Solving Friedmann equations

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Abstract

The project aims to comprehend the intricate dynamics of the universe through the application of numerical methods, specifically utilizing the odeint solver to solve Friedmann equations. Various cosmological models, including the Big Crunch, expanding models, Einstein-de Sitter model, and ΛCDM model, are examined within this framework. Through meticulous computational analysis, the research endeavors to elucidate the complex interactions governing the evolution of the universe, thereby providing valuable insights into its historical progression and potential future trajectories. This research not only contributes to the theoretical framework of cosmology but also offers insights into the fundamental nature of our existence within the cosmos.

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Declaration

I confirm that the work contained in this project report has been composed solely by myself. All sources of information have been specifically acknowledged and all verbatim extracts are distinguished by quotation marks.

Signed .... Date ....

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Chapter 1

Introduction

The Friedmann equations stand as cornerstones of modern cosmology, holding particular significance in the study of inflationary universes. These equations, derived from Einstein's theory of general relativity, provide a mathematical framework for understanding the dynamics of the cosmos on large scales. They offer profound insights into the evolution of the universe, encompassing its expansion, curvature, and energy content.

The aim of this study is to employ numerical methods to solve the Friedmann equations and obtain the scale factor as a function of time for all possible geometries of the universe. To achieve this goal, we adopt a modeling approach wherein the contents of the universe are represented as a perfect fluid—a simplification that enables us to focus on density (ρ) and pressure (p) as the primary determinants of its dynamics. Within this framework, the energy-momentum tensor takes on a straightforward form, facilitating the reduction of Einstein's equations to a set of two differential equations (1.1), (1.2). These equations allow us to compute the scale factor $R(t)$ as a function of density ρ, cosmic fluid pressure p, curvature parameter k, and cosmological constant Λ.

The density of matter emerges as a crucial parameter governing the universe's evolution. Its gravitational self-attraction exerts a gravitational pull that decelerates expansion—a relationship that underscores the fundamental interplay between matter density and cosmic dynamics. In modeling the perfect fluid, we utilize a state equation relating pressure p and energy density ρ_e, commonly expressed as $p = w\rho_e$, with w representing the equation-of-state parameter. For matter, radiation, and dark energy, w assumes values of 0, $\frac{1}{3}$, and -1, respectively.

In the following discourse, we present a comprehensive exploration of our findings, detailing the computational methodologies employed, the cosmological models investigated, and the insights garnered. This research not only contributes to the theoretical foundations of cosmology but also holds implications for broader inquiries into the nature of the cosmos and humanity's place within it.

1.1 Freedmann equations

1.2 Explanation of Friedmann Equations

The Friedmann equations are a set of equations derived from Einstein's field equations in general relativity. According to **chen2014friedmann**, they describe the evolution of the universe on large scales, particularly in terms of its expansion dynamics. These equations were first formulated by Alexander Friedmann in 1922 and later refined by George Lemaître in 1927 and independently by Howard P. Robertson in 1928. They are fundamental to modern cosmology and provide the mathematical framework for understanding the dynamics of an expanding universe.

The Friedmann equations can be expressed in terms of the scale factor $R(t)$, which quantifies the expansion of the universe as a function of time t. There are two main forms of the Friedmann equations, often referred to as the first and second Friedmann equations.

1.2.1 First Friedmann Equation

The first Friedmann equation, also known as the energy conservation equation, relates the rate of change of the scale factor $R(t)$ to the energy density ρ and pressure p of the various components (matter, radiation, dark energy) present in the universe. It can be written as:

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{1.1}$$

where:

- G is the gravitational constant,
- k is the curvature parameter representing the spatial curvature of the universe (positive for closed, negative for open, and zero for flat),
- Λ is the cosmological constant (representing the energy density of empty space),
- $\dot{R}$ represents the time derivative of the scale factor $a(t)$,

- ρ is the total energy density of the universe (including contributions from matter, radiation, and dark energy).

The terms on the right-hand side of the equation represent, respectively, the contribution of matter, curvature, and the cosmological constant to the expansion dynamics of the universe.

1.2.2 Second Friedmann Equation

The second Friedmann equation is derived from the first Friedmann equation and provides information about the acceleration (or deceleration) of the cosmic expansion. It is expressed as:

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.2}$$

where:

- $\ddot{R}$ represents the second derivative of the scale factor $R(t)$,
- p is the total pressure exerted by the various components of the universe.

This equation shows how the acceleration of the cosmic expansion is influenced by the total energy density ρ and pressure p, as well as the cosmological constant Λ.

In summary, the Friedmann equations provide a mathematical description of the dynamic evolution of the universe, accounting for its expansion, curvature, and energy content. They are essential tools in cosmology for understanding the large-scale structure and evolution of the cosmos.

1.3 Motivation

The Friedmann equations serve as fundamental tools in understanding the dynamics of the universe on large scales, particularly in the context of its expansion. The first of these equations yields the second derivative of the scale factor, which expresses acceleration or deceleration of expansion—a crucial insight into the evolving nature of cosmic structure. However, even with the simplifying assumptions mentioned, such as homogeneity and the perfect fluid model, density and pressure alone are insufficient to fully determine the geometry and dynamics of the Universe. The cosmological constant plays a pivotal role in this regard. Its impact on model outcomes will become apparent in the results presented below, underscoring the intricate interplay between fundamental parameters in shaping our understanding of the cosmos.

Chapter 2

Composition of the Universe

The constituents of the universe are characterized by their state equations, which dictate their dilution properties.

2.1 Cold Matter

In cosmology, "cold matter" refers to matter with low thermal motion compared to the speed of light ($v_{\mathrm{th}} \ll c$) and negligible pressure compared to its energy density ($p \ll \rho c^2$). This type of matter is non-relativistic, meaning its kinetic energy is much smaller than its rest mass energy.

The state equation for cold matter simplifies to $p = 0$ due to its negligible pressure. This allows us to approximate the density term ρc^2 as the dominant factor in the Friedmann equations.

