4) $F(x,t) = \int_c^t f(x,t)dt \to \phi(x)$ unif on $[a,b]$

(5) $\underset{= \int_c^t f_{,1}(x,t)\,dt}{\frac{\partial F}{\partial x}(x,t)} \to \psi(x)$ unif as $t \to d_-$

Then $\phi'(x) = \psi(x)$ i.e

$\frac{d}{dx} \int_c^d f(x,t)\,dt \,(= \psi(x)) \longleftarrow \int_c^t \frac{\partial f}{\partial x}(x,t)dt$ as $t \to d_-$

Pf. $\frac{\partial f}{\partial x} \to \psi$ unif as $t \to d_-$ $\therefore \psi(x)$ is cont. on $[a,b]$

For any $\alpha \in [a,b]$ $\int_\alpha^x \frac{\partial F}{\partial x}(\xi, t)\,d\xi \longrightarrow \int_a^x \psi(\xi)\,d\xi$ $G(x) = \int_a^x \psi(\xi)\,d\xi$ $G'(x) = \psi(x)$

$= F(x,t) - F(\alpha, t) \to \phi(x) - \phi(\alpha)$ $H(x) = \phi(x) - \phi(\alpha)$ $H'(x) = \phi'(x)$

简单来说，我们要保证 $\int_c^t f(x,s)ds$，$\int_c^t \frac{\partial}{\partial x} f(x,s)ds$ conv. unif.

Proposition 6.2.4 (Cauchy criterion). *A necessary and sufficient condition for the improper integral (5) depending on the parameter $y \in Y$ to converge uniformly on a set $E \subseteq Y$ is that for every $\varepsilon > 0$, there exists a neighborhood U of ω, such that*

$$\left| \int_{b_1}^{b_2} f(x,y)dx \right| < \varepsilon \quad \text{whenever } b_1, b_2 \in U,\ y \in E. \tag{7}$$

Example 6.2.5. The improper integral $\int_a^\omega f(x,y)dx$ depending on a parameter $y \in Y$ converges uniformly on a set $E \subset Y$ if and only if

$$\lim_{b<\omega,\, b\to\omega} F^\omega(b) = 0,$$

where

$$F^\omega(b) = \sup_{y\in E} \left| \int_b^\omega f(x,y)dx \right|.$$

Proposition 6.2.6 (Dirichlet test). *Assume that*

(a) there exists a constant M such that

$$\left| \int_a^b f(x,y)dx \right| \leqslant M, \quad \forall a < b < \omega,\ y \in [c,d];$$

(b) for each $y \in [c,d]$, $g(x,y)$ is a monotone function of x and converges to zero as $x \to \omega$, uniformly in $y \in [c,d]$.

Then the improper integral

$$\int_a^\omega f(x,y)g(x,y)dx$$

converges uniformly for $y \in [c,d]$.

Proposition 6.2.7 (Abel test). *Assume that*

(a) the improper integral

$$\int_a^\omega f(x,y)dx$$

converges uniformly for $y \in [c,d]$;

(b) for each $y \in [c,d]$, $g(x,y)$ is a monotone function of x, and there exists a constant M such that

$$|g(x,y)| \leqslant M \quad \text{for all } x \in [a,\omega),\quad y \in [c,d].$$

Then the improper integral

$$\int_a^\omega f(x,y)g(x,y)dx$$

converges uniformly for $y \in [c,d]$.

Proposition 6.2.9. *Assume that*

(a) the functions $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous on the set

$$\{(x,y) \in \mathbb{R}^2 : a \leqslant x < \omega,\quad c \leqslant y \leqslant d\},$$

(b) the integral

$$F(y) = \int_a^\omega f(x,y)dx$$

converges for $y \in [c,d]$,

(c) the integral

$$\Phi(y) = \int_a^\omega \frac{\partial f}{\partial y}(x,y)dx$$

converges uniformly for $y \in [c,d]$.

