Solving Friedmann equations

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A report submitted as part of the requirements for the course
Numerical Methods
at the International Centre for Theoretical Sciences - Tata Institute of Fundamental Research
Benguluru, India

May 2024

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Abstract

The project aims to comprehend the intricate dynamics of the universe through the application of numerical methods, specifically utilizing the odeint solver to solve Friedmann equations. Various cosmological models, including the Big Crunch, expanding models, Einstein-de Sitter model, and ΛCDM model, are examined within this framework. Through meticulous computational analysis, the research endeavors to elucidate the complex interactions governing the evolution of the universe, thereby providing valuable insights into its historical progression and potential future trajectories. This research not only contributes to the theoretical framework of cosmology but also offers insights into the fundamental nature of our existence within the cosmos.

Acknowledgements

I express my sincere gratitude to Prof. Ajith Parameswaran and Prof. Prayush Kumar for their invaluable guidance, support, and expertise throughout the duration of this research. Their insightful supervision significantly enriched the quality and depth of this work.

I would also like to extend my appreciation to ICTS-TIFR for providing a conducive research environment and resources essential for the completion of this project.

Special thanks to Yuvraj Sharma and Fabien Dournac for their contributions and assistance, which greatly facilitated the progress of this research endeavor. Their dedication and collaboration were instrumental in achieving the objectives of this research.

Declaration

I confirm that the work contained in this project report has been composed solely by myself. All sources of information have been specifically acknowledged and all verbatim extracts are distinguished by quotation marks.

Signed .... Date ....

Bhuvaneshwari Kashi

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Chapter 1

Introduction

he Friedmann equations, derived from Einstein's theory of general relativity, provide a mathematical framework for understanding the dynamics of the cosmos on large scales. They offer insights into the evolution of the universe, including its expansion, curvature, and energy content.

This study aims to numerically solve the Friedmann equations to obtain the scale factor as a function of time for different universe geometries. By modeling the universe as a perfect fluid (**chen2014friedmann**), we simplify the equations to focus on density (ρ) and pressure (p) as primary determinants of its dynamics. The equations allow us to compute the scale factor $R(t)$ considering density, pressure, curvature (k), and the cosmological constant (Λ).

The density of matter plays a crucial role in governing the universe's evolution, with its gravitational self-attraction decelerating expansion. The first Friedmann equation relates the rate of change of the scale factor to energy density, pressure, curvature, and the cosmological constant.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{1.1}$$

The second equation provides information about acceleration or deceleration of cosmic expansion.

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.2}$$

In summary, the Friedmann equations provide a mathematical description of the dynamic evolution of the universe, accounting for its expansion, curvature, and energy content. They are fundamental tools in cosmology for understanding the large-scale structure and evolution of the cosmos.

1.1 Motivation

The Friedmann equations are essential in understanding the universe's dynamics, particularly its expansion. However, density and pressure alone are insufficient to determine the universe's geometry and dynamics. The cosmological constant's impact underscores the intricate interplay between fundamental parameters, shaping our understanding of the cosmos.

Chapter 2

Cosmological Parameters

2.1 Hubble Constant

The Hubble constant H_0 represents the current rate of expansion of the universe, expressed as $\left(\frac{R'}{R}\right)_0$. It quantifies how fast the universe is expanding at the present moment, typically in units of km/s/Mpc.

2.2 Deceleration Parameter

The deceleration parameter q_0 measures the rate of slowing down of the universe's expansion. It is defined as $-\left(\frac{R''}{R}\frac{1}{(R')^2}\right)_0$, indicating whether the universe is decelerating ($q_0 > 0$), accelerating ($q_0 < 0$), or at a critical point ($q_0 = 0$).

2.3 Density Parameters

Density parameters compare the density of various components to the critical density.

2.3.1 Critical Density

The critical density $\rho_{\text{critical},0}$ is the threshold density for a geometrically flat universe, given by $\frac{3H_0^2}{8\pi G}$.

2.3.2 Density Parameter for Matter

The density parameter Ω_{0m} quantifies the contribution of matter to the total density of the universe, defined as $\frac{\rho_{\text{mat},0}}{\rho_{\text{critical},0}}$.

2.3.3 Density Parameter for Radiation

The density parameter Ω_{0r} represents the contribution of radiation to the total density of the universe, calculated as $\frac{\rho_{r,0}}{\rho_{\text{critical},0}}$.

2.3.4 Reduced Cosmological Constant

The reduced cosmological constant $\Omega_{0\Lambda}$ evaluates the role of the cosmological constant (Λ) in the universe's dynamics, expressed as $\frac{\Lambda c^2}{3H_0^2}$.