The evolution of cold matter density (ρ_{mat}) with respect to the scale factor ($R(t)$) is described by the equation:

$$\rho_{\mathrm{mat}} = \rho_{\mathrm{mat},0} \left(\frac{R_0}{R} \right)^3 \tag{2.1}$$

where:

- $\rho_{\mathrm{mat},0}$ is the present density of cold matter,
- R_0 is the present scale factor.

This equation indicates that the density of cold matter decreases as the universe expands (R increases). The R^{-3} dependence signifies that the volume of the universe increases cubically with expansion, leading to dilution of matter density.

In summary, cold matter, characterized by its low thermal motion and negligible pressure, contributes significantly to the mass density of the universe. Its density decreases with the expansion of the universe according to the R^{-3} dilution law.

2.2 Radiation

Radiation in cosmology refers to energy in the form of massless particles, typically photons, moving at the speed of light. Understanding the behavior of radiation involves considering its energy density (ρ_r) and pressure (p_r), which are related by its equation of state.

The equation of state for radiation is derived from thermodynamics and the relativistic effects of particles moving at the speed of light. It takes the form:

$$p_r = \frac{1}{3}\rho_r c^2 \tag{2.2}$$

This equation expresses the fact that radiation exerts pressure due to its momentum, and this pressure is related to its energy density. The factor of 1/3 arises from relativistic effects and is characteristic of radiation.

To understand how radiation evolves with the expansion of the universe, we consider its dilution properties. As the universe expands, the volume containing radiation also increases, leading to a decrease in its energy density. The dilution of radiation with the expansion of the universe is described by the equation:

$$\rho_r = \rho_{r,0}\left(\frac{R_0}{R}\right)^4 \tag{2.3}$$

Here, $\rho_{r,0}$ represents the present-day energy density of radiation, R_0 is the present-day scale factor, and R is the scale factor at any given time. This equation indicates that the energy density of radiation decreases as the inverse fourth power of the scale factor.

Combining the equation of state with the dilution equation allows us to track how the pressure and energy density of radiation evolve over cosmic time. This is crucial for understanding the role of radiation in the early universe, such as during the era of primordial nucleosynthesis and cosmic microwave background radiation.

2.3 Dark Matter

Dark matter is a hypothesized form of matter that does not emit, absorb, or interact with electromagnetic radiation, making it invisible and detectable only through its gravitational effects on visible matter. In cosmology, dark matter plays a crucial role in shaping the large-scale structure of the universe. Its existence is inferred from various astrophysical observations, such as the rotation curves of galaxies, gravitational lensing, and the dynamics of galaxy clusters.

In mathematical terms, the presence of dark matter is incorporated into the Friedmann equations, which describe the dynamics of the universe. The contribution of dark matter to the energy density of the universe (ρ_{DM}) is included as a separate term alongside other components such as radiation (ρ_r), baryonic matter (ρ_{b}), and dark energy (ρ_Λ).

The total energy density of the universe (ρ_{total}) is given by the sum of these components:

$$\rho_{\text{total}} = \rho_{\text{DM}} + \rho_r + \rho_{\text{b}} + \rho_\Lambda \tag{2.4}$$

In the Friedmann equations, the density parameter Ω_{DM} represents the ratio of the dark matter density to the critical density of the universe (ρ_{crit}), which determines the geometry of the universe:

$$\Omega_{\text{DM}} = \frac{\rho_{\text{DM}}}{\rho_{\text{crit}}} \tag{2.5}$$

where

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G} \tag{2.6}$$

Here, H represents the Hubble parameter, which quantifies the rate of expansion of the universe, and G is the gravitational constant.

The presence of dark matter affects the evolution of the universe by contributing to its overall mass density, influencing the rate of cosmic expansion and the formation of large-scale structures such as galaxies and galaxy clusters. While its exact nature remains elusive, dark matter remains a central puzzle in modern cosmology, prompting ongoing research efforts to uncover its properties and implications for our understanding of the cosmos.

2.4 Dark Energy, Vacuum Energy, and Quintessence

2.4.1 Dark Energy

Dark energy is a mysterious form of energy that drives the accelerated expansion of the universe. In the Friedmann equations, its presence is often attributed to a cosmological constant (Λ):

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \tag{2.7}$$

where:

- $\dot{a}$ is the time derivative of the scale factor $a(t)$,
- G is the gravitational constant,
- ρ is the total energy density of the universe,
- k is the curvature parameter representing the spatial curvature of the universe.

The term Λ represents the energy density of empty space and drives the accelerated expansion observed in the universe.

2.4.2 Vacuum Energy

Vacuum energy arises from quantum field theory, where the ground state of a quantum field possesses energy even in the absence of particles. It can be represented by the energy density ρ_{vacuum} and exerts negative pressure ($p = -\rho_{\text{vacuum}}$). In the Friedmann equations, vacuum energy contributes similarly to the cosmological constant:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{8\pi G}{3}\rho_{\text{vacuum}} \tag{2.8}$$

This equation reflects the dynamic influence of vacuum energy on the expansion of the universe.

2.4.3 Quintessence

Quintessence is a hypothetical form of dark energy characterized by a dynamic scalar field. Its dynamics are governed by an additional scalar field equation, typically represented by a potential $V(\phi)$. The energy density and pressure of quintessence can be expressed as:

$$\rho_{\text{quintessence}} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad \text{and} \quad p_{\text{quintessence}} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2.9}$$

In the Friedmann equations, the contribution of quintessence influences the overall dynamics of cosmic expansion alongside other components.

Chapter 3

Cosmological Parameters

3.1 Hubble Constant

The Hubble constant, denoted as H_0, represents the current rate of expansion of the universe. Mathematically, it is defined as the derivative of the scale factor $R(t)$ with respect to time t, evaluated at the present time t_0. Therefore, the Hubble constant is given by:

$$H_0 = \left(\frac{R'}{R}\right)_0 \tag{3.1}$$

where $R(t)$ is the scale factor representing the size of the universe as a function of time, and R' represents its derivative with respect to time. The subscript 0 indicates that the value is evaluated at the present time.