Then the integral

$$\int_a^\omega f(x,y)dx$$

converges uniformly on the whole set $[c,d]$, $F(y)$ is differentiable, and

$$F'(y) = \int_a^\omega \frac{\partial f}{\partial y}(x,y)dx.$$

Example $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ x>0$ $\qquad \Gamma(x)\sim\sqrt{2\pi}\,x^{x+\frac12}e^{-x}$

$\Gamma(x+1)=x\Gamma(x)$ integration by parts

$\Gamma(n+1)=n!$

$\dfrac{n!}{\sqrt{2\pi}\,n^{n+\frac12}e^{-n}}\to 1$ as $n\to\infty$ Stirling's formula

11.25 Mon.

Def. A seq of fcns. $\{f_n\}$ is said to conv. unif. to f on S if

$$\underbrace{\sup_{x\in S}|f_n(x)-f(x)|}_{\|f_n-f\|\ (\text{sup norm})}\to 0 \quad \text{as } n\to\infty$$

Since $\mathbb{R}$ is a complete metric space, the Cauchy criterion can be used to give an equivalent alternative formulation for uniform convergence: $(f_n)_{n\in\mathbb{N}}$ converges uniformly on E (in the previous sense) if and only if for every $\epsilon>0$, there exists a natural number N such that

$$x\in E, m,n\geq N \implies |f_m(x)-f_n(x)|<\epsilon.$$

In yet another equivalent formulation, if we define

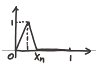
$$d_n=\sup_{x\in E}|f_n(x)-f(x)|,$$

uniformly, then how quickly the functions f_n approach f is "uniform" throughout E in the following sense: in order to guarantee that $f_n(x)$ differs from $f(x)$ by less than a chosen distance ϵ, we only need to make sure that n is larger than or equal to a certain N, which we can find without knowing the value of $x\in E$ in advance. In other words, there exists a number $N=N(\epsilon)$ that could depend on ϵ but is *independent of x*, such that choosing $n\geq N$ will ensure that $|f_n(x)-f(x)|<\epsilon$ *for all* $x\in E$. In contrast, pointwise convergence of f_n to f merely guarantees that

for all $x\in E$. In contrast, pointwise convergence of f_n to f merely guarantees that
for any $x\in E$ given in advance, we can find $N=N(\epsilon,x)$ (i.e., N could depend on the values of both ϵ and x) such that, *for that particular* x, $f_n(x)$ falls within ϵ of $f(x)$ whenever $n\geq N$ (and a different x may require a different, larger N for $n\geq N$ to guarantee that $|f_n(x)-f(x)|<\epsilon$).

$d(f,g)=\sup_{x\in S}|f(x)-g(x)|=\|f-g\|$

Example $\quad S=[0,1]$



$f_n(x)\to f(x)\equiv 0$ as $n\to\infty$ ptwise NOT unif.

Prop. $\quad S=I$ an interval

(1) Suppose that $\{f_n\}$ conv. unif. to f. Where f_n is cont. on I. Then f is cont. on I

(2) Suppose I is bdd & $f_n\in\mathscr{C}(I)$ (i.e. cont.) with $f_n\to f$ unif. as $n\to\infty$. Then

$g_n(x)=\int_c^x f_n(t)\,dt\to \overset{g(x)}{\int_c^x f(t)\,dt}$ unif on I (where $c\in I$ is arbitrary but fixed)

(3) Suppose that $f_n\in\mathscr{C}^1 I$ (i.e. cont. diff) & $f_n\to f$ ptwise. If $\overset{g_n(x)}{f_n'}\to g$ unif, then $f\in\mathscr{C}(I)$ & $f'=g$

(pf) (2) $\left|\int_c^x f_n(t)dt-\int_c^x f(t)dt\right|\leq\int_c^x|f_n-f|\leq\|f_n-f\|_I\cdot|x-c|\leq\|f_n-f\|_I\cdot|I|\cdot 2\to 0$

(3) $\int_c^x f_n'(t)\,dt=f_n(x)-f_n(c)$

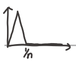
$\downarrow \qquad\qquad \downarrow$

$\int_c^x g(t)\,dt = f(x)-f(c)\Rightarrow g(x)=f'(x)$

$\Downarrow$

$f(x)=\int_c^x g(t)\,dt+f(c)$ so $f(x)$ is differentiable

Example of (2)



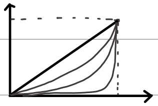
$f_n\to 0$ ptwise, not unif. $\quad \int_0^1 f_n(x)dx=\frac12\to\frac12 \quad g(x)=\frac12 \quad f(x)\equiv 0 \quad f'(x)\neq g(x)$

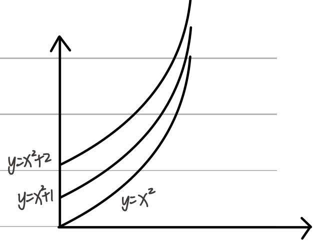
Def Let $\mathcal{F}$ be a family of fcns from I to $\mathbb{R}$. We say that $\mathcal{F}$ is equi-continuous at a pt $x_0\in I$

fcns in the family have same degree of continuity if $\forall \varepsilon>0\ \exists \delta>0$, s.t. $|f(x)-f(x_0)|<\varepsilon\ \forall |x-x_0|<\delta\ \&\ \forall f\in\mathcal{F}$

Example $f_n(x)=x^n\ \ x\in[0,1]$ Is $\{f_n\}$ equi-continuous?