Chapter 3

Implementation

3.1 Importance of `odeint`

According to **dias2019selected**, the `odeint` function holds significant importance in solving ordinary differential equations (ODEs) numerically, particularly in scenarios where analytical solutions are either difficult or impossible to obtain, such as in cosmological models described by the Friedmann equations (**faraoni1999solving**). These equations involve complex interactions between various components of the universe, including matter, radiation, dark energy, and curvature. By utilizing `odeint`, researchers can efficiently integrate these differential equations, providing insights into the dynamics and fate of the universe under different cosmological scenarios. Moreover, `odeint` offers flexibility and versatility in handling a wide range of differential equations, allowing for the incorporation of diverse physical parameters and initial conditions (**bovy2015galpy**). In the case of the provided equations, `odeint` enables the numerical solution of the Friedmann equations, accounting for the effects of matter, radiation, cosmological constant, and curvature on the evolution of the scale factor. This numerical approach not only facilitates the exploration of various cosmological models but also aids in comparing theoretical predictions with observational data. Ultimately, the use of `odeint` empowers researchers to conduct comprehensive analyses of the universe's evolution, shedding light on fundamental questions in cosmology and astrophysics.

3.2 Numerical Solving

Equations (1.1) and (1.2) involve the first and second derivatives of the scale factor (**dournac˙friedmann**). The objective is to combine these equations to formulate a second-order differential equation in matrix form, which can then be solved using the `odeint` function. To streamline the process, we introduce the normalized scale factor $y(t) = \frac{R(t)}{R_0}$.

$$y'' = -y\frac{H_0^2}{2}\left(\Omega_{0m}y^3 + 2\Omega_{0r}y^4 - 2\Omega_{0\Lambda}\right) \tag{3.1}$$

Similarly, equation (1.2) is represented by:

$$y = \Omega_{0m}\left(y'^2\frac{H_0^2}{2} - \Omega_{0r}y^2 - y^2\Omega_{0\Lambda} - \Omega_{0k}\right)^{-1} \tag{3.2}$$

Here, Ω_{0k} is defined as:

$$\Omega_{0k} = -\frac{kc^2}{R_0^2 H_0^2} \tag{3.3}$$

We substitute the expression for y obtained from equation (3.2) into equation (3.1), resulting in:

$$y'' = -\frac{H_0^2}{2}\left(\Omega_{0m}y^3 + 2\Omega_{0r}y^4 - 2\Omega_{0\Lambda}\right)\Omega_{0m}\left(y'^2\frac{H_0^2}{2} - \Omega_{0r}y^2 - y^2\Omega_{0\Lambda} - \Omega_{0k}\right)^{-1} \tag{3.4}$$

Furthermore, by using equation (3.2), we can derive the relationship:

$$\Omega_{0m} + \Omega_{0r} + \Omega_{0\Lambda} + \Omega_{0k} = 1 \tag{3.5}$$

This equation is pivotal in determining the value of Ω_{0k} as a function of Ω_{0m}, $\Omega_{0\Lambda}$, and Ω_{0r}. The subsequent equation aids in interpreting the results, particularly regarding the sign of the deceleration parameter q_0:

$$q_0 = \frac{1}{2}\sum_i (1 + 3w_i)\Omega_{0i} \tag{3.6}$$

By considering the contributions from matter (with $w = 0$), radiation ($w = \frac{1}{3}$), and the cosmological constant ($w = -1$), we can simplify to:

$$2q_0 = \Omega_{0m} + 2\Omega_{0r} - 2\Omega_{0\Lambda} \tag{3.7}$$

Using the above equations yields in graphs as shown in 4

Chapter 4

Results

4.1 Hyperbolic Geometry (k=-1)

In hyperbolic geometry, represented by $k = -1$, the universe exhibits a distinctive negative curvature, indicative of an open spatial geometry with infinite extent **schmidt2012nobel**. The expansion of such a universe continues indefinitely over time, as depicted in Figure 4.1. The dynamics of this expansion are influenced by cosmological parameters such as Ω_{1m}, Ω_{1r}, Ω_{1l}, and Ω_{1k}, where Ω_{1m} represents matter density, Ω_{1r} represents radiation density, Ω_{1l} represents dark energy density or the cosmological constant, and Ω_{1k} denotes spatial curvature. For this scenario, Ω_{1m} is 0.3, Ω_{1r} is 1×10^{-4}, Ω_{1l} is 0.0, and Ω_{1k} is calculated accordingly. The deceleration parameter q_0 for this model is approximately 0.15, indicating a gradual slowdown in the expansion rate over time.