In simpler terms , the Hubble constant tells us how fast the universe is expanding at the present moment. It has units of inverse time, typically expressed in kilometers per second per megaparsec (km/s/Mpc), where a megaparsec (Mpc) is a unit of distance used in cosmology.

3.2 Deceleration Parameter

The deceleration parameter, denoted as q_0, measures the rate at which the expansion of the universe is slowing down. It is defined as follows:

$$q_0 = -\left(\frac{R''}{R}\frac{1}{(R')^2}\right)_0 \tag{3.2}$$

where:

- R represents the scale factor of the universe,
- R' and R'' represent the first and second derivatives of the scale factor with respect to time t, respectively,
- The subscript 0 denotes the present value.

The deceleration parameter q_0 can be interpreted as follows:

- If $q_0 > 0$, the expansion of the universe is decelerating, indicating a universe dominated by matter and/or radiation.
- If $q_0 < 0$, the expansion of the universe is accelerating, indicating the presence of a cosmological constant or dark energy driving the accelerated expansion.
- If $q_0 = 0$, the expansion of the universe is neither accelerating nor decelerating, implying a universe on the border between matter-dominated and dark energy-dominated eras.

The deceleration parameter q_0 provides valuable insights into the dynamics and evolution of the universe, shedding light on the dominant components influencing its expansion at the present epoch.

3.3 Density Parameters

Density parameters provide insights into the composition of the universe by comparing the density of various components to the critical density.

3.3.1 Critical Density

The critical value of density, denoted as $\rho_{\text{critical},0}$, represents the threshold density required for the universe to be geometrically flat. It is given by the expression:

$$\rho_{\text{critical},0} = \frac{3H_0^2}{8\pi G} \tag{3.3}$$

where H_0 is the Hubble constant and G is the gravitational constant. This critical density serves as a reference point for understanding the overall density of the universe.

3.3.2 Density Parameter for Matter

The density parameter Ω_{0m} quantifies the contribution of matter to the total density of the universe. It is defined as the ratio of the density of matter ($\rho_{\text{mat},0}$) to the critical density ($\rho_{\text{critical},0}$):

$$\Omega_{0m} = \frac{\rho_{\text{mat},0}}{\rho_{\text{critical},0}} \tag{3.4}$$

The value of Ω_{0m} indicates whether matter dominates the density of the universe ($\Omega_{0m} > 1$), or if the universe is relatively empty ($\Omega_{0m} < 1$).

3.3.3 Density Parameter for Radiation

Similarly, the density parameter Ω_{0r} represents the contribution of radiation to the total density of the universe. It is given by the ratio of the density of radiation ($\rho_{r,0}$) to the critical density ($\rho_{\text{critical},0}$):

$$\Omega_{0r} = \frac{\rho_{r,0}}{\rho_{\text{critical},0}} \tag{3.5}$$

The value of Ω_{0r} characterizes the significance of radiation in the early universe, with smaller values reflecting its diluted contribution over time.

3.3.4 Reduced Cosmological Constant

The reduced cosmological constant $\Omega_{0\Lambda}$ provides a measure of the energy density contributed by the cosmological constant (Λ). It is calculated as the ratio of Λc^2 to $3H_0^2$:

$$\Omega_{0\Lambda} = \frac{\Lambda c^2}{3H_0^2} \tag{3.6}$$

This parameter (mentioned by **freedman1998measuring**)helps evaluate the role of the cosmological constant in shaping the overall dynamics of the universe.

Chapter 4

Implementation

4.1 Importance of `odeint`

According to **dias2019selected**, the `odeint` function holds significant importance in solving ordinary differential equations (ODEs) numerically, particularly in scenarios where analytical solutions are either difficult or impossible to obtain. In the context of cosmological models, such as those described by the Friedmann equations (**faraoni1999solving**), `odeint` plays a crucial role in simulating the evolution of the universe over time. These equations involve complex interactions between various components of the universe, including matter, radiation, dark energy, and curvature. By utilizing `odeint`, researchers can efficiently integrate these differential equations, providing insights into the dynamics and fate of the universe under different cosmological scenarios.

Moreover, `odeint` offers flexibility and versatility in handling a wide range of differential equations, allowing for the incorporation of diverse physical parameters and initial conditions (**bovy2015galpy**). In the case of the provided equations, `odeint` enables the numerical solution of the Friedmann equations, accounting for the effects of matter, radiation, cosmological constant, and curvature on the evolution of the scale factor. This numerical approach not only facilitates the exploration of various cosmological models but also aids in comparing theoretical predictions with observational data. Ultimately, the use of `odeint` empowers researchers to conduct comprehensive analyses of the universe's evolution, shedding light on fundamental questions in cosmology and astrophysics.

4.2 Numerical Solving

Equations (1.1) and (1.2) involve the first and second derivatives of the scale factor (**dournac˙friedmann**). The objective is to combine these equations to formulate a second-order differential equation in matrix form, which can then be solved using the `odeint` function. To streamline the process, we introduce the normalized scale factor $y(t) = \frac{R(t)}{R_0}$.

$$y'' = -y\frac{H_0^2}{2}\left(\Omega_{0m}y^3 + 2\Omega_{0r}y^4 - 2\Omega_{0\Lambda}\right) \tag{4.1}$$

Similarly, equation (1.2) is represented by:

$$y = \Omega_{0m}\left(y'^2\frac{H_0^2}{2} - \Omega_{0r}y^2 - y^2\Omega_{0\Lambda} - \Omega_{0k}\right)^{-1} \tag{4.2}$$