No! take $\varepsilon=\frac{1}{2}$ (No such δ)





Suppose $\{f_n\}$ is equi-cont. $\exists\delta>0$. s.t. $|f_n(x)-f_n(1)|<\frac{1}{2}\ \ \forall |x-1|<\delta, \forall n$

$|x^n-1|<\frac{1}{2} \Rightarrow x^n>(1-\delta)^n>\frac{1}{2} \Rightarrow n<\frac{\ln 1-\delta}{\ln 2}$ take $N=[\frac{\ln 1-\delta}{\ln 2}]+1$ $|f_N(x)-f_N(1)|\geq\frac{1}{2}$, so such δ doesn't exist

Non-uniformity of convergence: The convergence is not uniform, because we can find an $\epsilon>0$ so that no matter how large we choose N, there will be values of $x\in[0,1]$ and $n\geq N$ such that $|f_n(x)-f(x)|\geq\epsilon$. To see this, first observe

(2) We say that $\mathcal{F}$ is equi-continuous on I if it is equi-continuous at every pt. in I

Rk If $\mathcal{F}$ is equi-continuous on $I \Rightarrow \forall\varepsilon>0, \exists\delta>0$ s.t. $|f(x)-f(y)|<\varepsilon\ \ \forall|x-y|<\delta\ \ \&\ \forall f\in\mathcal{F}$ X

$f_n(x)=x^2+n\quad \{f_n\}\quad I=[0,\infty)$

we need to add the condition compactness

Def. let $\mathcal{F}$ be a family of fcns from I to $\mathbb{R}$ We say that $\mathcal{F}$ is unif. bdd if $\exists M>0$ s.t. $|f(x)|\leq M$ $\forall x\in I\ \&\ \forall f\in\mathcal{F}$

$f_1(x)\equiv1\quad f_2(x)\equiv2\ \cdots\ f_n(x)\equiv n\quad \{f_n\}$ is equi-cont but not unif-bdd

☆☆☆☆☆ compactness for function space (metric space & seperable) interval

Ascoli-Arzela Let $\{f_n\}$ be an equi-cont. & unif bdd family of fcns from I to $\mathbb{R}$. Then

(i) $\exists$ a subseq (compactness) $\{f_{n_k}\}$ conv. to a cont. fcn f on I

(ii) if I is compact, then the conv. in (i) is unif.

Let $f_n:[0,\infty)\to R$, given by $f_n(x)=\sin(\sqrt{x+4\pi^2n^2})$. Prove that (f_n) is equicontinuous and equibounded, but has no uniformly convergent subsequence.

(Pf) (i) Let $\{x_1,x_2,\cdots x_n\cdots\}$ be a countable dense of I (e.g. rationals in I) Since $\{f_n(x_1)\}$ is bdd,

$\exists$ a conv. subseq. $\{f_{11}(x_1), f_{12}(x_1),\cdots f_{1n}(x_1)\cdots\}=\{f_{1n}(x_1)\}$ So we choose $\{f_{1n}\}$

Consider $\{f_{1n}(x_2)\}$ it's bdd conv. at x_2, also at x_1

$\therefore$ it has a conv. subseq.

