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MAT4220: Partial Differential Equations

Class Quiz 3 – 2025 Fall

Duration: 5–10 minutes

(1) What is definition of a periodic space $\mathbb{T}$? Any intuition (specific example) for this space?

(2) What's the relation of the Fourier full series and sine/cosine series?

(3) What is the definition of pointwise convergence, L^2-convergence and uniform convergence for a sequence of functions $\{f_n\}$ defined on $[a, b]$?

(4) Can you tell me about any connections between the convergences mentioned in the previous problem?

(5) What are the three convergence theorems for full Fourier series?

— End —

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MAT4220: Partial Differential Equations

Class Quiz 4 – 2025 Fall

Duration: 5–10 minutes

(1) Let $f \in C^m(\mathbb{T})$, what is the relation between $c_k[f^{(m)}]$ and $c_k[f]$? (Here c_k denotes the Fourier full series coefficient.)

(2) Reversely, how does the decay of $c_k[f]$ implies the regularity of f?

(3) The heat equation on the $(0,\infty) \times \mathbb{T}$ admits the series solution

$$u(t,x) = \sum_{k\in\mathbb{Z}} c_k e^{-k^2 t} e^{ikx}, \qquad c_k \in \mathbb{C}.$$

(a) Show that for $t > 0$ and if the sequence $\{c_k\}_{k\in\mathbb{Z}}$ is uniformly bounded, then

$$u(t,x) \in C^\infty\big((0,\infty) \times \mathbb{T}\big).$$

(b) Suppose that

$$\lim_{t \to 0^+} u(t, \cdot) = h \quad \text{in } L^2(\mathbb{T}),$$

for some given function $h \in L^2(\mathbb{T})$. What are the coefficients c_k in terms of h?

(c) Suppose that the coefficients $\{c_k\}$ satisfy

$$\sum_{k \in \mathbb{Z}} k^2 |c_k|^2 < \infty.$$

First, check that h is well-defined through pointwise limit:

$$h(x) = \lim_{t \to 0^+} \sum_{k \in \mathbb{Z}} c_k e^{-k^2 t} e^{ikx}.$$

What is the optimal regularity for h? That is, does $h \in C^1(\mathbb{T})$ or $C^0(\mathbb{T})$?

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MAT4220: Partial Differential Equations

Class Quiz 5 – 2025 Fall

Strong and weak maximum principle, MVP

(1) What is a subharmonic function? What does it mean in one dimension?

(2) State the weak and strong maximum principles for subharmonic functions. Address their difference.

(3) State two versions of the mean value property for harmonic functions in $\mathbb{R}^n$.

(4) Let $\Omega \subset \mathbb{R}^n$ is bounded and smooth domain. Prove that there is a smooth solution to the boundary value problem

$$\begin{cases} \Delta u = f(x), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n}\big|_{\partial\Omega} = h(x), \end{cases}$$

only if

$$\int_\Omega f(x)\,dx = \int_{\partial\Omega} h(x)\,dS.$$

(5) Let u be a harmonic and smooth function in a domain $\Omega \subset \mathbb{R}^3$, x_0 is a point in Ω.
(a) For a ball $B_\varepsilon(x_0) \subset \Omega$ with $\varepsilon > 0$, prove that

$$\int_{\partial B_\varepsilon(x_0)} u(x) \frac{\partial}{\partial n}\left(\frac{1}{|x - x_0|}\right) dS = -4\pi u(x_0).$$

(b) **(True or False)** Since $u = u(x)$, $v(x) := \dfrac{1}{|x - x_0|}$ are harmonic functions in Ω, from Green's second identity,

$$\int_{\partial\Omega} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS = 0.$$

— End —

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MAT4220: Partial Differential Equations

Class Quiz 6 – 2025 Fall

Weak derivative and weak solution

(1) What is the definition of the weak derivative of a given function $u : \mathbb{R}^n \to \mathbb{R}$?

(2) Does the weak derivative of the function

$$f(x) = |x|, \qquad x \in \mathbb{R}$$

exist? If exist, find it; if no, justify why?

(3) Sketch the idea how to get a weak solution formulation of the following traffic problem:

$$\frac{\partial u}{\partial t} + (1 - 2u)\frac{\partial u}{\partial x} = 0, \qquad u|_{t=0} = g.$$

And state the form of weak solution.

(4) What is the Rankine–Hugoniot relation?

(5) **(True or False)** Let Ω be a domain in $\mathbb{R}^n$. If $u \in H_0^1(\Omega)$, then $u \in C(\Omega)$.

— End —

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MAT4220: Partial Differential Equations

Class Quiz 7 – 2025 Fall

Variational methods & energy methods

(1) Let Ω be a bounded domain in $\mathbb{R}^n$, $F : \mathbb{R} \to \mathbb{R}_+$ is a given smooth function. Consider the energy

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + F(u) \right) dx.$$

Suppose that $w \in C^\infty$ is a global minimizer of the energy subject to given boundary data $w = h$ on $\partial\Omega$.

(a) Derive the equation w satisfies. Hint: for any $\varphi \in \mathcal{D}(\Omega)$, consider

$$g(t) = \mathcal{E}(w + t\varphi).$$

(b) What is $w \in H^1(\Omega)$ only?

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(2) Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Given m, f, ϕ, ψ, consider the initial boundary value problem

$$\begin{cases} \partial_t u = \Delta u + mu + f(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = \phi(x), \quad u|_{\partial\Omega}(x,t) = \psi(x). \end{cases}$$

(a) What does it mean the solution to the problem is unique?

(b) Let $\Omega = (a,b) \subset \mathbb{R}$, $\Delta = \partial_x^2$ and $m = 0$. Prove the following stability subject to initial and boundary data: for $k = 1, 2$, if u_k solves the problem with data ϕ_k, ψ_k, then for any $T > 0$ and all $t \in (0,T)$,

$$\max_{\Omega} |u_1 - u_2|(t) \le C\Big(\max_{\Omega} |\phi_1 - \phi_2| + \max_{\partial\Omega} |\psi_1 - \psi_2| \Big).$$

Show that the stability implies uniqueness.

(c) Let $m \le 0$, is there a similar stability subject to the initial data under $L^2(\Omega)$-norm? Justify your answer. What about the case $m > 0$?