Here, Ω_{0k} is defined as:

$$\Omega_{0k} = -\frac{kc^2}{R_0^2H_0^2} \tag{4.3}$$

We substitute the expression for y obtained from equation (4.2) into equation (4.1), resulting in:

$$y'' = -\frac{H_0^2}{2}\left(\Omega_{0m}y^3 + 2\Omega_{0r}y^4 - 2\Omega_{0\Lambda}\right)\Omega_{0m}\left(y'^2\frac{H_0^2}{2} - \Omega_{0r}y^2 - y^2\Omega_{0\Lambda} - \Omega_{0k}\right)^{-1} \tag{4.4}$$

Furthermore, by using equation (4.2), we can derive the relationship:

$$\Omega_{0m} + \Omega_{0r} + \Omega_{0\Lambda} + \Omega_{0k} = 1 \tag{4.5}$$

This equation is pivotal in determining the value of Ω_{0k} as a function of Ω_{0m}, $\Omega_{0\Lambda}$, and Ω_{0r}. The subsequent equation aids in interpreting the results, particularly regarding the sign of the deceleration parameter q_0:

$$q_0 = \frac{1}{2}\sum_i(1 + 3w_i)\Omega_{0i} \tag{4.6}$$

By considering the contributions from matter (with $w = 0$), radiation ($w = \frac{1}{3}$), and the cosmological constant ($w = -1$), we can simplify to:

$$2q_0 = \Omega_{0m} + 2\Omega_{0r} - 2\Omega_{0\Lambda} \tag{4.7}$$

Using the above equations as in Appendix A yields in graphs as shown in 5

Chapter 5

Results

5.1 Hyperbolic geometry (k=1)

In hyperbolic geometry expansion of the universe (**schmidt2012nobel**), represented by $k = -1$, the graph illustrates a distinctive curvature characteristic. Unlike flat or positively curved geometries, hyperbolic geometry exhibits a negative curvature, indicative of an open universe. This geometric configuration suggests that the universe's spatial extent is infinite, expanding indefinitely over time.

The cosmological parameters influencing this hyperbolic expansion scenario are denoted by Ω_{1m}, Ω_{1r}, Ω_{1l}, and Ω_{1k}. Specifically, Ω_{1m} represents the density parameter for matter, while Ω_{1r} denotes the density parameter for radiation. Ω_{1l} signifies the density parameter associated with the cosmological constant or dark energy. The parameter Ω_{1k} is attributed to the spatial curvature of the universe. In this particular case, Ω_{1m} is 0.3, Ω_{1r} is 1×10^{-4}, Ω_{1l} is 0.0, and Ω_{1k} is derived from the equation $1-\Omega_{1m}-\Omega_{1r}-\Omega_{1l}$. These parameters collectively influence the overall dynamics and geometry of the expanding universe within a hyperbolic framework.

The plotted graph 5.1 showcases the eternal expansion of the universe within a hyperbolic geometry framework. Despite the boundless expansion, the rate of this expansion gradually diminishes as cosmic time progresses. This deceleration phenomenon is quantified by the deceleration parameter q_0, which in this scenario is approximately 0.15. A positive value of q_0 signifies deceleration, indicating that the cosmic expansion is gradually slowing down over time.

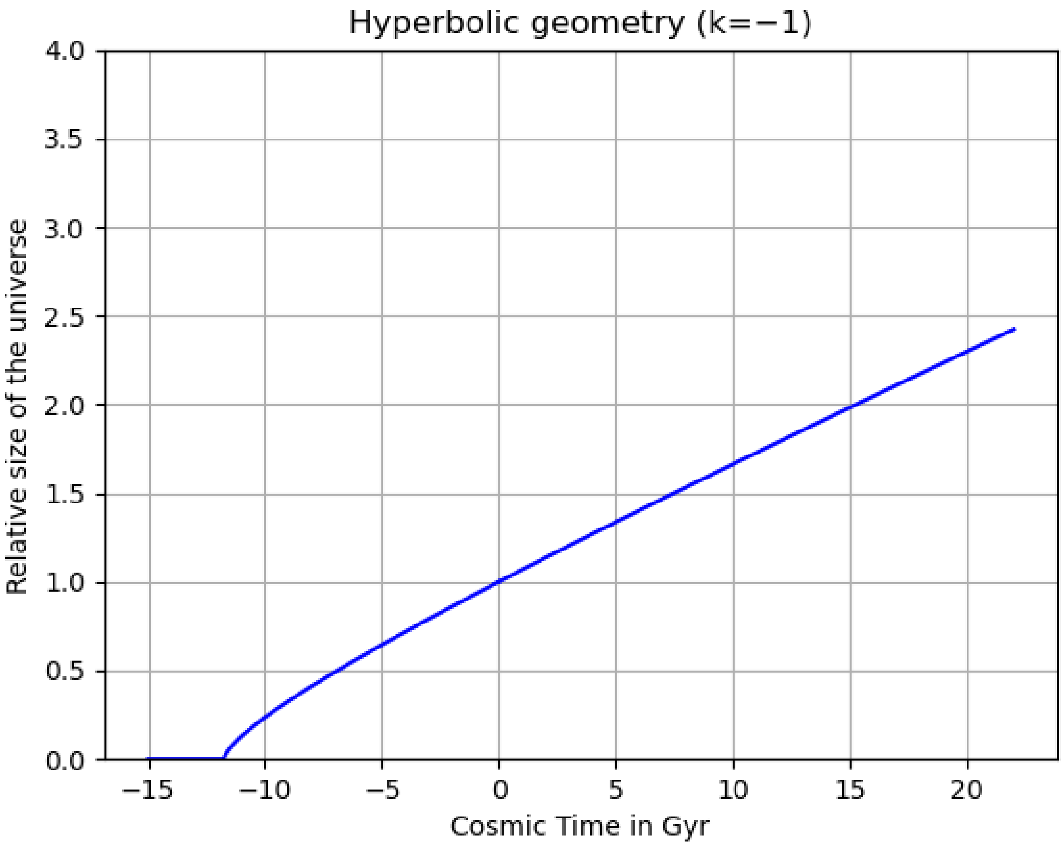

Figure 5.1: Hyperbolic geometry

5.2 The Collapsing Universe

The "Big Crunch" is a theoretical scenario in cosmology that describes the possible fate of the universe if it is closed and finite in extent, with positive curvature (k=1). In this scenario, the expansion of the universe, which began with the Big Bang, is not eternal. Instead, the gravitational attraction between matter and energy eventually overcomes the outward momentum of the expansion, causing the universe to cease its expansion and start collapsing in on itself (**steinhardt2002cyclic**).