$\{f_{21}(x_2), f_{22}(x_2)\cdots, f_{2n}(x_2)\cdots\}=\{f_{2n}\}$

Similarly $\{f_{2n}(x_3)\}$ is bdd & therefore has conv. subseq. $\to$ conv. at x_3, & also at x_1, x_2

$\{f_{31}(x_3)\cdots f_{3n}(x_3)\cdots\}=\{f_{3n}\}$

Proceeding is this manner, we have $\{f_{3n}\} \subseteq \{f_{2n}\} \subseteq \{f_{1n}\} \subseteq \{f_n\}$

$f_{11}(x_1),\ f_{12}(x_1) \cdots$ "Diagonal process"

$f_{21}(x_2),\ f_{22}(x_2) \cdots$ Pick the subseq $\{f_{nn}\}$ & $\{f_{nn}(x_k)\}_n$ conv. at x_k for every k

$f_{31}(x_3),\ f_{32}(x_3) \cdots$ because it is "essentially" a subseq. of $\{f_{kn}(x_k)\}$

$\vdots$

$f_{n1}(x_n),\ f_{n2}(x_n) \cdots$

$\vdots$

claim $\{f_{nn}(\tilde{x})\}$ conv. at every $\tilde{x} \in I$. Fix $\tilde{x} \in I$. By the equi-continuity $\forall \varepsilon > 0,\ \exists \delta > 0,$ s.t $|f_{nn}(\tilde{x}) - f_{nn}(y)| < \varepsilon/3$ $(*)$

for $|y - \tilde{x}| < \delta,\ \forall n$

Since $\{x_n\}$ is dense, $\exists\, x_k$ s.t $|\tilde{x} - x_k| < \delta$ & $|f_{nn}(x_k) - f_{mm}(x_k)| < \varepsilon/3$ $\forall\, m, n \geq N$ (as $\{f_{nn}(x_k)\}$ conv.)

$$|f_{nn}(\tilde{x}) - f_{mm}(\tilde{x})| \leq \underbrace{|f_{nn}(\tilde{x}) - f_{nn}(x_k)|}_{\text{equi-cont.}} + \underbrace{|f_{nn}(x_k) - f_{mm}(x_k)|}_{\text{conv.}} + \underbrace{|f_{mm}(x_k) - f_{mm}(\tilde{x})|}_{\text{equi-cont.}} \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

对于所点来说

i.e. $\{f_{nn}(\tilde{x})\}$ conv. as $n \to \infty$ call the limit as $f(\tilde{x}) = \lim_{n\to\infty} f_{nn}(\tilde{x})$ def at every pt. so ptwise conv.

$\forall \varepsilon > 0. \exists \delta > 0.$ s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_{nn}(x)|}_{\substack{<\varepsilon/3 \\ \text{by ptwise conv.}}} + \underbrace{|f_{nn}(x) - f_{nn}(x_0)|}_{\substack{<\varepsilon/3 \\ \text{by cont.}}} + \underbrace{|f_{nn}(x_0) - f_{nn}(x)|}_{\substack{<\varepsilon/3 \\ \text{by ptwise conv.}}}$$ when n large

The continuity of f follows from $(*)$ by letting $n, m \to \infty$

11.27 wedn

(ii) I: compact

$D = \sup\{|x-y| : x, y \in I\}$

$K = \sup\{|f(x) - f(y)| : x, y \in I, f \in F\}$

$\varphi(r) = \sup\{|f(x) - f(y)| : x, y \in I \ \&\ |x-y| < r,\ f \in F\}$

Clearly: $\cdot\ \varphi(r) \equiv K \quad \forall\, r \geq D$

$\cdot\ \varphi$ is non-decreasing

$\cdot\ |f(x) - f(y)| \leq \varphi(r) \quad \forall f \in F,\ |x-y| < r$

lemma $\lim_{r \to 0} \varphi(r) = 0$

(pf.) Given $\varepsilon > 0$

Equi-cont. $\Rightarrow \forall x \in I,\ \exists\, \delta\ (= \delta(x))$ s.t.

$|f(x) - f(y)| < \varepsilon/2 \quad \forall\, |y - x| < \delta$

Since the family **F** is equicontinuous, for this fixed ε and for every x in I, there is an open interval U_x containing x such that

$$|f(s) - f(t)| < \frac{\varepsilon}{3}$$

for all $f \in \mathbf{F}$ and all s, t in I such that $s, t \in U_x$.

The collection of intervals U_x, $x \in I$, forms an open cover of I. Since I is closed and bounded, by the Heine–Borel theorem I is compact, implying that this covering admits a finite subcover $U_1, \ldots, U_J$. There exists an integer K such that each open interval U_j, $1 \leq j \leq J$, contains a rational x_k with $1 \leq k \leq K$. Finally, for any $t \in I$, there are j and k so that t and x_k belong to the same interval U_j. For this choice of k,

$$|f_n(t) - f_m(t)| \leq |f_n(t) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

for all $n, m > N = \max\{N(\varepsilon, x_1), \ldots, N(\varepsilon, x_K)\}$. Consequently, the sequence $\{f_n\}$ is uniformly Cauchy, and therefore converges to a continuous function, as claimed. This completes the proof.