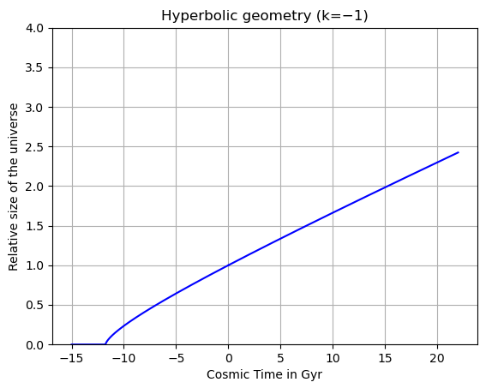

Figure 4.1: Hyperbolic geometry (k=-1)

4.2 The Collapsing Universe

The concept of the "Big Crunch" describes a theoretical scenario where a universe with positive curvature ($k = 1$) and finite extent eventually ceases its expansion and collapses inward under gravitational attraction **steinhardt2002cyclic**. This scenario is depicted in Figure 4.2, where the universe undergoes initial expansion followed by a contraction phase. As matter and energy converge, densities rise, leading to a singularity akin to the conditions at the universe's inception.

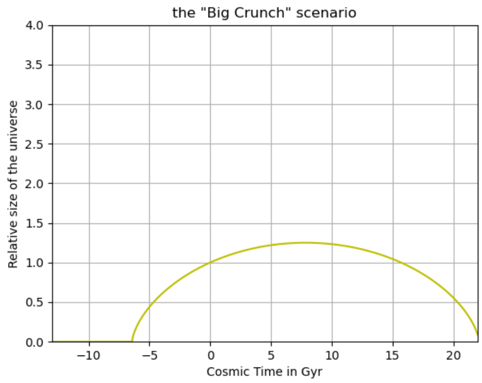

Figure 4.2: The Collapsing Universe (The Big Crunch)

4.3 Einstein-de Sitter Model

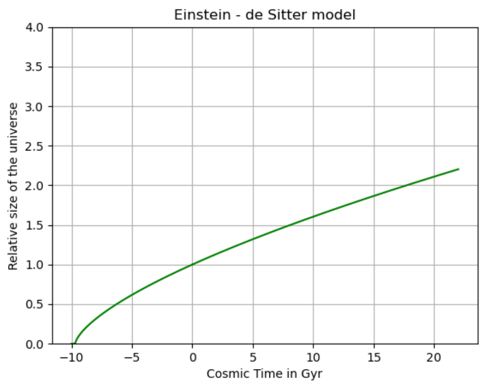

Figure 4.3: Einstein-de Sitter model

The Einstein-de Sitter model, proposed by Albert Einstein and Willem de Sitter, describes a spatially flat ($k = 0$), matter-dominated universe **janssen2016einstein**. In this model, matter density (Ω_{4m}) exactly matches the critical density ($\Omega_{4m} = 1$), resulting in an eternal but decelerating expansion ($q_0 = 0.5$). Figure 4.3 illustrates the perpetual expansion of the universe, with matter playing a crucial role in shaping its dynamics.

4.4 Lambda Cold Dark Matter (ΛCDM) Model

The prevailing cosmological model, ΛCDM, combines the Big Bang theory with Einstein's general theory of relativity **mavromatos2022lambda**. In this model, the universe is assumed to be homogeneous and isotropic, with dark energy (Λ) driving its accelerated expansion. Figure 4.4 depicts the evolution of the universe within the ΛCDM framework, characterized by matter density (Ω_{2m}), dark energy density (Ω_{2l}), and nearly flat spatial geometry ($\Omega_{2k} \approx 0$). The negative deceleration parameter ($q_0 \approx -0.55$) suggests an accelerating expansion, potentially leading to a "Big Rip" scenario where the universe tears apart at cosmic scales.

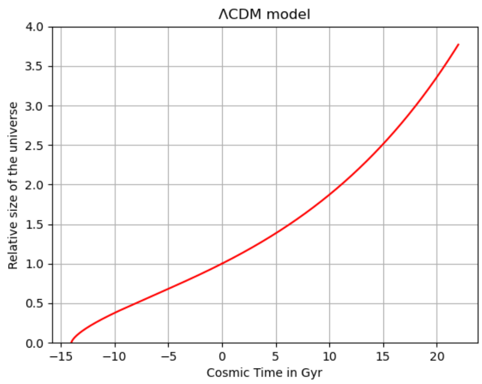

Figure 4.4: (ΛCDM) model

Chapter 5

Conclusion

5.1 Conclusions

In conclusion, solving the Friedmann equations using tools like `odeint` enables efficient exploration of cosmological models. These equations, rooted in general relativity, offer insights into the universe's evolution. Numerical methods, especially `odeint`, play a crucial role, facilitating accurate simulations. Our study examined hyperbolic, collapsing, Einstein-de Sitter, and ΛCDM models, each providing unique perspectives on cosmic dynamics. Through simulations, we unveiled the interplay of cosmological parameters and their implications. Such numerical approaches are invaluable for understanding the universe's complexities.

5.2 Future Work

Future research can enhance numerical methods and explore exotic cosmological models. Integrating observational data and refining theoretical frameworks remain crucial. Sensitivity analyses will elucidate key parameters' impacts, guiding future observations and model refinements. By pursuing these avenues, we can deepen our understanding of cosmology and uncover new insights into the universe's nature.

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