As the collapse progresses, galaxies, stars, and other cosmic structures would be drawn closer together due to gravitational forces. The universe would become increasingly dense and hot as matter and energy are compressed into a smaller volume. Eventually, this process would lead to a singularity—a point of infinite density and temperature—similar to the conditions at the moment of the Big Bang.

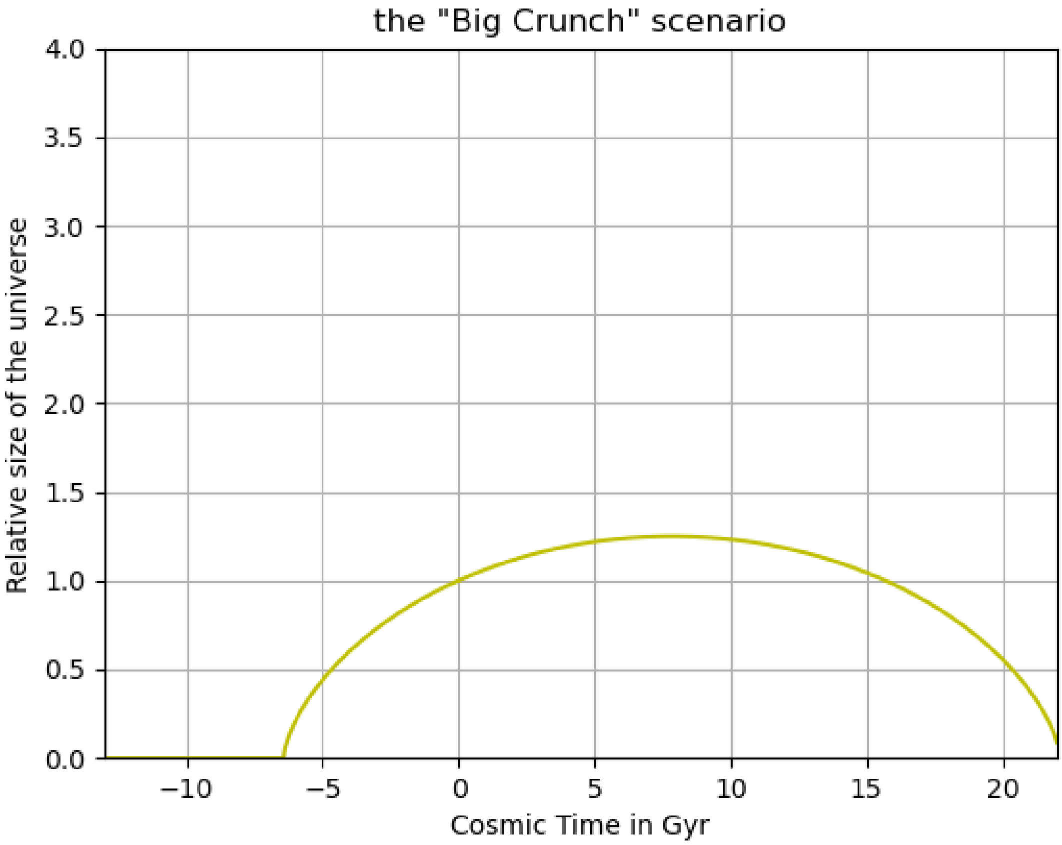

Figure 5.2: The Collapsing Universe (The Big crunch)

The concept of the Big Crunch is based on solutions to Einstein's equations of general relativity, which describe how matter and energy warp the fabric of spacetime. These solutions suggest that the fate of the universe depends on its overall density, the distribution of matter and energy, and the geometry of spacetime.

The Big Crunch scenario stands in contrast to other possible fates of the universe, such as eternal expansion (in the case of a flat or negatively curved universe) or the "Big Freeze" or "Heat Death" (where the universe continues expanding indefinitely and eventually becomes cold and dark).

The curve depicted in the graph 5.2 represents a spherical universe, characterized by positive curvature (k=1). Initially, this universe undergoes a phase of expansion, consistent with standard cosmological models. However, unlike scenarios where expansion continues indefinitely, this model predicts a subsequent contraction phase known as the "Big Crunch." During the Big Crunch, the universe collapses inward under the influence of gravitational forces, potentially leading to a singularity or a highly dense state.

The parameters Ω_{3m}, Ω_{3r}, Ω_{3l}, and Ω_{3k} represent the densities of matter, radiation, dark energy (cosmological constant), and curvature, respectively. In the context of the Big Rip scenario, the dominance of dark energy $\Omega 3_l$ over matter and radiation accelerates the universe's expansion, leading to its eventual demise in a cosmic tear known as the Big Rip.

5.3 Einstein-de Sitter model

The Einstein-de Sitter model is a cosmological model proposed by Albert Einstein and Willem de Sitter in 1932. It describes as in **janssen2016einstein**, a homogeneous and isotropic universe based on Einstein's theory of general relativity and assumes that the universe is filled with matter but devoid of any cosmological constant (i.e., $\Lambda = 0$). This model represents a spatially flat ($k = 0$) universe with quasi-Euclidean geometry ($\Omega_{0k} \approx 0$) and a zero cosmological constant ($\Omega_{0\Lambda} = 0$).

In the Einstein-de Sitter model, the density of matter (Ω_{4m}) exactly matches the critical density ($\Omega_{4m} = 1$), defining the characteristic features of the model. This condition ensures that the universe's expansion is precisely balanced between continued expansion and gravitational attraction, resulting in an eternal but decelerating expansion ($q_0 = 0.5$).