$\{B_{\delta(x)}(x) : x \in I\}$ is an open cover for I

Since I is cpt. $\exists$ lebesgue # $\rho > 0$ s.t.

$\forall y \in I \quad \exists \bar{x} \in I$ s.t. $\quad B_\rho(y) \subseteq B_{\delta(\bar{x})}(\bar{x})$

$\forall x, y \in I$ with $|x-y| < \rho \quad \exists \bar{x} \in I$ s.t. $\quad x \in B_\rho(y) \subseteq B_{\delta(\bar{x})}(\bar{x})$

$|f(x) - f(y)| \le |f(x) - f(\bar{x})| + |f(\bar{x}) - f(y)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

i.e. $\forall \varepsilon > 0 \ \exists \rho > 0$ s.t. $\psi(\rho) < \varepsilon$ — instead f is cont. on compact set $|f(x)-f(y)| < \varepsilon \ |x-y| < \eta$ take $\min\{\eta, r\}$

Pf of (ii) Lemma $\Rightarrow \forall \varepsilon > 0, \exists r > 0$ s.t. $|f_{nn}(x) - f_{nn}(y)| < \varepsilon/3$ if $|x-y| < r$ ($\Rightarrow |f(x) - f(y)| \le \varepsilon/3 \ \forall |x-y| < r$ letting $n \to \infty$)

$\{B_r(x) : x \in I\}$ is an open cover for I. I is cpt $\Rightarrow \exists$ finite subcover $\{B_r(x_1), \cdots, B_r(x_m)\}$

$f_{nn}(x_j) \to f(x_j) \quad j = 1, \cdots, m$ as $n \to \infty$

$\Rightarrow \exists N$ s.t. $|f_{nn}(x_j) - f(x_j)| < \varepsilon/3 \quad \forall n \ge N \quad j = 1, \cdots, m$

$\forall y \in I \quad y \in B_r(x_j)$ for some j

$$|f_{nn}(y) - f(y)| \le \underbrace{|f_{nn}(y) - f_{nn}(x_j)|}_{\text{by choice of } r} + \underbrace{|f_{nn}(x_j) - f(x_j)|}_{\text{Ptwise conv.}} + \underbrace{|f(x_j) - f(y)|}_{\text{unif cont.}} \le \varepsilon$$

diagonal process | compactness

on the compact set I, the functional space is compact iff it is equi-cont and unif. bdd

totally bdd + closed $\Leftrightarrow$ compact

Theorem 2.7 *A subset of $C^0[a,b]$ is compact iff it is closed, bounded and equicontinuous.*

$\mathcal{C}[0,1]$. The set of all cont fcns on $[0,1]$

Th'm The sol of all nowhere differentiable (but cont) fcns $\mathcal{N}$ on $[0,1]$ is a subset of 2^{nd} category of $\mathcal{C}[0,1]$

(In particular $\mathcal{N}$ is dense in $\mathcal{C}[0,1]$)

"$d(f,g) = \sup_{0 \le x \le 1} |f(x) - g(x)| (= \|f-g\|)$"

(pf) Strategy : To show $\mathcal{C}[0,1] \setminus \mathcal{N}$ is a set of 1^{st} category

Let $C([0,1],\mathbb{R})$ the space of continuous function $f : [0,1] \to \mathbb{R}$ with the supremum(uniform convergence) metric and let $\mathbb{B} \subset C([0,1],\mathbb{R})$ be the subset of continuous nowhere differentiable functions. I have to show that B contains a countable intersection of dense open sets.

In order to do that, we consider the set:

$$A_n := \{f \in C([0,1],\mathbb{R}) : \forall t \in [0,1] \ \exists h \ s.t \ |\frac{f(t+h) - f(t)}{h}| > n\}$$

And then, if we prove:

1. A_n is open in $C([0,1],\mathbb{R})$
2. A_n is dense in $C([0,1],\mathbb{R})$

Then we can conclude that $\mathbb{B}$ contains a countable intersection of open dense subsets. Finally, this means that the set $\mathbb{B}$ is dense because of the Baire's category theorem.

To achieve that, we'll show that set of fcns that have a finite derivative at some pt.

is a set of 1st category

RHS derivative $\quad h \to 0 \quad f(x) \le N$ for some x

Let $E_n = \{ f \in \mathscr{C}[0,1] \mid \exists\, x \in [0, 1-\frac{1}{n}]$ s.t. $\left| \frac{f(x+h)-f(x)}{h} \right| \le n \quad \forall h \in (0, 1-x) \}$ 存在x_0有右导数，且右导数小于等于n & $\sup\limits_{x \in (x_0,1)} |f(x)-f(x_0)| \le n(x-x_0)$

Then $\bigcup\limits_{n=1}^{\infty} E_n \supseteq$ The set of all fcns that have a finite RHS derivative at some pt. in $[0,1)$)

Claim 1: E_n is closed (in $\mathscr{C}[0,1]$)

前提是 E_n closed

Claim 2: E_n is nowhere dense (or $\mathscr{C}[0,1] \setminus E_n$ is dense)

$d(f_{m_k}, f) = \| f_{m_k} - f \| \to 0$

(pf of claim 1): Let $\{f_m\}$ be a seq. in E_n s.t. $f_m \rightrightarrows f$ in $\mathscr{C}[0,1]$ we need to show: $f \in E_n$ (n: fixed)

$$f_m \in E_n \Rightarrow \exists\, x_m \in [0, 1-\tfrac{1}{n}] \text{ s.t. } |f_m(x_m+h) - f_m(x_m)| \le nh \quad \forall h \in (0, 1-x_m)$$

$$\underset{\in f_{m_k}}{x_{m_k}} \to \bar{x} \in [0, 1-\tfrac{1}{n}] \text{ for some subseq } \{x_{m_k}\}$$

$$\forall h \in (0, 1-\bar{x}) \text{ fixed} \Rightarrow h \in (0, 1-x_{m_k}) \text{ for } m_k \text{ large}$$

$$|f(\bar{x}+h) - f(\bar{x})| \le |f(\bar{x}+h) - f(x_{m_k}+h)| + \underbrace{|f(x_{m_k}+h) - f_{m_k}(x_{m_k}+h)|}_{\le \|f_{m_k} - f\|} + |f_{m_k}(x_{m_k}+h) - f_{m_k}(x_{m_k})| +$$

$$\underbrace{|f_{m_k}(x_{m_k}) - f(x_{m_k})|}_{\le \|f_{m_k} - f\|} + |f(x_{m_k}) - f(\bar{x})|$$

$$\le \underbrace{|f(\bar{x}+h) - f(x_{m_k}+h)|}_{\text{unif. cont.} \to 0} + 2\underbrace{\|f_{m_k} - f\|}_{\to 0} + nh + \underbrace{|f(x_{m_k}) - f(\bar{x})|}_{\to 0 \text{ unif. cont}}$$

$$\le nh$$

pf of claim 2 It suffices to show:

any open set in $\mathscr{C}[0,1]$ must contain a fcn in $\mathscr{C}[0,1] \setminus E_n$

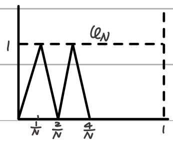
by the def. of dense $\forall f(x) \in \mathscr{C}[0,1]$, $\exists \{P_n(x)\} \in P$ s.t. $\{P_n(x)\} \rightrightarrows f(x)$ i.e. $\bar{P} = \mathscr{C}[0,1]$

By Weierstrass Approx th'm (Polynomials are dense in $\mathscr{C}[0,1]$) we only have to show that

$\forall$ ball contains pts in $\mathscr{C}[0,1] \setminus E_n$ & pts in f

for all smooth fcns f & $\forall \varepsilon > 0$, $B_\varepsilon(f) \cap (\mathscr{C}[0,1] \setminus E_n) \ne \phi$

$\forall$ balls contains pts in $\mathscr{C}[0,1] \setminus E_n$



$M = \sup\limits_{0 \le x \le 1} |f'(x)|$ Choose N large $\frac{1}{2}\varepsilon N > M + n$

Let $g(x) = f(x) + \frac{\varepsilon}{2} \varphi_N(x)$ $\quad \|g - f\| \le \frac{\varepsilon}{2} < \varepsilon$ i.e. $g \in B_\varepsilon(f)$