The graph in 5.3 depicted in the figure corresponds to the Einstein-de Sitter model, illustrating its essential characteristics. The model predicts an eternal expansion of the

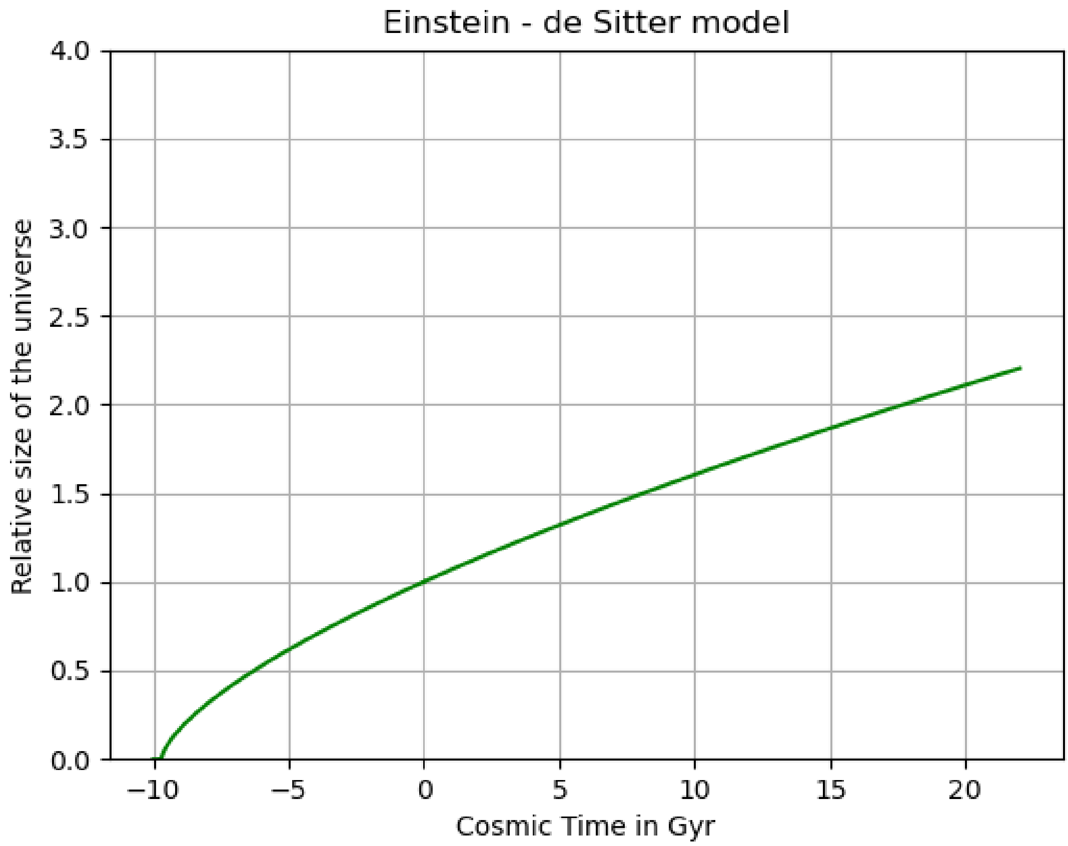

Figure 5.3: Einstein de-Sitter model

universe with a gradual decrease in the expansion rate over time due to the influence of matter. While the Einstein-de Sitter model provided valuable insights into the dynamics of the universe, subsequent observational data, particularly from studies of type Ia supernovae and the cosmic microwave background radiation, has suggested that the actual universe may exhibit a more complex behavior. Nonetheless, the Einstein-de Sitter model remains a cornerstone in cosmology and has significantly contributed to our understanding of the universe's evolution and structure.

5.4 Lambda Cold Dark Matter (ΛCDM) model

The Lambda Cold Dark Matter (ΛCDM) model is the prevailing cosmological model used to describe the evolution and structure of the universe. It combines elements of the Big Bang theory with the framework of Einstein's general theory of relativity. In this model, the universe is assumed to be homogeneous and isotropic on large scales, following the cosmological principle.

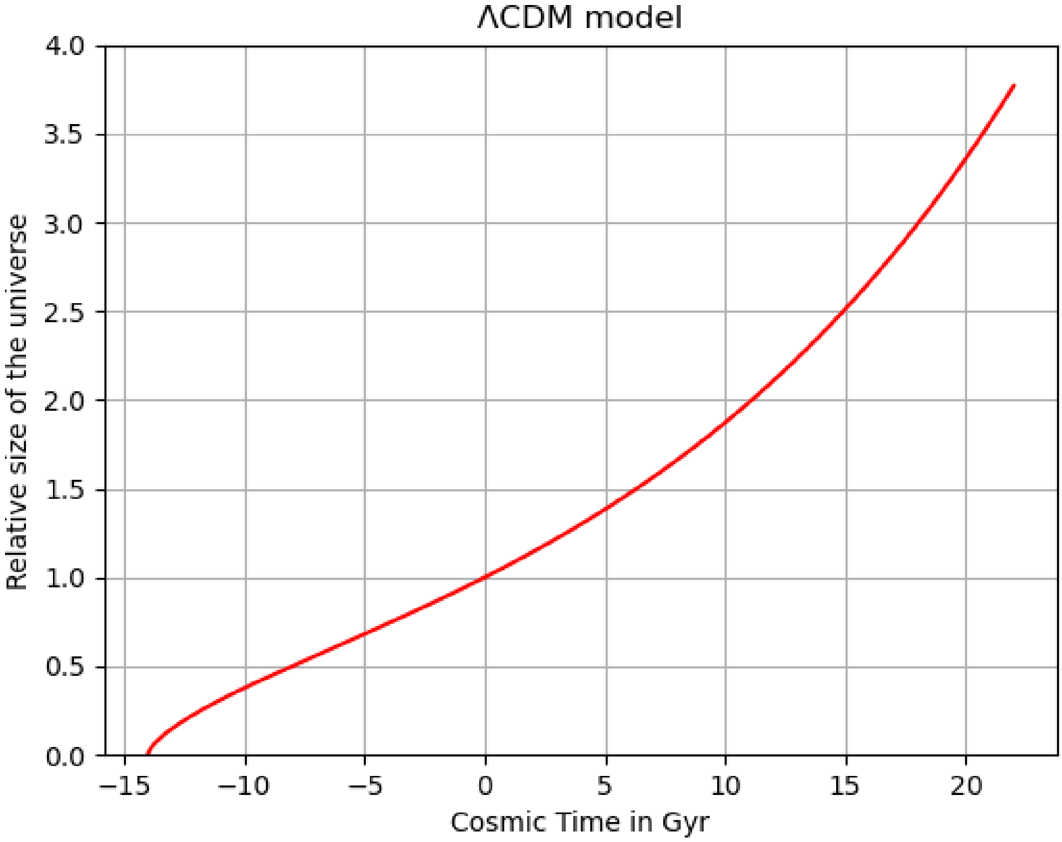

Figure 5.4: (ΛCDM) model

The term "Lambda" refers to the cosmological constant (Λ), a concept introduced by Albert Einstein to counterbalance the gravitational attraction of matter and achieve a static universe. In the ΛCDM model, the cosmological constant represents a form of dark energy distributed uniformly throughout space. This dark energy is associated with a negative pressure that causes the expansion of the universe to accelerate over time.

Additionally, the "Cold Dark Matter" component of the model refers to a type of dark matter characterized by particles moving at non-relativistic speeds (hence "cold")(**mavromatos2022lambda**). Dark matter is hypothesized to interact only through gravity and does not emit, absorb, or reflect electromagnetic radiation, making it invisible and detectable only through its gravitational effects.

In this model, the density parameter for matter (Ω_{0m}) is approximately 0.3, indicating that matter contributes significantly to the universe's total energy density. Conversely, the density parameter for dark energy ($\Omega_{0\Lambda}$) is approximately 0.7, suggesting that dark energy dominates the universe's energy budget.

The nearly flat geometry ($\Omega_{0k} \approx 0$) of the universe, as indicated by observational data, aligns with the predictions of the ΛCDM model. This quasi-Euclidean geometry implies that the universe's spatial curvature is close to zero, reflecting a near-perfectly flat universe on large scales. Such a geometry is consistent with observations and theoretical expectations, supporting the validity of the ΛCDM model.

In the context of the ΛCDM model, the deceleration parameter (q_0) is estimated to be approximately -0.55. This negative value suggests that the universe's expansion is not only continuing but also accelerating over time. This acceleration, driven primarily by dark energy, has profound implications for the fate of the universe. In particular, the accelerating expansion could lead to a scenario known as the "Big Rip," where the universe expands at an ever-increasing rate as shown in graph 5.4. Eventually, this expansion could become so rapid that it tears apart astrophysical objects and even fundamental particles, resulting in the dislocation of structures on cosmic scales.

The specific cosmological parameters (Ω_{2m}, Ω_{2r}, Ω_{2l}, Ω_{2k}) associated with this scenario are as follows: $\Omega_{2m} = 0.3$, $\Omega_{2r} = 1 \times 10^{-4}$, $\Omega_{2l} = 0.7$, and $\Omega_{2k} = 1 - \Omega_{2m} - \Omega_{2r} - \Omega_{2l}$. These parameters collectively define the energy density contributions from matter, radiation, dark energy, and spatial curvature, respectively, within the ΛCDM framework.

Chapter 6

Conclusion

6.1 Conclusions

In conclusion, the Friedmann equations serve as fundamental tools in unraveling the structure and evolution of the universe (**lemaitre1934evolution**). These equations, rooted in Einstein's theory of general relativity, encapsulate the dynamics of cosmic expansion and provide valuable insights into the cosmos's intricate workings. By numerically solving the Friedmann equations, using powerful tools like scipy's `odeint` function, researchers can explore and analyze diverse cosmological scenarios with unprecedented accuracy and efficiency.

The significance of `odeint` cannot be overstated in this context. As a robust numerical solver, `odeint` enables the precise computation of complex differential equations, allowing scientists to model and simulate various cosmological models accurately. Its versatility and reliability make it an indispensable tool in modern cosmological research, facilitating the exploration of vast expanses of theoretical space and driving advancements in our understanding of the universe.

In this study, we examined four distinct cosmological models, each offering unique insights into the universe's behavior. From the hyperbolic geometry, indicative of an open and eternally expanding universe, to the collapsing universe scenario, portraying a cyclical pattern of expansion and contraction, we explored a diverse array of cosmic possibilities. The Einstein-de Sitter model, representing a critical density universe on the brink of collapse, and the Lambda Cold Dark Matter (ΛCDM) model, embodying the prevailing cosmological paradigm with its blend of dark energy and dark matter, further enriched our understanding of cosmic evolution.

Through meticulous numerical simulations and data visualization, we showcased the intricate interplay of cosmological parameters and their profound implications for the universe's fate. These endeavors underscore the indispensable role of numerical methods in unraveling the mysteries of the cosmos and advancing our comprehension of the universe's grand tapestry. Indeed, numerical methods stand as invaluable tools in the scientist's toolkit, facilitating the exploration and understanding of various scientific phenomena across disciplines.

6.2 Future Work

Looking ahead, there are several avenues for future exploration and refinement in our study of cosmological models using numerical methods:

- Continual advancements in numerical methods, including enhancements to existing solvers like `odeint`, could further improve the accuracy and efficiency of cosmological simulations. Exploring alternative numerical techniques and algorithms may uncover novel approaches for tackling complex cosmological scenarios.
- While we considered four cosmological models in this study, there exist numerous other theoretical frameworks and scenarios worthy of investigation (**peebles1994evolution**). Future research could delve into exotic models, such as those incorporating modified gravity theories or alternative forms of dark energy, expanding the scope of our understanding of cosmic evolution.
- Integrating observational data from ongoing and upcoming cosmological surveys, such as the Dark Energy Survey and the Vera C. Rubin Observatory's Legacy Survey of Space and Time, could enhance the fidelity of our simulations. Comparing numerical predictions with observational constraints can provide valuable validation and refinement of theoretical models.
- Investigating the sensitivity of cosmological models to variations in key parameters, such as the Hubble constant, matter density, and dark energy equation of state, could shed light on the robustness of current theoretical frameworks. Sensitivity analyses can identify critical parameters that significantly impact model predictions and guide future observational efforts.