$\forall x \in [0, 1-\frac{1}{n}]$ $\exists h \in (0, 1-x)$

s.t. $\left| \frac{g(x+h) - g(x)}{h} \right| \ge \frac{\varepsilon}{2} \left| \frac{\varphi_N(x+h) - \varphi_N(x)}{h} \right| - \left| \frac{f(x+h) - f(x)}{h} \right| \ge \frac{\varepsilon}{2} N - M > n \qquad \therefore g \notin E_n$

Definition. Let S be a convex set in $\mathbb{R}^N$ and let $f: S \to \mathbb{R}^1$ be a real-valued function. We say that *f is a* **convex function on** S if and only if

$$f[\lambda x^1 + (1-\lambda)x^2] \leqslant \lambda f(x^1) + (1-\lambda) f(x^2)$$

for all $x^1, x^2 \in S$ and for all λ such that $0 \leqslant \lambda \leqslant 1$. Note that *convex functions are not defined if the domain is not a convex set.*

Definition. A set S in $\mathbb{R}^N$ is **convex** if and only if every point on the straight line segment $\overline{p_1 p_2}$ is in S whenever p_1 and p_2 are in S (see Figure 15.1).

Theorem 15.17. *Let S be a convex set in $\mathbb{R}^N$ and suppose that $f: S \to \mathbb{R}^1$ is a convex function. Then f is continuous on $S^{(0)}$, the interior of S. Furthermore, if $|f(x)| \leqslant M$ on S, then*

$$|f(x^1) - f(x^2)| \leqslant \frac{2M}{\delta}|x^1 - x^2| \tag{15.13}$$

where x^1, x^2 are such that the balls $B(x^1, \delta)$ and $B(x^2, \delta)$ are in S.

Theorem 15.18. *Let S be a convex set in $\mathbb{R}^N$ and suppose that $f: S \to \mathbb{R}^1$ is convex. Let x^0 be an interior point of S.*

(i) *Then there are real numbers $a_1, a_2, \ldots, a_N$ such that*

$$f(x) \geqslant f(x^0) + \sum_{i=1}^{N} a_i(x_i - x_i^0), \; x \in S.$$

(ii) *If $f \in C^1$ on $S^{(0)}$, then*

$$a_i = \left.\frac{\partial f}{\partial x_i}\right|_{x=x^0}.$$

(iii) *If $f \in C^2$ on $S^{(0)}$, then the convexity of f on $S^{(0)}$ is equivalent to the inequality*

$$\sum_{i,j=1}^{N} \left(\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{x=x^0} \right) \lambda_i \lambda_j \geqslant 0 \quad \textit{for all } \lambda \in \mathbb{R}^N \textit{ and all } x^0 \in S^{(0)}.$$

positive definite

function space equipped with sup $\| \ \|$ equi-cont.

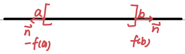
Dec 2^{nd} Taylor th'm Compactness $\boxed{\text{A-A th'm}}$ distance fcn Baire Category th'm unif conv. diff under integral

Fundamental Th'm of Calculus

$f: \mathbb{R} \to \mathbb{R}$ $\mathcal{C}^1$

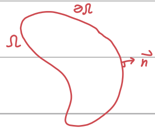
$\int_{\partial[a,b]} f \vec{n} = (+) f(b) + (-) f(a)$

$\int_a^b f'(t)\, dt = f(b) - f(a)$



$\partial[a,b] = \{a, b\}$

outward normal



Divergence Th'm D: domain $\subseteq \mathbb{R}^3$

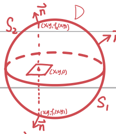
$F: \mathbb{R}^3 \to \mathbb{R}^3$ $F = \langle M, N, P \rangle$ $\begin{cases} M(x,y,z) \\ N(x,y,z) \\ P(x,y,z) \end{cases}$

$F(x,y,z) = M$

$= \iint_{\partial D} (Mn_1 + Nn_2 + Pn_3)\, ds$

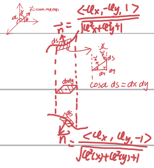
$\iiint_D \operatorname{div} \vec{F}\, dv = \iint_{\partial D} \vec{F} \cdot \vec{n}\, ds$

$\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$



$\vec{n} = \langle n_1, n_2, n_3 \rangle$

$\vec{n}$: unit outward norm



$\iint_{\partial D} Pn_3\, ds = \iint_R P\, dx\, dy$

(1) $\vec{F} \in C^1(\bar{D})$

(2) Denote the projection of D onto the xy-plane by R. Every line prep. to a pt. (x,y) in R will intersect ∂D at exactly 2 pts denoted by $f_2(x,y) > f_1(x,y)$