By pursuing these avenues for future work, researchers can advance our understanding of cosmology, refine theoretical models, and uncover new insights into the nature and evolution of the cosmos.

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Appendix A

Python Code

- The code imports the necessary libraries:
  - numpy for numerical computations,
  - odeint from scipy.integrate for solving ordinary differential equations (ODEs), and
  - matplotlib.pyplot for plotting.
- The actual Hubble value (H0), starting and ending times , and initial Omega parameters are defined.

```
# Actual Hubble value (m.Gyr^-1)
H0 = 67400 / (3.09 * 1e22) * (3600 * 24 * 365 * 1e9)

# Starting and ending time
t_begin = [-1.0e-5, 1e-5]
t_final = [-9, 22]

# Initial Omega parameters for different models

Omega1_m = 0.3
Omega1_r = 1e-4
Omega1_l = 0.0
Omega1_k = 1 - Omega1_m - Omega1_r - Omega1_l


Omega2_m = 0.3
Omega2_r = 1e-4
Omega2_l = 0.7
Omega2_k = 1 - Omega2_m - Omega2_r - Omega2_l

```

```
            Omega3_m = 5.0
            Omega3_r = 1e-4
            Omega3_l = 0.0
            Omega3_k = 1 - Omega3_m - Omega3_r - Omega3_l


            Omega4_m = 1.0
            Omega4_r = 1e-4
            Omega4_l = 0.0
            Omega4_k = 1 - Omega4_m - Omega4_r - Omega4_l
```

- Define a function hubble to represent the system of differential equations. It takes the current state y, time t, and Omega parameters as arguments and returns the derivatives of the scale factor.

```
            def hubble(y, t, Omega_m, Omega_r, Omega_l, Omega_k):
                # Actual Hubble value
                H0 = 67400 / (3.09 * 1e22) * (3600 * 24 * 365 * 1e9)
                return [y[1],(-(H0**(2)/2)*(Omega_m/y[0]**(3)+2*Omega_r/y
                    [0]**(4)-2*Omega_l))*Omega_m*(1/(H0**(2))*y[1]**(2)-Omega_r
                    /y[0]**(2)-Omega_l*y[0]**(2)-Omega_k)**(-1)]
```

- Setting Initial Conditions: Initial conditions for the scale factor and its derivative are set based on the chosen Omega parameters.

```

        # Initial scale factor and derivated scale factor for Model 1
        a1_0 = 1.0
        a1_prim_0 = H0 * ((Omega1_m/a1_0) + (Omega1_r/(a1_0 ** 2)) + Omega1_k +
             Omega1_l * (a1_0**2))**(0.5)
        init1 = [a1_0, a1_prim_0]

        # Initial scale factor and derivated scale factor for Model 2
        a2_0 = 1.0
        a2_prim_0 = H0 * ((Omega2_m/a2_0) + (Omega2_r/(a2_0 ** 2)) + Omega2_k +
             Omega2_l * (a2_0**2))**(0.5)
        init2 = [a2_0, a2_prim_0]

        # Initial scale factor and derivated scale factor for Model 3
        a3_0 = 1.0
        a3_prim_0 = H0 * ((Omega3_m/a3_0) + (Omega3_r/(a3_0 ** 2)) + Omega3_k +
             Omega3_l * (a3_0**2))**(0.5)
        init3 = [a3_0, a3_prim_0]

        # Initial scale factor and derivated scale factor for Model 4
        a4_0 = 1.0
```

```
    a4_prim_0 = H0 * ((Omega4_m/a4_0) + (Omega4_r/(a4_0 ** 2)) + Omega4_k + Omega4_l * (a4_0**2))**(0.5)
    init4 = [a4_0, a4_prim_0]
```

- The `odeint` function is employed to solve the differential equations for each set of Omega parameters (Omega1_m, Omega1_r, etc.). The solutions are computed for both forward and backward time intervals. The resulting scale factor solutions are stored in $y1$, $y2$, $y3$, and $y4$.
- Plots the scale factor against cosmic time for the set of Omega parameters.

```
for i in range(2):
    # Differential systems solving
    t1 = np.linspace(t_begin[i], t_final[i], 5000)
    t2 = t1.copy()
    t3 = t1.copy()
    t4 = t1.copy()
    y1 = odeint(syseq, init1, t1, args=(Omega1_m, Omega1_r, Omega1_l, Omega1_k), rtol=1e-10, atol=1e-10)
    #y2 = odeint(syseq, init2, t2, args=(Omega2_m, Omega2_r, Omega2_l, Omega2_k), rtol=1e-10, atol=1e-10)
    #y3 = odeint(syseq, init3, t3, args=(Omega3_m, Omega3_r, Omega3_l, Omega3_k), rtol=1e-10, atol=1e-10)
    #y4 = odeint(syseq, init4, t4, args=(Omega4_m, Omega4_r, Omega4_l, Omega4_k), rtol=1e-10, atol=1e-10)
    plt.figure(1)
    plt.plot(t4, y4[:, 0], 'y')

# Plot settings
plt.xlabel('Cosmic Time in Gyr')
plt.ylabel('Relative size of the universe')
# Add the title according to the model
plt.title('Hyperbolic geometry (k=1)')
plt.grid(True)
plt.ylim([0, 4])
# Save the plot as a PNG file
plt.savefig('hyperbolic_geometry(k=1).png')
plt.show()
```