(3) Similiar assumptions for yz-plane & zx-plane

$\partial D = S$ can be represented by $z = \varphi(x,y)$ $(x,y) \in R$

$\iiint_D \frac{\partial P}{\partial z}\, dv = \iint_S Pn_3\, ds$

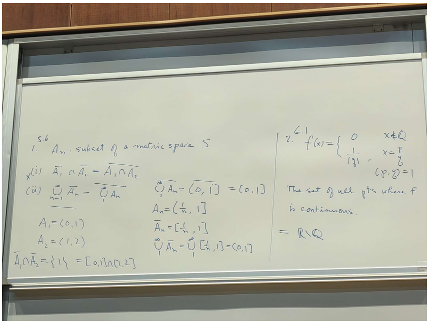
$\varphi(x,y) - z = 0$ $\vec{n} = \pm \frac{\langle \varphi_x, \varphi_y, -1 \rangle}{\sqrt{\varphi_x^2 + \varphi_y^2 + 1}}$

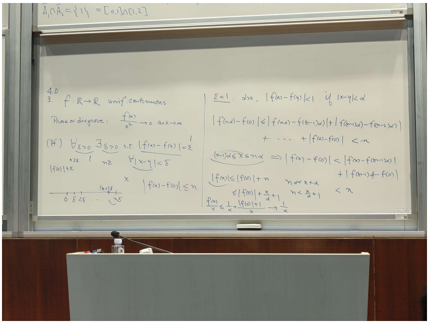
$\iiint_D \frac{\partial P}{\partial z}\, dv = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial P}{\partial z}\, dz \right) dx\,dy = \iint_R [P(x,y,f_2(x,y)) - P(x,y,f_1(x,y))]\, dx\,dy$

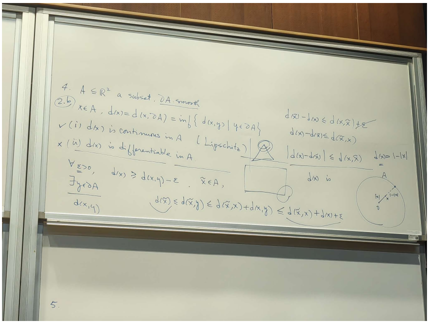
$\iint_R Pn_3\, ds = \iint_{S_2} Pn_3\, ds + \iint_{S_1} Pn_3\, ds = \iint_R P(x,y,f_2(x,y)) \cdot \frac{1}{\sqrt{(\frac{\partial f_2}{\partial x})^2 + (\frac{\partial f_2}{\partial y})^2 + 1}}\, ds + \iint_R P(x,y,f_1(x,y)) \cdot \frac{1}{\sqrt{(\frac{\partial f_1}{\partial x})^2 + (\frac{\partial f_1}{\partial y})^2 + 1}}\, ds = \iint_R P(x,y,f_2(x,y))\, dx\,dy - \iint_R P(x,y,f_1(x,y))\, dx\,dy$

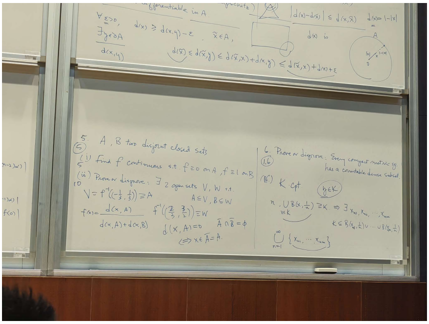
M. Spivak $\int_{\partial D} w = \int_D \partial w$

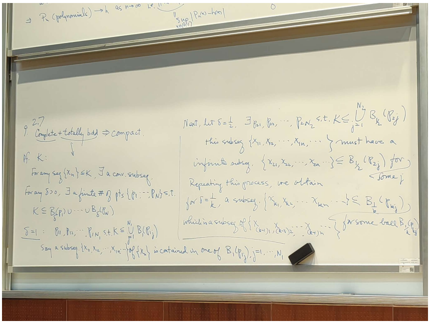
12.4 wedn review











Taylor th'm

Implicit fcn th'm

Diff. under Integral sign (Not test Riemann in final)

✓ Ascoli-Arzela th'm

✓ Weierstrass Approx. th'm

✓ Baire Catogory th'm

cont. but not monotone anywhere